‘Essays on Strategic Voting’

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Amongst friends and family, who in different occasions and with various intensities have shown interest in my work, and to whom I am grateful to, I wish to highlight the support given by my grandparents.

This thesis is dedicated to them.

Abstract

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In this thesis we extend the study of strategic voting to two frameworks that are novel to the literature. First, by analysing a four-party competition with purely instrumentally motivated voters (Part I); second, by focusing on a three-party competition where voters have instrumental and expressive motivations (Part II). We aim to explore an existing gap in the literature and, in particular, to investigate the possibility of a voting equilibrium with partial strategic voting and its stability. The three sub-models studied in Part I (including essay 1) and the model in Part II (including essays 2, 3 and 4) focus on the case of large electorates and include public uncertainty. This distinguishes them from Cox (1997), where no real uncertainty exists as the electorate gets large, and Myatt (2006), which includes both public and private information. Essays 2 and 3 present and explore the theoretical framework and implications for the model chosen for Part II and essay 4 applies it to the 1997 UK General Election.

From essay 1 we obtain the result that in a single-ballot simple-plurality election there is a tendency towards the Duvergerian equilibrium in a four-party model. Also, an equilibrium with partial strategic voting is never stable. From essays 2 and 3 three main results arise: a Duvergerian equilibrium is never possible; a stable equilibrium always exists; and more information leads to less strategic voting – contrary to Myatt (2006). Both the impossibility of any Duvergerian equilibrium and the possibility of a stable interior equilibrium in multiple cases are central to our theory of voting that includes an expressiveness component. The simulations in essay 4 suggest that a very low level of expressiveness is needed to obtain a level of strategic voting compatible with the findings in Fisher (2004). The theory predicts the impossibility of some constituency results that are in fact frequently observed in British elections.

Cox (1997), *Making Votes Count*. Cambridge, Cambridge University Press.
Fisher (2004),), *Definition and Measurement of Tactical Voting: the Role of Rational Choice*, British Journal of Political Science, 34(1), 152-66.
Myatt (2006), *On the Theory of Strategic Voting*, Review of Economics Studies, Blackwell Publishing vol. 74(1), pages 255-281.

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Errata for Essay 1

Due to the unfortunate impossibility of recovering the file used to write Essay 1 in this thesis, an *errata* regarding this chapter in including forefront.

Below is a numbered list of corrections to Essay 1.

First I include what appears in Essay 1; below it what should have been there instead.

In parenthesis are, in this order, the page number and the paragraph the correction refers to.

An overall change is that the reference to Fisher (2002) must be instead read as Fisher (2004).

Likewise, Myatt (2004) should be read as Myatt (2006).

1. (1, 4th)

"- the incumbent party – usually the one enjoying highest prior support."

"- the candidate holding the elected office prior to the election and the one enjoying the highest support"

2. (2, 5th)

"is whether a four-party system allow other than the Duvergerian equilibria to occur."

"is whether a system where four parties exist exogenously – which is the case, for instance, of presidential elections in Portugal – can allow for a situation where more than two candidates will obtain any votes."

3. (3, 2nd)

"statement that plurality rule"

"statement that single-ballot simple plurality rule"

4. (3, 5th)

"to vote in a candidate"

"to vote for a candidate"

5. (3, 8^{th})

“bounded rationality.”

“bounded rationality. Our approach also excludes cases of bounded rationality, the inclusion of which would not only be unnecessary and detrimental to the study of what changes in the results predicted by the model when the minimum amount of assumptions necessary for some change does change; but also pose significant problems later on when doing an empirical study. Instrumental rationality is a valid assumption to model voting’s choice, even if there would be some difficulty in finding a “pure” instrumentally rational voter. The assumption’s strength rests in the clarity with which it suggests that the main, if not sole concern of the voters is how they may be relevant in changing the outcome of an election, in our case, a single-ballot simple plurality rule election.”

6. (5, A4)

“votes equal to *n*.”

“votes equal to *n*. Voters have only one vote for one party.”

7. (6, A10)

“functions”

“functions, and the utility they obtain, mentioned in A8, is only achieved if the respective candidate is elected.”

8. (6, 9^{th})

“Assumptions A11 and A12 seem quite natural, while A13”

“Assumption A11 is a modelling choice, which is done to keep the analysis somewhat contained. Assumption A12 seems quite natural, while A13”

9. (7, 2^{nd})

“otherwise be wasted”

“otherwise be wasted – here meant in a “prior” sense, i.e., votes are considered to be wasted if there is 100% probability (or, even better, absolute certainty) that they would not influence the election in any scenario.”

10. (10, end)

“50% chance”

“50% chance, when both q_c and q_D are lower than $q_{M/2}$, as illustrated in Figure 6, page 28.”

11. (13, 4th)

“that when the electoral grows”

“that when the electorate grows”

12. (13, 6th)

“non-Duvergerian Myerson”

“non-Duvergerian equilibrium Myerson”

13. (15, 2nd)

“to vote in leading”

“to vote in the leading”

14. (16, 2nd)

“This is the what leads”

“This is what leads”

15. (17, 3rd)

“Myatt’s model truly a model”

“Myatt’s model is truly a model”

16. (21, 4th)

“not seen such ‘simple point’”.

“not seen such ‘simple point’. In our view this is indeed a simple point and Besley and Coate (1997) could have mention that an option like that was available, and certainly a stronger choice in theoretical grounds, but not necessarily so when other considerations have to be taken into account, such as when empirical studies are done."

17. (26, 3rd)

"'Battle of Sexes' game structure'"

"'Battle of Sexes' game structure', in that (i) the game is symmetric, (ii) there are gains for the voters when there is coordination in either of the two possible voting equilibrium, though the payoffs differ depending on which takes place, and (iii) no gains occur if no coordination takes place.

18. (34, 6th)

"$\alpha_{AC} < \alpha < \alpha_{AB}$"

"$\alpha_{AC} < \alpha < \alpha_{AB}$"

19. (67, 3rd))

Claim 3) is later commented and refined in the discussion in essay 3.

20. (71, 2nd)

"In Figure 24 (…) sincerely."

"In Figure 24 (…) sincerely. This illustrates a hypothetical scenario where everyone votes sincerely – it does not suggest that everyone voting sincerely is optimal, rather, it is just a starting point for the analysis of the possible equilibrium."

21. (75, 3rd)

"If i)"

"In i)

22. (87, end)

"Our conclusions are then"

"Our conclusions from this essay are then"

23. (88, 7th)

“shows that the existence of *private information* is indeed a necessary condition to obtain that.”

“suggests that the existence of *private information* is indeed a necessary condition to obtain that. This issue needs to be pursued further.”

24. (91, 5th)

“In the Northern Ireland, the distinction”

“In the Northern Ireland, and as of 2004, the distinction”

– *Essays on Strategic Voting* –

Essay 1

"Strategic Voting in a Four-Party Model"

I. Introduction

In this first essay we propose to study the phenomenon known as strategic voting, in a four-party model, in a one-shot election, using the plurality rule.

Our first original contribution is the analysis of a political system with four parties.

There is a vast literature on the three-party model – inspired by the English and American political systems, where there are two dominant parties – but not on the four-party one, which is the case for many countries[1]. At first glance, one may be led to think it would be trivial to extend the analysis to one more party – wouldn't a three-dimensional framework be enough to generalise the conclusions? In fact, they are very different, because they are strategically different: a four-party model can be very different game from the three-party one.

The three-party system basically involves a problem of coordination between the supporters of two parties that wish to defeat a third one – the incumbent party – usually the one enjoying highest support[2]. The coordination problem in this electoral game is similar to the well-known 'Battle of Sexes' game, since (i) both voters prefer to coordinate and choose similarly to the other and (ii) different voters prefer different (pure-strategy) equilibria.

With four parties, one immediate difference is that there are many ways in which the strategic interactions can take place. For instance, if we assume that there is one incumbent and three challenging parties, we have a problem of coordination between three players, which is strategically different from the case where there are only two. Alternatively, if we assume the existence of two incumbents, there are many different ways in which strategic voting can be modelled, which need not be similar to the 'Battle of Sexes' one.

Given that the three-party model assumes the behaviour of the supporters of one party to be fixed, it is a model where only m = 2 different players groups are involved. In contrast, a four-party model could involve m = 3 different player groups (if we assume only one incumbent) or m = 4 (if we assume no incumbent). So, the argument that a three-party model would be general since it involved a dimension of m = 3 is flawed – it is often a *de facto* model with a dimension of m = 2. Importantly, the four-party model, with a dimension of m = 3 (or m = 4), is eligible to such position[3].

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[1] One of the motivations for this essay is the understanding of the Portuguese case, which is basically a four-party system. Because our work is purely theoretical, we relegate a short description of that to the Appendix, which also includes a very brief description of the Irish case.

[2] This is particularly true for UK elections.

[3] We do not attempt to prove this claim, nor is it a goal of our reserach. As it will be clearer later on, the model with four parties is rich and complex enough to allow for very different interactions to occur. If, on the one hand, its richness could lead us to think that it may indeed be generalisable, on the other hand, its complexity makes the number of possible configurations for a four-party model be very large. This

Our second original contribution regards the way we modelled the uncertainty that voters have about the parties' support.

The need to model uncertainty in a new way has a twofold motivation: overcoming the problem in Cox's model while being different – and simpler – than Myatt's.

Cox's (1987, 1994) model includes uncertainty that is only apparent: as the electorate gets large, his model becomes one of public certainty, since there is common knowledge about the uncertainty in the model, which vanishes to 0 as n tends to ∞[4].

Because our approach stands between these two, we will be able to better understand some of the aspects driving the different equilibria in each of them, as well as their stability. Myatt's framework, with a private information setting, seems, within models where voters are purely instrumentally motivated, necessary to obtain a stable equilibrium involving partial strategic voting[5]. The results from essays 2 and 3, in Part II of this thesis show that including an expressive component in voter's preferences allows for a stable interior equilibrium to occur.

An important concern is the number of parties that get any votes in equilibrium. The three-party literature has long debated Duverger's (1954) proposition that "simple-majority single-ballot system favours the two-party system". He expected plurality-rule elections to generate a tendency towards bipartism. An important question is whether four-party systems allow other than the Duvergerian equilibria to occur.

Our results show that an equilibrium where only three parties survive is almost always possible and always stable. This is a non-Duvergerian result. However, it is important to note that here we study only a one-shot election. It is possible that a repeated game could lead to further erosion of some candidate's votes, making the Duvergerian outcome more likely.

We also get the result that the Duvergerian outcome is almost always possible and, when it occurs, always stable. There are also knife-edge cases where all parties survive, either involving no strategic voting or just partial strategic voting – and these are always unstable equilibria.

Overall, then, a stable prediction in our model will involve at most three parties getting votes in equilibrium. Plurality rule seems to be a strong force pushing towards the erosion of some of the candidate's support, but not so strong as to make the Duvergerian equilibrium the only possible outcome.

---

imposes a need to simplify and study only some of those many configurations, not enabling us to get the full and final picture of what we could call a "general four-party model".

[4] Where n is the number of voters in the electorate.

[5] Myatt (2006) proves the result regarding sufficiency.

1. Strategic Voting - an overview

Strategic or *tactical* voting has been debated for a long time. Farquharson (1969) distinguished *sincere* from *sophisticated voting*, the latter meaning that individuals vote in a way that best suits their interests - which need not be a short-sighted sincere vote.

It is natural to associate strategic voting with *plurality rule*, which elects the *m* candidates obtaining the largest vote totals. This inspired the Duverger's statement that plurality rule favours a two-party system - later baptised as Duverger's Law by Riker (1982), who quotes an early and eloquent description of the phenomenon of strategic voting, due to Henry Droop (1869):

> «As success depends upon obtaining a majority of the aggregate votes of all the electors, an election is usually reduced to a contest between the two most popular candidates... Even if other candidates go to the poll, the electors usually find out that their votes will be thrown away, unless given in favour of one or other of the parties between whom the election really lies.»

We propose the following definition for strategic voting:

Strategic voting takes place whenever an instrumentally rational voter chooses voluntarily to vote in a candidate other than his favourite.

We restrict the definition to instrumentally rational voters, who only care about the final outcome of the election. A definition of strategic or tactical voting similar to ours is that proposed by Fisher (2002):

> «A tactical voter is someone who votes for a party they believe is more likely to win than their preferred party, to best influence who wins in the constituency.»

Our definition is, in some sense, broader than Fisher's. We only define *strategic voting* as opposed to *sincere voting*, when it comes from instrumentally rational voters that vote *voluntarily*. We do not explicitly refer to the voters' expectations or intentions, which Fisher does, when he uses the expressions "they believe is more likely to win" and "to best influence". This seems to imply that a voter may have to calculate a subjective probability of winning for the candidates, which excludes some cases of bounded rationality.

A natural question that many non-economists (and even some economists) recurrently ask is "What does voting and elections have to do with economics? Shouldn't they be studied by politicians and sociologists?".

Mudambai (2000) presents three reasons pointed earlier by Schram (1991) on why economists should be interested in the study of elections:

> «First of all, let us recall the definition of Economics given by Samuelson (1976, pg.9): "Economics is the study of how people and society end up choosing, with or without the use of money, to employ scarce resources that could have alternative

uses". The study of elections seems to fall within this definition: "the scarce productive resources are those used by the government and those employed in the act of voting, and one of the ways in which people and society end up choosing is by voting" (Schram, 1991, pg.21). The second and third reasons are interrelated and based on the fact that these studies provide explanations for and theories about government behaviour. Government behaviour may have effects on the entire economy, and conversely, the entire economy may be expected to have an influence on voting behaviour during elections.»

2. Terminology

The term *natural* will be often used to mean any *prior* situation. For example, we will say that there are 30% of *natural* supporters of party B if there are 30% of voters for whom B is the favourite candidate, i.e., the candidate that a voter would most like to see win the election; hence, the *natural* candidate to vote for.

The terms *strategic voting* and *tactical voting* will be used interchangeably, though the first will be prevalent. We will use *candidate* and *party* as synonyms, as well.

Regarding the terminology used to characterise the different equilibria we find, we will label as *full-Duvergerian* (F-DE) those equilibria where *only two* parties get *any* votes in equilibrium. The opposite case where *all four* parties get *some* votes in equilibrium is labelled as *non-Duvergerian* (N-DE). The intermediate case where *exactly three* parties get *some* votes is called *semi-Duvergerian* (S-DE).

Stated perhaps in a more appealing way, the F-DE will imply that only two parties 'survive', while the other two 'disappear', in the sense that they get zero votes. The S-DE will mean that only one party disappears, while in the N-DE case all parties survive - albeit with a support in equilibrium that may differ from their natural support.

3. Structure of the document

In Section II we introduce the general assumptions of our model and obtain three different ways to model strategic voting in a four-party system.

Section III is devoted to a brief critical review of some of the literature that we found most relevant for our work.

In Section IV we analyse the equilibria for each of the models, where each model is critically accessed in a separate way.

Section V concludes.

II. The Model

We can describe our model in 14 assumptions, which we next present, grouping them into relevant categories.

1. The game and its rules

A1. There are *four parties* (or *candidates*), whose location is fixed. Parties A and B are on one side of the political spectrum and parties C and D are on the other side. We will call A and D the *extremist* parties and B and C the *moderate* parties. This doesn't have any significant meaning but will be used to facilitate the visualisation of the model.

A2. There are n voters[6].

A3. There is a *one-shot* election.

A4. *Abstention* is not possible, implying a total of votes equal to n.

A5. The winner is the candidate receiving the highest number of votes, according to the *plurality rule*.

A6. Ties are broken *equiprobably*. This means that when k candidates are tied in the lead, the winner is decided through a random device, where each of the k candidates has a probability of winning of $1/k$.

2. Preferences

A7. Each voter has a *favourite* (or *preferred*) party j, j = A, B, C, D. The voters are grouped according to this criterion and we will say they *belong to the same group J*, or, equivalently, that they are *J type voters* (or *type voters J*). The *prior* support of party j will be the proportion q_j of voters that belong to group j - so we have $q_A + q_B + q_C + q_D = 1$. We standardise the utility each voter has for his preferred party to 1.

A8. Some voters' groups will be allowed to have a non-zero utility for one - and only one - other party, while having a standardised 0 utility for the remaining two parties. This is the minimum set-up allowing for the possibility of strategic voting with four parties. Besides, it is rich enough to get a variety of results, and so we do not extend it beyond that.

The utility such a second preferred candidate brings to voter i is independently drawn from a uniform distribution between 0 and 1, and is described by the parameter $\beta_i \sim U[0, 1]$[7]. By

6 We focus our study on the behaviour of the electorate when n becomes arbitrarily large (i.e., as $n \to \infty$).

7 The assumption that voters' preferences for a second candidate are uniformly distributed on [0, 1] is with

"a voter of type $ß_i$ belonging to group *J*" we mean a voter whose favourite candidate is *j* and whose second preferred candidate brings an utility of $ß_i$.

A9. Voters are *instrumentally rational*, that is, they only care about the winner of the election.

A10. Voters have *von-Neumman Morgenstern* utility functions.

Given Assumptions A9 and A10, voters will maximise their expected utility, by taking into account only the events where they may have an effect on the winning candidate - these are *pivotal events*, when a voter casts a *pivotal* (or *decisive*) vote.

We propose three different ways to analyse strategic voting, which are three different ways of modelling voters' preferences. These are exhaustive when we impose the following assumptions (further to A1 and A7-A9):

A11. Strategic voting can only occur in a symmetric way on each side of the political spectrum.

A12. Strategic voting can only occur to a party immediately next to the preferred one[8].

A13. A party cannot receive strategic votes from more than one group.

Assumptions A11 and A12 seem quite natural, while A13 is made mostly for simplicity[9]. It is worth emphasising that the three models proposed are the *only* possible models when assumptions A1, A7-A9 and A11-A13 are used[10].

We illustrate the three proposed models next. The arrows represent the direction that strategic voting may take and the dashed line the symmetry axis.

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little loss of generality while greatly simplifying the conditions required in equilibrium. It is with little loss of generality because each voter *i* has a standardised utility of 1 for his preferred candidate and 0 for the (two) candidates he dislikes, so it is natural to assume that a candidate that is somewhere in between the two will bring a utility in the interval [0, 1]. It is greatly simplifying to use an uniform distribution because we expect voters to use a *cut-off* rule in equilibrium, voting strategically if and only if $ß_i > ß^*_i$. Then, the proportion of strategic votes will simply be $(1 - ß^*_i)$, instead of a more complex formula (perhaps) involving integrals.

8 Note that in Assumption A8 we only imposed that each voter could only vote strategically for *one* other party, but that could be a party not immediately next to his favourite, which we impose here.

9 It is certainly true that in the real world parties may receive strategic votes from more than one group of voters, even when they are instrumentally rational. This is one assumption that may be relaxed in future research. An interesting case would be a merge of Models I and II (see next pages), where each of the moderate parties could get strategic votes from the supporters of the two parties next to them. This is likely to be the Portuguese case (see Appendix).

10 There would be a fourth possible model, but it would be redundant because the labelling moderate / extremist is not important. Such model would involve the supporters of the moderate parties having a second preference for the extremist parties on their side. So, strategic voting could only occur from the moderate to the extremist parties. This is redundant to our Model I.

Model I

In Model I, strategic voting can occur only within each side of the political spectrum, and only in one direction, from the extremist party to the moderate party, which we illustrate in the next figure.

Figure 1. *The strategic scenario in Model I*

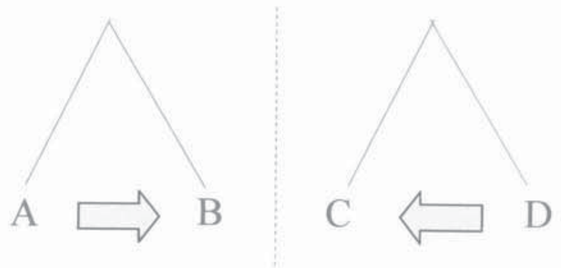

Table 1. *Voters' preferences in Model I*

Voter Type	Utility for candidate				Prior Probability
	A	B	C	D	
AB	1	$ß_A \sim U[0,1]$	0	0	q_{AB}
B	0	1	0	0	q_B
C	0	0	1	0	q_C
DC	0	0	$ß_D \sim U[0,1]$	1	q_{DC}

If we think about a benchmark scenario where all voters from all groups would consider a strategic vote, this model would seem appropriate when the supporters of the moderate parties could *credibly commit* to a sincere vote. This would perhaps be more likely to happen when the moderate parties enjoy a considerably higher support than the extremists, by that way becoming *focal*, in the sense of capturing votes that would otherwise be wasted.

This is our simplest model, because there are only two groups of voters, and they have opposite preferences.

Model II

In Model II, strategic voting can occur only between the two moderate parties, in both directions.

Figure 2. *The strategic scenario in Model II*

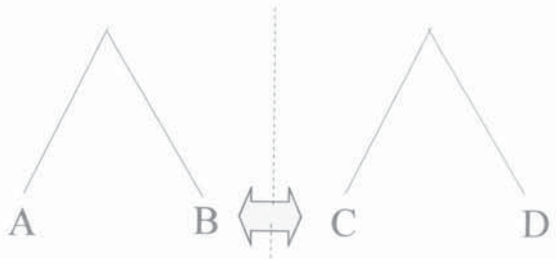

Table 2. *Voters' preferences in Model II*

Voter Type	Utility for candidate				Prior Probability
	A	B	C	D	
A	1	0	0	0	q_A
BC	0	1	$ß_B \sim U[0,1]$	0	q_{BC}
CB	0	$ß_C \sim U[0,1]$	1	0	q_{CB}
D	0	0	0	1	q_D

Our second model would seem appropriate when the extremist party supporters would never vote for other than their favourite. This would perhaps be more likely to happen when the extremist parties enjoy a significant support.

As we will see, this can be interpreted as a three-party model, since one of the extremist parties will be irrelevant. In other words, Model II *encompasses* the three-party model and we will draw a comparison with the related literature.

Model III

In Model III, strategic voting can occur within each side of the political spectrum, in both directions.

The fact that the extremist parties may receive strategic votes from the moderate supporters makes all parties be in a symmetric position. In this sense, this model involves a pair of symmetric games - each on each side of the political system.

This is the most complex of the three proposed frameworks, naturally involving a higher number of possible equilibria, as we shall see.

Figure 3. *The strategic scenario in Model III*

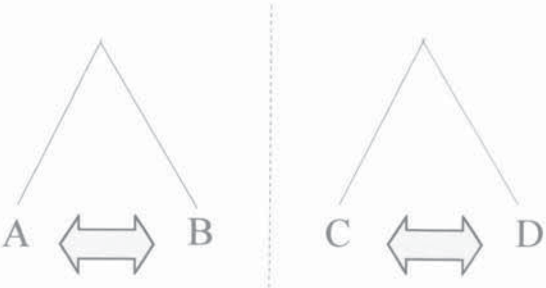

Table 3. *Voters' preferences in Model III*

Voter Type	Utility for candidate				Prior Probability
	A	B	C	D	
AB	1	$ß_A \sim U[0,1]$	0	0	q_{AB}
BA	$ß_B \sim U[0,1]$	1	0	0	q_{BA}
CD	0	0	1	$ß_C \sim U[0,1]$	q_{CD}
DC	0	0	$ß_D \sim U[0,1]$	1	q_{DC}

Since in this model, all parties can receive strategic votes, it is *as if* none of the voters could commit not to cast a strategic vote. This model would perhaps be more likely to happen when both parties on each side share a similar support.

Model I can be seen as a special case of Model III, when the uniform distribution of ß is reduced to a point mass at ß = 0 for type *BA* and *CD* voters.

3. Information

A14. As already mentioned, the way in which we model voters' information is critical to our work, in that it is an alternative to Cox's (1987, 1994) and Myatt's (2004) models, overcoming some limitations of the former while being simpler than the latter.

In our models, each voter will have information regarding: i) his own type; and ii) the distribution of all types of voters, while not knowing the exact type of any particular one.

The information regarding his own type tells each voter: 1) to which group he belongs to; and 2) what is the utility he has for a second preferred candidate. For instance, in Model I, a natural supporter of party B will be informed that he belongs to group *B*, and that he has a 0 utility for *all* other candidates; a supporter of party D will be informed that he belongs to group *DC* and that he has a utility $ß_D$ for his second preferred candidate, C.

Also, each voter has information regarding the *distribution* of all types of voters. For instance, in Model I, a voter will know that there is a proportion q_{AB} of voters whose preferred candidate is A, and that have a utility of $ß_A \sim U[0, 1]$ for party B. He will also know that there is a proportion q_B of voters whose preferred candidate is B, who never vote for any other party.

The way we bring uncertainty to our model is by assuming that the prior support of the *two* moderate candidates is uncertain, while the extremists' is certain[11].

To be precise, we assume that the prior support of parties A and D and the *sum* of that for parties B and C are known with certainty. These are, respectively, $\mathbf{q}_{AB}$, $\mathbf{q}_{DC}$ and $[\mathbf{q}_B + \mathbf{q}_C]$. We shall call the latter $\mathbf{q}_M$, so $\mathbf{q}_M = [\mathbf{q}_B + \mathbf{q}_C]$.

The uncertainty comes from the fact that the individual support of the moderate parties is not known with certainty. We define the parameter $\alpha = [\mathbf{q}_B / \mathbf{q}_M]$ to represent the proportion of supporters of the moderate parties whose first preference goes to party B.

Since we introduce uncertainty in a *one-dimensional* way - through the parameter α - we will be able to represent the [stochastic] outcome of the election in a very helpful graphic framework, which we introduce in Figure 4.

In that figure, we see that when n is large, party C wins when $\alpha < 0.5$ while party B wins when $\alpha > 0.5$[12]. If the draw is $\alpha = 0.5$, there is a tie between B and C and each party wins with a 50% chance.

11 Introducing uncertainty about the prior support of only two parties is reach enough to cover *all* possible types of equilibria - this will be clearer later on, when we analyse Model III. Extending the uncertainty to three or even four parties would only bring complexity into the discussion, with no added value in terms of results or intuition; we opted for simplicity - following Occam's razor.

12 Only when $n \to \infty$ can we say that a tie occurs exactly for $\alpha = 0.5$, since we can then ignore the near-ties and also not worry about the integer problem. Next section will cover this with more detail.

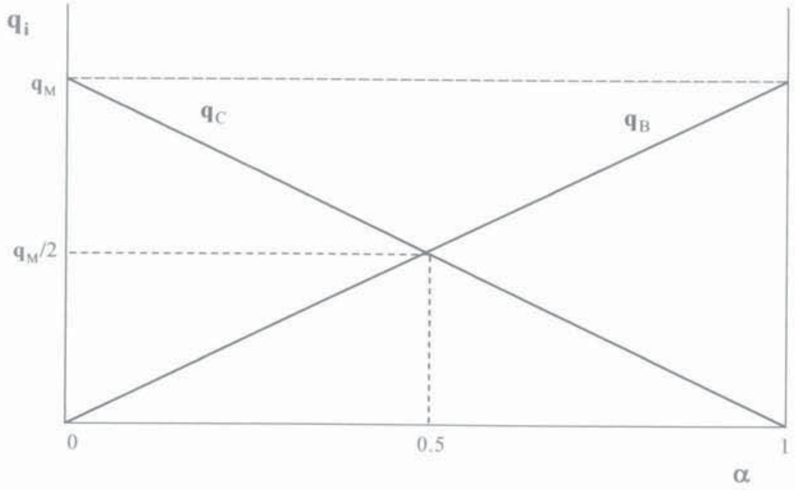

Figure 4 – The support of each of the moderate parties as a function of α.

The uncertainty regarding α can be described through a distribution function F(α), satisfying the usual three properties: F(0) = 0, F(1) = 1 and F(α) is a non-decreasing function, that is, F'(α) = f(α) ≥ 0, where f(α) is the probability density function of α.

We will further assume that f(α) > 0 for all α in [0, 1]. This will allow any (pivotal) event that occurs for α in [0, 1] to have a strictly positive likelihood. We do not impose symmetry, but we do impose *quasi-concavity* on f(α). We also assume f(α) to be *continuous* in [0, 1]. We illustrate a possible example for f(α) in Figure 5.

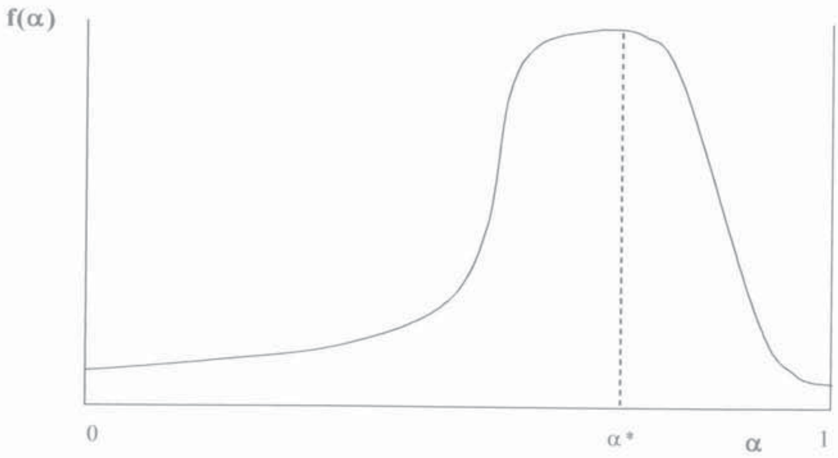

Figure 5 – Example of a probability density function f(α), describing the likelihood of the different values that α can take in [0, 1].

All the information above described is *a priori* common knowledge to all voters. That is, everyone knows that everyone knows, and so on, *ad infinitum*: i) his own type; and ii) the distribution of all other voters' preferences which, crucially, includes knowing that the support of the moderate parties depends on the parameter α, whose density function is $f(\alpha)$.

In other words, all voters have the same *prior beliefs* regarding the preference distribution across the electorate. Importantly, we assume that all voters hold their prior beliefs, so that they all have identical posterior beliefs. This is tantamount to assume that they ignore their own type as a source of information. We thus assume a model of public information, as opposed to a private information one.

Given that all voters share the same opinions about the support of the candidates, a natural way to interpret and indeed justify the way we model uncertainty is to imagine that there is an opinion poll which is publicly announced and whose information is believed by everyone to become common knowledge[13].

The density function $f(\alpha)$ could then be interpreted as the posterior distribution arising from an estimation of the true proportion α. The result from an estimator $\underline{\alpha}$ would be the estimate α^*. The distribution of $\underline{\alpha}$ would have a maximum at $\alpha^*$and more probability mass around it the lower its variance[14/15]. Naturally, the larger the sample, the lower its variance would be: in the limit, were the whole population to be sampled, the true proportion of voters supporting B would be known with certainty. Then, we would have to assume that such census would be sufficiently costly to do, to exclude the possibility of existing any mass points in the distribution $f(\underline{\alpha})$.

Despite the above *possible* explanation for how $f(\alpha)$ *could* be generated, we will assume that $f(\alpha)$ is *exogenous* to our model[16].

To summarise, we introduce uncertainty by assuming that there is common knowledge about the following:

1) $\mathbf{q}_M = [\mathbf{q}_B + \mathbf{q}_C]$ - known with certainty;

2) $\alpha = [\mathbf{q}_B / \mathbf{q}_M]$ - known to take a value in [0, 1] according to $f(\alpha)$.

Since this assumption is an innovation compared to other approaches taken in the literature, we dedicate our next section to a (brief) critical review of the strategic voting literature that is more relevant to our work. Our emphasis will be on stressing the different assumptions on some models, and the extent to which they influence the final results.

13 This is a strong assumption, that requires not only that all voters have access to this information (for instance, that they have a TV license, and watch TV when the results from the opinion poll are announced) but also that they are sure that everyone else also has access to this information, and so on.

14 The use of maximum likelihood estimation is sufficient for this to be true.

15 To be more precise, we should say "the *estimated* variance of the *estimator* $\underline{\alpha}$, evaluated at the *estimated value* α^*".

16 For a setting where information is endogenously generated see Fey (1997) and Myatt (2004).

III. Critical review of the literature

1. Review of the literature on the three-party model

The literature on strategic voting is vast, and its focus has been the analysis of a three-party model with a plurality rule.

Early models as McKelvey and Ordershook (1972) and Hoffman (1982) used a decision-theoretic framework in their analyses, where all voters consider the probabilities of certain outcome as exogenous. So, each voter considers whether or not to vote strategically, assuming that others will vote sincerely. The equilibrium may entail many voters voting strategically, while none has accounted for that, which makes it a non *rational expectations* outcome.

Cox (1987) is the first to propose a model where the probabilities of certain outcomes are made endogenous, in a game-theoretical framework. He uses the Bayesian equilibrium concept to close the model, which includes preferences, beliefs and voter intentions. His main contribution was to show that strategic voting is the norm rather than the exception, and that this would favour a leading candidate, while hurting the trailing one.

Palfrey (1989), using a model similar to Cox's, claims to prove mathematically Duverger's Law, by showing that when the electoral grows without bound, only the two candidates get any votes, except for some *knife-edge* cases, where a third candidate would survive. He believes his work to be an internally consistent explanation of Duvergers' Law in terms of strategic voting alone.

Myerson and Weber (1993) describe a non-Duvergerian equilibrium that is not just a *knife-edge* case, thus concluding that Palfrey had overstated his results. They conclude by saying that "Duverger's Law cannot be derived exclusively from analyses of voting equilibria. (...) Any derivation of Duverger's Law would seem to require some additional assumption of dynamic stability or persistence, (...) to eliminate equilibria of the type just illustrated".

Fey (1997) responds to the above claim, arguing that the non-Duvergerian Myerson and Weber found is unstable. He examines how opinion polls can coordinate voters on particular Duvergerian equilibria and argues that coordination on a three-party equilibria is impossible.

Myatt (2002, 2004) proposes a model where there is stable multi-candidate support. This is achieved by assuming that the popular support for each challenger is uncertain and that each voter has both private and public information. As he says, the fact that "the uniquely stable voting equilibrium entails only limited strategic voting and hence partial coordination (...) is due to the surprising presence of negative feedback: An *increase* in strategic voting by others actually *reduces* the incentives for an individual to vote strategically".

2. Comparing the different models

Myatt (2002, 2004) stands out from the other papers by assuming uncertainty regarding the support of some of the candidates, while previous models all assume - explicitly or not - certainty regarding such support, when n is large[17].

To illustrate the differences and similarities to our work, we choose Myatt's and Cox's models. The former is chosen for its singularity, while the latter is chosen as a representative of the other type of approach, for its clarity and also for its breakthrough character.

First, we turn our attention to Cox's model.

Cox's model can be described through the following assumptions (see Cox 1994):

1. Voters are instrumentally rational;
2. Voters have incomplete information about all other voters' preferences, which is modelled assuming that there is a function G(.) describing voters' preferences;
3. Voters have common knowledge about the preference distribution G(.);
4. Individual voter preferences are drawn independently from the distribution G(.);
5. The *natural* support of each candidate becomes certain when $n \rightarrow \infty$[18];
6. Voters have rational expectations;
7. Voters' expectations are publicly generated, implying they all have the same expectations.

Within this framework, the apparent uncertainty introduced through the distribution of voters' preferences G(.) will disappear when n is large, due to statistical independence and the Law of Large Numbers, and this will lead an infinite incentive to vote strategically, and thus the Duvergerian equilibrium. This process is best described in Myatt (2002):

> «Unfortunately, the game-theoretic treatments share a common feature: Individual preferences and voting decisions are drawn *independently* from a *commonly known* distribution. It is this, and this alone, that leads to strictly Duvergerian predictions. To see why, consider the relative probability of pivotal events. Mathematically, as the electorate grows large, the probability of a pivotal event involving the leading challenger becomes infinitely larger than the probability of such an event involving the trailing candidate. Equivalently, and perhaps more intuitively, *if* a pivotal event occurs, then it *almost always* involves the leading challenger. Any instrumental voter will, therefore, switch her vote to the leader. Game-theoretic reasoning is unnecessary for a Duvergerian conclusion: As long as the electorate is large and voting decisions are drawn independently from a commonly known distribution, almost all instrumental voters will switch, even before they account for strategic switching by others.»

17 To be more precise, he assumes uncertainty only regarding the *two* challenging candidates. The support of the incumbent party is known, and so the supporters of the other two parties face a coordination game to defeat him. Since the incumbent's support is known with certainty, this is then a qualified majority game between the two challengers.

18 Though Cox doesn't state it as one of its assumptions, that is the case. We explicitly include it here to stress the difference between his model and ours.

Our assumptions are similar to Cox's in everything except for the fifth assumption, since in our model the common beliefs [about the voters' preferences] and common expectations [about the outcome of the election] have some uncertainty embodied in them, even when n is large.

This is what makes it possible to avoid the infinite incentive to vote in leading party that always arises in Cox's model.

To illustrate with an example, while in both models there would be common knowledge that the preferences of the natural supporters of parties B and C would be independently drawn from a (uniform) distribution, in Cox's there would also be common knowledge that the support of party B was 32%, while in our model voters would only know party B's support could take a value in [22%, 40%], and that such uncertainty would still exist regardless of the size of the electorate.

We can better understand the importance of the uncertainty over α with the use of the following definition.

Definition 1. *Conditional on the realisation of α, the probability that a randomly selected voter will vote for candidate A, B, C or D is denoted as π_A, π_B, π_C, or π_D, respectively.*

The Duvergerian result can be avoided only because *unconditional* on the realisation of α voters will *not* be sure about π_i, i = A, B, C, D.

If voters were sure about all parties' results, our model would be like Cox's, with no real uncertainty for n large, which would happen if $f(\alpha)$ converged to a unique mass point. Voting decisions would then be drawn independently from a multinomial distribution, and the probability of a tie between the two leading candidates would become infinitely larger than any other pivotal probability.

For instance, if the voting probabilities satisfy $\pi_A > \pi_B > \pi_C > \pi_D$, the probability of a pivotal event between 'A and B' would become infinitely larger than any other pivotal probability, so that all voters would have an incentive to vote either for A or for B, leading to the Duvergerian equilibria where only parties A and B would get any votes[19].

This happens because when n tends to infinity, the Law of Large Numbers will make the voting results exactly match the referred voting probabilities.

We now turn to Myatt's model.

19 There could be a non-Duvergerian equilibrium in the *knife-edge(s)* case where two (or more) of the π_i are tied. Furthermore, there can also be no strategic voting at all, when the voter has a 0 utility for both front-runners.

Myatt (2002) introduces uncertainty about the constituency support of the parties by assuming that each voter knows his own (relative) preference but is not able to observe the decomposition in their *common* and *idiosyncratic* parts. Since the common part can be interpreted as the relative preference of the median voter, that determines the exact prior support for each of the two challenging candidates. Regardless of the size of the electorate, everyone is uncertain about the median of the distribution.

Naturally, each voter will have some beliefs about such common component; these are brought to the model by assuming each voter observes a *private* signal that allows him to update the common and diffuse prior belief about such component, according to Bayes' rule. This is the what leads to the *private information* character of Myatt's approach.

In our model, the preferences of all voters are assumed to be common knowledge, so that everyone will always be certain about how all types of voters will vote. The uncertainty comes from the fact that the prior distribution of types depends on the draw of some distribution function $F(\alpha)$. This means that all voters know how voters of a certain type will vote, *but they do not know what proportion of the electorate such voters represent*; they only know the distribution over such proportion.

Our approach is different than Myatt's because in our model there is common knowledge about the uncertainty regarding the prior support of some parties, while in Myatt's that does not happen - rather, each voter forms a different belief about the (uncertain) 'constituency support'.

In other words, our model is *'a model of public information with uncertainty about the prior support of each candidate'*, where there is common knowledge about such prior support and the preferences of each voter type, and all voters hold the same beliefs about the distribution of support for all candidates.

Myatt's model is *'a model of (both) private and public information with uncertainty about the prior support of each candidate'*, in that each voter observes some signals privately, which leads to different beliefs about the candidates' support, despite the fact that each starts with a common and diffuse prior.

To illustrate with an example, while in our model all voters would agree that the support of party B would be in [22%, 42%], in Myatt's, each voter would have its own particular opinion about what that distribution would be. Some voters would believe it could take a value in [23%, 37%] while other voters could consider the interval [31%, 52%], perhaps because they had had a sample more biased towards party B.

We summarise the previous differences between the three approaches in Table 4, by choosing the exact levels of information that differentiate each approach from the other two.

Table 4. *Comparing the assumptions of the models*

Certainty regarding	Cox's (1994)	Ours	Myatt (2004)
The preferences of *a particular* voter	No	No	No
The *distribution* of voters' preference	Yes	Yes	No
The *natural support* of each of the parties when *n* is large	Yes	No	No

The different character of these three models can be better understood in the frame proposed in the next table.

Table 5. *The different character of the three models*

	Certainty	Uncertainty
Public	Cox's	Ours
Private	---[20]	Myatt's

When *n* gets arbitrarily large, Cox's model becomes a model of *public certainty*.

On the other extreme, Myatt's model truly a model of *private uncertainty*.

Our model lies in between, since it is a model of *public uncertainty*.

In the next section we analyse the different models we have introduced.

Section IV should be read sequentially, since a thorough description of the procedure to find and characterise the equilibria is done for Model I, while being somewhat abbreviated for Models II and III, to avoid unnecessary repetition.

20 "Private certainty" is somewhat paradoxical. There can't be certainty when any information is only held privately. If a voter has *private* information about the preferences of any other voters, that implies that other voters do *not* have that same information - or, better, that they do not *necessarily* have so. They *may* have the same information but they will *not* know it is the *same*. Hence, they *may* hold different *beliefs*, which is tantamount to say that there is *uncertainty*. Certainty requires *common knowledge* of the relevant information, and this is opposed, by definition, to players having private information.

IV. Equilibrium analysis

1. The equilibrium - an overview

We will use a *game-theoretical* rather than a *decision-theoretical* approach. This means that each voter thinks strategically - by considering whether he should cast a vote for his first or second preferred candidate - and takes into account that others consider voting strategically as well.

We have assumed voters to be expected utility maximisers and instrumentally rational, which together implies that they will only be concerned with the identity of the winner. Consequently, they will only need to evaluate the probabilities of the events where there is a dispute for the first place; these are the *pivotal events*.

We recall that in any Nash equilibrium two conditions have to be met:

i) *optimal behaviour*: each agent, in equilibrium, acts in a way that is a best response to all other agents' actions:

ii) *rational expectations*: in equilibrium, all the beliefs are fulfilled.

In the context of our model of strategic voting, this means, respectively, that[21]:

i) each agent decides whether or not to vote strategically, taking into consideration what all other voters do, by assessing what are the probabilities of the pivotal events involving their first and/or second most preferred candidates;

ii) in equilibrium, the beliefs held by the voters will be fulfilled, subject to the restrictions imposed by the uncertainty over α, i.e., the voting results will be as expected, but they may entail some uncertainty[22].

Next, we consider how the probabilities of the pivotal events should be calculated.

2. The pivotal probabilities

In our approach, we take the view of a single voter that considers the other (n - 1) voters behaviour when making his choice. For him, a pivotal event will be an event where the votes of the remaining voters lead to a *tie* or a *near-tie* for the lead. Only in these cases can a single vote influence the outcome of the election, by, respectively, *break* or *force* a tie[23].

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21 We will provide a more precise description of the equilibrium notion we use - the Bayesian-Nash voting equilibrium - when we analyse Model I.

22 To put it in another way, voters will know - *prior to a draw from the distribution $f(\alpha)$ being made* - the distribution of the results for all parties, after taking into account the strategic votes that take place. In this sense, there will be self-fulfillement of expectations regarding the *distribution* of the parties' results - prior to the the draw from $f(\alpha)$ being known.

23 'Breaking a tie' always leads to the victory of the party for which a vote was cast, while 'forcing a tie' leads

All the non-pivotal events are thus irrelevant for an instrumentally rational voter. Moreover, and contrary to common intuition, the absolute probability of influencing the outcome is completely irrelevant - only the conditional probabilities of being pivotal matter. In Myatt's words (2002b):

> «An instrumental voter considers the relative chances of different pairs of parties being involved in a tie (or near-tie) for the lead. In effect they take it for granted that they are pivotal when deciding who to vote for. Instead of asking, "What are the chances of being pivotal?" as supposed by standard intuition, they ask, "Who will I be pivotal between supposing that I am pivotal?"».

We illustrate this idea with a very simple example.

Imagine there are three parties, X, Y and Z, and a voter that has a utility of 6, 3 and 1, respectively, for each of the parties. The voter will never vote for Z, since it is ranked lowest, but may vote sincerely for X, or strategically for Y. Let's assume that there are only two pivotal events: one where X and Z (exactly) tie, and one where Y and Z (exactly) tie. The voter will then realise that in such events any vote he casts will be decisive about who wins the election.

Should he vote for X or Y? It depends. Let's say that the probability of a tie between X and Z is 0.002%, while that between Y and Z is 0.01%. The expected utility of voting for X is 0.002% * 6 = 0.00012, while a vote in Y gives 0.01% * 3 = 0.0003. He is better off voting for Y, even though Y gives him a lower utility, because he is pivotal for Y with a *sufficiently* higher probability, such that the expected utility is highest when he votes for Y[24].

The analysis for our model is slightly more complex. In fact, with four parties, further to two-way ties and near-two ties, there are three-way ties, near-three way ties, four-way ties and near four way ties. Myatt (2002b) shows that in a three-party model the near two-way ties and two-way ties are asymptotically equivalent as $n \to \infty$, which also applies to three-way and near three-way ties. He also shows that as $n \to \infty$, the probability of a three-way tie vanishes at rate n^{-2}, whereas the probability of a two-way tie vanishes at rate n^{-1}; this means that the three-way ties (and hence the near three-way ties) will be asymptotically irrelevant and only the two-way ties need be considered.

These conclusions can be replicated to our model, since we're using the same tie-breaking rule, which makes the nature of the pivotal outcomes identical in everything except for the number of parties[25].

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to a case where the k parties involved in the (forced) tie win with a probability of $1/k$, following the equiprobable tie-breaking rule we have adopted. This is the usual definition of a *tie* and a *near tie*.

24 We could also use the conditional probabilities. He knows that - *if* he is pivotal - he *could* be pivotal for X with probability 1/6 or for Y with probability 5/6, following a simple application of Bayes' rule. The relative likelihood of a pivotal event involving Y compared to X is (5/6) / (1/6) = 5.

25 Interestingly, because all voters have a zero utility from a win by both their third and fourth preference candidates, each voter needs only evaluate three outcomes: the voting results for his two preferred candidates and the maximum of the voting results for his third and fourth preference candidates. In this sense, our approach is in fact equivalent to that in Myatt's three-party model, and so a direct parallel is

The rationale can be extended to the four-way and near four-way ties, which also be asymptotically irrelevant since they would vanish at rate n^{-3}. The only relevant ties are, thus, the two-way and near two-way ties.

Definition 2. *Conditional on the realisation of α, we denote the number of votes that parties A, B, C, and D have, amongst the remaining (*n - 1*) voters, as* a, b, c *and* d, *respectively.*

To illustrate the pivotal events, take an *AB* type voter in Model I (which has a utility 1 for his first choice A, a utility $ß_B$ for his second choice B and a zero utility for C and D). One of the pivotal events that he would consider is the two-way tie between parties A and B, which we denote as Pr[a = b], where we implicitly mean Pr[a = b > max (c + 1, d + 1)]. The near two-way tie would happen, for instance, if party B got one more vote than A amongst the other (*n* - 1) voters, so a vote for A would force a tie between them. We can write this as Pr[a = b - 1]. Since they are asymptotically equivalent, Pr[a = b] ≈ Pr[a = b - 1] as $n \to \infty$.

Similarly, the four-way tie would happen with probability Pr[a = b = c = d], while one of the possible near four-way ties would take place with probability Pr[a + 1 = b = c = d], where a vote in A would force a four-way tie. One three-way tie would occur with probability Pr[a = b = c > d + 1], and a near three-way with Pr[a + 1 = b = c > d + 1].

We will provide a detailed description of the pivotal probabilities when we describe the optimal behaviour of voters in Model I.

In some scenarios, a voter will not need to calculate any probabilities of pivotal events taking place. This happens when none of the parties he likes can may win, regardless of the behaviour of the electorate. If a party can never win the race, voting for him will be a *weakly dominated strategy* for any instrumental voter; we analyse this issue next.

3. The (weakly) dominance argument

Since all voters are rational, we naturally rule out any equilibrium that involves the use of *weakly dominated strategies*. A strategy is said to be weakly dominated if it never yields a better outcome and sometimes yields a worst outcome. A rational voter would never choose to play a weakly dominated strategy, since his goal is to maximise his expected utility.
Also, because we assume common knowledge of rationality, we can take the next step and proceed to the *iterated elimination of weakly dominated strategies*. Dhillon and Lockwood (2000) describe the rationale behind this procedure in an elegant way:

> «First, we argue below that eliminating weakly dominated strategies is very reasonable in the plurality rule game; it simply amounts to no-one voting for her worst-ranked alternative. But, there is nothing to stop voters going a step further and recalculating

possible.

which strategies are weakly dominated for them given that *other* voters will not use weakly dominated strategies. In other words, if we *iteratively* eliminate weakly dominated strategies, it is possible that we could substantially narrow down the set of possible outcomes in the plurality voting game. Indeed, it is possible that so many strategies could be eliminated via iterated deletion that the remaining strategies can generate only one outcome: that is, the plurality voting game could be *dominance-solvable*.»

Some authors, like Besley and Coate (1997) argue only for the elimination of weakly dominated strategies, while others as De Sinopoli and Turrini (2002) and the supra cited argue that such is not enough, that iterated elimination of weakly dominated strategies should also be used. It is worth pointing that De Sinopoli and Turrini (2000) say that such procedure is a 'simple point', without mentioning why could have Besley and Coate (1997) not seen such 'simple point'.

In fact, the distance between the two is nothing else than the well-known difference between assuming *rationality* and *common knowledge of rationality* for all players in a game. This difference is particularly relevant in a voting game, where the number of players and the amount of information to be processes may be considerably large.

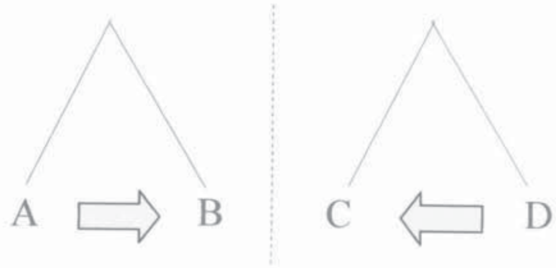
In our case, the results are heavily dependent on the common knowledge assumption; a departure from it would lead to a model with private information, which is not our aim here.

4.I Equilibrium analysis for Model I

4.I.1 Description of the model

The following information is a reminder of the structure of Model I.

Figure 1. *The strategic scenario in Model I (reminder)*



Strategic voting can occur only within each side of the political spectrum, and only in one direction: from the extremist party to the moderate one.

Table 1. *Voters' preferences in Model I (reminder)*

Voter Type	Utility for candidate				Prior Probability
	A	B	C	D	
AB	1	$\beta_A \sim U[0,1]$	0	0	q_{AB}
B	0	1	0	0	q_B
C	0	0	1	0	q_C
DC	0	0	$\beta_D \sim U[0,1]$	1	q_{DC}

We introduce uncertainty by assuming there is common knowledge about the following:

$$\mathbf{q}_M = [\mathbf{q}_B + \mathbf{q}_C]$$

$$\alpha = [\mathbf{q}_B / \mathbf{q}_M] \text{ in } [0, 1] \text{ with density function } f(\alpha)$$

$$\beta_i \sim U[0, 1],\ i = A, D$$

Voter types *AB* and *DC* have a preference for their second candidate that is represented, respectively, by the parameters β_A and β_D, both uniformly distributed between 0 and 1.

4.I.2 Voters' optimal strategies

Following the previous discussion on the weakly dominated strategies, we know that no voter will ever vote for one of his two disliked candidates, which by construction give him a zero utility - his choice is then reduced to the first and second candidates.

Take an *AB* voter, that has a utility 1 for his first choice A and utility $ß_A$ for his second choice B. Consider a scenario in which he abstains from voting, and consider the effect of a vote for A. As we know, only the pivotal events matter for an instrumentally rational voter, so a vote for A, rather than abstention, can only have an effect when A is involved in a tie. The voter will evaluate the pivotal probabilities based on the votes of the other n - 1 voters, and because n is large, we will ignore three-way and four-way ties.

First, there could be a tie between A and B. As described earlier, we denote the probability of such event as Pr[a = b], where we implicitly mean Pr[a = b > max {c + 1, d + 1}]. Given that ties are broken randomly, a vote for A rather than abstention yields a change in expected utility of

$$\Pr[a = b] \times [1 - (1 + ß_A)/2] + \Pr[a = b - 1] \times [(1 + ß_A)/2 - ß_A)] \approx \Pr[a = b] \times (1 - ß_A),$$

since the probability of a near two-way tie is asymptotically equivalent to that of a two-way tie, as we explained in the last section. So, for a large n, we have $\Pr[a = b] \approx \Pr[a = b - 1]$.

Next, there can be a tie between A and C. The change in expected utility, relative to abstention, would then be

$$\Pr[a = c] \times [1 - 1/2] + \Pr[a = c - 1] \times 1/2 \approx \Pr[a = c].$$

Finally, there can be a tie between A and D. The change in expected utility, relative to abstention, would be

$$\Pr[a = d] \times [1 - 1/2] + \Pr[a = d -1] \times 1/2 \approx \Pr[a = d].$$

Thus, the total change in expected utility, relative to abstention, is approximately (with again the approximation becoming accurate as $n \to \infty$)

$$(1 - ß_A) \Pr[a = b] + \Pr[a = c] + \Pr[a = d].$$

Notice that this expression is positive, and hence a voter would rather vote for A than abstain[26].

Next, we must consider the change in expected utility stemming from a vote for B, again relative to abstention.

26 Indeed, abstaining is a weakly dominated strategy in this game, simply because the act of voting carries no physical cost; so a voter weakly prefers to vote for his preferred candidate than to abstain.

For ties involving A and B, this change is

$$Pr[a = b] \times [ß_A - (1 + ß_A)/2] + Pr[a = b - 1] \times [(1 + ß_A)/2 - 1)] \approx Pr[a = b] \times (ß_A - 1).$$

For ties involving B and C, we obtain a change of (approximately) $ß_A \times Pr[b = c]$, and similarly an effect of $ß_A \times Pr[b = d]$ for ties involving B and D. The total change in expected utility is, then,

$$(ß_A - 1)\ Pr[a = b] + ß_A\ (Pr[a = c] + Pr[a = d]).$$

Contrary to the expression obtained for a vote in A, this may well be negative, if $Pr[a = b]$ is too large relative to the other two pivotal probabilities. The intuition is immediate: if amongst all possible pivotal events involving B, the one involving A - the candidate that gives him highest utility - has a *considerably higher* probability than the other two, it would be better to abstain than to vote for B.

An *AB* voter will choose to vote for either A or B. It will be optimal to vote for A when

$$(1 - ß_A)\ Pr[a = b] + Pr[a = c] + Pr[a = d] \geq (ß_A - 1)\ Pr[a = b] + ß_A\ (Pr[a = c] + Pr[a = d])$$

$$\Leftrightarrow 2(1 - ß_A)\ Pr[a = b] + Pr[a = c] + Pr[a = d] \geq ß_A\ (Pr[a = c] + Pr[a = d])$$

$$\Leftrightarrow 2\ Pr[a = b] + Pr[a = c] + Pr[a = d] \geq\ ß_A\ (2\ Pr[a = b] + Pr[a = c] + Pr[a = d])$$

$$\Leftrightarrow ß_A \leq \{2\ Pr[a = b] + Pr[a = c] + Pr[a = d]\}\ /\ \{2\ Pr[a = b] + Pr[b = c] + Pr[b = d]\}.$$

Not surprisingly, we get an expression that tells us that an *AB* voter will vote for his natural candidate if the utility for his second preferred candidate is *sufficiently* low.

Consider the right hand side of the last inequality. The numerator includes all the pivotal probabilities that involve party A, but $Pr[a = b]$ is multiplied by a coefficient of 2. This highlights the fact that a tie between A and B is, in some sense, of twice the importance. Similarly, the denominator includes all the pivotal probabilities that involve party B, where $Pr[a = b]$ has twice the weight as well.

The ratio on the right hand side of the above expression is called the *likelihood ratio*. It expresses the *likelihood* of a pivotal event involving party A - *relative* to the *likelihood* of a pivotal event involving party B.

Definition 3. *Define, whenever possible, the likelihood ratio* $\lambda_A = \{2\ Pr[a = b] + Pr[a = c] + Pr[a = d]\}\ /\ \{2\ Pr[a = b] + Pr[b = c] + Pr[b = d]\}$.

The likelihood ratio will not be defined when both the numerator and denominator are equal to zero, since a ratio of 0/0 is undetermined. This will happen when there are no pivotal events involving either of the two preferred candidates[27]. Whenever the likelihood ratio is defined, we can summarise the condition for optimal behaviour of *AB* type voters in the following way.

Lemma 1.a. *Optimality condition for* AB *type voters*

	A	if	$\beta_A < \lambda_A$
Vote for:	B	if	$\beta_A > \lambda_A$
	A *or* B[28]	if	$\beta_A = \lambda_A$

This is a simple *cut-off* rule that divides the natural supporters of party A in those voting strategically and sincerely: in equilibrium, an *AB* voter votes sincerely *iff* $\beta_A \leq \lambda_A$. He votes for A if the *relative* preference for A against B exceeds the *relative* likelihood of his single vote influencing the outcome for B against A.

Namely, we know that a *necessary* condition for him to vote strategically is that Pr[b = c] + Pr[b = d] > Pr[a = c] + Pr[a = d][29]; in other words, there must be a higher probability that the second-choice candidate ties with a disliked candidate. Intuitively, a *lower* relative preference for A (a *higher* β_A), will make the condition for a strategic vote to be cast be more easily satisfied.

The type voters *B* and *C* always vote for their favourite parties so in a sense we can ignore the optimality conditions for their behaviour, for simplicity.

Type voters *DC* face the exact same problem as *AB* type voters, so we can replicate all the results we used for *AB* voters. The condition for their optimal behaviour is:

Lemma 1.b. *Optimality condition for* DC *type voters*

	D	if	$\beta_D < \lambda_D$
Vote for:	C	if	$\beta_D > \lambda_D$
	D *or* C	if	$\beta_D = \lambda_D$

27 Therefore implying that all strategies are admissable, as they cannot be excluded on grounds of rationalisability.

28 We assume that an indifferent voter will vote for his preferred candidate.

29 Otherwise the likelihood ratio would always be higher than 1, which would exclude the possibility of any strategic vote, since $\beta_A > 1$ is impossible.

Whenever defined, the two likelihood ratios can be written as follows:

$$\lambda_A = \{2 \Pr[a = b] + \Pr[a = c] + \Pr[a = d]\} / \{2 \Pr[a = b] + \Pr[b = c] + \Pr[b = d]\}$$

$$\lambda_D = \{2 \Pr[d = c] + \Pr[d = b] + \Pr[d = a]\} / \{2 \Pr[d = c] + \Pr[c = b] + \Pr[c = a]\}$$

We see that λ_A and λ_D have identical term in their formulas. In fact, all pivotal probabilities enter the two formulas except for Pr[a = b] and Pr[d = c], which comes from the fact that the two preferred candidates of voters *AB* and *DC* do not have any party in common.

This stands as a key difference between our model and the general three-party model. In the latter, the supporters of the non-incumbent parties consider voting for the *same* two candidates, which brings two critical strategic features: i) a likelihood ratio that is the inverse of that for the other voter; and ii) a 'Battle of Sexes' game structure.

Importantly, in our model the two likelihood ratios are not completely dependent, which constitutes an important *strategic* difference between the two approaches[30].

4.I.3 The Bayesian-Nash voting equilibrium

We know that the likelihood ratios can take any positive value, while the preference parameter β_i can only take values between 0 and 1.

This fact along with Lemmas 1.a and 1.b enables us to summarise the condition required for optimal behaviour of voters in equilibrium.

Condition 1.I. ***Optimal behaviour in equilibrium (Model I)***

In an equilibrium, types B *and* C *voters vote respectively for B and C, and types* AB *and* DC *voters use the following strategy:*

AB *type: "Vote for A if* $\beta_A \le \beta^*_A$ *and vote for B if* $\beta_A > \beta^*_A$*", where*

$$\beta^*_A = \min\{1, \lambda_A\}$$

DC *type: "Vote for D if* $\beta_D \le \beta^*_D$ *and vote for C if* $\beta_D > \beta^*_D$*", where*

$$\beta^*_D = \min\{1, \lambda_D\}$$

30 This makes us suspect that equilibria other than the Duvergerian may arise. As explained previously, in Cox's model, when *n* gets arbitrarily large the incentive to vote strategically becomes infinitely large. In other words, one of the likelihood ratios will tend to infinity, while the other will *necessarily* tend to zero (since they're the inverse of each other). In our model, however, because there is no *direct* link between the two ratios, there is not such a *strong* implication between the behaviour of two groups of voters.

A voting equilibrium will be reached when the strategies of all players are not only *optimal* but also *self-fulfilling*, or *consistent*. We formalise the latter in the following condition.

Condition 2.I. ***Rational expectations in equilibrium (Model I)***

$\pi_A = q_{AB}\, ß^*_A$

$\pi_B = q_B + q_{AB}\,(1 - ß^*_A) = q_{AB}\,(1 - ß^*_A) + \alpha\, q_M$

$\pi_C = q_C + q_{DC}\,(1 - ß^*_D) = [q_{DC}\,(1 - ß^*_D) + q_M] - \alpha\, q_M$

$\pi_D = q_{DC}\, ß^*_D$

Recall that, in general, none of the probabilities π_A, π_B, π_C, and π_D need be *determined* in equilibrium, i.e., they need not take a value that is known with certainty. In Model I, only the probabilities π_B and π_C are *stochastic* in equilibrium, as they depend on the prior support of parties B and C - and these involves some uncertainty, represented by α. This is highlighted in the expressions for π_B and π_C above. Importantly, this is what allows our model to depart from the undesirable characteristic of Cox's model, where, following Myatt, "uncertainty is only apparent".

There is an obvious interdependency between the two conditions. From condition 2, we know that the cutpoints ($ß^*_A$, $ß^*_D$) will determine the support for parties A and D, which will be known with certainty in equilibrium. They will also influence the results for B and C, though these will be uncertain in equilibrium, since they also depend on the draw from the density $f(\alpha)$. Since they cutpoints ($ß^*_A$, $ß^*_D$) will influence each of the parties' support in equilibrium, they will affect the pivotal probabilities and, consequently, the likelihood ratio, which enters in condition 1.

An equilibrium is reached when the 'loop' between these two conditions is closed, which is described in the equilibrium notion that we will use.

Definition 4. *A Bayesian-Nash voting equilibrium (denoted as BNVE) is a pair* ($ß^*_A$, $ß^*_D$) *such that: i) every voter is optimising given everyone else's strategies; and ii) every voter expects the equilibrium to take place, and that is commonly known*[31]. *In other words,* ($ß^*_A$, $ß^*_D$) *is a pair satisfying conditions 1 and 2.*

31 In Myerson and Weber's words (1993): «A voting equilibrium arises when the voters in an electorate, acting in accordance with both their preferences for the candidates and their perceptions of the relative chances of various pairs of candidates being in contention for victory, generate an election result that justifies their perceptions».

4.I.4 Study of possible equilibria

Our approach will be focused on describing the different equilibria. We draw extensively on graphical illustration, from which intuition can be more easily built.

Next we describe the different equilibria - case by case.

Case 1: $q_{DC} \leq q_{AB} \leq q_M/2$[32]

This case is described in Figure 6 below, assuming that all voters vote sincerely.

In the *x*-axis we plot α, which can vary between 0 and 1. In the *y*-axis we represent the voting results that each party gets for a certain vector (β_A, β_D), which need not be an equilibrium one - these are π_i, i = A, B, C, D[33].

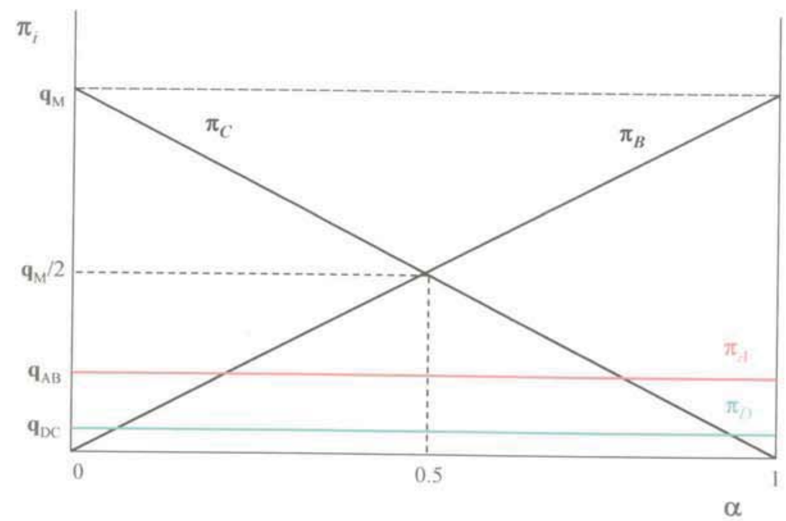


Figure 6 – The case where $q_{AB} < q_M/2$ – Model I

If all voters vote sincerely, party A obtains votes from (and only from) all the *AB* voters, which are a proportion q_{AB} of the electorate. This proportion does not depend on α, since we assumed the prior support for the extremist parties to be known with certainty. Therefore, π_A is a flat line in our graphic: $\pi_A = q_{AB}$. A similar reasoning applies to party D; since its prior support is independent of α as well, party D obtains a result $\pi_D = q_{DC}$.

32 We will assume - without loss of generality - that $q_{DC} \leq q_{AB}$.

33 So, we plot the results as probabilities, and not as sum of votes. Recall definition 1 to check the definition of π_i.

Party B would obtain votes from *B* voters only. These votes are stochastic, and depend positively on α. Recall that we have defined $\alpha = [\mathbf{q}_B / \mathbf{q}_M]$, and so the prior support of party B is $\mathbf{q}_B = \mathbf{q}_M \alpha$. Since party B only gets sincere votes, we have $\pi_B = \mathbf{q}_B = \alpha\, \mathbf{q}_M$, and so its voting results are a line with slope $\mathbf{q}_M$ and a 0 intercept. A similar reasoning applies to party C, whose voting result is $\pi_C = \mathbf{q}_C = \mathbf{q}_M (1 - \alpha) = \mathbf{q}_M - \alpha\, \mathbf{q}_M$, so, with a slope - $\mathbf{q}_M$ and an intercept $\mathbf{q}_M$.

When all voters vote sincerely, π_B and π_C cross at $\alpha = 0.5$, where they are equal to $\mathbf{q}_M/2$. In case 1, parties A and D have a prior support lower than $\mathbf{q}_M/2$, so their lines are always below the maximum between those of B and C, which implies that they can never win the election.

Given that both parties A and D can *never* win the election, it is a weakly dominated strategy for voters *AB* and *DC* to vote sincerely. They will both vote strategically and so the unique equilibrium arising in case 1 in the F-DE $(\beta^*_A, \beta^*_D) = (0, 0)$. In other words, the game here is *dominance-solvable*, since it entails only one possible prediction. The equilibrium is shown in Figure 7.

