On some portfolio optimisation problems under pathwise constraints

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Chapter 1

Introduction

Portfolio optimisation theory is a well-established field of research, which originates from the fundamental results in Markowitz's mean-variance theory and the theory of expected utility. Alongside the thriving sophisticated mathematical problems arising in the field, it also provides a natural economical interpretations of the bahaviour of a portfolio maximiser. One of the main features of investor's behaviour is a degree of her so-called risk aversion that measures to which extent investor is keen on accepting risks in a favour of extra premium. The risk aversion naturally suits the portfolio theory as investor is said to maximise her expected utility of terminal wealth where the utility function is assumed to be concave and nondecreasing. More precisely, for an investor with smooth utility function U we calculate her risk aversion as

$$r(x) := -\frac{xU''(x)}{U'(x)}, \quad x > 0.$$

The criterion under discussion appears as follows:

$$\text{maximise\_over\_}V \quad \mathbb{E}U(V_T),$$

where V_T is the investor's wealth process at time T.

The main critique of the expected utility criterion lays in its time-inconsistency cf. Guasoni and Robertson [28]. As the optimal strategy depends on the time horizon T (unless $r(x) \equiv 0$) it is not possible to track the optimal strategy dynamically as time passes. Instead, one can look at distant time horizons and analyse an asymptotically optimal strategy of a risk-averse investor. One of the tractable criteria being used to understand and model the bahaviour of an investor on long horizons is an asymptotic growth rate of expected utility of wealth criterion, which is defined as follows:

$$\text{maximise\_over\_}V \quad \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}U(V_T).$$

Inline with [12, 25] we consider maximisation of the asymptotic growth rate of the expected utility of wealth. The idea to look at the long-run optimality has proven to be a powerful tool in solving various portfolio optimisation problems explicitly. It was used by Grossman and Vila [24] in a problem with leverage constraints and by Dumas and Luciano [15] in presence of transaction costs. It is close to the objectives studied in the **risk-sensitive control**, see Section 3.6 of Chapter 3. We refer to Guasoni and Robertson [28] for a more detailed discussion.

Another alternative criterion is to look at the optimal consumption-investment strategies and maximise the lifetime expected utility of consumption. There is an extensive literature on the problem since the seminal papers by Merton [44, 45]. The problem was considered for an incomplete market model by Lehoczky, Karatzas and Shreve and Xu [35], Cox and Huang [9].

1.1 Constrained optimisation and Azéma-Yor processes

The constraints within the portfolio optimisation problems appear in various forms depending on the setup of the problem. One of the well-analyzed classes of constraints is a convex portfolio constraint, when the proportion of the wealth in risky stocks takes its values in a given closed, convex subset of $\mathbb{R}^d$. In paper by Cvitanic and Karatzas [11] the duality result was obtained representing the optimal constrained portfolio via maximiser in auxiliary unconstrained problem. For an incomplete market results were obtained in a special case by Karatzas, Lehoczky, Shreve and Xu [35].

Another class of constraints are pathwise constraints. They are generally harder to deal with as the duality in a sense of Follmer and Kramkov (see [23]) does not seem to work. One well-known in the literature constraint is a drawdown constraint. It is typically imposed on the wealth process and requires it to be always above a given function of its up-to-date maximum. In other words, for wealth process $(V_t)_{t\geq 0}$ and a given function $w(x)$ we assume

$$V_t \geq w(\sup_{0\leq s\leq t} V_s), \quad t \geq 0.$$

The classical example of a drawdown function is a linear drawdown, i.e. $w(x) = \alpha x$ for some $0 < \alpha < 1$.

The drawdown constraint appears in portfolio optimisation theory starting from the seminal paper by Grossman and Zhou [25] who found the maximum of the growth rate of the expected utility subject to a linear drawdown constraint. Their result was based on the forward approach and the solution was obtained by solving the corresponding dynamic programming equation. Later this result was generalised and, moreover, simplified in [12] where the probabilistic arguments were used to connect the problem with an auxiliary unconstrained one. The main contribution of our work is that in Chapter 3 the equivalence result is obtained for a general semimartingale market, general utility function U and general drawdown function w. Essentially, it is proved that the value function of the drawdown asymptotic constrained problem with utility function U, drawdown function w is equal to the rate of growth of the expected utility without constraint but for a modified utility function which is given explicitly in terms of U and w. The optimal wealth processes are also linked explicitly.

The main tool for obtaining the stated result is in a use of the so-called Azéma-Yor processes, which were introduced in [2]. The Azéma-Yor processes are extensively studied in the recent paper by Carraro, El Karoui and Obłój [5]. In this paper the connection is built between the class of non-negative semimartingales and the class of semimartingales which satisfy the drawdown constraint.

Another pathwise constraint that naturally appears in financial mathematics is a so-called floor constraint which requires the wealth process of an investor to be greater or equal than a prespecified process. The floor process is referenced in literature in some cases as an American capital guarantee (see, for example, [18]). This constraint has practical meaning and is observed on the real market in the form of different insurance products which guarantee investor's wealth to dominate a certain floor process. The final horizon problem of maximising utility of wealth under a floor constraint was studied in [18]. The core result of that paper is in linking the optimal wealth process with the Max-Plus decomposition of a supermartingale (see [38] for more details). For the special case of a floor process the solution is an Azéma-Yor process.

In Chapter 4 we extend the result from [18] but for the incomplete market model with price deflators with a restriction on the class of deflators. In the general setting we also solve the growth rate of the expected utility problem subject to a floor constraint in the same chapter. The result appears to be intriguing: the constraint does not affect the optimal rate of growth. By the moment this result was obtained we became aware of the paper by J. Sekine [53] with similar results for the diffusion driven market model and for a power utility function.

1.2 Utility of consumption subject to drawdown constraint

The problem of maximising lifetime utility of consumption in the presence of constraints was considered in seminal papers. In [11] the convex constraint was imposed on the wealth process. The problem was extended by Hua and Pages in [30], El Karoui, Jeanblanc in [17] to account for a labour income. The case of the proportional transaction costs was considered in [1, 13, 43].

The case of a linear drawdown constraint was considered by Roche [51]. The solution was obtained heuristically. The more recent paper by Elie and Touzi [20] examined the case of a zero interest rates but for a fairly general utility function. By means of viscosity theory and by solving the dynamic programming equation the solution was obtained in a closed form.

Chapter 5 deals with the lifetime consumption subject to the drawdown constraint. The contribution of the result with respect to [20] is that the drawdown function is not necessary linear. By means of Azéma-Yor processes an equivalent problem is derived to the initial one. The latter is tackled in a similar way as [20] except for several proofs of lemmas and propositions.

1.3 Comments

Chapters 3 to 5 are intended to be standalone papers with the only change that material covered in Chapter 2 will be recalled in each paper. In addition, Chapter 3 was submitted to Finance and Stochastics and is in the second stage of the review.

Chapter 2

Azéma–Yor processes

To be able to formulate our results we need to introduce the so–called Azéma–Yor processes and recall their properties established in Carraro, El Karoui and Obłój [5]. This will equip us with the tools necessary to relate Problems 1, 2 and their solutions in an explicit manner. We specialise here to the context of the present work and refer to [5] for the general statements.

Proposition 2.0.1 (Carraro, El Karoui and Obłój [5])**.** *Let F' be a locally bounded function, $F(x) = F(x_0) + \int_{x_0}^{x} F'(u)\mathrm{d}u$, and (X_t) a max-continuous $(\mathcal{F}_t)$–semimartingale. The associated Azéma–Yor process $M^F(X)$ is given via*

$$M_t^F(X) := F(\overline{X}_t) - F'(\overline{X}_t)(\overline{X}_t - X_t) = F(X_0) + \int_0^t F'(\overline{X}_u)\mathrm{d}X_u, \ t \geq 0, \tag{2.1}$$

where $\overline{X}_t := \sup_{u \leq t} X_u$. Further

- *if $F' \geq 0$ then $\overline{M_t^F(X)} = F(\overline{X}_t)$, $t \geq 0$,*
- *if $F' > 0$ then $M^K(M^F(X)) = X$ with $K = F^{-1}$ the inverse of F,*
- *if F is concave then $M_t^F(X) \geq F(X_t)$, $t \geq 0$.*

Proof. The equivalence of definitions in (2.1) can be established for smooth enough function F' by Ito's formula, noticing that $\int_0^t (\overline{X}_t - X_t) dF'(\overline{X}_t)$. This can be generalised to all bounded F' by the monotone class theorem. Indeed, let $\mathcal{S}$ be a π-system generating $\mathcal{F}$, then (2.1) holds for $F'(x) = \mathbf{1}_{[a,b]}(x)$; let $F_n'(x)$ be a sequence of functions increasing pointwise to a bounded function $F'(x)$ and (2.1) holds for F_n', then by the monotone convergence theorem (2.1) also holds for F'. Finally, by localisation argument (2.1) can be extended to all locally bounded F'.

The first property from the Proposition follows by noticing that $F(\overline{X}_t) - F'(\overline{X}_t)(\overline{X}_t - X_t)$ achieves its maximum on $X_t = \overline{X}_t$ for a fixed $\overline{X}_t$.

Second property can be obtained by a direct calculation.

And the last property is obtained by noticing that $F(y) - F(x) \geq F'(y)(y-x)$ for $y \geq x \geq 0$ as F is a concave function. $\square$

The above combines Definition 2.1 and Proposition 2.2 in [5] while the last property is clear (see also Proposition 4.12 point c) in [5]). For our purposes, the crucial property of Azéma–Yor processes is that they automatically satisfy a drawdown property. In fact from (2.1), when $X \geq 0$ and $F' \geq 0$, we see that $M^F(X)$ satisfies w-DD, i.e. $M_t^F(X) \geq w(\overline{M^F(X)}_t)$ for $t \geq 0$ with $w(x) = x - K(x)/K'(x)$, $K := F^{-1}$. Crucially, we can start with w, solve the ODE for K and hence find the suitable F.

Proposition 2.0.2 (Carraro, El Karoui and Obłój [5]). *Let $w : [a^*, \infty] \to \mathbb{R}$ be a nondecreasing function such that $y - w(y) > 0$ is locally bounded and locally bounded away from zero for $y \geq a^*$. Define*

$$K(x) := a \exp\left(\int_{a^*}^{x} \frac{1}{u - w(u)} \mathrm{d}u\right), x \geq a^* > 0, \tag{2.2}$$

which is continuous and strictly increasing and has a well defined inverse $F := K^{-1} : [a, \infty) \to [a^*, \infty)$. For a nonnegative max-continuous semimartingale $(X_t), X_0 = a$ the drawdown equation

$$dY_t = (Y_{t-} - w(\overline{Y}_t))\frac{dX_t}{X_{t-}}, \quad t \geq 0, \tag{2.3}$$

has a strong, pathwise unique, max-continuous solution which satisfies $w-DD$ constraint and $Y_0 = a^*$, given by $Y_t = M_t^F(X)$, where F is an inverse of K.

Conversely, given (Y_t) a max-continuous semimartingale satisfying $w-DD$ constraint with $Y_0 = a^*$, there exists a pathwise unique max-continuous semimartingale (X_t), $X_0 = a$, which solves (2.3).

Finally, K is convex and F is concave.

Proof. Properties of K follow by a straightforward differentiation. Observe that as $x - w(x) > 0$ is bounded and bounded away from zero we obtain $K(x) < \infty$ for all $x > a^*$ and $K(\infty) = \infty$ so that the inverse F is well defined on $[a,\infty)$ and $F(\infty)=\infty$. The process $Y=M^F(X)$ is well defined and max-continuous since $X_t\geq 0$. We apply Theorem 3.4 in [5] with $\zeta:=\infty$. We conclude that Y satisfies the required w–drawdown property for all $t\geq 0$ and (2.3). In particular $\min\{Y_t,Y_{t-}\}>0$.

Likewise, for the converse, Y satisfies the w-DD condition for all times and hence we apply Theorem 3.4 in [5] with $\zeta=\infty$.

Finally, the last statement is clear since when w is non-decreasing then, by direct differentiation, so is K'. □

By (2.1), in the above it is sufficient to consider $F(x)$ for $x\geq a$ since $\overline{X_t}\geq X_0=a$. We are free to define F on $[0,a)$ in any way without affecting Y_t. For completeness we specify one such extension by extending F for all positive x as follows

$$F(x):=\begin{cases} K^{-1}(a) & \text{if } x\geq a \\ F'(a+)(x-a)+a & \text{if } 0\leq x<a \end{cases} \tag{2.4}$$

so that $F(0)=w(F(a))=w(a^*)>0$ and F is increasing and concave on $[0,\infty)$ if w is nondecreasing. We write K_w,F_w when we want to stress the dependence on the drawdown function w.

Example 2.0.1. *Consider the linear drawdown constraint $w(x)=\alpha x$, $\alpha\in(0,1)$. Then $K(v)=v_0(v/v_0)^{1/(1-\alpha)}$ and $F(x)=v_0(x/v_0)^{1-\alpha}$. For $X\geq 0$ max-continuous semimartingale, Proposition 2.0.2 gives us that $Y:=M^F(X)$ satisfied $w-DD$ constraint and an explicit calculation using* (2.1) *gives*

$$Y_t=M_t^F(X)=v_0^{\alpha}\left(\alpha\overline{X}_t^{1-\alpha}+(1-\alpha)\overline{X}_t^{-\alpha}X_t\right),\ t\geq 0,$$

with an analogous expression for X in terms of Y. More precisely,

$$X_t=M_t^K(Y)=\frac{v_0^{-\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(X_t-\alpha\overline{X}_t\right)\overline{X}_t^{-\frac{\alpha}{1-\alpha}}$$

Remark 2.0.1. *The main statement in Proposition 2.0.2 may be reformulated as saying that the mapping $X\to Y:=M^F(X)$ is a bijection between the class of nonnegative semimartingales and the class of semimartingales which satisfy $w-DD$ constraint, and the inverse is given by $X\to M^K(Y)$.*

Remark 2.0.2. *We consider here, similarly to [25, 12], drawdown constraint imposed on the discounted wealth D_tX_t, where D_t is a discount factor. In practice this may be adequate if the so-called hurdle rate is* *present, see [26]. However in many situations one is interested in avoiding drawdowns for the actual wealth process* Y_t*. Suppose for simplicity that all the assets are continuous and consider a positive wealth process* X*. Using our methodology we could consider* $Y = M^F(X)$ *which would indeed satisfy the w-DD constraint, but would not be a wealth process of a self-financing portfolio.*

More specifically, assume $D_t = \exp(-\int_0^t R_u \mathrm{d}u)$ *for some positive adapted process* R*. Then,*

$$\begin{aligned}
\mathrm{d}\left(D_t M_t^F(V)\right) &= D_t F'(\overline{V}_t)\mathrm{d}V_t + M_t^F(V)\mathrm{d}D_t \\
&= F'(\overline{V}_t)\mathrm{d}(D_t V_t) + \left(F(\overline{V}_t) - F'(\overline{V}_t)\overline{V}_t\right)\mathrm{d}D_t \\
&= -R_t\left(F(\overline{V}_t) - F'(\overline{V}_t)\overline{V}_t\right)D_t\mathrm{d}t + F'(\overline{V}_t)\sum_{i=1}^d \pi_t^i \mathrm{d}(D_t S_t^i).
\end{aligned}$$

This provides the dynamics of a wealth process of a consumption and investment strategy, where the rate of instantaneous consumption is $R_t\left(F(\overline{V}_t) - F'(\overline{V}_t)\overline{V}_t\right)$*.*

The above calculation suggests that drawdown constraints imposed on the undiscounted wealth should be considered in the context of maximisation of utility of consumption. We note however that it is not clear how to build bijection akin to Proposition 2.0.2 above and what the adequate sets of wealth processes should be. We believe this is a challenging topic for further studies.

The methodology based on Azéma–Yor processes, as introduced above, is perfectly suited for our analysis. We note however that it is also possible to study drawdowns, and in particular laws related to the first time a drawdown occurs, by considering $X_t = (X_t - \overline{X}_t) + \overline{X}_t$ and applying methods of processes of class Σ, see e.g. Cheridito, Nikeghbali and Platen [7].

Chapter 3

Portfolio optimisation under non-linear drawdown constraints in a semimartingale financial model

This chapter has been submitted to Finance and Stochastics journal and is at the second stage of the review.

3.1 Introduction

We study portfolio optimisation subject to drawdown constraints. A drawdown constraint specifies that the investor's wealth V_t has to remain above a given function w of its maximum to date: $V_t > w(\sup_{u \leq t} V_u)$. The motivating example is the case of a linear w, when the current wealth is always greater than a fixed fraction of its past maximum. Such features are often embedded in investment opportunities available in the financial markets. From the investor's perspective, they offer a partial protection of the realised gains, where the past maximum is viewed as a natural reference point. For a manager who is trading clients' money avoiding large drawdowns is crucial – typically many investors have a **stop-loss** provision and a large drawdown would result in a sudden withdrawal of capital from the fund, see Chekhlov et al. [6].

This problem was originally introduced by Grossman and Zhou [25] who considered a power utility investor in a Black-Scholes market who faces a linear drawdown constraint and maximises the long-term (asymptotic) growth rate of the expected utility of her wealth. Grossman and Zhou [25] applied the **forward approach** and solved the problem using the dynamic programming principle. Later Cvitanić and Karatzas [12] generalised the setting in [25] to a complete n-dimensional market with deterministic coefficients. Using martingale theory they were able to link the solution to the optimisation problem with the drawdown constraint to an unconstrained problem which they could solve using the **dual approach** as in Karatzas, Lehoczky and Shreve [34]. The initial motivation for our work was to see if a similar link between the constrained and unconstrained problems could be established in a much greater generality.

In this chapter, we effectively solve the long-run continuous time portfolio optimisation problem with drawdown constraints. More precisely, the main contribution of the chapter is an equivalence result: the w-drawdown constrained problem with utility U has the same value function as an unconstrained portfolio optimisation problem but with utility $U \circ F_w$, where F_w is given explicitly in terms of w. Moreover, the optimal wealth process for the drawdown constrained problem is obtained as an explicit pathwise transformation of the optimal wealth process for unconstrained problem. These results hold in an abstract semimartingale model and the investor is endowed with a generic utility function U and a drawdown constraint w. Specifically, we only assume that wealth processes are max-continuous (i.e. have a continuous running supremum), that U either behaves like a logarithm or dominates a power function, and has a finite asymptotic elasticity as in Kramkov and Schachermayer [40], and that $w(x)/x \in (0,1)$ is bounded away from 0 and 1.

In the general setting of this chapter there is usually little hope to solve explicitly portfolio optimisation problems. Adding a drawdown constraint, which is a path-dependent constraint on the admissible investment strategies, appears to significantly increase the complexity. Rather surprisingly, our results show that this is not the case: the constrained problem is just as easy, or just as hard, as the analogue portfolio optimisation problem with no constraints. Since we consider the long-run optimality, Guasoni and Robertson [28] show that the latter can be solved in rather general diffusion setting, see Section 3.6.3.

This chapter relies in an essential way on the so-called Azéma-Yor processes. They effectively provide us with a bijection between non-negative wealth processes and the wealth processes which satisfy a given drawdown constraint. Azéma-Yor martingales have initially appeared in [2] where they were used to solve the Skorokhod embedding problem. Carraro, El Karoui and Obłój [5] introduced a more general class of Azéma-Yor processes and studied them from an SDE perspective. In particular they investigated their properties in relation to drawdown constraints. These results provided crucial insights for our work. In fact, methods of Cvitanić and Karatzas [12] can be expressed using Azéma-Yor processes simplifying greatly their proofs, see Section 3.6.2 below.

Without the methods presented in this chapter the drawdown constraints are in general very hard to study and we are not aware of any works investigating them in the generality similar to the one considered here. Nevertheless, they have received some attention in the financial literature, which one would expect given their practical significance. Magdon-Ismail and Atiya [42] derived results linking the maximum drawdown to the mean return. Chekhlov, Uryasev and Zabarankin [6] analysed discrete-time portfolio optimisation where the investors maximises the expected return subject to risk constraints expressed in terms of drawdowns. They reduce the problem to a linear programming problem which can then be solved numerically.

In the continuous time framework, apart from the early contributions in [25] and [12], drawdown constraints have recently been considered in setups with consumption. Roche [51] investigated maximisation of expected utility of consumption over infinite time horizon for a power utility and under a linear drawdown constraint. Elie and Touzi [20] generalised this to a general class of utility functions in the setting of zero interest rates obtaining explicit representation of the solution. Subsequently, Elie [19] analysed the problem of maximising the expected utility of consumption and terminal wealth on a finite time horizon. He did not have explicit formulae but rather represented the value function as the unique (discontinuous) viscosity solution of the Hamilton-Jacobi-Bellman equation. All the above works considered only the Black-Scholes market. It is not clear at present if, and to what extent, our methods extend to such problems.

Finally, we mention that Vecer [54] analysed options on drawdowns as a more effective way against portfolio losses than put or lookback options. Further analysis of option sensitivities to drawdowns was presented in Pospisil and Vecer [49].

The chapter is organised as follows. Firstly, in Section 3.2, we introduce the financial market, give definitions and formulate the main portfolio optimisation problems of interest. In Section 2, we recall the relevant results on Azéma-Yor processes. Section 3.3 presents the main result and its proof. It considers the problem with uniform units: the wealth in both the utility function and the drawdown constraint is discounted by the same numeraire. In Section 3.4, we provide our results for utility of "wealth in dollars" but subject to drawdown condition on the discounted wealth, as in [12, 25]. This requires stronger asymptotic assumptions on U and w as well as deterministic interest rates. Section 3.5 is devoted to the drawdown constrained optimisation problem with an asymptotically logarithmic utility. Finally, in Section 3.6 different examples are presented. We first consider a general market model which admits price deflators as in Karatzas and Kardaras [33] and give sufficient conditions for finiteness of the value function. Then in Section 3.6.2 we specialise to the complete market with deterministic coefficients and give explicit solutions, extending results in [12]. Finally, Section 3.6.3 provides an explicit solution for an incomplete market model.

The Appendix contains some technical lemmas needed in the proofs but which are of independent interest. In particular we show continuity of the value function – the long-term (asymptotic) growth rate of the expected utility of wealth – in the utility function U and its invariance under perturbation of U on some initial interval $[0, x_0]$.

3.2 Financial market model

We consider a general financial market model with no frictions. The dynamics of d risky assets are represented by a vector $S = (S^1, \ldots, S^d)$ of semimartingales defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions. For simplicity we assume $\mathcal{F}_0$ is trivial. S^0 is the riskless asset (money market account) which is a non-decreasing adapted process with $S^0_0 = 1$. The discount factor is denoted $D_t = 1/S^0_t$. The only restriction we impose is that all assets are **max-continuous**, i.e. $(\sup_{u \le t} S^i_u)_{t \ge 0}$ is a continuous process, $i = 0, 1, \ldots, d$.

In this market agents are allowed to invest by trading in the usual self-financing way. However, we restrict their wealth processes to be max-continuous and positive.

Definition 3.2.1. *An adapted semimartingale* (V_t) *is called a* ***wealth process*** *if it is strictly positive,* $V_t > 0$ *and* $V_{t-} > 0$ *for all* $t \ge 0$ *a.s.,* $(D_t V_t)$ *is max-continuous and there exists an* $(\mathcal{F}_t)$*–predictable process* $\pi = (\pi^1_t, \ldots, \pi^d_t)$ *such that* $D_t V_t = V_0 + \sum_{i=1}^d \int_0^t \pi^i_u \mathrm{d}(D_u S^i_u)$*, where the integrals are assumed to be well-defined.*

The set of wealth processes with $V_0 = v_0$ *is denoted* $\mathcal{A}_{S^0}(v_0)$*.*

The choice of the bond S^0 for units is customary. We say that (N_t) is a **numeraire** if it is a wealth process with $N_0 = 1$. One can see that if (V_t) is a wealth process then there exists an $(\mathcal{F}_t)$–predictable process $\pi = (\pi^0_t, \ldots, \pi^d_t)$ such that $V_t/N_t = V_0 + \sum_{i=0}^d \int_0^t \pi^i_u \mathrm{d}(S^i_u/N_u)$. In analogue to the above, the set of strictly positive processes V which admit such a representation and satisfy (V_t/N_t) is max-continuous, is denoted $\mathcal{A}_N(v_0)$. We note that when all assets S^i are continuous then $\mathcal{A}_N(v_0) = \mathcal{A}_{S^0}(v_0)$. Likewise, if we consider the case when S^i have only negative jumps, which are inaccessible and unbounded, then $\mathcal{A}_{S^0}(v_0)$ corresponds to wealth processes with no short selling restriction and is equal to $\mathcal{A}_N(v_0)$ for a continuous numeraire N.

Note that so far we have not assumed "no-arbitrage" in either strong sense of existence of an equivalent martingale measure $\mathbb{Q}$, or in a weaker sense of existence of a benchmark asset N for which all V/N are supermartingales, see Section 3.6. Neither have we made any strong assumptions on the integrability of (π_t) which would make wealth processes $\mathbb{Q}$–martingales. Instead we consider utility maximisation in a general setting. We are interested in maximising the long-term asymptotic growth rate of the expected utility of wealth. More precisely we consider the following

Problem 1. *Given a numeraire (N_t) and a function U compute*

$$\begin{aligned} \mathrm{CER}_{U,N}(v_0) &= \sup_{V\in\mathcal{A}_N(v_0)} \mathcal{R}_U(V/N), \\ \textit{where}\quad \mathcal{R}_U(V/N) &= \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}\left[U\left(V_T/N_T\right)\right], \end{aligned} \tag{3.1}$$

along with the optimal wealth process which achieves the supremum.

In (3.1), and throughout the chapter, we extend log to $\mathbb{R}\setminus\{0\}$ via $\log(x) = -\log(-x)$. Note that $\mathcal{R}_{U_1}(X) \geq \mathcal{R}_{U_2}(X)$ if $U_1 \geq U_2$ are two functions of the same sign. Naturally, the problem only makes sense under some assumptions on U. We say that U is a **utility function** if $U : (0,\infty) \to \mathbb{R}$ is non-decreasing and concave. In Section 3.3 we consider the above problem for a utility function U which is either strictly positive or strictly negative. Utility functions with behaviour similar to logarithm are treated in Section 3.5 where (3.1) is modified into (3.13).

The idea to look at the growth rate of the expected utility goes back to Dumas and Luciano [15], Grossman and Vila [24] and Grossman and Zhou [25]. Similar criterion also appears in the **risk-sensitive control** literature e.g. Fleming and Sheu [22], see also Section 3.6.3. It is designed to capture the long-horizon optimality and is often a more tractable criterion than the fixed-horizon utility maximisation of terminal wealth, cf. Guasoni and Robertson [28]. We note however that it may fail to provide strategies optimal on a finite time horizon, as discussed by Klass and Nowicki [39] in the context of drawdown constraints introduced below. CER above stands for **Certainty Equivalent Rate** and is interpreted as the critical safe rate – if the investor was offered such (or higher) rate of growth via other investment opportunities she would be happy to abandon the market and move to the alternative investment opportunities. Note that $V_t = v_0 N_t$ is an admissible wealth process so that $\mathrm{CER}_{U,N} \geq 0$.

Utility maximisation problem above can be solved in number of fairly general setups, see Section 3.6 below. The aim of this chapter is to show a direct link between the solution to this problem and the solution to a seemingly more complex problem with a pathwise drawdown constraint.

Definition 3.2.2. *We say that w is a* ***drawdown function*** *if it is non-decreasing and*

$$\exists \alpha_1, \alpha_2 : 0 < \alpha_1 \leq w(x)/x \leq \alpha_2 < 1, \quad x \geq 0. \tag{3.2}$$

*We say that (V_t) satisfies the $w-$**drawdown** (w-DD) condition relative to a numeraire N if*

$$\min\{V_{t-}/N_{t-}, V_t/N_t\} > w(\sup_{u \leq t} V_u/N_u), \quad t \geq 0.$$

The set of $(V_t) \in \mathcal{A}_N(v_0)$ which satisfy w-DD relative to N is denoted $\mathcal{A}_N^w(v_0)$.

We often refer to the processes from the class $\mathcal{A}_{S^0}^w(v_0)$ simply as processes which satisfy w-DD condition. The canonical example of a drawdown function is: $w(x) = \alpha x$. To the best of our knowledge this is the only example which has been considered in the literature, including [12, 19, 20, 25, 51]. We consider a general possibly non-linear drawdown constraint. In particular, Definition 3.2.2 allows for a piece-wise constant[1] function w which has the effect that drawdown constraint is updated discretely at times when the wealth process reaches a new threshold.

