

On surgery presentations of bordism bicategories



Filippos Ilarion Sytilidis

St John's College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2025

To my parents. Without their love and support, this work would never have been possible.

Acknowledgments

Writing this thesis has been an incredible journey. Unlike the manifolds that appear later, this journey wasn't always smooth. But it was a journey that left me fulfilled, taught me to embrace imperfection, and helped me grow. By the end of it, I am pleased to find that doing mathematics as an adult has once again brought me the excitement and joy I first associated with mathematics as a child.

I am, above all, grateful to God for everything—but in particular, for bringing into my life some truly amazing people, without whom this journey would have been impossible, or at the very least far less fun. Among them, there are some I would especially like to acknowledge.

I owe sincere thanks to my supervisors for their guidance and support. Chris Douglas first introduced me to the problem that ultimately gave rise to this thesis. Throughout our many conversations—often exploring unpromising directions before the right ideas emerged—he listened with patience and insight. His enthusiasm for spontaneous running and climbing breaks made even the most intense phases of writing genuinely enjoyable. Bruce Bartlett graciously agreed to co-supervise me before we had ever met in person. He devoted many hours to clarifying relevant background material and met with me regularly—whether over Zoom or in beautiful Stellenbosch—to help shape both the mathematics and its exposition. His suggestion to take Juhász's paper seriously, and his encouragement to engage deeply with Cerf theory, were especially pivotal to this work. At the beginning of my PhD, I was also co-supervised by Ulrike Tillmann, to whom I am truly grateful. Her mentorship and support, as well as our many insightful meetings, had a great impact on how I think about mathematics.

I am especially grateful to Luciana Bonatto for being a wonderful friend, for welcoming me into the European Talbot community, and for regularly meeting with me over the past year to talk through my thesis. Her insights helped me notice subtleties I had previously overlooked, but most importantly, she encouraged me to do mathematics in person at a time when I was feeling quite isolated.

I am grateful to André Henriques, András Juhász, Chris Schommer-Pries, and Thomas Wasserman for their thoughtful feedback as examiners at the various stages of my PhD. I owe particular thanks to Chris Schommer-Pries for laying much of the foundational work on which this thesis builds, and for the warm welcome during my visit to South Bend while we discussed it.

I am grateful to Peter Kronheimer, my undergraduate mentor—it was a privilege to be his student. Special thanks also to Dennis Gaitsgory, Mike Hopkins, Sander Kupers, and Haynes Miller for teaching the topology courses that made me want to be part of this mathematical community.

I am thankful to all (honorary) members of the Oxford topology group for fostering such a wholesome and supportive community. Special thanks to Christoph Weis for being a caring friend and for introducing me to climbing, coding, and so many things in between; to Alice Kerr for being a wonderful housemate and graciously taste-testing my focaccia experiments; to Adele Jackson for all the Currydor trips; and to Jane Tan for co-organizing the cake seminar with me. I’m grateful to Christoph Dorn for joining many of my early climbing outings, and to Jan Steinebrunner for enthusiastically sharing so much mathematics—especially during my first year. Many thanks to Sam Fisher, Monika Kudlinska, and Patrick Nairne for being a lovely cohort to begin the PhD journey with. Finally, I’m deeply grateful to Adrià Marín Salvador, Nivedita, Sofía Marlasca Aparicio, Glen Lim, Thomas Wasserman and Thibault Décoppet for creating the kind of community within the topology group where I felt most at home for the bulk of my PhD.

I am grateful to the friends I’ve made since coming to Oxford: Casey Adam, for all the long walks and for always being there—even with an ocean between us; Archana Ramesh, for being endlessly caring, with just the right amount of teasing; Mona Barkat, for the many walks and deep conversations; Tesfahiwet Yemane, for bringing so much kindness into the world; Munazza Ebtikar, for all the hot chocolate outings; and Alaa Baazaoui, Romain Fournier, Charlotte Hand, Lorika Shkreli, and Mingying Huang, for all the regular waffles.

I am also grateful to the friends I made before Oxford: Natalia Pacheco-Tallaj, for sharing so much mathematics during our undergraduate years—without which I might never have ended up in topology; Luke Melas-Kyriazi, for always being up for a spontaneous adventure; Dan Stefan Eniceicu, for making me just a little more competitive; Mirac Suzgun, for being the kindest soul; and Stella Kyratzi, without whom I would probably be speaking even more slowly than I already do.

I was lucky to meet all the friends I made in Stellenbosch, especially Nzaganya Edson, Dario Trinchieri, Hosana Ranaivo, and Brian Muchiri. Thank you all for making it a warm and welcoming second academic home.

I will always be thankful to my teachers in Greece. Special thanks to Dionisis Takouridis for being my first maths teacher and believing in me all the way; Timos Panagiotopoulos for introducing me to the world of Euclidean geometry; and Eleni Mitsiou, whose support and mentorship kicked off my mathematical journey.

Τέλος, τίποτα από όλα αυτά δεν θα ήταν δυνατό χωρίς την οικογένειά μου. Η μαμά και ο μπαμπάς μου είναι οι πιο υποστηρικτικοί και στοργικοί γονείς που θα μπορούσα να έχω—προσαρμόζοντας τη ζωή τους σε διάφορες στιγμές για να καλλιεργήσουν το πάθος μου για τα μαθηματικά, ζητώντας μόνο να φροντίζω τον εαυτό μου ως αντάλλαγμα. Η αδερφή μου είναι ταυτόχρονα αδερφή και αληθινή φίλη, με την οποία μπορώ να μοιραστώ τα πάντα. Και η γιαγιά μου είναι μια αστείρευτη πηγή αγάπης. Ξέρω ότι ίσως να μην καταλάβετε ποτέ το περιεχόμενο αυτής της διατριβής, αλλά έγινε δυνατή μόνο χάρη σε εσάς.

This thesis was supported through the Ioan and Rosemary James Scholarship by St. John's College and EPSRC grant no. 2433258.

Abstract

The d -dimensional bordism $(2, 1)$ -category has as objects closed $(d-1)$ -dimensional manifolds, as 1-morphisms d -dimensional cobordisms, and as 2-morphisms isotopy classes of diffeomorphisms.

In Part I of this thesis, we obtain generators-and-relations presentations of the oriented and unoriented d -dimensional bordism $(2, 1)$ -categories. These presentations, denoted $\text{Sur}_{(d-1; d; \sim)}^{\text{or}}$ and $\text{Sur}_{(d-1; d; \sim)}$ respectively, consist of Morse and Cerf graphics suitably decorated with surgery data.

In Part II, we discuss what it means for a 2-categorical presentation to be skeletal. We introduce a general skeletalization procedure and apply it to $\text{Sur}_{(1; 2; \sim)}^{\text{or}}$. This yields a skeletal presentation of the oriented 2-dimensional bordism $(2, 1)$ -category, which is, in particular, finite.