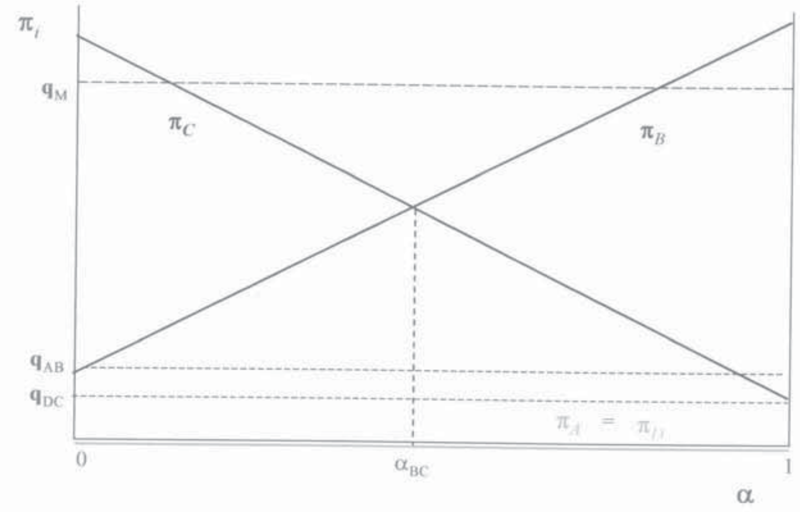


Figure 7 – The F-DE $(\beta^*_A, \beta^*_D) = (0, 0)$ – Model I

When voters *AB* vote strategically, party A line suffers a parallel downward shift. In this case, since *all AB* voters switch their vote to B, that shift goes till zero, coinciding with the *x*-axis: we have $\pi_A = 0$. Similarly, $\pi_D = 0$, since *all DC* voters switch their votes as well. The strategic votes go to the moderate parties B and C, whose lines thus suffer an upward parallel shift of, respectively, $\mathbf{q}_{AB}$ and $\mathbf{q}_{DC}$.

Definition 5. *Define α_{ij} as the draw from $f(\alpha)$ for which curves π_i and π_j intersect.*

Another way to describe how the equilibrium arises is by noting that both supporters *AB* and *DC* realise they can only be pivotal between 'B and C'. Conditional on the existence of a pivotal event, that would be between 'B and C' with probability 1 as $n \to \infty$, by the Law of Large Numbers. Hence, each of them has an infinite incentive to vote strategically for their second preferred candidate.

From the above figure, we see that the pivotal event between 'B and C' occurs when $\alpha = \alpha_{BC}$. When $n \to \infty$, the probability of a pivotal event between parties B and C tends to zero, while the likelihood ratios for both voters are proportional to $0/f(\alpha_{BC}) = 0$[34].

We need to verify that the two conditions required for equilibrium are met.

According to condition 1, the zero likelihood ratio implies $\beta^*_i = 0$, and so this condition is clearly satisfied, since all voters vote strategically in equilibrium. Condition 2 is trivially satisfied, as parties A and D get zero votes and the moderate parties get all votes from the voters on their side of the political arena. Thus, $(\beta^*_A, \beta^*_D) = (0, 0)$ is a Bayesian-Nash voting equilibrium. Because it involves only two parties getting any votes, this is a full-Duvergerian one.

Further, it can take place for all possible values of $\mathbf{q}_{AB}$ and $\mathbf{q}_{DC}$, since it involves self-fulfilling expectations that are optimal in equilibrium: if all extremist voters decide to vote strategically, then, *once that equilibrium is commonly expected to be played*, no one will have - *individually* - an incentive to deviate.

To put it in another way, once voters realise that the F-DE implies total erosion of his favourite party support, voting sincerely becomes a weakly dominated strategy, since such party can never win the election. In this sense, the F-DE is *always possible*.

We summarise all this information in the following way:

BNVE I.1 $(\beta^*_A, \beta^*_D) = (0, 0)$ | F-DE | *always possible.*

The winner of this election will depend on the draw of the uniform distribution. Party C wins if $0 \leq \alpha < \alpha_{BC}$ and party B wins if $\alpha_{BC} < \alpha \leq 1$, so, B wins with probability $1 - F(\alpha_{BC})$ and C with probability $F(\alpha_{BC})$; if $\alpha = \alpha_{BC}$ there is a tie and each of them wins with a probability of 50%.

Case 2: $\mathbf{q}_{DC} \leq \mathbf{q}_M/2 < \mathbf{q}_{AB} \leq \mathbf{q}_M/2 + \mathbf{q}_{DC}/2$

Figure 8 illustrates case 2, in the scenario where all voters vote strategically.

34 Recall that for a continuous function - which we assumed $f(\alpha)$ to be - the probability of it taking any particular value is zero. Also, recall we have assumed that $f(\alpha) > 0$ for all α in [0, 1].

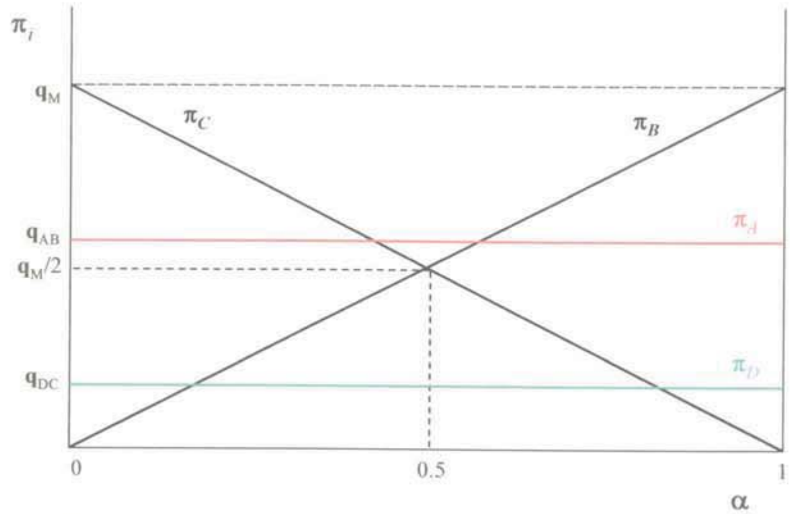


Figure 8 – The case where $\mathbf{q}_{DC} < \mathbf{q}_M/2 < \mathbf{q}_{AB}$ – Model I

Figure 8 is in everything identical to Figure 6, except for A's line. All curves in Figure 8 represent the prior support of each of the parties: parties B and C curves have the same slopes and intercepts as in Figure 6, which means they still intercept when $\alpha = 0.5$, with a result of $\mathbf{q}_M/2$. Party D has a prior support lower than $\mathbf{q}_M/2$, so its curve lies below the interception of B and C's curves; this means party D can never win.

The difference between case 1 and case 2 is that, differently from party D, party A has a chance of winning the election, since its curve lies above the interception between B and C's curves, as we have assumed $\mathbf{q}_{AB} > \mathbf{q}_M/2$.

Again using the concept of weakly dominated strategies, we see that voting sincerely is not a weakly dominated strategy for *AB* voters, but it is for *DC* voters, as party D can never win. So, the later will always switch their vote to party C, leading to a parallel shift in curves π_C and π_D of, respectively, $\mathbf{q}_{DC}$ and - $\mathbf{q}_{DC}$; this is described in Figure 9.

Case 2 assumes that $\mathbf{q}_{AB} < \mathbf{q}_M/2 + \mathbf{q}_{DC}/2$. This means that after accounting for the strategic votes that party C receives from *DC* voters, party A has no longer a chance of winning, since its line will then lie below the new interception point of curves π_B and π_C. Even though voting for A is not - in the game - a weakly dominated strategy for *AB* voters, it becomes so once they take into account that *DC* voters always vote strategically for C.

This illustrates the iterated elimination of weakly dominated strategies: first, we can only be sure that *DC* voters never vote for D, but once such strategy is excluded voting sincerely becomes a weakly dominated strategy for *AB* voters as well.

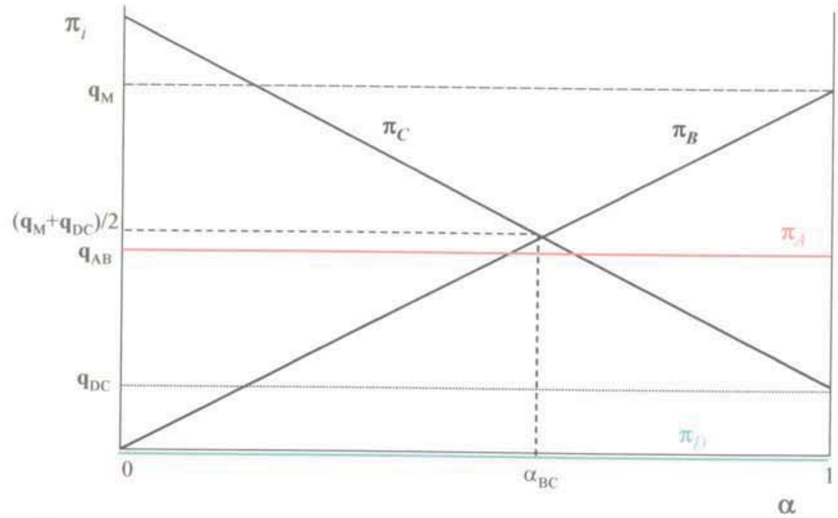


Figure 9 – The case where $q_{DC} < q_M/2 < q_{AB} < (q_M + q_{DC})/2$: A can never win if *DC* voters vote for C, leading to the F-DE $(\beta^*_A, \beta^*_D) = (0, 0)$ – Model I

Therefore, all voters vote strategically, and again we reach a F-DE, similar to that in case 1. The illustration is naturally identical to that in Figure 7, where the lines for B and C suffer an upward shift due to the strategic votes received, while A and D get zero votes. Similarly as well are conditions 1 and 2 satisfied.

The equilibrium found is identical to BNVE I.1 and the outcome of the election is stochastic in the same way described for such equilibrium.

Compared to case 1, the game in case 2 is still dominance-solvable (leading to the uniqueness of the F-DE) though it requires two steps in the process of iteration of weakly dominated strategies.

The above illustrates the importance of the assumption of common knowledge of rationality, compared to an assumption of rationality only. If *AB* voters were rational but weren't certain that *DC* voters were rational too, they could not be sure that *DC* voters would never vote for D; if this were the case, they couldn't exclude voting for A on grounds of rationality. Only with common knowledge of rationality can they exclude other players' weakly dominated strategies.

Case 3: $q_{DC} \le q_M/2 < q_M/2 + q_{DC}/2 \le q_{AB}$

This is a more interesting case, since party A can win, even after taking into account the strategic votes from *DC* voters to party C.

We illustrate this in Figure 10, assuming *AB* voters vote sincerely.

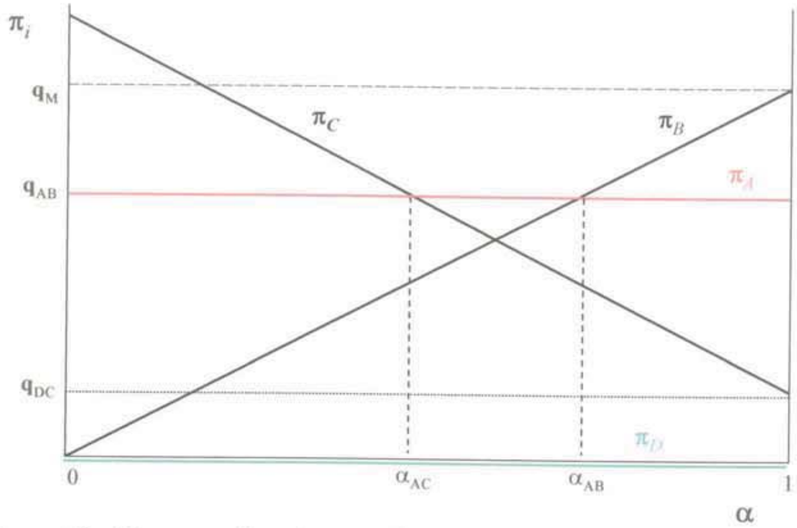


Figure 10 – The case where $(q_{DC}+q_M)/2 < q_{AB}$: A can win and so a S-DE $(\beta^*_A, \beta^*_D) = (1, 0)$ is always possible, regardless of $f(\alpha)$ – Model I

Because party A has a chance of winning the election, it is not a weakly dominated strategy for *AB* voters to vote sincerely. Could this be an equilibrium?

We see that if all *AB* voters vote sincerely, there are two pivotal events: one involving a tie between parties 'A and C' and the other between 'A and B'. each of these occurs, respectively, when $\alpha = \alpha_{AC}$ and $\alpha = \alpha_{AB}$. Then, an *AB* voter will be *forced* to vote for A in the scenario illustrated in Figure 10, simply because A is present in *both* events and brings a higher utility than any of the other candidates. Voting for A would be the best choice in *any* of the two events, and so an *AB* voter doesn't need to make any trade-off: he will always vote for A.

In other words, *if any* of the *AB* voters expects *all* other *AB* voters to vote sincerely, then his optimal best response will be to vote sincerely - and so this expectation is self-fulfilled. Formally, when all *AB* voters vote sincerely, the likelihood ratio considered by any such voter has the following expression:

$$\lambda_A = \{2\ \Pr[a = b] + \Pr[a = c]\} / 2\ \Pr[a = b]\ =\ 1 + 0.5\ \{\Pr[a = c] / \Pr[a = b]\} > 1,$$

and so, from condition 1 we get $\beta^*_A = 1$, which means that indeed it is optimal to vote sincerely. Condition 1 is also satisfied for *DC* voters since they can be pivotal for C but not for D, leading to a likelihood ratio equal to zero and $\beta^*_D = 0$, which is in accordance with all of them voting strategically. Condition 2 is trivially satisfied for both voters.

Therefore, there is an equilibrium where all *AB* voters vote sincerely and all *DC* voters vote strategically; it is $(\beta^*_A, \beta^*_D) = (1, 0)$. This entails three parties getting votes in equilibrium, and so it is a S-DE, according to the label we have proposed.

Further, this S-DE only requires that $\mathbf{q}_{AB} \geq \mathbf{q}_M/2 + \mathbf{q}_{DC}/2$: no condition is required for the *DC* voters' behaviour to be optimal, since when they vote strategically for C that becomes an optimal individual behaviour for any of them, as D would have no chance of winning; for *AB* voters' behaviour to be optimal, we require that the prior support of party D is not so high as to make it impossible for party A to win after the strategic votes for C are taken into account[35].

BNVE I.2 $(\beta^*_A, \beta^*_D) = (1, 0)$ | S-DE | *requires* $[\mathbf{q}_{AB} \geq \mathbf{q}_M/2 + \mathbf{q}_{DC}/2]$.

The outcome of the election when the BNVE I.2 is played will again be stochastic: party C wins if $0 \leq \alpha < \alpha_{AC}$, party A wins if $\alpha_{AC} < \alpha < \alpha_{AB}$, and party B wins if $\alpha_B < \alpha \leq 1$. In case of a tie, each party involved in it has an equal chance of winning.

We have thus far shown that: i) there exists a unique F-DE, with $(\beta^*_A, \beta^*_D) = (0, 0)$, regardless of the prior support that each candidate has; and ii) there may exist a S-DE if the prior support of the extremist party getting some votes in equilibrium is sufficiently high, with $(\beta^*_A, \beta^*_D) = (1, 0)$.

The remaining possibility is a N-DE, which requires all parties to get some votes in equilibrium. For that to happen, we require that (β^*_A, β^*_D) be such that $0 < \beta^*_i \leq 1$, for i = A, D. This can only happen if both parties A and D can win the election with some positive probability; if this weren't the case, all the pivotal probabilities involving the preferred candidate would be zero and the likelihood ratios would be zero as well[36]. We also know that the votes for these parties are not stochastic, but rather known with certainty (they are flat lines in our graphical representation). So, the N-DE can only take place when $\pi_A = \pi_D$.

Case 4: $\mathbf{q}_M/2 < \mathbf{q}_{AB} = \mathbf{q}_{DC} < \mathbf{q}_M$

In Figure 11 we illustrate case 4, assuming that all voters vote sincerely.

We see that there is a two-way tie between parties A and D in the *interval* $\alpha_{AC} < \alpha < \alpha_{AC}$. So, the pivotal event has a *real probability* of taking place, which is $F(\alpha_{AB}) - F(\alpha_{AC})$. This implies that *AB* voters need only consider this event, as the tie with parties C and B is irrelevant - even when *n* is finite. The same reasoning applies to *DC* voters. Consequently, when both *AB* and *DC* voters expect all others to vote strategically, they will have an infinite incentive to vote sincerely, implying $\lambda_A = \lambda_D = 1$. Then, from condition 1, the vector $(\beta^*_A, \beta^*_D) = (1,1)$ is a possible equilibrium.

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35 In the boundary case where $\mathbf{q}_{AB} = \mathbf{q}_M/2 + \mathbf{q}_{DC}/2$, the three curves (for parties A, B and C) would intersect at the same point. In this case, there would be only one pivotal event, which would be a three-way tie. Each of the candidates would win with a probability of 1/3 and so any *AB* voter would maximise his expected utility by voting in the candidate that gives him highest utility, which is A. Hence, this boundary case also satisfies the conditions for a S-DE to occur.

36 Assuming the likelihood ratios would not be 0/0. This is not a strong assumption since it only requires that each of the moderate parties has a positive probability of winning the election.

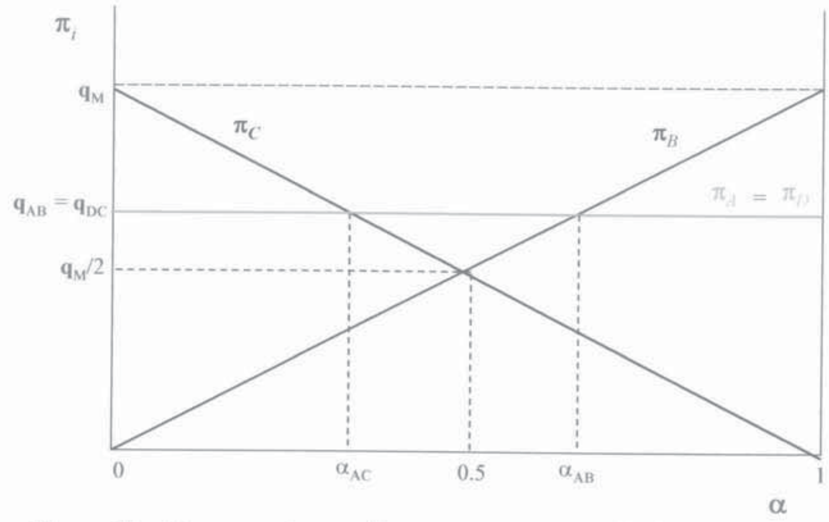

Figure 11 – The case where $q_M/2 < q_{DC} = q_{AB} < q_M$, a *knife-edge* case that makes the N-DE $(ß^*_A, ß^*_D) = (1, 1)$ possible – Model I

Because all voters vote sincerely we call it a *frontier* non-Duvergerian equilibrium [denoting it as N-DE (F)], in contrast with an *interior* non-Duvergerian equilibrium [N-DE (I)], which would describe a case where there is *partial* strategic voting, requiring $0 < ß^*_i < 1$. We next consider the remaining case where $q_M \leq q_{AB} = q_{DC}$.

Case 5: $q_M \leq q_{AB} = q_{DC}$

This is illustrated in Figure 12. Here, there is a tie between A and D for *all* values that α can take, since the prior support of each of the extremist parties is higher than the sum of the support of the two moderate parties.

As in case 4, if *AB* and *DC* voters believe that everyone will vote sincerely, it will optimal to vote sincerely as well. The likelihood ratios are again infinite if all voters vote sincerely, which is consistent, since infinite likelihood ratios imply $ß^*_A = ß^*_D = 1$, from condition 1. Thus, the N-DE (F) $(ß^*_A, ß^*_D) = (1,1)$ is again possible.

We can now fully describe the conditions involving the third type of equilibrium we have found for Model I.

BNVE I.3 $(ß^*_A, ß^*_D) = (1, 1)$ | N-DE (F) | *requires* $[q_{AB} = q_{DC} \geq q_M/2]$.

Would it be possible to have an equilibrium with only *partial* strategic voting? As we will informally show, it is not possible to have a N-DE (I) in Model I.

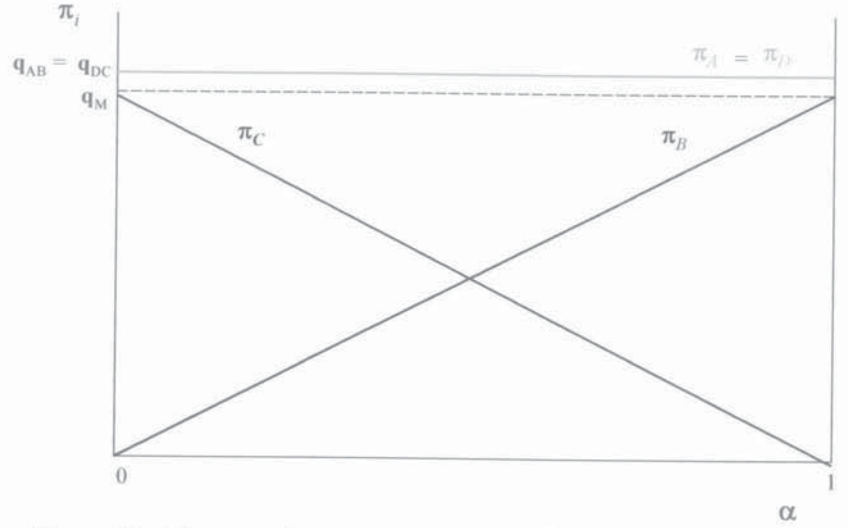

Figure 12 – The case where $q_{DC} = q_{AB} > q_M$, a *knife-edge* case that makes the N-DE $(ß^*_A, ß^*_D) = (1, 1)$ possible – Model I

Partial strategic voting requires that $0 < ß^*_A < 1$, in the case it occurred for *AB* type voters. From condition 1, we know this would happen only if $\lambda_A = ß^*_A$ and hence it would require that $0 < \lambda_A < 1$. However, this is not possible as we can easily see geometrically: because π_A is a flat curve and π_B has a positive slope, we know that in equilibrium party B can only be involved in one pivotal event, while party A can be involved in more than one. Moreover, since π_A is a flat line and A must in equilibrium have a chance of winning the election, it must be true that the pivotal event involving B also involves A. This is sufficient to show that - *in Model I* - a *positive* likelihood ratio λ_A must be *equal* or *higher* than 1[37].

An alternative explanation would be the following.

If A is to have a chance of winning an election, only three possible configurations are possible: i) it is above curves π_B or π_C (as in Figure 12); or ii) it intercepts π_B and π_C at different points (as in Figures 10 and 11); or iii) it intercepts π_B and π_C in the same point, i.e., the three curves intercept at the same exact point.

In case i) the likelihood ratio is infinite, since party B doesn't have a chance of winning, but A has. In case ii) the likelihood ratio is strictly greater than 1, because A is involved in two pivotal events, but B is only in one of those two. In case iii) the likelihood ratio is exactly equal to 1, since the unique pivotal event involves A and B.

The conclusion is that in *none* of the three *exhaustive* possible cases can the likelihood ratio be in the interior of [0, 1]; so, an interior equilibrium is *never* possible.

37 Naturally, the likelihood ratio can be 0 - this happens in the case of a F-DE or S-DE. Here, since we are studying the possibility of an *interior* equilibrium, we condition the likelihood ratio on being *positive*.

How is this related to the assumptions of our model?

We have seen that in all equilibria voters from the same group always vote in the same way. This is a consequence of assuming a *commonly known certainty* about the prior support of the parties whose supporters may vote strategically, because it necessarily leads to an infinite incentive for *all* voters to vote either strategically or sincerely, implying that voters from the same group all cast *identical* votes.

Myatt's (2002) model makes it possible to have an equilibrium with partial strategic voting, exactly by not assuming such common knowledge; rather, in his model voters have private information and hence will hold different beliefs that will permit that they behave *differently* in equilibrium, even when they share the same favourite candidate.

We summarise the three possible Bayesian-Nash equilibria in Model I in the following table, which highlights the possibility of having multiple equilibria[38].

Table 6. *The set of possible equilibria in Model I*

Case	Conditions required	Possible Equilibria (Y/N)		
		F-DE	S-DE (F)	N-DE (F)
a	$\mathbf{q}_{DC} < \mathbf{q}_{AB} < \mathbf{q}_M/2 + \mathbf{q}_{DC}/2$	Y	N	N
b	$\mathbf{q}_M/2 + \mathbf{q}_{DC}/2 \leq \mathbf{q}_{AB}$	Y	Y	N
c	$\mathbf{q}_M/2 \leq \mathbf{q}_{DC} = \mathbf{q}_{AB} < \mathbf{q}_M$	Y	N	Y
d	$\mathbf{q}_M \leq \mathbf{q}_{DC} = \mathbf{q}_{AB}$	Y	Y	Y

There is always more than one possible equilibrium except when $\mathbf{q}_{DC} \leq \mathbf{q}_{AB} < \mathbf{q}_M/2 + \mathbf{q}_{DC}/2$, when only the F-DE is possible. As we have argued, the F-DE is always possible. For a $\mathbf{q}_{AB}$ sufficiently high, a S-DE will always be possible as well. A N-DE (F) can occur only when $\mathbf{q}_{AB} = \mathbf{q}_{DC}$, and as long as $\mathbf{q}_{AB}$ is not too small.

A natural question to ask regards the stability of the equilibria described above. We deal with this in the next section.

4.I.5 Analysis of stability of the equilibria

Besides describing the possible equilibria, it is very important to study their stability properties, since an unstable equilibrium is in a sense a weaker equilibrium.

---

38 We are still assuming [without loss of generality] that $\mathbf{q}_{DC} \leq \mathbf{q}_{AB}$ always holds, so that when they are not equal, it is implicit that $\mathbf{q}_{DC} < \mathbf{q}_{AB}$.

In our approach to the problem of stability of the equilibria we will use a notion of *partial stability*, which we clarify next.

Definition 6. *Define a partially [un]stable equilibrium* ($ß_A$, $ß_D$) = ($ß^*_A$, $ß^*_D$) *to be such that when holding* $ß_D = ß^*_D$ *constant, the likelihood function depending on* $ß_A$ *and* $ß_D = ß^*_D$ *will be locally [un]stable around* $ß_A = ß^*_A$. *The likelihood function will be said to be locally stable* iff *there exists at least one* $\underline{\varepsilon} > 0$ *such that, for all* $\varepsilon < \underline{\varepsilon}$, *any* $ß_A$ *in* $B(ß^*_A, \varepsilon)$[39] *will originate a convergent path to* $ß^*_A$[40].

Simply put, if there is any deviation from the equilibrium from *one* group of extremist voters - *while keeping the behaviour of the other extremist group constant* - that deviation must lead back to the equilibrium. This requires that after a deviation has occurred, the incentives lead to a *negative* feedback. For instance, if in equilibrium everyone is expected to vote sincerely and one voter deviates and votes strategically, the likelihood ratio resulting from such behaviour must provide *higher* incentives to vote sincerely than before, to avoid an effect of *contamination.*

This *resembles* in some way the notion of a *trembling-hand perfect equilibrium*, since an equilibrium can only be stable if it is immune to the 'mistakes' that *some* voters may make. This is embodied in the above definition, since we demanded that when $ß_A$ takes any value different from $ß^*_A$ that doesn't lead to a divergent path away from the equilibrium. In fact, a $ß_A$ different from $ß^*_A$ *could* be interpreted as a 'mistake' in equilibrium - in light of the *trembling-hand* refinement - and in this sense the equilibrium would be stable if it didn't lead the other voters to change their behaviour.

The F-DE and S-DE are analysed together in Figure 13, which is drawn holding $ß^*_B = 0$ constant and varying $ß_A$. We know from the equilibrium conditions for the F-DE that when $ß_A = 0$ the likelihood is also 0, which we can see from the picture on the next page. What happens when $ß_A$ starts increasing from $ß_A = 0$?

Going back to Figure 7, we see that there would be an upward parallel shift in π_A, while π_B would shift downwards by the same amount. For a *small* $ß_A$ this would still leave π_A below the intersection point of curves B and C, and so party A would *never* have a chance of winning the election, while B would. This implies that the likelihood ratio for *AB* voters would still be 0 for a *small* $ß_A$. Thus, the F-DE is a stable equilibrium.

In Figure 10 we illustrated the S-DE, assuming that $\mathbf{q}_M/2 + \mathbf{q}_{DC}/2 \leq \mathbf{q}_{AB}$ holds. In equilibrium, the likelihood ratio λ_A is higher than 1, since A can be involved in two pivotal events, while B can only be in one. When $ß_A$ starts decreasing from $ß_A = 1$ we see that π_A would still intersect π_B and π_C for different values of α, so there would still exist two pivotal

39 A *ball* centred in $ß^*_A$ with radius ε.

40 For simplicity, we will simply say that an equilibrium is *(un)stable* and not *partially (un)stable.*

events involving A and only one involving B, which keeps $\lambda_A > 1$[41].

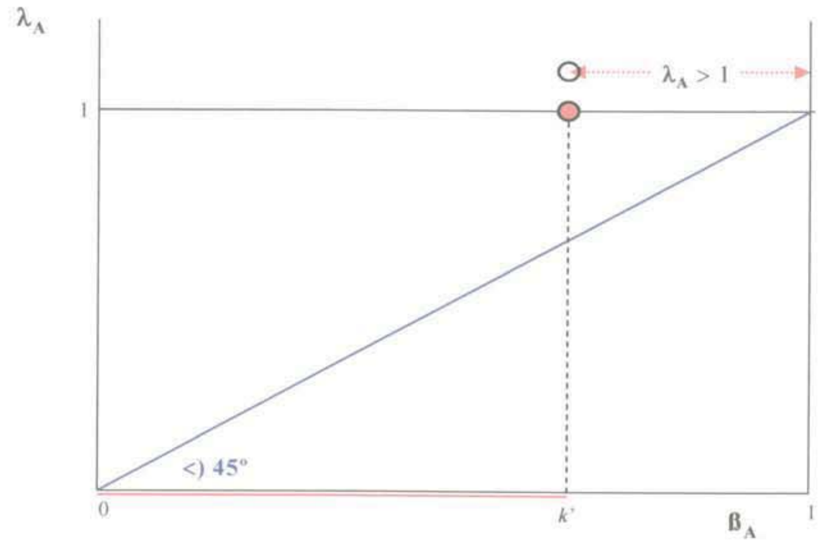

Figure 13 – The stability of both the F-DE $(\beta^*_A, \beta^*_B) = (0, 0)$ and the S-DE $(\beta^*_A, \beta^*_B) = (1, 0)$ for a parametrised $\beta^*_B = 0$ – Model I

This would occur until the curves intersected in the same point, when $\beta_A = k'$[42]. In such scenario, *AB* voters would need consider only one pivotal event, which is the three-way tie between A, B and C. Since such event involves parties A and B, the likelihood ratio would be equal to 1. From $\beta_A = k'$, if *one* more *AB* voter voted strategically, the curve π_A would be below the intersection point between π_B and π_C and party A would thus have no chance of winning. Therefore, the likelihood would suddenly go from 1 to 0, and hence the discontinuity we see in the function represented above.

The likelihood ratio would then remain zero for all values of β_A in $[0, k')$, which also confirms the stability of the F-DE.

We now turn our attention to the N-DE, whose stability analysis is illustrated in Figure 14, this time holding $\beta^*_B = 1$ constant, and still varying β_A.

41 The likelihood ratio λ_A can take any values when $k' < \beta_A < 1$, as long as $\lambda_A > 1$, which follows from the assumption that $f(\alpha) > 0$ for all α. Also, it need not be a continuous function.

42 The value k' is such that the three curves intersect at the same point. When there are $(1 - k')$ *AB* voters voting strategically, $\pi_B = (1 - k')\,\mathbf{q}_{AB} + \mathbf{q}_B$ and so, when curves π_B and π_C intersect, the support for both parties B and C is equal to $[\mathbf{q}_M + \mathbf{q}_{DC} + (1 - k')\,\mathbf{q}_{AB}]/2$. Since $\pi_A = k'\,\mathbf{q}_{AB}$, the three curves intersect when π_A is equal to that level of support, and so k' solves $(1 - k')\,\mathbf{q}_{AB} + \mathbf{q}_B = [\mathbf{q}_M + \mathbf{q}_{DC} + (1 - k')\,\mathbf{q}_{AB}]/2$, from which we get $k' = 1/3\,[1 + (\mathbf{q}_M + \mathbf{q}_{DC})/\mathbf{q}_{AB}]$.

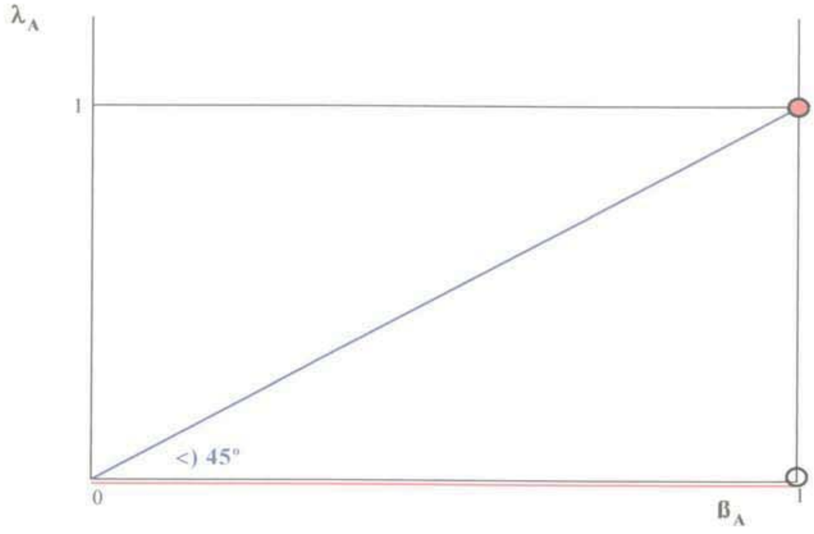

Figure 14 – The instability of the N-DE $(\beta^*_A, \beta^*_B) = (1, 1)$ for a parametrised $\beta^*_B = 1$ – Model I

The analysis for the stability of this equilibrium is very simple. As we know, the N-DE involves a *knife-edge* case, requiring $\mathbf{q}_{AB} = \mathbf{q}_{DC}$, which we see from the fact that π_A and π_D are coincident lines in Figures 11 and 12. Given this, we see that if β_A is just slightly below 1, π_A will be below π_D and thus party A won't have any chance of winning.

In Figure 11, where party B *may* win the election, this will imply that the likelihood ratio is defined and immediately drops to 0 - thus, the discontinuity at $\beta_A = 1$. So, from a N-DE, it is only required that *one* voter switches to party B in order to bring *infinite* incentives for all others to switch as well. This *positive effect* is the cause of instability, since any individual deviation becomes a self-feeding process, ending up with all voters voting strategically.

In the case illustrated in Figure 12, a shift to party B would leave the likelihood ratio undefined (equal to 0/0), since party B could neither win. In this case all strategies are rationalisable, and so we can't say much about it, except that everything can happen. This is, however, a very uninteresting case, only presented for sake of completeness.

4.I.6 Conclusions for Model I

Model I is our simplest from the three proposed frameworks to analyse strategic voting and yet it provides some interesting insights.

First, we see that the F-DE can always occur and is always stable. In this equilibrium only two parties survive, naturally those that are immune to strategic voting from their natural supporters.

This result is in favour of Duverger's Law, since it shows that a four-party model can - in a single one-shot election - reduce the number of parties with effective support to two.

From a point of view of a politician or opinion maker, all that is required for this equilibrium to occur is to create (credible) expectations that a *large enough* proportion of both extremist voters will indeed vote strategically. Since the F-DE is stable, such an expectation would lead the other extremist voters to behave in the same manner, without fearing any *surprising* effect[43].

Second, it is possible to have a N-DE, where all voters vote for their preferred parties and all parties to survive. However, this is not a stable equilibrium and, moreover, it requires a (highly unlikely) *knife-edge* case.

Our N-DE is in the spirit of Cox's non-Duvergerian equilibrium, where strategic voting has a positive effect on itself, creating a self-feeding process that converges to the Duvergerian outcome[44]. The N-DE is very demanding in terms of the coordination that is required from the voters, since any mistake would make the equilibrium fall apart. The coordination problem also arises in the three-party literature, where it is similar to that present in the 'Battle of Sexes' game. The fulfilled expectations assumption is used to side step this issue.

Thirdly, it is possible that three parties survive. The S-DE can always occur as long as party A has a sufficiently high support, such that it can sometimes win the election, after the strategic votes from *DC* voters to C have been taken into account. Not only this equilibrium does *not* depend on the specification for $f(\alpha)$ but also it is *always stable*.

Hence, the S-DE must be considered as another plausible prediction for the outcome of the game, in addition to the F-DE. There are two possible S-DE (F), if both A and D may win after the strategic votes from the supporters of the other extremist party have been taken into account.

We must acknowledge, however, that since this prediction involves three parties getting results in a one-shot election, in case we extended our analysis to a repeated game, would be likely that a Duvergerian result could be obtained. In this sense, the S-DE, though not a Duvergerian one, should be interpreted with some precaution, given the fact that we're analysing a one-shot game only.

Finally - and given all previous comments - we have yet to consider the additional problem of *equilibrium selection*, since there may be more than one possible equilibrium for a given distribution of prior support. Myerson and Weber (1993) describe some issues that arise when there are multiple equilibria, namely regarding the power of the *focal arbiters* in making an equilibrium focal or not[45].

43 In the sense that, since the equilibrium is stable, there would be a convergence towards the F-DE that involved 'no risk' of diverging to any other outcome.

44 In fact, it could be the case that only one of the groups changed his behaviour, leading to a S-DE and not a F-DE.

45 In their words, «A *focal arbiter* is an individual whose opinions and statements about the candidates command wide attention, giving him influence upon the outcome of the election by making focal an

These interesting comments are made in the context of a three-party model, but apply to ours as well:

> «What is the political significance of the multiplicity of equilibria? The existence of a large set of equilibria in an electoral situation provides great political influence to political leaders and other individuals who have access to the mass media; that is, the existence of multiple equilibira does not merely mean that outcomes are difficult to predict theoretically. The multiplicity of equilibria may also have substantive implications for the distribution of power in society.
>
> Schelling (1960) argues that in games with multiple Nash equilibria, anything that tends to focus the players' attention on any one equilibrium may lead each player to expect the others to act in accordance with that equilibrium. With such expectations, each player does best for himself by also acting in accordance with the equilibrium; thus, the expectations are fulfilled. An equilibrium realised because of a focusing of attention is said to be focal.
>
> In an electoral setting, if the voters come to believe some given prediction about the aggregate distribution of votes in an election, then their individual voting optimisation decisions can confirm this prediction (...) if and only if the prediction corresponds to a voting equilibrium. If there are multiple voting equilibria, then there is more than one prediction about the outcome of the election that could be fulfilled by rational voter behaviour if believed by the electorate. An individual who could focus the voters' expectations would then be in a position to pick any voting equilibrium and make a self-fulfilling prophecy.»

Overall, there are 4 possible equilibria in Model I: one F-DE, two S-DE and one N-DE.

Considering only the stable equilibria, we can exclude the N-DE one. The F-DE and S-DE are both possible when $\mathbf{q}_M/2 + \mathbf{q}_{DC}/2 \leq \mathbf{q}_{AB}$. If we recall that this is *the* interesting case, where party A can win after taking account of the strategic voting from the supporters of the other extremist party, we can in fact say that - *given* that we consider "*the* interesting case" - there is *always* a problem of selecting between the F-DE and the S-DE in Model I.

The F-DE gives no chance of party A winning the election, while S-DE does. However, in the F-DE party B wins with higher probability than in S-DE. This trade-off makes it not obvious what will the expected utility of any *AB* voter from the two equilibria be. The difficulty in tackling the problem of multiplicity of equilibria is that we can *not* analyse the behaviour of *AB* voters *as if* they *jointly* decided in which equilibria to coordinate but rather we must analyse it from the perspective of a *single AB* voter.

Adding to this difficulty, we believe that a voter should be inclined to choose the *risk-dominant* strategy, as opposed to the Pareto-dominant one[46]. This analysis is left for future work.

---

equilibrium in which a candidate he supports has a significant chance of winning (or one he condemns does not). (...) To become a focal arbiter, an individual does not need good judgement in evaluating candidates or even to be perceived as having good judgement. It is only necessary that he be able to get his views prominently reported to the public».

46 See Carlsson and van Damme (1993).

4.II Equilibrium analysis for Model II

4.II.1 Description of the model

The following information is a reminder of the structure of Model II, where strategic voting can occur only between the two moderate parties, in both directions.

Figure 2. *The strategic scenario in Model II (reminder)*

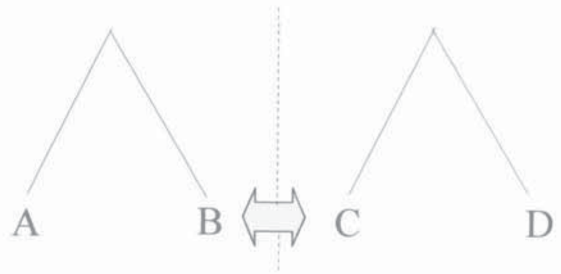

Table 2. *Voters' preferences in Model II (reminder)*

Voter Type	Utility for candidate				Prior Probability
	A	B	C	D	
A	1	0	0	0	q_A
BC	0	1	$ß_B \sim U[0,1]$	0	q_{BC}
CB	0	$ß_C \sim U[0,1]$	1	0	q_{CB}
D	0	0	0	1	q_D

We introduce uncertainty by assuming there is common knowledge about the following:

$$\mathbf{q}_M = [\mathbf{q}_B + \mathbf{q}_C]$$

$$\alpha = [\mathbf{q}_B / \mathbf{q}_M] \text{ in } [0, 1] \text{ with density function } f(\alpha)$$

$$ß_i \sim U[0,1],\ i = B, C$$

In this model, only the supporters of the moderate parties may vote strategically, and only to the other moderate party. The uncertainty is modelled similarly to Model I, regarding only the prior support of B and C. This, together with the fact that any strategic votes occur only for parties B or C, implies that the voting results for parties A and D will in equilibrium be equal to their prior support.

The fact that the *equilibrium support* of parties A and D is always known with certainty implies that *BC* and *CB* voters will need only consider the extremist party that enjoys greatest support amongst the two - the other becomes irrelevant and is thus ignored[47].

We then see that Model II is *strategically equivalent* to the model of three-party competition that has been most studied in the strategic voting literature, since it turns into a "defeat the extremist" game for the supporters of the moderate parties - a game where they try to coordinate in order to defeat the "incumbent" extremist party.

The fact that Model II involves a *de facto* three-party competition gives an additional interest to our analysis, as we will be able to *directly* compare our conclusions [and, perhaps more important, what drives them] with those arising from such literature.

4.II.2 Voters' optimal strategies and equilibrium conditions[48]

Voters that consider voting strategically in Model II - *BC* and *CB* types - will adopt a similar cut-off rule as their counterparts *AB* and *DC* from Model I: each voter *i* of type *BC* or *CB* will chose to vote for its favourite candidate if $ß_i < ß^*_i$, for i = B,C, where each $ß^*_i$ satisfies the two conditions for a Bayesian-Nash voting equilibrium.

Voters of type *BC* and *CB* will consider voting strategically for C and B, respectively, and this implies that their likelihood ratios are the inverse of one another. This is a well-known result from the three-party literature, where the supporters of the challenging parties share the same two preferred candidates (naturally in reverse order)[49].

The formulas for their likelihood ratios are similar to those presented for Model I, which follows from the fact that voters *BC* and *CB* similarly equate the probabilities of being pivotal for any of their two preferred candidates. Their likelihood ratios are:

$$\lambda_B = \{2 \Pr[b = c] + \Pr[b = a] \} / \{2 \Pr[b = c] + \Pr[c = a]\}$$

$$\lambda_C = \{2 \Pr[b = c] + \Pr[c = a]\} / \{2 \Pr[b = c] + \Pr[b = a] \}$$[50].

Note that all voters share the same opinion of the electoral situation, which leads them to share the same strategic incentives. This in turn implies that only one group of moderate voters can face strategic incentives; it is not possible that both voters vote strategically in equilibrium. This is clear from the relation $\lambda_C = 1/\lambda_B$, which makes it impossible to have both likelihood ratios lower than 1. In fact, only three cases are possible: i) $\lambda_C > 1 > \lambda_B$; ii) $\lambda_B > 1 > \lambda_C$; or iii) $\lambda_C = 1 = \lambda_B$.

47 As in Model I, we assume without loss of generality that $\pi_D < \pi_A$, so that party D becomes irrelevant.

48 We assume the reader to have read the description for Model I and so the description for Model II will have less detail, to avoid repetition.

49 See, for instance, Myatt (2004).

50 Note that party D can never be involved in a pivotal event, since we assume $q_D < q_A$. So, the probabilities Pr[b = d] and Pr[c = d] are *always* equal to 0, the reason why they do not appear in the formulas.

The optimality conditions are identical to those of Model I, with the necessary adaptations.

Condition 1.II. ***Optimal behaviour in equilibrium (Model II)***

In an equilibrium, types A *and* D *voters vote respectively for A and D, and types* BC *and* CB *voters use the following strategy:*

BC *type: "Vote for B if* $ß_B \leq ß^*_B$ *and vote for C if* $ß_B > ß^*_B$*", where*

$$ß^*_B = \min \{1, \lambda_B\}$$

CB *type: "Vote for C if* $ß_C \leq ß^*_C$ *and vote for B if* $ß_C > ß^*_C$*", where*

$$ß^*_C = \min \{1, \lambda_C\}$$

Condition 2.II. ***Rational expectations in equilibrium (Model II)***

$$\pi_A = q_A$$

$$\pi_B = q_{BC}\, ß^*_B + q_{CB}\,(1 - ß^*_C) = q_M\,[\alpha\, ß^*_B + (1 - \alpha)\,(1 - ß^*_C)]$$

$$= q_M\,(1 - ß^*_C) + \alpha\, q_M\,(ß^*_B + ß^*_C - 1)$$

$$\pi_C = q_{CB}\, ß^*_C + q_{BC}\,(1 - ß^*_B) = q_M\,[\alpha\,(1 - ß^*_B) + (1 - \alpha)\, ß^*_C]$$

$$= q_M\, ß^*_C + \alpha\, q_M\,(1 - ß^*_B - ß^*_C)$$

$$\pi_D = q_D$$

Note that the expression $[\alpha\,(ß^*_B) + (1 - \alpha)\,(1 - ß^*_C)]$ is nothing else than the overall fraction of the moderate voters that vote for B. This fraction is naturally stochastic, since it depends on the draw from the density $f(\alpha)$. We define it next.

Definition 7. *Define* p *as the overall fraction of the moderate voters that vote for B in equilibrium, so* $p = [\alpha\,(ß^*_B) + (1 - \alpha)\,(1 - ß^*_C)]$. *Thus,* $\pi_B = p\, \mathbf{q}_M$.

We now proceed to the description of the different equilibria and the analysis of their stability. Differently from Model I, we present *both* analyses together for each of the equilibrium we find.

4.II.3 Study of possible equilibria *and* their stability

Even though this particular model is strategically identical to the three-party one, where it only makes sense to refer to the 'Duvergerian' and 'non Duvergerian' equilibria, we will keep our proposed labelling.

A first remark is that in Model II it is not possible to have only two parties getting votes in equilibrium, since - by construction - parties A and D always get votes and at least one of the parties B and C must get some votes in equilibrium. So, there can never exist a F-DE in Model II.

We therefore need only study the S-DE and N-DE, but baring in mind that these correspond, respectively, to the "Duvergerian" and "non-Duvergerian" equilibria from the three-party literature.

Case 1: S-DE | $(\beta^*_B, \beta^*_C) = (1, 0)$

We start with some comments on the graphical representation of voting results in Model II.

The votes that party B gets in equilibrium will depend on three variables: i) the uncertainty expressed by $f(\alpha)$; ii) the votes that are strategically cast for C; and iii) the strategic votes that are received from *CB* voters.

When $\alpha = 1$, party B gets $\mathbf{q}_M * \beta_B$ votes, reflecting the fact that the draw over α gives him a prior support of $\mathbf{q}_M$ votes, from which a proportion of $[1 - \beta_B]$ strategic votes has to be subtracted. Since party C gets no prior support when $\alpha = 1$, any strategic votes from *CB* supporters need not be considered.

When $\alpha = 0$, party B doesn't get any prior support and only receives any hipothethical strategic votes from *CB* voters, which sum to $\mathbf{q}_M * [1 - \beta_C]$.

Then, the slope of party B's curve is simply $\mathbf{q}_M [\beta_B + \beta_C - 1]$, as it is clearly seen from the expression for π_B in condition 2.II. For instance, if none of the supporters of either party B and C voted strategically [i.e., if $\beta_B = \beta_C = 1$], party B's curve would have slope equal to $\mathbf{q}_M$; this is identical to the curve used in Model I, where these voters were not allowed to vote strategically.

A S-DE involves, by definition, only one of the parties B or C getting votes in equilibrium; if such party is B, then all voters of type *CB* must vote strategically for B while none of the voters *BC* can vote strategically for C. This implies, respectively, that $\beta^*_C = 0$ and $\beta^*_B = 1$, so the pair $(\beta^*_B, \beta^*_C) = (1, 0)$ is the unique candidate for a S-DE.

This is the famous "Duvergerian" result in the three-party literature, where one of the challengers disappears due to a complete switch of votes to the other party. We illustrate this scenario in Figures 15.a and 15.b.

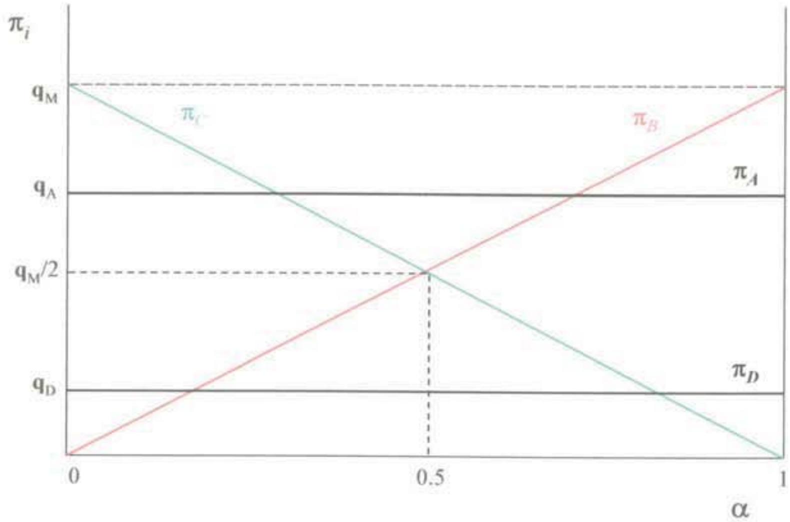

Figure 15.a – The case where $\mathbf{q}_M$, the moderate parties' total support, exceeds that for any of the extremist parties – $\mathbf{q}_{DC} < \mathbf{q}_{AB} < \mathbf{q}_M$ – Model II

Figure 15.a is drawn assuming all voters vote sincerely. Figure 15.b illustrates the case where the moderate supporters coordinate on party B. This will imply that party B gets all the moderate votes - regardless of the draw of the uniform distribution - which adds up to $\mathbf{q}_M$. Party C gets zero votes. The extremist parties, as said before - and as it is always the case in Model II - get a result equal to their prior support.

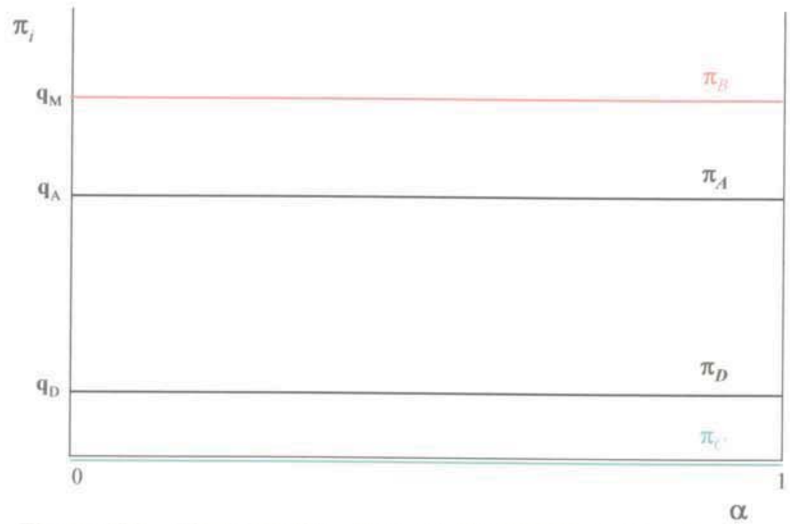

Figure 15.b – The S-DE $(\beta^*_B, \beta^*_C) = (1, 0)$ where the moderate voters coordinate on party B – Model II. This coordinated equilibrium is the "Duvergerian equilibrium" from the three-party literature.

From Figure 15.b, we see that there are no pivotal events: full coordination from the moderate voters ensures that party B is *always* the winner. The fact that there are no pivotal events prevents us from using the likelihood ratios, since these are not defined. However, a simple argument can be made to see that this behaviour is indeed optimal. In equilibrium, party C can *never* win the election, and so it is a *weakly dominated strategy* for *each* particular voter to vote for C, which implies all moderate voters must vote for party B.

The previous rational requires that party B has a chance of winning the election. Since its voting results are a flat line, this requires simply that the proportion of moderate voters is at least equal to the proportion of supporters of party A - or in general the extremist party with highest support. Analytically, we get

$$\mathbf{q}_M \geq \text{Max}\ \{\mathbf{q}_A, \mathbf{q}_D\} \Rightarrow \mathbf{q}_M \geq \text{Max}\ \{1 - \mathbf{q}_M - \mathbf{q}_D, 1 - \mathbf{q}_M - \mathbf{q}_A\} \Rightarrow$$

$$\Rightarrow \mathbf{q}_M \geq 1 - \mathbf{q}_M - \text{Min}\ \{\mathbf{q}_D, \mathbf{q}_A\} \Rightarrow \mathbf{q}_M \geq 0.5\ [1 - \text{Min}\ \{\mathbf{q}_D, \mathbf{q}_A\}].$$

The last expression clearly shows that the game being played is a *simple majority game* when we consider an electorate that excludes the extremist party with least votes. When such condition is met, $(\beta^*_B, \beta^*_C) = (1, 0)$ is in fact an equilibrium, which we state next[51].

BNVE II.1 $(\beta^*_B, \beta^*_C) = (1, 0)$ | S-DE | *requires* $[\mathbf{q}_M \geq \mathbf{q}_A]$.

We describe the stability properties of this equilibrium in Figure 16.

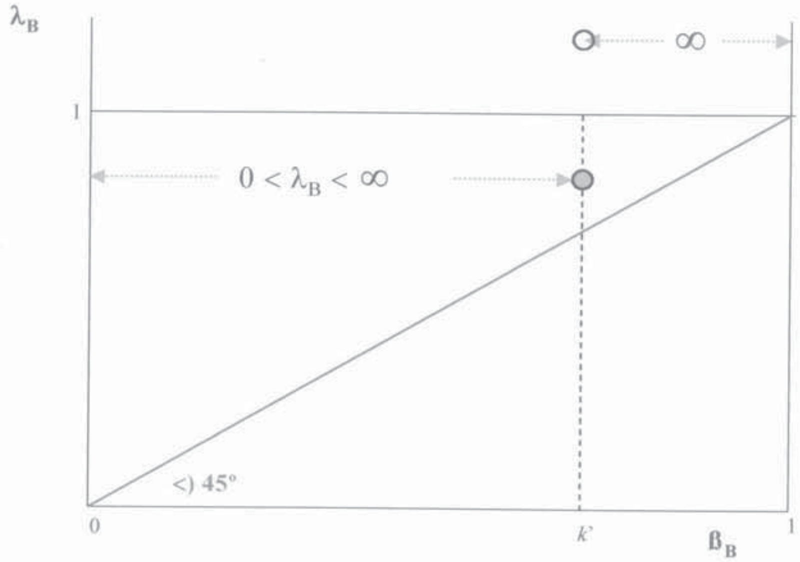

Figure 16 – The stability of the S-DE $(\beta^*_B, \beta^*_C) = (1, 0)$ for a parametrised $\beta^*_C = 0$ – Model II

51 So is $(\beta^*_B, \beta^*_C) = (0, 1)$. We exclude redundant equilibria in our description to simplify the discussion.

The above figure is drawn holding $\beta^*_C = 1$ constant and allowing β_B to vary. We know that when $\beta^*_B = 1$ the likelihood ratio for *BC* voters is infinite, since party C gets zero votes, which can be seen in Figure 15.b. If a proportion *s* of *BC* voters deviates from this equilibrium, by switching to C, π_B would then have a negative slope of $-s\ \mathbf{q}_M$, while π_C would have a positive slope of $s\ \mathbf{q}_M$, and both would keep their intercepts. So, for a small *s*, party B will still not have a chance of winning and the likelihood ratio will still be infinite. This will give incentives to vote sincerely for B and thus will be a *negative effect*: strategic voting will lead to *less* strategic voting. Hence, this equilibrium is stable.

We also see that, as *s* grows (i.e., as more voters vote strategically), eventually π_C will cross π_A and, consequently, the likelihood ratio will suddenly change from *infinite* to a *finite* number[52]. This will happen when the proportion *s* is be equal to $(1 - k')$ in the figure above. As *s* grows, the likelihood ratio would still be defined and always greater than zero[53].

Case 2: N-DE | $(\beta^*_B, \beta^*_C) = (1, k)$ with $0 < k \leq 1$

A N-DE must involve all parties getting some votes in equilibrium, so it is not possible to have a likelihood ratio equal to 0, as that would mean a complete erosion of one candidate's support. Moreover, since $\lambda_C = 1/\lambda_B$, only two cases are possible: either both likelihood ratios are equal to 1 [implying, from condition 1, that $\beta^*_B = \beta^*_C = 1$] or one of them is equal to *k* in (0, 1) and the other to $1/k$ [implying that $\beta^*_B = k$ and $\beta^*_C = 1$]. As we have labelled them in Model I, the first type is a *frontier* N-DE, while the second is an *interior* N-DE.

Consider Figures 17.a and 17.b, which illustrate a potential interior equilibrium, where all *BC* voters vote sincerely and a proportion $(1 - k)$ of *CB* voters votes strategically for B.

When $\alpha = 0$, party B gets no prior support and obtains result equal to $(1 - k) * \mathbf{q}_M$, the strategic votes coming from *CB* voters, whose proportion of the electorate is $\mathbf{q}_M$.

If $\alpha = 1$, party B gets a voting result equal to its prior support, $\mathbf{q}_M$, since no strategic votes occur, as party C gets no prior support.

Thus, B's line has a slope of $\mathbf{q}_M\ k$, which can be confirmed from condition 2.II.

Analogously, party C gets 0 votes when $\alpha =$ is 1, and gets $\mathbf{q}_M\ k$ votes if $\alpha = 0$. Its line has a slope of $- \mathbf{q}_M\ k$.

Clearly, if the two curves do not intersect - as it is the case in Figure 17.a - party C could never win, implying a likelihood ratio equal to zero. Hence, any *k* in [0.5, 1] cannot be part of a N-DE, as it doesn't allow party C to have a chance of winning the race, which is a necessary condition to get *some* votes in equilibrium from voters that are instrumentally rational.

52 Since the denominator will no longer be zero, but a positive number, as we have assumed $f(\alpha) > 0$.

53 This is not necessary for the stability result and is provided only for completeness.

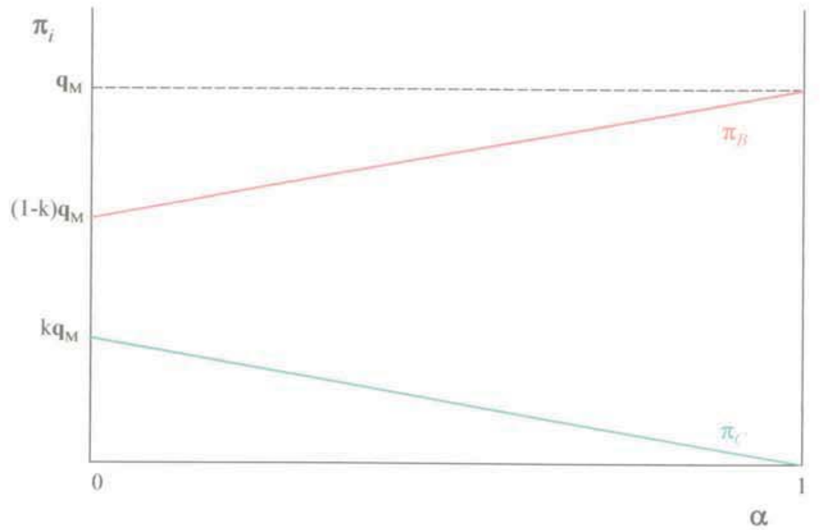

Figure 17.a – The case where party C could never win the election, when a proportion $k < 0.5$ of *CB* voters votes sincerely – Model II

Figure 17.b illustrates the case where $0.5 \leq k \leq 1$. We can write the equations for the two parties' (stochastic) voting results as:

$$\pi_B = \mathbf{q}_M (1 - k) + \alpha \, \mathbf{q}_M k$$

$$\pi_C = \mathbf{q}_M k - \alpha \, \mathbf{q}_M k$$

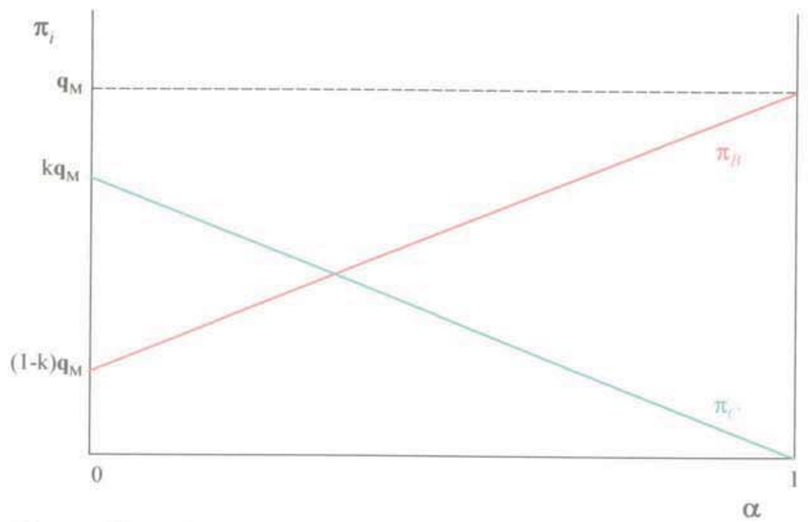

Figure 17.b – The case where party C could win the election, when a proportion $k \geq 0.5$ of *CB* voters votes sincerely – Model II

To check the possible equilibria, it is useful to distinguish two different cases:
1) $\mathbf{q}_A < \mathbf{q}_M/2$ - party A can never win the election (Figure 18.a)
2) $\mathbf{q}_A \geq \mathbf{q}_M/2$ - party A can win the election (Figure 18.b)

Case 2.1 $\mathbf{q}_M > 2\mathbf{q}_A$

We illustrate this case in Figure 18.a, assuming all voters vote sincerely, which makes parties B and C the only candidates to win the election.

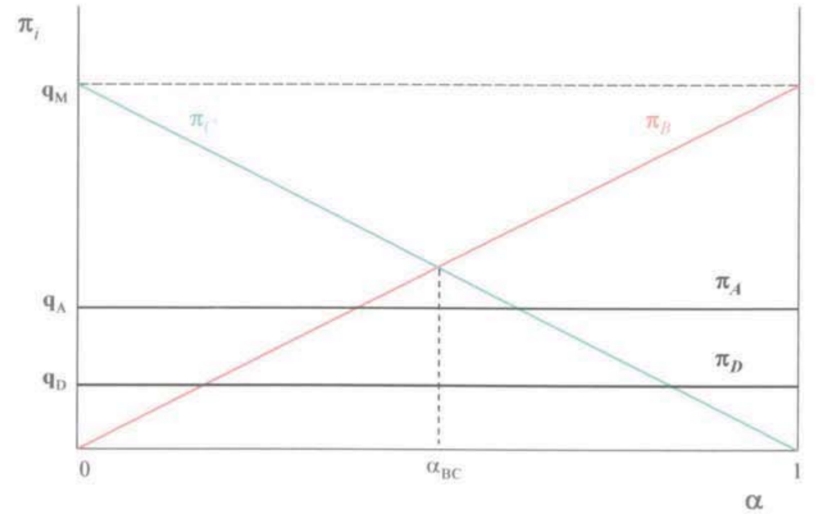

Figure 18.a – The case where the N-DE $(ß^*_B, ß^*_C) = (1, 1)$ is always possible, regardless of $f(\alpha)$ – Model II

Both *CB* and *BC* voters realise that there is only one pivotal event, between 'B and C'. Because this is the only pivotal event, and it involves both parties B and C, the likelihood ratios of both voters are equal to 1, and so an interior equilibrium is not possible. When party A can never win the election, the N-DE $(ß^*_B, ß^*_C) = (1, 1)$ is the only possible N-DE.

Another intuitive way to explain this result is that when the total support for the two moderate parties is higher than twice the support of party A, *everyone knows* that *everyone voting sincerely* is an equilibrium, and one which prevents party A from winning the race - regardless of the draw from $f(a)$. We next present the condition for this equilibrium to exist.

BNVE II.2 $(ß^*_B, ß^*_C) = (1, 1)$ | N-DE (F) | *when* $[\mathbf{q}_M > 2\mathbf{q}_A]$ *is always possible.*

This equilibrium is stable, which we show in the figure below.

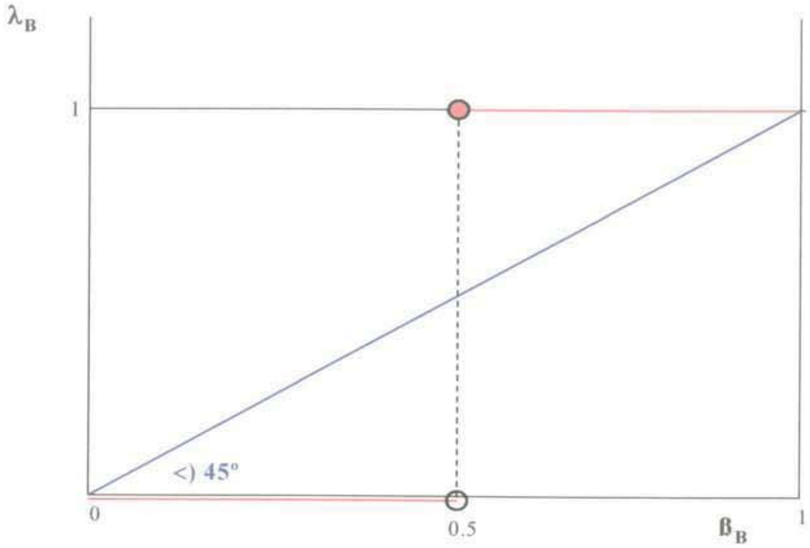

Figure 19 – The stability of the N-DE $(ß^*_B, ß^*_C) = (1, 1)$ when $q_M > 2q_A$ for a parametrised $ß^*_C = 1$ – Model II

The reason why this is a stable N-DE is that if some of the *BC* voters switched to C - assuming all *CB* voters vote sincerely - that would lead to a change in the slopes of their curves, but there would still exist only one pivotal event, when π_B and π_C intersected each other. Hence, the likelihood ratio would still be equal to 1, and this is sufficient for stability.