Problem 2. *Given a numeraire (N_t), a drawdown function w and a function U compute*

$$\mathrm{CER}_{U,N}^w(v_0) = \sup_{V \in \mathcal{A}_N^w(v_0)} \mathcal{R}_U(V/N) \tag{3.3}$$

along with the optimal wealth process which achieves the supremum.

---

[1] More precisely, we can take w to be piece-wise constant on $[v_0, \infty)$ for any $v_0 > 0$ and for the drawdown problem we only consider $w(x)$ for x greater than the initial capital.

This is a greatly generalised version of the problem introduced by Grossman and Zhou [25] and analysed later by Cvitanić and Karatzas [12]. Firstly, we allow for an almost arbitrary utility function U and not just the power utility. Secondly, we consider a general possibly non-linear drawdown constraint. Finally, we work in a general semimartingale financial market model and not a complete Black-Scholes-like model. Working in such a generality we can not hope for an explicit solution to Problem 2 as in [12, 25]. What we obtain is an explicit formula for the value function and the optimal investment strategy in Problem 2 in terms of the value function and the optimal investment strategy in Problem 1 but with a suitably modified utility function.

Note that Problem 2 has unified units: both the drawdown and the utility are applied to wealth in units of N. In [12, 25] the drawdown is relative to $N = S^0$ but the reward functional is taken of the wealth in dollars: $\mathcal{R}_U(V)$. This introduces further inhomogeneity and is solved in Section 3.4 under additional assumptions.

Finally we note that we consider only wealth processes which are strictly positive but, in view of Lemma 3.7.1, this is not restrictive and we could allow wealth processes which become zero from some point onwards. Likewise, we impose strict drawdown condition but this could be relaxed to allow the drawdown constraint to become binding from some time onwards. Theorem 3.3.1 shows that such wealth processes would never be optimal for problems of long-run utility maximisation considered in this chapter.

3.3 Main results

We are now ready to formulate our main results. The essence of the results is simple and explicit: the w–drawdown problem with a utility function U has the same value as the unconstrained problem with the utility function $U \circ F_w$: $\mathrm{CER}^w_{U,N} = \mathrm{CER}_{U\circ F_w,N}$, where w and F_w are related by (2.2) and (2.4) with $a = a^* = v_0$. Further, the optimal wealth process is given by $NM^{F_w}(V^*/N)$, where V^* is the optimal wealth for the unconstrained problem. We impose

Assumption 3.3.1. *Assume that, for some $\varepsilon > 0$, U satisfies either*

$$\frac{U(x)}{x^{\varepsilon}} \xrightarrow[\mathrm{x}\to\infty]{} \infty, \quad \text{\textit{and U is strictly positive on} } (0,\infty),$$

or

$$U(x)x^{\varepsilon} \xrightarrow[\mathrm{x}\to\infty]{} 0, \quad \textit{and } U \textit{ is strictly negative on } (0,\infty).$$

This insures that our utility functions are of constant sign and they dominate a power utility. We will further assume that they admit positive finite Asymptotic Elasticity in the sense of Kramkov and Schachermayer [40]. Throughout the chapter a **utility function** simply means a non-decreasing concave function and U'_+ denotes the right derivative, $U'_+(x) := \lim_{y\downarrow x} \frac{U(y)-U(x)}{y-x}$.

Theorem 3.3.1. *Let w be a drawdown function, N a numeraire, $v_0 > 0$ and U a utility function satisfying Assumption 3.3.1 and*

$$\limsup_{x\to\infty} \frac{xU'_+(x)}{|U(x)|} = \gamma \in (0,\infty). \tag{3.4}$$

Recall that K_w is given by (2.2) and let F_w be its inverse extended to $[0,\infty)$ as in (2.4) or in any other way which preserves monotonicity and concavity. Assume that, for some $\delta > 0$, $\mathrm{CER}_{G,N}(v_0) < \infty$ where $G(x) = U \circ F_w(x)$ when $U < 0$ and $G(x) = (U \circ F_w(x))^{1+\delta}$ when $U > 0$. Then

$$\mathrm{CER}^{w}_{U,N}(v_0) = \mathrm{CER}_{U\circ F_w,N}(v_0) < \infty$$

and if $V^* \in \mathcal{A}_N(v_0)$ achieves the maximum in the unconstrained problem then $NM^{F_w}(V^*/N) \in \mathcal{A}^w_N(v_0)$ achieves the maximum in the w-drawdown constrained problem.

Remark 3.3.1. *Using Proposition 2.0.2 from Chapter 2 we obtain that for any $V \in \mathcal{A}_N(v_0)$ the process $X := NM^{F_w}(V/N) \in \mathcal{A}^w_N(v_0)$. Indeed, we see that X satisfies Definition 3.2.1 with*

$$\pi^X_t = \Big(F(\overline{V_t/N_t}) - F'(\overline{V_t/N_t})\overline{V_t/N_t}\Big)\pi^N_t + F'(\overline{V_t/N_t})\pi^V_t,$$

with the obvious (vector) notation. Conversely, for $X \in \mathcal{A}^w_N(v_0)$ the process $V := NM^{K_w}(X/N) \in \mathcal{A}_N(v_0)$.

Remark 3.3.2. *It will be clear from the proof, and in particular from Lemma 3.7.2 in the Appendix, that when $U < 0$ we have in fact $\mathrm{CER}^w_{U,N} < \infty$ if and only if $\mathrm{CER}_{U\circ F_w,N} < \infty$.*

Remark 3.3.3. *We note first that for $U \geq 0$ Assumption 3.3.1 implies (3.4). Indeed, we have $\frac{U(x)}{x^{\varepsilon}} \to \infty$, for some $\varepsilon > 0$, which implies that there exists a sequence $x_k \to \infty$ such that $\left(\frac{U(x)}{x^{\varepsilon}}\right)'_+ |_{x=x_k} =$*

$\frac{U(x_k)}{x_k^{\varepsilon+1}}\left(\frac{x_kU'_+(x_k)}{U(x_k)}-\varepsilon\right) > 0$. *Thus,* $\limsup_{x\to\infty}\frac{xU'_+(x)}{U(x)} \geq \varepsilon$. *On the other hand, by concavity,* $U(x) \geq U(x)-U(0) \geq xU'_+(x)$ *so that* $\limsup_{x\to\infty}\frac{xU'_+(x)}{U(x)} \leq 1$.

The reverse is not true. This can be seen by considering a U *which alternates between linear and log-like behaviours. Specifically, consider* U_0 *with* $U_0(0) = 0$, $U_0'(0+) = 1$, $U_0'(x)$ *being continuous, with* $U_0'(x) = U_0'(x_{2k})$ *for* $x \in [x_{2k}, x_{2k+1}]$ *and* $U_0'(x) = (x - x_{2k+1} + U_0'(x_{2k+1})^{-1})^{-1}$ *for* $x \in [x_{2k+1}, x_{2k+2}]$ *where* $x_0 = 0$, $x_1 = 1$ *and*

$$\begin{aligned} x_{2k} &:= \inf\left\{x \geq x_{2k-1}+1 : U_0(x) \leq x^{1/k}\right\}, \quad k \geq 1, \\ x_{2k+1} &:= \inf\left\{x \geq x_{2k}+1 : \frac{xU_0'(x)}{U_0(x)} \geq 1-1/k\right\}, \quad k \geq 1. \end{aligned} \tag{3.5}$$

Choice of x_i *guarantees that* U_0 *does not dominate any positive power of* x *asymptotically, but has a strictly positive asymptotic elasticity. More precisely, the asymptotic elasticity is equal to one as it is bounded from above by 1 since* U *is a positive utility function and* $\lim_{k\to\infty}\frac{x_{2k+1}U'(x_{2k+1})}{U(x_{2k+1})} = 1$.

For $U < 0$ *we need both assumptions, because, for instance,* $U(x) = -\mathrm{e}^{-x}$ *satisfies Assumption 3.3.1 but not* (3.4) *and* $-\frac{1}{U_0(x)}$ *is a utility function which satisfies* (3.4) *but not Assumption 3.3.1.*

Remark 3.3.4. *Observe that* CER *in Problems 1 and 2 are invariant under a multiplication of* U *by a positive constant. Further, for a positive utility function* U, *they are invariant under a constant shift of* U *which preserves the sign. More precisely, write* $C = \mathrm{CER}_{U,N}(v_0)$ *and let* $\kappa > 0$. *For any* $\delta > 0$, $V \in \mathcal{A}_N(v_0)$, *taking* T *large enough we have*

$$\log \mathbb{E}[U(V_T/N_T)+\kappa] \leq \log(\mathrm{e}^{T(C+\delta)}+\kappa) \leq \log(2\mathrm{e}^{T(C+\delta)}).$$

This yields $\mathcal{R}_{U+\kappa}(V/N) \leq C+\delta$ *and letting* $\delta \searrow 0$ *we have* $\mathrm{CER}_{U+\kappa,N}(v_0) = C$. *Finally, both problems are invariant under changes of* U *in the neighbourhood of zero. This is clear under the drawdown constraint since* U *is never evaluated on* $x \in (0, w(v_0))$. *By Theorem 3.3.1 it is also true for the unconstrained problem. A direct argument for this is given in the proof of Lemma 3.7.1 in the Appendix.*

Proof of Theorem 3.3.1. Let $V \in \mathcal{A}_N(v_0)$ and $X := NM^F(V/N)$ which is in $\mathcal{A}_N^w(v_0)$ by Proposition 2.0.2. Now, by Proposition 2.0.1 we obtain directly

$$U\left(\frac{X_t}{N_t}\right) = U\left(M_t^F\left(\frac{V}{N}\right)\right) \geq U\left(F\left(\frac{V_t}{N_t}\right)\right), \ t \geq 0,$$

which readily implies $\mathcal{R}_U(X/N) \geq \mathcal{R}_{U\circ F}(V/N)$. Taking supremum over all $V \in \mathcal{A}_N(v_0)$ we conclude

$$\mathrm{CER}^w_{U,N}(v_0) \geq \mathrm{CER}_{U\circ F,N}(v_0)\ .$$

It follows also that if we had equality and the right hand side was attained by a wealth process V^* then the left hand side is attained by $NM^F(V^*/N)$, as required.

It remains to establish the reverse inequality. The idea of the proof is to consider a sequence on unconstrained problems whose value functions all dominate $\mathrm{CER}_{U\circ F,N}(v_0)$ and converge to $\mathrm{CER}^w_{U,N}(v_0)$. We will do this by relaxing the drawdown constraint w to w_n and considering utility functions $U \circ F_{w_n}$. Let

$$w_n(x) := x - (1+\frac{1}{n})\frac{K(x)}{K'(x)} = w(x) - \frac{1}{n}\frac{K(x)}{K'(x)} = (1+\frac{1}{n})w(x) - \frac{1}{n}x, \quad x \geq v_0, \tag{3.6}$$

where the equalities follow from $w(x) = x - K(x)/K'(x)$. We take n large enough so that $w_n(x)$ satisfies (3.2) but we note that w_n may fail to be globally non-decreasing on $(0,\infty)$.

It follows by a direct computation that

$$K_n(x) := K_{w_n}(x) = v_0 \exp\left(\int_{v_0}^{x} \frac{1}{u - w_n(u)}\mathrm{d}u\right) = v_0^{\frac{1}{1+n}}\left(K(x)\right)^{\frac{n}{1+n}}, \quad x \geq v_0,$$

and $w(x) = x - K_n(x)/K_n'(x)$, where $K_n'(v_0)$ is understood as $K_n'(v_0+)$. Consider $(X_t) \in \mathcal{A}^w_N(v_0)$ and let $Y^n_t := N_t M^{K_n}_t(X/N)$ which is an element of $\mathcal{A}_N(v_0)$ by Proposition 2.0.2. Using (2.1) and the drawdown property of X we obtain

$$\frac{Y^n_t}{N_t} \geq K_n\left(\overline{\left(\frac{X}{N}\right)_t}\right) - K_n'\left(\overline{\left(\frac{X}{N}\right)_t}\right)\left(\overline{\left(\frac{X}{N}\right)_t} - w\left(\overline{\left(\frac{X}{N}\right)_t}\right)\right) = h_n\left(\overline{\left(\frac{X}{N}\right)_t}\right),$$

where $h_n(x)$ is defined for $x \geq v_0$ as

$$\begin{aligned}
h_n(x) :&= K_n(x) - K_n'(x)(x - w(x)) = K_n(x)\left(1 - \frac{K_n'(x)}{K_n(x)}(x - w(x))\right)\\
&= K_n(x)\left(1 - \frac{x - w(x)}{x - w_n(x)}\right) = K_n(x)\left(1 - \frac{1}{1+1/n}\right)\\
&= \frac{1}{1+n}K_n(x),
\end{aligned}$$

where we used (3.6) and $w_n(x) = x - K_n(x)/K_n'(x)$.

Let $F_n(v)$ be the inverse of $K_n(v)$, for $v \geq v_0$ and extended to $[0,\infty)$ via (2.4). Explicitly, we have $F_n(v) = F\left(v_0^{-1/n} v^{\frac{1+n}{n}}\right)$, $v \geq v_0$ and $F_n(v) = F(v) - \frac{1}{n}(F(v_0) - F(v))$, $v \in [0, v_0)$. F_n is continuous and strictly increasing on $[0,\infty)$ and we take n large enough so that $F_n(0) > 0$.

Observe that for $x > v_0$:

$$x\frac{F_n'(x)}{F_n(x)} = \frac{x}{K_n'(F_n(x))F_n(x)} = \frac{F_n(x) - w_n(F_n(x))}{F_n(x)} \le 1 \tag{3.7}$$

and hence

$$x\left(U \circ F_n(x)\right)_+' = F_n(x)U_+'(F_n(x))\frac{xF_n'(x)}{F_n(x)} \le F_n(x)U_+'(F_n(x)).$$

Using (3.4) we conclude that

$$\limsup_{x\to\infty} \frac{x\left(U \circ F_n(x)\right)_+'}{U \circ F_n(x)} \le \limsup_{x\to\infty} \frac{F_n(x)U_+'(F_n(x))}{U \circ F_n(x)} = \limsup_{y\to\infty} \frac{yU_+'(y)}{U(y)} = \gamma,$$

where we used the fact that F is a strictly increasing continuous map of $[v_0, \infty)$ onto itself so any sequence $y_m \to \infty$ can be represented as $y_m = F_n(x_m)$ for $x_m = K_n(y_m) \to \infty$ as $m \to \infty$.

Similarly, since w_n satisfies (3.2) we see from the above that $xF_n'(x)/F_n(x) \ge \alpha_n$ for all $x \ge v_0$ and some $\alpha_n \in (0, 1)$. This allows us to conclude that $U \circ F_n$ has positive finite Asymptotic Elasticity:

$$\alpha_n\gamma \le \limsup_{x\to\infty} \frac{x\left(U \circ F_n(x)\right)_+'}{U \circ F_n(x)} \le \gamma.$$

Applying Lemma 3.7.3 to $U \circ F_n$ and $x_0 = v_0/(n + 1)$ we deduce that there exists $\gamma_n \in \mathbb{R}$, $\gamma_n \neq 0$ such that for all $x \ge v_0/(n + 1)$ and all $\lambda \ge 1$ we have $U \circ F_n(\lambda x) \le \lambda^{\gamma_n} U \circ F_n(x)$. Set $\lambda = (1 + n)$ and $x = K_n\left(\overline{\left(\frac{X}{N}\right)_t}\right)/\lambda$. Observe that $\overline{\left(\frac{X}{N}\right)_t} \ge v_0$ and that $K_n(v_0) = v_0$ so $x \ge v_0/(n + 1)$. Combining all the above, we obtain

$$\begin{aligned} U \circ F_n\left(\frac{Y_t^n}{N_t}\right) &\ge U \circ F_n\left(h_n\left(\overline{\left(\frac{X}{N}\right)_t}\right)\right) = U \circ F_n\left(\frac{1}{1+n}K_n\left(\overline{\left(\frac{X}{N}\right)_t}\right)\right) \\ &\ge \left(\frac{1}{1+n}\right)^{\gamma_n} U\left(\overline{\left(\frac{X}{N}\right)_t}\right) \ge \left(\frac{1}{1+n}\right)^{\gamma_n} U\left(\frac{X_t}{N_t}\right). \end{aligned}$$

The factor of $(1 + n)^{-\gamma}$ disappears when we apply $\frac{1}{t}\log$ and let $t \to \infty$:

$$\mathcal{R}_{U\circ F_n}(Y^n/N) \ge \mathcal{R}_U(X/N). \tag{3.8}$$

Taking supremum over $X \in \mathcal{A}_N^w(v_0)$ we conclude that

$$\mathrm{CER}_{U\circ F_n,N}(v_0) \ge \mathrm{CER}_{U,N}^w(v_0)$$

and thus we indeed have a sequence on unconstrained problems with value functions all dominating $\mathrm{CER}_{U,N}^w(v_0)$. It remains to argue that they converge to $\mathrm{CER}_{U\circ,N}(v_0)$ and for this we use Lemma 3.7.2 in the Appendix.

For $v \geq v_0$ we have $F(v) \leq F_n(v) = F\left(v_0^{-1/n} v^{\frac{1+n}{n}}\right)$ and for $v \in [0, v_0)$ we have $c_n F(v) \leq F_n(v) \leq F(v)$ where $c_n = 1 + \frac{1}{n}\frac{w(v_0)-v_0}{w(v_0)}$ and we take $n > n_0$ chosen such that $c_n > c_0 := c_{n_0} > 0$. Fix $\epsilon \in (0,1)$ and note that $v_0^{-1/n} \leq \epsilon^{-1/n} < \epsilon$. Together the above give us

$$c_0 F(v) \leq F_n(v) \leq F(v^{1+1/n}/\epsilon), \quad v \geq \epsilon. \tag{3.9}$$

Thanks to (3.4), we can apply Lemma 3.7.3 to U with $x_0 = c_0 F(\epsilon)$ to see that there exists a non-zero $\gamma' \in \mathbb{R}$ such that $U(\frac{1}{c_0} c_0 F(v)) \leq c_0^{-\gamma'} U(c_0 F(v))$, for all $v \geq \epsilon$. We thus obtain

$$c_0^{\gamma'} U \circ F(v) \leq U(c_0 F(v)) \leq U \circ F_n(v) \leq U \circ F\left(\frac{1}{\epsilon} v^{1+1/n}\right), \quad v \geq \epsilon.$$

Finally observe that $F_n(x) \to F(x)$ for all $x > 0$ so that $U \circ F_n \to U \circ F$ pointwise. Together with $\mathrm{CER}_{G,N} < \infty$, Lemma 3.7.2 now yields

$$\mathrm{CER}_{U \circ F_n, N}(v_0) \xrightarrow[\mathrm{n} \to \infty]{} \mathrm{CER}_{U \circ F, N}(v_0) < \infty$$

which concludes the proof. □ □

From Lemma 3.7.1 we immediately have

Corollary 3.3.1. *Under the assumptions of Theorem 3.3.1 we have for any* $v > 0$

$$\mathrm{CER}^w_{U,N}(v) = \mathrm{CER}^w_{U,N}(1) = \mathrm{CER}_{U \circ F_w, N}(1) = \mathrm{CER}_{U \circ F_w, N}(v) < \infty$$

and if $V^* \in \mathcal{A}_N(1)$ *achieves* $\mathrm{CER}_{U \circ F_w, N}(1)$ *then* vV^* *achieves* $\mathrm{CER}_{U \circ F_w, N}(v)$ *and* $NM^{F_w}(vV^*/N) \in \mathcal{A}^w_N(v)$ *achieves* $\mathrm{CER}^w_{U,N}(v)$.

3.4 Utility of wealth in dollars

We turn now to the inhomogeneous problem as considered by Grossman and Zhou [25]. We seek to maximise the utility of wealth $U(V_T)$, but the drawdown constraint is imposed on the discounted wealth process V_t/S^0_t. Note that in the case of a linear constraint, $w(x) = \alpha x$, this is equivalent to saying that the drawdown constraint is growing at a hurdle rate equal to the riskless rate, see Guasoni and Obłój [26]. In analogy to Problems 1, 2 we define

$$\mathrm{CER}_U(v_0) = \sup_{V \in \mathcal{A}_{S^0}(v_0)} \mathcal{R}_U(V), \quad \mathrm{CER}^w_U(v_0) = \sup_{V \in \mathcal{A}^w_{S^0}(v_0)} \mathcal{R}_U(V). \tag{3.10}$$

As previously, by considering $V = v_0 S^0$ we see that in both cases CER ≥ 0. In order to be able to relate these problems we essentially need to go back to the homogenates case when $\mathcal{R}_U(V)$ is replaced by $\mathcal{R}_U(V/S^0)$ and for this we need to be able to factor the discounting in and out of the reward functional $\mathcal{R}_U$. This is possible when U is a power utility, w is linear and S^0 is deterministic as in [12] and [25]. Here we need to assume this holds asymptotically.

Assumption 3.4.1. *Assume the following three conditions hold*

(i) U *is either strictly positive or strictly negative on* $(0, \infty)$ *and the following limit exists* $\frac{xU'_+(x)}{U(x)} \to \gamma \in (-\infty, 1) \setminus \{0\}$ *as* $x \to \infty$*,*

(ii) the following limit exists $\frac{w(x)}{x} \to \alpha \in (0, 1)$ *as* $x \to \infty$*,*

(iii) S^0 *is deterministic and the following limit exists* $r^* := \lim_{T\to\infty} \frac{\log S^0_T}{T}$*.*

The first assumption is a strengthened version of the finite asymptotic elasticity of Kramkov and Schachermayer [40] which we assumed earlier in (3.4). It follows from Lemma 3.7.4 in the Appendix that it implies Assumption 3.3.1 holds. The second condition above is in fact equivalent to saying that K in (2.2) also has such (converging) finite asymptotic elasticity. This is immediate since $xK'(x)/K(x) = x/(x - w(x))$. We denote the CRRA (power) utility with $H^{(p)}(x) = \frac{x^p}{p}$, $p \leq 1$. We assume $p \neq 0$, which is the case of logarithmic utility treated below in Section 3.5. Finally, we denote $w_\alpha(x) = \alpha x$ the linear drawdown function.

We assumed above that the interest rates are deterministic and that asymptotically U is a power utility and w is linear. Comparing with the setup in Section 3.3 these are strong assumptions and our main contribution, relative to [12], is that we work in a general max-continuous semimartingale market.

Theorem 3.4.1. *Let* U *be a utility function and* w *a drawdown function for which Assumption 3.4.1 holds. Assume further that* $\mathrm{CER}_{H^{(\gamma(1-\alpha)(1+\delta))}}(v_0) < \infty$ *for some* $\delta > 0$*. Then*

$$\mathrm{CER}^w_U(v_0) = \mathrm{CER}^{w_\alpha}_{H^{(\gamma)}}(v_0) = \mathrm{CER}_{H^{(\gamma(1-\alpha))}}(v_0) + |\gamma|\alpha r^* < \infty$$

and if $V^* \in \mathcal{A}_{S^0}(v_0)$ *achieves the maximum in the unconstrained problem then* $S^0 M^{F_w}(V^*/S^0) \in \mathcal{A}^w_{S^0}(v_0)$ *achieves the maximum in the* w*-drawdown constrained problem, where* F_w *is as in Theorem 3.3.1.*

Remark 3.4.1. *Theorem 1.1 of Grossman and Zhou [25] and Theorem 5.1 of Cvitanić and Karatzas [12] are consequences of the above statement. Namely, they specialise to* $w = w_\alpha$, $U = H^{(\gamma)}$ *with* $\gamma \in (0,1)$ *and a particular (deterministic, constant coefficients) market setup. Standard techniques allow then to compute explicitly* $\mathrm{CER}_{H^{(\gamma(1-\alpha))}}(v_0)$ *and find the optimal wealth process* V^**, see Section 3.6.2. Therein we also discuss how various objects in [12] and in our work relate explaining how methods of [12] helped us develop intuition behind this paper.*

Remark 3.4.2. *Similarly to Theorem 3.3.1, it follows from Lemma 3.7.2 that when* $\gamma < 0$ *the equality* $\mathrm{CER}_U^w = \mathrm{CER}_{H^{(\gamma(1-\alpha))}} + |\gamma|\alpha r^*$ *holds without assuming that* $\mathrm{CER}_{H^{(\gamma(1-\alpha)(1+\delta))}}$ *is finite.*

Proof. Recall that $\mathcal{R}_{U_1}(X) \leq \mathcal{R}_{U_2}(X)$ for two functions $U_1 \leq U_2$ of the same sign. Consider $X \in \mathcal{A}^w_{S^0}(v_0)$ and a small $\varepsilon > 0$. As $X_t \geq w(v_0) > 0$, we can apply Lemma 3.7.4 in the Appendix to obtain

$$\mathcal{R}_U(X) \leq \mathcal{R}_{H^{(\gamma(1+\varepsilon))}}(X) = \mathcal{R}_{H^{(\gamma(1+\varepsilon))}}(X/S^0) + |\gamma|(1+\varepsilon)r^*, \tag{3.11}$$

the last equality following since S^0 is deterministic.

Recall $K = K_w$ defined in (2.2). Note that $xK'(x)/K(x) = x/(x - w(x))$ and hence that Assumption 3.4.1 implies

$$\lim_{x\to\infty} \frac{xK'(x)}{K(x)} = \lim_{x\to\infty} \frac{x}{x - w(x)} = \lim_{x\to\infty} \frac{1}{1 - w(x)/x} = \frac{1}{1-\alpha}.$$

$F : [v_0, \infty) \to [v_0, \infty)$ is the inverse of K_w and we have $\lim_{x\to\infty} xF'(x)/F(x) = 1 - \alpha$. Lemma 3.7.4 then implies that $F(x) \leq c_2 H^{((1-\alpha)(1+\varepsilon))}$ for some $c_2 \geq 1$ and all $x \geq v_0/2$. In consequence, for any $Y \in \mathcal{A}_{S^0}(v_0)$ with $Y/S^0 \geq v_0/2$

$$\begin{aligned}\mathcal{R}_{H^{(\gamma(1+\varepsilon))}\circ F}(Y/S^0) &\leq \mathcal{R}_{H^{(\gamma(1-\alpha)(1+\varepsilon)^2)}}(Y/S^0)\\ &= \mathcal{R}_{H^{(\gamma(1-\alpha)(1+\varepsilon)^2)}}(Y) - |\gamma|(1-\alpha)(1+\varepsilon)^2 r^*,\end{aligned} \tag{3.12}$$

which is finite for ε small enough by assumption.

By a similar reasoning and using the assumption of the theorem we conclude that $\mathrm{CER}_{G,S^0}(v_0) < \infty$ for $G(x) = (H^{(\gamma(1+\varepsilon))} \circ F(x))^{1+\delta}$ with $\delta = \varepsilon \mathbf{1}_{\gamma>0}$ and for ε small enough. Applying Theorem 3.3.1 and combining (3.11) with (3.12) we obtain

$$\begin{aligned}\mathrm{CER}^{w}_{U}(v_0) &\leq \mathrm{CER}^{w}_{H^{(\gamma(1+\varepsilon))},S^0}(v_0) + |\gamma|(1+\varepsilon)r^*\\ &\leq \mathrm{CER}_{H^{(\gamma(1-\alpha)(1+\varepsilon)^2)}}(v_0) + |\gamma|\alpha r^* + \varepsilon|\gamma|r^*(1-(2+\varepsilon)(1-\alpha)).\end{aligned}$$

Taking $\varepsilon \to 0$ and invoking Lemma 3.7.2 yields "$\leq$" inequalities in the desired equality. The reverse inequalities are obtained in an analogous manner but exploiting the lower bound in Lemma 3.7.4. Finally, the above also shows we can replace w by w_α. $\square$ $\square$

Remark 3.4.3. *Similarly to Theorem 3.3.1 and Corollary 3.3.1,* $\mathrm{CER}^{w}_{U}(v_0)$ *above does not depend on* v_0 *and the optimal strategy for the unconstrained problem scales linearly in the initial wealth.*

Corollary 3.4.1. *In the setup of Theorem 3.4.1 we have*

$$\mathrm{CER}_{H^{(\gamma(1-\alpha))}}(v_0) = \mathrm{CER}_{U\circ F}(v_0).$$

This Corollary follows from the proof of Theorem 3.4.1. Naturally, similar statements can be made relating in general CER for power utility and for U which satisfies (i) in Assumption 3.4.1. This is not surprising in the light of the results on the so-called turnpike theorems. In this stream of literature authors study the convergence of the value function and the optimal strategy for the Merton problem of maximising utility of terminal wealth as the horizon T tends to infinity. In particular, Hubermann and Ross [32] argue that, in the case of a complete discrete market, the convergence of optimal strategies is equivalent to the convergence of the relative risk aversion, i.e. $-\frac{xU''(x)}{U'(x)} \to 1-\gamma$, which is essentially (i) in Assumption 3.4.1. Huang and Zariphopoulou [31] study the problem for a continuous time complete market model with deterministic coefficients, as in Section 3.6.2. They find sufficient conditions on U for the optimal strategy to converge to the optimal strategy coming from the problem with a power utility. The analysis in a recent paper of Guasoni and Robertson [28] includes also incomplete markets. In comparison, our results apply in a much more general context but are also much weaker. Problem 1 looks at maximising the long-term asymptotic growth rate of the expected utility and the above Corollary shows that the resulting value function is the same when two utility functions have the same asymptotic behaviour. It does not say anything precise about finite horizon utility maximisation and its convergence.