Contents

I	Introduction	viii
I.1	What this thesis is about	viii
I.2	Background and related work	x
I.3	Results and methods	xxi
I.4	Future directions	xxv
I	From bordisms to surgery diagrams	1
1	Morse and Cerf theory	2
1.1	Whitney topologies	3
1.2	Thom’s jet transversality theorem	6
1.3	Morse functions	10
1.4	Cerf’s fibration lemma	28
1.5	Cerf paths	32
1.6	Elementary Cerf paths	37
1.7	Cerf homotopies	54
1.8	Elementary Reidemeister III homotopies	67
2	Surgery presentations of bordism categories	71
2.1	Linear diffeomorphism diagrams	72
2.2	Linear surgery diagrams	75
2.3	Bordism categories equipped with Morse functions	81
2.4	Bordism categories equipped with Morse data	85
2.5	The equivalence between $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{(\text{or})}$	91
3	Surgery presentations of bordism bicategories	103
3.1	2-presentations	104
3.2	Planar diffeomorphism diagrams	112

3.3	2-presentation of planar surgery diagrams	129
3.4	Bordism $(2, 1)$ -categories equipped with Cerf paths	149
3.5	Bordism $(2, 1)$ -categories equipped with Cerf data	158
3.6	The equivalence between $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$	174
II	From surgery presentations to finite presentations	200
4	Skeletal 2-presentations	201
4.1	Definition	201
4.2	Skeletalization procedure	207
5	Constructing a finite presentation of $\text{Bord}_{01\sim}^{(\text{or})}$	218
5.1	Skeletalization of $\text{Sur}_{01\sim}^{\text{or}}$	218
5.2	Skeletalization of $\text{Sur}_{01\sim}$	233
6	Constructing a finite presentation of $\text{Bord}_{12\sim}^{\text{or}}$	245
6.1	Skeletalization of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$	245
6.2	Skeletalization of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$	263
6.3	Skeletalization of $\text{Sur}_{12\sim}^{\text{or}}$	281
A	The $\mathcal{F}^2$ presentation of $\text{Bord}_{12\sim}^{\text{or}}$	298
	References	304

Chapter I

Introduction

I.1 What this thesis is about

Over the past three decades, 3-dimensional topological quantum field theory (TQFT) has emerged as a central area of study, revealing profound connections between low-dimensional topology, representation theory, and higher category theory. Despite significant progress and rich developments, the foundational problem of classifying extended 3-dimensional TQFTs remains unresolved. This thesis is primarily motivated by this classification problem and presents an important stepping stone towards its resolution.

Building on Witten’s foundational work connecting quantum field theory with low-dimensional topology, and Segal’s axiomatization of conformal field theories, Atiyah [Ati88] defined a topological quantum field theory as a symmetric monoidal functor from a ‘geometric’ source category of manifolds and cobordisms to an ‘algebraic’ target category of vector spaces and linear maps. Although Atiyah’s definition was not originally framed in this language, this formulation has since become standard, as it provides a cohesive framework for assigning data on $(d-1)$ -dimensional manifolds and d -dimensional bordisms, respecting natural gluing laws. Further building on Atiyah’s work, Lawrence [Law93] and Freed [Fre94] defined extended TQFTs, embodying gluing laws for manifolds of higher codimension, leading to the cobordism hypothesis of Baez-Dolan [BD95] and Lurie [Lur09]. The cobordism hypothesis provides a deep and elegant classification of fully extended—or local— n -dimensional TQFTs, which assign data to k -dimensional manifolds for all k between 0 and n .

An analogous classification of general non-local TQFTs remains largely out of reach. However, such a classification seems plausible for “almost local” n -dimensional TQFTs, which assign data to k -dimensional manifolds for k between 1 and n . In the case of $n = 2$, these are the non-extended 2-dimensional TQFTs, and it is a folklore

theorem that they are classified by symmetric Frobenius algebras. The case of interest for this thesis is the next step in complexity: extended 3-dimensional TQFTs, namely symmetric monoidal functors

$$\mathrm{Bord}_{123}^{\mathcal{S}} \rightarrow \mathcal{C}.$$

Here, the source is a symmetric monoidal bicategory of closed 1-dimensional manifolds, 2-dimensional bordisms and diffeomorphism classes of 3-dimensional bordisms with corners, potentially equipped with some tangential structure $\mathcal{S}$, and the target is some chosen delooping $\mathcal{C}$ of the category of vector spaces. In this thesis, we will either omit any additional tangential structure, or set $\mathcal{S}$ to be an orientation.

In a broader sense, TQFTs can be viewed in analogy with representations of a group G , namely functors

$$BG \rightarrow \mathrm{Vect}_k.$$

Here, BG is a category whose only object has an automorphism group isomorphic to G . Classifying the representations of G often involves identifying a presentation of G in terms of generators and relations, which induces a corresponding presentation $\mathcal{P}$ of the category BG . Representations of G can then be described by specifying $\mathcal{P}$ -shaped data in Vect_k . Extending this analogy to TQFTs, we are naturally led to consider generators-and-relations presentations of bordism categories as a route to classification.

Indeed, this is how the simplest low-dimensional TQFTs are classified. For example, the 1-dimensional oriented bordism category $\mathrm{Bord}_{01}^{\mathrm{or}}$ has a straightforward presentation as a symmetric monoidal category illustrated below in Figure [I.1](#). This presentation allows us to conclude that oriented 1-dimensional TQFTs are classified by the choice of a vector space V , its dual V^* , and evaluation and coevaluation maps that witness their duality.

This analogy with group representations, along with its early success in classifying low-dimensional TQFTs, motivates a deeper study of a generators-and-relations presentation of $\mathrm{Bord}_{123}^{\mathrm{or}}$. Here we will focus on a slightly easier problem. $\mathrm{Bord}_{123}^{\mathrm{or}}$ has an interesting subcategory, namely $\mathrm{Bord}_{12\sim}^{\mathrm{or}}$, the symmetric monoidal $(2, 1)$ -category of closed, oriented 1-manifolds, oriented 2-dimensional cobordisms and isotopy classes of orientation preserving diffeomorphisms. Following Tillmann [\[Til98\]](#), the symmetric monoidal functors

$$\mathrm{Bord}_{12\sim}^{\mathrm{or}} \rightarrow \mathcal{C}$$

Generating Objects: $+\bullet \quad -\bullet$

Generating Morphisms: $\begin{array}{c} + \\ - \end{array} \smile \quad \frown \begin{array}{c} - \\ + \end{array}$

Generating Relations:

$$\begin{array}{c} + \\ - \end{array} \begin{array}{c} \smile \\ \frown \end{array} = \begin{array}{c} + \\ - \end{array} \text{---} \begin{array}{c} + \\ - \end{array}$$

Figure I.1: The presentation of $\text{Bord}_{01}^{\text{or}}$ as a symmetric monoidal category.

are known as topological modular functors and have also been extensively studied in the literature.¹ A generators-and-relations presentation of $\text{Bord}_{12\sim}^{\text{or}}$ is, thus, both a quest of independent interest and a natural step towards a generators-and-relations presentation of $\text{Bord}_{123}^{\text{or}}$. The ultimate goal of this thesis is to obtain a *finite* such presentation of manageable complexity.