The intuition is that a small deviation from the N-DE wouldn't change the incentives, since the pivotal event considered would still be unique and involve B and C, implying an incentive for their supporters to vote sincerely.

Going back to Figures 17.a and 17.b, we see that if more than half of the voters vote strategically, the two curves won't intersect. In this case party B would have no chance of winning - because its curve would be always below C's. So, the likelihood ratio would suddenly change to 0 when $ß_B = 0.5$, as it is clear from the figure above.

Case 2.2 $\mathbf{q_A < q_M \leq 2q_A}$

This is the more interesting case, since party A may also win the election. We separate the analysis in two parts: the first is dedicated to the N-DE (F) and the second to the N-DE (I).

1) The N-DE (F)

As we noted before, each of the moderate parties faces a qualified majority game where they must obtain a proportion γ of the total moderate votes to defeat the incumbent A. In our game, $\gamma = (\mathbf{q_A} / \mathbf{q_M})$. For example, if party A has 40% of support, while the moderate parties enjoy 50% of support, $\gamma = 4/5 = 80\%$. This means that any of the moderate parties can only defeat party A if it gets at least 80% of the moderate votes. In our model, this would occur if the draw from the distribution were higher than 80% or lower than 20%, guaranteeing a victory to parties B and C, respectively. We illustrate the potential equilibrium where all voters vote sincerely in Figure 18.b.

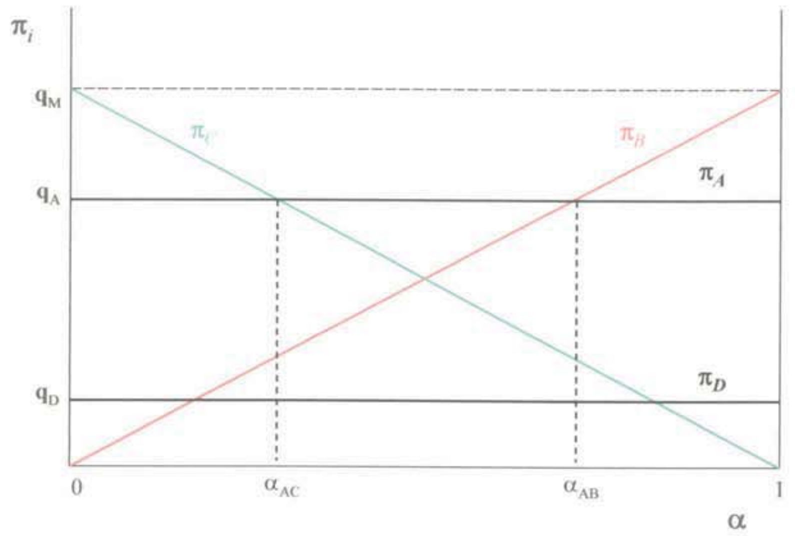

Figure 18.b – The case where the N-DE $(\beta^*_B, \beta^*_C) = (1, 1)$ is possible only when $f(\alpha)$ is [locally] symmetric – Model II

We know, from condition 1.II that a N-DE (F) is only possible if $\lambda_C = \lambda_B = 1$. In a scenario where *all other* voters are expected to vote sincerely, *a BC* voter will note the existence of two pivotal events, one between 'A and C' and the other between 'A and B'. As an instrumental voter, he will only care about the probabilities of those two pivotal events, which are Pr[a = c] and Pr[a = b]. When n is large, these probabilities will be proportional to $f(\alpha_{AB})$ and $f(\alpha_{AC})$, respectively, and so the likelihood ratio λ_B = Pr[b = a] / Pr[c = a] will be evaluated as $f(\alpha_{AB}) / f(\alpha_{AC})$.

The fact that the likelihood ratio is in fact equal to $f(\alpha_{AB}) / f(\alpha_{AC})$, when $n \rightarrow \infty$, requires some clarification, which we present next.

First, we write down the expressions for the critical values α_{AC} and α_{AB}.

i) α_{AC} is such that $\pi_C = \pi_A$ $\Rightarrow$ $\alpha_{AC} [1 - \beta^*_B]\, \mathbf{q}_M + [1 - \alpha_{AC}]\, \beta^*_C\, \mathbf{q}_M = \mathbf{q}_A$

$$\Rightarrow \alpha_{AC} = [(\mathbf{q}_A / \mathbf{q}_M) - \beta^*_C] / [1 - \beta^*_B - \beta^*_C]$$

$$\Rightarrow \alpha_{AC} = [\beta^*_C - \gamma] / [\beta^*_B + \beta^*_C - 1];$$

ii) α_{AB} is such that $\pi_B = \pi_A \quad \Rightarrow \quad \alpha_{AB}\, ß^*_B\, \mathbf{q}_M + [1 - \alpha_{AB}]\, [1 - ß^*_C]\, \mathbf{q}_M = \mathbf{q}_A$

$$\Rightarrow \quad \alpha_{AB} = [(\mathbf{q}_A / \mathbf{q}_M) + ß^*_C - 1] / [ß^*_B + ß^*_C - 1])$$

$$\Rightarrow \quad \alpha_{AB} = [ß^*_C - (1 - \gamma)] / [ß^*_B + ß^*_C - 1]).$$

Then, recalling from definition 3 that $\pi_B = p\, \mathbf{q}_M$, we can write the distribution function for p

$$G(p) = \Pr[\alpha\, (ß^*_B) + (1 - \alpha)\, (1 - ß^*_C) \leq p]$$

$$= \Pr[\alpha\, [(ß^*_B) - (1 - ß^*_C)] \leq p - (1 - ß^*_C)]$$

$$= \Pr[\alpha \leq [p + ß^*_C - 1] / [ß^*_B + ß^*_C - 1]$$

$$= F([p + ß^*_C - 1] / [ß^*_B + ß^*_C - 1]).$$

Differentiating to obtain the density over p, we get

$$g(p) = G'(p) = f([ß^*_C - (1 - p)] / [ß^*_B + ß^*_C - 1]) \times [1 / (ß^*_B + ß^*_C - 1)].$$

In the case we're studying, there are two pivotal events, one between 'A and B', and the other between 'A and C'. These occur, respectively, when $\pi_B = \pi_A$ and $\pi_C = \pi_A$. Dividing both sides by $\mathbf{q}_M$ we obtain the equivalent expressions $p = \gamma$ and $(1 - p) = \gamma$, respectively. In other words, the two pivotal events involve $p = \gamma$ and $p = (1 - \gamma)$.

Then, a *BC* voter calculates the likelihood ratio $\boldsymbol{\lambda}_{\mathbf{B}} = \Pr[b = a] / \Pr[c = a]$, when n is large, as $g(\gamma) / g(1 - \gamma)$, which is equal to

$$\frac{f([ß^*_C - (1 - \gamma)] / [ß^*_B + ß^*_C - 1]) \times [1 / (ß^*_B + ß^*_C - 1)]}{f([ß^*_C - \gamma\,] / [ß^*_B + ß^*_C - 1]) \times [1 / (ß^*_B + ß^*_C - 1)]}.$$

We see that the two terms multiplying the density function cancel out and we obtain

$$\frac{f(\alpha_{AB})}{f(\alpha_{AC})}.$$

Importantly, such terms cancel out *regardless of the values that* $ß^*_B$ *and* $ß^*_C$ *take*. In other words, the likelihood ratio in Model II will always equal to $f(\alpha_{AB}) / f(\alpha_{AB})$, when $n \rightarrow \infty$, independently of the pattern of strategic voting that takes place.

The reason behind this is very simple, and was indeed already mentioned: the absolute value of the curve π_B is always equal to that for π_C in Model II. This happens because the sum of votes for parties B and C must *always* add up to a *constant* - $\mathbf{q}_M$ - regardless of the pattern of strategic voting that takes place *and* of α. So, the slopes of π_B and π_C are the negative of one another, implying they have the same absolute value.

The N-DE (F) thus require that $f(\alpha_{AB}) = f(\alpha_{AC})$, when $n \to \infty$, to satisfy $\lambda_C = \lambda_B = 1$. However, we see that the critical points α_{AC} and α_{AB} are at the exact same distance from the median point $\alpha = 0.5$; so $\alpha_{AC} = 1 - \alpha_{AB}$. It is then a *sufficient* condition for the N-DE to exist that the $f(\alpha)$ be symmetric in the interval [0, 1]. This means i.e., that $f(\alpha) = f(1 - \alpha)$, so that $f(\alpha_{AB}) = f(1 - \alpha_{AB}) = f(\alpha_{AC})$[54].

The intuition is that all voters know that there may be a pivotal event between 'A and B' or between 'A and C', and they also know that, *given that there is a pivotal event,* it will be equally likely that such event involves party B or C when n is large, since $f(\alpha_{AB}) = f(\alpha_{AC})$.

BNVE II.3.1 $(\beta^*_B, \beta^*_C) = (1, 1)$ | N-DE (F) | *when* $[\mathbf{q}_A < \mathbf{q}_M < 2\mathbf{q}_A]$ *requires symmetric* $f(\alpha)$.

In this case - when $\mathbf{q}_A > \mathbf{q}_M/2$ - the N-DE is not stable, as we illustrate in Figure 20.

The instability of this equilibrium follows from the fact that $f(\alpha)$ is symmetric and quasi-concave.

In Figure 18.b, imagine some *BC* voters deviate and vote for C. This would result in a new likelihood ratio $f(\alpha'_{AB}) / f(\alpha'_{AC})$, where $\alpha'_{AB} > \alpha_{AB}$ and $\alpha'_{AC} > \alpha_{AC}$. Assuming (strict) quasi-concavity and symmetry around $\alpha = 0.5$ implies that that $f(\alpha)$ has a (global) maximum at $a = 0.5$. Consequently, we have that $f(\alpha'_{AB}) < f(\alpha_{AB})$ and $f(\alpha'_{AC}) > f(\alpha_{AC})$. So, the likelihood ratio resulting from such deviation would be less than one, as it is equal to $f(\alpha'_{AB}) / f(\alpha'_{AC}) < f(\alpha_{AB}) / f(\alpha_{AC}) = 1$[55].

Since the likelihood ratio resulting from a deviation is - *in the margin* - below 1, this would give marginal incentives to vote strategically, again in a self-feeding process away from the equilibrium. Similarly to the previous case, if the proportion of voters switching to C is sufficiently high, party B will not have a chance to win the race, implying that the likelihood ratio turns to 0. In this case - because the support of party A is higher - the required proportion of strategic votes will be just (1 - *k*'), less than 50%.

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54 The *necessary* condition is that the density function is locally symmetric, i.e., that $f(\alpha) = f(1 - \alpha)$ in *some* ball centred at $\alpha = \alpha_{AB}$.

55 Again, this needn't take any particular value nor be continuous; the curve is merely illustrative.

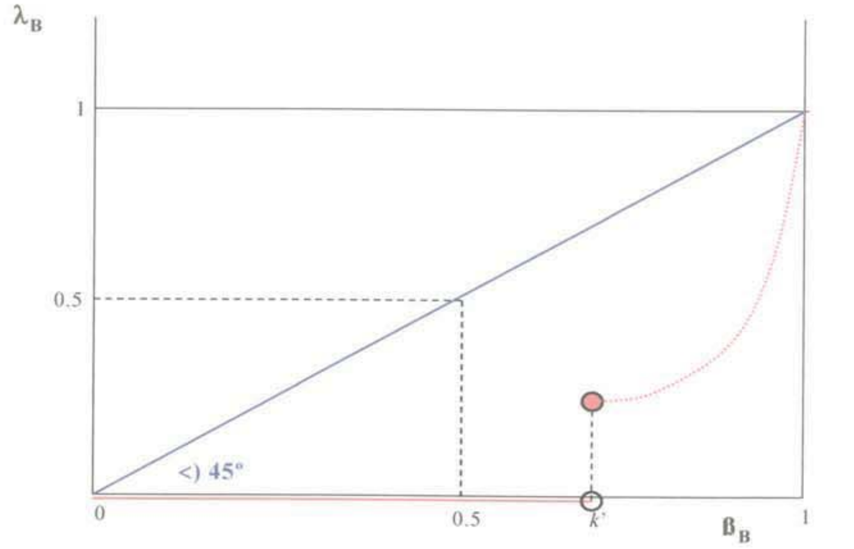

Figure 20 – The instability of the N-DE $(ß^*_B, ß^*_C) = (1, 1)$ when $q_M < 2q_A$ for a parametrised $ß^*_C = 1$ – Model II

2) The N-DE (I)

To start with, take the case where the density function is not symmetric, biased towards party B, so that $f(\alpha_{AC}) < f(\alpha_{AB})$. What would happen if all voters voted strategically? We see that this scenario would imply a likelihood ratio $\lambda_C = f(\alpha_{AC}) / f(\alpha_{AB}) < 1$. How would voters respond in such hypothetical scenario?

Voters *CB* would realise that - conditional on there being a pivotal event - it would be more likely that they were pivotal for party B than for party C. Hence, some voters - those with higher preference for party B - would want to switch their votes for B. If we were to take a decision-theoretic approach - where each voter considers whether or not to vote strategically *while* assuming all others to vote sincerely - such proportion would be exactly $[1 - \lambda_C] = [1 - f(\alpha_{AC}) / f(\alpha_{AB})]$.

We illustrate the behaviour of a decision-theoretic electorate in Figure 21, which reads as follows.

If all voters vote sincerely [i.e., if $k = 1$] the resulting likelihood ratio is λ_{CB} (1); this is I in Figure 21. A decision-theoretic voter *i* would decide whether or not to vote strategically by evaluating the likelihood ratio that results when all other voters vote sincerely. If all voters went through the same thinking process, we would expect the outcome to be II.

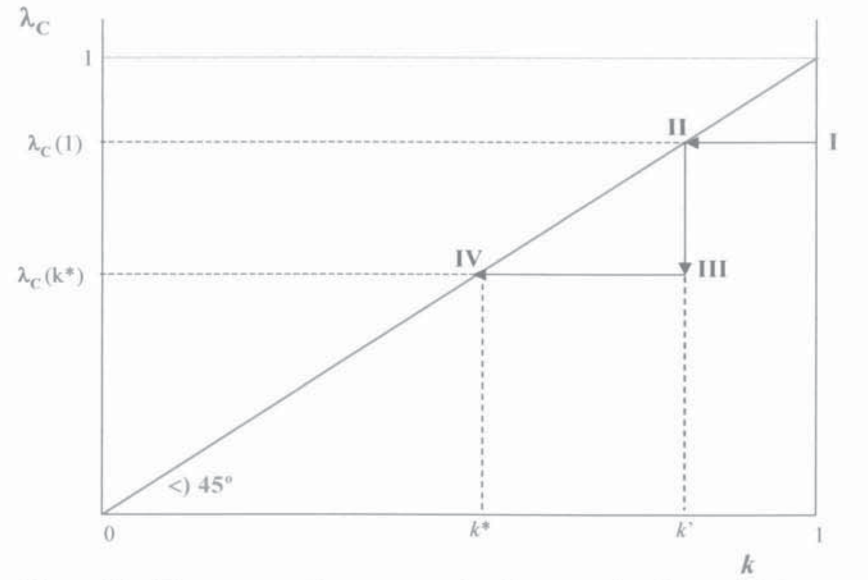

Figure 21 – The process of convergence leading to an interior equilibrium at the fixed point k^* – Model II

Then, the process would repeat itself again: if it was expected that only k' were voting sincerely, a decision-theoretic voter would compute the likelihood ratio that results from this - λ_C (k') - and consider point III in the graphic. Again, if all voters do the same, we would end up in IV. This process would repeat itself until we reached an equilibrium - *if any* - which would be a fixed point. In the above graphic, that would be possible for $k = k^*$, since $\lambda_C (k^*) = k^*$.

Differently from the previous description, we assume that voters are game-theoretical and not decision-theoretic. This means that they can go through all the *step-by-step* process and anticipate the equilibrium to be played, should it in fact exist[56].

We can only guarantee the existence of a fixed point for $k = 0$, because when all voters vote strategically the probability of being pivotal for their favourite party is 0, so that the likelihood ratio is 0 as well[57]. In other words, everyone voting sincerely is a self-fulfilling behaviour.

We illustrate the possibility for an interior N-DE in Figures 22.a, 22.b and 22.c.

In Figure 22.a, we see the critical values of α for which a pivotal event may occur when a proportion of k' votes sincerely: α'_{AC} and α'_{AB}.

56 Very much like two duopolists in a one-shot game play the Cournot-Nash equilibrium, without needing several steps to get there.

57 We're ignoring, for simplicity, the case where the likelihood ratio is 0/0 and hence not defined, when the probability of being pivotal for the second favourite candidate is 0 as well.

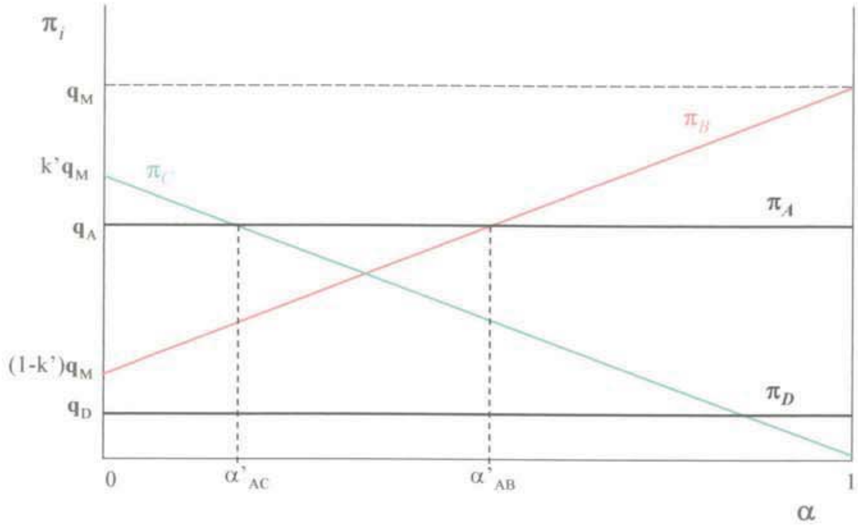

Figure 22.a – The case where an interior N-DE $(\beta^*_B, \beta^*_C) = (1, k')$ is possible, entailing *partial* strategic voting in equilibrium – Model II

In Figure 22.b, we plot the density function $f(\alpha)$ and check the values it takes when such draws occur - $f(\alpha'_{AC})$ and $f(\alpha'_{AB})$, respectively.

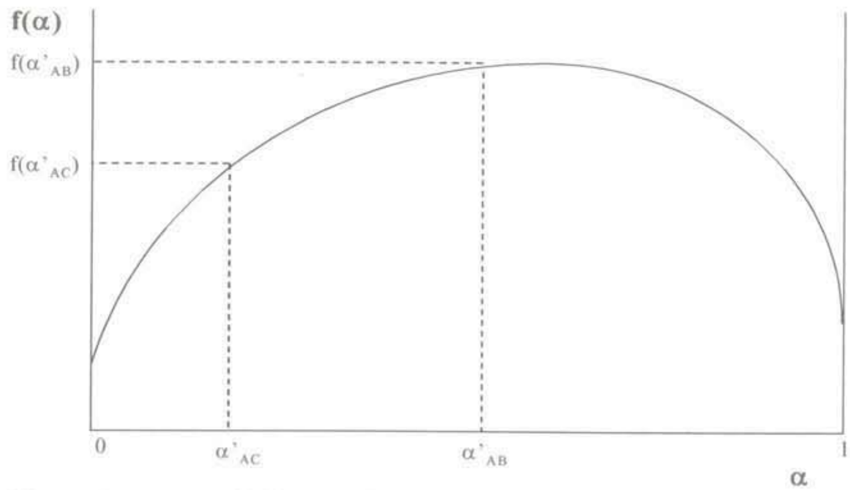

Figure 22.b – The likelihood of the different pivotal events in the interior N-DE $(\beta^*_B, \beta^*_C) = (1, k')$ – Model II

Finally, in Figure 22.c we illustrate the fixed-point argument: when n is large there is an equilibrium if the function $\lambda_C(k)$ crosses the 45° line, when $k' = \lambda_C(k') = f(\alpha'_{AC}) / f(\alpha'_{AB})$.

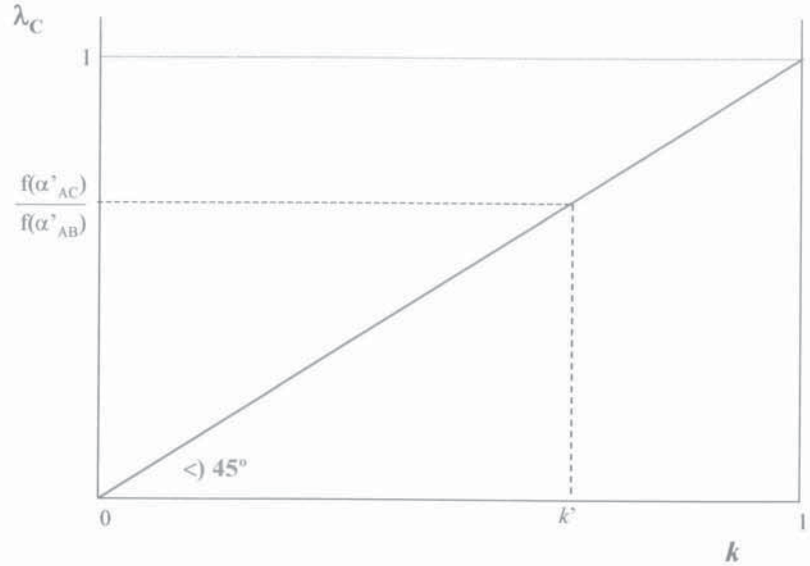

Figure 22.c – The fixed point at $k = k'$, allowing an interior equilibrium to occur – Model II

We cannot guarantee the uniqueness of an interior equilibrium, since it is possible to have more than one fixed point in (0,1). Neither can we guarantee that an equilibrium exists, since a fixed-point argument cannot be used[58].

Next, we present an example for $f(\alpha)$ that illustrates the possibility of a N-DE (I).

4.II.4 Special case for f(α): the standardised Beta-distribution

Since α is restricted to the interval [0, 1], it is natural to use the standardised Beta-distribution, whose support lies in the same interval. With only two parameters - a and b - this function is very general. It is symmetric when $a = b$, and biased towards the left (right) when $a > b$ ($a < b$). It is quasi-convex when $a, b \leq 1$ and quasi-concave when $a, b \geq 1$. When $a = b = 1$, it is the uniform distribution. Since we assumed $f(\alpha)$ to be quasi-concave, we will present some examples where $a, b > 1$, both symmetric and biased.

The probability density function of a standardised Beta-distribution is given by the formula

$$f(\alpha) = [\text{constant}]\ \alpha^{a-1}\ (1 - \alpha)^{b-1}$$

where [constant] = $\Gamma(a + b) / \Gamma(a)\,\Gamma(b)$, and $\Gamma(.)$ is the Gamma-function[59].

58 The reason why we cannot apply a fixed point argument in this case is because the likelihood function $\lambda_C(k)$ is not defined in a compact set: for a k in [0.5, 1], the likelihood ratio $\lambda_C(k)$ may take any positive value and need not be bounded to the interval [0.5, 1], so we cannot guarantee the existence of a fixed point.

59 The expected value and variance of a Beta random variable X with parameters a and b are given,

From the expression for $f(\alpha)$ we see one important additional advantage of this distribution: the [constant] term will cancel out when we compute the likelihood ratios, where we will simply obtain a polynomial expression.

We will now illustrate how an analytical solution for the N-DE (I) $(ß^*_B, ß^*_C) = (1, k')$ could be found, when $a = b = 3$. In this case, the density function is symmetric around the average point 0.5 and $f(\alpha)$ = [constant] $[\alpha\ (1 - \alpha)]^2$. This is also interesting because it shows that the N-DE (I) does not require an asymmetric distribution to occur.

First, we need to describe the interception points as a function of k'. Note that the votes C obtains will be a proportion k' of its stochastic prior support, while the votes for party B will be the sum of its prior support and any strategic votes it gets, which are a proportion $[1 - k']$ of C's prior support.

i) α'_{AC} is such that $\pi_C = \pi_A \quad \Rightarrow \quad [1 - \alpha'_{AC}]\ k'\ \mathbf{q}_M = \mathbf{q}_A \quad \Rightarrow \quad \alpha'_{AC} = 1 - \mathbf{q}_A / [\mathbf{q}_M\ k']$;

ii) α'_{AB} is such that $\pi_B = \pi_A \quad \Rightarrow \quad \alpha'_{AB}\ \mathbf{q}_M + [1 - \alpha'_{AB}]\ [1 - k']\ \mathbf{q}_M = \mathbf{q}_A$

$$\Rightarrow \quad \alpha'_{AB} = [\mathbf{q}_A - \mathbf{q}_M\ (1 - k')] / [\mathbf{q}_M\ k'].$$

Second, the likelihood ratio is $\lambda_C = f(\alpha'_{AC}) / f(\alpha'_{AB})$, and for a Beta distribution with parameters a and b it becomes

$$\lambda_C = [\alpha'_{AC} / \alpha'_{AB}]^{(a-1)} \times [(1 - \alpha'_{AC}) / (1 - \alpha'_{AB})]^{(b-1)}.$$

Lastly, an interior equilibrium requires the existence of a fixed-point k', i.e., that $k' = \lambda_C(k')$ for some k' in (0.5, 1).

We can guarantee that there exists a fixed-point in (0.5, 1) by noting that $\lambda_C(0.5) = 1$, that $\lambda_C(1) = 0$ and finally, that $\lambda_C(k')$ is a continuous function in (0.5, 1), which follows from the fact that it is a ratio of polynomials, with a non-zero denominator in the relevant domain for k'. Together, these three characteristics ensure us that there must be a k' in (0.5, 1) such that $\lambda_C(k') = k'$, when the curve $\lambda_C(k')$ crosses the 45° line.

The solution to this fixed point problem cannot in general be made explicit, but it is easily solvable with common software. We next present six possible cases which illustrate the general solution for different combinations of the parameters a and b for $a \geq 2$ and $b \geq 1$. All examples assume that $\mathbf{q}_M = 45\%$ and $\mathbf{q}_A = 30\%$, which satisfies one of the necessary conditions for a N-DE (I) to be possible, $\mathbf{q}_A < \mathbf{q}_M < 2\mathbf{q}_A$. Also, this avoids the *knife-edge* case where $\mathbf{q}_A = \mathbf{q}_D$, which is impossible when $\mathbf{q}_M = 45\%$ and $\mathbf{q}_A = 30\%$.

In each of the examples, we plot the likelihood ratio as a function of the proportion of *CB* voters that vote sincerely. We only use values for k in the interval [0.5, 1] since we know

respectively, by the formulae $E(X) = a / (a + b)$ and $Var(X) = ab / [(a + b)^2 (a + b + 1)]$.

there cannot be an interior equilibrium with less than 50% of *CB* voters voting sincerely[60]. A fixed point exists when this curve intercepts the 45° line.

Case 1: $a = b = 2$

Since $a = b$, the density function is symmetric. Because a and b are both equal to 2, it has the special form $f(\alpha) = [\text{constant}]\ \alpha\ (1 - \alpha)$ - a negative quadratic function in α. In this case, there is only one fixed point - $k = 1$ - which we can see from Figure 23.a below.

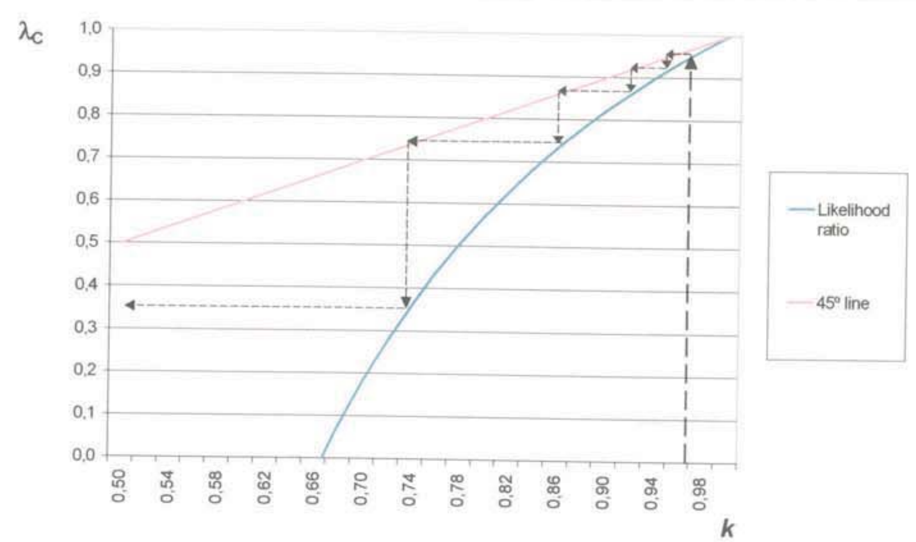

Figure 23.a - Case 1: $a = b = 2$. Only the N-DE (F) is possible, but it is not stable.

The equilibrium above, which is a N-DE (F), is not stable: any deviation from $k = 1$ will diverge to a complete switch of votes to party B.

From the dynamics that are involved in our system of two equations[61] there will exist a stable equilibrium when: i) the slopes of the two curves have opposite signs; and ii) the sum of their absolute value is *less than* 2 at the equilibrium point. Since one of the curves in our system is the 45° line, these two conditions simply require that the likelihood function - *evaluated at the equilibrium point* - has a negative slope whose absolute value is *less than* 1.

In the above case, the curve has a positive slope, and so the equilibrium can't be stable.

60 Recall the discussion about the Figures 17.a and 17.b. In any *interior* equilibrium, there must be at least half of *CB* voters voting sincerely, otherwise, party B would *always* have less votes than party C, and hence, voting for C would be a weakly dominated strategy. When $k < 50\%$, the only equilibrium possible is for $k = 0$, when there is complete coordination on party B.

61 Recall our analysis of Figure 21.

Case 2: $a = 2$ and $b = 3$

In this case, the density function is asymmetric, with an expect value of 2/5 < 1/2, so it is biased towards party C. Because it is asymmetric, a N-DE (F) is not possible, but in this case a N-DE (I) is, where approximately 74.3% of *CB* voters vote sincerely. In other words, there is *partial strategic voting*, coming from around 25.7% of *CB* voters.

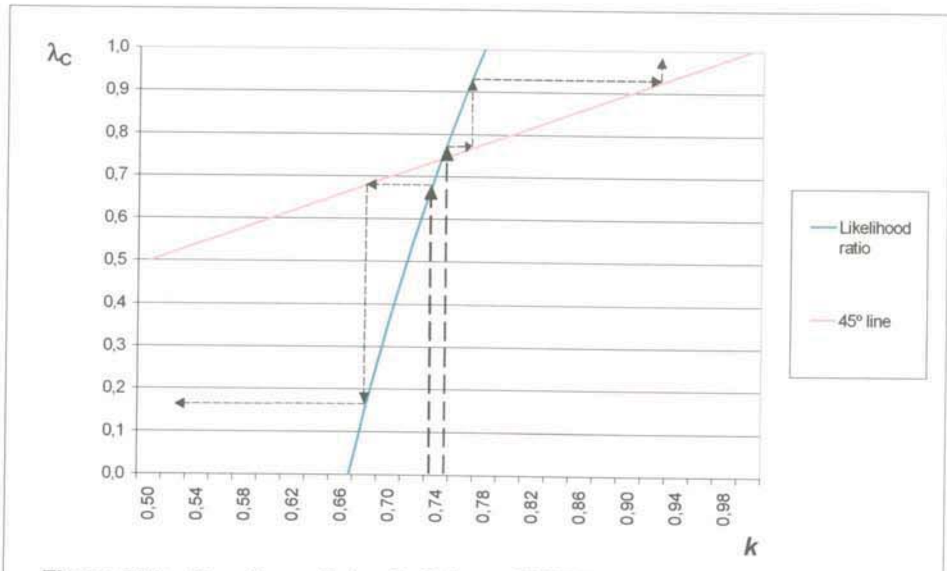

Figure 23.b - Case 2: $a = 2$, $b = 3$. Only one N-DE (I) is possible, but it is not stable.

It is interesting to note that there is a bias favouring party C and yet strategic voting moves towards B. Essentially, the underlying bias for C is offsetting the bandwagon effect towards B. A bias towards C means that there is a higher probability that *CB* voters' share is more than 50% when compared to that for *BC* voters. A bandwagon effect means that when C is expected to be victorious that will lead more voters to vote for C. This is reflected in the fact that the likelihood ratio as a function of k has a positive slope - the higher the proportion of voters voting sincerely for C the higher the likelihood ratio will be, implying fewer incentives to vote strategically. The two effects together will imply the possibility of an interior equilibrium, where they are exactly traded-off; this is the fixed point in the previous figure.

Similarly to the discussion on the N-DE in Model I, this equilibrium would require extreme coordination[62].

We see that this equilibrium is not stable since there are divergent paths in either directions. Again, this results from the positive slope of the likelihood function at the equilibrium point.

62 Recall the conclusions for Model I (pgs. 40-42).

Case 3: $a = b = 3$

Here, the density function is symmetric, and there are two equilibria, one when $k \sim 57.5\%$ and the other when $k = 1$.

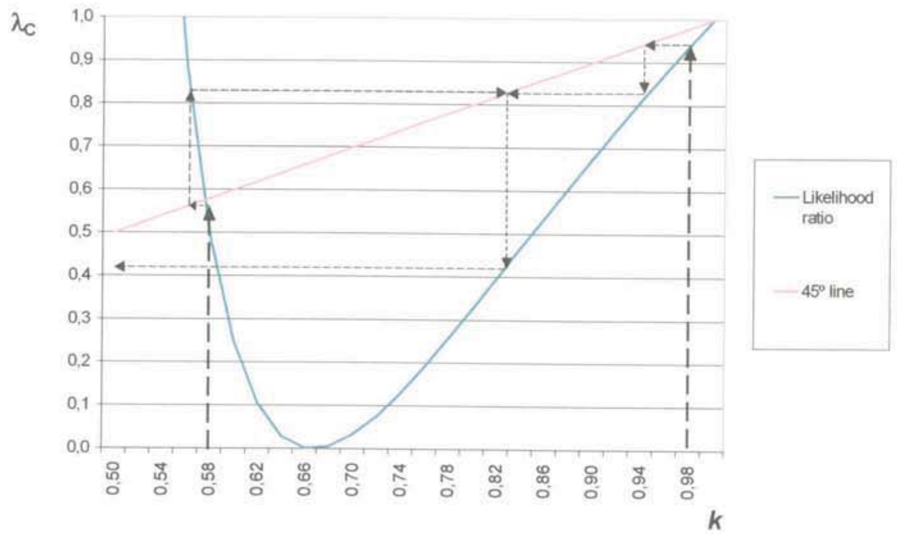

Figure 23.c - Case 3: $a = b = 3$. Both the N-DE (F) and one N-DE (I) are possible, but none is stable.

Once again, the equilibrium at $k = 1$ can't be stable. The interior equilibrium would be stable only if the slope at such point was less than 1 in absolute value, which is clearly not the case. We can confirm this by looking at the divergent paths that a deviation from either of the equilibria originates.

Case 4: $a = 3$ and $b = 2$

Here, the density function is asymmetric. The likelihood ratio function has only one fixed point, at $k \sim 55.2\%$, so a N-DE (I) is possible.

This interior equilibrium is not stable, since any deviation will diverge from it.

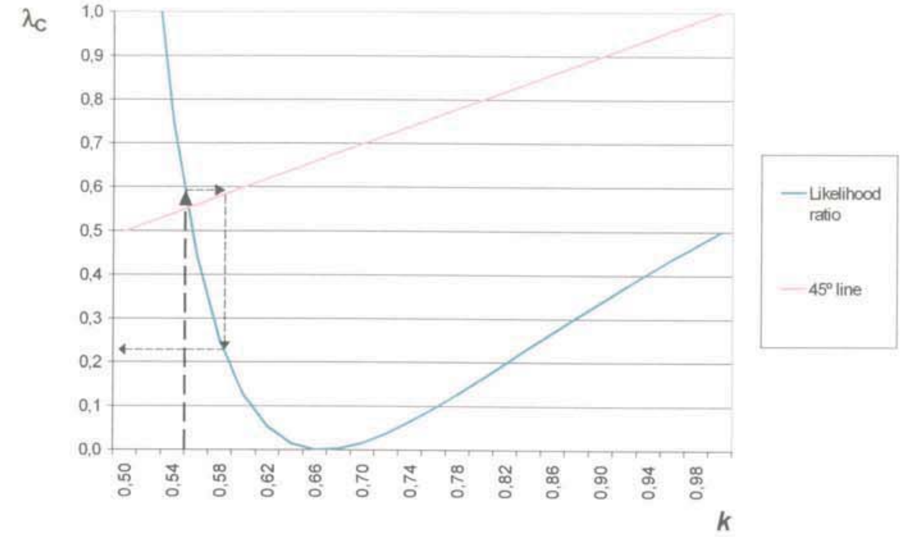

Figure 23.d - Case 4: a = 3, b = 2. Only one N-DE (I) is possible, but it is not stable.

Case 5: $a = 3$ and $b = 4$

In this case, there are two interior fixed points, for $k \sim 59.5\%$ and $k \sim 82.5\%$.

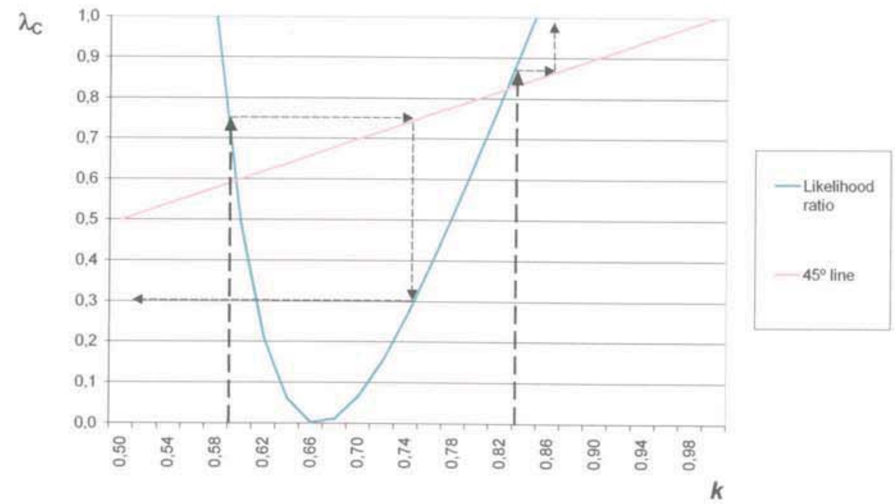

Figure 23.e - Case 5: a = 3, b = 4. Two N-DE (I) are possible, but none is stable.

For the same reasons used before, none of these equilibria is stable.

Example 6: $a = 4$ and $b = 3$

In our last example, the likelihood ratio function has no fixed points in [0.5, 1], implying that no N-DE is possible.

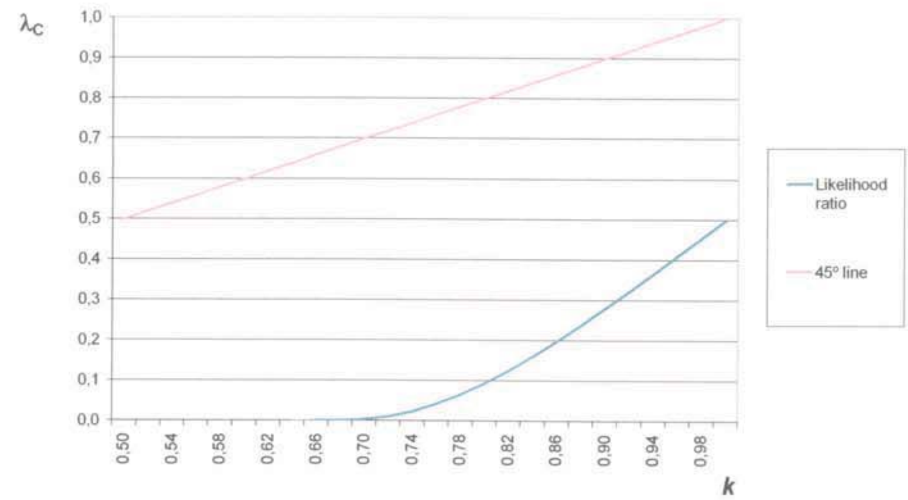

Figure 23.f - Case 6: $a = 4$, $b = 3$. No equilibria are possible.

Interpretation of the previous results using the standardised Beta distribution

The previous examples show that all the four combinations of the two types of N-DE - *interior* (I) and *frontier* (F) - can occur for some parameters a and b; they are:

i) *neither* I *nor* F (case 6);

ii) I, *but not* F (cases 2, 4 and 5);

iii) F, *but not* I (case 1);

iv) *both* I *and* F (case 3).

However, none of the equilibria we found is stable.

Next, we summarise the equilibria we found for the particular case of a standardised Beta-distribution.

BNVE II.3.2 $(\beta^*_B, \beta^*_C) = (1, 1)$ | N-DE (F) | *for a standardised Beta distribution - and when* $[\mathbf{q}_A < \mathbf{q}_M < 2\mathbf{q}_A]$ *- requires* a = b.

BNVE II.4 (β^*_B, β^*_C) = (1, *k'*) | N-DE (I) | *for a standardised Beta distribution, requires* [$\mathbf{q}_A < \mathbf{q}_M < 2\mathbf{q}_A$] *and either of three:*

i) b > a;
ii) b = a *and* a *'even';*
iii) b < a and a *'odd'.*

In the next table we describe the possible equilibria that can take place in Model II.

Table 7. *The set of possible equilibria in Model II*

Case	Conditions required	Possible Equilibria (Y/N/M[63])			
		F-DE	S-DE (F)	N-DE (F)	N-DE (I)
a	$\mathbf{q}_M > 2\mathbf{q}_A$	N	Y	Y	N
b	$\mathbf{q}_A < \mathbf{q}_M < 2\mathbf{q}_A$	N	Y	M/Y[64]	M

4.II.5 Conclusions for Model II

The overall conclusions for this model are the following.

First, a F-DE is never possible - at least three parties 'survive'. This is not a very important result as it is just a consequence of the way we constructed it.

Second, a S-DE (F) is always possible and it is always stable. This is not surprising if we recall this is the Duvergerian result in the three party literature. There are two S-DE, since coordination in each of the moderate parties will be a possible equilibrium.

Third, the N-DE (F) - where all voters vote sincerely - is always possible when party A can *never* win and is also stable [when $\mathbf{q}_M > 2\mathbf{q}_A$]. However, in the more interesting case where party A *can* win [when $\mathbf{q}_A < \mathbf{q}_M < 2\mathbf{q}_A$] the N-DE is possible if $f(\alpha)$ is a symmetric density function, but is always unstable.

This is the case studied in the three-party literature. Our results are similar to Cox's, whose non-Duvergerian equilibrium Fey (1997) showed to be unstable. As explained before, this arises from the fact that there is a *positive effect* following a deviation from an equilibrium where all voters vote sincerely: as soon as one voter switches his vote, all other voters will have incentives to vote strategically as well.

63 Y - yes; N - no; M - maybe, i.e., depending on $f(\alpha)$.
64 In case *b*, both a frontier and interior non-Duvergerian equilibria can occur. For the N-DE (F) to be possible it is a sufficient condition that $f(\alpha)$ be symmetric.

Fourth, a N-DE (I) - entailing partial strategic voting - is possible when $\mathbf{q}_A < \mathbf{q}_M < 2\mathbf{q}_A$ but is not stable[65]. Though we only showed some examples for the Beta-distribution, we suspect this to be true in general for our present model.

The reason why we believe this equilibrium to be *in general* unstable is that our model is one of *public uncertainty*. This implies that all voters *commonly know* any scenario; even if it involves some uncertainty, they would all agree on such uncertainty. So, all voters are able to figure out what *everyone thinks* about a deviation from the equilibrium, since they all share the same information and all are 'equally rational'. Since a deviation from these equilibria will generate a *positive effect* - and all voters can anticipate *and* be certain about such effect - it will be common knowledge that such equilibria are unstable. The coordination problem would again be very severe.

Myatt (2004) describes an equilibrium involving partial strategic voting that is stable. Critically, voters in his model have private information and hold different beliefs about the constituency support of all parties. As we said, Myatt's is a model of *private uncertainty*. Because voters have different beliefs and do not (necessarily) agree on the scenario in place, it is not *necessarily* true that a deviation from equilibrium will replicate itself. In fact, the paper shows that there is a *negative effect*: the more strategic voting there is the less the incentives are to vote strategically. This happens because when there are more voters voting strategically, there is an *increasing risk* that strategic voting will involve switching to the *wrong* candidate.

Our claim, then, can be summarised in three arguments:

1) A *necessary* condition for the existence of a *stable* interior equilibrium is the presence of a *negative effect*;

2) A *necessary* condition for the existence of a *negative effect* is the presence of *private information*.

3) A *necessary* condition for the existence of *private information* is the presence of *private uncertainty*[66].

We are then led to believe that Myatt (2004) indeed uses the *necessary* and *sufficient* conditions to obtain such type of equilibria.

In this sense, we think that our model - because it can be placed between Cox's and Myatt's in terms of its assumptions about voters' information - is useful to highlight the fact that private uncertainty *may* be indeed a *necessary* condition to obtain a stable interior equilibrium, in addition to the *sufficiency* proved in Myatt (2004).

---

65 This claim can be applied only to the distribution studied but the intuition for the result seems more general, as we argue after.

66 As we argued in Section III.2 (recall footnote 20, pg. 17), private information is not compatible with certainty, and so private information indeed requires *private uncertainty*.

The necessity follows from the fact that the list of possible combinations in Table 5 is exhaustive. So, in addition to the fact that Myatt's result is possible *as long as* there is private information (*sufficiency*), we argue that it can be possible *only when* private information is indeed present (*necessity*)[67].

Finally, we observe the possibility of multiple equilibria in Model II. There are at most 4 equilibria: two S-DE, one N-DE (F) and one (or more) N-DE (I). However, in case *b*, which is the more interesting case, only the S-DE is stable, so we restrict the number of stable predictions to the two possible S-DE. The same comments made for Model I regarding the multiplicity of equilibria apply here.

67 To be very clear, we do *not* believe *nor* indeed claim to have proven what we stated; rather, we give our opinion based on intuitive reasoning, hoping that further work may help clarifying this important issue.

4.III Equilibrium analysis for Model III

4.III.1 Description of the model

The following information is a reminder of the structure of Model III, where strategic voting can occur only within each side of the political spectrum, but possibly in any direction.

Figure 3. *The strategic scenario in Model III (reminder)*

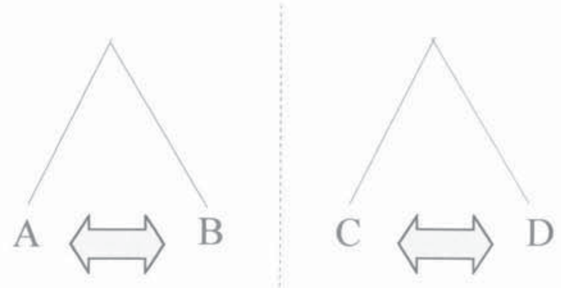

Table 3. *Voters' preferences in Model III (reminder)*

Voter Type	Utility for candidate				Prior Probability
	A	B	C	D	
AB	1	$ß_A \sim U[0,1]$	0	0	q_{AB}
BA	$ß_B \sim U[0,1]$	1	0	0	q_{BA}
CD	0	0	1	$ß_C \sim U[0,1]$	q_{CD}
DC	0	0	$ß_D \sim U[0,1]$	1	q_{DC}

We introduce uncertainty by assuming there is common knowledge about the following:

$$\mathbf{q}_M = [\mathbf{q}_{BA} + \mathbf{q}_{CD}]$$

$$\alpha = [\mathbf{q}_B / \mathbf{q}_M] \text{ in } [0, 1] \text{ with density function } f(\alpha)$$

$$ß_i \sim U[0, 1],\ i = A, B, C, D$$

In this model all voters consider voting strategically for the other party on the same side of the political spectrum. Uncertainty only concerns the prior support of the two moderate parties. However, if an extremist party receives any strategic votes from the moderate supporters its outcome will be uncertain, since it will then depend on the draw from $f(\alpha)$.

4.III.2 Voters' optimal strategies and equilibrium conditions

In this model there are four likelihood ratios to be considered, λ_A, λ_B, λ_D and λ_C. Analogously to the Model II, in Model III we have $\lambda_B = 1/\lambda_A$ and $\lambda_C = 1/\lambda_D$; their expressions are:

$$\lambda_A = \{2\ Pr[a = b] + Pr[a = c] + Pr[a = d]\} / \{2\ Pr[a = b] + Pr[b = c] + Pr[b = d]\}$$

$$\lambda_B = \{2\ Pr[a = b] + Pr[b = c] + Pr[b = d]\} / \{2\ Pr[a = b] + Pr[a = c] + Pr[a = d]\}$$

$$\lambda_C = \{2\ Pr[c = d] + Pr[c = a] + Pr[c = b]\} / \{2\ Pr[c = d] + Pr[d = a] + Pr[d = b]\}$$

$$\lambda_D = \{2\ Pr[c = d] + Pr[d = a] + Pr[d = b]\} / \{2\ Pr[c = d] + Pr[c = a] + Pr[c = b]\}$$

Condition 1.III. ***Optimal behaviour in equilibrium (Model III)***

In an equilibrium, types AB, BA, CD *and* DC *voters use the following strategy:*

AB *type: "Vote for A if* $ß_A \leq ß^*_A$ *and vote for B if* $ß_A > ß^*_A$*", where*

$$ß^*_A = \min\{1, \lambda_A\}$$

BA *type: "Vote for B if* $ß_B \leq ß^*_B$ *and vote for A if* $ß_B \geq ß^*_B$*", where*

$$ß^*_B = \min\{1, \lambda_B\}$$

CD *type: "Vote for C if* $ß_C \leq ß^*_C$ *and vote for D if* $ß_C > ß^*_C$*", where*

$$ß^*_C = \min\{1, \lambda_C\}$$

DC *type: "Vote for D if* $ß_D \leq ß^*_D$ *and vote for C if* $ß_D > ß^*_D$*", where*

$$ß^*_D = \min\{1, \lambda_D\}$$

Condition 2.III. ***Rational expectations in equilibrium (Model III)***

$$\pi_A = q_{AB}\, ß^*_A + q_{BA}\,(1 - ß^*_B) = q_{AB}\, ß^*_A + \alpha\, q_M\,(1 - ß^*_B)$$

$$\pi_B = q_{BA}\, ß^*_B + q_{AB}\,(1 - ß^*_A) = q_{AB}\,(1 - ß^*_A) + \alpha\, q_M\, ß^*_B$$

$$\pi_C = q_{CD}\, ß^*_C + q_{DC}\,(1 - ß^*_D) = [q_{DC}\,(1 - ß^*_D) + q_M\, ß^*_C] - \alpha\, q_M\, ß^*_C$$

$$\pi_D = q_{DC}\, ß^*_D + q_{CD}\,(1 - ß^*_C) = [q_{DC}\, ß^*_D + q_M\, ß^*_C] - \alpha\, q_M\,(1 - ß^*_C)$$

4.III.3 Study of possible equilibria

We start by describing how different patterns of strategic voting will influence the voting results of the four parties and their graphical representation.

In Figure 24 we illustrate the case where $\mathbf{q}_M/2 < \mathbf{q}_{AB} < \mathbf{q}_M$ and everyone votes sincerely.

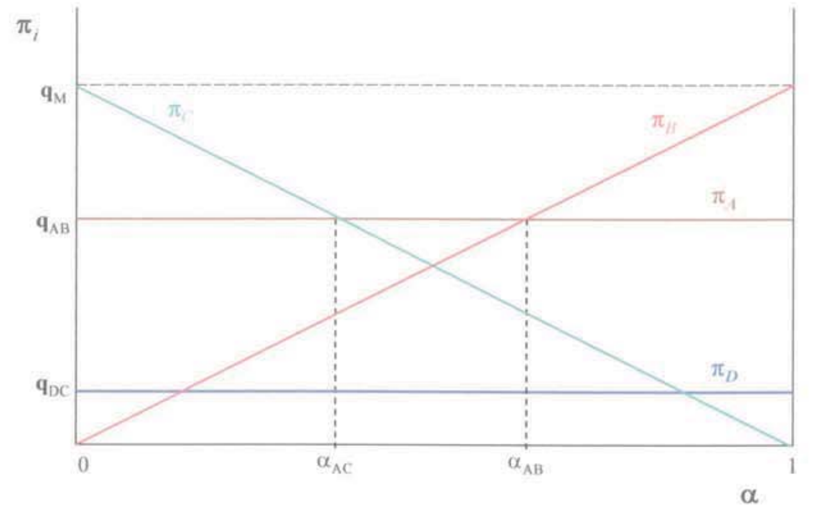

Figure 24 – The scenario where all voters would vote sincerely – Model III

In this model all voters can vote strategically, which makes it richer, since it allows for the possibility of three or four parties facing stochastic results in equilibrium.

If a proportion $(1 - k_1)$ of *AB* voters votes strategically, there is a downward and upward parallel shift on curves π_A and π_B, respectively, by the amount of strategic votes $(1 - k_1)\ \mathbf{q}_{AB}$. We illustrate this in Figure 25.a.

In Figure 25.b, we illustrate the case where a proportion $(1 - k_2)$ of *BA* voters votes strategically, which will change the slope of π_A and π_B, but not their intercept: π_A will have a slope of k_2, while π_B will have a slope of $(1 - k_2)$. The fact that π_A has a positive slope means that the prior support of party A will then depend on α, which is intuitive: if the number of *BA* voters is stochastic, and a proportion of them votes for A, such proportion will be stochastic as well.

The interesting feature is that in each of these cases there is uncertainty regarding the voting results of three parties, and not just two. Naturally, if *CD* voters also votes strategically for D, we would have uncertainty regarding the results of all parties.

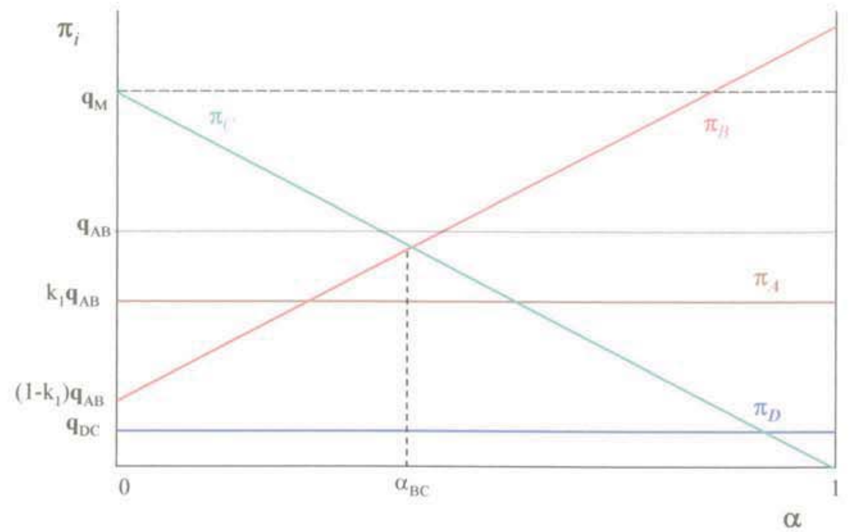

Figure 25.a – The case where a proportion (1 - k_1) of *AB* voters votes for B, leading to a paralell shift in their lines – Model III

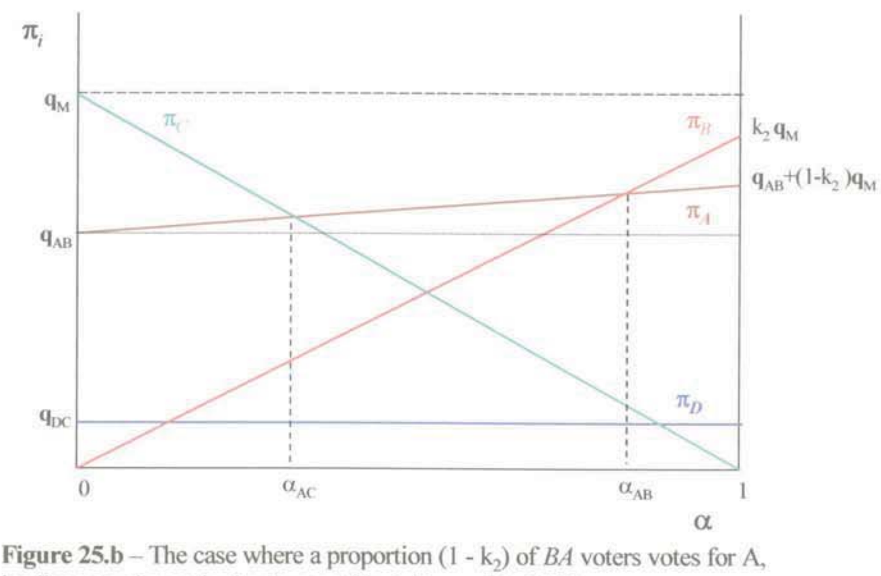

Figure 25.b – The case where a proportion (1 - k_2) of *BA* voters votes for A, leading to a change in the slopes of their lines – Model III

It is not possible, however, that both *AB* and *BA* vote strategically in equilibrium. This follows from condition 1.III: if $ß^*_A$ were between 0 and 1, so would be λ_A, implying a λ_B larger than 1 and consequently $ß^*_B = 1$. So, *BA* voters cannot vote strategically in equilibrium when *AB* voters do.

If $ß^*_A = 0$ and $ß^*_B = 1$, π_B would be a line with intercept $\mathbf{q}_{AB}$ and slope 1; a geometrically identical line would arise from $ß^*_A = 1$ and $ß^*_B = 0$, which would then be π_A.

This means that whenever there is an equilibrium involving *all* voters of one type voting strategically, there will also be an equilibrium where *all* voters of the other type that is on the same side vote strategically.

The intuition is that a complete coordination between voters *AB* and *BA*, whether in party A or B, always guarantees i) the same *fixed* number of votes and ii) the same *marginal* votes as α increases. The *fixed* votes come from *AB* voters, whose proportion of the electorate is known with certainty - this will be the intercept value of the line of the party on which coordination takes place. The *marginal* votes come from *BA* voters, whose propotion is stochastic - this will be the slope of the referred line.

There are many possible equilibria in this model, since strategic voting can occur for any group of voters. To simplify the presentation of our results, we propose the following definition, whose goal is merely to help framing the analysis.

Definition 8. *Define option* I.J *as representing the vector* ($ß^*_A$, $ß^*_B$, $ß^*_C$, $ß^*_D$) *combining option* I *for* ($ß^*_A$, $ß^*_B$) *and option* J *for* ($ß^*_C$, $ß^*_D$), *where* I *and* J *are in* {1, 2, 3, 4 , 5}, *where "1", "2", "3", "4" and "5'" stand for, respectively, the following pairs: (1, 1); (k, 1), with $0 < k < 1$; (1, k), with $0 < k < 1$; (0, 1); and (1, 0).*

Since options "4" and "5" lead to a geometrically identical result, we need only study one of them - say 4. We thus get a 4x4 matrix of possible equilibria, with 16 cases to study. However, given the symmetry of the game in the two sides of the political spectrum, we anticipate option I.J to replicate option J.I, and so we need only consider one of the two.This narrows the number of different cases to 10. There will be an exception to the previous point, since options 4.1 and 1.4 do not satisfy it, as we shall see. Therefore, there are 11 possible cases to be considered. In the next table we describe all possible combinations of equilibria in Model III, shading the 11 cases we shall consider.

Table 8. *Possible equilibrium vectors* ($ß^*_A$, $ß^*_B$, $ß^*_C$, $ß^*_D$) in *Model III*

Cases I/J	1	2	3	4	5
1	(1, 1, 1, 1)	(1, 1, 1, k_2)	(1, 1, k_2, 1)	(1, 1, 1, 0)	(1, 1, 0, 1)
2	(k_1, 1, 1, 1)	(k_1, 1, 1, k_2)	(k_1, 1, k_2, 1)	(k_1, 1, 1, 0)	(k_1, 1, 0, 1)
3	(1, k_1, 1, 1)	(1, k_1, 1, k_2)	(1, k_1, k_2, 1)	(1, k_1, 1, 0)	(1, k_1, 0, 1)
4	(0, 1, 1, 1)	(0, 1, 1, k_2)	(0, 1, k_2, 1)	(0, 1, 1, 0)	(0, 1, 0, 1)
5	(1, 0, 1, 1)	(1, 0, 1, k_2)	(1, 0, k_2, 1)	(1, 0, 1, 0)	(1, 0, 0, 1)

We start by analysing the options that are along the diagonal of that matrix.

1 - Option 1.1: N-DE (F) | ($ß^*_A$, $ß^*_B$, $ß^*_C$, $ß^*_D$) = (1, 1, 1, 1)

Notice that this example involves all voters voting sincerely, and so it is *geometrically* identical to the N-DE (F) in Model I, which we illustrated in Figure 11.

However, when compared with that in Model I, the N-DE (F) in Model III demands further conditions to be met in equilibrium, since there are *four* groups of voters that *may* vote strategically, whereas in Model I there were only *two*. The N-DE (F) of Model III requires that the choice of the supporters of parties B and C to vote sincerely be optimal, while in Model I that was *necessarily* the case[68].

The implication of this is that all four likelihood ratios must be equal to 1, which in turn requires that all parties have a chance of winning and also implies that lines π_A and π_D must be horizontal. Because their curves are flat, parties A and D can only have a chance of winning if $\pi_A = \pi_D$. Then, from condition 2.III and the fact that $ß^*_A = ß^*_D = 1$, we must have that $\mathbf{q}_{DC} = \mathbf{q}_{AB}$.

The four curves can either intercept at the exact same point, or not. In either case, both *BA* and *CD* voters will only consider one pivotal event, since their curves have a non-zero slope and cross the coincident curves π_A and π_D only once. Since such pivotal event always involves their two preferred candidates, their likelihood ratio will be equal to 1.

However, when π_A intercepts π_B and π_C in different points, voters *AB* would consider two pivotal events involving their favourite candidate, but only one event involving the second favourite. This would imply a likelihood ratio strictly higher than 1[69].

Thus, the N-DE (F) will only be possible when the four curves intercept at the exact same point, which requires $\mathbf{q}_{DC} = \mathbf{q}_{AB} = \mathbf{q}_M/2$, or $\mathbf{q}_{DC} = \mathbf{q}_{AB} = \mathbf{q}_M/2 = 25\%$. This is a unique *knife-edge* case, where there is a 50/50 spit between the support for *both* extremist parties and *both* moderate parties, and where the extremist parties enjoy the exact same support.

BNVE III.1 ($ß^*_A$, $ß^*_B$, $ß^*_C$, $ß^*_D$) = (1, 1, 1, 1) | N-DE (F) | *requires* [$\mathbf{q}_{AB} = \mathbf{q}_{DC} = 25\%$].

The N-DE (F) in Model III is unstable, for the same reasons used for its counterpart in Model I[70].

68 Recall the assumption in Model I that the supporters of the two moderate parties always vote sincerely, simply because the utility they get for the other three candidates is - by construction - zero.

69 Recall that we can be sure of this because we have assumed $f(\alpha) > 0$ for all α. For a more detailed description, see the discussion of the N-DE in Model I.

70 See the discussion of Figure 14 (pg. 40).

2 - **Option 2.2:** N-DE (I) | ($ß^*_A$, $ß^*_B$, $ß^*_C$, $ß^*_D$) = (k_1, 1, 1, k_2) with 0 < k_1, k_2 < 1

In this example, a proportion of *AB* and *DC* voters votes strategically, but all *BA* and *CD* voters vote sincerely. This implies that parties A and D results are flat lines. Furthermore, since we have assumed that k_1, $k_2 > 0$ - requiring non-zero likelihood ratios in equilibrium - both parties A and D must have a chance of winning the election. The two previous arguments together imply that $\pi_A = \pi_D > 0$.

Again, we must consider two possible cases: i) their support is below the intersection of curves π_A and π_B; or ii) the four curves intersect in the same point.

If i), because the flat lines intersect each of the other two curves, the extremist party voter will consider two pivotal events involving his favourite candidate, one of which (and only one) involves the second favourite candidate. This implies a likelihood ratio higher than 1, which contradicts our assumptions, since from condition 1 we know that the likelihood ratios for *AB* and *DC* voters must be equal to k_1 and k_2, respectively, and these are assumed to be strictly less than 1.

If ii), there is only one pivotal event, which involves the four parties, implying that all likelihood ratios are equal to 1, which also contradicts our assumption that k_1, $k_2 < 1$.

Hence, this N-DE is *not* possible.

3 - **Option 3.3:** N-DE (I) | ($ß^*_A$, $ß^*_B$, $ß^*_C$, $ß^*_D$) = (1, k_1, k_2, 1) with 0 < k_1, k_2 < 1

This option involves both *BA* and *CD* supporters voting strategically for the extremist parties A and D, respectively. As we have pointed out before, this will imply that the extremist parties result will have a stochastic component.

In other words, this equilibrium entails uncertainty about the results of all parties. This is a very important point that needs to be stressed, since it critically differentiates Model III from Models I and II, where - by construction - uncertainty could only regard the voting outcome of the moderate parties.

Geometrically, the fact that uncertainty influences the results of all parties reflects itself in the fact that curves π_A and π_D also have a non-zero slope. We illustrate this example in Figure 26.

Party A receives a proportion (1 - k_1) of party B's (stochastic) prior support, so the intercept of curve π_A is $\mathbf{q}_{AB}$ but the slope is (1 - k_1) $\mathbf{q}_M$, while the slope of B's result is k_1 $\mathbf{q}_M$. In this equilibrium, both parties A and B must have a chance of winning the election - these are necessary conditions for the interior equilibrium we're studying to occur, and are embodied in condition 1.III. Their study also helps to build some intuition about the strategic interactions of Model III, through a graphical analysis.

