

Faster Exponential-time Algorithms for Approximately Counting Independent Sets*

Leslie Ann Goldberg[†]

John Lapinskas[‡]

David Richerby[§]

26 August 2021

Abstract

Counting the independent sets of a graph is a classical #P-complete problem, even in the bipartite case. We give an exponential-time approximation scheme for this problem which is faster than the best known algorithm for the exact problem. The running time of our algorithm on general graphs with error tolerance ε is at most $O(2^{0.2680n})$ times a polynomial in $1/\varepsilon$. On bipartite graphs, the exponential term in the running time is improved to $O(2^{0.2372n})$. Our methods combine techniques from exact exponential algorithms with techniques from approximate counting. Along the way we generalise (to the multivariate case) the FPTAS of Sinclair, Srivastava, Štefankovič and Yin for approximating the hardcore partition function on graphs with bounded connective constant. Also, we obtain an FPTAS for counting independent sets on graphs with no vertices with degree at least 6 whose neighbours' degrees sum to 27 or more. By a result of Sly, there is no FPTAS that applies to all graphs with maximum degree 6 unless $P = NP$.

1 Introduction

The problem of counting the independent sets of a bipartite graph was originally shown to be #P-complete by Provan and Ball [17]. Clearly, this implies that counting the independent sets of an arbitrary graph is also #P-complete. The problem of enumerating independent sets goes back even further [1].

Given that there is unlikely to be a polynomial-time algorithm for counting independent sets, much work [3, 4, 5, 8, 11, 15, 22, 25] has focussed on developing exponential algorithms that are as fast as possible. The fastest known algorithm for counting independent sets is due to Gaspers and Lee [11], with running time $O(2^{0.3022n})$, where n denotes the number of vertices of the graph G . The algorithm of Gaspers and Lee builds on a long history of improvements, as detailed in Table 1.

Given the #P-completeness of *exactly* counting independent sets, one might wonder whether there are faster algorithms for approximation. Let $Z(G)$ denote the number of independent sets of a graph G and let #IS be the problem of computing $Z(G)$, given a graph G .

*The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013) ERC grant agreement no. 334828. The paper reflects only the authors' views and not the views of the ERC or the European Commission. The European Union is not liable for any use that may be made of the information contained therein.

[†]Department of Computer Science, University of Oxford, UK.

[‡]Department of Computer Science, University of Bristol, UK.

[§]School of Computer Science and Electronic Engineering, University of Essex, UK.

Algorithm	Running time
Brute force enumeration	$O(2^n) \text{poly}(n)$
Dubois [5] and Zhang [25]	$O(2^{0.6943n})$
Dahllöf, Jonsson and Wahlström [3]	$O(2^{0.4057n})$
Dahllöf, Jonsson and Wahlström [4]	$O(2^{0.3290n})$
Fürer and Kasiviswanathan [8]	$O(2^{0.3174n})$
Wahlström [22]	$O(2^{0.3077n})$
Junosza-Szaniawski and Tuczynski [15]	$O(2^{0.3068n})$
Gaspers and Lee [11]	$O(2^{0.3022n})$
This paper (general graphs)	$O(2^{0.2680n}) \text{poly}(1/\varepsilon)$ approximation
This paper (bipartite graphs)	$O(2^{0.2372n}) \text{poly}(1/\varepsilon)$ approximation

Table 1: A history of exponential-time algorithms for counting independent sets.

An ε -approximation of $Z(G)$ is a number N satisfying $(1 - \varepsilon)Z(G) \leq N \leq (1 + \varepsilon)Z(G)$. An *approximation scheme* for $\#IS$ is an algorithm that takes as input a graph G and an error tolerance $\varepsilon \in (0, 1)$ and outputs an ε -approximation of $Z(G)$. It is a *fully polynomial-time* approximation scheme (FPTAS) if its running time is at most a polynomial in $|V(G)|$ and $1/\varepsilon$. Weitz [23] gave an FPTAS for approximating $Z(G)$ on graphs of degree at most 5. However, Sly [20] showed that, unless $P = NP$, there is no FPTAS on graphs of degree at most 6.

Although there is unlikely to be an FPTAS for $\#IS$, our main result is that there is an approximation scheme whose running time is faster than the best known algorithms for exactly counting independent sets. In particular (see Theorem 1), we give an approximation scheme for $\#IS$ whose running time, given an n -vertex graph G and an error tolerance ε , is at most $O(2^{0.2680n})$ times a polynomial in $1/\varepsilon$.

The problem of approximately counting the independent sets of a *bipartite* graph, denoted $\#BIS$, is a canonical problem in approximate counting. It is known [6] to be complete in a large complexity class with respect to approximation-preserving reductions. Many important approximate counting problems have been shown to be $\#BIS$ -hard or $\#BIS$ -equivalent [13, 9, 2, 10] so resolving its complexity is a major open problem in approximate counting.

Our second result (also given in Theorem 1) is that there is an approximation scheme for $\#BIS$ whose running time is faster than the best known approximation scheme for general graphs. In particular, we give an approximation scheme whose running time is at most $O(2^{0.2372n})$ times a polynomial in $1/\varepsilon$.

Theorem 1. *There is an approximation scheme for $\#IS$ which, given an n -vertex input graph with error tolerance ε , runs in time $O(2^{0.2680n}) \text{poly}(1/\varepsilon)$. There is an approximation scheme for $\#BIS$ with running time $O(2^{0.2372n}) \text{poly}(1/\varepsilon)$.*

Our improvement for approximate counting over the best exact-counting algorithm has two sources. The first is a technical analysis for bipartite graphs; we defer discussion of this to the proof sketch in Section 1.1. The second is a substantially faster handling of sparse instances for both bipartite and non-bipartite graphs. It is not at all surprising that this is possible, since there is an FPTAS for instances with maximum degree at most 5 [23] but it is $\#P$ -complete to solve 3-regular instances exactly [24] (even if they are bipartite). However, it is necessary to do substantially more than to use [23] as a black box.

Recall that the result of [23] is tight in the sense that it is NP-hard to approximate $Z(G)$ when G is a 6-regular graph [20]. Nevertheless, the result of [23] has been improved in several directions [14, 16, 18]. The most relevant of these in our context is the result of [18] which replaces the maximum degree bound with a weaker bound on a natural measure of average degree called the *connective constant*, which we now define. Given a graph G , a vertex v , and a positive integer ℓ , let $N_G(v, \ell)$ denote the number of simple paths of length ℓ starting from v in G .

Definition 2. ([19]). Let \mathcal{F} be a family of graphs and let κ be a positive real number. The connective constant of \mathcal{F} is at most κ if there are real numbers a and c such that, for every $G \in \mathcal{F}$, every vertex $v \in V(G)$, and every integer $\ell \geq a \log |V(G)|$, we have $\sum_{i=1}^{\ell} N_G(v, i) \leq c\kappa^{\ell}$.

As an example, let \mathcal{F} be a family of graphs such that the maximum degree of any graph in \mathcal{F} is d . It is easy to see that the connective constant of \mathcal{F} is at most $d - 1$. However, for some such families \mathcal{F} it is even smaller.

Sinclair, Srivastava, Štefankovič and Yin [18] give an FPTAS for $Z(G)$ which works on any family of graphs with connective constant at most 4.141. These are the “sparse instances” that we referred to earlier. Our approach is to use this FPTAS for sparse instances as a base case in our exponential-time algorithm for counting independent sets. We are left with two key difficulties. The first issue is that most previous fast exponential-time algorithms for counting independent sets exploit the ability to add “weights” to vertices; amongst other things, this allows for quick removal of degree-1 vertices so it substantially improves the running time. However, the result of [18] requires all vertices to have the same weight, precluding the use of such techniques. We therefore give a more general, multivariate version of the theorem of [18] (which may also be useful in other contexts).

In order to describe the generalisation, we need some notation. Let $\mathcal{I}(G)$ denote the set of independent sets of a graph G . We define a *weighted graph* $\mathcal{G} = (G, w_+, w_-, W)$ to be a tuple consisting of a graph G , two functions w_+ and w_- assigning positive integer weights to the elements of $V(G)$, and a positive integer W . We then define the partition function

$$Z(\mathcal{G}) = W \sum_{I \in \mathcal{I}(G)} \prod_{v \in I} w_+(v) \prod_{v \in V(G) \setminus I} w_-(v).$$

Modulo normalisation (which will be useful for our recursive instances), this is the multivariate hard-core partition function of G .

We need two more definitions in order to state our result. Given a positive real number λ , we say that a weighted graph \mathcal{G} is λ -balanced if, for every vertex v of G , we have $w_+(v) \leq \lambda w_-(v)$. “ λ -balance” is just a way of bounding how much more weight accrues for being in an independent set, as opposed to out of it. The result of [18, Theorem 1.1] applies to the (usual) univariate hard-core model where every vertex has $w_+(v) = \lambda$ and $w_-(v) = 1$. They give an FPTAS which works for any family \mathcal{F} with connective constant at most κ as long as $\lambda < \lambda_c(\kappa)$ where $\lambda_c(\kappa) = \kappa^{\kappa} / (\kappa - 1)^{\kappa+1}$ is the critical activity for the hard-core model on an infinite κ -ary tree. This leads to our last definition here. If $\lambda < \lambda_c(\kappa)$ we say that κ is *subcritical with respect to λ* .

Theorem 3. Let λ be a positive real number and let \mathcal{F} be a family of graphs whose connective constant is at most a quantity which is subcritical with respect to λ . There is an FPTAS for $Z(\mathcal{G})$ on λ -balanced weighted graphs $\mathcal{G} = (G, w_+, w_-, W)$ such that $G \in \mathcal{F}$.

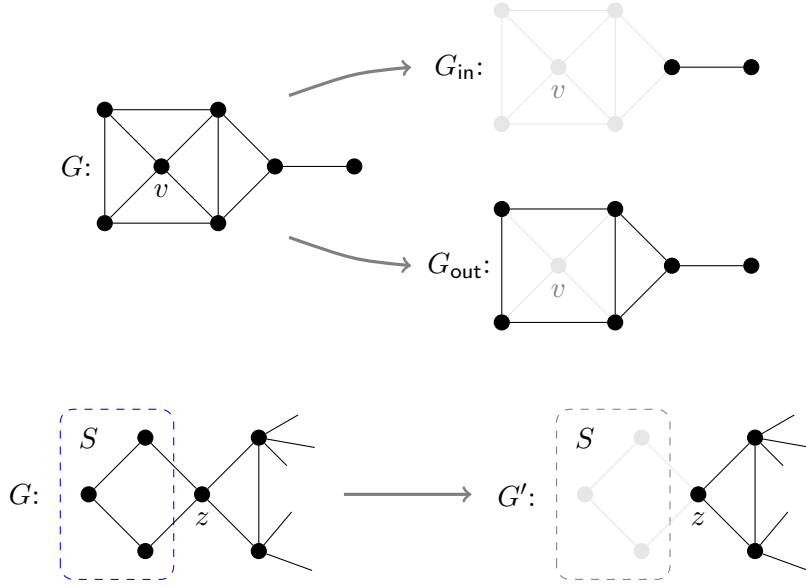


Figure 1: Top: *branching* on a vertex $v \in V(G)$. Any independent set of G either contains v and has the form $\{v\} \cup S$, where S is an independent set of $G_{\text{in}} = G - v - \Gamma(v)$ or does not contain v and is an independent set of $G_{\text{out}} = G - v$. Bottom: *pruning* a set $S \subseteq V(G)$ with $|\Gamma_G(S)| = 1$. We form the graph G' by deleting S and updating the weights of the vertex z to compensate.

Thus, Theorem 3 gives an FPTAS up to the critical point for the multivariate hard-core model, allowing our algorithm to use vertex weight in order to remove degree-1 vertices (amongst other things). The next issue is how to use this FPTAS to get a faster running time. For this, we define the *2-degree* of a vertex as the sum of its neighbours' degrees, and we use Theorem 3 to prove the following result, which may also be of independent interest.

Theorem 4. *There is an FPTAS for $Z(G)$ on graphs G in which every vertex of degree at least 6 has 2-degree at most 26.*

Theorem 4 implies that, given an instance, either we can solve it in polynomial time, or we can find a vertex of degree at least 6 and 2-degree at least 27. Most previous work in the area is already built around finding vertices with high degree and high 2-degree (often called “size”), so we can then use standard techniques to establish an improved bound on the running time, enabling us to prove Theorem 1.

1.1 Proof Sketch

Given a graph $G = (V, E)$, a vertex $v \in V$, and a subset X of V , let $\Gamma_G(v)$ denote the set of neighbours of v in G , and let $d_G(v) = |\Gamma_G(v)|$ and $\Gamma_G(X) = \bigcup_{v \in X} \Gamma_G(v) \setminus X$. Let $\mathcal{G} = (G, w_+, w_-, W)$ be a weighted graph, and write $G = (V, E)$. The two main building blocks of our algorithm are shown in Figure 1. First, we can *branch* on a vertex v , applying the relation

$$Z(\mathcal{G}) = w_-(v) Z(\mathcal{G} - v) + w_+(v) \prod_{x \in \Gamma_G(v)} w_-(x) Z(\mathcal{G} - v - \Gamma_G(v))$$

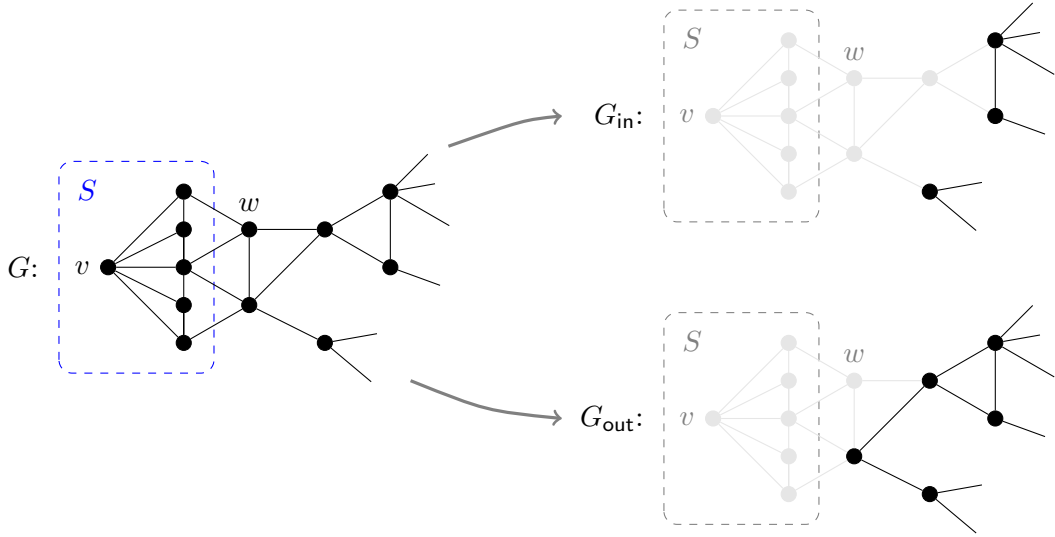


Figure 2: A vertex v with $|\Gamma_G^2(v)| = 2$. Rather than branching on v , we branch on w and prune the set S .

to transform \mathcal{G} into two smaller instances, then solve the two smaller instances recursively. Second, if a set $S \subseteq V$ satisfies $|\Gamma_G(S)| \leq 1$, then we can apply *pruning* (sometimes known as *multiplier reduction*) to remove S from \mathcal{G} in $O(2^{|S|})$ time; if $|\Gamma_G(S)| = 1$, say $\Gamma_G(S) = \{z\}$, then the weight on z will change in the process (see Definition 28). In particular, this allows us to easily remove vertices of degree 1; for this reason, we will typically assume that input graphs have minimum degree 2.

A slightly simplified version of the algorithm is then as follows. For all $v \in V$, we write $\Gamma_G^2(v)$ for the set of vertices at distance exactly two from v . If we can determine $Z(\mathcal{G})$ in polynomial time, we do so. Otherwise, if G contains a vertex v of degree at least 11, we branch on v . Otherwise, we take v to be a vertex of maximum 2-degree subject to $d_G(v) \geq 2|E|/|V|$. If (after some suitable transformations to G) $|\Gamma_G^2(v)| \geq 3$, then we branch on v . If $|\Gamma_G^2(v)| = 2$, then the choice of the branching-vertex is more complicated. In the simplest case, we branch on a vertex in $\Gamma_G^2(v)$ and apply pruning as shown in Figure 2. Finally, if $|\Gamma_G^2(v)| \leq 1$, we apply pruning to remove $\{v\} \cup \Gamma_G(v)$ from \mathcal{G} .

To analyse the running time of this algorithm, we use the potential-based method described in e.g. [7, Chapter 6]. This method requires us to define a non-negative potential function f on weighted graphs \mathcal{G} with the property that every time our algorithm branches, yielding two instances \mathcal{G}_{out} and \mathcal{G}_{in} , we have $f(\mathcal{G}_{\text{out}}), f(\mathcal{G}_{\text{in}}) < f(\mathcal{G})$. (In particular, if $f(\mathcal{G}) = 0$, then we must be able to approximate $Z(\mathcal{G})$ in polynomial time without further branching.) The running time of the algorithm is then bounded in terms of the ensemble of possible values for the pair $(f(\mathcal{G}) - f(\mathcal{G}_{\text{out}}), f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}))$. We take f to be of roughly the following form (see Definition 45). Let s be a positive integer. Taking $f_1, \dots, f_s: \mathbb{R}^2 \rightarrow \mathbb{R}$ to be linear functions (*slices*), and $-1 = k_0 < k_1 < \dots < k_s = \infty$ to be real *boundary points*, we set

$$f(\mathcal{G}) = f_i(|E|, |V|) \text{ whenever } 2|E|/|V| \in (k_{i-1}, k_i].$$

We require that f is continuous, and that for all $i \in [s]$ and all $m, n \in \mathbb{R}$, $f_i(m, n) = \min\{f_j(m, n) : j \in [s]\}$. This latter requirement will allow us to reduce the analysis of possible

values of $(f(\mathcal{G}) - f(\mathcal{G}_{\text{out}}), f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}))$ to an analysis of $(f_i(\mathcal{G}) - f_i(\mathcal{G}_{\text{out}}), f_i(\mathcal{G}) - f_i(\mathcal{G}_{\text{in}}))$ for each $i \in [s]$, subject to $2|E|/|V| \in (k_{i-1}, k_i]$; since each slice f_i is linear, this is much more tractable. We can then turn this analysis of possible branches into a near-optimal choice of f_1, \dots, f_s and k_1, \dots, k_s relatively easily with the help of a computer.

The ideas described above are classical, and were originally introduced in [4]. The novel parts of our argument are contained in the following three steps.

Step 1: If G has average degree in $(k_i, k_{i+1}]$, prove that G contains a vertex v with degree at least $2|E|/|V|$ and 2-degree increasing with i .

Step 2: Prove Theorem 4, i.e. that if G does not contain a vertex w with degree at least 6 and 2-degree at least 27 then we can approximate $Z(G)$ in polynomial time.

Step 3: Analyse the possible branches of the algorithm given that G contains both the vertex v guaranteed by Step 1 and the vertex w guaranteed by Step 2.

We deal with Step 1 in Section 3, and our approach largely follows that of [4]. Using a discharging argument, they showed that any graph with high average degree d must contain a vertex with degree at least d and “associated average degree” at least d (see Lemma 5). They then used a computer search to bound the minimum 2-degree of any vertex with associated average degree at least d . However, because our algorithm runs significantly faster, our analysis needs to work with graphs of maximum degree 10 rather than graphs of maximum degree 5, and this makes a naïve computer search prohibitively difficult; we require some preliminary analysis to narrow the search space, culminating in Lemma 11.

For Step 2, we first give an FPTAS for families \mathcal{F} of suitably-weighted graphs with small connective constant (Theorem 3, proved in Section 4). The work of Sinclair et al. [18] applies only to graphs with uniform weights on vertices, and pruning requires the use of differing vertex weights, so we cannot simply quote their result. Instead, we simulate vertex weights in the unweighted model using gadgets, and then apply [18] as a black box to the resulting graph. The gadgets themselves are standard — the main difficulty is in proving that the family formed by adding gadgets to graphs in \mathcal{F} has the same connective constant as \mathcal{F} itself. With Theorem 3 in hand, we then show in Section 5 that the family \mathcal{D} of graphs with minimum degree at least two and with no vertices of degree at least 6 and 2-degree at least 27 has connective constant small enough for Theorem 3 to apply, thereby proving Theorem 26. This theorem contains almost everything needed to prove Theorem 4, though the proof of the latter is deferred until Section 6.2 where suitable pruning has been introduced to remove isolated vertices and vertices of degree 1. The proof of Theorem 26 uses a potential function to bound the number of leaves of the graph’s self-avoiding walk trees, in a way that could easily be repurposed to prove other bounds on the connective constant based on local structure; see Lemma 25.

For Step 3, our analysis is significantly more complicated than that of [4]. The main difficulty is a technical one. If \mathcal{G} has average degree 2, then since it also has minimum degree 2, it must be 2-regular and so we can obtain $Z(\mathcal{G})$ in polynomial time. As such, we would like to set $f(\mathcal{G}) = 0$, which means taking the first slice of f to be $f_1(m, n) = c(m - n)$ for an appropriate constant $c > 0$. But this has an undesirable consequence: In maintaining the invariant that G has minimum degree 2, we will frequently remove isolated vertices and tree components from G . Doing so may actually increase the value of $f(\mathcal{G})$, which we are not allowed to do. Thus in analysing our branching operations, we must be very careful to account for the effect of removing all the isolated vertices and tree components that we create.

In the non-bipartite case, as in the analyses of [4] and [8], this turns out not to matter too much. Let v be a vertex with some 2-degree D , and suppose $|\Gamma_G^2(v)| \geq 3$ so that we branch on v . Recall that we are concerned with the possible values of $(f_i(\mathcal{G}) - f_i(\mathcal{G}_{\text{out}}), f_i(\mathcal{G}) - f_i(\mathcal{G}_{\text{in}}))$ for a choice of i depending on the average degree of G ; we denote this pair by $\partial = (\partial_{\text{out}}, \partial_{\text{in}})$. (Roughly speaking, the larger ∂_{out} and ∂_{in} are, the more progress we make by branching on v . Thus, the larger these values are in the worst case, the stronger the bound we secure on the algorithm's running time.) In general, we will have $f_i(\mathcal{G}) = \sigma_i|V| + \rho_i|E|$ for some $\rho_i \geq 0$ and $\sigma_i \geq -\rho_i$, so ∂ is determined by the number of vertices and edges lost in passing to \mathcal{G}_{out} and \mathcal{G}_{in} . Since $\rho_i \geq 0$, losing more edges always increases ∂ and thereby makes the branch better, but losing more vertices may make the branch worse if $\sigma_i < 0$. In passing to \mathcal{G}_{in} , we always lose at least $D - |E(G[\Gamma_G(v)])|$ edges, and it turns out that even when $\sigma_j < 0$, the worst cases arise when $G[\Gamma_G(v)]$ contains many edges (see Lemma 48(ii)). In the bipartite setting, however, $G[\Gamma_G(v)]$ never contains any edges, and when $\sigma_i < 0$ the worst cases turn out to arise when we lose as many vertices as possible. In particular, any vertex in $\Gamma_G^2(v)$ whose neighbourhood is contained in $\Gamma_G(v)$ will become isolated on passing to \mathcal{G}_{in} , and will need to be removed to keep the minimum degree at least two; the worst cases are those in which $\Gamma_G^2(v)$ contains many such vertices (see Lemma 48(ii)).

A careful analysis of all possible branches, and their effects on a slice f_i of our potential function f , is the meat of Section 6. We also set out the algorithm in detail, with some minor modifications to improve the worst-case branches when $|\Gamma_G^2(v)| = 2$ and to make the branching analysis easier, and bring the results of previous sections together to bound its running time in terms of f (see Corollaries 49 and 50). Finally, in Section 7 we set out our choice of f and explain the code included to verify that it is valid, and in Section 8 we prove Theorem 1.

2 Definitions and notation

Appendix A contains an index of the notation used in this paper. For all positive integers n , we write $[n]$ for the set $\{1, \dots, n\}$. All our logarithms are base e unless specified otherwise.

