

Simulation of a simple particle system interacting through hitting times

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Abstract

We develop an Euler-type particle method for the simulation of a McKean–Vlasov equation arising from a mean-field model with positive feedback from hitting a boundary. Under assumptions on the parameters which ensure differentiable solutions, we establish convergence of order $1/2$ in the time step. Moreover, we give a modification of the scheme using Brownian bridges and local mesh refinement, which improves the order to 1. We confirm our theoretical results with numerical tests and empirically investigate cases with blow-up.

Keywords: McKean–Vlasov equations, particle method, timestepping scheme, Brownian bridge.

1 Introduction

There has been a recent surge in interest in mean-field problems, both from a theoretical and applications perspective. We focus on models where the interaction derives from feedback on the system when a certain threshold is hit. Application areas include electrical surges in networks of neurons and systemic risk in financial markets. As analytic solutions are generally not known, numerical methods are inevitable, but still lacking.

We therefore propose and analyse numerical schemes for the simulation of a specific McKean–Vlasov equation which exhibits key features of these models, namely

$$Y_t = Y_0 + W_t - \alpha L_t, \quad t \in [0, T], \quad (1)$$

$$L_t = \mathbb{P}(\tau \leq t), \quad t \in [0, T], \quad (2)$$

$$\tau = \inf\{t \in [0, T] : Y_t \leq 0\}, \quad (3)$$

where $\alpha, T \in \mathbb{R}_+$, W a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, on which is also given an \mathbb{R}_+ -valued random variable Y_0 independent of W . The non-linearity arises from the dependence of L_t in (1) on the law of Y . More specifically, if $t \rightarrow L_t$ has a derivative p_τ , which is then the density of the hitting time τ of zero,

$$dY_t = dW_t - \alpha p_\tau(t) dt, \quad t \in (0, T),$$

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so that the drift depends on the law of the path of Y .

Theoretical properties of (1)–(3) have been studied in Hambly et al. (2018), who prove the existence of a differentiable solution $(L_t)_{0 \leq t < t_*}$ up to an “explosion time” t_* . Conversely, they show that L cannot be continuous for all t for α above a threshold determined by the law of Y_0 . Such systemic events where discontinuities occur are also referred to as “blow-ups” in the literature.

The question of the constructive solution, however, remained open. Examples of a numerical solution computed with Algorithm 1 introduced in Section 2 are shown in Figure 1. The left plot shows the formation of a discontinuity in the loss function $t \rightarrow L_t$ for increasing α , with $Y_0 \sim \text{Gamma}(1.5, 0.5)$. The density of Y_T for T before and after the shock is displayed in the right panel.

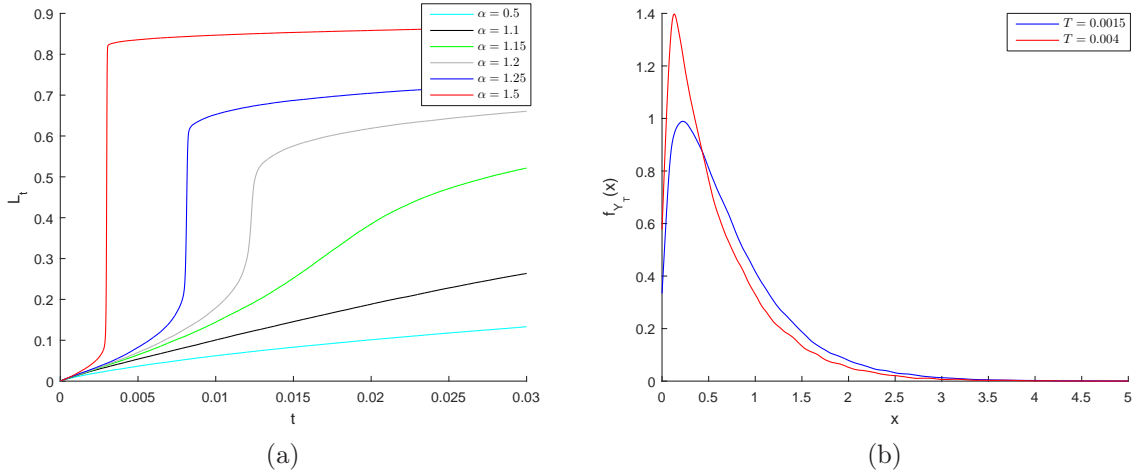


Figure 1: (a) L_t for different α near the jump; (b) distribution of Y_T for $Y_T > 0$ before and after the jump. Fitted by kernel density estimation with normal kernel for $N = 10^7$.

A similar model has been studied in Nadtochiy and Shkolnikov (2017), where the authors consider $\log(1 - L_t)$ instead of $-L_t$ in (1). Our numerical scheme can be applied in principle to this problem, but we concentrate the analysis on (1)–(3) in this paper.

One motivation for studying these equations comes from mathematical finance, in particular, systemic risk. A large interconnected banking network can be approximated by a particle system with interactions by which the default of one firm, modeled as the hitting of a lower default threshold of its value, causes a downward move in the firm value of others. More details can be found in Hambly et al. (2018) and Nadtochiy and Shkolnikov (2017). This model can also be viewed as the large pool limit of a structural default model for a pool of firms where interconnectivity is caused by mutual liabilities, such as in Lipton (2016).

An earlier version of this problem is found in neuroscience, where a large network of electrically coupled neurons can be described by McKean–Vlasov type equations (Cáceres et al. (2011); Carrillo et al. (2013); Delarue et al. (2015b,a)). If a neuron’s potential reaches some fixed threshold, it jumps to a higher potential level and sends a signal to

other neurons. This feedback leads to the following equations

$$X_t = X_0 + \int_0^t b(X_s) ds + \alpha \mathbb{E}[M_t] + W_t - M_t, \quad (4)$$

$$M_t = \sum_{k \geq 1} \mathbb{1}_{[0,t]}(\tau_k), \quad \tau_k = \inf\{t > \tau_{k-1} : X_{t-} \geq 1\}, \quad (5)$$

where $X_0 < 1$ a.s. The similarity to (1)–(3) is seen by noticing that $L_t = \mathbb{E}[\mathbb{1}_{[0,t]}(\tau)]$ and M_t in (5) is constant between hitting times, however, while in (4) an upper boundary is hit and after that the value resets to zero, in our model we are interested in hitting the zero boundary (from above), and after hitting the particle’s value remains zero.

While McKean–Vlasov equations are an active area of current research, to our knowledge, there is only a fairly small number of papers where their simulation is studied rigorously and none of these encompass the models above.

Early works include Bossy and Talay (1997), which proves convergence (with order 1/2 in the timestep and inverse number of particles) of a particle approximation to the distribution function for the measure ν_t of X_t in the classical McKean–Vlasov equation

$$X_t = X_0 + \int_{\mathbb{R}} \beta(X_t, u) \nu_t(du) dt + \int_{\mathbb{R}} \alpha(X_t, u) \nu_t(du) dW_t \quad (6)$$

with sufficient regularity. The proven rate in the timestep is improved to 1 in Antonelli et al. (2002) using Malliavin calculus techniques.

More recently, multilevel simulation algorithms have been proposed and analysed: Ricketson (2015) considers the special case of an SDE whose coefficients at time t depend on X_t and the expected value of a function of X_t ; Szpruch et al. (2017) study a method based on fixed point iteration for the general case (6). An alternative variance reduction technique by importance sampling is given in Dos Reis et al. (2018).

The system (1)–(3) above does not fall into the setting of (6) due to the extra path-dependence of the coefficients through the hitting time distribution. In this paper, we therefore propose and analyse a particle scheme for (1)–(3) with an explicit timestepping scheme for the nonlinear term. We simulate N exchangeable particles at discrete time points with distance h , whereby at each time point t we use an estimator of L_{t-h} from the previous particle locations to approximate (3). We prove the convergence of the numerical scheme up to some time T as the time step h goes to zero and number of particles goes to infinity. We also proved that the scheme can be extended up to the explosion time under certain conditions on the model parameters. The order in h for this standard estimator is 1/2. Next, we use Brownian bridges to better approximate the hitting probability, similar to barrier option pricing (Glasserman (2013)). In this case, the convergence order improves to $(1 + \beta)/2$, where $\beta \in (0, 1]$ is the Hölder exponent of the density of the initial value Y_0 (e.g., 0.5 in the example from Figure 8 below). The order can be improved to 1 for all $\beta \in (0, 1]$ by non-uniform time-stepping.

A main contribution of the paper is the first provably convergent scheme for equations of the type (1)–(3). The analysis uses a direct recursion of the error and regularity results proven in Hambly et al. (2018). This has the advantage that sharp convergence orders – i.e., consistent with the numerical tests – can be given, but also means that it seems difficult to apply the analysis directly to variations of the problem where such results are not available. Nonetheless, the method itself is natural and applicable in principle to other settings such as those outlined above.

The rest of the paper is organized as follows. In Section 2 we list the running assumptions and state the main results of the paper; in Sections 3.1 and 3.2 we prove the uniform convergence of the discretized process; in Section 3.3, we show the convergence of Monte Carlo particle estimators with an increasing number of samples; in Section 3.4 we prove the convergence order for the scheme with Brownian bridges; in Section 4 we give numerical tests of the schemes; finally, in Section 5, we conclude.

2 Assumptions and main results

We begin by listing the assumptions. The first one, Hölder continuity at 0 of the initial density, is key for the regularity of the solution. The Hölder exponent will also limit the rate of convergence of the discrete time schemes.

Assumption 1. *We assume that Y_0 has a density f_{Y_0} supported on \mathbb{R}_+ such that*

$$f_{Y_0}(x) \leq Bx^\beta, \quad x \geq 0, \quad (7)$$

for some $\beta \in (0, 1]$.

Under Assumption 1, we can refer to Theorem 1.8 in Hambly et al. (2018) for the existence of a unique, differentiable solution $t \rightarrow L_t$ for (1)–(3) up to time

$$t_* := \sup \{t > 0 : \|L\|_{H^1(0,t)} < \infty\} \in [0, \infty],$$

and a corresponding \hat{B} such that for every $t < t_*$

$$L'_t \leq \hat{B}t^{-\frac{1-\beta}{2}} \text{ a.e.} \quad (8)$$

This estimate admits a singularity of the rate of losses at time 0, however, what is actually observed in numerical studies (see Figure 1, left) is that the loss rate is bounded initially but then has a sharp peak for small β (and then especially for large α).