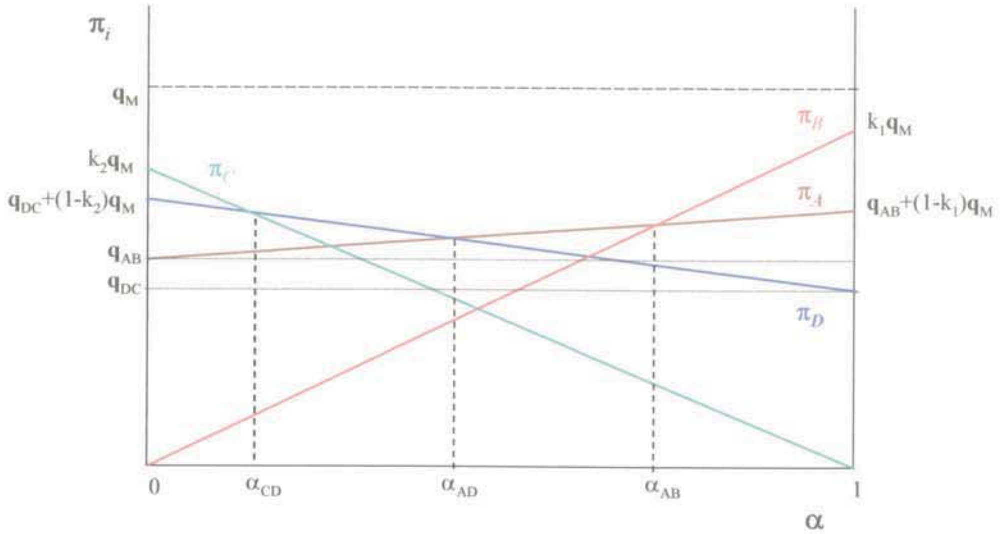

Figure 26 – The case where partial strategic voting occurs on both sides, from the moderate to the extremist parties: $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (1, k_1, k_2, 1)$, an interior N-DE – Model III

For B to have a chance of winning, it is required that π_A is not always above π_B. This must be true when $[\pi_B - \pi_A]$ takes its maximum value, which happens when $\alpha = 1$, where we get the expression $k_1\ \mathbf{q}_M > (1 - k_1)\ \mathbf{q}_M + \mathbf{q}_{AB}$, or $k_1 > 0.5 + \mathbf{q}_{AB} / 2\mathbf{q}_M$. Since the equilibrium assumes $k_1 < 1$, we only require $\mathbf{q}_{AB} < \mathbf{q}_M$, which we assume to hold.

For A to have a chance of winning, we require π_A to be above the intersection point between π_B and π_D[71]. Curves π_B and π_D cross when $k_1\, \alpha_{BC}\, \mathbf{q}_M = \mathbf{q}_{DC} + (1 - k_2)\ (1 - \alpha_{BC})\ \mathbf{q}_M$, from which we would obtain an expression for $\alpha_{BC}\ (k_1, k_2, \mathbf{q}_M, \mathbf{q}_{DC})$. We would then replace α_{BC} (.) on the relevant condition - $\pi_A(\alpha_{BC}) > \pi_B(\alpha_{BC})$ - to obtain one of the (inequality) conditions that must be satisfied in equilibrium.

Similarly, for D to have a chance of winning, we require π_D to be above the intersection point between π_C and π_A, which would lead us to a second necessary condition for the equilibrium to take place.

All the information contained in the two previous requirements can be - *implicitly* - summarised as[72]

i) $\alpha_{CD} < \alpha_{AD} < \alpha_{AB}$.

71 Note that because $k_1 > 1/2$, the slope of B's curve is higher than A's, and this makes it a necessary condition that A's curve be above the intersection between B's and C's curves.

72 'Implicitly' in the sense that they summarise what we get from an intuitive analysis of the strategic scenario obtained through a graphical analysis.

Condition 1.III is satisfied for *AB* and *DC* voters, since they consider two pivotal events, one that includes both their first and second preferred candidate, and the other that includes only their preferred candidate. As mentioned many times, this implies a likelihood ratio strictly greater than 1, which is compatible with all voters voting sincerely.

Condition 1.III also imposes - *explicitly* - that the likelihood ratios for *BA* and *CD* voters be equal to the proportion of their sincere votes, since such proportion is strictly less than 1. For them, the favourite candidate is involved in only one pivotal event, while the second favourite is involved in two. Then, their likelihood ratios will be less than one, but this is not sufficient for the equilibrium to occur: they must be precisely equal to the proportion of sincere votes. In other words, there must be a fixed point of the likelihood ratio as a function of the proportion of sincere votes. The conditions required are, then

ii) $\lambda_B = k_1 \;\Rightarrow\; 2\,\Pr[a = b] / \{2\,\Pr[a = b] + \Pr[a = d]\} = k_1$;

iii) $\lambda_C = k_2 \;\Rightarrow\; 2\,\Pr[c = d] / \{2\,\Pr[c = d] + \Pr[a = d]\} = k_2$.

Similarly to the case explained in Model II, when $n \to \infty$ these probabilities will be proportional to the density function evaluated at the critical point, *multiplied* by the inverse of the absolute value of the slope of the relevant curves. The absolute value of the slope of curves π_A, π_B, π_C and π_D are, respectively, $\mathbf{q}_M ß^*_B$, $\mathbf{q}_M (1 - ß^*_B)$, $\mathbf{q}_M ß^*_C$ and $\mathbf{q}_M (1 - ß^*_C)$, so the relevant expressions, when $n \to \infty$, become

ii) $\lambda_B = k_1 \;\Rightarrow\; \{2\,f(\alpha_{AB}) / ß^*_B\} / \{[2\,f(\alpha_{AB}) / ß^*_B] + [f(\alpha_{AD}) / (1 - ß^*_B)]\} = k_1$;

iii) $\lambda_C = k_2 \;\Rightarrow\; \{2\,f(\alpha_{CD}) / ß^*_C\} / \{[2\,f(\alpha_{CD}) / ß^*_C] + [f(\alpha_{AD}) / (1 - ß^*_C)]\} = k_2$.

Note that $\Pr[a = d]$ enters both the expressions for k_1 and k_2 in the same way, but takes different values when $n \to \infty$. The reason for this is that only the slope of the curve which is affected by the behaviour of the relevant voter type matters. So, in k_2 - which concerns *CD* voters' behaviour - it only matters the slope of π_D, while in k_1 it is π_A's slope that has to be taken into account.

This equilibrium is thus *possible*, conditional on the conditions presented before being satisfied. There may multiple equilibria of this kind, as there may be none[73].

Further, we expect it to be unstable, for the same reasons pointed out in Model II, regarding any interior equilibrium.

<u>**BNVE III.2**</u> $(ß^*_A, ß^*_B, ß^*_C, ß^*_D) = (1, k_1, k_2, 1)$ with $0 < k_1, k_2 < 1$ | N-DE (I) | *requires above conditions i), ii) and iii) to be satisfied.*

73 As we mentioned before, we will not provide an example for a specific density function, since it would add very little to the examples used for Model II.

4 - Option 4.4: F-DE | ($ß^*_A, ß^*_B, ß^*_C, ß^*_D$) = (0, 1, 1, 0)

This case is naturally possible, as there will be only one pivotal event involving the parties that get any votes - B and C in this case - so the concentration of votes on those parties becomes self-fulfilling. This is identical to the F-DE we found in Model I, which is illustrated in Figure 7.

Geometrically, the curves π_B and π_C intercept at one point, say $\alpha = \alpha^*$, and the likelihood ratio considered by any of the extremist voters is simply proportional to $0/f(\alpha^*) = 0$, as required from condition 1.III.

Hence, this equilibrium is always possible - and it is also stable.

BNVE III.3 ($ß^*_A, ß^*_B, ß^*_C, ß^*_D$) = (0, 1, 1, 0) | F-DE | *always possible.*

In Model III, there are four possible F-DE, since there can be coordination on each of the two parties of each of the two sides of the political system. The fact that on each side voters may coordinate on either of the two parties ressembles again a 'Battle of Sexes' game, where voters wish to coordinate in one of the parties.

In this sense, the F-DE of Model III is *close to* a *pair* of 'Battle of Sexes' games[74].

5 - Option 4.3: S-DE (I) | ($ß^*_A, ß^*_B, ß^*_C, ß^*_D$) = (0, 1, k_2, 1) with $0 < k_2 < 1$

In this case, all *AB* voters vote strategically for B, while some of *CD* supporters vote strategically for D. Then, $\pi_A = 0$, while π_B suffers an upward parallel shift equal to $\mathbf{q}_{DC}$; π_C and π_D will change similarly to option 3.3. We illustrate this in the Figure 27.

The strategic votes from *AB* voters for party B become self-fulfilling. The partial strategic voting from *CD* voters to party D can be part of an equilibrium as long as party B's curve crosses parties C and D lines in such a way that, when $n \to \infty$, two conditions are met:

i) $\alpha_{BD} > \alpha_{CD}$;

ii) $\lambda_C = k_2 \Rightarrow \{2 f(\alpha_{CD}) / ß^*_C\} / \{[2 f(\alpha_{CD}) / ß^*_C] + [f(\alpha_{BD}) / (1 - ß^*_C)]\} = k_2$.

Similarly, so will option 3.4 be possible, when similar conditions are satisfied. Again, we expect it to be unstable.

BNVE III.4 ($ß^*_A, ß^*_B, ß^*_C, ß^*_D$) = (0, 1, k_2, 1) with $0 < k_1, k_2 < 1$ | S-DE (I) | *requires above conditions i) and ii) to be satisfied.*

74 However, Model III cannot be considered *globally* as a pair of 'Battle of Sexes' game, since this would require the extremist and moderate supporters [on each side] to be in a symmetric position. This is not the case in our model because we assume the prior support of the moderate parties to be stochastic while the extremists' is not.

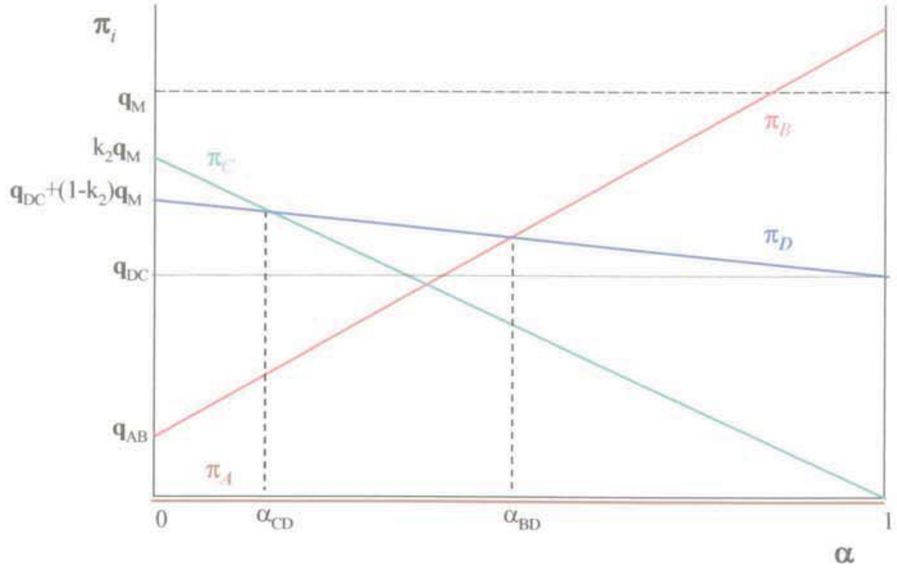

Figure 27 – The case where partial strategic voting occurs only on one side, $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (0, 1, k_2, 1)$, an interior S-DE – Model III

6 - Option 1.3: N-DE (I) | $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (1, 1, k_2, 1)$ with $0 < k_2 < 1$

In this case, all voters vote sincerely, except for a proportion of $(1 - k_2)$ of *CD* voters that switch their votes to party D. Party A's curve is again a flat line at $\mathbf{q}_{AB}$, while B's has a slope of $\mathbf{q}_M$.

The only way to guarantee that the likelihood ratios for *AB* and *BA* voters are equal to 1 is if there is only one pivotal event where A and B are present. This is only possible when curves π_A, π_B and π_D intercept at the same point. This *knife-edge* equilibrium is illustrated in Figure 28.

This equilibrium will *not* be possible when $n \to \infty$.

Recalling our initial discussion on the pivotal probabilities, we know that when the electorate gets arbitrarily large, the events involving three-way ties will become dominated by two-way ties[75]. So, *CD* voters will in fact consider only the two-way tie involving parties C and D, and hence their likelihood ratio will be equal to 1; so $k_2 < 1$ is not possible.

75 See pages 18-20 for a discussion of the pivotal probabilities.

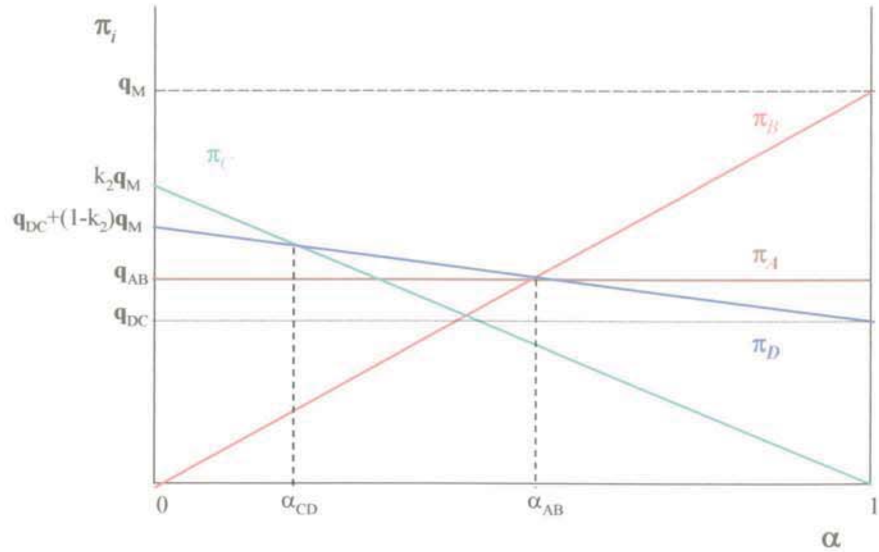

Figure 28 – The case where $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (1, 1, k_2, 1)$, which is *not* an equilibrium - Model III

7 - Option 2.3: N-DE (I) | $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (k_1, 1, k_2, 1)$ with $0 < k_1, k_2 < 1$

In this case, π_A is a flat line, equal to $k_1 \mathbf{q}_{DC}$. Since $k_1 < 1$, party A must have some chance of winning. However, if it crosses any other curves only once, that would be a *knife-edge* scenario similar to the case presented in Figure 28, where it would intercept π_D and π_B in the same point. Then, the only pivotal event would involve parties A and B, and so the likelihood ratio would be $1 > k_1$, which wouldn't satisfy condition 1.III. If it crossed any other curves twice, it would be similar to the case illustrated in Figure 21, where the likelihood ratio is strictly higher than 1, hence not satisfying condition 1.III either.

Thus, this equilibrium is *not* possible; similarly, neither will case 3.2 be.

8 - Option 4.1: S-DE (F) | $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (0, 1, 1, 1)$

In this case, all *AB* voters vote strategically and hence party B has a minimum of $\mathbf{q}_{AB}$ votes, to which will be added any stochastic votes it may obtain. It is also true that *DC* and *CD* voters vote sincerely, which implies that party D has a total of $\mathbf{q}_{DC}$ votes.

Because we have assumed - without loss of generality - that $\mathbf{q}_{DC} < \mathbf{q}_{AB}$, party D always gets less votes than B does. Hence, it becomes a weakly dominated strategy for *DC* voters to vote sincerely, after they take into account that all *AB* voters are switching their votes to B, since party D could never win. Thus, voting sincerely cannot be part of an equilibrium.

Therefore, this case is *not* possible.

9 - Option 1.4: S-DE (F) | $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (1, 1, 1, 0)$

Despite option 4.1 not being possible, this option is, as voting sincerely is not a weakly dominated strategy for *AB* voters. This is identical to the S-DE (F) of Model I, illustrated in Figure 10, which is possible when $\mathbf{q}_{DC} > \mathbf{q}_{AB}$.

Importantly, Model III is very different from Model I: in Model III the supporters of party B *can* vote strategically, while that was not possible in Model I. This means that the we additionally have to check the conditions for optimal behaviour of these voters.

If we look at Figure 10, we see that there are two pivotal events involving A and only one involving B. In Model III, *BA* voters would in such scenario face a likelihood ratio lower than 1, which is incompatible with them voting sincerely. The equilibrium we are studying requires that $\lambda_A = \lambda_B = 1$. This can only be possible if there is a *knife-edge* case and the three curves intercept at the same point, which is illustrated in Figure 29[76].

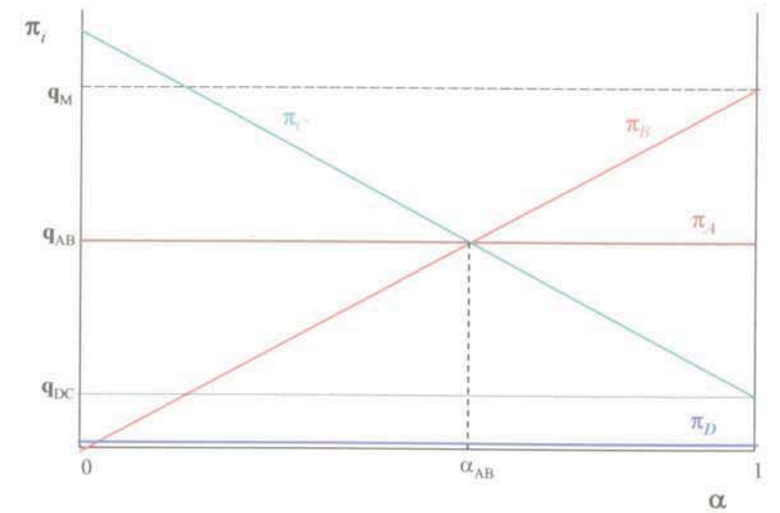

Figure 29 – The *knife-edge* case of a S-DE $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (1, 1, 1, 0)$ - Model III

Since π_A and π_B intercept at $\alpha = \alpha_{AB}$, we have that $\alpha_{AB} = \mathbf{q}_{AB} / \mathbf{q}_M$. Further to that, we require that party D's prior support be such that party C's curve will cross this same point, which is equivalent to $\mathbf{q}_{DC} + (1 - \alpha_{AB})\, \mathbf{q}_M = \mathbf{q}_{AB}$, implying that $\mathbf{q}_{DC} = 2\, \mathbf{q}_{AB} - \mathbf{q}_M$.

76 In this case there is only one pivotal event, which is a three-way tie. Recall that a three-way tie becomes dominated by a two-way tie as *n* gets large *but*, importantly, here the three-way tie *must* be considered, since it is *unique*.

Consequently, this equilibrium is only possible for the *knife-edge* cases involving $\mathbf{q}_{AB} = 1/3$ and $\mathbf{q}_{DC} = 2/3 - \mathbf{q}_{M}$. We see that this equilibrium requires *stricter* conditions than the S-DE in Model I did, since here we demand that the sincere votes from *BA* voters be optimal, whereas in Model I that behaviour was given.

BNVE III.5 $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (1, 1, 1, 1)$ | S-DE (F) | *requires* $[\mathbf{q}_{AB} = 1/3]$.

It is clearly unstable, since any deviation from *AB* voters would lead to a self-feeding process towards a total shift for party B.

10 - Option 2.4: S-DE (I) | $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (k_1, 1, 1, 0)$ with $0 < k_1 < 1$

Similarly to option 2.3, this option will *not* be possible because the likelihood ratio considered by the *AB* voters, *given that it must be positive*, will be equal or greater to 1, which can't satisfy condition 1.III[77].

11 - Option 2.1: N-DE (I) | $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D) = (k_1, 1, 1, 1)$ with $0 < k_1 < 1$

This is *not* possible, for the exact same reasons used for option 2.1.

From the 11 options analysed, we see that there are 5 *non-redundant* different equilibria, and 7 *redundant* ones, adding up to a total of 12 possible equilibria. This is summarised in the next table, where the non-redundant equilibria are shaded.

Table 9. *The 5 non-redundant and 7 redundant equilibrium vectors* $(\beta^*_A, \beta^*_B, \beta^*_C, \beta^*_D)$ *in Model III*

Cases I/J	1	2	3	4	5
1	(1, 1, 1, 1)	--	--	(1, 1, 1, 0)	(1, 1, 0, 1)
2	--	--	--	--	--
3	--	--	$(1, k_1, k_2, 1)$	$(1, k_1, 1, 0)$	$(1, k_1, 0, 1)$
4	--	--	$(0, 1, k_2, 1)$	(0, 1, 1, 0)	(0, 1, 0, 1)
5	--	--	$(1, 0, k_2, 1)$	(1, 0, 1, 0)	(1, 0, 0, 1)

We describe the conditions required for each of the 5 non-redundant equilibria to occur in the next table, similarly to the other two models, still assuming that $\mathbf{q}_{DC} \leq \mathbf{q}_{AB}$.

77 It must be positive since party A must have a chance of winning, in order to justify the hypothesis that he gets some sincere votes in equilibrium.

Table 10. *The set of different equilibria in Model III*

Case	Conditions required	Possible Equilibria (Y/N/M)				
		F-DE	S-DE (F)	S-DE (I)	N-DE (F)	N-DE (I)
a	$\mathbf{q}_{DC} = 1/4 = \mathbf{q}_{AB} = \mathbf{q}_M/2$	Y	N	M	Y	M
b	$1/3 = \mathbf{q}_{AB} < \mathbf{q}_M$	Y	Y	M	N	M
c	$\mathbf{q}_{AB} < \mathbf{q}_M$	Y	N	M	N	M

4.I.4 Conclusions for Model III

Model III can be said to be the richest from the three proposed, as it allows all voters to vote strategically. Its results should be compared to those of Model I, since this is a special case of Model III, where the moderate supporters always vote sincerely.

On the one hand, Model III allows the possibility of more equilibria when compared to Model I.

The fact that there may be uncertainty about the four parties' results generates the possibility of *interior* equilibria: both a S-DE (I) and a N-DE (I) can occur in Model III, which was not possible in Model I. If the voting results weren't uncertain, there would always exist an incentive for all voters to vote in the same way, and this is why uncertainty is critical in determining the outcome[78].

The uncertainty gives the 'extra bit' that allows voters from the same group to behave differently. The equilibria involving partial strategic voting is an interesting result, but it rests on the possibility of a *knife-edge* case involving strong conditions regarding $f(\alpha)$; any of these equilibria would require extreme coordination, but they are, nevertheless, possible. However, the fact that the uncertainty is *commonly known* implies that any interior equilibrium will not be stable, since any deviation would be self-reinforced.

On the other hand, the equilibria that were possible in Model I are 'less likely' to occur in Model III - in the sense of involving stricter conditions.

If we look at the N-DE (F), we see that in Model I it only required $\mathbf{q}_{AB} = \mathbf{q}_{DC} \geq \mathbf{q}_M/2$, while in Model III it requires $\mathbf{q}_{AB} = \mathbf{q}_{DC} = \mathbf{q}_M/2$.

If we look at the S-DE (F), we see that in Model I it only required $\mathbf{q}_{AB} \geq \mathbf{q}_M/2 + \mathbf{q}_{DC}/2$, while in Model III it requires $\mathbf{q}_{AB} = 1/3$.

78 Or, perhaps better, in determining the set of possible outcomes. The fact that an interior equilibrium requires uncertainty about the voting result of the party implies that partial strategic voting in Model III can only come from the moderate supporters.

So, while in Model I the N-DE (F) and the S-DE (F) required only *inequality* conditions on the prior support of the parties, in Model III they require *equality* conditions - which are much stricter.

Finally, the F-DE is naturally identical in both models - and, so, it is always possible in Model III as well. This should be no surprise to us because of the self-fulfilling character of this kind of equilibrium. If all voters from one group are to switch completely to another party, that is not only self-fulfilling for such group, but also makes it optimal for the supporters of the party receiving those strategic votes to stick to their preferred candidate.

Thus, the extra constraint in Model III that comes from the fact that the behaviour of the moderate supporters must also be optimal is *not binding*. The F-DE is - by its nature - a stable equilibrium.

However, there is one important difference imposed by the stricter conditions required in Model III: here, the F-DE is a stronger prediction than in Model I, since all the other possible equilibria are *knife-edge* cases, and further to that, they are all unstable.

From the 11 possible equilibria in Model III, there are only 4 stable predictions, which are the *four* F-DE.

This reinforces the importance of considering this model to be close to a pair of 'Battle of Sexes' game, where the coordination problem gets an even higher importance. This leads us back again to the comments made in Model I regarding the problem of selecting an equilibrium by making it focal to all voters. Here, the power of a *focal arbiter* would be even greater than in Model I[79].

The previous conclusion is in favour of Duverger's Law - more than it was in Model III, which could seem somewhat paradoxical. Our interpretation is that when uncertainty is commonly known among the electorate, the more uncertainty there is - *in the sense that more parties have voting results that are stochastic* - the stronger is the incentive to coordinate in the two front-runners.

79 See page 42.

5. Overview on Models I, II and III

We have seen that in each model there are different possible equilibria, with different stability properties. It is interesting to analyse how these relates to the strategic interactions that take place in each of them.

Model I is a game where the extremist parties are in somewhat a disadvantage, as their supporters can vote strategically for the moderate parties, but the reverse is not true. This leads to a tendency for at least one of them to disappear.

Model II is the opposite of Model I in the sense that in Model II the parties that can be considered to be in somewhat a disadvantage are the moderate ones. As we showed, Model II is strategically equivalent to a model of three-party competition, which is basically a 'Battle of Sexes' game where the two moderate parties must coordinate to defeat the extremist party with highest support.

Further to this difference, there are two other important differences between Models I and II that we wish to stress.

The first difference comes from the fact that we modelled uncertainty in a *asymmetric* way, by assuming that only the prior support of the moderate parties is stochastic.

The consequence of this is that in Model I a *stable prediction* must involve *at least* one of the extremist parties getting zero votes. In fact, the voting results of parties A and D are commonly known with certainty in equilibrium. Then, two cases may be possible: i) parties A and D do not tie; or ii) parties A and D do tie.

In the first case, the supporters of the party getting least votes will have an *infinite* incentive to vote strategically. In the second case, they may vote sincerely but such equilibrium is *unstable*, since any deviation will lead away from it.

The second difference comes from the fact that in Model I the voters that *may* vote strategically do *not* have any common party in their two preferred candidates, which they do in Model II.

This implies that in Model II the equilibrium behaviour of the voters that may vote strategically is *directly* interdependent, which makes the likelihood ratios be the inverse of each other. In Model I, however, they are only *partially* interdependent, making it possible to have both voters voting strategically. If Model II resembles the 'Battle of Sexes' game, in Model I it is not even adequate to use the term 'coordination', since the voters have completely opposite preferences. It is then more appropriate to associate Model I to an *implicit* bargaining situation, where each group of supporters of the extremist parties evaluates how likely the other group is to cast a sincere or a strategic vote.

Model III is the richest model, where all parties can suffer or benefit from strategic voting, depending on whether that comes from its or other party supporters.

In this sense, Model III is a pair of symmetric games - a pair of 'Battle of Sexes' games - with the only difference that only the moderate parties have a stochastic prior support.

Interestingly - because the extremist parties can receive strategic votes from the supporters of the moderate parties - there is a possibility that the result of three or even four parties will be stochastic in equilibrium. This is an important difference to Models I and II, which allows for the possibility of having equilibria with partial strategic voting in either side of the political arena. However, those equilibria are not stable.

We summarise the equilibria we found for the three models in Table 11.

Since we have offered an extensive analysis of all the equilibria found for each of the three models individually, we will restrict our attention to an overview on them.

Table 11. *The set of different equilibria in Models I, II and III*

	Types of Equilibrium (Y/N; S/NS[80])				
	F-DE	**S-DE (F)**	**S-DE (I)**	**N-DE (F)**	**N-DE (I)**
Model I	**Y-S**	**Y-S**	N	**Y-NS**	N
Model II	N	**Y-S**	N	**Y-NS**	**Y-NS**
Model III	**Y-S**	**Y-S**	**Y-NS**	**Y-NS**	**Y-NS**

The N-DE (I) is only possible in Models II and III. As we said, this steems from the fact that *uncertainty* about the party whose supporters vote strategically is a *necessary* condition to have an interior equilibrium. They are, however, *unstable*, because a *necessary* condition for stability is that the uncertainty is *not commonly known*.

The N-DE (F) is possible in all models, but, as in the previous equilibrium, it involves a *knife-edge* case that leads to instability. This equilibrium has a special importance because it involves the truthful revelation of the [first] preferences of each voter in an election. The fact that it is not stable has a strong implication: in any plurality rule election, the election results will *always* be expected to involve *some* strategic voting. And, in this sense, the results can *not* be interpreted as being a mirror image of the true preferences of the electorate.

The S-DE (I) is only possible in Model III, and is unstable as well.

The S-DE (F) is always possible and always stable, regardless of the framework considered. In other words, if everyone expects one party to get no votes in equilibrium, that is a stable self-fulfilling prediction.

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80 Y - Yes; N- No; S - Stable; NS - Not stable.

The F-DE is always possible and stable. There is an exception for Model II, but which is not serious, since that is excluded by construction.

Naturally, this brings up the problem of equilibrium selection. We have seen that in Models I and II there are 4 possible equilibria, whereas in Model III there are 12. If we restrict our attention to the stable equilibria, we see that there are: i) 3 stable predictions in Model I [one F-DE and two S-DE]; ii) 2 stable predictions in Model II [two S-DE]; and iii) 8 stable predictions in Model III [four F-DE and four S-DE].

So, the problem is significant in Models I and II and severe in Model III. In either of these, the problem of equilibrium selection is one of knowing which will be the party (or parties) to be 'sacrificed' in equilibrium, in the sense of not getting any votes.

In Model I, at least one - but perhaps both - of the extremist parties must disappear.

In Model II, exactly one moderate party must disappear.

In Model III, either one or two parties disappear, but never from the same side.

The prediction in Model II corresponds to the well-known Duvergerian outcome from the three-party literature, where the supporters of the challenger parties coordinate in only one of them.

Overall, then, we obtain a strong tendency towards the elimination of *at least one* party.

Our conclusions are then in consonance with Duverger's Law - which predicted a *tendency* to bipartism in a system under a plurality-rule election.

V. Conclusion

In this paper we introduced the analysis of strategic voting in a four-party model.

We showed that a four-party model can be *strategically different* from the vastly studied three-party one, as strategic voting can be brought in many different ways to a model of four-party competition. Given this, we analysed only a *cohort* of the possible combinations.

We proposed three different frameworks - Models I, II, and III - an *exhaustive* sub-set if we consider some additional assumptions. A natural next step to this would be to allow voters to consider voting for three candidates and not just two.

We also modelled uncertainty in a way that is *novel* to the literature.

We used a model of *public uncertainty*, where there is true uncertainty even when the electorate is arbitrarily large, but which is *common knowledge* to everyone.

We claimed that our approach closes the existing gap between the mainstream approach initiated by Cox (1987), where uncertainty is only apparent, and Myatt's (2204) approach, which uses a private information setting.

We then concluded that our model of *public uncertainty* is *not sufficient* to obtain a stable equilibrium with multi-candidate support. This conclusion, together with Myatt's, shows that the existence of *private information* is indeed a *necessary* condition to obtain that.

A stable prediction in our model involves *at least* one party not getting any votes in equilibrium. It is possible to have a stable equilibrium with three parties, so a Duvergerian result *need not* occur in our model. However, this outcome should be interpreted with some caution, because we restricted our attention to a static *one-shot* election. Introducing a dynamic model would be likely to reinforce the Duvergerian prediction.

Having this in mind, we must acknowledge that a model like ours - where voters hold the same beliefs - leads to results in strong support of Duverger's Law. The reason for this is simply that in any equilibrium where all parties survive strategic voting has a *positive effect* on itself - because everything in the model is commonly known.

One other important feature of our model is the multiplicity of equilibria, which raises the problem of *equilibrium selection*. This is a crucial issue for the political actors, as it involves the possibility of manipulating voters' expectations and attempting to make some preferred outcome become focal to everyone.

It is our wish that future work will bring other new and important insights on the impact that strategic voting may have on the political realm. The growing importance of and accessibility to information in modern's societies is an additional motivation to study these issues that may certainly interfere with people's everyday life.

APPENDIX

As we mentioned in the Introduction, this Appendix is self-contained and *not* relevant for our analysis, which is purely theoretical.

We decided to include it to give an opportunity chance to the interested reader to learn something about the Portuguese political system, one nice example of a four-party competition. We also include a very brief description of the Irish case.

The four dominant parties in the Portuguese Parliament are described in Table A1.

Table A1. *Portuguese parties*

	Left	**Right**
Moderate	PS	PSD
Extremist	PCP	PP

__Legend__:

PS - *Partido Socialista* (Socialist Party)
PCP - *Partido Comunista Português* (Portuguese Communist Party)
PSD - *Partido Social Democrata* (Social Democratic Party)
PP - *Partido Popular* (Popular Party)

Portugal is a young political democracy, with only 30 years of age. After the long-lasting dictatorial period when the country was governed by Salazar (1928-68) and Caetano (1968-74), a peaceful military revolution in 1974 set the country in a new era of freedom and democracy.

Three parties were created that same year: the socialist party, PS; the social-democratic party, PPD-PSD; and the Christian-democratic party, CDS (which changed to PP recently). The communist party, PCP, was the only party that already existed; it had grown in the clandestinity, holding strong connections with the soviet counterpart since the early 20's.

From the 80's onwards, the electoral support for each party has been relatively stable, with the extremist parties PCP and PP enjoying a support of around 8-10%, while the moderate parties enjoy a support of around 35-45%[81]. The split of votes for the left-wing parties (PS and PCP) and the right-wing parties (PSD and PP) has been close to a 50-50 split.

The Portuguese parliamentary elections occur every four years, where the 230 members of

81 Here, we mean the effective result in the legislative elections. Naturally, the outcome may have some strategic voting embodied in it.

the Parliament are chosen. There are 20 constituencies, each of which is allocated a number of seats[82]. Each vote counts only at the constituency level, and the members are chosen according to the proportional representation Hondt system. The party obtaining the highest number of seats is called upon to govern the country.

The presidential election runs every five years and is decided by the *run-off* rule[83]. This works as follows: if any candidate obtains more than 50% of the votes in the first round, he is declared the winner; if that doesn't happen, the two candidates with highest support proceed to a second round, where the winner is the candidate with highest number of votes

In our model we use the plurality rule, and this is much closer to the run-off rule than to any rule involving proportional representation. For this reason, any use of our work to comment on the Portuguese political system should be restricted to presidential elections[84].

The political debate about strategic voting in Portuguese politics is an intense one. It is a rather inevitable thing in all Portuguese elections - and certainly in the parliamentary ones - that *all* parties ask their supporters to cast a *useful vote* in their party, as opposed to a *wasted vote* in the party next to them[85]. Since the elections are basically decided between the two moderate parties[86], such parties ask the supporters of the extremist party on their side to cast a *useful* vote on their parties, to defeat the other moderate party.

The extremist parties not only demand that *their* supporters do *not* vote strategically, but also ask the supporters of the moderate party on their side cast a *useful* vote for them. The arguments vary from i) electing more members to the parliament to defend their own causes, ii) preventing the moderate party from having an absolute majority[87], iii) making them have a stronger position in case of an ex-post coalition, among many others.

Interestingly, strategic voting has also been reported to occur from the supporters of one moderate party to another. The reason people do this is to increase the probability of one of them having an absolute majority, which would bring more stability to the government[88].

82 To be precise, there are 18 constituencies in continental Portugal, and 2 other that correspond to the two Atlantic islands - Madeira and Azores.

83 Also known as *majority rule, run-off* election.

84 Despite the superior interest that parliamentary elections have (over the presidential ones), one cannot analyse a model using a proportionality rule without having first studied the case for the plurality rule.

85 The expression in portuguese is *voto útil*, which translates as *useful vote*. Politicians rarely use the expression "strategic voting"; political analysts somewhat more often use it. In Portugal, the word *strategic* either is not really comprehended by some people or it has a bad connotation; people interpret it as something *dishonourable*, as if you had betrayed your party with a very unscrupulous motive. The expression "useful vote" is more accepted, because people interpret it with some sort of resignation or commiseration: someone casts a *useful* vote when they "see no other option than that".

86 There may be post-elections coalitions, but the prime-minister always comes from one of the moderate parties, and in that sense the election is decided between the two moderate parties.

87 This happens when a party obtains more than half of the seats in the parliament, enabling him to pass all the resolutions in the parliament, since they are done according to a simple majority rule (exception made to constitutional changes, which require a support of 2/3).

88 This has gain more and more importance after the 1999 elections, where the socialists got exactly 130 seats. The years that follow involved a very criticised on-going *log-rolling* practise and the Prime Minister ended up resigning after the local elections held in 2001, where the socialists lost many important counties.

Naturally, this desire (for some moderate party to obtain an absolute majority) generates a *bandwagon effect*, following Simon (1954), since the leading candidate in a published opinion poll would be likely to capture more votes than those announced, reinforcing his advantage. This also highlights the critical role that opinion polls may have in influencing the final result of an election.

One other example where a four-party model could be useful is the political system in Northern Ireland, which we describe very briefly.

The four parties with parliamentary representation are described in table A2.

Table A2. *Irish parties*

	Unionists	**Nationalists**
Moderate	UUP	SDLP
Extremist	DUP	SF

Legend:

UUP - Ulster Unionist Party

DUP - Democratic Unionist Party

SDLP - Social Democratic and Labour Party

SF - Sin Feinn[89]

In Northern Ireland, the distinction between the two sides of the political spectrum must be made using the labelling nationalists / unionists and not right / left, as it was the Portuguese case. This demonstrates the usefulness of taking a general approach and avoids labelling what needn't be labelled, which is the approach we will take.

Northern Ireland has 18 seats in the House of Commons (in a total of 651 seats). Ulster Unionists holds 6 seats, the Democratic Unionist Party holds 5 seats, Sin-Feinn holds 4 seats, and the Labour Democrats and Labour Party holds 3 seats[90].

Besides the parties with parliamentary representation, there are other 12 smaller parties in Northern Ireland.

Elections are based on the Hondt system, similarly to Portugal.

In the subsequent parliamentary elections, both moderate parties stressed the importance of an absolute majority, while PP asked their voters to vote sincerely, to increase the need for the PSD to form a coalition with them in order to obtain a total of more than 130 seats - which indeed happen.

89 Though the DNP (Democratic Nationalist Party) is considered to be politically nastier than SF, SF is taken to be the most radical, as it has or has had an army arm, the IRA.

90 Sin Feinn, in protest, have refused to take their seats at Westminster.

– *Essays on Strategic Voting* –

Essay 2

"Do You Follow Your Heart or Your Head? A Voting Model with Expressive and Instrumental Motivations"

1. INTRODUCTION

Every human action can be said to have an *input* or an *output* motivation. A motivation is input driven when the cause of an action is found in the action *per se*. The motivation is output driven when it depends on the *results* of the action. This instrumental motivation is present across different subjects in the social sciences, from economics to sociology and politics. An action is instrumental when it is an instrument to obtain something and not an end in itself. An action can be an end in itself for several reasons, such as an ethical motivation, a desire to express something or the will to fulfill some beliefs about oneself. In our model of voting, a non-instrumental motivation comes from an expressive desire: the voter wishes to express his preferences and attachment to the candidate he prefers. Whether the source of the input-driven motivation is an expressive desire or some other (perhaps ethical) motivation, we will refer to this aspect of a voter's behavior as an expressive motivation.

The distinction between instrumental and non-instrumental motivations is not always easy. For instance, you may exercise to lose weight or to get fit (these are outputs) but you can also exercise just for the pleasure you derive while doing it (this is an input). One could always say, of course, that the pleasure derived from such activity is an end in itself—meaning that you would exercise for the purpose of obtaining that pleasure.[91] We think it is useful to avoid this tautology and differentiate the two motivations, since it is almost always, if not always possible—but of little use, in terms of real explanatory power—to say that everything we do has an instrumental motivation, many times based

[91]Examples are abundant. The choice of upgrading the safety of your sports car after having a child seems clearly instrumental. But why would someone contribute anonymously to a charity organization *without* telling anyone about that? You can argue that you contribute for the *result* of (contributing to) saving someone's life or to *obtain* some conscious relief, but you are more likely to have had a non-instrumental motivation, making a donation because you feel you *had to*.

on the fact that there is (or there can always be) an output driven reason behind every action, be it a sense of moral duty, psychic pleasure, or any other. Justifying actions from a cost-benefit perspective is always possible *a posteriori*, but this jeopardizes the understanding we have on how individuals *really* decide, and it is something we wish to avoid.

Brennan and Lomasky (1993) criticize public choice for its assumptions and methodology, namely for considering that the behavior of the 'political actor' can be fully understood at light of the 'economic agent', which implies that a complete analogy between 'politics' and the 'market' is adequate. As they put it:

> "And public choice cannot claim that input-oriented (expressive) elements do not exist, since otherwise it can provide no account of why voters vote in the numbers they do. (...) More than 'fitting the facts' is required of a theory: it must also genuinely explain in the sense of rendering intelligible, the facts it fits."

To explain the nature of expressive preferences, they draw a comparison between the voter and the sporting fan, which is a good example of how such underlying motivation is present in many different contexts:

> "The fan's actions are purely expressive. They arise from a desire to express feelings, without any necessary implication that the desired outcome will be brought about thereby. Revealing a preference is a direct consumption activity, yielding benefits to the individual in and of itself (which goes against the separation in utility functions)."

A question that may puzzle many economists, given their fixation on purely instrumental motivations, is ”Why do people vote for a certain party when it faces no chance of electing any representative?” Brennan and Lomasky’s answer to this is the following:

> “… thus identification with a party forms part of the citizen’s self-image, initially inherited through the family but then strengthened with its duration as adult voters rely more and more on their partisanship to help them make sense of the continuing barrage of political information which assay’s them. (...) The American voter accepts the Republican Party’s arguments because he is a Republican, and not (as the Rational Choice model would have it) the other way around.”

In their description of what they propose as three functions of political thinking—understanding, social adjustment and self-expression—, the expressive element is worth being highlighted:

> “… to the extent and content of political thinking reflect the need to express self. Many aspects of self can be expressed in politics—not just personality and its conflicts but philosophy, morality, esteem and core values. The radical child of conservative parents expresses emerging independence in political form. The authoritarian expresses repressed anger in a desire for strong government; the environmentalist expresses ecological values by supporting the green party.”

Green and Shapiro (1994) criticise many aspects of rational choice theory, and propose a broader approach to the study of political behavior, that we substantially subscribe:

> “ If social sciences were viewed less as a prizefight between competing theoretical perspectives (...) and more as a joint venture in which explanations condition and argument one another, the partisan impulses that give rise to methodologically deficient research might be held in check. The question would change from “Whether or not Rational Choice theory?” to something more fruitful: “How does rationality interact with other facets of human nature and organizations to produce the politics that we seek to understand?”.

This essay presents a model of costly voting. The cost of voting in our model is not physical but psychological. Whenever the voter decides not to vote for his favorite candidate, he incurs in a cost—an *opportunity cost*—that represents the loss of utility from not having expressed his attachment to the party he prefers. In other words, costly voting takes place whenever a strategic vote is cast. According to Fisher (2005), ”a tactical voter is someone who votes for a party they believe is more likely to win than their favorite party, to best influence who wins in the constituency”. A Duvergerian equilibrium—where only two parties obtain any support in equilibrium—will be impossible in our model: in such a scenario any voter casting a strategic vote would be better off switching to his favorite candidate, saving the opportunity cost of a non-sincere vote.

Borgers (2004) and Goeree and Grosser (2004) also study costly voting, but the cost in their model is physical and not psychological.[92] Their voting game involves two candidates and allows for abstention, while we have three candidates but no abstention.

[92]Goeree and Grosser (2004) is an extension of Borgers (2005), the later being the published and updated version of an earlier paper that inspired the former here referred.

In their models, therefore, the rational voter has to decide whether or not to abstain, given that he would always vote sincerely if he ever votes. This excludes the possibility of strategic voting and makes both models little comparable to ours in any meaningful sense.

There is a vast literature on strategic voting. Myatt (2006) and Cox (1999) are the most relevant papers to our work, as they also focus on the analysis of the possible equilibria and their stability and their assumptions on how uncertainty comes up in the model lie in the two possible extremes. Myatt (2006) proposes a model where voters observe a signal and hold different posterior beliefs about the result of the election, despite having common prior beliefs about the (uncertain) parties' popularity . This 'private information' component allows for a stable equilibrium with partial strategic voting to exist.[93]

Cox (1999) proposes a model where uncertainty is not only common knowledge, but vanishes as the electorate gets large. This makes the non Duvergerian equilibrium be an unstable knife-edge case. Mendes (2004) is somewhere in between the two previous models: it does not include an element of private information but there is true uncertainty when the electorate gets large. The fact that there is common knowledge about everything in the model, including the (uncertain) parties' popularity, renders it impossible to have a stable interior equilibrium. In common with Myatt (2006) and Cox (1999), voters in Mendes (2004) are purely instrumental. We extend Mendes (2004) to include an expressive element in a model where uncertainty is publicly known.

[93]Partial strategic voting refers to a situation where, in equilibrium, a fraction of a group of voters votes strategically.

2. THE MODEL

There are three parties in our model: Left (L), Middle (M) and Right (R). Abstention is ruled out. Every voter's choice is to cast a vote for one of the three parties. The winner is decided according to simple plurality rule.[94] Ties are broken equiprobably. There is no second round. The election is a one-shot game.[95]

The share of the electorate who supports party R is fixed and equal to λ. We assume that $\lambda > \frac{1}{3}$ and $\lambda < \frac{1}{2}$, so that party R is an incumbent that can be defeated. Voters who dislike R can be split into categories LM and ML. These refer, respectively, to voters whose first preference is L and M but who consider voting strategically for M and L. The overall proportions of LM and ML voters are $\alpha(1-\lambda)$ and $(1-\alpha)(1-\lambda)$. So, among those who dislike party R their proportions are, respectively, α and $(1-\alpha)$.

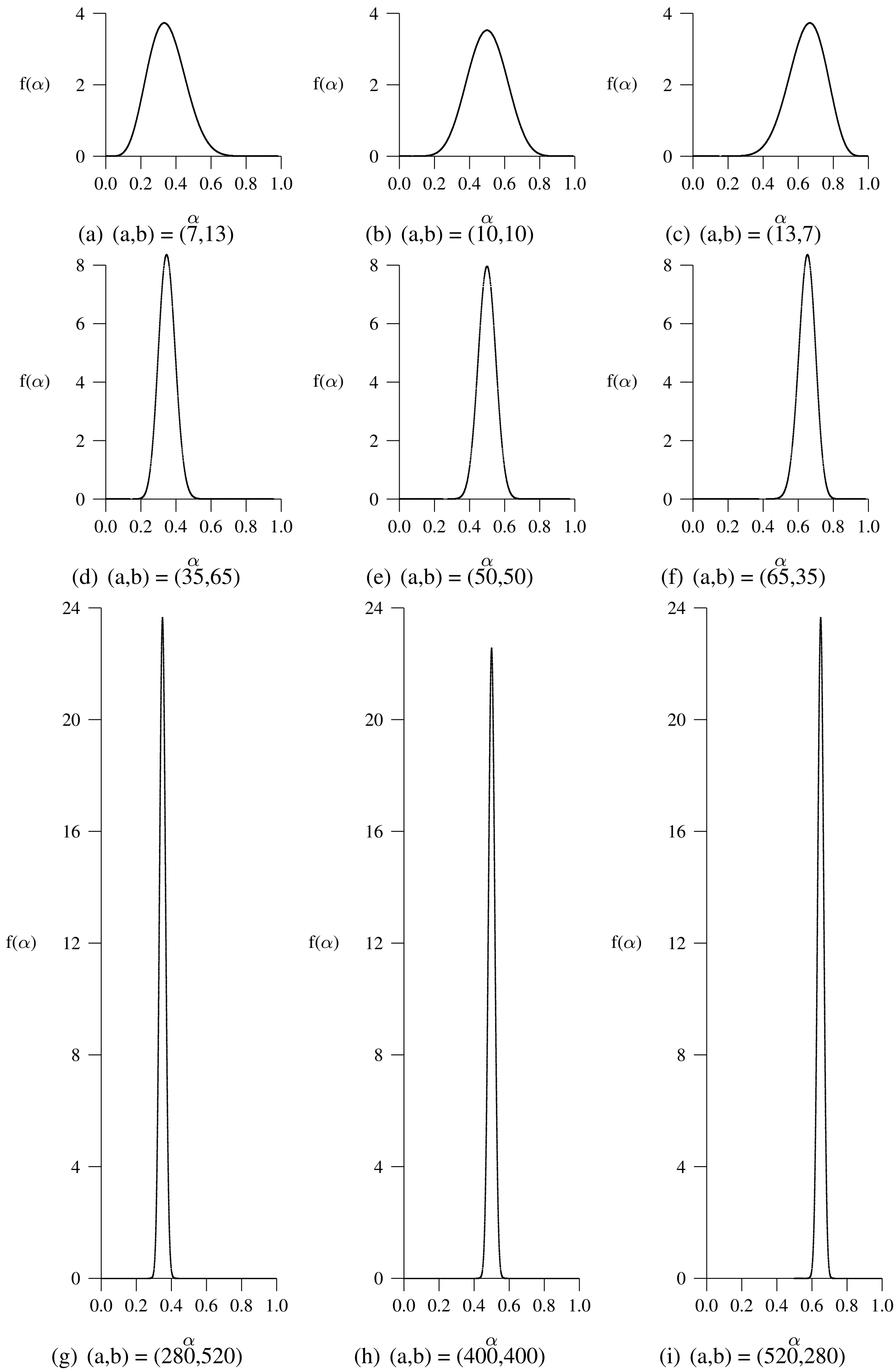
The proportion λ is known with certainty and is common knowledge to everyone.[96] We introduce uncertainty via α. This proportion is distributed according to a density function $f(\alpha)$ that is common knowledge to all voters. As in Mendes (2004), we can call this common or public uncertainty, since there is true uncertainty when the electorate gets large, but no private information involved in it: all prior and posterior beliefs held by voters are identical, which follows from the fact that no private signal is observed by anyone. A natural way for these prior beliefs to come into place is by having a commonly known opinion poll.[97]

[94]So, the winner is the party obtaining the highest percentage of votes (even if that is less than 50%).
[95]Strictly speaking, this implies that there is no 'past' nor 'future'. Of course, without a past politics would probably be meaningless—but the assumption should be understood in the relevant context. One example of a dynamic game is given in Piketty (2000), where "voters care about current decision-making (they are "strategic"), but they also care about communicating their views about their most preferred candidate, so as to influence future elections, through an impact on voters' opinion and/or party positioning.
[96]In other words, everyone knows λ, everyone knows that everyone knows λ, and so on, *ad infinitum*, following Aummann (1976).
[97]See previous footnote.

Even though we base most of the analysis in this chapter on a generic $f(\alpha)$ function, it will be useful to consider a specific distribution for α, to have a better insight about the partial effects of changing some of the parameters in the model. The standardized Beta-distribution $f(\alpha, a, b, 0, 1)$ seems a natural candidate: besides being defined for the interval $[0, 1]$—a necessary requirement for electoral results, as they are expressed as percentages—it can be derived as the posterior distribution that is obtained from binomial draws when a sample of size $(a + b)$ is observed and an update is then made. The bell-shape of a standardized Beta-distribution implies that the extreme values are unlikely to occur and that both $f(0)$ and $f(1)$ are equal to 0. We illustrate some examples of it in Figure 30 in the next page, where we vary the sample size from small (20) to medium (100) to large (800), showing, for each example, the symmetric case, along with two asymmetric cases.

Since voters can not abstain, it will never be rational to vote for their least preferred candidate. Thus, voters from groups ML and LM can both only vote for either M or L. Each of them has two components from which utility is driven, an expressive and an instrumental one. As it will be made clearer further on, it is not possible to have strategic voting occurring in both directions—common knowledge of the electoral situation would render that irrational for the voters of one of the two groups from where strategic voting would be arising. Given this—and without loss of generality—we restrict our analysis to the case where strategic voting may occur from ML voters, towards party L.

Each of the ML voters can be indexed by type i, his utility function being $U_i(X, Y)$, where X stands for the votes allocated from an expressive motivation and Y the votes



(a) (a,b) = (7,13)
(b) (a,b) = (10,10)
(c) (a,b) = (13,7)
(d) (a,b) = (35,65)
(e) (a,b) = (50,50)
(f) (a,b) = (65,35)
(g) (a,b) = (280,520)
(h) (a,b) = (400,400)
(i) (a,b) = (520,280)

FIGURE 30. Examples of a standardized Beta-distribution

allocated from a strategic motivation. Voting expressively means voting for one's favorite party. Since casting a vote for the least preferred candidate can never be optimal, strategic voting in a three party model is equivalent to voting for the second most preferred candidate. In the context of our model, we can therefore—and equivalently—say that X are the votes for party M and Y the votes for party L. Voters only have one vote to cast, which means the analysis boils down to comparing $U_i(1, 0)$ to $U_i(0, 1)$.

Intuitively, X is providing an input driven utility, something similar to consumption utility in other contexts. Voters get a positive direct utility from expressing their political preference sincerely in the individual ballots, regardless of the election's outcome. We can represent this expressive utility by $v_i(X)$. Since there is only one vote to be cast, we need to look only at $v_i(1)$ and $v_i(0)$. Without loss of generality, we standardize $v_i(0) = 0$ and write $v_i(1) = e$, $e \sim (0, \infty)$ being the benefit arising from an expressive vote, assumed to be the same for all voters.

Apart from the expressive utility $v_i(X)$, the voter can also obtain some instrumental utility. This will be contingent on the result of the election, and independent of the way he votes. If the expressive utility depends on the 'input' (the vote itself), the instrumental utility depends on the 'output' (the outcome of the election). If party M wins the race, the instrumental utility is standardized to 1. If L is the winner, the utility for voter i will be β_i—indexing his type. We assume β_i to lie between 0 and 1, according to some probability distribution $h(\beta)$. A voter with $\beta_i = 0$ will get no utility from a victory of his second preferred candidate, while a voter with $\beta_i = 1$ obtains the same instrumental utility from L as from M (but not the same overall utility, since an ML voter obtains $e > 0$ if he casts a vote for M). Overall, it is sufficient to introduce variability in one parameter, so the assumption that $e > 0$ is the same for all voters is not restrictive, given that we are varying β_i. Also note that, differently from the previous chapter in this thesis, a voter with $\beta_i = 1$ is not someone a priori indifferent between the two candidates because of the expressive component in his or her preferences.

The instrumental utility also includes a parameter reflecting a paternalistic concern for the outcome of the election, which will depend on the size of the electorate. The larger the electorate, the more a voter will care about the result of the election. The average concern per voter will be decreasing. The intuition for the inclusion of this element is that electoral results have strong implications for the economy and for the everyday lives of fellow citizens.[98] A voter is likely to care, to some extent, about the effect of political decisions on the community he lives in. We encapsulate the concern for others in the term $c(n) \geq 1$.[99] We assume $c'(n) > 0$ and $c''(n) < 0$, so that the concern is increasing with the electoral size but at a decreasing rate, as we said before. At least to some, this concern will seem altruistic. We could label it as altruistic-paternalistic, since there is a concern for others' welfare, even if it is based on one's own preferences. Essentially the concern is paternalistic, since it is the voter's view that is weighted in his own calculus. This element plays an important role in the weighting of the benefits of voting sincerely or strategically, as we will see later on.

We can summarise the utility each voter gets in Table 12 below.

Vote / Winner	L	M	R
L	β_i	1	0
M	β_i+e	1+e	e

TABLE 12. Utility for ML Voters

[98]From education to health, civil liberties to war intervention, to mention a few.
[99]If $c(n) = 1$, there is no (extra) altruistic concern, and we have the standard model, with no 'leverage' on the instrumental utility in the model.

For the instrumental component of his utility, a voter must take into account the probabilities of both L and M being involved in a pivotal event, which is any event where the two parties with highest number of votes are involved in a tie or a near tie. We write the pivotal probabilities generically as $\Pr[L, R]$ and $\Pr[M, R]$. For a large n, we can overcome the discreteness problem and take the pivotal event to take place when one of the contenders obtains a share equal to λ, the proportion that party R gets. The two possible pivotal events occur at the critical draws α_L and α_M, with likelihoods $f(\alpha_L)$ and $f(\alpha_M)$. So, the relative likelihood of each of the two possible pivotal events is $f(\alpha_L)$ and $f(\alpha_M)$. We shall call L the 'most likely contender' if $f(\alpha_L) > f(\alpha_M)$ and the 'least likely contender' if $f(\alpha_L) < f(\alpha_M)$. If $f(\alpha_L) = f(\alpha_M)$, each party is a 'equally likely contender'.

A diagrammatical exposition may be helpful here. There is a pivotal event between parties L (M) and R when $R_L = \lambda$ (($R_M = \lambda$)). In Figure 31, this is simply the intersection between lines R_L (R_M) and λ. We shall label, generically, α_L (α_M) as the value of α that makes party L (M) be involved in a pivotal event with party R. Since α is uncertain, so will R_L and R_M be—since they depend on it—, and, therefore, so will the outcome of the election be.

In Figure 31 in the next page we can see how the percentage of votes each party gets depends on the draw of the distribution. The lines R_L and R_M represent the results obtained by parties L and M, respectively. If $\alpha = 0$, there are no supporters of party L, so $R_L = 0$ and the supporters of party M are in proportion $(1 - \lambda)$ of the electorate, so $R_M = (1 - \lambda)$. Likewise, if $\alpha = 1$, then $R_L = (1 - \lambda)$ and $R_M = 0$. If a proportion $k < 1$ of ML voters decides to vote sincerely (for M) while the remaining

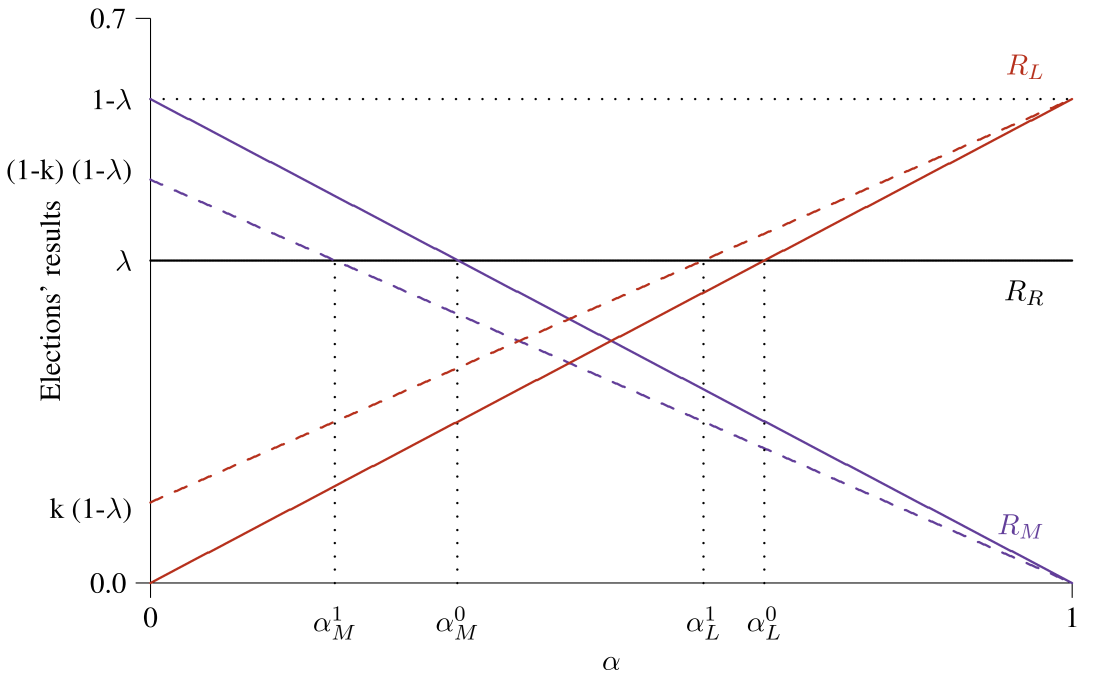


FIGURE 31. The election's (uncertain) results

$(1-k)$ voters decide to vote strategically (for L), the results for parties L and M are, respectively, $R_L = \alpha(1-\lambda) + (1-k)(1-\alpha)(1-\lambda)$ and $R_M = k(1-\alpha)(1-\lambda)$.

The proportion of votes for party R is the same regardless of the value of α, since we assume its hardcore supporters will never consider a strategic vote.[100] Therefore, $R_R = \lambda$, implying that the maximum support any of the two contenders can get is $(1-\lambda)$.

We will use diagrams with a similar structure to the one described in Figure 31 above for mainly two purposes. First, to find out, for each β^*, what are the critical values α_L, α_M, $f(\alpha_L)$ and $f(\alpha_M)$ which will enter the decision process of each voter; second, to check (how) the (uncertain) outcome of the election (depends on the draw of α). The rationale is the same used on essay 1, in that case based on Figure 7[91]. Recall that β^* describes the proportion of voters from one vote group that vote sincerely. This

[100] Also, and as mentioned before, voters from categories LM and ML will never vote for R since R is their least preferred candidate and voting for the least preferred candidate is a dominated strategy.
[91] See page 33.

proportion will influence the critical values α_L, α_M, $f(\alpha_L)$ and $f(\alpha_M)$ that determine the incentive ration, which must be compared to the initial value β^* in order to assess the presence of an equilibrium. If β^* goes down—that is, if strategic voting towards party L increases—then both α_L and α_M will go down, as we can see in the diagram. If a proportion k of ML voters decides to vote strategically for party L, the pivotal events shift from α_L^0 to α_L^1 and from α_M^0 to α_M^1. In the above diagram, we can see how the values of $f(\alpha_M)$ and $f(\alpha_L)$ change when α_L and α_M change. In the margin, the sign of this change will depend on whether the initial value is to the left or to the right of the mode, whereas the size of the change will mostly depend on the shape of the distribution, which is critically influenced by the prior asymmetry and the (implicit, or not) sample size in the distribution.[101]

Eventually, if β^* goes down enough, α_M will no longer be possible, which would mean that there would be only one pivotal event (involving party L). In Figure 31, if β^* becomes sufficiently low, R_L will shift upwards, eventually not crossing λ and making it impossible to have any pivotal events. In such case, party L would win the election with probability 1. The winner of the election can be found by looking at the upper contour of the three curves R_L, R_M and R_R. If there is no pivotal event, party L is the winner with certainty. If there is only one pivotal event, party R is the winner if $\alpha < \alpha_L$ and party L wins otherwise. If there are two pivotal events, party M will win for $\alpha < \alpha_M$, party R for $\alpha_M < \alpha < \alpha_L$ and party L for $\alpha > \alpha_L$.[10]

[101]Whether or not it is implicit depends on how the distribution is ”rationalised” or explained within the model, that is, how the information becomes commonly known to each and every voter.

[10]Recall the ties are broken equiprobably. As the electorate gets large, the distinction between a tie and a near tie becomes irrelevant—the discreteness issue vanishes away.

3. Optimal behavior

Any voter's choice is between allocating a sincere vote to their favorite candidate and a strategic vote for their second favorite candidate. The game can be seen as a coordination game where both voters LM and ML wish to defeat the commonly disliked candidate R and must coordinate their votes to some extent in order to succeed.[103] Our model involves *costly voting*. If a voter decides to cast a strategic vote, he misses the expressive utility he would get by voting for the party closest to his heart. This is, as we mentioned before, a *psychological* (not *physical*) opportunity cost, not to be neglected.

As in the model of the first chapter of this thesis (in Party I), the equilibrium will entail a cut-off rule that splits each of the categories of voters, ML and LM, between those that vote for their favorite candidate and those who vote strategically. This cut-off rule will determine the overall support for each contender and, consequently, the likelihood of each being involved in a pivotal event with party R. Every voter's choice will depend on his type, reflected in β_i. Two conditions have to be meet in equilibrium: optimality and rational expectations. The first requires each agent to be optimising given his set of beliefs. The second requires that the beliefs of all voters are met in equilibrium. Together, they guarantee that no one has an incentive to deviate from the equilibrium and that the equilibrium will actually occur when everyone expects that to happen. Note that we are not primarily concerned with predicting how voters will "really" behave, but how they "have to" behave if a voting equilibrium is to take place. Thus, our approach is focused on equilibrium conditions, which is common in the theory of elections.

[103]Each will succeed when it achieves a qualified majority of $\lambda/(1-\lambda)$. Because of the way uncertainty is introduced in the model, a party does not need full-coordination to win the election. For instance, if everyone votes sincerely, each of the three parties have a chance of winning the election.

Common knowledge of uncertainty implies that each voter will use the same estimate for the probability of any possible pivotal event in any given electoral situation. It also implies that there can *not* exist strategic voting occurring in both directions, since from rational expectations it must be true that everyone has the same beliefs in equilibrium, and this implies that at most there is an incentive for one of the groups to vote strategically. Broadly speaking, this will happen when the likelihood of a pivotal event involving the other party is sufficiently high to compensate for the psychological cost of not voting expressively. This makes it clear why strategic voting can not occur in both directions in equilibrium: voters from the other group would never vote strategically when, further to having an opportunity cost of doing so, such choice would not be sensible even in purely instrumental grounds (given that their party would, in such hypothetical situation, be more likely to be involved in a pivotal event).

An ML voter of type β_i will never vote for party R—his choice is between the preferred party M and the second preferred party L. To clearly distinguish the utility coming from the expressive and from the instrumental motivation, we use the following labels:

(1) $U_i(1,0)$ is the overall utility arising from a vote for M;

(2) $v_i(1)$ is the expressive utility arising from a vote for M;

(3) $E_i[s(1,0)]$ is the instrumental utility arising from a vote for M: the expected utility of influencing the outcome of the election;

(4) $\Pr[X,Y]$ is the probability of a pivotal event between parties X and Y taking place.

We can write the overall utility of each vote as:

$$U_i(\textit{vote for M}) = U_i(1,0) = v_i(1) + E_i[s(1,0)] = e + c(n)\Pr[M,R]$$

and

$$U_i(\textit{vote for L}) = U_i(0,1) = v_i(0) + E_i[s(0,1)] = 0 + c(n)\beta_i \Pr[L,R]$$

It is optimal to vote for M when $U_i(1,0) > U_i(0,1)$, for which we obtain the following expression:

Condition 1. $e \geq c(n)\{\beta_i \Pr[L,R] - \Pr[M,R]\}$

As we restrict out attention to large electorates, we are interested in n becoming large. When $n \to \infty$, the following results will be useful:

(1) $\lim_{n\to\infty} c(n)/n = c \geq 1$

(2) $\lim_{n\to\infty} \Pr[L,R] = f(\alpha_L)/[nH(\beta^*)]$

(3) $\lim_{n\to\infty} \Pr[M,R] = f(\alpha_M)/[nH(\beta^*)]$

The first result says that the average concern for each citizen converges to $c \geq 1$ as the electorate gets large.[104] If $\lim_{n\to\infty} c(n)/n$ exists, it will be equal to $\lim_{n\to\infty} c'(n)$, from L'Hopital's rule.