Let G be a graph. We write $n(G)$ for the number of vertices in G and $m(G)$ for the number of edges in G . We will always assume $V(G) = [n(G)]$. For a vertex $v \in V(G)$ and a set $X \subseteq V(G)$, we write $\Gamma_G(v, X) = \{w \in X \mid \{v, w\} \in E(G)\}$ and $d_G(v, X) = |\Gamma_G(v, X)|$. If X is not specified, it defaults to $V(G)$. In addition, we write $d_G^2(v) = \sum_{\{v, w\} \in E(G)} d_G(w)$; this is the same as the 2-degree of v , as defined in the introduction. We write $\Gamma_G^2(v)$ for the set of vertices at distance exactly 2 from v ; note that, unless v has no neighbours, $d_G^2(v) > |\Gamma_G^2(v)|$. For a set $S \subseteq V(G)$, we write $\Gamma_G(S) = \bigcup_{v \in S} \Gamma_G(v) \setminus S$. We will omit the subscripts from all of these definitions when G is clear from context. We also write $\delta(G)$ for the minimum degree of G and $\Delta(G)$ for the maximum degree of G . The length of a path is the number of edges it contains.

We define a weighted graph in a slightly more general way than what we described in the introduction. The generality helps to simplify the notation for our recursive instances. A *weighted graph* $\mathcal{G} = (G, w_+, w_-, W)$ is a tuple consisting of a graph G , two positive integer functions w_+ and w_- whose domains contain $V(G)$, and a positive integer W . For any $\lambda \in \mathbb{R}_{>0}$, we say that \mathcal{G} is λ -balanced if $w_+(v) \leq \lambda w_-(v)$ for all $v \in V(G)$. Sometimes, we will take w_+ and w_- to be the constant function with value 1 on domain $V(G)$ — we denote this by $\mathbf{1}$. For all $X \subseteq V(G)$, we define $w_+(X) = \prod_{x \in X} w_+(x)$ and $w_-(X) = \prod_{x \in X} w_-(x)$. We

adopt the convention that empty products are equal to one; in particular, $w_+(\emptyset) = w_-(\emptyset) = 1$. We write $\mathcal{I}(G)$ for the set of independent sets in G , and define

$$Z(\mathcal{G}) = Z(G, w_+, w_-, W) = W \sum_{I \in \mathcal{I}(G)} w_+(I) w_-(V(G) \setminus I).$$

Thus $Z(\mathcal{G})$ is the (multivariate) hardcore partition function of G under weights w_+ and w_- . By convention, if G has no vertices then it has one independent set — the empty set — so $Z(\mathcal{G}) = W$. Note that if $w_+(v) = w_-(v) = 1$ for all $v \in V(G)$, then $Z(\mathcal{G}) = W|\mathcal{I}(G)|$.

For all vertex sets X , we write $G - X = G[V(G) \setminus X]$, and for all $v \in V(G)$ we write $G - v = G - \{v\}$. We extend this notation to weighted graphs in the natural way, so that $\mathcal{G} - X = (G - X, w_+, w_-, W)$ and $\mathcal{G} - v = (G - v, w_+, w_-, W)$. Likewise, for all $X \subseteq V(G)$ we take $\mathcal{G}[X] = (G[X], w_+, w_-, W)$.

For $x, y \in \mathbb{R}_{\geq 0}$ and $\varepsilon \in (0, 1)$, x is an ε -approximation to y if $(1 - \varepsilon)y \leq x \leq (1 + \varepsilon)y$.

3 Bounds on local structure from high average degree

In this section, we bound the “worst-case” branch, given a lower bound on the average degree of our input graph. For this our main tool will be as follows. Following Dahllöf, Jonsson and Wahlström [4] and subsequent authors [8, 21], given a graph G with average degree k , for each $v \in V(G)$ we define

$$\begin{aligned} \alpha(v) &= d_G(v) + |\{w \in \Gamma_G(v) \mid d_G(w) < k\}| \\ \beta(v) &= 1 + \sum_{\substack{w \in \Gamma_G(v) \\ d_G(w) < k}} \frac{1}{d_G(w)}. \end{aligned}$$

The *associated average degree* of a vertex v is then given by $\text{aad}(v) = \alpha(v)/\beta(v)$.

3.1 Translating average degree to high associated average degree

The following is essentially Lemma 6 of Dahllöf *et al.* [4], but translated from #2-SAT to #IS under the usual correspondence. We give a full proof for completeness and so that the reader doesn’t have to translate notation.

Lemma 5 ([4]). *Let $G = (V, E)$ be a graph with average degree k . There is some $v \in V$ with $d_G(v) \geq k$ and $\text{aad}(v) \geq k$.*

Proof. In this proof, we will write $d(v) = d_G(v)$ for all $v \in V(G)$. Let $X = \{v \in V \mid d(v) \geq k\}$. There is at least one vertex with at-least-average degree, so $X \neq \emptyset$. Let $A = \sum_{v \in X} \alpha(v)$ and $B = \sum_{v \in X} \beta(v)$.

For a vertex v , let $d_{\geq k}(v) = |\{w \in \Gamma(v) \mid d(w) \geq k\}|$ and write

$$f(v) = \begin{cases} d_{\geq k}(v) & \text{if } d(v) < k \\ d(v) & \text{if } d(v) \geq k. \end{cases}$$

We can view each $v \in V$ with $d(v) \geq k$ as contributing $d(v)$ to A , and each vertex with $d(v) < k$ as contributing 1 to A for each neighbour with degree $\geq k$. Therefore, $A = \sum_{v \in V} f(v)$. By similar reasoning, $B = \sum_{v \in V} f(v)/d(v)$.

For $i \geq 1$, let n_i be the number of degree- i vertices in G . For $i \geq k$, let $n'_i = n_i$ and, for $1 \leq i < k$, let $n'_i = |\{v \in V \mid d(v) < k \text{ and } d_{\geq k}(v) = i\}|$. There may be vertices with $d(v) < k$ and $d_{\geq k}(v) = 0$, so $\sum_{1 \leq i < k} n'_i \leq \sum_{1 \leq i < k} n_i$. Writing $S = \sum_{1 \leq i < k} (n_i - n'_i)$, we have

$$A = \sum_{i \geq 1} i n'_i = \sum_{i \geq 1} i n_i - \sum_{1 \leq i < k} i (n_i - n'_i) = 2|E| - \sum_{1 \leq i < k} i (n_i - n'_i) \geq 2|E| - kS = k(|V| - S).$$

For $i \geq k$, there are n'_i vertices of degree i , each of which contributes exactly 1 to B . For $1 \leq i < k$, there are n'_i vertices v with $d_{\geq k}(v) = i$ and $d(v) < k$, each of which contributes at most 1 to B . Therefore,

$$B \leq \sum_{i \geq 1} n'_i = \sum_{i \geq 1} n_i - \sum_{1 \leq i < k} (n_i - n'_i) = |V| - S.$$

Therefore, $A \geq kB$. If $\alpha(v) < k\beta(v)$ for all $v \in X$, then $A < kB$, contradicting what we have just derived. Therefore, X contains a vertex v with $\alpha(v)/\beta(v) \geq k$, and $d(v) \geq k$ by the definition of X . \square

3.2 Using high associated average degree to lower-bound 2-degree

We now lower-bound $d_G^2(v)$ in terms of $\text{aad}(v)$. We will find it useful to work with $\alpha(v)$ and $\beta(v)$ as numerical functions, as follows.

Definition 6. Let $d \in \mathbb{Z}_{>0}$ and $k \in \mathbb{R}_{>0}$ with $d \geq k$. For all vectors $\mathbf{x} \in \mathbb{Z}_{>0}^d$, we define

$$\begin{aligned} a_k(\mathbf{x}) &= d + |\{i \mid x_i < k\}| \\ b_k(\mathbf{x}) &= 1 + \sum_{i: x_i < k} \frac{1}{x_i} \\ \text{aad}_k^*(\mathbf{x}) &= a_k(\mathbf{x})/b_k(\mathbf{x}). \end{aligned}$$

Associated average degree and aad^* correspond naturally. Let $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{Z}_{>0}^d$. If G is a graph with average degree k , and $v \in V(G)$ has degree d and neighbours with degrees x_1, \dots, x_d , we have $\alpha(v) = a_k(\mathbf{x})$, $\beta(v) = b_k(\mathbf{x})$ and $\text{aad}(v) = \text{aad}_k^*(\mathbf{x})$.

We use associated average degree to guarantee vertices of high 2-degree.

Definition 7. For any real numbers $k \geq 2$, let $K = \lfloor k \rfloor + 1$ be the smallest integer that is larger than k . Let $D_2(k)$ be the minimum integer z such that there is an integer $d \geq K$ and a tuple $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{Z}_{\geq 2}^d$ with $\sum_{i=1}^d x_i = z$ and $\text{aad}_K^*(\mathbf{x}) > k$.

Lemma 8. For any real number $k \geq 2$, let $K = \lfloor k \rfloor + 1$. Every graph G with minimum degree at least 2 and average degree in $(k, K]$ contains a vertex with degree at least K and 2-degree at least $D_2(k)$.

Proof. Let $A > k$ be the average degree of G . By Lemma 5, G contains a vertex v such that $d_G(v) \geq A > k$ and $\text{aad}(v) \geq A > k$. Since $d_G(v)$ is an integer, it is at least K (which is part of what is required in the lemma statement).

Let $\mathbf{x} = (x_1, \dots, x_{d_G(v)})$ be the degrees of v 's neighbours in G . By the definition of aad^* , $\text{aad}(v) = \text{aad}_A^*(\mathbf{x})$.

Since the elements of \mathbf{x} are integers, K is the smallest integer that is larger than k , and $A \in (k, K]$, $x_i < A$ iff $x_i < K$. Thus, $\text{aad}_A^*(\mathbf{x}) = \text{aad}_K^*(\mathbf{x})$.

To finish, we wish to show that $d_G^2(v) = \sum_i x_i$ is at least $D_2(k)$. From the definition of $D_2(k)$, it suffices to show that $\text{aad}_K^*(\mathbf{x}) > k$, which we have shown. \square

3.3 Efficiently computing the function $D_2(k)$ that we used to bound 2-degree

In the remainder of this section, we work towards computing the function $D_2(k)$. Suppose $k \geq 2$ and let $K = \lfloor k \rfloor + 1$. Note that

$$\text{aad}_K^*(\underbrace{K, \dots, K}_{K \text{ times}}) = K > k,$$

so $D_2(k) \leq K^2$. In principle, one could find $D_2(k)$ by exhaustive search through all tuples of positive integers with total at most K^2 , but this would take an unpleasantly long time for the values we wish to calculate, so we will restrict the search space. This leads us to the following definition.

Definition 9. For any real number $k \geq 2$, let $K = \lfloor k \rfloor + 1$. We say that a positive integer z is *suitable* for k if either $z \geq K^2$ or there exist integers d and s such that the following hold:

- (S1) $K \leq d \leq z/2$;
- (S2) $0 \leq s \leq d - 1$;
- (S3) $Ks + 2(d - s) \leq z \leq Ks + (K - 1)(d - s)$; and
- (S4) Writing $q = \lfloor (z - Ks)/(d - s) \rfloor$, $d_1 = (z - Ks) \bmod (d - s)$, and $d_0 = d - s - d_1$,

$$\frac{d + d_0 + d_1}{1 + d_0/q + d_1/(q + 1)} > k.$$

Note that d and q are integers that are at least 2. Also, s , d_0 and d_1 are non-negative integers. The point of Definition 9 is Lemma 11, which says that $D_2(k)$ is the minimum value of z such that k is suitable for z . We will use the following weaker result in the proof of Lemma 11.

Lemma 10. For any real number $k \geq 2$, and any positive integer z which is suitable for k , we have $D_2(k) \leq z$.

Proof. By the definition of $D_2(k)$, it suffices to exhibit a vector $\mathbf{x} \in \mathbb{Z}_{\geq 2}^d$ (for some $d \geq K$) with $\sum_i x_i = z$ and $\text{aad}_K^*(\mathbf{x}) > k$. Let $K = \lfloor k \rfloor + 1$.

Case 1: $z \geq K^2$. In this case, we set $d = K$ and take \mathbf{x} to be the vector with $x_1, \dots, x_{K-1} = K$ and $x_K = (z - K(K - 1))$. Since $z \geq K^2$, we have $x_K \geq K$, and so $\text{aad}_K^*(\mathbf{x}) = d = K > k$. Since $K > k \geq 2$, we have $\mathbf{x} \in \mathbb{Z}_{\geq 2}^K$. Finally, we have $\sum_i x_i = z$ as required, so the result follows.

Case 2: $z \leq K^2 - 1$. In this case, since z is suitable for k , we can take d , s , q , d_0 and d_1 as in Definition 9. We take \mathbf{x} to be the vector with $x_1, \dots, x_{d_0} = q$, $x_{d_0+1}, \dots, x_{d_0+d_1} = q + 1$, and $x_{d_0+d_1+1}, \dots, x_{d_0+d_1+s} = K$. Then we have

$$\begin{aligned} \sum_{i=1}^{d_0+d_1+s} x_i &= qd_0 + (q + 1)d_1 + Ks = q(d_0 + d_1) + d_1 + Ks = q(d - s) + d_1 + Ks \\ &= \left\lfloor \frac{z - Ks}{d - s} \right\rfloor (d - s) + (z - Ks) \bmod (d - s) + Ks = (z - Ks) + Ks = z, \end{aligned}$$

as required.

By (S3) we have $z - Ks \leq (K - 1)(d - s)$, so $q \leq K - 1$. If $q \leq K - 2$, then all entries in $x_1, \dots, x_{d_0+d_1}$ are at most $K - 1$. If instead $q = K - 1$, then we must have $z - Ks = (K - 1)(d - s)$ and hence $d_1 = 0$; thus once again all entries in $x_1, \dots, x_{d_0+d_1}$ are at most $K - 1$. All entries in $x_{d_0+d_1+1}, \dots, x_d$ are equal to K , so in both cases we have

$$\text{aad}_K^*(\mathbf{x}) = \frac{d + d_0 + d_1}{1 + d_0/q + d_1/(q + 1)}.$$

By (S4), this is strictly greater than k as required.

By (S3) we have $z - Ks \geq 2(d - s)$, so $q \geq 2$ and $\mathbf{x} \in \mathbb{Z}_{\geq 2}^{d_0+d_1+s} = \mathbb{Z}_{\geq 2}^d$ as required. Finally, (S1) implies that $d \geq K$ as required, so the result follows. \square

Lemma 11. *For any real number $k \geq 2$, $D_2(k)$ is the minimum integer z which is suitable for k .*

Proof. By Lemma 10, the minimum integer z which is suitable for k is at least $D_2(k)$, so it suffices to show that $D_2(k)$ itself is suitable for k . Let $z = D_2(k)$; then by Definition 7, there is an integer $d \geq K$ and a tuple $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{Z}_{\geq 2}^d$ with $\sum_{i=1}^d x_i = z$ and $\text{aad}_K^*(\mathbf{x}) > k$. Fix such a d and \mathbf{x} .

Claim 1. We may choose \mathbf{x} to ensure that for all $x_i, x_j \leq K - 1$, we have $|x_i - x_j| \leq 1$.

Proof: Suppose that there are elements $2 \leq x_i < x_j \leq K - 1$ with $x_j \geq x_i + 2$. Consider the tuple \mathbf{x}' formed from \mathbf{x} by replacing x_i and x_j with $x'_i = x_i + 1$ and $x'_j = x_j - 1$. It is clear that $x'_j = x_j - 1 > x_i \geq 2$, so $\mathbf{x}' \in \mathbb{Z}_{\geq 2}^d$, and that $\sum_{i=1}^d x'_i = \sum_{i=1}^d x_i = z$.

To finish proving the claim, we will show that $\text{aad}_K^*(\mathbf{x}') \geq \text{aad}_K^*(\mathbf{x})$. This means that we can consider \mathbf{x}' in place of \mathbf{x} , repeating as necessary until Claim 1 is satisfied. We have $x_i, x'_i, x'_j < x_j \leq K - 1$, so $a_K(\mathbf{x}') = a_K(\mathbf{x})$. Let $f(y)$ be the function given by $f(y) = \frac{1}{y} - \frac{1}{y+1}$. Then

$$b_K(\mathbf{x}') = b_K(\mathbf{x}) - \frac{1}{x_i} - \frac{1}{x_j} + \frac{1}{x_i + 1} + \frac{1}{x_j - 1} = b_K(\mathbf{x}) - f(x_i) + f(x_j - 1) \leq b_K(\mathbf{x}),$$

since $0 < x_i \leq x_j - 1$ and $f(y)$ is decreasing for $y > 0$. Hence $\text{aad}_K^*(\mathbf{x}') = a_K(\mathbf{x}')/b_K(\mathbf{x}') \geq a_K(\mathbf{x})/b_K(\mathbf{x}) = \text{aad}_K^*(\mathbf{x})$, as required.

Claim 2. For all $i \in [d]$, $x_i \leq K$.

Proof: Suppose for contradiction that for some $i \in [d]$, $x_i > K$. Consider the tuple $\mathbf{x}' \in \mathbb{Z}_{\geq 2}^d$ formed from \mathbf{x} by replacing x_i with $x'_i = K$. By the definition of aad^* , $\text{aad}_K^*(\mathbf{x}') = \text{aad}_K^*(\mathbf{x}) > k$. But the sum of the elements in \mathbf{x}' is smaller than z , contradicting $z = D_2(k)$.

Claim 3. z is suitable for k .

Proof: Let $q = \min\{x_i : i \in [d]\}$. By Claims 1 and 2, we can choose \mathbf{x} so that its coordinates lie in the set $\{q, q + 1, K\}$. Let s be the number of coordinates of \mathbf{x} equal to K , let d_0 be the number of coordinates of \mathbf{x} equal to q , and let $d_1 = d - d_0 - s$. (Thus d_1 is the number of coordinates of \mathbf{x} equal to $q + 1$ unless $q + 1 = K$, in which case $d_1 = 0$.)

If $s = d$, then we have $z = Kd \geq K^2$, and so z is suitable for k ; suppose instead $0 \leq s \leq d - 1$ (so (S2) is satisfied). Since $\sum_{i=0}^d x_i = z$, we have

$$q = \left\lfloor \frac{z - Ks}{d - s} \right\rfloor \text{ and } d_1 = (z - Ks) \bmod (d - s),$$

so q , d_0 and d_1 are exactly as in Definition 9. Moreover, since either $q + 1 \leq K - 1$ or $d_1 = 0$, we have

$$\frac{d + d_0 + d_1}{1 + d_0/q + d_1/(q + 1)} = \frac{a_K(\mathbf{x})}{b_K(\mathbf{x})} = \text{aad}_K^*(\mathbf{x}) > k,$$

so (S4) is satisfied. Since $x \in \mathbb{Z}_{\geq 2}^d$, we have $z = \sum_{i=1}^d x_i \geq Ks + 2(d - s)$, and by Claim 1, we have $z = \sum_{i=1}^d x_i \leq Ks + (K - 1)(d - s)$; hence (S3) is satisfied. Finally, we have $d \geq K$ and (S3) implies that $z \geq 2d$, so (S1) is satisfied. Hence z is suitable for k , and the result follows. \square

Corollary 12. *For any real number $k \geq 2$, writing $K = \lfloor k \rfloor + 1$, we have $2K \leq D_2(k) \leq K^2$.*

Proof. $D_2(k) \leq K^2$ follows immediately from Lemma 11 and the fact that K^2 is always suitable for K , and $D_2(k) \geq 2K$ follows immediately from Lemma 11 and (S1). \square

Given Lemma 11 and Corollary 12, we can compute $D_2(k)$ for any real number $k \geq 2$ by considering increasing integers $2K \leq z \leq K^2$ and checking whether z is suitable for k . Mathematica code to do this is included in Appendix D as the functions `isSuitable[]` and `Dtwo[]`.

4 Approximating $Z(\mathcal{G})$ when the connective constant is small

In this section, we prove Theorem 3. That is, for a positive real number λ and a family \mathcal{F} of graphs whose connective constant is subcritical with respect to λ , we give an FPTAS for $Z(\mathcal{G})$ on λ -balanced weighted graphs $\mathcal{G} = (G, w_+, w_-, W)$ such that $G \in \mathcal{F}$. This FPTAS provides the base case for our main algorithm (see Theorem 1).

4.1 Definitions and Notation

We start by recalling some ideas from the introduction, and giving more detailed notation.

Definition 13 ([12]). Let G be a graph and let $v \in V(G)$. The *self-avoiding walk tree* of G with root v is the following tree $T_{\text{SAW}}(v, G)$. The vertices are the self-avoiding walks (simple paths) from v in G . The root is the trivial path v . Given any vertex P , the children of P are the simple paths that extend P by one edge. $N_G(v, \ell)$ denotes the number of length- ℓ self-avoiding walks in G starting from v so it is the number of depth- ℓ vertices in $T_{\text{SAW}}(v, G)$, where the root is considered to have depth 0.

Recall the following definitions from the introduction.

Definition 2. ([19]). Let \mathcal{F} be a family of graphs and let κ be a positive real number. The connective constant of \mathcal{F} is at most κ if there are real numbers a and c such that, for every $G \in \mathcal{F}$, every vertex $v \in V(G)$, and every integer $\ell \geq a \log |V(G)|$, we have $\sum_{i=1}^{\ell} N_G(v, i) \leq c\kappa^{\ell}$.

Definition 14. Let κ and λ be positive real numbers. κ is said to be *subcritical w.r.t.* λ if $\lambda < \lambda_c(\kappa) := \kappa^{\kappa}/(\kappa - 1)^{\kappa+1}$.

Note that $\lambda_c(\kappa)$ is decreasing in κ , so if κ is subcritical w.r.t. some λ and $\kappa' < \kappa$, then κ' is also subcritical w.r.t. λ .

4.2 Reduction to approximating the univariate hardcore partition function

In the *univariate hardcore model*, we are given a graph G and a real parameter $\lambda > 0$ and we wish to compute the partition function

$$Z_\lambda(G) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.$$

Thus, $\lambda = 1$ corresponds to counting independent sets and, for any $\lambda > 0$, $Z_\lambda(G) = Z(\mathcal{G})$ where $\mathcal{G} = (G, w_+, \mathbf{1}, 1)$ and w_+ is the constant function which assigns weight λ to every vertex.

Note that, as λ decreases, the condition that a family \mathcal{F} has subcritical connective constant w.r.t. λ becomes weaker, but the requirement that $w_+(v) \leq \lambda w_-(v)$ becomes more restrictive.

We will prove Theorem 3 by reducing the problem of approximating $Z(\mathcal{G})$ on λ -balanced weighted graphs $\mathcal{G} = (G, w_+, w_-, W)$ such that G comes from a family whose connective constant is subcritical with respect to λ to the problem of approximating the partition function of the (univariate) hardcore model in the subcritical case, using the following theorem of Sinclair *et al.*

Theorem 15 ([18]). *Fix a family \mathcal{F} of graphs and $\lambda \in \mathbb{R}_{>0}$ such that \mathcal{F} has subcritical connective constant w.r.t. λ . There is an FPTAS $\text{ApproxZUni}_{\lambda, \mathcal{F}}(G, \varepsilon)$ for $Z_\lambda(G)$ for input graphs $G \in \mathcal{F}$.*

4.3 Defining the gadgets for the reduction

We first define the gadgets we will use in our reduction. The construction is relatively simple — given a weighted input graph $\mathcal{G} = (G, w_+, w_-, W)$, we form an instance of the hardcore model by attaching appropriately chosen depth-2 trees to each vertex of G . However, to apply Theorem 15, we must bound the connective constant of the family of all possible hardcore instances that we may construct (see Lemma 18). For this reason, we use the following notation.

Definition 16. For any graph $G = (V, E)$, a *weight map* for G is a map ϕ from V to finite (possibly empty) multisets of positive integers. The *realisation* of ϕ is the graph G' formed from G as follows. Let $\{T_{v,t} \mid v \in V, t \in \phi(v)\}$ be a multiset of vertex-disjoint stars, where $T_{v,t}$ has t leaves. Take the disjoint union of G and the $T_{v,t}$'s, then join the centre of each star $T_{v,t}$ to v by an edge.

An example of the construction is given in Figure 3. We will apply Theorem 15 to a family of graphs defined as in Definition 17, using Lemma 18 to argue that the connective constant is subcritical.

Definition 17. Let \mathcal{F} be a family of graphs and let $y \in \mathbb{Z}_{>0}$. For any graph $G = (V, E)$, let $\Phi_y(G)$ be the set of all possible weight maps ϕ for G such that, for all $v \in V$, $\sum_{i \in \phi(v)} (i+1) \leq y|V|^2$. We define \mathcal{F}_y^+ to be the set of all realisations of weight maps in $\bigcup_{G \in \mathcal{F}} \Phi_y(G)$.

The choice of $y|V|^2$ in the above definition is not fundamental; the following lemma remains true, with minor modifications to the proof, for any choice of polynomial in $|V|$.