Integrating (8), we have for future reference a bound on L_t ,

$$L_t \leq \tilde{B}t^{\frac{1+\beta}{2}}, \quad (9)$$

where $\tilde{B} = 2\hat{B}/(1 + \beta)$.

The following assumption will be used to control the propagation of the discretisation error, by bounding the density (especially at 0) of the running minimum of Y and its approximations.

Assumption 2. *We assume that $T < \min(T^*, t_*)$, where T^* is defined by*

$$\alpha B \left[\sqrt{\frac{2T^*}{\pi}} + \alpha \tilde{B}(T^*)^{\frac{1+\beta}{2}} \right]^\beta = 1, \quad (10)$$

with B and \tilde{B} the smallest constants such that (7) and (9) hold for given β .

In the following, we assume that Assumptions 1 and 2 hold. Consider a uniform time mesh $0 = t_0 < t_1 < \dots < t_n = T$, where $t_i - t_{i-1} = h$, and a discretized process, for $1 \leq i \leq n$,

$$\tilde{Y}_{t_i} = Y_0 + W_{t_i} - \alpha \tilde{L}_{t_i}, \quad (11)$$

$$\tilde{L}_{t_i} = \mathbb{P}(\tilde{\tau} < t_i), \quad (12)$$

$$\tilde{\tau} = \min_{0 \leq j \leq n} \{\tilde{Y}_{t_j} \leq 0\}. \quad (13)$$

We extend \tilde{L}_{t_i} to $[0, T]$ by setting $\tilde{L}_s = \tilde{L}_{t_{i-1}}$ for $t_{i-1} < s < t_i$.

The first theorem, proven in Section 3.1, shows that \tilde{L}_t converges uniformly to L_t .

Theorem 1. *Consider \tilde{L}_{t_i} from (11)–(13) and L_t from (1)–(3). Then, for any $\delta > 0$, there exists $C > 0$ independent of h such that*

$$\max_{i \leq n} |\tilde{L}_{t_i} - L_{t_i}| \leq Ch^{\frac{1}{2}-\delta}. \quad (14)$$

We now propose a particle simulation scheme for (11)–(13) in Algorithm 1.

Algorithm 1 Discrete time Monte Carlo scheme for simulation of the loss process

Require: N — number of Monte Carlo paths

Require: n — number of time steps: $0 < t_1 < t_2 < \dots < t_n$

- 1: Draw N samples of Y_0 (from initial distribution) and W (a Brownian path)
 - 2: Define $\hat{L}_0 = 0$
 - 3: **for** $i = 1 : n$ **do**
 - 4: Estimate \tilde{L}_{t_i} by $\hat{L}_{t_i}^N = \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{\min_{j < i} \hat{Y}_{t_j}^{(k)} \leq 0\}}$
 - 5: **for** $k = 1 : N$ **do**
 - 6: Update $\hat{Y}_{t_i}^{(k)} = Y_0^{(k)} + W_{t_i}^{(k)} - \alpha \hat{L}_{t_i}^N$
 - 7: **end for**
 - 8: **end for**
-

In Section 3.3, we prove convergence in probability of Algorithm 1 as $N \rightarrow \infty$.

Theorem 2. *For all $i \leq n$,*

$$\hat{L}_{t_i} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \tilde{L}_{t_i}. \quad (15)$$

Next, we improve our scheme by using a Brownian bridge strategy to estimate the hitting probabilities. In order to do this, we consider the process

$$\check{Y}_t = Y_0 + W_t - \alpha \check{L}_t, \quad t \in [t_i, t_{i+1}), \quad (16)$$

$$\check{L}_t = \mathbb{P}(\check{\tau} < t_i), \quad t \in [t_i, t_{i+1}), \quad (17)$$

$$\check{\tau} = \inf_{0 \leq s \leq T} \{\check{Y}_s \leq 0\}. \quad (18)$$

Then, for each Brownian path $(W_t^{(k)})_{t \geq 0}$, we compute $\bar{Y}_t^{(k)} = Y_0^{(k)} + W_t^{(k)} - \alpha \bar{L}_t^N$ in (t_i, t_{i+1}) , where \bar{L}_t^N is an N -sample estimator of \check{L}_{t_i} given below. Hence, using Brownian

bridges, we compute

$$\begin{aligned}
p_{t_i}^{(k)} &= \mathbb{P} \left(\inf_{s < t_i} \bar{Y}_s^{(k)} > 0 \mid \bar{Y}_0^{(k)}, \dots, \bar{Y}_{t_i}^{(k)} \right) \\
&= \prod_{j=1}^i \mathbb{P} \left(\inf_{s \in [t_{j-1}, t_j)} \bar{Y}_s^{(k)} > 0 \mid \bar{Y}_{t_{j-1}}^{(k)}, \bar{Y}_{t_j}^{(k)} \right) \\
&= \prod_{j=1}^i \left(1 - \exp \left(-\frac{2(\bar{Y}_{t_{j-1}}^{(k)} \vee 0)(\bar{Y}_{t_j}^{(k)} \vee 0)}{h} \right) \right).
\end{aligned}$$

Thus, a natural choice for $\bar{L}_{t_i}^N$ is

$$\bar{L}_{t_i}^N = \frac{1}{N} \sum_{k=1}^N \left(1 - p_{t_i}^{(k)} \right). \quad (19)$$

As a result, the new algorithm with the Brownian bridge modification is the following.

Algorithm 2 Discrete time Monte Carlo scheme with Brownian bridge

Require: N — number of Monte Carlo paths

Require: n — number of time steps: $0 < t_1 < t_2 < \dots < t_n$

- 1: Draw N samples Y_0 (from the initial distribution) and W (a Brownian path)
 - 2: **for** $i = 1 : n$ **do**
 - 3: Estimate \bar{L}_{t_i} using (19)
 - 4: **for** $k = 1 : N$ **do**
 - 5: Update $\bar{Y}_{t_i}^{(k)} = Y_0^{(k)} + W_{t_i}^{(k)} - \alpha \bar{L}_{t_i}^N$
 - 6: **end for**
 - 7: **end for**
-

The convergence rate for (17) is given as follows and proven in Section 3.4.

Theorem 3. Consider \check{L}_t from (16)–(18) and L_t from (1)–(3), $\beta \in (0, 1]$ from Assumption 1. Then, there exists $C > 0$ independent of h such that

$$\max_{i \leq n} |\check{L}_{t_i} - L_{t_i}| \leq Ch^{\frac{1+\beta}{2}}. \quad (20)$$

We will later give a result with variable time steps which achieves rate 1 for all β .

3 Convergence results

3.1 Convergence of the timestepping scheme

In this section we prove Theorem 1. Then, in Section 3.2, under a modification of Assumption 2, we formulate and prove an improvement of this theorem which extends the applicable time interval.

The proof is based on induction on the error bound over the timesteps, which requires an error estimate of the hitting probability after discretisation (Lemmas 1 and 2), a sort of consistency, plus a control of the resulting misspecification of the barrier through a bound on the density of the running minimum (Lemma 3), a kind of stability.

As we have crude estimates on the densities, using only Assumption 1 but no sharper bound on, and regularity of, f_{Y_0} , or any regularity of the distribution of $\inf_{s \leq t}(W_s - \alpha L_s)$ or its approximations, the time T^* until which the numerical scheme is shown to converge will by no means be sharp. Indeed, in our numerical tests (see Section 4) we did not encounter any difficulties for any t , even $t > t_*$.

We first formulate these auxiliary results which we will use to prove Theorems 1, 3, and 4. The proofs are given in Appendix A.

First, we modify Proposition 1 from Asmussen et al. (1995), where an analogous result is shown for standard Brownian motion, without the presence of L and \tilde{L} (i.e., $\alpha = 0$).

Lemma 1. *Define $Y_t^* = Y_0 + W_{h\lfloor \frac{t}{h} \rfloor} - \alpha L_{h\lfloor \frac{t}{h} \rfloor}$ and $\tilde{Y}_t^* = Y_0 + W_{h\lfloor \frac{t}{h} \rfloor} - \alpha \tilde{L}_{h\lfloor \frac{t}{h} \rfloor}$ for $t < t_*$. Then, as $h \rightarrow 0$,*

$$\frac{1}{\sqrt{h}} \sup_{s \leq t} (Y_s - Y_s^*) \rightarrow_d \sqrt{2 \log \frac{t}{h}}, \quad (21)$$

and

$$\frac{1}{\sqrt{h}} \sup_{s \leq t} (\tilde{Y}_s - \tilde{Y}_s^*) \rightarrow_d \sqrt{2 \log \frac{t}{h}}. \quad (22)$$

The next lemma deduces the convergence rate of the hitting probabilities on the mesh.

Lemma 2. *Consider the processes Y and \tilde{Y} . Then, for any $\delta > 0$ there exist $\gamma, \tilde{\gamma} > 0$, independent of h and i , such that*

$$0 \leq \mathbb{P} \left(\min_{j < i} Y_{t_j} > 0 \right) - \mathbb{P} \left(\inf_{s < t_i} Y_s > 0 \right) \leq \gamma h^{\frac{1}{2} - \delta} \quad (23)$$

and

$$0 \leq \mathbb{P} \left(\min_{j < i} \tilde{Y}_{t_j} > 0 \right) - \mathbb{P} \left(\inf_{s < t_i} \tilde{Y}_s > 0 \right) \leq \tilde{\gamma} h^{\frac{1}{2} - \delta}, \quad (24)$$

where $t_i < t_*$.

Finally, we bound the probability that the running minimum is close to the boundary.

Lemma 3. *Consider $Z_i = \inf_{s \leq t_i} Y_s$, $\bar{Z}_i = \min_{j < i} Y_{t_j}$, and $\tilde{Z}_i = \min_{j < i} \tilde{Y}_{t_j}$ for some $i \leq n$ and $t_i < t_*$. Then, Z_i , \bar{Z}_i , and \tilde{Z}_i each have a density, denoted φ_i , $\bar{\varphi}_i$, and $\tilde{\varphi}_i$, respectively, with*

$$\max(\varphi_i(z), \bar{\varphi}_i(z), \tilde{\varphi}_i(z)) \leq B \left[(z \vee 0) + \sqrt{\frac{2t_i}{\pi}} + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}} \right]^\beta, \quad (25)$$

where β , B are from (7) and \tilde{B} is from (9).