The second and third results above describe the appropriate formulas for the pivotal probabilities. $H(\beta)$ is the cumulative function obtained from the density function $h(\beta)$ evaluated at β, the probability that $\beta_i \leq \beta$. We obtain the two expressions in the following way. Call the proportion of voters amongst ML and LM voters that vote for L p, so $p = \alpha + (1-k)(1-\alpha)$, where k is the proportion of ML voters voting sincerely for M and $(1-k)$ the proportion voting strategically for L. Let $g(p)$ and $G(p)$

[104]Since this term reflects an additional concern for the result of the election, it doesn't make sense to consider that the average concern would be lower than 1. Alternatively—and perhaps more intuitively—, think of $c(n) = 1 + \hat{c}(n)$, where $c(n)$ would reflect the incremental effect coming from the concern for others. Then, we would restrict $\hat{c}(n) \geq 0$, which is equivalent to assuming that $c(n) \geq 1$.

be, correspondingly, the density and cumulative functions for p. We can anticipate that in equilibrium there will be a cut-off rule, such that some proportion of the ML voters will vote in some way, and the remaining proportion will vote the other way.[105] The cut-off value is the equilibrium value β^*. Therefore, we can replace k by $\Pr[\beta \geq \beta^*]$ to obtain:

$$p = \alpha + (1-\alpha)\Pr[\beta \geq \beta^*] = \alpha + (1-\alpha)[1 - H(\beta^*)]$$

and

$$G(p) = \Pr[\alpha + (1-\alpha)(1 - H(\beta^*)) \leq p] = \Pr[\alpha \leq \frac{p - 1 + H(\beta^*)}{H(\beta^*)}]$$

When $p = \lambda$, the right-hand side of the last expression is simply α_L, since $G(\lambda)$ gives the probability that party L gets less than the required support to be in a tie with party R.[106] This is clear from Figure 2 that we have seen before. For a generic p, we can use $\alpha[p] = \frac{p-1+H(\beta^*)}{H(\beta^*)}$ and rewrite the CDF function as $G(p) = F(\frac{\alpha[p]-1+H(\beta^*)}{H(\beta^*)})$. To obtain the density function we need to differentiate the CDF, obtaining $g(p) = f(\alpha[p])\frac{1}{H(\beta^*)}$. If $H(\beta)$ is uniformly distributed—i.e., if $H(\beta) = \beta$—the previous expression simplifies to $g(p) = \frac{f(\alpha[p])}{\beta}$. Taking these insights into account, we can rewrite the previous condition as $e \geq c(n)\{\beta_i \frac{f(\alpha_L)}{nH(\beta^*)} - \frac{f(\alpha_M)}{nH(\beta^*)}\}$ for a generic $H(\beta)$ distribution, which simplifies to:

Condition 2. $e \geq \frac{c(n)}{n}\frac{1}{H(\beta^*)}\{\beta_i f(\alpha_L) - f(\alpha_M)\}$

[105] This is always the case when the agents are modeled as having different continuous types, following the Bayesian-Nash literature.

[106] Which is the same as asking about the probability that α is between 0 and α_L (given that α_L is the point at which they tie and that R_L is an increasing function in α).

This condition expresses the intuition of the previous one: voting for M is optimal if e is sufficiently large. We can express it in terms of β_i:

Condition 3. $\beta_i \leq [\frac{e}{c}H(\beta^*) + f(\alpha_M)]/f(\alpha_L)$

Call the right hand-side of the above expression $\tilde{\beta}$:

Definition 1. $\tilde{\beta} = [\frac{e}{c}H(\beta^*) + f(\alpha_M)]/f(\alpha_L)$

Our strategic voting equilibrium will entail the cut-off rule that is common in Bayesian-Nash games. This dictates that if a player has a draw that is above the cut-off value he will vote in some way, and if the draw is below that same level, he will vote in some other possible way. Here, since the choice is dual, the cut-off value will be unique—call it β^*. This is the value that leaves a voter indifferent between voting for M or L. Whenever $\tilde{\beta} = [\frac{e}{c}H(\beta^*) + f(\alpha_M)]/f(\alpha_L)$ is in $[0, 1]$, we must have $\beta^* = \tilde{\beta}$. But $\tilde{\beta}$ can be higher than 1, though β^* is restricted to the domain for β_i, which is $[0, 1]$. We can then define another useful variable, $\hat{\beta}$. This is the ”optimal threshold” that we will use in several diagrams, and is it helpful to use $\hat{\beta}$ and not β^* since an equilibrium will be an intersection of the 45o line with $\hat{\beta}$.

$\hat{\beta} = Min\{\tilde{\beta}, 1\}$

We can summarise the conditions for optimal behavior of the ML and LM voters as:[107]

[107]We are also assuming, for simplicity (but intuitively), that when the voter is indifferent between two candidates he will cast a vote for his favorite. Throughout the paper, whenever unspecified, all β's will refer to ML voters.

Condition 4. *(for ML voters) If* $f(\alpha_L) = 0$*, vote for M. If* $f(\alpha_L) > 0$*, vote for M if* $\beta_i \leq \beta_M^*$*, and vote for L if* $\beta_i > \beta_M^*$*, where*

$$\beta_M^* = \hat{\beta}_M = \min\{\tilde{\beta}_M, 1\} \quad \textit{and} \quad \tilde{\beta}_M = \frac{1}{f(\alpha_L)}\left[\frac{e}{c}H(\beta_M^*) + f(\alpha_M)\right]$$

Condition 5. *(for LM voters) If* $f(\alpha_M) = 0$*, vote for L. If* $f(\alpha_M) > 0$*, vote for L if* $\beta_i \leq \beta_L^*$*, and vote for M if* $\beta_i > \beta_L^*$*, where*

$$\beta_L^* = \hat{\beta}_L = \min\{\tilde{\beta}_L, 1\} \quad \textit{and} \quad \tilde{\beta}_L = \frac{1}{f(\alpha_M)}\left[\frac{e}{c}H(\beta_L^*) + f(\alpha_L)\right]$$

The difference between our model and a purely instrumental one is that in the latter the relevant likelihood ratio would be $\tilde{\beta} = f(\alpha_M)/f(\alpha_L)$. Not surprisingly, given that the difference between such model and ours comes from the inclusion of an expressive component in our model, we expect to arrive at such expression by setting $e = 0$ in our model, which does happen. By doing that the term $\frac{e}{c}$ would disappear, so the right hand side of the expression would be $f(\alpha_M)/f(\alpha_L)$.

We also have two conditions concerning the rational expectations requirement for the ML voters, which basically explicit the relation between the level of strategic voting and the values of α for which the pivotal events occur, which we will next make explicit. Each of the two possible pivotal events can only take place for a certain range of values of β^*. The pivotal event involving party L will be possible as long as strategic voting is not so high as to make it win the election with certainty—so β^* must not be too low. Graphically, the line R_L in Figure 31 must cross the threshold λ for a pivotal event to be possible. The pivotal event involving party M will only be possible if strategic voting is sufficiently low—β^* must be sufficiently high. As we can clearly see from Figure 31, understand intuitively and confirm analytically, as we do next, the value of λ influences this greatly. The higher the λ, the lower β^* can be while still having a pivotal event. Intuitively, a higher λ means a higher need of coordination to defeat the incumbent, and the higher the level of strategic voting, the lower the β^* will be.

Both conditions below refer to the case where $f(\alpha_L) > 0$, since when $f(\alpha_L) = 0$, every ML voter will vote expressively, a behavior which will in itself entail the property of a self-fulfilling prophecy, satisfying the rational expectations condition. We also conclude that for some pivotal event to be possible, β^* must lie in the interval $[\frac{1-2\lambda}{1-\lambda}, 1]$. $(1 - 2\lambda)/(1 - \lambda)$ is the lowest value of β^* that, for a certain λ, allows at least one pivotal event to occur, in our case between parties R and M. Call it β_{inf}. Party M will only be involved in a pivotal event if the strategic voting is sufficiently small, if β^* is higher than $\frac{\lambda}{1-\lambda}$. The two possible—and conditional on the value of β^*—intersections are:

Condition 6. $\alpha_L = 1 + \frac{2\lambda-1}{(1-\lambda)\beta^*}$, *for* $\beta^* > \beta_{\text{inf}} = \frac{1-2\lambda}{1-\lambda} = 1 - \gamma$

Condition 7. $\alpha_M = 1 - \frac{\lambda}{1-\lambda}\frac{1}{\beta^*}$, *for* $\beta^* > \beta_{\text{sup}} = \frac{\lambda}{1-\lambda} = \gamma$

Note the interesting relationship between the values that β^* can take and the qualified majority parameter, γ. β^* can never be lower than γ and there are two pivotal events only when $\beta^* > \gamma$. For instance, consider $\lambda = 41\%$, so $\gamma = 69.5\%$. In this case the proportion of strategic voting can not be higher than 30.5% if both parties are to have a chance of being involved in a pivotal event. At most strategic voting can reach 69.5% of the relevant group, as β^* can never be lower than 30.5%.

A diagrammatical description of our voting equilibrium is helpful here. We want to find a β^* that satisfies both the optimality and rational expectations condition. Beginning with the later, a β^* determines, from Figure 31, whether α_L and α_M exist and what values do they take. In turn, from Figure 30, the intersection(s) determine, once more conditionally on their existence, $f(\alpha_L)$ and $f(\alpha_M)$. These values enter the optimality condition and determine $\hat{\beta}$. We have found an equilibrium whenever we can "close the circle", whenever $\beta^* = \hat{\beta}$. We can think of repeated iterations of this process when we search for an equilibrium. (In fact, that is what a computer program does—as we will see in the third chapter of Part II).

Note the relevance of the paternalistic concern $c(n)$ in Condition 2. Without it, the expression on the right side will tend to zero as n grew to infinity, and therefore an equilibrium with strategic voting would be impossible just by including an expressive motivation. This would not be the case for finite electorates, but to be congruent with the analysis done on essay 1, we opt to analyse what happens to the voting equilibria when the electorate gets large. At light of this, we can see the inclusion of a paternalistic concern as a means to overcome a restriction that due to the choice of "taking the limit" of the probabilities involved in the voting calculus. Had we chose not to focus on large electorates, and the concern would not be necessary (though it would not change qualitatively the analysis). The absolute value of c is not relevant on itself, rather it is the value of e/c that is, which is patent in Definition 1.

It is important to distinguish three theoretical situations, relating to the number of intersections with the λ line. There can be either none, one or two intersections. We elaborate on these in the section "Characterization of equilibria".

We prove the existence by showing that when an equilibrium with sincere voting does not exist, then at least one equilibrium with some strategic voting is possible. This is illustrated in Figure 33, where we plot $\hat{\beta}$ as a function of β^*—$\hat{\beta}(\beta^*)$. The dependence between $\hat{\beta}(\beta^*)$ and β^* comes from $\tilde{\beta}$ (β^*), since $\tilde{\beta}$ (β^*) depends on β^* via its impact on α_L and α_M, determining $f(\alpha_L)$ and $f(\alpha_M)$, both of which enter the expression for β^*. There is a voting equilibrium when $\hat{\beta}(\beta^*) = \beta^*$.[108]

To use a fixed point argument, we need to check continuity of the relevant function, $\hat{\beta}(\beta^*)$.[109] For $\beta^* < \beta_{\text{inf}}$, there is no pivotal event, so no voter has any incentive to vote strategically, implying that $\hat{\beta} = 1$ for β^* in $[0, \beta_{\text{inf}})$. Thus, the function $\hat{\beta}(\beta^*)$ is continuous in $[0, \beta_{\text{inf}})$. For $\beta^* > \beta_{\text{sup}}$, two pivotal events are possible, so both $f(\alpha_L)$ and $f(\alpha_M)$ are (strictly) positive. Since $f(\alpha)$ is a continuous function and neither $f(\alpha_L)$ nor $f(\alpha_M)$ take a zero value in the relevant interval, $\hat{\beta}(\beta^*)$ will be continuous in $(\beta_{\text{sup}}, 1]$. In $(\beta_{\text{inf}}, \beta_{\text{sup}})$, there is only one pivotal event which involves party L, so we have $f(\alpha_M) = 0$. Since $f(\alpha_L)$ will be positive in the interval and $f(\alpha)$ is continuous, $\hat{\beta}(\beta^*)$ will also be continuous in $(\beta_{\text{inf}}, \beta_{\text{sup}})$.

We also need to check continuity at the critical points β_{sup} and β_{inf}. There will be continuity at each of these points if and only if we have left- and right-continuity at them. As we will see, a necessary condition for them to be continuous is to have $f(0) = 0$. This is a plausible assumption, guaranteed by most distribution functions,

[108] If there is an interior equilibrium—i.e., one that involves $\beta^* < 1$—then it also implies that $\tilde{\beta}(\beta^*) = \beta^*$, but that need not be true if there is an equilibrium for $\beta^* = 1$.

[109] Note that, since $\hat{\beta}(\beta^*) = Min\{1, \tilde{\beta}(\beta^*)\}$, continuity of $\hat{\beta}(\beta^*)$ implies that $\tilde{\beta}(\beta^*)$ must be continuous when $\tilde{\beta}(\beta^*) < 1$, which is not required for $\tilde{\beta}(\beta^*) > 1$, since in that case $\hat{\beta}(\beta^*)$ would be equal to 1 and hence continuous, regardless of the value of $\tilde{\beta}(\beta^*) > 1$.

but must nevertheless be seen as a restriction.[110] Starting with β_{sup}, right-continuity at it exists if, as $\beta^* \to \beta_{\text{sup}}$ from above, $f(\alpha_M) \to f(0)$. This will always be guaranteed, since $f(\alpha_M)$ is continuous and, as $\beta^* \to \beta_{\text{sup}}$ from above, $\alpha_M \to 0$.[111] Left-continuity, however, requires $f(0) = 0$. This is because in $(\beta_{\text{inf}}, \beta_{\text{sup}})$ there is only one pivotal event, which is, for the relevant function $\hat{\beta}(\beta^*)$, equivalent to having $f(\alpha_M) = 0$. Since at β_{sup} there is a qualitative change—from the absence of $f(\alpha_M)$ to the presence of $f(\alpha_M) = f(0)$—left-continuity requires $f(0) = 0$.

As for β_{inf}, and since $\hat{\beta}(\beta) = 1$ for all values of $\beta^* < \beta_{\text{inf}}$, left-continuity requires $\hat{\beta}(\beta_{\text{inf}}) = 1$. This, in turn, requires $\tilde{\beta} \geq 1$, or $\frac{1}{f(\alpha_L)} \left[\frac{e}{c}H(\beta^*) + f(\alpha_M)\right] \geq 1$, where $f(\alpha_M) = 0$ and $f(\alpha_L) = f(0)$, since the only pivotal event involves party L, and happens for $\alpha_L = 0$. Here, a sufficient but not necessary condition for $\frac{1}{f(0)} \left[\frac{e}{c}H(\beta^*)\right] \geq 1$ to be true is to have $f(0) = 0$. The necessary condition is for $f(0)$ to be sufficiently small: $f(0) \leq \left[\frac{e}{c}H(\beta^*)\right]$ would suffice. As for right-continuity, and similarly to the case for β_{sup}, this is automatically assured by the fact that $f(\alpha)$ is continuous.

Now that we have seen the required conditions for $\hat{\beta}(\beta)$ to be continuous in $[0, 1]$, we can use a simple fixed point argument. If a sincere equilibrium does not exist, it must be true that $\hat{\beta}(1) < 1$. But, if $\hat{\beta}(1) < 1$, $\hat{\beta}(\beta_{\text{inf}}) = 1$, $\beta_{\text{inf}} < 1$ and $\hat{\beta}(\beta^*)$ is continuous in $[\beta^*_{\text{inf}}, 1]$, then, there exists at least one fixed point in $[\beta^*_{\text{inf}}, 1)$, which is to say, there exists at least one voting equilibrium in $[\beta^*_{\text{inf}}, 1)$. If a sincere equilibrium exists, then at least one voting equilibrium exists. Therefore, a voting equilibrium is always guaranteed to exist.[112]

[110]For instance, the uniform distribution does not satisfy this assumption.

[111]Recall that in the limit point, β_{sup}, there are two pivotal events, and the one involving party M happens when $\alpha_M = 0$.

[112]The fact that we only mention one side of the equilibrium is without loss of generality. If an interior equilibrium exists (say) involving strategic voting from ML voters to party L, then voters LM have a strict incentive to vote sincerely: with $\hat{\beta}_M(1) < 1$, it is not possible to have all ML voters voting

5.1. **The Duvergerian equilibrium:** $(\beta^*_{L,}\beta^*_M) = (\mathbf{1}, \mathbf{0})$**.** A 'Duvergerian equilibrium' involves only two parties getting any votes in equilibrium[92]. This follows Duverger's insight that plurality rule favors a two-party system.[113] This equilibrium is *not* possible in our model—that is, in fact, one of the main results of this chapter. This happens for the simple reason that every voter faces an opportunity cost of voting. Voters derive utility from the very act of expressing their views, regardless of the outcome of the election. The psychological cost involved is the loss of utility incurred by not voting for one's preferred candidate, which is enough to exclude such type of equilibrium.

The Duvergerian equilibrium, as we have seen in Part I of this thesis, would imply that no pivotal events exist, so there wouldn't be any intersections with the λ line in Figure 31. Likewise, if there were no intersections with such line—coming from a sufficiently high level of strategic voting—, the only sustainable equilibrium will be a Duvergerian one, where strategic voting towards one of the contenders is not partial but full. The reason for this is that partial strategic voting could not be optimal when no pivotal events are possible, if the motivations are either purely instrumental or a mix of instrumental and positive expressive ones. Within such context, both situations

sincerely in equilibrium, since that would not satisfy the optimality condition. When $\hat{\beta}_M(1) < 1$, we necessarily have $\hat{\beta}_L(1) > 1$, so, all LM voters voting sincerely is a self-fulfilling behavior if the condition holds. This, in turn, reinforces the idea that an interior equilibrium where ML voters vote strategically (and LM voters vote sincerely, since strategic voting can only occur in one direction) would be possible. This illustrates why considering a generic $\hat{\beta}(.)$ in our analysis—as opposed to looking at both $\hat{\beta}_M(.)$ and $\hat{\beta}_L(.)$— is without loss of generality.

[92]Recall the discussion on pages 2 and 3.

[113]Riker (1982) labels this claim as Duverger's Law, and quotes an early and eloquent description of the phenomenon of strategic or tactical voting, due to Henry Droop (1860), which is worth revisiting: "As success depends upon obtaining a majority of the aggregate votes of all the electors, an election is usually reduced to a contest between the two most popular candidates... Even if other candidates go to the poll, the electors usually find out that their votes will be thrown away, unless given to in favor of one or other of the parties between whom the election really lies."

are therefore equivalent: any voting equilibrium that involves no pivotal event (no intersection with λ line in Figure 31) must be a Duvergerian equilibrium, with full shift of votes.

To see why is the Duvergerian equilibrium not possible in our model, imagine it was. In such scenario, with a full shift of votes from ML and LM voters to one of the contenders—and regardless of the draw of α—the outcome of the election would be known with certainty: the incumbent, enjoying a support of λ, would be defeated by the party on which coordination had occurred, since it would get a support of $1 - \lambda > \lambda$.[114] With a winner that is known with certainty, no pivotal event can occur. In a purely instrumental model of voting, this would always be a possible equilibrium, involving a likelihood ratio equal to zero for the group that is voting strategically, making it perfectly rational, for *all* types of voters in that group, to vote in such way. Any shared belief about the Duvergerian outcome would be, in such a context, a self-fulfilling prophecy. In our model, however, every voter gets some consumption utility by simply voting for his favorite party. Therefore, from an hypothetical situation where a Duvergerian equilibrium takes place, any voters would individually have an incentive to deviate and vote sincerely, in that way automatically getting an utility of $e > 0$, higher than the (zero expected) utility he could get from voting strategically (since the probability of being pivotal the outcome would still be zero in both cases, if only one voter changes his vote).

In short, a Duvergerian equilibrium is not possible whenever $e > 0$. This is a simple and yet powerful result. It basically says that the two-party prediction will not hold—or will, at least, be weakened—if voters have some non-instrumental component in

[114]Recall that $\frac{1}{3} < \lambda < \frac{1}{2}$.

their preferences. From a political science perspective, it is common to attribute some explanatory power to the hypothesis that the average voter feels some discomfort—a 'psychological cost'—if he does not vote for the party he feels a greater attachment to. This does not imply that strategic voting and instrumental considerations are not important—it only points out to further considerations that may help understanding voting behavior. As we suggested before, our model should be seen as an add-on to the common focus on instrumental motivations, potentially encompassing some of those approaches.

5.2. **The sincere equilibrium:** $(\beta^*_{L,}\beta^*_M) = (\mathbf{1}, \mathbf{1})$**.** In the 'sincere equilibrium', all voters vote for their preferred party. The sincere behavior from ML and LM voters will originate two pivotal events, as it is clear from Figure 2. In order for $(\beta^*_{L,}\beta^*_M) = (1, 1)$ to be an equilibrium, we need the incentive ratios for both voters to be equal or higher than 1. This will satisfy the optimality conditions already presented. The critical values of α that lead to each pivotal event are obtained from the rational expectations condition. Taking into account that $(\beta^*_{L,}\beta^*_M) = (1, 1)$, they are $\alpha_L = \frac{\lambda}{1-\lambda}$ and $\alpha_M = \frac{1-2\lambda}{1-\lambda}$. Further considering that when $\beta^* = 1$, $H(\beta^*) = 1$, the two conditions for optimality are:

$$\hat{\beta}_L \geq 1 \quad \Longrightarrow \quad [\frac{e}{c} + f(\alpha_L)]/f(\alpha_M) \geq 1 \quad \Longrightarrow \quad \frac{e}{c} \geq f(\alpha_M) - f(\alpha_L)$$

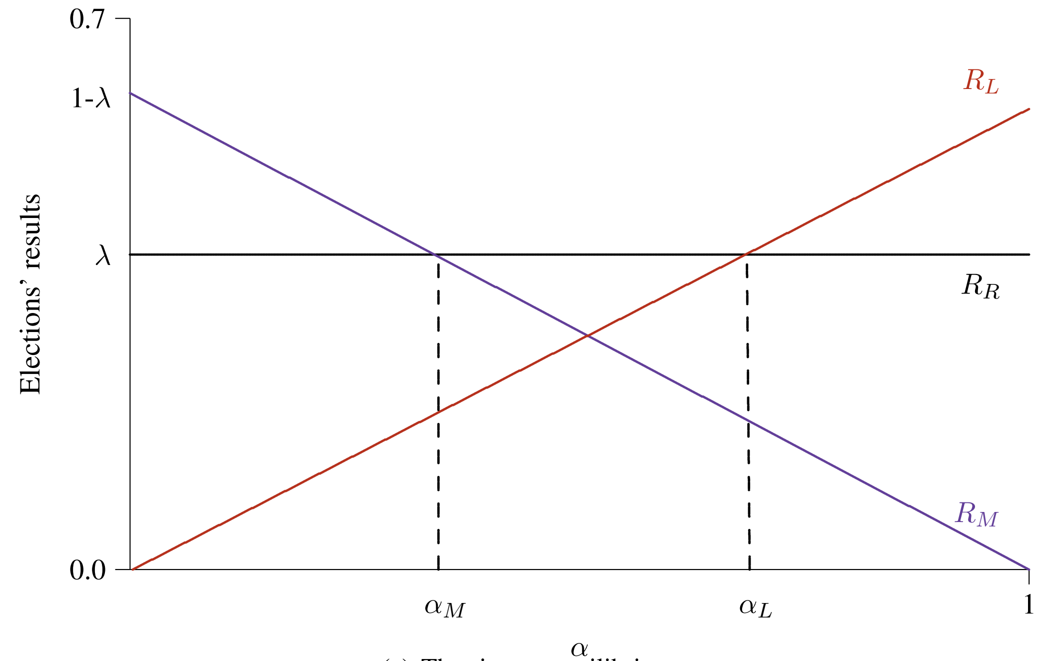
and

$$\hat{\beta}_M \geq 1 \quad \Longrightarrow \quad [\frac{e}{c} + f(\alpha_M)]/f(\alpha_L) \geq 1 \quad \Longrightarrow \quad \frac{e}{c} \geq f(\alpha_L) - f(\alpha_M)$$

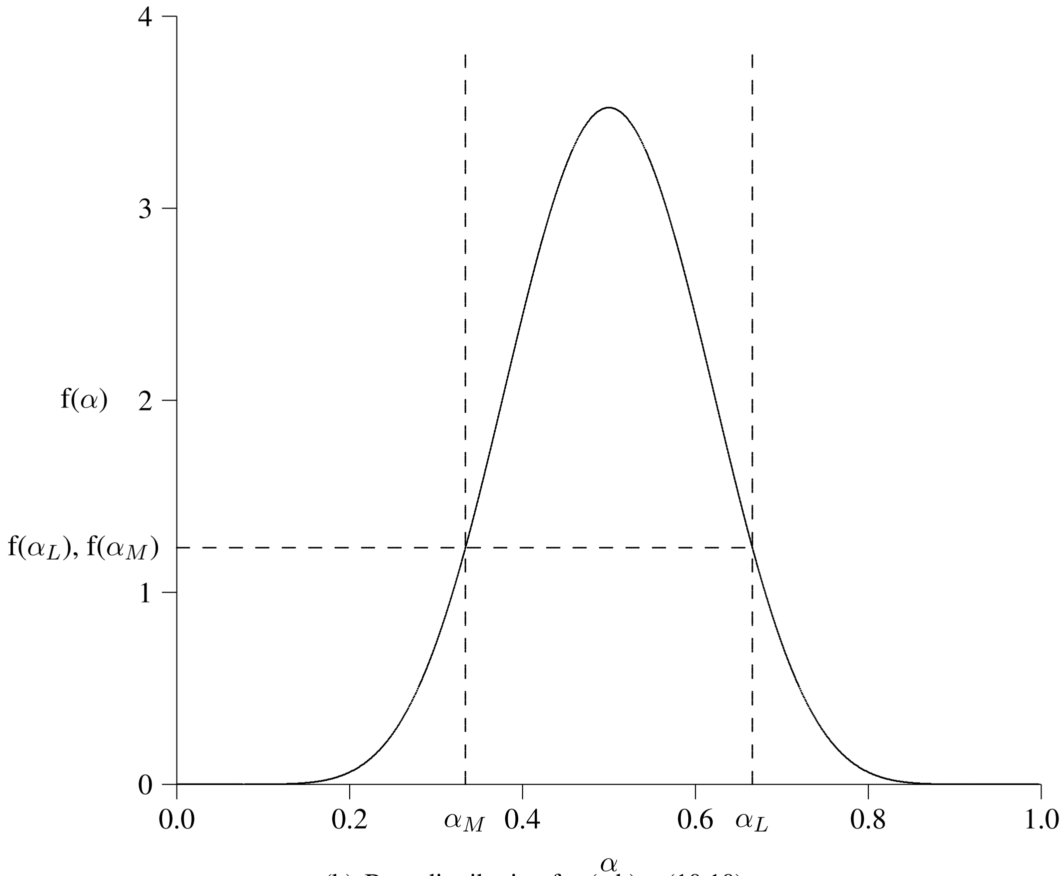
Intuitively, it is optimal for each voter not to vote strategically as long as β_i is sufficiently small, i.e., as long as he does not care too much about his second preferred party. We can immediately draw one conclusion: if the distribution $f(\alpha)$ is symmetric, implying that $f(\alpha) = f(1-\alpha)$, this equilibrium is always possible, since we would get $f(\alpha_L) = f(\alpha_M)$, given that $\alpha_L = 1 - \alpha_M$ when $(\beta^*_{L,}\beta^*_M) = (1, 1)$. The two conditions would boil down to one, $\frac{e}{c} \geq 0$, which is always true since $e, c > 0$.[115] Thus, symmetry is a sufficient condition for an equilibrium with no strategic voting to be possible. This is true regardless of what $H(\beta)$ is.

What can we say more generically? If $f(\alpha_L) \neq f(\alpha_M)$, then either $f(\alpha_L) - f(\alpha_M) < 0$ or $f(\alpha_M) - f(\alpha_L) < 0$, implying that at least one of the inequalities is surely satisfied, since $\frac{e}{c} > 0$. More broadly, the conditions will be satisfied for $\frac{e}{c}$ not too small and when the difference between the densities of the two pivotal events is not too large. In other words, there must be a combined sufficiently high incentive to vote expressively and a sufficiently low incentive not to vote strategically. The latter requires that the probability of the most preferred party being involved in a pivotal event must be sufficiently close to that of the other party, so that the incentive to vote expressively prevails. We illustrate the sincere equilibrium in Figure 32, where, as we said, $\alpha_L = 1 - \alpha_M$ and, given the symmetry in $f(\alpha)$, $f(\alpha_M = f(\alpha_M)$. As it is clear from that diagram, a sincere equilibrium always requires two intersections with the λ line, but it is not true that having two intersections (i.e., two pivotal events) implies that a sincere equilibrium takes place.

[115] Also noting that $f(\alpha_M) = f(\alpha_L) \neq 0$.



(a) The sincere equilibrium



(b) Beta-distribution for (a,b) = (10,10)

FIGURE 32. Illustration of the sincere equilibrium

5.3. **The interior equilibrium:** $(\beta^*_{L,}\beta^*_M) = (\mathbf{1}, \mathbf{k})$**, with** $\mathbf{k} < \mathbf{1}$**.** In the 'interior equilibrium', there is a proportion $(1 - k)$ of ML voters that vote strategically for party L, while the remaining proportion K of LM voters vote sincerely. In an equilibrium with β^*, the proportion of LM voters that vote sincerely is equal to $H(\beta^*)$. Recall the optimality condition:

$$k = \beta^* = \frac{1}{f(\alpha_L)} \left[\frac{e}{c} H(\beta^*) + f(\alpha_M) \right]$$

To find out whether and in what conditions this equilibrium is possible, we need to know the cumulative distribution $H(\beta^*)$, which gives a measure of the relative preferences between the first and second candidate as to what concerns the instrumental part of the utility function. The most natural candidate is the uniform distribution, since it does not discriminate between high, medium and low levels of attachment to the second favorite party, by assuming that the proportion of voters for each β_i in $[0, 1]$ is the same. The cumulative distribution function for an uniform distribution in $[0, 1]$ is $H(\beta) = \beta$, so the optimality condition simplifies to an expression that is clearer and more useful to us:

$$\beta^* \left[f(\alpha_L) - \frac{e}{c} \right] = f(\alpha_M)$$

Two cases must be considered, depending on the number of pivotal events. If there is only one pivotal event, then $f(\alpha_M) = 0$, and from the optimality expression we get that $f(\alpha_L) = \frac{e}{c}$. In this case we would then have a recursive system of equations: given the exogenous $\frac{e}{c}$, we would need to find the value of α_L such that $f(\alpha_L) = \frac{e}{c}$ and then, from the rational expectations condition, find out the β^* that would lead to such α_L. Finally, we would need to confirm that such β^* indeed generates only one and not two pivotal events, which requires that $\frac{1-2\lambda}{1-\lambda} < \beta^* < \frac{1}{1-\lambda}$. A necessary condition for this equilibrium to occur is that $\frac{e}{c}$ is lower than $f(mode)$, where $mode = \frac{a-1}{a+b-2}$ is the mode of a standardized Beta-distribution with parameters a and b. This need not be satisfied for the case where there are two pivotal events, since the possibility of M being involved in a pivotal event allows $\frac{e}{c}$ to be larger than $f(mode)$.

Intuitively, from the bell-shape of the Beta-distribution function and the fact that we assume continuity of the function $f(\alpha)$ in $[0, 1]$, we can suspect that at most there can be three interior equilibria. In fact, the function $\tilde{\beta}(\beta^*) \equiv \tilde{\beta} = [\frac{e}{c}\beta^* + f(\alpha_M)]/f(\alpha_L)$ will be, in the interval $[0, 1]$, a section of an inverted bell-shape or quasi-convex function. When there is only one pivotal event, $f(\alpha_M) = 0$, so we have $\tilde{\beta} = [\frac{e}{c}\beta^*]/f(\alpha_L)$. Since $\frac{e}{c}\beta^*$ is a linear increasing function in β^* and $f(\alpha_L)$ is quasi-convex in β^*, $\tilde{\beta}(\beta^*)$ will be the ratio of a linear and a bell-shape function in β^*, resulting in an inverted bell-shape function in β^*.[116] So, $\hat{\beta}(\beta^*) = Min\{\tilde{\beta}(\beta^*), 1\}$—the relevant function to look for a fixed point—will still have an inverted bell-shape in the $[0, 1] \times [0, 1]$ box.

If there are two pivotal events, implying that $f(\alpha_L), f(\alpha_M) > 0$, $\frac{e}{c}$ must be sufficiently low for the left-hand side to be positive too. This is intuitive: there can only be strategic voting if the expressive benefit, standardized by the concern for others, is not too high, given the likelihood of the pivotal events involved. Also, and since β^* must be lower than 1, $f(\alpha_M)$ must be sufficiently lower than $f(\alpha_L)$. More precisely, we must have $f(\alpha_L) - f(\alpha_M) > \frac{e}{c}$. Otherwise, a too high $f(\alpha_M)$ would lead the the marginal ML voter to stick to his more natural choice of voting for M, thus breaking the equilibrium.[117]

[116]This is a consequence of $f(\alpha_L)$ being quasi-convex in α_L and $\alpha_L = \frac{1}{1-\lambda}\beta^* - 1$ being linear in β^*.

[117]The ‘marginal’ here refers to the ML voter that would be indifferent between voting for M or L.

If there are two pivotal events, we must consider the additional term $f(\alpha_M)/f(\alpha_L)$. Since $\alpha_M = 1 - \frac{\lambda}{1-\lambda}\frac{1}{\beta^*}$, α_M is a monotonic function of β^*, though not linear. It is not immediate what should the shape of $f(\alpha_M)/f(\alpha_L)$ be—in fact, we believe it can not be proved generically. After having done a very large number of simulations for $\tilde{\beta}(\beta^*)$, using different values of a and b, varying both the asymmetry and the sample size implicit in the Beta-distribution, and not having found one single case where $\tilde{\beta}(\beta^*)$ was not clearly quasi-convex, we take that to be the case. This result implies that there can only be three intersections of $\tilde{\beta}(\beta^*)$ with the 45o line in $(0, 1)$, and, consequently, at most three equilibria[93]. The case were there are four possible equilibria is extremely rare—see one example in Figure 47 in essay 3, a case of a clear quasi-concave yet not concave function. For this reason with keep it aside, for sake of simplification and clarity in the exposition of our arguments. So, unless otherwise stated, we assume that at most there can be three equilibria: a sincere equilibrium, and interior stable and an interior unstable equilibrium.

In the **sincere** equilibrium, $\beta^* = 1$. The interior equilibria can be distinguished by the respective slope of $\tilde{\beta}(\beta^*)$ at that point. Because of continuity, and given the slope of the 45o line (equal to 1), one intersection must have a slope that is lower than 1, and the other a slope that is higher than 1. These are labeled, respectively, the **interior stable** and **interior unstable** equilibria. The sincere, interior stable and interior unstable equilibria—which we label, respectively, as β^*_s,$\beta^*_{i_s}$ and $\beta^*_{i_u}$—are illustrated in Figure 33, where the distinction between $\tilde{\beta}$ and $\hat{\beta}$ also becomes clear.A small deviation from the interior stable equilibrium will lead to a negative effect on itself—leading back to

[93]The number of equilibria is necessarily odd, and can be either one or three. If there is one equilibrium that can be either a sincere or a interior (stable) equilibrium. There can also be three equilibria, either a conjunction of one sincere and two interior equilibria (one stable and one unstable) or three interior equilibria (one stable and two unstable). The latter is the rare case illustrated in Figure 47.

the original equilibrium—while any deviation from the interior unstable equilibrium will lead to a positive effect, away from it.

The reasoning for this is that a small deviation in strategic voting is not able to create enough ”momentum” to shift the equilibrium, as the incentive to act in an opposite way is stronger, leading back to it. This is related to the shape of the incentive ratio function. Consider the interior stable equilibrium first, $\beta^*_{i_s}$. If there is a deviation because a few marginal voters decide to vote strategically, this leads to $\hat{\beta}$ becoming greater than the proportion of voters voting sincerely, since the incentive ratio function is above the 45o line. Given the linearity of $H(\beta)$, this means that for some voters that are voting strategically—originally, voters for whom β_i is initially higher than $\beta^*_{i_s}$— it is now better to vote sincerely, since $\hat{\beta}$ has gone high enough to make it rational to vote sincerely, given that β_i is lower than the new incentive ratio, $\hat{\beta}$. This always holds simply because the incentive ratio function is to the left of the unstable interior equilibrium, above the 45o line, implying that no matter what proportion of voters deviates from the initial equilibrium, voting strategically, the proportion of voters that, in the new (hypothesised or not) situation desires to change the vote from a strategic vote to a sincere vote is larger.

Likewise, if there is a deviation to the right—with some voters shifting from a sincere vote to a strategic vote—, there would be a larger proportion now better off shifting from a strategic to a sincere vote, since the individual preference parameter would be higher than the new incentive ratio. In this sense, there is a stable equilibrium whenever the slope of the incentive function is lower than 1. If the slope is negative, it is immediately obvious that stability is satisfied, if the slope is between 0 and 1 this is still accomplished (this is the case of the ”second interior stable equilibrium”, illustrated in Figure 47 in essay 3). It is sufficient, for a linear $H(\beta)$, that the 45o line is crossed from above.

An analogous reasoning shows why it is necessary and sufficient for the interior unstable equilibrium, $\beta^*_{i_u}$, to occur at an intersection with the 45o line that occurs from below. If some voters deviate and vote strategically, the new incentive ratio will go down more than proportionally to that deviation, so that the liquid effect is that some additional voters would be better off voting strategically, since their individual preference parameter β_i is now higher than the new and relevant incentive ratio. Likewise, if there is a deviation towards more sincere voting, the new incentive ratio goes sufficiently high to lead even more voters to vote sincerely, since now their individual parameters are lower than the relevant incentive ratio.

In the more interesting case of an interior equilibrium, we know that a pre-condition for it to exist is that $\beta^* > \beta_{\text{inf}} = \frac{1-2\lambda}{1-\lambda}$. Given this, there is one pivotal event when:

$\beta^* > \frac{\lambda}{1-\lambda}$

In this case, given that $f(\alpha_M) = 0$, the optimality condition simplifies to:

$$\beta^* \left[f(\alpha_L) - \frac{e}{c} \right] = 0)$$

Or:

$f(\alpha_L) = \frac{e}{c}$

For example, if $\frac{e}{c} = 0.1$, $\lambda = 0.40$, $a = 21$ and $b = 9$, $f(\alpha) = 0.1$ occurs for $\alpha = 0.454$, which determines $\beta^* = 0.611$, within the limits of a unique pivotal event to be possible.

There are two pivotal events if, further to $\beta^* > \beta_{\text{inf}} = \frac{1-2\lambda}{1-\lambda}$:

$\beta^* < \frac{\lambda}{1-\lambda}$

In this case, the optimal condition is:

$$\beta^* \left[f(\alpha_L) - \frac{e}{c} \right] = f(\alpha_M)$$

Another example: for $\frac{e}{c} = 1$, $\lambda = 0.37$, $a = 21$ and $b = 9$, we would find (in the aforementioned sheet) that $\alpha_L = 0.552$ and $f\alpha_M = 0.362$, so that $f(\alpha_L) = 1.007$ and $f(\alpha_M) = 0.004$, determining that $\beta^* = 0.9210$, which is in the range of admissible values for an equilibrium with two pivotal events.

6. Comparative statics

There are four different parameters in our model that we will analyse in order to understand the impact that they have on the possible interior equilibria:

(1) λ, the support of the incumbent (the degree of coordination required to defeat him);

(2) $a+b$, the sample size of the opinion poll (the quality of information possessed);

(3) a/b, the asymmetry in the distribution of the support of parties L and M (the bias in the prior support, reflected in distribution $f(\alpha)$);

(4) $\frac{e}{c}$, the degree of expressiveness (a measure of the 'loyalty' to one's favorite party).

Whenever we vary one of the above parameters, the inverted bell-shape function $\tilde{\beta}(\beta^*)$ illustrated in Figure 34 will change. In this figure, $\tilde{\beta}(\beta^*)$, which we also refer to simply

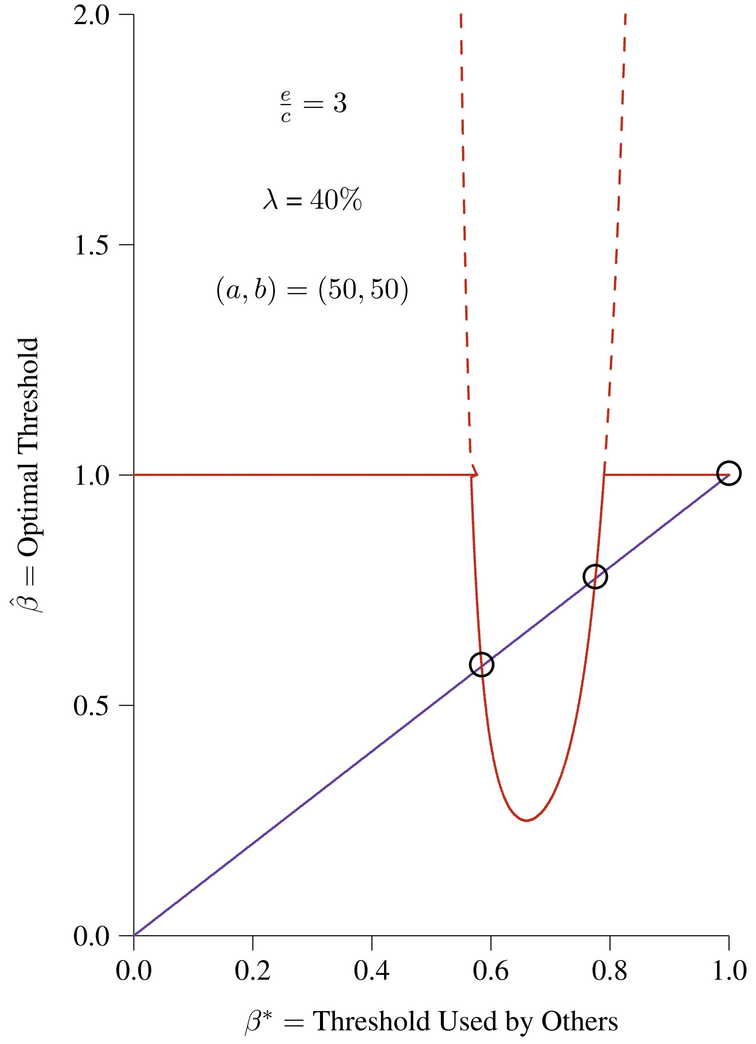


FIGURE 33. The three possible equilibria

as $\tilde{\beta}$, is represented in the curved line. This, in turn, is part of the curve $\tilde{\beta}$, which is the quasi-convex function, including the dashed line.

In studying comparative statics, our interest will lie on how the levels of strategic voting in the interior equilibria change with small variations in each of those parameters, and also on how the knife-edge conditions that determine whether the sincere equilibrium is possible or not. The eight interesting cases that result from the intersection of $\tilde{\beta}(\beta^*)$ and the limits of the square box $[0, 1] \times [0, 1]$ are illustrated in the next two pages, as parts of Figure 5, and can be characterized in the following way:

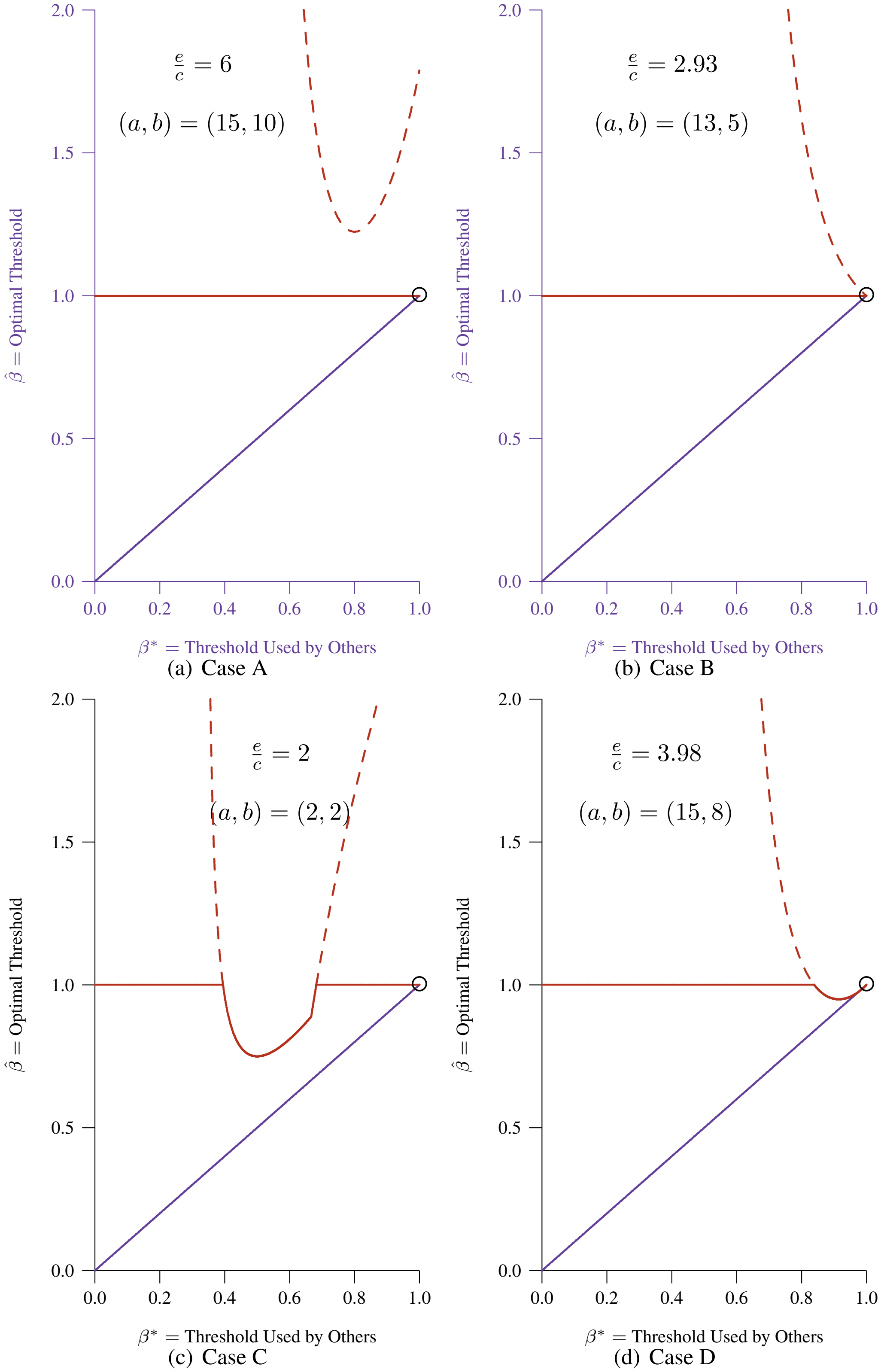
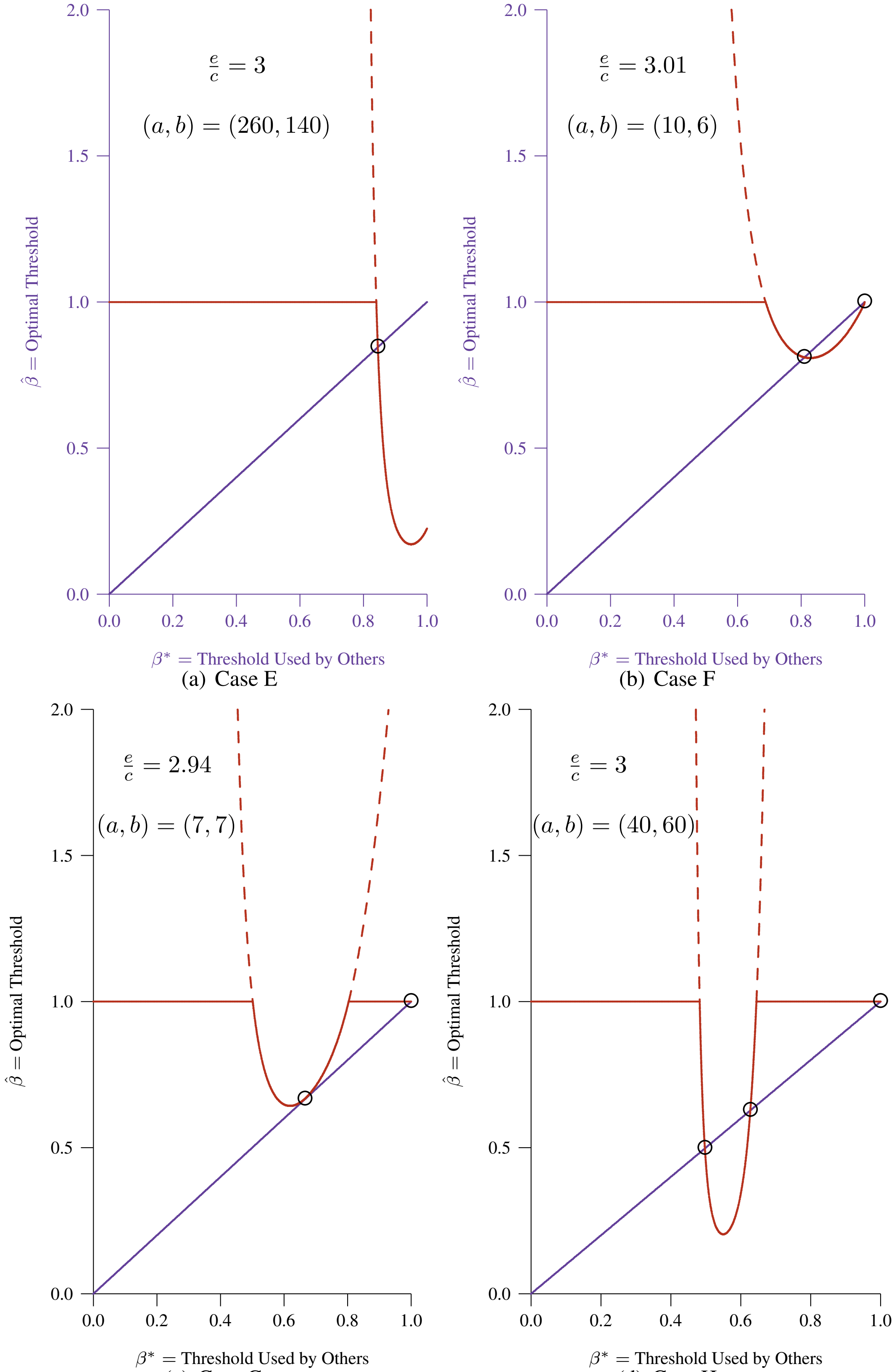


FIGURE 34. Illustration of the eight possible cases, $\lambda = 40\%$



(a) Case E

(b) Case F

(c) Case G

(d) Case H

Case A (Figure 5a): $\tilde{\beta}(\beta^*)$ does not intersect the box at any point;

Case B (Figure 5b): $\tilde{\beta}(\beta^*)$ only intersects the box at the point $(1, 1)$;

Case C (Figure 5c): $\tilde{\beta}(\beta^*)$ intersects the box twice without intersecting either the 45o line or the point $(1, 1)$;

Case D (Figure 5d): $\tilde{\beta}(\beta^*)$ intersects the box twice, one of which is the point $(1, 1)$, and without intersecting the 45o line;

Case E (Figure 5e): $\tilde{\beta}(\beta^*)$ only intersects the 45o line, and at one point;

Case F (Figure 5f): $\tilde{\beta}(\beta^*)$ intersects the 45o line twice, one of which at the point $(1, 1)$;

Case G (Figure 5g): $\tilde{\beta}(\beta^*)$ is tangent to the 45o line;

Case H (Figure 5h): $\tilde{\beta}(\beta^*)$ intersects the 45o line twice, none of which at the point $(1, 1)$;

If we change the focus from the different possible intersections between the function $\tilde{\beta}(\beta^*)$ and the box $[0, 1] \times [0, 1]$ and instead we look at the different types of equilibria that are possible for each set of parameters, we arrive at six scenarios, which are:

Scenario I (Cases A and D): only the sincere equilibrium is possible, which is *not* a knife-edge case and is stable;

Scenario II (Cases B and C): only the sincere equilibrium is possible, which is a knife-edge case and is stable;

Scenario III (Case E): only the interior stable equilibrium is possible;

Scenario IV (Case F): both the interior stable and the sincere equilibria are possible, the sincere one being an unstable knife-edge case;

Scenario V (Case G): a knife-edge tangent interior equilibrium is possible and unstable, along with the stable sincere equilibrium;

Scenario VI (Case H): both the interior stable and interior unstable equilibria are possible, as well a stable sincere equilibrium.

Each of the six different scenarios will require the four parameters in our model to satisfy some conditions. We need to understand how each of them influences the inverted bell-shape of the $\hat{\beta}(\beta^*)$ function and consequently the set of possible equilibria. Whenever we change a parameter, we want to see whether strategic voting is going up or down. To see this, we consider the proportion of voters voting strategically to be fixed (recall Figure 31) and see what happens to the incentive ratio. To keep the description reasonably contained, we assume that the starting point for the comparative statics analysis is a symmetric distribution—that is, we assume $a = b$.[118]

1. λ (the incumbent's support)

For each level of asymmetry and sample size, increasing λ implies that a higher degree of coordination is needed to defeat the incumbent, as we see in Figure 31. When the incumbent's support is higher, An increase in λ leads to an upward parallel shift in the line representing the support of party R and a downward shift in the line representing the maximum support that each of the two contenders can obtain, $1 - \lambda$. These shifts will make α_L and α_M change. It is useful to split the analysis into two cases. If there is only one pivotal event, the optimality conditions involve a recursive system with the two following equations:

[118]Implying that the mode is equal to the mean equal to the median equal to 0.5.

$$f(\alpha_L) = \frac{e}{c} \qquad \beta^* = \frac{(2\lambda - 1)}{(1-\lambda)} \frac{1}{(\alpha_L - 1)}$$

The first condition implies that α_L must be the same if we are to remain in equilibrium. In other words, strategic voting must adjust in such a way that the intersection between the relevant lines (see Figure 31) occurs at the α_L. From the second condition we see that, if α_L is to remain the same and λ goes up, then β^* goes down. The result is intuitive: strategic voting in equilibrium will increase in response to an increase in the incumbent's support. Straightforward differentiation of the second condition, keeping in mind that α_L is fixed and lower than 1, shows immediately that β^* must decrease in response to an increase in λ when only one pivotal event is possible.

If there are two pivotal events, α_L goes up and α_M goes down as λ increases and we cannot be sure about what happens to $f(\alpha_L)$ and $f(\alpha_M)$, since α_L and α_M depend on the equilibrium value β^*. If the distribution is symmetric and β^* is sufficiently close to 1, $f(\alpha_L)$ will see a larger drop than $f(\alpha_M)$, since its value after the change will be closer to the mode, implying a stepper slope. In that case, the incentive to vote sincerely will be higher, so the new equilibrium will entail more strategic voting. However, if the distribution is sufficiently biased towards party M, the incentive ratio may go down, making strategic voting more attractive. This would lead to a lower level of strategic voting in the new equilibrium, or a higher β^*. This is the case of ”wrong switching”—”wrong” in the sense that party M has an advantage in terms of the uncertain prior support but yet strategic voting occurs to party L. If the need for coordination is higher, then it is possible that the ”wrong” strategic voting decreases, because the decrease in the likelihood of M being involved in a pivotal event is higher than the one relating to L, making the overall ratio go down.

As we can see from the examples in Figure 35 in the next page—where we keep $\frac{e}{c} = 3$ and $(a, b) = (10, 10)$ while varying λ, which takes the values 37%, 40% and 43%— we can see when λ increases, the function $\hat{\beta}(\beta^*)$ shifts to the left, so there is more strategic voting in both the stable and unstable interior equilibria. Also, if initially there only exists one (therefore stable) interior equilibrium, the increase in the incumbent support makes the sincere equilibrium more likely, since strategic voting is a less attractive option.

In short, this confirms what we would intuitively expect: a higher support for the incumbent party makes it more difficult to defeat it and strategic voting increases in response to that. Where the Conservatives have a higher support, all else remaining the same, we would expect more strategic votes to occur in equilibrium.

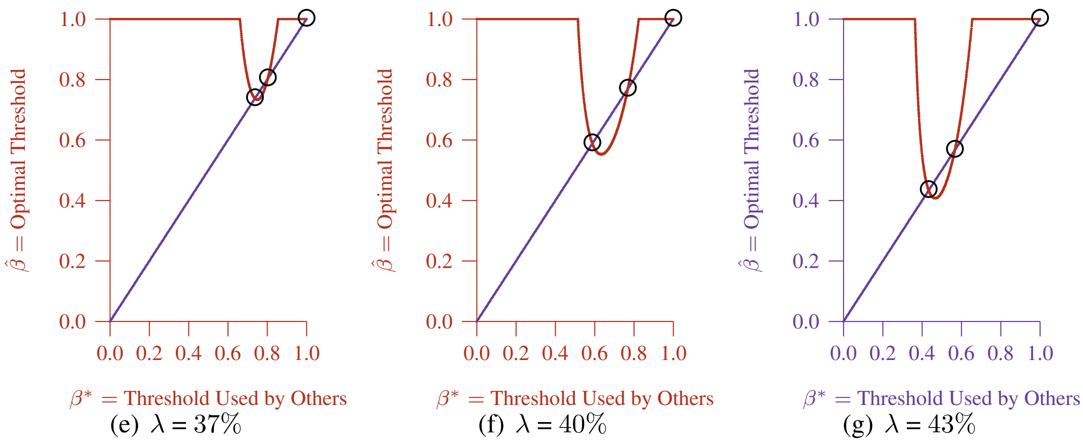


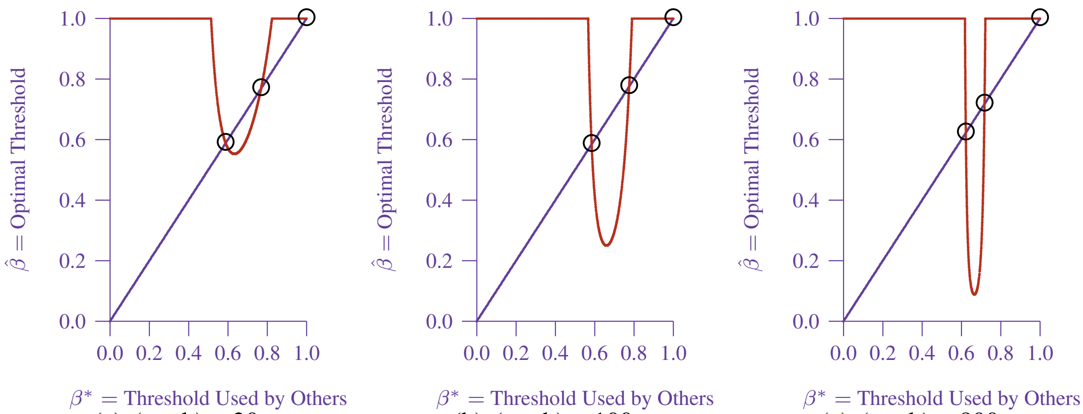
FIGURE 35. The effect of changing λ for $\frac{e}{c}$ = 3 and (a,b) = (10,10)

2. $a + b$ (the degree of information)

When the sample size goes up, the voters are more informed and so the degree of uncertainty regarding the prior support of the candidates is lower. This translates itself into a distribution with more probability mass around the mode. The effect of this on the level of strategic voting is not trivial, and it depends on how far from the mode the values of α_L and α_M are. If there is only one pivotal event and α_L is sufficiently close to the mode, then $f(\alpha_L)$ will go up as the sample size increases and the new equilibrium would entail less strategic voting, so that the pivotal event would occur for a higher α_L and a lower $f(\alpha_L)$, satisfying the optimality condition. If, however, $f(\alpha_L)$ goes down as the sample size increases, due to α_L being sufficiently far apart from the mode, then strategic voting would increase in the new equilibrium.

If there are two pivotal events, the conclusion will in general be the same. The pivotal event involving party M will occur for a very low α_M, so it must almost always be the case that $f(\alpha_M)$ goes down. In this case, it will generally be true that α_L is close to the mean, and therefore β^* goes up. The effect is simply reinforced. If, however, α_L is far from the mean, they will have opposite effects, but the latter will tend to dominate, making β^* go down, due to the bell-shape of the beta-distribution.

As before, we can think about what happens to the inverted bell-shape of the incentive ratio function as the information goes up. This curve will be more concentrated around its mode, therefore making the effect on the interior equilibria depend on whether we look at the stable or unstable one. The interior stable equilibrium will have less strategic voting as information goes up. We can see this in Figure 36 below, where we vary the sample size $(a + b)$ from 20 to 100 to 800, while keeping $\frac{e}{c} = 3$, $\lambda = 40\%$ and $\frac{a}{b} = 1$. The effect is obvious: the distribution becomes more concentrated around its mean.



(a) (a + b) = 20
(b) (a + b) = 100
(c) (a + b) = 800

FIGURE 36. The effect of changing (a + b), for $\frac{e}{c}$ = 3, λ = 40% and $\frac{a}{b}$ = 1

This is an interesting result, contrasting with Myatt's (2005), which precludes a positive effect of information on strategic voting: in his model, when the purely instrumental voters have more information, they feel more confident about their bets and strategic voting will increase; when there is less information, the higher uncertainty makes the purely instrumental voter be more prudent, and strategic voting decreases, to avoid switching to the wrong contender. However, in our model voters are not purely instrumental. This is why more information in general leads to less strategic voting in the stable equilibrium. Less uncertainty means that the pivotal event(s) are less likely to take place, and therefore less people will vote strategically. In the unstable equilibrium, on the contrary, more information leads to more strategic voting.

The exceptions to this happen only when the level of information is very low and the degree of expressiveness sufficiently high for the equilibrium to occur at a high level of β^* and with a very flat curve. Then, as information goes up, the incentive ratio function concentrates more around its lowest point—which then would be well outside the $[0, 1]x[0, 1]$ box—but the new crossing with the 45o line will happen at a lower value of β^*. An interesting case where the function rotates around the same intersection happens for $\lambda = 40\%$, $\frac{e}{c} = 3$ and $\frac{a}{a+b} = 75\%$. For $a+b < 14$ there is only a sincere equilibrium. For $a+b = 14$ there is an interior equilibrium: $\beta^* = 97.90\%$, with a very flat slope. As the information increases from $a+b = 14$ to $a+b = 36$, strategic voting increases: at $a+b = 36$ we have $\beta^* = 85.60\%$. From $a+b = 36$ to $a+b = 46$ the level of strategic voting remains the same, with the slope increasing as the incentive function becomes less disperse. From $a+b = 47$ to $a+b = 57$ the equilibrium remains constant at 85.70%, then it goes up, with strategic voting decreasing. Essentially, these cases require a considerably high degree of expressiveness, a low level of information and a degree of asymmetry that is proportional to the support of the incumbent.

Finally, as information goes up, we are more likely to see the sincere equilibrium taking place, as the distribution concatenates around the mean, the same happening with the function $\hat{\beta}(\beta^*)$.

3. $\frac{a}{b}$ (the asymmetry in the prior distribution)

If we change the asymmetry in the distribution, increasing the bias towards party L—in other words, increasing a while keeping $a + b$ constant—we will get less strategic voting in equilibrium, i.e., higher levels for β^*. This will happen regardless of the number of pivotal events. The reason is that an increase in the bias towards party L makes strategic voting less needed to defeat the incumbent. The intuition is that additional support for the party that has a better position as a challenger will compensate for—and therefore dispense—some strategic voting. Less voters will have to go i Diagrammatically, it implies a shift in the distribution towards the right. The new equilibrium will occur for higher levels of α_M and α_L, making $f(\alpha_M)$ and $f(\alpha_M)$ closer to the values they had in the initial equilibrium. Then, if the α's are higher, β^* will have to be higher. The intuition is that since party L is relatively more likely to have a higher support, less ML voters will feel inclined to vote strategically. Simply, it is not necessary to have so much strategic voting to have a pivotal event between L and R taking place, and therefore an equilibrium condition will only be met for a higher β^*. This is confirmed by looking at $\hat{\beta}(\beta^*)$, which will move to the right when asymmetry towards party L increases, making both the stable and unstable interior equilibria occur for a higher value of β^*, implying less strategic voting. That is illustrated in Figure 37, where we keep $\frac{e}{c} = 3$, $\lambda = 40\%$ and $(a+b) = 120$, while changing $\frac{a}{b}$ from $\frac{2}{3}$ to 1 to $\frac{3}{2}$. As $\frac{a}{b}$ goes up, the curve clearly shifts to the right. Also, as we can see from the third diagram in Figure 8, as the asymmetry gets higher we get closer to the case where the sincere equilibrium is a knife-edge case.

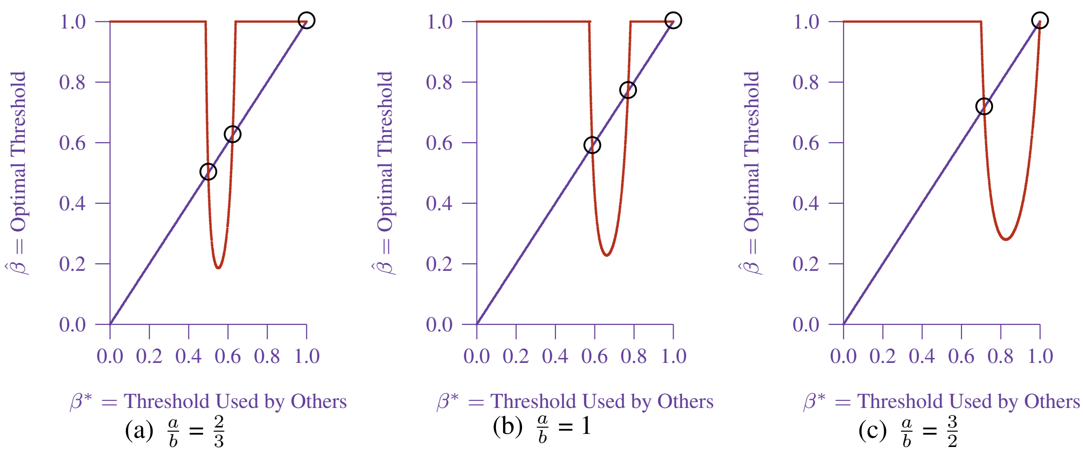


FIGURE 37. The effect of changing $\frac{a}{b}$, for $\frac{e}{c} = 3$, $\lambda = 40\%$ and (a + b) = 120

We can also think the other way round. If the asymmetry towards party M goes up, the function $\hat{\beta}(\beta^*)$ goes to the left, and there is more strategic voting in any of the existing interior equilibria. In this case what we can say is that if an interior equilibrium is to take place, it requires more strategic voting, which is understandable, given the lower prior support of the party receiving the votes, that still has to achieve the same percentage of votes as before if it is to defeat the incumbent.

Naturally, as the asymmetry towards party M goes up, the sincere equilibrium is more likely to occur (think of moving from right to left in Figure 37). If party M enjoys a very low prior support, then only the interior equilibrium is sustainable—this is the Scenario I we described before. As party M starts to enjoy a higher support, with the function $\hat{\beta}(\beta^*)$ moving to the left, we have scenarios III, IV and VI.

An interesting comparison we can do is between the level of asymmetry, $\frac{a}{a+b}$ and the degree of coordination required, $\frac{\lambda}{1-\lambda}$. If the asymmetry is much larger than the required coordination, then party L is so likely to get a high support that voting tactically cannot be optimal—a vote is very unlikely to make a difference, and so you are better off voting to your favorite candidate. As the asymmetry goes down, and starts getting closer to the level of coordination required to defeat the incumbent, γ, then the interior equilibrium is unique. If the asymmetry is sufficiently lower than that level, than the three equilibria will be possible: a sincere equilibrium is possible, because party M enjoys sufficient prior support; nevertheless, some pattern of strategic voting can be optimal too.

