Finite groups satisfying a certain 3-local condition

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Contents.

Chapter I.

Introduction and statement of results.

The results derived in this thesis, centre broadly round the final target of classifying all finite groups possessing a self-centralizing, elementary Abelian, Sylow 3-subgroup of order nine. This larger problem may be viewed as a consequent step, following upon the heels of the classification, due to W. Feit and J.G. Thompson of all finite groups with a self-centralizing subgroup of order three [9]. The result due to Feit and Thompson is crucial to our analysis, and we quote it directly:-

Theorem (Feit and Thompson [9])

Let G be a finite group which contains a self-centralizing subgroup of order three. Then one of the following statements is true:-

(i) G contains a nilpotent normal subgroup N, such that G/N is isomorphic to either Z_3 or D_6.

(ii) G contains a normal subgroup N, which is a 2-group, such that G/N is isomorphic to A_5.

(iii) G is isomorphic to PSL(2,7).

We now present a brief account of our basic problem. Let us suppose that G denotes a finite group possessing a self-centralizing, elementary Abelian, Sylow 3-subgroup, P, of order nine. An analysis of the possible structure of G may begin with considerations of the structure of $N_G(P)/C_G(P)$. As P acts fixed point freely on $O_{3'}(G)$, it follows, using a theorem due to Martineau [19], that $O_{3'}(G)$ is soluble, of Fitting height at most two. We denote $\bar{G} = G/O_{3'}(G)$. Then $\bar{G}$ also has a self-centralizing, elementary Abelian, Sylow 3-subgroup of order nine, and $N_{\bar{G}}(\bar{P})$ is isomorphic to $N_G(P)$.

Thus it is convenient, in this initial analysis, to suppose that $O_{3'}(G) = 1$.

Now $N_G(P)/C_G(P)$ acts as a group of automorphisms of P, and is thus naturally isomorphic to a subgroup of $GL(2,3)$. In fact, the following possibilities for the isomorphism type of $N_G(P)/C_G(P)$ may arise:-

(i) $N_G(P)/C_G(P) = 1$. Applying Burnside's transfer theorem, G has a normal 3-complement, $O_{3'}(G)$. Since we have assumed that $O_{3'}(G) = 1$, it follows that $G = P$.

(ii) $N_G(P)/C_G(P)$ is isomorphic to Z_2. This possibility lends itself to an application of the results due to Smith and Tyrer [24]. In particular, it follows that $O^3(G) \neq G$, or that G is 3-soluble of 3-length one.

(iii) $N_G(P)/C_G(P)$ is isomorphic to V_4. If we suppose that $P = \langle x,y \mid x^3 = y^3 = [x,y] = 1\rangle$, then we may suppose also that $N_G(P) = \langle a,b,P \mid a^2 = b^2 = [a,b] = 1,\ a^{-1}xa = x^{-1},\ a^{-1}ya = y,\ b^{-1}xb = x,\ b^{-1}yb = y^{-1}\rangle$.

In [17], G. Higman shows that there is no finite simple group falling into this category. Note that any group satisfying the present hypotheses has precisely three conjugate classes of elements of order three.

Theorem A of this thesis classifies all finite groups satisfying the hypotheses of this section. In particular, our chosen group G is isomorphic to one of:-

$D_6 \times D_6$: $D_6 \times A_5$: $D_6 \times PSL(2,7)$
H_1 : $A_5 \times A_5$: $A_5 \times PSL(2,7)$
H_2 : : $PSL(2,7) \times PSL(2,7)$.

Here H_1 denotes an extension of Z_3 by S_5, and H_2 an extension of Z_3 by $PGL(2,7)$.

(iv) $N_G(P)/C_G(P)$ is isomorphic to Z_4. If we suppose that $P = \langle x,y \mid x^3 = y^3 = [x,y] = 1\rangle$, then we may suppose also that $N_G(P) = \langle c,P \mid c^4 = 1,\ c^{-1}xc = y,\ c^{-1}yc = x^{-1}\rangle$.

In his doctoral thesis [14], P.G. Henry proves that our chosen group G is isomorphic to one of $N_G(P)$, A_6 or A_7.

Note that any group satisfying the present hypotheses has precisely two conjugate classes of elements of order three.

(v) $N_G(P)/C_G(P)$ is isomorphic to D_8. If we suppose that $P = \langle x,y \mid x^3 = y^3 = [x,y] = 1\rangle$, then we may suppose also that $N_G(P) = \langle a,b,P \mid a^4 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$, where we also know that $a^{-1}xa = y^{-1}$, $a^{-1}ya = x$, $b^{-1}xb = x$, $b^{-1}yb = y^{-1}$.

Theorem B of this thesis begins an attack on this problem. Note that any group satisfying the present hypotheses has precisely two conjugate classes of elements of order three. The only known finite simple groups satisfying the present hypotheses, are A_8 and $GL(5,2)$. (For further information, see Appendix.)

(vi) $N_G(P)/C_G(P)$ is isomorphic to Q_8. Note that any group satisfying the present hypotheses has a unique conjugate class of elements of order three. In his doctoral thesis, [14] P.G. Henry proves that our chosen group G is isomorphic to one of:-

$N_G(P)$, M_{22},
$PSL(3,q)$, $q \equiv 4, 7\ (9)$
$PSU(3,q^2)$, $q \equiv 2, 5\ (9)$ $q > 2$.

(vii) $N_G(P)/C_G(P)$ is isomorphic to Z_8. The regular action of the Sylow 3-automiser ensures that any group satisfying the present hypotheses has a unique conjugate class of elements of order three. In this instance, very little else is known.

(viii) $N_G(P)/C_G(P)$ is isomorphic to the semidihedral group of order sixteen. The semi-regular action of the Sylow 3-automiser ensures that any group satisfying the present hypotheses, has a unique conjugate class of elements of order three. There is very little else known in this instance.

After this preliminary discussion, we may now state our results:-

Theorem A.

Let G be a finite group possessing a self-centralizing Sylow 3-subgroup of order nine, and precisely three conjugate classes of elements of order three. Then there is a soluble normal subgroup N of G, such that N has Fitting height at most two, and G/N is isomorphic to a group in the following list:-

$$\begin{array}{ccccc} D_6 \times D_6 & : & D_6 \times A_5 & : & D_6 \times PSL(2,7) \\ H_1 & : & A_5 \times A_5 & : & A_5 \times PSL(2,7) \\ H_2 & : & & : & PSL(2,7) \times PSL(2,7). \end{array}$$

Here, H_1 denotes an extension of Z_3 by S_5, and H_2 an extension of Z_3 by PGL(2,7).

Theorem B.

Let G be a finite simple group possessing a self-centralizing, elementary Abelian, Sylow 3-subgroup, P, of order nine, with $N_G(P)/P$ isomorphic to D_8. Then one of the following occurs:-

(i) G is isomorphic to A_8

(ii) G is isomorphic to GL(5,2)

(iii) G has at least three conjugate classes of involutions.

We now present a brief plan of the thesis. In Chapters II and III, we detail (respectively) those group theoretic, and those character theoretic results which we shall later employ in the proofs of our theorems.

Chapter IV is concerned (in general) with the construction of the principal 3-block of a finite group G possessing a self-centralizing Sylow 3-subgroup, P, which is elementary Abelian of order nine. In Chapter V, a proof of Theorem A is presented. Essentially, Theorem A is proved by a classification (highly character theoretic) of the possible minimal normal subgroups of a finite group, G, which is taken as a counter-example of least possible order, to the required result. In Chapter VI, a proof of Theorem B is to be found. We outline here our proof of Theorem B. If we take G to denote a counterexample to the theorem, then it follows that G has at most two conjugate classes of involutions. In fact G must have precisely two conjugate classes of involutions.

We choose a Sylow 3-subgroup P of G, with
$P = \langle x,y \mid x^3 = y^3 = [x,y] = 1\rangle$, and
$N_G(P) = \langle a,b,P \mid a^4 = b^2 = 1,\ a^{-1} = b^{-1}ab,\ a^{-1}xa = y^{-1},\ a^{-1}ya = x,\ b^{-1}xb = x,\ b^{-1}yb = y^{-1}\rangle$.

Now $C_G(x) = \langle x\rangle \times A$, for some subgroup A of G, where it is known that A has a self-centralizing subgroup of order three. Fusion arguments prove that all involutions of $C_G(x)$ are conjugate in $N_G(x)$. The previously quoted result due to Feit and Thompson is then applied to show that either A is isomorphic to A_5 or to PSL(2,7), or that A has twice odd order. Analogous considerations apply to $C_G(xy)$. We next choose Q to be a 2-subgroup of G, of maximal possible order subject to being normalized by P. The above constraints on the structures of $C_G(x)$ and $C_G(xy)$ force $|Q| \leq 256$. An analysis of the structure of $N_G(Q)$ enables us to determine the isomorphism type of G. The proof relies heavily on the classification due to B. Beisiegel [2], and V. Stingl [26] of finite simple groups possessing a Sylow 2-subgroup of order at most 2^{10}.

In Chapter VII, we discuss the general status of our results and indicate the possibilities for further development in this general area.

Notation

The notation used in this thesis is for the most part standard, as used in Gorenstein's book [10]. We will however require to use the following additional, or occasionally alternative notation:-

$\#(g^{\bullet}h^{\bullet} = a)$	Here a,g,h, are fixed elements of a group G. Then this symbol denotes the number of solutions in G of the equation $g_1h_1 = a$, with g_1 conjugate to g, and h_1 conjugate to h.
$O^{\pi}(G)$	If G is a finite group, then $O^{\pi}(G) \trianglelefteq G$, and $G/O^{\pi}(G)$ is a π-group (Here π denotes a specified set of primes.). If also $K \trianglelefteq G$ and G/K is a π-group, then $O^{\pi}(G) \trianglelefteq K$.
$\langle k,m,n\rangle$	The polyhedral group with presentation $\langle x,y,z \mid x^k = y^m = z^n = xyz = 1\rangle$.
r(G)	The sectional 2-rank of the finite group G. (This is the maximal 2-rank of any section of a Sylow 2-subgroup of G.)
D_{2m}	The dihedral group of order 2m.
V_4	The elementary Abelian group of order 4.
QD_{16}	The semi-dihedral group of order 16.
$SU(n,q^2)$	The special unitary group of dimension n, over the field of q^2 elements.
$PSU(n,q^2)$	The projective special unitary group of dimension n ~~over~~ the field of q^2 elements.

Chapter II.

Assumed Group Theoretic Results.

In this chapter, we state those results which we shall require in the course of this work. Several of these are well known, and are included merely because of their frequent relevance. All groups discussed here may be assumed finite, unless this is clearly not the case:-

§2A Preliminary Results.

Proposition 2.1 (Frattini argument: Gorenstein [10] 1.3.7)

If $H \trianglelefteq G$, and P is a Sylow p-subgroup of H, then $G = N_G(P)H$.

Proposition 2.2 (Burnside's Lemma: Gorenstein [10] 7.1.1)

If P is a Sylow p-subgroup of G, then two normal subsets of P are conjugate in G if and only if they are conjugate in $N_G(P)$.

Proposition 2.3 (Burnside's Transfer Theorem: Gorenstein [10] 7.4.3)

If a Sylow p-subgroup of G lies in the centre of its normalizer in G, then G has a normal p-complement.

Proposition 2.4 (Gorenstein [10] 5.3.15)

Let A be a p'-group of automorphisms of a p-group P, and let H be an A-invariant normal subgroup of P. Then $C_{P/H}(A)$ is the image in P/H of $C_P(A)$.

Proposition 2.5 (Gorenstein [10] 5.3.16)

Let P be a p-group, and Q a non-cyclic Abelian q-group of automorphisms of P. Then

$$P = \prod_{x \in Q-\{1\}} C_P(x)$$

(Here p and q are distinct primes)

Proposition 2.6 (Gorenstein [10] 7.4.4)

If the Sylow p-subgroup P of G is Abelian, then $P \cap G' = P \cap (N_G(P))'$, and $G/O^3(G)$ is isomorphic to $P \cap Z(N_G(P))$.

Proposition 2.7 (Gaschütz)

If the normal Abelian subgroup A has finite exponent k and finite index in the group G, then G splits over A if and only if for each prime p dividing k, each Sylow p-subgroup P splits over $P \cap A$.

Proposition 2.8 The infinite group $\langle 3,3,3 \rangle$ possesses a normal Abelian subgroup of index three.

Proof (The proof presented here is adapted from that found in [9])

We may suppose that $\langle 3,3,3 \rangle := \langle x,y \mid x^3 = y^3 = (xy)^3 = 1 \rangle$. Since $(xy)^3 = 1$, it follows that $xyx = y^{-1}x^{-1}y^{-1}$. Substituting $y = y^{-2}$ and $x = x^{-2}$, we derive that:-

$$(1) \qquad xy^{-1}y^{-1}x = y^{-1}xxy^{-1}.$$

It follows that $[xy^{-1}, y^{-1}x] = 1$.

Conjugating the relation (1) by x and by x^2, we obtain the further relations:-

$$y^{-1}x \cdot x^{-1}y^{-1}x^{-1} = x^{-1}y^{-1}x^{-1} \cdot y^{-1}x$$

and $$x^{-1}y^{-1}x^{-1} \cdot xy^{-1} = xy^{-1} \cdot x^{-1}y^{-1}x^{-1}.$$

Thus the subgroup $H = \langle xy^{-1}, y^{-1}x, x^{-1}y^{-1}x^{-1} \rangle$ is Abelian. Moreover H is normalized by x, and since xy^{-1} lies in H, it follows that H is also normalized by y. Thus $H \trianglelefteq \langle 3,3,3 \rangle$. The group $\langle 3,3,3 \rangle$ may be mapped homomorphically onto a <u>non-Abelian</u> group of order twenty seven. Thus H is a proper subgroup of $\langle 3,3,3\rangle$. Since xy^{-1} lies in H, and x does not lie in H, we observe that H is of index three in $\langle 3,3,3\rangle$.

Corollary 2.9

In any finite group, if g,h,k are elements of order three such that $ghk = 1$, the group $\langle g,h,k\rangle$ has an Abelian normal subgroup of index three.

§2B Classification Theorems and Deeper Results.

The following result will be crucial to our analysis:-

Proposition 2.10 (Feit-Thompson [9])

Let G be a finite group which contains a self-centralizing subgroup of order three. Then one of the following statements is true:-

(i) G contains a nilpotent normal subgroup N, such that G/N is isomorphic to either Z_3 or D_6.

(ii) G contains a normal subgroup N which is a 2-group, such that G/N is isomorphic to $SL(2,4)$. (In fact, Theorem 8.2 of [16] shows that N must be elementary Abelian, a direct sum of minimal normal subgroups of G of order 2^4, each of which may be identified with a two-dimensional vector space over $GF(4)$, in such a way that the action of G/N on the subgroup is given by the usual action of $SL(2,4)$ as a group of matrices.)

(iii) G is isomorphic to $PSL(2,7)$.

<u>Proposition 2.11</u> (Smith-Tyrer [24])

Let G be a finite group, P a Sylow p-subgroup of G (some odd prime p). Suppose further that P is Abelian and that $|N_G(P) : C_G(P)| = 2$. Then:-

(a) If G is perfect, then P is cyclic

(b) If P is non-cyclic, then $O^p(G) \neq G$, or G is p-soluble of of p-length one.

<u>Proposition 2.12</u> (Gorenstein-Harada [11])

Let G denote a finite simple group of sectional 2-rank at most four. Then the structure of G is of known type.

<u>Proposition 2.13</u> (Gorenstein-Walter [12])

Let G be a finite simple group with dihedral Sylow 2-subgroups. Then G is isomorphic to A_7 or to PSL(2,q) (some odd $q > 3$).

<u>Proposition 2.14</u> (Alperin, Brauer, Gorenstein [1])

Let G be a finite simple group with semidihedral Sylow 2-subgroups. Then one of the following occurs:-

(i) G is isomorphic to PSL(3,q) $q \equiv 3(4)$

(ii) G is isomorphic to $PSU(3,q^2)$ $q \equiv 1(4)$

(iii) G is isomorphic to M_{11}.

<u>Proposition 2.15</u> (Suzuki [30])

Let S be a 2-group containing a self-centralizing elementary Abelian subgroup of order four . Then S is dihedral or semidihedral.

<u>Proposition 2.16</u> (Harada [13])

Let S be a 2-group containing a self-centralizing elementary Abelian subgroup of order eight. Then $r(S) \leq 4$.

Proposition 2.17 (Harada [13])

Let G be a finite group containing an elementary Abelian subgroup A of order sixteen. Suppose:-

(i) A is a Sylow 2-subgroup of $C_G(A)$ and

(ii) $N_G(A)/C_G(A)$ is isomorphic to A_6 or to A_7.

Then $r(G) \leq 4$.

Proposition 2.18 (Stroth [28])

Let G be a finite simple group and Q an elementary Abelian subgroup of G of order sixteen. Suppose further that Q is a Sylow 2-subgroup of $C_G(Q)$ and that $N_G(Q)/C_G(Q)$ is isomorphic to S_6. Then the structure of G is of known type.

Proposition 2.19 (Beisiegel-Stingl [2] and [26])

Let G be a finite simple group with Sylow 2-subgroup of order at most 2^{10}. Then the structure of G is of known type.

Note A full list of finite simple groups satisfying the hypotheses of Proposition 2.19, may be found in the Appendix.

Proposition 2.20 (Martineau [19])

Let G be a finite group admitting an elementary Abelian fixed point free group of automorphisms V of order r^2 (some prime r). Then G has a normal subgroup F, such that both F and G/F are nilpotent.

Proposition 2.21 (Henry [14])

Let G be a finite group with a self-centralizing, elementary Abelian Sylow 3-subgroup P of order nine, and suppose that $N_G(P)/P$ is isomorphic to Z_4. Then $G/O_{3'}(G)$ is isomorphic to A_6 or A_7, or $G/O_{3'}(G)$ has a normal Sylow 3-subgroup.

§2C Classification Theorems based on Character Theoretic Considerations.

Proposition 2.22 (Dornhoff [8] p. 144)

Let G be a finite non-Abelian subgroup of GL(2,C), then one of the following holds:-

(i) G has a normal Abelian subgroup of index two.

(ii) G/Z(G) is isomorphic to one of A_4, S_4, or A_5, and Z(G) consists of scalar matrices.

Proposition 2.23 (Blichfeldt [3])

Let G be a finite simple group with an irreducible complex representation of degree 4. Then G is isomorphic to one of A_5, A_6, PSL(2,7) or PSU($4,2^2$).

Proposition 2.24 (Wales [31])

Suppose that the finite simple group G has an irreducible complex representation of degree 7. Then G is isomorphic to one of PSL(2,13), PSL(2,8), A_8, PSL(2,7), PSU($3,3^2$) or PSp(6,2).

Chapter III.

Assumed Character Theoretic Results.

Our proof of Theorem A will require a fundamental knowledge of the principles of modular character theory. In broad outline, this chapter details the results required in Chapters IV and V.

We assume that the basic theory of non-modular characters is known. From this theory, we shall require but three results:-

Proposition 3.1 If G is a finite group, then G has precisely $|G : G'|$ linear characters.

Proposition 3.2 For any elements a,b,c, of the finite group G

$$\# \, (a^{\bullet} b^{\bullet} = c) = \frac{|G|}{|C_G(a)|\,|C_G(b)|} \sum_{\iota} \frac{\chi(a)\chi(b)\overline{\chi(c)}}{\chi(1)}$$

the sum being taken over a complete set of irreducible characters of G

Proposition 3.3 (Clifford's theorem: Curtis & Reiner p.343)

Suppose that the finite group G has a normal subgroup H. Let M denote an irreducible KG-module, where K is an arbitrary field. Then M_H is a completely reducible KH-module, and the irreducible KH-submodules of M_H are all conjugates of each other.

Our approach to modular character theory centres around a study of the blocks of characters. We choose a prime p, and partition the set of elements of our finite group G into disjoint subsets, the p-sections of G. We also partition the set of irreducible characters of G into disjoint subsets, the p-blocks of G. We may then study the values of the characters in a fixed p-block, on the elements in a fixed p-section.

Each element x of G may be uniquely expressed as a product $x = x_p r$ of a p-element x_p, and a p-regular element r lying in $C(x_p)$. A p-section S, consists of the elements x of G whose p-factor x_p, is conjugate to a fixed p-element u of G. Thus S is a union of conjugacy classes of G, and one of these classes (namely that containing u), consists of p-elements. Each element x of S is conjugate to elements of the form ur, where r is a p'-element of $C(u)$. Hence we know the value of a character χ on S, if we know the values $\chi(ur)$.

We may now define the p-blocks of G. We will denote by Ω, the field obtained from the field of rational numbers, by adjoining the $|G|$-th roots of unity. For any element x in G, it is known that $\frac{|G|}{|C(x)|} \frac{\chi(x)}{\chi(1)}$ is an algebraic integer in Ω. (For any irreducible character χ of G.) We shall say that the irreducible characters χ_i, χ_j, lie in the same block, if and only if, for all elements x in G

$$\left(\frac{|G|}{|C(x)|}\left(\frac{\chi_i(x)}{\chi_i(1)} - \frac{\chi_j(x)}{\chi_j(1)}\right)\right)^* = 0$$

Here $(*)$ denotes the residue class modulo a fixed prime divisor of p in the field Ω. We observe that this is an equivalence relation on the set of irreducible characters of G. The equivalence classes under this relation will be called the p-blocks of G, and in particular, we define the principal p-block, $B_o(G)$, to be that p-block which contains the principal character of G.

Consider a p-block B of G. We must now explain what is meant by a basic set for B. Let Z denote the ring of integers. Let G^o denote the set of p-regular elements of G. If χ is a character of G, let $\chi|G^o$ denote the restriction of χ to G^o. If B is a p-block, then the linear combinations of the restrictions $\chi|G^o$ of the characters χ of B with integral coefficients, form a Z-module M_B. Any Z-basis of M_B is a basic set for B.

If B is a p-block, and u is a p-element, then for χ_i in B, and for p-regular elements r of C(u), we have

$$\chi_i(u) = \sum_{b \in Bl(C(u),B)} \sum_{\phi \in [b]} d(\chi_i,\phi)\phi(r).$$

Here Bl(C(u),B) is a specific set of p-blocks b of C(u), defined for example in [4]. For each b in Bl(C(u),B), [b] denotes a basic set for b. The generalized decomposition numbers $d(\chi_i,\phi)$ are algebraic integers in Ω, which do not depend on r. If u has order p^m, they belong to the field obtained from the rational field by adjunction of a primitive p^m-th root of unity.

We shall more frequently modify the above notation to write

$$(1) \qquad \chi_i(ur) = \sum_j d^u_{ij}\phi_j(r).$$

The decomposition numbers satisfy the following properties:-

Proposition 3.4 (Curtis & Reiner [7] 90.2, 90.3)

(a) If a, b, are non-conjugate p-elements of G, then

$$\sum_i d^a_{ij}\,\overline{d^b_{ik}} = 0$$

(b) Otherwise $\sum_i d^a_{ij}\,\overline{d^a_{ik}} = c^a_{jk}$ for all j,k

((c^a_{jk}) is the Cartan matrix related to $C_G(a)$.)