We are now in a position to prove Theorem 1.

Proof of Theorem 1. We shall proceed by induction. For $t_0 = 0$, we have $L_0 = \tilde{L}_0$; for t_1 , we have $\min_{j < 1} \tilde{Y}_{t_j} = Y_0$ and $\mathbb{P}(Y_0 > 0) = 1$, hence,

$$|\tilde{L}_{t_1} - L_{t_1}| \leq \left| 1 - \mathbb{P} \left(\inf_{s < t_1} Y_s > 0 \right) \right|,$$

which can be estimated using Lemma 2, (23).

For $i > 1$, assume now that we have shown $\tilde{L}_{t_j} = L_{t_j} - \tilde{C}_j h^{\frac{1}{2}-\delta}$ for $j < i$, where $\tilde{C}_j \geq 0$ as $\tilde{L}_{t_j} \leq L_{t_j}$. We split the error into two contributions,

$$\begin{aligned} |\tilde{L}_{t_i} - L_{t_i}| &= \left| \mathbb{P} \left(\min_{j < i} \tilde{Y}_{t_j} > 0 \right) - \mathbb{P} \left(\inf_{s < t_i} Y_s > 0 \right) \right| \\ &\leq \left| \mathbb{P} \left(\min_{j < i} \tilde{Y}_{t_j} > 0 \right) - \mathbb{P} \left(\min_{j < i} Y_{t_j} > 0 \right) \right| + \left| \mathbb{P} \left(\min_{j < i} Y_{t_j} > 0 \right) - \mathbb{P} \left(\inf_{s < t_i} Y_s > 0 \right) \right|. \end{aligned}$$

We can use Lemma 2, (23), for the second term. For the first term,

$$\begin{aligned} \mathbb{P} \left(\min_{j < i} \tilde{Y}_{t_j} > 0 \right) - \mathbb{P} \left(\min_{j < i} Y_{t_j} > 0 \right) &= \mathbb{P} \left(\min_{j < i} \left(Y_{t_j} + \alpha \tilde{C}_j h^{\frac{1}{2}-\delta} \right) > 0 \right) - \mathbb{P} \left(\min_{j < i} Y_{t_j} > 0 \right) \\ &\leq \mathbb{P} \left(\min_{j < i} Y_{t_j} > -\alpha \max_{j < i} \tilde{C}_j h^{\frac{1}{2}-\delta} \right) - \mathbb{P} \left(\min_{j < i} Y_{t_j} > 0 \right) \\ &= \bar{F}_i(0) - \bar{F}_i \left(-\alpha \max_{j < i} \tilde{C}_j h^{\frac{1}{2}-\delta} \right) \\ &\leq \alpha \max_{j < i} \tilde{C}_j h^{\frac{1}{2}-\delta} \sup_{\theta \in [0,1]} \bar{\varphi}_i \left(-\theta \alpha \max_{j < i} \tilde{C}_j h^{\frac{1}{2}-\delta} \right), \end{aligned}$$

where $\bar{F}_i(x)$ and $\bar{\varphi}_i(x)$ are the CDF and pdf of $\min_{j < i} Y_{t_j}$.

Then, using Lemma 3, we have

$$\bar{\varphi}_i \left(-\theta \frac{\alpha}{2} \max_{j < i} \tilde{C}_j h^{\frac{1}{2}-\delta} \right) \leq B \left[\sqrt{\frac{2t_i}{\pi}} + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}} \right]^\beta,$$

as $-\theta \frac{\alpha}{2} \max_{j < i} \tilde{C}_j h^{\frac{1}{2}-\delta} < 0$. As a result, we have the following inequality for \tilde{C}_i ,

$$\tilde{C}_i \leq \alpha \max_{j < i} \tilde{C}_j B \left[\sqrt{\frac{2t_i}{\pi}} + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}} \right]^\beta + \gamma,$$

hence \tilde{C}_i is bounded independent of i and h by Assumption 2. By induction we get (14). \square

3.2 Extension of the result in time

The following result extends the applicability of Theorem 1 up to the explosion time t_* under certain conditions on the parameters, as specified precisely in (27) below.

We shall adapt Theorem 1 in Borovkov and Novikov (2005), which states: For a Lipschitz function f with Lipschitz constant K , and g such that $\sup_{s \leq t} |f(s) - g(s)| \leq \varepsilon$,

$$|\mathbb{P}(\exists s \in [0, t] : W_s < f(s)) - \mathbb{P}(\exists s \in [0, t] : W_s < g(s))| \leq \left(2.5K + \frac{2}{\sqrt{t}} \right) \varepsilon.$$

By Remark 2 in Borovkov and Novikov (2005), this result can be improved for a non-decreasing function g . Indeed, retracing the steps in their proof and using monotonicity, one finds easily the slightly better bound

$$|\mathbb{P}(\exists s \in [0, t] : W_s < f(s)) - \mathbb{P}(\exists s \in [0, t] : W_s < g(s))| \leq 2 \left(K + \frac{1}{\sqrt{t}} \right) \varepsilon.$$

In our case, we cannot directly apply the result with $f(s) = -Y_0 + \alpha L_s$ and $g(s) = -Y_0 + \alpha \tilde{L}_s$ as f is not guaranteed to be Lipschitz at $s = 0$. But, along the lines of the proof of Theorem 1 in Borovkov and Novikov (2005), the above result can be modified as follows:

Lemma 4. *For a non-decreasing function g which is Lipschitz with constant K on $[T^*, T]$, and a function f such that $f(t) = g(t)$ for $t \leq T^*$ and $\sup_{s \leq T} |f(s) - g(s)| \leq \varepsilon$,*

$$|\mathbb{P}(\exists s \in [0, T] : W_s < f(s)) - \mathbb{P}(\exists s \in [0, T] : W_s < g(s))| \leq 2 \left(K + \frac{1}{\sqrt{T}} \right) \varepsilon. \quad (26)$$

Theorem 4. *Assume T^* from (10) satisfies*

$$2\alpha \left(\hat{B}(T^*)^{-\frac{1-\beta}{2}} + \frac{1}{\sqrt{T^*}} \right) < 1. \quad (27)$$

Then Theorem 1 holds for any $T < t_$ (and not only up to T^* as per Assumption 2).*

Proof. We first split again the error by

$$\begin{aligned} |\tilde{L}_{t_i} - L_{t_i}| &= \left| \mathbb{P} \left(\min_{j < i} \tilde{Y}_{t_j} > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\ &\leq \left| \mathbb{P} \left(\inf_{s \leq t_i} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right| + \left| \mathbb{P} \left(\min_{j < i} \tilde{Y}_{t_j} > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} \tilde{Y}_s > 0 \right) \right|. \end{aligned} \quad (28)$$

The second term can be estimated from Lemma 2, (24). Again, we shall then proceed by induction. We have already shown that $|\tilde{L}_{t_i} - L_{t_i}| \leq C_i h^{\frac{1}{2}-\delta}$ for $t_i \leq T^*$ according to Theorem 1.

Consider $T^* < t_i \leq T$. Assume we have shown that $|L_{t_j} - \tilde{L}_{t_j}| \leq C_j h^{\frac{1}{2}-\delta}$ for $j < i$. We want to derive C_i such that $|L_{t_i} - \tilde{L}_{t_i}| \leq C_i h^{\frac{1}{2}-\delta}$ and where all C_i are bounded independent of h . First, consider an intermediate point $s \in (t_{i-1}, t_i)$, then

$$L_s - \tilde{L}_s \leq L_{t_{i-1}} + K(s - t_{i-1}) - \tilde{L}_{t_{i-1}} \leq L_{t_{i-1}} - \tilde{L}_{t_{i-1}} + Kh,$$

with K the Lipschitz constant of L , and thus

$$\sup_{s \leq t_i} |L_s - \tilde{L}_s| \leq \max \left(\max_{j < i} |L_{t_j} - \tilde{L}_{t_j}| + Kh, |L_{t_i} - \tilde{L}_{t_i}| \right). \quad (29)$$

Now we show that $|L_{t_i} - \tilde{L}_{t_i}| \leq C_{t_i} h^{\frac{1}{2}-\delta}$. Consider

$$l_t = \begin{cases} L_t, & t \leq T^*, \\ \tilde{L}_t, & t > T^*, \end{cases}$$

and $Y_t^l = Y_0 + W_t - \alpha l_t$. Then,

$$\begin{aligned} &\left| \mathbb{P} \left(\inf_{s \leq t_i} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\ &\leq \left| \mathbb{P} \left(\inf_{s \leq t_i} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s^l > 0 \right) \right| + \left| \mathbb{P} \left(\inf_{s \leq t_i} Y_s^l > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right|. \end{aligned} \quad (30)$$

To estimate the first term, we can write

$$\begin{aligned}
\left| \mathbb{P} \left(\inf_{s \leq t_i} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s^l > 0 \right) \right| &= \mathbb{E} \left[\mathbb{1}_{\{\exists t \in [0, t_i] : Y_t^l \leq 0, \forall s \in [0, t_i] \tilde{Y}_s > 0\}} \right] \\
&= \mathbb{E} \left[\mathbb{1}_{\{\exists t \in [0, T^*] : Y_t^l \leq 0, \forall s \in [0, t_i] \tilde{Y}_s > 0\}} \right] \\
&\leq \mathbb{E} \left[\mathbb{1}_{\{\exists t \in [0, T^*] : Y_t^l \leq 0, \forall s \in [0, T^*] \tilde{Y}_s > 0\}} \right] \\
&= \left| \mathbb{P} \left(\inf_{s \leq T^*} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq T^*} Y_s^l > 0 \right) \right|,
\end{aligned}$$

where we have used in the second line that since $l_t = \tilde{L}_t$ for $t > T^*$, hitting after T^* does not affect the difference. For $t \leq T^*$, $l_t = L_t$. Then, using Theorem 1,

$$\left| \mathbb{P} \left(\inf_{s \leq t_i} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s^l > 0 \right) \right| \leq \left| \mathbb{P} \left(\inf_{s \leq T^*} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq T^*} Y_s > 0 \right) \right| \leq \bar{C}^{T^*} h^{\frac{1}{2}-\delta}.$$

For the second term in (30), L_t is Lipschitz on $[T^*, T]$ with $K = \hat{B}(T^*)^{-\frac{1-\beta}{2}}$ and we can apply (26). Thus,

$$\left| \mathbb{P} \left(\inf_{s \leq t_i} \tilde{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \leq 2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right) \sup_{s \leq t_i} |L_s - \tilde{L}_s| + \bar{C}^{T^*} h^{\frac{1}{2}-\delta} \quad (31)$$

$$\leq 2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right) \max \left(\max_{j < i} |L_{t_j} - \tilde{L}_{t_j}| + Kh, |L_{t_i} - \tilde{L}_{t_i}| \right) + \bar{C}^{T^*} h^{\frac{1}{2}-\delta}, \quad (32)$$

using (29). Moreover, by (24) and (32), (28) can be written as

$$|L_{t_i} - \tilde{L}_{t_i}| \leq 2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right) \max \left(\max_{j < i} |L_{t_j} - \tilde{L}_{t_j}| + Kh, |L_{t_i} - \tilde{L}_{t_i}| \right) + \bar{\gamma} h^{\frac{1}{2}-\delta},$$

where $\bar{\gamma} = \tilde{\gamma} + \bar{C}^{T^*}$.