Lemma 18. *Let $\kappa \in \mathbb{R}_{\geq 1}$, $\lambda \in \mathbb{R}_{>0}$ and $y \in \mathbb{Z}_{\geq 1}$. Let \mathcal{F} be a family of graphs with connective constant at most κ , subcritical w.r.t. λ . Then \mathcal{F}_y^+ also has subcritical connective constant w.r.t. λ .*

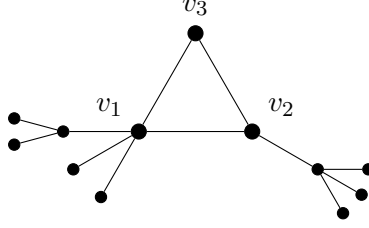


Figure 3: An example of the realisation of a weight map ϕ for a triangle with vertices v_1 , v_2 and v_3 . Here $\phi(v_1) = \{0, 0, 2\}$, $\phi(v_2) = \{3\}$, and $\phi(v_3) = \emptyset$.

Proof. Fix $\gamma > 1$ such that $\gamma\kappa$ is subcritical w.r.t. λ . We show that \mathcal{F}_y^+ has connective constant at most $\gamma\kappa$.

By the definition of κ (Definition 2), there exist real numbers a and c such that for all $G \in \mathcal{F}$, all $v \in V(G)$, and all $\ell \geq a \log |V(G)|$, we have $\sum_{i=1}^{\ell} N_G(v, i) \leq c\kappa^\ell$. Let $a' = \max\{a, 8/\log \gamma\}$ and $c' = 40y^4(1+c)$; we will show that for all $G' \in \mathcal{F}_y^+$, all $v \in V(G')$, and all $\ell \geq a' \log |V(G')|$, we have $\sum_{i=1}^{\ell} N_{G'}(v, i) \leq c'(\gamma\kappa)^\ell$. For ease of notation, let $n = |V(G)|$ and $n' = |V'(G)|$.

Let $G' \in \mathcal{F}_y^+$, let $v \in V(G')$. Let $G \in \mathcal{F}$ and $\phi \in \Phi_y(G)$ be such that G' is a realisation of ϕ . Observe that, for all positive integers i , any self-avoiding walk in G' of length i must stay almost entirely within G , only entering the attached trees $T_{w,t}$ for at most two steps at the start (if $v \in V(G') \setminus V(G)$) and at most two steps at the end. Since $\phi \in \Phi_y(G)$, for all $w \in V(G)$ we have

$$\left| \bigcup_{t \in \phi(w)} V(T_{w,t}) \right| = \sum_{t \in \phi(w)} (t+1) \leq yn^2, \quad (1)$$

so there are at most yn^2 choices for each step outside $V(G)$. Thus, for all $i \geq 4$,

$$\begin{aligned} N_{G'}(v, i) &\leq N_G(v, i) + 2yn^2 N_G(v, i-1) + 3y^2 n^4 N_G(v, i-2) \\ &\quad + 2y^3 n^6 N_G(v, i-3) + y^4 n^8 N_G(v, i-4) \\ &\leq 3y^4 n^8 \sum_{j=i-4}^i N_G(v, j). \end{aligned}$$

Now, suppose that $0 \leq i \leq 3$. Note that, by (1), $n' \leq yn^2 + n$. For all $v \in V(G')$,

$$N_{G'}(v, i) \leq (n')^i \leq (yn^2 + n)^i \leq 8y^3 n^6 < 8y^4 n^8 \leq 8y^4 n^8 \sum_{j=0}^i N_G(v, j).$$

Therefore, for all $0 \leq i \leq n$,

$$N_{G'}(v, i) \leq 8y^4 n^8 \sum_{j=\max\{0, i-4\}}^i N_G(v, j).$$

For any ℓ , Each term $N_G(v, k)$ appears at most five times in $\sum_{i=1}^{\ell} \sum_{j=\max\{0, i-4\}}^i N_G(v, j)$, so

$$\sum_{i=1}^{\ell} N_{G'}(v, i) \leq 40y^4 n^8 \sum_{i=0}^{\ell} N_G(v, i).$$

For $\ell \geq a' \log n' \geq a \log n$ and $\kappa \geq 1$, it follows that

$$\sum_{i=1}^{\ell} N_{G'}(v, i) \leq 40y^4 n^8 (1 + c\kappa^\ell) \leq c' n^8 \kappa^\ell.$$

By the definition of a' , we have $n^8 \leq \gamma^{a' \log n} \leq \gamma^\ell$, so $\sum_{i=1}^{\ell} N_{G'}(v, i) \leq c'(\gamma\kappa)^\ell$. Thus, the connective constant of \mathcal{F}_y^+ is at most $\gamma\kappa$, which is subcritical. \square

4.4 The approximation algorithm

We now state the algorithm of Theorem 3. Throughout, we write $\Lambda_t = 1 + \lambda(1 + \lambda)^{-t}$ for any $t \in \mathbb{Z}_{\geq 0}$. Note that, for all $\lambda, t > 0$, we have $1 < \Lambda_{t+1} < \Lambda_t$. Recall that $\mathcal{G} = (G, w_+, w_-, W)$ is λ -balanced if $w_+(v) \leq \lambda w_-(v)$ for all $v \in V(G)$.

Algorithm $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$: *The parameters are \mathcal{F} , a family of graphs, and $\lambda \in \mathbb{R}_{>0}$ such that \mathcal{F} has subcritical connective constant w.r.t. λ ; \mathcal{F} and λ are not part of the input. The inputs are $\mathcal{G} = (G, w_+, w_-, W)$, a non-empty λ -balanced n -vertex weighted graph with $G \in \mathcal{F}$, and a rational number $0 < \varepsilon < 1$. The output is an ε -approximation to $Z(\mathcal{G})$.*

- (i) If $\varepsilon \leq 3n\lambda(1 + \lambda)^{-n}$, then calculate $Z(\mathcal{G})$ exactly by brute force and return it.
- (ii) Form a weighted graph $\mathcal{G}' = (G', w_+, w_-, W')$ from \mathcal{G} as follows: for each $v \in V(G)$ with $w_+(v) \leq \frac{\varepsilon}{3n} w_-(v)$, delete v and multiply W by $w_-(v)$. Let W' be the final value of W .
- (iii) For all $v \in V(G')$, calculate $a_{0,v}, \dots, a_{n,v}$ as follows. Initially, set $x_v \leftarrow \lambda w_-(v)/w_+(v)$. Then, for $t = 0$ to n , let $a_{t,v} = \lfloor \log_{\Lambda_t} x_v \rfloor$, and update $x_v \leftarrow x_v / \Lambda_t^{a_{t,v}}$.
- (iv) Define a weight map ϕ for G' by mapping each vertex v to the multiset consisting of exactly $a_{t,v}$ copies of t for all $t \in \{0, \dots, n\}$. Construct the realisation G'' of ϕ .
- (v) Return $\text{ApproxZUni}_{\lambda, \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+}(G'', \varepsilon/3) \cdot W' \lambda^{-n} \prod_{v \in V(G')} (w_+(v) / \prod_{i=0}^n (1 + \lambda)^{i a_{i,v}})$.

4.5 Building up to the proof of correctness

It is immediate that $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$ outputs $Z(\mathcal{G})$ when $\varepsilon \leq 3n\lambda(1 + \lambda)^{-n}$. When $\varepsilon > 3n\lambda(1 + \lambda)^{-n}$, we prove correctness in three steps, as detailed in the proof of Theorem 3. We show that $Z(\mathcal{G})$ is roughly $Z(\mathcal{G}')$ (see Lemma 19) and that $Z_\lambda(G'')$ is roughly $Z(\mathcal{G}')/C$ where

$$C = W' \lambda^{-n} \prod_{v \in V(G')} \left(w_+(v) / \prod_{i=0}^n (1 + \lambda)^{i a_{i,v}} \right)$$

is easily computed (Lemma 21). Finally, we will show that $G'' \in \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+$ and that $\mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+$ has subcritical connective constant w.r.t. λ , so that $\text{ApproxZUni}_{\lambda, \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+}(G'', \varepsilon/3)$ is roughly $Z_\lambda(G'')$ in step (v) (Lemma 20). We conclude that the output is roughly $Z(\mathcal{G})$, as required (see the proof of Theorem 3). This last part of the argument is the reason we bound ε below in step (i) and pass from \mathcal{G} to \mathcal{G}' in step (ii).

Lemma 19. *Let $\lambda > 0$ and let \mathcal{F} be a family of graphs with subcritical connective constant w.r.t. λ . Let $\mathcal{G} = (G, w_+, w_-, W)$ be a non-empty λ -balanced n -vertex weighted graph with $G \in \mathcal{F}$. Let $3n\lambda(1 + \lambda)^{-n} < \varepsilon < 1$ be rational. Then after step (ii) of $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$, we have $Z(\mathcal{G}) \geq Z(\mathcal{G}') \geq (1 - \varepsilon/3)Z(\mathcal{G})$.*

Proof. Write $\mathcal{G}' = (G', w_+, w_-, W')$ as in $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$, and write $S = V(G) \setminus V(G')$. Observe that $W' = W w_-(S)$, so

$$Z(\mathcal{G}') = W' \sum_{I \in \mathcal{I}(G')} w_+(I) w_-(V(G') \setminus I) = W \sum_{\substack{I \in \mathcal{I}(G) \\ I \subseteq V(G')}} w_+(I) w_-(V(G) \setminus I). \quad (2)$$

It follows immediately that

$$Z(\mathcal{G}') \leq W \sum_{I \in \mathcal{I}(G)} w_+(I) w_-(V(G) \setminus I) = Z(\mathcal{G}),$$

as required. Moreover, since for all $v \in S$ we have $w_+(v) \leq \frac{\varepsilon}{3n} w_-(v)$,

$$\begin{aligned} w_-(S) &= \prod_{v \in S} w_-(v) \geq \prod_{v \in S} \frac{w_-(v) + w_+(v)}{1 + \varepsilon/3n} = \frac{1}{(1 + \varepsilon/3n)^{|S|}} \sum_{J \subseteq S} w_+(J) w_-(S \setminus J) \\ &\geq (1 - \varepsilon/3n)^{|S|} \sum_{J \subseteq S} w_+(J) w_-(S \setminus J) \geq (1 - \varepsilon/3) \sum_{J \subseteq S} w_+(J) w_-(S \setminus J). \end{aligned}$$

It follows from (2) that

$$\begin{aligned} Z(\mathcal{G}') &\geq (1 - \varepsilon/3) W \sum_{\substack{I \in \mathcal{I}(G) \\ I \subseteq V(G')}} w_+(I) w_-(V(G') \setminus I) \sum_{J \subseteq S} w_+(J) w_-(S \setminus J) \\ &\geq (1 - \varepsilon/3) W \sum_{I \in \mathcal{I}(G)} w_+(I) w_-(V(G) \setminus I) = (1 - \varepsilon/3) Z(\mathcal{G}). \quad \square \end{aligned}$$

We collect the relevant properties of the $a_{t,v}$'s and x_v 's in the following lemma.

Lemma 20. *Let $\lambda > 0$ and let \mathcal{F} be a family of graphs with subcritical connective constant w.r.t. λ . For every non-empty λ -balanced n -vertex weighted graph $\mathcal{G} = (G, w_+, w_-, W)$ with $G \in \mathcal{F}$, and every rational ε satisfying $3n\lambda(1 + \lambda)^{-n} < \varepsilon < 1$, the following are true after step (iv) of $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$:*

$$(i) \text{ for all } v \in V(G'), \left(1 - \frac{\varepsilon}{3n}\right) \frac{\lambda w_-(v)}{w_+(v)} \leq \prod_{t=0}^n \Lambda_t^{a_{t,v}} \leq \frac{\lambda w_-(v)}{w_+(v)},$$

$$(ii) \ G'' \in \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+.$$

Proof. Towards (i), fix $v \in V(G')$. For all $0 \leq t \leq n+1$, let

$$x_{t,v} = \frac{\lambda w_-(v)}{w_+(v)} \prod_{i=0}^{t-1} \Lambda_i^{-a_{i,v}}.$$

Thus, $x_{0,v}$ is the initial value of x_v in step (iii) of the algorithm, and for all $i \in [n+1]$, $x_{i,v}$ is the value of x_v at the end of the iteration of the for loop in which $t = i - 1$. In particular, at the end of step (iii), $x_v = x_{n+1,v}$.

By definition, for all $0 \leq t \leq n$ we have $a_{t,v} = \lfloor \log_{\Lambda_t} x_{t,v} \rfloor$ and $x_{t+1,v} = x_{t,v} \Lambda_t^{-a_{t,v}}$, so $1 \leq x_{t+1,v} < \Lambda_t$. By taking $t = n$, this implies that

$$1 \leq \frac{\lambda w_-(v)}{w_+(v)} \prod_{i=0}^n \Lambda_i^{-a_{i,v}} \leq \Lambda_n, \quad \text{so} \quad \prod_{i=0}^n \Lambda_i^{a_{i,v}} \leq \frac{\lambda w_-(v)}{w_+(v)} \leq \Lambda_n \prod_{i=0}^n \Lambda_i^{a_{i,v}},$$

and we have the upper bound of (i). Recall that $\Lambda_t = 1 + \lambda(1 + \lambda)^{-t}$. For any $x > 0$, $1/(1+x) \geq 1-x$, so $1/\Lambda_n \geq 1 - \lambda(1 + \lambda)^{-n} \geq 1 - \varepsilon/3n$, establishing the lower bound of (i).

It remains to show that $\mathcal{G}'' \in \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+$. Recall that, for all $1 \leq t \leq n+1$, we have $1 \leq x_{t,v} < \Lambda_{t-1}$. For any $z > 0$, $\log(1+z) \leq z$ and, for any $z \in (0, 1)$, $\log(1+z) \geq z/2$. For all $\lambda > 0$ and all $t \geq 1$, $0 < \lambda(1 + \lambda)^{-t} < 1$. Therefore, for all $t \geq 1$,

$$a_{t,v} \leq \lfloor (\log \Lambda_{t-1}) / \log \Lambda_t \rfloor \leq \lfloor 2(1 + \lambda) \rfloor.$$

By the construction of G' in step (ii), we have $w_+(v)/w_-(v) > \varepsilon/3n > \lambda(1 + \lambda)^{-n}$. Thus $a_{0,v} = \lfloor \log_{1+\lambda} \lambda w_-(v)/w_+(v) \rfloor \leq n$. It follows that

$$\begin{aligned} \sum_{t \in \phi(v)} (t+1) &= \sum_{t=0}^n a_{t,v}(t+1) \leq n + \lfloor 2(1 + \lambda) \rfloor \sum_{i=1}^n (i+1) \\ &= n + \frac{1}{2} \lfloor 2(1 + \lambda) \rfloor n(n+3) \leq \lfloor 5(1 + \lambda) \rfloor n^2, \end{aligned}$$

so $G'' \in \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+$, as required. \square

Lemma 21. *Let $\lambda > 0$ and let \mathcal{F} be a family of graphs with subcritical connective constant w.r.t. λ . Let $\mathcal{G} = (G, w_+, w_-, W)$ be a non-empty λ -balanced n -vertex weighted graph with $G \in \mathcal{F}$. Let $3n\lambda(1 + \lambda)^{-n} < \varepsilon < 1$ be rational. Then in step (iv) of $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$,*

$$(1 - \varepsilon/3)Z(\mathcal{G}') \leq Z_\lambda(G'') W' \lambda^{-n} \prod_{v \in V(G')} \left(w_+(v) / \prod_{i=0}^n (1 + \lambda)^{i a_{i,v}} \right) \leq Z(\mathcal{G}'). \quad (3)$$

Proof. Let $\mathcal{G}' = (G', w_+, w_-, W')$ be as in step (ii). For notational convenience, let $M = Z_\lambda(G'') W' \lambda^{-n} \prod_{v \in V(G')} (w_+(v) / \prod_{i=0}^n (1 + \lambda)^{i a_{i,v}})$ be the middle term of (3).

Let T be a star with t leaves. The total weight of independent sets in T is $Z_\lambda(T) = \lambda + (1 + \lambda)^t$, and the total weight of independent sets in T that do not include T 's centre is $(1 + \lambda)^t$. Therefore,

$$Z_\lambda(G'') = \sum_{I \in \mathcal{I}(G'')} \lambda^{|I|} \left(\prod_{v \in I} \prod_{i=0}^n (1 + \lambda)^{i a_{v,i}} \right) \left(\prod_{v \in V(G'') \setminus I} \prod_{i=0}^n (\lambda + (1 + \lambda)^i)^{a_{v,i}} \right).$$

This gives

$$\begin{aligned} M &= W' \lambda^{-n} \sum_{I \in \mathcal{I}(G')} \lambda^{|I|} \left(\prod_{v \in I} w_+(v) \right) \left(\prod_{v \in V(G') \setminus I} w_+(v) \prod_{i=0}^n \left(\frac{\lambda + (1 + \lambda)^i}{(1 + \lambda)^i} \right)^{a_{i,v}} \right) \\ &= W' \lambda^{-n} \sum_{I \in \mathcal{I}(G')} \lambda^{|I|} \left(\prod_{v \in I} w_+(v) \right) \left(\prod_{v \in V(G') \setminus I} w_+(v) \prod_{i=0}^n \Lambda_i^{a_{i,v}} \right). \end{aligned} \quad (4)$$

By Lemma 20, it follows that

$$M \leq W' \lambda^{-n} \sum_{I \in \mathcal{I}(G')} \lambda^{|I|} \left(\prod_{v \in I} w_+(v) \right) \left(\prod_{v \in V(G') \setminus I} \lambda w_-(v) \right) = Z(G'),$$

since the λ 's cancel. For the lower bound, (4) similarly implies that $M \geq (1 - \varepsilon/3n)^n Z(G')$, which is at least $(1 - \varepsilon/3) Z(G')$. \square

4.6 The proof of the theorem

We are now ready to prove Theorem 3, which we restate for convenience.

Theorem 3. *Let λ be a positive real number and let \mathcal{F} be a family of graphs whose connective constant is at most a quantity which is subcritical with respect to λ . There is an FPTAS for $Z(\mathcal{G})$ on λ -balanced weighted graphs $\mathcal{G} = (G, w_+, w_-, W)$ such that $G \in \mathcal{F}$.*

Proof. We will show that $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$ is the desired FPTAS.

Let λ, \mathcal{F} and \mathcal{G} be as in the theorem statement. Let ε be an error tolerance in $(0, 1)$ and let $n = |V(G)|$. If $\varepsilon \leq 3n\lambda(1 + \lambda)^{-n}$, then $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$ runs in time $\text{poly}(2^n) = \text{poly}(1/\varepsilon)$. Suppose instead that $\varepsilon > 3n\lambda(1 + \lambda)^{-n}$. Then steps (i)–(iv) all take time $\text{poly}(n, 1/\varepsilon)$. Moreover, by Lemma 20 $G'' \in \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+$, and $\mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+$ has subcritical connective constant w.r.t. λ by Lemma 18. Thus by Theorem 15, step (v) also takes time $\text{poly}(n, 1/\varepsilon)$. Thus in all cases, $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$ runs in time $\text{poly}(n, 1/\varepsilon)$ as required.

By Lemma 19, $Z(\mathcal{G}) \geq Z(G') \geq (1 - \varepsilon/3)Z(\mathcal{G})$. By Lemma 21, it follows that

$$Z(\mathcal{G}) \geq Z_\lambda(G'') W' \lambda^{-n} \prod_{v \in V(G')} \left(w_+(v) / \prod_{i=0}^n (1 + \lambda)^{i a_{i,v}} \right) \geq (1 - \varepsilon/3)^2 Z(\mathcal{G}).$$

By the correctness of $\text{ApproxZ}_{\lambda, \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+}^{\text{Uni}}$ (Theorem 15), it follows that

$$\begin{aligned} (1 + \varepsilon/3)Z(\mathcal{G}) &\geq \text{ApproxZ}_{\lambda, \mathcal{F}_{\lfloor 5(1+\lambda) \rfloor}^+}^{\text{Uni}}(G'', \varepsilon/3) \cdot W' \lambda^{-n} \prod_{v \in V(G')} \left(w_+(v) / \prod_{i=0}^n (1 + \lambda)^{i a_{i,v}} \right) \\ &\geq (1 - \varepsilon/3)^3 Z(\mathcal{G}). \end{aligned}$$

Thus the output of $\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$ lies in $(1 \pm \varepsilon)Z(\mathcal{G})$, as required. \square

5 The connective constant and local structure

In this section, we will give an FPTAS for $Z(\mathcal{G})$ in the case where the input weighted graph \mathcal{G} has an underlying graph G which comes from the following family.

Definition 22. Let \mathcal{D} be the family of graphs G such that $\delta(G) \geq 2$ and, for all $x \in V(G)$ with $d(x) \geq 6$, $d^2(x) \leq 26$.

Suppose we are given a weighted graph $\mathcal{G} = (G, w_+, w_-, W)$ with $\delta(G) \geq 2$ and, for all $v \in V(G)$, $w_+(v) \leq w_-(v)$ (i.e., \mathcal{G} is 1-balanced). Then either we can approximate $Z(\mathcal{G})$ relatively quickly using the FPTAS from this section, or \mathcal{G} contains a vertex of degree at

least 6 and 2-degree at least 27 — which our main algorithm will exploit. Thus \mathcal{D} will serve as the base case for our algorithm (see Section 6). We will show that \mathcal{D} has subcritical connective constant w.r.t. $\lambda = 1$, so that the existence of an FPTAS follows immediately from Theorem 3; see Theorem 26.

Recall from Section 4 that to show \mathcal{D} has subcritical connective constant, we must bound the number of leaves in the self-avoiding walk trees $T_{\text{SAW}}(G, v)$ of graphs $G \in \mathcal{D}$. To this end, we first introduce some notation and ancillary lemmas.

Let $T = (V, E, r)$ be a tree (V, E) with root $r \in V$. We write $L(T)$ for the set of T 's leaves. For any $v \in V$, we write $C(v)$ for the set of v 's children (i.e., v 's neighbours that are not on the unique path from v to r ; we take $C(r) = \Gamma(r)$), and $D(v)$ for v 's depth (i.e., the length of the path from v to r , so $D(r) = 0$).

Definition 23. Let $T = (V, E, r)$ be a rooted tree and let $\kappa \geq 1$. A function $\theta: V \rightarrow \mathbb{R}$ is κ -decreasing for T if, for every $x \in V$, $1/\kappa \leq \theta(x) \leq 1$ and, for every $x \in V \setminus \{r\}$,

$$\sum_{y \in C(x)} \theta(y) \leq \kappa \theta(x). \quad (5)$$

The specific requirement that $\theta(x) \geq 1/\kappa$ is for convenience. We will use a family of κ -decreasing functions for trees in a particular family, and any positive lower bound on the values taken by all of these functions would suffice.

Lemma 24. Let G be a graph and let $\kappa > 1$. Suppose that, for every $v \in V(G)$, there is a κ -decreasing function for $T_{\text{SAW}}(G, v)$. Then $\Delta(G) \leq \kappa^2 + 1$.

Proof. Towards a contradiction, suppose that $d_G(u) > \kappa^2 + 1$ for some $u \in V(G)$. $d_G(u) > 1$ so u has a neighbour v . Let θ be a κ -decreasing function for $T_{\text{SAW}}(G, v)$. Now consider the node x of $T_{\text{SAW}}(G, v)$ corresponding to the walk vu in G . For each $w \in \Gamma_G(u) \setminus \{v\}$, x has a child corresponding to the walk vuw , so it has more than κ^2 children. Therefore, $\sum_{y \in C(x)} \theta(y) > \kappa^2 \frac{1}{\kappa} \geq \kappa \theta(x)$, contradicting (5). \square

The following lemma is a useful way to bound connective constants.

Lemma 25. Let \mathcal{F} be a family of graphs. Let $\kappa > 1$ and suppose that, for every $G \in \mathcal{F}$ and every $v \in V(G)$, there is a κ -decreasing function for $T_{\text{SAW}}(G, v)$. Then the connective constant of \mathcal{F} is at most κ .

Proof. Let $G \in \mathcal{F}$, let $v \in V(G)$, let $T = T_{\text{SAW}}(G, v)$, let r be T 's root and let θ be κ -decreasing for T . For any $i \geq 0$, let T_i be the subgraph of T induced by vertices at distance at most i from r . We claim that, for all $i \geq 0$,

$$S_i := \sum_{w \in L(T_i)} \theta(w) \kappa^{-D(w)} < 2\kappa.$$

We have $S_0 = \theta(r) \leq 1 < 2\kappa$. By Lemma 24, $d_G(v) \leq \kappa^2 + 1$, so $|L(T_1)| \leq \kappa^2 + 1$ and $S_1 \leq (\kappa^2 + 1)/\kappa < 2\kappa$. Finally, suppose $S_i < 2\kappa$ for some $i \geq 1$ and consider any leaf w of T_i . If w is a leaf in T_{i+1} , then it contributes the same amount to S_i and S_{i+1} . If w is not a leaf in T_{i+1} , then it does not contribute to S_{i+1} but all of its children do. Their contribution is

$$\sum_{y \in C(w)} \theta(y) \kappa^{-D(y)} = \kappa^{-(D(w)+1)} \sum_{y \in C(w)} \theta(y) \leq \kappa^{-(D(w)+1)} \kappa \theta(w),$$

where the inequality is by (5). But $\kappa^{-D(w)} \theta(w)$ is exactly the contribution of w to T_i , so $S_{i+1} \leq S_i < 2\kappa$ as claimed.