We shall also require the following result:-

Proposition 3.5 Suppose that x is a p-element of G and that $C_G(x)$ is p-nilpotent, then:-

(a) For all p-regular s in $C_G(x)$, $\chi_i(sx) = \chi_i(x)$, for all χ_i in $B_o(G)$

(b) $$\sum_{\chi_i \text{ in } B_o(G)} |\chi_i(x)|^2 = |P| ,$$ where P is a Sylow p-subgroup of $C_G(x)$.

We now need to introduce the concept of the defect group of a block. We say that a character χ_j in a block B has <u>height 0</u> if $|\chi_j(1)|_p \leq |\chi(1)|_p$ for all χ in B.

If $|G|_p/|\chi_j(1)|_p = p^d$, we call d the <u>defect</u> of B. There is at least one conjugate class K of G satisfying the following two properties:-

(a) $|G|_p / |K|_p = p^d$

(b) For any χ in B and any x in K $\left(\frac{|G|}{|C(x)|} \frac{\chi(x)}{\chi(1)}\right)^* = 0$.

Given this fact, then each Sylow p-subgroup of $C(x)$ (any x in K) is a <u>defect group</u> of B. We note the following:-

<u>Proposition 3.6</u> (Brauer [4] p. 106)

(i) The defect groups of a block B of defect d have order p^d.

(ii) The defect groups of a block B are determined uniquely up to conjugacy

(iii) If U is normal p-subgroup of G, then U is contained in all defect groups of each block.

<u>Proposition 3.7</u> (Brauer [6], Theorem 8 and corollary)

Let B be a p-block of a finite group G, and suppose that B has defect group S. We have a fixed system of elements of S

$$s_o = 1, s_1, \dots, s_m ,$$

which represent the different conjugate classes of G which meet S. Let s be a p-element of G, and v a p-regular element of $C_G(s)$. If s is not conjugate to any s_i $(0 \leq i \leq m)$ then $\chi(sv) = 0$, for all irreducible characters χ in B.

Proposition 3.8 (Brauer [5], Theorem 3)

Let G be a finite group with a Sylow p-subgroup of order p^n (some odd prime p). If the degree of any one character of the p-block B of G is divisible by p^{n-1}, the same is true for all characters of B.

Proposition 3.9 (Dornhoff [8], Lemma 66.1)

Assume P is a p-subgroup of the group G, such that $G = P\,C_G(P)$; set $|P| = p^b$, $\bar{G} = G/P$. Let $c = (c_{ij})$ be the Cartan matrix related to G, $\bar{c} = (\bar{c}_{ij})$ the Cartan matrix related to $\bar{G}$. Then $c_{ij} = p^b \bar{c}_{ij}$, for all i,j.

Proposition 3.10 ([18] p. 206)

Given any element a of a finite group G, then a is a p-element if and only if $(\chi(a) - \chi(1))^{**} = 0$, for all irreducible characters χ of G. (Here ** denotes congruence modulo the radical of (p) in the field generated by the characters.)

Proposition 3.11 (Dornhoff [8], Theorem 65.2 p. 397)

Let G be a finite group, and suppose that $B_0(G)$ denotes the principal p-block of G, for some prime p. Then

$$O_{p'}(G) = \cap \ker \chi \quad (\chi \text{ in } B_0(G))$$

Proposition 3.12 (Brauer's first main theorem, [7] p. 625)

There is a one to one correspondence between the blocks of G with defect group H, and the blocks of $N(H)$ with defect group H.

[Suppose that $G_1 \leq G$, and let B be a block of G_1. Under certain circumstances, we may assign to B a corresponding block $\hat{B}$ of G, using the Brauer correspondence. This is the correspondence referred to in the above proposition.]

Proposition 3.13 (Brauer's second main theorem, [8] p.384)

Let u be a p-element of G. Then as previously (in (1))

$$\chi_i(ur) = \sum_j d^u_{ij} \phi(r).$$

(Here r is a 3'-element of $C_G(u)$). Suppose that $d^u_{ij} \neq 0$, and let B denote the block of $C_G(u)$ containing ϕj. Then $\hat{B}$ is defined and χ_i lies in $\hat{B}$.

Proposition 3.14 (Brauer's third main theorem, [23] p.163)

Let Q be a p-subgroup of G, and let H be a subgroup of G, such that $QC_G(Q) \leq H$. Let B be a block of H with Q as defect group. Then $\hat{B}$ is the principal block of G if and only if B is the principal block of H.

Chapter IV

The Principal 3-Block

Section 1

We begin this chapter with intent of determining the principal 3-block of a finite group G, with a self-centralizing Sylow 3-subgroup, $P = \langle x,y \mid x^3 = y^3 = [x,y] = 1\rangle$, and Sylow 3-automiser $N_G(P)/P$ isomorphic to $Z_2 \times Z_2$. As will become apparent, the results which we obtain are slightly more general. In fact the calculations found in this chapter would enable us to establish the principal 3-block of any finite group with a self-centralizing, elementary Abelian, Sylow 3-subgroup of order nine, in which each 3-element is conjugate to its own inverse.

If we now suppose that

$N_G(P) = \langle a,b,P \mid a^2 = b^2 = [a,b] = 1,\ a^{-1}xa = x^{-1},\ a^{-1}ya = y,\ b^{-1}xb = x,\ b^{-1}yb = y^{-1}\rangle$,

Then the "conjugate classes of elements of P in G" are simply $\{x,x^{-1}\}$, $\{y,y^{-1}\}$ and $\{xy,xy^{-1},x^{-1}y,x^{-1}y^{-1}\}$. We now discuss the structure of the subgroups $C_G(x)$, $C_G(y)$ and $C_G(xy)$.

Lemma 4.1 $C_G(xy)$ is 3-nilpotent.

Proof From Proposition 2.7, it follows that $C_G(xy) = \langle xy\rangle \times K$, for some subgroup K of G. As $C_G(P) = P$, a Sylow 3-subgroup of K is self-centralizing in K. The structure of K is thus specified by Proposition 2.10. Since $C_{N_G(P)}(xy) = P$, it follows that the group K has a normal 3-complement. Thus $C_G(xy)$ has a normal 3-complement.

Lemma 4.2 $C_G(x) = \langle x\rangle \times A$, where one of the following holds:-

(i) $A/O_{3'}(A)$ is isomorphic to D_6.

(ii) $A/O_2(A)$ is isomorphic to $SL(2,4)$, and $O_2(A)$ is elementary Abelian, a direct sum of minimal normal subgroups of A, of order 2^4, each of which may be identified with a two-dimensional vector space over $GF(4)$, in such a way that the action of $A/O_2(A)$ on $O_2(A)$ is given by the usual action of $SL(2,4)$ as a group of matrices.

(iii) A is isomorphic to $PSL(2,7)$.

Proof From Proposition 2.7, it follows that $C_G(x) = \langle x \rangle \times A$, for some subgroup A of G. As $C_G(P) = P$, a Sylow 3-subgroup of A is self-centralizing in A. The structure of A is thus specified by Proposition 2.10. Since $C_{N_G(P)}(x) = \langle P,b \rangle$, it follows that A is not 3-nilpotent, and thus A has one of the above described structures.

Notes

(i) The possible structures for $C_G(y)$ are exactly analogous to the possible structures for $C_G(x)$, and are determined in a parallel fashion.

(ii) In considering the structure of $C_G(x)$ $(C_G(y))$, in the case in which $A/O_2(A)$ is isomorphic to $SL(2,4)$, we shall often tacitly invoke the fact that $SL(2,4)$ is isomorphic to A_5.

We may now begin to obtain information about the principal 3-block of G.

Section 2

We shall determine the values taken by the irreducible characters in the principal 3-block of G, on the 3-section of G containing x.

Lemma 4.3 $Bl(C_G(x), B_o(G)) = \{B_o(C_G(x))\}$

Proof Let v be a 3'-element of $C_G(x)$. Then as in the last chapter,

$$\chi_i(xv) = \sum_j d^x_{ij}\, \phi_j(v)$$

for each character χ_i in $B_0(G)$, where the $\{\phi_j\}$ are irreducible characters of $C_G(x)$. We know from Proposition 3.13, that if $d^x_{ij} \neq 0$, and ϕ_j lies in the block b_1 of $C_G(x)$, then $\hat{b}_1$ is defined, and χ_i lies in $\hat{b}_1$.

Now $\langle x\rangle \trianglelefteq C_G(x)$, so that Proposition 3.6, applies to tell us that $\langle x\rangle$ lies in all defect groups of each block of $C_G(x)$. Thus if b_2 is a block of $C_G(x)$, then b_2 has defect group $\langle x\rangle$ or P. From Proposition 3.14, it follows that if $\hat{b}_2 = B_0(G)$, then b_2 has defect group P. However, we know from Proposition 3.12, that $B_0(C_G(x))$ is the sole block of $C_G(x)$ with defect group P. It follows that whenever $d^x_{ij} \neq 0$, then ϕ_j lies in $B_0(C_G(x))$. We have thus obtained the required result.

The above lemma is important because it tells us that, in determining the values taken by the irreducible characters in the principal 3-block of G, on the 3-section of G containing x, the only block of $C_G(x)$ whose Cartan matrix (with respect to some basic set which we must specify) we need to determine is the principal 3-block.

We now determine a suitable basic set for the principal 3-block of $C_G(x)$, and calculate the corresponding Cartan matrix. Now $C_G(x) = \langle x\rangle \times A$, where A has one of the structures described in Lemma 4.2. We may suppose that $\langle y,b\rangle \leq A$. In fact (Proposition 3. 5), the principal 3-block of A consists of precisely three irreducible characters, all of degree prime to three. As $C_A(y) = \langle y\rangle$, these are the only characters of A of degree coprime to three.

(Proposition 3.10). Each of these characters contains $O_{3'}(A)$ in its kernel (Proposition 3.11). These characters of A yield upon composition with the natural homomorphism of A onto $A/O_{3'}(A)$, distinct irreducible characters of $A/O_{3'}(A)$ of unaltered degree.

We exhibit (corresponding to the three cases of Lemma 4.2) possible fragments of the principal 3-block of A.

(i) $A/O_{3'}(A)$ is isomorphic to D_6.

We obtain, in this instance, the following fragment of character table:-

χ	1	y	b	k
1_A	1	1	1	1
ϕ_A	1	1	-1	1
ζ_A	2	-1	0	2

Here, k is an element of $O_{3'}(A)$

(ii) $A/F(A)$ is isomorphic to A_5

We obtain, in this instance, the following fragment of character table:-

χ	1	y	b	k
1_A	1	1	1	1
ϕ_A	4	1	0	4
ζ_A	5	-1	1	5

Here, k is an element of $F(A)$

(iii) A is isomorphic to PSL(2,7)

In this instance our fragment of character table is a part of the character table of PSL(2,7).

χ	1	y	b
1_A	1	1	1
ϕ_A	7	1	-1
ζ_A	8	-1	0

In each of these cases, the characters $\{1_A, \phi_A\}$ form a basic set for $B_o(A)$, with corresponding Cartan matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

We pass now from the principal 3-block of A, to the principal 3-block of $C_G(x)$. All the irreducible characters of $C_G(x)$ are of the form $\chi_{\langle x \rangle} \otimes \chi_A$, for some irreducible characters $\chi_{\langle x \rangle}$ of $\langle x \rangle$, χ_A of A.

Using Proposition 3.9 (Dornhoff [8], Lemma 66.1), we deduce that the characters $1_{\langle x \rangle} \otimes 1_A$, and $1_{\langle x \rangle} \otimes \phi_A$, form a basic set for the principal 3-block of $C_G(x)$, with corresponding Cartan matrix $\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$. We shall denote $1_{C_G(x)} = 1_{\langle x \rangle} \otimes 1_A$ and $\phi_{C_G(x)} = 1_{\langle x \rangle} \otimes \phi_A$, and may also drop these new subscripts, where more convenient. We draw up the corresponding fragment of the character of $C_G(x)$.

(i) $A/O_{3'}(A)$ is isomorphic to D_6.

χ	1	y	b
1	1	1	1
ϕ	1	1	-1

(ii) $A/F(A)$ is isomorphic to A_5.

χ	1	y	b
1	1	1	1
ϕ	4	1	0

(iii) A is isomorphic to PSL(2,7)

χ	1	y	b
1	1	1	1
ϕ	7	1	-1

We must now use the information obtained in the above paragraphs, to calculate a fragment of the character table of G.

Since $Bl(C_G(x), B_0(G)) = \{B_0(C_G(x))\}$, and since the Cartan matrix for $B_0(C_G(x))$ (with respect to the basic set $\{1,\emptyset\}$) is $\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$, it follows that if χ_i is an irreducible character in $B_0(G)$, and v is a 3'-element of $C_G(x)$, then

$$\chi_i(xv) = d^x_{i1} + d^x_{i2}\emptyset(v),$$

where the generalized decomposition numbers $\{d^x_{ij}\}$ satisfy

$$\sum_i d^x_{i1}\,\overline{d^x_{i1}} = \sum_i d^x_{i2}\,\overline{d^x_{i2}} = 6 \quad \text{and} \quad \sum_i d^x_{i1}\,\overline{d^x_{i2}} = 3.$$

The d^x_{ij} are algebraic integers in the field $Q(\omega)$, where ω is a primitive cube root of unity. Thus, using W. Narkiewicz ([20], p. 63), we may write

$$d^x_{ij} = a^x_{ij} + \omega b^x_{ij},$$

for some rational integers a^x_{ij}, b^x_{ij}. It will be convenient in the following paragraphs, to omit the superscripts and write $d_{ij} = a_{ij} + \omega b_{ij}$.

<u>Lemma 4.4</u> The generalized decomposition numbers, d_{ij}, are rational integers.

<u>Proof</u> $\sum_i d_{i1}\overline{d_{i1}} = 6$. Thus $\sum_i (a_{i1} + \omega b_{i1})(\overline{a_{i1} + \omega b_{i1}}) = 6$. Then $0 \leq (a^2_{i1} - a_{i1}b_{i1} + b^2_{i1}) \leq 6$. Thus $0 \leq a^2_{i1} + b^2_{i1} \leq 6 + a_{i1}b_{i1}$. Now $0 \leq (a_{i1} - b_{i1})^2$. Thus $a_{i1}b_{i1} \leq \frac{1}{2}(a^2_{i1} + b^2_{i1})$.

It follows that $0 \leq a^2_{i1} + b^2_{i1} \leq 6 + \frac{1}{2}(a^2_{i1} + b^2_{i1})$. Whence we deduce that $0 \leq a^2_{i1} + b^2_{i1} \leq 12$. This implies that $|a_{i1}| \leq 3$ and that $|b_{i1}| \leq 3$. Parallel arguments yield that $|a_{i2}| \leq 3$ and that $|b_{i2}| \leq 3$.

As x is conjugate to x^{-1} in G, it follows that $\chi_i(x)$ is a rational integer.

$$\begin{aligned}\chi_i(x) &= d_{i1} + d_{i2}\emptyset(1)\\ &= (a_{i1} + \omega b_{i1}) + \emptyset(1)(a_{i2} + \omega b_{i2})\end{aligned}$$

As $\chi_i(x)$ is a rational integer, we may equate the coefficient of ω, in the above, to zero. Thus we obtain that $b_{i1} = -\emptyset(1)b_{i2}$.

We know from our previous work that $\emptyset(1)$ is one of $1,4,7$. The condition that $|b_{i1}| \leq 3$ implies immediately in the final two cases, that $b_{i1} = b_{i2} = 0$.

Thus it remains to treat the case where $\emptyset(1) = 1$. This corresponds to the case in which $A/O_{3'}(A)$ is isomorphic to D_6. In this instance, the element xb of $C_G(x)$, is conjugate to its own inverse in $N_G(x)$. It follows that $\chi_i(xb)$ is a rational integer.

Now $$\begin{aligned}\chi_i(xb) &= d_{i1} + d_{i2}\emptyset(b)\\ &= (a_{i1} + \omega b_{i1}) + \emptyset(b)(a_{i2} + \omega b_{i2})\\ &= (a_{i1} + \omega b_{i1}) - (a_{i2} - \omega b_{i1})\end{aligned}$$

As $\chi_i(bx)$ is a rational integer, we may equate the coefficient of ω, in the above, to zero. Thus we obtain that $b_{i1} = 0$.

The proof of the lemma is now complete.

Since $\sum_i d_{i1}^2 = \sum_i d_{i2}^2 = 6$, it follows now that $|d_{ij}| \leq 2$ (for all i; $j = 1,2$).

It will suffice, for our present purposes, to calculate the values $\chi_i(xv)$ (here v denotes a $3'$-element of $C_G(x)$) up to sign only. It follows, that, with respect to a suitable re-ordering of the χ_i. (In the final ordering, χ_o will denote the principal character of G.), we may assume that either:-

(a) $d_{i1} = 1 \quad 0 \leq i \leq 5$ or (b) $d_{o1} = d_{11} = 1$

$d_{i1} = 0$ otherwise

$d_{21} = 2$

$d_{i1} = 0$ otherwise.

Similarly the only possible sets of non-zero values of $|d_{i2}|$, are $\{1,1,1,1,1,1\}$ and $\{1,1,2\}$.

We now define the intersection of the columns of the generalized decomposition numbers corresponding to x, to be the set of those rows of entries for which neither d_{i1} nor d_{i2} is zero. We observe that the only contributions to the sum $\sum_i d_{i1}d_{i2} = 3$ come from terms in the intersection. The intersection must have order 2,3,4 or 5, since we know that $d_{o1} = 1$ and $d_{o2} = 0$, and furthermore that there are at most six values of i for which $d_{i1} \neq 0$.

Lemma 4.5 If χ_i is an irreducible character in $B_o(G)$, then $\chi_i(x) \not\equiv 0(3)$.

Proof Suppose that for some χ_i in $B_o(G)$, we have that $\chi_i(x) \equiv 0(3)$. It follows from the Proposition 3.10, that $\chi_i(1) \equiv 0(3)$. Then applying Proposition 3.8, $\chi_j(1) \equiv 0(3)$, for all irreducible characters χ_j in $B_o(G)$. However, we already know that $\chi_o(1) = 1$.

We deal firstly with the situation (a), in which $d_{i1} = 1$ $(0 \leq i \leq 5)$ and $d_{i1} = 0$ (otherwise).

Case (i) The intersection has order 2

Since $\sum_i d_{i1}d_{i2} = 3$, and $|d_{ij}| < 3$, the following table exhibits the sole possibility.

χ	xv
χ_0	1
χ_1	1
χ_2	1
χ_3	1
χ_4	$1 + 2\phi(v)$
χ_5	$1 + \phi(v)$
χ_6	$\phi(v)$

(The table is exact, within the convention that each row is correct up to sign only.)

(Here v denotes a $3'$-element of $C_G(x)$.)

We recall that $\phi(1) \equiv 1(3)$. We observe that $\chi_4(x) \equiv 0(3)$. This case may be thus ruled out by an application of Lemma 4.5.

<u>Case (ii)</u> <u>The intersection has order 3</u>

Since $\sum_i d_{i1}d_{i2} = 3$, and $|d_{ij}| < 3$, the following table exhibits the sole possibility:-

χ	xv
χ_0	1
χ_1	1
χ_2	1
χ_3	$1 + \phi(v)$
χ_4	$1 + \phi(v)$
χ_5	$1 + \phi(v)$
χ_6	$\phi(v)$
χ_7	$\phi(v)$
χ_8	$\phi(v)$

(This table is exact, within the convention that each row is correct up to sign only.)

(Here v denotes a $3'$-element of $C_G(x)$.)

<u>Case (iii)</u> <u>The intersection has order 4</u>

In this instance, all the entries in the column (d_{i2}) must have absolute value 1, and since $\sum_i d_{i1}d_{i2} = 3$, we obtain directly that this case cannot arise.

Case (iv) **The intersection has order 5**

In this instance, all the entries in the column (d_{i2}) must have absolute value 1, and since $\sum_i d_{i1}d_{i2} = 3$, the only possibility is that presented in the following table:-

χ	xv
χ_o	1
χ_1	$1 - \phi(v)$
χ_2	$1 + \phi(v)$
χ_3	$1 + \phi(v)$
χ_4	$1 + \phi(v)$
χ_5	$1 + \phi(v)$
χ_6	$\phi(v)$

(This table is exact, within the convention that each row is correct up to sign only.)

(Here v denotes a 3'-element of $C_G(x)$.)

We recall that $\phi(1) \equiv 1(3)$.

We observe that $\chi_1(x) \equiv 0(3)$. This case may be thus ruled out by an application of Lemma 4.5.

We now treat the situation (b), in which case we know that $d_{o1} = d_{11} = 1$, $d_{21} = 2$, and $d_{i1} = 0$ (for $i > 2.$). Since $d_{01} = 1$ and $d_{02} = 0$, and since $\sum_i d_{i1}d_{i2} = 3$, it follows that the interscetion must have order two.

The following tables illustrate the only possibilities in this situation:-

χ	xv
χ_o	1
χ_1	$1 - \phi(v)$
χ_2	$2 + 2\phi(v)$
χ_3	$\phi(v)$

χ	xv
χ_o	1
χ_1	$1 + \phi(v)$
χ_2	$2 + \phi(v)$
χ_3	$\phi(v)$
χ_4	$\phi(v)$
χ_5	$\phi(v)$
χ_6	$\phi(v)$

χ	xv
χ_o	1
χ_1	$1 + \phi(v)$
χ_2	$2 + \phi(v)$
χ_3	$2\phi(v)$

(The tables are exact, within the convention that each row is correct up to sign only.)

(In the above tables, v denotes a $3'$-element of $C_G(x)$.)

Now note, as previously, that since $\phi(1) \equiv 1(3)$, there is, in each of the above cases a character χ in $B_o(G)$ with $\chi(x) \equiv 0(3)$. Thus Lemma 4.5 enables us to rule out these possibilities.

Summary We have determined the values taken by the irreducible characters in the principal 3-block of G, on the 3-section of G containing x. We obtain (with the convention that each row is correct up to sign) the following character table fragment:-

χ	xv
χ_o	1
χ_1	1
χ_2	1
χ_3	$\phi(v)$
χ_4	$\phi(v)$
χ_5	$\phi(v)$
χ_6	$1 + \phi(v)$
χ_7	$1 + \phi(v)$
χ_8	$1 + \phi(v)$

(Here v denotes a $3'$-element of $C_G(x)$.)

Notes 1. An exactly analogous procedure enables us to determine the values taken by the irreducible characters in the principal 3-block of G, on the 3-section of G containing y.

2. This section improves on the methods used by Prince [22], Lemma 3.5.

Section 3

Maintaining the notation of the previous sections, we now determine the action of the χ_i $(0 \leq i \leq 8)$, on the elements of the 3-sections of G containing y and xy.

Let u be a 3'-element of $C_G(y)$, then we already possess the following information:-

(4.6) $\sum_{i=0}^{8} \chi_i(xv)\overline{\chi_i(yu)} = 0$ (Proposition 3.7)

(4.7) $\chi_i(y) \equiv \chi_i(x) \equiv \chi_i(1)\ (3)$ $0 \leq i \leq 8$ (Proposition 3.10)

(4.8) $C_G(y) = \langle y \rangle \times B$, where the group B has one of the following structures:-

(i) $B/O_{3'}(B)$ is isomorphic to D_6.

(ii) $B/O_2(B)$ is isomorphic to A_5.

(iii) B is isomorphic to PSL(2,7).