Taking into account (27), we have

$$|L_{t_i} - \tilde{L}_{t_i}| \leq \max \left(\left(2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right) \max_{j < i} C_j + \bar{\gamma} + \tilde{\varepsilon} \right) h^{\frac{1}{2}-\delta}, \frac{\bar{\gamma}}{1 - 2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right)} h^{\frac{1}{2}-\delta} \right),$$

where $\tilde{\varepsilon} = 2\alpha \left(K + \frac{1}{\sqrt{T^*}} \right) Kh^{\frac{1}{2}+\delta}$.

As a result, we have the following inequality for C_i ,

$$C_i \leq \max \left(\left(2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right) \max_{j < i} C_j + \bar{\gamma} + \tilde{\varepsilon} \right), \frac{\bar{\gamma}}{1 - 2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right)} \right).$$

Since the Lipschitz constant $K \leq \hat{B}(T^*)^{-\frac{1-\beta}{2}}$, using (27), $2\alpha \left(K + \frac{1}{\sqrt{t_i}} \right) < 1$. Hence, C_i is bounded independent of h , and by induction we get the result. \square

3.3 Monte Carlo simulation of discretized process

In this section, we prove the convergence in probability of

$$\hat{L}_{t_i} = \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{\min_{j < i} \hat{Y}_{t_j}^{(k)} \leq 0\}}$$

in Algorithm 1 to \tilde{L}_{t_i} as $N \rightarrow \infty$. We note that we cannot directly apply the law of large numbers, as the summands are dependent through $\hat{L}_{t_j}^N, j < i$. However, we see below that the dependence diminishes (i.e., the covariance goes to zero) as $N \rightarrow \infty$, which easily gives convergence, albeit without a Central Limit Theorem-type error estimate or a rate for the variance.

First, we formulate an auxiliary lemma.

Lemma 5. *Consider $i \leq n$. Assume for all $j < i$*

$$\hat{L}_{t_j}^N \xrightarrow{\mathbb{P}} \tilde{L}_{t_j}. \quad (33)$$

Then,

$$\mathbb{E}[\hat{L}_{t_i}^N] \xrightarrow{N \rightarrow \infty} \tilde{L}_{t_i} \quad (34)$$

$$\mathbb{V}[\hat{L}_{t_i}^N] \xrightarrow{N \rightarrow \infty} 0. \quad (35)$$

The proof is given in Appendix B. Now we can deduce the convergence instantly.

Proof of Theorem 2. The proof is immediate by induction. The statement is true for $i = 0$. Now take $i \geq 1$. By Lemma 5, there exists N^* such that for all $N > N^*$,

$$|\mathbb{E}[\hat{L}_{t_i}^N] - \tilde{L}_{t_i}| \leq \frac{\varepsilon}{2}.$$

Thus, by Chebyshev's inequality, we have

$$\mathbb{P}(|\hat{L}_{t_i}^N - \tilde{L}_{t_i}| > \varepsilon) \leq \mathbb{P}\left(|\hat{L}_{t_i}^N - \mathbb{E}[\hat{L}_{t_i}^N]| > \frac{\varepsilon}{2}\right) \leq \frac{4\mathbb{V}[\hat{L}_{t_i}^N]}{\varepsilon^2}.$$

Using again Lemma 5, we have that $\mathbb{V}[\hat{L}_{t_i}^N] \xrightarrow{N \rightarrow \infty} 0$. Hence,

$$\hat{L}_{t_i}^N \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \tilde{L}_{t_i},$$

for i and by induction we have proved the theorem. \square

3.4 Brownian bridge convergence improvement

In this section, we prove Theorem 3, which ascertains the uniform convergence (in t) of $\check{\check{L}}$ to L at the improved rate.

Proof of Theorem 3. We shall proceed by induction. For $i = 0$, we have that $\check{\check{L}}_0 = L_0 = 0$. Assume we have shown that $|\check{\check{L}}_{t_j} - L_{t_j}| \leq C_j h^{\frac{1+\beta}{2}}$ for all $j < i$ with some $C_j > 0$, and we want to estimate $|\check{\check{L}}_{t_i} - L_{t_i}|$. First, for $j > 0$ we have

$$\sup_{t_j \leq s < t_{j+1}} |\check{\check{L}}_s - L_s| \leq |\check{\check{L}}_{t_j} - L_{t_j}| + \hat{B} h^{\frac{1+\beta}{2}} \leq (C_j + \hat{B}) h^{\frac{1+\beta}{2}}, \quad (36)$$

since $L'_\zeta \leq \hat{B}\zeta^{-\frac{1-\beta}{2}}$, while for $j = 0$, $\sup_{0 \leq s < t_1} |\check{L}_s - L_s| \leq L_{t_1} \leq \tilde{B}h^{\frac{1+\beta}{2}}$. Now consider

$$\begin{aligned}
|\check{L}_{t_i} - L_{t_i}| &= \left| \mathbb{P} \left(\inf_{s \leq t_i} \check{Y}_s > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\
&= \left| \mathbb{P} \left(\inf_{s \leq t_i} (Y_s + \alpha(L_s - \check{L}_s)) > 0 \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\
&\leq \left| \mathbb{P} \left(\inf_{s \leq t_i} Y_s > -\alpha \sup_{s < t_i} |\check{L}_s - L_s| \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\
&\leq \mathbb{P} \left(\inf_{s \leq t_i} Y_s > -\alpha \max_{j < i} (C_j + \hat{B})h^{\frac{1+\beta}{2}} \right) - \mathbb{P} \left(\inf_{s \leq t_i} Y_s > 0 \right) \\
&\leq \alpha \max_{j < i} (C_j + \hat{B}) \sup_{\theta \in [0,1]} \varphi \left(-\theta \alpha \max_{j < i} (C_j + \hat{B})h^{\frac{1+\beta}{2}} \right) h^{\frac{1+\beta}{2}},
\end{aligned}$$

where $\varphi_i(x)$ is the density of $\inf_{s < t_i} Y_s$.

Then, using Lemma 3, we have

$$C_i \leq \alpha B \max_{j < i} (C_j + \hat{B}) \left[\sqrt{\frac{2t_i}{\pi}} + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}} \right]^\beta \leq \gamma \sum_{k=0}^i (\alpha B)^k \prod_{j=1}^k \left[\sqrt{\frac{2t_j}{\pi}} + \alpha \tilde{B} t_j^{\frac{1+\beta}{2}} \right]^\beta,$$

where $\gamma = \alpha B \hat{B} \left[\sqrt{\frac{2T}{\pi}} + \alpha \tilde{B} T^{\frac{1+\beta}{2}} \right]^\beta$.

Thus, C_i is bounded independent of h and i by (10). By induction we get (20). \square

The proof of Theorem 3 indicates that the order is limited by the behaviour of L for small t . The next result shows that a locally refined time mesh achieves convergence order 1 for all β .

Corollary 1. *Consider a non-uniform time mesh $t_i = (ih)^{\frac{2}{1+\beta}}$ for $0 \leq i \leq n$ with $h = T^{\frac{1+\beta}{2}}/n$. Then, there exists $C_1 > 0$, independent of h , such that*

$$\max_{i \leq n} |\check{L}_{t_i} - L_{t_i}| \leq C_1 h.$$

Proof. We shall proceed by induction as in the proof of Theorem 3 above. We now have the time step

$$t_{j+1} - t_j = ((j+1)h)^{\frac{2}{1+\beta}} - (jh)^{\frac{2}{1+\beta}} = \frac{2}{1+\beta} h^{\frac{2}{1+\beta}} (j + \xi)^{\frac{1-\beta}{1+\beta}},$$

for some $\xi \in (0, 1)$, and, for $j > 0$,

$$L'_\zeta \leq \hat{B} t_j^{-\frac{1-\beta}{2}} = \hat{B} (jh)^{-\frac{1-\beta}{1+\beta}}.$$

Hence, for $j > 0$,

$$\sup_{t_j \leq s < t_{j+1}} |\check{L}_s - L_s| \leq |\check{L}_{t_j} - L_{t_j}| + L'_\zeta (t_{j+1} - t_j) \leq |\check{L}_{t_j} - L_{t_j}| + \bar{B}h, \quad (37)$$

for some $\bar{B} > 0$ independent of h and j . We treat $j = 0$ separately and obtain directly

$$\sup_{0 \leq s < t_1} |\check{L}_s - L_s| \leq L_{t_1} \leq \tilde{B}h.$$

Repeating the remaining steps of the proof of Theorem 3, we get the result. \square

4 Numerical experiments

In this section, we demonstrate that the proven convergence orders in the timestep of $1/2$, $(1 + \beta)/2$, and 1 for the different methods are indeed sharp in the case of regular solutions; that the empirical variance of N -sample estimators is $1/N$; and that the method also converges experimentally in the presence of blow-up. To show this, we study three test cases with varying regularity of the initial data and of the loss function.

4.1 Lipschitz initial data and no blow-up

In our first experiment, we choose Y_0 such that $\frac{1}{Y_0} \sim \exp(\lambda)$, which guarantees that the density decays exponentially near zero. We take $\lambda = 1$ in our experiments, and pick the parameters in (1) to be $\alpha = 0.8$, $T = 2$. The solution is found to be continuous.