I.2 Background and related work

I.2.1 General background

The study of modular functors and 3-dimensional extended TQFTs has a rich history, dating back to the late 1980s and early 1990s. Foundational works in the area include contributions from Bakalov and Kirillov [BK01], Lyubashenko [Lyu95], Kerler and Lyubashenko [KL01], Turaev [Tur94], Reshetikhin and Turaev [RT91], Hennings [Hen96], and Walker [Wal91]. A key focus of these early studies has been the construction of modular functors and extended 3-dimensional TQFTs using modular tensor categories.

More recent work by Bartlett, Douglas, Schommer-Pries, and Vicary [BDSPV15] offers a comprehensive overview of all this development, which the interested reader is encouraged to consult. Crucially, their work provides the primary motivation for this thesis, establishing a classification of extended 3-dimensional TQFTs in terms of modular tensor categories, assuming a generators-and-relations presentation of

¹One usually defines topological modular functors to be functors out of a certain central extension of $\text{Bord}_{12\sim}^{\text{or}}$, but we will not include any further details about this here.

$\text{Bord}_{123}^{\text{or}}$. Rigorously establishing the presentation used in [BDSPV15] has been a guiding principle throughout this project.

At present, four notable examples of bordism bicategories have explicit, useful generators-and-relations presentations:

1. Baez and Langford [BL03] provided a presentation of the 2-dimensional tangle bicategory, comprising points in $\mathbb{R}^2$, braids in $\mathbb{R}^3$, and knotted surfaces in $\mathbb{R}^4$. Their construction builds on the combinatorial description of knotted surfaces developed by Carter, Rieger, and Saito [CRS97], which itself relies on Morse–Cerf theory.
2. Schommer-Pries [SP14] established presentations of the oriented and unoriented 2-dimensional bordism bicategories $\text{Bord}_{012}^{(\text{or})}$, which he used to classify extended 2-dimensional (oriented) TQFTs.²
3. Pstragowski [Pst14] extended the work of Schommer-Pries by constructing a presentation of the framed 2-dimensional bordism bicategory $\text{Bord}_{012}^{\text{fr}}$ and using it to prove (a version of) the cobordism hypothesis in dimension 2.

I.2.2 Presentation of $\text{Bord}_{012}^{(\text{or})}$

Schommer-Pries’ techniques have profoundly influenced subsequent work and form a key conceptual foundation for this thesis. We briefly review the core ideas here, slightly tailoring the exposition to align with the approach adopted in our work.

Equipping bordisms with planar diagrams

The starting point is 2-Morse theory, which plays a central role by enabling the decomposition of 2-dimensional bordisms with corners into elementary pieces in two directions. A 2-Morse function on a smooth manifold M is, roughly speaking, a generic map $f : M \rightarrow \mathbb{R}^2$ that has only fold and cusp singularities, and such that its post-composition with the projection $p_x : \mathbb{R}^2 \rightarrow \mathbb{R}$ onto the x -axis is a Morse function.³ We can construct a bicategory $\text{Bord}_{012}^{2\text{-Morse}, (\text{or})}$ whose objects are (oriented) 0-dimensional manifolds, 1-morphisms are (oriented) 1-dimensional bordisms equipped with Morse functions, and 2-morphisms are equivalence classes

²We use the notation $X^{(\text{or})}$ when we wish to simultaneously refer to the oriented and unoriented version of X .

³The Morse critical points of $p_x \circ f$ are called 2-Morse critical points of f .

of (oriented) 2-dimensional bordisms equipped with 2-Morse functions. One can then show that the forgetful functor

$$\mathrm{Bord}_{012}^{2\text{-Morse}, (or)} \rightarrow \mathrm{Bord}_{012}^{(or)}$$

is an equivalence of bicategories.

An important object associated with a Morse or 2-Morse function is its *singularity graphic*—that is, the collection of all its critical values in the real line or real plane, respectively. An example of a singularity graphic for a 2-Morse function is shown in Figure I.2. While the singularity graphic of a 2-Morse function encodes significant information, it is generally insufficient on its own to recover the original bordism up to equivalence. However, for 2-dimensional bordisms, the situation simplifies: since a 2-Morse function on a 2-dimensional manifold has 0-dimensional fibers, the additional data needed to reconstruct the bordism from the singularity graphic are purely combinatorial. Building on this observation, Schommer-Pries constructs a 2-category $\mathrm{Bord}_{012}^{\mathrm{PD}, (or)}$ in which:

- objects are (oriented) 0-dimensional manifolds,
- 1-morphisms are (oriented) 1-dimensional bordisms equipped with Morse functions and associated Morse singularity graphics decorated with combinatorial data, that we refer to as *linear diagrams*,
- 2-morphisms are equivalence classes of (oriented) 2-dimensional bordisms equipped with 2-Morse functions and associated 2-Morse singularity graphics decorated with combinatorial data, that we refer to as *planar diagrams*.

He then shows that the forgetful functor

$$\mathrm{Bord}_{012}^{\mathrm{PD}, (or)} \rightarrow \mathrm{Bord}_{012}^{2\text{-Morse}, (or)}$$

is an equivalence of bicategories.⁴

⁴Schommer-Pries actually only defines $\mathrm{Bord}_{012}^{\mathrm{PD}, (or)}$ and shows that the forgetful functor $\mathrm{Bord}_{012}^{\mathrm{PD}, (or)} \rightarrow \mathrm{Bord}_{012}^{(or)}$ is an equivalence. We break this into two steps here to better align with the structure of the exposition adopted later in this thesis.

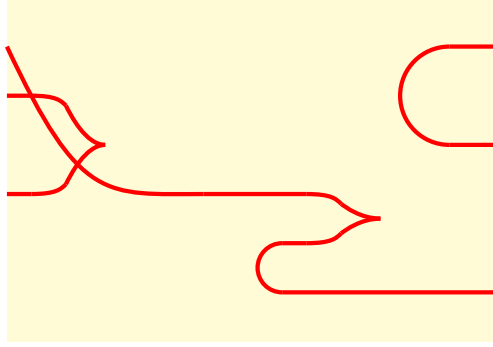


Figure I.2: Example of a 2-Morse singularity graphic.