We may suppose that $\langle x,a \rangle \leq B$. We obtain the following possible fragments of the character table of $C_G(y)$, corresponding to the cases (i), (ii), (iii) above.

(i)

χ	1	x	a
1	1	1	1
ψ	1	1	-1

(ii)

χ	1	x	a
1	1	1	1
ψ	4	1	0

(iii)

χ	1	x	a
1	1	1	1
ψ	7	1	-1

The only non-zero values of $\chi_i(yu)$ $(0 \leq i \leq 8)$, are, up to sign $\{1, 1, 1, \psi(u), \psi(u), \psi(u), 1+\psi(u), 1+\psi(u), 1+\psi(u)\}$.

We begin with the following fragment of the character table of G:-

χ	xv	yu
χ_0	1	1
χ_1	1	?
χ_2	1	?
χ_3	$\phi(v)$	?
χ_4	$\phi(v)$	?
χ_5	$\phi(v)$	?
χ_6	$1 + \phi(v)$	?
χ_7	$1 + \phi(v)$	?
χ_8	$1 + \phi(v)$	?

The remaining entries in the column (yu), are known up to sign to be $\{1, 1, \psi(u), \psi(u), \psi(u), 1+\psi(u), 1+\psi(u), 1+\psi(u)\}$.

From (4.6), we know that $\sum_{i=0}^{8} \chi_i(xv)\,\overline{\chi_i(yu)} = 0$; this equation holds moreover for all 3'-elements v of $C_G(x)$ and for all 3'-elements u of $C_G(y)$. Thus we derive a set of equations of the form

$$k_1 + k_2\phi(v) + k_3\phi(v)\,\overline{\psi(u)} + k_4\overline{\psi(u)} = 0.$$

Case (iii) $A/O_3(A)$ is isomorphic to D_6,

B is isomorphic to $PSL(2,7)$.

We know the values $\phi(1)$, $\phi(b)$, $\psi(1)$, $\psi(a)$. We thus obtain four simultaneous equations:-

(a) $k_1 + k_2 + 7k_3 + 7k_4 = 0$ (v = 1 ; u = 1)

(b) $k_1 + k_2 - k_3 - k_4 = 0$ (v = 1 ; u = a)

(c) $k_1 - k_2 - 7k_3 + 7k_4 = 0$ (v = b ; u = 1)

(d) $k_1 - k_2 + k_3 - k_4 = 0$ (v = b ; u = a)

From (a) and (b), we obtain that $k_1 + k_2 = 0$, and that $k_3 + k_4 = 0$. From (c) and (d), we obtain that $k_1 - k_2 = 0$, and that $k_3 - k_4 = 0$. Thus it follows in this instance that $k_j = 0$ $(1 \leq j \leq 4)$.

Case (iv) $A/F(A)$ is isomorphic to A_5,

$B/O_3(B)$ is isomorphic to D_6.

By interchanging the rôles of x and y, and with suitable consequent amendments, this case is covered by case (ii) (above).

Case (v) $A/F(A)$ is isomorphic to A_5,

B is isomorphic to $PSL(2,7)$.

We know the values $\phi(1)$, $\phi(b)$, $\psi(1)$, $\psi(a)$. We thus obtain four simultaneous equations:-

(a) $k_1 + 4k_2 + 28k_3 + 7k_4 = 0$ (v = 1 ; u = 1)

(b) $k_1 + 4k_2 - 4k_3 - k_4 = 0$ (v = 1 ; u = a)

(c) $k_1 + 7k_4 = 0$ (v = b ; u = 1)

(d) $k_1 - k_4 = 0$ (v = b ; u = a)

From (c) and (d), we deduce easily that $k_1 = k_4 = 0$. By consequent substitution in (a) and (b), we obtain $k_2 = k_3 = 0$.

Case (vi) $A/F(A)$ is isomorphic to A_5,

$B/F(B)$ is isomorphic to A_5.

We know the values $\phi(1)$, $\phi(b)$, $\psi(1)$, $\psi(a)$. We thus obtain four simultaneous equations:-

(a) $k_1 + 4k_2 + 16k_3 + 4k_4 = 0$ $(v = 1 ;\ u = 1)$

(b) $k_1 + 4k_2 = 0$ $(v = 1 ;\ u = a)$

(c) $k_1 + 4k_4 = 0$ $(v = b ;\ u = 1)$

(d) $k_1 = 0$ $(v = b ;\ u = a)$

Back-substitution yields immediately that $k_1 = k_2 = k_3 = k_4 = 0$.

Case (vii) A is isomorphic to $PSL(2,7)$,

$B/O_{3'}(B)$ is isomorphic to D_6.

By interchanging the rôles of x and y, with suitable consequent amendments, this case is covered by case (vii) (above).

Case (viii) A is isomorphic to $PSL(2,7)$,

$B/F(B)$ is isomorphic to A_5.

By interchanging the rôles of x and y, with suitable consequent amendments, this case is covered by case (v) (above).

Case (ix) A is isomorphic to $PSL(2,7)$,

B is isomorphic to $PSL(2,7)$.

We know the values $\phi(1)$, $\phi(b)$, $\psi(1)$, $\psi(a)$. We thus obtain four simultaneous equations:-

(a) $k_1 + 7k_2 + 49k_3 + 7k_4 = 0$ $(v = 1 ;\ u = 1)$

(b) $k_1 + 7k_2 - 7k_3 - k_4 = 0$ $(v = 1 ;\ u = a)$

(c) $k_1 - k_2 - 7k_3 + 7k_4 = 0$ $(v = b ;\ u = 1)$

(d) $k_1 - k_2 + k_3 - k_4 = 0$ $(v = b ;\ u = a)$

From (a) and (b), we obtain that $k_1 + 7k_2 = 0$ and that $7k_3 + k_4 = 0$. From (c) and (d), we obtain that $k_1 - k_2 = 0$ and that $k_3 - k_4 = 0$. Thus it follows in this instance that $k_j = 0$ $(1 \leq j \leq 4)$.

We now may proceed with our construction of a larger fragment of the character table of G.

We begin with the known fragment of the character table of G.

χ	xv	yu
χ_0	1	1
χ_1	1	?
χ_2	1	?
χ_3	$\phi(v)$	?
χ_4	$\phi(v)$	?
χ_5	$\phi(v)$	?
χ_6	$1 + \phi(v)$	?
χ_7	$1 + \phi(v)$	?
χ_8	$1 + \phi(v)$	?

<u>Step 1</u> In the sum $\sum_{i=0}^{8} \chi_i(xv)\overline{\chi_i(yu)}$, we may equate the coefficient k_1 (of unity) to zero (with the previous notation). We already have a contribution of 1 to this coefficient. We seek to eliminate this contribution. We know from (4.7), that $\chi_i(y) \equiv \chi_i(x)$ (3), and since $\psi(1) \equiv \phi(1) \equiv 1(3)$, we are able to derive two further possibilities

(1a)

χ	xv	yu
χ_o	1	1
χ_1	1	
χ_2	1	$-1-\psi(v)$
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	

(1b)

χ	xv	yu
χ_o	1	1
χ_1	1	
χ_2	1	
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	

<u>Step 2</u>

(i) <u>Table (1a)</u> In the sum $\sum_{i=o}^{8} \chi_i(xv)\,\overline{\chi_i(yu)}$, we may equate the coefficient k_4 (of $\overline{\psi(u)}$) to zero (with the previous notation). We already have a contribution of -1 to this coefficient. We seek to eliminate this contribution. There are two possibilities:-

(2a)

χ	xv	yu
χ_o	1	1
χ_1	1	$\psi(u)$
χ_2	1	$-1-\psi(u)$
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	

(2b)

χ	xv	yu
χ_o	1	1
χ_1	1	
χ_2	1	$-1 - \psi(u)$
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	$1 + \psi(u)$

(ii) Table (1b) In the sum $\sum_{i=o}^{8} \chi_i(xv)\overline{\chi_i(yu)}$, we may equate the coefficient k_2 (of $\phi(v)$) to zero (with the previous notation). We already have a contribution of -1 to this coefficient. We seek to eliminate this contribution. There are two possibilities:-

(2c)

χ	xv	yu
χ_o	1	1
χ_1	1	
χ_2	1	
χ_3	$\phi(v)$	1
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	

(2d)

χ	xv	yu
χ_o	1	1
χ_1	1	
χ_2	1	
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	$1 + \psi(u)$

Step 3

(i) Table (2c) In case (2c), we may complete the table directly. The present contribution in the table (2c), to the coefficient k_1, is zero. Up to sign, the entry $1 + \psi(u)$ appears three times in the column (yu). In order to ensure that $k_1 = 0$, one such entry must appear in the row corresponding to χ_4 or χ_5. The other two entries must necessarily have opposite signs. The table is then easily completed and is found to have the following form:-

χ	xv	yu
χ_0	1	1
χ_1	1	$\psi(u)$
χ_2	1	$-1-\psi(u)$
χ_3	$\phi(v)$	1
χ_4	$\phi(v)$	$\psi(u)$
χ_5	$\phi(v)$	$-1-\psi(u)$
χ_6	$1+\phi(v)$	-1
χ_7	$1+\phi(v)$	$-\psi(u)$
χ_8	$1+\phi(v)$	$1+\psi(u)$

This fragment of character table is certainly a possibility. In fact we show that it is the only possible fragment.

In what remains of this sub-section, we shall suppose that the case (2c) does not arise.

(ii) <u>Table (2a)</u> As table (2c) does not arise, there are now two possible character table fragments to discuss in this section.

(A)

χ	xv	yu
χ_0	1	1
χ_1	1	$\psi(u)$
χ_2	1	$-1-\psi(u)$
χ_3	$\phi(v)$	1
χ_4	$\phi(v)$	1
χ_5	$\phi(v)$	
χ_6	$1+\phi(v)$	
χ_7	$1+\phi(v)$	
χ_8	$1+\phi(v)$	

(B)

χ	xv	yu
χ_0	1	1
χ_1	1	$\psi(u)$
χ_2	1	$-1-\psi(u)$
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1+\phi(v)$	-1
χ_7	$1+\phi(v)$	-1
χ_8	$1+\phi(v)$	

Case A This possibility may be directly ruled out. The present contribution to the coefficient k_1, in table A, is zero. The entry $1 + \psi(u)$, must appear twice more (up to sign) in the column (yu). The congruence relation $\chi_i(x) \equiv \chi_i(y)$ (3) applies to tell us that we cannot reduce the coefficient k_1 to zero.

Case B This possibility may be directly ruled out. The present contribution to the coefficient k_1 in table A is -2. If $\chi_8(yu) = 1 + \psi(u)$, then $k_1 = -1$, otherwise $k_1 = -2$. It follows that $k_1 \neq 0$, and so this case cannot arise.

In what remains of this section we shall suppose that case (2a) does not arise.

(iii) Table (2b) In table (2b), the present contribution to the coefficient k_1 is 1. We seek to eliminate this contribution. There are two possibilities:-

(A)

χ	xv	yu
χ_0	1	1
χ_1	1	$-1-\psi(u)$
χ_2	1	$-1-\psi(u)$
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(u)$	$1 + \psi(u)$

(B)

χ	xv	yu
χ_0	1	1
χ_1	1	
χ_2	1	$-1-\psi(u)$
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	$1+\psi(u)$

Case A This possibility may be directly ruled out. Since the contribution in the present table, to the coefficient k_2 (of $\phi(v)$) is 1, it is visibly impossible to complete the table in such a way that $k_2 = 0$.

Case B We are now assuming that case (2c) does not arise, therefore either $\chi_1(yu) = 1$ or $\chi_7(yu) = -1$. It is in both cases then impossible to reduce the coefficient k_1 to zero.

(iv) Table (2d) As (2c) does not arise, there are now two possible character table fragments to discuss in this section:-

(A)

χ	xv	yu
χ_0	1	1
χ_1	1	1
χ_2	1	
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	
χ_8	$1 + \phi(v)$	$1 + \psi(u)$

(B)

χ	xv	yu
χ_0	1	1
χ_1	1	
χ_2	1	
χ_3	$\phi(v)$	
χ_4	$\phi(v)$	
χ_5	$\phi(v)$	
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	-1
χ_8	$1 + \phi(v)$	$1 + \psi(u)$

Case A This possibility may be directly ruled out. The present contribution to the coefficient k_1 in table A is 2. The entry $1 + \psi(u)$ must appear twice more (up to sign) in the column (yu). The congruence relation $\chi_i(x) \equiv \chi_i(y)$ (3) applies to tell us that we cannot reduce the coefficient k_1 to zero.

Case B This possibility may be directly ruled out. The present contribution to the coefficient k_1 in table B is zero. Using the congruence relation $\chi_i(x) \equiv \chi_i(y)$ (3), we derive immediately the following fragment of character table:-

χ	xv	yu
χ_0	1	1
χ_1	1	$\psi(u)$
χ_2	1	$\psi(u)$
χ_3	$\phi(v)$	$\psi(u)$
χ_4	$\phi(v)$	$-1 - \psi(u)$
χ_5	$\phi(v)$	$-1 - \psi(u)$
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	-1
χ_8	$1 + \phi(v)$	$1 + \psi(u)$

In fact in this fragment, we see that the coefficient k_2 of $\phi(v)$ in the sum $\sum_{i=o}^{8} \chi_i(xv)\,\overline{\chi_i(yu)}$ is non-zero. (in fact $k_2 = -3$) It follows that this fragment cannot occur.

<u>Summary</u> We now know the action of the characters χ_i $(0 \leq i \leq 8)$ on the 3-section of G containing yu. We have obtained the following fragment of character table:-

χ	xv	yu
χ_0	1	1
χ_1	1	$\psi(u)$
χ_2	1	$-1 - \psi(u)$
χ_3	$\phi(v)$	1
χ_4	$\phi(v)$	$\psi(u)$
χ_5	$\phi(v)$	$-1 - \psi(u)$
χ_6	$1 + \phi(v)$	-1
χ_7	$1 + \phi(v)$	$- \psi(u)$
χ_8	$1 + \phi(v)$	$1 + \psi(u)$

We seek now to determine the action of the characters χ_i, $(0 \leq i \leq 8)$, on the 3-section of G containing xy. We know from Lemma 4.1, that $C_G(xy)$ is 3-nilpotent. Then from Proposition 3.5, we deduce that:-

(a) For all 3-regular elements s of $C_G(xy)$, $\chi_i(xys) = \chi_i(xy)$ $(0 \leq i \leq 8)$

(b) $\sum_{i=0}^{8} |\chi_i(xy)|^2 = |P| = 9.$

Since xy is conjugate to $(xy)^{-1}$ in G, it follows that $\chi_i(xy)$ is a rational integer, and thus from (b) that $\chi_i(xy) = \pm 1$. Since $\chi_i(xy) \equiv \chi_i(x)$ (3), we may immediately deduce the following larger fragment of the character table of G:-

χ	1	xv	yu	xys
χ_0	1	1	1	1
χ_1	a_1	1	$\psi(u)$	1
χ_2	a_2	1	$-1 - \psi(u)$	1
χ_3	a_3	$\phi(v)$	1	1
χ_4	a_4	$\phi(v)$	$\psi(u)$	1
χ_5	a_5	$\phi(v)$	$-1 - \psi(u)$	1
χ_6	a_6	$1 + \phi(v)$	-1	-1
χ_7	a_7	$1 + \phi(v)$	$-\psi(u)$	-1
χ_8	a_8	$1 + \phi(v)$	$1 + \psi(u)$	-1

The character degrees a_i $(1 \leq i \leq 8)$, are at present unknown, however the standard column orthogonality relations allow us to determine linear relationships between the a_i.

We have the following orthogonality relations:-

(i) $\sum_{i=o}^{8} \chi_i(xv)\, \chi_i(1) \equiv 0$

(ii) $\sum_{i=o}^{8} \chi_i(yu)\, \chi_i(1) \equiv 0$

(iii) $\sum_{i=o}^{8} \chi_i(xy)\, \chi_i(1) = 0.$

We derive the following equations:-

(i) $1 + a_1 + a_2 + a_3\phi(v) + a_4\phi(v) + a_5\phi(v) + a_6(1 + \phi(v)) + a_7(1 + \phi(v)) + a_8(1 + \phi(v)) \equiv 0$

(ii) $1 + a_1\psi(u) + a_2(-1-\psi(u)) + a_3 + a_4\psi(u) + a_5(-1-\psi(u)) - a_6 - a_7\psi(u) + a_8(1 + \psi(u)) \equiv 0$

(iii) $1 + a_1 + a_2 + a_3 + a_4 + a_5 - a_6 - a_7 - a_8 = 0.$

Since the forms (i) and (ii) represent a set of equations (as v ranges over the 3-regular elements of $C_G(x)$, and as u ranges over the 3-regular elements of $C_G(y)$), we may in (i) and (ii), equate the coefficients of $\phi(v)$ and $\psi(u)$ to zero. We thus obtain five linear relationships between the (a_i).

(a) $1 + a_1 + a_2 + a_6 + a_7 + a_8 = 0$

(b) $a_3 + a_4 + a_5 + a_6 + a_7 + a_8 = 0$

(c) $1 - a_2 + a_3 - a_5 - a_6 + a_8 = 0$

(d) $a_1 - a_2 + a_4 - a_5 - a_7 + a_8 = 0$

(e) $1 + a_1 + a_2 + a_3 + a_4 + a_5 - a_6 - a_7 - a_8 = 0$

Step 1 We eliminate a_8

From (e), $\underline{a_8 = 1 + a_1 + a_2 + a_3 + a_4 + a_5 - a_6 - a_7.}$

Substituting this value for a_8 in the preceding equations, we obtain:-

(a) $2 + 2a_1 + 2a_2 + a_3 + a_4 + a_5 = 0$

(b) $1 + a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 = 0$

(c) $2 + a_1 + 2a_3 + a_4 - 2a_6 - a_7 = 0$

(d) $1 + 2a_1 + a_3 + 2a_4 - a_6 - 2a_7 = 0$

<u>Step 2</u> <u>We eliminate a_7 and a_5</u>

From (c), $\underline{a_7 = 2 + a_1 + 2a_3 + a_4 - 2a_6}$

From (a) $\underline{a_5 = -2 - 2a_1 - 2a_2 - a_3 - a_4}$

Substituting these values in our equations, we obtain:-

(b) $- 3 - 3a_1 - 3a_2 = 0$

(d) $- 3 - 3a_3 + 3a_6 = 0.$

Thus we have that $\underline{a_2 = - 1 - a_1}$ and that $\underline{a_6 = 1 + a_3}$.

By back substitution, we now obtain:-

$a_5 = - a_3 - a_4$; $a_7 = a_1 + a_4$; $a_8 = - 1 - a_1 - a_3 - a_4$.

Our character table fragment now has the following form:-

χ	1	xv	yu	xys
χ_o	1	1	1	1
χ_1	a_1	1	$\psi(u)$	1
χ_2	$-1-a_1$	1	$-1 - \psi(u)$	1
χ_3	a_3	$\phi(v)$	1	1
χ_4	a_4	$\phi(v)$	$\psi(u)$	1
χ_5	$-a_3-a_4$	$\phi(v)$	$-1 - \psi(u)$	1
χ_6	$1+a_3$	$1 + \phi(v)$	-1	-1
χ_7	a_1+a_4	$1 + \phi(v)$	$- \psi(u)$	-1
χ_8	$-1-a_1-a_3-a_4$	$1 + \phi(v)$	$1 + \psi(u)$	-1

As the character table fragment is correct only within the convention that each row is correct up to sign, there seems little point in retaining minus signs in column (1). We now re-organize the above table, to obtain (within the usual convention), the following final form:-

χ	1	xv	yu	xys
χ_o	1	1	1	1
χ_1	j	$\phi(v)$	1	1
χ_2	m	1	$\psi(u)$	1
χ_3	n	$\phi(v)$	$\psi(u)$	1
χ_4	j+1	$1 + \phi(v)$	-1	-1
χ_5	m+n	$1 + \phi(v)$	$- \psi(u)$	-1
χ_6	m+1	-1	$1 + \psi(u)$	-1
χ_7	j+n	$- \phi(v)$	$1 + \psi(u)$	-1
χ_8	1+j+m+n	$-1 - \phi(v)$	$-1 - \psi(u)$	1

Chapter V

Theorem A

In this chapter, we discuss the structure of a finite group G, possessing a self-centralizing Sylow 3-subgroup P, of order nine, and having precisely three conjugate classes of elements of order three.

We prove the following theorem:-

Theorem A

Let G be a finite group, possessing a self-centralizing Sylow 3-subgroup of order nine, and precisely three conjugate classes of elements of order three. Then there is a soluble normal subgroup, N of G, with G/N isomorphic to a group in the following list:-

$$\begin{array}{ccccc} D_6 \times D_6 & : & D_6 \times A_5 & : & D_6 \times PSL(2,7) \\ H_1 & : & A_5 \times A_5 & : & A_5 \times PSL(2,7) \\ H_2 & & & : & PSL(2,7) \times PSL(2,7). \end{array}$$

Here H_1 will denote an extension of Z_3 by S_5, and H_2 will denote an extension of Z_3 by PGL(2,7).

Suppose that G is a finite group satisfying the hypotheses of Theorem A. We first restate some of the results and some of the notation of chapter IV.

We suppose that $P = \langle x,y \mid x^3 = y^3 = [x,y] = 1\rangle$ is a Sylow 3-subgroup of G, with normaliser

$$N_G(P) = \langle a,b,P \mid a^2 = b^2 = [a,b] = 1,\ a^{-1}xa = x^{-1},\ a^{-1}ya = y,\ b^{-1}xb = x,\ b^{-1}yb = y^{-1}\rangle$$

The following facts were derived in chapter IV:-

(a) (Lemma 4.2) $C_G(x) = \langle x \rangle \times A$, where one of the following holds:-

(i) $A/O_3(A)$ is isomorphic to D_6

(ii) $A/O_2(A)$ is isomorphic to A_5

(iii) A is isomorphic to $PSL(2,7)$.

We shall say that $C_G(x)$ is of <u>type 1</u>, <u>type 2</u>, or <u>type 3</u>, according as (i), (ii) or (iii) holds (respectively).

Note that $C_G(y) = \langle y \rangle \times B$, where the structure of B is determined in exactly the same way as the structure of A.

(b) The principal 3-block of G has the following form (with the convention that each row is correctly determined up to sign only):-

χ	1	x	y	xy
χ_0	1	1	1	1
χ_1	j	$\phi(1)$	1	1
χ_2	m	1	$\psi(1)$	1
χ_3	n	$\phi(1)$	$\psi(1)$	1
χ_4	j+1	$1 + \phi(1)$	-1	-1
χ_5	m+n	$1 + \phi(1)$	$-\psi(1)$	-1
χ_6	m+1	-1	$1 + \psi(1)$	-1
χ_7	j+n	$-\phi(1)$	$1 + \psi(1)$	-1
χ_8	1+j+m+n	$-1 - \phi(1)$	$-1 - \psi(1)$	1

In this table, ϕ is a character of $C_G(x)$, and $\phi(1) = 1, 4$, or 7, corresponding (respectively) as $C_G(x)$ is of type 1, type 2 or type 3. Similarly ψ is a character of $C_G(y)$, and $\psi(1) = 1, 4$, or 7, corresponding (respectively) as $C_G(y)$ is of type 1, type 2 or type 3. The derivation of the characters ϕ and ψ, is fully explained in

Chapter IV, as also is the construction of the above character table fragment.