We perform numerical simulations using Algorithms 1 and 2 with $N = 2 \times 10^5$ particles and different time meshes varying from 50 to 3200 points. To estimate the error, we consider the difference $|\tilde{L}_T^{2n} - \tilde{L}_T^n|$ between the solutions with n and $2n$ timesteps, respectively, computed with the same paths. The results, presented in Figure 2, agree with the theory: for Algorithm 1, we get the convergence rate $\frac{1}{2}$, and for Algorithm 2, the rate is 1, because the initial distribution is regular enough around 0, i.e. $\beta = 1$ in (1). We also investigate the convergence rate of \hat{L}_{t_i} to \tilde{L}_{t_i} and \bar{L}_{t_i} to \check{L}_{t_i} empirically. In order

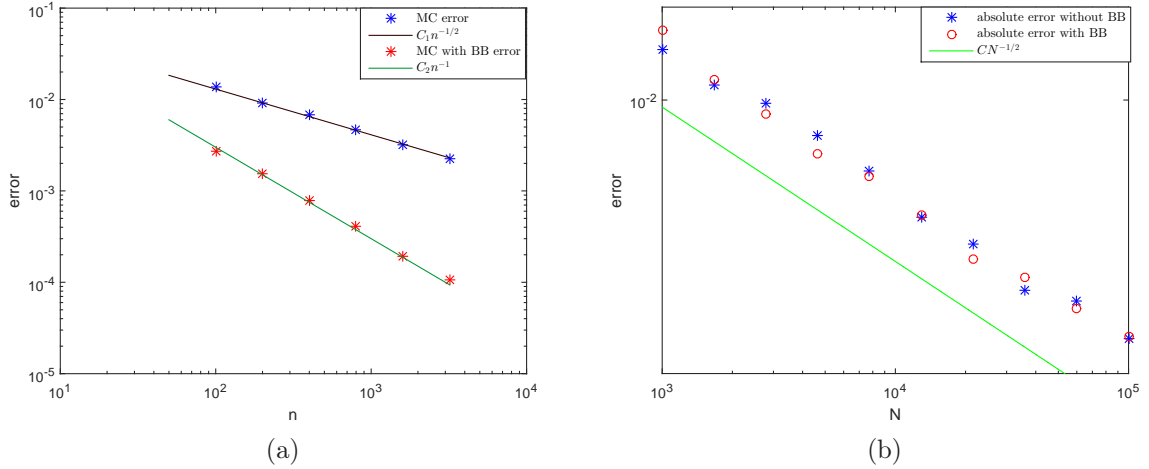


Figure 2: Error of the loss process at $t = T$ for $\frac{1}{Y_0} \sim \exp(1)$: (a) for increasing number n of timesteps; (b) for increasing number N of samples, both for Algorithms 1 and 2.

to compute the benchmark solution, we used $N = 5 \times 10^7$ particles. From the results we conclude that both Algorithm 1 and 2 have the convergence rate $\frac{1}{2}$ in N .

To illustrate the dependence on the parameter α , we include in Figure 3 plots for L_t and L'_t for different values of α . We evaluate L'_t numerically using a central finite difference approximation. In order to reduce the Monte Carlo noise, we increase N to 5×10^7 and reduce n to 200.

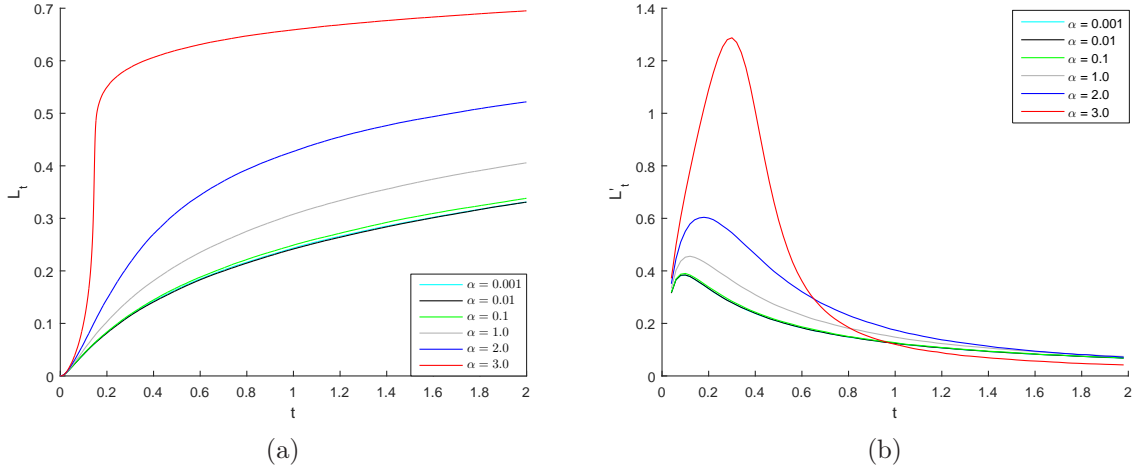


Figure 3: (a) L_t and (b) L'_t for different values of α .

4.2 Hölder 1/2 initial data and no blow-up

In another example, we again consider $\alpha = 0.8$ and $T = 2$, but choose $Y_0 \sim \text{Gamma}(1 + \beta, 1/2)$, i.e. with density

$$f_{Y_0}(x) = \frac{x^\beta e^{-x/2}}{\Gamma(1 + \beta) 2^{1+\beta}},$$

such that we have that $f_{Y_0}(x) \leq Cx^\beta$ for $x > 0$ and some $C > 0$. We choose $\beta = \frac{1}{2}$. The solution is again found to be continuous.

We perform numerical simulations using Algorithm 1, and Algorithm 2 on uniform and non-uniform meshes varying from 50 to 3200 points and with $N = 2 \times 10^5$ particles. The results are presented in Figure 4. As predicted by the theory, for Algorithm 1 we get the convergence rate $\frac{1}{2}$, for Algorithm 2 on uniform meshes rate $\frac{1+\beta}{2} = \frac{3}{4}$, and for Algorithm 2 on non-uniform meshes rate 1. As in the previous example, we also investigate the

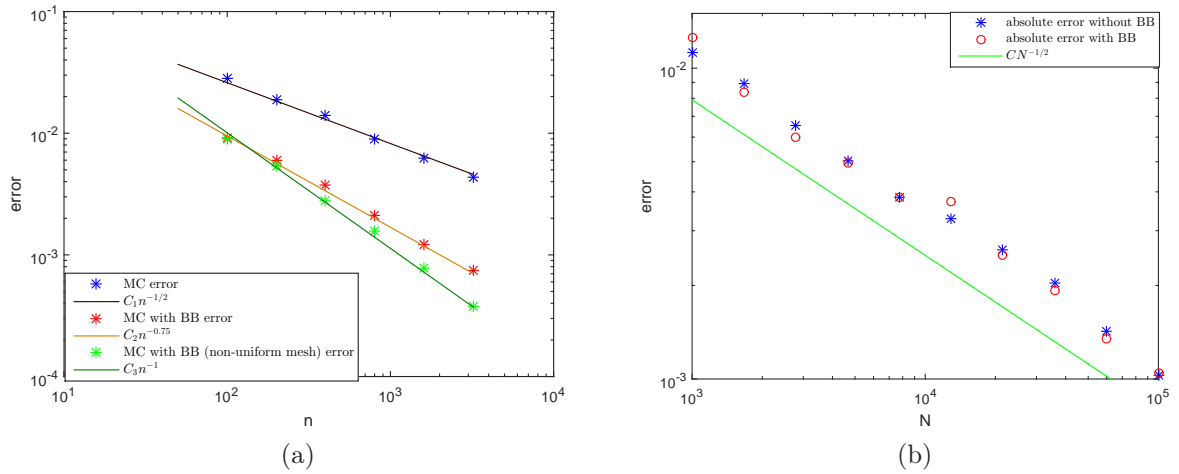


Figure 4: Error of the loss process at $t = T$ for $Y_0 \sim \text{Gamma}(3/2, 1/2)$: (a) for increasing number n of timesteps; (b) for increasing number N of samples, both for Algorithms 1 and 2.

convergence rate in N empirically. These results also confirm $\frac{1}{2}$ convergence rate in N for both Algorithms 1 and 2. In Figure 5 we present the dependence of L_t and L'_t on the parameter α . As in the previous example, we use $N = 5 \times 10^7$ and $n = 200$.

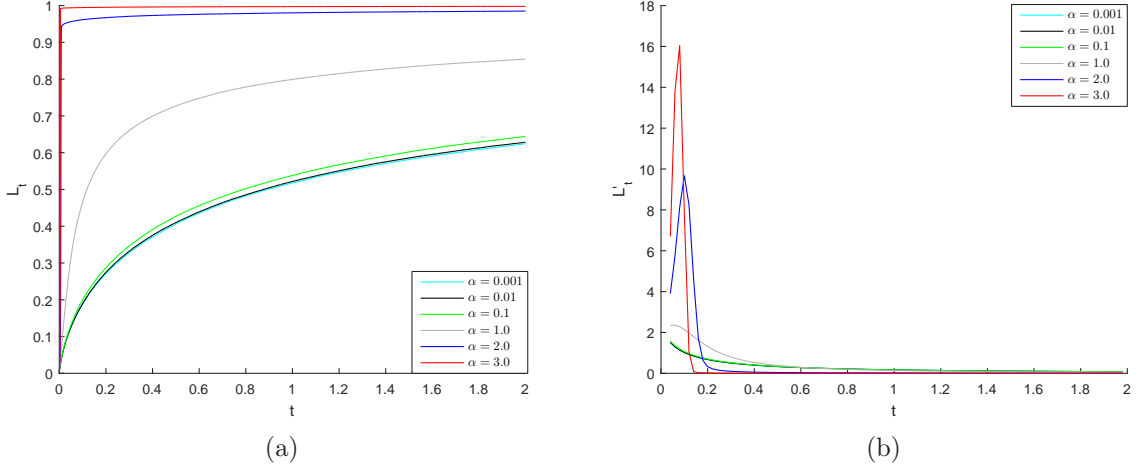


Figure 5: (a) L_t and (b) L'_t for different values of α .

4.3 Hölder 1/2 initial data and blow-up

In our third example, we illustrate possible jumps arising in the solution for sufficiently large values of α . We consider $\alpha = 1.5$, $T = 0.008$ and choose Y_0 as in the previous example, $Y_0 \sim \text{Gamma}(1 + \beta, 1/2)$. Note that the blow-up happens already for very small t due to the interplay of the mass close to 0 for Y_0 and the relatively large α .

