Supplementary Information: Application of catastrophe theory to multicolor-laser-field-assisted scattering

Dino Habibović[1], Thomas Rook[2], Dejan B. Milošević[1,3]

[1]Faculty of Science, University of Sarajevo, Zmaja od Bosne 35, Sarajevo, 71000, Bosnia and Herzegovina.
[2]Department of Physics, University of Oxford, Clarendon Laboratory, Parks Road, Oxford, OX1 3PU, United Kingdom.
[3]Academy of Sciences and Arts of Bosnia and Herzegovina, Bistrik 7, Sarajevo, 71000, Bosnia and Herzegovina.

Supplementary Note 1: Analytic bifurcation sets for the $r\omega$–$s\omega$ field

We consider a bichromatic $r\omega$–$s\omega$ linearly polarized driving field with integer $s = n > r$, and $\xi_r = 1$, $\xi_s = \xi$, $\phi_r = 0$, $\phi_s = \phi$. In Fig. 4a)-d) for fixed values of the parameters r, s, and a_0, in the control parameters ξ and ϕ plane, we presented in polar coordinates numerical solutions of the system of equations $\mathcal{S}' = 0$ and $\mathcal{S}'' = 0$. For this field, the parametrized action can be written as

$$\begin{aligned}\mathcal{S}(\varphi;\xi,\phi,a_0) \;=\; & (k_\mathrm{f}-k_\mathrm{i})A_0[a_0\varphi+\sin(r\varphi)/r^2 \\ & +\xi\sin(s\varphi+\phi)/s^2]/\omega. \end{aligned} \tag{S1}$$

The number of real solutions of the stationary-phase (SP) equation $\mathcal{S}' = 0$ depends on three parameters: the relative phase ϕ, the field amplitude ratio ξ, and the parameter a_0. Our system of equations $\mathcal{S}' = 0$ and $\mathcal{S}'' = 0$ has the form

$$a_0 + \frac{1}{r}\cos(r\varphi) + \frac{\xi}{s}\cos(s\varphi+\phi) = 0, \tag{S2}$$

$$\sin(r\varphi) + \xi\sin(s\varphi+\phi) = 0. \tag{S3}$$

We will now show how this system can be solved analytically in a parametrized form $\xi(\varphi)$, $\phi(\varphi)$, $\varphi \in [0, 2\pi]$. Introducing new variable χ such that $s\varphi + \phi = s\chi$ and the notation $x = r\phi/s$, $s_j = \sin(j\chi)$, $c_j = \cos(j\chi)$, $j = r, s$, equations (S2) and (S3) can be solved over the control parameter ξ, with the result

$$\xi = -\frac{s}{r}\frac{ra_0 + c_r \cos x + s_r \sin x}{c_s} = -\frac{s_r \cos x - c_r \sin x}{s_s}. \tag{S4}$$

From equation (S4) it follows that

$$a \sin x + b \cos x + sra_0 s_s = 0, \tag{S5}$$

where

$$a = ss_r s_s + rc_r c_s, \quad b = sc_r s_s - rs_r c_s, \tag{S6}$$

so that the required solution for ϕ, with the notation $A = \sqrt{a^2 + b^2}$, $\tan\psi = b/a$, is

$$\phi = -\frac{s}{r}\left[\psi + \arcsin(rsa_0 s_s/A)\right], \tag{S7}$$

where a proper branch of the arcus function should be chosen. Therefore, our curve in the (ξ, ϕ) plane, parametrized by the variable $\chi = \varphi + \phi/s$, $\varphi \in [0, 2\pi]$, is given by equations (S4) and (S7).

In order to find the high codimension catastrophe points of LAS in bichromatic fields we solve the system of four equations $\mathcal{S}' = 0$, $\mathcal{S}'' = 0$, $\mathcal{S}''' = 0$, and $\mathcal{S}^{(\mathrm{IV})} = 0$. The first two equations are given by (S2) and (S3) while the remaining two equations are

$$r \cos(r\varphi) + s\xi \cos(s\varphi + \phi) = 0, \tag{S8}$$

$$r^2 \sin(r\varphi) + s^2\xi \sin(s\varphi + \phi) = 0. \tag{S9}$$

Using equations (S3) and (S9) and substituting $-(r^2/s^2)\sin(r\varphi)$ for $\xi\sin(s\varphi + \phi)$, one finds that $(1 - r^2/s^2)\sin(r\varphi) = 0$. For arbitrary values of r and s, this equation is satisfied if $\varphi = j\pi/r$, $j = 0, 1, \ldots, 2r$, $\varphi \in [0, 2\pi]$. Then using equations (S2) and (S8) and substituting $-(r/s^2)\cos(r\varphi)$ for $(\xi/s)\cos(s\varphi + \phi)$, we obtain

$$a_0 = \left(\frac{r}{s^2} - \frac{1}{r}\right)\cos(r\varphi). \tag{S10}$$

From this equation the values of a_0 for which a swallowtail catastrophe exits may be extracted. It follows straightforwardly to calculate the corresponding values of ξ and ϕ. This allows us to find the most degenerate critical point in parameter space, represented by the yellow point in Fig. 3. In this case, $r = 1$ and $s = 2$, so $\varphi = \pi$ and $a_0 = 3/4$, and hence $\phi = 0$ and $\xi = 1/2$ are the parameter values corresponding to the swallowtail catastrophe.