We now begin our proof of Theorem A.

Proof

Lemma 5.1 Let G be a finite group satisfying the hypotheses of Theorem A. Then G is not simple.

Proof The original proof of this fact is due to G. Higman, and appears in [17], in outline. We present the basic ideas here, in somewhat expanded form.

From Proposition 2.8, we deduce that:-

$$\#(x^\bullet x^\bullet = y) = \#(x^\bullet x^\bullet = xy) = \#(y^\bullet y^\bullet = x) = \#(y^\bullet y^\bullet = xy) = 0.$$

We obtain the four corresponding character theoretic relations:-

$$\Sigma \frac{\chi^2(x)\chi(y)}{\chi(1)} = \Sigma \frac{\chi^2(x)\chi(xy)}{\chi(1)} = \Sigma \frac{\chi^2(y)\chi(x)}{\chi(1)} = \Sigma \frac{\chi^2(y)\chi(xy)}{\chi(1)} = 0$$

Each of the sums must in principle be taken over all the irreducible characters of G. We show next that the contribution to these sums from characters outside $B_o(G)$ is zero.

(i) We may omit in each of these sums, any character χ for which two of $\chi(x)$, $\chi(y)$ and $\chi(xy)$ are zero. Thus (Proposition 3.7), we may omit any character belonging to a 3-block of G, whose defect group is a proper subgroup of P.

(ii) Since $C_G = P$, and $N_G(P)$ is isomorphic to $D_6 \times D_6$, it follows (Proposition 3.12), that $B_o(G)$ is the only 3-block of G with defect group P.

Taking (i) and (ii) together, it is clear that we may restrict the above sums to the principal 3-block of G. We thus obtain the following four equations:-

1. $$\sum_{i=o}^{8} \frac{\chi_i^2(x)\chi_i(y)}{\chi_i(1)} = 0$$

$$1 + \frac{(\phi(1))^2}{j} + \frac{(\psi(1)}{m} + \frac{(\phi(1))^2\psi(1)}{n} - \frac{(1+\phi(1))^2}{j+1} - \frac{(1+\phi(1))^2\psi(1)}{m+n}$$

$$+ \frac{(1+\psi(1))}{m+1} + \frac{(\phi(1))^2(1+\psi(1))}{j+n} + \frac{(1+\phi(1))^2(-1-\psi(1))}{1+j+m+n} = 0$$

2. $$\sum_{i=o}^{8} \frac{\chi_i^2(x)\chi_i(xy)}{\chi_i(1)} = 0$$

$$1 + \frac{(\phi(1))^2}{j} + \frac{1}{m} + \frac{(\phi(1))^2}{n} + \frac{(1+\phi(1))^2(-1)}{j+1} + \frac{(1+\phi(1))^2(-1)}{m+n}$$

$$+ \frac{(-1)}{m+1} + \frac{(\phi(1))^2(-1)}{j+n} + \frac{(1+\phi(1))^2}{1+j+m+n} = 0$$

3. $$\sum_{i=o}^{8} \frac{\chi_i^2(y)\chi_i(x)}{\chi_i(1)} = 0$$

$$1 + \frac{\phi(1)}{j} + \frac{(\psi(1))^2}{m} + \frac{\phi(1)(\psi(1))^2}{n} + \frac{(1+\phi(1))}{j+1} + \frac{(1+\phi(1))\psi(1))^2}{m+n}$$

$$- \frac{(1+\psi(1))^2}{m+1} - \frac{\phi(1)(1+\psi(1))^2}{j+n} - \frac{(1+\phi(1))(1+\psi(1))^2}{1+j+m+n} = 0$$

4. $$\sum_{i=o}^{8} \frac{\chi_i^2(y)\chi_i(xy)}{\chi_i(1)} = 0$$

$$1 + \frac{1}{j} + \frac{(\psi(1))^2}{m} + \frac{(\psi(1))^2}{n} - \frac{1}{j+1} - \frac{(\psi(1))^2}{m+n} - \frac{(1+\psi(1))^2}{m+1}$$

$$- \frac{(1+\psi(1))^2}{j+n} + \frac{(1+\psi(1))^2}{1+j+m+n} = 0$$

We shall frequently refer to these equations as the "degree equations". This terminology stems from the fact that in the above equations the only unknowns are the character degrees j,m,n; thus the equations may be used to determine the degrees of some of the irreducible characters of G.

The degree equations are used to show that there is no finite simple group satisfying the hypotheses of Theorem A. Without loss of generality (by interchanging x and y if necessary), there are precisely six different cases to consider, according as the types of $C_G(x)$ and $C_G(y)$ vary (thus according as $\phi(1)$ and $\psi(1)$ vary). These cases are:-

$C_G(x)$	$C_G(y)$	$\phi(1)$	$\psi(1)$	
type 1	type 1	1	1	Case I
type 1	type 2	1	4	Case II
type 1	type 3	1	7	Case III
type 2	type 2	4	4	Case IV
type 2	type 3	4	7	Case V
type 3	type 3	7	7	Case VI

In the first three cases, the degree equations are exactly soluble (that is to say that the values of j,m,n may be determined) by hand. It turns out in each of these cases that $j = 1$. Thus in none of the first three cases is it possible to find a finite simple group G satisfying the hypotheses of Theorem A.

In the final three cases, the degree equations have been investigated by D. Taylor and G. Higman. Computation, the bulk of which was carried out by computer, has shown that in these three cases there exists no finite simple group satisfying the hypotheses of Theorem A.

Later in this chapter, we shall require to solve explicitly the degree equations corresponding to cases I, II and III.

Let G be a finite group satisfying the hypotheses of Theorem A. Then $G/O_{3'}(G)$ satisfies the identical hypotheses, and from Lemma 5.1, $G/O_{3'}(G)$ is not simple. We now proceed to obtain the possible minimal normal subgroups of $G/O_{3'}(G)$.

Lemma 5.2 Let H be a finite group with $|H|_3 = 3$, and suppose that H admits a group $\langle \alpha,\beta \mid \alpha^3 = \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1}\rangle$ of automorphisms. Suppose further that $H_* = H\langle \alpha,\beta\rangle$ has a self-centralizing Sylow 3-subgroup of order nine, and precisely three conjugate classes of elements of order three, and that $C_{H_*}(\alpha)$ is of type 1. Then H is soluble.

Proof We take H to be a counterexample to the lemma, of least possible order.

Step 1 H is simple.

If this is not the case, then as $|H|_3 = 3$, H is not characteristically simple. We choose a minimal non-trival normal $\langle \alpha,\beta\rangle$-invariant subgroup H_1 of H. If H_1 is a 3'-group, then since H_* has a self-centralizing elementary Abelian Sylow 3-subgroup of order nine, it follows (Proposition 2.20) that H_1 is soluble. Moreover the hypotheses of the lemma in this instance go over to H/H_1. The minimality of H implies now that H/H_1 is soluble. It follows that H is soluble, and thus that H is no counterexample to our lemma. Thus we must assume that $|H_1|_3 = 3$.

Let S be an $\langle \alpha,\beta\rangle$-invariant Sylow 3-subgroup of H_1. If $N_{H_1}(S) \neq C_{H_1}(S)$, then the hypotheses of the lemma go over to H_1. The minimality of H implies in this case that H_1 is soluble. If $N_{H_1}(S) = C_{H_1}(S)$, then Burnside's transfer theorem (Proposition 2.3), applies to tell us that H_1 has a normal 3-complement, $O_{3'}(H_1)$. The minimality of H_1 implies that $H_1 = S$. In particular, H_1 is soluble.

It follows in all instances that H_1 is soluble. The Frattini argument yields that $H = H_1N_H(S)$. Note that $N_H(S)$ is soluble. Thus H is itself soluble, and so H is no counterexample to our lemma.

Step 2 H_* has precisely two linear characters.

We have that $H_* = H\langle\alpha,\beta\rangle$. Since H is simple, it follows that $H_*' = H\langle\alpha\rangle$. Thus $|H_* : H_*'| = 2$, and H_* has precisely two linear characters.

Step 3 (Final step)

We now solve the degree equations for H_* (In fact this is precisely case I of Lemma 5.1).

Our principal 3-block for H_* is (with the usual convention that each row is correct only up to sign):-

χ	1	x	α	$x\alpha$
χ_0	1	1	1	1
χ_1	j	1	1	1
χ_2	m	1	1	1
χ_3	n	1	1	1
χ_4	$j+1$	2	-1	-1
χ_5	$m+n$	2	-1	-1
χ_6	$m+1$	-1	2	-1
χ_7	$j+n$	-1	2	-1
χ_8	$1+j+m+n$	-2	-2	1

Here $\langle x\rangle$ is taken to denote a Sylow 3-subgroup of H.

The four degree equations are:-

1. $$1 + \frac{1}{j} + \frac{1}{m} + \frac{1}{n} - \frac{4}{j+1} - \frac{4}{m+n} + \frac{2}{m+1} + \frac{2}{j+n} - \frac{8}{1+j+m+n} = 0$$

2. $$1 + \frac{1}{j} + \frac{1}{m} + \frac{1}{n} - \frac{4}{j+1} - \frac{4}{m+n} - \frac{1}{m+1} - \frac{1}{j+n} + \frac{4}{1+j+m+n} = 0$$

3. $$1 + \frac{1}{j} + \frac{1}{m} + \frac{1}{n} + \frac{2}{j+1} + \frac{2}{m+n} - \frac{4}{m+1} - \frac{4}{j+n} - \frac{8}{1+j+m+n} = 0$$

4. $$1 + \frac{1}{j} + \frac{1}{m} + \frac{1}{n} - \frac{1}{j+1} - \frac{1}{m+n} - \frac{4}{m+1} - \frac{4}{j+n} + \frac{4}{1+j+m+n} = 0$$

From 1. and 2. we obtain $\frac{1}{m+1} + \frac{1}{j+n} = \frac{4}{1+j+m+n}$.

Denote $A = m+1$ and $B = j+n$. Then $\frac{1}{A} + \frac{1}{B} = \frac{4}{A+B}$, and so $(A-B)^2 = 0$.

Thus $m+1 = j+n$.

Similarly from 3. and 4. we obtain $j+1 = m+n$.

Thus $j+1 = m+n = j+2n-1$, and so $n = 1$ and $j = m$. Then substitution in 1. yields:- $1 + \frac{1}{j} = \frac{4}{j+1}$; and so $j = 1$.

Thus $j = m = n = 1$.

But this implies that H_* has four distinct linear characters. This result contradicts step 2. Thus H_* is soluble, as is H.

This completes the proof of the lemma.

Lemma 5.3 Let H be a finite group with $|H|_3 = 3$, and suppose that H admits a group $\langle \alpha, \beta \mid \alpha^3 = \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1}\rangle$ of automorphisms. Suppose further that $H_* = H\langle\alpha,\beta\rangle$ has a self-centralizing Sylow 3-subgroup of order nine, and precisely three conjugate classes of elements of order three, and that $C_{H_*}(\alpha)$ is of type 2. Then $H/O_{3'}(H)$ is isomorphic to A_5, and $O_{3'}(H)$ is soluble.

Proof We take H to be a counterexample to the lemma, of least possible order.

Step 1 H is simple.

If this is not the case, then since $|H|_3 = 3$, H is not characteristically simple. We choose a minimal non-trivial normal $\langle\alpha,\beta\rangle$-invariant subgroup H_1 of H. If H_1 is a 3'-group, then since H_* has a self-centralizing elementary Abelian Sylow 3-subgroup of order nine, it follows (Proposition 2.20) that H_1 is soluble. Moreover the hypotheses of the lemma in this instance go over to H/H_1. The minimality of H now implies that $H/H_1 / O_{3'}(H/H_1)$ is isomorphic to A_5 and that $O_{3'}(H/H_1)$ is soluble. It follows that $H/O_{3'}(H)$ is isomorphic to A_5 and that $O_{3'}(H)$ is soluble. Thus in this instance, H is no counterexample to our lemma. Thus we must assume that $|H_1|_3 = 3$.

Let S be an $\langle\alpha,\beta\rangle$-invariant Sylow 3-subgroup of H_1. If $N_{H_1}(S) = C_{H_1}(S)$, then by Burnside's transfer theorem (Proposition 2.3), H_1 has a normal 3-complement $O_{3'}(H_1)$. The minimality of H_1 yields that $O_{3'}(H_1) = 1$, and thus that $H_1 = S$. Now note that $N_H(S)$ is soluble. Thus H is soluble. This is an impossibility, since $C_{H_*}(\alpha)$ is of type 2.

Thus $N_{H_1}(S) \neq C_{H_1}(S)$, and so the hypotheses of the lemma now go over to H_1. By the minimality of H it follows that $H_1/O_{3'}(H_1)$ is isomorphic to A_5, where $O_{3'}(H_1)$ is soluble. From the minimality of H_1 it follows that H_1 is isomorphic to A_5. Now the Frattini argument yields that $H = H_1N_H(S)$. It now follows that $H = H_1 \times A$, for some nilpotent group A. Thus in this instance, H is no counterexample to our lemma.

Step 2 (The degree equations)

We now solve the degree equations for H_*. (In fact this is precisely case II of Lemma 5.1).

Our prinipal 3-block for H_* is (with the usual convention that each row is correct only up to sign):-

χ	1	x	α	$x\alpha$
χ_0	1	1	1	1
χ_1	j	1	1	1
χ_2	m	1	4	1
χ_3	n	1	4	1
χ_4	$j+1$	2	-1	-1
χ_5	$m+n$	2	-4	-1
χ_6	$m+1$	-1	5	-1
χ_7	$j+n$	-1	5	-1
χ_8	$1+j+m+n$	-2	-5	1

Here $\langle x\rangle$ is taken to denote a Sylow 3-subgroup of H. Note that at this stage, we could infer directly from the structure of H_*, that $j = 1$. We prefer instead to obtain the most general solution of the degree equations. These equations are:-

1. $$1+\frac{1}{j}+\frac{4}{m}+\frac{4}{n}-\frac{4}{j+1}-\frac{16}{m+n}+\frac{5}{m+1}+\frac{5}{j+n}-\frac{20}{1+j+m+n}=0$$

2. $$1+\frac{1}{j}+\frac{1}{m}+\frac{1}{n}-\frac{4}{j+1}-\frac{4}{m+n}-\frac{1}{m+1}-\frac{1}{j+n}+\frac{4}{1+j+m+n}=0$$

3. $$1+\frac{1}{j}+\frac{16}{m}+\frac{16}{n}+\frac{2}{j+1}+\frac{32}{m+n}-\frac{25}{m+1}-\frac{25}{j+n}-\frac{50}{1+j+m+n}=0$$

4. $$1+\frac{1}{j}+\frac{16}{m}+\frac{16}{n}-\frac{1}{j+1}-\frac{16}{m+n}-\frac{25}{m+1}-\frac{25}{j+n}+\frac{25}{1+j+m+n}=0$$

Combining 3. and 4. we obtain:-

(a) $\frac{1}{j+1}+\frac{16}{m+n}=\frac{25}{1+j+m+n}$. Denote $A = j+1$ and $B = m+n$.

Then $\frac{1}{A}+\frac{16}{B}=\frac{25}{A+B}$, whence $(4A-B)^2 = 0$, and so $4A = B$.

Thus $4(j+1) = m+n$.

Using (a) and making obvious substitutions, we obtain the following "revised" equations:-

1. $1 + \frac{1}{j} + \frac{4}{m} + \frac{4}{n} - \frac{48}{m+n} + \frac{5}{m+1} + \frac{5}{j+n} = 0$

2. $1 + \frac{1}{j} + \frac{1}{m} + \frac{1}{n} - \frac{84}{5(m+n)} - \frac{1}{m+1} - \frac{1}{j+n} = 0$

3. $1 + \frac{1}{j} + \frac{16}{m} + \frac{16}{n} - \frac{25}{m+1} - \frac{25}{j+n} = 0$

4. $4(j+1) = m+n.$

Taking one third of the sum of equation 1. with five times equation 2., we obtain:-

5. $2 + \frac{2}{j} + \frac{3}{m} + \frac{3}{n} - \frac{44}{m+n} = 0$

Taking one sixth of the sum of equation 3. with five times equation 1., we obtain:-

6. $1 + \frac{1}{j} + \frac{6}{m} + \frac{6}{n} - \frac{40}{m+n} = 0$

From equations 5. and 6. we obtain $\frac{1}{m} + \frac{1}{n} = \frac{4}{m+n}$. It follows that $m = n$. Thus $j+1 = \frac{1}{2}m$.

Now from 5. $2 + \frac{2}{j} + 3(\frac{1}{m} + \frac{1}{n}) - \frac{44}{m+n} = 0,$

thus $1 + \frac{1}{j} = \frac{8}{m} = \frac{4}{j+1}$

Whence $j = 1$, $m = 4$ and $n = 4$.

Thus χ_2 is an irreducible character of H_* of degree four. The restriction of χ_2 to H yields a character of H.

Since $\chi_2(x) = 1$, it follows that this restricted character has as component, some non-principal irreducible character λ of H. Using Clifford's theorem (Proposition 3.3), we deduce that $\lambda(1) = 2$ or $\lambda(1) = 4$.

Case (i) $\lambda(1) = 2$. Then by Proposition 2.22, it follows that H is isomorphic to A_5.

Case (ii) $\lambda(1) = 4$. We may at this stage refer to the work of Blichfeldt (Proposition 2.23), to conclude that H must be isomorphic to one of A_5, A_6, PSL(2,7), or PSU$(4,2^2)$.

Now $|A_6| = 360$, and $|PSU(4,2^2)| = 25,920$, both these orders are divisible by 9, and since $|H|_3 = 3$, it follows that these two groups cannot occur. From our specifications of $C_{H_*}(\alpha)$, it follows that 5 divides $|H_*|$. Therefore the only possibility is that H is isomorphic to A_5.

We conclude in all instances that H is no counterexample to the lemma. The proof of the lemma is therefore complete.

Lemma 5.4 Let H be a finite group with $|H|_3 = 3$, and suppose that H admits a group $\langle \alpha,\beta \mid \alpha^3 = \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1}\rangle$ of automorphisms. Suppose further that $H_* = H\langle\alpha,\beta\rangle$ has a self-centralizing Sylow 3-subgroup of order nine, and precisely three conjugate classes of elements of order three, and that $C_{H_*}(\alpha)$ is of type 3. Then $H/O_{3'}(H)$ is isomorphic to PSL(2,7), and $O_{3'}(H)$ is soluble.

Proof We take H to be a counterexample to the lemma, of least possible order.

Step 1 H is simple.

If this is not the case, then since $|H|_3 = 3$ H is not characteristically simple. We choose a minimal non-trivial normal $\langle\alpha,\beta\rangle$-invariant subgroup of H_1 of H. If H_1 is a $3'$-group, then since H_* has a self-centralizing elementary Abelian Sylow 3-subgroup of order nine, it follows (Proposition 2.20), that H_1 is soluble. Moreover the hypotheses of the lemma in this instance go over to H/H_1. The minimality of H now implies that $H/H_1 \,/\, O_{3'}(H/H_1)$ is isomorphic to $PSL(2,7)$, and that $O_{3'}(H/H_1)$ is soluble. It follows that $H/O_{3'}(H)$ is isomorphic to $PSL(2,7)$, and that $O_{3'}(H)$ is soluble. Thus in this instance, H is no counterexample to our lemma. Thus we must assume that $|H_1|_3 = 3$.

Let S be an $\langle\alpha,\beta\rangle$-invariant Sylow 3-subgroup of H_1. If $N_{H_1}(S) = C_{H_1}(S)$, then by Burnside's transfer theorem (Proposition 2.3), H_1 has a normal 3-complement $O_{3'}(H_1)$. The minimality of H_1 yields that $O_{3'}(H_1) = 1$, and thus that $H_1 = S$. Now note that $N_H(S)$ is soluble. Thus H is soluble. This is an impossibility, since $C_{H_*}(\alpha)$ is of type 3.

Thus $N_{H_1}(S) \neq C_{H_1}(S)$, and so the hypotheses of the lemma now go over to H_1. By the minimality of H, it follows that $H_1/O_{3'}(H_1)$ is isomorphic to $PSL(2,7)$, where $O_{3'}(H_1)$ is soluble. From the minimality of H_1, it follows that H_1 is isomorphic to $PSL(2,7)$. Now the Frattini argument yields that $H = H_1N_H(S)$. It now follows that $H = H_1 \times A$ for some nilpotent group A. Thus in this instance, H is no counterexample to our lemma.

<u>Step 2</u> (The degree equations)

We now solve the degree equations for H_* (in fact this is precisely case III of Lemma 5.1).

Our principal 3-block for H_* is (with the usual convention that each row is correct only up to sign):-

χ	1	x	α	$x\alpha$
χ_0	1	1	1	1
χ_1	j	1	1	1
χ_2	m	1	7	1
χ_3	n	1	7	1
χ_4	$j+1$	2	-1	-1
χ_5	$m+n$	2	-7	-1
χ_6	$m+1$	-1	8	-1
χ_7	$j+n$	-1	8	-1
χ_8	$1+j+m+n$	-2	-8	1

Here $\langle x\rangle$ is taken to denote a Sylow 3-subgroup of H. Note that this stage, we could infer directly from the structure of H_*, that $j = 1$. We prefer instead to obtain the most general solution of the degree equations. These equations are:-

1. $1 + \frac{1}{j} + \frac{7}{m} + \frac{7}{n} - \frac{4}{j+1} - \frac{28}{m+n} + \frac{8}{m+1} + \frac{8}{j+n} - \frac{32}{1+j+m+n} = 0$

2. $1 + \frac{1}{j} + \frac{1}{m} + \frac{1}{n} - \frac{4}{j+1} - \frac{4}{m+n} - \frac{1}{m+1} - \frac{1}{j+n} + \frac{4}{1+j+m+n} = 0$

3. $1 + \frac{1}{j} + \frac{49}{m} + \frac{49}{n} + \frac{2}{j+1} + \frac{98}{m+n} - \frac{64}{m+1} - \frac{64}{j+n} - \frac{128}{1+j+m+n} = 0$

4. $1 + \frac{1}{j} + \frac{49}{m} + \frac{49}{n} - \frac{1}{j+1} - \frac{49}{m+n} - \frac{64}{m+1} - \frac{64}{j+n} + \frac{64}{1+j+m+n} = 0$

(a) From 3. and 4. we obtain $\frac{1}{j+1} + \frac{49}{m+n} = \frac{64}{1+j+m+n}$.

Denote $A = j+1$ and $B = m+n$.

Then $\frac{1}{A} + \frac{49}{B} = \frac{64}{A+B}$, whence $(7A-B)^2 = 0$, and so $7A = B$.

Thus $7(j+1) = m+n$.