With the lack of convergence theory for discontinuous $(L_t)_{t \geq 0}$, we apply Algorithms 1 and 2, and empirically estimate the error. In Figure 6 (a) we show \tilde{L}_t computed using Algorithm 1 for different n ; in Figure 6 (b) the numerical error as a function of t for specific n ; and in Figure 7 (a) and (b) we estimate the convergence rate for different t .

Figure 6 (a) shows that a fairly fine resolution is needed to capture the discontinuity and its timing, but that all meshes predict the size of the jump well. This is further illustrated in Figure 6 (b), which shows the build-up of the error before the jump, the lack of uniform convergence due to the displacement of the jump on different meshes, and the relatively constant error after the jump.

In Figure 7 (a) we estimate the convergence order at $T = 0.002$, i.e. before the jump. By regression, we get 0.53 (0.47, 0.59) for Algorithm 1 and 0.80 (0.72, 0.88) for Algorithm 2, where the 95% confidence interval is in brackets, which agrees with the theory for continuous L_t . In Figure 7 (b), for the error at $T = 0.008$, i.e. after the jump, we get 0.93 (0.81, 1.05) for Algorithm 1 and 1.03 (0.92, 1.14) for Algorithm 2. The faster convergence may result from the almost constancy of the losses after the jump.

To get an overall picture of the accuracy, we suggest the following metrics to measure the “closeness” of the computed solutions to L_t :

1. $d_1(L_t, \tilde{L}_t) = \int_0^T |L_t - \tilde{L}_t| dt$. This is practically computed by numerical integration.

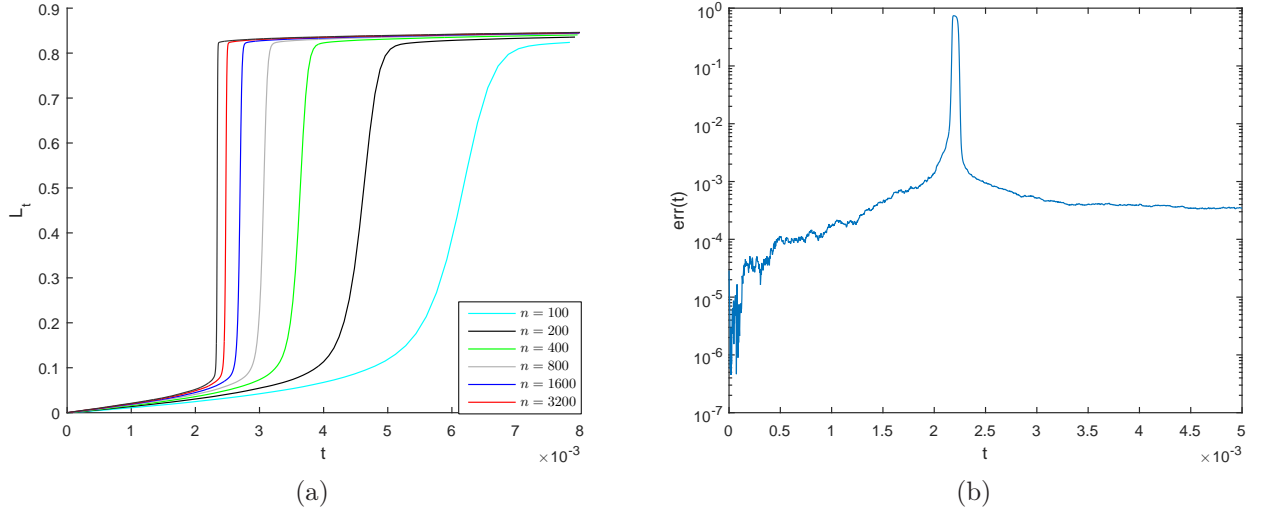


Figure 6: (a) Loss process computed using Algorithm 1 for different n ; (b) error as a function of t .

2. $d_2(L_t, \tilde{L}_t) = |t_* - \tilde{t}_*|$, where t_* and \tilde{t}_* are the jump times for L_t and \tilde{L}_t , respectively. They are approximated by the points with the steepest gradient $j_* = \text{argmax}_j(\tilde{L}_{t_j} - \tilde{L}_{t_{j-1}})$, $t_* = j_* h$.
3. $d_3(L_t, \tilde{L}_t) = \sup_{t \in [0, T]} |L_t^{-1} - \tilde{L}_t^{-1}|$, where L_t^{-1} and \tilde{L}_t^{-1} are the corresponding inverse functions. The inverse functions are found by the chebfun toolbox (Driscoll et al. (2014)), which automatically splits functions into intervals of continuity.

In the absence of the exact L_t , we use again the difference between quantities computed using $2n$ and n points, for the same Monte Carlo paths, as a proxy.

We present the results in Figure 8 (a) for Algorithm 1 and in Figure 8 (b) for Algorithm 2. For metric 1 we have convergence rate 0.68 (0.63, 0.73) and 0.76 (0.71, 0.81), for metric 2 we have 0.70 (0.65, 0.80) and 0.76 (0.72, 0.80), and for metric 3 we have 0.60 (0.52, 0.68) and 0.71 (0.63, 0.79) for Algorithms 1 and 2, respectively, where the 95% confidence interval are in brackets. We observe that the convergence rate for Algorithm 1 is somewhat better than 0.5, while for Algorithm 2 it is around 0.75 for all three metrics.

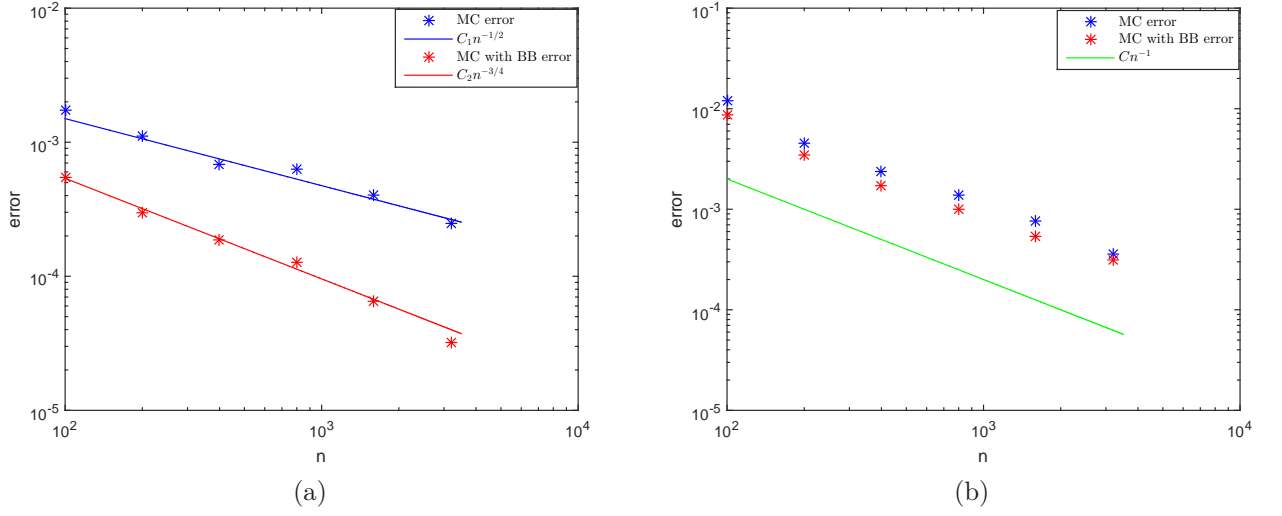


Figure 7: Convergence rate: at (a) $T = 0.001$, (b) $T = 0.008$.

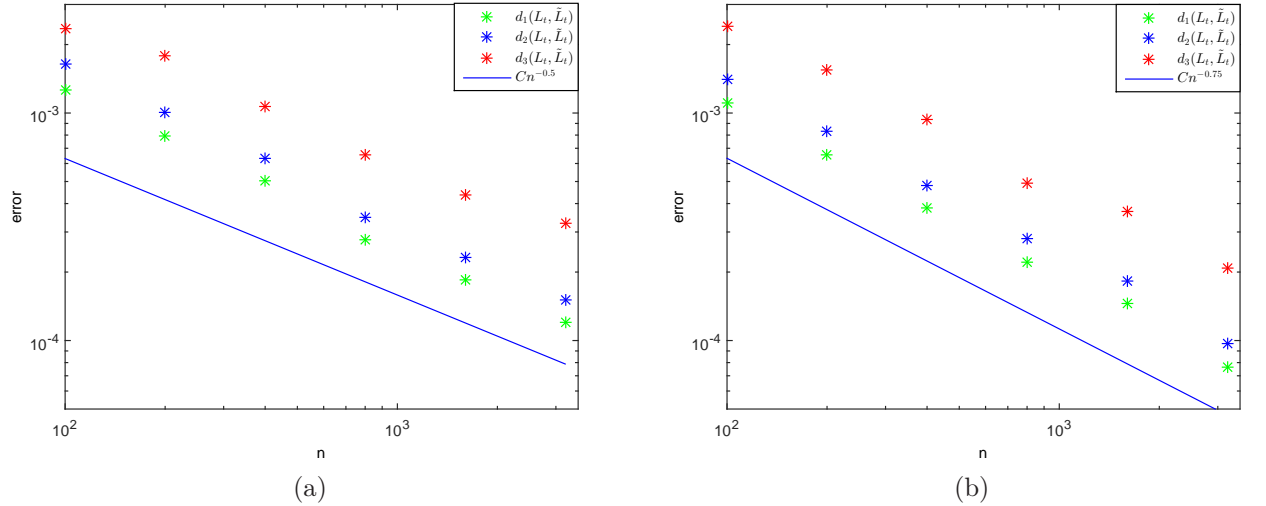


Figure 8: Convergence rate for $d_1(L_t, \tilde{L}_t)$, $d_2(L_t, \tilde{L}_t)$, $d_3(L_t, \tilde{L}_t)$ for (a) Algorithm 1, (b) Algorithm 2.

5 Conclusions and outlook

We have developed particle methods with explicit timestepping for the simulation of (1)-(3). Convergence with a rate up to 1 in the timestep is shown under a condition on the model parameters and time horizon, when the loss function is differentiable. Experimentally, the method also converges in the blow-up regime, and the variance of estimators is inversely proportional to the number of samples.