3.5 Logarithmic Utility

So far we have only considered utility functions with constant sign and which dominated a power utility, as in Assumption 3.3.1. In this section, we consider utility functions akin to $U(x) = \log x$. The results are very close in spirit to ones in the previous two sections, but in fact require less technicalities in the proofs. First we need to introduce a modified version of maximisation criterion in (3.1).

Problem 3. *Given a numeraire (N_t), a drawdown function w and function U compute*

$$\begin{aligned}\widetilde{\mathrm{CER}}_{U,N}(v_0) &= \sup_{V\in\mathcal{A}_N(v_0)} \widetilde{\mathcal{R}}_U(V/N),\\ \widetilde{\mathrm{CER}}^{w}_{U,N}(v_0) &= \sup_{V\in\mathcal{A}^w_N(v_0)} \widetilde{\mathcal{R}}_U(V/N),\\ \text{where}\quad \widetilde{\mathcal{R}}_U(V/N) &= \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[U\left(V_T/N_T\right)\right],\end{aligned} \tag{3.13}$$

along with the optimal wealth processes which achieve the supremum.

Theorem 3.5.1. *Let N be a numeraire, w a drawdown function and U a utility function satisfying*

$$\limsup_{x\to\infty} xU'_+(x) < \infty \quad \text{and} \quad \liminf_{x\to\infty}\frac{U(x)}{\log(x)} > 0.$$

Let F_w be as in Theorem 3.3.1 then

$$\widetilde{\mathrm{CER}}^{w}_{U,N}(v_0) = \widetilde{\mathrm{CER}}_{U\circ F_w,N}(v_0)$$

and if V^* achieves the maximum in the unconstrained problem then $NM^{F_w}(V^*/N)$ achieves the maximum in the w-drawdown constrained problem.

Remark 3.5.1. *The equality between value functions in particular states that they are either both finite or both infinite.*

Proof. The first part of the proof is identical to the first part of the proof of Theorem 3.3.1 and yields

$$\widetilde{\mathrm{CER}}^{w}_{U,N}(v_0) \geq \widetilde{\mathrm{CER}}_{U\circ F,N}(v_0).$$

We also obtain that $NM^{F_w}(V^*/N)$ is optimal for constrained problem when V^* is optimal for unconstrained.

It follows from the assumptions that for any $y > 0$ there exists γ such that

$$xU'_+(x) < \gamma < \infty, \quad x \geq y.$$

Applying Lemma 3.7.3 to $e^{U(x)}$, we deduce that

$$U(\lambda x) \leq U(x) + \gamma \log \lambda, \ \lambda > 1, \ x \geq y. \tag{3.14}$$

Write $F = F_w$ and define w_n, K_n, F_n as in the proof of Theorem 3.3.1 and recall the computation in (3.7). It follows that $xU \circ {F_n}'_+(x) < \gamma$ for $x \geq v_0$. The same reasoning holds with F in place of F_n. We deduce that (3.14) holds with $U \circ F$ in place of U and γ' instead of γ and likewise for $U \circ F_n$ and γ_n.

Let $X \in \mathcal{A}_N^w(v_0)$, $Y^n = NM^{K_n}(X/N)$ and recall from the proof of Theorem 3.3.1 that

$$\frac{Y_t^n}{N_t} \geq \frac{1}{1+n} K_n \left(\overline{\left(\frac{X}{N}\right)}_t \right).$$

Using (3.14) for $U \circ F_n$ and $y = v_0/(n+1)$, $\lambda = n+1$, noting that $K_n(\overline{X/N}) \geq v_0$, we obtain

$$\begin{aligned} U\left(\overline{X/N}_t\right) &\leq U \circ F_n \left(\frac{1}{1+n} K_n(\overline{X/N}_t) \right) + \gamma_n \log(n+1) \\ &\leq U \circ F_n \left(\frac{Y_t^n}{N_t} \right) + \gamma_n \log(n+1). \end{aligned}$$

Dividing by t, taking log and passing to the limit $t \to \infty$ yields:

$$\widetilde{\mathcal{R}}_{U \circ F_n}(Y^n/N) \geq \widetilde{\mathcal{R}}_U(\overline{X/N}) \geq \widetilde{\mathcal{R}}_U(X/N).$$

Taking supremum over $X \in \mathcal{A}_N^w(v_0)$ we conclude that

$$\widetilde{\mathrm{CER}}_{U \circ F_n, N}(v_0) \geq \widetilde{\mathrm{CER}}^w_{U,N}(v_0).$$

It remains to establish the convergence in n on the LHS. An analogous argument to Lemma 3.7.1 shows that it suffices to consider $V \in \mathcal{A}_N(v_0)$ such that $V/N \geq v_0/2$ to compute $\widetilde{\mathrm{CER}}_{U \circ F_n, N}(v_0)$. The bound in (3.9) gives

$$U(c_0 F(v)) \leq U \circ F_n(v) \leq U \circ F \left(\frac{1}{\epsilon} v^{1+1/n} \right), \quad v \geq \epsilon$$

for arbitrary $\epsilon \in (0,1)$ and some $0 < c_0 < 1$. Taking $\epsilon < v_0/2$ and using (3.14) for U and $U \circ F$ with $y = \min\{\epsilon, c_0 F(\epsilon)\}$ we obtain

$$U \circ F(v) + \gamma \log c_0 \leq U \circ F_n(v) \leq U \circ F(v) + \gamma' \log(v^{1/n}/\epsilon), \quad v \geq \epsilon. \tag{3.15}$$

Finally, consider $\log F(x)/x$ for large x and let $z = F(x)$. Then, using (3.2),

$$\frac{\log F(x)}{\log x} = \frac{\log z}{\log K(z)} = \frac{\log z}{\log v_0 + \int_{v_0}^{z} \frac{du}{u-w(u)}} \geq \frac{\log z}{\log v_0 + \frac{1}{1-\alpha_2}\log z/v_0},$$

which can be made arbitrary close to $1-\alpha_2 > 0$ by considering z large enough. Using the assumption on U we conclude that

$$\liminf_{x\to\infty} \frac{U\circ F(x)}{\log x} = \liminf_{x\to\infty} \frac{U\circ F(x)}{\log F(x)} \frac{\log F(x)}{\log x} > 0.$$

It follows that for some positive constants c, c_1, $cU\circ F(x) + c_1 \geq \log(x)$ for all $x \geq \epsilon$. Combined with (3.15), this shows that

$$\widetilde{\mathrm{CER}}_{U\circ F,N}(v_0) \leq \widetilde{\mathrm{CER}}_{U\circ F_n,N}(v_0) \leq (1 + c\gamma'/n)\widetilde{\mathrm{CER}}_{U\circ F,N}(v_0).$$

Taking $n \to \infty$ establishes the desired convergence. $\square$ $\square$

We close this section with a result similar to Theorem 3.4.1. The definitions of $\widetilde{\mathrm{CER}}_U$, $\widetilde{\mathrm{CER}}^w_U$ should be clear and the proof follows closely the arguments in Section 3.4 and we omit it for the sake of brevity.

Theorem 3.5.2. *Let U be a utility function with $xU'_+(x) \to \gamma \in (0,\infty)$ as $x \to \infty$ and w a drawdown function such that (ii) and (iii) in Assumption 3.4.1 hold. Then*

$$\widetilde{\mathrm{CER}}^w_U(v_0) = \gamma(1-\alpha)\widetilde{\mathrm{CER}}_{\log}(v_0) + \gamma\alpha r^*$$

and $S^0 M^F(V^*/S^0) \in \mathcal{A}^w_{S^0}(v_0)$ achieves the maximum in the drawdown constrained problem if $V^* \in \mathcal{A}_{S^0}(v_0)$ achieves the maximum in the unconstrained problem.

3.6 Examples

We discuss now some examples. Our aim is twofold. First, we want to give an example of a rather general setup in which sufficient conditions can be found which guarantee finiteness of CER for the unconstrained problem, as assumed in Theorem 3.3.1. Second, we want to discuss specific examples when the unconstrained, and hence also the drawdown constrained, portfolio optimisation problem is solved explicitly. In particular we relate our results and methods to the ones in [12].

3.6.1 Market with price deflators

We start by assuming existence of a price deflator (or a state price density) process. In the setup of Section 3.2 we further assume that all S_t^i are continuous and that there exists a $\mathbb{P}$-local martingale (Z_t), $Z_t > 0$ for all $t \geq 0$, such that $(Z_t D_t S_t^i)$ are $\mathbb{P}$-local martingales, $i = 1, \ldots, d$. Note that we do not necessarily assume that (Z_t) is a true martingale and hence that an equivalent martingale measure exists. Our setup is in fact analogous to the most general setup in which stochastic portfolio optimisation is considered, see Fernholz and Karatzas [21]. Note that if $(V_t) \in \mathcal{A}(v_0) = \mathcal{A}_N(v_0)$ then

$$\mathrm{d}(Z_t D_t V_t) = D_t(V_t - \pi_t S_t)\mathrm{d}Z_t + \pi_t \mathrm{d}(Z_t D_t S_t),$$

so that $(Z_t D_t V_t)$ is a positive $\mathbb{P}$-local martingale and hence a supermartingale. Karatzas and Kardaras [33] show that the existence of (Z_t) is equivalent to the NUPBR condition (No Unbounded Profit With Bounded Risk). This condition is weaker than the usual NFLVR condition from Delbaen and Schachermayer [14] and allows for some (very mild) arbitrage opportunities, see examples constructed in [33]. Recall that $H^{(p)}(x) = \frac{1}{p}x^p$.

Lemma 3.6.1. *Let N be a numeraire. The following implications hold for any $p < 1$, $p \neq 0$, and $v_0 > 0$*

$$\mathcal{R}_{H^{(-p/(1-p))}}(DZ) < \infty \quad \Longrightarrow \quad \mathrm{CER}_{H^{(p)}}(v_0) < \infty,$$

$$\mathcal{R}_{H^{(-p/(1-p))}}(DZN) < \infty \quad \Longrightarrow \quad \mathrm{CER}_{H^{(p)},N}(v_0) < \infty.$$

Proof. Let $(V_t) \in \mathcal{A}(v_0)$ so that $(Z_t D_t V_t)$ is a $\mathbb{P}$-local martingale, as above. For $p < 0$ we have

$$\begin{aligned} \mathbb{E}[V_T^p] &= \mathbb{E}[(D_T Z_T)^{-p}(Z_T D_T V_T)^p] \\ &\geq \left(\mathbb{E}[(D_T Z_T)^{-\frac{p}{1-p}}]\right)^{(1-p)} (\mathbb{E}[Z_T D_T V_T])^p \geq v_0^p \left(\mathbb{E}[(D_T Z_T)^{-\frac{p}{1-p}}]\right)^{(1-p)}, \end{aligned}$$

where we used reversed Hölder's inequality and the fact that a non-negative local martingale is a supermartingale. The inequalities above are reversed when we divide both sides by $\frac{1}{p} < 0$ and the claim follows. The case $p \in (0, 1)$ is even more straightforward – it suffices to reverse the inequalities in the above. The case with numeraire is entirely analogous. □ □

The lemma above gives an example of sufficient conditions to apply Theorems 3.3.1 and 3.4.1 since they both require that $\mathrm{CER}_{G,N}$ or CER_G is finite. For the latter G is a power utility function and Lemma

3.6.1 applies directly. For the former we would need to bound G by a power utility.

Naturally, in the current very general setup there might be little hope to compute CER_U or find the optimal wealth process. However, one might expect this to be the simplest portfolio optimisation problem to solve. The strength of our results is to show that solving the seemingly much more complex problem with drawdown constraint on wealth paths is in fact equally simple (or hard).

Karatzas and Kardaras [33, Theorem 4.12] also show that the existence of (Z_t) is equivalent to the existence of a benchmark numeraire $\tilde{N}$ such that $V/\tilde{N}$ is a supermartingale for any $V \in \mathcal{A}(v_0)$, see also Christensen and Larsen [8]. This readily implies that $\widetilde{\mathrm{CER}}_{\log}(v_0) = \widetilde{\mathcal{R}}_{\log}(\tilde{N})$. Indeed, considering $V \in \mathcal{A}(v_0)$ and applying Jensen's inequality gives

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\log V_T \le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\log \tilde{N}_T + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\log \frac{V_T}{\tilde{N}_T} \le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\log \tilde{N}_T.$$

This observation essentially goes back to Bansal and Lehmann [3]. In a no-arbitrage complete market model (Z_t) is the density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ where $\mathbb{Q}$ is the equivalent martingale (risk-neutral) measure. Completeness means that $(D_t Z_t)^{-1}$ is an admissible wealth process and thus the benchmark numeraire. In particular, in the setting of Theorem 3.5.2, we then have

$$\widetilde{\mathrm{CER}}^w_U(v_0) = \gamma r^* - \gamma(1-\alpha)\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\log Z_T.$$

It may be natural to start modelling by simply requiring that the benchmark numeraire $\tilde{N}$ exists. This is pursued in the so-called **benchmark approach**, see Platen and Heath [48]. It is clear that Lemma 3.6.1 remains true in this approach when we replace DZ by $1/\tilde{N}$.

3.6.2 Complete market model with deterministic coefficients

We consider now the classical complete financial market model with deterministic coefficients. Let $W_t = (W_t^1, \ldots, W_t^d)'$ be a standard d-dimensional Brownian motion and $(\mathcal{F}_t)$ the augmentation of its natural filtration. Here $'$ denotes vector transpose. $S_t^0 = \exp(\int_0^t r_u \mathrm{d}u)$ is deterministic and $\frac{1}{T}\int_0^T r_u \mathrm{d}u \to r^* \ge 0$ as $T \to \infty$. Each asset follows dynamics given by

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_t^i \mathrm{d}t + \sum_{j=1}^d \sigma_t^{ij}\mathrm{d}W_t^j, \quad S_0^i = s_0^i > 0$$

where μ_t^i and σ_t^{ij} are bounded deterministic functions and σ_t is invertible. Recall Definition 3.2.1 of wealth process and let $\tilde{\pi}_t^i := \pi_t^i S_t^i / V_t$ be the proportion of wealth invested in the i^{th} asset so that $\mathrm{d}(D_t V_t) = \sum_{i=1}^d \tilde{\pi}_t^i V_t \frac{\mathrm{d}(D_t S_t^i)}{S_t^i}$. The market price of risk is given as $\theta_t := \sigma^{-1}(\mu_t - r_t \mathbb{I})$, where $\mathbb{I}$ is a d-dimensional vector with all entries equal to one. We assume θ_t is also bounded and that

$$||\theta^*||^2 := \lim_{T\to\infty} \frac{1}{T} \int_0^T ||\theta_u||^2 \mathrm{d}u \quad \text{is well defined and finite.}$$

The state price density

$$Z_t := \exp\left\{ -\int_0^t \theta_u' \mathrm{d}W_u - \frac{1}{2} \int_0^t ||\theta_u||^2 \mathrm{d}u \right\}$$

is a $\mathbb{P}$–martingale which defines the unique risk neutral measure $\mathbb{Q}$, $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = Z_t$. To solve $\mathrm{CER}_{H^{(p)}}$ one first considers the problem of maximising the expected utility of wealth at a given horizon T. The solution is obtained using, by now standard, convex duality arguments, see Karatzas, Lehoczky and Shreve [34] or Karatzas and Shreve [37, pp. 97–118]. The optimal wealth process V^* is described explicitly via

$$\tilde{\pi}_t^* = \frac{1}{1-p} \theta_t' \sigma_t^{-1} \tag{3.16}$$

and in particular it is independent of the time horizon T. We conclude that it is also optimal for the long-term asymptotic growth rate optimisation. Taking limit of the value functions for the finite horizon problem we obtain

$$\mathrm{CER}_{H^{(p)}}(v_0) = \mathcal{R}_{H^{(p)}}(V^*) = |p| r^* + \frac{|p|}{2(1-p)} ||\theta^*||^2.$$

Note that the difference of a factor $|p|$ when compared to [25, 12] is immediate since they consider $\frac{1}{|p|}\mathcal{R}_{H^{(p)}}(V)$ instead of $\mathcal{R}_{H^{(p)}}(V)$.

Applying Theorem 3.4.1 for a utility function U and a drawdown function w, which satisfy Assumption 3.4.1, we obtain

$$\mathrm{CER}_U^w(v_0) = \mathrm{CER}_{H^{(\gamma(1-\alpha))}}(v_0) + |\gamma|\alpha r^* = |\gamma| \left(r^* + \frac{(1-\alpha)}{2(1-\gamma(1-\alpha))} ||\theta^*||^2 \right)$$

which is achieved by the optimal wealth process $X = S^0 M^F(V^*/S^0)$. Using (2.3) for $(D_t X_t)$ we see that

$$\mathrm{d}(D_t X_t) = \left(D_t X_t - w(\overline{D_t X_t})\right) \sum_{i=1}^d \left(\frac{1}{1-\gamma(1-\alpha)} \theta_t' \sigma_t^{-1} \right)^i \frac{\mathrm{d}(D_t S_t^i)}{D_t S_t^i}.$$

In particular, we recover Theorem 5.1 in [12] by taking $U = H^{(\gamma)}$, $\gamma \in (0,1)$ and $w(x) = \alpha x$. It is insightful to understand better how the objects in [12] relate to the tools of our work. In fact the Auxiliary problem introduced and solved in [12] is nothing else but $\mathrm{CER}_{U\circ F}(v_0) = \mathrm{CER}_{H^{(\gamma(1-\alpha))}}(v_0)$. Indeed, the process N^{π}_{α} defined in (4.1) therein is simply $S^0 M^K(X/S^0)$ and $\hat{\pi}_t = \frac{1}{1-\gamma(1-\alpha)}\theta'_t\sigma_t^{-1}$.

3.6.3 Incomplete market example

In a recent paper Guasoni and Robertson [28] solve the unconstrained portfolio optimisation problem for an investor with a power utility in a rather general diffusion model. Our results allow to solve w–drawdown constrained problem in their setting. The solution in [28] is involved and we do not cite the details here for the sake of brevity. Instead we propose to study an application of Theorem 3.4.1 in an incomplete market example adapted from the risk-sensitive control approach in Fleming and Sheu [22].

Consider a market with constant interest rate r and one risky asset $S(t)$ evolving according to

$$\begin{aligned} \frac{\mathrm{d}S(t)}{S(t)} &= (\mu_1 + \mu_2 x(t))\mathrm{d}t + \sigma \mathrm{d}W^1_t + \rho \mathrm{d}W^2_t, \\ \mathrm{d}x(t) &= bx(t)\mathrm{d}t + \mathrm{d}W^1_t, \end{aligned}$$

where W^1, W^2 are two independent Brownian motions and $x(t)$ has an interpretation of an economical factor.

In Theorem 3.1 in [22] the authors provide a link between Problem 1 with a power utility function $H^{(\gamma)}$ and a viscosity solution of the dynamic programming equation. In Theorem 4.1 the optimal investment policy is found. We refer the reader to [22] for further details of the method. In the last section of their paper Fleming and Sheu consider Vasicek interest rate model with a single stock and give an explicit solution to the utility maximisation problem. Our model above is slightly different but we are still able to use their solution.

The difference with Fleming and Sheu [22] example is that the interest rate is given by $r(t) = r$ in our work and by $r(t) = \lambda x(t) - \frac{b_1}{b_2}$ in theirs, which requires us to change some coefficients in final formulae in [22]. More precisely, assume $\gamma < 0$ and $\mu_2^2 \geq \sigma^2 (K^{(\gamma)})^2$ where $K^{(\gamma)}$ is defined below. Then the value function is equal to

$$\mathrm{CER}_{H^{(\gamma)}}(v_0) = \frac{1}{2}K^{(\gamma)} + \frac{1}{2}|\eta|^2 + 1/2\frac{\gamma}{1-\gamma}\frac{(\mu_1 + \sigma\eta)^2}{\sigma^2 + \rho^2} + |\gamma| r,$$

where

$$\eta = -\frac{\gamma}{1-\gamma}\frac{\mu_2 + K^{(\gamma)}\sigma\mu_1}{(D^{(\gamma)} + K^{(\gamma)}E^{(\gamma)})(\sigma^2+\rho^2)}$$

and

$$\begin{aligned}
E^{(\gamma)} &= 1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}, \\
K^{(\gamma)} &= -\frac{b + \frac{\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\mu_2\sigma}{1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}} - \frac{1}{\sqrt{1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}}} \\
&\quad \cdot \left(-\frac{\gamma}{1-\gamma}\frac{\mu_2^2}{\sigma^2+\rho^2} + \frac{(b + \frac{\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2})^2}{1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}} \right)^{1/2}, \\
D^{(\gamma)} &= -\sqrt{1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}} \left(-\frac{\gamma}{1-\gamma}\frac{\mu_2^2}{\sigma^2+\rho^2} + \frac{(b + \frac{\gamma}{1-\gamma}\frac{\mu_2\sigma}{\sigma^2+\rho^2})^2}{1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}} \right)^{1/2}.
\end{aligned}$$

And the optimal investment policy $\tilde{\pi}_t$, which is fraction of wealth invested in risky asset at time t, is given by

$$\tilde{\pi}_t = D^{(\gamma)}x(t) + a^{(\gamma)},$$

for some constant $a^{(\gamma)}$.

In the setting of Theorem 3.4.1 we obtain that

$$\mathrm{CER}_U^w(v_0) = \frac{1}{2}K^{(\gamma(1-\alpha))} + \frac{1}{2}|\eta|^2 + 1/2\frac{\gamma(1-\alpha)}{1-\gamma(1-\alpha)}\frac{(\mu_1+\sigma\eta)^2}{\sigma^2+\rho^2} + |\gamma|(1-\alpha)r,$$

where $\gamma < 0$ and $\alpha \in (0,1)$ are defined in Assumption 3.4.1. Note that we could also consider $\gamma \in (0,1)$ under the additional parameter restriction which makes appropriate $K^{(\cdot)}$ well defined.

3.7 Appendix

We state and prove here lemmas used in the proofs in the main body of the chapter. In fact the first two lemmas are of independent interests. Lemma 3.7.1 shows that computing CER_U it is enough to consider wealth processes which dominate a given fraction of the numeraire. Lemma 3.7.2 studies convergence of $\mathrm{CER}_{U_n} \to \mathrm{CER}_U$ as $U_n \to U$.

Lemma 3.7.1. *Let U be a continuous non-decreasing function with a well defined locally bounded right* *derivative* U'_+. *Assume* U *is either positive or negative and satisfies*

$$\limsup_{x\to\infty} \frac{xU'_+(x)}{|U(x)|} < \infty. \tag{3.17}$$

Then for any $v_0 > 0$ *and any numeraire* N

(i) for any $0 < y < v_0$

$$\begin{aligned} \sup_{V\in\mathcal{A}_N(v_0)} \mathcal{R}_U(V/N) &= \sup_{V\in\mathcal{A}_N(v_0),\ V/N\geq y} \mathcal{R}_U(V/N), \\ \sup_{V\in\mathcal{A}_{S^0}(v_0)} \mathcal{R}_U(V) &= \sup_{V\in\mathcal{A}_{S^0}(v_0),\ V\geq y} \mathcal{R}_U(V); \end{aligned}$$

(ii)

$$\begin{aligned} \mathrm{CER}_{U,N}(1) &= \mathrm{CER}_{U,N}(v_0), \\ \mathrm{CER}_U(1) &= \mathrm{CER}_U(v_0). \end{aligned}$$

Proof. First, for $V \in \mathcal{A}_N(v_0)$ and some $0 < \varepsilon < 1$ consider the process $\tilde{V}_t = \varepsilon v_0 N_t + (1-\varepsilon)V_t \geq \varepsilon v_0 N_t$, $t \geq 0$. As U satisfies (3.17) and $\frac{\tilde{V}_t}{N_t} \geq \varepsilon v_0$ we are able to use Lemma 3.7.3 to deduce that for some non-zero $\gamma \in \mathbb{R}$

$$(1-\varepsilon)^\gamma U\left(\frac{V_t}{N_t}\right) \leq (1-\varepsilon)^\gamma U\left(\frac{1}{1-\varepsilon}\frac{\tilde{V}_t}{N_t}\right) \leq U\left(\frac{\tilde{V}_t}{N_t}\right),$$

where we used $\frac{1}{1-\varepsilon}\frac{\tilde{V}}{N} \geq \frac{V}{N}$. Taking expectation, applying $\frac{1}{T}\log$ and taking limit as $t \to \infty$, we deduce that

$$\mathcal{R}_U(V/N) \leq \mathcal{R}_U(\tilde{V}/N). \tag{3.18}$$

Thus, taking $\varepsilon = y/v_0$ we obtain

$$\sup_{V\in\mathcal{A}_N(v_0)} \mathcal{R}_U(V/N) \leq \sup_{V\in\mathcal{A}_N(v_0),\ \text{s.t.} V/N\geq y} \mathcal{R}_U(V/N)$$

and the reverse inequality is trivial.

Similarly, for $V \in \mathcal{A}_{S^0}(v_0)$ considering $\tilde{V}_t = \varepsilon v_0 S^0_t + (1-\varepsilon)V_t \geq \varepsilon v_0 S^0_t \geq \varepsilon v_0$ and using analogous arguments we obtain that

$$\mathcal{R}_U(\tilde{V}) \geq \mathcal{R}_U(V).$$

Taking $\varepsilon = y/v_0$ we conclude that the second equality in (i) also holds.

To show (ii), consider $V \in \mathcal{A}_N(v_0)$ such that $V/N \geq y$ for some $y > 0$. Note that $V^1 := \frac{1}{v_0} V \in \mathcal{A}_N(1)$. If $v_0 > 1$, Lemma 3.7.3 and monotonicity of U yield, for some $\gamma \in \mathbb{R} \setminus \{0\}$,

$$U(V_t^1/N_t) \leq U(V_t/N_t) = U(v_0 V_t^1/N_t) \leq v_0^\gamma U(V_t^1/N_t), \quad t \geq 0.$$

If $0 < v_0 < 1$ we obtain similarly

$$v_0^\gamma U(V_t^1/N_t) = v_0^\gamma U\left(\frac{1}{v_0} V_t/N_t\right) \leq U(V_t/N_t) \leq U(V_t^1/N_t).$$

It follows that

$$\sup_{V \in \mathcal{A}_N(v_0),\ V/N \geq y} \mathcal{R}_U(V/N) = \sup_{V^1 \in \mathcal{A}_N(1),\ V^1/N \geq y/v_0} \mathcal{R}_U(V^1/N)$$

The first equality in (ii) now follows from (i) and the second one is analogous. □ □

Lemma 3.7.2. *Let N be a numeraire, U_n, U, all nondecreasing functions of the same sign, continuous with a well defined locally bounded right derivative, satisfying Assumption 3.3.1 and* (3.17). *Assume further that for some $c, c_1 > 0$ and some $0 < \nu < 1$*

$$\forall\ \delta > 0\ \exists n_\delta\ \forall n \geq n_\delta \quad c_1 U(x) \leq U_n(x) \leq U(cx^{1+\delta}),\ x \geq \nu.$$

If U_n, U are negative we have, for any $v_0 > 0$,

$$\begin{gathered} \mathrm{CER}_{U_n,N}(1) \xrightarrow[\text{n}\to\infty]{} \mathrm{CER}_{U,N}(1), \\ \mathrm{CER}_{U_n}(1) \xrightarrow[\text{n}\to\infty]{} \mathrm{CER}_{U}(1). \end{gathered} \tag{3.19}$$

If U_n, U are positive the above holds assuming that $\mathrm{CER}_{G,N}(1) < \infty$ *and* $\mathrm{CER}_G(1) < \infty$ *respectively, where $G(x) := U(x)^{1+\delta}$ for some $\delta > 0$. Consequently, we then have* $\mathrm{CER}_{U,N}(1) < \infty$ *and* $\mathrm{CER}_U(1) < \infty$ *respectively.*

Proof. We prove both statements in (3.19) simultaneously. They follow respectively by taking $\xi = \frac{V}{N}$, $V \in \mathcal{A}_N(1)$ and $\xi = V \in \mathcal{A}_{S^0}(1)$ in what follows. Observe that, by Lemma 3.7.1 it is sufficient to consider $\xi \geq \nu$.