$N(v, i)$ is the number of vertices in T with depth i so

$$N(v, i) \leq |L(T_i)| \leq \kappa \sum_{w \in L(T_i)} \theta(w) \leq \kappa^{i+1} \sum_{w \in L(T_i)} \theta(w) \kappa^{-D(w)} = \kappa^{i+1} S_i < 2\kappa^{i+2}.$$

So, for all $\ell \geq 1$,

$$\sum_{i=1}^{\ell} N(v, i) < \sum_{i=1}^{\ell} 2\kappa^{i+2} < \frac{2\kappa^3}{\kappa - 1} \kappa^{\ell}.$$

In particular, this holds for all $\ell \geq \log |V(G)|$, so the connective constant of \mathcal{F} is at most κ . \square

We are now in a position to prove the main result of the section.

Theorem 26. *There is an FPTAS $\text{BaseCount}(\mathcal{G}, \varepsilon)$ for $Z(\mathcal{G})$ on 1-balanced weighted graphs $\mathcal{G} = (G, w_+, w_-, W)$ with $G \in \mathcal{D}$.*

Proof. We use 4.141-decreasing functions to show that \mathcal{D} has connective constant at most 4.141. This is subcritical w.r.t. $\lambda = 1$, so the algorithm $\text{ApproxZ}_{1, \mathcal{D}}$ is an FPTAS as shown in the proof of Theorem 3 and the result follows.

Let $G \in \mathcal{D}$. The restrictions on $\delta(G)$ and maximum 2-degree imply that $\Delta(G) \leq 13$. Let $v \in V(G)$ and let $T = T_{\text{SAW}}(G, v)$.

Let $\psi(d, p)$ be the following function for integers $d, p \geq 2$.

$$\begin{aligned} \psi(2, p) &= 0.245 \text{ for all } p \geq 2 & \psi(d, 2) &= 1 & \text{for all } d \geq 6 \\ \psi(3, p) &= 0.456 \text{ for all } p \geq 2 & \psi(d, 3) &= 0.941 \text{ for all } d \geq 6 \\ \psi(4, p) &= 0.647 \text{ for all } p \geq 2 & \psi(d, p) &= 0.889 \text{ for all } d \geq 6, p \geq 4. \\ \psi(5, p) &= 0.859 \text{ for all } p \geq 2 \end{aligned}$$

Define the function $\theta: V(T) \rightarrow \mathbb{R}$ by setting $\theta(x) = 1$ if x is the root of T and, otherwise, setting $\theta(x) = \psi(d_T(x), d_T(p(x)))$, where $p(x)$ is x 's parent in T .

We claim that θ is 4.141-decreasing for T , from which the result follows by Lemma 25 and Theorem 3. It is immediate that $1/4.141 < 0.242 \leq \theta(x) \leq 1$ for all $x \in V(T)$ so it remains to show that (5) holds for every $x \in V(T) \setminus \{r\}$. We verify this with the Mathematica program given in Appendix B.

Consider a vertex $x \in V(T) \setminus \{r\}$. Let $d_T(x) = d$. For each $i \in \{2, \dots, 13\}$, let $n[i]$ be the number of children of x with degree i , and let $p = d_T(p(x))$. Then (5) becomes

$$\sum_{i=2}^{13} n[i] \psi(i, d) \leq 4.141 \psi(d, p). \quad (6)$$

The program attempts to find some combination of $n[2], \dots, n[13] \in \{0, \dots, 12\}$ and $d, p \in \{2, \dots, 13\}$ that can arise from a vertex in T and that does not satisfy (6), i.e., according to Definition 22 it checks every combination that has $1 + \sum_i n[i] = d$ and that has $d \leq 5$ or $p + \sum_i i n[i] \leq 26$. The program outputs “{}”, indicating that every instantiation of (6) is satisfied. The program uses exact arithmetic, with no numerical approximations, so this is a rigorous check. Therefore, by Lemma 25, the connective constant of \mathcal{D} is at most 4.141 as claimed. \square

6 Approximating $Z(\mathcal{G})$ for arbitrary graphs

6.1 Preprocessing

To analyse our algorithm's running time, we use a continuous piecewise-linear potential function whose linear “slices” have the following form.

Definition 27. A function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a *good slice* if it is of the form $f(m, n) = \rho m + \sigma n$ for some $\rho \geq 0$ that satisfies $\rho \geq -\sigma$ and which is not identically zero (that is, it is not the case that $\sigma = \rho = 0$). Given any good slice f , we also use the name f to denote the function from graphs to real numbers given by $f(G) = f(|E(G)|, |V(G)|)$. Likewise, if $\mathcal{G} = (G, w_+, w_-, W)$ is a weighted graph, we write $f(\mathcal{G}) = f(G)$.

Note that we allow σ to be negative as long as $\sigma \geq -\rho$, and indeed our final potential function will contain a slice with $\sigma = -\rho$ (see Section 7). This essentially encodes a well-known trick: we can remove vertices of degree at most 1 from our input graph $\mathcal{G} = (G, w_+, w_-, W)$ in polynomial time (see Corollary 32). If after doing this, G has average degree 2, then G must be 2-regular, so we can compute $Z(\mathcal{G})$ in polynomial time. Thus we can safely set $f(G) = 0$ when $|E(G)| = |V(G)|$.

Unfortunately, taking $\sigma = -\rho$ introduces technical difficulties. In order for f to be useful for analysing the running time of our algorithm, we require that whenever the algorithm makes recursive calls with instances \mathcal{G}_{in} and \mathcal{G}_{out} and original input \mathcal{G} , we have $f(\mathcal{G}_{\text{in}}) < f(\mathcal{G})$ and $f(\mathcal{G}_{\text{out}}) < f(\mathcal{G})$. If G contains a tree component T , then repeatedly removing degree-1 vertices from G will remove all of T . We then have

$$f(G - V(T)) = f(G) - \rho|E(T)| - \sigma|V(T)| = f(G) + \rho > f(G).$$

Thus removing degree-1 vertices may increase G 's potential, so we must be very careful to keep track of how many tree components we create as we branch. For this reason, our preprocessing doesn't just remove degree-1 vertices. Instead, it prunes G in other ways to decrease the number of tree components that are formed. Our main operation incorporates what is often referred to *multiplier reduction* in the literature [4], and is given below. At the end of the section we will describe a further, simpler operation of removing tree components.

Definition 28. Let $\mathcal{G} = (G, w_+, w_-, W)$ be a weighted graph. Let S be a subset of $V(G)$ satisfying $|\Gamma_G(S)| \leq 1$. If $\Gamma_G(S) = \emptyset$, then $\text{Prune}(\mathcal{G}, S)$ is the weighted graph $(G - S, w_+, w_-, Z(G[S], w_+, w_-, W))$. If instead $\Gamma_G(S) = \{v\}$, then let $S' = S \cap \Gamma_G(v)$ and let $S'' = S \setminus \Gamma_G(v)$. We define $\text{Prune}(\mathcal{G}, S)$ to be the weighted graph $(G - S, w'_+, w'_-, W)$, where

$$w'_+(x) = \begin{cases} w_+(v) w_-(S') Z(G[S''], w_+, w_-, 1) & \text{if } x = v, \\ w_+(x) & \text{otherwise,} \end{cases}$$

$$w'_-(x) = \begin{cases} w_-(v) Z(G[S], w_+, w_-, 1) & \text{if } x = v, \\ w_-(x) & \text{otherwise.} \end{cases}$$

Lemma 29. Let $\mathcal{G} = (G, w_+, w_-, W)$ be a 1-balanced weighted graph, and let $S \subseteq V(G)$ with $|\Gamma_G(S)| \leq 1$. Then $Z(\text{Prune}(\mathcal{G}, S)) = Z(\mathcal{G})$ and $\text{Prune}(\mathcal{G}, S)$ is 1-balanced.

Proof. If $\Gamma_G(S) = \emptyset$, then the result is immediate, so suppose instead that $\Gamma_G(S) = \{v\}$ for some $v \in V(G)$. Write $\mathcal{G}' = \text{Prune}(\mathcal{G}, S) = (G - S, w'_+, w'_-, W)$. Partitioning sets $I \in \mathcal{I}(\mathcal{G})$

according to whether or not $v \in I$, we have

$$Z(\mathcal{G}) = w_+(v) w_-(\Gamma_G(v)) Z(\mathcal{G} - v - \Gamma_G(v)) + w_-(v) Z(\mathcal{G} - v).$$

Similarly, using the fact that $\Gamma_G(S) = \{v\}$ and Definition 28,

$$\begin{aligned} Z(\mathcal{G}) &= w_+(v) w_-(\Gamma_G(v)) Z(G[S''], w_+, w_-, 1) Z(\mathcal{G} - S - v - \Gamma_G(v)) \\ &\quad + w_-(v) Z(\mathcal{G} - S - v) Z(G[S], w_+, w_-, 1) \\ &= w'_+(v) w_-(\Gamma_G(v) \setminus S) Z(\mathcal{G} - S - v - \Gamma_G(v)) + w'_-(v) Z(\mathcal{G} - S - v) = Z(\mathcal{G}'). \end{aligned}$$

It remains to show that $\text{Prune}(\mathcal{G}, S)$ is 1-balanced. We have $w_+(x) \leq w_-(x)$ for all $x \in V(G)$ and we must show that $w'_+(x) \leq w'_-(x)$ for all $x \in V(G')$. So, let $x \in V(G')$. If $x \neq v$ then $w'_+(x) = w_+(x) \leq w_-(x) = w'_-(x)$ and we are done. Otherwise, $x = v$. We have

$$\begin{aligned} w_-(S') Z(G[S''], w_+, w_-, 1) &= \sum_{\substack{I \in \mathcal{I}(G[S]) \\ I \cap S' = \emptyset}} w_+(I) w_-(S \setminus I) \leq \sum_{I \in \mathcal{I}(G[S])} w_+(I) w_-(S \setminus I) \\ &= Z(G[S], w_+, w_-, 1), \end{aligned}$$

and $w_+(v) \leq w_-(v)$ by hypothesis. Therefore, $w'_+(v) \leq w'_-(v)$. \square

In order to compute the weighted graph $\text{Prune}(\mathcal{G}, S)$ we need to evaluate $Z(\mathcal{G}[S])$ and possibly $Z(\mathcal{G}[S''])$. For this, we will use the following brute-force algorithm.

Algorithm ExactCount(\mathcal{G}, Y): *The input is a weighted graph $\mathcal{G} = (G, w_+, w_-, W)$ and a subset Y of $V(G)$ such that $\mathcal{G} - Y$ is a forest.*

- (i) For all $I \in \mathcal{I}(G[Y])$, compute $Z(\mathcal{G} - Y - \Gamma_G(I))$.
- (ii) Return $\sum_{I \in \mathcal{I}(G[Y])} w_+(I) w_-(Y \setminus I) Z(\mathcal{G} - Y - \Gamma_G(I))$.

Observation 30. *If $\mathcal{G} = (G, w_+, w_-, W)$ is a weighted graph and Y is a subset of $V(G)$ such that $\mathcal{G} - Y$ is a forest then $\text{ExactCount}(\mathcal{G}, Y)$ returns $Z(\mathcal{G})$ and has running time $2^{|Y|} \text{poly}(|V(G)|)$.*

6.2 A digression

As noted in Section 5 our main algorithm will use the FPTAS from Theorem 26 which approximates $Z(\mathcal{G})$ on 1-balanced weighted graphs $\mathcal{G} = (G, w_+, w_-, W)$ with $G \in \mathcal{D}$. (Recall from Definition 22 that \mathcal{D} is the family of graphs with minimum degree at least 2 such that every vertex with degree at least 6 has 2-degree at most 26.)

In the introduction we stated Theorem 4, which is essentially an unweighted version of Theorem 26. The only difference is that, in Theorem 4, inputs are allowed to have vertices with degree less than 2. Given the preprocessing that we have just done, we can use Theorem 26 to prove Theorem 4, which we restate for convenience.

Theorem 4. *There is an FPTAS for $Z(G)$ on graphs G in which every vertex of degree at least 6 has 2-degree at most 26.*

Proof. Consider a graph G in which every vertex of degree at least 6 has 2-degree at most 26. We will show how to construct a 1-balanced weighted graph $\mathcal{G} = (G', w_+, w_-, W)$ with $G' \in \mathcal{D}$ such that $Z(\mathcal{G}) = Z(G)$. We can then use the FPTAS from Theorem 26.

The construction is simple. Let $\mathcal{G} = (G, \mathbf{1}, \mathbf{1}, 1)$. Now repeat the following. If the minimum degree of \mathcal{G} is at least 2, we are finished. Otherwise, let S be the set containing any vertex of G whose degree is less than 2. By Lemma 29, $Z(\text{Prune}(\mathcal{G}, S)) = Z(\mathcal{G})$ and $\text{Prune}(\mathcal{G}, S)$ is 1-balanced. Since \mathcal{G} has the property that any vertex of degree at least 6 has 2-degree at most 26, so does $\text{Prune}(\mathcal{G}, S)$. Replace \mathcal{G} with $\text{Prune}(\mathcal{G}, S)$. \square

6.3 More preprocessing

Our preprocessing algorithm will prune its input graph by removing non-empty induced subgraphs of the following form.

Definition 31. We say that a graph is a *near-forest* if it is the empty graph (with no vertices) or if it contains a vertex v such that $G - v - \Gamma_G(v)$ is a forest.

In this definition (and everywhere else) we consider the empty graph to be a forest. Thus, a forest is a near-forest. The point of Definition 31 is the following corollary.

Corollary 32. Let $\mathcal{G} = (G, w_+, w_-, W)$ be an n -vertex weighted graph with maximum degree at most Δ . Let $X \subseteq V(G)$ span a near-forest in G with $|\Gamma_G(X)| \leq 1$. Then $\text{Prune}(G, X)$ can be computed in time at most $2^\Delta \text{poly}(n)$.

Proof. If $G[X]$ is a forest, let $S = \emptyset$. Otherwise, there exists $v \in X$ such that $G[X] - v - \Gamma_G(v)$ is a forest; take $S = \{v\} \cup \Gamma_G(v)$. Thus $G[X] - S$ is a forest, so by Observation 30, for all $Y \subseteq X$ we have $Z(\mathcal{G}[Y]) = \text{ExactCount}(\mathcal{G}[Y], S)$. Again by Observation 30, it follows that $Z(\mathcal{G}[Y])$ can be computed in time $2^\Delta \text{poly}(n)$ for all $Y \subseteq X$. In particular, this allows us to compute $\text{Prune}(\mathcal{G}, X)$ in $\text{poly}(n)$ time using Definition 28. \square

The output of our preprocessing algorithm will be of the following form.

Definition 33. A graph G is *reduced* if for all non-empty subsets X of $V(G)$ such that $G[X]$ is a near-forest, $|\Gamma_G(X)| \geq 2$.

Note that a reduced graph G is either empty or satisfies $\delta(G) \geq 2$, and it has no components which are near-forests. We now set out our preprocessing algorithm and prove its correctness.

Algorithm Reduce(\mathcal{G}): The input is a non-empty weighted graph \mathcal{G} with no tree components. For each weighted graph \mathcal{G}_i that we construct in this algorithm, we write $\mathcal{G}_i = (G_i, w_+^i, w_-^i, W_i)$.

- (i) Set $i = 1$ and $\mathcal{G}_1 = \mathcal{G}$.
- (ii) If either G_i or some subgraph $G_i - v$ (where $v \in V(G_i)$) contains a component C which is a near-forest, then:
 - (a) Let $\mathcal{G}_{i+1} = \text{Prune}(\mathcal{G}_i, V(C))$.
 - (b) Increment i and go to (ii).
- (iii) Otherwise, return \mathcal{G}_i .

Lemma 34. *Let $\mathcal{G} = (G, w_+, w_-, W)$ be a 1-balanced n -vertex weighted graph with no tree components and maximum degree at most Δ . Then $\text{Reduce}(\mathcal{G})$ returns a 1-balanced reduced weighted graph \mathcal{H} such that*

- $Z(\mathcal{G}) = Z(\mathcal{H})$,
- \mathcal{H} has maximum degree at most Δ , and
- for any good slice f , $f(\mathcal{H}) \leq f(\mathcal{G})$.

The running time is $2^\Delta \text{poly}(n)$.

Proof. Observe that each graph G_i is produced by repeated calls to **Prune** (which meet the requirements of Definition 28). Thus each \mathcal{G}_i has maximum degree at most Δ and, by Lemma 29, is 1-balanced and satisfies $Z(G_i) = Z(\mathcal{G})$. Suppose that G_i does not satisfy the condition in step (ii) of Algorithm **Reduce**. We want to show, using Definition 33, that it is reduced. To see this, note that if $X \subseteq V(G_i)$ spans a non-empty near-forest in G_i with $|\Gamma_{G_i}(X)| \leq 1$, then $G_i[X]$ also contains a near-forest component; thus the algorithm continues and constructs G_{i+1} from G_i . We conclude that \mathcal{H} is reduced. Note that $|V(G_1)| > |V(G_2)| > \dots > |V(\mathcal{H})|$, so step (ii) is iterated at most n times. By Corollary 32, each iteration takes time at most $2^\Delta \text{poly}(n) = \text{poly}(n)$, so our stated time bound follows.

It remains only to prove that $f(\mathcal{H}) \leq f(\mathcal{G})$ for all good slices f . Write $f(m, n) = \rho m + \sigma n$ and $\mathcal{H} = \mathcal{G}_s$. Let C_1, \dots, C_r be the components of G , and for all $i \in [r]$ and $j \in [s]$, let $C_{i,j} = G_j[V(C_i) \cap V(G_j)]$ be the remaining part of C_i in G_j . Thus

$$f(\mathcal{G}) - f(\mathcal{H}) = \sum_{i=1}^r (f(C_i) - f(C_{i,s})). \quad (7)$$

We will show that each term of this sum is non-negative.

Since each application of **Prune** deletes a vertex set with at most one neighbour, it can never remove a separator; thus for all $i \in [r]$, either $C_{i,s}$ is empty or $C_{i,j}$ is a component of G_j for all $j \in [s]$. If $C_{i,s}$ is empty, then since \mathcal{G} contains no tree components by hypothesis and f is a good slice, we have

$$f(C_i) - f(C_{i,s}) = \rho e(C_i) + \sigma |C_i| \geq (\rho + \sigma) |C_i| \geq 0.$$

Suppose instead $C_{i,j}$ is a component of G_j for all $j \in [s]$. Then for all $2 \leq j \leq s$, either $C_{i,j} = C_{i,j-1}$ or $C_{i,j}$ is formed from $C_{i,j-1}$ by deleting a connected set sending at least one edge into $V(C_{i,j})$. In either case, at least as many edges are deleted as vertices. Thus

$$\begin{aligned} f(C_i) - f(C_{i,s}) &= \sum_{j=2}^s (f(C_{i,j-1}) - f(C_{i,j})) \\ &= \sum_{j=2}^s \left(\rho (e(C_{i,j-1}) - e(C_{i,j})) + \sigma (|C_{i,j-1}| - |C_{i,j}|) \right) \\ &\geq \sum_{j=2}^s (\rho + \sigma) (|C_{i,j-1}| - |C_{i,j}|) \geq 0. \end{aligned}$$

The result therefore follows from (7). \square

Having shown how to reduce graphs using the pruning operation we now define the final operation, tree removal.

Definition 35. Let G be a graph with tree components T_1, \dots, T_r . Then we define the *tree removal* of G by $\text{TR}(G) = G - V(T_1) - \dots - V(T_r)$. Moreover, if $\mathcal{G} = (G, w_+, w_-, W)$ is a weighted graph, we define the *tree removal* of \mathcal{G} by

$$\text{TR}(\mathcal{G}) = \left(\text{TR}(G), w_+, w_-, W \cdot \prod_{i=1}^r \text{ExactCount}(\mathcal{T}_i, \emptyset) \right),$$

where $\mathcal{T}_i = (T_i, w_+, w_-, 1)$.

The following observation is immediate from the definition and Observation 30.

Observation 36. Let \mathcal{G} be a weighted graph with underlying graph G and let $\text{TR}(\mathcal{G})$ have underlying graph H . Then,

- (i) $H \subseteq G$,
- (ii) $Z(\text{TR}(\mathcal{G})) = Z(\mathcal{G})$,
- (iii) if \mathcal{G} is 1-balanced then \mathcal{H} is also 1-balanced, and
- (iv) $\text{TR}(\mathcal{G})$ can be computed in time $\text{poly}(|V(G)|)$. □

6.4 Graph decompositions

To describe our algorithm, we use two decompositions, which we call the standard decomposition and the extended decomposition. Recall from Sections 6.1 and 6.3 that it will be important for us to keep track of the tree components formed as we proceed; for this reason, we introduce some notation for the local structure of a graph around a vertex.

Definition 37. Let G be a connected graph, and let $v \in V(G)$. We define the *standard decomposition* of G from v to be the tuple $(\Gamma_v, S, X, H_1, \dots, H_k; T_1, \dots, T_\ell)$, where:

- $\Gamma_v = \Gamma_G(v)$;
- H_1, \dots, H_k are the non-tree components of $G - v - \Gamma_v$;
- T_1, \dots, T_ℓ are the tree components of $G - v - \Gamma_v$;
- $S = (V(H_1) \cup \dots \cup V(H_k)) \cap \Gamma_G^2(v)$;
- $X = \{v\} \cup \Gamma_v \cup (V(T_1) \cup \dots \cup V(T_\ell))$.

An example of the standard decomposition is illustrated in Figure 4. Note that a (non-empty) connected graph G is a near-forest if and only if there exists $v \in V(G)$ such that $S = \emptyset$ in the standard decomposition of G from v .

Lemma 38. Let G be a connected reduced graph, and let $v \in V(G)$. Then in the standard decomposition of G from v , $|S| \geq 2$.

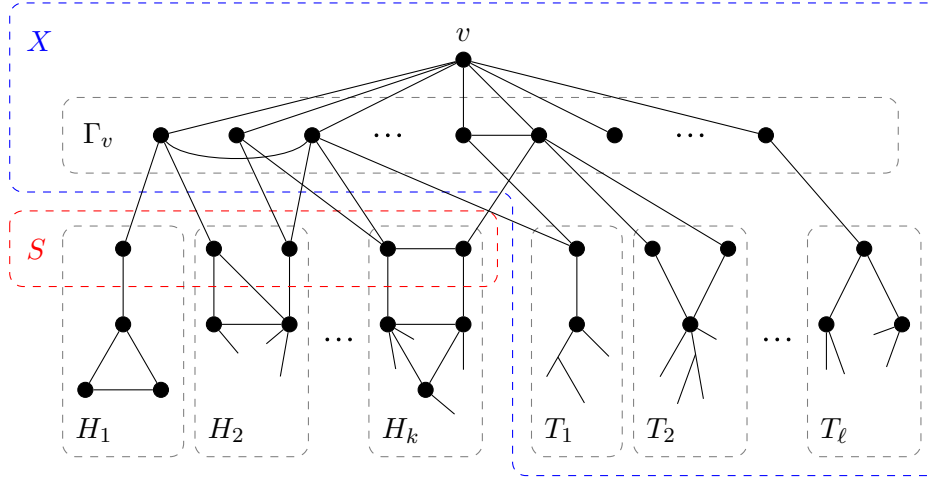


Figure 4: An example of the standard decomposition of a connected graph G from a vertex v .

Proof. Observe that in the standard decomposition of G from v , $G[X] - v - \Gamma_G(v) = T_1 \cup \dots \cup T_\ell$ is a forest, so $G[X]$ itself is a near-forest. Thus by the definition of reducedness (Definition 33), it follows that $|S| = |\Gamma_G(X)| \geq 2$. \square

Our main branching operation will be the intuitive one — we pick a vertex $v \in V(G)$ and partition sets $I \in \mathcal{I}(G)$ according to whether or not $v \in I$, giving

$$Z(\mathcal{G}) = w_-(v) Z(\mathcal{G} - v) + w_+(v) w_-(\Gamma_G(v)) Z(\mathcal{G} - v - \Gamma_G(v)).$$

Note that this equation holds even if $\mathcal{G} - v - \Gamma_G(v)$ contains no vertices. After picking v , we then recurse on $\mathcal{G} - v$ and $\mathcal{G} - v - \Gamma_G(v)$. (Informally, we call this *branching on v* , and we call $\mathcal{G} - v$ and $\mathcal{G} - v - \Gamma_G(v)$ the resulting *branches*.)