4. $\frac{e}{c}$ (the level of expressiveness)

Finally, when the degree of expressiveness $\frac{e}{c}$ increases, strategic voting becomes less attractive overall. Recall that e reflects the level of expressive concern for the preferred party, while the paternalistic concern c is present in the model to allow us to study the case of large electorates, as opposed to finite ones. As e/c increases, the voters feel more strongly for their preferred parties, and the impact of this on the stable interior equilibrium will be a decrease in the level of strategic voting. This is what we expect intuitively. Diagramatically, the change is quite similar to what happens when we change the sample size. If we think about the incentive ratio curve $\hat{\beta}(\beta^*)$, an increase in $\frac{e}{c}$ is equivalent to an upward shift in it. Strategic voting will increase for the interior unstable equilibrium and decrease for the interior stable one. The impact of this change on the intersections between $\hat{\beta}(\beta^*)$ and the 45o line would be to make the lower β^* go up and the higher β^* go down. Again, the conclusions seem to make sense. All else constant, a higher degree of expressiveness will lead to less strategic voting in the interior stable equilibrium.

We illustrate this in Figure 38, where we vary the expressiveness ratio from 1 to 3 to 5, keeping $\lambda = 40\%$ and $(a, b) = (50, 50)$ constant. It is clear that the main effect is to shift the curve upwards, while also making it slightly more concentrated around its minimum.

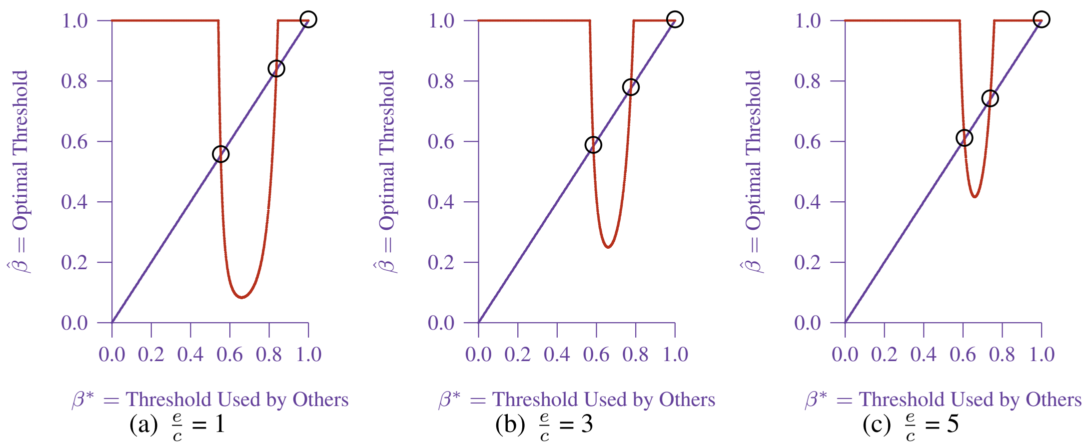


FIGURE 38. The effect of changing $\frac{e}{c}$, for λ = 40% and (a,b) = (50,50)

We have commented before on the stability of the two interior equilibria. This section adds a more formal treatment of the question of stability. There are two main views on how to analyse stability in a voting game. The first focuses on analyzing the effect of changing the equilibrium in a continuous (or marginal) manner, the second focuses on discrete changes. If we look at continuous variations, an equilibrium β^* will be stable if $\partial\hat{\beta}/\partial\beta^* < 1$, and unstable if $\partial\hat{\beta}/\partial\beta^* > 1$. The intuition is straightforward, as we explained before: if $\partial\hat{\beta}/\partial\beta^* < 1$, an increase in strategic voting—meaning a lower β^*, now taken to be the proportion of voters voting sincerely, not necessarily in an equilibrium situation—will lead to an increase in $\hat{\beta}$, therefore higher than the new β^*. This implies that some voters would be better off voting sincerely, until we reached the initial situation. More specifically, voters for which β_i is lower than $\hat{\beta}$ and higher than β^* will have an incentive to vote strategically. As long as $\hat{\beta}$ is higher than β^* there will be some voters who would be better off voting sincerely, making β^* go up and $\hat{\beta}$ go down. This adjustment will only stop once an equilibrium is reached, where $\hat{\beta} = \beta^*$. This analysis is based on variations that are small enough not to make other stable equilibria the point to which votes will gravitate too.

If we look at discrete variations we need to look at the dynamics of the model. There can be situations where we go from one equilibrium to another in cycles or where we diverge, not reaching any stable equilibrium. It is hard to obtain general results with the discrete approach. It also seems more appropriate to consider the continuous approach if we think about the small unit of analysis that we would consider when thinking about a deviation—the voter.[119]

If we take the continuous approach, a sufficient and necessary condition for stability is $\partial\hat{\beta}/\partial\beta^* < 1$, which guarantees that at least one stable equilibrium exists. The reason for this is that there will be two or more intersections with the 45o line and, from continuity of the function $\hat{\beta}(\beta^*)$, at least one of them will have a slope that is lower than 1. If there is only one intersection, it must have a slope lower than 1. Finally, if there are no intersections with the 45o line, only the sincere equilibrium will be possible, and in such case it will always be stable.

To analyse the stability of the sincere equilibrium we need to consider two possible cases: $\tilde{\beta}(1) > 1$ and $\tilde{\beta}(1) = 1$. When $\tilde{\beta}(1) > 1$, a marginal deviation from some voters, from a sincere to a strategic vote will lower $\tilde{\beta}$ but won't make it lower than 1. This is what happens in cases A, C, G and H in Figures 34. When $\tilde{\beta}(1) = 1$, stability will depend on how much the incentive ratio changes. If $\partial\tilde{\beta}/\partial\beta^* \mid_{\beta^*=1} > 1$, the equilibrium is stable, since after the deviation the incentive ratio $\hat{\beta}$will be higher than the new proportion of ML voters voting sincerely, creating an incentive for some voters to vote sincerely, which causes a negative effect leading towards the initial equilibrium—cases B (where $\partial\tilde{\beta}/\partial\beta^* \mid_{\beta^*=1} < 0$) and D (where $0 < \partial\tilde{\beta}/\partial\beta^* \mid_{\beta^*=1} < 1$) illustrate the two possibilities. If $\partial\tilde{\beta}/\partial\beta^* \mid_{\beta^*=1} < 1$, the sincere equilibrium is unstable, and so strategic voting would happen until a stable interior equilibrium was reached. From the above description, we know that the curve $\hat{\beta}$ will have to cross the 45o line from

[119] We could consider a larger unit of analysis—such as a family or a group of people—but the approach would not be too appropriate, since the commitments would not be binding, given that choice would not be verifiable. Having the voter as the basic unit of analysis in our voting game seems the natural option. This is also the approach used in essay 1.

below, which means that there is one intersection where the slope is lower than 1, therefore guaranteeing that an interior stable equilibrium exists (see case F).

So, when a sincere equilibrium exists, either it is stable, or, if it is not stable, then a stable interior equilibrium must exist. If a sincere equilibrium does not exist, we can be sure that a stable interior equilibrium exists (this is case E). We conclude, then, that there is always at least one stable equilibrium and, at most, two—which is illustrated in case H.

We can obtain an explicit expression for stability to hold by doing a total differentiation of $\hat{\beta}$ in respect to β^*. Recalling that $\hat{\beta} = [\frac{e}{c}\beta^* + f(\alpha_M)]/f(\alpha_L)$, we get[120]:

$$\frac{d\hat{\beta}}{d\beta^*} = \frac{\partial\hat{\beta}}{\partial\beta^*} + \frac{\partial\hat{\beta}}{\partial\alpha_M}\frac{\partial\alpha_M}{\partial\beta^*} + \frac{\partial\hat{\beta}}{\partial\alpha_L}\frac{\partial\alpha_L}{\partial\beta^*} = \frac{e}{c}\frac{1}{f(\alpha_L)} + \frac{f'(\alpha_M)}{f(\alpha_L)}\frac{\partial\alpha_M}{\partial\beta^*} - \frac{[\frac{e}{c}\beta^* + f(\alpha_M)]f'(\alpha_L)}{f(\alpha_L)^2}\frac{\partial\alpha_L}{\partial\beta^*}$$

From the rational expectation conditions we know that, if there are two pivotal events, $\alpha_L = 1 + \frac{2\lambda-1}{(1-\lambda)\beta^*}$ and $\alpha_M = 1 - \frac{\lambda}{1-\lambda}\frac{1}{\beta^*}$, from which we derive $\frac{\partial\alpha_L}{\partial\beta^*} = \frac{(1-2\lambda)}{(1-\lambda)}\frac{1}{[\beta^*]^2}$ and $\frac{\partial\alpha_M}{\partial\beta^*} = \frac{\lambda}{(1-\lambda)}\frac{1}{[\beta^*]^2}$. Thus we can further write the above expression as:

$$\frac{d\hat{\beta}}{d\beta^*} = \frac{e}{c}\frac{1}{f(\alpha_L)} + \frac{f'(\alpha_M)}{f(\alpha_L)}\frac{(1-2\lambda)}{(1-\lambda)[\beta^*]^2} - \frac{[\frac{e}{c}\beta^* + f(\alpha_M)]f'(\alpha_L)}{f(\alpha_L)^2}\frac{\lambda}{(1-\lambda)}\frac{1}{[\beta^*]^2}$$

Or:

$$\frac{d\hat{\beta}}{d\beta^*} = \frac{\frac{e}{c}}{f(\alpha_L)} + \frac{f'(\alpha_M)(1-2\lambda)f(\alpha_L) - [\frac{e}{c}\beta^* + f(\alpha_M)]f'(\alpha_L)\lambda}{(1-\lambda)[f(\alpha_L)\beta^*]^2}$$

[120]Recall the assumption that $H(\beta^* = \beta^*$.

In an equilibrium with two pivotal events the optimality condition implies that $\frac{e}{c}\beta^* + f(\alpha_M) = \beta^* f(\alpha_L)$, so we get the expression:

$$\frac{d\hat{\beta}}{d\beta^*} = \frac{\frac{e}{c}}{f(\alpha_L)} + \frac{f'(\alpha_M)(1-2\lambda) - \beta^* f'(\alpha_L)\lambda}{(1-\lambda)f(\alpha_L)[\beta^*]^2}$$

It is not possible to simplify this expression generically, but we can look at the particular case when there is only one pivotal, setting $\frac{\partial\hat{\beta}}{\partial\alpha_M}\frac{\partial\alpha_M}{\partial\beta^*} = 0$, to obtain a simpler expression:

$$\frac{d\hat{\beta}}{d\beta^*} = \frac{\partial\hat{\beta}}{\partial\beta^*} + \frac{\partial\hat{\beta}}{\partial\alpha_L}\frac{\partial\alpha_L}{\partial\beta^*} = \frac{\frac{e}{c}}{f(\alpha_L)} - \frac{\frac{e}{c}\beta^* f'(\alpha_L)\lambda}{(1-\lambda)[f(\alpha_L)\beta^*]^2}$$

Also note that when there is only one pivotal $\frac{e}{c} = f(\alpha_L)$, so $\frac{\frac{e}{c}}{f(\alpha_L)} = 1$. Since stability requires and is guaranteed by $\frac{d\hat{\beta}}{d\beta^*} < 1$, a stable equilibrium must satisfy:

$$\frac{\frac{e}{c}\beta^* f'(\alpha_L)\lambda}{(1-\lambda)[f(\alpha_L)\beta^*]^2} > 0$$

Since the denominator is always positive, we need $\frac{e}{c}\beta^* f'(\alpha_L)\lambda > 0$. This requires $f'(\alpha_L) > 0$. So a sufficient and necessary condition for stability is for the Beta-distribution $f(\alpha)$ to have a positive derivate at the point α_L. This is intuitive. If $f'(\alpha_L) > 0$, an increase in strategic voting (a lower β^*) will lead to a new pivotal event, occurring for a lower α_L, with a lower $f(\alpha_L)$. This would mean that the pivotal event involving party L is less likely to happen, providing a lower incentive to vote strategically, which prevents a divergent result. When there are two pivotal events, the intuition is the same: there will be stability when α_L and α_M change in a way that makes it—in the margin—more attractive for voters to vote in the opposite way

8. Equilibrium selection

The problem of multiple equilibria arises a question about equilibrium selection, which we have addressed before[94]. In the current model, where we have a possibility of at least one but possibly two stable equilibria, an immediately sensible criterion for equilibrium selection should be the stability of the equilibrium. Whenever there is a stable and an unstable equilibrium, a sensible prediction would support the stable equilibrium, whether it be interior or sincere. But what in the case where there are two stable equilibria (and, necessarily, a third unstable one)? As we mentioned before, there is a very uncommon case where there can be two interior stable equilibria and one interior unstable. This happens for an extremely small set of parameters, as it will be clearer in the next chapter. The pervading scenario involves one interior and one sincere equilibrium. At least two important arguments can be put forward here.

On the one hand, the sincere equilibrium can be argued to be focal, given the nature of the voting game: voters have and commonly know others have an expressive concern, and this component of ”loyalty” could be reasonably justify that voters would more likely coordinate on a sincere equilibrium than on an interior equilibrium when both are available and stable. Such ”focality” could arise in the same way described in essay 1. Important figures in the political sphere could influence expectations and lead people to vote according to the equilibrium suggested by them.

The second argument would suggest the equilibrium with a higher range of (prior) beliefs leading towards it would be picked. When there is a sincere and an interior stable

[94]See pages 41 and 42, essay 1, for some important comments about the issue of multiplicity of equilibria in a political context.

equilibrium, the sincere equilibrium will (or, perhaps better, would, since the nature of this exercise is hypothetical) be reached whenever the beliefs regarding the proportion of sincere voting are higher than the proportion that characterizes the unstable interior equilibrium—$\beta^*_{i_u}$. So the range of beliefs supporting the sincere equilibrium would be $[1 - \beta^*_{i_s}]$. Likewise, whenever the hypothetised support is lower than that characterising the unstable interior equilibrium, the stable prediction would be the interior stable equilibrium. The range of beliefs supporting this would be $[\beta^*_{i_s} - \beta_{\min}]$, where $\beta_{\min}$ represents the lowest β^* that would generate a value of the incentive function strictly below 1. Figure 33 is helpful in visualing the implications of this criterion. In that figure $\beta^*_{\min} = 0.58$ and $\beta^*_{i_s} = 0.78$. In this case the range of support for the sincere equilibrium would be 0.22, which is bigger than 0.20, the range of support for the interior stable equilibrium, and so that would be a stronger prediction. This second argument relies on the assumption that the support for the parties is uniformly distributed, which is one of our assumptions. The argument would hold if that was not the case, it would only be necessary to adapt the calculation to achieve a proper estimate for the range of support, attending to the distribution function of β_i. We could also calculate the percentages of support that each voting equilibrium had by diving the referred numbers by the overall distance between the relevant extremes, equal to $[1 - \beta_{\min}]$.

This second argument is more quantitative and can, to some extent, by linked to the idea behind ”risk dominance” being a criterion for selecting a Nash equilibrium. The enormous difference here between the two is that risk dominance applies to the behavior of one agent one, and involves calculating the range of values for some relevant parameter on the model that would lead to one or the another Nash equilibrium, while in the argument we presented we would be looking at range of values that would determine different choices of many agents, and in that, a different voting equilibria.

Both arguments could have a place and their relative strength would depend on the context of the election.

9. COMPARISON WITH OTHER MODELS

Three papers we should to compare our results with are Myatt (2006), Cox (1999) and Mendes (2004), the essay in Part I of this thesis. All three focus on the positive analysis of strategic voting in a three-party competition where voters are purely instrumental.

Myatt (2006) proposes a model where voters have both private and public information. Each voter shares a common prior, but then observes a signal which he updates to construct his own belief about the support of the parties. Thus, in this model uncertainty is not common knowledge to everyone, which means it is possible for voters to hold different beliefs in equilibrium. The main conclusion of this paper is that a Duvergerian outcome can be avoided: it is possible to have an equilibrium with partial strategic voting that is stable. This stability comes from a negative effect that strategic voting has on itself: when strategic voting increases, the uncertainty about the outcome of the election is higher, and voters get relatively more skeptical about not casting a vote to their favorite candidate, fearing that they would choose in the wrong direction. One clear difference is that the Duvergerian outcome is possible in Myatt (2006) but not in our model, since a complete coordination in one party would render a deviation profitable, as $\frac{e}{c} > 0$ is immediately attainable.

Cox (1999) looks at a model where there is some uncertainty, but which vanishes away completely when the electorate approaches infinity. The Duvergerian outcome is possible, and the non-Duvergerian equilibrium is unstable and a knife-edge case. With these two outcomes in mind, Cox (1997) proposes a bi-modality hypothesis, suggesting that in the distribution of first to second losers in all constituencies should have two modes, at zero and at one. The next chapter will look closer into this hypothesis.

In contrast with Cox's result, our model does not allow the Duvergerian equilibrium to take place, so, there would be no spike around zero, but in the interval $(0, 1)$, corresponding to the interior stable equilibrium. In Part I of this thesis we study a model which is somewhere in between Myatt's and Cox's, when we consider a game with effectively only three parties competing (the second model in essay 1, see page 46). In such model there is uncertainty when the electorate approaches infinity, but one that is commonly known—in contrast with Myatt (2006). So, it allows for a Duvergerian and a sincere equilibrium, though not a stable one. It also makes it possible to have an interior equilibrium, which, however, is never stable.

This chapter extends the model from the previous chapter, by including a non-instrumental component in voters' preferences. This is sufficient to provide a set of new and interesting new results, and adding more complexities to it would not allow us to isolated the effects of either of the extensions done. To be fruitful, an extension must be as simple as possible as long as an interesting difference in the results arises from it.

The first result is that the sincere equilibrium, where every voter votes for his favorite party, will be possible in general, and stable. This is a consequence of the psychological and opportunity 'cost of voting', which distinguishes it from all the three cited papers. A second result is that an interior stable equilibrium will be possible, which only happens in Myatt (2006). This shows that private information is only necessary to have such result within a model with purely instrumental motivations. It is not necessary overall, though, one of the ways of achieving that being a model with expressive motivations. This result is theoretically interesting and we will want to see how it may or may not explain the UK 1997 General Election, in essay 4 of this thesis. A third result, which also goes against the conclusions in Myatt (2006), is that when information goes up, there will be less strategic voting in the interior stable equilibrium.[121] In Myatt's model, more information leads to more strategic voting, as the voters are more confident that they are shifting their votes in the right direction. In our model, however—and because there is an expressive component—, more information will almost always decrease the incentive to vote strategically, as the pivotal event will be relatively less likely to occur.

Another strand of papers that are relevant includes Picketty (2000), Morelli (2004) and Razin (2003). Morelli (2004) studies implications for party formation and policy outcomes arising from different electoral rules, namely proportional representation and plurality rule, in a multi-district model. In his paper, parties provide a coordination device to voters during the elections, and candidates are separable from the parties, arising after the election. One of the main conclusions is that sincere voting induces more party formation and, somewhat contrary to intuition, strategic voting can be observed more often under proportional representation rule. The other two models are closer to ours in that they propose a sound explanation for why and how a non-instrumental

[121]Recall that in our model information is basically encapsulated in the sample size of an (implicit, or not) opinion poll and, when we consider a standardized Beta-distribution with parameters a and b, it is equal to $a + b$.

motivation may be present in voter's preferences. Picketty (2000), "Voting as Communicating", studies a model where voters trade-off two different interests: to influence the current election and to influence future elections, by communicating their views. This preference for "communicating" ones views is similar to our expressive motivation and is a sensible way to include the possibility of sincere voting in a model. The result on the first-stage period is not necessarily Duvergerian, and a conclusion from the author is that, for higher efficiency, electoral systems should favour elections with two rounds instead of one. Razin (2003) also explores a model where information in election results count more than solely determining its winner. Voters have an "election" motivation—to influence the winner of the election—but also a "signalling motivation". In his model, policy makers take into account the electoral results when choosing their policy. This creates an incentive, compared to models where voters are purely instrumental, to vote sincerely. The paper focuses on the efficiency of equilibria under this conditions, which departs significantly from our focus, but it is interesting to note another plausible justification for expressive voting.

10. Conclusions

We sum up the main results from our analysis in eight points:

(1) A voting equilibrium always exists;
(2) Three equilibria can occur: a sincere, an interior stable and an interior unstable equilibria;
(3) There will be at least one stable equilibrium and at most two;
(4) A stable sincere equilibrium (involving no strategic voting) is possible;
(5) A stable interior equilibrium (involving partial strategic voting) is possible;

(6) A Duvergerian equilibrium (involving full-shift to one of the contenders) is not possible;

(7) Regarding the comparative statics analysis, we observe that:

  (a) More information leads to less strategic voting;

  (b) A higher expressive effect lowers the level of strategic voting;

  (c) The higher the prior support of the party who supporters vote strategically, the higher the level of strategic voting;

  (d) The higher the support of the incumbent party the lower the level of strategic voting.

(8) As to the equilibrium selection problem, different arguments can be drawn about the best prediction to have when more than one stable equilibrium is possible. We highlight two of those. On the one hand, the sincere equilibrium can be argued to be focal, given the nature of the voting game; on the other hand, the interior stable equilibrium will, in general (as long as the asymmetry towards party M is not too large), have a higher range of (prior) beliefs leading towards it.

In conclusion, we can say that our model encapsulates a fairly commonsensical idea—that people have a psychological cost of not voting for their preferred party—, and provides a whole range of results that are in line with some commonly observed facts, including the existence of non-Duvergerian equilibria, both involving outcomes with sincere voting and with stable partial strategic voting. While the results from essay 1 largely support Duverger's Law, in this essay we describe the possibility of a stable non-Duvergerian equilibrium, which is due to the presence of an expressive motivation in voters' preferences. This means that even with single-ballot plurality rule elections it is possible to have a stable voting outcome where more than two candidates survive the election. This is a major contribution of our work.

The fact that part of our results are not in line with some of the literature regarding strategic voting adds to the understanding of the phenomenon of strategic voting. Our theory constitutes an alternative theory of voting, where expressive benefits play an important role in determining how to vote, along with instrumental motivations. It can (and ideally should) also be seen, from a wider perspective, as complementary to existing theories of voting, in the sense that it expands them, clarifying the implications of a model with mixed motivations but only public uncertainty. A natural extension of this work would be to use a model with mixed motivations while having both public and private information, as in Myatt (2006).

The next essay explores one particular implication from our theory: that outcomes tend to concentrate in two regions, with a gap in between where no outcomes are possible. This means we have an expanded bi-modality scenario, different from Cox's in its nature but close to what he predicted it would happen when some hypotheses were relaxed. In the fourth essay we use our model to bootstrap the strategic votes that may have taken place in the UK 1997 General Election.

– Essays on Strategic Voting –

Essay 3

“Rescuing the Bimodality Hypothesis”

> “Under Duvergerian equilibria, the SF ratio will be near zero. Under non-duvergerian equilibria, the SF ratio will be near unity. Thus, if one were to compute the ratio for a number of districts and plot the resulting distribution, one should find a spike at zero and a spike at one.” *Cox (1997)*

1. Introduction

This essay argues for a bimodality hypothesis, following Cox (1994, 1997). Cox's "bimodality hypothesis" is well known in the literature and his theoretical prediction is that the ratio of the second to the first loser's vote total – the “SF ratio” – should be exactly zero or one. Cox’s model involves a plurality rule three-party competition where voters have purely instrumental motivations and there is no real uncertainty as the electorate gets large.

A SF ratio equal to 0 is equivalent to a Duvergerian result, since the third candidate gathers no votes at all. In Cox’s model, the Duvergerian equilibrium is always possible, regardless of the initial support of each of the contenders: it only requires a common expectation that everyone will coordinate their votes in a certain contender for that to be a possible outcome.

Our model has two critical differences to Cox’s: voters’ preferences have not only an instrumental component, but also an expressive one; and there is real uncertainty regarding the final outcome of the election even as the electorate gets large. Differently from Myatt (2006), we assume that all information is publicly known, in other words, we exclude private information from the model.

The results from our analysis, building on the theoretical model proposed in the previous essay, suggest a bimodality hypothesis, but one that is less strict than Cox’s. Our hypothesis implies, less restrictively, that there will be a “no zone” area for the ratio of the electoral results for the two contenders ("Rf" in our model – a variable we will introduce shortly), but this no zone area will not be the interval]0.5,1[, but something strictly within it. The referred ratio (Rf) will be concentrated in two different regions – loser to 0.5 and to 1 –, and it is based on this that we talk about an "extended bimodality hypothesis": though our ratio can take more than two values, those values are in some sense an "extension", an "elongation" of the extreme values 0.5 and 1.

In this essay we focus on the theoretical study of what this new bimodality result implies. In the next essay we will look at the 1997 UK General Election to see whether our theory could provide some explanation to what was observed.

2. The Bimodality Hypothesis

We illustrate Cox's bimodality hypothesis in Figure 39 in page 151, but before addressing that we clarify some of the notation we will use throughout the essay. The subscript "i" refers to the "initial" or prior support of each candidate; the subscript "f" refers to the "final" or posterior support of each candidate. "A" refers to the observed leading contender; "B" refers to the observed trailing contender.

Af is the result (in number of effective votes) of the observed first contender after votes are cast.

Bf is the result (in number of effective votes) of the observed second contender after votes are cast.

Ai is the support (counted as the number of potential votes) of the observed first contender before votes are cast.

Bi is the support (counted as the number of potential votes) of the observed second contender before votes are cast.

From here we construct four ratios: the first two related to Cox's terminology; the last two related to our terminology.

SFf = Bf/Af

SFi = Bi/Ai

Rf = Af/(Af + Bf)

Ri = Ai/(Ai + Bi)

Cox's SF ratio refers to the final situation, so, using the terminology just proposed, Cox's SF ratio is equal to Bf/Af. It is important to note that the Cox's analysis is focused on the final (or observed) results. This is not the case in our model, where it is vital to consider the expectations about the initial (or prior) support of each candidate.

Figure 39 in the next page reads as follows. On the horizontal axis we have the SFi ratio (equal to Bi/Ai) and on the vertical axis the SFf ratio (equal to Bf/Af), which can both take values between 0 and 1. The black line – in this case only a dot, at point (1,1) – describes all the combinations that can sustain an equilibrium where everyone votes sincerely. The red line describes all combinations that allow for strategic voting to occur in equilibrium.

Looking at Figure 39, we see that Bf/Af can take the value 0 for whatever Bi/Ai in [0,1], and can take the value 1 only when Bi/Ai = 1. Bi/Ai = 1 is a knife-edge case where both contenders have, initially, exactly the same support. Bf/Af = Bi/Ai means that there is no strategic voting, so everyone is voting sincerely. In other words, the 45o (dashed) line represents all the outcomes where no strategic voting takes place. Therefore, within Cox's model a sincere equilibrium is only possible for Bi/Ai = 1 and Bi/Ai = 0.

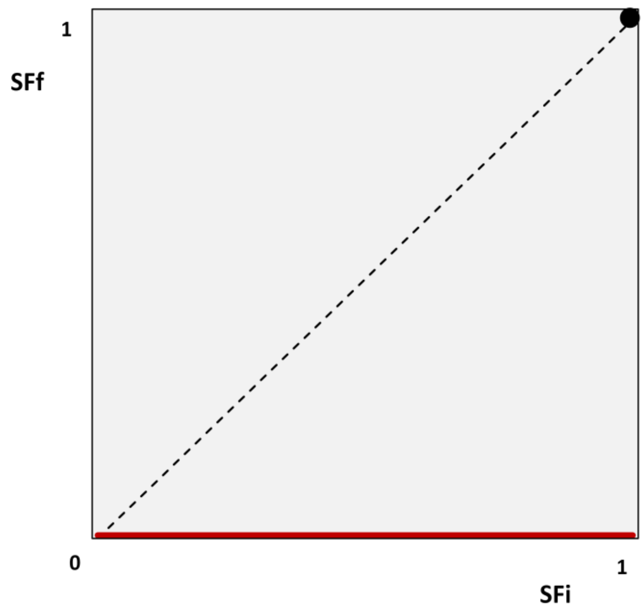


Figure 39 – Cox's hypothesis in the (SFi, SFf) axis

Bi/Ai = 0 is, however, meaningless – this would mean that no one supported the third candidate in the first place, so we would be analysing a two-party competition. The meaningful equilibrium with sincere voting in Cox's model happens when Bi/Ai = 1. As Cox himself suggests, this would be (in his model) a highly unstable knife-edge outcome. Such instability comes largely from the information structure of his model, as we have seen in Part I of this thesis[123].

The Duvergerian equilibrium occurs whenever Bf/Af = 0 and the sincere equilibrium happens when Bf/Af = 1. Cox's bimodality hypothesis, strictly considered, says that the distribution of the SF ratio (in our terminology, the SFf ratio) will have only distribution mass at 0 and 1. More loosely speaking, it suggests that the distribution should show two clear peaks around 0 and 1 and very small values for other values.

[123] See pages 14-17.

Before looking on to whether Cox's prediction seems to be sustained by the data or not, we illustrate his hypothesis in a way that is directly comparable to the one we adopt on our model. This requires two adjustments, a simple relabeling and a more substantive change.

Starting with the relabeling, instead of focusing on the ratio SFj = Bj/Aj, for j = i, f, we will use the ratio Rj = Aj/(Aj + Bj). Rj represents the share of the first contender among the electorate that dislikes the incumbent at moment j. This is more useful than SFj for two reasons.

First, it highlights an essential feature of these models of strategic voting: that the game being played between the dislikers of the incumbent is a qualified majority game, where one of the contenders wins the election if it obtains more than $\lambda/(1 - \lambda)$ of the votes cast by the incumbent's dislikers, that is, if $Rf = Af/(Af + Bf) > \gamma = \lambda/(1 - \lambda)$[124]. By using Rf we can directly and intuitively compare it to γ, an essential parameter in our model. With SFf there would be no obvious value to compare to.

Second, Ai and Bi will be related to *a* and *b* – the parameters of the Beta-distribution in our model[125]. Ri = Ai/(Ai + Bi) is related to the average of the Beta-distribution, a/(a+b), in a way that will become more obvious later on.

With this relabeling in mind, we illustrate Cox's hypothesis in Figure 40 in the next page, this time with Ri and Rf as the axes. Being aware of what underlies Rf and Ri makes it easy to see why we should read the diagram by first looking at the vertical axis. This is because both Af and Ai (whatever party underlies that labelling) refer to the contender with highest observed support – and that information comes from observing the final situation.

[124] See Myatt, D. and Fisher, S. (2002).
[125] See pages 98-100.

Figure 40 reads as follows: Rf can take the value 1 for any value of Ri. These are the Duvergerian equilibria. The non-Duvergerian (in this case, a sincere equilibrium), is only possible for Rf = 0.5 = Ri[126].

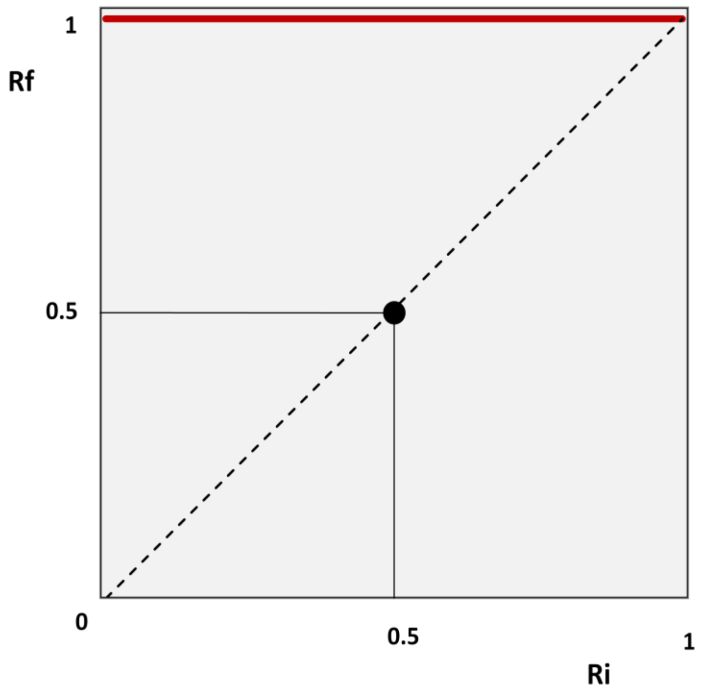


Figure 40 – Cox's hypothesis in the (Ri,Rf) axis

We are left with the second and more substantive adjustment to Cox's approach, in order to make it more comparable to what we will do in this essay. In Cox's model there is no assumption about the direction of strategic voting. Strategic voting can occur towards the party with least support – in the "wrong direction" –, what we could call a "wrong switch". As virtue of being a voting equilibrium, as long as voters expect that to happen, that may indeed happen. This allows for a ratio of Bf/Af = 1, in Figure 39, being originated by a ratio

[126] Note that theoretically it is not possible to have Rf < 0.5.

of Bi/Ai < 0.5, in other words, for strategic voting to occur towards the party with least initial support. In Figure 40, likewise, we can see that a (posterior ratio) Rf = 1 can be originated by a prior ratio Ri < 0.5.

In this essay we restrict strategic voting to occur towards the contender enjoying the highest prior support. Since, by construction, we must have that Rf > 0.5, ruling out "wrong switching" of votes implies that Ri > 0.5. This makes it impossible to have the observed first contender starting in disadvantage to the observed second contender.

This can be seen as somewhat restrictive, particularly taking into account that there are reported cases of wrong switching[127]. However, these are residual and the simplification it brings is significant. Furthermore, such simplification is a requirement for the analysis we do in the next essay, and having both essays coherent in this matter is essential.

Figure 41 in the next page illustrates Cox's hypothesis with the restriction of ruling out wrong switching after our relabeling, using Ri and Rf as axes, with Ri in [0.5,1].

Figure 42 in the next page summarizes the distribution of Rf in the 1997 UK General Election. The first impression we get from it is that a theory that argues for the existence of two peaks around 0.5 and 1 has little explanatory power regarding this particular election. Roughly speaking, the distribution seems to be uniform between 50% and 80%, with higher values between 80% and 90% and no values above 90%. The fact that no values close to 100% are observed means that a full Duvergerian outcome did not take place in any of the constituencies. The distribution is clearly not bimodal, but single peaked. It is as if we had a truncated uniform distribution, where the values above 90% are concatenated between 80% and 90%.

[127] See Fisher (2001).

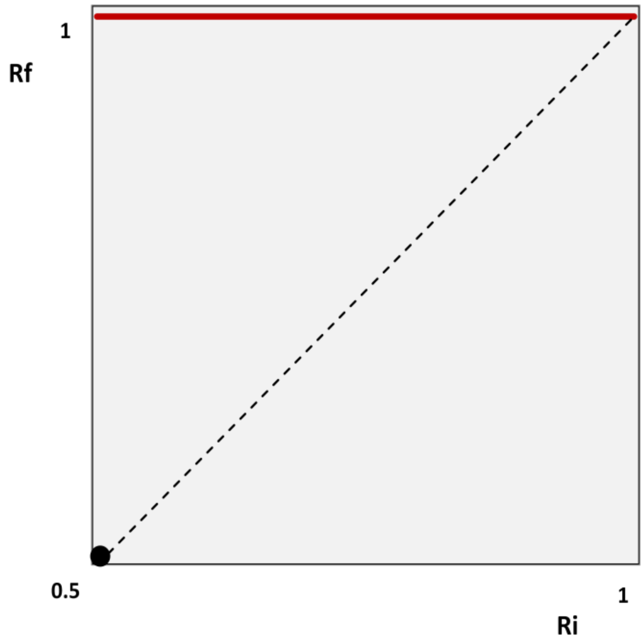


Figure 41 – Cox's hypothesis ruling out "wrong switching"

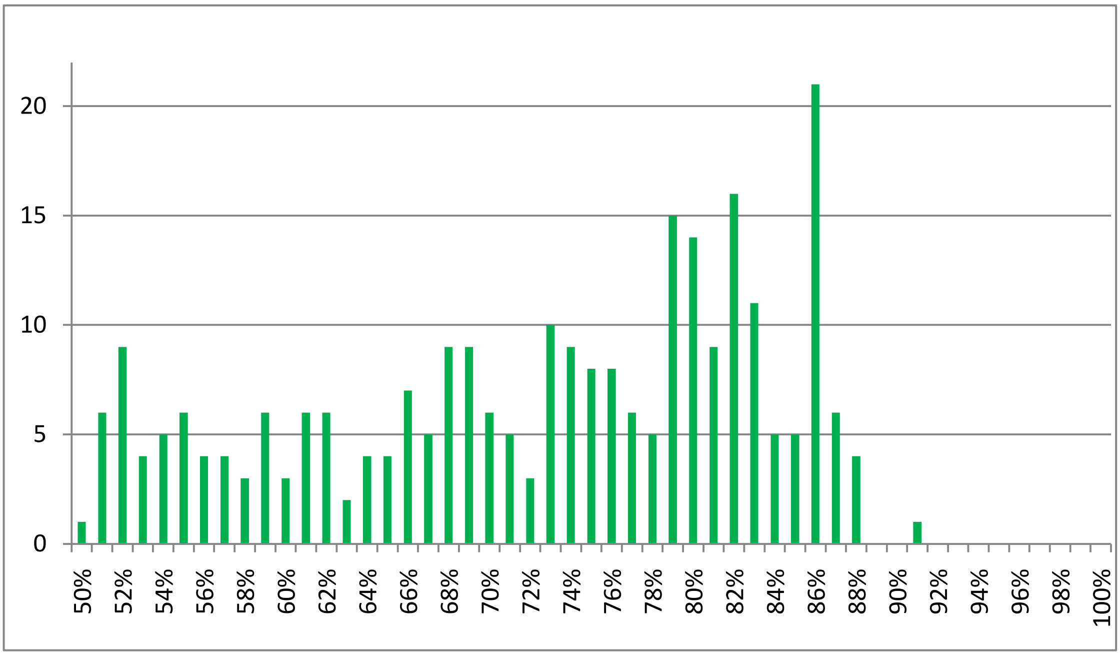


Figure 42 – Distribution of Rf in 1997 UK General Election

Cox's hypothesis does not seem to be supported by this election's data. There is a clear mode in the distribution and, yet, all other frequencies are more than just marginal "noise". Cox is not too strict in presenting his theory – he does point out that frictions in the real world would make the observed values differ from the theoretical predictions. In his own words:

> Allowing for some frictions in the model – e.g., some noninstrumental voters, some disagreement about which candidates are trailing and which are front-running – the prediction is softened. The SF ratio should either be close to unity (when second losers are so close in the polls to first losers that they do not lose their support due to strategic voting) or close to zero (when second losers are sufficiently far behind first losers that strategic voting kicks in and they are reduced to their non-instrumental support level, which I assume to be close to zero for most candidates). The SF distribution, in other words, should be bimodal. *Cox (1997)*

Our model provides some light to what may be behind a distribution that looks mostly uniform, but with a peak at very high values, though not immediately close to 1. To motivate the differences between ours and Cox's model, imagine a situation where everyone believes that one of the two contenders enjoys a prior support of 60% of the electorate that may vote strategically. Could there be a sincere equilibrium? In Cox's model, the answer is no. The fact that there is no real uncertainty in the model would only make it possible if voters had some non instrumental motivation, which they don't.

We argued in Part I of this thesis that having some real uncertainty is vital for *any* model. In that sense, the fact that we have *both* real uncertainty – albeit with public information, differently from Myatt (2007) – *and* expressiveness motivations really should count as one, since the former is imperative.

Our model formalises the two frictions mentioned by Cox in the above quote. The results we obtain are not just slightly different from his: they change the perspective from which you look at this problem. In a model with an expressive component, a sincere equilibrium is something that you can expect will occur, given the “attraction” that the sincere (expressive) vote has for the voters (which is, of course, not absolute). On the other hand, the Duvergerian outcome is not possible.

The predictions of our model in terms of the values of the Rf ratio that can be observed depend on the values of the parameters. A value of 1 is never possible, since each party must always get some votes. A value on the 45o line, where Rf = Ri, is possible for many combinations and represents a sincere equilibrium. This is more likely the lower the degree of coordination needed to defeat the incumbent and the higher the level of expressiveness. The effect of the level of information available is more complex. A more detailed comparative statics analysis will be done in section 7. Next we recap the main features of our model and explain the two types of equilibria it allows for.

3. The Model

A brief recap of the four critical variables in our model: they are λ, e/c, (a+b) and a/(a+b). λ stands for the support of the incumbent party (the Conservative Party, C); e/c is the ratio of the expressiveness benefit *e* over the altruistic-paternalistic concern for other voters *c*, as the electorate gets large; (a+b) reflects the level of information each voter (and the electorate, since all information is commonly known) has, and can be seen as a proxy for the size of an opinion poll whose results would be publicly known; a/(a+b) indicates the degree of asymmetry in the (known) distribution of the (uncertain) prior support of the contenders[128].

We restrict our attention to the eligible constituencies – where the incumbent gets a share of votes between 1/3 and 1/2. We assume the prior support of the Conservative Party to be equal to its observed support, i.e., we exclude the possibility of strategic voting from its supporters and towards that party.

In Table 13, we describe the correspondence of values of λ to the critical ratio $\lambda / (1 - \lambda)$. The values of λ with highest frequencies in the 1997 UK General Election – 34%, 38%, 41%, 45% and 48% – are values for which the ratio $\lambda / (1 - \lambda)$ is approximately 50%, 60%, 70%, 80% and 90%.

Label	λ	γ	$\sim\gamma$
Very low	34%	52%	50%
Low	38%	61%	60%
Medium	41%	71%	70%
High	45%	82%	80%
Very high	48%	92%	90%

Table 13 – Values for λ

In terms of interpreting the values for (a+b), and given that they arise from a Beta-distribution with parameters *a* and *b*, as explained in the previous essay, they refer to the size of hypothetical or real sample of people from whose stated opinions voters draw information about the whole electorate. Given that opinions in groups are not uncorrelated, samples

[128] In this essay their identity does not matter. In the next essay they are, since we look at the 1997 UK General Election, the Labour Party (L) and the Liberal Democratic Party (LD).

larger than 5, 10, 30 and 100 when opinions are correlated would be needed if we wanted to get a level of information corresponding to samples of sizes 5, 10, 30 and 100 of uncorrelated opinions. Regarding the sizes chosen, we can think of a size 5 as representing a level of information restricted to the closest relatives; a size of 10 could further include close friends; a size of 30 would allow for more acquaintances from work and other spheres; a size of 100 would be close to a larger scale opinion poll.

The uncertainty of the underlying distribution of the two contenders, which critically differentiates our model from Cox's, is incorporated in the model via *a* and *b*, the parameters of the relevant Beta-distribution. The asymmetry of such distribution is summarised in a/(a+b). None of the voters can know for sure what the leading contender's real underlying support, Ai, is. As uncertainty vanishes, namely through the sample size increasing, we know that Ri = Ai/(Ai+Bi) will approach a/(a+b). Both Ri and a/(a+b) will, by definition, be in the interval [0.5,1]. Since we exclude strategic voting towards the second contender, Rf > Ri.

The ratio e/c reflects both an (input-driven, non instrumental) expressiveness concern from the voters, *e*, and an (output-driven, instrumental) altruistic-paternalistic concern for the results of the election on the whole electorate, *c*, which we introduced in the previous essay. As described in Table 14, the parameter e/c can take six values. For instance, if e/c = 0.5, the expressive benefit of voting in our favourite candidate is half the value of the instrumental-altruistic concern for each member of the electorate if the less preferred candidate wins, as the electorate gets large. If e/c = 3, then the expressive benefit is three times larger than the instrumental-altruistic concern for other voters, and so on.

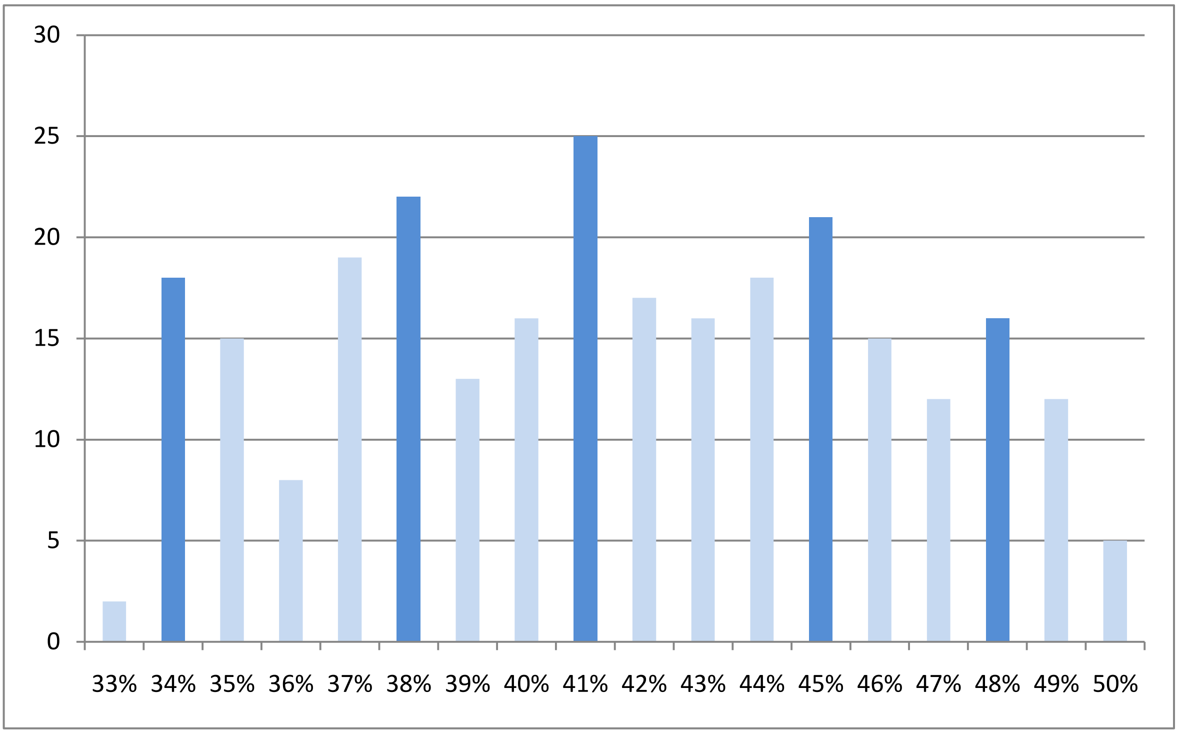


Figure 43 – Number of constituencies (Y) per value obtained by

The Conservative Party (X) in the 1997 UK General Election

Label	**(a + b)**
Very low	5
Low	10
Medium-low	30
Medium-high	100

Table 14 – Values for (a + b)

We observed (but do not include here) that for values of e/c = 10 or e/c = 100 it would be almost impossible to obtain an interior equilibrium, even for favourable combinations of the other parameters. The expressiveness component would become comparatively so large that no one would be inclined to vote strategically. Likewise, an e/c that is much lower than 0.01 (for instance 0.001 or 0.0001) would never allow for a sincere voting equilibrium, so it would also be uninteresting for our model, in the sense that it would not provide sufficient differentiation to models or purely instrumental motivations. It would, in that case, essentially be a model of pure instrumental motivations with some noise, with no real challenge to the known results coming out of such models, therefore of little interest. We include a range of values that allows us to explore the implications of a theoretically richer framework where instrumental and expressiveness concerns have a role to play, described in Table 15 below.

Label	**e / c**
Very low	0.01
Low	0.1
Medium-low	0.5
Medium-high	1
High	2
Very high	3

Table 15 – Values for e/c

4. The two types of equilibria

There are two types of equilibria in our model: a sincere equilibrium, where everyone votes for their preferred candidate, and an interior equilibrium, where some voters do not vote for their preferred candidate, choosing to vote strategically.

There is an equilibrium whenever there is a fixed point in the space (β^,β*)[129].β* represents the percentage of supporters of the trailing contender that vote sincerely. Thus, (1 – β *) is the percentage of strategic voting amongst those supporters. β~ is the "incentive ratio" for the voters to vote strategically[130]. Whenever β~ > β*, there will be some voters who would prefer to vote sincerely, making the percentage of sincere voters – in equilibrium reflected in β* – higher. Analogously, when β~ < β* some voters would prefer to vote strategically, making β* go down. Only when β~ = β* no one would wish to change their vote, and a voting equilibrium would exist. Recall that:

β~ = [e/c H(β∗) + f(αM)] / f(αL)

β^ = Min{β~,1}

We use β^ to simplify the analysis, since β* is in [0,1]. An equilibrium with only sincere voting is equivalent to β* = 1. (In which case Rf = Ri). Strategic voting implies and is implied by β* being lower than 1. For each combination of the four parameters in our model we have a situation that may allow for one or both of these equilibria. In the four diagrams below, we illustrate different possibilities that may arise within our model, with β* represented in the X-axis and β^ in the Y-axis.

In the case illustrated in the next page, in Figure 44, there is only one equilibrium, with sincere voting, given that the line B^ does not intercept the 45o line for any value of β* that is below 1. Consider the case where β* = 0.8. For that value, we see that β^ > β*. This means that if 20% of the supporters of the third contender voted strategically, this would generate an incentive ratio higher than 80%. Thus, all the voters that (hypothetically) voted

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[129] See Figure 33 in page 126.
[130] Recall that β^ = Min{β ~,1}.

strategically and for which βi < β^ would prefer to vote sincerely[131]. If they did, then B^ would still be above β*, and the adjustment would only end at β* = 1, a point where no strategic voting would occur.

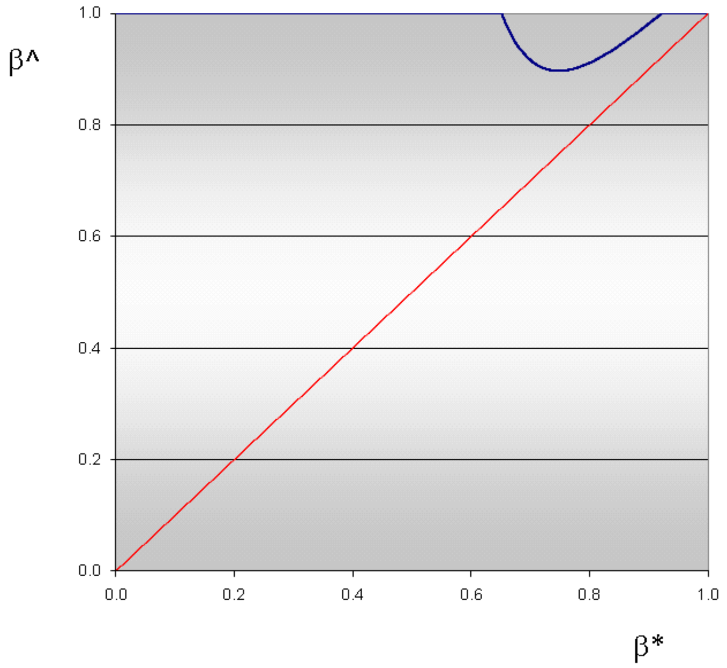

Figure 44 – Situation for λ = 38%, (a+b) = 5, e/c = 1, a/(a+b) = 70%

In the second case, illustrated in Figure 45, the only interception between β^ and the 45o line occurs for a value lower than 1, so we have a unique interior equilibrium.

A third case, illustrated in Figure 46, allows for three equilibria to take place: the sincere and interior stable equilibrium, plus an unstable interior equilibrium.

131 Where βi reflects the preference for the second favourite party (see A8, essay 1, page 5).

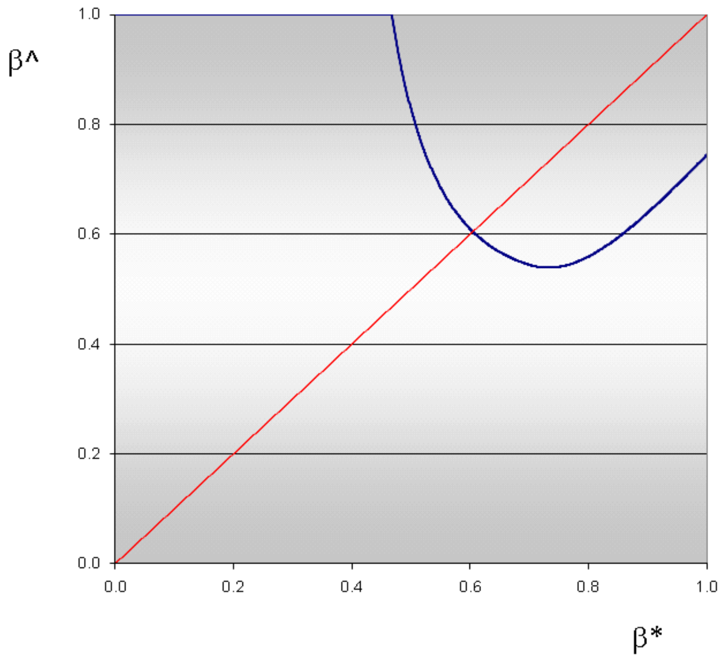

Figure 45 – Situation for λ = 41%, (a+b) = 5, e/c = 1, a/(a+b) = 70%

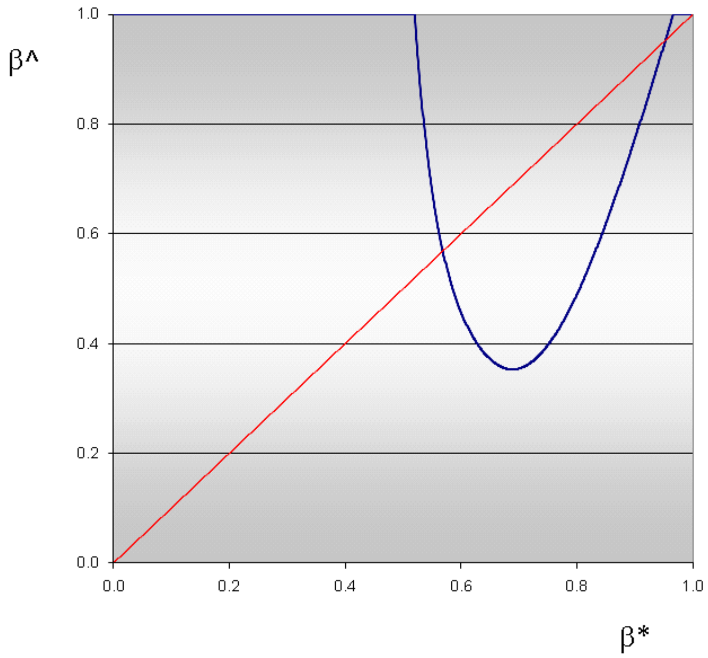

Figure 46 – Situation for λ = 41%, (a+b) = 5, e/c = 1, a/(a+b) = 55%

In Figure 47 we illustrate the unlikely case where three interior equilibria are possible, two of them – the ones where the 45o line is intersected from below – being stable, a case which makes clear the quasi-concave but not necessarily concave shape of the β^ function, which is related to the quasi-convex shape of the Beta-distribution function.

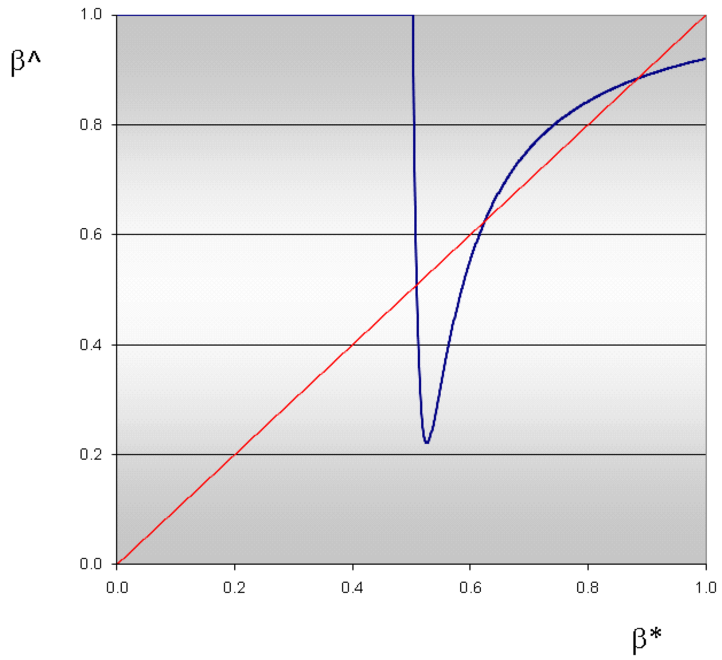

Figure 47 – Situation for, λ = 34%, (a+b) = 5, e/c = 0.01, a/(a+b) = 65%

Note that the above exercise asks a forward-looking question: given a certain set of parameters, what outcomes – set of possible equilibria, ratio of the observed support for the two contenders – can be generated from them? A related, backward-looking question would consist in asking, for a certain outcome observed – again, the ratio of the observed support the two contenders got –, what equilibria could underlie it and what prior set of parameters could have originated such outcomes (ratio and equilibria).

That question motivates the next essay, when we calculate the observed results and try to guess what may have happened in terms of strategic voting. In this essay our concern is mostly forward-looking, however, both perspectives will come up later in this essay.

5. Our model versus Cox's

Diagrammatical illustrations are useful to visualise and understand the differences between our and Cox's models. In Figure 48 below we illustrate the possible equilibria for a certain combination of the parameters in our model and for different values of Ri. The green line, γ, defines the level above which the leading contender would defeat the incumbent. The red line represents the interior equilibria. The black line depicts the sincere equilibria. The red dash line highlights the lowest value of Rf that allows for an interior equilibrium to occur. The black dash line highlights the value of Rf above which a sincere equilibrium is, in the margin, no longer possible (being again – and always – possible for higher values of Rf, closer to 1).

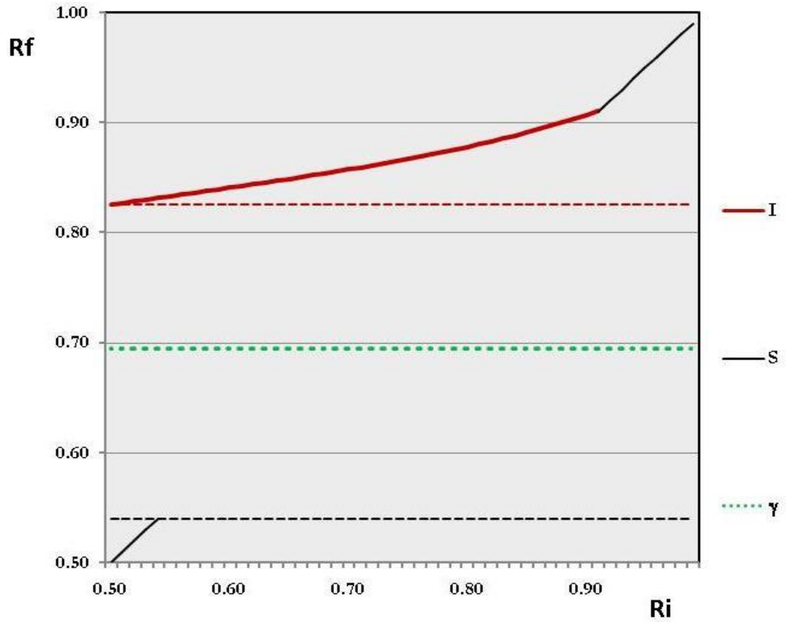

Figure 48 – The electoral situation for λ = 41%, (a + b) = 5 and e/c = 0.5

The main feature of our model is highlighted in Figure 49, representing the same electoral situation: the distribution of Rf takes values in two intervals, close to 0.5 and close to 1, with values of Rf in between the two intervals never being observed. The interior equilibria, along the red line, are not possible in Cox's model. The sincere equilibria, along the black line, are possible for many values of Rf in our model, but only for the knife-edge case Rf = 1 in Cox's model. Compared to Cox's strict prediction, where Rf can only take the values 0.5 and 1, as in Figure 41, our model is more permissive, allowing for values close to 0.5 and 1.

We see that whenever Ri is close to 1 the gap between the first and second contender is very large and voting sincerely is the only choice that can lead to a voting equilibrium. A strategic vote would, in this situation, be extremely unlikely to change something, so securing some utility by voting expressively is wiser. The higher the level of the incumbent, the more critical will a strategic vote be, so we expect the interval concerning the values of Ri that make sincere voting possible to shrink. This will be clearer in the comparative statics section.

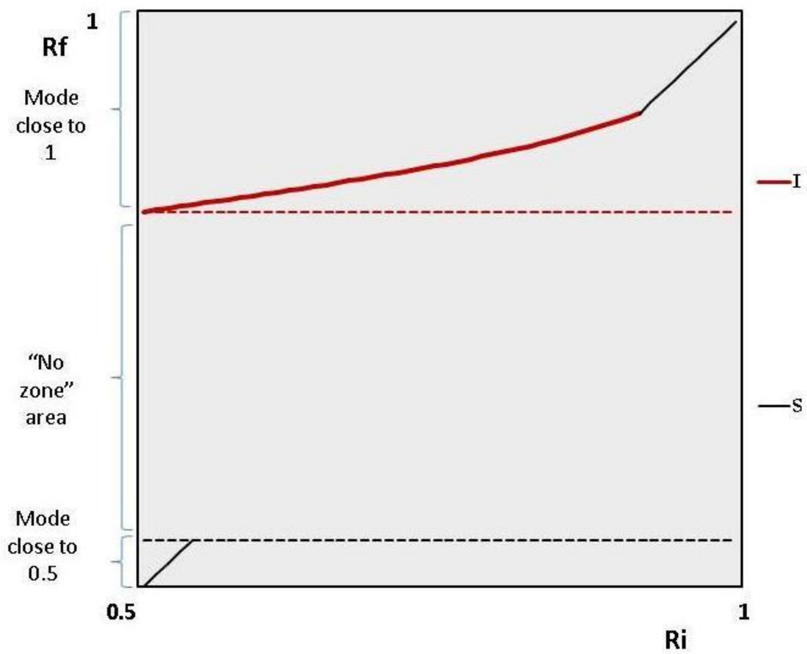

Figure 49 – Illustrating the "extended" bimodality hypothesis

Figure 49 also suggests that, if our model is correct, any combination of parameters will create a (different) “no zone” area, where no values of Rf should be observed. In which way? To answer that we must explore the connection between Ri and Rf in equilibrium.

Recall that:

Rf = Af / (Af + Bf)

Ri = Ai / (Ai + Bi)

Af = Ai + (1 – β*) Bi

Bf = β* Bi

From these we obtain the following expression for Rf:

Rf = [Ai + ((1 – β*) Bi] / [Ai + Bi] = Ri + [(1 – β*) Bi / (Ai + Bi)]

The above expression simplifies to:

Rf = Ri + (1 – β*) (1 - Ri) = 1 - β* + β* Ri

This can be further simplified to two equations, one focusing how Rf changes with β*, (for an implicit Ri); the second one focusing on how Rf changes with Ri (for an implicit β*)[132]. They are:

Rf = 1 + (Ri – 1) **β***

[132] The “implicit” coming from the fact that we’re depicting a relationship in equilibrium, and given the dependency of β* on Ri.

Rf = (1 – β*) + β* **Ri**

Strategic voting is related to the vertical distance between the red line and the 45o line, as illustrated in Figure 50, using green arrows. However, we can not simply say that when such vertical distance increases strategic voting increases. The vertical distance is equal to [Rf – Ri], where both are equilibrium values.

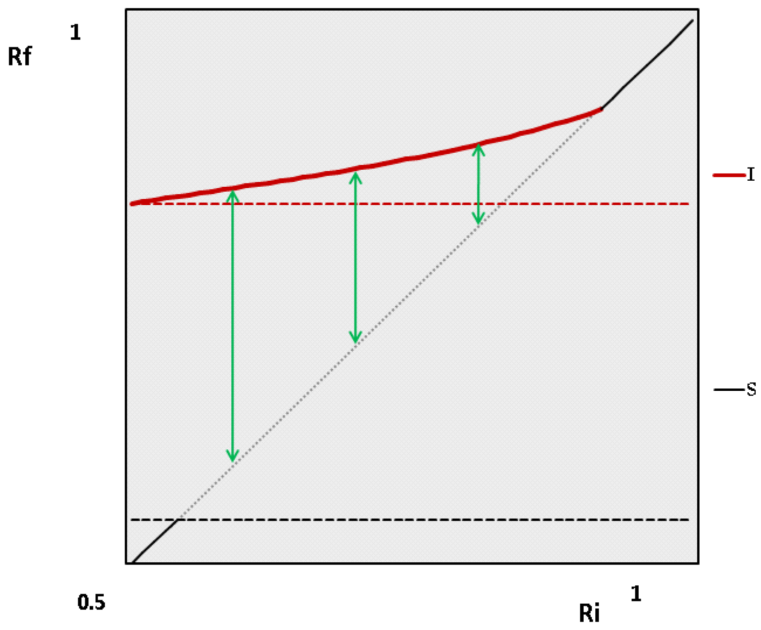

Figure 50 – Strategic voting in (Ri, Rf) axes

From the previous equation, we obtain **[Rf – Ri]** = 1 – β* + β* Ri – Ri, or:

[Rf – Ri] = (1 – β*) (1 – Ri)

As Ri decreases, (1 – Ri) increases, and so (Rf – Ri) will increase, as long as (1 – β*) does not decrease too much. A better way to look at this is to note that:

dRf / dRi = β*

In Figures 49 and 50 we see that the slope increases very marginally as Ri goes up, indicating higher strategic voting when Ri is little, that is, when the advantage of the first contender is not too large. Such slope increases as Ri increases, suggesting that strategic voting decreases as the prior advantage of the first over the second contender decreases. This makes sense since the closer the two contenders are the higher the strategic voting would need to be to make it optimal not to vote sincerely.

The comparative statics section makes it clear that, apart from a few exceptions, strategic voting decreases as Ri increases. Looking at Figure 45 in page 164, we can see that as Ri increases (which is to say, as a/(a+b) increases[133]), the incentive ratio function moves to the right. Such movement would continue until strategic voting decreases in the interior equilibria up the point where it becomes zero – making the sincere equilibrium the only possible outcome.

What types of equilibria can we observe, and how do they depend on the levels of Ri and Rf? We answer this question in Figures 51 and 52, which give, respectively, a forward-looking and a backward-looking perspective on the voting equilibria for the following parameter values: $\lambda = 41\%$, $(a + b) = 5$ and $e/c = 1$.

The forward-looking perspective starts by focusing on Ri and asks, for each Ri, what type of equilibria can arise from it. This is the approach underlying Figures 44, 45, 46 and 47: there we plug in the values for the four parameters of our model and see whether an interior and/or a sincere equilibrium is/are possible; in Figures 51 and 52 we vary Ri – which is equivalent, in this exercise, to varying a/(a+b) while keeping the other three parameters fixed –, implicitly find what B* are possible, then plug them in the equation $Rf = (1 - \beta^*) + \beta^* Ri$ and plot the corresponding Rf.

[133] The relationship between Ri and a/(a+b) will be clearer in section 7 of this essay.

As we can see from Figure 51, for very low levels of Ri both types of equilibria are possible. For Ri close to one, only the sincere equilibrium is possible. For values in between, only the interior equilibrium is possible. We see that from a forward-looking perspective both equilibria are only simultaneously possible for Ri close to 0.5[134]. The intuition is that if both contenders have a prior support that does not differ too much, a sincere equilibrium is possible, but strategic voting can be sustainable too, if sufficient voters switch their votes (recall that strategic voting decreases – in proportion – as Ri increases, along the red line). The interior equilibrium remains possible until Ri is so high that only the sincere equilibrium can be part of an equilibrium, as the chance of making a difference, instrumentally speaking, is not enough to counter the expressive benefit immediately attainable by a sincere vote.