Assume that for n and K large enough and any $\xi \geq \nu$ we have

$$\begin{aligned} \mathcal{R}_{U_n}(\xi) &= \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}[U_n(\xi_T)] = \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}[U_n(\xi_T)\,\mathbf{1}_{\xi_T \leq K^T}], \\ \mathcal{R}_{U}(\xi) &= \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}[U(\xi_T)] = \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}[U(\xi_T)\,\mathbf{1}_{\xi_T \leq K^T}]. \end{aligned} \tag{3.20}$$

Take $\delta > 0$, large K, T, n so that the assumptions yield

$$\begin{aligned} c_1 U(\xi_T)\mathbf{1}_{\xi_T \le K^T} \le U_n(\xi_T)\mathbf{1}_{\xi_T \le K^T} &\le U(c\xi_T^{1+\delta})\mathbf{1}_{\xi_T \le K^T} \le U(cK^{\delta T}\xi_T)\mathbf{1}_{\xi_T \le K^T} \\ &\le (cK^{\delta T})^{\gamma} U(\xi_T)\mathbf{1}_{\xi_T \le K^T}, \end{aligned}$$

where we used Lemma 3.7.3 to obtain the last inequality. Recall that we defined $\log x = -\log(-x)$ for $x < 0$. Taking expectation, applying $\frac{1}{T}\log$ and taking the limit as $T \to \infty$ in the above we conclude, thanks to (3.20), that

$$\mathcal{R}_U(\xi) \le \mathcal{R}_{U_n}(\xi) \le \mathcal{R}_U(\xi) + |\gamma|\delta \log K.$$

This is true for n large enough and any ξ and hence also when we take supremum over ξ. We deduce (3.19) taking $\delta \to 0$.

It remains to argue (3.20). We will prove this for separately depending on the sign of U. Consider first $U_n, U \ge 0$. Assumption 3.3.1 implies that there exists $\tilde{c} > 0$ and $\varepsilon > 0$ such that $\nu \le x < \tilde{c}U(x)^{1/\varepsilon}$. For any $\delta' > 0$, using Lemma 3.7.3, we obtain

$$\begin{aligned} U(x^{1+\delta'})^{1+\delta'} \le &\quad U\left(\left(\tfrac{x}{\nu}\right)^{\delta} x\right)^{1+\delta} \le \left(\left(\tfrac{x}{\nu}\right)^{\gamma\delta'} U(x)\right)^{1+\delta'} \\ \le &\quad \left(\tfrac{\tilde{c}}{\nu}\right)^{\gamma\delta'(1+\delta')} U(x)^{1+\delta'(1+\frac{\gamma'(1+\delta')}{\varepsilon})}, \quad x \ge \nu. \end{aligned}$$

From the proof of Lemma 3.7.3 it is clear that U and γ have the same sign and we conclude that for some $\delta' \le \delta$ we have

$$U(x^{1+\delta'})^{1+\delta'} \le c_2 U(x^{1+\delta}) = c_2 G(x), \quad x \ge \nu, \tag{3.21}$$

for some $c_2 > 0$.

Using Chebyshev's inequality we obtain

$$\mathbb{P}\left(\xi_T \ge K^T\right) \le \frac{\mathbb{E}U_n(\xi_T)}{U_n(K^T)} \le \frac{\mathbb{E}U_n(\xi_T)}{\tilde{c}^{-\varepsilon}c_1 K^{\varepsilon T}}.$$

In the last inequality we used again the fact that $U_n(x) \ge c_1 U(x)$ for $x \ge \nu$ and that $U(x) \ge \tilde{c}^{-\varepsilon}x^{\varepsilon}$. Take $n > n_{\delta'}$ with δ' as in (3.21). Combining the above and using twice Hölder's inequality with $p = 1 + \delta'$,

$1/p + 1/q = 1$, we obtain

$$\begin{aligned}\mathbb{E}[U_n(\xi_T)\mathbf{1}_{\xi_T \geq K^T}] &\leq (\mathbb{E}U_n(\xi_T)^p)^{\frac{1}{p}}\,\mathbb{P}(\xi_T \geq K^T)^{\frac{1}{q}} \leq (\mathbb{E}U_n(\xi_T)^p)^{\frac{1}{p}}\left(\frac{EU_n(\xi_T)}{\tilde{c}^{-\varepsilon}c_1K^{\varepsilon T}}\right)^{\frac{1}{q}} \\ &\leq \frac{(\mathbb{E}U_n(\xi_T)^p)^{\frac{1}{p}\left(1+\frac{1}{q}\right)}}{\tilde{c}^{-\varepsilon/q}c_1^{1/q}K^{\varepsilon T/q}} \leq \frac{\left(\mathbb{E}U(c\xi_T^{\delta'+1})^{1+\delta'}\right)^{\frac{1}{p}\left(1+\frac{1}{q}\right)}}{\tilde{c}^{-\varepsilon/q}c_1^{1/q}K^{\varepsilon T/q}}.\end{aligned}$$

Let C_G denote $\mathrm{CER}_{G,N}(1)$ or $\mathrm{CER}_G(1)$ depending on whether we consider $\xi = V/N$ or $\xi = V$. Let γ' be the constant resulting from Lemma 3.7.3 applied with $x_0 = v^{1+\delta'}$. We can then continue the above chain of inequalities

$$\begin{aligned}\frac{\left(\mathbb{E}U(c\xi_T^{\delta'+1})^{1+\delta'}\right)^{\frac{1}{p}\left(1+\frac{1}{q}\right)}}{\tilde{c}^{-\varepsilon/q}c_1^{1/q}K^{\varepsilon T/q}} &\leq \frac{\left(\max\{c,1\}^{\gamma'}\mathbb{E}U(\xi_T^{\delta'+1})^{1+\delta'}\right)^{\frac{1}{p}\left(1+\frac{1}{q}\right)}}{\tilde{c}^{-\varepsilon/q}c_1^{1/q}K^{\varepsilon T/q}} \\ &\leq c_3\frac{\exp((C_G+\kappa)(1/p+1/pq)T)}{K^{\varepsilon T/q}} \\ &= c_3\exp\left(\left((C_G+\kappa)\left(\frac{1}{p}+\frac{1}{pq}\right)-\frac{\varepsilon}{q}\log K\right)T\right),\end{aligned}$$

where to get the second inequality we used (3.21) and the fact that for any $\kappa > 0$, for T large enough, we have $\mathbb{E}G(\xi_T) \leq \exp((C_G(1)+\kappa)T)$. c_3 is a positive constant which is can be made explicit from the above computation. For K large enough, the above is decreasing exponentially in T. Combining the two displays above we conclude that for any $\kappa > 0$, $n > n_{\delta'}$, K large enough and all T large enough we have $\mathbb{E}[U_n(\xi_T)\mathbf{1}_{\xi_T \geq K^T}] \leq \kappa$ and hence

$$\mathbb{E}[U_n(\xi_T)] \geq \mathbb{E}[U_n(\xi_T)\mathbf{1}_{\xi_T \leq K^T}] \geq \mathbb{E}[U_n(\xi_T)] - \kappa \geq \mathbb{E}[U_n(\xi_T)]\left(1 - \frac{\kappa}{c_1U(\nu)}\right),$$

where we wrote $\kappa = \frac{\kappa}{\mathbb{E}[U_n(\xi_T)]}\mathbb{E}[U_n(\xi_T)] \leq \frac{\kappa}{U_n(\nu)}\mathbb{E}[U_n(\xi_T)]$ and used the assumption $U_n \geq c_1U$. The first equality in (3.20) now follows by taking expectations, applying $\frac{1}{T}\log$ and letting $T \to \infty$. Analogous, but simplified, arguments to the above yield the second equality in (3.20).

It remains to argue (3.20) when $U_n, U < 0$. We detail the arguments for U_n and the first equality in (3.20). Obviously $0 \geq U_n(\xi_T)\mathbf{1}_{\xi_T \leq K^T} \geq U_n(\xi_T)$ so (3.20) holds if $\zeta_n := \mathcal{R}_{U_n}(\xi) = \infty$. Assume now that $\zeta_n < \infty$ and note also that $\zeta_n \geq 0$ since $\xi \geq \nu$. Using Assumption 3.3.1 on U we see that there exists $\varepsilon > 0$ such that $0 > U(x) \geq -\tilde{c}x^{-\varepsilon}$, $x \geq \nu$. This yields $\mathbb{E}[U_n(\xi_T)\mathbf{1}_{\xi_T \geq K^T}] \geq c_1U(K^T) \geq -c_1\tilde{c}K^{-T\varepsilon}$. It follows that

$$\mathbb{E}[U_n(\xi_T)] \le \mathbb{E}[U_n(\xi_T)\mathbf{1}_{\xi_T \le K^T}] \le \mathbb{E}[U_n(\xi_T)] + c_1\tilde{c}K^{-T\varepsilon}$$
$$= \mathbb{E}[U_n(\xi_T)]\left(1 - \frac{c_1\tilde{c}K^{-T\varepsilon}}{\mathbb{E}|U_n(\xi_T)|}\right) \le \mathbb{E}[U_n(\xi_T)](1 - c_4\mathrm{e}^{-(\varepsilon \ln K - \zeta_n - \kappa)T}),$$

where $c_4 = c_1\tilde{c}$, we took $\kappa > 0$ arbitrary and T large enough. Taking $K > \exp((\zeta_n + \kappa)/\varepsilon)$, applying $\frac{1}{T}\log$ and letting $T \to \infty$ we see that (3.20) holds.

□

The following Lemma is a slight extension of the first part of Lemma 6.3 in Kramkov and Schachermayer [40].

Lemma 3.7.3. *Let* $U : (0, \infty) \to \mathbb{R}$ *be a continuous nondecreasing function, either strictly positive or strictly negative, with a well defined and locally bounded right derivative and which satisfies* (3.17). *Then for any* $x_0 > 0$ *there exists* $\gamma \in \mathbb{R} \setminus \{0\}$ *such that*

$$U(x) \le U(\lambda x) \le \lambda^\gamma U(x) \quad \text{for all } \lambda > 1,\ x \ge x_0.$$

Proof. Let $x_0 > 0$. From (3.17), the fact that U is monotone and of constant sign, and U'_+ is locally bounded, there exists non-zero $\gamma \in \mathbb{R}$ such that

$$\frac{xU'_+(x)}{\gamma U(x)} < 1 \quad x \ge x_0,$$

where γ has the same sign as U.

Fix $x \ge x_0$ and define functions $F(\lambda) := U(\lambda x)$ and $G(\lambda) := \lambda^\gamma U(x)$ for $\lambda > 1$. Then, $F(1) = G(1)$ and $F'_+(1) = xU'_+(x) < \gamma U(x) = G'_+(1)$. Hence, $F(\lambda) > G(\lambda)$ for $\lambda \in (1, 1+\varepsilon)$ for some $\varepsilon > 0$. Assume that $F < G$ for some point in $(0, 1)$ then from continuity of F and G there exists a point $\lambda^* > 1$ such that $F(\lambda^*) = G(\lambda^*)$ and $F'_+(\lambda^*) \ge G'_+(\lambda^*)$, but

$$F'_+(\lambda^*) = xU'_+(\lambda^* x) < \frac{\gamma}{\lambda^*}U(\lambda^* x) = \frac{\gamma}{\lambda^*}F(\lambda^*) = \frac{\gamma}{\lambda^*}G(\lambda^*) = G'_+(\lambda^*),$$

which leads to contradiction. □

Lemma 3.7.4. *Suppose* U *is a utility function which satisfies the first condition of Assumption 3.4.1. Then for any* $c > 0, \varepsilon > 0$, *there exist* $c_1, c_2 > 0$ *such that*

$$c_1 H^{(\gamma(1-\varepsilon))}(x) \le U(x) \le c_2 H^{(\gamma(1+\varepsilon))}(x), \quad x \ge c.$$

Proof. The assumptions on U mean that there exists $x_0 > 0$ such that

$$\frac{yU'_+(y)}{U(y)} \in (\gamma(1-\varepsilon), \gamma(1+\varepsilon)) \text{ for } y \in [x_0, \infty).$$

For $x \geq x_0$ we express $U(x)$ as

$$U(x) = U(x_0) \exp\left\{ \int_{x_0}^{x} \frac{yU'_+(y)}{U(y)} \frac{\mathrm{d}y}{y} \right\}. \tag{3.22}$$

In the case of positive U the claim follows by taking

$$c_1 := \min\left\{ \frac{U(x)}{H^{(\gamma(1-\varepsilon))}(x)}, x \in [c, y_1] \right\}$$

and $c_2 := \max\left\{ \frac{U(x)}{H^{(\gamma(1+\varepsilon)}(x))}, x \in [c, y_1] \right\}$. For a negative U we interchange max and min in definitions of c_1, c_2. $\square$

Chapter 4

Floor constraint

4.1 Introduction

We are studying a dynamic asset allocation problem of an investor whose preferences are characterised via a utility function. She aims to insure a specified minimum value for the wealth of her portfolio given as a floor process, which is also referred in literature as an American capital guarantee. This constraint on the wealth of an investor is motivated by different insurance products available in a real-world financial market commonly named as a floor. These products guarantee a pre-specified minimum wealth of the investor's portfolio.

This chapter deals with the problem of maximising the asymptotic rate of growth of the expected utility of terminal wealth subject to a floor constraint. The main result states that for any reasonable floor process there exists a wealth which maximises the Certainty Equivalent Rate (CER, see Chapter 3 for details) for an unconstrained problem and dominates any given fraction of this floor process. The outcome of the result is that the floor constraint does not affect the optimal value of the long-term optimisation problem, which provides a certain critique of maximisation of an asymptotic growth rate of expected utility. We construct the optimal wealth process as well as the strategy explicitly.

We want to notice that by the time the main result of this chapter was derived we found out about the paper by Sekine [53], who deals with the same problem and obtains similar results for power utility functions on diffusion driven markets.

In this chapter we also provide a connection between the optimisation subject to a floor constraint and the drawdown constraint for a long-term investor. We use the equivalence results from Chapter 3, where the unconstrained problem was explicitly linked to drawdown constrained problem.

The long-term behavior of an investor whose preferences are characterised via a utility function are extensively studied in seminal papers. In particular, the behavior of the optimal strategy as horizon tends to infinity was studied in discrete market setting ([46], [41], [29]) and continuous market setting ([10], [31], [16], [27]). These results are of the class of turnpike theorems and examine the necessary and sufficient conditions for an optimal strategy for finite horizon to converge as the horizon tends to infinity.

Black and Jones [4] and Perold and Sharpe [47] examine the automated strategies including the Constant Proportion Portfolio Insurance method which guarantees that discounted value of the wealth dominates a pre-specified final floor. More generally, the principle of fund separation is studied. In a recent paper by Guasoni and Robertson [27], it is shown that as the horizon becomes long an investor tends to trade according to fund separation principle.

One of the first works on the optimal portfolio allocation problems subject to pathwise constraints was written by Grossman and Zhou [25]. The authors considered a drawdown constraint imposed on the discounted wealth process, which guarantees that the wealth will not fall below a prescribed percentage of the up to date maximum. In the context of the complete Black-Scholes market with constant coefficients they obtained an explicit representation of the optimal strategy for the long term investor who maximises her certainty equivalent rate, or rate of growth of the expected utility of wealth. Their paper was simplified and generalised in [12], where the problem was solved by first introducing an auxiliary unconstrained problem which is easier to deal with and, then, connecting the problems explicitly. More generally, this connection was examined in Chapter 3, where the equivalence result is established. This result explicitly links constrained and unconstrained problems in a general semimartingale market.

In [18], the authors studied the floor constraint imposed on investor's wealth in the context of finite horizon utility of terminal wealth maximisation. A solution is obtained for the case of complete market and the optimal wealth is given in terms of the max-plus decomposition of the floor process. In this chapter we extensively use the approach established in [18] to deal with the finite horizon utility maximisation problem with the specific floor for which it is possible to obtain the solution in an explicit manner by means of so-called Azéma–Yor processes which were originally introduced in [2]. The properties of Azéma–Yor processes in relation to constrained processes were recently studied in [5].

A recent paper by Guasoni and Robertson [28] describes the concept of long-run optimality of the wealth process. The wealth process of a long-term investor is called long-run optimal if the rate of growth of value functions for finite horizon problems of an investor who aims to maximise her expected utility of terminal wealth converges to the certainty equivalent rate. We extend this notion to the framework of constrained optimisation. Namely, we provide conditions for the wealth which solves long-term optimisation problem subject to a constraint to be long-run optimal.

The chapter is organised as follows. First, we introduce a general market setup and solve the long-term investor problem in Section 4.2 . In Section 4.3 the long-run property is studied in the constrained framework. Finally, in Section 4.4 we solve the finite horizon utility maximisation problem subject to a floor constraint of a special form.

4.2 Optimal portfolio of a long-term investor subject to floor constraint

4.2.1 Market and problem formulation

We consider a frictionless market defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with the dynamics of n risky assets given as a vector of strictly positive semimartingales $S_t = (S_t^1, .., S_t^n)$. The riskless asset is a predictable non-decreasing process (S_t^0) with $S_0^0 = 1$. We define $D_t := 1/S_t^0$ to be a discounting factor.

The investor is assumed to trade on this market in the usual self-financing way. More precisely, we define the set of all admissible investment strategies according to the following definition of admissible wealth process.

Definition 4.2.1. *An adapted semimartingale (V_t) is called a **wealth process** if it is non-negative and there exists an $(\mathcal{F}_t)$–predictable process $\pi = (\pi_t^1, \dots, \pi_t^d)$ such that $D_t V_t = V_0 + \sum_{i=1}^d \int_0^t \pi_u^i \mathrm{d}(D_u S_u^i)$, where the integrals are assumed to be well-defined.*

The set of wealth processes with $V_0 = v_0$ is denoted $\mathcal{A}(v_0)$.

We say that (N_t) is a **numèraire** if it is a strictly positive wealth process with $N_0 = 1$. One can see that (V_t) is a wealth process if and only if it is non-negative, and there exists an $(\mathcal{F}_t)$–predictable process $\pi = (\pi^0_t, \dots, \pi^d_t)$ such that $V_t/N_t = V_0 + \sum_{i=0}^d \int_0^t \pi^i_u \mathrm{d}(S^i_u/N_u)$.

For a given adapted non-negative semimartingale process $(G_t)_{t\geq 0}$ we consider a subclass of all wealth processes defined as follows

$$\mathcal{A}_G(v_0) := \{(V_t) \in \mathcal{A}(v_0) : \quad V_t \geq G_t, \quad t \geq 0\}.$$

4.2.2 Main result

In this section we introduce the infinite horizon problem of an investor who aims to maximise her long-term growth rate of utility of terminal wealth subject to a floor constraint. Our main theorem provides the connection between constrained and unconstrained problems. We are able to show that the floor constraint does not affect the value function for a wide class of the floor processes considered. More precisely, for any floor process which admits at least one wealth process dominating it, there exists an optimal solution to the unconstrained problem which dominates any given fraction of this floor process.

We consider function U which satisfies the following

Assumption 4.2.1. *Function U is non-decreasing and it is either strictly positive, or it is strictly negative. It satisfies*

$$\limsup_{x\to\infty} \frac{xU'_+(x)}{|U(x)|} < \infty \tag{4.1}$$

We now state our main result.

Theorem 4.2.1. *Consider a concave non-decreasing function U which satisfies Assumptions 4.2.1, let $v_0 > 0$ be an initial wealth. Consider a floor process $(G_t)_{t\geq 0} \in \mathcal{A}(v_0)$ then for any $0 < \varepsilon < 1$*

$$\sup_{V\in\mathcal{A}(v_0)} \mathcal{R}_U(V) = \sup_{V\in\mathcal{A}_{(1-\varepsilon)G}(v_0)} \mathcal{R}_U(V), \qquad \textit{where} \tag{4.2}$$

$$\mathcal{R}_U(V) = \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\left[U\left(V_T\right)\right].$$

Moreover, right hand side in (4.2) is maximised by $\hat{V}_t := \varepsilon\hat{\xi}_t + (1-\varepsilon)G_t$, where $(\hat{\xi}_t)_{t\geq 0}$ maximises the left hand side and $\hat{\xi}_t \geq c > 0$ a.s. for some $c \in \mathbb{R}^+$.

In the definition of $\mathcal{R}_U$, and throughout the chapter, we extend log to $\mathbb{R} \setminus \{0\}$ via $\log(x) = -\log(-x)$.

Remark 4.2.1. *The condition that the optimal wealth process for an unconstrained problem is bounded away from zero is not restrictive and, in fact, such wealth process always exists. Let* $(\hat{\eta}_t)_{t\geq 0} \in \mathcal{A}(v_0)$ *be optimal for unconstrained problem then for any* $\delta \in (0,1)$ *we consider* $\xi_t := \delta v_0 S^0_t + (1-\delta)\hat{\eta}_t{}_t \geq \delta$ *which belongs to* $\mathcal{A}(v_0)$ *and, whence, using Lemma 3.7.3 from Chapter 3 we obtain for some* $\gamma' > 0$*:*

$$U(\xi_t) \geq (1-\delta)^{\gamma'} U(\frac{\delta}{1-\delta} v_0 S^0_t + \hat{\eta}^1_t) \geq (1-\delta)^{\gamma'} U(\hat{\eta}^1_t).$$

Thus, $\mathcal{R}(\xi) \geq \mathcal{R}(\hat{\eta})$*, which concludes the optimality of* ξ*.*

In a similar way, considering a wealth process $\psi_t := \delta v_0 N_t + (1-\delta)\hat{\eta}_t \geq \delta$ *we obtain* $\mathcal{R}(\psi/N) \geq \mathcal{R}(\hat{\eta}/N)$*.*

Corollary 4.2.1. *Consider a concave non-decreasing function* U *which satisfies Assumptions 4.2.1; let* $v_0 > 0$ *be an initial wealth and let* N *be a numéraire. Consider a floor process* $(G_t)_{t\geq 0} \in \mathcal{A}(v_0)$ *then for any* $\varepsilon > 0$

$$\sup_{V \in \mathcal{A}(v_0)} \mathcal{R}_U(V/N) = \sup_{V \in \mathcal{A}_{(1-\varepsilon)G}(v_0)} \mathcal{R}_U(V/N), \tag{4.3}$$

Moreover, right hand side in (4.3) *is maximised by* $\tilde{V}_t := \varepsilon\hat{\psi}_t + (1-\varepsilon)G_t$*, where* $(\hat{\psi}_t)_{t\geq 0}$ *maximises the left hand side and* $\hat{\psi}_t \geq c > 0$ *a.s. for some* $c \in \mathbb{R}^+$*.*

Proof. The proofs of Corollary 4.2.1 and Theorem 4.2.1 consist of similar steps, therefore, we prove them together in the sequel.

Consider a process $\hat{V}_t := \varepsilon\hat{\xi}_t + (1-\varepsilon)G_t$. It is a wealth process from the class $\mathcal{A}(v_0)$. This in fact turns out to be the exact same strategy as used by Sekine in [53].

From nonnegativity of $\hat{\xi}$ we deduce

$$\hat{V}_t \geq (1-\varepsilon)G_t.$$

Therefore, $(\hat{V}_t)_{t\geq 0} \in \mathcal{A}_{(1-\varepsilon)G}(v_0)$.

Similarly, one can check that $(\tilde{V}_t)_{t\geq 0} \in \mathcal{A}_{(1-\varepsilon)G}(v_0)$.

Now, we need to show the optimality of wealth processes $\hat{V}$ and $\tilde{V}$. We know that if $\theta^1_t \geq \theta^2_t$ for all $t \geq 0$ then $\mathcal{R}(\theta^1) \geq \mathcal{R}(\theta^2)$. This is due to the fact that U is non-decreasing. And as $\hat{V}_t \geq \varepsilon\hat{\xi}_t$ we deduce that $\mathcal{R}(\hat{V}) \geq \mathcal{R}(\varepsilon\hat{\xi})$ and $\mathcal{R}(\tilde{V}/N) \geq \mathcal{R}(\varepsilon\hat{\psi}/N)$.

Using Lemma 3.7.3 from Chapter 3 for function U which satisfies Assumption 4.2.1, we obtain that for any $x_0 > 0$ there exists $\gamma \in \mathbb{R}$ such that

$$U(x) \leq U(\lambda x) \leq \lambda^\gamma U(x) \quad \text{for any} \quad \lambda > 1, x \geq x_0.$$

As $\hat{\xi}_t \geq c$ and $\hat{\psi}_t \geq cN_t$ for some $c > 0$, we deduce that $U(\varepsilon\hat{\xi}_t) \geq \varepsilon^\gamma U(\hat{\xi}_t)$ for $x_0 = c\varepsilon$ by Lemma 3.7.3 from Chapter 3. Thus, $\mathcal{R}(\varepsilon\hat{\xi}) = \mathcal{R}(\hat{\xi})$. Similarly, $\mathcal{R}(\varepsilon\hat{\psi}/N) = \mathcal{R}(\hat{\psi}/N)$. □

Remark 4.2.2. *From the proof of Corollary 4.2.1 and Theorem 4.2.1 we obtain that*

$$\limsup_{T\to\infty} \frac{1}{T}\Big(\log \mathbb{E}U(\hat{\xi}_T) - \log \mathbb{E}U(\hat{V}_T)\Big) = 0,$$
$$\limsup_{T\to\infty} \frac{1}{T}\Big(\log \mathbb{E}U(\hat{\psi}_T/N_T) - \log \mathbb{E}U(\tilde{V}_T/N_T)\Big) = 0,$$

Proof. Indeed, we have that

$$\limsup_{T\to\infty} \frac{1}{T}\Big(\log \mathbb{E}U(\hat{\xi}_T) - \log \mathbb{E}U(\hat{V}_T)\Big) \geq \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}U(\hat{\xi}_T) - \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}U(\hat{V}_T) = 0$$

On the other hand, $\hat{V}_t \geq \varepsilon\hat{\xi}_t$, whence, using Lemma 3.7.3 from Chapter 3 we obtain

$$\begin{aligned}&\limsup_{T\to\infty} \frac{1}{T}\Big(\log \mathbb{E}U(\hat{\xi}_T) - \log \mathbb{E}U(\hat{V}_T)\Big)\\ &\leq \limsup_{T\to\infty} \frac{1}{T}\Big(\log \mathbb{E}U(\hat{\xi}_T) - \log \varepsilon^\gamma \mathbb{E}U(\hat{\xi}_T)\Big) = 0\end{aligned}$$

□

Remark 4.2.3. *Combining main result of this chapter and Theorem 4.2.1 from Chapter 3 we are able to connect in an explicit manner all three problems: the unconstrained problem, the problem with a drawdown constraint and the problem with the floor constraint.*

More precisely, assume that the stock process $(S_t)_{t\geq 0}$ is continuous.

Proposition 4.2.1. *Let w be a drawdown function, N a numéraire, $v_0 > 0$ and U a nondecreasing concave function with finite asymptotic elasticity satisfying Assumption 4.2.1 and Assumption 3.3.1 from Chapter 3.*

Consider K_w with $F_w = K_w^{-1}$ defined in (2.1) with $a = a^* = v_0$ and consider any extension of F_w on $\mathbb{R}^+$ which preserves monotonicity and concavity with $F_w(0) \geq 0$. Assume that, for some $\delta > 0$, $\sup_{V\in\mathcal{A}(v_0)} \mathcal{R}_\psi(V/N) < \infty$ where $\psi(x) = U \circ F_w(x)$ when $U < 0$ and $\psi(x) = (U \circ F_w(x))^{1+\delta}$ when $U > 0$. Consider a floor process $(G_t)_{t\geq 0} \in \mathcal{A}(v_0)$. Then, for any $\varepsilon > 0$

$$\sup_{V\in\mathcal{A}(v_0)} \mathcal{R}_{U\circ F_w}(V/N) = \sup_{V\in\mathcal{A}^{(1-\varepsilon)G}(v_0)} \mathcal{R}_{U\circ F_w}(V/N) = \sup_{V\in\mathcal{A}_{w,N}(v_0)} \mathcal{R}_U(V/N) < \infty \tag{4.4}$$

and if $(V_t^*) \in \mathcal{A}(v_0)$ is bounded away from zero and achieves the maximum in the unconstrained problem then $\varepsilon V_t^* + (1-\varepsilon)G_t$ achieves the maximum in the problem with floor constraint, and $N_t M_t^{F_w}(V^*/N) \in \mathcal{A}_N^w(v_0)$ achieves the maximum in the w-drawdown constrained problem.