Supplementary Note 2: High-codimension catastrophes for trichromatic fields

For $n = 3$ our n-color linearly polarized laser field becomes a trichromatic $r\omega$–$s\omega$–$u\omega$ field. Choosing $\xi_r = 1$, $\phi_r = 0$, with the notation $\xi = \xi_s$, $\phi = \phi_s$, $\rho = \xi_u$, $\psi = \phi_u$, the field and the SP equation take the form

$$E(\varphi) = E_0[\sin(r\varphi) + \xi \sin(s\varphi + \phi) + \rho \sin(u\varphi + \psi)], \tag{S1}$$

$$a_0 + \frac{1}{r}\cos(r\varphi) + \frac{\xi}{s}\cos(s\varphi + \phi) + \frac{\rho}{u}\cos(u\varphi + \psi) = 0. \tag{S2}$$

Therefore, we have five control parameters: ξ, ϕ, ρ, ψ, and a_0. This means that it is conceivable that there is a codimension five catastrophe, where six stationary points coalesce for some point in the parameter space. The location of this point may be found by solving the system of equations given by $d^n\mathcal{S}(\varphi;\xi,\phi,\rho,\psi,a_0)/d\varphi^n = 0$, $n \in \{1,2,3,4,5,6\}$. Equation (S2) represents the condition $\mathcal{S}' = 0$, while for other equations of this system we get

$$\begin{aligned}
\sin(r\varphi) + \xi\sin(s\varphi+\phi) + \rho\sin(u\varphi+\psi) &= 0,\\
r\cos(r\varphi) + s\xi\cos(s\varphi+\phi) + u\rho\cos(u\varphi+\psi) &= 0,\\
r^2\sin(r\varphi) + s^2\xi\sin(s\varphi+\phi) + u^2\rho\sin(u\varphi+\psi) &= 0,\\
r^3\cos(r\varphi) + s^3\xi\cos(s\varphi+\phi) + u^3\rho\cos(u\varphi+\psi) &= 0,\\
r^4\sin(r\varphi) + s^4\xi\sin(s\varphi+\phi) + u^4\rho\sin(u\varphi+\psi) &= 0.
\end{aligned} \tag{S3}$$

Although solving the system of equations (S2) and (S3) appears to be a daunting task, the structure of these trigonometric equations allows us to do this analytically. Using the equations for $n = 4$ and $n = 6$, the functions $\sin(s\varphi + \phi)$ and $\sin(u\varphi + \psi)$ can be expressed via the function $\sin(r\varphi)$ as

$$\begin{aligned}
\xi\sin(s\varphi+\phi) &= -\frac{r^2}{s^2}\frac{u^2-r^2}{u^2-s^2}\sin(r\varphi),\\
\rho\sin(u\varphi+\psi) &= -\frac{r^2}{u^2}\frac{s^2-r^2}{s^2-u^2}\sin(r\varphi).
\end{aligned} \tag{S4}$$

Introducing this into the equation for $n = 2$, we obtain

$$\sin(r\varphi)\left[1 - \frac{r^2}{u^2-s^2}\left(\frac{u^2-r^2}{s^2} + \frac{r^2-s^2}{u^2}\right)\right] = 0. \tag{S5}$$

For arbitrary values of r, s, u, this equation is satisfied if $\varphi = j\pi/r$, $j = 0, 1, \ldots, 2r$, $\varphi \in [0, 2\pi]$. Similarly, using the stationarity equations for $n = 3, 5$, we get

$$\xi\cos(s\varphi+\phi) = -\frac{r}{s}\frac{u^2-r^2}{u^2-s^2}\cos(r\varphi),$$

$$\rho\cos(u\varphi+\psi) \;=\; -\frac{r}{u}\frac{s^2-r^2}{s^2-u^2}\cos(r\varphi), \tag{S6}$$

so that, using the stationarity equation for $n=1$, we derive the condition

$$\cos(r\varphi)\left[-\frac{1}{r}+\frac{r}{u^2-s^2}\left(\frac{u^2-r^2}{s^2}+\frac{r^2-s^2}{u^2}\right)\right]=a_0, \tag{S7}$$

using which we determine the values of a_0 for which the six stationary points coalesce. It follows straightforwardly to calculate the corresponding values of ξ, ϕ, ρ, and ψ. This allows us to determine the critical point in parameter space represented in Fig. 5. In this case we have $r=1$, $s=2$, $u=3$, so that equations (S5) and (S7) are satisfied for $\varphi=\pi$ and $a_0=2/3$. Using these values and equation (S4) we get $\phi=\psi=0$, while using equation (S6) we find that $\xi=0.8$ and $\rho=0.2$.

In principle, this procedure may be generalized for an m-color field, to solve $d^n\mathcal{S}(\varphi;\mathcal{C})/dt^n=0$, for $n\in\mathbb{N}_{<2m}$, and find the point in the $(2m-1)$-dimensional parameter space where $2m$ stationary points will coalesce. Namely, the phase of the first field component can be chosen to be equal to zero, while its amplitude can be absorbed in E_0, so that the dimension of the parameter space is $2(m-1)+1$, where $2(m-1)$ stands for for the phases and amplitudes of the field components except the first one, while 1 stands for the parameter a_0.