This opens up several theoretical and practical questions. The efficiency of the method could be significantly improved by a simple application of multilevel simulation (Szpruch et al. (2017); Ricketson (2015)). If the variance can be shown to behave on the finer levels as suggested by the numerical tests, combined with the proven result on the time stepping

bias, the computational complexity for root-mean-square error ϵ would be brought down to ϵ^{-2} , from ϵ^{-3} as observed presently (ϵ^{-2} from the number of samples and ϵ^{-1} from the number of timesteps).

For the particular system (1)–(3), it is conceivable to apply a particle method without time stepping of the form

$$Y_t^{(i)} = Y_0^{(i)} + W_t^{(i)} - \alpha \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{\tau_{(j)} \leq t\}},$$

where $Y_0^{(i)}$ and $W^{(i)}$ are N i.i.d. copies of Y_0 and W , and $\tau_{(j)}$, $1 \leq j \leq N$, is the order statistic of the hitting times of zero. Then, for $\tau_{(i)} < t < \tau_{(i+1)}$, conditional on the information up to $\tau_{(i)}$, the law of $Y_t^{(j)} - Y_{\tau_{(i)}}^{(j)}$ for those $N - i$ particles j which have not yet hit, is identical to that of independent standard Brownian motions with constant drift $-\alpha i/N$. So if we simulate those independent hitting times of $-Y_{\tau_{(i)}}^{(j)}$ for each and then take the smallest one to be $\tau_{(i+1)}$, this inductive construction satisfies the correct law. The complexity is then $O(N^2)$, and combined with the observed variance of $O(1/N)$, this gives a complexity of ϵ^{-4} for error ϵ . Another disadvantage is that this method with exact sampling is restricted to the constant parameter case, while the time stepping scheme can more easily be extended to variable coefficients.

Theoretically, one would like guaranteed convergence also in the blow-up regime. This requires the choice of an appropriate metric – the Skorokhod distance may be suitable, considering the analysis in Delarue et al. (2015b) and the tests in Section 4.3. It also requires verification that the limit in the case of blow-up is a so-called “physical solution” as defined in Hambly et al. (2018); Nadtochiy and Shkolnikov (2017); Delarue et al. (2015b), which on the one hand results from a specific sequential realisation of the losses in the case of blow-up (Nadtochiy and Shkolnikov (2017); Delarue et al. (2015b)), and on the other hand is the right-continuous solution with the smallest jump size (Hambly et al. (2018)).

Lastly, it would be interesting to investigate the extension to the models in Nadtochiy and Shkolnikov (2017) and Delarue et al. (2015b) in more detail.

Acknowledgements

We thank Alex Lipton, Andreas Søjmark, and Sean Ledger for useful comments and interesting discussions.

Appendices

A Proof of Lemmas in Section 3.1

A.1 Proof of Lemma 1

Proof of Lemma 1. Writing

$$\begin{aligned} \sup_{s \leq h \lfloor \frac{t}{h} \rfloor} (Y_s - Y_s^*) &= \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \sup_{0 \leq s \leq h} (W_{h(k-1)+s} - W_{h(k-1)} - \alpha(L_{h(k-1)+s} - L_{h(k-1)})) \\ &= \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \sup_{0 \leq s \leq h} (W_{h(k-1)+s} - W_{h(k-1)} - \alpha s L'_{h(k-1)+\theta s}) \end{aligned}$$

for some $\theta \in [0, 1]$, the last expression can be estimated from both sides,

$$\begin{aligned} \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \left\{ \sup_{0 \leq s \leq h} (W_{h(k-1)+s} - W_{h(k-1)}) - \alpha \sup_{0 \leq s \leq h} h L'_{h(k-1)+\theta s} \right\} &\leq \\ \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \sup_{0 \leq s \leq h} (W_{h(k-1)} - W_{h(k-1)+s} - \alpha s L'_{h(k-1)+\theta s}) & \\ \leq \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \left\{ \sup_{0 \leq s \leq h} (W_{h(k-1)} - W_{h(k-1)+s}) - \alpha \inf_{0 \leq s \leq h} s L'_{h(k-1)+\theta s} \right\}. & \end{aligned}$$

Since $0 \leq L'_s \leq \hat{B}s^{-\frac{1-\beta}{2}}$,

$$\begin{aligned} \sup_{s \leq h \lfloor \frac{t}{h} \rfloor} (Y_s - Y_s^*) &\geq \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \left\{ \sup_{0 \leq s \leq h} (W_{h(k-1)+s} - W_{h(k-1)}) - \alpha \hat{B}h(h(k-1))^{-\frac{1-\beta}{2}} \right\} \\ &= {}_d \sqrt{h} \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \left\{ \sup_{0 \leq s \leq 1} W_s^{(k)} - \alpha \hat{B}\sqrt{h}(h(k-1))^{-\frac{1-\beta}{2}} \right\}, \end{aligned}$$

and

$$\begin{aligned} \sup_{s \leq h \lfloor \frac{t}{h} \rfloor} (Y_s - Y_s^*) &\leq \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \left\{ \sup_{0 \leq s \leq h} (W_{h(k-1)+s} - W_{h(k-1)}) \right\} \\ &= {}_d \sqrt{h} \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \left\{ \sup_{0 \leq s \leq 1} W_s^{(k)} \right\}, \end{aligned}$$

where $W^{(1)}, W^{(2)}, \dots$ are i.i.d. copies of W .

Taking into account that $\sqrt{h}(h(k-1))^{-\frac{1-\beta}{2}} \rightarrow 0$, as $h \rightarrow 0$, and applying the same arguments as in Proposition 1 in Asmussen et al. (1995), we get (21).

Similar,

$$\begin{aligned} \sup_{s \leq h \lfloor \frac{t}{h} \rfloor} (\tilde{Y}_s - \tilde{Y}_s^*) &= \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \sup_{0 \leq s \leq h} (W_{h(k-1)+s} - W_{h(k-1)}) \\ &= \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \sup_{0 \leq s \leq h} (W_{h(k-1)+s} - W_{h(k-1)}) = {}_d \sqrt{h} \max_{1 \leq k \leq \lfloor \frac{t}{h} \rfloor} \left\{ \sup_{0 \leq s \leq 1} W_s^{(k)} \right\}, \end{aligned}$$

where $W^{(1)}, W^{(2)}, \dots$ are i.i.d. copies of W .

Again, applying the same arguments as in Proposition 1 in Asmussen et al. (1995), we get (22). \square

A.2 Proof of Lemma 2

Proof of Lemma 2. We first prove (23). It is obvious that

$$\mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) \geq \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right).$$

Also,

$$\mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) = \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) + \mathbb{P}\left(\left\{\inf_{s<t_i} Y_s \leq 0\right\} \cap \left\{\min_{j<i} Y_{t_j} > 0\right\}\right).$$

Thus,

$$\begin{aligned} \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) &= \mathbb{P}\left(\left\{\inf_{s<t_i} Y_s \leq 0\right\} \cap \left\{\min_{j<i} Y_{t_j} > 0\right\}\right) \\ &\leq \mathbb{P}\left(\min_{j<i} Y_{t_j} - \inf_{s<t_i} Y_s \geq \varepsilon\right) + \mathbb{P}\left(\min_{j<i} Y_{t_j} \in (0, \varepsilon), \min_{j<i} Y_{t_j} - \inf_{s<t_i} Y_s < \varepsilon\right), \end{aligned}$$

for any $\varepsilon > 0$.

Consider the first term. Using Markov's inequality

$$\mathbb{P}\left(\min_{j<i} Y_{t_j} - \inf_{s<t_i} Y_s \geq \varepsilon\right) \leq \frac{\mathbb{E}\left[(\min_{j<i} Y_{t_j} - \inf_{s<t_i} Y_s)^p\right]}{\varepsilon^p},$$

for any $p \geq 1$.

Using Lemma 1,

$$\frac{1}{\sqrt{h}} \left(\min_{j<i} Y_{t_j} - \inf_{s<t_i} Y_s \right) \rightarrow_d \sqrt{2 \log \frac{t_i}{h}}$$

thus

$$\frac{\mathbb{E}\left[(\min_{j<i} Y_{t_j} - \inf_{s<t_i} Y_s)^p\right]}{\varepsilon^p \cdot h^{p/2}} \rightarrow \frac{(2 \log \frac{t_i}{h})^{p/2}}{\varepsilon^p}.$$

For the second term

$$\begin{aligned} \mathbb{P}\left(\min_{j<i} Y_{t_j} \in (0, \varepsilon), \min_{j<i} Y_{t_j} - \inf_{s<t_i} Y_s < \varepsilon\right) &\leq \mathbb{P}\left(\min_{j<i} Y_{t_j} \in (0, \varepsilon)\right) = \\ &\mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > \varepsilon\right) = \bar{F}_i(\varepsilon) - \bar{F}_i(0) \leq \varepsilon \sup_{\theta \in [0,1]} \bar{\varphi}_i(\theta \varepsilon), \end{aligned}$$

where $\bar{F}_i(x)$ and $\bar{\varphi}_i(x)$ are the CDF and PDF of the process $\min_{j<i} Y_{t_j}$. We also note that $\bar{\varphi}_i(\theta \varepsilon)$ is bounded according to Lemma 3.

Combining both terms, we have

$$\mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \leq A \frac{h^{p/2}}{\varepsilon^p} + B\varepsilon.$$

Minimising the right-hand side over ε , we find $\varepsilon = h^a$ with $a = \frac{p}{2(p+1)}$. As a result, choosing $p \rightarrow \infty$, we get that for any $\delta > 0$, there exists $\gamma > 0$, such that

$$\mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \leq \gamma h^{\frac{1}{2}-\delta}.$$

For (24), we use identical steps and the corresponding estimates in Lemma 1 and Lemma 3 for \tilde{Y} instead of Y .