From planar diagrams to the surgery presentation

The previous section establishes a chain of equivalences

$$\mathrm{Bord}_{012}^{\mathrm{PD}, (\mathrm{or})} \xrightarrow{\sim} \mathrm{Bord}_{012}^{2\text{-Morse}, (\mathrm{or})} \xrightarrow{\sim} \mathrm{Bord}_{012}^{(\mathrm{or})}$$

reducing the problem of constructing a presentation of $\mathrm{Bord}_{012}^{(\mathrm{or})}$ to that of finding a presentation of $\mathrm{Bord}_{012}^{\mathrm{PD}, (\mathrm{or})}$, a bicategory that has more structure we may hope to exploit. We can reduce the problem even further by leveraging the fact that planar diagrams contain enough information to reconstruct the original bordism up to equivalence. This leads to the 2-category $\mathrm{PD}^{(\mathrm{or})}$ whose objects are (oriented) 0-dimensional manifolds, 1-morphisms are linear diagrams and 2-morphisms are equivalence classes of planar diagrams. The forgetful functor

$$\mathrm{PD}^{(\mathrm{or})} \xleftarrow{\sim} \mathrm{Bord}_{012}^{\mathrm{PD}, (\mathrm{or})}$$

is an equivalence and we can explicitly construct an inverse $\mathrm{PD}^{(\mathrm{or})} \rightarrow \mathrm{Bord}_{012}^{\mathrm{PD}, (\mathrm{or})}$ leading to the chain of equivalences

$$\mathrm{PD}^{(\mathrm{or})} \xleftarrow{\sim} \mathrm{Bord}_{012}^{\mathrm{PD}, (\mathrm{or})} \xrightarrow{\sim} \mathrm{Bord}_{012}^{2\text{-Morse}, (\mathrm{or})} \xrightarrow{\sim} \mathrm{Bord}_{012}^{(\mathrm{or})}.$$

This significantly simplifies the problem of constructing a presentation of $\mathrm{Bord}_{012}^{(\mathrm{or})}$: whereas the bordism bicategory originally involves data in dimensions 0, 1, and 2, the 2-category $\mathrm{PD}^{(\mathrm{or})}$ relies entirely on combinatorial (i.e., 0-dimensional) data and the task now reduces to finding a presentation of $\mathrm{PD}^{(\mathrm{or})}$. To proceed, however, we must first establish a precise notion of what it means to present a symmetric monoidal bicategory via generators and relations—a nontrivial task, given the substantial work required to define symmetric monoidal bicategories in the first place.

To this end, Schommer-Pries introduced a stricter version of a symmetric monoidal bicategory, called an *unbiased semistrict symmetric monoidal 2-category*, and showed that every symmetric monoidal bicategory is equivalent to one. These categories feature many trivial coherence 2-cells and admit a graphical calculus that closely parallels the diagrams in $\text{PD}^{(\text{or})}$. Moreover, their structure allows for an easy-to-state definition of a presentation of an unbiased semistrict symmetric monoidal 2-category. Such a presentation $\mathcal{P}$ consists of generating data $(G_0, G_1, G_2; R)$, where G_i denotes the generating i -cells and R denotes the generating relations.⁵ These data generate an unbiased semistrict symmetric monoidal 2-category denoted by $F\mathcal{P}$. We refer to such presentations as *unbiased semistrict symmetric monoidal 2-presentations* or simply as *2-presentations*, since they are the only kinds of presentations of 2-categories considered in this thesis.⁶

Exploiting the resemblance between the diagrams in $\text{PD}^{(\text{or})}$ and the graphical calculus for unbiased semistrict symmetric monoidal 2-categories, Schommer-Pries shows that $\text{PD}^{(\text{or})}$ itself has the structure of such a 2-category. With a precise definition of presentation for these 2-categories in hand, we now specify one for $\text{PD}^{(\text{or})}$, the 2-presentation of (oriented) surgery diagrams denoted $\text{Sur}_{012}^{(\text{or})}$.⁷ Since the morphisms in $\text{PD}^{(\text{or})}$ are decorated singularity graphics, the generators of its presentation must be decorated *building blocks* of such graphics. These building blocks arise from the singularity graphics of *simple* Morse and 2-Morse functions:⁸

Definition I.2.1.

- (a) A Morse function is *simple* if it has at most one critical point.
- (b) A 2-Morse function is *simple* if it has at most one of the following:
 - a cusp critical point;
 - a pair of Morse critical points with the same critical value;
 - a 2-Morse critical point.

We call the resulting singularity graphics *simple Morse* and *simple 2-Morse* graphics, respectively; the latter are illustrated in Figure I.3 below.

⁵Throughout this thesis, we use the terms object, 1-morphism, and 2-morphism interchangeably with 0-cell, 1-cell, and 2-cell, respectively.

⁶These presentations are termed unbiased semistrict symmetric monoidal 3-computads in [SP14].

⁷This intermediate presentation does not appear explicitly in [SP14]; it is introduced here because it aligns more closely with the exposition followed in this thesis.

⁸The following definitions of “simplicity” are introduced here solely for exposition and will not be used later in the thesis.

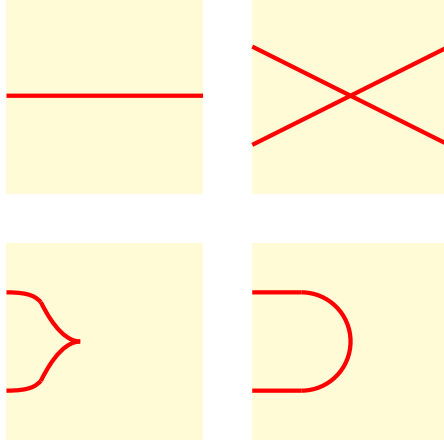


Figure I.3: The simple 2-Morse graphics.

The 2-presentation $\text{Sur}_{012}^{(\text{or})}$ is given by:

- **Generating 0-cells:** connected (oriented) 0-manifolds;
- **Generating 1-cells:** simple Morse graphics decorated with combinatorial data;
- **Generating 2-cells:** simple 2-Morse graphics decorated with combinatorial data;
- **Generating relations:** local moves between decorated 2-Morse singularity graphics (see Figure I.4 for examples of the corresponding moves between undecorated graphics).

Henceforth we work with $F\text{Sur}_{012}^{(\text{or})}$ and omit further mention of $\text{PD}^{(\text{or})}$, as the two 2-categories are equivalent and $F\text{Sur}_{012}^{(\text{or})}$ is more naturally expressed in terms of generators and relations.

From the surgery presentation to a finite presentation

While $\text{Sur}_{012}^{(\text{or})}$ is a useful presentation, it can be further simplified. In the language introduced later in this thesis, $\text{Sur}_{012}^{(\text{or})}$ is not a *skeletal 2-presentation*; that is, it includes generating 1-cells that become equivalent in $F\text{Sur}_{012}^{(\text{or})}$ via elementary operations valid in any unbiased semistrict symmetric monoidal 2-category.⁹ However, it is straightforward to construct a skeletal 2-presentation $\mathcal{P}^{(\text{or})}$ equivalent to $\text{Sur}_{012}^{(\text{or})}$,

⁹Two 1-morphisms in a 2-category $\mathcal{C}$ are *equivalent* if they are connected by an *invertible* 2-morphism in $\mathcal{C}$.