Using the standard decomposition of G from v , we will bound the efficiency of this branching operation in terms of $d_G(v)$, $d_G^2(v)$ and $|S|$ (see Lemma 42). The bounds in Lemma 42 are weaker when S is small. By Lemma 38 we have $|S| \geq 2$; if $|S| = 2$, say $S = \{x, y\}$, then we use an alternative strategy introduced in [4] and branch on y rather than v . In both branches, the neighbourhood of what remains of X is then either \emptyset or $\{x\}$, so we can quickly remove it with **Prune**. In [4], the authors consider arbitrary graphs, and in this setting the strategy is effective enough that the $|S| = 2$ case is no longer the limiting factor in the running time analysis. However, if G is bipartite, then the bounds of Lemma 42 are stronger and this is no longer true. What we do is construct an “extended decomposition” (see Definition 39 below) which allows us to branch on a different vertex z , rather than on y . This gives better results in the bipartite case (see Appendix C). Crucially, in the bipartite case, branching on z lets us improve what would otherwise be the least efficient branch when $|S| = 2$, so it improves the results. It will turn out that this is only an issue when $d_{G-X}(y) = 1$; in this case, we will extend y into a path in $G - X$, and branch on the other endpoint z of the path.

Suppose, then, that G is connected, and $|S| = 2$ in the standard decomposition of G from $v \in V(G)$. Let $S = \{x, y\}$. By Definition 37, no component of $G - X$ in the standard decomposition is a tree so the component containing y is not an isolated vertex or path. If $d_{G-X}(y) = 1$, $G - X$ must contain a path from y to some z with $d_{G-X}(z) > 2$. This allows us to make the following definition.

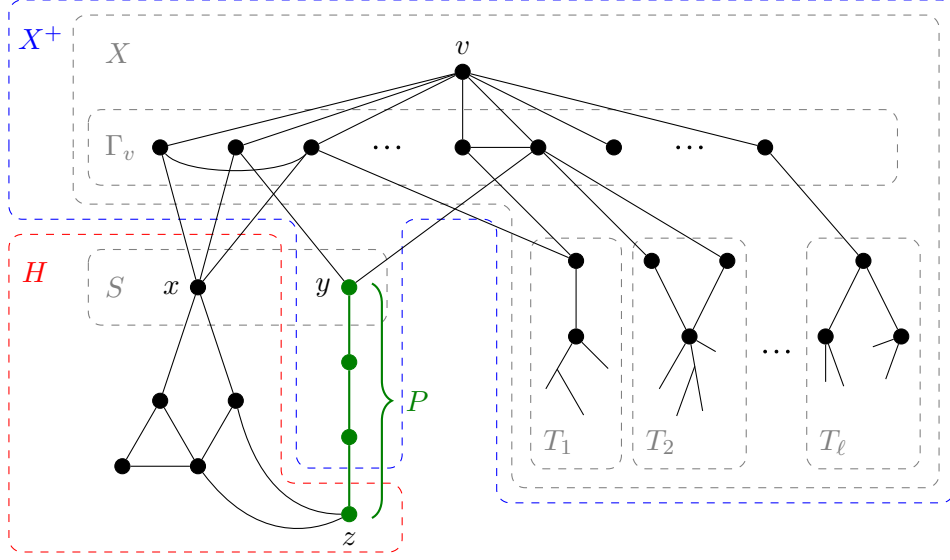


Figure 5: An example of the extended decomposition of a connected graph G from a vertex v . Relevant elements of the graph's standard decomposition are shown in pale grey.

Definition 39. Let G be a connected graph, and let $v \in V(G)$. Suppose $|S| = 2$ in the standard decomposition of G from v . Then we define the *extended decomposition* of G from v to be the standard decomposition from v , together with the tuple (X^+, P, x, y, z, H) , where:

- $\{x, y\} = S$, where $x < y$ (recall that $V(G) = \{1, \dots, n\}$);
- if $d_{G-X}(y) \geq 2$, then $z = y$; otherwise, z is the (unique) closest vertex to y in $G - X$ with $d_{G-X}(z) > 2$;
- P is the (unique) y - z path in $G - X$;
- $X^+ = X \cup V(P) \setminus \{z\}$;
- $H = G - X^+$.

When $|S| = 2$ in the extended decomposition from v , we will branch on z and then prune what remains of X^+ in both branches. This will be efficient enough that the worst case in the analysis will be $|S| = 3$ even when G is bipartite (see Lemma 44). An example of the extended decomposition is given in Figure 5 — we justify the depiction of P and z in the following lemma.

Lemma 40. Let G be a connected reduced graph, let $v \in V(G)$, and suppose that the standard decomposition of G from v has $|S| = 2$. Then in the extended decomposition of G from v , we have $x \notin V(P)$ and $d_H(z) \geq 2$.

Proof. Suppose, towards a contradiction, that $x \in V(P)$ as in Figure 6. This implies that $d_{G-X}(y) = 1$. Let P' be the sub-path of P from y to x and let $U = (X \cup V(P')) \setminus \{x\}$. $G[U]$ is a near-forest — delete v and $\Gamma_G(v)$ to obtain the forest $(P' - x) \cup T_1 \cup \dots \cup T_\ell$ — but $\Gamma_G(U) = \{x\}$, contradicting the hypothesis that G is reduced.

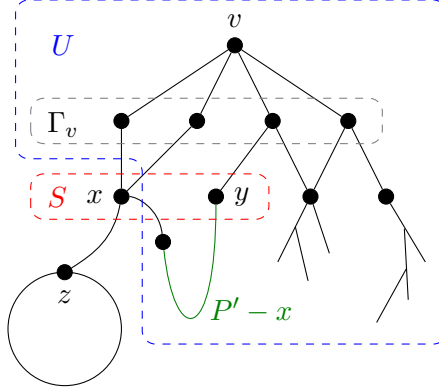


Figure 6: A supposed extended decomposition that leads to a contradiction in the proof of Lemma 40.

Therefore, $z \neq x$. If $z = y$ then $X^+ = X$, so $d_H(z) = d_{G-X}(y) \geq 2$. Otherwise, $z \notin \{x, y\} = \Gamma_G(V(H))$, so $d_H(z) = d_{G-X}(z) - 1 \geq 2$. \square

6.5 The main approximation algorithm

Algorithm Count(\mathcal{G}, ε): *The input is a non-empty 1-balanced weighted graph $\mathcal{G} = (G, w_+, w_-, W)$ with no tree components and a rational number $0 < \varepsilon < 1$.*

(i) If $\Delta(G) \geq 11$:

(a) Let v be the lexicographically least vertex of degree at least 11 in G and let

$$\begin{aligned} \mathcal{G}_{\text{out}} &= (G_{\text{out}}, \cdot, \cdot, W_{\text{out}}) \leftarrow \text{TR}(\mathcal{G} - v), \\ W'_{\text{out}} &\leftarrow w_-(v), \\ C_{\text{out}} &\leftarrow \text{If } G_{\text{out}} \text{ is empty then } W_{\text{out}} \text{ else } \text{Count}(\mathcal{G}_{\text{out}}, \varepsilon), \\ \mathcal{G}_{\text{in}} &= (G_{\text{in}}, \cdot, \cdot, W_{\text{in}}) \leftarrow \text{TR}(\mathcal{G} - v - \Gamma_G(v)), \\ W'_{\text{in}} &\leftarrow w_+(v) \cdot w_-(\Gamma_G(v)), \text{ and} \\ C_{\text{in}} &\leftarrow \text{If } G_{\text{in}} \text{ is empty then } W_{\text{in}} \text{ else } \text{Count}(\mathcal{G}_{\text{in}}, \varepsilon). \end{aligned}$$

(b) Return $W'_{\text{out}} \cdot C_{\text{out}} + W'_{\text{in}} \cdot C_{\text{in}}$.

(ii) $\mathcal{G} \leftarrow \text{Reduce}(\mathcal{G})$.

(iii) If G has no vertices of degree at least 6 and 2-degree at least 27, return **BaseCount**(\mathcal{G}, ε).

(iv) If G has average degree at most 5, then let $v \in V(G)$ be the lexicographically least vertex of degree at least 6 and 2-degree at least 27. Otherwise, let $v \in V(G)$ be the lexicographically least vertex which maximises $d_G^2(v)$ subject to $d_G(v) \geq 2|E(G)|/|V(G)|$.

(v) Let G_v be the component of G containing v , and consider the standard decomposition

of G_v from v . If $|S| \geq 3$, let

$$\begin{aligned} \mathcal{G}_{\text{out}} &= (G_{\text{out}}, \cdot, \cdot, W_{\text{out}}) \leftarrow \text{TR}(\mathcal{G} - v), \\ W'_{\text{out}} &\leftarrow w_-(v), \\ \mathcal{G}_{\text{in}} &= (G_{\text{in}}, \cdot, \cdot, W_{\text{in}}) \leftarrow \text{TR}(\mathcal{G} - v - \Gamma_G(v)), \text{ and} \\ W'_{\text{in}} &\leftarrow w_+(v) \cdot w_-(\Gamma_G(v)). \end{aligned}$$

(vi) If instead $|S| = 2$, consider the extended decomposition of G_v from v and let

$$\begin{aligned} \mathcal{G}_{\text{out}} &= (G_{\text{out}}, \cdot, \cdot, W_{\text{out}}) \leftarrow \text{TR}(\text{Prune}(\mathcal{G} - z, X^+)), \\ W'_{\text{out}} &\leftarrow w_-(z), \\ \mathcal{G}_{\text{in}} &= (G_{\text{in}}, \cdot, \cdot, W_{\text{in}}) \leftarrow \text{TR}(\text{Prune}(\mathcal{G} - z - \Gamma_G(z), X^+ \setminus \Gamma_G(z))), \text{ and} \\ W'_{\text{in}} &\leftarrow w_+(z) \cdot w_-(\Gamma_G(z)). \end{aligned}$$

(vii) Let

$$\begin{aligned} C_{\text{out}} &\leftarrow \text{If } G_{\text{out}} \text{ is empty then } W_{\text{out}} \text{ else } \text{Count}(\mathcal{G}_{\text{out}}, \varepsilon) \text{ and} \\ C_{\text{in}} &\leftarrow \text{If } G_{\text{in}} \text{ is empty then } W_{\text{in}} \text{ else } \text{Count}(\mathcal{G}_{\text{in}}, \varepsilon). \end{aligned}$$

Return $W'_{\text{out}} \cdot C_{\text{out}} + W'_{\text{in}} \cdot C_{\text{in}}$.

Lemma 41. *Let $\mathcal{G} = (G, w_+, w_-, W)$ be a non-empty 1-balanced n -vertex weighted graph with no tree components. Let $0 < \varepsilon < 1$. Then,*

- (i) $\text{Count}(\mathcal{G}, \varepsilon)$ returns an ε -approximation of $Z(\mathcal{G})$;
- (ii) Within time $\text{poly}(n, 1/\varepsilon)$, $\text{Count}(\mathcal{G}, \varepsilon)$ either terminates or computes the arguments \mathcal{G}_{in} and \mathcal{G}_{out} to its recursive calls;
- (iii) At each recursive call $\text{Count}(\mathcal{H}, \varepsilon)$, \mathcal{H} is non-empty and 1-balanced and has no tree components.

Proof. We first prove part (iii). The argument of a recursive call is either \mathcal{G}_{in} or \mathcal{G}_{out} . These are computed from \mathcal{G} by deleting vertices and calling some subset of **Prune**, **Reduce** and **TR**. It is immediate from the definition that deleting vertices preserves the property of being 1-balanced; **Prune**, **Reduce** and **TR** maintain 1-balance by Lemma 29, Lemma 34 and Observation 36, respectively. The last operation is always a call to **TR**, which removes any tree components by construction, so the result follows.

We now prove (i) and (ii). First, note that the input decreases in size on each recursive call, so **Count** must terminate.

If $\Delta(G) \geq 11$, then steps (i)(a) and (i)(b) are executed. By partitioning sets $I \in \mathcal{I}(G)$ according to whether or not $v \in I$, we have

$$\begin{aligned} Z(\mathcal{G}) &= w_-(v) Z(\mathcal{G} - v) + w_+(v) w_-(\Gamma_G(v)) Z(\mathcal{G} - v - \Gamma_G(v)) \\ &= W'_{\text{out}} Z(\mathcal{G} - v) + W'_{\text{in}} Z(\mathcal{G} - v - \Gamma_G(v)). \end{aligned}$$

By Observation 36, $Z(\mathcal{G}) = W'_{\text{out}}Z(\mathcal{G}_{\text{out}}) + W'_{\text{in}}Z(\mathcal{G}_{\text{in}})$. By part (iii) of the current lemma, \mathcal{G}_{in} and \mathcal{G}_{out} are 1-balanced and have no tree components, so the recursive calls to **Count** are valid. Thus by the recursive correctness of **Count**, $\text{Count}(\mathcal{G}, \varepsilon) \in (1 \pm \varepsilon)Z(\mathcal{G})$ as required. \mathcal{G}_{in} and \mathcal{G}_{out} are computed in time $\text{poly}(n)$ by Observation 36.

Otherwise, $\Delta(G) \leq 10$. By Lemma 34, step (ii) runs in time $2^{\Delta(G)} \text{poly}(n) = \text{poly}(n)$ and, after this step, \mathcal{G} is reduced and 1-balanced and $Z(\mathcal{G})$ has not changed. Recall from Definition 22 that \mathcal{D} is the family of all graphs with minimum degree at least 2 which contain no vertices of degree at least 6 and 2-degree at least 27. Since G is reduced, $\delta(G) \geq 2$. Thus if G has no vertices of degree at least 6 and 2-degree at least 27, then $G \in \mathcal{D}$ and step (iii) runs in time $\text{poly}(n, 1/\varepsilon)$ and is correct by Theorem 26.

Step (iv) can clearly be computed in time $\text{poly}(n)$.

In step (v), G_v is a component of a reduced graph, so it is reduced. If $|S| \geq 3$, then correctness and polynomial running time follow exactly as for step (i). Otherwise, by Lemma 38, $|S| \geq 2$, so we execute step (vi). As in step (i), we have

$$Z(\mathcal{G}) = W'_{\text{out}}Z(\mathcal{G} - z) + W'_{\text{in}}Z(\mathcal{G} - z - \Gamma_G(z)).$$

Note that $\Gamma_{G-z}(X^+) = \{x\}$, and $\Gamma_{G-z-\Gamma_G(z)}(X^+ \setminus \Gamma_G(z)) \subseteq \{x\}$. Hence by Lemma 29 and Observation 36, it follows that $Z(\mathcal{G}) = W'_{\text{out}}Z(\mathcal{G}_{\text{out}}) + W'_{\text{in}}Z(\mathcal{G}_{\text{in}})$. Note that \mathcal{G}_{out} and \mathcal{G}_{in} are 1-balanced and have no tree components by part (iii) of the current lemma, so the recursive calls to **Count** are valid. Thus, by the recursive correctness of **Count**, $\text{Count}(\mathcal{G}, \varepsilon) \in (1 \pm \varepsilon)Z(\mathcal{G})$ as required.

It remains to show that, in step (vi), \mathcal{G}_{in} and \mathcal{G}_{out} can be constructed in time $\text{poly}(n)$. Since $|S| = 2$, $\Gamma_{G-z}(X^+) = \{x\}$, so $|\Gamma_{G-z}(X^+)| = 1$. Moreover, $G[X^+] - v - \Gamma_G(v) = T_1 \cup \dots \cup T_\ell \cup (P - z)$ is a forest, so $G[X^+]$ is a near-forest. Thus by Corollary 32, $\text{Prune}(G - z, X^+)$ can be computed in time $2^{\Delta(G)} \text{poly}(n) = \text{poly}(n)$, so \mathcal{G}_{out} can be computed in time $\text{poly}(n)$ by Observation 36. Similarly, $G[X^+ \setminus \Gamma_G(z)]$ is a near-forest with $\Gamma_{G-z-\Gamma_G(z)}(X^+ \setminus \Gamma_G(z)) \subseteq \{x\}$, so \mathcal{G}_{in} can also be computed in time $\text{poly}(n)$. \square

To bound the running time of **Count**, we first show that the value of any good slice must decrease by a substantial amount whenever **Count** makes recursive calls. We will accomplish this in Lemma 48. (Recall from Definition 27 that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a good slice if it is of the form $f(m, n) = \rho m + \sigma n$ for some $\rho \geq 0$ and $\rho \geq -\sigma$.) We will use the following lemma for the case where **Count** executes step (v).

Lemma 42. *Let \mathcal{G} be a weighted graph whose underlying graph G is connected and reduced. Let $v \in V(G)$, and suppose $f(m, n) = \rho m + \sigma n$ is a good slice. Let $\mathcal{G}_{\text{out}} = \text{TR}(\mathcal{G} - v)$ and $\mathcal{G}_{\text{in}} = \text{TR}(\mathcal{G} - v - \Gamma_G(v))$, and consider the standard decomposition of G from v . Then*

$$\begin{aligned} f(\mathcal{G}) - f(\mathcal{G}_{\text{out}}) &= \rho d_G(v) + \sigma, \\ f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) &\geq \rho \left\lceil \frac{d_G^2(v) + d_G(v) + |S|}{2} \right\rceil + \sigma(1 + d_G(v)). \end{aligned}$$

Furthermore, if G is bipartite, then

$$f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) \geq \begin{cases} \rho d_G^2(v) + \sigma(1 + d_G(v)) & \text{if } \sigma \geq 0, \\ \rho d_G^2(v) + \sigma \left\lceil \frac{d_G^2(v) + d_G(v) + 2 - |S|}{2} \right\rceil & \text{if } \sigma < 0. \end{cases}$$

Proof. Let $G_{\text{out}} = \text{TR}(G - v)$ be the underlying graph of \mathcal{G}_{out} , and let $G_{\text{in}} = \text{TR}(G - v - \Gamma_G(v))$ be the underlying graph of \mathcal{G}_{in} . Since G is reduced, whenever $Y \subseteq V(G)$ spans a tree we have $|\Gamma_G(Y)| \geq 2$; in particular, $G - v$ has no tree components, so $G_{\text{out}} = \text{TR}(G - v) = G - v$. It follows immediately that $f(G) - g(G_{\text{out}}) = \rho d_G(v) + \sigma$ as required.

Observe that by the definition of the standard decomposition, we have

$$G_{\text{in}} = G - v - \Gamma_G(v) - V(T_1) - \dots - V(T_\ell).$$

Hence, writing m_1 for the number of edges in $G[\Gamma_G(v)]$, and writing m_2 for the number of edges between $\Gamma_G(v)$ and $\Gamma_G^2(v)$, we have

$$f(G) - f(G_{\text{in}}) = \rho(d_G(v) + m_1 + m_2) + \sigma(1 + d_G(v)) + \sum_{i=1}^{\ell} f(T_i).$$

For all i , since f is a good slice and T_i is a tree, we have $f(T_i) = \rho(|V(T_i)| - 1) + \sigma|V(T_i)| \geq \sigma$. Moreover, $d_G^2(v) = d_G(v) + 2m_1 + m_2$. It follows that

$$f(G) - f(G_{\text{in}}) \geq \rho(d_G^2(v) - m_1) + \sigma(1 + d_G(v) + \ell). \quad (8)$$

G is reduced, so $|\Gamma_G(V(T_i))| \geq 2$ for all $i \in [\ell]$, and each T_i sends at least two edges to $\Gamma_G(v)$. Each vertex in S also sends at least one edge to $\Gamma_G(v)$, so

$$m_2 \geq 2\ell + |S|. \quad (9)$$

We now prove our first inequality. By (9),

$$m_1 = \frac{d_G^2(v) - d_G(v) - m_2}{2} \leq \frac{d_G^2(v) - d_G(v) - |S|}{2} - \ell \Rightarrow m_1 \leq \left\lfloor \frac{d_G^2(v) - d_G(v) - |S|}{2} \right\rfloor - \ell.$$

It follows from (8) that

$$f(G) - f(G_{\text{in}}) \geq \rho \left(\left\lceil \frac{d_G^2(v) + d_G(v) + |S|}{2} \right\rceil + \ell \right) + \sigma(1 + d_G(v) + \ell).$$

Since f is a good slice, the result now follows, as $\rho\ell + \sigma\ell \geq 0$.

Finally, we prove our second inequality. Suppose that G is bipartite, which implies that $m_1 = 0$. If $\sigma \geq 0$, then the result follows immediately from (8), so suppose $\sigma < 0$. Then by (9), we have

$$\ell \leq \frac{m_2 - |S|}{2} = \frac{d_G^2(v) - d_G(v) - |S|}{2} \Rightarrow \ell \leq \left\lfloor \frac{d_G^2(v) - d_G(v) - |S|}{2} \right\rfloor.$$

It therefore follows from (8) that

$$f(G) - f(G_{\text{in}}) \geq \rho d_G^2(v) + \sigma \left\lfloor \frac{d_G^2(v) + d_G(v) + 2 - |S|}{2} \right\rfloor,$$

so the result follows. \square

It is more difficult to analyse **Count** in the case where step (vi) is executed, so we require the following ancillary lemma before proving our bound (which appears as Lemma 44).

Lemma 43. *Let H be a graph, let $z \in V(H)$ with $d_H(z) \geq 2$, and suppose $H - z$ has no tree components. Let m_1 be the number of edges in $H[\Gamma_H(z)]$, let m_2 be the number of edges in H between $\Gamma_H(z)$ and $\Gamma_H^2(z)$, and let r be the number of tree components of $H - z - \Gamma_H(z)$. Then $m_1 + m_2 \geq r + 2$.*

Proof. We may assume that H is connected; if not, work within the component that contains z , noting that none of the discarded components are trees. Let $H^- = H - z - \Gamma_H(z)$. Let the tree components of H^- be T_1, \dots, T_r . Each component of H^- sends at least one edge to $\Gamma_H(z)$ in H , so H^- has at most m_2 components and, in particular, $m_2 \geq r$. If $m_2 \geq r + 2$ then $m_1 + m_2 \geq r + 2$, so we are done. Two cases remain.

Case 1: $m_2 = r$. Suppose $m_2 = r$. If $m_1 \geq 2$, then we are done. Otherwise, $H[\Gamma_H(z)]$ is a forest. Each T_i sends exactly one edge to $\Gamma_H(z)$ in H , so $H - z$ is a forest, which is a contradiction.

Case 2: $m_2 = r + 1$. Suppose $m_2 = r + 1$. If $m_1 \geq 1$, we are done, so suppose that $m_1 = 0$.

First, suppose that the only components of H^- are the trees T_1, \dots, T_r . Some T_i receives exactly two edges, e_1 and e_2 , from $\Gamma_H(z)$ in H ; the others receive exactly one edge each. The graph $H - z - e_1$ is a forest: it consists of $\Gamma_H(z)$, which is an independent set because $m_1 = 0$, plus the trees T_1, \dots, T_r , plus one edge from each T_j to $\Gamma_H(z)$. It contains at least two components, because $d_H(z) \geq 2$. Adding e_1 to this graph gives $H - z$, which must contain at least one tree component, contradicting the hypothesis.