Using (a) and making obvious substitutions, we obtain the following "revised" equations:-

1. $1 + \frac{1}{j} + \frac{7}{m} + \frac{7}{n} - \frac{84}{m+n} + \frac{8}{m+1} + \frac{8}{j+n} = 0$

2. $1 + \frac{1}{j} + \frac{1}{m} + \frac{1}{n} - \frac{57}{2(m+n)} - \frac{1}{m+1} - \frac{1}{j+n} = 0$

3. $1 + \frac{1}{j} + \frac{49}{m} + \frac{49}{n} - \frac{64}{m+1} - \frac{64}{j+n} = 0$

4. $7(j+1) = m+n.$

Taking one third of the sum of equation 1. with eight times equation 2., we obtain:-

5. $3 + \frac{3}{j} + \frac{5}{m} + \frac{5}{n} - \frac{104}{m+n} = 0$

Taking one third of the sum of equation 3. with eight times equation 1., we obtain:-

6. $3 + \frac{3}{j} + \frac{35}{m} + \frac{35}{n} - \frac{224}{m+n} = 0$

From equations 5. and 6. we obtain $\frac{1}{m} + \frac{1}{n} = \frac{4}{m+n}$.
It follows that $m = n$. Thus $j+1 = \frac{2}{7}m$.

Now from 5.

$$3 + \frac{3}{j} + 5\left(\frac{1}{m} + \frac{1}{n}\right) - \frac{104}{m+n} = 0.$$

Thus $1 + \frac{1}{j} = \frac{14}{m} = \frac{4}{j+1}$. Whence $j = 1$, $m = 7$ and $n = 7$.

Thus χ_2 is an irreducible character of H_* of degree 7. The restriction of χ_2 to H yields a character of H. Since $\chi_2(x) = 1$, it follows that this restricted character has as component some non-prinicpal irreducible character λ of H. Using Clifford's theorem (Proposition 3.3), we deduce that $\lambda(1) = 7$. We may apply the previously quoted result due to Wales (Proposition 2.24) to deduce that H is isomorphic to one of PSL(2,13), PSL(2,8), A_8,

$PSL(2,7)$, $PSU(3,3^2)$ or $PSp(6,2)$. Of these groups, only the groups $PSL(2,13)$ and $PSL(2,7)$ do not have order divisible by nine. If H is isomorphic to $PSL(2,13)$, then $C_H(x)$ is isomorphic to $Z_3 \times Z_2$, and thus $C_H(x)$ is centralized by α. It follows that this case does not arise. Thus H is isomorphic to $PSL(2,7)$. The proof of the lemma is complete.

We may now proceed with the proof of Theorem A. We take G to denote a finite group satisfying the hypotheses of Theorem A. Since G has a self-centralizing, elementary Abelian, Sylow 3-subgroup of order nine, the group $O_{3'}(G)$ is necessarily soluble of Fitting height at most two (Proposition 2.20). Moreover the group $\bar{G} = G/O_{3'}(G)$ also satisfies the hypotheses of Theorem A. From Lemma 5.1, we know that $\bar{G}$ is not simple. We choose a minimal non-trivial normal subgroup $\bar{G}_1$ of $\bar{G}$. There are two cases to consider, according as $|\bar{G}_1|_3 = 9$ or $|\bar{G}_1|_3 = 3$:-

<u>Case 1</u> $|\bar{G}_1|_3 = 9$.

We choose a Sylow 3-subgroup $\bar{P}$ of $\bar{G}_1$. By the Frattini argument, $\bar{G} = \bar{G}_1 N_{\bar{G}}(\bar{P})$. As $\bar{G}_1 \neq \bar{G}$, it follows that $N_{\bar{G}}(\bar{P}) \not\leq \bar{G}_1$. Thus we have that $|N_{\bar{G}_1}(\bar{P}) : \bar{P}| = 1$ or $|N_{\bar{G}_1}(\bar{P}) : \bar{P}| = 2$.

(a) If $|N_{\bar{G}_1}(\bar{P}) : \bar{P}| = 1$, then $\bar{P} = N_{\bar{G}_1}(\bar{P})$. It follows using Proposition 2.3, that $\bar{G}_1$ has a normal 3-complement $O_{3'}(\bar{G}_1)$. Since $O_{3'}(\bar{G}) = 1$, we have that $O_{3'}(\bar{G}_1) = 1$, and thus $\bar{G}_1 = \bar{P}$. Whence we obtain that $\bar{G} = N_{\bar{G}}(\bar{P})$. But then $\bar{G}$ is isomorphic to $D_6 \times D_6$, in which instance $\bar{P}$ is not a minimal normal subgroup of $\bar{G}$.

(b) If $|N_{\bar{G}_1}(\bar{P}) : \bar{P}| = 2$, then it follows using the Smith-Tyrer theorem that $\bar{G}_1$ is not simple. Since $\bar{G}_1$ is characteristically simple, and $|\bar{G}_1|_3 = 9$, the only other possibility is that $\bar{G}_1$ is isomorphic to a direct product of two isomorphic simple groups. This latter possibility is ruled out by the condition that $|N_{\bar{G}_1}(\bar{P}) : \bar{P}| = 2$.

<u>Case 2</u> $\quad$ <u>$|\bar{G}_1|_3 = 3$</u>

It follows directly that $\bar{G}_1$ is simple. We now refer to the Lemmata 5.2, 5.3 and 5.4, to deduce that $\bar{G}_1$ is isomorphic to one of Z_3, A_5 or $PSL(2,7)$.

(i) We suppose firstly that $\bar{G}_1$ is isomorphic to Z_3. Then $\bar{G}_1 = \langle \bar{x} \mid \bar{x}^3 = 1\rangle$. Thus $\bar{G} = N_{\bar{G}}(\bar{x})$, and $|\bar{G} : C_{\bar{G}}(\bar{x})| = 2$. Using Lemma 4.2 and the fact that $O_{3'}(\bar{G}) = 1$, it follows that $C_{\bar{G}}(\bar{x})$ is isomorphic to one of $Z_3 \times D_6$, $Z_3 \times A_5$ and $Z_3 \times PSL(2,7)$.

Thus $\bar{G}$ is isomorphic to one of the following groups:- $D_6 \rtimes D_6$, $D_6 \times A_5$, $D_6 \times PSL(2,7)$, H_1, H_2. Here H_1 will denote an extension of Z_3 by S_5 and H_2 will denote an extension of Z_3 by $PGL(2,7)$.

We may now suppose that $\bar{G}$ does not contain a normal subgroup of order three.

(ii) We suppose now that $\bar{G}_1$ is isomorphic to A_5. Now $\bar{G} = N_{\bar{G}}(\bar{G}_1)$. Thus, using Lemma 4.2, $\bar{G}$ must contain a subgroup of index two or less given by $\bar{G}_1 \rtimes C_{\bar{G}}(\bar{G}_1)$. Using Lemma 4.2, together with the facts that $O_{3'}(\bar{G}) = 1$ and that $\bar{G}$ has no normal subgroups of order three, it follows now that $\bar{G}$ is isomorphic to $A_5 \times A_5$ or to $A_5 \times PSL(2,7)$.

We may now suppose that $\bar{G}$ does not contain a normal subgroup isomorphic to A_5.

(iii) We suppose now that $\bar{G}_1$ is isomorphic to $PSL(2,7)$. Now $\bar{G} = N_{\bar{G}}(\bar{G}_1)$. Thus, using Lemma 4.2, $\bar{G}$ must contain a subgroup of index two or less, given by $\bar{G}_1 \times C_{\bar{G}}(\bar{G}_1)$. Using Lemma 4.2 together with the facts that $O_{3'}(\bar{G}) = 1$, and that $\bar{G}$ has no normal subgroups isomorphic to Z_3 or to A_5, it follows now that $\bar{G}$ is isomorphic to $PSL(2,7) \times PSL(2,7)$.

The proof of Theorem A is now complete.

Chapter VI

Theorem B

In this chapter, we consider the possible structure of a finite group G, possessing an elementary Abelian, self-centralizing Sylow 3-subgroup P, of order nine, with $N_G(P)/C_G(P)$ isomorphic to D_8. We shall prove the following theorem:-

Theorem B

Let G be a finite simple group possessing a self-centralizing, elementary Abelian, Sylow 3-subgroup P of order nine, with $N_G(P)/C_G(P)$ isomorphic to D_8. Then either:-

(i) G is isomorphic to A_8 or to $GL(5,2)$, or

(ii) G has at least three conjugate classes of involutions.

Section I

In the first section of this chapter, we shall take G to denote a finite group with Sylow 3-subgroup

$$P = \langle x,y \mid x^3 = y^3 = [x,y] = 1\rangle$$

and Sylow 3-normaliser

$$N_G(P) = \langle a,b,P \mid a^{-1}xa = y^{-1},\ a^{-1}ya = x,\ b^{-1}xb = x,\ b^{-1}yb = y^{-1},\ a^4 = b^2 = b^{-1}aba = 1\rangle$$

It follows that G has precisely two conjugate classes of elements of order three, with representatives x and xy.

Lemma 6.1 $C_G(x) = \langle x\rangle \times A$, where one of the following holds:-

(i) $A/O_{3'}(A)$ is isomorphic to D_6 and $O_{3'}(A)$ is nilpotent of class at most two.

(ii) $A/O_2(A)$ is isomorphic to $SL(2,4)$, and $O_2(A)$ is elementary Abelian, a direct sum of minimal normal subgroups of A, of order 2^4, each of which may be identified with a two-dimensional vector space over $GF(4)$, in such a way that the action of $A/O_2(A)$ on $O_2(A)$ is given by the usual action of $SL(2,4)$ as a group of matrices.

(iii) A is isomorphic to $PSL(2,7)$.

Proof This lemma is really a reiteration of Lemma 4.2.

Notes

(i) The possible structures for $C_G(xy)$ are exactly analogous to the possible structures for $C_G(x)$, and are determined in a parallel fashion.

(ii) In considering the structure of $C_G(x)$ $(C_G(xy))$, in the case in which $A/O_2(A)$ is isomorphic to $SL(2,4)$, we shall often tacitly invoke the fact that $SL(2,4)$ is isomorphic to A_5.

Lemma 6.2 No involution of $C_G(x)$ is conjugate to any involution of $C_G(xy)$.

Proof Suppose that we can find an involution b_1 in $C_G(x)$, such that for some involution c_1 in $C_G(xy)$, there is an element g in G with

$$c_1^g = b_1 .$$

Then b_1 lies in $C_G(x) \cap C_G((xy)^g)$. Since $C_G(P) = P$, it follows that $\langle x\rangle$ and $\langle (xy)^g\rangle$ are both Sylow 3-subgroups of $C_G(b_1)$. But this implies that x is conjugate to xy in G. It is therefore impossible to choose an element b_1 with the above properties. The proof of the lemma is thus complete.

<u>Corollary 6.3</u> G has more than one conjugate class of involutions.

<u>Proof</u> From Lemma 6.2, it follows in particular that the involutions b and ab are not conjugate in G.

<u>Lemma 6.4</u> If G possesses precisely two conjugate classes of involutions, then all involutions of $C_G(x)$ are conjugate in $N_G(x)$.

<u>Proof</u> We suppose that G possesses precisely two conjugate classes of involutions. From Lemma 6.2, no involution of $C_G(x)$ is conjugate to any involution of $C_G(xy)$. Thus all involutions of $C_G(x)$ are conjugate in G. Choose involutions u and v of $C_G(x)$. Then there is an element g in G with $u^g = v$. Since $C_G(P) = P$, it follows that $\langle x\rangle$ and $\langle x\rangle^g$ are both Sylow 3-subgroups of $C_G(v)$. Thus $\langle x\rangle^m = \langle x\rangle^g$ for some m in $C_G(v)$.

Now $u^{gm^{-1}} = v^{m^{-1}} = v$ and gm^{-1} lies in $N_G(x)$. The proof of the lemma is complete.

<u>Lemma 6.5</u> If G possesses precisely two conjugate classes of involutions, then $C_G(x) = \langle x\rangle \times A$, where one of the following holds:-

(i) $A/O_{3'}(A)$ is isomorphic to D_6, and $O_{3'}(A)$ is nilpotent of odd order and of class at most two.

(ii) A is isomorphic to A_5.

(iii) A is isomorphic to $PSL(2,7)$.

<u>Proof</u> The basic structure of $C_G(x)$ has already been determined in Lemma 6.1. From Lemma 6.4, if G possesses precisely two conjugate classes of involutions, then all involutions of $C_G(x)$ are

conjugate in $N_G(x)$. Under these conditions, it follows now that $O_2(C_G(x)) = 1$. The proof of the lemma is thus complete.

Note If G possesses precisely two conjugate classes of involutions then it is possible to show, in an exactly parallel manner that $C_G(xy) = \langle xy \rangle \times B$, where B has one of the three structures attributed to the group A in Lemma 6.5.

Section 2

In this section, we shall prove the following lemma:-

Lemma 6.6 Suppose that the non-trivial 2-group Q admits as a non-trivial, fixed point free group of automorphisms, the elementary Abelian 3-group $P = \langle x,y \mid x^3 = y^3 = [x,y] = 1 \rangle$.

Suppose further that for any non-trivial element z in P, either $C_Q(z) = 1$ or $C_Q(z)$ is isomorphic to $Z_2 \times Z_2$. Then $|Q| = 4$, 16, 64 or 256, corresponding as 2, 4, 6 or 8 elements of $P-\{1\}$ act with fixed points on Q. Moreover if $|Q| = 4$ or 16, then Q is elementary Abelian.

Proof We know from Proposition 2.5, that $Q = \prod_{z \in P-\{1\}} C_Q(z)$.
It thus follows that $Q = C_Q(x)C_Q(y)C_Q(xy)C_Q(xy^{-1})$, and so $|Q| \leq 256$.
If z_1, z_2 are distinct elements of the set $\{x, y, xy, xy^{-1}\}$ then as $C_Q(P) = 1$, it follows that $C_Q(z_1) \cap C_Q(z_2) = 1$. Moreover the groups $C_Q(x)$, $C_Q(y)$, $C_Q(xy)$, $C_Q(xy^{-1})$ are all P-invariant. It follows that if Q is Abelian, then $|Q| = 4$, 16, 64 or 256.
If Q is non-Abelian, the previous arguments now yield that each lower central factor of Q has order an integral power of four.
It now follows that $|Q| = 4$, 16, 64 or 256.

(i) If $|Q| = 4$, then Q is isomorphic to $Z_2 \times Z_2$, and $Q = C_Q(z)$, for some non-trivial z in P. Since $C_Q(P) = 1$, it follows that the only elements of $P-\{1\}$ acting with non-trivial fixed point subgroup on Q, are z and z^{-1}.

(ii) $\underline{|Q| = 16}$. If Q is non-Abelian, then $Q/\Phi(Q)$ has order four. Denote $\bar{Q} = Q/\Phi(Q)$. Then $\bar{Q} = C_{\bar{Q}}(z) = \overline{C_Q(z)}$ for some element z in $P-\{1\}$. Thus $Q = \langle C_Q(z), \Psi(Q)\rangle = C_Q(z)$. This does not arise, and it follows that Q is elementary Abelian. Revising our notation if necessary, we may suppose that

$$Q = C_Q(x) \times C_Q(y).$$

Suppose that z is chosen to denote one of xy or xy^{-1}. Then z acts fixed point freely on $C_Q(x)$ and on $Q/C_Q(x)$. It follows now that z acts fixed point freely on Q itself.

(iii) $\underline{|Q| = 64}$. If Q is non-Abelian, then $Q/\Phi(Q)$ has order four or sixteen. Denote $\bar{Q} = Q/\Phi(Q)$.

(a) If $|\bar{Q}| = 4$, then $\bar{Q} = C_{\bar{Q}}(z) = \overline{C_Q(z)}$ for some element z in $P-\{1\}$. Thus $Q = \langle C_Q(z), \Phi(Q)\rangle = C_Q(z)$. This does not arise.

(b) Suppose now that $|\bar{Q}| = 16$. Revising our notation if necessary, we may suppose that $\bar{Q} = C_{\bar{Q}}(x) \times C_{\bar{Q}}(y)$, and that $\Phi(Q) = C_Q(xy)$. In this instance, xy^{-1} acts fixed point freely on $\Phi(Q)$ and on $\bar{Q}$. It follows that xy^{-1} acts fixed point freely on Q.

Now we have to suppose that Q is elementary Abelian. Revising our notation if necessary, we may suppose that $Q = C_Q(x) \times C_Q(y) \times C_Q(xy)$. In this instance, xy^{-1} acts fixed point freely on $C_Q(xy)$ and on $Q/C_Q(xy)$. It follows that xy^{-1} acts fixed point freely on Q itself.

(iv) $|Q| = 256$. Since $Q = \prod_{z \in P-\{1\}} C_Q(z)$, it follows immediately that no element of $P-\{1\}$ may act fixed point freely on Q.

The proof of the lemma is complete.

Section 3

In this section we shall complete the proof of Theorem B. We shall suppose, throughout this section, that G denotes a finite simple group satisfying the hypotheses of Theorem B. We shall retain the notation of Section 1.

If G is a counterexample to the theorem, then in particular, G has less than three conjugate classes of involutions. Moreover, from Corollary 6.3, it follows in this instance, that G has precisely two conjugate classes of involutions. At this point, the structure of $C_G(x)$ and $C_G(xy)$ are at least partially determined by Lemma 6.5.

In what follows, we shall suppose that G satisfies the hypotheses of Theorem B, and moreover that G has precisely two conjugate classes of involutions.

We shall say that $C_G(x)$ $(C_G(xy))$ is of type 1, type 2 or type 3, according as $C_G(x)$ $(C_G(xy))$ is classified (respectively) under part (i), part (ii) or part (iii) of Lemma 6.5.

Proposition 6.7

Let G be a finite simple group satisfying the hypotheses of Theorem B. Suppose further that G has precisely two conjugate classes of involutions, and that $C_G(x)$ is of type 2 and $C_G(xy)$ of type 1. Then G is isomorphic to A_8.

Proof We take Q to denote a 2-subgroup of G, chosen of maximal possible order, subject to being normalized by P. From Proposition

2.5, we know that $Q = \prod_{z \in P-\{1\}} C_Q(z)$. From the structures of $C_G(x)$ and $C_G(xy)$, it follows that $C_Q(xy) = C_Q(xy^{-1}) = 1$, and that $Q = C_Q(x)C_Q(y)$. From Lemma 6.6, we deduce that Q is elementary Abelian of order four or sixteen.

Case 1 $|Q| = 4$

By suitably adapting our notation, we may suppose that $Q = C_Q(x)$. We denote $N = N_G(Q)$, and proceed to consider the structure of $N_N(P)$. Now we recall that $N_G(P) = \langle a,b,P\rangle$, where

$$a^{-1}xa = y^{-1};\ a^{-1}ya = x;\ b^{-1}xb = x;\ b^{-1}yb = y^{-1}.$$

It is clear that the elements a, a^3, ab, a^3b do not normalize Q, since they each conjugate $C_G(x)$ into $C_G(y)$. Moreover since $C_Q(x)$ is a Sylow 2-subgroup of $C_G(x)$, it follows that the element b cannot lie in $N_N(P)$.

Therefore the only possibilities for $N_N(P)$ are given by:-

(i) $N_N(P) = P$ (ii) $N_N(P) = P\langle a^2\rangle$ (iii) $N_N(P) = P\langle a^2b\rangle$.

(i) If $N_N(P) = P$, then by Burnside's transfer theorem (Proposition 2.3), N has a normal 3-complement, $O_{3'}(N)$. Now P must normalize a Sylow 2-subgroup of $O_{3'}(N)$. It follows from the maximality of Q, that Q is a Sylow 2-subgroup of $O_{3'}(N)$, and thus also of N. Since Q is a Sylow 2-subgroup of $N_G(Q)$, it follows that Q is a Sylow 2-subgroup of G. But the Sylow 2-subgroups of G are non-Abelian, since G contains the dihedral 2-subgroup $\langle a,b\rangle$. This cannot arise.

(ii) $N_N(P) = P\langle a^2\rangle$

In this instance, we deduce from Proposition 2.11, that N is 3-soluble of 3-length one, and that $N/O_{3'}(N)$ is isomorphic to $P\langle a^2\rangle$. Furthermore, using the maximality of Q, and the fact that P must normalize a Sylow 2-subgroup of $O_{3'}(N)$, we obtain that Q is a Sylow 2-subgroup of $O_{3'}(N)$.

We show now that a^2 cannot centralize Q. Suppose, for a contradiction, that a^2 centralizes Q. We choose some non-trivial element k of Q. It follows that:-

$$k^y = k^{ya^2} = k^{a^2y^{-1}} = k^{y^{-1}} .$$

Conjugating both sides of this equation by y^{-1}, we obtain $k = k^y$. This is impossible, since y acts fixed point freely on Q. Since a^2 does not centralize Q, we now deduce that Q is a Sylow 2-subgroup of $C_G(Q)$. We may now apply Proposition 2.15, to see that G has dihedral or semidihedral Sylow 2-subgroups. Using Proposition 2.13 and Proposition 2.14, we deduce that there is no finite simple group satisfying the present hypotheses.

(iii) $\underline{N_N(P) = P\langle a^2b\rangle}$

Using Proposition 2.6, we note that $P \cap O^3(N) = \langle x\rangle$. Denote $\bar{N} = N/O_{3'}(N)$. As $C_G(x)$ is isomorphic to $Z_3 \times A_5$, and since $C_N(x)$ contains a non-trivial normal 2-subgroup, Q, it follows that $\langle\bar{x}\rangle$ is self-centralizing in $O^3(\bar{N})$. Thus, using the result due to Feit and Thompson (Proposition 2.10), we deduce that $O^3(\bar{N})$ is isomorphic to one of D_6, A_5 or PSL(2,7). The maximality of Q is now used to show that Q is a Sylow 2-subgroup of $O_{3'}(N)$, and that $O^3(\bar{N})$ is isomorphic to D_6. It follows that $N/O_{3'}(N)$ is isomorphic to $P\langle a^2b\rangle$.

We now show that a^2b centralizes Q. As a^2b is an involution acting on the 2-group Q, it follows that a^2b must centralize some non-trivial element k of Q. But then $k^y = k^{a^2by} = k^{ya^2b}$, and so a^2b also centralizes k^y. But

$\langle y\rangle$ acts regularly on the set of non-trivial elements of Q, and thus $k \neq k^y$. It now follows that $\langle k,k^y\rangle = Q$, and thus that a^2b centralizes Q. Denote $S = Q\langle a^2b\rangle$. Since $Q \leq S$, we have that $C_G(S) \leq C_G(Q) \leq N$. Thus, since S is a Sylow 2-subgroup of N, we know that S is a Sylow 2-subgroup of $C_G(S)$. We now use Proposition 2.16, to deduce that $r(G) \leq 4$. Thus using Proposition 2.12, we may recognize the isomorphism type of G. There is no finite group satisfying our prsent hypotheses.

Case 2 $|Q| = 16$

In this instance we know from our previous work that $Q = C_Q(x) \times C_Q(y)$. We denote $N = N_G(Q)$. We shall consider the structure of $N_N(P)$.

Since $C_Q(x)$ is a Sylow 2-subgroup of $C_G(x)$, it follows that the element b cannot lie in N. Similarly, since $C_Q(y)$ is a Sylow 2-subgroup of $C_G(y)$, the element a^2b cannot lie in N. Therefore the only possibilities for $N_N(P)$ are given by:-

(i) $N_N(P) = P$ (ii) $N_N(P) = P\langle a^2\rangle$ (iii) $N_N(P) = P\langle ab\rangle$

(iv) $N_N(P) = P\langle a^3b\rangle$ (v) $N_N(P) = P\langle a\rangle$ (vi) $N_N(P) = P\langle a^2,ab\rangle$.