4.3 Long-run properties

4.3.1 Definition of long-run optimality

In this section we aim to study the behaviour of the value function of the finite horizon problem when the horizon becomes distant. In number of papers, the behavior of the optimal strategy as horizon tends to infinity was studied. Namely, the turnpike theorems (see ([46], [41], [29] for discrete market setting and [10], [31], [16], [27] for continuous market setting) for details) state that the optimal strategy for the finite horizon problem converges as the horizon tends to infinity. In the classical case when the utility function satisfies certain asymptotic properties and the market is Black-Scholes, it was shown that optimal strategies converge to a CPPI-type strategy.

In [28], long-run optimality of the investment strategies is studied. An investment strategy is called long-run optimal if the rate of growth of the value function for finite horizon problem converges to a certainty equivalent rate. In this section, we establish a long-run optimality results in the constrained framework, i.e. when there is either a floor constraint, or a drawdown constraint imposed on the wealth process.

First, we define classes of admissible wealth processes up to time horizon T.

Definition 4.3.1. *An adapted semimartingale $(V_t)_{t\geq 0}$ is called a* ***wealth process*** *if it is strictly positive and there exists an $(\mathcal{F}_t)$–predictable process $\pi = (\pi_t^1, \ldots, \pi_t^d)$ such that $D_t V_t = V_0 + \sum_{i=1}^d \int_0^t \pi_u^i \mathrm{d}(D_u S_u^i)$ for all $t \geq 0$, where the integrals are assumed to be well-defined.*

For a given numeraire N the set of wealth processes with $V_0 = v_0$ and with $(V_t/N_t)_{t\geq 0}$ being max-continuous (i.e. with continuous running maximum) is denoted $\mathcal{A}_N(v_0)$.

For a given floor process $(G_t)_{0\leq t\leq T}$ we define a subclass of $\mathcal{A}_N(v_0)$ as

$$\mathcal{A}_{G,N}(T)(v_0) := \{V \in \mathcal{A}_T(v_0) \quad \text{s.t.} \quad V_t \geq G_t, \quad 0 \leq t \leq T\}.$$

For a given drawdown function and numéraire N we define

$$\mathcal{A}^w_N(T)(v_0) := \{V \in \mathcal{A}(v_0) \quad \text{s.t.} \quad V_t/N_t \geq w(\overline{V/N}_t) \quad 0 \leq t \leq T\}.$$

We simply write $\mathcal{A}^w_N(v_0)$ and $\mathcal{A}_{G,N}(v_0)$ when $T = \infty$.

Definition 4.3.2. *Consider a nondecreasing function U and numéraire N. A wealth process $\hat{V} \in \mathcal{B}$ is called $\mathcal{B}$-long-run optimal if*

$$\limsup_{T\to\infty} \left(\frac{1}{T} \log \sup_{V\in\mathcal{B}_T} \mathbb{E}U(V_T/N_T) - \frac{1}{T} \log \mathbb{E}U(\hat{V}_T/N_T) \right) = 0, \tag{4.5}$$

where $\mathcal{B}_T = \mathcal{A}_N(v_0)$, $\mathcal{A}_{G,N}(T)(v_0)$ or $\mathcal{A}^w_N(T)(v_0)$, and $\mathcal{B} = \mathcal{A}_N(v_0)$, $\mathcal{A}_{G,N}(v_0)$ or $\mathcal{A}^w_N(v_0)$, respectively.

Remark 4.3.1. *One similarly defines a long-run optimal wealth process in the case of inhomogeneous problems with an undiscounted wealth process.*

For the necessity of this definition and intuition behind we refer to [28] where the authors define a *certainty equivalent loss* $l : [0, \infty) \to \mathbb{R}$ of a wealth process $(V_t)_{t\geq 0}$ by

$$\mathbb{E}U(e^{l_T T} V_T/N_T) = \sup_{Y\in\mathcal{A}_N(v_0)} \mathbb{E}U(Y_T/N_T).$$

One can see that for a process $V \in \mathcal{A}_N(v_0)$ such that $V/N \geq c > 0$: if $l_T(V) \to 0$ as $T \to \infty$ then V is $\mathcal{A}_N(v_0)$-long-run optimal.

Indeed, using definition of l we obtain

$$\sup_{V\in\mathcal{A}_N(v_0)} \mathbb{E}U(V_T/N_T) = \mathbb{E}U(e^{l_T T} V_T/N_T).$$

Since $V/N \geq c > 0$ we are able to use Lemma 3.7.3 from Chapter 3 to derive for some $\gamma \in \mathbb{R}$ that $U(e^{l_T T} V_T/N_T) \leq e^{\gamma l_T T} U(V_T/N_T)$. Thus,

$$\limsup_{T\to\infty} \left(\frac{1}{T} \log \sup_{V\in\mathcal{A}_N(v_0)} \mathbb{E}U(V_T/N_T) - \frac{1}{T} \log \mathbb{E}U(V_T/N_T) \right) \leq \limsup_{T\to\infty} \gamma l_T = 0$$

In other words, vanishing certainty equivalent loss is a sufficient condition for long-run optimality.

Long-run optimal wealth process allows to keep asymptotically optimal growth rate of expected utility as well as it stays close to optimal for long finite horizons.

Our definition generalises this concept to a constrained setup. We are able to obtain necessary and sufficient conditions under appropriate assumptions for constrained long-run optimality. The definition of long-run optimality from [28] implies our definition. In fact it coincides when a utility function is power and the certainty equivalent rate is achieved with a limit.

4.3.2 Floor constraint

In this subsection we aim to construct the long-run optimal portfolio for a problem with a floor constraint from the long-run optimal portfolio for an unconstrained problem.

The main result for this section is the following:

Theorem 4.3.1. *Let* $(\hat{X}_t)_{t\geq 0}$ *be* $\mathcal{A}_N(v_0)$*-long-run optimal for some numéraire* N *and utility function* U *which satisfies Assumption 4.2.1. Then, for any* $0<\varepsilon<1$ *and any floor process* $(G_t)_{t\geq 0}\in\mathcal{A}_N(v_0(1-\varepsilon))$ *the process* $\hat{V}_t := \varepsilon\hat{X}_t + G_t$ *is* $\mathcal{A}_{G,N}(v_0)$*-long-run optimal.*

Proof. By Remark 4.2.2 one obtains

$$\begin{aligned}0 &\leq \limsup_{T\to\infty}\left(\frac{1}{T}\log\sup_{V\in\mathcal{A}_{G,N}(T)(v_0)}\mathbb{E}U(V_T/N_T) - \frac{1}{T}\log\mathbb{E}U(\hat{V}_T/N_T)\right)\\ &= \limsup_{T\to\infty}\left(\frac{1}{T}\log\sup_{V\in\mathcal{A}_{G,N}(T)(v_0)}\mathbb{E}U(V_T/N_T) - \frac{1}{T}\log\mathbb{E}U(\hat{X}_T/N_T)\right).\end{aligned}$$

As $\mathcal{A}_{G,N}(T)(v_0)\subset\mathcal{A}_N(v_0)$ we deduce that

$$\begin{aligned}&\limsup_{T\to\infty}\left(\frac{1}{T}\log\sup_{V\in\mathcal{A}_{G,N}(T)(v_0)}\mathbb{E}U(V_T/N_T) - \frac{1}{T}\log\mathbb{E}U(\hat{V}_T/N_T)\right)\\ &\leq \limsup_{T\to\infty}\left(\frac{1}{T}\log\sup_{V\in\mathcal{A}_N(v_0)}\mathbb{E}U(V_T/N_T) - \frac{1}{T}\log\mathbb{E}U(\hat{X}_T/N_T)\right).\end{aligned}$$

The right hand side is equal to zero as X is $\mathcal{A}_N(v_0)$-long-run optimal. Thus, we conclude that $\hat{V}$ is $\mathcal{A}_{G,N}(v_0)$-long-run optimal. □

4.3.3 Drawdown constraint

This subsection discusses the long-term optimality of the solution to the drawdown constrained problem.

Our main result on the long-run optimality for a drawdown constrained case can be stated as follows.

Theorem 4.3.2. *Recall the setup of Theorem 3.3.1 from Chapter 3 with a utility function* U*, a drawdown function* w *and numeraire* N *we then have:*

i) If $(\hat{\xi}_t)_{t\geq 0} \in \mathcal{A}_N(v_0)$ *is* $\mathcal{A}_N(v_0)$*-long-run optimal with utility function* $U \circ F_w$*, numéraire* N*, and if* $\limsup_{T\to\infty} \frac{1}{T} \log \sup_{V \in \mathcal{A}_N(v_0)} \mathbb{E} U \circ F_w \left((V_T/N_T)^{1+\delta} \right) < \infty$ *for some* $\delta > 0$*, then* $N_t M_t^{F_w}(\hat{\xi}/N)$ *is* $\mathcal{A}_N^w(v_0)$*-long-run optimal with utility function* U*, numéraire* N *and drawdown function* w*.*

ii) Assume that $N_t M_t^{F_w}(\hat{\xi}/N)$ *is* $\mathcal{A}_N^w(v_0)$*-long-run optimal with utility function* U*, numéraire* N *and drawdown function* w*, where* $(\hat{\xi}_t)_{t\geq 0}$ *achieves the certainty equivalent rate in the problem with utility* $U \circ F_w$*, numéraire* N *and initial wealth* v_0 *(see Chapter 3) with*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E} U \circ F_w(\hat{\xi}_T/N_T) = \liminf_{T\to\infty} \frac{1}{T} \log \mathbb{E} U \circ F_w(\hat{\xi}_T/N_T)$$

then $\hat{\xi}$ *is* $\mathcal{A}_N(v_0)$*-long-run optimal with utility function* $U \circ F_w$ *and numéraire* N*.*

Proof. i) Let us notice that

$$\sup_{V \in \mathcal{A}_N^w(T)(v_0)} \mathbb{E} U(V_T/N_T) \leq \sup_{Y \in \mathcal{A}_N(v_0)} \mathbb{E} U \circ F_w(\overline{Y/N}_T) \leq \sup_{Y \in \mathcal{A}_N^{w_\varepsilon}(T)(v_0)} \mathbb{E} U \circ F_w(\overline{Y/N}_T),$$

where $w_\varepsilon(x) = \varepsilon x$. And, therefore, using explicit expression for a wealth process from $\mathcal{A}_N^{w_\varepsilon}(T)(v_0)$ we can write for some c_ε

$$\sup_{V \in \mathcal{A}_N^w(T)(v_0)} \mathbb{E} U(V_T/N_T) \leq \sup_{X \in \mathcal{A}_N(v_0)} \mathbb{E} U \circ F_w(c_\varepsilon M_T^{F_{w_\varepsilon}}(X/N)^{\frac{1}{1-\varepsilon}}),$$

where $F_{w_\varepsilon}(u) = v_0^\varepsilon u^{1-\varepsilon}$. Now, as $M_t^{F_{w_\varepsilon}}(X/N) \in \mathcal{A}_N(v_0)$ for any $X \in \mathcal{A}_N(v_0)$ we obtain

$$\sup_{V \in \mathcal{A}_N^w(T)(v_0)} \mathbb{E} U(V_T/N_T) \leq \sup_{Y \in \mathcal{A}_N(v_0)} \mathbb{E} U \circ F_w(c_\varepsilon (Y_T/N_T)^{\frac{1}{1-\varepsilon}}). \tag{4.6}$$

By Lemma 3.7.3 from Chapter 3 we get that for any $x_0 > 0$ there exists $\gamma \in \mathbb{R}$ such that for all $x \geq x_0$ and all $c > 1$:

$$U \circ F_w(y) \leq U \circ F_w(cy) \leq c^\gamma U \circ F_w(y).$$

Now,

$$\begin{aligned}
&\sup_{Y\in\mathcal{A}_N(v_0)} \mathbb{E}U\circ F_w(c_\varepsilon(Y_T/N_T)^{\frac{1}{1-\varepsilon}}) \le \sup_{Y\in\mathcal{A}_N(v_0)} \mathbb{E}U\circ F_w(c_\varepsilon(1+Y_T/N_T)^{\frac{1}{1-\varepsilon}}) \le \\
&\tilde{c}_\varepsilon \sup_{Y\in\mathcal{A}_N(v_0)} \mathbb{E}U\circ F_w\left(\left(\frac{N_T+Y_T}{2N_T}\right)^{\frac{1}{1-\varepsilon}}\right) \qquad (4.7)\\
&\le \tilde{c}_\varepsilon \sup_{X\in\mathcal{A}_N(v_0)} \mathbb{E}U\circ F_w\left(\left(\frac{X_T}{N_T}\right)^{\frac{1}{1-\varepsilon}}\right), \qquad (4.8)
\end{aligned}$$

where the last inequality was obtained by noting that $(1/2N_t + 1/2Y_t)_{0\le t\le T} \in \mathcal{A}_N(v_0)$ for all $Y \in \mathcal{A}_N(v_0)$.

On the other hand, using the property of Azéma–Yor processes for the concave function F_w, we get

$$\mathbb{E}U(M_T^{F_w}(\hat{\xi}/N)) \ge \mathbb{E}U\circ F_w(\hat{\xi}_T/N_T). \tag{4.9}$$

Using (4.6), (4.8) and (4.9) we obtain for any $0<\varepsilon<1/2$

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\left[\log \sup_{V\in\mathcal{A}_N^w(T)(v_0)} \mathbb{E}U(V_T/N_T) - \log\mathbb{E}U(M_T^{F_w}(\hat{\xi}/N))\right]\\
&\le \limsup_{T\to\infty}\frac{1}{T}\left[\log \sup_{X\in\mathcal{A}_N(v_0)} \mathbb{E}U\circ F_w(X_T/N_T) - \log\mathbb{E}U\circ F_w(\hat{\xi}_T/N_T)\right] +\\
&+\limsup_{T\to\infty}\frac{1}{T}\left[\log \sup_{X\in\mathcal{A}_N(v_0)} \mathbb{E}U\circ F_w((X_T/N_T)^{\frac{1}{1-\varepsilon}}) - \log \sup_{X\in\mathcal{A}_N(v_0)} \mathbb{E}U\circ F_w(X_T/N_T)\right].
\end{aligned}$$

Now, we will show that there exists $K>0$ such that for all $\tilde{U}$ such that $\tilde{U}(x) \le U\circ F_w(x^{1+\delta})$ for all $x\ge 1$, where $\delta>0$ is defined above, we have

$$\tilde{c} := \limsup_{T\to\infty}\frac{1}{T}\left[\log \sup_{V\in\mathcal{A}_N(v_0)} \mathbb{E}\tilde{U}(V_T/N_T) - \log \sup_{V\in\mathcal{A}_N(v_0)} \mathbb{E}\tilde{U}(V_T/N_T)\mathbf{1}_{V_T/N_T\le K^T}\right] = 0 \tag{4.10}$$

Indeed, let $(V_t^T)_{t\ge0}$ achieves the supremum in $\sup_{V\in\mathcal{A}_N(v_0)}\mathbb{E}\tilde{U}(V_T/N_T)$ (if such process does not exist then choose such $(V_t^T))_{t\ge0}$ that $\mathbb{E}U(V_T^T)$ differs from the supremum on less than $1/T$), then

$$0 \le \tilde{c} \le \limsup_{T\to\infty}\frac{1}{T}\left[\log\mathbb{E}\tilde{U}(V_T^T/N_T) - \log\mathbb{E}\tilde{U}(V_T^T/N_T)\mathbf{1}_{V_T^T/N_T\le K^T}\right].$$

Now, the argument follows the lines of the proof of Lemma 3.7.2 from Chapter 3, where we set $\xi_T := V_T^T/N_T$ and we set $C_G = \limsup_{T\to\infty}\frac{1}{T}\log\sup_{V\in\mathcal{A}_N(v_0)}\mathbb{E}U\circ F_w\left((V_T/N_T)^{1+\delta}\right)$. And we get that $\tilde{c}=0$.

Thus, using (4.10) for $\tilde{U} = U \circ F_w(x^{\frac{1}{1-\varepsilon}})$ and for $\tilde{U} = U \circ F_w(x)$ we obtain

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\left[\log \sup_{V\in\mathcal{A}_N^w(T)(v_0)} \mathbb{E}U(V_T/N_T) - \log \mathbb{E}U(M_T^{F_w}(\hat{\xi}/N))\right] \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\left[\log \sup_{X\in\mathcal{A}_N(v_0)} \mathbb{E}U \circ F_w((X_T/N_T)^{\frac{1}{1-\varepsilon}})\mathbf{1}_{X_T/N_T\leq K^T} - \log \sup_{X\in\mathcal{A}_N(v_0)} \mathbb{E}U \circ F_w(X_T/N_T)\mathbf{1}_{X_T/N_T\leq K^T}\right] \\
&\leq \frac{\varepsilon}{1-\varepsilon}|\gamma| \log K
\end{aligned} \tag{4.11}$$

where we used that there exists some $\gamma > 0$ such that $U \circ F_w(x^{\frac{1}{1-\varepsilon}}) \leq x^{\gamma\varepsilon/(1-\varepsilon)} U \circ F_w(x)$ for $x \geq 1$ by Lemma 3.7.3 from Chapter 3.

The righthandside of (4.11) tends to zero as $\varepsilon \to 0$ and, therefore, we get that

$$\limsup_{T\to\infty} \frac{1}{T}\left[\log \sup_{V\in\mathcal{A}_N^w(T)(v_0)} \mathbb{E}U(V_T/N_T) - \log \mathbb{E}U(M_T^{F_w}(\hat{\xi}/N))\right] = 0$$

ii) Indeed,

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\left[\log \sup_{Y\in\mathcal{A}_N(v_0)} \mathbb{E}U \circ F_w(Y_T/N_T) - \log \mathbb{E}U \circ F_w(\hat{\xi}_T/N_T)\right] \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\left[\log \sup_{X\in\mathcal{A}_N^w(T)(v_0)} \mathbb{E}U(X_T/N_T) - \log \mathbb{E}U(M_T^{F_w}(\hat{\xi}/N))\right] \\
&+ \limsup_{T\to\infty} \frac{1}{T}\left[\log \mathbb{E}U(M_T^{F_w}(\hat{\xi}/N)) - \log \mathbb{E}U \circ F_w(\hat{\xi}_T/N_T)\right] = \mathrm{CER}_U^w(v_0) - \mathrm{CER}_{U\circ F_w}(v_0) = 0,
\end{aligned}$$

where in last equation we used the fact that $\limsup(A - B) \leq \limsup A - \liminf B = \limsup A - \limsup B$ when $\limsup B = \liminf B$.

□

4.3.4 Example: complete market model with deterministic coefficients

We consider now the classical complete financial market model with deterministic coefficients. $W_t = (W_t^1, \ldots, W_t^d)'$ is a standard d-dimensional Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ is an augmentation of its natural filtration. Here $'$ denotes vector transpose. $S_t^0 = \exp(\int_0^t r_u \mathrm{d}u)$ is deterministic and $\frac{1}{T}\int_0^T r_u \mathrm{d}u \to r^* \geq 0$ as $T \to \infty$. Each asset follows dynamics given by

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_t^i \mathrm{d}t + \sum_{j=1}^{d} \sigma_t^{ij} \mathrm{d}W_t^j, \quad S_0^i = s_0^i > 0$$

where μ_t^i and σ_t^{ij} are bounded deterministic functions and σ_t is invertible. Recall Definition 4.2.1 of wealth process and let $\tilde{\pi}_t^i := \pi_t^i S_t^i / V_t$ be the proportion of wealth invested in the i^{th} asset so that $\mathrm{d}(D_t V_t) =$

$\sum_{i=1}^{d} \tilde{\pi}_t^i V_t \frac{\mathrm{d}(D_t S_t^i)}{S_t^i}$. The market price of risk is given as $\theta_t := \sigma^{-1}(\mu_t - r_t \mathbb{I})$, where $\mathbb{I}$ is a d-dimensional vector with all entries equal to one. We assume θ_t is also bounded and that

$$||\theta^*||^2 := \lim_{T\to\infty} \frac{1}{T} \log \int_0^T ||\theta_u||^2 \mathrm{d}u \quad \text{is well defined and finite.}$$

The state price density

$$Z_t := \exp\left\{ -\int_0^t \theta_u' \mathrm{d}W_u - \frac{1}{2} \int_0^t ||\theta_u||^2 \mathrm{d}u \right\}$$

is a $\mathbb{P}$–martingale which defines the unique risk neutral measure $\mathbb{Q}$, $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = Z_t$.

We point out that $\mathcal{A}_N(v_0)$ is independent from N since all wealth processes are continuous. Therefore, we simply write $\mathcal{A}(v_0)$ and $\mathcal{A}^w(T)(v_0)$.

We consider the problem of maximising the expected utility of discounted wealth at a given horizon T this corresponds to considering the numéraire $N_t = S_t^0$. The solution is obtained using, by now standard, convex duality arguments, see Karatzas, Lehoczky and Shreve [34] or Karatzas and Shreve [37, pp. 97–118]. And the value function for a utility function $U_p(x) = x^p/p$ equals

$$\begin{aligned} V(v_0, T, p) &= \sup_{V\in\mathcal{A}(v_0)} \mathbb{E}U_p(D_T V_T) = U_p(v_0)\left(\mathbb{E}Z_T^{-\frac{p}{1-p}}\right)^{1-p} \\ &= U_p(v_0) \exp\{\frac{p}{2(1-p)} \int_0^t ||\theta_u||^2 du\}. \end{aligned}$$

Moreover, the optimal wealth process V^* which is characterised via $\tilde{\pi}_t^* = \frac{1}{1-p}\theta_t' \sigma_t^{-1}$ is independent of the horizon T and, therefore, is long-run optimal for the unconstrained problem.

Considering the linear drawdown constraint, $w(x) = \alpha x$, one can check that the assumptions of the first part in Theorem 4.3.2 hold, and, therefore, we can deduce long-run optimality of the wealth process $X_t = S_t^0 M_t^{F_w}(V^*/S^0)$ for w-drawdown constrained problems, where V^* solves the unconstrained problem. Moreover, the following asymptotics of the finite horizon problem with drawdown constraint is obtained using Theorem 4.3.2

$$\log \sup_{V\in\mathcal{A}^w(T)(v_0)} \mathbb{E}U_p(D_T V_T) = T\left(\frac{|p|(1-\alpha)}{2(1-p(1-\alpha))}||\theta^*||^2 + o(1)\right)$$

This asymptotic result can be sharpened as follows:

Proposition 4.3.1.

$$\log \sup_{V\in\mathcal{A}^w(T)(v_0)} \mathbb{E}U_p(D_TV_T) = \frac{|p|(1-\alpha)}{2(1-p(1-\alpha))}\int_0^T ||\theta_t||^2dt + O\left(\log T\right).$$

Proof. With no loss of generality we put $v_0 = 1$. By property of the Azéma–Yor processes for a concave function F_w, we obtain

$$\sup_{V\in\mathcal{A}^w(T)(1)} \mathbb{E}U_p(D_TV_T) \geq \sup_{X\in\mathcal{A}(1)} \mathbb{E}U_p\circ F_w(D_TX_T) = (1-\alpha)V(1,T,p(1-\alpha))$$

For $\varepsilon > 0$, consider function $K_\varepsilon(x) = x^{\frac{1}{1-\varepsilon}}$ which has a well-defined inverse $F_\varepsilon(y) = y^{1-\varepsilon}$. Then

$$\sup_{X\in\mathcal{A}(1)} \mathbb{E}U_p\circ F_w\circ K_\varepsilon(D_TX_T) \geq \sup_{X\in\mathcal{A}^{w_\varepsilon}(T)(1)} \mathbb{E}U_p\circ F_w\circ K_\varepsilon(D_TX_T), \tag{4.12}$$

where $w_\varepsilon(x) = \varepsilon x$ and where we used that $\mathcal{A}^{w_\varepsilon}(T)(1) \subseteq \mathcal{A}(1)$.

The right hand side of (4.12) can be rewritten as

$$\sup_{X\in\mathcal{A}^{w_\varepsilon}(T)(1)} \mathbb{E}U_p\circ F_w\circ K_\varepsilon(D_TX_T) = \sup_{Y\in\mathcal{A}(1)} \mathbb{E}U_p\circ F_w\circ K_\varepsilon(M_T^{F_\varepsilon}(DY))$$

where we used the bijection between classes $\mathcal{A}^{w_\varepsilon}(T)(1)$ and $\mathcal{A}(1)$ obtained by means of Azéma-Yor processes.

Using the drawdown constraint property of $M_T^{F_\varepsilon}(DY)$ we deduce

$$\begin{aligned}&\sup_{X\in\mathcal{A}(1)} \mathbb{E}U_p\circ F_w\circ K_\varepsilon(D_TX_T) \geq \varepsilon^{\frac{p(1-\alpha)}{1-\varepsilon}} \sup_{Y\in\mathcal{A}(1)} \mathbb{E}U_p\circ F_w(\overline{DY}_T)\\ &\geq \varepsilon^{\frac{p(1-\alpha)}{1-\varepsilon}} \sup_{V\in\mathcal{A}^w(T)(1)} \mathbb{E}U_p(D_TV_T).\end{aligned}$$

Thus, we obtain inequality

$$(1-\alpha)V\left(1,T,p(1-\alpha)\right) \leq \sup_{V\in\mathcal{A}^w(T)(1)} \mathbb{E}U_p(D_TV_T) \leq \frac{1-\alpha}{1-\varepsilon}\varepsilon^{-\frac{p(1-\alpha)}{1-\varepsilon}} V\left(1,T,\frac{p(1-\alpha)}{1-\varepsilon}\right)$$

Taking logarithm we obtain

$$\begin{aligned}&\log\frac{1}{p} + \frac{|p|(1-\alpha)}{2(1-p(1-\alpha))}\int_0^T ||\theta_t||^2dt \leq \log \sup_{V\in\mathcal{A}^w(T)(1)} \mathbb{E}U_p(D_TV_T)\\ &\leq \log\frac{1}{p} + \frac{|p|(1-\alpha)}{2(1-p(1-\alpha)-\varepsilon)}\int_0^T ||\theta_t||^2dt - \frac{p(1-\alpha)}{1-\varepsilon}\log\varepsilon.\end{aligned}$$

Now, taking $\varepsilon = \frac{1}{T}$ we obtain the required asymptotics.

□

4.4 Finite horizon utility maximisation subject to a floor constraint

In this section we turn to the problem of maximising an expected utility of terminal wealth problem on a finite horizon subject to a floor constraint of the special form.

We start by assuming the continuity of the stock process S.

Now, for a given floor process $(G_t)_{0\leq t\leq T}$, we define the subclass of $\mathcal{A}(v_0)$ as

$$\mathcal{A}_G(T)(v_0) = \{V \in \mathcal{A}_T(v_0) \text{ s.t. } V_t \geq G_t, \quad 0 \leq t \leq T\}.$$

For a stopping time $T < \infty$, we aim to find

$$\sup_{V\in\mathcal{A}_G(T)(v_0)} \mathbb{E}U(V_T)$$

along with the optimal wealth process which achieves the supremum.

We start by characterising the market. We assume existence of a price deflator (or a state price density) process. For a general semimartingale market, we will use the setup of Karatzas and Kardaras [33]; in accordance with their definition of price deflators, we assume that

$$\mathcal{Z}_\mathcal{A}(v_0) := \{Z \geq 0 \;|Z_0 = 1, Z_T > 0, \text{ and } ZV \text{ is a loc. mart. } \forall V \in \mathcal{A}(v_0)\} \neq \varnothing \tag{4.13}$$

Assumption 4.4.1. *For some* $0 \neq p < 1$ *there exists a price deflator* $(Z_t)_{t\geq 0} \in \mathcal{Z}_\mathcal{A}(v_0)$ *such that* $\xi_t = \frac{Z_t^{-1/(1-p)}}{\mathbb{E}Z_t^{-p/(1-p)}}$ *is an admissible wealth process, i.e.* $\xi \in \mathcal{A}(1)$.

Remark 4.4.1. *In Assumption 4.4.1 we implicitly assumed that* $\mathbb{E}Z_t^{-p/(1-p)} < \infty$ *for all* $t \geq 0$.