□

A.3 Proof of Lemma 3

Proof of Lemma 3. We can rewrite $Z_i = Y_0 + V_i$, where $V_i = \inf_{s \leq t_i} (W_s - \alpha L_s)$. As Y_0 and V_i are independent, using convolution, we have

$$\varphi_i(z) = \int_{-\infty}^{+\infty} f_{Y_0}(z-v) \mathbb{P}(V_i \in dv).$$

Using the fact that $V_i \leq 0, Y_0 \geq 0$, and (7), we have

$$\begin{aligned} \varphi_i(z) &= \int_{-\infty}^{z \wedge 0} f_{Y_0}(z-v) \mathbb{P}(V_i \in dv) \leq B \int_{-\infty}^{z \wedge 0} (z-v)^\beta \mathbb{P}(V_i \in dv) \\ &= BF_{V_i}(z \wedge 0) \int_{-\infty}^{z \wedge 0} (z-v)^\beta \frac{\mathbb{P}(V_i \in dv)}{F_{V_i}(z \wedge 0)}, \end{aligned} \quad (38)$$

where F_{V_i} is the CDF of V_i . It is obvious that $F_{V_i}(z) > 0$ for all $z \leq 0$. Indeed, $F_{V_i}(z) \geq \mathbb{P}(V_i \leq z) \geq \mathbb{P}(\inf_{s \leq t_i} W_s \leq z) > 0$.

Since $\beta \in (0, 1]$, for all z the function $(z - \cdot)^\beta : (-\infty, z \wedge 0) \rightarrow (0, \infty)$ is concave and, using Jensen's inequality for the proper probability measure $\frac{\mathbb{P}(V_i \in dv)}{F_{V_i}(z \wedge 0)}$ on $(-\infty, z \wedge 0)$,

$$\begin{aligned} \varphi_i(z) &\leq BF_{V_i}(z \wedge 0)^{1-\beta} \left(\int_{-\infty}^{z \wedge 0} (z-v)^\beta \mathbb{P}(V_i \in dv) \right)^\beta \\ &= BF_{V_i}(z \wedge 0)^{1-\beta} \left(\mathbb{E}[(z - V_i) \mathbb{1}_{\{V_i \leq z\}}] \right)^\beta, \end{aligned} \quad (39)$$

where $(V_i \leq z) \Leftrightarrow (V_i \leq z \wedge 0)$ as $V_i \leq 0$. Since $t \rightarrow L_t$ is non-decreasing, inserting V_i ,

$$\begin{aligned} \varphi_i(z) &\leq BF_{V_i}(z \wedge 0)^{1-\beta} \left(\mathbb{E}[(z - \inf_{s \leq t_i} \{W_s\} + \alpha L_{t_i}) \mathbb{1}_{\{\inf_{s \leq t_i} \{W_s\} - \alpha L_{t_i} \leq z\}}] \right)^\beta \\ &\leq BF_{V_i}(z \wedge 0)^{1-\beta} \left(\mathbb{E}[(z - \inf_{s \leq t_i} \{W_s\} + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}}) \mathbb{1}_{\{\inf_{s \leq t_i} \{W_s\} \leq z + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}}\}}] \right)^\beta. \end{aligned} \quad (40)$$

By simple integration of the density of the running minimum of Brownian motion,

$$\mathbb{E}[(\xi - \inf_{s \leq t_i} W_s) \mathbb{1}_{\{\inf_{s \leq t_i} W_s \leq \xi\}}] = \left(2\Phi\left(\frac{\xi}{\sqrt{t_i}}\right) \wedge 1 \right) \xi + \sqrt{\frac{2t_i}{\pi}} e^{-\frac{\xi^2}{2t_i}}, \quad (41)$$

and, taking $\xi = z + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}}$, we get from (40) for $z > 0$,

$$\varphi_i(z) \leq B \left[z + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}} + \sqrt{\frac{2t_i}{\pi}} \right]^\beta,$$

and, similarly, for $z \leq 0$,

$$\varphi_i(z) \leq B \left[\alpha \tilde{B} t_i^{\frac{1+\beta}{2}} + \sqrt{\frac{2t_i}{\pi}} \right]^\beta.$$

Combing the last two equations, we finally get (25) for φ_i .

Using similar arguments for \bar{Z}_i and \tilde{Z}_i , using $\inf_{s \leq t_i} W_s \leq \min_{j < i} W_{t_j}$ and $L_{t_i} \geq \tilde{L}_{t_i}$, respectively, we get the analogous estimates for $\bar{\varphi}_i$ and $\tilde{\varphi}_i$, and hence (25). \square

B Proof of Lemma 5, Section 3.3

Proof of Lemma 5. We can write

$$\mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > 0\}}] = \mathbb{E}[\mathbb{1}_{\{\min_{j<i} (Y_0^{(k)} + W_{t_j}^{(k)} - \alpha \tilde{L}_{t_j} + \alpha(\tilde{L}_{t_j} - \hat{L}_{t_j}^N)) > 0\}}],$$

and estimate

$$\mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}}] \leq \mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > 0\}}] \leq \mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}}]. \quad (42)$$

We evaluate left- and right-hand side of the last equation separately. We start with the right-hand side, and define for brevity $\tilde{A}_i^{(k)} = \{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}$. Consider some $\varepsilon > 0$. Then,

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\tilde{A}_j^{(k)}}] &= \mathbb{E}[\mathbb{1}_{\tilde{A}_j^{(k)} \cap (\{\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) \leq \varepsilon\} \cup \{\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) > \varepsilon\})}] = \\ &\quad \mathbb{E}[\mathbb{1}_{\tilde{A}_j^{(k)} \cap \{\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) \leq \varepsilon\}}] + \mathbb{E}[\mathbb{1}_{\tilde{A}_j^{(k)} \cap \{\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) > \varepsilon\}}] \leq \\ &\quad \mathbb{P}\left(\tilde{A}_j^{(k)} \cap (\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) \leq \varepsilon)\right) + \mathbb{P}\left(\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) > \varepsilon\right) \leq \\ &\quad \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \varepsilon\right) + \mathbb{P}\left(\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) > \varepsilon\right) = \\ &\quad \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j}^{(k)} > 0\right) + \left(\mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \varepsilon\right) - \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j}^{(k)} > 0\right)\right) + \mathbb{P}\left(\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) > \varepsilon\right) \leq \\ &\quad 1 - \tilde{L}_{t_i} + \alpha \varepsilon \sup_{\theta \in [0,1]} \tilde{\varphi}_i(-\theta \alpha \varepsilon) + \mathbb{P}\left(\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) > \varepsilon\right), \end{aligned}$$

where $\tilde{\varphi}_i(x)$ is the pdf of $\min_{j<i} \tilde{Y}_{t_j}^{(k)}$, which is bounded according to Lemma 3.

Using (33) and properties of convergence in probability, we have

$$\max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) \xrightarrow{\mathbb{P}} 0.$$

Thus, the last term goes to 0 with $N \rightarrow \infty$. Considering $\varepsilon \rightarrow 0$, we get

$$\lim_{N \rightarrow \infty} \mathbb{E}[\mathbb{1}_{\tilde{A}_j^{(k)}}] \leq 1 - \tilde{L}_{t_i}. \quad (43)$$

Similar, denote by $\bar{A}_i^{(k)} = \{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}$, then

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\bar{A}_j^{(k)}}] &= \mathbb{E}[\mathbb{1}_{\bar{A}_j^{(k)} \cap (\{\min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) \geq -\varepsilon\} \cup \{\min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) < -\varepsilon\})}] = \\ &\quad \mathbb{E}[\mathbb{1}_{\bar{A}_j^{(k)} \cap \{\min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) \geq -\varepsilon\}}] + \mathbb{E}[\mathbb{1}_{\bar{A}_j^{(k)} \cap \{\min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) < -\varepsilon\}}] \geq \\ &\quad \mathbb{P}\left(\bar{A}_j^{(k)} \cap \{\min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N) \geq -\varepsilon\}\right) \geq \\ &\quad \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j}^{(k)} > \alpha \varepsilon\right) \geq 1 - \tilde{L}_{t_i} - \alpha \varepsilon \inf_{\theta \in [0,1]} \tilde{\varphi}_i(\theta \alpha \varepsilon), \end{aligned}$$

where $\tilde{\varphi}_i(x)$ is the pdf of $\min_{j<i} \tilde{Y}_{t_j}^{(k)}$, which is bounded according to Lemma 3. Thus,

$$\lim_{N \rightarrow \infty} \mathbb{E}[\mathbb{1}_{\bar{A}_j^{(k)}}] \geq 1 - \tilde{L}_{t_i}. \quad (44)$$

Combining (43), (44), and (42), we immediately get (34).

Consider now the variance

$$\mathbb{V}[\hat{L}_{t_i}^N] = \frac{1}{N^2} \left(\sum_{k=1}^N \mathbb{V}[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}}] + \sum_{k \neq l} \text{cov} \left(\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}}, \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right) \right). \quad (45)$$

The covariance can be estimated by

$$\begin{aligned} \text{cov} \left(\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}}, \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right) = \\ \mathbb{E} \left[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}} \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right] - \mathbb{E} \left[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}} \right] \mathbb{E} \left[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right]. \end{aligned} \quad (46)$$

As above and because of exchangeability, for the second term of the last expression,

$$\mathbb{E} \left[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}} \right] \mathbb{E} \left[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right] \xrightarrow{N \rightarrow \infty} (1 - \tilde{L}_{t_i})^2. \quad (47)$$

Now we consider the first term of (46). Similar to above, it can be estimated by

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}} \mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(l)} > -\alpha \min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}}] \leq \\ \mathbb{E} \left[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}} \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right] \\ \leq \mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}} \mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(l)} > -\alpha \max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}}]. \end{aligned} \quad (48)$$

Similar to (43) and (44), one can show that

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}} \mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(l)} > -\alpha \min_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}}] &\geq (1 - \tilde{L}_{t_i})^2, \\ \lim_{N \rightarrow \infty} \mathbb{E}[\mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(k)} > -\alpha \max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}} \mathbb{1}_{\{\min_{j<i} \tilde{Y}_{t_j}^{(l)} > -\alpha \max_{j<i} (\tilde{L}_{t_j} - \hat{L}_{t_j}^N)\}}] &\leq (1 - \tilde{L}_{t_i})^2. \end{aligned}$$

Hence,

$$\mathbb{E} \left[\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}} \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right] \xrightarrow{N \rightarrow \infty} (1 - \tilde{L}_{t_i})^2. \quad (49)$$

Thus, combining (47) and (49), we get

$$\text{cov} \left(\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}}, \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right) \xrightarrow{N \rightarrow \infty} 0.$$

Considering the first term of (45), it is easy to see that each variance is bounded by 1. As a result, we have

$$\mathbb{V}[\hat{L}_{t_i}^N] \leq \frac{1}{N} + \max_{k \neq l} \left| \text{cov} \left(\mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} > 0\}}, \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(l)} > 0\}} \right) \right| \xrightarrow{N \rightarrow \infty} 0.$$

□

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