(i) $N_N(P) = P$

Applying Burnside's transfer theorem (Proposition 2.3), we deduce, in this case, that N has a normal 3-complement, $O_{3'}(N)$. Now P normalizes a Sylow 2-subgroup of $O_{3'}(N)$, and thus the maximality of Q implies that Q is a Sylow 2-subgroup of $O_{3'}(n)$. It follows now that Q is a Sylow 2-subgroup of $N_G(Q)$ and hence also of G. But G already contains the non-Abelian 2-subgroup $\langle a,b\rangle$, therefore G does not possess Abelian Sylow 2-subgroups, and consequently this case does not arise.

(ii) $N_N(P) = P\langle a^2\rangle$

In this instance, $N_G(Q)/C_G(Q)$ is isomorphic to a subgroup of $GL(4,2)$. The group $GL(4,2)$ is itself isomorphic to A_8. Denote $\bar{N} = N_G(Q)/C_G(Q)$. We deduce from Proposition 2.11, that $\bar{N}$ is 3-soluble of 3-length one, and that $\bar{N}/O_{3'}(\bar{N})$ is isomorphic to $P\langle a^2\rangle$. The group $GL(4,2)$ has a self-centralizing, elementary Abelian Sylow 3-subgroup of order nine and precisely two conjugate classes of elements of order three. A 3-element of $GL(4,2)$ has centralizer isomorphic either to $Z_3 \times A_5$ or to $Z_3 \times D_6$, according to the conjugate class to which it belongs. Thus the only $3'$-subgroups of $GL(4,2)$ which are normalized by a Sylow 3-subgroup thereof are 2-subgroups. It follows from the above, together with the maximality of Q, that $O_{3'}(\bar{N}) = 1$, and that $\bar{N}$ is isomorphic to $P\langle a^2\rangle$. Extending the arguments given in case 1 (ii), we may suppose that the action of a^2 on Q is as follows:-

$$C_Q(x) = \langle a_1,a_2\rangle \qquad C_Q(y) = \langle a_3,a_4\rangle$$
$$a^2a_1a^2 = a_2 \qquad a^2a_3a^2 = a_4.$$

Here, the groups $\langle a_1,a_2\rangle$, $\langle a_3,a_4\rangle$ are elementary Abelian of order four. We denote $S = Q\langle a^2\rangle$. In fact, S is isomorphic to the wreath product $(Z_2 \times Z_2) \wr Z_2$. We observe that Q is the unique elementary Abelian subgroup of S of index two. Thus Q is a characteristic subgroup of S, and so $N_G(S) \leq N_G(Q)$. By Proposition 2.2, central elements of S conjugate in G, must be conjugate in $N_G(S)$. Note that $Z(S) = \langle a_1a_2,a_3a_4\rangle$. We deduce from the known structure of N, together with the fact that $N_G(S) \leq N$, that the distinct involutions of $Z(S)$ lie in distinct conjugate classes in G. This latter fact contradicts our present assumption that G has precisely two conjugate classes of involutions.

(iii) $\underline{N_N(P) = P\langle ab\rangle}$

We denote $\bar N = N_G(Q)/C_G(Q)$. Arguing as in case 2 (ii), we obtain that $\bar N$ is isomorphic to a subgroup of $GL(4,2)$. The structure of $GL(4,2)$ implies that $\bar P$ normalizes no non-trivial subgroups of $\bar N$ of order prime to 2 and 3. The maximality of Q now implies that $\bar P$ normalizes no non-trivial 3'-subgroup of $\bar N$. It follows now that $O_{3'}(\bar N) = 1$. Using Proposition 2.6, we deduce that $P \cap O^3(N) = \langle xy^{-1}\rangle$. Now we know that $O_{3'}(C_{\underline{O^3(\bar N)}}(\overline{xy^{-1}}))$ is $\bar P$-invariant, and thus from the above arguments $\langle xy^{-1}\rangle$ is self-centralizing in $\bar N$. Using Proposition 2.10, together with the maximality of Q, we deduce that $\bar N$ is isomorphic to $P\langle ab\rangle$. Now ab conjugates $C_G(x)$ into $C_G(y)$. We may assume that the action of ab on Q is as follows:-

$$C_Q(x) = \langle a_1,a_2\rangle \qquad C_Q(y) = \langle a_3,a_4\rangle$$

$$aba_1ab = a_3 \qquad aba_2ab = a_4.$$

Here the groups $\langle a_1,a_2\rangle$, $\langle a_3,a_4\rangle$ are elementary Abelian of order four. It follows as in case 2 (ii), with in this case $S = Q\langle ab\rangle$, that $N_G(S) \leq N_G(Q)$. Thus S is a Sylow 2-subgroup of G of order 2^5. The fusion arguments of case 2 (ii) are however no longer applicable, since $Z(S) = \langle a_1a_3,a_2a_4\rangle$, and the involutions of $Z(S)$ are permuted by $\langle xy\rangle$. However we do know that S is a Sylow 2-subgroup of G. Thus from Proposition 2.19, we derive a list of the possible isomorphism types for G. There is no finite simple group satisfying our present hypotheses.

(iv) $\underline{N_N(P) = P\langle a^3b\rangle}$

With a suitable revision of notation, the proof of case 2 (iii) may be adapted to deal with this case also.

(v) $\underline{N_N(P) = P\langle a\rangle}$

We denote $\bar{N} = N_G(Q)/C_G(Q)$. We know as in case 2 (ii), that $\bar{N}$ is isomorphic to a subgroup of $GL(4,2)$. Note that using the maximality of Q, it follows that Q is a Sylow 2-subgroup of $C_G(Q)$. It follows now from Proposition 2.21 that $\bar{N}/O_{3'}(\bar{N})$ is isomorphic to one of $P\langle a\rangle$, A_6 or A_7. It follows as in case 2 (ii) that $O_{3'}(\bar{N}) = 1$. Thus $\bar{N}$ is itself isomorphic to one of $P\langle a\rangle$, A_6 or A_7. In the latter two cases, we may identify the isomorphism type of G using Proposition 2.17. We may now assume that $\bar{N} = \overline{P\langle a\rangle}$. We may now also assume that a has the following action on Q:-

$$C_Q(x) = \langle a_1,a_2\rangle \qquad\qquad C_Q(y) = \langle a_3,a_4\rangle$$

$$a^{-1}a_1a = a_3,\quad a^{-1}a_2a = a_4,\quad a^{-1}a_3a = a_2,\quad a^{-1}a_4a = a_1.$$

Here the groups $\langle a_1,a_2\rangle$, $\langle a_3,a_4\rangle$ are elementary Abelian of order four. Denote $S = Q\langle a\rangle$. Observe that S is of order 2^6 and that Q is a characteristic subgroup of S, since it is the unique normal elementary Abelian subgroup of order 2^4. Thus $N_G(S) \leq N_G(Q)$, and S is a Sylow 2-subgroup of G. We obtain from Proposition 2.19, a list of the possible structures of G.

The only finite simple group classified under this section is A_8.

(vi) $\underline{N_N(P) = P\langle a^2,ab\rangle}$

We denote $\bar{N} = N_G(Q)/C_G(Q)$. We know as in case 2 (ii), that $\bar{N}$ is isomorphic to a subgroup of $GL(4,2)$. We also know, using Theorem A, together with the maximality of Q, that $\bar{N}/O_{3'}(\bar{N})$ is isomorphic to $P\langle a^2,ab\rangle$. It follows precisely as in <u>case</u> 2 (ii) that $O_{3'}(\bar{N}) = 1$.

We may now assume that the group $\langle a^2, ab\rangle$ has the following action on Q:-

$$C_Q(x) = \langle a_1, a_2\rangle \qquad C_Q(y) = \langle a_3, a_4\rangle$$

$$a^2a_1a^2 = a_2 \qquad a^2a_3a^2 = a_4$$

$$aba_1ab = a_3 \qquad aba_2ab = a_4$$

Here $\langle a_1, a_2\rangle$ and $\langle a_3, a_4\rangle$ are elementary Abelian subgroups of Q of order four. We shall denote $S = Q\langle a^2, ab\rangle$. Observe that $Z(S) = \langle a_1a_2a_3a_4\rangle$, and that $Z_2(S) = \langle a_1a_2, a_3a_4, a_1a_3\rangle$. Now choose a Sylow 2-subgroup T of $N_G(S)$. Then T must normalize $Z_2(S)$. Denote $T_1 = C_T(Z_2(S))$. Then $Q \leq T_1$, and $N_{T_1}(Q) = Q$. It follows that $T_1 = C_T(Z_2(S)) = Q$. Since T normalizes $Z_2(S)$, it is clear that T normalizes $C_T(Z_2(S))$. Thus T normalizes Q. But S is a Sylow 2-subgroup of $N_G(Q)$, and thus $S = T$. It follows, since S is a Sylow 2-subgroup of $N_G(S)$, that S is a Sylow 2-subgroup of G. Since $|S| = 2^6$, we may use Proposition 2.19, to identify the isomorphism type of G. In fact there is no finite simple group satisfying our present hypotheses.

The proof of the Proposition is now complete.

Proposition 6.8

There is no finite simple group G satisfying the hypotheses of Theorem B, together with the following conditions:-

(a) G has precisely two conjugate classes of involutions.

(b) $C_G(x)$ is of type 3 and $C_G(xy)$ is of type 1.

Proof We take G to denote a counterexample to the above proposition. We take Q to denote a 2-subgroup of G, chosen of maximal possible order subject to being normalized by P. From Proposition 2.5, we know that $Q = \prod_{z \in P-\{1\}} C_Q(z)$. From the known structures of $C_G(x)$ and $C_G(xy)$, we deduce that $C_Q(xy) = C_Q(xy^{-1}) = 1$, and that $Q = C_Q(x)C_Q(y)$. From Lemma 6.6, it follows that Q is elementary Abelian of order four or sixteen.

Case 1 $|Q| = 4$

By suitably adapting our notation, we may suppose that $Q = C_Q(x)$. We denote $N = N_G(Q)$, and proceed to determine the structure of $N_N(P)$. We now recall that

$$N_G(P) = \langle a,b,P\rangle, \text{ where}$$
$$a^{-1}xa = y^{-1};\quad a^{-1}ya = x;\quad b^{-1}xb = x;\quad b^{-1}yb = y^{-1}.$$

It is clear that the elements a, a^3, ab, a^3b do not normalize Q, since they each conjugate $C_G(x)$ into $C_G(y)$. Therefore the only possibilities for $N_N(P)$ are given by:-

(i) $N_N(P) = P$ (ii) $N_N(P) = P\langle a^2\rangle$ (iii) $N_N(P) = P\langle a^2b\rangle$

(iv) $N_N(P) = P\langle b\rangle$ (v) $N_N(P) = P\langle a^2,b\rangle$

The cases (i), (ii), and (iii) may be treated exactly as in Proposition 6.7 case 1 (i), (ii) and (iii).

(iv) $N_N(P) = P\langle b\rangle$

Using Proposition 2.6, we deduce that $P \cap O^3(N) = \langle y\rangle$. Since Q is elementary Abelian of order four, and as y acts fixed point freely on Q, we may deduce that $O^3(N)/C_{O^3(N)}(Q)$ is isomorphic to $\langle y,b\rangle$. It now follows that $N/O_{3'}(N)$ is isomorphic to $P\langle b\rangle$, and that $O_{3'}(N)$ centralizes Q. Since P must normalize a Sylow 2-subgroup of $O_{3'}(N)$, it follows that Q is a Sylow 2-subgroup of $O_{3'}(N)$. Thus Q is a Sylow 2-subgroup of $C_G(Q)$. We may now apply Proposition 2.15, to see that G has dihedral or semidihedral Sylow 2-subgroups. Using Propositions 2.13 and 2.14, we deduce that there is no finite simple group satisfying the present hypotheses.

(v) $N_N(P) = P\langle a^2,b\rangle$

The possible structures for $N/O_{3'}(N)$ are determined by Theorem A. Using the maximality of Q, we deduce that $N/O_{3'}(N)$ is isomorphic to $P\langle a^2,b\rangle$. Now P must normalize a Sylow 2-subgroup of $O_{3'}(N)$. It follows from the maximality of Q, that Q is a Sylow 2-subgroup of $O_{3'}(N)$. The arguments used in Proposition 6.7 case 1 (ii) and (iii) may be applied in this instance to determine the action of $\langle a^2,b\rangle$ on Q. We deduce as previously, that a^2b centralizes Q, and that a^2 does not centralize Q. Denote $S_1 = Q\langle a^2b\rangle$. Then S_1 is a Sylow 2-subgroup of $C_G(Q)$. Since $Q \leq S_1$, it follows that $C_G(S_1) \leq C_G(Q)$, and thus that S_1 is a Sylow 2-subgroup of $C_G(S_1)$. We now use Proposition 2.16, to deduce that $r(G) \leq 4$. Thus using Proposition 2.12, we may recognize the isomorphism type of G. There is no finite simple group satisfying our present hypotheses.

Case 2 $|Q| = 16$

In this instance we know from our previous work that $Q = C_Q(x) \times C_Q(y)$. We denote $N = N_G(Q)$. We shall consider the structure of $N_N(P)$. The possibilities for $N_N(P)$ are given by:-

(i) $N_N(P) = P$
(ii) $N_N(P) = P\langle a^2\rangle$
(iii) $N_N(P) = P\langle ab\rangle$
(iv) $N_N(P) = P\langle a^3b\rangle$
(v) $N_N(P) = P\langle a\rangle$
(vi) $N_N(P) = P\langle a^2,ab\rangle$
(vii) $N_N(P) = P\langle b\rangle$
(viii) $N_N(P) = P\langle a^2b\rangle$
(ix) $N_N(P) = P\langle a^2,b\rangle$
(x) $N_N(P) = P\langle a,b\rangle$.

The cases (i), (ii), (iii), (iv), (v) and (vi) may be treated exactly as in Proposition 6.7 case 2.

(vii) $N_N(P) = P\langle b\rangle$

Using Proposition 2.6, we deduce that $P \cap O^3(N) = \langle y\rangle$. Denote $\bar{N} = N_G(Q)/C_G(Q)$. It follows that $\bar{N}$ is isomorphic to a subgroup of $GL(4,2)$. Arguing as in Proposition 6.7 case 2 (ii), we obtain that $O_{3'}(\bar{N}) = 1$. Now we know that $O_{3'}(C_{O^3(\bar{N})}(\bar{y}))$ is $\bar{P}$-invariant. It follows from our knowledge of the 3-local structure of $GL(4,2)$, that $O_{3'}(C_{O^3(\bar{N})}(\bar{y}))$ is a 2-group. The maximality of Q now forces $O_{3'}(C_{O^3(\bar{N})}(\bar{y})) = 1$. It follows that $\langle\bar{y}\rangle$ is self-centralizing in $O^3(\bar{N})$. Using Proposition 2.10 together with the maximality of Q, and the fact that $O_{3'}(\bar{N}) = 1$, we obtain that $\bar{N}$ is isomorphic to $P\langle b\rangle$.

The maximality of Q now enables us to deduce that Q is a Sylow 2-subgroup of $C_G(Q)$. Denote $S = Q\langle b\rangle$. Then if S_1 is any Sylow 2-subgroup of G containing S, it follows that $N_{S_1}(Q) = S$. Thus Lemma 8.3 of [21], applies to tell us that $r(S_1) \leq 4$. Thus using [11] we may recognize the isomorphism type of G. There is no finite simple group satisfying our present hypotheses.

(viii) $N_N(P) = P\langle a^2b\rangle$

With a suitable revision of notation, the proof of case 2 (vii) may be adapted to deal with this case also.

(ix) $N_N(P) = P\langle a^2,b\rangle$

We show in this section, that G has a Sylow 2-subgroup of order at most 2^9. The possible structures for $N/O_{3'}(N)$ are determined by Theorem A. Using the maximality of Q, we deduce that $N/O_{3'}(N)$ is isomorphic to $P\langle a^2,b\rangle$. Now P must normalize a Sylow 2-subgroup of $O_{3'}(N)$. It follows from the maximality of Q, that Q is a Sylow 2-subgroup of $O_{3'}(N)$. Denote $S = Q\langle a^2, b\rangle$. Then S is a Sylow 2-subgroup of N. The arguments used in Proposition 6.7 case 1 (ii) and (iii) may be adapted in this instance to determine the action of $\langle a^2, b\rangle$ on Q. We may suppose that the following action is known:-

$$\begin{array}{lll} C_Q(x) = \langle a_1, a_2\rangle & C_Q(y) = \langle a_3, a_4\rangle & \\ a^2 a_1 a^2 = a_2 & a^2 a_3 a^2 = a_4 & \\ b a_1 b = a_2 & b a_3 b = a_3 & b a_4 b = a_4. \end{array}$$

Here $\langle a_1, a_2\rangle$ and $\langle a_3, a_4\rangle$ are elementary Abelian groups of order four. The following is a list of all those normal elementary Abelian subgroups of S of order 2^4:-

$$\begin{array}{ll} Q = & \langle a_1, a_2, a_3, a_4\rangle \\ A := & \langle a^2, b, a_1 a_2, a_3 a_4\rangle \\ B := & \langle b, a_3, a_4, a_1 a_2\rangle \\ C := & \langle a^2 b, a_1, a_2, a_3 a_4\rangle \end{array}$$

Let T denote a Sylow 2-subgroup of $N_G(S)$. If $T = S$, then S is a Sylow 2-subgroup of $N_G(S)$, and thus also of G. It follows, since $|S| = 2^6$, that G is of known isomorphism type (Proposition 2.19). There is no finite simple group satisfying these hypotheses. Thus we have to suppose that $T \neq S$.

Now the group T/S acts by conjugation on the set $\{Q, A, B, C\}$ of groups. In particular, if $T \neq S$, then since S is a Sylow 2-subgroup of $N_G(Q)$, it follows that no element of T/S can fix Q. Thus T/S is a subgroup of the symmetric group on $\{Q, A, B, C\}$ and $|T : S| \leq 4$.

Since Q is a Sylow 2-subgroup of $C_G(Q)$, it follows that

$Z(T) \leq Q$ indeed that $Z(T) \leq Z(S)$. We know that $Z(S) = \langle a_1a_2, a_3a_4\rangle$.

Subcase (a)

We shall initially suppose that $Z(S) = Z(T)$. In particular, the subgroups $\langle a_1a_2\rangle$ and $\langle a_3a_4\rangle$ are normal subgroups of T. Thus we obtain that

$$Z\left(\frac{S}{\langle a_1a_2\rangle}\right) \trianglelefteq \frac{T}{\langle a_1a_2\rangle} \quad \text{and that} \quad Z\left(\frac{S}{\langle a_3a_4\rangle}\right) \trianglelefteq \frac{T}{\langle a_3a_4\rangle} .$$

Taking pre-images in T, we deduce that:-

$$(*) \qquad D := \langle b, a_1, a_2, a_3a_4\rangle \trianglelefteq T \quad \text{and that} \quad E := \langle a^2b, a_1a_2, a_3, a_4\rangle \trianglelefteq T.$$

(1) The action of T/S is semi-regular

Proof Denote $\bar{T} = T/Z(T)$. If g is an element of $T-S$, then $\bar{g}$ permutes the groups $\bar{Q}$, $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{g}$ normalizes the groups $\langle \bar{b}, \bar{a}_1\rangle$ and $\langle \overline{a^2b}, \bar{a}_3\rangle$. We tabulate the possible ways in which such an element g may operate with fixed points on the set $\{Q, A, B, C\}$ of groups. The table also shows how each possibility is to be ruled out:-

Possibility	$Q^g = Q$	$A^g = A$	$B^g = B$	$C^g = C$
Step 1	-	$(A\cap D)^g = (A\cap D)$	$(B\cap D)^g = (B\cap D)$	$(C\cap E)^g = (C\cap E)$
Step 2	-	$B^g = B$	$A^g = A$	$A^g = A$
Step 3	-	$(B\cap E)^g = (B\cap E)$	← see col. 2	← see col. 2
Step 4	-	$Q^g = Q$	-	-

(2) T/S is not cyclic of order four

Proof If T/S is cyclic of order four, then we can find an element g in $T-S$ satisfying one of the following three properties:-

(a) $Q^g = A$; $A^g = B$; $B^g = C$; $C^g = Q$

(b) $Q^g = A$; $A^g = C$; $C^g = B$; $B^g = Q$

(c) $Q^g = B$; $B^g = A$; $A^g = C$; $C^g = Q$.

Case (a) We shall once again work in $\bar{T} = T/Z(T)$.

As previously, g must normalize the groups $\bar{D}$ and $\bar{E}$. We find it helpful to derive a contradiction with the aid of a diagram:-

$$\begin{tikzcd}
Q \arrow[d, "g"'] & \bar{a}_1 \arrow[dr, "g"] & \bar{a}_3 & \overline{a_1a_3} \\
\bar{A} \arrow[d, "g"'] & \overline{a^2b} & \bar{b} & \overline{a^2} \\
\bar{B} \arrow[d, "g"'] & \bar{b} & \bar{a}_3 & \overline{a_3b} \\
\bar{C} \arrow[uuu, bend left=60, "g"] & \overline{a^2b} & \bar{a}_1 \arrow[d, "?"] & \overline{a^2ba_1} \\
 & & &
\end{tikzcd}$$

As $D^g = D$, and $Q^g = A$, it follows that $\bar{a}_1^g = \bar{b}$. However $\bar{a}_1$ lies in $\bar{C}$ and $\bar{C}^g = \bar{Q}$, and $\bar{b}$ does not lie in $\bar{Q}$. Thus we have derived a contradiction. This case does not arise.

Case (b) We shall once again work in $\bar{T} = T/Z(T)$. As previously, g must normalize the groups $\bar{D}$ and $\bar{E}$. We find it helpful to derive a contradiction with the aid of a diagram:-

$$\begin{tikzcd}
\bar{Q} \arrow[d, "g"'] & \bar{a}_1 & \bar{a}_3 \arrow[dl, "g"'] & \overline{a_1a_3} \\
\bar{A} \arrow[d, "g"'] & \overline{a^2b} & \bar{b} & \overline{a^2} \\
\bar{C} \arrow[d, "g"'] & \overline{a^2b} & \bar{a}_1 & \overline{a^2ba_1} \\
\bar{B} \arrow[uuu, bend left=60, "g"] & \bar{b} & \bar{a}_3 \arrow[d, "?"] & \overline{a_3b} \\
 & & &
\end{tikzcd}$$

As $E^g = E$ and $Q^g = A$, it follows that $\bar{a}_3^g = \overline{a^2b}$. However $\bar{a}_3$ lies in $\bar{B}$ and $\bar{B}^g = \bar{Q}$, and $\overline{a^2b}$ does not lie in $\bar{Q}$. Thus we have derived a contradiction. This case does not arise.