Otherwise, H^- has exactly $r + 1$ components: the trees T_1, \dots, T_r and one non-tree C . Each of these receives exactly one edge from $\Gamma_H(z)$ in H . Since $d_H(z) \geq 2$, there is some $y \in \Gamma_H(z)$ that is not adjacent to C in H . The component of $H - z$ that contains y is a tree, contradicting the hypothesis. \square

Lemma 44. *Let \mathcal{G} be a weighted graph whose underlying graph G is connected and reduced. Let $v \in V(G)$, and suppose $f(m, n) = \rho m + \sigma n$ is a good slice. Consider the standard decomposition of G from v , and suppose $|S| = 2$. Consider the extended decomposition of G from v , and let*

$$\begin{aligned}\mathcal{G}_{\text{out}} &= \text{TR}(\text{Prune}(G - z, X^+)), \\ \mathcal{G}_{\text{in}} &= \text{TR}(\text{Prune}(G - z - \Gamma_G(z), X^+ \setminus \Gamma_G(z))).\end{aligned}$$

Then

$$\begin{aligned}f(\mathcal{G}) - f(\mathcal{G}_{\text{out}}) &\geq \rho(1 + d_G(v)) + \sigma, \\ f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) &\geq \rho \left\lceil \frac{d_G^2(v) + d_G(v) + 4}{2} \right\rceil + \sigma(1 + d_G(v)).\end{aligned}$$

Furthermore, if G is bipartite, then

$$f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) \geq \begin{cases} \rho(1 + d_G^2(v)) + \sigma(1 + d_G(v)) & \text{if } \sigma \geq 0, \\ \rho d_G^2(v) + \sigma \left\lceil \frac{d_G^2(v) + d_G(v) - 2}{2} \right\rceil & \text{if } \sigma < 0. \end{cases}$$

Proof. Let $G_{\text{out}} = \text{TR}(G - z - X^+)$ be the underlying graph of \mathcal{G}_{out} , and let $G_{\text{in}} = \text{TR}(G - z - \Gamma_G(z) - X^+)$ be the underlying graph of \mathcal{G}_{in} . We first show that $H - z$ has no tree components. Indeed, suppose for a contradiction that C is a tree component of $H - z$. Since

G is reduced, we have $|\Gamma_G(V(C))| \geq 2$. Since $\Gamma_H(V(C)) \subseteq \{z\}$, it follows that C sends an edge into X^+ , so $x \in V(C)$. Then $C^+ := G[X^+ \cup V(C)]$ is a component of $G - z$, and $C^+ - v - \Gamma_G(v) = T_1 \cup \dots \cup T_\ell \cup (P - z) \cup C$ is a forest, so C^+ is a near-forest. Moreover, $\Gamma_G(V(C^+)) = \{z\}$; this contradicts the fact that G is reduced. Thus $H - z$ has no tree components, so

$$G_{\text{out}} = \text{TR}(H - z) = H - z.$$

By Lemma 40, we have $d_H(z) \geq 2$. Since f is a good slice, it follows that

$$f(G_{\text{out}}) = f(H) - \rho d_H(z) - \sigma \leq f(H) - \rho. \quad (10)$$

Writing m for the number of edges between X^+ and $V(H)$, we have

$$\begin{aligned} f(G) - f(H) &= (f(G) - f(G - v)) + (f(G - v) - f(H)) \\ &= (\rho d_G(v) + \sigma) + (f(G[X^+] - v) + \rho m). \end{aligned} \quad (11)$$

Let D_1, \dots, D_R be the tree components of $G[X^+] - v$, and let D'_1, \dots, D'_s be the non-tree components. Since f is a good slice, we have

$$f(G[X^+] - v) = \sum_{i=1}^R f(D_i) + \sum_{i=1}^s f(D'_i) \geq \sigma R. \quad (12)$$

Since G is reduced, $G - v$ has no tree components, so each of D_1, \dots, D_R must be joined to H by at least one edge. Thus $m \geq R$, so (11) and (12) imply

$$f(G) - f(H) \geq \rho d_G(v) + \sigma + \sigma R + \rho m \geq \rho d_G(v) + \sigma.$$

The claimed bound on $f(\mathcal{G}_{\text{out}})$ therefore follows from (10).

Now let r be the number of tree components of $H - z - \Gamma_H(z)$, let m_1 be the number of edges internal to $\Gamma_H(z)$ in H , and let m_2 be the number of edges between $\Gamma_H(z)$ and $\Gamma_H^2(z)$ in H . Since f is a good slice, we have

$$\begin{aligned} f(G_{\text{in}}) &= f(\text{TR}(H - z - \Gamma_H(z))) \leq f(H - z - \Gamma_H(z)) - \sigma r \\ &= f(H) - \rho(d_H(z) + m_1 + m_2) - \sigma(1 + d_H(z) + r). \end{aligned}$$

Recall that we have already shown that $H - z$ has no tree components and $d_H(z) \geq 2$; thus by Lemma 43 applied to H and z , we have $m_1 + m_2 \geq r + 2$. Since f is a good slice, it follows that

$$f(G_{\text{in}}) \leq f(H) - \rho. \quad (13)$$

Observe that by Lemma 40, H is formed from $G - X$ by removing a (possibly empty) induced path $P - z$. Since f is a good slice, it follows that $f(H) \leq f(G - X) = f(\text{TR}(G - v - \Gamma_G(v)))$. Since $|S| = 2$, it follows by Lemma 42 that

$$f(G) - f(H) \geq \rho \left\lceil \frac{d_G^2(v) + d_G(v) + 2}{2} \right\rceil + \sigma(1 + d_G(v))$$

in all cases and, if G is bipartite, then

$$f(G) - f(H) \geq \begin{cases} \rho d_G^2(v) + \sigma(1 + d_G(v)) & \text{if } \sigma \geq 0, \\ \rho d_G^2(v) + \sigma \left\lfloor \frac{d_G^2(v) + d_G(v)}{2} \right\rfloor & \text{if } \sigma < 0. \end{cases}$$

Thus by (13) and the fact that f is a good slice, we obtain the required bounds on $f(\mathcal{G}_{\text{in}})$. \square

To analyse the running time of **Count**, we will incorporate good slices into a piecewise-linear potential function of the following form.

Definition 45. Let s be a positive integer, let $-1 = k_0 < k_1 < \dots < k_s = \infty$, and let $f_1, \dots, f_s: \mathbb{R}^2 \rightarrow \mathbb{R}$ be linear functions. Then the *pre-potential* with *boundary points* k_0, \dots, k_s and *slices* f_1, \dots, f_s is the function $f: \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}$ defined as follows. If $n = 0$, then $f(m, n) = 0$. Otherwise, $f(m, n) = f_i(m, n)$ whenever $k_{i-1} < 2m/n \leq k_i$. We say f is a *valid pre-potential* if it satisfies the following properties.

- (VP1) f_1, \dots, f_s are good slices. That is, $f_i(m, n) = \rho_i m + \sigma_i n$ where $\rho_i \geq 0$, $\sigma_i + \rho_i \geq 0$, and it is not the case that $\sigma_i = \rho_i = 0$.
- (VP2) $f_s(m, n) = \sigma_s n$ for some $\sigma_s > 0$.
- (VP3) Every integer in $[6, k_{s-1}]$ is a boundary point.
- (VP4) For all $n > 0$ and $m \geq 0$, $f(m, n) = \min_{j \in [s]} f_j(m, n)$.

The *potential* corresponding to f is the map f^+ on the class of all graphs given by

$$f^+(G) = \begin{cases} f_s(|E(G)|, |V(G)|) & \text{if } \Delta(G) \geq 11, \\ f(|E(G)|, |V(G)|) & \text{otherwise.} \end{cases}$$

If $\mathcal{G} = (G, w_+, w_-, W)$ is a weighted graph, we write $f^+(\mathcal{G}) = f^+(G)$. We say f^+ is a *valid potential* if it is constructed this way from a valid pre-potential f .

We now set out bounds on the behaviour of a valid potential function as the algorithm **Count** from Section 6.5 branches. Suppose G is a reduced graph with average degree in $(k_i, k_{i+1}]$ for some $i \in \{0, \dots, s-1\}$, and suppose that $\Delta(G) < 11$ and G contains at least one vertex of degree at least 6 and 2-degree at least 27 so that v is defined in step (iv) of **Count**. Then we have two cases:

- If G has average degree at most 5, then we will have $d_G^2(v) \geq 27$.
- Otherwise, v is a vertex which maximises $d_G^2(v)$ subject to $d_G(v) \geq 2|E(G)|/|V(G)| \in (\max\{k_i, 5\}, k_{i+1}]$. Since G is reduced it has minimum degree at least 2, so $d_G^2(v) \geq 2d_G(v) \geq 2k_i$. Moreover, by (VP3), we have $k_{i+1} \leq \lfloor \max\{k_i, 5\} \rfloor + 1$; thus by Lemma 8 applied with $k = \max\{k_i, 5\}$, it follows that G contains a vertex with degree at least $2|E(G)|/|V(G)|$ and 2-degree at least $D_2(k_i)$, and so $d_G^2(v) \geq D_2(k_i)$.

We therefore define the following quantity.

Definition 46. For all $k < 5$, let $D'_2(k) = 27$. For all $k \geq 5$, let $D'_2(k) = \max\{2k, D_2(k)\}$.

Observation 47. Suppose that k_0, \dots, k_s are the boundary points of a valid pre-potential f . In the execution of **Count**(\mathcal{G}, ε), if \mathcal{G} has average degree in $(k_i, k_{i+1}]$ for some $i \in \{0, \dots, s-1\}$ and step (iv) is executed, then $d_G^2(v) \geq D'_2(k_i)$.

Lemma 48. Let f be a valid pre-potential with boundary points k_0, \dots, k_s and slices f_1, \dots, f_s where, for each $i \in \{1, \dots, s\}$, $f_i = \rho_i m + \sigma_i n$. Let $0 < \varepsilon < 1$, let G_0 be a graph, and let $\mathcal{G}_0 = \text{TR}(G_0, \mathbf{1}, \mathbf{1}, 1)$. Suppose that \mathcal{G}_0 is non-empty. Then when **Count**($\mathcal{G}_0, \varepsilon$) is executed, the following properties hold for every recursive call **Count**(\mathcal{G}, ε), where $\mathcal{G} = (G, w_+, w_-, W)$ and $n = |V(G)|$.

- (i) $f^+(\mathcal{G}) \geq 0$, with equality only if $\text{Count}(\mathcal{G}, \varepsilon)$ halts in time $\text{poly}(n, 1/\varepsilon)$ with no further recursive calls.
- (ii) If $\text{Count}(\mathcal{G}, \varepsilon)$ makes recursive calls to $\text{Count}(\mathcal{G}_{\text{out}}, \varepsilon)$ and $\text{Count}(\mathcal{G}_{\text{in}}, \varepsilon)$, then $f^+(\mathcal{G}_{\text{out}}) < f^+(\mathcal{G})$ and $f^+(\mathcal{G}_{\text{in}}) < f^+(\mathcal{G})$. Moreover, the following stronger bounds hold. If $\Delta(G) \geq 11$, then

$$\begin{aligned} f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{out}}) &\geq f_s(0, 1), \\ f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) &\geq f_s(0, 12). \end{aligned}$$

Otherwise, there exist $j \in [s]$ and a positive integer x satisfying $\max\{6, \lfloor k_{j-1} \rfloor + 1\} \leq x \leq 10$ such that

$$\begin{aligned} f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{out}}) &\geq f_j(x, 1), \\ f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) &\geq f_j\left(\left\lceil \frac{x + D'_2(k_{j-1}) + 3}{2} \right\rceil, 1 + x\right), \end{aligned}$$

and, if G_0 is bipartite, then

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \begin{cases} f_j(D'_2(k_{j-1}), 1 + x) & \text{if } \sigma_j \geq 0, \\ f_j(D'_2(k_{j-1}), \lfloor \frac{x + D'_2(k_{j-1}) - 1}{2} \rfloor) & \text{if } \sigma_j < 0. \end{cases}$$

Proof. Property (i): Since G has no tree components and is non-empty, we have $|E(G)| \geq |V(G)| > 0$; thus by (VP1) we have $f_i(G) \geq 0$ for all i , so $f^+(\mathcal{G}) \geq 0$. Suppose $f^+(\mathcal{G}) = 0$. By (VP2), it follows that $\Delta(G) \leq 10$, so step (ii) is executed. If $\text{Reduce}(\mathcal{G})$ is empty then no recursive call is made, so the algorithm therefore halts in time $\text{poly}(n, 1/\varepsilon)$. Assume that $\text{Reduce}(\mathcal{G})$ is non-empty. Let i be such that $f^+(\mathcal{G}) = f_i(\mathcal{G})$; then by (VP4) and Lemma 34, we have

$$f^+(\text{Reduce}(\mathcal{G})) \leq f_i(\text{Reduce}(\mathcal{G})) \leq f_i(\mathcal{G}) = f^+(\mathcal{G}) = 0. \quad (14)$$

Moreover, $\text{Reduce}(\mathcal{G})$ is reduced by Lemma 34, so it has no tree components; it follows that $f^+(\text{Reduce}(\mathcal{G})) = 0$. By (VP1), it follows that $\text{Reduce}(\mathcal{G})$ has the same number of edges as vertices. Again since $\text{Reduce}(\mathcal{G})$ is reduced, it has no vertices of degree less than 2, so it must be 2-regular. But any such graph is a disjoint union of cycles, and cycles are near-forests, contradicting the fact that $\text{Reduce}(\mathcal{G})$ is reduced.

Property (ii): Note that $D'_2(x) \geq 2x$ for all $x > 0$, so by (VP1) and (VP2) the bounds $f^+(\mathcal{G}_{\text{in}}) < f^+(\mathcal{G})$ and $f^+(\mathcal{G}_{\text{out}}) < f^+(\mathcal{G})$ are indeed implied by the subsequently stated bounds. We split into cases depending on where \mathcal{G}_{in} and \mathcal{G}_{out} are defined.

Case 1: Step (i) is executed. In this case, $\Delta(G) \geq 11$. Recursive calls are made only if \mathcal{G}_{out} and \mathcal{G}_{in} , respectively, are non-empty. By (VP4), and (VP2),

$$\begin{aligned} f^+(\mathcal{G}_{\text{out}}) &\leq f_s(\mathcal{G}_{\text{out}}) \leq f_s(G - v) = f_s(G) - \sigma_s, \\ f^+(\mathcal{G}_{\text{in}}) &\leq f_s(\mathcal{G}_{\text{in}}) \leq f_s(G - v - \Gamma_G(v)) = f_s(G) - (1 + d_G(v))\sigma_s \leq f_s(G) - 12\sigma_s, \end{aligned}$$

as required.

Case 2: Step (v) is executed. Write $\mathcal{G}' = \text{Reduce}(\mathcal{G})$. Assume that \mathcal{G}' is non-empty (otherwise, there are no recursive calls). \mathcal{G}_0 is 1-balanced by definition, so \mathcal{G} is 1-balanced and has no

tree components by Lemma 41(iii). Moreover, G has maximum degree at most 10 since step (v) is executed. Thus by Lemma 34, $\Delta(\mathcal{G}') \leq \Delta(\mathcal{G}) \leq 10$. Since \mathcal{G} has no tree components, as in (14) we have $f^+(\mathcal{G}') \leq f^+(\mathcal{G})$.

Let $j \in [s]$ be such that $k_{j-1} < 2|E(\mathcal{G}')|/|V(\mathcal{G}')| \leq k_j$, so that $f^+(\mathcal{G}') = f_j(\mathcal{G}')$. Write G_v for the component of \mathcal{G}' containing v , as in $\text{Count}(\mathcal{G}, \varepsilon)$. Then by (VP4), we have

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq f^+(\mathcal{G}') - f_j(\mathcal{G}_{\text{in}}) = f_j(\mathcal{G}') - f_j(\text{TR}(\mathcal{G}' - v - \Gamma_{G_v}(v))).$$

By Lemma 34, \mathcal{G}' is reduced and therefore has no tree components. Thus by (VP1),

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq f_j(G_v) - f_j(\text{TR}(G_v - v - \Gamma_G(v))).$$

Again by (VP1), it follows by Lemma 42 (applied with $G = G_v$ and $f = f_j$) that

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \rho_j \left\lceil \frac{d_{G_v}^2(v) + d_{G_v}(v) + |S|}{2} \right\rceil + \sigma_j(1 + d_{G_v}(v)), \quad (15)$$

and, if G_v is bipartite, then

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \begin{cases} \rho_j d_{G_v}^2(v) + \sigma_j(1 + d_{G_v}(v)) & \text{if } \sigma_j \geq 0, \\ \rho_j d_{G_v}^2(v) + \sigma_j \left\lfloor \frac{d_{G_v}^2(v) + d_{G_v}(v) + 2 - |S|}{2} \right\rfloor & \text{if } \sigma_j < 0. \end{cases} \quad (16)$$

\mathcal{G}' has average degree in $(k_{j-1}, k_j]$ by the definition of j , so $d_{G_v}^2(v) \geq D'_2(k_{j-1})$ by Observation 47. Moreover, since f_j is a good slice, the right-hand sides of (15) and (16) are increasing functions of $d_{G_v}^2(v)$. We may therefore replace $d_{G_v}^2(v)$ by $D'_2(k_{j-1})$. Moreover, since step (v) is executed, we must have $|S| \geq 3$ in the standard decomposition of G_v from v . Thus (15) and (16) give

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \rho_j \left\lceil \frac{D'_2(k_{j-1}) + d_{G_v}(v) + 3}{2} \right\rceil + \sigma_j(1 + d_{G_v}(v)). \quad (17)$$

If G_0 is bipartite, then so is G_v . Therefore, when G_0 is bipartite,

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \begin{cases} \rho_j D'_2(k_{j-1}) + \sigma_j(1 + d_{G_v}(v)) & \text{if } \sigma_j \geq 0, \\ \rho_j D'_2(k_{j-1}) + \sigma_j \left\lfloor \frac{D'_2(k_{j-1}) + d_{G_v}(v) - 1}{2} \right\rfloor & \text{if } \sigma_j < 0. \end{cases} \quad (18)$$

By our choice of v in step (iv) of **Count**, we have $d_{G_v}(v) \geq \max\{6, \lfloor k_{j-1} \rfloor + 1\}$, and since step (i) was not executed, we have $d_{G_v}(v) \leq 10$. Thus the lemma is satisfied on taking $x = d_{G_v}(v)$.

Similarly, by (VP1), (VP4) and Lemma 42 we have

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{out}}) \geq f_j(G_v) - f_j(\text{TR}(G_v - v)) \geq \rho_j d_{G_v}(v) + \sigma_j.$$

As above, the lemma follows on taking $x = d_{G_v}(v)$.

Case 3: Step (vi) is executed. Exactly as in Case 2, write $\mathcal{G}' = \text{Reduce}(\mathcal{G})$ and take $j \in [s]$ such that $f^+(\mathcal{G}) = f_j(\mathcal{G})$. By (VP4), (VP1), and Lemma 44 applied to G_v ,

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \rho_j \left\lceil \frac{d_{G_v}^2(v) + d_{G_v}(v) + 4}{2} \right\rceil + \sigma_j(1 + d_{G_v}(v)),$$

and, for bipartite G_v ,

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \begin{cases} \rho_j(1 + d_{G_v}^2(v)) + \sigma_j(1 + d_{G_v}(v)) & \text{if } \sigma_j \geq 0, \\ \rho_j d_{G_v}^2(v) + \sigma_j \left\lfloor \frac{d_{G_v}^2(v) + d_{G_v}(v) - 2}{2} \right\rfloor & \text{if } \sigma_j < 0. \end{cases}$$

As before, the right-hand sides are increasing functions of $d_{G_v}^2(v)$, since f_j is a good slice. Observation 47, as in Case 2 it follows that

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \rho_j \left\lceil \frac{D'_2(k_{j-1}) + d_{G_v}(v) + 4}{2} \right\rceil + \sigma_j(1 + d_{G_v}(v))$$

and, for bipartite G_0 ,

$$f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{in}}) \geq \begin{cases} \rho_j(1 + D'_2(k_{j-1})) + \sigma_j(1 + d_{G_v}(v)) & \text{if } \sigma_j \geq 0, \\ \rho_j D'_2(k_{j-1}) + \sigma_j \left\lfloor \frac{D'_2(k_{j-1}) + d_{G_v}(v) - 2}{2} \right\rfloor & \text{if } \sigma_j < 0. \end{cases}$$

Note that these bounds are stronger than (17) and (18), respectively, so the result follows as in Case 2. Similarly, by (VP1), Lemma 44 and Observation 47 we have $f^+(\mathcal{G}) - f^+(\mathcal{G}_{\text{out}}) > \rho_j d_{G_v}(v) + \sigma_j$. As in Case 2, these bounds are of the required form on taking $x = d_{G_v}(v)$. \square

6.6 Analysis of recursion

We will use the theory of branching factors to bound the running time of **Count**; see e.g. [7, Chapter 2.1] for an overview. We briefly recall the salient points. For all integers $b \geq 2$ and all $a_1, \dots, a_b \geq 0$, we write $\tau(a_1, \dots, a_b)$ for the unique positive solution in x of $\sum_{i=1}^b x^{-a_i} = 1$. Observe that $a_1, \dots, a_b > 0$ and $\tau(a_1, \dots, a_b) < c$ precisely when $c^{-a_1} + \dots + c^{-a_b} \leq 1$.

Suppose we have a non-negative potential function for instances of a problem, and wish to use it to bound a recursive algorithm's running time. Suppose the non-recursive part of the algorithm runs in polynomial time. Let $b, p_1, \dots, p_b \geq 1$ be such that the algorithm always recurses in one of b possible ways, and that the i 'th possible branch splits the instance into p_i parts. Suppose that if the original instance has potential x , then the j 'th part of the i 'th possible branch always has potential at most $x - y_{i,j}$ for some $y_{i,j} > 0$. Finally, suppose that our potential is only zero on instances which the algorithm solves without recursing further. Under these circumstances, the following recurrence holds for the worst-case running time $T(x)$ of a potential- x instance:

$$T(x) \leq \begin{cases} \max\{\sum_{j=1}^{p_i} T(x - y_{i,j}) : i \in [b]\} + \text{poly}(n) & \text{if } x > 0, \\ \text{poly}(n) & \text{otherwise.} \end{cases} \quad (19)$$

The solution to this recurrence is

$$T(x) = O^*(\max_i \{\tau(y_{i,1}, \dots, y_{i,p_i})\}^x). \quad (20)$$

We now use (20) and Lemma 48 to obtain expressions for the running time of **Count** in terms of valid pre-potential functions. We will then set out our choices of these functions, and prove their validity, in Section 7.

Corollary 49. *Let G be an n -vertex graph which is not a forest. Let f be a valid pre-potential with boundary points k_0, \dots, k_s and good slices f_1, \dots, f_s , writing $f_i(m, n) = \rho_i m + \sigma_i n$. For all $i \in [s]$ and all $d \geq 6$, let*

$$T_{i,d} = \tau\left(f_i(d, 1), f_i\left(\left\lceil \frac{d + D'_2(k_{i-1}) + 3}{2} \right\rceil, 1 + d\right)\right),$$

and

$$T = \max\left(\{T_{i,d} \mid i \in [s], \max\{6, \lfloor k_{i-1} \rfloor + 1\} \leq d \leq 10\} \cup \{\tau(\sigma_s, 12\sigma_s)\}\right).$$

If $T \leq 2$, then $\text{Count}(\text{TR}(G, \mathbf{1}, \mathbf{1}, 1), \varepsilon)$ has running time $O(2^{\sigma_s n}) \text{poly}(n, 1/\varepsilon)$.

Proof. Let $\mathcal{G}_0 = \text{TR}(G, \mathbf{1}, \mathbf{1}, \varepsilon)$. Since G is not a forest, \mathcal{G}_0 is non-empty. For all $x \geq 0$, let $R_\varepsilon(x)$ be the worst-case running time of $\text{Count}(\mathcal{G}, \varepsilon)$ on any weighted graph \mathcal{G} with $f^+(\mathcal{G}) \leq x$ which arises as a recursive call in the evaluation of $\text{Count}(\mathcal{G}_0, \varepsilon)$. For each $i \in [s]$, let $z_i = \max\{6, \lfloor k_{i-1} \rfloor + 1\}$. If recursive calls are made to $\text{Count}(\mathcal{G}_{\text{out}}, \varepsilon)$ and $\text{Count}(\mathcal{G}_{\text{in}}, \varepsilon)$ then, by Lemma 48(ii), either

$$\begin{aligned} f^+(\mathcal{G}_{\text{out}}) &\leq f^+(\mathcal{G}) - f_s(0, 1) < f^+(\mathcal{G}), \\ f^+(\mathcal{G}_{\text{in}}) &\leq f^+(\mathcal{G}) - f_s(0, 12) < f^+(\mathcal{G}), \end{aligned}$$

or there exist $i \in [s]$ and an integer d with $z_i \leq d \leq 10$ such that

$$\begin{aligned} f^+(\mathcal{G}_{\text{out}}) &\leq f^+(\mathcal{G}) - f_i(d, 1) < f^+(\mathcal{G}), \\ f^+(\mathcal{G}_{\text{in}}) &\leq f^+(\mathcal{G}) - f_i\left(\left\lceil \frac{d + D'_2(k_{i-1}) + 3}{2} \right\rceil, 1 + d\right) < f^+(\mathcal{G}). \end{aligned}$$

(If, e.g., only $\text{Count}(\mathcal{G}_{\text{out}}, \varepsilon)$ is called, then the bounds on $f^+(\mathcal{G}_{\text{out}})$ still hold.)

It follows by the above equations and Lemma 48(i) that the conditions of (19) are satisfied by taking our potential to be f^+ , the $y_{i,j}$'s to be the right-hand sides of the above equations, and the p_i 's to be the number of recursive calls in the i 'th possible branch (which is at most 2). For brevity, let $t_{i,d,\text{out}} = f_i(d, 1)$, and $t_{i,d,\text{in}} = f_i(\lceil (d + D'_2(k_{i-1}) + 3)/2 \rceil, 1 + d)$. Thus (19) implies the following recurrence for $R_\varepsilon(x)$:

$$R_\varepsilon(x) \leq \begin{cases} \max\left(\{R_\varepsilon(x - f_s(0, 1)) + R_\varepsilon(x - f_s(0, 12))\} \cup \{R_\varepsilon(x - t_{i,d,\text{out}}) + R_\varepsilon(x - t_{i,d,\text{in}}) : i \in [s], z_i \leq d \leq 10\}\right) & \text{if } x > 0, \\ \text{poly}(n, 1/\varepsilon) & \text{otherwise.} \end{cases}$$

It follows by (20) that $R_\varepsilon(x) = O(T^x) \text{poly}(n, 1/\varepsilon)$. We have $T \leq 2$ by hypothesis, so $R_\varepsilon(x) = O(2^x) \text{poly}(n, 1/\varepsilon)$.