Now, we are able to formulate the main theorem.

Theorem 4.4.1. *Let us consider function* $U_p(x) = \frac{1}{p}x^p$ *for* $0 \neq p < 1$ *and let Assumption 4.4.1 hold for* p. *Consider a wealth process* $(N_t)_{t\geq 0}$ *such that* $T := \inf\{t \geq 0 : N_t/\xi_t = 0\} < \infty$ *a.s., where* ξ *is defined in Assumption 4.4.1, and* N/ξ *is a max-continuous process, i.e. with continuous running maximum.*

For a given non-decreasing non-negative concave function F *with* $F(v_0) = v_0$, *define the floor process* $G_t = \xi_t F(\frac{N_t}{\xi_t})$. *Then for* $\phi(x) := F(x) - F'(x)x$, *we obtain*

$$\sup_{V\in\mathcal{A}_G(T)(v_0)} \mathbb{E}U(V_T) = \frac{1}{p}\left(\mathbb{E}Z_T^{-p/(1-p)}\right)^{1-p} \mathbb{E}\xi_T\phi(\sup_{0\leq s\leq T} N_s),$$

and the optimal wealth process is $\hat{V}_t = v_0\xi_t M_t^F(N/\xi)$.

Proof. We will use the methodology established in [18], where the authors solved a similar problem by first changing measure and then using the so-called max-plus decomposition (see [18] for more details) of the floor process to derive the solution in terms of the max-plus representation. We also use some results from [38] on connection between max-plus decompositions and Azéma–Yor processes.

Consider a measure $\mathbb{Q}$ defined by its Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t/D_t,$$

Further, consider a measure $\mathbb{Q}^\xi$ given by

$$\frac{d\mathbb{Q}^\xi}{d\mathbb{Q}}|_{\mathcal{F}_t} = D_t\xi_t,$$

Changing the measure, we get

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}V_T^p = \mathbb{E}^{\mathbb{Q}}\frac{D_T}{Z_T}V_T^p = \mathbb{E}^{\mathbb{Q}^\xi}\frac{1}{\xi_T Z_T}V_T^p = \mathbb{E}^{\mathbb{Q}^\xi}\frac{1}{\xi_T^{1-p}Z_T}\left(\frac{V_T}{\xi_T}\right)^p\\ &= \left(\mathbb{E}Z_T^{-p/(1-p)}\right)^{1-p}\mathbb{E}^{\mathbb{Q}^\xi}\left(\frac{V_T}{\xi_T}\right)^p.\end{aligned}$$

Considering ξ as a numéraire, we deduce that $\left(\frac{V_t}{\xi_t}\right)_{0\le t\le T}$ is a $\mathbb{Q}^\xi$-local martingale for any $V \in \mathcal{A}(v_0)$.

Our problem simplifies to the following problem: find

$$\max_{V^\xi}\mathbb{E}^{\mathbb{Q}^\xi}(V_T^\xi)^p, \qquad \text{s.t.}$$

V^ξ is $\mathbb{Q}^\xi$-local martingale, $V_t \ge F(N_t^\xi)$ for $0 \le t \le T$, where we denoted $N_s^\xi := N_s/\xi_s$.

We use the results achieved in [38]. Namely, it is shown that the maximum in the problem above is achieved by the process $v_0M_t^F(N^\xi)$ (which does not depend on the utility function U).

Finally, $\hat{V}_t = \xi_t v_0 M_t^F(N^\xi) = \xi_t v_0 M_t^F(N/\xi)$ and
$\mathbb{E}U(\hat{V}_T) = \frac{1}{p}\left(\mathbb{E}Z_T^{-p/(1-p)}\right)^{1-p}\mathbb{E}\xi_T\phi(\sup_{0\le s\le T}N_s)$.

$\square$

Incomplete market example

We consider a market consisting of one observable and one tradable stock. We assume that the interest rate is zero for notational simplicity.

The dynamics of S_t^1, S_t^2 with S^1 tradable and S^2 observable is given by

$$dS_t^1/S_t^1 = \mu dt + \sigma dW_t^1,$$

$$dS_t^2/S_t^2 = \rho_1 dW_t^1 + \rho_2 dW_t^2,$$

where W_t^1, W_t^2 are independent standard Brownian Motions and $\sigma, \rho_1, \rho_2 > 0$.

One can note that

$$\mathcal{Z}_{\mathcal{A}}(v_0) := \{(Z^{(\lambda)}) : \quad dZ_t^{(\lambda)}/Z_t^{(\lambda)} = -\frac{\mu}{\sigma} dW_t^1 + \lambda dW_t^2, Z_0^{(\lambda)} = 1 \text{ with } , \lambda \in \mathbb{R}\}$$

Let us consider $Z_t = Z_t^{(0)}$, one can check that Assumption 4.4.1 holds for Z_t with the replicating porfolio

$$\pi_t = -\frac{\mu}{\sigma^2(1-p)} Z_t^{-1/(1-p)}/S_t^1.$$

Now, let us consider portfolio $\pi_t^N = -\frac{1}{(T-t)S_t^1}$ where T is a deterministic time horizon. Then, the corresponding wealth process N_t is given by

$$N_t = \exp\left(-\mu \log \frac{T}{T-t} - \frac{t\sigma^2}{2T(T-t)} + \sigma \int_0^t \frac{dW_s^1}{T-s}\right).$$

And, thus, $T = \inf\{t : N_t/\xi_t = 0\}$ $\mathbb{P}$-a.s.

Now, consider the function $F(x) = x^\gamma$ for $0 < \gamma < 1$.

The floor constraint can then be rewritten as $G_t = \xi_t^{1-\gamma} N_t^\gamma$.

Using Theorem 4.4.1, we obtain that the optimal value is

$$\sup_{V \in \mathcal{A}_G(T)(v_0)} \mathbb{E}U(V_T) = \frac{1-p}{p} \exp\{\frac{1}{2}\frac{\mu^2 p}{1-p} T\} \mathbb{E}\left[\xi_T(\sup_{0 \le s \le T} N_s)^p\right].$$

One can consider for some $0 < \varepsilon < 1$

$$F(x) := \begin{cases} x^{\frac{1}{1+\varepsilon}} & \text{if } 0 \le x \le 1 \\ \frac{1}{1+\varepsilon} x + \frac{\varepsilon}{1+\varepsilon} & \text{if } x > 1 \end{cases} \tag{4.14}$$

The floor process then becomes more tractable:

$$G_t = \xi_t^{\frac{\varepsilon}{1+\varepsilon}} N_t^{\frac{1}{1+\varepsilon}} \mathbf{1}_{N_t \le \xi_t} + \left(\frac{1}{1+\varepsilon} N_t + \frac{\varepsilon}{1+\varepsilon} \xi_t\right) \mathbf{1}_{N_t \ge \xi_t} \ge \frac{1}{1+\varepsilon} N_t.$$

The optimal strategy becomes a linear combination of N and ξ. Namely,

$$\hat{V}_t = \xi_t v_0 M_t^{F_\varepsilon}(N/\xi) = \frac{\varepsilon}{1+\varepsilon}\xi_t + \frac{1}{1+\varepsilon}N_t.$$

And, the value function equals

$$\sup_{V \in \mathcal{A}_G(T)(v_0)} \mathbb{E}U(V_T) = \frac{\varepsilon}{1+\varepsilon}\frac{1-p}{p}\exp\{\frac{\mu^2(1+p)}{2(1-p)}T\}.$$

Chapter 5

Utility of consumption

5.1 Introduction

We study dynamic portfolio allocation problem in the context of utility of consumption maximisation. We are concerned with investment strategies whose wealth process is required to dominate a predefined non-decreasing function of its up-to-date maximum. This portfolio constraint is called a drawdown constraint and in literature typically the linear drawdown constraints are considered in the context of portfolio optimisation problems. More precisely, for some function $w(x)$ we require the wealth process $(V_t)_{t\geq 0}$ to satisfy

$$V_t > w\left(\sup_{0\leq s\leq t} V_s\right), \quad t \geq 0.$$

The motivation for this constraint comes from the practice. Potential investors choosing a fund manager look at several characteristics of her portfolio. The drawdown is one of these characteristics.

As a fully pathwise the drawdown constraint requires a treatment different from usual methods for convex portfolio constraints. The drawdown constraint was originally used by Grossman and Zhou in [25] in the context of long-term growth rate of expected utility of wealth maximisation. The problem was solved by means of dynamic programming equation and closed-form expression was found for a solution in a market driven by a geometric Brownian motion. Later, this problem was extended to the case of a market with deterministic coefficients by [12]. The authors simplified and generalised the results of Grossman and Zhou. The idea was to use an auxiliary unconstrained problem to tackle the drawdown constrained problem. In the end the solution to constrained problem was expressed explicitly as a transformation of the solution to an unconstrained problem with a different utility function. These ideas inspired further research and in Chapter

3 the equivalence results are obtained for a general semimartingale market, a general utility function and a general drawdown function. The result showed that the value function for the drawdown constrained problem is equal to the value function of the unconstrained problem with a modified utility function. Moreover, the optimal wealth process for the constrained problem can be expressed as an explicit transformation of the optimal wealth process for an unconstrained problem.

In this chapter we analyse consumption-investment strategies of an investor who maximises expected integrated utility of consumption subject to a drawdown constraint. This problem was studied by Elie and Touzi in [20] and by Roche in [51]. In [51] the problem was solved in the case of market driven by geometric Brownian motion and for power utility functions. In [20] the results were obtained for the same market but for a general utility function by tackling the dynamic programming equation arising in the system. The solution to this equation was found by means of a dual equation. Our work greatly relies on the results obtained by Elie and Touzi in [20]. By means of Azéma-Yor processes we build the connection between the constrained problem and auxiliary unconstrained problem. Latter is solved by means of dynamic programming equation for a general drawdown function.

By analyzing the solution to the equivalent unconstrained problem we derive that it also solves the utility of consumption problem with a capacity constraint. By capacity constraint here we mean that the wealth process is required to stay below a certain constant. This can be motivated by considering a fund manager who faces nonlinear taxation on P&L and, therefore, is keen on keeping her wealth below a certain threshold.

The work is organised as follows. Section 5.2 characterises the general financial market for which the equivalence result holds. In Section 5.3 the problem is introduced, Azéma-Yor processes are defined, and the equivalence result is proved. Section 5.4 is devoted to the dynamic programming equation driving the system as well as the verification theorem with a proof. The dual equation is solved in Section 5.5. Finally, in Section 5.7 the problem with capacity constraint is solved.

5.2 Financial market

We consider a general financial market $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is an observable probability measure. We define an $\mathcal{F}_t$-measurable semimartingale $(S_t)_{t\geq 0}$ to represent the dynamics of the stock process.

We also assume interest rate to be equal to zero in the sequel.

An investor on the market trades according to a predictable portfolio $\pi_t = (\theta_t^0, \theta_t)$, and consumes each infinitesimal moment with rate C_t. The strategy is called self-financing in this case if $d(\theta_t S_t + \theta_t^0) = -C_t dt + \theta_t dS_t$. The self-financing trading strategy is fully characterised by pair process (C_t, θ_t) and initial wealth x_0.

The corresponding wealth process $(X_t^{C,\theta})$ of a pair (C_t, θ_t) with initial wealth x_0 satisfies the following stochastic differential equation:

$$\begin{aligned} dX_t^{C,\theta} &= -C_t dt + \theta_t dS_t, \\ X_0^{C,\theta} &= x_0. \end{aligned}$$

We will simply write X_t instead of $X_t^{C,\theta}$ when there is no confusion with the notation.

Now we will characterise the class of all admissible strategies.

Definition 5.2.1. *A self-financing consumption-investment strategy (C_t, θ_t) with initial wealth x_0 is called admissible if its corresponding wealth process is non-negative and max-continuous. We denote the class of all admissible wealth processes by $\mathcal{A}(x_0)$.*

Let us introduce a drawdown constraint.

Definition 5.2.2. *A nondecreasing function $w : [0, \infty) \to \mathbb{R}$ such that $y - w(y)$ and $1/(y - w(y))$ are locally bounded away from zero is called a drawdown function.*

The process X_t is said to satisfy w-drawdown constraint if $\min\{X_t, X_{t-}\} > w(\overline{X}_t)$ for all $t \geq 0$, where $\overline{X}_t = \sup_{0\leq s\leq t} X_s$.

We denote by $\mathcal{A}^w$ the subclass of $\mathcal{A}^0$ consisting of strategies with corresponding discounted wealth process satisfying a drawdown constraint. Precisely,

$$\mathcal{A}^w(x_0) = \{(C_t, \theta_t) \in \mathcal{A}(x_0) : X_t^{C,\theta} > w(\overline{X_t^{C,\theta}}), t \geq 0\}$$

We simply write X_t *instead of* $\min\{X_t, X_{t-}\}$ *since* X *is max-continuous.*

5.3 Problem formulation and proposed solution

To formulate a problem we need to specify the utility function. We only restrict the utility function to be concave and nondecreasing in wealth (i.e. $U(t,x)$ is concave and nondecreasing in x). We refer reader to [37] for the details on utility functions.

The problem which we examine is the following.

Problem 4. *For a utility function* U*, a drawdown function* w *and initial wealth* x_0 *find the supremum of*

$$\mathcal{R}(C,\theta) := \mathbb{E}\int_0^\infty U(t, C_t)dt$$

over a class of investment strategies (C_t, θ) *from* $\mathcal{A}^w(x_0)$*.*

Let us also introduce another problem

Problem 5. *For a utility function* U*, nonincreasing function* f *and initial wealth* x_0 *find the supremum of*

$$\mathcal{R}_1^f(C,\theta) = \mathbb{E}\int_0^\infty U(t, C_t f(\overline{X^{C,\theta}}_t))dt$$

over a class $\mathcal{A}(x_0)$*, where* $X^{C,\theta}$ *is a corresponding wealth process of* (C,θ)*.*

The main contribution of the following proposition is in providing an equivalence of two above problems.

We introduce the following functions: $K(x) := x_0 \exp\{\int_{x_0}^x \frac{du}{u-w(u)}\}$ which is increasing and has a well defined inverse $F := K^{-1}$ (F is well-defined as K is nondecreasing); $f = F', k = K'$.

Proposition 5.3.1. *For a given concave nondecreasing function* U*, drawdown function* w *and initial wealth* x_0 *the following holds*

$$\sup_{(C,\theta)\in\mathcal{A}^w(x_0)} \mathcal{R}(C,\theta) = \sup_{(C,\theta)\in\mathcal{A}(x_0)} \mathcal{R}_1^f(C,\theta),$$

where function f *is defined above.*

Moreover, optimal consumption-investment strategies are connected by explicit relations via

$$(\tilde{C}_t, \tilde{\theta}_t) = (\hat{C}_t/k(\overline{X_t}, \hat{\theta}_t/k(\overline{X_t})),$$

where $(\tilde{C}_t, \tilde{\theta}_t)$ *optimises left hand side, and* $(\hat{C}_t, \hat{\pi}_t)$ *right hand side, and* X_t *is the corresponding wealth process of strategy* $(\tilde{C}_t, \tilde{\theta}_t)$*, and function* k *is defined above.*

The proof of the proposition is a direct implication of the following lemma, which builds a one-to-one correspondence between classes $\mathcal{A}^w(x_0)$ and $\mathcal{A}(x_0)$ using so-called Azéma-Yor processes.

Lemma 5.3.1. *Let the strategy* (C_t, θ_t) *of a class* $\mathcal{A}^w(x_0)$ *have the wealth process* X*, then the process* $Y_t = M_t^K(X)$ *represents the investment strategy* $(C_t/f(\overline{Y_t}), \theta_t/f(\overline{Y_t}))$ *of a class* $\mathcal{A}(x_0)$*.*

And vice versa, if $(C_t, \theta_t) \in \mathcal{A}(x_0)$*, then* $(C_t/k(\overline{X_t}), \theta_t/k(\overline{X_t}))$ *is of a class* $\mathcal{A}^w(x_0)$*.*

Proof. For $(C_t, \theta_t) \in \mathcal{A}^w(x_0)$ we have

$$dX_t = -C_t dt + \theta_t dS_t, \quad X_0 = x_0.$$

We define $Y_t = M_t^K(X)$. $Y_t \geq 0$ as X satisfies w-drawdown constraint, also it is a max-continuous process. As $F = K^{-1}$ we get $X_t = M_t^F(Y)$. Then

$$dY_t = \frac{1}{f(\overline{Y}_t)} dX_t = -\frac{C_t}{f(\overline{Y}_t)} dt + \frac{\theta_t}{f(\overline{Y}_t)} dS_t.$$

Therefore, $(Y_t)_{t \geq 0}$ is a wealth process with corresponding portfolio and consumption processes $(C_t/f(\overline{Y_t}), \theta_t/f(\overline{Y_t}))$. Similarly, one gets the reverse statement in the proposition. □

5.4 Viscosity solution approach to the auxiliary problem

In this section we apply the framework of viscosity solution technique established in [20] for the optimisation problem which is different from the one considered in [20] and, therefore, our approach looks at the initial problem at a slightly different angle. We also obtain the result for a more general setup. Precisely, we obtain the solution for a general drawdown function, whereas in [20] the result is obtained for a linear drawdown function.

We assume the stock process to follow Black-Scholes dynamics:

$$dS_t/S_t = \sigma(dW_t + \lambda dt),$$

where $\sigma > 0$ is an instant volatility and $\lambda \in \mathbb{R}$ is a constant risk premium.

Our purpose is to show that starting from a transformed problem and solving a partial differential equation one gets the solution to the original problem consistent with the results from [20].

We start with the dynamic programming equation driving the system.

We assume that $U(t,x) = e^{-\beta t}U(x)$ for some positive β.

We also assume that function $w(x)$ is a continuously differentiable function which implies that f is also continuously differentiable.

Moreover, we consider a utility function $U(x)$ such that

$$U : \mathbb{R}_+ \to \mathbb{R} \quad C^1, \text{increasing, strictly concave, and} \tag{5.1}$$

$$\text{satisfying } |U(0)| < \infty, U'(0+) = \infty \text{ and } U'(\infty) = 0.$$

Define the value function of Problem 5 in the following way

$$u(x,z) = \sup_{(C,\theta)\in\mathcal{A}(x,y)} \mathbb{E}\int_0^\infty e^{-\beta t} U\left(C_t f(\overline{Y_t})\right) dt.$$

where $(x,z) \in \mathcal{D} := \{(a,b) : 0 < a \leq b\}$.

Here and throughout the paper we will use the notation $\mathcal{A}(x,y)$ for consumption-investment strategies with the wealth process starting at x and the supremum of the wealth process starting at y.

Remark 5.4.1. *Let $\tilde{u}(x,z)$ denote the value function of Problem 4, i.e.*

$$\tilde{u}(x,z) := \sup_{(C,\theta)\in\mathcal{A}^w(x,y)} \mathbb{E}\int_0^\infty e^{-\beta t} U\left(C_t\right) dt,$$

where we denote $\mathcal{A}^w(x,y) := \{(C,\theta) \in \mathcal{A}(x,y) : \quad X_t^{C,\theta} \geq \overline{X^{C,\theta}}_t \quad t \geq 0\}$. Then

$$\tilde{u}(x,z) = u(K(z) - k(z)(z-x), K(z)).$$

Proof. The proof is analogous to proof of Proposition 5.3.1. ☐

We denote by u_x and u_{xx} first and second order partial derivatives of u with respect to first argument. Similarly, we denote by u_z the partial derivative with respect to second argument.

The dynamic programming equation for u is related to the second-order operator

$$\mathcal{L}u := -\beta u + \sup_{C\geq 0,\theta\in\mathbb{R}} \left[\mathcal{L}^{C,\theta}u + U\left(Cf(z)\right)\right],$$

where

$$\mathcal{L}^{C,\theta}u := (\theta\sigma\lambda - C)u_x + \frac{\theta^2\sigma^2}{2}u_{xx},$$

and $f(x)$ is defined in previous section.

We denote by V the Fenchel-Legendre dual transform of U defined by

$$V(y) := \sup_{x\geq 0}\left(U(x) - xy\right), \quad y \geq 0. \tag{5.2}$$

We claim the following

Theorem 5.4.1. *Let $\hat{u}$ be a $C^0(\overline{\mathcal{D}}) \cap C^{2,1}(\mathcal{D})$ function.*

(i) If $\hat{u}$ satisfies $-\mathcal{L}\hat{u} \geq 0$ on $\mathcal{D}$ and $-\hat{u}_z(z,z) \geq 0$ for $z > 0$, then $\hat{u} \geq u$.

(ii) Assume in addition that:

1. $\mathcal{L}\hat{u} = 0$ on $\mathcal{D}$ and $\min\{\beta\hat{u} - V(\frac{\hat{u}_x}{f(z)}), -\hat{u}_z\}(z,z) = 0$ for $z > 0$.

2. $\mathcal{L}\hat{u} = -\beta\hat{u} + U(\hat{C}f(z)) + \mathcal{L}^{\hat{C},\hat{\theta}}\hat{u}$ on $\mathcal{D}$ for some continuous functions $\hat{C}$ and $\hat{\theta}$ satisfying

$$\hat{C}(0,z) = \hat{\theta}(0,z) = 0, \quad z \geq 0, \tag{5.3}$$

and such that the stochastic differential equation

$$d\hat{X}_t = -\hat{C}(\hat{X}_t, \hat{Z}_t)dt + \hat{\theta}(\hat{X}_t, \hat{Z}_t)(dW_t + \lambda dt), \quad t \geq 0, \hat{Z}_t = z \wedge \hat{X}^*_t$$

has for any initial condition $(\hat{X}_0, \hat{Z}_0) = (x,z) \in \mathcal{D}$, a unique strong solution $(\hat{X}, \hat{Z})$.

3. For every sequence of bounded stopping times $(\tau_n)_{n\geq 1}$ with $\tau_n \to \infty$ a.s., we have

$$\liminf_{n\to\infty} \mathbb{E}\left[e^{-\beta\tau_n}\hat{u}(\hat{X}_{\tau_n}, \hat{Z}_{\tau_n})\right] = 0$$

Then, for $(x,z) \in \overline{\mathcal{D}}$, we have $\hat{u}(x,z) = u(x,z)$, and $(C^*_t, \theta^*_t) := (\hat{C}, \hat{\theta})(\hat{X}_t, \hat{Z}_t)$ is an optimal consumption-investment strategy in $\mathcal{A}(x,z)$.

Lemma 5.4.1. *For $z \geq 0$ we have $u(0,z) = U(0)/\beta$, and the corresponding optimal consumption-investment strategies are given by $(C^*, \theta^*) = 0$.*

Proof. We notice that for $(C,\theta) \in \mathcal{A}(0,z)$, the corresponding wealth process X is bounded from below. Let $\mathbb{P}^0$ be the equivalent martingale measure which turns the process $\{W^0_t := W_t + \lambda t, t \geq 0\}$ into a $\mathbb{P}^0$-Brownian motion by the Girsanov theorem. Due to consumption being nonnegative one gets that $\{\int_0^t \sigma\theta_r dW_r^0, t \geq 0\}$ is a $\mathbb{P}^0$-supermartingale as a local martingale bounded from below.

Therefore, we obtain

$$\mathbb{E}^{\mathbb{P}^0}\left[\int_0^t C_r dr\right] \leq \mathbb{E}^{\mathbb{P}^0}[-X_t] \leq 0,$$

where the last inequality is due to nonnegativity of the process X. This implies, that the only sustainable consumption process is zero.

Then, we deduce that

$$\int_0^t \sigma\theta_r dW_r^0 = X_t \geq 0,$$

which immediately implies the required result. □

Proof. (Proof of Theorem 5.4.1)

Firstly, by $-\mathcal{L}\hat{u} \geq 0$ we obtain

$$\beta\hat{u} \geq \sup_{C \geq 0, \theta = 0}\left[\mathcal{L}^{C,\theta}\hat{u} + U\left(Cf(z)\right)\right] = V\left(\hat{u}_x/f(z)\right) \geq U(0),$$

since V is a decreasing function and $V(\infty) = U(0)$. By Lemma 5.4.1, we get $\hat{u} \geq u$ on $\overline{\mathcal{D}}\backslash\mathcal{D} = \{(0,z) : z \geq 0\}$. Then, in case ii), combining $-\mathcal{L}\hat{u} = 0$ with (5.3) we deduce that $\hat{u} = U(0)/\beta$ on $\overline{\mathcal{D}} \setminus \mathcal{D}$, and the statement of the theorem is trivial on $\overline{\mathcal{D}} \setminus \mathcal{D}$.

From now on we fix a pair $(x, z) \in \mathcal{D}$.

i) Let (C, θ) be an arbitrary admissible consumption-investment strategy in $\mathcal{A}(x, z)$, call $(X, Z) := (X^{x,C,\theta}, Z^{x,C,\theta})$ the corresponding wealth process and its running maximum with initial conditions $(X_0, Z_0) = (x, z)$, and define the nondecreasing sequence of stopping times

$$\tau_n := n \wedge \inf\{t > 0 : X_t < n^{-1}\}.$$

Denoting by τ_∞ its limit in the a.s. sense, from the same argument as in the proof of Lemma 5.4.1 it follows that

$$\mathbf{1}_{\{\tau_\infty < \infty\}} \int_{\tau_\infty}^{\infty} C_t dt = 0. \tag{5.4}$$

Observe now that, by Ito's formula, we obtain

$$e^{-\beta\tau_n}\hat{u}(X_{\tau_n}, Z_{\tau_n}) = \hat{u}(x, z) + M_{\tau_n}^n + \int_0^{\tau_n} e^{-\beta t}\left(\hat{u}_z(X_t, Z_t)dZ_t + \left(\mathcal{L}^{C_t,\theta_t}\hat{u} - \beta\hat{u}\right)(X_t, Z_t)dt\right),$$

where

$$M_t^n := \int_0^{t\wedge\tau_n} e^{-\beta t}\theta_t\sigma\hat{u}_x(X_t,Z_t)dW_t, \quad t\geq 0, n>0.$$

From $-\hat{u}_z(z,z)\geq 0$ and from the fact that Z increases only when $X_t = Z_t$ we deduce that

$$-\int_0^{\tau_n} e^{-\beta t}\hat{u}_z(X_t,Z_t)dZ_t \geq 0.$$

Now, combining this with the fact that $\mathcal{L}^{C,\theta}\hat{u}-\beta\hat{u}+U(Cf(z))=\mathcal{L}\hat{u}\leq 0$ we obtain

$$\hat{u}(x,z)\geq e^{-\beta\tau_n}\hat{u}(X_{\tau_n},Z_{\tau_n})+\int_0^{\tau_n} e^{-\beta t}U(C_tf(Z_t))dt - M_{\tau_n}^n. \tag{5.5}$$

Now, $\hat{u}_x(X_t,Z_t)$ is continuous on $[0,\tau_n]$ by definition of τ_n. And as $\int_0^n \theta_t^2 dt<\infty$ a.s. we obtain that M^n is a local martingale. Using (5.4) one derives that from (5.5) M^n is uniformly bounded from below.

Thus, it is a supermartingale and taking expectation in (5.5) we obtain

$$\hat{u}(x,z)\geq \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta t}U(C_tf(Z_t))dt+\frac{U(0)}{\beta}e^{-\beta\tau_n}\right].$$

Using monotone convergence theorem, we deduce

$$\begin{aligned}\hat{u}(x,z) \qquad &\geq \mathbb{E}\left[\int_0^{\tau_\infty} e^{-\beta t}U(C_tf(Z_t))dt+\frac{U(0)}{\beta}e^{-\beta\tau_\infty}\mathbf{1}_{\tau_\infty<\infty}\right]\\ &=\mathbb{E}\left[\int_0^{\infty} e^{-\beta t}U(C_tf(Z_t))dt\right]\end{aligned}$$

by (5.4). And as $(C,\theta)\in\mathcal{A}(x,z)$ is arbitrary we conclude that $\hat{u}(x,z)\geq u(x,z)$.

ii) We set $(C_t^*,\theta_t^*):=(\hat{C},\hat{\theta})(\hat{X}_t,\hat{Z}_t)$, where we introduced $\hat{C},\hat{\theta},\hat{X},\hat{Z}$ in condition (ii-2). Since wealth process X is strictly positive, the sequence of bounded stopping times

$$\hat{\tau}_n := n\wedge\inf\{t>0:\hat{X}_t<1/n \text{ or } \hat{Z}_t>n\}\to\infty \text{ a.s}$$

as $n\to\infty$.