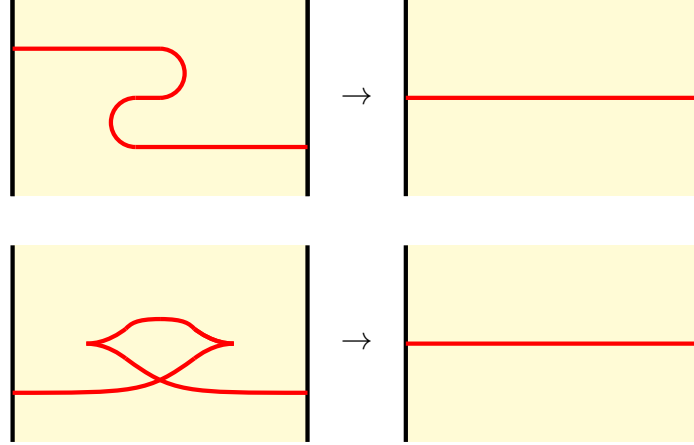


Figure I.4: Examples of moves between 2-Morse singularity graphics: 2-Morse cancellation and swallowtail.

which, in fact, turns out to be finite.¹⁰ For reference, the generating cells of $\mathcal{P}$ are shown in Figure I.5. The following zigzag of equivalences

$$F\mathcal{P}^{(\text{or})} \xrightarrow{\sim} F\text{Sur}_{012}^{(\text{or})} \xleftarrow{\sim} \text{Bord}_{012}^{\text{PD}, (\text{or})} \xrightarrow{\sim} \text{Bord}_{012}^{2\text{-Morse}, (\text{or})} \xrightarrow{\sim} \text{Bord}_{012}^{(\text{or})}.$$

therefore provides a finite generators-and-relations presentation of $\text{Bord}_{012}^{(\text{or})}$.

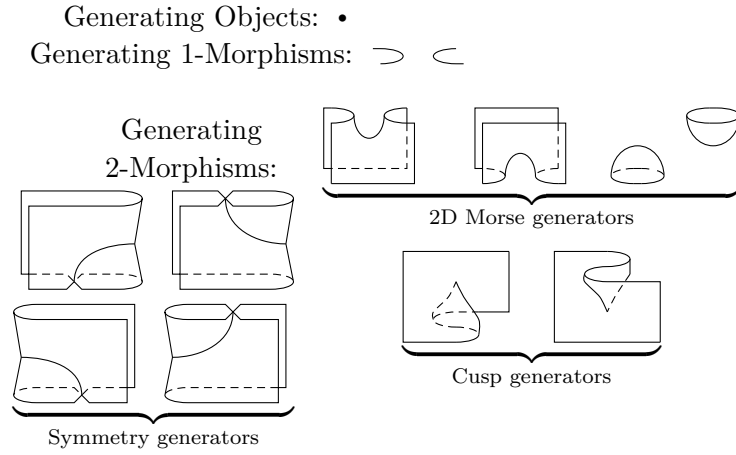


Figure I.5: Generating cells for the presentation of Bord_{012} constructed by Schommer-Pries (from [SP14]).

¹⁰Two presentations are equivalent if they generate equivalent categories.

I.2.3 Presentation of $\text{Bord}_{(d-1; d)}^{(\text{or})}$

The preceding framework extends to other bordism bicategories and underlies the approach of this thesis, but the details become subtler in higher dimensions. When $d > 2$, a d -dimensional bordism with corners can still be recovered—up to equivalence—from a 2-Morse singularity graphic decorated with additional data; however, the extra data required are no longer purely combinatorial. Instead, they originate from surgery and determining their precise form is not immediately straightforward.

Additional insight comes from Juhász’ work [Juh18], which is somewhat orthogonal to that of Schommer-Pries: it provides a presentation of the (oriented) d -dimensional bordism *1-category* $\text{Bord}_{(d-1; d)}^{(\text{or})}$ for *every dimension* d .¹¹ Chapter 2 examines Juhász’ construction and proof in detail; for now, we outline the main ideas, tailoring the exposition to align with the approach adopted in our work.

Equipping bordisms with Morse data

The first step towards obtaining a presentation of $\text{Bord}_{(d-1; d)}^{(\text{or})}$ is to use Morse functions to decompose d -dimensional bordisms into elementary pieces. This leads to the definition of the category $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$, whose objects are closed, (oriented) $(d-1)$ -manifolds and whose morphisms are equivalence classes of (oriented) d -dimensional bordisms equipped with Morse functions. With a suitably chosen equivalence relation, the forgetful functor

$$\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})} \xrightarrow{F^{\text{Morse}}} \text{Bord}_{(d-1; d)}^{(\text{or})}$$

becomes an equivalence—precisely the 1-categorical counterpart of the above established equivalence

$$\text{Bord}_{012}^{2\text{-Morse}, (\text{or})} \rightarrow \text{Bord}_{012}^{(\text{or})}.$$

A Morse function determines a singularity graphic, but, as in the 2-categorical case, the graphic alone does not suffice to reconstruct the bordism up to equivalence. To remedy this, the singularity graphic must be decorated with surgery data: the critical points are labeled with embeddings of framed attaching spheres that encode the corresponding handle attachments. Furthermore, if the Morse function has no critical points, the bordism is a mapping cylinder, and although its singularity graphic is empty, it must still be decorated with the data of the underlying diffeomorphism.

¹¹For specific values of d , we will abbreviate the notation for the corresponding bordism category. For instance, $\text{Bord}_{(1; 2)}^{(\text{or})}$ will be written simply as $\text{Bord}_{12}^{(\text{or})}$. Similar abbreviations will be used for bordism bicategories and surgery presentations without further comment.

These extra surgery data play exactly the same role here that the combinatorial decorations played for 2-Morse graphics in $\text{PD}^{(\text{or})}$.