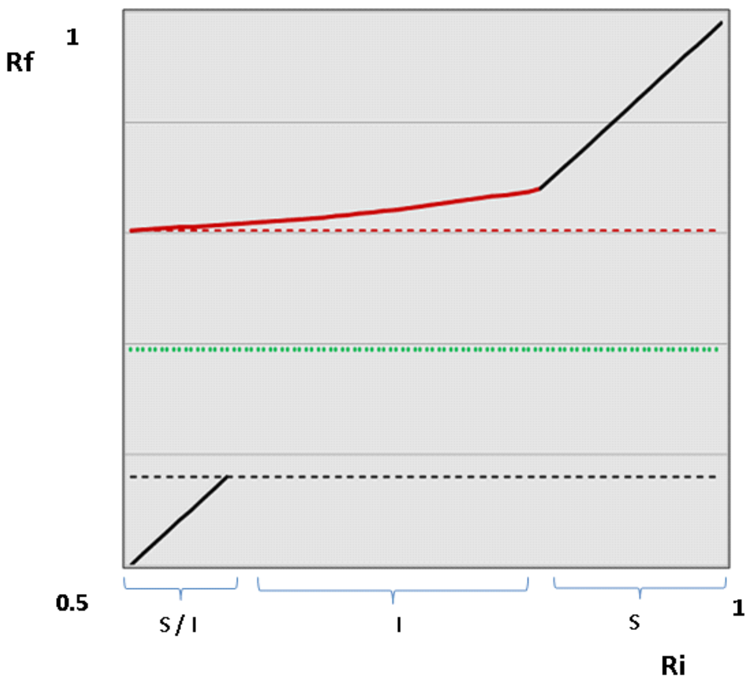

Figure 51 – Equilibria from a forward-looking view [λ=41%,(a+b)=5,e/c=1]

[134] The figures chosen are – qualitatively speaking – representative of the dynamics of our model. A quick look at the comparative statics section, which shows 120 diagrams for different combinations of parameters, makes this clear. The differences in slope (for the interior equilibria curves) and in the range of values allowed for Rf changes, but the overall "portrait" is – apart from a few exceptions – the one we referred to in here.

Using a Excel simple program to confirm the existence of an equilibrium and to calculate, when it exists, the interior equilibrium for a certain combination of the parameters, we confirm that for a/(a+b) = 55% (which is to say, for Ri = 55%) in our diagram, both equilibria are possible, the interior equilibrium involving β* = 46,3% and leading to Rf = 80.6%. If a/(a+b) = 65%, only an interior equilibrium with β* = 56.5% and Rf = 81.3% is possible. Note how Rf increases only marginally, despite Ri going up 10 percentage points.

This is clear from the diagram and happens because the proportion of sincere voting goes up from 46.3% to 56.5%. So strategic voting decreases in a way that prevents Rf from taking more values – the reason why there is a "no zone" area in our diagrams. The reason for this is that when one contender's prior advantage goes up the incentive to vote strategically goes down, and these opposite effects almost cancel each other in terms of Rf, restricting its possible values to a smaller interval. If a/(a+b) = 75%, then β* = 69.7% and Rf = 82.6%. If a/(a+b) = 85%, the interior equilibrium is no longer possible.

The backward-looking perspective involves asking, for a certain Rf, what pair or pairs (β*, Ri) could have generated it, so we look for what types of equilibria may give rise to an observed Rf. This is the approach we will be using in the next essay to guess what could underlie the observed results in the 1997 UK General Election.

From Figure 52, we see that a Rf close to 0.5 can only arise from a sincere voting pattern. Likewise for a Rf close to 1. For high values of Rf, sufficiently away from 1, Rf would arise from an interior equilibrium. Generally, only one equilibrium is possible from a backward-looking perspective for any Rf, but some exceptions exist, as illustrated in Figure 53. A few more cases can be found by inspecting Figures 55 to 59.

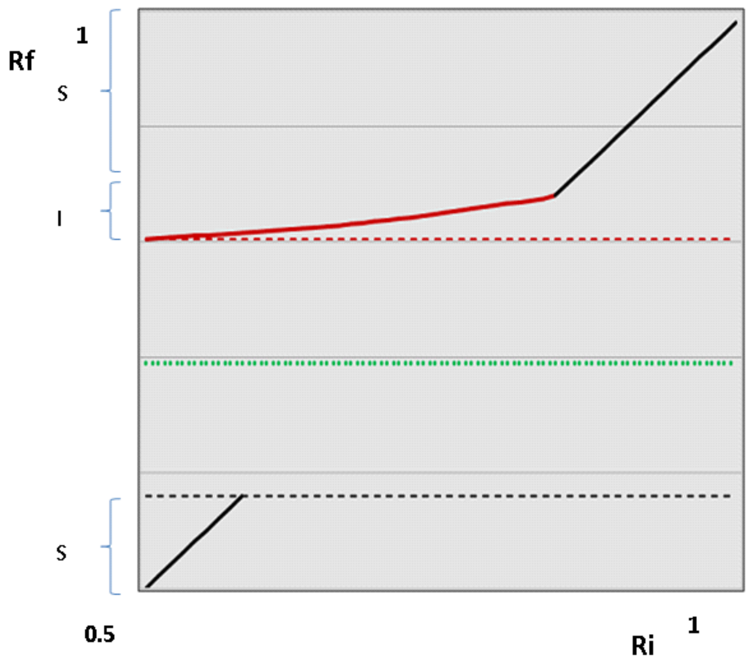

Figure 52 – Equilibria from a backward-looking view [λ=41%,(a+b)=5,e/c = 1]

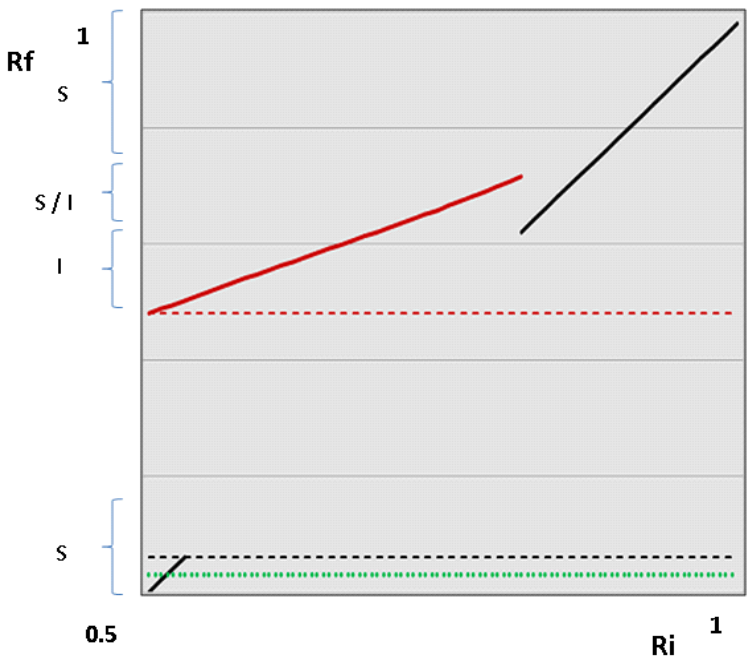

Figure 53 – Equilibria from a backward-looking view [λ=34%,(a+b)=10,e/c=0.1]

6. Success of strategic voting

A natural question to ask is whether and when is strategic voting successful, assuming voters are rational. In Figure 54 we highlight the two relevant areas that dictate whether coordination amongst the voters who dislike the incumbent was successful or not in defeating it. Recall that the green line represents $\gamma = \lambda / (1 - \lambda)$, the qualified majority that one of the contenders needs to achieve in order to win the election. That happens when Rf > γ, otherwise the incumbent wins.

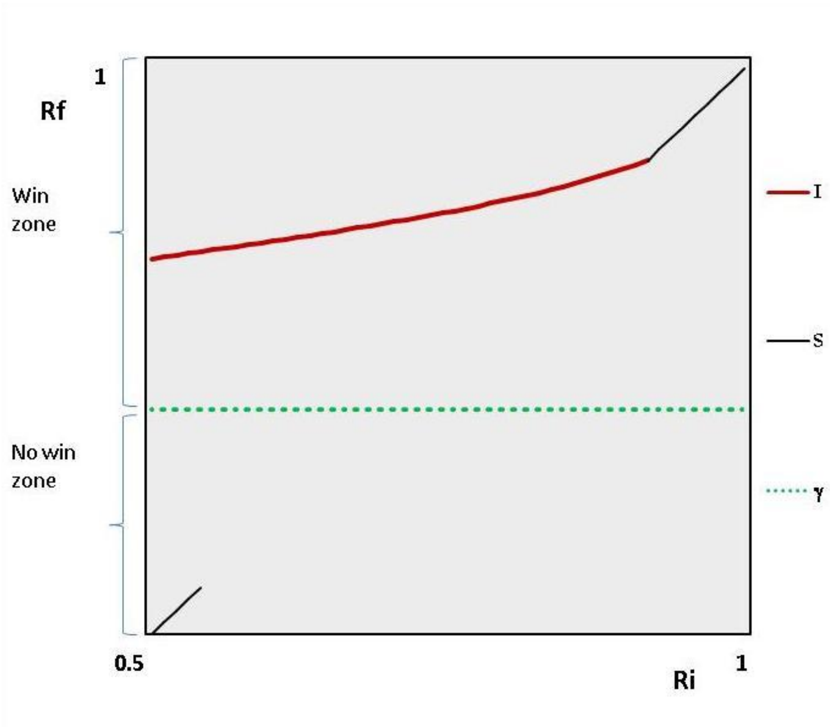

Figure 54 – Winning and no winning zones for the leading contender

In the case above whenever there is strategic voting in equilibrium the contender wins. When sincere voting occurs in equilibrium this may or may not be the case: it depends on the value of Ri. So is it true that it is always best to vote strategically, since this guarantees that the incumbent is defeated? The answer is no – mostly because the question is wrongly constructed. The unit of analysis in our model, as in most models of voting, is the individual.

When an individual voter decides, he only equates his individual costs and benefits of voting, instrumental and/or expressive; whatever may be relevant for him or her[135]. Looking at Figure 54 and seeing whether an equilibrium with strategic voting leads to the incumbent loosing the election involves doing something from a different perspective: the aggregate result of all the individual decisions being made.

The rationality of the individual choice comes before that. Whenever both equilibria are possible outcomes – which requires Ri to be sufficiently close to 1 – each voter must choose whether to vote strategically or not. If the possible interior voting equilibrium involves a value of β*, we know that voters who have an individual preference parameter βi < β* will in any case vote sincerely. For voters who have a higher instrumental utility regarding the victory of the second contender the option to vote strategically is not immediately dismissible.

We can look at the utilities the voters obtain from an ex-ante and an ex-post point a view when both equilibria are possible (for sufficiently low Ri), where we restrict the ex-post point of view to be one where a voting equilibrium takes place. Ex-post, we know that a sincere equilibrium, in the case where two equilibria are possible, is not successful in terms of defeating the incumbent, so the ex-post utility is equal to *e*, while the ex-ante utility is *e* + *c*.Pr(B,.), where *e* reflects the expressiveness benefit, *c* the altruistic-paternalistic concern, Bi is the individual preference parameter and Pr(B,.) is the probability of a pivotal event involving the trailing candidate (labelled as "B", by definition) and the incumbent.

The ex-post utility of voting strategically, given that an equilibrium with strategic voting must lead to the incumbent being defeated, is *c*.βi, while the ex-ante utility is *c*.βi.Pr(A,.),

[135] In our case they include an altruistic-paternalistic concern – *c* – but they still decide as individuals, not as groups or teams, as, according (for instance) to Michal Bacharach – see Bacharach (2006) – may be relevant in some social situations.

where βi is the individual preference parameter and Pr(A,.) is the probability of a pivotal event involving the leading contender and the incumbent.

The rationality of a vote is *exclusively* related to the optimality of voting that way from an ex-ante perspective. The voter needs to assess the expected utility of each vote and decide accordingly, as we explained in the previous essay. The condition for a sincere vote to be optimal is:

$$e + c \, \mathbf{Pr(B,.)} > c \, \beta \mathbf{i} \, \mathbf{Pr(A,.)}$$

This simplifies to:

$$e > c \, [\beta \mathbf{i} \, \mathbf{Pr(A,.)} - \mathbf{Pr(B,.)}]$$

A sincere vote, from an ex-post perspective (and, again, in the particular case where both equilibria are possible), is better than a strategic vote when:

$$e > c \, \beta \mathbf{i}$$

We can rearrange the expression regarding the ex-ante utility as:

$$e > c \, \beta \mathbf{i} + c \, \{\beta \mathbf{i} \, [\mathbf{Pr(A,.)} - 1 \,] - \mathbf{Pr(B,.)}]\}$$

Since Pr(A,.) is lower than 1 and Pr(B,.) can not be negative, the expression between brackets must be positive, as is *c*. So, whenever a sincere vote is better than a strategic vote from an ex-post point of view, it is also better from an ex-ante point of view (which intuitively makes sense), but not the other way around. It is possible that a sincere vote is more attractive from an ex-ante perspective even if ex-post that is not the case.

One key thing to keep in mind is that we are not and we *can not* say that strategic voting is better because it always leads to a victory – it only leads to a victory when it happens in equilibrium, and for it to happen there are a lot of conditions that must be met. The rationality of each voter's choice must be checked ex-ante. Another essential thing not to be forgotten is that uncertainty is only solved in the end – Ri is not known until after everyone voted.

In Figure 55 below we illustrate a rare example where strategic voting in equilibrium may or may not involve a defeat of the incumbent, depending on the value of Ri. Since, as we just said, Ri is uncertain, so is the defeat of the incumbent when a strategic voting is the outcome of the election. The fact that Ri is uncertain means that ex-ante voters do not know where they are in that diagram.

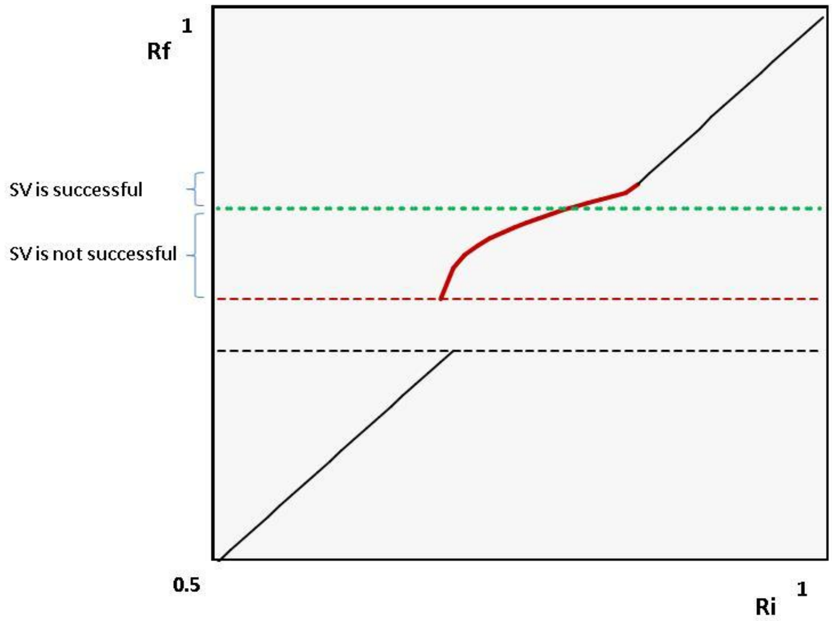

Figure 55 – Electoral situation for λ = 45%, (a+b) = 5, e/c = 2

In Figure 55, if some of the voters vote strategically, resulting in an equilibrium with partial strategic voting, they are not sure that the incumbent is defeated – in any case the rationality of their vote is assessed before this is known. In Figure 54, differently, they do know that if a strategic voting is the outcome the incumbent does not win the election.

All voters know λ, e/c and (a+b). Voters also know a/(a+b) – in other words, they know *a* and *b* –, that determines the shape and asymmetry of the distribution of α, which gives the proportion of the leading contender's supporters amongst those who dislike the incumbent. This is, of course, Ri. Ri is restricted to values between 0.5 and 1 but this does not influence the Beta-distribution, which can take any value between 0 and 1. If we label the parties as – say – L and M – as in the previous essay –, and assume strategic voting occurs towards party L, we know that the proportion of L supporters can vary between 0 and 1. It's only after that that we decide that we will only focus our attention in the case where there is no "wrong" switching, that is, in the case where L is the candidate with highest prior support. This is not the same as truncating or conditioning the initial distribution by imposing that $\alpha > 0.5$. Simply, we are only looking at the cases where that happens, without imposing that such is always the case.

So, even though a/(a+b) is the best estimate for Ri, Ri can take many more values. When we use a/(a+b) as a proxy for Ri, as we did before, it is because we need a value for Ri to include in a formula and a/(a+b) is the best choice. But note that Ri is not known for sure until all votes are counted. It is already determined, of course, but it is unknown. It is not that the proportion of "true supporters" of each of the contenders may change; it's just that it is not known with complete certainty – only its distribution is known, even if the draw has already taken place.

What we can say, regarding the interior equilibrium – in a situation like that illustrated in Figure 54, which is representative of almost all cases –, is that if it did happen in equilibrium, the incumbent must have been defeated. It is a conclusion conditional on having fulfilled the equilibrium conditions, and not a recommendation for individual voters – or a group of them – to vote strategically in order to achieve such outcome.

Our approach, which focuses on studying the possible outcomes for a particular voting is common in the theory of voting and theory of games, prioritising the description of the possible outcomes over the evaluation of how they may arise – in the "real world"[136].

7. Comparative statics

In essay 2 we looked at how strategic voting – summarised in B* – changes as the parameters in the model change. In this essay we look at how Rf changes – or, perhaps better, how Rf (Ri) changes in equilibrium – when the parameters change.

In our model the bimodality hypothesis is tantamount to saying that some values of Rf will never be possible – corresponding to the "no zone" area introduced in Figure 49. The sets of such (impossible) values vary according to the parameters we start with, which makes a comparative statics analysis interesting. One of our focuses will be looking at how the boundaries of this no zone change as the parameters change.

We arrange all the information in five different tables, each corresponding to the five different values of λ we highlight: 34%, 38%, 41%, 45% and 48%. In each table there are

[136] Though an interesting topic, it is beyond the scope of this thesis. The interested reader is referred to the work, amongst others, of Thomas Schelling, Michael Bacharach and Robert Sugden in stressing the importance of context, tradition, social conventions, framing, team-reasoning, bounded rationality and other aspects that challenge the framework of "classical" game theory, namely its assumptions of methodological individualism, unbounded rationality and context -independence (or "–irrelevance").

four rows and six columns, resulting in 24 entries per table. The rows indicate the level of information – (a+b) takes the values 5, 10, 30 and 100 – and the columns give us the expressiveness concern – e/c varying from 0.01 to 0.1, 0.5, 1, 2 and 3. Overall there are, therefore, 120 individual diagrams presented in five tables.

Some comments are valid for all the 120 individual diagrams. First, whenever Ri = 0.5, we confirm that a sincere equilibrium is always possible[137]. In some cases, an interior equilibrium will be possible too. We also know that if, for a certain combination of parameters, an interior equilibrium is possible for some Ri*, it will also be possible for all Ri < Ri*, in turn implying that it is possible for Ri = 0.5, the symmetric case. Recalling the equation Rf = (1 – β*) + β* Ri = 1 - β* (1- Ri), we see that, for any given Ri, the higher the proportion of strategic voting (the lower the β*), the higher Rf is, generating a gap between the interior equilibrium line and the 45o line, implying that some values of Rf will not be possible.

When Ri is close to 1 a sincere equilibrium will be the only possible equilibrium. The reason for this is that the support of the trailing contender is so low that a few strategic votes would not be sufficient to critically change the relevant likelihoods of having a pivotal event. Given the expressiveness concern, it would be better to stick with voting sincerely. As Ri goes down, the support of the trailing contender becomes more relevant and strategic voting can make a bigger difference, so an interior equilibrium eventually becomes possible. In many cases, further to the symmetric case Ri = 0.5, a sincere equilibrium will also be possible for values of Ri close to 0.5.

In short, sincere equilibria can occur for values of Ri in [o.5,x] and in [y,1], with y > x, (y < 1 and x ≥ 0.5). Since in a sincere equilibrium Ri = Rf, the same intervals apply to Rf. How *x*

[137] A result derived in essay 2 (see page 118).

and y change with the parameters will be analysed later in this section. If $x = y$, the sincere equilibrium is possible for all values of Ri and in general no interior equilibrium exists, but there are a few exceptions to this, when both equilibria are possible for the same Ri. If $x \neq y$, a sincere equilibrium is not possible in]x,y[, implying that an interior equilibrium must exist in that interval, which in turn makes it also possible for all values of Ri in [0.5,x] – and, therefore, overall in the interval [0.5,y]. As for Rf, in all sincere equilibria Rf = Ri, so the range of values for which it is possible follows Ri. In any interior equilibrium Rf > Ri, so Rf will never be equal 0.5. In general there is continuity at Ri = y, implying that Rf = y for the interior equilibrium (and for the sincere equilibrium, by definition). Again, there are a few exceptions to this[138].

In Table 16 we collect the diagrams for λ = 34%, the case where the need to coordinate is the lowest possible in our game, given the restriction that $\lambda > 1/3$[139]. The lower need of coordination to defeat the incumbent intuitively makes us suspect that the sincere equilibria will be possible for a wider range of values of Ri than it happens with higher values of λ, which we confirm by comparing this table to the remaining four.

For (a+b) = 5, the sincere equilibrium is the unique outcome for e/c in {0.5, 1, 2, 3}. If e/c = 0.1, we can have some interior equilibria. Note that a red line close to the 45o implies very little strategic voting taking place. For e/c = 0.1, we see that, from a backward-looking perspective, it is possible that one outcome may have originated from two different interior equilibria, since an horizontal line would cross the red line twice for some observed values of Rf. This is a very exceptional case, clearly influenced by the fact that λ is so low.

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[138] See the case where λ = 38%, (a+b) = 10 and e/c = 2 in Table 18 (page 187).

[139] Approximated to the percentage unit.

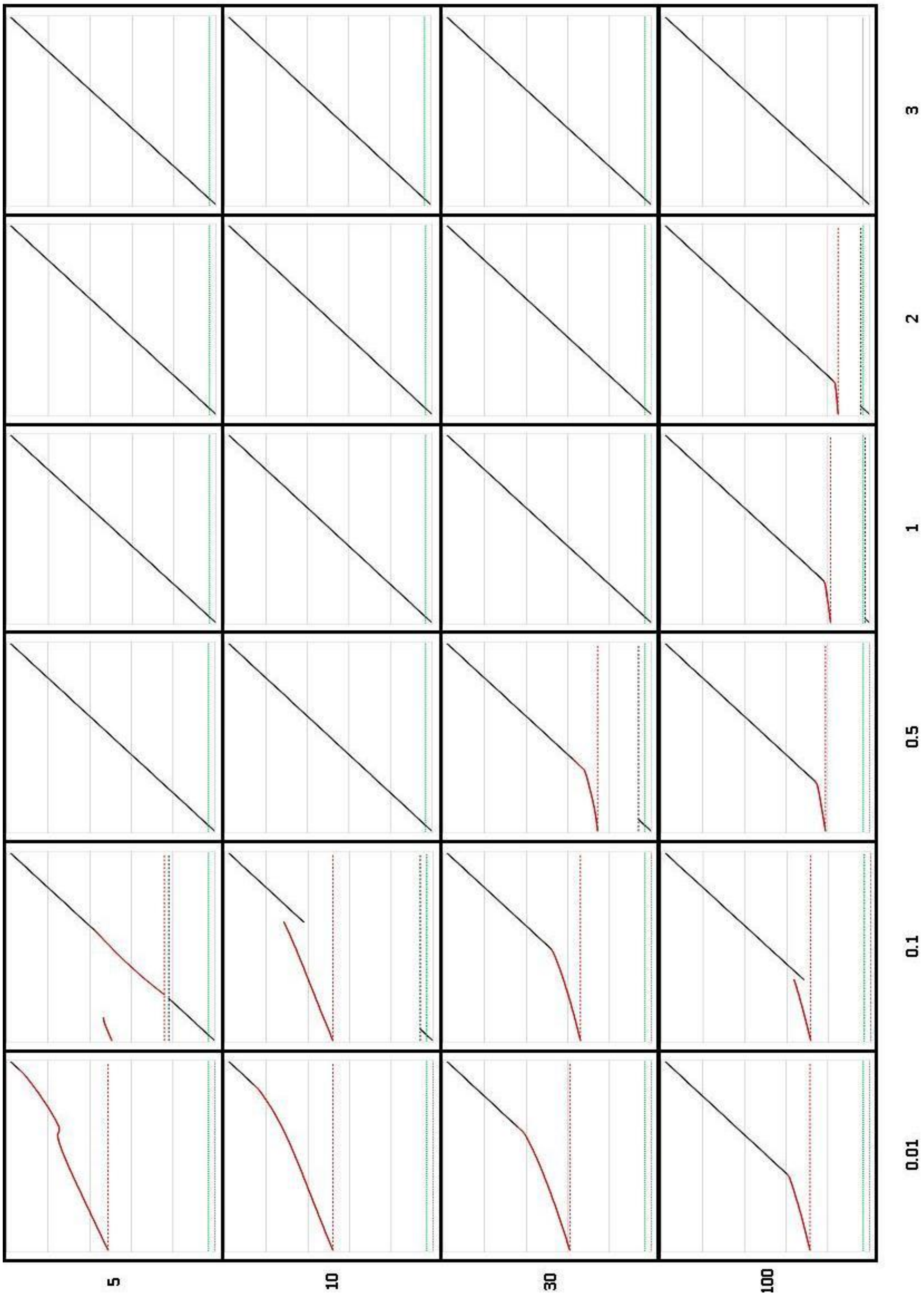

Table 16 – λ = 34%

When e/c = 0.01, a sincere equilibrium is only possible for very high levels of Ri. In this case, the degree of expressiveness is so low that only a very comfortable advantage in terms of prior support from one of the contenders can allow for sincere voting to take place. Note how low the green line is in this diagram: it is very easy for the leading contender to win this election.

If we look at the column e/c = 0.01, we see that as information gets larger, and for values of Ri close to 1, a sincere equilibrium is possible for a wider range of values of Ri. This makes sense: for a certain high value of Ri – implying that the third contender has a low chance of winning –, more information means that the voters are more certain that the leading contender has a clear advantage over the trailing contender and therefore a sincere equilibrium can be more easily supported.

If the information is low, then the voter is not so sure that the leading contender is a convincing winner. Compared to a situation where information is higher, a voter reaps the same benefit from voting sincerely, but a strategic vote becomes more appealing, since there is a higher chance that such vote would be pivotal. In the previous essay we saw that as the information increases, the function β^ (β*) becomes steeper, making a sincere equilibrium more likely[140]. Less information means that such function is more "spread out", making the sincere equilibrium more difficult to occur. This intuition applies to all columns in the diagrams.

Reading the rows, we see that as e/c increases the sincere equilibrium can occur for a higher range of values of Ri. This is expected, since the stronger the benefits of a single sincere vote – all else constant – the more likely each voter is to vote sincerely, and thus the higher

[140] See Figure 36 (page 133).

is the range of values leading to such type of equilibrium. This also applies to the four tables to come.

Some other aspects we will want to look at are the range of values of Ri close to 0.5 for which a sincere equilibrium is possible; the range of values of Rf for which there is no equilibrium; the range of values of Rf for which there is an interior equilibrium. In other words, we will be interested in knowing how the intervals in Figures 51 and 52 are affected by the variables of our model.

For λ = 38%, and comparing to λ = 34%, the sincere equilibrium is the unique equilibrium for a much lower number of situations: only for e/c = 3 and (a+b) =5 or (a+b) = 10 and for e/c = 2 and (a+b) = 5 is that the case. The green line goes up considerably and the range of situations where the incumbent is defeated decreases, as we would expect.

For any of the 24 diagrams in Table 17, and comparing to each equivalent diagram in Table 16, the interior equilibrium is possible for a larger range of values of Ri and, for each value of Ri, it implies a higher value of Rf, comprising, in other words, a higher level of strategic voting (a lower β* for each Ri). Again, this is expected, particularly if we take into account the comparative statics analysis from the previous essay.

All else constant, an increase in λ implies that, whenever there is an interior equilibrium, the level of strategic voting is higher. This is necessary to hold it optimal to vote strategically in equilibrium. If strategic voting hadn't change, for an equal λ, the relevant ratio would make it more attractive to vote sincerely. Only by having a higher percentage of supporters of the trailing contender voting strategically can this ratio change in the direction of equilibrium – "compensating" for what changes when λ increases.

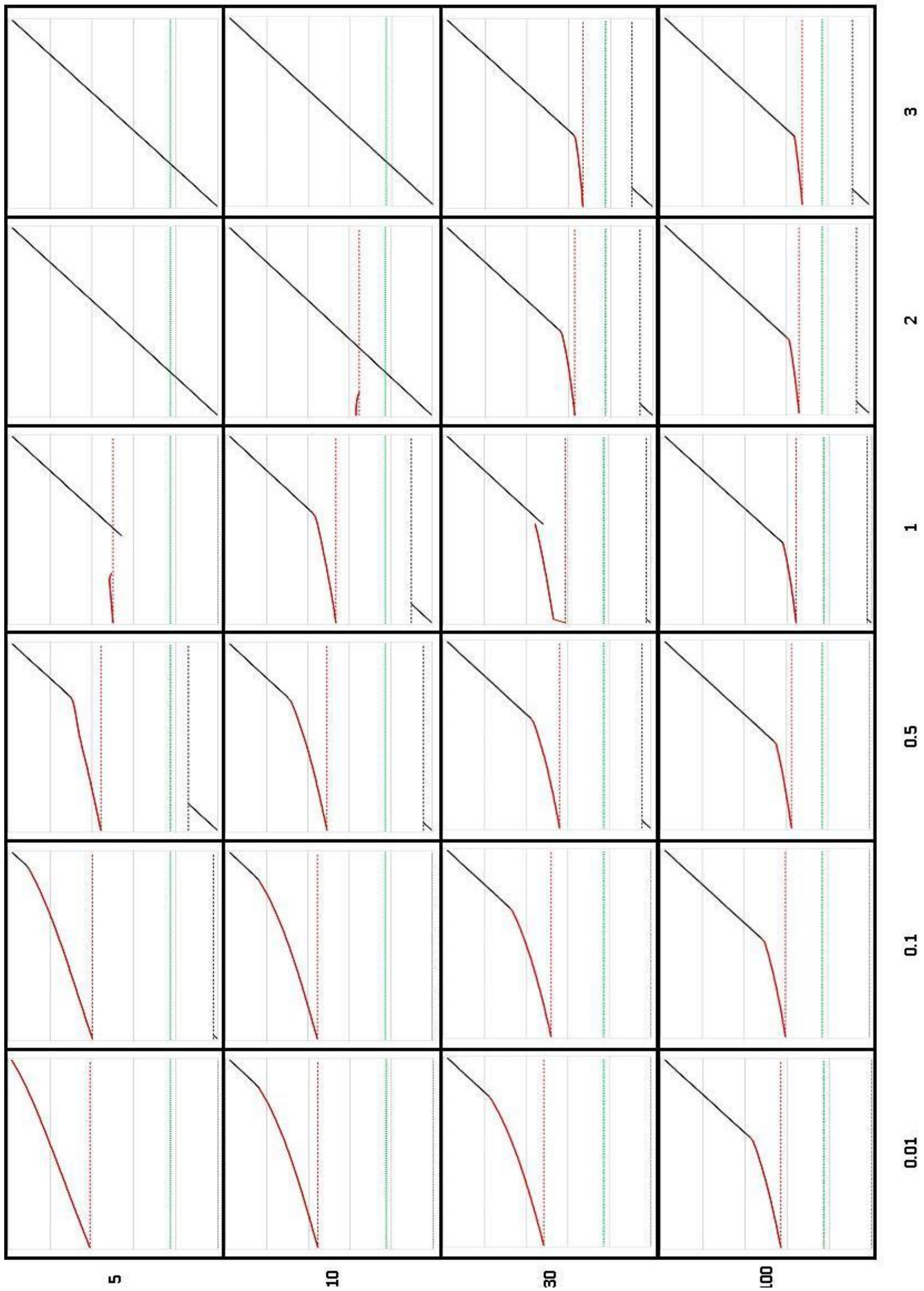

Table 17 – λ = 38%

The fact that an interior equilibrium is possible for a higher range of values of Ri is also related to this. Consider a high “turning value” of Ri for λ = 34%, where an interior equilibrium stops being possible, where the sincere equilibrium becomes the unique optimal choice (the right end of the red line in any of the individual diagrams in Table 17). If λ increases, all else constant, the sincere equilibrium is not possible anymore, as marginally the trade-off between voting sincerely and strategically shifts towards voting strategically. So the red line extends to higher values of Ri.

This effect is, as we would expect, valid for higher values of λ, as we can observe from Tables 18, 19 and 20 in the following pages. As λ increases, in each analogous diagram the red line expands upwards and to the right.

An implication of this, along with what we mentioned about the role of information, is that for higher values of λ and low levels of information, only the interior equilibrium is possible for any values of Ri apart from Ri = 50%, when a sincere voting pattern can always be optimal. In terms of empirical implications, the suggestion is that wherever the incumbent enjoys a great support and the level of information is low, any misbalance between the support of the two contenders will lead to a pattern of strategic voting – and one where the share of supporters voting strategically is higher the larger is the advantage of the incumbent candidate.

We also observe that, for each λ, the range of values of Ri in the interval [o.5,x] that allow a sincere equilibrium to take place increases as e/c goes up and, all else constant, it increases as λ increases. Combined with the previous comment on the effect on the interior voting line, this implies that the “no zone” shifts upwards. The effect on the range of values that are not observable for Rf varies with the parameters, but both boundaries of its interval go up.

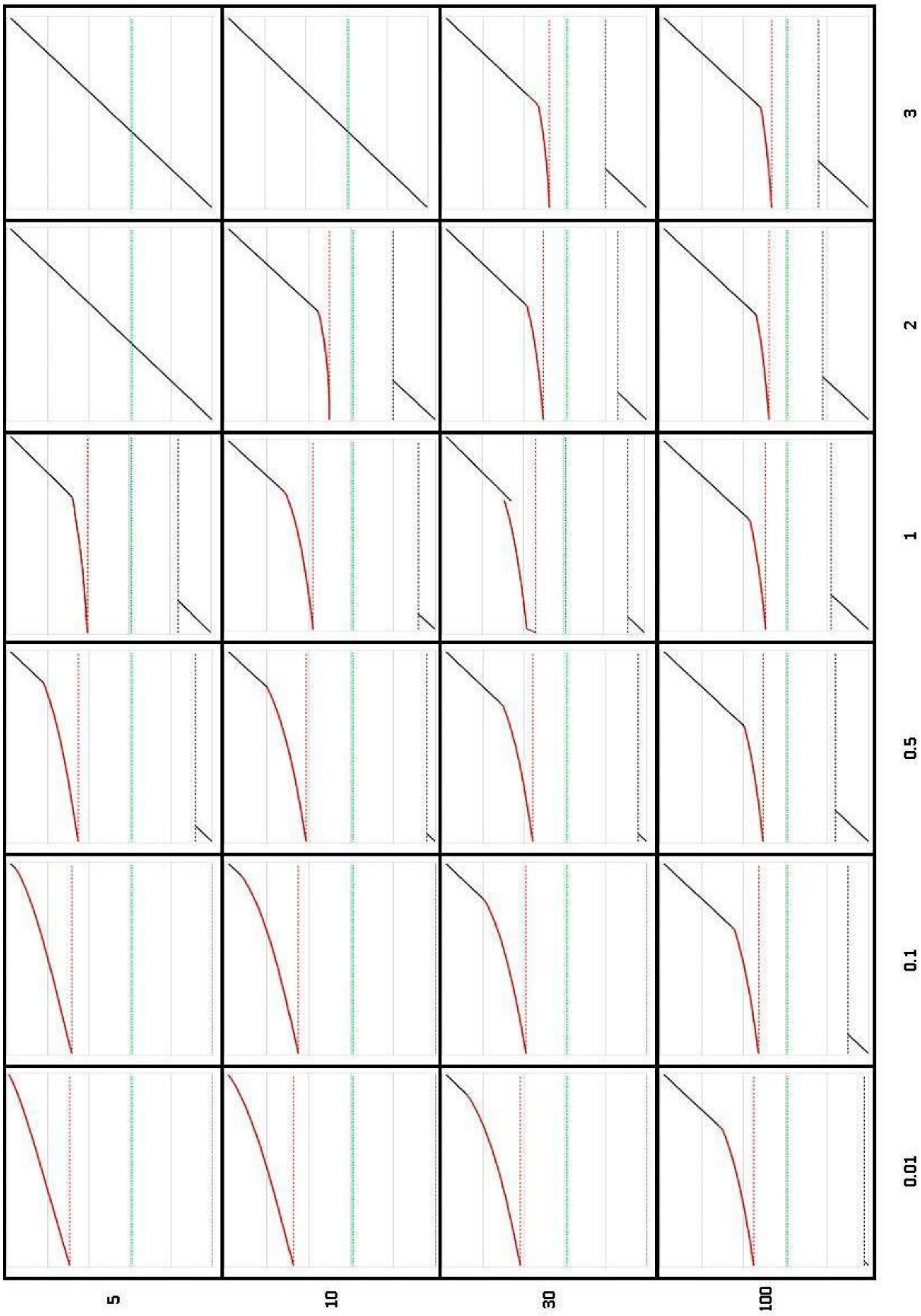

Table 18 – λ = 41%

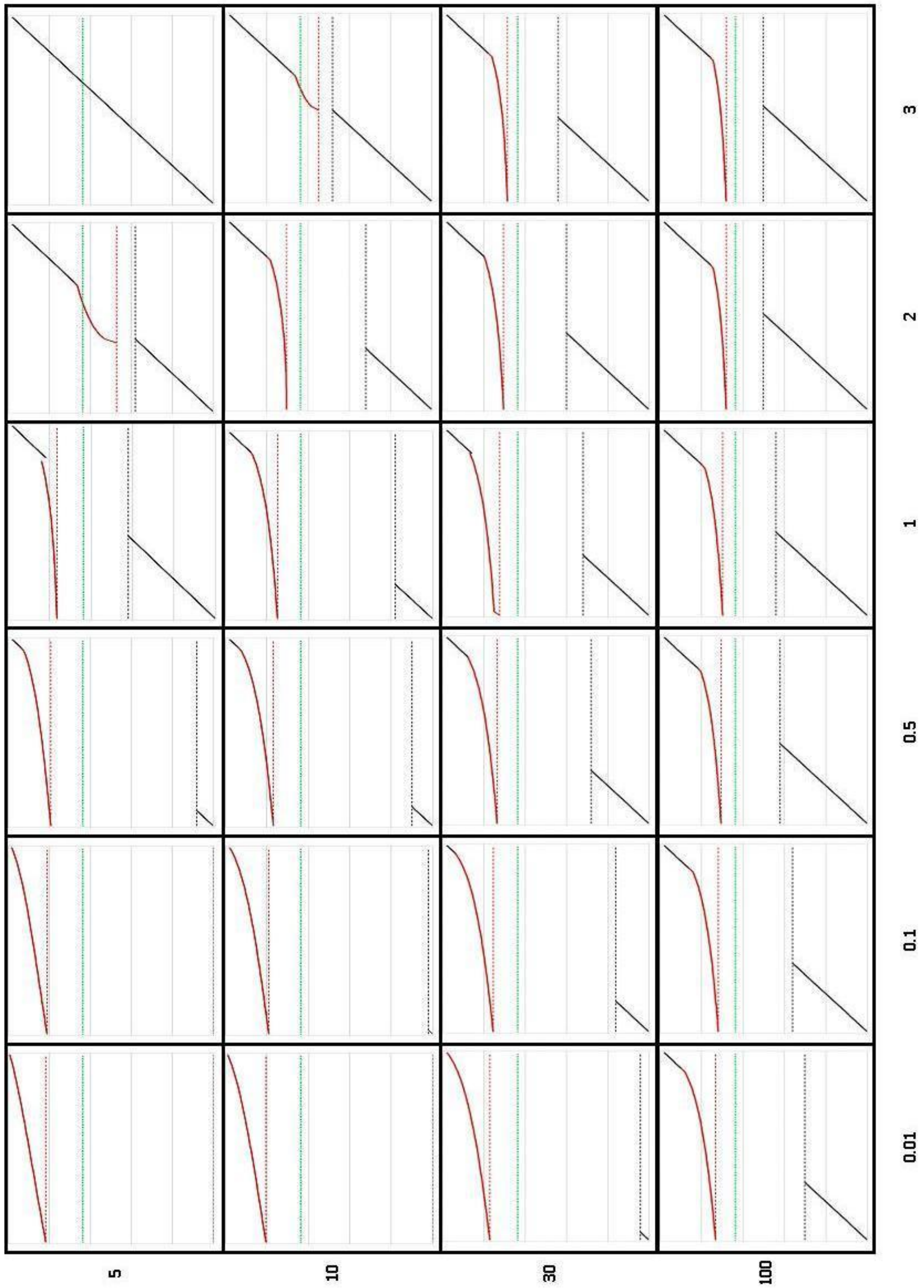

Table 19 – λ = 45%

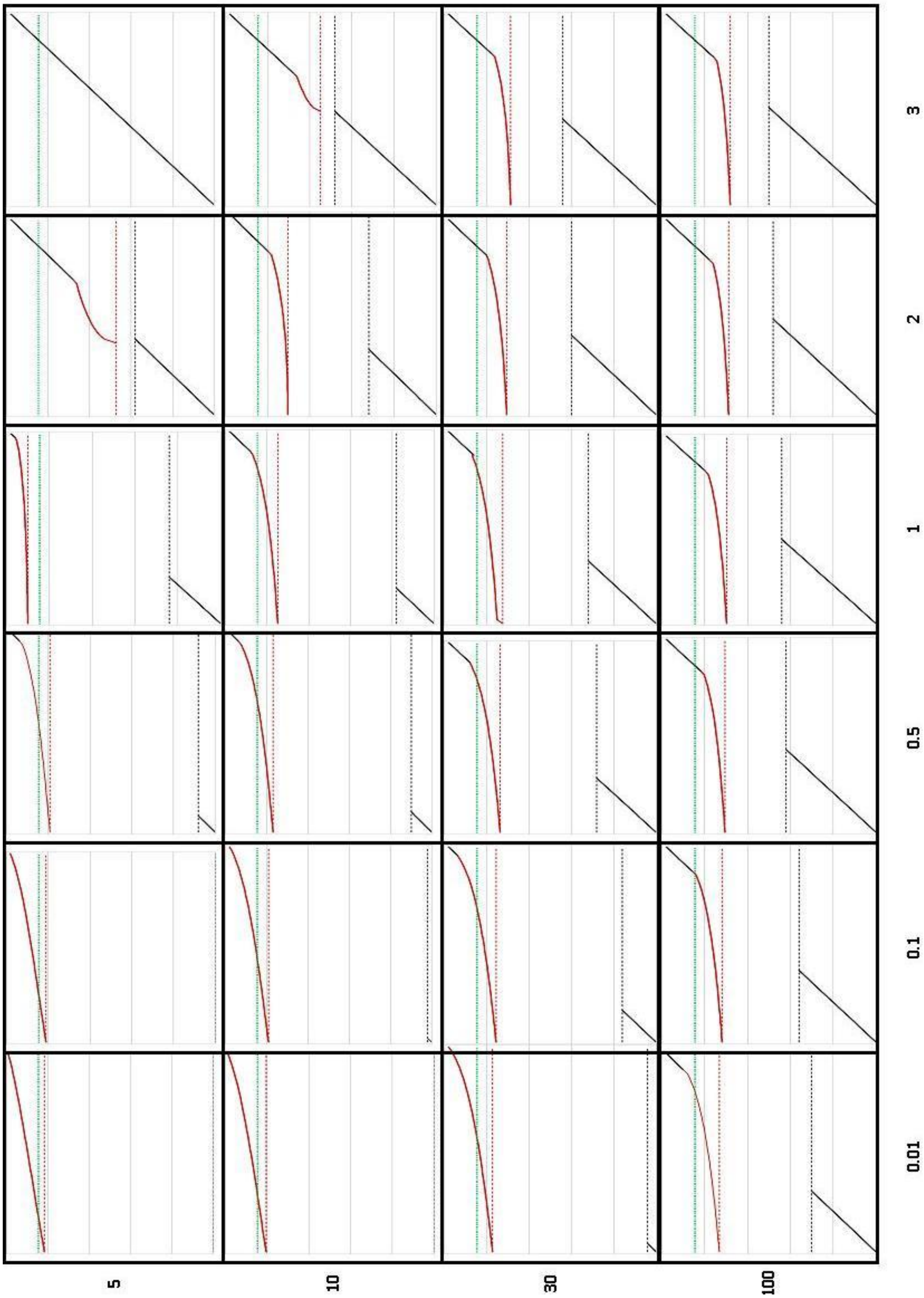

Table 20 – λ = 48%

The effect on the range of values of this “no zone” is diverse. Consider the diagram at the low left-hand corner, for values of (a + b) = 100 and e/c = 0.01. As λ increases from 34% to 38% and to 41%, the increase in *x* is, respectively, null and very marginal, while the increase in the lowest bound of the red line – the value of **Rf** that we obtain from the equation **Rf = 1 - β* (1- Ri)** by considering **Ri** = 0.5 and **β*** the value of the interior equilibrium when Ri = 0.5 – is substantial, making the overall gap increase. When λ increases from 41% to 45% and 48%, on the contrary, the increase in *x* is clearly larger than the increase in the lowest bound of the red line, leading to a decrease in the gap.

For other rows, we generally observe that an increase in λ from 41% to 45% to 48% leads to a decrease in this gap, which does not happen for lower values of λ. The logic behind it is that values of λ that are high and very high (i.e., 45% and 48%) make the trade-off between a strategic vote and an instrumental vote lead towards the sincere vote. Even though what matters is the relative probability of casting a pivotal vote, and not the absolute probability, a much higher λ, in a model where there is a fixed benefit for voting sincerely – *e* – makes it more appealing to vote sincerely.

Related to this is the fact that the red line becomes flatter as λ increases. We have seen that the derivate of this curve is β*, which has two implications. First, as λ increases, and all else constant, an interior equilibrium will involve higher levels of strategic voting; second, the share of supporters of the second candidate voting strategically will be not just higher for each level of Ri, but also have less variance.

Finally, all else constant, a higher λ implies that all levels of Ri that allowed for an interior equilibrium – labelled here as Ii – will still allow it. In other words, the interval [0.5,Ii] – representing the values of Ri which allow for an interior equilibrium, for a value of λi – expands with λi (i.e., Ii is strictly increasing with λi).

8. Implications for policy making

The issue of multiplicity of equilibria and equilibrium selection is a vast topic and we comment on it only briefly here. As mentioned in the first essay of this thesis, "focal arbiters", in the words of Myerson and Weber (1993) can play an important role in the selection of one voting equilibrium, by influencing voter's choices via providing a strong and credible prediction[141] about what the outcome may be[142].

Simon (1954) also proposes a simple yet powerful idea: that opinion polls – and, more generally, any source of information that is used to build expectations about an equilibrium of a game – can greatly influence the outcome of an election. His focus is on how bandwagon and underdog effects can affect the outcome of an election, but the main intuition behind it is helpful for many analyses of equilibrium selection problems in political science contexts.

The fact that there are multiple equilibria in may cases within our model creates room for campaigning in the real world. We find multiplicity of equilibria particularly for values of Ri close to 0.5, when both contenders are not far from each other in terms of prior support. This in line with common intuition: campaigning makes sense when there is no clear front-runner amongst the two contenders, since in such situation strategic voting becomes essential to defeat the incumbent. Thus, generating "momentum" towards one candidate is essential. As this effect – if successful – feeds on itself, clear and well timed investment can make all the difference. Contributing to the rationality of such campaigning is the fact that – within our

[141] Or they wouldn't be "focal arbiters".
[142] See pages 40-42 in essay 1.

model – whenever coordination is successful and a voting equilibrium results from it, the incumbent candidate will almost always be defeated[143].

Apart from a case where λ is very small, an equilibrium with sincere voting will almost always lead to a defeat of the leading contender, unless Ri is high. But when Ri is high the equilibrium is unique, and in such case campaigning would not be needed. However, Ri is uncertain! We do not know whether Ri is high or low with certainty. Opinion polls and other sources of information can be gathered and used to influence voter's perceptions of the underlying hard core support of each of the candidates.

Whenever Ri is believed to be sufficiently high (and even the pollsters do not know the true Ri – that is to say, the draw of α –, but they are more likely to have a more reliable guess than the average citizen), most efforts should probably be concentrated on informing voters of that, as the interior equilibrium would not be possible. However, such equilibrium can occur as Ri goes down, and depending on the level of uncertainty, different policies can be developed to maximise voting shares for some party, given the different scenarios involving Ri, the best strategy for each scenario, its benefits and costs and, importantly, the coherence of the message sent to an electorate[144].

In any case, whenever there is an effort to incentivize strategic voting resources should be focused on the voters that value highly the party they may be voting strategically for – which is not surprising. In a voting equilibrium with partial strategic voting there will be voters voting sincerely, and they are not a random selection from the supporters of the second

[143] We can not guarantee, of course, that campaigning will generate a voting equilibrium as we characterise it.

[144] It would be unconceivable that a campaign would give "contingent" information to the voters, as if the point was to convey statistical information to them. Any message delivered would need to result from, among other things, equating the possibilities of different Ri, but in the end come up with a clear, straightforward message.

contender, but those who have lower instrumental utility if the winner is their second preferred candidate, in their case the leading contender[145].

Any policy that aims to influence voter's choices, and assuming voter's rationality, must be targeted at the right level, electorally speaking. In our model, given the plurality rule at a constituency level and the instrumental motivation of the voters, it would be important to campaign at a constituency level, choosing, in a country wide perspective, to invest according to the expected gains from each constituency. Because voters in our model are not purely instrumentally motivated, there would also be scope for campaigning at a broad level, aiming to influence the way they weigh their identification with their favourite party.

9. The bimodality hypothesis in the electoral simplex

Table 20 illustrates the main implication of our extended bimodality hypothesis in an electoral simplex. Considering the percentages obtained by the three candidates – the incumbent obtaining λ, the leading contender Af and the trailing contender Bf –, we draw the limit curves of the "no zone" area in Figure 49. The curves will change as e/c and/or (a+b) change. For each combination of these two parameters, the curved lines will be in any case upward sloping, reflecting the fact that, as λ increases, the lower and upper limits of that "no zone" interval will also increase.

The horizontal lines in the simplex remind us that λ must be in [1/3, 1/2]. One way to read the simplex is to draw an horizontal line between those two limiting lines (not drawn in Figure 56). Such line would cross the limits of the no zone area, giving us the values of Af – and, implicitly Rf – that would not possible for each λ. By moving such horizontal curve up

[145] Described by the individual parameter Bi.

and down, λ will be changing, respectively, down and up, getting the same type of information for each of them.

The left hand-side of the simplex is not a replication of the right-hand side simply because we use Af and Bf, instead of labelling the parties by their names, in the horizontal axis of the simplex. Since Af reflects the result for the leading contender, regardless of its identity, no result where Bf > Af is admissible. A simplex with the percentages for each of the parties, where their identities would matter, would be symmetric in terms of "no zone" area, given the irrelevance of the labelling in such case.

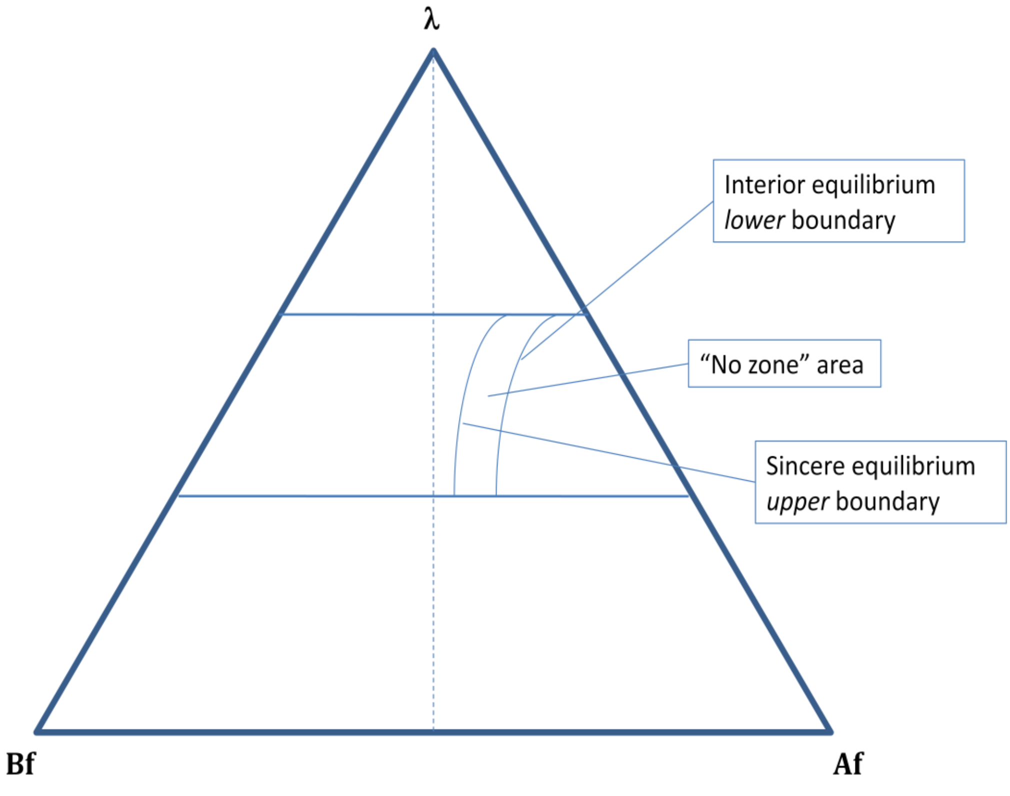

Figure 56 – Bimodality hypothesis in the electoral simplex

The role of a/(a+b) is peculiar in this simplex. Here there is no need to use a/(a+b) as a proxy for Ri. Rather, a/(a+b), since it determines the shape and asymmetry of the Beta-distribution, will determine the relative likelihood of Af (and Bf), which does not depend on the value of λ. The change in (a+b) will have the effect that we already mentioned in the comparative statics section: it will make the no zone shrink. This will be a parametric variation of the simplex here used.

A simple way to have a feeling on whether our theory seems to provide a good explanation to what we observe in reality would be to see whether the actual mapping of the results in such a simplex suggests that there may be a "no zone" area. Of course such inspection would have the implication of taking e/c and (a+b) to be identical in all constituencies, which may seem somewhat strong. However, there is no particular reason to think – particularly without seeming arbitrary – that it should be differently.

The reason why this was not done will be clearer in the next essay. It basically has to do with the first results we obtained from our simulations, which suggested a somewhat poor fit for the model for different parameters, as well as a considerably disparity in the no zone area depending on the values of the parameters adopted. This altogether made it more sensible to us to investigate the issue through a numerical and exhaustive case by case approach.

10. Conclusions

It is interesting to compare the results from this model with those obtain in the second model studied in Part I of this thesis, which can be summarized as follows[146]. There are four parties, two of which – the "extreme" parties – have a fixed support. The two "centre" parties can enjoy strategic voting from their supporters. Given the fixed support of the two extremist parties – and the fact that there is common knowledge about that –, the model is,

[146] See page 66 for the conclusions on Model II of essay 1.

in fact, one of three-party competition, where the supporters of the centre parties try to defeat the leading extremist party. (We keep in mind the *de facto* three-party competition in the comments to come.)

In that model, it is possible to observe an outcome where all voters vote sincerely. More specifically, if the extremist party does *not* have a chance of winning the election (requiring that its support is *lower* than half the combined prior support of the centre parties), such equilibrium is always possible and stable. However, in the more interesting case when that extremist party does have a chance of winning the election (requiring that its support is *higher* than half the combined prior support of the centre parties), that sincere equilibrium is only possible if the density function $f(\alpha)$ is symmetric and, moreover, it is never stable.

In contrast with that model, where voters only have instrumental motivations, the model studied in this and in the previous essay, including instrumental and expressive motivations, makes it possible to have a stable sincere equilibrium when the incumbent can win the election[147]. A sincere equilibrium is stable when the incentive ratio function as a slope higher than 1 at the point $\beta^* = 1$. When this is not the case, an interior stable equilibrium will always exist.

In short, including an expressiveness benefit in the model expands the cases where a sincere equilibrium can be stable. Given that the model used in essay 1 is purely instrumental, it is not surprising that a stable sincere equilibrium is not possible: any deviation from the knife-edge equilibrium makes it worthwhile for everyone else to change their votes. In contrast, the model in essays 2 and 3, by introducing an expressive benefit, brings some “cushion” to the sincere equilibrium: it is no longer necessarily a knife-edge case, it can be supported by a wide range of values. On top of that, it can be a stable equilibrium.

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[147] See pages 138-141.

An equilibrium with partial strategic voting is possible in Model II of essay 1 but it is never stable[148]. That is not the case in the model studied in essays 2 and 3. Here, an interior stable equilibrium is possible and, whenever the interior equilibrium is unique or there are at least two interior equilibria, we can be sure that at least one of them will be stable.

In essay 1, page 68, we wrote that the conclusions from the model used in that essay suggested that Myatt's model – by including private information – seemed necessary to achieve a stable interior equilibrium within a framework of purely instrumentally motivated voters. We can now say that one other way to achieve that kind of result, while still excluding private information, is to allow voters to derive utility directly from the way they vote, regardless of the (expected) outcome of the election. This means that Myatt's inclusion of private information is a sufficient "add-on" to allow a stable interior equilibrium within a framework of purely instrumentally motivated voters, but another sufficient "add-on" is the one we propose: the inclusion of an element of expressiveness in voters' preferences.

In short, the model that underlies the classical Duvergerian prediction can be extended in two ways to achieve an outcome that defies it, within the appropriate framework of a system where three parties compete. One such way is including private information (Myatt's); another way is to expand on voters' preferences, from purely instrumental to mixed preferences, where an element of expressiveness exists. Naturally, other ways to extend the basic model and achieve a stable non Duvergerian equilibrium may exist.

[148] See Table 11 in page 86 for a summary of results for the three models in essay 1.

Essay 4

“Expressive and Instrumental Motivations in the UK 1997 General Election – a Simulation”

1. Introduction

Based on the theoretical model described in essays 2 and 3 of this thesis, in this last essay we propose to do a simulation exercise for the UK 1997 General Election. This exercise is both valuable in itself and also a way of informally testing our theory – getting a feeling of whether what comes out from matching the data with the predictions from our theory seems to support it or not.

The simulation exercise we do consists basically in finding, from the observed results each of the three parties obtained, and according to our assumptions, what could have been their initial or prior support, i.e., the percentage of voters who rank them first in their preferences (which we assume remains stable during the electoral act). The difference between the two, if any, would be due to strategic voting, which we wish to estimate.

The main reason to adopt this approach, as opposed to, say, testing directly the comparative statics of the model is that we were interested not only in testing our model but in obtaining clear figures for the overall level of strategic voting for different combinations of the parameters in our model.

The strategic votes are calculated at a constituency level and then added up together. Fisher (2004) gives an estimate for the overall level of strategic voting amongst the risk population of 24.4%. The estimate at which Fisher arrived had not restricted strategic voting to the constituencies in which the Conservatives obtained between one third and a half of the votes, it also included strategic voting to and from the conservatives. The concept of risk population is clear from the quote below, taken from Fisher (2004):

> Despite the problem of an imperfect definition, the concept and use of a risk population is still important. The definition here is a pragmatic one and does not commit us to the claim that those tactical voters outside the risk population cannot be consistent

> Duvergerian tactical voters. The important point about the risk population is that it enables the analyst to compare the decision to vote tactically among people that could reasonably have voted tactically and only those people. The risk population here includes most of the voters who could reasonably have vote tactically and relatively few for whom it would be unreasonable to vote tactically. (*page 6*)

One of the main goals in this exercise is to estimate a percentage of strategic voting over the relevant population that is comparable to Fisher's. By choosing different combinations of our parameters, we will obtain 24 comparable statistics.

In our model, the "risk population" in each constituency consists of the supporters of the trailing contender in that constituency. Since the trailing contender may have a different identify in different constituencies, we need to add up the data at a constituency level to achieve an overall comparable result. By comparing those estimates to Fisher's we can get a sense of what values (or combinations of values, to be more precise) seem to me more realistic – if our theory is to be a reasonable representation of reality[149]. Despite this, it is not our intention to try to "perfectly" calibrate our model to achieve the aforementioned figure.

We observe, as expected, that the overall estimated level of strategic voting changes considerably with the parameters included in our model, in line with the theoretical predictions from the previous essay. Our choice of presenting summary statistics over a number of simulations in this essay, rather than, say, taking the values of those parameters that lead to the highest proportion of equilibrium prediction and an overall level of tactical voting that is reasonably close to the survey estimate is taken due to poor fit. This will be more clear later on.

[149] If in the end, we conclude it is not, then having compared our results to a reliable estimate is inherently flawed, but assessing that reliability one step is to do such comparison, which requires assuming the model acceptably explains the referred election.

Another result we are interested in is the “efficacy” of strategic voting. Strategic voting is effective, in our interpretation, when it is critical in determining the winner of the election, i.e., when the winner swings from the incumbent to one of the contenders. Myatt (2006) provides some estimates for the number of constituencies that, according to his model, may have changed hands due to strategic voting. We perform a similar exercise.

Finally, we are also interested in seeing how the mix of sincere and interior equilibria changes as the parameters change, since one of the interesting features of our model, in contrast with others, such as Myatt (2006), which only allows for equilibria with partial strategic voting, is that it can accommodate both a sincere and an interior equilibrium in different scenarios.

One of the main implications of our theory, thoroughly explored in the previous essay, is that a range of values for Rf is not possible. The range of values depends on the parameters of the model, as we have seen in the comparative statics analysis in the last essay[150], but that is a general feature of our theory. The implication for the empirical work we do in this essay is that we should not observe certain values of Rf in the data. Whenever we do values of Rf in the no zone area we will not be able to find any equilibria for them. The simulation exercise we propose to do will be impossible, by construction. Such cases will be referred to as “missing constituencies”, given that they are not and they can not be captured by our model.

Any missing constituency is a challenge to our theory. Why do we find that we don’t find any equilibrium for some constituencies? It may be that some or all voters in that particular missing constituency do not have an expressive benefit as high as we assume; or that they have private information, instead of or complementing the public information; or that the

[150] See Tables 16-20 in the pages ahead.

parameters are not the same across the whole country; or that voters perhaps are not optimising, even if their preferences and information are in line with our model; or that a voting equilibrium does not occur because expectations are not met. There are several individual reasons why we could be observing some missing constituencies. Naturally, such could also arise from a combination of any of them.

The interest of the calibration exercise lies considerably in testing a model that fills a gap in the strategic voting literature. In that perspective, whatever results come out of it – supportive or less supportive –, they bring light to the understanding of the phenomenon of strategic voting.

2. The simulation – theory

Out of 529 constituencies, we focus on the 270 constituencies where the percentage result obtained by the Conservative Party, labelled as λ, was between 1/3 and 1/2. Following the previous essay, the percentage of votes obtained by the party with highest votes between the Labour and the Liberal Democratic Party is Af; Bf is the percentage of votes obtained by the party with least votes; both refer to percentages amongst their vote total. The identity of the two contenders is therefore not relevant in our exercise.

The program we developed identifies the highest observed result between the two contenders.[151] The constituencies where the Conservative Party did not get a result between 1/3 and 1/2 are excluded since they do not fit in our model. As mentioned in the previous

[151] As mentioned in the Acknowledgements section, the simulation here described is based on MATLAB programming. The programming was all done by Tessa Bold, under my direction of how to identify both interior and sincere equilibrium, as well as checking for their stability. Naturally, this was an iterative process, as we learnt along the way, readjusting several times the program to more efficiently and accurately identify the possible equilibria for each simulation.

essay, we assume strategic voting occurs in the "right direction", which implies both that it occurred towards the contender with the highest observed result and also that such contender had an advantage in terms of prior support (meaning that the asymmetry in the Beta distribution favoured it). This is a necessary requirement to keep the (already significantly large) set of results reasonably contained.

When the Conservative Party obtains more than half of the votes, there is no room in our theory for assuming strategic voting may have happened in that constituency, since this would be useless. If the Conservative Party obtains less than half of the votes, that means that in the particular constituency to which it refers the party cannot be taken to be the "incumbent". If this study was purely statistical it could make sense to study all constituencies, regardless of the votes the Conservatives have, but there are important reasons not to do this, since our study cannot forget the political context involved in the study we do.

In the 1997 UK General Election the Conservative Party was the previous elected party and was taken by both the Labour and the Liberal Democratic parties to be the "party to defeat". Such defeat depended on the overall number of seats gained, for which all constituencies matter. To study strategic voting on the level of the constituencies (the correct level, given the single-ballot simple plurality rule), while having in mind the political meaning that comes out from the landscape of the political competition amongst the three parties in 1997, requires that we consider the support for the Conservative Party as fixed, since this was the party in power. We use the word "fixed" in the sense that all conservative voters are hard-core supporters that would never vote for another party, while non-supporters of the Conservative Party would never vote for it strategically. In our model, this translates in a zero utility for all conservative supporters from a victory of either a Labour or a Liberal

Democrat candidate in the relevant constituency, as well as a zero utility for Labour or Liberal Democrat supporters for voting in the Conservative party.

The labels Af and Bf refer to the percentages at a constituency level, so each of them is used in this essay to refer to *any* constituency, or *no* constituency in particular. Since $\lambda > 1/3$, there are only two possible rankings for each of the 270 selected constituencies (ignoring the case of ties):

λ > Af > Bf

Or:

Af > λ > Bf

In the first the incumbent wins; in the second case it is defeated. In our model, each of these situations may or may not include strategic voting.

We recall two important equations for the simulation exercise:

Rf = Af/(Af + Bf)

Rf = 1 + β* (Ri – 1)

The first equation gives the definition of Rf: it is an identity. The second equation describes Rf taking into account that strategic voting takes place. B* reflects the proportion of supporters of the second contender that voted sincerely. The percentage of strategic voting in the "risk population", following Fisher's definition, is, therefore, equal to (1 – β*). A sincere equilibrium is equivalent to having β* = 1 and Rf = Ri. If there is an interior equilibrium, Rf will be higher than Ri and, for the same Ri, will be higher the lower β* is.

The simulation exercise consists in, upon looking at a triplet (λ, Af, Bf) – which implies a certain Rf – and certain choices of (a+b) and e/c, searching for a pair of values for β* and Ri such that the equation Rf = 1 + β* (Ri – 1) is satisfied, along with the relevant conditions that relate β* with Ri. These conditions refer to the optimal behaviour of each individual voter, first presented in essay 1 of Part II of this thesis. Next we recall the essential features behind them.