Using conditions i) and ii) of the theorem we obtain together with $dZ_t=0$ when $X_t<Z_t$, by Ito

$$\hat{u}(x,z)=e^{-\beta\hat{\tau}_n}\hat{u}(\hat{X}_{\hat{\tau}_n},\hat{Z}_{\hat{\tau}_n})+\int_0^{\hat{\tau}_n} e^{-\beta t}U(C_t^*f(\hat{Z}_t))dt-\hat{M}_n,$$

where

$$\hat{M}_n := \int_0^{\hat{\tau}_n} e^{-\beta t}\sigma\left[\hat{\theta}\hat{u}_x\right](\hat{X}_t,\hat{Z}_t)dW_t, \quad n\geq 0.$$

From continuity of $\hat{u}_x$ and $\hat{\theta}$ on $\mathcal{D}$, it follows that $(\hat{\theta}\hat{u}_x)(\hat{X},\hat{Z})$ is uniformly bounded on $[0,\hat{\tau}_n]$. Thus, $\mathbb{E}\hat{M}_n = 0$ and

$$\hat{u}(x,z) = \mathbb{E}\left[e^{-\beta\hat{\tau}_n}\hat{u}(\hat{X}_{\hat{\tau}_n},\hat{Z}_{\hat{\tau}_n})\right] + \mathbb{E}\left[\int_0^{\hat{\tau}_n} e^{-\beta t}U(C_t^* f(\hat{Z}_t)dt\right]$$

As $(\hat{\tau}_n)_{n\geq 1}$ is a sequence of bounded stopping times and it converges to ∞ a.s., we obtain using condition iii) from the theorem

$$\hat{u}(x,z) = \mathbb{E}\left[\int_0^{\infty} e^{-\beta t}U(C_t^* f(\hat{Z}_t)dt\right]$$

Combining with i) we obtain $\hat{u} = u$.

□

5.5 Dual dynamic programming equation

As we know from Theorem 5.4.1 u solves the partial differential equation

$$\mathcal{L}u(x,z) = 0 \quad \text{for } (x,z) \in \mathcal{D}, \tag{5.6}$$

on the domain of $\mathcal{D} := \{(x,z) : 0 < x \leq z\}$ with boundary conditions

$$\min\{\beta u - V\left(u_x/f(z)\right), -u_z\}(z,z) = 0 \quad \text{for} \quad z > 0, \tag{5.7}$$

where $V(x)$ is a convex conjugate of U given by (5.2).

We first observe that condition (i) from Theorem 5.4.1 implies

$$\beta u - V(u_x/f(z)) - \theta\sigma\lambda u_x - \frac{1}{2}\theta^2\sigma^2 u_{xx} \geq 0 \quad \text{for all } \theta \in \mathbb{R}.$$

Hence, any function u satisfying condition (i) must be concave in x. Then, operator $\mathcal{L}^{C,\theta}$ achieves its maximum on

$$\hat{C} := -V'(u_x/f(z)) = (U')^{-1}(u_x/f(z)),$$
$$\hat{\theta} := -\frac{\lambda}{\sigma}\frac{u_x}{u_{xx}}.$$

and, therefore,

$$\mathcal{L}u = -\beta u + V(u_x/f(z)) - \frac{\lambda^2}{2}\frac{u_x^2}{u_{xx}}.$$

Now, we define a dynamic programming problem dual to this one.

Let us introduce the Fenchel-Legendre transforms of the value function u with fixed z, i.e.

$$v(y,z) := \sup_{0\le x\le z} (u(x,z) - xy)$$

In case of $f(x) = 1$ for $x \ge 0$ function $u^0(x) := u(x,z)$ is independent of variable z and its dual v^0 can be obtained explicitly in terms of the density of the risk-neutral measure. This can be seen by the following formal argument. Assuming that u^0 is smooth and satisfies Inada conditions ($(u^0)'(0) = \infty$, $(u^0)'(\infty) = 0$) we obtain

$$v^0(y) = u^0(\left[(u^0)'\right]^{-1}(y)) - y\left[(u^0)'\right]^{-1}(y) \quad y \ge 0$$

and $v^0(y) = \infty$ for $y < 0$. Substituting into the dynamic programming equation (5.6) we obtain that v^0 solves on $(0,\infty)$ the linear parabolic partial differential equation

$$\mathcal{L}^* v^0(y) := v^0(y) - y v^0_y(y) - \frac{1}{\gamma} y^2 v^0_{yy}(y) = \frac{1}{\beta} V(y), \tag{5.8}$$

where we introduced the constant

$$\gamma := \frac{2\beta}{\lambda^2}.$$

The probabilistic intuition for that allows us to rewrite

$$v^0(y) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} V(e^{\beta t} Y_t)dt\right], \quad \text{where } Y_t := y\exp\{-\lambda W_t - \frac{1}{2}\lambda^2 t\}.$$

This result is known in financial mathematics and can be rigorously proved by probabilistic arguments; see, e.g., Theorem 9.21 in Sect. 9, Chap. 3 of [37].

In this complete market setting, it is worth noticing that v^0 solves a linear PDE. This is the key observation in order to find a candidate solution for the consumption-investment Problem 5.

We are going to derive the solution to the dual partial differential equation obtained by means of Fenchel-Legendre transform within the next three steps.

Step 1. The PDE satisfied by the dual function v.

Define functions $\phi(z) := u_x(z,z)$ and $\psi(z) := u_x(0,z)$ for $z > 0$.

For any $z > 0$, we expect that $\phi(z) \le \psi(z)$ as we want to find a concave function $u(\cdot, z)$. Denoting by $h(\cdot, z)$ the inverse of $u_x(\cdot, z)$, from the definition of the dual function v it follows that

$$v(y,z) = u(h(y,z),z) - h(y,z)y, \quad \text{if } u_x(h(y,z),z) = y \in [\phi(z), \psi(z)]\,, \tag{5.9}$$

$$v(y,z) = u(z,z) - yz, \quad \text{if } y \le \phi(z), \tag{5.10}$$

$$v(y,z) = u(0,z), \quad \text{if } y \ge \psi(z).$$

Notice that $h(y,z) = -v_y(y,z)$ by the classical property of the Fenchel-Legandre transformation. When $y \in [\phi(z), \psi(z)]$, we obtain by a change of variables in (5.6)

$$\mathcal{L}^* v(y,z) = \frac{1}{\beta} V(y/f(z)), \quad \text{for } \phi(z) < y < \psi(z), \tag{5.11}$$

where $\mathcal{L}^*$ is a linear operator defined in (5.8).

Using the definition of ϕ and ψ, from the Fenchel duality one obtains

$$v_y(\phi(z),z) = -z \quad \text{for } z > 0, \tag{5.12}$$

$$v_y(\psi(z),z) = 0 \quad \text{for } z > 0, \tag{5.13}$$

We also notice that:

- for $\phi(z) < y < \psi(z)$, we have

$$\beta u - V(u_x) = \beta(v(u_x) - u_x v_y(u_x)) - V(u_x/f(z)) = (\lambda^2 y^2 v_{yy}/2) \circ u_x.$$

- and for $y < \phi(z)$, we have from (5.10) that $v_z(y,z) = u_x(z,z) + u_z(z,z) - y$, and, thus, $v_z(\phi(z),z) = u_z(z,z)$ by the continuity of v which we get by its convexity and the definition of function ϕ.

Hence, the boundary condition (5.7) is transformed into

$$\min\{y v_{yy}, -v_z\}(\phi(z),z) = 0 \quad \text{for } z > 0. \tag{5.14}$$

In the next step we examine functions $v(y,z), \phi(z), \psi(z)$ which satisfy the ODE (5.11) subject to boundary conditions (5.12)-(5.14). We further assume that

$$\psi(z) = \infty \text{ for every } z \geq 0,$$

which is due to the economic intuition of the problem: when the wealth process reaches zero, the investor can neither consume nor invest for the entire future, see Lemma 5.4.1, and therefore the marginal utility in this situation is expected to be infinite, since $U'(0) = \infty$. Notice, that this does not mean that our argument is not rigorous since in this section we are looking for a candidate solution only. We prove in the verification theorem that candidate solution is a true solution.

Step 2: General solution of the ODE (5.11)*, subject to* (5.12)-(5.13)

We treat the equation (5.11) as a ordinary differential equation on function $v(y, z)$ where z is a parameter.

The homogeneous equation, defined by ignoring the RHS of the equation (5.11) has a general solution spanned on the basis of $w(y, z) = y$ and $w(y, z) = y^{-\gamma}$. This solutions are linearly independent since their Wronskian equals $W = -(\gamma + 1)y^{-\gamma}$.

By changing variable $y = \log \tilde{y}$ and examining the linear system one obtains the solution using standard method of solving linear nonhomogeneous equations

$$v(y, z) = a(z)y - b(z)y^{-\gamma} + \frac{2}{\lambda^2(1+\gamma)} y \int_{\phi(z)}^{\infty} V(\xi/f(z))\xi^{\gamma-1}(y \vee \xi)^{-(1+\gamma)} d\xi,$$

where the integral is finite as $V(\infty) = U(0) < \infty$.

By boundary conditions we obtain that $a(z) = 0$ and

$$b(z) = \frac{1}{\gamma}\phi(z)^{1+\gamma} \left(-z - \frac{2}{\lambda^2(1+\gamma)} \int_{\phi(z)}^{\infty} V(\xi/f(z))\xi^{-2} d\xi \right).$$

Updating function v we obtain

$$v(y, z) = \frac{y}{\gamma} \left(\frac{\phi(z)}{y} \right)^{1+\gamma} \left(z + \frac{2}{\lambda^2(1+\gamma)} \int_{\phi(z)}^{\infty} V(\xi/f(z))\xi^{-2} d\xi \right) + \tag{5.15}$$
$$\frac{2}{\lambda^2(1+\gamma)} \left(y^{-\gamma} \int_{\phi(z)}^{y} V(\xi/f(z))\xi^{\gamma-1} d\xi + y \int_{y}^{\infty} V(\xi/f(z))\xi^{-2} d\xi \right)$$

Step 3: Determining the marginal utility at the maximum $\phi(z)$ We now utilize the boundary condition (5.14) to determine the function ϕ:

$$\phi(z)v_{yy}(\phi(z),z) = (1+\gamma)z - \frac{2}{\lambda^2 f(z)}\int_{\phi(z)}^{\infty} -V'(\xi/f(z))\xi^{-1}d\xi$$

and

$$v_z(\phi(z),z) = \frac{1}{\gamma}\phi(z) + \frac{\phi'(z)}{\gamma}\left[(1+\gamma)z - \frac{2}{\lambda^2 f(z)}\int_{\phi(z)}^{\infty} -V'(\xi/f(z))\xi^{-1}d\xi\right] \tag{5.16}$$

$$+\frac{2\phi(z)f'(z)}{\lambda^2\gamma f^2(z)}\int_{\phi(z)}^{\infty} -V'(\xi/f(z))\xi^{-1}d\xi. \tag{5.17}$$

We rewrite it as

$$\begin{aligned} v_z(\phi(z),z) \qquad &= \frac{1}{\gamma}\phi(z)\left(1+(1+\gamma)\frac{zf'(z)}{f(z)}\right) \\ &+\left(\frac{\phi'(z)}{\gamma} - \frac{\phi(z)f'(z)}{\gamma f(z)}\right)\phi(z)v_{yy}(\phi(z),z) \end{aligned}$$

Assuming that $\phi(z) > 0$ and $(\phi(z)/f(z))' < 0$ for all $z > 0$. We obtain

$$\min\{\phi(z)v_{yy}(\phi(z),z), -v_z(\phi(z),z)\} = -\frac{1}{\gamma}\phi(z)\left(1+(1+\gamma)\frac{zf'(z)}{f(z)}\right)^+$$
$$-\left(\frac{\phi'(z)}{\gamma} - \frac{\phi(z)f'(z)}{\gamma f(z)}\right)\phi(z)v_{yy}(\phi(z),z).$$

Assumption 5.5.1. *We make an assumption that $f(z)$ is such that $1+(1+\gamma)\frac{zf'(z)}{f(z)}$ is either nonnegative or nonpositive for all $z > 0$, which in terms of w is*

$$\frac{w(x)}{x} - \frac{1}{1+\gamma} \geq 0 \quad x \geq 0 \ (or \ \leq 0 \quad x \geq 0 \ ).$$

Case A: $1+(1+\gamma)\frac{zf'(z)}{f(z)} \leq 0$ for $z \geq 0$.

We obtain

$$(1+\gamma)zf(z) - \frac{2}{\lambda^2}\int_{\phi(z)}^{\infty} -V'(\xi/f(z))\xi^{-1}d\xi = 0 \tag{5.18}$$

And, therefore,

$$\phi(z) = f(z)\kappa^{-1}\left((1+\gamma)zf(z)\right),$$

where

$$\kappa(x) = \frac{2}{\lambda^2}\int_{x}^{\infty} -V'(\xi)\xi^{-1}d\xi.$$

Case B: $1+(1+\gamma)\frac{zf'(z)}{f(z)} \geq 0$ for $z \geq 0$.

We get that $v_z(\phi(z), z) = 0$. Explicitly,

$$\frac{1}{\gamma}\phi(z) + \frac{(1+\gamma)}{\gamma}\phi'(z)z - \frac{2}{\lambda^2\gamma}\left(\frac{\phi(z)}{f(z)}\right)' \int_{\phi(z)/f(z)}^{\infty} -V'(\xi)\xi^{-1}d\xi = 0, \tag{5.19}$$

where $\phi(z)/f(z)$ is a decreasing function.

Example: This equation can be solved explicitly in terms of function κ for the case of linear $w(x)$ (or $f(x)$ being power function). Let $w(x) = \alpha x$, then $f(x) = x^{-\alpha}$. Then,

$$(1+\gamma)\rho^{1-\alpha}(t) - \frac{1-\alpha(1+\gamma)}{1-\alpha}t\left(\rho^{1-\alpha}(t)\right)' = \kappa(t),$$

where $\frac{\phi(\rho(t))}{f(\rho(t))} = t$. We rewrite it as

$$\left(\rho^{1-\alpha}(t)t^{-\frac{(1+\gamma)(1-\alpha)}{1-\alpha(1+\gamma)}}\right)' = -\frac{1-\alpha}{1-\alpha(1+\gamma)}\kappa(t)t^{-\frac{(1+\gamma)(1-\alpha)}{1-\alpha(1+\gamma)}-1}$$

and we are able to solve this equation as far as we know κ.

5.5.1 Candidate solution to the consumption-investment problem

We now derive an explicit candidate solution for our consumption-investment problem.

We know that

$$u(x,z) = v \circ h(x,z) + xh(x,z), \tag{5.20}$$

where $h(.,z)$ is the inverse of $-v_y(.,z)$ given by

$$\begin{aligned} -v_y(y,z) \quad &= \left(\frac{\phi(z)}{y}\right)^{1+\gamma}\left(z + \frac{2}{\lambda^2(1+\gamma)f(z)}\int_{\phi(z)}^{\infty} V'(\xi/f(z))\xi^{-2}d\xi\right) \\ &+ \frac{2}{\lambda^2(1+\gamma)f(z)}\int_{\phi(z)}^{\infty}\frac{-V'(\xi/f(z))}{\xi}\left(\frac{y\wedge\xi}{y}\right)^{1+\gamma}d\xi \end{aligned} \tag{5.21}$$

Thus, we rewrite

$$-v_y(y,z) = -\frac{\gamma}{y}v(y,z) + \frac{2}{\lambda^2}\int_y^{\infty} V(\xi/f(z))\xi^{-2}d\xi$$

Plugging $h(x,z)$ instead of y we obtain

$$x = \frac{\gamma}{h(x,z)}v \circ h(x,z) - \frac{2}{\lambda^2}\int_{h(x,z)}^{\infty} V(\xi/f(z))\xi^{-2}d\xi. \tag{5.22}$$

Thus,

$$v \circ h(x,z) = \frac{1}{\gamma} x h(x,z) + \frac{2}{\lambda^2 \gamma} h(x,z) \int_{h(x,z)}^{\infty} V(\xi/f(z)) \xi^{-2} d\xi.$$

From this using (5.20) we deduce that

$$u(x,z) = x h(x,z) \frac{1+\gamma}{\gamma} + \frac{1}{\beta} h(x,z) \int_{h(x,z)}^{\infty} V(\xi/f(z)) \xi^{-2} d\xi. \tag{5.23}$$

The candidate optimal consumption-investment strategy is obtained by means of the maximisers in dynamic programming equation as

$$\hat{C}(x,z) = -V'(h(x,z)/f(z)), \quad (x,z) \in \mathcal{D}. \tag{5.24}$$

By noticing that $u_x/u_{xx}(x,z) = h(x,z) v_{yy}(h(x,z),z)$ and that $v_{yy}(y,z) = -\frac{1+\gamma}{y} v_y(y,z) + \frac{2}{y\lambda^2 f(z)} \int_y^\infty -V'(\xi/f(z))\xi^{-2} d\xi$ we obtain using that h is an inverse of $-v_y$

$$v_{yy}(h(x,z),z) = \frac{(1+\gamma)x}{h(x,z)} + \frac{2}{h(x,z)\lambda^2 f(z)} \int_{h(x,z)}^{\infty} -V'(\xi/f(z)) \xi^{-2} d\xi.$$

Finally, using that $\hat{\theta} = -\frac{\lambda}{\sigma}\frac{u_x}{u_{xx}}$ we obtain

$$\hat{\theta}(x,z) = \frac{\lambda}{\sigma}(1+\gamma)x - \frac{2}{\lambda \sigma f(z)} \int_{h(x,z)}^{\infty} -V'(\xi/f(z)) \xi^{-1} d\xi, \quad (x,z) \in \mathcal{D}. \tag{5.25}$$

5.5.2 Power utility case

Now, let us consider particular case of power function $U(x) = x^p/p$ and $V(y) = y^{-q}/q$ where $\frac{1}{p} - \frac{1}{q} = 1$. Let $w(x)/x \geq 1/(1+\gamma)$. Assume with no loss of generality that $f(1) = 1$.

As power utility function inherits homogeneity property, we get $u(x,z) = u(x/z,1)f(z)^p$. Therefore, we expect $\hat{C}(x,z)$ to satisfy a similar property, namely, $\hat{C}(x,z) = z\hat{C}(x/z,1)$. By the same argument we obtain that $-V'(h(x,z)/f(z))/x$ is a function of a single variable x/z. It follows that we can write

$$h(x,z) = f(z) \left(x \hat{h}^{-1}\left(\frac{x}{z}\right)\right)^{-1+p}, \tag{5.26}$$

for some $\hat{h}$.

Now, we notice that function $\phi(z)$ rewrites by (5.18) as

$$\phi(z) = \left(\frac{\lambda^2(1+q)(1+\gamma)}{2}\right)^{-1+p} f^p(z) z^{-1+p}$$

Thus, using expression for ϕ and plugging (5.26) into (5.21) for $z = 1$ we obtain

$$\xi = \frac{c_1}{\hat{h}(\xi)} \left(\frac{1 - \frac{c_2}{\hat{h}(\xi)}}{1 - \frac{c_2}{c_1}} \right)^{\frac{\lambda^2}{2(1-p)^2} c_2^{-1}},$$

and

$$c_1 = \frac{\lambda^2(1+\gamma)}{2(1-p)} \quad \text{and} \quad c_2 = \frac{\lambda^2(\gamma - p - p\gamma)}{2(1-p)^2}$$

The value function for Problem 4 then is

$$\hat{u}(x, z) = u\left(K(z) - k(z)(z - x), K(z)\right),$$

and the consumption

$$\tilde{C}(x, z) = \frac{1}{k(z)} \left(K(z) - k(z)(z - x)\right) \hat{h}^{-1} \left(\frac{x - w(z)}{z - w(z)} \right).$$

Thus, we get results which are consistent with [20]. Moreover, we derive slightly generalised results as $w(x)$ is not linear only.

5.6 The main result

In this section we verify that the candidate solution (5.23)-(5.25) satisfies the conditions from Theorem 5.4.1. We first recall the notion of an asymptotic elasticity introduced by Kramkov and Schachermayer [40].

For a smooth function U we define

$$AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}.$$

Theorem 5.6.1. *Let utility function U satisfies* (5.1)*, moreover, let $AE(U) < \frac{\gamma}{1+\gamma}$; assume that equation* (5.19) *has a well-defined smooth decreasing solution $\phi(z)/f(z)$. We also assume that f is nonincreasing continuously differentiable function which satisfies Assumption 5.5.1.*

Then, u coincides with the candidate defined in (5.23)-(5.25)*. Moreover, u is a $C^0(\overline{\mathcal{D}}) \cap C^{2,1}(\mathcal{D})$ function, and for any initial data $(\hat{X}_0, \hat{Z}_0) = (x, z) \in \overline{\mathcal{D}}$, the stochastic differential equation*

$$d\hat{X}_t = -\hat{C}(\hat{X}_t, \hat{Z}_t)dt + \hat{\theta}(\hat{X}_t, \hat{Z}_t)\sigma(dW_t + \lambda dt), \quad \hat{Z}_t := z \vee \hat{X}_t^*,$$

has a unique strong solution with values $(\hat{X}_t, \hat{Z}_t) \in \mathcal{D}$ a.s.; moreover, the pair process $(C^*, \theta^*) := (\hat{C}, \hat{\theta})(\hat{X}, \hat{Z}) \in \mathcal{A}(x, z)$ is a solution to the Problem 5.

This theorem is proved by verifying that the candidate u satisfies all the conditions of the verification Theorem 5.4.1.

5.6.1 Regularity of the candidate value function

First we prove regularity properties of the candidate solution u.

Proposition 5.6.1. *The candidate value function u is in $C^0(\overline{\mathcal{D}}) \cap C^{2,1}(\mathcal{D})$ and satisfies*

$$\mathcal{L}u = 0 \text{ on } \mathcal{D}, \quad \min\{\beta u - V(u_x), -u_z\}(z,z) = 0 \text{ for } z > 0. \tag{5.27}$$

The proof of this proposition relies on the following result on the regularity of function h defined as an inverse of $-v_y$.

Lemma 5.6.1. *The function h is in $C^1(\mathcal{D})$, and, for any $(x,z) \in \mathcal{D}$, we have*

$$\frac{h_x(x,z)}{h(x,z)} = -\left((\gamma+1)x + \frac{\gamma}{\beta f(z)} \int_{h(x,z)}^{\infty} V'(s/f(z))/s ds\right)^{-1}, \tag{5.28}$$

$$\frac{h_z(x,z)}{h(x,z)} = \left(-\frac{\gamma}{x} v_z(x,z) + \frac{2}{\lambda^2}\frac{-f'(z)}{f^2(z)} \int_x^{\infty} -V'(\xi/f(z))\xi^{-2} d\xi\right) \frac{h_x(x,z)}{h(x,z)}. \tag{5.29}$$

Proof. By construction, the function h defined as the inverse of $-v_y$ satisfies

$$h(-v_y(y,z),z) = y \text{ for } y \geq \phi(z), \quad \text{and } -v_y(h(y,z),z) = y \text{ for } (x,z) \in \mathcal{D}. \tag{5.30}$$

By definition, the function $-v_y(\cdot,z)$ and its inverse $h(\cdot,z)$ are C^1 and decreasing for any $z > 0$. Direct computation then leads to (5.28). We also observe that $-v_y$ is in $C^{1,1}(\{(y,z), y \geq \phi(z)\})$ and that

$$\begin{aligned} -v_{yz}(y,z) &= \left(-\frac{\gamma}{y} v(y,z) - \frac{2}{\lambda^2}\int_y^{\infty} V(\xi/f(z))\xi^{-2} d\xi\right)'_z \\ &= -\frac{\gamma}{y} v_z(y,z) + \frac{2}{\lambda^2}\frac{-f'(z)}{f^2(z)} \int_y^{\infty} -V'(\xi/f(z))\xi^{-2} d\xi \geq 0, \end{aligned} \tag{5.31}$$

where we used that $v_z(x,z) \leq 0$ for $z \geq 0$. We also observe that $v_z(x,z)$ is a continuous function due to (5.15).

Therefore, $-v_y$ and h are increasing in z and $\phi(z) = h(z,z)$ is decreasing;

recall that $\phi > 0$ with $(\phi/f)'(z) < 0$ for $z > 0$. In order to prove that $h \in C^1(\mathcal{D})$, we shall prove that h is differentiable in each variable with continuous partial derivatives.

1. In this step, we show that $h \in C^0(\mathcal{D})$, which implies that $h_x \in C^0(\mathcal{D})$ by (5.28). For $(x,z) \in \mathcal{D}$, we study separately two alternative cases:

- If $x < z$, for l' small enough, $(x, z+l') \in \mathcal{D}$, and we deduce that

$$-v_y(h(x,z+l'),z) - x = v_y(h(x,z+l'),z) - v_y(h(x,z+l'),z+l') \to 0 \text{ as } l' \to 0.$$

Therefore, since $h(x,z+l') \geq \phi(z)$ from the monotonicity of h, combining (5.30) and the continuity of $h(\cdot,z)$, we obtain

$$h(x,z+l') - h(x,z) = -v_y(h(-v_y(x,z+l'),z),z) - h(x,z) \to 0 \text{ as } l' \to 0.$$

Moreover, notice that $(x+l, z+l') \in \mathcal{D}$ for $l \leq l'$, and we have

$$h(x+l,z+l') - h(x,z) = h_x(x_l, z+l')l + h(x,z+l') - h(x,z)$$

for some $x_l \in [x, x+l]$. Now, since h is monotonic in both variables, from (5.28) we deduce that h and h_x are bounded on any compact subset $\mathcal{D}$ containing (x,z). Therefore, combining last two equalities, we deduce that h is continuous at (x,z).

- If $x = z$, we have, for any l and l' satisfying $(z+l, z+l') \in \mathcal{D}$, that

$$h(z+l,z+l') = h_x(z_l, z+l')(l'-l) + \phi(z+l')$$

for some $z_l \in [z+l, z+l']$.

Therefore, similar arguments as above combined with the continuity of ϕ lead to the continuity of h on $\mathcal{D}$.

2. We know prove that h is differentiable with respect to z with continuous partial derivatives. Take $(x,z) \in \mathcal{D}$ and l' such that $(x, z+l') \in \mathcal{D}$. Combining $h(x,z) \geq \phi(z+l')$ with (5.30), we deduce

$$\begin{aligned}
&\frac{1}{l'}\{h(x,z+l') - h(x,z)\} = \frac{1}{l'}\{h(x,z+l') - h(-v_y(h(x,z),z+l'),z+l')\} \\
&= h_x(x_{l'}, z+l')\frac{1}{l'}\{-v_y(h(x,z),z) + v_y(h(x,z),z+l')\}
\end{aligned}$$

for some $x_{l'} \in [x, x+l']$. Since $h_x \in C^0(\mathcal{D})$ and $-v_{yz}(h(x,z),\cdot)$ is continuous, we obtain

$$\frac{1}{h'}\{h(x,z+h') - h(x,z)\} \to h_x(x,z)v_{yz}(h(x,z),z) \text{ as } h' \to 0.$$

Finally, combining (5.30) and (5.31), simple computations lead to (5.29), and h_z inherits the continuity of h on D. □

Proof of Proposition 5.6.1.

Proof. We only need to prove the regularity properties of the candidate value function u, since (5.27) is satisfied by the construction of u in Sect. 5.5.

By Lemma 5.6.1, $h \in C^1(\mathcal{D})$, and from (5.23) we deduce that $u \in C^1(\mathcal{D})$. Direct computation leads to $u_x = h$ on D, and therefore $u \in C^{2,1}(\mathcal{D})$. We now prove that $u \in C^0(\overline{\mathcal{D}})$.

By (5.22) we deduce that

$$xh(x,z) = \gamma v \circ h(x,z) + \frac{2}{\lambda^2}\int_{h(x,z)}^{\infty} V(\xi/f(z))\xi^{-2}d\xi$$

From $-v_y(\infty,z) = 0$ using monotonicity of v_y we deduce that $h(x,z) \to \infty$ as $x \to 0$. Now, notice that $V(y) \to U(0)$ as $y \to \infty$ and we obtain

$$xh(x,z) = \gamma v \circ h(x,z) + \frac{2U(0)}{\lambda^2 h(x,z)} + o(1), \quad x \to 0$$

From (5.15) we deduce that $v(y,z) \to 0$ as $y \to \infty$ and we finally obtain

$$xh(x,z) \to 0$$

By similar argument one can show that

$$\frac{1}{\beta}h(x,z)\int_{h(x,z)}^{\infty} V(\xi/f(z))\xi^{-2}d\xi \to \frac{U(0)}{\beta}$$

Therefore, using (5.23) we obtain that

$$u(x,z) \to U(0)/\beta \quad \text{as } x \to 0.$$

□

5.6.2 The wealth process with optimal feedback policy

Given an initial condition $(x, z) \in \mathcal{D}$, we consider the stochastic differential equation

$$d\hat{X}_t = -\hat{C}(\hat{X}_t, \hat{Z}_t)dt + \hat{\theta}(\hat{X}_t, \hat{Z}_t)\sigma(dW_t + \lambda dt), \tag{5.32}$$

where we used the previous notation $\hat{Z}_t := z \vee \hat{X}^*_t$, $t \geq 0$.