To obtain the necessary decorations on the singularity graphic associated with a given Morse function f , we first need to choose appropriate *Morse data* for f . This choice includes a Morse coordinate chart around each critical point, together with a compatible gradient-like vector field. They lead to the definition of the category $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$, whose objects are closed (oriented) $(d-1)$ -manifolds and whose morphisms are equivalence classes of (oriented) d -dimensional cobordisms equipped with both a Morse function and associated Morse data. With a suitably chosen equivalence relation, the forgetful functor

$$\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}} \xrightarrow{F^{\text{data}}} \text{Bord}_{(d-1; d)}^{\text{Morse, (or)}}$$

becomes an equivalence—the 1-categorical counterpart of the above established equivalence

$$\text{Bord}_{012}^{\text{PD, (or)}} \rightarrow \text{Bord}_{012}^{2\text{-Morse, (or)}}.$$

From bordisms with Morse data to the surgery presentation

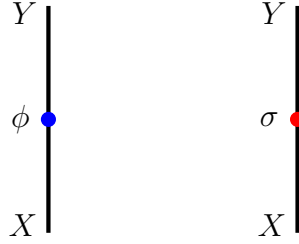
The previous section established a chain of equivalences

$$\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}} \xrightarrow{F^{\text{data}}} \text{Bord}_{(d-1; d)}^{\text{Morse, (or)}} \xrightarrow{F^{\text{Morse}}} \text{Bord}_{(d-1; d)}^{(\text{or})}$$

reducing the problem of constructing a presentation of $\text{Bord}_{(d-1; d)}^{(\text{or})}$ to that of finding a presentation of $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$. A Morse function equipped with Morse data determines a Morse graphic decorated with diffeomorphism and surgery data. This leads to the definition of the presentation $\text{Sur}_{(d-1; d)}^{(\text{or})}$, specified as follows:

- **Generating objects:** closed, connected (oriented) $(d-1)$ -manifolds;
- **Generating morphisms:** simple Morse graphics decorated with diffeomorphism and surgery data;
- **Generating relations:** local moves between decorated Morse graphics, arising from Cerf theory.

The generating morphisms in $\text{Sur}_{(d-1; d)}^{(\text{or})}$ fall into two types, represented by the following diagrams:



Here, the blue dot labeled ϕ corresponds to a diffeomorphism from X to Y , and the red dot labeled σ represents surgery data associated to a handle attachment.

Passing between Morse data and decorated singularity graphics yields functors

$$F \operatorname{Sur}_{(d-1; d)}^{(\text{or})} \xrightleftharpoons[B]{S} \operatorname{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$$

that are inverse equivalences of categories. Combined with the previously established equivalences, this gives a chain

$$F \operatorname{Sur}_{(d-1; d)}^{(\text{or})} \xrightleftharpoons[B]{S} \operatorname{Bord}_{(d-1; d)}^{\text{Morse data, (or)}} \xrightarrow{F^{\text{data}}} \operatorname{Bord}_{(d-1; d)}^{\text{Morse, (or)}} \xrightarrow{F^{\text{Morse}}} \operatorname{Bord}_{(d-1; d)}^{(\text{or})}$$

realizing $\operatorname{Sur}_{(d-1; d)}^{(\text{or})}$ as a presentation of $\operatorname{Bord}_{(d-1; d)}^{(\text{or})}$. This construction serves as the 1-categorical analogue of the chain of equivalences

$$F \operatorname{Sur}_{012}^{(\text{or})} \xrightleftharpoons{\quad} \operatorname{Bord}_{012}^{\text{PD, (or)}} \xrightarrow{\sim} \operatorname{Bord}_{012}^{2\text{-Morse, (or)}} \xrightarrow{\sim} \operatorname{Bord}_{012}^{(\text{or})}.$$

which realizes $\operatorname{Sur}_{012}^{(\text{or})}$ as a presentation of $\operatorname{Bord}_{012}^{(\text{or})}$.

From the surgery presentation to a finite presentation

In parallel with the bicategorical framework described earlier, the equivalence between $F \operatorname{Sur}_{(d-1; d)}^{(\text{or})}$ and $\operatorname{Bord}_{(d-1; d)}^{(\text{or})}$ enables a significant simplification: while the definition of $\operatorname{Bord}_{(d-1; d)}^{(\text{or})}$ involves d -dimensional data, the presentation $\operatorname{Sur}_{(d-1; d)}^{(\text{or})}$ is given entirely in terms of $(d-1)$ -dimensional data. Nevertheless, we may hope to simplify further. The presentation $\operatorname{Sur}_{(d-1; d)}^{(\text{or})}$ is not skeletal, and in many cases, passing to a skeletal presentation offers conceptual and computational advantages. Here, we define a presentation $\mathcal{P}$ of a symmetric monoidal category to be skeletal if (1) no two distinct generating objects are connected by an invertible morphism in $F\mathcal{P}$, and (2) no two generating morphisms are equivalent up to composition with a symmetric group action.

Juhász constructs a skeletal presentation equivalent to $\operatorname{Sur}_{12}^{\text{or}}$, recovering the well-known presentation of $\operatorname{Bord}_{12}^{\text{or}}$ and hence the classification of oriented 2-dimensional

TQFTs in terms of symmetric Frobenius algebras. Moreover, he constructs a skeletal presentation equivalent to $\text{Sur}_{23}^{\text{or}}$, yielding the first classification of (non-extended) 3-dimensional TQFTs in terms of a new algebraic structure he calls a J -algebra. We will not pursue these constructions further in this thesis.

I.2.4 Bordism categories and pseudo-isotopy

Juhász’s work builds heavily on Cerf’s foundational techniques developed in [Cer73]. For example, a central idea in [Juh18] is the equivalence relation imposed on the morphisms of $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$, which identifies two Morse functions on the same bordism if there exists a *Cerf path* between them—a path of smooth functions with only Morse and cusp singularities. The existence of a Cerf path between any two Morse functions, which follows from the genericity of Cerf paths, ensures that this equivalence relation is sufficient to make the forgetful functor

$$F^{\text{Morse}} : \text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})} \rightarrow \text{Bord}_{(d-1; d)}^{(\text{or})}$$

an equivalence of categories.

Cerf’s methods are central to this thesis and will be reviewed in detail in Chapter 1. Their relevance here is not incidental: the pseudo-isotopy problem that Cerf addressed in [Cer73] is deeply connected to bordism bicategories and their surgery presentations. A pseudo-isotopy may be viewed as a non-level-preserving isotopy—specifically, a diffeomorphism $M \times [0, 1] \rightarrow M \times [0, 1]$ that restricts to the identity on $M \times 0$. In this sense, pseudo-isotopies represent 2-cells in $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$ between 1-cells corresponding to mapping cylinders. Cerf studied conditions under which every pseudo-isotopy on a manifold M is isotopic to an actual isotopy.

To formalize this, consider the bordism bicategory $\text{Man}_{(d-1; \sim_2)}^{(\text{or})}$ whose objects are closed (oriented) $(d - 1)$ -dimensional manifolds, 1-morphisms are (orientation-preserving) diffeomorphisms, and 2-morphisms are isotopy classes of isotopies. The inclusion functor

$$\text{Man}_{(d-1; \sim_2)}^{(\text{or})} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$$

raises the pseudo-isotopy problem, which essentially asks when this functor is full on 2-cells. Cerf’s approach involves assigning geometric data—specifically, Cerf graphics enhanced with surgery data—to isotopy classes of pseudo-isotopies, and then analyzing when this data can be simplified. If the associated Cerf graphic can be simplified so that it contains no critical values, then the pseudo-isotopy is isotopic to an isotopy. Although Cerf did not phrase his results in the language of bordism

bicategories, much of his work can be interpreted as laying the foundations for the study of their surgery presentations.