Case (c) We shall once again work in $\bar{T} = T/Z(T)$. As previously, g must normalize the groups D and E. We find it helpful to derive a contradiction with the aid of a diagram:-

$$\begin{tikzcd}
\bar{Q} \arrow[d, "g"'] & \bar{a}_1 & \bar{a}_3 & \overline{a_1a_3} \\
\bar{B} \arrow[d, "g"'] & \bar{b} \arrow[d, "g"] & \bar{a}_3 & \overline{a_3b} \\
\bar{A} \arrow[d, "g"'] & \overline{a^2b} & \bar{b} \arrow[dl, "g"] & \overline{a^2} \\
\bar{C} \arrow[uuu, bend left=60, "g"] & \overline{a^2b} & \bar{a}_1 & \overline{a^2ba_1}
\end{tikzcd}$$

Now $\bar{b}$ lies in $\bar{B} \cap \bar{A}$. Thus $\bar{b}^g$ lies in $\bar{A} \cap \bar{C}$. It follows that $\bar{b}^g = \overline{a^2b}$. However we know that $D^g = D$, and that $\bar{b}$ lies in $\bar{D}$ although $\overline{a^2b}$ does not lie in $\bar{D}$. Thus we have derived a contradiction. This case does not arise.

(3) We may suppose that T/S is not cyclic of order two.

If T/S is cyclic of order two, then we can find an element g in $T-S$ satisfying one of the following three properties:-

(a) $Q^g = A$; $B^g = C$

(b) $Q^g = B$; $A^g = C$

(c) $Q^g = C$; $A^g = B$.

Case (a) We shall denote $\bar{T} = T/Z(T)$. Combining the information given by (a), with the fact that $D^g = D$ and $E^g = E$, we obtain that

$$\bar{a}^{2^g} = \overline{a_1a_3} \; ; \qquad \bar{b}^g = \bar{a}_1 \; ; \qquad \bar{a}_3^g = \overline{a^2b}.$$

Now we observe that $Z(\bar{T}) = \langle \overline{a^2ba_3}, \overline{ba_1} \rangle$. It follows now that $\bar{S}$ is the unique elementary Abelian subgroup of $\bar{T}$ of order 2^4. Thus $\bar{S}$ is a characteristic subgroup of $\bar{T}$.

Since $Z(T)$ is a characteristic subgroup of T, it follows now that S is a characteristic subgroup of T. Thus we obtain that $N_G(T) \leq N_G(S)$. Since T is a Sylow 2-subgroup of $N_G(S)$, it follows that T is a Sylow 2-subgroup of $N_G(T)$. Thus T is a Sylow 2-subgroup of G of order 2^7. Then using Proposition 2.19, we may recognize the isomorphism type of G. There is no finite simple group satisfying the present hypotheses.

<u>Case (b)</u> We shall denote $\bar{T} = T/Z(T)$. Combining the information given by (b) with the fact that $D^g = D$ and $E^g = E$, we obtain that

$$\bar{a}^{2^g} = \overline{a^2ba_1} \; ; \qquad \bar{b}^g = \bar{a}_1 \; ; \qquad \bar{a}_3^g = \bar{a}_3 .$$

In this instance $Z(\bar{T}) = \langle \overline{a^2b}, \overline{ba_1}, \bar{a}_3 \rangle$. Let T_1 denote a Sylow 2-subgroup of $N_G(T)$. Then T_1 normalizes the group $Z_2(T)$. We know from the above that

$$Z_2(T) = \langle a^2b, ba_1, a_3, a_4, a_1a_2 \rangle .$$

Thus T_1 must also normalize $\Omega_1(Z_2(T))$. We know that $\Omega_1(Z_2(T)) = E = \langle a^2b, a_1a_2, a_3, a_4 \rangle$.

<u>Remark 1</u> Denote $Q_o = \langle a_3, a_4 \rangle$, and consider $N_G(Q_o)$. Now $N_{N_G(Q_o)}(P) = P\langle a^2,b \rangle$, since $P\langle a^2,b \rangle$ already normalizes Q_o, and since a conjugates y into x and so cannot normalize Q_o. Thus Theorem A applies to yield the structure of $N_G(Q_o)/O_{3'}(N_G(Q_o))$.

Using the maximality of Q, and since $Q_o \leq O_{3'}(N_G(Q_o))$, it follows immediately that $N_G(Q_o)/O_{3'}(N_G(Q_o))$ is isomorphic to one of:-

(i) $D_6 \times D_6$ (ii) $D_6 \times A_5$ (iii) $D_6 \times PSL(2,7)$

(iv) H_1 (v) H_2.

Since a^2b is known to centralize $\langle a_1, a_2\rangle$, the final case cannot occur. In the first four cases, the maximality of Q implies that S is a Sylow 2-subgroup of $N_G(Q_o)$.

Using Remark 1, it follows that $C_T(\langle a_3, a_4\rangle) \leq S$. It follows now that $C_T(\Omega_1(Z_2(T))) \leq S$. Thus $C_T(\Omega_1(Z_2(T))) = \langle b, a_1, a_2, a_3a_4\rangle$. Whence we obtain that $S = \Omega_1(Z_2(T))C_T(\Omega_1(Z_2(T)))$. Since T_1 is known to normalize $\Omega_1(Z_2(T))$, it follows that T_1 normalizes $C_T(\Omega_1(Z_2(T)))$. Thus T_1 must normalize S. But T is a Sylow 2-subgroup of $N_G(S)$, so that we must have that $T_1 = T$. It follows that T is a Sylow 2-subgroup of $N_G(T)$, and thus also of G. As $|T| = 2^7$, we may apply Proposition 2.19, to recognize the isomorphism type of G. There is no finite simple group satisfying our present hypotheses.

Case (c) With a suitable revision of notation (for instance interchanging x and y), the proof of case (b) may be adapted to deal with this case also. Thus we may now suppose that T/S is not cyclic. It follows now that T/S is elementary Abelian of order four.

(4) The final case: T/S is elementary Abelian of order four.

As before, we shall denote $\bar{T} = T/Z(T)$. Using the results of the previous paragraphs, we may now suppose that:-

$$\bar{T} = \langle \bar{g}, \bar{h}, \bar{a}^2, \bar{b}, \bar{a}_1, \bar{a}_3\rangle,$$

where $Q^g = B$ and $A^g = C$ and $Q^h = A$ and $B^h = C$ and g^2 and h^2 both lie in S. Arguing as in (3) cases (a) and (b), we obtain that:-

$$\bar{a}^{2^g} = \overline{a^2ba_1} \quad : \quad \bar{b}^g = \bar{a}_1 \quad : \quad \bar{a}_3^g = \bar{a}_3$$

$$\bar{a}^{2^h} = \overline{a_1a_3} \quad : \quad \bar{b}^h = \bar{a}_1 \quad : \quad \bar{a}_3^h = \overline{a^2b} \quad .$$

The following is a list of all those normal subgroups of $\bar{T}$ of order 2^4, which are possibly elementary Abelian:-

$$\bar{S} = \langle \bar{a}^2, \bar{b}, \bar{a}_1, \bar{a}_3 \rangle \qquad \bar{S}_1 := \langle \bar{g}, \overline{ba_1}, \bar{a}_3, \overline{a^2b} \rangle$$

$$\bar{S}_2 := \langle \overline{gh}, \bar{b}, \bar{a}_1, \overline{a^2ba_3} \rangle \qquad \bar{S}_3 := \langle \bar{g}, \bar{h}, \overline{a^2ba_3}, \overline{ba_1} \rangle \ .$$

We show that it is possible to assume that each of the above groups is elementary Abelian. If this is not the case, then there are four cases to consider:-

(a) If none of $\bar{g}$, $\bar{h}$, $\overline{gh}$ are involutions, then $\bar{S}$ is the unique normal elementary Abelian subgroup of $\bar{T}$ of order 2^4. Thus $\bar{S}$ is a characteristic subgroup of $\bar{T}$. Since $Z(T)$ is a characteristic subgroup of T, it follows now that S is a characteristic subgroup of T, and thus that $N_G(T) \leq N_G(S)$. But T is a Sylow 2-subgroup of $N_G(S)$ and hence also of $N_G(T)$. Thus T is a Sylow 2-subgroup of G of order 2^8. Using Proposition 2.19, we may identify G. There is no finite simple group satisfying the present hypotheses.

(b) If $\bar{g}$ is not an involution, then it follows, since $\bar{g}$ lies in $\bar{S}_1$ and $\bar{S}_3$, that either $\overline{gh}$ is an involution, or that $\bar{S}$ is again the unique elementary Abelian normal subgroup of $\bar{T}$ of order 2^4. In the latter case, the arguments of (a) (above) apply. We must therefore presume that $\overline{gh}$ is an involution. Choose a Sylow 2-subgroup T_1 of $N_G(T)$. If $T_1 = T$, then T is a Sylow 2-subgroup of G of order 2^8. Using Proposition 2.19, to recognize the possible isomorphism type of G, we deduce that there is no finite simple group satisfying the present hypotheses. Thus we may suppose that $T_1 \neq T$. Since $\bar{S}$ and $\bar{S}_2$ are the only elementary Abelian, normal subgroups of $\bar{T}$ of order 2^4, it follows, using the fact that T is a Sylow 2-subgroup of $N_G(S)$, that $|T_1 : T| = 2$, and that any element k of T_1-T must interchange $\bar{S}$ and $\bar{S}_2$. We choose an element k of T_1-T. From the above, it follows that k normalizes $\overline{S \cap S_2}$. Denote $B_1 = S \cap S_2 = \langle b, a_1, a^2ba_3, a_3a_4, a_2 \rangle$. It follows that k must normalize $\Omega_1(B_1)$. Now we know that $\Omega_1(B_1) = \langle b, a_1, a_2, a_3a_4 \rangle$. It follows next that k must normalize $C_T(\Omega_1(B_1))$. Arguments analogous to those used in Remark 1, show that S is a Sylow 2-subgroup of $N_G(\langle a_1, a_2 \rangle)$. It follows that $C_T(\Omega_1(B_1)) \leq S$. Thus k normalizes $C_S(\Omega_1(B_1))$. We know that $C_S(\Omega_1(B_1)) = \langle a^2b, a_1a_2, a_3, a_4 \rangle$. It follows now that k normalizes the group $\Omega_1(B_1) \cdot C_S(\Omega_1(B_1))$. However $\Omega_1(B_1) \cdot C_S(\Omega_1(B_1)) = S$. Thus k must normalize S. This contradicts our choice of k, and thus the analysis of this subcase is complete.

(c) We suppose now that $\overline{gh}$ is not an involution. With a suitable revision of notation, the proof of (b) may be adapted to deal with this case also.

(d) We have finally to suppose that both $\bar{g}$ and $\overline{gh}$ are involutions, and that $\bar{h}$ is not an involution. Choose a Sylow 2-subgroup T_1 of $N_G(T)$. We may suppose, as for example in (b), that $T_1 \neq T$. Now T_1/T permutes the set $\{\bar{S}, \bar{S}_1, \bar{S}_2\}$ of groups, since this set is a full set of normal, elementary Abelian subgroups of $\bar{T}$ of order 2^4. No element of T_1-T normalizes S. By suitably altering our notation, if necessary, we may assume that there is an element k in T_1-T which interchanges $\bar{S}$ and $\bar{S}_2$. The arguments of (b) now hold in this instance also.

We may now assume that the groups $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$ are all elementary Abelian. The situation is now familiar, in the sense that the groups $\bar{T}$ and S are isomorphic. As previously, we choose a Sylow 2-subgroup T_1 of $N_G(T)$. We may once again suppose that $T_1 \neq T$. Observe that T_1/T acts by conjugation on the set $\{\bar{S}, \bar{S}_1, \bar{S}_2, \bar{S}_3\}$ of subgroups of $\bar{T}$. Observe also that no non-trivial element of T_1/T may normalize $\bar{S}$, since T is a Sylow 2-subgroup of $N_G(S)$. The elements of T_1/T have order at most four.

We shall denote $\bar{\bar{T}} = \bar{T}/Z(\bar{T}) = T/Z_2(T)$. Now $Z_2(T) = \langle a^2ba_3,\ ba_1,\ a_1a_2,\ a_3a_4\rangle$. Thus we obtain that

$$\bar{\bar{S}} = \langle \bar{\bar{a}}^2, \bar{\bar{a}}_1\rangle \qquad \bar{\bar{S}}_1 = \langle \bar{\bar{g}}, \overline{\overline{a^2a_1}}\rangle$$

$$\bar{\bar{S}}_2 = \langle \overline{\overline{gh}}, \bar{\bar{a}}_1\rangle \qquad \bar{\bar{S}}_3 = \langle \bar{\bar{g}}, \bar{\bar{h}}\rangle \ .$$

We now determine the structure of T_1/T.

(A) Suppose firstly that we may choose k in T_1-T, such that $\bar{S}^k = \bar{S}_3$ and $\bar{S}_1^k = \bar{S}_1$ and $\bar{S}_2^k = \bar{S}_2$.

Since k normalizes $\bar{S}_1$, and interchanges $\bar{S}$ and $\bar{S}_3$, we deduce that k must normalize the group $\langle\overline{\overline{a^2a_1g}}\rangle$.

Since k normalizes $\bar{S}_2$, and interchanges $\bar{S}$ and $\bar{S}_3$, we deduce that k must normalize the group $\langle\overline{\overline{gha_1}}\rangle$. It follows that k normalizes the group $\overline{\langle a^2a_1g,\ gha_1,\ a^2ba_3,\ ba_1\rangle}$. Denote $\bar{A}_1 = \overline{\langle a^2a_1g,\ gha_1,\ a^2ba_3,\ ba_1\rangle}$. Thus k normalizes $\bar{A}_1$. It follows that k must normalize $\bar{A}_1'$. Now $\bar{A}_1' = \langle\overline{a^2ba_3}\rangle$. It follows that k normalizes the group $Z(\bar{T}/\overline{\langle a^2ba_3\rangle})$. Taking pre-images in $\bar{T}$, we obtain that k must normalize the group $\langle\overline{gh},\ \bar{a}_3,\ \overline{a^2b},\ \overline{ba_1}\rangle$. We recall that k normalizes the group $\bar{S}_2$, where $\bar{S}_2 = \langle\overline{gh}, \bar{a}_1,\ \overline{a^2ba_3},\ \bar{b}\rangle$. It follows now that k must centralize $\overline{\overline{gh}}$. But $\overline{\overline{gh}}$ lies in $\bar{\bar{S}}_3$ and $\bar{\bar{S}}^k = \bar{\bar{S}}_3$ although $\overline{\overline{gh}}$ does not lie in $\bar{\bar{S}}$. Therefore this choice of k is not possible.

(B) Suppose now that we may choose k in $T_1 - T$ such that $\bar{S}^k = \bar{S}_2$ and such that k^2 lies in T. In this instance, k normalizes the group $\overline{S \cap S_2}$. Now $\overline{S \cap S_2} = \langle \bar{a}_1, \overline{a^2ba_3}, \bar{b}\rangle$. Thus k normalizes the group $\langle a_1, a_2, b, a^2ba_3, a_3a_4\rangle$. Denote $B_1 = \langle a_1, a_2, b, a^2ba_3, a_3a_4\rangle$. Then k normalizes $\Omega_1(B_1)$, and $\Omega_1(B_1) = \langle b, a_1, a_2, a_3a_4\rangle$. The arguments used in (b) now apply to show that k normalizes S. This case is not possible.

(C) Suppose now that we may choose k in $T_1 - T$ such that $\bar{S}^k = \bar{S}_1$, and such that k^2 lies in T. With a suitable revision of notation, the proof of (B) may be adapted to deal with this case also.

(D) Suppose now that T_1/T contains a cycle kT of order four, acting regularly on our set of subgroups of $\bar{T}$. It follows from (A), (B), and (C), that $\bar{S}^{k^2} = \bar{S}_3$ and that $\bar{S}_1^{k^2} = \bar{S}_2$. The fact that kT has order four implies that $\bar{k}$ as an element of $\bar{T}_1$, must centralize $Z(\bar{T})$. The arguments used in (2) case (c) may now be adapted to deal with this case also.

(E) It follows now that $|T_1 : T| = 2$, and that if k is an element of $T_1 - T$, then $\bar{S}^k = \bar{S}_3$ and $\bar{S}_1^k = \bar{S}_2$.

Remark 2 $Z(\bar{T})$ is a characteristic subgroup of $\bar{T}_1$.

Proof If we suppose that this not the case, then certainly $Z(\bar{T}) \neq Z(\bar{T}_1)$, and thus we may suppose that $Z(\bar{T}_1) < Z(\bar{T})$. Thus $Z(\bar{T}_1) < \langle \overline{a^2ba_3}, \overline{ba_1}\rangle$. If $Z(\bar{T}_1) = \langle \overline{a^2ba_3}\rangle$, then k also normalizes $Z(\bar{T}/\overline{\langle a^2ba_3\rangle})$. Taking pre-images in $\bar{T}$, we obtain that k must normalize the group $\langle \overline{gh}, \bar{a}_3, \overline{a^2b}, \overline{ba_1}\rangle$. Denote $\bar{C}_1 = \langle \overline{gh}, \bar{a}_3, \overline{a^2b}, \overline{ba_1}\rangle$. It follows now that k normalizes $C_{\bar{T}}(\bar{C}_1)$. Now $C_{\bar{T}}(\bar{C}_1) = \langle \bar{g}, \bar{b}, \overline{a^2ba_3}, \bar{a}_1\rangle$. Moreover k must also normalize $(C_{\bar{T}}(\bar{C}_1))'$. However we know that $(C_{\bar{T}}(\bar{C}_1))' = \langle \overline{ba_1}\rangle$. It follows now that k centralizes the group $\langle \overline{ba_1}\rangle$. Since $Z(\bar{T}_1) = \langle \overline{a^2ba_3}\rangle$, we have a contradiction.

Parallel arguments may be invoked to dismiss the case in which $Z(\bar{T}_1) = \langle \overline{ba_1} \rangle$.

Therefore we have to suppose that $Z(\bar{T}_1) = \langle \overline{a^2a_1a_3} \rangle$. This forces $Z_2(\bar{T}_1) = Z(\bar{T})$, and thus $Z(\bar{T})$ is a characteristic subgroup of $\bar{T}_1$.

<u>Remark 3</u> $Z_2(T)$ is a characteristic subgroup of T_1.

<u>Proof</u> If $Z(T_1) = Z(T)$, then using Remark 2, $Z(\bar{T})$ is a characteristic subgroup of $\bar{T}_1$, and thus it follows that $Z_2(T)$ is a characteristic subgroup of T_1, as required. Thus we have to suppose that $1 \neq Z(T_1) < Z(T)$. In this situation, it follows that $Z_2(T_1) \leq Z_2(T)$. If $Z_2(T_1) = Z_2(T)$, then there is no difficulty. Suppose that $Z(T_1) = \langle a_1a_2 \rangle$. Then T_1 normalizes $Z(T/\langle a_1a_2 \rangle)$. Thus T_1 normalizes the group $\langle b, a_1, a_2, a_3a_4 \rangle$. We recall our previous notation that $D = \langle b, a_1, a_2, a_3a_4 \rangle$. In this instance, T_1 must also normalize $C_T(D)$. Applying Remark 1, we obtain that $C_T(D) = \langle a^2b, a_1a_2, a_3, a_4 \rangle$. We recall our previous notation that $E = \langle a^2b, a_1a_2, a_3, a_4 \rangle$. Thus T_1 must normalize the group E'. However $E' = \langle a_3a_4 \rangle$. This contradicts our assumption that $Z(T_1) = \langle a_1a_2 \rangle$. A parallel argument serves to treat the case in which $Z(T_1) = \langle a_3a_4 \rangle$.

This forces $Z(T_1) = \langle a_1a_2a_3a_4 \rangle$. It follows directly now from the definition of $Z_2(T_1)$, that $Z_2(T_1) = \langle a_1a_2, a_3a_4 \rangle = Z(T)$.

Thus $Z_2(T)$ is a characteristic subgroup of T_1, as required.

<u>Conclusion</u> We consider the group $\bar{\bar{T}}_1 = T_1/Z_2(T)$. This group is specified by $\bar{\bar{T}}_1 = \langle \bar{\bar{k}}, \bar{\bar{a}}^2, \bar{\bar{a}}_1, \bar{\bar{g}}, \bar{\bar{h}} \rangle$. Since $\bar{S}^k = \bar{S}_3$ and $\bar{S}_1^k = \bar{S}_2$, we may deduce that k acts as follows on $\bar{\bar{T}}$:-

$$\bar{\bar{a}}^{2^k} = \bar{\bar{h}} \qquad \text{and} \qquad \bar{\bar{a}}_1^k = \bar{\bar{g}} .$$

It follows that $\bar{\bar{T}}$ is the unique elementary Abelian normal subgroup of $\bar{\bar{T}}_1$, of order 2^4. Thus $\bar{\bar{T}}$ is a characteristic subgroup of $\bar{\bar{T}}_1$. From Remark 3, we know that $Z_2(T)$ is a characteristic subgroup of T_1. It follows that T is a characteristic subgroup of T_1, and hence that $N_G(T_1) \leq N_G(T)$. But T_1 is a Sylow 2-subgroup of $N_G(T)$. Thus T_1 is a Sylow 2-subgroup of $N_G(T_1)$, and therefore also of G, of order 2^9. Thus we may use Proposition 2.19, to recognize the possible isomorphism type of G. There is no finite simple group satisfying our present hypotheses.

Our analysis of subcase (a) is therefore complete.

Subcase (b)

We now suppose that $Z(T) \neq Z(S)$.

(1) We suppose firstly that $Z(T) = \langle a_1a_2\rangle$. We know that $S \trianglelefteq T$. Thus

$$Z\left(\frac{S}{\langle a_1a_2\rangle}\right) \trianglelefteq \frac{T}{\langle a_1a_2\rangle}$$

Taking pre-images in T, we obtain that $\langle b, a_1, a_2, a_3a_4\rangle \trianglelefteq T$. We recall our previous notation, that $D = \langle b, a_1, a_2, a_3a_4\rangle$. Now $C_S(D) \trianglelefteq T$, and $C_S(D) = \langle a^2b, a_1a_2, a_3, a_4\rangle$. We recall our previous notation that $E = \langle a^2b, a_1a_2, a_3, a_4\rangle$. It follows now that $E' \trianglelefteq T$. Since $E' = \langle a_3a_4\rangle$, we obtain a contradiction to our suppostion that $Z(T) = \langle a_1a_2\rangle$.

(2) A parallel argument applies, if we suppose next that $Z(T) = \langle a_3a_4\rangle$.

(3) We must finally suppose that $Z(T) = \langle a_1a_2a_3a_4\rangle$. It follows directly that $Z_2(T) = \langle a_1a_2, a_3a_4\rangle$.

(A) The action of T/S on the set $\{Q, A, B, C\}$ of groups may be assumed to be semi-regular.

Proof Denote $\bar{T} = T/Z_2(T)$. If g is an element of $T-S$, then g permutes the groups $\{\bar{Q}, \bar{A}, \bar{B}, \bar{C}\}$. We tabulate the ways in which such an element g may operate with fixed points on the set $\{Q, A, B, C\}$ of groups. The table also shows, when it is clear from the information to hand, how some of the possibilites may be ruled out:-

Possibility	Reasons for non-occurrence
$Q^g = A$ $B^g = B$ $C^g = C$	See below
$Q^g = B$ $A^g = A$ $C^g = C$	b lies in A, so b^g lies in A b lies in B, so b^g lies in Q $A \cap Q = Z_2(T)$
$Q^g = C$ $A^g = A$ $B^g = B$	a^2b lies in A, so $(a^2b)^g$ lies in A a^2b lies in C, so $(a^2b)^g$ lies in Q $A \cap Q = Z_2(T)$

Thus we are left with the case in which $Q^g = A$, $B^g = B$ and $C^g = C$. From this knowledge, we deduce the action of g on $\bar{T}$:-

$$\bar{a}^{2^g} = \overline{a_1a_3} \qquad \bar{b}^g = \bar{a}_3 \qquad \bar{a}_1^g = \overline{a^2b}$$

Now T/S is isomorphic to a 2-subgroup of the symmetric group on $\{Q, A, B, C\}$. The above paragraphs allow us to deduce that gS is the only non-trivial element of T/S which acts with fixed points on the set $\{Q, A, B, C\}$ of groups. It follows now that $|T : S| = 2$.