By Observation 36, calculating \mathcal{G}_0 from G_0 takes time $\text{poly}(n)$, so our overall running time is $O(2^{f^+(G_0)}) \text{poly}(n, 1/\varepsilon)$. Moreover, by (VP4) and (VP2), we have $f^+(\mathcal{G}_0) \leq f_s(\mathcal{G}_0) = \sigma_s n$. The result therefore follows. \square

We obtain an analogous result for bipartite graphs. The proof is identical, except for using the bounds from the bipartite case of Lemma 48(ii).

Corollary 50. *Let G be an n -vertex bipartite graph which is not a forest. Let f be a valid pre-potential with boundary points k_0, \dots, k_s and good slices f_1, \dots, f_s , writing $f_i(m, n) = \rho_i m + \sigma_i n$. For all $i \in [s]$ and all $d \geq 6$, let*

$$T_{i,d} = \begin{cases} \tau(f_i(d, 1), f_i(D'_2(k_{i-1}), 1+d)) & \text{if } \sigma_i \geq 0, \\ \tau\left(f_i(d, 1), f_i\left(D'_2(k_{i-1}), \left\lfloor \frac{d+D'_2(k_{i-1})-1}{2} \right\rfloor\right)\right) & \text{otherwise,} \end{cases}$$

and

$$T = \max\left(\{T_{i,d} \mid i \in [s], \max\{6, \lfloor k_{i-1} \rfloor + 1\} \leq d \leq 10\} \cup \{\tau(\sigma_s, 12\sigma_s)\}\right).$$

If $T \leq 2$, then $\text{Count}(\text{TR}(G, \mathbf{1}, \mathbf{1}, 1), \varepsilon)$ has running time $O(2^{\sigma_s n}) \text{poly}(n, 1/\varepsilon)$.

7 A valid pre-potential

We include two valid pre-potentials, both in approximate form as Appendices E and F and in exact form as ancillary files bi-potential.csv and potential.csv (available on the arXiv). We use these, along with Corollaries 49 and 50, to bound the running time of $\text{Count}(\mathcal{G}, \varepsilon)$. To prove that they are valid, we will use the following lemma. (Note that property (iv) is equivalent to requiring that f be continuous.)

Lemma 51. *Let f be a pre-potential with slices f_1, \dots, f_s and boundary points $-1 = k_0 < k_1 < \dots < k_s = \infty$, writing $f_i(m, n) = \rho_i m + \sigma_i n$ for all $i \in [s]$. Then f is a valid pre-potential if the following properties all hold.*

- (i) $\rho_s = 0$ and $\sigma_s > 0$.
- (ii) For all $i \in [s-1]$, $\rho_i + \sigma_i \geq 0$
- (iii) For all $i \in [s-1]$, at least one of ρ_i, σ_i is non-zero.
- (iv) For all $i \in [s-1]$, $\rho_i \geq \rho_{i+1}$ and $\sigma_i \leq \sigma_{i+1}$.
- (v) Every integer in $[6, k_{s-1}]$ is a boundary point.
- (vi) For all $i \in [s-1]$, $\rho_i = \rho_{i+1} + 2(\sigma_{i+1} - \sigma_i)/k_i$.

Proof. Property (VP1) of Definition 45 follows from properties (i), (ii), (iii) and (iv) of the lemma statement. Property (VP2) follows from property (i). Property (VP3) is property (v). To establish property (VP4), consider $i \in [s]$. Suppose that n is a positive integer and m is a non-negative integer satisfying $k_{i-1} < 2m/n \leq k_i$. By property (vi), for all $j > i$ we have

$$\rho_i - \rho_j = \sum_{\ell=i}^{j-1} (\rho_\ell - \rho_{\ell+1}) = \sum_{\ell=i}^{j-1} \frac{2}{k_\ell} (\sigma_{\ell+1} - \sigma_\ell) \leq \frac{2}{k_i} \sum_{\ell=i}^{j-1} (\sigma_{\ell+1} - \sigma_\ell) = \frac{2}{k_i} (\sigma_j - \sigma_i).$$

By property (iv) and the choice of n and m , it follows that

$$f_j(m, n) - f_i(m, n) = -(\rho_i - \rho_j)m + (\sigma_j - \sigma_i)n \geq (\sigma_j - \sigma_i) \left(n - \frac{2}{k_i}m\right) \geq 0.$$

Thus for all $j > i$, $f_j(m, n) \geq f_i(m, n)$. Similarly, by property (vi), for all $j < i$ we have

$$\rho_j - \rho_i = \sum_{\ell=j}^{i-1} (\rho_\ell - \rho_{\ell+1}) = \sum_{\ell=j}^{i-1} \frac{2}{k_\ell} (\sigma_{\ell+1} - \sigma_\ell) \geq \frac{2}{k_{i-1}} \sum_{\ell=j}^{i-1} (\sigma_{\ell+1} - \sigma_\ell) = \frac{2}{k_{i-1}} (\sigma_i - \sigma_j),$$

so property (iv) and the choice of n and m imply that for all $j < i$,

$$f_j(m, n) - f_i(m, n) = (\rho_j - \rho_i)m - (\sigma_i - \sigma_j)n \geq (\sigma_i - \sigma_j) \left(\frac{2}{k_{i-1}} m - n \right) \geq 0.$$

Thus for all $j < i$, $f_j(m, n) \leq f_i(m, n)$. We conclude that (VP4) holds as required. \square

8 Conclusion

We can now prove our main theorem.

Theorem 1. *There is an approximation scheme for #IS which, given an n -vertex input graph with error tolerance ε , runs in time $O(2^{0.268n}) \text{poly}(1/\varepsilon)$. There is an approximation scheme for #BIS with running time $O(2^{0.2372n}) \text{poly}(1/\varepsilon)$.*

Proof. Consider an input consisting of an n -vertex graph G and an error tolerance $\varepsilon \in (0, 1)$.

Let $\mathcal{G} = \text{TR}(G, \mathbf{1}, \mathbf{1}, 1)$. The approximation scheme computes \mathcal{G} and then runs **Count**(\mathcal{G}, ε). By Lemma 41(i), this outputs an ε -approximation of $Z(\mathcal{G})$, and by Observation 36 and the definition of \mathcal{G} we have $Z(G) = Z(\mathcal{G})$, so the algorithm outputs an approximation to $Z(G)$, as required. By Observation 36, we can compute \mathcal{G} in polynomial time, so it remains to bound the running time of **Count**.

Recall from Section 3 that the Mathematica function **Dtwo**[**k**] in Appendix D outputs $D_2(k)$ given a rational argument $k \geq 2$. Thus, if the input file **input** contains a potential function f in the specified CSV format, the function **Validate**[**input**] given in Appendix D first checks whether the conditions of Lemma 51 hold, and hence whether f is a valid pre-potential. If f is valid, then the run-time is $O(2^{\sigma_s n}) \text{poly}(n, 1/\varepsilon)$ by Corollary 49 for general graphs and Corollary 50 for bipartite graphs. For general graphs, $\sigma_s < 0.2680$, giving a running time of $O(2^{0.2680n}) \text{poly}(1/\varepsilon)$; for bipartite graphs, $\sigma_s < 0.2372$. As explained at the beginning of Appendix D, the code checks that the pre-conditions of these corollaries are satisfied and then outputs the relevant bounds. \square

Our general running time for approximate counting is $O(2^{0.2680n})$ times a polynomial in $1/\varepsilon$, which is better than the current best-known running time of $O(2^{0.3022n})$ for exact counting [11], and we have further improved on this when the input is bipartite. In the process of resolving our base case, we generalised an FPTAS of Sinclair et al. [18] for approximating the partition function of the hardcore model on graphs with low connective constant, so that it also works on graphs with arbitrary vertex weights. This may be useful in other contexts. For the recursive part of the algorithm, our approach followed the standard methods of the field, but uncovered interesting worst-case behaviour in bipartite graphs which required a more careful analysis to handle correctly.

References

- [1] C. Bron and J. Kerbosch. Algorithm 457: Finding all cliques in an undirected graph. *Comm. ACM*, 16:575–577, 1973. [1](#)
- [2] Andrei A. Bulatov, Martin E. Dyer, Leslie Ann Goldberg, Mark Jerrum, and Colin McQuillan. The expressibility of functions on the boolean domain, with applications to counting CSPs. *J. ACM*, 60(5):32:1–32:36, 2013. [2](#)
- [3] V. Dahllöf, P. Jonsson, and M. Wahlström. Counting satisfying assignments in 2-SAT and 3-SAT. In *Proc. Computing and Combinatorics, 8th Intl. Conference (COCOON 2002)*, pages 535–543. Springer, 2002. [1](#), [2](#)
- [4] V. Dahllöf, P. Jonsson, and M. Wahlström. Counting models for 2SAT and 3SAT formulae. *Theoretical Computer Science*, 332(1–3):265–291, 2005. [1](#), [2](#), [6](#), [7](#), [8](#), [21](#), [26](#), [45](#)
- [5] O. Dubois. Counting the number of solutions for instances of satisfiability. *Theoretical Computer Science*, 81:49–64, 1991. [1](#), [2](#)
- [6] Martin E. Dyer, Leslie Ann Goldberg, Catherine S. Greenhill, and Mark Jerrum. The relative complexity of approximate counting problems. *Algorithmica*, 38(3):471–500, 2004. [2](#)
- [7] Fedor V. Fomin and Dieter Kratsch. *Exact Exponential Algorithms*. Springer, 2010. [5](#), [37](#)
- [8] M. Fürer and S. P. Kasiviswanathan. Algorithms for counting 2-SAT solutions and colorings with applications. Technical Report 33, Electronic Colloquium on Computational Complexity, 2005. [1](#), [2](#), [7](#), [8](#)
- [9] Andreas Galanis, Leslie Ann Goldberg, and Mark Jerrum. Approximately counting h -colorings is #BIS-hard. *SIAM J. Comput.*, 45(3):680–711, 2016. [2](#)
- [10] Andreas Galanis, Daniel Stefankovic, Eric Vigoda, and Linji Yang. Ferromagnetic potts model: Refined #BIS-hardness and related results. *SIAM J. Comput.*, 45(6):2004–2065, 2016. [2](#)
- [11] S. Gaspers and E. J. Lee. Faster graph coloring in polynomial space. In *Proc. Computing and Combinatorics, 23rd Intl. Conference (COCOON 2017)*, pages 371–383. Springer, 2017. Full version: ArXiv CoRR abs/1607.06201. [1](#), [2](#), [40](#)
- [12] C. D. Godsil. Matchings and walks in graphs. *J. Graph Theory*, 5(3):285–297, 1981. [12](#)
- [13] Leslie Ann Goldberg and Mark Jerrum. Approximating the partition function of the ferromagnetic potts model. *J. ACM*, 59(5):25:1–25:31, 2012. [2](#)
- [14] M. Jenssen, P. Keevash, and W. Perkins. Algorithms for #BIS-hard problems on expander graphs. In *Proc. 30th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA 2019)*, pages 2235–2247. SIAM, 2019. [3](#)

- [15] K. Junosza-Szaniawski and M. Tuczynski. Counting independent sets via Divide Measure and Conquer method. CoRR abs/1503.08323, 2015. [1](#), [2](#)
- [16] J. Liu and P. Lu. FPTAS for #BIS with degree bounds on one side. In *Proc. 47th Annual ACM Symposium on Theory of Computing (STOC 2015)*, pages 549–556. ACM, 2015. [3](#)
- [17] J. Scott Provan and Michael O. Ball. The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM J. Comput.*, 12(4):777–788, 1983. [1](#)
- [18] A. Sinclair, P. Srivastava, D. Štefankovič, and Y. Yin. Spatial mixing and the connective constant: Optimal bounds. In *Proc. 26th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA 2015)*, pages 1549–1563. SIAM, 2015. Full version: ArXiv CoRR abs/1410.2595. [3](#), [6](#), [13](#), [40](#)
- [19] A. Sinclair, P. Srivastava, and Y. Yin. Spatial mixing and approximation algorithms for graphs with bounded connective constant. In *Proc. 54th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2013)*, pages 300–309. IEEE Computer Society, 2013. Full version: ArXiv CoRR abs/1308.1762. [3](#), [12](#)
- [20] A. Sly. Computational transition at the uniqueness threshold. In *Proc. 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS 2010)*, pages 287–296. ACM, 2010. Full version: ArXiv CoRR abs/1005.5584. [2](#), [3](#)
- [21] M. Wahlström. *Algorithms, Measures and Upper Bounds for Satisfiability and Related Problems*. PhD thesis, Linköpings universitet, 2007. [8](#)
- [22] M. Wahlström. A tighter bound for counting max-weight solutions to 2SAT instances. In *Proc. 3rd Intl. Conference on Parameterized and Exact Computation (IWPEC 2008)*, pages 202–213. Springer, 2008. [1](#), [2](#)
- [23] D. Weitz. Counting independent sets up to the tree threshold. In *Proc. 38th Annual ACM Symposium on Theory of Computing (STOC 2006)*, pages 140–149. ACM, 2006. [2](#), [3](#)
- [24] M. Xia, P. Zhang, and W. Zhao. Computational complexity of counting problems on 3-regular planar graphs. *Theoretical Computer Science*, 384:111–125, 2007. [2](#)
- [25] W. Zhang. Number of models and satisfiability of sets of clauses. *Theoretical Computer Science*, 155:277–288, 1996. [1](#), [2](#)

A Index of Notation and Terms

$[n]$	$\{1, \dots, n\}$	Sec. 2	p. 7
$\alpha(v), \beta(v)$	Used in associated average degree	Sec. 3	p. 8
$\Gamma_G(S)$	$\bigcup_{v \in S} \Gamma_G(v) \setminus S$	Sec. 2	p. 7
$\Gamma_G(v, X)$	$\{w \in X \mid \{v, w\} \in E(G)\}$	Sec. 2	p. 7
$\Gamma_G^2(v)$	Vertices at distance 2 from v	Sec. 2	p. 7
Γ_v	$\Gamma_G(v)$ in standard decomposition	Def. 37	p. 25
$(\Gamma_v, S, X, H_1, \dots, H_k; T_1, \dots, T_\ell)$	Standard decomposition of G w.r.t. vertex v	Def. 37	p. 25
$\delta(G), \Delta(G)$	Minimum and maximum degrees	Sec. 2	p. 7
κ	Connective constant	Def. 2	p. 3
λ	Parameter of univariate hard-core model	Sec. 4	p. 13
λ -balanced	$w_+(v) \leq \lambda w_-(v)$	Sec. 2	p. 7
λ_t	$\lambda(1 + \lambda)^{-t}$	Sec. 4	p. 15
ρ, σ	Coefficients of slice of a potential function	Def. 27	p. 21
$\tau(a_1, \dots, a_b)$	Positive root of $\sum_{i=1}^b x^{-a_i} = 1$	Sec. 6.6	p. 37
ϕ	Weight map for G	Def. 16	p. 13
$a_k(\mathbf{x})$	Used in numerical associated average degree	Def. 6	p. 9
$\text{aad}(v)$	Associated average degree	Sec. 3	p. 8
$\text{aad}_k^*(\mathbf{x})$	Numerical associated average degree	Def. 6	p. 9
$\text{ApproxZ}_{\lambda, \mathcal{F}}(\mathcal{G}, \varepsilon)$	Approximates $Z(\mathcal{G})$ when κ is subcritical	Sec. 4	p. 15
$b_k(\mathbf{x})$	Used in numerical associated average degree	Def. 6	p. 9
$\text{BaseCount}(\mathcal{G}, \varepsilon)$	FPTAS for base case	Sec. 5	p. 20
branch	Transform into smaller instances $\mathcal{G}_{\text{out}}, \mathcal{G}_{\text{in}}$	Sec. 1	p. 4
$C(v)$	Set of children of vertex v in a tree	Sec. 5	p. 19
$\text{Count}(\mathcal{G}, \varepsilon)$	Main approximation algorithm	Sec. 6.5	p. 28
$D(v)$	Depth of vertex v in a tree	Sec. 5	p. 19
\mathcal{D}	Base case graph family handled in Sec. 5	Def. 22	p. 18
$d_G(v)$	Degree of vertex v in graph G	Sec. 1	p. 4
$d_G(v, X)$	$ \Gamma_G(v, X) $	Sec. 2	p. 7
$d_G^2(v)$, 2-degree of v	$\sum_{\{v, w\} \in E(G)} d_G(w)$	Sec. 2	p. 7
$D_2(k)$	Function used to bound 2-degree	Def. 7	p. 9
$D'_2(k)$	Adjusted version of $D_2(k)$	Def. 46	p. 34
κ -decreasing	Used to bound connective constant	Def. 23	p. 19
$\text{ExactCount}(\mathcal{G}, Y)$	Exact computation when $\mathcal{G} - Y$ is a forest	Sec. 6	p. 22
f	Slice of a potential function	Def. 27	p. 21
f_1, \dots, f_s	Slices of a pre-potential	Def. 45	p. 34
f^+	Potential made from pre-potential f	Def. 45	p. 34
\mathcal{F}_y^+	Set of gadgets (realisations of weight maps)	Def. 17	p. 13

$\mathcal{G} = (G, w_+, w_-, W)$	Weighted graph	Sec. 2	p. 7
G'	Realisation of a weight map	Def. 16	p. 13
\mathcal{G}'	Reduce (\mathcal{G})	Lem. 48	p. 35
$G - X, \mathcal{G} - X$	Graph made by removing X	Sec. 2	p. 8
$\mathcal{G}[X]$	Induced weighted graph	Sec. 2	p. 8
good slice	A slice with $\rho \geq 0$, $\rho \geq -\sigma$ and $\rho\sigma \neq 0$	Def. 27	p. 21
H	$G - X^+$ in extended decomposition	Def. 39	p. 27
H_1, \dots, H_k	Non-tree components in standard decomposition	Def. 37	p. 25
$\mathcal{I}(G)$	Independent sets of G	Sec. 2	p. 8
k_1, \dots, k_s	Boundary points of a pre-potential	Def. 45	p. 34
$L(T)$	Set of tree T 's leaves	Sec. 5	p. 19
$m(G)$	Number of edges of G	Sec. 7	p. 7
$n(G)$	Number of vertices of G	Sec. 7	p. 7
$N_G(v, \ell)$	Number of length- ℓ simple paths from v in G	Def. 13	p. 12
near-forest	$G - v - \Gamma_G(v)$ a forest for some v	Def. 31	p. 23
P	y - z path in extended decomposition	Def. 39	p. 27
pre-potential	Function used to define a potential	Def. 45	p. 34
Prune (\mathcal{G}, S)	Weighted graph based on deleting S	Def. 28	p. 21
Reduce (\mathcal{G})	Produces a reduced graph from \mathcal{G}	Sec. 6.3	p. 23
reduced	$G[X]$ a near forest implies $ \Gamma_G(X) \geq 2$	Def. 33	p. 23
S	$\Gamma_G^2(v) \cap \bigcup_i V(H_i)$ in standard decomposition	Def. 37	p. 25
S'	$S \cap \Gamma_G(v)$ (in Prune)	Def. 28	p. 21
S''	$S \setminus \Gamma_G(v)$ (in Prune)	Def. 28	p. 21
κ subcritical wrt λ	$\lambda < \lambda_c(\kappa) = \kappa^\kappa / (\kappa - 1)^{\kappa+1}$	Def. 14	p. 12
suitable, (S1)–(S4)	Used to pin down $D_2(k)$	Def. 9	p. 10
$T(x)$	Running time when instance potential is x	Def. 6.6	p. 37
T_1, \dots, T_ℓ	Tree components in standard decomposition	Def. 37	p. 25
$T_{\text{SAW}}(v, G)$	Self-avoiding walk tree	Def. 13	p. 12
TR (\mathcal{G})	Graph made by deleting tree components	Def. 35	p. 25
$w_+(X), w_-(X)$	$\prod_{x \in X} w_+(x), \prod_{x \in X} w_-(x)$	Sec. 2	p. 7
(VP1)–(VP4)	Properties of a valid pre-potential	Def. 45	p. 34
x, y	$S = \{x, y\}$ in standard/extended decomposition	Def. 39	p. 27
X	$\{v\} \cup \Gamma_v \cup \bigcup_i V(T_i)$ in standard decomposition	Def. 37	p. 25
X^+	$X \cup V(P) \setminus \{x\}$ in extended decomposition	Def. 39	p. 27
(X^+, P, x, y, z, H)	Extended decomposition of G	Def. 39	p. 27
z	Last vertex of P in extended decomposition	Def. 39	p. 27
$Z(\mathcal{G})$	Partition function	Sec. 2	p. 8
$Z_\lambda(G)$	Univariate hard-core partition function	Sec. 4	p. 13

B Mathematica code for Section 5

The following Mathematica code is used to rigorously prove Theorem 26. `FindInstance` attempts to find integer solutions to the given equations over the given variables. It returns “{}”, indicating (with certainty) that there are no counterexamples to the claim in the theorem.

```
psi[d_, p_] :=
  If[d == 2, 245/1000,
    If[d == 3, 456/1000,
      If[d == 4, 647/1000,
        If[d == 5, 859/1000,
          (* Else, d >= 6 *)
            If[p == 2, 1,
              If[p == 3, 941/1000,
                (*Else, p >= 4*) 889/1000
              ]
            ]
          ]
        ]
      ]
    ]
  ]]]

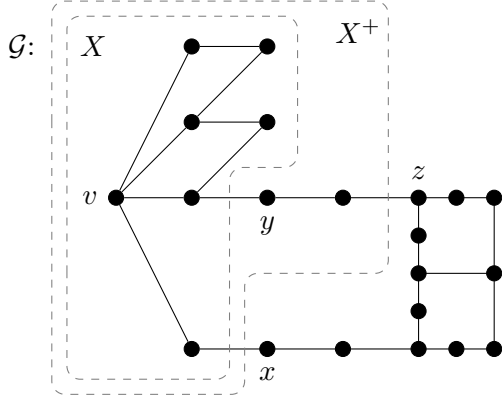
FindInstance[
  (* Equations *)
    0 <= n[2] <= 12 && 0 <= n[6] <= 12 && 0 <= n[10] <= 12 &&
    0 <= n[3] <= 12 && 0 <= n[7] <= 12 && 0 <= n[11] <= 12 &&
    0 <= n[4] <= 12 && 0 <= n[8] <= 12 && 0 <= n[12] <= 12 &&
    0 <= n[5] <= 12 && 0 <= n[9] <= 12 && 0 <= n[13] <= 12 &&
    2 <= p <= 13 && 2 <= d <= 13 &&
    d == 1 + Sum[n[i], {i, 2, 13}] &&
    (d <= 5 || p + Sum[i*n[i], {i, 2, 13}] <= 26) &&
    Sum[n[i] psi[i, d], {i, 2, 13}] > (4141/1000) psi[d, p],
  (* Variables *)
    {n[2], n[3], n[4], n[5], n[6], n[7], n[8], n[9], n[10], n[11], n[12], n[13],
      p, d},
  (* Domain *)
    Integers]
```

C Better branching with the extended decomposition

Consider the (bipartite) graph \mathcal{G} shown in Figure 7, with underlying graph G . Observe that for all $\emptyset \subset X \subset V(G)$ we have $|\Gamma_G(X)| \geq 2$, and that G itself contains two disjoint cycles whose distance is greater than 2, so G is therefore not a near-forest. It follows that G is reduced. Let $f(m, n) = m - n$, which is a good slice. We have $f(\mathcal{G}) = 25 - 21 = 4$. Writing $f(m, n) = \rho m + \sigma n$ as in Definition 27, we have $\rho = 1$ and $\sigma = -1$. For the calculations below, note that $d_G(v) = 4$ and $d_G^2(v) = 10$.

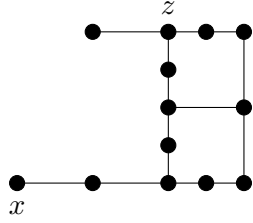
The strategy of [4] is to branch on y and prune what remains of X , as shown in Figure 7(a). This yields instances \mathcal{G}_{out} and \mathcal{G}_{in} with $f(\mathcal{G}_{\text{out}}) = 14 - 13 = 1$ and $f(\mathcal{G}_{\text{in}}) = 13 - 12 = 1$, so the value of f decreases by 3 in each branch.