I.3 Results and methods

The first part comprises Chapters 1–3 and pursues two main objectives:

1. To define the 2-presentation $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$, which serves both as a higher-dimensional analogue of the invertible part of $\text{Sur}_{012}^{(\text{or})}$ and as a categorification of $\text{Sur}_{(d-1; d)}^{(\text{or})}$.
2. To establish the following equivalence:

Theorem I.3.1. *The symmetric monoidal bicategories $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$ are equivalent.*

The construction of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ and the comparison with $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$ builds on the work of Schommer-Pries, Juhász, and Cerf. Chapter 1 introduces the relevant background from Morse–Cerf theory, with particular emphasis on *elementary birth-death paths*, which play a central role later. Chapter 2 presents Juhász’ result establishing the equivalence between $F\text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{(\text{or})}$.

Chapter 3 then proves Theorem I.3.1. In Section 3.3, we define the 2-presentation of planar surgery diagrams $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ as follows:

- **Generating 0-cells:** closed, connected (oriented) $(d - 1)$ -manifolds;
- **Generating 1-cells:** simple Morse graphics decorated with diffeomorphism and surgery data;
- **Generating 2-cells:** *simple Cerf graphics* decorated with diffeomorphism and surgery data;¹²
- **Generating relations:** local moves between decorated Cerf graphics, arising from the study of generic homotopies between Cerf paths.

The equivalence between $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$ and $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ is established via a two-step process, mirroring the framework applied earlier to obtain surgery presentations of $\text{Bord}_{012}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{(\text{or})}$.

¹²Simple Cerf graphics are the singularity graphics of *simple Cerf paths*, i.e., Cerf paths with at most one of the following: a single cusp critical point or a single pair of Morse critical points with the same critical value.

Equipping bordisms with Cerf data

The first step is to construct the $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$ of closed (oriented) $(d - 1)$ -dimensional manifolds, (oriented) d -dimensional bordisms equipped with Morse functions, and equivalence classes of (oriented) mapping cylinders equipped with Cerf paths, where the equivalence relation includes Cerf homotopies. We show that the forgetful functor

$$F^{\text{Cerf}} : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$$

is an equivalence. This is the subject of Section 3.4.

The second step is to enhance the framework with Cerf data. This includes Morse data associated to 1-cells, as well as Cerf data, which are their 1-parameter analogues, associated to 2-cells. This enhancement allows us to construct the $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})}$. A key point here is that not all Cerf paths admit Cerf data—only the so-called *elementary* Cerf paths do. These elementary Cerf paths and their associated data are carefully defined in Chapter 1. Based on Cerf’s study of relative homotopy between general and elementary Cerf paths, we deduce that the forgetful functor

$$F^{\text{data}} : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$$

is also an equivalence. This is discussed in Section 3.5.

From bordisms with Cerf data to the surgery presentation

Equivalence classes of bordisms equipped with Cerf data correspond bijectively to equivalence classes of Cerf graphics decorated with diffeomorphism and surgery data. This correspondence gives rise to functors

$$F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})} \xrightleftharpoons[B]{S} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})}$$

which we show to be equivalences.¹³ This result is established in Section 3.6.

Accordingly, Part I establishes the following chain of equivalences:

$$F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})} \xrightleftharpoons[B]{S} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})} \xrightarrow{F^{\text{data}}} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})} \xrightarrow{F^{\text{Cerf}}} \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}.$$

¹³We do not show that S is a *symmetric monoidal* functor, although we suspect this to be the case. Hence, we only establish that B is a symmetric monoidal equivalence and that B and S are inverses as “plain” functors.

This realizes $\text{Sur}_{(d-1; d)}^{(\text{or})}$ as a presentation of $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$ and serves as the categorification of the chain

$$F \text{Sur}_{(d-1; d)}^{(\text{or})} \xrightleftharpoons[B]{S} \text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}} \xrightarrow{F^{\text{data}}} \text{Bord}_{(d-1; d)}^{\text{Morse, (or)}} \xrightarrow{F^{\text{Morse}}} \text{Bord}_{(d-1; d)}^{(\text{or})},$$

that realizes $\text{Sur}_{(d-1; d)}^{(\text{or})}$ as a presentation of $\text{Bord}_{(d-1; d)}^{(\text{or})}$.

I.3.1 Part II: From surgery presentations to finite presentations

The second part comprises Chapters 4–6 and pursues two main objectives:

1. To define the notion of a skeletal presentation for an unbiased semistrict symmetric monoidal 2-category, and to establish a general skeletalization procedure applicable to the presentations considered in this thesis.
2. To establish the following theorem via this skeletalization procedure:

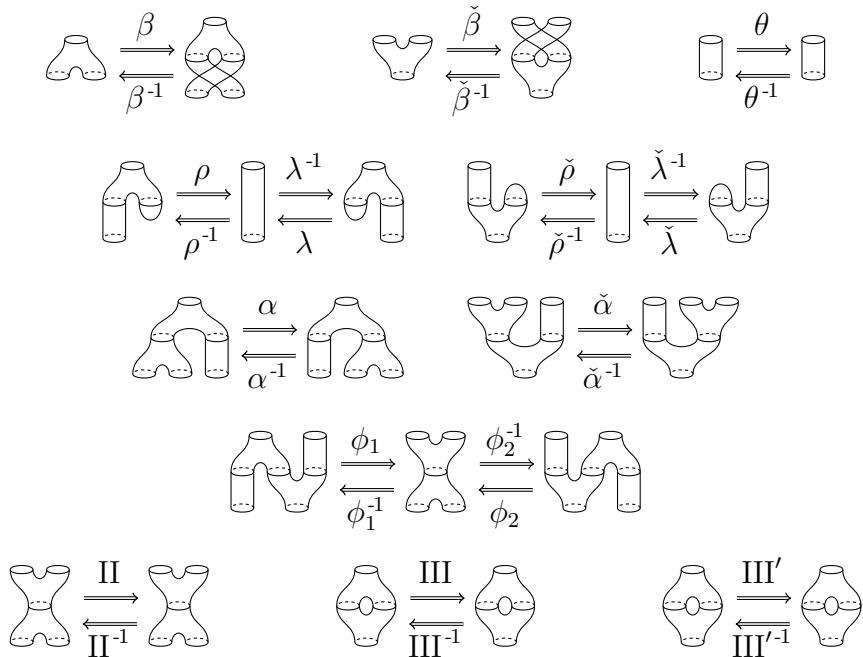
Theorem I.3.2. *$\text{Sur}_{12\sim}^{\text{or}}$ is equivalent to a finite skeletal 2-presentation, which consists of*