Proposition 5.6.2. *The stochastic differential equation* (5.32) *has a unique strong solution* $(\hat{X}, \hat{Z})$ *for any initial condition* $(x, z) \in \overline{\mathcal{D}}$*. Moreover,*

(i) $(\hat{C}, \hat{\theta})$ *satisfies* (5.3)*, and the pair process* $(C^*, \theta^*) := (\hat{C}, \hat{\theta})(\hat{X}_t, \hat{Z}_t)$ *is in* $\mathcal{A}(x, z)$.

(ii) If $zf'(z)/f(z) \leq -1/(1+\gamma)$ *for all* $z \geq 0$*, the running maximum is flat, i.e.,* $\hat{Z}_t = z$ *for every* $t \geq 0$.

This result is based on the following lemmas whose proofs are reported later on.

Lemma 5.6.2. *If* $zf'(z)/f(z) \leq -1/(1+\gamma)$ *for all* $z \geq 0$*, then, for every fixed* $z > 0$*,* $\hat{C}(\cdot, z)$ *and* $\theta(\cdot, z)$ *are locally Lipschitz on* $(0, z)$ *and* $\hat{\theta}(x, z) = O(\sqrt{z-x})$ *near the ray* $\{(z, z), z \geq 0\}$.

Lemma 5.6.3.

$$\limsup_{z\to\infty} \sup_{0\leq x\leq z} \frac{|\hat{\theta}(x, z)|}{z} < \infty$$

Proof of Proposition 5.6.2

Proof. We first observe that in order to prove item (i) of the Proposition, it is sufficient to show that $(C^*, \theta^*) \in \mathcal{A}(x, z)$. Indeed, from the continuity of the functions $\hat{C}$ and $\hat{\theta}$, it follows that

$$\int_0^T \hat{C}(\hat{X}_t, \hat{Z}_t)dt + \int_0^T \left|\hat{\theta}(\hat{X}_t, \hat{Z}_t)\right|^2 dt < \infty \quad \text{a.s. for every } T > 0.$$

We now prove the remaining claims of the proposition by considering the two alternative situations.

Case 1: Let $zf'(z)/f(z) > -1/(1+\gamma)$ for all $z \geq 0$.

We have that $\hat{C}$ and $\hat{\theta}$ are locally Lipschitz as continuously differentiable functions.

1. In this step, we show that the stochastic differential equation

$$d\tilde{X}_t = \hat{\theta}(\tilde{X}_t, \tilde{Z}_t)\sigma(dW_t + \lambda dt), \quad \tilde{X}_0 = x_0 \tag{5.33}$$

has a unique strong solution. To see this, we use a localisation argument.

First, observe that if $\hat{\theta}$ is Lipschitz, we directly estimate, for $t \geq 0$ and $x, y \in C^0(\mathbb{R}_+)$, and for $G(t, (x_u)_{u \leq t}) = \sigma\hat{\theta}(x_t, \overline{x}_t)$

$$|G(t,x) - G(t,y)| \leq K\left(|x(t) - y(t)| + |z \vee x^*(t) - z \vee y^*(t)|\right) \leq 2K|x-y|_t^*$$

where $K > 0$ is the Lipschitz constant of $\sigma\hat{\theta}$, and where we denote $|u|_t^* = \sup_{s \leq t} |u(s)|$. This proves that G is functional Lipschitz in the sense of Protter [50]. The existence and uniqueness of a strong solution to (5.33) follows from Theorem 7 in Sect. 9, Chap. 3 in [50].

Now we build a solution as follows. We define globally Lipschitz θ_N to be equal to $\hat{\theta}$ on $\{(x,z) \in \mathcal{D}, z \leq N\}$. Thus by previous observation there is a unique solution X^N to (5.33) with $\hat{\theta}$ changed to θ^N. By uniqueness of a solution X^N and X^{N+1} agree until $\tau_N := \inf\{t : |X^{N+1}| \geq N\}$. Hence, $X_t = X_t^N$ for $t \leq \tau_N$.

Now, clearly, for $t \leq \tau_N$

$$X_t = X_t^N = X_0 + \int_0^t \hat{\theta}(X_t, \overline{X}_t)\sigma(dW_t + \lambda dt),$$

so, provided we can prove that $\mathbb{P}(\sup \tau_N = \infty) = 1$, then X is a solution.

By Lemma 5.6.3 we deduce that there exists $C > 0$ such that

$$|\hat{\theta}(x,z)| \leq C(1+z)$$

By Lemma 11.5 in Sect. 11, Chap. 2 in [52] we deduce that for some constant C which depends only on T, for $0 \leq t \leq T$

$$\mathbb{E}\overline{X}_t^2 \leq C\{X_0^2 + \mathbb{E}\int_0^t \theta_s^2\sigma^2(\lambda^2+1)ds\}.$$

Using last two inequalities we obtain for $0 \leq t \leq T$,

$$\rho_t := \mathbb{E}(\overline{X}_{t\wedge\tau_N})^2 \leq \gamma\left[1 + \mathbb{E}\int_0^{t\wedge\tau_N} \overline{X}_s ds\right], \tag{5.34}$$

where γ depends on T only.

By Gronwall's Lemma 11.11 (5.34) we obtain that

$$\rho_t \leq \gamma e^{\gamma t}.$$

Thus,

$$\mathbb{P}(\tau_N < T) \leq N^{-2}\rho_T \leq N^{-2}\gamma e^{\gamma T} \to 0 \quad \text{as } N \to \infty$$

and so $\mathbb{P}(\sup \tau_N = \infty) = 1$. Uniqueness is carried by argument in line of Corollary 11.10 in Sect. 11, Chap. 2 in [52], which shows that any solution must agree with X^N on $[0, \tau_N]$.

2. Since $\hat{C}$ is locally Lipschitz on $\mathcal{D}$, and $\hat{C}(0, z) = 0$ for $z > 0$, a similar argument to the above step 1 shows that local existence and uniqueness hold for the stochastic differential equation (5.32). Recalling that $\hat{C} \geq 0$, it follows that $0 \leq \hat{X} \leq \tilde{X}$, which shows that there is no explosion of the local solution. Hence $(\hat{X}, \hat{Z})$ is the unique strong global solution to (5.32).

Case 2: Let $zf'(z)/f(z) \leq -1/(1+\gamma)$. The first crucial observation in this case is that $\hat{\theta}(z, z) = 0$ by (5.25) and the definition of ϕ. Since $\hat{C}$ is nonnegative, this shows that any possible solution of the stochastic differential equation (5.32) exhibits a flat component $\hat{Z}_t = z$ for every $t \geq 0$. We are then reduced to studying stochastic differential equation

$$d\hat{X}_t = -\hat{C}(\hat{X}_t, z)dt + \hat{\theta}(\hat{X}_t, z)\sigma(dW_t + \lambda dt), \tag{5.35}$$

where $z > 0$ is now a fixed parameter. In the present setting we only know that $\hat{C}(\cdot, z)$ and $\hat{\theta}(\cdot, z)$ are locally Lipschitz. However, since $\hat{C}(0, z) = \hat{\theta}(0, z) = 0$ for $z > 0$ it follows that any possible solution of (5.35) must satisfy $0 \leq \hat{X} \leq z$. In view of these bounds, we only need to prove the local existence and uniqueness for (5.35). But, according to Lemma 5.6.2, the coefficients of the stochastic differential equation satisfy the conditions of Proposition 5.2.13 in [36], which concludes the proof.

□

Proof of Lemma 5.6.3

Proof.

$$\hat{\theta}(x, z)/z = \frac{\lambda}{\sigma}(1+\gamma)x/z - \frac{2}{\lambda\sigma z f(z)}\int_{h(x,z)}^{\infty} -V'(\xi/f(z))\xi^{-1}d\xi, \quad (x, z) \in \mathcal{D}.$$

Consider for some $(x,z) \in \mathcal{D}$

$$\left|\hat{\theta}(x,z)/z - \frac{\lambda}{\sigma}(1+\gamma)x/z\right| = \frac{2}{\lambda\sigma z f(z)} \int_{h(x,z)}^{\infty} -V'(\xi/f(z))\xi^{-1} d\xi$$
$$\leq \frac{2}{\lambda\sigma z f(z)} \int_{\phi(z)}^{\infty} -V'(\xi/f(z))\xi^{-1} d\xi = \frac{\lambda(1+\gamma)}{\sigma} - \frac{\lambda}{\sigma}\phi(z) v_{yy}(\phi(z),z) \leq \frac{\lambda(1+\gamma)}{\sigma}$$

Therefore,

$$\left|\hat{\theta}(x,z)/z\right| \leq \frac{\lambda}{\sigma}(1+\gamma)x/z + \frac{\lambda(1+\gamma)}{\sigma},$$

and we conclude the proof. ☐

Proof of Lemma 5.6.2

Proof. For any fixed $z > 0$, from Lemma 5.6.1 we deduce that $\hat{\theta}(\cdot, z)$ and $\hat{C}(\cdot, z)$ are C^1 and therefore locally Lipschitz on $(0, z)$. Observe now that, combining (5.25) with the definition of h, we get

$$\hat{\theta}(x,z) = \frac{2}{\lambda\sigma f(z)} \int_{\phi(z)}^{h(x,z)} \frac{-V'(\xi/f(z))}{\xi} \left(\frac{\xi}{h(x,z)}\right)^{\gamma} d\xi.$$

From $\phi(z) v_{yy}(\phi(z), z) = 0$ and definition of h we obtain

$$z - x = \frac{2}{\lambda^2(1+\gamma) f(z)} \int_{\phi(z)}^{h(x,z)} -V'(\xi/f(z))\xi^{-1} \left[1 - \left(\frac{\xi}{h(x,z)}\right)^{1+\gamma}\right] d\xi.$$

To conclude, we see that, near $x = z$, we have $h(x,z) \sim \phi(z)$ and therefore

$$\frac{\theta(x,z)^2}{z-x} \sim -\frac{2(1+\gamma)V'(\phi(z)/f(z))}{\sigma^2} \frac{\frac{h(x,z)}{\phi(z)} - 1}{\left(\frac{h(x,z)}{\phi(z)}\right)^{1+\gamma} - 1} \sim -\frac{2V'(\phi(z)/f(z)}{\sigma^2 f(z)}$$

☐

5.6.3 Transversality condition

We finally turn to the proof of the transversality condition of Theorem 5.4.1.

Proposition 5.6.3. *Given $(x,z) \in \mathcal{D}$, let $(\hat{X}, \hat{Z})$ be the unique strong solution to the stochastic differential equation (5.32). Then, for every sequence of bounded stopping times $(\tau_n)_{n\geq 1}$ with $\tau_n \to \infty$ a.s., we have*

$$\liminf_{n\to\infty} \mathbb{E}\left[e^{-\beta\tau_n} u(\hat{X}_{\tau_n}, \hat{Z}_{\tau_n})\right] = 0.$$

The proof of this result is organised as follows. Lemma 5.6.4 shows that the transversality condition holds under a convenient growth condition on the candidate value function.

Lemma 5.6.4. *Let* $p : \overline{\mathcal{D}} \to \mathbb{R}$ *be such that there exist* $K > 0$ *and* $0 < \delta < \gamma/(1+\gamma)$ *satisfying*

$$|p(x,z)| \leq K(1 + x^{\delta})), \quad (x,z) \in \overline{\mathcal{D}}.$$

Given $(x,z) \in \overline{\mathcal{D}}$, *let* $(C,\theta) \in \mathcal{A}(x,z)$ *be a consumption-investment strategy, and set* $(X,Z) := (X^{C,\theta}, Z^{C,\theta})$. *Assume in addition that* $\pi_t := \theta_t/X_t$ *is uniformly bounded on* $\mathbb{R}_+ \times \Omega$.

Then, for every sequence of bounded stopping times $(\tau_n)_{n\geq 1}$ *with* $\tau_n \to \infty$ *a.s., we have*

$$\liminf_{n\to\infty} \mathbb{E}\left[e^{-\beta\tau_n} p(X_{\tau_n}, Z_{\tau_n})\right] = 0.$$

Proof. Denoting $c := C/X$ from the condition of the lemma and definition of X it follows that, for $t > 0$,

$$|p(X_t, Z_t)| \leq K\left(1 + N_t \exp\{-\int_0^t \delta\left(c_r - \lambda\sigma\pi_r + (1-\delta)\frac{(\sigma\pi_r)^2}{2}\right) dr\}\right),$$

where $N := \mathcal{E}(\int_0^{\cdot} \sigma\delta\pi_t dW_t)$ is the exponential martingale defined by the dynamics

$$dN_t = N_t \sigma\delta\pi_t dW_t;$$

recall that π is uniformly bounded. Direct computation shows that

$$\begin{aligned}
\eta_s &:= \beta + \delta\left(c_s - \lambda\sigma\pi_s + (1-\delta)\frac{(\sigma\pi_s)^2}{2}\right) \\
&\geq \frac{\lambda^2}{2}\left[\gamma + \delta\left((1-\delta)\left(\frac{\sigma\pi_s}{\lambda} - \frac{1}{1-\delta}\right)^2 - \frac{1}{1-\delta}\right)\right] \\
&\geq \frac{\lambda^2}{2}\left(\gamma - \frac{\delta}{1-\delta}\right) =: 2\eta > 0,
\end{aligned}$$

since $\delta < \gamma/(1+\gamma)$ by the condition of the lemma. Therefore,

$$\mathbb{E}\left[e^{-\beta\tau_n} p(X_{\tau_n}, Z_{\tau_n})\right] \leq K\mathbb{E}\left[e^{-\beta\tau_n} + e^{-2\eta\tau_n} N_{\tau_n}\right]. \tag{5.36}$$

Furthermore, for any $\varepsilon > 0$, from the Hölder inequality it follows that

$$\begin{aligned}
\mathbb{E}\left[e^{-2\eta\tau_n} N_{\tau_n}\right] \quad &\leq ||e^{-2\eta\tau_n + \frac{\varepsilon}{2}\int_0^{\tau_n} |\sigma\delta\pi_t|^2 dt}||_{L^{1+\varepsilon^{-1}}} \\
&\times \mathbb{E}\left[\mathcal{E}\left((1+\varepsilon)\sigma\delta\int_0^{\cdot} \pi_t dW_t\right)_{\tau_n}\right]^{(1+\varepsilon)^{-1}} \\
&= ||e^{-2\eta\tau_n + \frac{\varepsilon}{2}\int_0^{\tau_n} |\sigma\delta\pi_t|^2 dt}||_{L^{1+\varepsilon^{-1}}} \\
&\leq ||e^{-(2\eta + \frac{\varepsilon}{2}\sigma^2\delta^2 ||\pi_t||_{\infty}^2)\tau_n}||_{L^{1+\varepsilon^{-1}}}.
\end{aligned}$$

Hence, for a sufficiently small $\varepsilon > 0$, from (5.36) we deduce that

$$\mathbb{E}\left[e^{-\beta\tau_n} p(X_{\tau_n}, Z_{\tau_n})\right] \leq K\left(\mathbb{E}e^{-\beta\tau_n} + ||e^{-\eta\tau_n}||_{L^{1+\varepsilon^{-1}}}\right),$$

and the required result follows from the dominated convergence theorem by sending n to infinity.

□

Proof of Proposition 5.6.3

Proof. Since U has finite asymptotic elasticity $p := AE(U) < \gamma/(1+\gamma)$ then its Fenchel-Legandre transform V satisfies

$$\limsup_{y\to 0} -V'(y)y^{\frac{1}{1-p}} < \infty \qquad \text{and} \qquad \limsup_{y\to 0} V(y)y^{\frac{p}{1-p}} < \infty.$$

Let us recall now that

$$-v_y(y,z) = -\frac{\gamma}{y}v(y,z) + \frac{2}{\lambda^2}\int_y^\infty V(\xi/f(z))\xi^{-2}d\xi$$

We rewrite this differential equation in the following form

$$\begin{aligned} v(y,z) &= c(y,z)y^\gamma, \\ c'_y(y,z) &= -\frac{2}{\lambda^2}y^{-\gamma}\int_y^\infty V(\xi/f(z))\xi^{-2}d\xi. \end{aligned}$$

We notice that $|c'_y(y,z)| \leq L\frac{2}{\lambda^2}y^{-\gamma}f(z)^{p/(1-p)}\int_y^\infty \xi^{-p/(1-p)-2}d\xi$ for some large enough L.

Thus, $|c'_y(y,z)| \leq \frac{2}{\lambda^2}Lf(z)^{p/(1-p)}y^{-\gamma-p/(1-p)-1}$.

Now, $c(\infty,z) = 0$ as $v(\infty,z) = 0$, therefore, $c(y,z) \leq \frac{2}{\lambda^2(\gamma+p/(1-p))}Lf(z)^{p/(1-p)}y^{-\gamma-p/(1-p)}$. Therefore, $v(y,z) \leq \frac{2}{\lambda^2(\gamma+p/(1-p))}Lf(z)^{p/(1-p)}y^{-p/(1-p)}$.

This implies, that its Fenchel-Legandre transform u satisfies

$$u(x,z) = \min_{y\geq\phi(z)}(v(y,z)+xy) \leq q^{-\frac{q}{1+q}}(1+q)\left(\frac{2}{\lambda^2(\gamma+p/(1-p))}L\right)^{\frac{1}{1+q}} f(z)^{\frac{p}{(1-p)(1+q)}}x^p.$$

And as $f(z)$ is decreasing and $z \geq x$, when $x \to \infty$ we obtain that $u(x,z) \leq K(1+x^p)$, where $p = AE(U) < \frac{\gamma}{1+\gamma}$.

Finally, we use Lemma 5.6.4 to obtain transversality condition, just noticing in addition that $\hat{\theta}(x,z)/x \leq \lambda(1+\gamma)/\sigma$.

□

5.7 Optimisation problem with capacity constraint on wealth

In this section we analyze the optimisation problem of an investor maximising expected utility of discounted consumption subject to a static constraint imposed on a wealth process at all times. Precisely, we assume that the wealth of investor is not allowed to exceed a certain predefined level. One natural interpretation of the constraint appears when considering an investor facing nonlinear taxation rule of the P&L. In this case she might tend to keep her wealth below a certain threshold.

We consider the market from Section 5.4.

Firstly, we introduce the class of constrained investment strategies by the following

$$\mathcal{A}_K(v_0) := \{(C,\pi) \in \mathcal{A}(v_0) : \quad X_t^{C,\pi} \leq K, \quad t \geq 0\}.$$

Consider the following problem.

Problem 6. *For an utility function U a capacity level K, initial wealth v_0 find the supremum of*

$$\mathcal{R}(\pi, C) := \mathbb{E}\int_0^\infty e^{-\beta t} U(C_t)dt$$

over a class of investment strategies (π_t, C_t) from $\mathcal{A}_K(v_0)$.

Proposition 5.7.1. *Assume that $v_0 < K$ then*

$$\begin{aligned} \sup_{(\pi_t,C_t)\in\mathcal{A}_K(v_0)} \mathcal{R}(\pi, C) &= \mathcal{R}(\pi^*, C^*) = \\ \sup_{(\pi_t,C_t)\in\mathcal{A}(v_0)} \mathcal{R}_1^f(\pi, C) &= \mathcal{R}_1^f(\pi^*, C^*), \end{aligned}$$

where $f(z) = (z/K)^{-1/(1+\gamma)}$.

Proof. By Proposition 5.6.2 we obtain that the optimal wealth for the drawdown contrained problem has a flat maximum when $\alpha \geq \frac{1}{1+\gamma}$. By 5.3.1 we deduce that for $f(z) = (z/K)^{-1/(1+\gamma)}$ the optimal wealth has also a flat maximum (that is because a maximum of Azéma-Yor process with increasing F is a function of maximum of an underlying semimartingale).

Now, as $\mathcal{A}_K(v_0) \subset \mathcal{A}(v_0)$ and $\mathcal{R}_1^f(\pi, C) = \mathcal{R}(\pi, C)$ for any $(\pi_t, C_t)_{t\geq 0} \in \mathcal{A}_K(v_0)$ we obtain the result. □

Bibliography

[1] Optimal investment and consumption with transaction costs. Annals of Applied Probability 4, 3 (1994), 609–692.

[2] Azéma, J., and Yor, M. Une solution simple au problème de Skorokhod. In Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), C. Dellacherie, P.-A. Meyer, and M. Weil, Eds., vol. 721 of Lecture Notes in Math. Springer, Berlin, 1979, pp. 90–115.

[3] Bansal, R., and Lehmann, B. N. Growth-optimal portfolio restrictions on asset pricing models. Macroeconomic Dynamics 1, 02 (1997), 333–354.

[4] Black, F., and Jones, R. Simplifying portfolio insurance. The Journal of Portfolio Management 14 (1987), 48–51.

[5] Carraro, L., Karoui, N. E., and Obłój, J. On Azéma-Yor processes, their optimal properties and the Bachelier-Drawdown equation. Ann. Probab. 40, 1 (2012), 372–400.

[6] Chekhlov, A., Uryasev, S., and Zabarankin, M. Drawdown measure in portfolio optimization. International J. Theoretical and Applied Finance 8, 1 (2005), 13–58.

[7] Cheridito, P., Nikeghbali, A., and Platen, E. Processes of class (Σ), last passage times and drawdowns. SIAM Journal on Financial Mathematics (2012, to appear). arXiv:0910.5493.

[8] Christensen, M. M., and Larsen, K. No arbitrage and the growth optimal portfolio. Stochastic Analysis & Applications 25, 1 (2007), 255 – 280.

[9] Cox, J. C., and fu Huang, C. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory 49, 1 (1989), 33–83.

[10] Cox, J. C., and fu Huang, C. A continuous-time portfolio turnpike theorem. Journal of Economic Dynamics and Control 16, 3-4 (1992), 491 – 507.

[11] Cvitanić, J., and Karatzas, I. Convex duality in constrained portfolio optimization. Annals of Applied Probability 2, 4 (1992), 767–818.

[12] Cvitanić, J., and Karatzas, I. On portfolio optimization under "drawdown" constraints. IMA Volumes in Mathematics and Its Applications 65, 3 (1994), 35–46.

[13] Davis, M. H., and Norman, A. R. Portfolio selection with transaction costs. Math. Oper. Res. 15, 4 (Oct. 1990), 676–713.

[14] Delbaen, F., and Schachermayer, W. A general version of the fundamental theorem of asset pricing. Mathematische Annalen 300, 1 (1994), 463–520.

[15] Dumas, B., and Luciano, E. An exact solution to a dynamic portfolio choice problem under transactions costs. Journal of Finance 46, 2 (1991), 577–595.

[16] Dybvig, P., Rogers, L., and Back, K. Portfolio turnpikes. Review of Financial Studies 12, 1 (1999), 165–195.

[17] El Karoui, N., and Jeanblanc-Picqu, M. Optimization of consumption with labor income. Finance and Stochastics 2 (1998), 409–440.

[18] El Karoui, N., and Meziou, A. Constrained optimization with respect to stochastic dominance: Application to portfolio insurance. Mathematical Finance 16, 1 (2006), 103–117.

[19] Elie, R. Finite time merton strategy under drawdown constraint: A viscosity solution approach. Applied Mathematics & Optimization 58, 3 (2008), 411–431.

[20] Elie, R., and Touzi, N. Optimal lifetime consumption and investment under a drawdown constraint. Finance and Stochastics 12, 3 (2008), 299–330.

[21] Fernholz, R., and Karatzas, I. Stochastic portfolio theory: an overview. In Handbook of Numerical Analysis, P. Ciarlet, Ed., vol. 15 of Handbook of Numerical Analysis. Elsevier, 2009, pp. 89–167.

[22] Fleming, W. H., and Sheu, S. J. Risk-sensitive control and an optimal investment model. Mathematical Finance 10, 2 (2000), 197–213.

[23] Fllmer, H., and Kramkov, D. Optional decompositions under constraints. Probability Theory and Related Fields 109 (1997), 1–25.

[24] Grossman, S., and Vila, J. Optimal dynamic trading with leverage constraints. Journal of Financial and Quantitative Analysis 27, 2 (1992), 151–168.

[25] Grossman, S. J., and Zhou, Z. Optimal investment strategies for controlling drawdowns. Mathematical Finance 3, 3 (1993), 241–276.

[26] Guasoni, P., and Obłój, J. The incentives of hedge fund fees and high-water marks.

[27] Guasoni, P., and Robertson, S. Static fund separation of long term investments. SSRN eLibrary (2011).

[28] Guasoni, P., and Robertson, S. Portfolios and risk premia for the long run. Ann. Appl. Probab. (to appear).

[29] Hakansson, N. H. Convergence to isoelastic utility and policy in multiperiod portfolio choice. Journal of Financial Economics 1, 3 (1974), 201 – 224.

[30] He, H., and Pags, H. F. Labor income, borrowing constraints, and equilibrium asset prices. Economic Theory 3 (1993), 663–696.

[31] Huang, C.-f., and Zariphopoulou, T. Turnpike behavior of long-term investments. Finance and Stochastics 3 (1999), 15–34.

[32] Huberman, G., and Ross, S. Portfolio turnpike theorems, risk aversion, and regularly varying utility functions. Econometrica 51, 5 (1983), 1345–1361.

[33] Karatzas, I., and Kardaras, C. The numéraire portfolio in semimartingale financial models. Finance and Stochastics 11 (2007), 447–493.

[34] Karatzas, I., Lehoczky, J. P., and Shreve, S. E. Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. SIAM J. of Control and Optimization 25 (1987), 1157–1586.

[35] Karatzas, I., Lehoczky, J. P., Shreve, S. E., and Xu, G.-L. Martingale and duality methods for utility maximization in an incomplete market. 702–730.

[36] Karatzas, I., and Shreve, S. E. Brownian motion and stochastic calculus. Springer-Verlag, 1991.

[37] Karatzas, I., and Shreve, S. E. Methods of mathematical finance, vol. 39 of Applications of Mathematics (New York). Springer-Verlag, New York, 1998.

[38] Karoui, N. E., and Meziou, A. Max-plus decomposition of supermartingales and convex order. application to american options and portfolio insurance. Ann. Probab. 36, 2 (2008), 647–697.

[39] Klass, M. J., and Nowicki, K. The grossman and zhou investment strategy is not always optimal. Statist. Probab. Lett. 74 (2005), 245–252.

[40] Kramkov, D., and Schachermayer, W. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. The Annals of Applied Probability 9, 3 (1999), 904–950.

[41] Leland, H. On turnpike portfolios. Mathematical methods of investment and finance (1972).

[42] Magdon-Ismail, M., and Atiya, A. Maximum drawdown. Risk 17, 10 (2004), 99–102.

[43] Magill, M., and Constantinides, G. M. Portfolio selection with transactions costs. Journal of Economic Theory 13, 2 (1976), 245–263.

[44] Merton, R. C. Lifetime portfolio selection under uncertainty: The continuous-time case. The Review of Economics and Statistics 51, 3 (1969), 247–57.

[45] Merton, R. C. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 4 (Dec. 1971), 373–413.

[46] Mossin, J. Optimal multiperiod portfolio policies. Journal of Business 41, 2 (1968), 215–229.

[47] Perold, A. F., and Sharpe, W. F. Dynamic strategies for asset allocation. Financial Analysts Journal 44, 1 (1988), 16–27.

[48] Platen, E., and Heath, D. A Benchmark Approach to Quantitative Finance. Springer, 2006.

[49] Pospisil, L., and Vecer, J. Portfolio sensitivity to changes in the maximum and the maximum drawdown. Quant. Finance 10, 6 (2010), 617–627.

[50] Protter, P. Stochastic Integration and Differential Equations. Springer, 1990.

[51] Roche, H. Optimal Consumption and Investment Strategies under Wealth Ratcheting. 2006.

[52] Rogers, L., and Williams, D. Diffusions, Markov processes, and martingales. J Wiley, 1987.

[53] Sekine, J. Long-term optimal portfolios with floor. To appear in Finance and Stochastics (2012).

[54] Vecer, J. Maximum drawdown and directional trading. Risk 19, 12 (2006), 88–92.