In the second essay of this thesis we assumed, without loss of generality, that strategic voting would occur from LM voters towards party M[152]. In this essay the assumption is that strategic voting can occur only towards the observed leading contender – be it L or M, to use the terminology of that essay, were we presented the following optimality conditions:

For ML voters:

Condition *If f(αL) = 0, vote for M. If f(αL) > 0, vote for M if βi ≤ β*M, and vote for L if βi > β*M, where B*M = β^M = min{β~M,1} and β~M = 1/ f(αL) [e/c H(β*M) + f(αM)]*

For LM voters:

Condition *If f(αM) = 0, vote for L. If f(αM) > 0, vote for L if βi ≤ β*L, and vote for M if βi > β*L, where B*L = β^L = min{β~L,1} and β~L = 1/ f(αM) [e/c H(β*L) + f(αL)]*

The conditions for optimal behavior in this essay require replacing L by A and M by B, since strategic voting occurs to party A – the contender with the highest observed support. So we get:

For BA voters:

[152] LM voters are voters whose preferred candidate is L and second preferred candidate is M, and therefore consider voting strategically for M. Likewise for ML voters.

Condition *If $f(\alpha A) = 0$, vote for B. If $f(\alpha A) > 0$, vote for B if $\beta i \leq \beta^*B$, and vote for A if $\beta i > \beta^*B$, where $\beta^*B = \beta^\wedge B = \min\{\beta\sim B,1\}$ and $\beta\sim B = 1/ f(\alpha A)\ [e/c\ H(\beta^*B) + f(\alpha B)]$*

For AB voters:

Condition *If $f(\alpha B) = 0$, vote for A. If $f(\alpha B) > 0$, vote for A if $\beta i \leq \beta^*A$, and vote for B if $\beta i > \beta^*A$, where $\beta^*A = \beta^\wedge A = \min\{\beta\sim A,1\}$ and $\beta\sim A = 1/ f(\alpha B)\ [e/c\ H(\beta^*A) + f(\alpha A)]$*

There is another important difference, though. Recall, from Figure 32 in essay 2 that αM and αL are the draws of the Beta distribution determining the percentage of LM voters who, for a certain level of strategic voting – not necessarily an equilibrium level, the labelling is general –, would lead, respectively, to a pivotal event between M and R and between L and R. In essay 2, both pivotal events are possible.

In the current essay this is not the case. The probabilities f(αA) and f(αB)) concern, respectively, the pivotal events involving the first contender and the second contender, from the perspective of a supporter of party B. So αA and αB are the draws of the Beta distribution that determine the percentage obtained by the second contender, labelled as B (and which can be the Labour or the Liberal Democratic Party), such that, respectively, party A and party B would be involved in a pivotal event.

The difference to essay 2 is that here only a pivotal event between A and C is *really* possible, while a pivotal event between B and C is merely hypothetical. However, this difference is not relevant on a substantive level: it results merely from the choice of labelling the relevant parties as A and B, from a post-electoral perspective, instead of adopting a pre-electoral perspective, by labelling them as L and M, as in the previous essay. The naming is mere cosmetics, but underlying it is the need to restrict the direction of strategic voting. The calculus of each voter is still the same; the only imposition in this essay is that voting occurs in the right direction.

The $\beta^* < 1$ that characterizes the equilibrium with partial strategic voting is the β^* relevant for the group of voters from which some strategic voting arises. For the other group, the incentive ratio is necessarily higher than 1, so we can ignore it[153]. The expressions that we need to keep in mind for the B supporters are:

$\beta\sim = [e/c\ \beta^* + f(\alpha B)] / f(\alpha A)$

$\beta\wedge = \text{Min}\ \{1, \beta\sim\}$

An equilibrium exists when $\beta\wedge = \beta^*$.

A sincere equilibrium implies that $\beta^* = 1$ and that $\beta\sim$ is equal or higher than 1, so that $\beta\wedge$ is not below than 1. This implies that $[e/c + f(\alpha B)] / f(\alpha A)$ is equal of higher than 1 in a sincere equilibrium. In an interior equilibrium we have that $\beta^* < 1$, so we also have $\beta^* = \beta\sim$, so $\beta^* = [e/c\ \beta^* + f(\alpha B)] / f(\alpha A)$, implying that $\beta^* = f(\alpha B) / [f(\alpha A) - e/c]$ and, therefore, since $\beta^* < 1$, $f(\alpha B) / [f(\alpha A) - e/c] < 1$.

In any interior equilibrium we have that $\beta^* = f(\alpha B) / [f(\alpha A) - e/c]$, which tells us that[154], all else constant, a higher pivotal probability involving the second contender would imply a higher β^*, or a lower level of strategic voting, which we would expect. Also, a higher level of e/c would imply a higher level of β^*, making sincere voting more appealing. In a purely instrumental model, $e/c = 0$, so any interior equilibrium in such a model would involve $\beta* =$

[153] If that was not the case, sincere voting would not be optimal for them, and we have seen before that, in our model, it is not possible to have in equilibrium strategic voting occurring to both parties.
[154] For some strange reason when this thesis was revised and reedited this bit of the text remained, strangely with this format. I was unable to edit it to a normal paragraph.

$f(\alpha B) / f(\alpha A)$. It is clearer where does the β^* come from, but what about Ri in the important expression $Rf = 1 + \beta^* (Ri – 1)$?

The way we link Ri to αB and αA is vital to our work. In practical terms, they come out from trying to find, for a pair of values for β^* and Rf, the resulting Ri and, from there and the simple equation $Ai + Bi + \lambda = 1$, guessing what Ai and Bi are, with the following simple formulas:

$Ai = (1 – \lambda) Ri$

$Bi = (1 – \lambda) (1 – Ri)$

Voters are uncertain about the support of what will be the first and the second contenders. They only know the parameters *a* and *b* of the Beta distribution $f(\alpha)$, where α reflects, in essay 2, the support in percentage of the party L amongst the LM and ML voters and, in this essay, the support in percentage of the second contender amongst the voters that dislike the incumbent party. The essential thing is that to retrapolate the observed results we need to assume that the voters “guessed correctly”, when they behaved in a way that generated β^* – the percentage of supporters of party B who voted sincerely. As we mentioned in the previous essay, this is common in any game-theoretical study of elections.

The best guess for the ratio of the prior support of party B over party A, from a prior perspective, is a/(a+b). So the best assumption we can make – and, crucially, we need to make one to be able to calculate the observed results – is that Ri = a/(a+b). This assumption is the best we can make and it is a better proxy the higher the degree of information those voters have, since the uncertainty about the distribution of the prior support of both parties goes down as the information goes up[155].

We now possess all the ingredients needed. From a pre-electoral point of view, we have four parameters: λ, e/c, (a+b) and a/(a+b). As we saw in essay 2, these parameters are sufficient to determine whether a sincere or an interior equilibrium is possible. If a sincere equilibrium occurs, and following the necessary restriction that Ri = a/(a+b), we have Rf = a/(a+b). If there is an interior equilibrium, then Rf = 1 + β* [a/(a+b) – 1].

From a post-electoral point of view – the relevant one to do this exercise –, we wish to see whether a combination of four variables that characterise the observed result – λ, e/c, (a+b) and Rf – may have been generated by a sincere equilibrium. If a sincere equilibrium is possible, we "guess" or retrapolate that Ri = a/(a+b), the value underlying the computation done, since Rf = a/(a+b) by construction. If an interior equilibrium is possible, we need to "discount" the strategic votes that took place to find a value of Ri equal to a/(a+b), where a/(a+b) = 1 + (Rf – 1) / β*.

Notice the important asymmetry that we have when there is an interior equilibrium. From a pre-electoral perspective, it is sufficient, having the four parameters mentioned two paragraphs above – the quadruplet [λ, e/c, (a+b), a/(a+b)] –, to look at a figure like Figure 34 to find out whether a sincere equilibrium, an interior equilibrium or both are possible[156]. However, to find out, from a post-electoral perspective, whether a certain quadruplet could have been generated by an interior equilibrium, we can not look at a figure like Figure 34, because in it the parameters are used as inputs and, critically, Rf cannot be an input for such figure. Rf is always the output, coming from a certain Ri together with the associated β* that characterises the given equilibrium.

[155] The relevant distribution has a lower variance – see Figure 30 on page 100.
[156] See page 128.

This point is made clearer in essay 2, by illustrating, in Figures 51, 52 and 53, the differences between a forward- and a backward-looking approach to understanding the equilibria observed. An example: looking at Figure 48, we can see that a value of Ri of 75% can only give rise to an interior equilibrium – generating a diagram like that illustrated in Figure 45. The resulting Rf would be somewhere between 80% and 90%. A value of Rf equal to 75% could not be observed, so a sincere equilibrium with Rf = 75% would not be possible.

We stated before that upon observing a certain value of Rf, it is very unlikely that such value could have arose by either a pattern of sincere or strategic voting – i.e., that behind the same Rf can be two different equilibria. The few exceptions involve a diagram that looks like the one in Figure 53, where for some values of Rf a parallel line would cross the red and the black lines. This is related to the backward-looking approach. Only 1.5% of the equilibria found could be either a sincere or an interior equilibria, for the same Rf[157].

A different question we may ask, which relates to the forward-looking approach, is whether an observed Rf could have been originated by a certain Ri, which, in turn, could have also originated a Rf different from the one observed. The answer is that it may happen sometimes, as we observe, for instance, in Figure 51, where the same Ri can generate two different values for Rf, corresponding to two different equilibria.

To illustrate this, consider the value Rf = 52% in Figure 48. In this case, only a sincere equilibrium is possible, and the same value of Ri could have generated an interior equilibrium with β^* = 35.7% and Rf = 82.9%. Likewise, if we had started with Rf = 83%, we see that only an interior equilibrium is possible, for Ri = 53% and β^* = 36.3%. A value for Ri of 53% also allows for a sincere equilibrium.

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[157] Each involving, of course, different Ri's and β^*'s.

In some sense the later is a more interesting question for the real world, while the former is mostly a computational curiosity. Knowing whether the same Rf could have been generated by two different values of Ri means that a sincere or an interior equilibria can be behind such value – but, theoretically (without, say, local post-electoral surveys), there is not much more to take from it apart from the possibility of different origins. In contrast, the later question of checking whether the same value of Ri can give rise to different values of Rf means that, upon observing one of those values of Rf, we know that there was an alternative behavior that could have generated a different Rf.

This has important policy implications, namely because there are two equilibria that can emerge and therefore there is room to influence voters' choices in order to coordinate them in one of them – as in Figure 51 for low values of Ri –, which we covered in the previous essay.

3. The simulation – the program

The program developed uses MATLAB and the reasoning behind it is the following: first, we allow (a+b) to take the values 5, 10, 30 and 100; and (e/c) to take the values 0.01, 0.1, 0.5, 1, 2 and 3. From this we obtain 24 different combinations, and we do one simulation exercise for each of them, so we obtain 24 outcomes, one for each pair of values of e/c and (a+b).

For each of these 24 combinations, we look at the relevant 270 constituencies, taking note of λ, the percentage obtained by the conservatives, and of the percentages obtained by the Labour and the Liberal Democratic Party, from which we obtain Rf. The goal is to find an equilibrium for each of the 270 constituencies in each of the 24 simulation exercises. The amount of data generated is quite significant.

Taking into account e/c and (a+b) – for each exercise – and λ and Rf – for each constituency, within that exercise –, we wish to investigate whether there is an Ri and β^* that could have lead to the observed Rf, given the triplet (e/c, (a+b), λ). The fourth parameter in our theory – a/(a+b) – is free to vary, which is essential. When we say we wish to investigate whether there exists an Ri that (along with a β^*) could have lead to the observed Rf, in fact we are asking the computer to consider different values of *a* and *b* that characterise a Beta-distribution – generating a value for a/(a+b) – and see what equilibria they would generate.

This is the idea behind Figures 44-47 in the previous essay. For a certain a/(a+b) – added to the already established, for each constituency within each exercise, e/c, (a+b) and λ – we will find some equilibrium and, by assuming Ri = a/(a+b), we get a value of Rf using the equation Rf = 1 + β^* (Ri – 1). What MATLAB does, in a quicker way than other computer programs, is to run through several values of a/(a+b), checking if they generate an equilibrium, determine the associated Rf, check if it matches satisfactorily with the observed Rf (taken from the electoral data) and, given this, tell us whether or not there is an equilibrium, and of what type.

There are two output tables that aggregate the essential data for this search and that help us understand the philosophy behind the programming. Both tables have values of a/(a+b) in the rows and λ in the columns. The row variable, a/(a+b), takes 50 values, varying from 0.50 to 0.99, in increments of 0.01. The column variable, λ, takes 166 values, varying from 33.4% and 49.9%, in increments of 0.1%. So each table has 8.300 entries. We decided that varying λ by only 1% would not give sufficient depth, given that it refers to the actual observed results of the Conservative party; also – and reinforced by the restriction just mentioned – we thought that expanding the values of a/(a+b) to further detail would make computations too slow. [158]

The first table is the β^* table, which indicates, for each a/(a+b) and λ, whether an interior equilibrium is possible and, if so, what value it takes. The second table is the Rf table, giving, whenever an interior equilibrium exists, the resulting Rf, from the formula Rf = 1 + β^* (a/(a+b) – 1).

The search for a sincere equilibrium is very easy and consists on using the observed Rf as an input for the program – i.e., using a/(a+b) = Rf – and seeing whether the incentive ratio when $\beta^* = 1$ is equal or higher than 1. If it is greater than 1 there is a sincere equilibrium, if it is lower than 1 a sincere equilibrium is not possible. Recall the expression for the incentive ratio β~:

β~= [(e/c) H(β*) + f(αB)] / f(αA)

To check the value of β~ in an hypothetical sincere equilibrium, we need only replace β^* = 1, recall that H(β) is the cumulative function for β, so H(1) = 1; and, from page 117, αA□= λ /(1 – λ) and αB = (1 – 2λ)/(1 – λ).

In short, for each pair {e/c,(a+b)} we ran one simulation, obtaining the following output:

(1) a β* table, indicating, in each of its entries, whether an interior equilibrium was possible or not, in the first case indicating the value found;

[158] Each of the 24 exercises took about 3 hours to run, and allowing a/(a+b) to have increments of 0.001 would make the time of computation 10 times larger.

(2) a Rf table, using the information from the β* table to give, whenever an interior equilibrium is detected, Rf, according to the formula Rf = 1 + β* (Ri – 1).;

(3) a simple column of dimension 1 x 270, with 1's and 0's indicating, respectively, whether a sincere equilibrium is possible or not for each of the 270 constituencies, given the observed electoral result.

We obtain 24 sets of the output data just described. Note that the computer program needs values *a* and *b* for the Beta distribution, and gets them indirectly from the decision on (a+b) and on a/(a+b)[159]. As we said, the search for the sincere equilibria is relatively simple, given that the computer uses the observed data – Rf and λ – as inputs. The search for an interior equilibrium is more complex and requires allowing for an approximation error, as we do not study a continuum of values for both a/(a+b) and λ.

[159] Which varies from 0.50 to 0.99, implying that for each row entry in β* and Rf tables the level of information is the same but the asymmetry of the underlying Beta-distribution is different.

B*	39.3%	39.5%	39.8%	40.3%	40.4%
0.5	0.4426	0.4357	0.4254	0.4079	0.4044
0.51	0.4471	0.4402	0.4297	0.4121	0.4085
0.52	0.4518	0.4448	0.4342	0.4164	0.4128
0.53	0.4566	0.4496	0.4389	0.4209	0.4172
0.54	0.4616	0.4545	0.4437	0.4255	0.4218
0.55	0.4668	0.4596	0.4487	0.4303	0.4266
0.56	0.4722	0.4649	0.4539	0.4352	0.4315
0.57	0.4778	0.4704	0.4592	0.4404	0.4366
0.58	0.4836	0.4761	0.4648	0.4457	0.4419
0.59	0.4896	0.482	0.4706	0.4513	0.4474
0.6	0.4959	0.4882	0.4766	0.457	0.4531
0.61	0.5023	0.4946	0.4828	0.463	0.459
0.62	0.5091	0.5012	0.4893	0.4693	0.4652
0.63	0.5161	0.5082	0.4961	0.4757	0.4716
0.64	0.5234	0.5154	0.5031	0.4825	0.4783
0.65	0.5311	0.5229	0.5105	0.4895	0.4853
0.66	0.539	0.5307	0.5181	0.4968	0.4926
0.67	0.5473	0.5389	0.5261	0.5045	0.5001
0.68	0.556	0.5474	0.5345	0.5125	0.5081
0.69	0.5651	0.5564	0.5432	0.5209	0.5164
0.7	0.5746	0.5658	0.5523	0.5297	0.5251
0.71	0.5846	0.5756	0.5619	0.5389	0.5342
0.72	0.5951	0.5859	0.572	0.5485	0.5438
0.73	0.6061	0.5967	0.5826	0.5587	0.5538
0.74	0.6177	0.6082	0.5937	0.5693	0.5644
0.75	0.6299	0.6202	0.6055	0.5806	0.5756
0.76	0.6428	0.6329	0.6179	0.5925	0.5874
0.77	0.6565	0.6464	0.631	0.6051	0.5999
0.78	0.671	0.6606	0.6449	0.6185	0.6131
0.79	0.6864	0.6758	0.6597	0.6327	0.6272
0.8	0.7028	0.6919	0.6755	0.6478	0.6422
0.81	0.7203	0.7092	0.6923	0.6639	0.6582
0.82	0.739	0.7276	0.7103	0.6812	0.6753
0.83	0.7591	0.7474	0.7297	0.6997	0.6937
0.84	0.7808	0.7688	0.7505	0.7197	0.7135
0.85	0.8043	0.7919	0.7731	0.7414	0.7349
0.86	0.8298	0.817	0.7976	0.7649	0.7582
0.87	0.8577	0.8444	0.8244	0.7905	0.7837
0.88	0.8882	0.8745	0.8538	0.8187	0.8116
0.89	0.922	0.9078	0.8863	0.8499	0.8425
0.9	0.9597	0.9448	0.9224	0.8846	0.8769
0.91	1	0.9865	0.9631	0.9235	0.9155
0.92	1	1	1	0.9678	0.9594
0.93					
0.94					
0.95					
0.96					
0.97					
0.98					
0.99					

Rf	39.3%	39.5%	39.8%	40.3%	40.4%
0.5	0.7787	0.7821	0.7873	0.796	0.7978
0.51	0.7809	0.7843	0.7894	0.7981	0.7998
0.52	0.7832	0.7865	0.7916	0.8001	0.8019
0.53	0.7854	0.7887	0.7937	0.8022	0.8039
0.54	0.7877	0.7909	0.7959	0.8043	0.806
0.55	0.7899	0.7932	0.7981	0.8064	0.808
0.56	0.7922	0.7954	0.8003	0.8085	0.8101
0.57	0.7946	0.7977	0.8025	0.8106	0.8123
0.58	0.7969	0.8	0.8048	0.8128	0.8144
0.59	0.7993	0.8024	0.8071	0.815	0.8166
0.6	0.8017	0.8047	0.8094	0.8172	0.8188
0.61	0.8041	0.8071	0.8117	0.8194	0.821
0.62	0.8065	0.8095	0.8141	0.8217	0.8232
0.63	0.809	0.812	0.8164	0.824	0.8255
0.64	0.8116	0.8145	0.8189	0.8263	0.8278
0.65	0.8141	0.817	0.8213	0.8287	0.8302
0.66	0.8167	0.8196	0.8238	0.8311	0.8325
0.67	0.8194	0.8222	0.8264	0.8335	0.835
0.68	0.8221	0.8248	0.829	0.836	0.8374
0.69	0.8248	0.8275	0.8316	0.8385	0.8399
0.7	0.8276	0.8303	0.8343	0.8411	0.8425
0.71	0.8305	0.8331	0.837	0.8437	0.8451
0.72	0.8334	0.8359	0.8398	0.8464	0.8477
0.73	0.8364	0.8389	0.8427	0.8492	0.8505
0.74	0.8394	0.8419	0.8456	0.852	0.8532
0.75	0.8425	0.845	0.8486	0.8548	0.8561
0.76	0.8457	0.8481	0.8517	0.8578	0.859
0.77	0.849	0.8513	0.8549	0.8608	0.862
0.78	0.8524	0.8547	0.8581	0.8639	0.8651
0.79	0.8559	0.8581	0.8615	0.8671	0.8683
0.8	0.8594	0.8616	0.8649	0.8704	0.8716
0.81	0.8631	0.8653	0.8685	0.8739	0.875
0.82	0.867	0.869	0.8721	0.8774	0.8784
0.83	0.8709	0.8729	0.876	0.881	0.8821
0.84	0.8751	0.877	0.8799	0.8848	0.8858
0.85	0.8794	0.8812	0.884	0.8888	0.8898
0.86	0.8838	0.8856	0.8883	0.8929	0.8938
0.87	0.8885	0.8902	0.8928	0.8972	0.8981
0.88	0.8934	0.8951	0.8975	0.9018	0.9026
0.89	0.8986	0.9001	0.9025	0.9065	0.9073
0.9	0.904	0.9055	0.9078	0.9115	0.9123
0.91	0.91	0.9112	0.9133	0.9169	0.9176
0.92	0.92	0.92	0.92	0.9226	0.9232
0.93					
0.94					
0.95					
0.96					
0.97					
0.98					
0.99					

Table 21 – Tables β* and Rf for e/c = 0.01 and (a+b) = 30

Also note that both the β* table and the Rf table do not require any information about the data observed in the constituencies; they are "purely theoretical", requiring only the use of the Beta distribution to get the kind of information conveyed in Figures 44 to 47.

The final step in the search for interior equilibria is to ask, within each of the 24 simulations made, whether a particular constituency has information consistent with what comes out from β* and Rf tables. In fact, the check up for the existence of an interior equilibrium only requires us to look at the Rf table. We do this by choosing the column that is closest to the observed λ and then see if there is a value of Rf sufficiently close to the observed Rf. We define "sufficiently close" to be any case where:

|Rf – Rf| < 0.01, where Rf here refers for the "estimated Rf".

Whenever we find a pair (Rf, Rf) in these conditions, the program will flag an interior equilibrium. After constructing the two tables, the program looks up at all the values in the Rf table and selects the one that, satisfying that condition, is closest to the value observed. Whenever an equilibrium is found, the program then looks up at the B* table and gives us the value of β* corresponding to the interior equilibrium detected.

The fact that Rf is not necessarily exactly equal to Rf implies that a/(a+b) underlying the β* found will not be equal to the Ri that we obtain when retrapolating Ri from the observed Rf and β*[160]. Tables 62 and 63 help us illustrate this point. The parameters involved in them are e/c = 0.01 and (a+b) = 30, the one that gives an overall level of strategic voting over the relevant population closest to the percentage in Fisher (2004), and look at some constituencies for which values of λ are between 39% and 41%.

[160] Ri being the "estimated Ri".

	Constituency	Votes	C	L	LD	λ	Af	Bf	B*	$\underline{Rf}$	Rf	$\underline{Ri}$	Ri	γ	SV
35	Battersea	46216	18687	24047	3482	40.4%	24047	3482	65.8%	87.5%	87.4%	81.0%	80.8%	67.9%	1808
50	Bexleyheath & Crayford	45860	18527	21942	5391	40.4%	21942	5391	41.3%	80.2%	80.3%	52.0%	52.2%	67.8%	7669
53	Birmingham Edgbaston	46957	18712	23554	4691	39.8%	23554	4691	55.2%	83.4%	83.4%	70.0%	69.9%	66.2%	3803
107	Burton	53907	21480	27810	4617	39.8%	27810	4617	64.5%	85.8%	85.8%	78.0%	77.9%	66.2%	2542
189	Devon North	54814	21643	5347	27824	39.5%	27824	5347	59.7%	83.9%	83.9%	73.0%	73.0%	65.2%	3614
280	Gravesham	51269	20681	26460	4128	40.3%	26460	4128	61.9%	86.4%	86.5%	78.0%	78.2%	67.6%	2546
310	Hemel Hempstead	53503	21539	25175	6789	40.3%	25175	6789	40.8%	19.6%	78.8%	50.0%	47.9%	67.4%	9855
419	Newbury	54364	21370	3107	29887	39.3%	29887	3107	96.0%	91.0%	90.6%	90.0%	90.2%	64.8%	130
507	Selby	54618	22002	25838	6778	40.3%	25838	6778	40.8%	79.6%	79.2%	50.0%	49.1%	67.5%	9839
533	Stafford	50378	20292	24606	5480	40.3%	24606	5480	45.7%	81.5%	81.8%	60.0%	60.1%	67.4%	6511

Table 22 – Examples of the simulation calculus, with e/c = 0.01 and (a+b) = 30

Table 22 includes observed values of λ, Af and Bf – from which we obtain Rf. The other variables are calculated in the following way. Consider the first constituency in Table 22, Battersea, a constituency with 46.216 votes counted, where the Labour came ahead of the Liberal Democratic Party and defeated the Conservatives with 52% of the overall votes.

In Battersea we observed λ = 40.4% and we get Rf = 87.4%. These two values are sufficient for us to check, using Table 21, whether there may have been an interior equilibrium or not. Looking at the right-hand table included in Table 21 (the Rf table), in the appropriate column, λ = 40.4%, we search for the number that is closest to 87.4% and within a margin of 0.01%. This number is 87.5%. Looking for the corresponding row, we see that the associated value of a/(a+b) for $\underline{\text{Rf}}$ = 87.5% is 81%. The next step is to look, in the β^* table, for the value in the entry with a/(a+b) = 81% and λ = 40.4%. the value found is B* = 65.8%, implying a percentage of strategic voting amongst the risk population of 34.2%.

This constituency can be characterised by β^* = 65.8% and a theoretical value of Ri equal to 81%, which we obtain from the right hand side table in Table 21. However, it is more interesting to use the observed Rf and the β^* value to calculate a more precise value for Ri ($\underline{\text{Ri}}$ in Table 22), using the equation Rf = 1 + β^* ($\underline{\text{Ri}}$ – 1). Ri and $\underline{\text{Ri}}$ are bound to difer whenever the theoretical value of Rf in Table 21 – $\underline{\text{Rf}}$ – is not exactly the same as the observed value Rf.

Both $\underline{\text{Rf}}$ and $\underline{\text{Ri}}$ are included in Table 22 to give an idea of how some important variables in our model can vary because of the approximations we do (namely, having a/(a+b) changing in unit percentage points only). As one would expect, the variations are not too large. The fact that some values of Rf are not observed contribute to this, since a variation in a/(a+b) will not affect Rf as much as it could were Rf more disperse than in our model.

The column SV, in Table 22, indicates the number of strategic votes that took place and is found by taking the difference between Bi and Bf, respectively the initial (and potential) number of votes and the effective number of votes obtained by the second contender, in this constituency the Liberal Democratic Party. Since Bf = β* Bi, the difference (Bf – Bi) is equal to Bi (1 – β*). We do not observe Bi, but we know that Bf is equal to β* Bi, since a proportion (1 – β*) of potential votes goes away to the first contender. From here we get Bi = Bf / β*, to get the final expression for the number of strategic votes, SV = Bf (1 – β*) / β*.

Finally, we include a column with γ, where γ = λ / (1 – λ) is the qualified majority percentage. The leading contender defeats the incumbent whenever Rf > γ. By comparing Ri to γ we can see whether or not strategic voting was necessary to defeat the incumbent. By comparing Rf to γ we see whether or not the leading incumbent ended up winning the election. For all constituencies in Table 22 we see that Rf > γ, so the party gathering the strategic votes always won. For some constituencies – when we observe Ri > γ –, the party with highest support would, in any case, and according to our model, win the election. Again, it is essential to bare in mind that voters face uncertainty: they are not sure that the initial overall support, in percentage, of the leading contender is Ri (1 – λ), what they know is that the such figure follows a Beta distribution with parameters *a* and *b*, where a/(a+b) = Ri and (a+b) reflects the level of information they possess. So it is possible that strategic voting occurs even if "on average" the supporters of the last contender expect the first contender to win, even if there are no strategic votes, because there is uncertainty[161].

Regarding the constituency Battersea, we can say that, considering the initial parameters e/c = 0.01 and (a+b) = 30, the lowest level of expressiveness and a high level of information,

---

[161] The fact that in this simulation the value of e/c is the lowest possible – 0.01 – also contributes to the relative attractiveness of a strategic vote versus a sincere vote (when compared to simulations with higher values of e/c).

and according to the assumptions in our model, there was an interior equilibrium where 34.2% of Liberal Democrat supporters voted strategically towards the Labour Party, succeeding in defeating the incumbent.

As a second example, consider Devon North, a constituency where the Liberal Democrats came ahead of the Labour and defeated the Conservatives with a share of 50.8% of the overall votes.

For this constituency, λ = 39.5% and Rf = 83.9%. Following the same procedure just described, we look at the column λ = 39.5% in the Rf table, in Table 21, to find the closest value to 83.9% that is within a margin of 1% of that value. That value is 83.89%, , found in row (a/a+b) = 73%. Turning to the β^* table, we get a corresponding value for β^* of 59.7%.

The number of strategic votes estimated for this constituency is 3.614 and we have that γ = 65.2% < Ri = 73.0%, indicating that even if there was no strategic voting the Liberal Democrats would have won the election. In this case we see that the $\underline{Rf}$ value found in the Rf table – 83.89% – is very close to the observed value of Rf – 83.88% –, the reason why Ri and $\underline{Ri}$ are also very close.

4. Results

The vast amount of information involved in the simulation exercise is summarized in Figures 57 to 61. The full program used in MATLAB can be found in the Appendix B. We present the results using a top down approach, beginning with the higher level statistics and then decomposing them in further detail.

In Figure 57 we show the number of constituencies for which our program found some equilibrium and the number for which it did not, which are labelled respectively as “missing” and “not missing” constituencies in the column bars. On top of the bars we have (a reminder of) the total number of the relevant constituencies – 270 – and, floating above it, the percentage of “success”, i.e., the percentage of constituencies for which the computer identified a possible equilibrium amongst all those analysed.

The lowest number of constituencies found in the 24 computations made was 72, for the combination (e/c,(a+b)) = (0.01,5), amounting to 27% of the relevant constituencies. The higher number observed was 269, more than 99% of the overall constituencies, for the combination (e/c,(a+b)) = (3,5). In terms of comparative statics, we see that for each (a+b) the number of constituencies found increases as the e/c increases. Apart from two exceptions – (a+b) = 5 and e/c = 2 and e/c = 3 –, we also see that for each e/c the number of constituencies found increases as (a+b) increases. The interpretation of these effects requires us to look at Figure 58, where we decompose the constituencies found between sincere and interior equilibria.

Before that, though, it is worth mentioning that, as we said in the previous essay, the fact that for *each* combination of parameters some constituencies are missing counts is not supportive of our theory. The more constituencies are not found the more we are inclined to believe that some of the premises in our model may not describe reality accurately – at least to what concerns the context of the 1997 UK General Election.

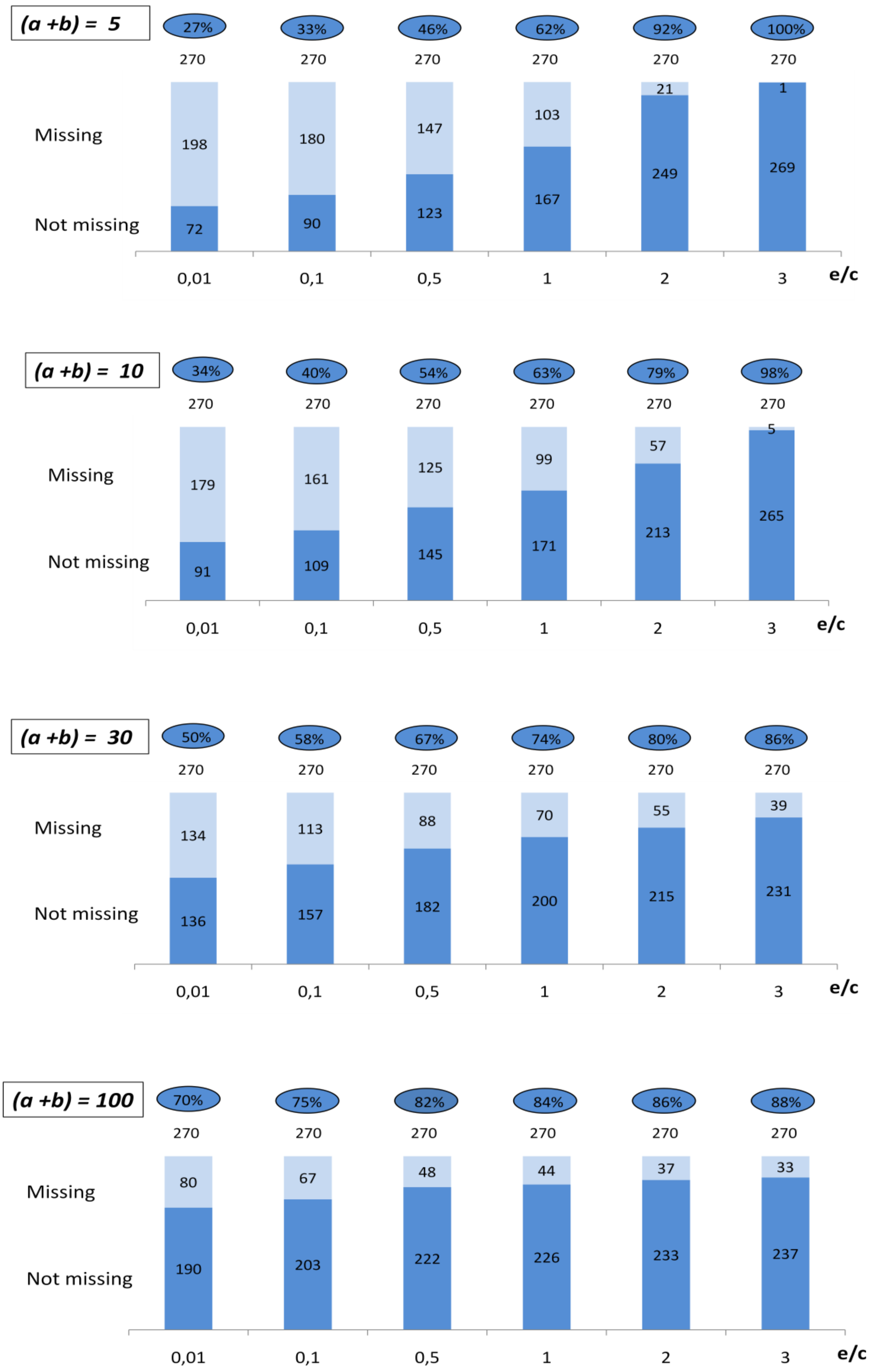

Figure 57 – Missing and not missing constituencies

It may be that voters do not have such an expressiveness concern; or they may have private information; or some of the parameters – degree of expressiveness and/or level of information – may vary from constituency to constituency. As stated upfront, the interest of this exercise lies considerably in getting a sense of whether a theory of voting with mixed motivations and public uncertainty is adherent to the reality or not. A negative answer, based on the analysis of the UK 1997 General Election, does not completely disprove the theory but raises important concerns and opportunities – such as applying it to a different reality and expanding it to a case of mixed information.

In Figure 58 we depict the composition of the constituencies found, indicating the percentages of sincere and interior equilibria found in the column bars, on top of which we recall, from Figure 57, the total number of constituencies found. The number floating above the bars refers to the number of interior equilibria found for each combination. This is the relatively more interesting number for us, since, the Duvergerian equilibrium not being possible, it is from the equilibria with partial strategic voting that we obtain strategic votes that add up to a total that will gives us a percentage of strategic voting in the risk population comparable to Fisher's.

We expect the number of sincere equilibria to go up with e/c, all else constant: a higher expressiveness concern makes it easier to sustain a sincere equilibrium; which indeed happens. We also observe that, for the same (a+b), not only does the number of sincere equilibria found goes up as e/c increases, but the percentages go up too. The numbers inside the elliptic curves show that the number of interior equilibria goes down as e/c goes up. Overall, and given the data from Figure 57, we know that the increase in the number of sincere equilibria found is higher than the decrease in the number of interior equilibria found, since the overall number of equilibria found goes up with e/c.

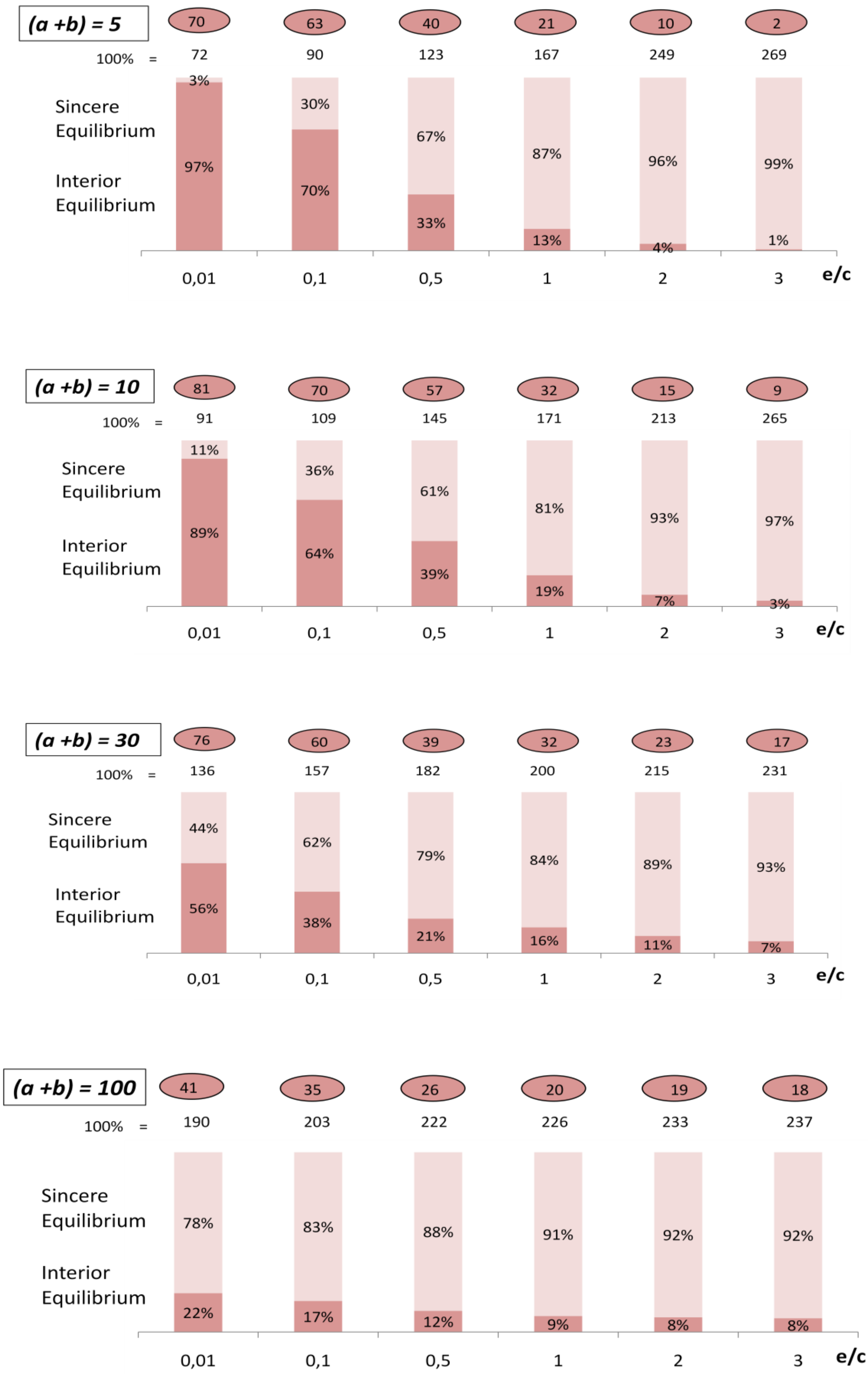

Figure 58 – Sincere and interior equilibria in non-missing constituencies

Adding up the data that results from the 24 simulations done for the 270 constituencies – an overall number of 6480 cases analysed –, we were able to find equilibria for 4461 of those cases, or 69% of the total. The 4461 cases found can be decomposed in 3585 sincere equilibria and 876 interior equilibria, respectively 79.7% and 20.3% of the 4461 found. Only 65 cases allowed for both a sincere and an interior equilibrium from a backward-looking perspective (meaning that the same Rf could have been generated by either equilibria, each of them having, therefore, different starting Ri values and implicit β^*, but resulting in the same Rf), which amounts to 1.5% of the total of 4461 constituencies found.

In Figure 59 we illustrate the "success" of strategic voting, here in the strict and more "usable", for an empirical study, sense that strategic voting is successful if it led to a change in the winner of the election that would have not happened if everyone had voted sincerely, regardless of that being a voting equilibrium or not. This is the idea behind the comparison made between Ri and γ, in Table 22, that allows us to assess whether strategic voting was critical or not in defeating the incumbent.

Overall, considering the 876 cases where an interior equilibrium was found, we get an average of 46% cases where strategic voting led to a constituency changing hands, compared to an hypothetical scenario of full sincere voting. This is the weighted average of the percentages shown in Figure 59, in the lower part of the column bar, indicating the percentage of change for each combination of e/c and (a+b). Keeping (a+b) constant, as e/c goes up we observe that, broadly speaking, the number of constituencies that change hands goes down while its percentage goes up.

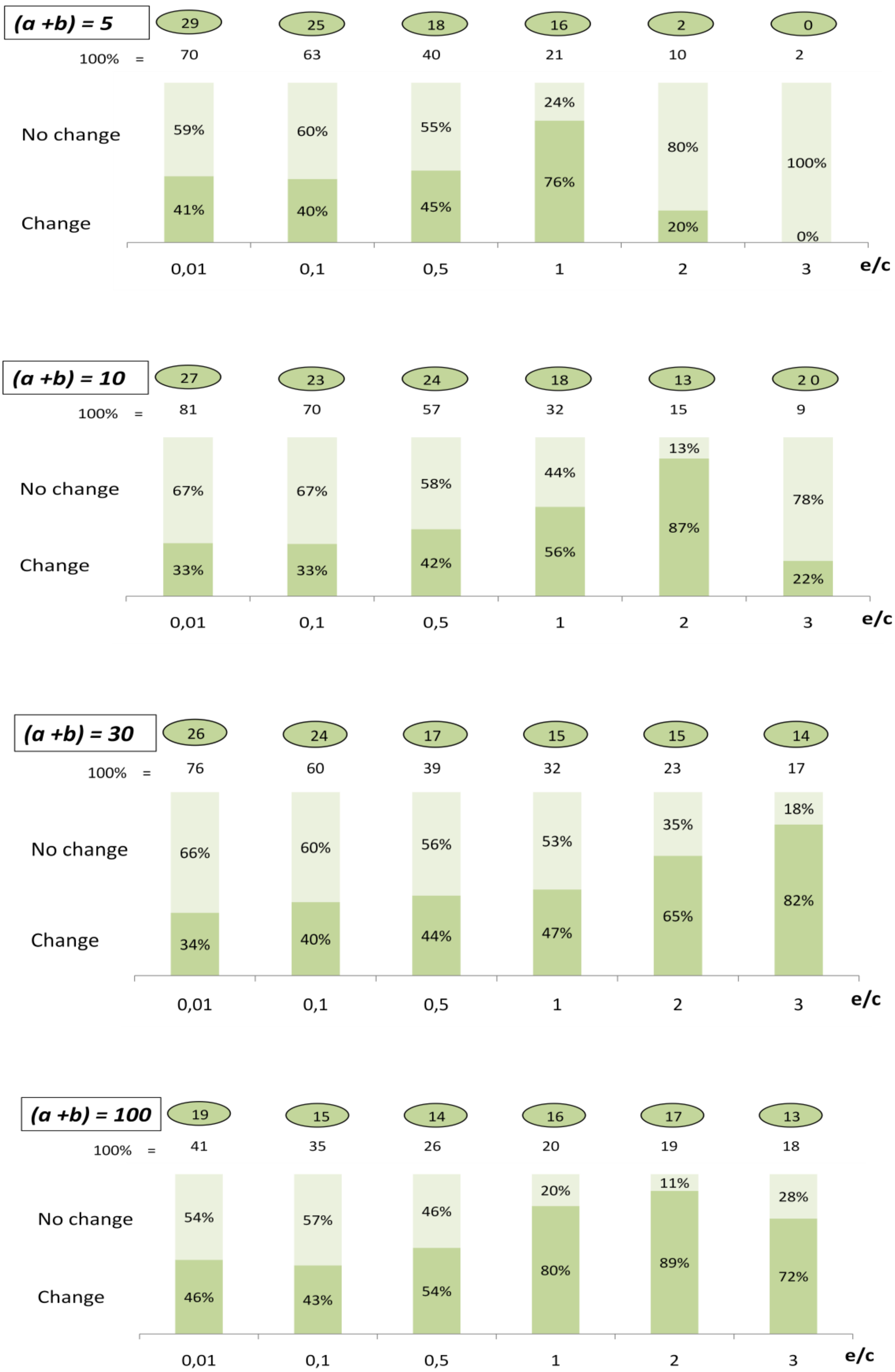

Figure 59 – Success of interior equilibria (change in constituencies)

The rational behind this is that, keeping (a+b) constant, the higher the e/c the higher the β* will be in any interior equilibrium. Therefore, there is a selection, as e/c goes up, towards having interior equilibria with higher values of β*, which on average means a higher likelihood of seeing the incumbent defeated. Despite the percentage going up, the drop in the constituencies where an interior equilibrium is detected is larger enough to lead to a decrease in the number of constituencies where strategic voting is pivotal in the sense above used. The variation both in the overall numbers and in percentages of the constituencies changing hands as (a+b) changes, keeping e/c the same, does not seem to follow a pattern.

There is no particular value that we should expect to see regarding the overall average for the statistic giving us the percentage of success of strategic voting in changing hands. The value we find is close to a half, but it bares no special meaning. The rationality of a strategic voting ex-ante does not imply a certain percentage of success ex-post, in terms of shifting votes, which is even clearer when we recall that in many (almost all) cases where sincere and strategic voting are possible for the same value of Ri (a value necessarily close to 0.5), a pattern of sincere voting always leads to the incumbent winning the election.

If we look at Tables 16-20 in the previous essay, we see that whether the red line (describing the interior equilibria) is above the "green line" (the level of Rf needed to defeat the incumbent) or not depends on λ. In our interpretation, strategic voting leads to a change in the constituency when not only the red line is above the green line, but the Ri that is found by (horizontally) looking at the red line is lower than γ. This guarantees that if there was no strategic voting the incumbent would not be defeated. Therefore, strategic voting is said to be, in this loose sense, pivotal, leading to a constituency changing hands (and whether or not the sincere voting constitutes an equilibrium for the calculated value Ri).

Again, recall that this is a different assertion from that made in page 209 (and explored in the previous essay), which takes a purely theoretical approach, demanding that, in order to tackle the issue of changing constituencies, the sincere equilibrium – as well as the observed interior equilibrium – has to be possible for us to consider that such constituency elegible for the analysis of whether or not strategic voting was successful. In short, there is a theoretical and an empirical approach to it, the later being the one we favour when dealing with the numbers, without forgetting the other (and more "correct" one, though not necessarily the most useful for all situations and aims) when we are more concerned with political implications of what comes out from our model.

In Figure 60 we decompose the constituencies changing hands to the Labour and to the Liberal Democratic Party. As expected, most of the changes occur towards the Labour Party. Averaging out the results for the 120 simulations, 82% of the time the change occurs towards Labour and 18% of the time towards the Liberal Democrats.

Last but not least, in Figure 61 we have the percentage of strategic voting in the risk population for the 120 simulations, and highlight the value, rounded to percentage units, proposed in Fisher (2004), 24%. Recall that in our model the "risk population" consists of the supporters of the trailing contender and is calculated by adding up the support of the third contender in all the constituencies where we find an equilibrium – sincere or interior, so each constituency that we find contributes for the denominator of this ratio.

The numerator needed to calculate that percentage involves adding up the strategic votes towards the second contender in all the constituencies where an interior equilibrium was detected. Therefore, this percentage can, in our model, be decomposed into two things: the percentage of interior equilibria overall and the (average) percentage of strategic voting in the constituencies where an interior equilibrium is found.

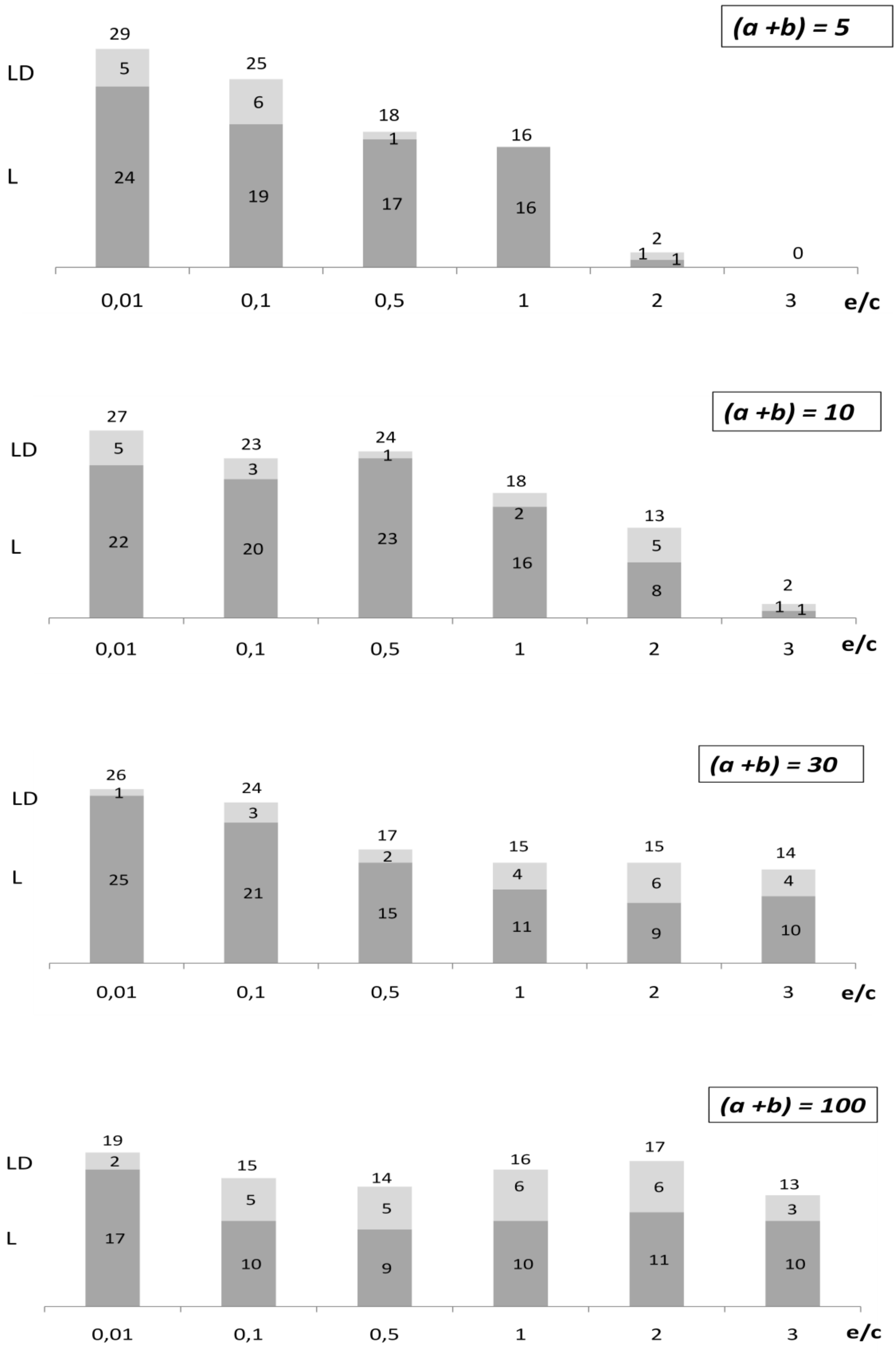

Figure 60 – Constituencies that changed hands

Figure 61 – Percentage of strategic voting over risk population

The percentage of strategic voting over the risk population varies, in our model, with the parameters (a+b) and e/c, as we see from Figure 61. The combination that gives a percentage closest to that is e/c = 0.01 and (a+b) = 30, with a percentage of 23%. This is a very low level of expressiveness and a high level of information (though not as high as the one we expect from a nation-wide opinion poll).

There is a significant drop in the percentage of risk population if e/c changes from 0.01 to 0.1 and to 0.5. Any value of e/c equal or higher than 1 leads to a percentage of strategic voting of less than 10%, with particularly low values for e/c = 2 and e/c = 3. This suggests that the expressiveness component cannot be too high. The percentage of strategic voting over the risk population is very sensitive to the degree of expressiveness, which indicates, following the previous comments – and this statistic is crucial in testing any voting model –, that e/c cannot be too high – even 0.1 may be too high.

We also see that the higher the degree of information the lower the percentage of strategic voting. In essay 2 we saw that the partial effect of an increase in information is almost always to decrease the percentage of strategic voting, via increase in β^* on each constituency[162]. Aggregating these individual effects results in the increase reflected in Figure 61.

In a broader context, however, the level of expressiveness e/c can be seen as not being so low, if we take into account that *e* is discounted by the value *c*, which indicates the altruistic-paternalistic concern for *each* voter in the constituency. As explained in the previous essay, values of e/c below 0.01 would make the model more strict in terms of a bimodality hypothesis, in the sense that the values of Rf allowed would be stricter in both sides: the range of values close to 0.5 would decrease because it would be harder to have a sincere

[162] See Figure 26 in page 133.

equilibrium with such low value for e/c (such as 0.001 or 0.0001) and the range of values closer to 1 would also go down. In other words, we can say that, not surprisingly, naturally, as e/c approaches zero our model comes closer to Cox's, and that would not be interesting to study. So, the somewhat high – seen in a broader context, outside our model – value of 0.01 for e/c is an interesting choice exactly because it allows us to test something beyond existing theories. The fact that our theory is more or less corroborated by the data should not change our pre-choice of expanding existing models to test alternative theories of voting.

For this combination of parameters (e/c = 0.01 and (a+b) = 30), only half of the constituencies were found to have an equilibrium – 136 out of 270 –, which means our results must be read with some caution. From the 136 constituencies found, an interior equilibrium was found for 76 of them (56%). Despite the fact that for a given combination of our parameters we do not find any equilibria for some constituencies, we can see that increasing e/c has a very high impact on the percentage of strategic voting over the risk population. Out of those 76 interior equilibria, 26 involve a change in constituencies, 25 from the Conservative to the Labour Party and 1 from the Conservative to the Liberal Democratic Party.

Myatt (2002) provides some estimates for the number of constituencies which would have changed hands -- 18 to 77 (out of 270), depending on the level of information, and according to his theory of purely instrumentally motivated voters, who have both public and private information Unfortunately, our numbers cannot be directly compared to Myatt's, because we do not find equilibria for all constituencies. It would be misleading to find a proportional number to compare with by multiplying the number found (26) by the ratio of the overall constituencies (270) over those found (136) – roughly 52.

That would require an assumption that the "sample" of constituencies found would be representative of the "universe" of the overall constituencies, which is not acceptable, because the constituencies that were not found fall on the "no zone" area, which is one of the essential characteristics of our theory – the extended bimodality hypothesis, implying that some values of Rf would never be observed.

Putting together the results obtained from the 24 simulations done, no equilibrium is found for 31% of the constituencies. The average overall level of strategic voting in the relevant population is 10.9%. We find that the higher the e/c, the lower is the number of constituencies not found and the lower is the percentage of strategic votes. In other words, these two statistics move in opposite ways and we cannot vary e/c to reach better results in both of them.

5. Conclusions

The essential characteristic of our theoretical model – that there is a vast "no zone" area for Rf, implying that such ratio can only be in the vicinities of 0.5 and 1 – is not convincingly supported by the election studied, given the large percentage of constituencies that fall in that area, for which no equilibrium can be found by the program developed. Overall, our theory does not seem to fit the data for the UK 1997 General Election very well.

Overall, considering all the 24 simulations we ran, 31% of the constituencies are not found, however, this statistic can be misleading, given that it is an average. For higher values of e/c it is easier to find more equilibria – which improves this statistic –, which will more than proportionally be sincere equilibria, and this lowers the percentage of statistic voting over the relevant population, an essential statistic for us.

The combination of parameters that gives a percentage of strategic voting over the risk population closest to that reported by Fisher – 24.4% – is e/c = 0.01 and (a+b) = 30, with a result of 23.0%. For this combination of parameters roughly 50% of the constituencies are not found by our model, which is of course a very large percentage.

The fact that we observed a significant number of "impossible" outcomes in our exercise, according to our model, means that something in it must be wrong – either everything in the model is correct but voters are not "smart" enough to play the right equilibrium; or everyone may be acting rationally but expectations are not met, meaning that the outcome is not a voting equilibrium; or some assumption in our model – namely the inclusion of an expressiveness component, the way we model information or the assumption that these two are the same across all constituencies – is wrong.

The first two possibilities would lead us nowhere, in the sense that "any" model may suffer from those problems. (This does not mean that they can not be a problem, rather that they should not, by their nature, be used as a panacea when comparing different models, since they would, perhaps with some – not too critical – differences, apply to all models.) The latter would imply, in any case, that our model is not capable of giving, in a simple snapshot, a good description of what may have been behind voter's motivations in the 1997 UK General Election.

On a broader perspective the results are not disappointing, though. To the extent that they come from a theoretical model not yet studied in the literature, that fills an important gap in terms of assumptions regarding the voters' information and motivations, an important contribution is made in what respects this particular election. It is a case where we can say that, due to its novelty in the literature, whatever news are good news. Something previously unknown is unveiled.

For the interested reader and researcher, a model involving both public and private information and expressive and instrumental motivations would be an interesting extension of the current model. It would also be fruitful to test the theory in other plurality rule elections that satisfy the main requirements of our model, namely having a clear candidate that is perceived by everyone as the incumbent and information that is largely publicly known.

APPENDIX 2 – MATLAB codes

This section includes the codes used in MATLAB to calculate the outcome from the 1997 UK General Election. They include the definition of relevant variables, the specification of the Beta-distribution function and the codes to detect both the sincere equilibria and the interior equilibria.

The sincere equilibria are detected by simply checking the value of the incentive function when $\beta^* = 1$.

The interior equilibria are found by asking the program to detect all the intersections of the incentive function with the 45o line and then choosing (if it exists) the intersection with the lowest derivative – in order to pick an interior stable equilibrium.

The outputs, as mentioned before, are a β^* table and the Rf table for each of the 24 simulations, along with a column with 1's and 0's for the sincere equilibria.

1 - Btilde.m

```
function[output]=Btilde(x,a,b,lambda,eoverc);

test=1+(2*lambda-1)/(1-lambda)*1/x;

if test>0

alphaL=test;

else

alphaL=-9999;

end

test=1-lambda/(1-lambda)*1/x;

if test>0

alphaM=test;

else

alphaM=-9999;

end

if alphaL<=-9998;

falphaL=-9999;

else

falphaL=alphaL^(a-1)*(1-alphaL)^(b-1)/(exp(gammaln(b)+gammaln(a)-
gammaln(a+b)));

end


if alphaM<=-9998;

falphaM=-9999;

else

falphaM=alphaM^(a-1)*(1-alphaM)^(b-1)/(exp(gammaln(b)+gammaln(a)-
gammaln(a+b)));

end

FAL=falphaL;

FAM=falphaM;
```

```
if FAL<=-9998;

output=-9999;

else

  if FAL>0.00001

      if FAM<=-9998

      output=eoverc*x/FAL;

      else

      output=(eoverc*x+FAM)/FAL;
      end

  else


  output=-9999;

  end

end
```

2 – Btildemin.m

```
function[output]=Btilde(x,a,b,lambda,eoverc);

test=1+(2*lambda-1)/(1-lambda)*1/x;

if test>0

alphaL=test;

else

alphaL=9999;

end


test=1-lambda/(1-lambda)*1/x;

if test>0

alphaM=test;

else

alphaM=9999;

end


if alphaL>=9998;

falphaL=9999;

else

falphaL=alphaL^(a-1)*(1-alphaL)^(b-1)/(exp(gammaln(b)+gammaln(a)-
gammaln(a+b)));

end


if alphaM>=9998;

falphaM=9999;

else
```

```
falphaM=alphaM^(a-1)*(1-alphaM)^(b-1)/(exp(gammaln(b)+gammaln(a)-
gammaln(a+b)));

end

FAL=falphaL;

FAM=falphaM;

if FAL>=9998;

output=9999;

else

  if FAL>0.00001

      if FAM>=9998

      output=eoverc*x/FAL;

      else

      output=(eoverc*x+FAM)/FAL;

      end

  else

  output=9999;

  end

end %%1
```

3 - Inspect.m

```
SINCEREVOTINGINSPECT1=SINCEREVOTING(:,:,1);

SINCEREVOTINGINSPECT2=SINCEREVOTING(:,:,2);

SINCEREVOTINGINSPECT3=SINCEREVOTING(:,:,3);

SINCEREVOTINGINSPECT4=SINCEREVOTING(:,:,4);

SINCEREVOTINGINSPECT5=SINCEREVOTING(:,:,5);

SINCEREVOTINGINSPECT6=SINCEREVOTING(:,:,6);


NONSINCEREVOTINGBETA1=NONSINCEREVOTING(:,:,1);

NONSINCEREVOTINGBETA2=NONSINCEREVOTING(:,:,2);

NONSINCEREVOTINGBETA3=NONSINCEREVOTING(:,:,3);

NONSINCEREVOTINGBETA4=NONSINCEREVOTING(:,:,4);

NONSINCEREVOTINGBETA5=NONSINCEREVOTING(:,:,5);

NONSINCEREVOTINGBETA6=NONSINCEREVOTING(:,:,6);

betastar1=betastar(:,:,1);

betastar2=betastar(:,:,2);

betastar3=betastar(:,:,3);

betastar4=betastar(:,:,4);

betastar5=betastar(:,:,5);

betastar6=betastar(:,:,6);

LaboverLabLD1=LaboverLabLD(:,:,1);

LaboverLabLD2=LaboverLabLD(:,:,2);

LaboverLabLD3=LaboverLabLD(:,:,3);

LaboverLabLD4=LaboverLabLD(:,:,4);

LaboverLabLD5=LaboverLabLD(:,:,5);

LaboverLabLD6=LaboverLabLD(:,:,6);
```

4 - Btildesmall.m

```
function[output]=Btildesmall(FAL,FAM,x,a,b,lambda,eoverc);

if FAL<=-9998;

output=-9999;

else

  if FAL>0.00001

      if FAM<=-9998

      output=eoverc*x/FAL;

      else

      output=(eoverc*x+FAM)/FAL;

      end

  else

  output=-9999;

  end

end
```

5 - Btildesolve.m

```
function[output]=Btildesolve(x,a,b,lambda,eoverc);

test=1+(2*lambda-1)/(1-lambda)*1/x;

if test>0

alphaL=test;

else

alphaL=9999;

end

test=1-lambda/(1-lambda)*1/x;

if test>0

alphaM=test;

else

alphaM=9999;

end

if alphaL>=9998;

falphaL=9999;

else

falphaL=alphaL^(a-1)*(1-alphaL)^(b-1)/(exp(gammaln(b)+gammaln(a)-
gammaln(a+b)));

end

if alphaM>=9998;

falphaM=9999;

else

falphaM=alphaM^(a-1)*(1-alphaM)^(b-1)/(exp(gammaln(b)+gammaln(a)-
gammaln(a+b)));

end

FAL=falphaL;
```

```
FAM=falphaM;
if FAL>=9998;
output=9999;
else
  if FAL>0.00001
      if FAM>=9998
      output=eoverc*x/FAL;
      else
      output=(eoverc*x+FAM)/FAL;
      end
  else
  output=9999;
  end
end
output;
output=min(output,1);
output=output-x;
```

6 – Tiago.m

```
EOVERCVECTOR=[0.01 0.1 0.5 1 2 3];
sumab=30;
for indexeoverc=1:length(EOVERCVECTOR)
eoverc=EOVERCVECTOR(indexeoverc);

LLAMBDA=[0.334:0.001:0.499];
for i=1:length(LLAMBDA)
for aoverab=50:99;
index=aoverab-49;
a=aoverab/100*sumab;
b=sumab-a;

lambda=LLAMBDA(i);

input=[0:0.01:1];

for k=1:length(input)
x0=input(k);
[x,fval]=fzero(@Btildesolve,x0,[],a,b,lambda,eoverc);
if x>=0 & x<1 & fval>=-0.001 & fval<=0.001
xx(k)=x;
else
xx(k)=-9999;
end
end
```

```
rr=find(xx>0);

if size(rr)>0

for h=1:length(rr)

V2=Btilde(xx(rr(h))+0.0001,a,b,lambda,eoverc);

V1=Btilde(xx(rr(h)),a,b,lambda,eoverc);

Derivative(h)=(V2-V1)/(0.0001);

end

[value,index2]=min(Derivative(:));

solution=xx(rr(index2));

else

solution=-9999;

end


[output]=Btilde(1,a,b,lambda,eoverc);

if output>=1

solutionsincere=1;

else

solutionsincere=0;

end


clear Derivative

clear xx

sincere(index,i,indexeoverc)=solutionsincere;



solution'

aoverab

i
```

```
LaboverLabLD(index,i,indexeoverc)=(a+b*(1-solution))./(a+b);

betastar(index,i,indexeoverc)=solution;

end

betastar(index,i,indexeoverc)

LaboverLabLD(index,i,indexeoverc)

end

end

save TESSADATA


load TESSADATA100


load TESSAVOTING

EOVERCVECTOR=[0.01 0.1 0.5 1 2 3];



NONSINCEREVOTING=zeros(length(DATAVOTING),1,length(EOVERCVECTOR));

NONSINCEREVOTING2=zeros(length(DATAVOTING),1,length(EOVERCVECTOR));

SINCEREVOTING=zeros(length(DATAVOTING),1,length(EOVERCVECTOR));

count=0;

for hh=1:length(DATAVOTING)

    if DATAVOTING(hh,3)==1

DATAVOTING(hh,4)=max(DATAVOTING(hh,5),
DATAVOTING(hh,6))/(DATAVOTING(hh,5)+DATAVOTING(hh,6));

    end

end

DATAVOTING(517,3)=0;

for indexeoverc=1:length(EOVERCVECTOR)

for hh=1:length(DATAVOTING)

if DATAVOTING(hh,3)==1
```

```
index4=round(DATAVOTING(hh,2)*1000-333);

index3=round(DATAVOTING(hh,4)*100)-49;

if sincere(index3,index4,indexeoverc)==1

SINCEREVOTING(hh,1,indexeoverc)=1;

else

SINCEREVOTING(hh,1,indexeoverc)=0;

end

[index5,index6]=find(LaboverLabLD(:,index4,indexeoverc)>=DATAVOTING(hh,
4)-0.01 & LaboverLabLD(:,index4,indexeoverc)<=DATAVOTING(hh,4)+0.01);


if size(index5)>0

diff=[];

hh

betastar(index5,index4,indexeoverc)

for k=1:min(length(index5),40)

   diff(k)=abs(DATAVOTING(hh,4)-
LaboverLabLD(index5(k),index4,indexeoverc))

end

[CC1,II1]=min(diff);


NONSINCEREVOTING(hh,1,indexeoverc)=betastar(index5(II1),index4,indexeov
erc);

NONSINCEREVOTING2(hh,1,indexeoverc)=index5(II1)+49;

if NONSINCEREVOTING(hh,1,indexeoverc)>0.9999

       NONSINCEREVOTING(hh,1,indexeoverc)=0;

       NONSINCEREVOTING2(hh,1,indexeoverc)=0;

end

end


end

end
```

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