Instead, we branch on z and prune what remains of X^+ , as in Figure 7(b). This gives

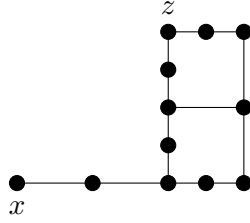


a) branching on y :

$$\mathcal{G}_{\text{out}} = \text{Prune}(\mathcal{G} - y, X):$$

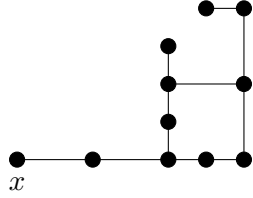


$$\mathcal{G}_{\text{in}} = \text{Prune}(\mathcal{G} - y - \Gamma_G(y), X \setminus \Gamma_G(y)):$$

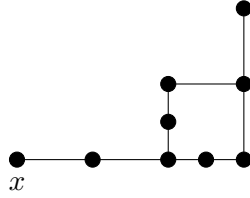


b) branching on z :

$$\mathcal{G}_{\text{out}} = \text{TR}(\text{Prune}(\mathcal{G} - z, X^+)):$$

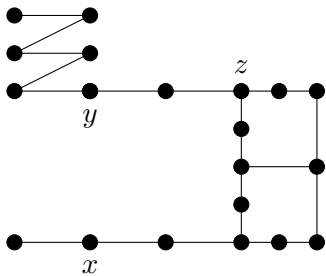


$$\mathcal{G}_{\text{in}} = \text{TR}(\text{Prune}(\mathcal{G} - z - \Gamma_G(z), X^+ \setminus \Gamma_G(z))):$$



c) branching on v :

$$\mathcal{G}_{\text{out}} = \text{TR}(\mathcal{G} - v):$$



$$\mathcal{G}_{\text{in}} = \text{TR}(\mathcal{G} - v - \Gamma_G(v)):$$

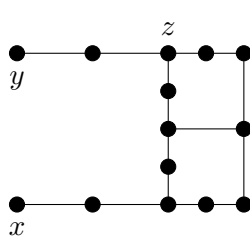


Figure 7: An example showing that our branching strategy using the extended decomposition improves on other branching strategies.

instances with $f(\mathcal{G}_{\text{out}}) = 11 - 11 = 0$ and $f(\mathcal{G}_{\text{in}}) = 8 - 8 = 0$, so f decreases by 4 in each branch. Note that this example shows that Lemma 44 is tight. It gives

$$\begin{aligned} f(\mathcal{G}) - f(\mathcal{G}_{\text{out}}) &\geq \rho(1 + d_G(v)) + \sigma = 1 + 4 - 1 = 4 \\ f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) &\geq \rho \lceil \tfrac{1}{2}(d_G^2(v) + d_G(v) + 4) \rceil + \sigma(1 + d_G(v)) \\ &= \lceil \tfrac{1}{2}(10 + 4 + 4) \rceil - 5 = 4, \end{aligned}$$

and, using the fact that \mathcal{G} is bipartite (and $\sigma < 0$),

$$\begin{aligned} f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) &\geq \rho d_G^2(v) + \sigma \lceil \tfrac{1}{2}(d_G^2(v) + d_G(v) - 2) \rceil \\ &= 10 - \lceil \tfrac{1}{2}(10 + 4 - 2) \rceil = 4. \end{aligned}$$

Finally, we could instead branch on v , as in Figure 7(c). In this case, there is nothing to prune. We obtain $f(\mathcal{G}_{\text{out}}) = 21 - 20 = 1$ and $f(\mathcal{G}_{\text{in}}) = 15 - 14 = 1$, so f decreases by 3 in each branch — the same decrease as branching on y . This also shows that Lemma 42 is tight. $S = \{x, y\}$ so the lemma gives

$$\begin{aligned} f(\mathcal{G}) - f(\mathcal{G}_{\text{out}}) &\geq \rho d_G(v) + \sigma = 4 - 1 = 3 \\ f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) &\geq \rho \lceil \tfrac{1}{2}(d_G^2(v) + d_G(v) + |S|) \rceil + \sigma(1 + d_G(v)) \\ &= \lceil \tfrac{1}{2}(10 + 4 + 2) \rceil - (1 + 4) = 3, \end{aligned}$$

and, since \mathcal{G} is bipartite,

$$\begin{aligned} f(\mathcal{G}) - f(\mathcal{G}_{\text{in}}) &\leq \rho d_G^2(v) + \sigma \lfloor \tfrac{1}{2}(d_G^2(v) + d_G(v) + 2 - |S|) \rfloor \\ &= 10 - \lfloor \tfrac{1}{2}(10 + 4 + 2 - 2) \rfloor = 3. \end{aligned}$$

In summary, we have shown that the bounds of Lemma 42 are tight for the strategy of bounding on v , that the bounds of Lemma 44 are tight for the strategy of branching on z . We also showed that branching on z is better in this case because it leads to a higher lower bound on $f(\mathcal{G}) - f(\mathcal{G}_{\text{out}})$.

D Mathematica code for Section 8 and auxiliary files

The Mathematica code in this section (also contained in the file `validator.nb` on the ArXiv) is used in Section 8 to prove Theorem 1 with the aid of our pre-potential functions (see files `bi-potential.csv` and `potential.csv`, also on the ArXiv). The function `Dtwo[k]` computes $D_2(k)$ using the method from Section 3. The function `Validate[]` takes as input the name of a file in CSV (comma-separated values) format that describes a pre-potential function which we will call f . If the first line of the file is exactly the string `Bipartite`, the potential function is interpreted as being for use with bipartite graphs; otherwise, it is for general graphs. The second, third and fourth lines of the file are, respectively, the list of values ρ_1, \dots, ρ_s , the list of values $\sigma_1, \dots, \sigma_s$ and the list of values k_1, \dots, k_{s-1} . To allow exact computation without numerical approximation, values are given as fractions, e.g., `12/13`.

`Validate[]` first checks that the input is a valid pre-potential as defined in Definition 45. To do this, it checks that the conditions of Lemma 51 hold and aborts if they do not.

`Validate[]` then verifies the conditions of Corollary 49 (in the general case) and Corollary 50 (in the bipartite case) to verify that f meets the conditions required to give a running

time of $O(s^{\sigma_s n}) \text{poly}(n, 1/\varepsilon)$. Again, it aborts if any check fails. In more detail, the following steps are taken for each $i \in [s]$.

- Compute $D'_2(k_{i-1})$ (stored in the variable **D2p**) according to Definition 46.
- For appropriate values of d , as given by Corollary 49 or 50, compute values **exp1** and **exp2** such that $T_{i,d} = \tau(\text{exp1}, \text{exp2})$. Thus, $\text{exp1} = f_i(d, 1)$, and **exp2** corresponds to some expression of the form $f_i(\cdot, \cdot)$ that depends on d, i and whether or not the input graph is bipartite. As observed in Section 6.6, if $2^{-\text{exp1}} + 2^{-\text{exp2}} > 1$, then $T_{d,i} > 2$ and, hence, T (as defined in Corollary 49 or 50 as appropriate) is greater than 2 and the code aborts.

In addition, the code aborts if $2^{-\sigma_s} + 2^{-12\sigma_s} > 1$. If so, $T > 2$, so the code aborts.

If the code has not aborted as a result of any of these checks, we conclude that $T \leq 2$ and, thus the code outputs that **Count** has running time $O(2^{\sigma_s n}) \text{poly}(n, 1/\varepsilon)$.

```
IsSuitable[k_, z_] := Module[ {d, K, s, q, d0, d1},
  K = Floor[k] + 1;
  If [z >= K^2, Return[True]];
  For [d = K, d <= z/2, d++,
    For [s = 0, s < d, s++,
      If [(z >= K s + 2 (d - s)) && ( z <= K s + (K-1)(d - s)),
        q = Floor[(z - K s)/(d - s)];
        d1 = Mod[z - K s, d-s];
        d0 = d - s - d1;
        If [(d+d0+d1)/(1 + d0/q + d1/(q + 1)) > k, Return[True]]];];
  Return[False]]
```

```
Dtwo[k_] := Module[{K, z},
  K = Floor[k] + 1;
  For[z = 2 K, z < K^2, z++,
    If[ IsSuitable[k, z], Return[z] ]]; Return[K^2];
```

```

Validate[input_] := (
  Quiet[SetDirectory[NotebookDirectory[]];
  file = Import[input, "CSV"];
  isBipartite = (file[[1]] == {"Bipartite"});
  rho = Map[ToExpression, file[[2]]];
  sigma = Map[ToExpression, file[[3]]];
  k = Map[ToExpression, file[[4]]];
  s = Length[rho];
  (* check that the input is a valid pre-potential *)
  For [ i = 1, i <= s - 2, i++, If[ k[[i]] >= k[[i + 1]], Print["k's not ascending"]; Abort[]];
  If[rho[[s]] != 0 || sigma[[s]] <= 0, Print["Fails Lemma 48(i)"]; Abort[]];
  For [i = 1, i <= s - 1, i++,
    If [rho[[i]] + sigma[[i]] < 0, Print["Fails Lemma 48(ii)"]; Abort[]];
    If [rho[[i]] == 0 && sigma[[i]] == 0, Print["Fails Lemma 48(iii)"]; Abort[]];
    If [ rho[[i]] < rho[[i + 1]] || sigma[[i]] > sigma[[i + 1]],
      Print["Fails Lemma 48(iv)"]; Abort[]];
  ];
  For [j = 6, j <= k[[s - 1]], j ++, If [! MemberQ[k, j], Print["Fails Lemma 48(v)"]; Abort[]];
  (* Establish the runtime guarantee via Corollary 46 or 47.  $T_{\{i,d\}} = \tau(\exp1, \exp2)$  *)
  For [i = 1, i <= s, i ++,
    If [i == 1 || k[[i - 1]] < 5 , D2p = 27, D2p = Max[2 k[[i - 1]], Dtwo[k[[i - 1]] ]]];
    For [d = If [i == 1, 6, Max[Floor[k[[i - 1]]] + 1, 6]], d <= 10, d ++,
      exp1 = (d rho[[i]] + sigma[[i]]); (* exp1 = f_i(d,1) *)
      If[ isBipartite,
        If[sigma[[i]] >= 0,
          exp2 = D2p * rho[[i]] + (1 + d) sigma[[i]], (* exp2=f_i(D2p,1+d) 1st part Cor 27*)
          exp2 = D2p * rho[[i]] + Floor[(d + D2p - 1) / 2] sigma[[i]], (*2nd part Cor 27*)
          exp2 = Ceiling[(d + D2p + 3) / 2] * rho[[i]] + (1 + d) sigma[[i]] ]; (* Cor 26 *)
        If[2^(-exp1) + 2^(-exp2) > 1,
          Print["Can't apply Cor 46/47 because  $T_{\{i,d\}} > 2$  for i =", i, " and d = ", d]; Abort[]];
        If[2^(-sigma[[s]]) + 2^(-12 sigma[[s]]) > 1,
          Print["Can't apply Cor 46/47 because  $\tau(\sigma_s, 12 \sigma_s) > 2$ "]; Abort[]];
      ];
    Print["Potential function is valid, running time  $O(2^{($ ",
      Ceiling[sigma[[s]], 0.0001], "n))*poly(1/eps),
      which is  $O($ ", Ceiling[2^sigma[[s]], 0.0001], "n))*poly(1/eps))."]];

```

Validate["bi-potential.csv"] ;

Validate["potential.csv"]

This code will output:

```

Potential function is valid, running time  $O(2^{0.2372n}) \cdot \text{poly}(1/\epsilon)$ ,
  which is  $O(1.1788^n) \cdot \text{poly}(1/\epsilon)$ .
Potential function is valid, running time  $O(2^{0.2680n}) \cdot \text{poly}(1/\epsilon)$ ,
  which is  $O(1.2042^n) \cdot \text{poly}(1/\epsilon)$ .

```

E A pre-potential function for bipartite graphs

The following table contains an approximation of our pre-potential function for bipartite graphs, rounded away from zero to five decimal places. We give an exact version in the ancillary file bi-potential.csv, available on the arXiv.

Boundary points k_0, \dots, k_s	Edge weights ρ_1, \dots, ρ_s	Vertex weights $\sigma_1, \dots, \sigma_s$
$k_0 = -1$		
$k_1 = 4$	$\rho_1 = 0.13168$	$\sigma_1 = -0.13168$
$k_2 = 4.18794$	$\rho_2 = 0.13168$	$\sigma_2 = -0.13168$
$k_3 = 4.23794$	$\rho_3 = 0.13168$	$\sigma_3 = -0.13168$
$k_4 = 4.28572$	$\rho_4 = 0.1241$	$\sigma_4 = -0.11562$
$k_5 = 4.33572$	$\rho_5 = 0.11588$	$\sigma_5 = -0.098$
$k_6 = 4.38572$	$\rho_6 = 0.10882$	$\sigma_6 = -0.0827$
$k_7 = 4.43572$	$\rho_7 = 0.10176$	$\sigma_7 = -0.06721$
$k_8 = 4.44445$	$\rho_8 = 0.0957$	$\sigma_8 = -0.05377$
$k_9 = 4.49445$	$\rho_9 = 0.0957$	$\sigma_9 = -0.05377$
$k_{10} = 4.55$	$\rho_{10} = 0.08753$	$\sigma_{10} = -0.03542$
$k_{11} = 4.57143$	$\rho_{11} = 0.08317$	$\sigma_{11} = -0.0255$
$k_{12} = 4.62143$	$\rho_{12} = 0.08294$	$\sigma_{12} = -0.02496$
$k_{13} = 4.66667$	$\rho_{13} = 0.07306$	$\sigma_{13} = -0.00213$
$k_{14} = 4.71667$	$\rho_{14} = 0.07306$	$\sigma_{14} = -0.00213$
$k_{15} = 5.10639$	$\rho_{15} = 0.07214$	$\sigma_{15} = 0.00004$
$k_{16} = 5.2174$	$\rho_{16} = 0.06856$	$\sigma_{16} = 0.00919$
$k_{17} = 5.33334$	$\rho_{17} = 0.06528$	$\sigma_{17} = 0.01774$
$k_{18} = 5.45455$	$\rho_{18} = 0.06226$	$\sigma_{18} = 0.02578$
$k_{19} = 5.5$	$\rho_{19} = 0.05947$	$\sigma_{19} = 0.03339$
$k_{20} = 5.55556$	$\rho_{20} = 0.05693$	$\sigma_{20} = 0.0404$
$k_{21} = 5.625$	$\rho_{21} = 0.05458$	$\sigma_{21} = 0.0469$
$k_{22} = 5.71429$	$\rho_{22} = 0.05242$	$\sigma_{22} = 0.053$
$k_{23} = 5.83334$	$\rho_{23} = 0.0504$	$\sigma_{23} = 0.05877$
$k_{24} = 6$	$\rho_{24} = 0.0485$	$\sigma_{24} = 0.06431$
$k_{25} = 6.08696$	$\rho_{25} = 0.03667$	$\sigma_{25} = 0.09978$
$k_{26} = 6.17648$	$\rho_{26} = 0.0354$	$\sigma_{26} = 0.10365$
$k_{27} = 6.26866$	$\rho_{27} = 0.03421$	$\sigma_{27} = 0.10734$
$k_{28} = 6.36364$	$\rho_{28} = 0.03308$	$\sigma_{28} = 0.11089$
$k_{29} = 6.46154$	$\rho_{29} = 0.032$	$\sigma_{29} = 0.11429$
$k_{30} = 6.5$	$\rho_{30} = 0.03099$	$\sigma_{30} = 0.11757$
$k_{31} = 6.54546$	$\rho_{31} = 0.03003$	$\sigma_{31} = 0.12068$
$k_{32} = 6.6$	$\rho_{32} = 0.02913$	$\sigma_{32} = 0.12362$
$k_{33} = 6.66667$	$\rho_{33} = 0.02828$	$\sigma_{33} = 0.12643$
$k_{34} = 6.76057$	$\rho_{34} = 0.02747$	$\sigma_{34} = 0.12913$
$k_{35} = 6.85715$	$\rho_{35} = 0.0267$	$\sigma_{35} = 0.13174$
$k_{36} = 7$	$\rho_{36} = 0.02596$	$\sigma_{36} = 0.13428$
$k_{37} = 7.07369$	$\rho_{37} = 0.01826$	$\sigma_{37} = 0.16123$
$k_{38} = 7.14894$	$\rho_{38} = 0.01779$	$\sigma_{38} = 0.16288$
$k_{39} = 7.22581$	$\rho_{39} = 0.01735$	$\sigma_{39} = 0.16448$

$k_{40} = 7.30435$	$\rho_{40} = 0.01692$	$\sigma_{40} = 0.16604$
$k_{41} = 7.38462$	$\rho_{41} = 0.0165$	$\sigma_{41} = 0.16755$
$k_{42} = 7.46667$	$\rho_{42} = 0.0161$	$\sigma_{42} = 0.16901$
$k_{43} = 7.5$	$\rho_{43} = 0.01572$	$\sigma_{43} = 0.17044$
$k_{44} = 7.53847$	$\rho_{44} = 0.01536$	$\sigma_{44} = 0.17181$
$k_{45} = 7.58334$	$\rho_{45} = 0.01501$	$\sigma_{45} = 0.17314$
$k_{46} = 7.63637$	$\rho_{46} = 0.01467$	$\sigma_{46} = 0.17441$
$k_{47} = 7.71429$	$\rho_{47} = 0.01435$	$\sigma_{47} = 0.17565$
$k_{48} = 7.79382$	$\rho_{48} = 0.01403$	$\sigma_{48} = 0.17685$
$k_{49} = 7.875$	$\rho_{49} = 0.01373$	$\sigma_{49} = 0.17803$
$k_{50} = 8$	$\rho_{50} = 0.01344$	$\sigma_{50} = 0.17918$
$k_{51} = 8.064$	$\rho_{51} = 0.00799$	$\sigma_{51} = 0.20096$
$k_{52} = 8.12904$	$\rho_{52} = 0.00784$	$\sigma_{52} = 0.20159$
$k_{53} = 8.19513$	$\rho_{53} = 0.00768$	$\sigma_{53} = 0.20222$
$k_{54} = 8.2623$	$\rho_{54} = 0.00754$	$\sigma_{54} = 0.20282$
$k_{55} = 8.33058$	$\rho_{55} = 0.00739$	$\sigma_{55} = 0.20342$
$k_{56} = 8.4$	$\rho_{56} = 0.00725$	$\sigma_{56} = 0.204$
$k_{57} = 8.47059$	$\rho_{57} = 0.00712$	$\sigma_{57} = 0.20457$
$k_{58} = 8.5$	$\rho_{58} = 0.00698$	$\sigma_{58} = 0.20513$
$k_{59} = 8.53334$	$\rho_{59} = 0.00686$	$\sigma_{59} = 0.20567$
$k_{60} = 8.57143$	$\rho_{60} = 0.00673$	$\sigma_{60} = 0.2062$
$k_{61} = 8.61539$	$\rho_{61} = 0.00661$	$\sigma_{61} = 0.20671$
$k_{62} = 8.68218$	$\rho_{62} = 0.0065$	$\sigma_{62} = 0.20721$
$k_{63} = 8.75$	$\rho_{63} = 0.00639$	$\sigma_{63} = 0.20769$
$k_{64} = 8.8189$	$\rho_{64} = 0.00628$	$\sigma_{64} = 0.20817$
$k_{65} = 8.88889$	$\rho_{65} = 0.00617$	$\sigma_{65} = 0.20865$
$k_{66} = 9$	$\rho_{66} = 0.00607$	$\sigma_{66} = 0.20911$
$k_{67} = 9.05661$	$\rho_{67} = 0.00199$	$\sigma_{67} = 0.22746$
$k_{68} = 9.11393$	$\rho_{68} = 0.00195$	$\sigma_{68} = 0.22761$
$k_{69} = 9.17198$	$\rho_{69} = 0.00192$	$\sigma_{69} = 0.22774$
$k_{70} = 9.23077$	$\rho_{70} = 0.00189$	$\sigma_{70} = 0.22788$
$k_{71} = 9.29033$	$\rho_{71} = 0.00187$	$\sigma_{71} = 0.22801$
$k_{72} = 9.35065$	$\rho_{72} = 0.00184$	$\sigma_{72} = 0.22815$
$k_{73} = 9.41177$	$\rho_{73} = 0.00181$	$\sigma_{73} = 0.22828$
$k_{74} = 9.47369$	$\rho_{74} = 0.00178$	$\sigma_{74} = 0.2284$
$k_{75} = 9.5$	$\rho_{75} = 0.00176$	$\sigma_{75} = 0.22853$
$k_{76} = 9.52942$	$\rho_{76} = 0.00173$	$\sigma_{76} = 0.22865$
$k_{77} = 9.5625$	$\rho_{77} = 0.00171$	$\sigma_{77} = 0.22877$
$k_{78} = 9.6$	$\rho_{78} = 0.00168$	$\sigma_{78} = 0.22889$
$k_{79} = 9.65854$	$\rho_{79} = 0.00166$	$\sigma_{79} = 0.229$
$k_{80} = 9.7178$	$\rho_{80} = 0.00163$	$\sigma_{80} = 0.22911$
$k_{81} = 9.77778$	$\rho_{81} = 0.00161$	$\sigma_{81} = 0.22923$
$k_{82} = 9.83851$	$\rho_{82} = 0.00159$	$\sigma_{82} = 0.22933$
$k_{83} = 9.9$	$\rho_{83} = 0.00157$	$\sigma_{83} = 0.22944$
$k_{84} = 10$	$\rho_{84} = 0.00155$	$\sigma_{84} = 0.22955$
$k_{85} = \infty$	$\rho_{85} = 0$	$\sigma_{85} = 0.23725$

F A potential function for general graphs

The following table contains an approximation of our pre-potential function for bipartite graphs, rounded away from zero to five decimal places. We give an exact version in the ancillary file `potential.csv`, available on the arXiv.

Boundary points k_0, \dots, k_s	Edge weights ρ_1, \dots, ρ_s	Vertex weights $\sigma_1, \dots, \sigma_s$
$k_0 = -1$		
$k_1 = 5.10639$	$\rho_1 = 0.13168$	$\sigma_1 = -0.13168$
$k_2 = 5.33334$	$\rho_2 = 0.11402$	$\sigma_2 = -0.08658$
$k_3 = 5.5$	$\rho_3 = 0.10004$	$\sigma_3 = -0.04931$
$k_4 = 5.625$	$\rho_4 = 0.08896$	$\sigma_4 = -0.01883$
$k_5 = 5.83334$	$\rho_5 = 0.08006$	$\sigma_5 = 0.0062$
$k_6 = 6$	$\rho_6 = 0.07243$	$\sigma_6 = 0.02845$
$k_7 = 6.08696$	$\rho_7 = 0.04523$	$\sigma_7 = 0.11007$
$k_8 = 6.26866$	$\rho_8 = 0.04157$	$\sigma_8 = 0.12119$
$k_9 = 6.46154$	$\rho_9 = 0.03833$	$\sigma_9 = 0.13135$
$k_{10} = 6.54546$	$\rho_{10} = 0.03542$	$\sigma_{10} = 0.14076$
$k_{11} = 6.66667$	$\rho_{11} = 0.0329$	$\sigma_{11} = 0.14901$
$k_{12} = 6.85715$	$\rho_{12} = 0.03066$	$\sigma_{12} = 0.15648$
$k_{13} = 7$	$\rho_{13} = 0.0286$	$\sigma_{13} = 0.16354$
$k_{14} = 7.07369$	$\rho_{14} = 0.0105$	$\sigma_{14} = 0.22689$
$k_{15} = 7.14894$	$\rho_{15} = 0.00994$	$\sigma_{15} = 0.22886$
$k_{16} = 7.22581$	$\rho_{16} = 0.00994$	$\sigma_{16} = 0.22886$
$k_{17} = 7.38462$	$\rho_{17} = 0.00942$	$\sigma_{17} = 0.23075$
$k_{18} = 7.5$	$\rho_{18} = 0.00892$	$\sigma_{18} = 0.23258$
$k_{19} = 7.58334$	$\rho_{19} = 0.00846$	$\sigma_{19} = 0.2343$
$k_{20} = 7.63637$	$\rho_{20} = 0.00804$	$\sigma_{20} = 0.2359$
$k_{21} = 7.71429$	$\rho_{21} = 0.00804$	$\sigma_{21} = 0.2359$
$k_{22} = 7.875$	$\rho_{22} = 0.00764$	$\sigma_{22} = 0.23743$
$k_{23} = 8$	$\rho_{23} = 0.00727$	$\sigma_{23} = 0.23891$
$k_{24} = \infty$	$\rho_{24} = 0$	$\sigma_{24} = 0.26796$