From the known action of g, we deduce that $\bar{S}$ is the unique normal, elementary Abelian subgroup of order 2^4. It follows that $\bar{S}$ is a characteristic subgroup of $\bar{T}$. Since $Z_2(T)$ is a characteristic subgroup of T, it follows that S is also a characteristic subgroup of T. Thus $N_G(T) \leq N_G(S)$. It follows now that T is a Sylow 2-subgroup of G of order 2^7. We may Proposition 2.19, to recognize the isomorphism type of G. There is no finite simple group satisfying our present hypotheses.

Thus the action of T/S is semi-regular, as required.

(B) T/S is not a cycle of order four.

Suppose that we can find an element k in $T-S$, such that k permutes the four groups Q, A, B, C, cyclically. It follows that k has order at least four. Since $Z_2(T)$ is elementary Abelian of order four, it is clear that k must centralize $Z_2(T)$. This latter fact contradicts our present assumptions on $Z(T)$.

(C) Suppose that we may choose an element g in $T-S$, with $Q^g = B$ and $A^g = C$. We denote, as previously, $\bar{T} = T/Z_2(T)$. We know that:-

$$\bar{T} \geq \langle \bar{g}, \bar{a}^2, \bar{b}, \bar{a}_1, \bar{a}_3 \rangle .$$

From the action of g, we deduce that:-

$$\bar{a}^{2^g} = \overline{a^2ba_1} \qquad \bar{b}^g = \bar{a}_1 \qquad \bar{a}_3^g = \bar{a}_3 .$$

In this instance, we obtain that $Z(\bar{T}) \leq \langle \overline{a^2b}, \bar{b}a_1, \bar{a}_3 \rangle$, and that g normalizes the group $\langle a^2b, ba_1, a_3, a_4, a_1a_2 \rangle$. Denote $F = \langle a^2b, ba_1, a_3, a_4, a_1a_2 \rangle$. Then $\Omega_1(F)$ is normalized by g, and we know that:-

$$\Omega_1(F) = \langle a^2b, a_3, a_4, a_1a_2 \rangle .$$

Moreover g must also normalize $C_S(\Omega_1(F))$, and we know that $C_S(\Omega_1(F)) = \langle b, a_1, a_2, a_3a_4 \rangle$. It follows that g normalizes the groups $(\Omega_1(F))'$ and $C_S(\Omega_1(F))'$. Since $(\Omega_1(F))' = \langle a_3a_4 \rangle$ and $C_S(\Omega_1(F))' = \langle a_1a_2 \rangle$, we conclude that g centralizes $Z_2(T)$.

Parallel arguments apply, in the case in which it is possible to choose h in $T-S$ with $Q^h = C$ and $A^h = B$, to demonstrate that h centralizes $Z_2(T)$.

(D) Since the action of T/S is semi-regular, and since T/S is not cyclic of order four, the assumption that $Z(T) = \langle a_1a_2a_3a_4\rangle$ together with the above section (C) indicates that $|T : S| = 2$, and that we may find an element k in $T-S$ with $Q^k = A$ and $B^k = C$. We shall denote $\bar{T} = T/Z_2(T)$.

We know that:-

$$\bar{T} = \langle \bar{k}, \bar{a}^2, \bar{b}, \bar{a}_1, \bar{a}_3\rangle.$$

From the action of k on the set $\{Q, A, B, C\}$ of groups, we deduce that:-

$$\bar{a}^{2^k} = \overline{a_1a_3} \ ; \qquad \bar{b}^k = \bar{a}_1 \ ; \qquad \bar{a}_3^k = \overline{a^2b} \ .$$

It follows in this instance, that $\bar{S}$ is the unique normal elementary Abelian subgroup of $\bar{T}$ of order 2^4. Thus $\bar{S}$ is a characteristic subgroup of $\bar{T}$. Since $Z_2(T)$ is a characteristic subgroup of T, it follows now that S is a characteristic subgroup of T, and hence that $N_G(T) \leq N_G(S)$. From the fact that T is a Sylow 2-subgroup of $N_G(S)$, we deduce that T is a Sylow 2-subgroup of $N_G(T)$, and thus of G. Since $|T| = 2^7$, we may use Proposition 2.19, to recognize the possible isomorphism type of G. There is no finite simple group satisfying our present hypotheses.

The case in which $N_N(P) = P\langle a^2,b\rangle$ is now complete.

(x) <u>$N_N(P) = P\langle a,b\rangle$</u>

Denote $\bar{N} = N_G(Q)/C_G(Q)$. We show that $\bar{N}$ is isomorphic to $P\langle a,b\rangle$. We know that $\bar{N}$ is isomorphic to a subgroup of $GL(4,2)$. The group $GL(4,2)$ is isomorphic to A_8. Moreover, $GL(4,2)$ has a self-centralizing elementary Abelian Sylow 3-subgroup of order nine, and precisely two conjugate classes of elements of order three. A 3-element of $GL(4,2)$ has centralizer isomorphic to $Z_3 \times A_5$, or to $Z_3 \times D_6$, according to the conjugate class to which it belongs. The Sylow 3-automiser of $GL(4,2)$ is isomorphic to D_8.

We know from the maximality of Q, that Q is a Sylow 2-subgroup of $C_G(Q)$. From the 3-local structure of $GL(4,2)$, it follows that either $O_{3'}(\bar{N})$ is trivial, or that $O_{3'}(\bar{N})$ is a 2-group. Using the maximality of Q, we deduce that $O_{3'}(\bar{N}) = 1$. We deduce from the structure of $C_G(x)$, that $C_{\bar{N}}(\bar{x})$ is isomorphic to $Z_3 \times D_6$. We deduce from the 3-local structure of $GL(4,2)$, and the maximality of Q, that $C_{\bar{N}}(\bar{x}y)$ is isomorphic to $Z_3 \times D_6$. We may now apply [22] Lemma 3.2, to see that either $\bar{N}$ is isomorphic to S_6 or that $\bar{N}$ is isomorphic to $P\langle a,b\rangle$. In the former case, the arguments of [28], apply. Thus we may suppose that $\bar{N}$ is isomorphic to $P\langle a,b\rangle\rangle$ As previously, we may deduce the action of the group $\langle a,b\rangle$ on Q:-

$$\begin{array}{llll}
C_Q(x) = \langle a_1,a_2\rangle & & C_Q(y) = \langle a_3,a_4\rangle & \\
a^{-1}a_1a = a_3 & & a^{-1}a_3a = a_2 & \\
a^{-1}a_2a = a_4 & & a^{-1}a_4a = a_1 & \\
b^{-1}a_1b = a_2 & & b^{-1}a_3b = a_3 & b^{-1}a_4b = a_4.
\end{array}$$

Here $\langle a_1,a_2\rangle$, $\langle a_3,a_4\rangle$ are elementary Abelian groups of order four. We shall denote $S_1 = Q\langle a^2,b\rangle$. We observe that $Z(Q\langle a,b\rangle) = \langle a_1a_2a_3a_4\rangle$, and that $Z_2(Q\langle a,b\rangle) = \langle a_1a_2,\ a_3a_4\rangle$. Let R denote a Sylow 2-subgroup of G, chosen so as to contain $Q\langle a,b\rangle$. It follows now that $Z_2(R) = Z_2(Q\langle a,b\rangle)$, and that $|R : C_R(Z_2(R))| = 2$.

We now return to consider the arguments of case (ix) (above). The arguments of case (ix) go through in this instance, if instead of considering the groups G and S, as previously, we now consider the groups $C_G(Z_2(R))$ and S_1 (respectively).

The only difficulty occurs in the transposition of Remark 1. Note that Remark 1 is visibly false in $C_G(Z_2(R))$ in this instance, although it remains true in G, and thus is again applicable. The adapted arguments of case (i) yield that $|C_R(Z_2(R))| \leq 2^9$. It follows now that $|R| \leq 2^{10}$. Thus from Proposition 2.19, we may deduce the isomorphism type of G. There is no finite simple group satisfying our present hypotheses.

<u>This completes the proof of the proposition.</u>

In the proofs of Proposition 6.7 and Proposition 6.8, once we have established the order of a Sylow 2-subgroup of the group in question, we turn in several instances to the fact that the group has precisely two conjugate classes of involutions, only in order to eliminate particular possibilities at that stage. The fact that the group is assumed to possess precisely two conjugate classes of involutions is used at no other point in the proofs. In fact from Proposition 2.19, we know the isomorphism type of all those finite simple groups possessing a Sylow 2-subgroup of order at most 2^{10}. The above fusion arguments are therefore in some sense superfluous.

We shall now return to our proof of Theorem B. We now suppose that G denotes a finite simple group satisfying the hypotheses of Theorem B. We preserve as much of the previous notation as possible. By Lemma 6.5, the structures of $C_G(x)$ and $C_G(xy)$ are determined. By interchanging the rôles of x and xy if necessary for notational purposes, there are precisely six cases to consider.

	$C_G(x)$	$C_G(xy)$
Case I	Type 1	Type 1
Case II	Type 1	Type 2
Case III	Type 1	Type 3
Case IV	Type 2	Type 2
Case V	Type 2	Type 3
Case VI	Type 3	Type 3

Proposition 6.7 treats case II.

Proposition 6.8 treats case III.

It thus remains to deal with the cases I, IV, V, VI.

Observe that in case I, both $C_G(x)$ and $C_G(xy)$ have twice odd order. An as yet unpublished result due to B. Stellmacher, allows the classification of all those finite simple groups, each of whose 3-elements has centralizer of order divisible by two but not by four. We deduce that there is no finite simple group satisfying our present hypotheses and classified under case I.

It remains to deal with the cases IV, V, VI. We cite a recent result due to O'Nan.

Theorem (O'Nan [21])

Let H be a finite group having no non-identity Abelian normal subgroups. Suppose that H has an elementary Abelian subgroup P of order nine, such that if a is any element of $P-\{1\}$, then there is an odd prime power q, such that $C_H(a)/\langle a\rangle$ is isomorphic to one of PSL(2,q), PGL(2,q) or $P\Sigma^*L(2,q)$.

Then H has a self-centralizing normal subgroup N which is isomorphic to one of:-

(i) PSU(3,5) (ii) PSL(3,7) (iii) M_{22} (iv) M_{23} (v) M_{24}
(vi) H_iS (vii) R. (Rudvalis group) (viii) PSL(5,2)
(ix) J_2 (x) $PS_p(4,4)$.

The above Theorem applies directly to the cases IV, V, VI. We deduce that if G is a finite simple group satisfying our present hypotheses, and classified under cases IV, V, VI, then in particular G is classified under case V, and G is isomorphic to PSL(5,2). Note that the group PSL(5,2) has precisely two conjugate classes of involutions.

The proof of Theorem B is complete.

Chapter VII

In Retrospect

Theorems A and B have now been formally proved. As regards Theorem A, one of the most plausible next logical steps might be an attempted classification of all finite groups with a Sylow 3-subgroup P, elementary Abelian of order nine, with $N_G(P)/C_G(P)$ isomorphic to $Z_2 \times Z_2$. On reflection, this problem is necessarily rather difficult, as it would imply a classification of all finite groups possessing a cyclic Sylow 3-subgroup of order three.

As regards Theorem B, there remains a great deal of scope. We first note that Proposition 6.7 and Proposition 6.8 may both be more concisely proved, if we quote the recent classification by G. Stroth, of all those finite simple groups possessing an elementary Abelian 2-subgroup of order sixteen, which is a Sylow 2-subgroup of its own centralizer ([27], [28], [29]). We would hope however that our methods developed in the proofs of the above propositions might find some other application and in particular that they might be useful in the further discussion of the structure of finite groups possessing a self-centralizing elementary Abelian Sylow 3-subgroup of order nine.

Let us consider now the more general problem of the classification of finite groups G which possess a self-centralizing elementary Abelian Sylow 3-subgroup P, of order nine. We shall further suppose that $O^3(G) = G$.

The following seems a reasonable hypothesis in this instance:-

(1) Every involution of $G-O_{3'}(G)$ is conjugate to some involution of $N_G(P)$.

This hypothesis has been proved true by P.G. Henry [14], in the cases where $N_G(P)/P$ is isomorphic to Z_4 or to Q_8.

If $N_G(P)/P$ is isomorphic to Z_8, recent results due to S.D. Smith, indicate that the hypothesis is true, and that G has a single conjugate class of involutions.

In the case in which $N_G(P)/P$ is isomorphic to D_8, it might be possible to use character theoretic methods to prove the hypothesis true (or otherwise). In fact we may already obtain one character theoretic relation in this instance, which may prove useful. Since the group $\langle 3,3,2\rangle$ is isomorphic to A_4, it follows that $\#(x^{\cdot}(xy)^{\cdot} = k) = 0$, for all involutions k of G. As in the proof of Theorem A, this translates into the character theoretic relation

$$\Sigma \quad \frac{\chi(x)\chi(xy)\chi(k)}{\chi(1)} = 0,$$

where this sum may be taken over the principal 3-block of G. The principal 3-block of G is determined using the methods of Chapter IV. Thus we have a 'degree equation'.

The analysis of the case in which $N_G(P)/P$ is isomorphic to D_8 is visibly incomplete. For other related results, broadly in this area, we would refer the reader to Lemma 3.2 and Lemma 3.6 of the paper [22] by Prince.

Finally, as regards the problem of classifying all finite groups possessing a self-centralizing elementary Abelian Sylow 3-subgroup of order nine, it seems reasonable to hope that the methods used in this thesis might also find some application the cases discussed in Chapter I, in which the Sylow 3-automiser is isomorphic to Z_8 or to Q D_{16}.

Appendix

(1) We list the only known finite simple groups G possessing an elementary Abelian Sylow 3-subgroup P, of order nine, with $N_G(P)/C_G(P)$ isomorphic to D_8. Such groups are:-

$PSL(4,q)$	$q \equiv -1(3)$	$q \not\equiv -1(9)$
$PSL(5,q)$	$q \equiv -1(3)$	$q \not\equiv -1(9)$
$PSU(4,q^2)$	$q \equiv 1(3)$	$q \not\equiv 1(9)$
$PSU(5,q^2)$	$q \equiv 1(3)$	$q \not\equiv 1(9)$
$PSp(4,q)$	$q \equiv -1(3)$	$q \not\equiv -1(9)$.

I have this list on the authority of lectures given by Professor Higman, in Oxford, in Michaelmas term 1974.

We now compile a sublist of the above list, including only those of the above groups which have a self-centralizing Sylow 3-subgroup. We suppose H to denote a group in our sublist.

(i) Suppose that the group H is isomorphic to $PSL(4,q)$ (some $q \equiv -1(3)$, $q \not\equiv -1(9)$).

Observe that $SL(4,q)$ contains a subgroup isomorphic to $SL(2,q) \times SL(2,q)$, given by the matrices of the following type:- $\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\right.$ where A and B are elements of $\left.SL(2,q)\right\}$.

It follows that $PSL(4,q)$ has a section isomorphic to $PSL(2,q) \times PSL(2,q)$. Since we require that our group H have a self-centralizing Sylow 3-subgroup, the quoted result due to Feit and Thompson (Proposition 2.10), forces $q = 5$ or $q = 2$. In fact from our construction, we require further that the above section be indeed a subgroup of H. This is only the case if $q = 2$.

The group $PSL(4,2)$ is isomorphic to A_8, and certainly satisfies our reuqirements. Furthermore, PSL(4,2) has two conjugate classes of involutions, whilst the centralizers of 3-elements in PSL(4,2) are isomorphic to $Z_3 \times D_6$ and $Z_3 \times A_5$.

(ii) Suppose that the group H is isomorphic to PSL(5,q) (some $q \equiv -1(3)$, $q \not\equiv -1(9)$). Arguing as in (i), we observe that H has a section isomorphic to PSL(2,q) × PSL(3,q). Using the result due to Feit and Thompson (Proposition 2.10), We obtain that $q = 2$.

The group PSL(5,2) does in fact have a self-centralizing Sylow 3-subgroup. Furthermore PSL(5,2) has two conjugate classes of involutions, and the centralizers of 3-elements in PSL(5,2) are isomorphic to $Z_3 \times A_5$ and $Z_3 \times$ PSL(2,7).

(iii) Suppose that the group H is isomorphic to $PSU(4,q^2)$ (some $q \equiv 1(3)$, $q \not\equiv 1(9)$). Arguing as in (i), we observe that H has a section isomorphic to $PSU(2,q^2) \times PSU(2,q^2)$. Since we may identify $PSU(2,q^2)$ with PSL(2,q), it follows using the quoted result (Proposition 2.10) due to Feit and Thompson, that we must have that $q = 7$. Thus we have to consider the group $PSU(4,7^2)$. In fact from our construction, we require further that the above section be indeed a subgroup of H. In the group $PSU(4,7^2)$, this is not the case.

This instance does not arise.

(iv) Suppose that the group H is isomorphic to $PSU(5,q^2)$ (some $q \equiv 1(3)$, $q \not\equiv 1(9)$). In this case $PSU(5,q^2)$ has a section isomorphic to $PSU(2,q^2) \times PSU(3,q^2)$. It follows, using the quoted result (Proposition 2.10) due to Feit and Thompson, that H cannot have a self-centralizing Sylow 3-subgroup. This case cannot therefore arise.

(v) Suppose that the group H is isomorphic to PSp(4,q) (some $q \equiv -1(3)$, $q \not\equiv -1(9)$). In this case, H has a section isomorphic to $PSp(2,q) \times PSp(2,q)$. Using the fact that PSp(2,q) is isomorphic to PSL(2,q), and employing the quoted result due to Feit and Thompson (Proposition 2.10), we deduce that $q = 2$ or $q = 5$. In fact from our construction, we require further that the above section be indeed a subgroup of H. Thus it remains to consider the case in which $q = 2$. The group PSp(4,2) does in fact have a self-centralizing Sylow 3-subgroup. However PSp(4,2) is not simple, being itself isomorphic to S_6.

(2) We list the finite simple groups with Sylow 2-subgroups of order at most 2^{10}. The list is taken directly from V. Stingl's doctoral thesis:-

Order of Sylow 2-subgroup	Groups occurring		
2^0	Z_p	All odd primes p	
2^1	Z_2		
2^2	$PSL(2,q)$	$q \equiv 3,5\ (8)$	$q > 3$
2^3	$PSL(2,q)$	$q \equiv 7,9\ (16)$	
	$PSL(2,8)$		
	$JR(q)$	$q \equiv 3,5\ (8)$	$q > 3$
	A_7		
2^4	$PSL(2,q)$	$q \equiv 15,17\ (32)$	
	$PSL(2,16)$		
	$PSL(3,q)$	$q \equiv 3\ (8)$	
	$PSU(3,q^2)$	$q \equiv 5\ (8)$	
	M_{11}		
2^5	$PSL(2,q)$	$q \equiv 33,35\ (64)$	
	$PSL(2,32)$		
	$PSL(3,q)$	$q \equiv 5\ (8)$	
	$PSL(3,q)$	$q \equiv 7\ (16)$	
	$PSU(3,q^2)$	$q \equiv 3\ (8)$	
	$PSU(3,q^2)$	$q \equiv 9\ (16)$	
2^6	$PSL(2,q)$	$q \equiv 63,65\ (128)$	
	$PSL(2,64)$		
	$PSL(3,q)$	$q \equiv 15\ (32)$	
	$PSL(3,4)$		
	$PSp(4,q)$	$q \equiv 3,5\ (8)$	
	$G_2(q)$	$q \equiv 3,5\ (8)$	
	$PSU(3,q^2)$	$q \equiv 17\ (32)$	
	$PSU(3,4^2)$		
	${}^3D_4(q^3)$	$q \equiv 3,5\ (8)$	
	${}^2B_2(8)$		
	A_8		
	A_9		
	M_{12}		

2^7	$PSL(2,q)$	$q \equiv 127,129\ (256)$
	$PSL(2,128)$	
	$PSL(3,q)$	$q \equiv 9\ (16)$
	$PSL(3,q)$	$q \equiv 31\ (64)$
	$PSU(3,q^2)$	$q \equiv 7\ (16)$
	$PSU(3,q^2)$	$q \equiv 33\ (64)$
	$PSL(4,q)$	$q \equiv 3,5\ (8)$
	$PSU(4,q^2)$	$q \equiv 3,5\ (8)$
	A_{10}	
	A_{11}	
	M_{22}	
	M_{23}	
	M_cL	
	J_2	
	J_3	
2^8	$PSL(2,q)$	$q \equiv 255,257\ (512)$
	$PSL(2,256)$	
	$PSL(3,q)$	$q \equiv 63\ (128)$
	$PSU(3,q^2)$	$q \equiv 65\ (128)$
	$PSp(4,q)$	$q \equiv 7,9\ (16)$
	$G_2(q)$	$q \equiv 7,9\ (16)$
	${}^3D_4(q^3)$	$q \equiv 7,9\ (16)$
	$PSp(4,4)$	
	L_yS	
2^9	$PSL(2,q)$	$q \equiv 511,513\ (1024)$
	$PSL(2,512)$	
	$PSL(3,8)$	
	$PSL(3,q)$	$q \equiv 127\ (256)$
	$PSL(3,q)$	$q \equiv 17\ (32)$
	$PSL(4,q)$	$q \equiv 7\ (16)$
	$PSL(5,q)$	$q \equiv 3\ (8)$
	$PSU(3,q^2)$	$q \equiv 129\ (256)$
	$PSU(3,q^2)$	$q \equiv 15\ (32)$
	$PSU(4,q^2)$	$q \equiv 9\ (16)$
	$PSU(5,q^2)$	$q \equiv 5\ (8)$

	$PSU(3,8^2)$	
	$P\Omega(7,q)$	$q \equiv 3,5\ (8)$
	$PSp(6,q)$	$q \equiv 3,5\ (8)$
	$PSp(6,2)$	
	A_{12}	
	A_{13}	
	O'N	
	H_iS	
2^{10}	$PSL(2,q)$	$q \equiv 1023,1025\ (2048)$
	$PSL(2,1024)$	
	$PSL(5,2)$	
	$PSL(3,q)$	$q \equiv 255\ (512)$
	$PSL(4,q)$	$q \equiv 9\ (16)$
	$PSU(3,q^2)$	$q \equiv 257\ (512)$
	$PSU(4,q^2)$	$q \equiv 7\ (16)$
	$PSU(5,2^2)$	
	$PSp(4,q)$	$q \equiv 15,17\ (32)$
	$G_2(q)$	$q \equiv 15,17\ (32)$
	${}^3D_4(q^3)$	$q \equiv 15,17\ (32)$
	$P\Omega^-(8,q^2)$	$q \equiv 3,5\ (8)$
	${}^2B_2(32)$	
	A_{14}	
	A_{15}	
	M_{24}	
	CO_3	
	He	

In this table, JR(q) denotes a group of Janko-Ree type.

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