(a) *A single generating 0-cell:* 

(b) *Four generating 1-cells:*



(c) *Twenty-eight generating 2-cells:*



(d) *Sixty-six generating relations.*

In Chapter 4, we introduce the concept of *skeletal presentations of unbiased semistrict symmetric monoidal 2-categories*. Being skeletal is a condition that, for example, ensures the presentation does not contain distinct generating 1-cells connected by an invertible 2-cell. We also introduce a general skeletalization procedure for such presentations. While this procedure is non-canonical and relies on several choices, it is broadly applicable and well-suited to practical computations. To illustrate its utility, we apply it to skeletalize $\text{Sur}_{01\sim}^{(\text{or})}$ in Chapter 5, recovering the invertible part of the presentation of $\text{Bord}_{012\sim}^{(\text{or})}$ given in [SP14]. Finally, in Chapter 6, we apply the skeletalization procedure to $\text{Sur}_{12\sim}^{\text{or}}$, which leads to a proof of Theorem I.3.2, thereby establishing a finite presentation of $\text{Bord}_{12\sim}^{\text{or}}$.

Remark I.3.3. Each generating cell in the skeletal presentation of Theorem I.3.2 corresponds to a generating cell of $\text{Sur}_{12\sim}^{(\text{or})}$, and likewise, each generating relation in the skeletal presentation arises from a corresponding relation in $\text{Sur}_{12\sim}^{(\text{or})}$. This correspondence is made explicit—albeit somewhat tediously—via the skeletalization procedure described in Chapter 4.

We identify all 66 generating relations in $\text{Sur}_{12\sim}^{(\text{or})}$ that contribute corresponding relations in the skeletal presentation and demonstrate that they are sufficient according to the skeletalization procedure, thereby establishing Theorem I.3.2. Of these, the explicit skeletalized forms have been determined in all but 14 cases. For the remaining 14, we have identified conjectural skeletalized forms, but at the time of submitting this thesis, their validity has not yet been rigorously confirmed. We intend to complete this verification shortly via a detailed application of the skeletalization procedure. The resulting relations together with the generating cells given in the statement of Theorem I.3.2 constitute the $\mathcal{F}^2$ *presentation*, found in Appendix A. Once the final 14 relations are verified, we will have completed the proof that the $\mathcal{F}^2$ presentation is equivalent to $\text{Sur}_{12\sim}^{(\text{or})}$, yielding the chain of equivalences

$$F\mathcal{F}^2 \xrightarrow{\sim} F\text{Sur}_{12\sim}^{(\text{or})} \xleftarrow[B]{S} \text{Bord}_{12\sim}^{\text{Cerf data, (or)}} \xrightarrow{F^{\text{data}}} \text{Bord}_{12\sim}^{\text{Cerf, (or)}} \xrightarrow{F^{\text{Cerf}}} \text{Bord}_{12\sim}^{(\text{or})}$$

and thereby realizing the $\mathcal{F}^2$ presentation as an explicit finite presentation of $\text{Bord}_{12\sim}^{(\text{or})}$.

Remark I.3.4. We briefly comment on the structure of the exposition. Both parts of the thesis follow a similar organizational pattern: each begins with a chapter introducing the necessary tools, then proceeds with a chapter that applies them to a known result, and finally ends with a chapter that uses them to prove the main theorem of that part.

I.4 Future directions

There are two immediate avenues for future investigation stemming from the work presented in this thesis.

The first is to define the presentation $\text{Sur}_{(d-1; d; d+1)}^{(\text{or})}$ and show that it generates a bicategory equivalent to $\text{Bord}_{(d-1; d; d+1)}^{(\text{or})}$. Skeletalizing $\text{Sur}_{123}^{\text{or}}$ should in turn recover the presentation of $\text{Bord}_{123}^{\text{or}}$ used in [BDSPV15]. The techniques developed here are expected to extend naturally to this case. Building on [SP14], we already know how to define the bicategory $\text{Bord}_{(d-1; d; d+1)}^{2\text{-Morse}, (\text{or})}$, whose 2-morphisms are equivalence classes of $(d+1)$ -dimensional bordisms equipped with 2-Morse functions. However, the definition of $\text{Bord}_{(d-1; d; d+1)}^{2\text{-Morse data}, (\text{or})}$, which would serve as the analogue of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})}$, remains incomplete, although partial progress has been made.

The second avenue involves applying the skeletalization procedure to $\text{Sur}_{23\sim}^{\text{or}}$ and $\text{Sur}_{234}^{\text{or}}$ (noting that the former can be pursued independently of the first avenue above). This is expected to yield manageable presentations of $\text{Bord}_{23\sim}^{\text{or}}$ and $\text{Bord}_{234}^{\text{or}}$, respectively. In particular, the former would provide a categorification of Juhász’s classification of non-extended 3-dimensional TQFTs in terms of J -algebras. While this appears feasible in principle, the details may prove to be quite intricate.

Part I

From bordisms to surgery diagrams

Chapter 1

Morse and Cerf theory

The aim of this chapter is to introduce the necessary geometric tools that will be used to obtain presentations of bordism (bi)categories in terms of surgery diagrams. In particular, we review essential facts regarding Morse functions, (elementary) Cerf paths, and Cerf homotopies. Cerf developed these tools in his work on the pseudo-isotopy problem [Cer73], which we will closely follow.

Informally, a Morse function is a “generic map” from a smooth manifold M to $\mathbb{R}$. According to the Morse lemma, it admits only isolated singularities with specific local behavior. Analogously, a Cerf path (respectively, homotopy) is a generic path (respectively, homotopy) of smooth real-valued functions on M . Cerf paths are paths of functions that are Morse at all but finitely many times. At those times, the functions may have singularities that are not of Morse type but still exhibit specific local behavior. Similarly, Cerf homotopies are homotopies of paths of smooth functions that consist of Cerf paths at all but finitely many times, where the failure of a path to be a Cerf path is tightly constrained. To make all of this precise, we will first introduce certain concepts from singularity theory.

For any two smooth manifolds M and N , we denote by $C^\infty(M, N)$ the set of all smooth maps from M to N . Likewise, we will denote by $C^r(M, N)$ the set of all maps from M to N , which are differentiable up to order r .

Definition 1.0.1. Let $f \in C^\infty(M, N)$ be a smooth map:

- (a) f is an **immersion** at the point $p \in M$ if the differential $df_p : T_p M \rightarrow T_{f(p)} N$ is injective. This presupposes that $\dim(M) \leq \dim(N)$.
- (b) f is a **submersion** at the point $p \in M$ if the differential df_p is surjective. This presupposes that $\dim(M) \geq \dim(N)$.

- (c) A point $p \in M$ is a **critical point** of $f \in C^\infty(M, N)$ if f is neither an immersion nor a submersion at p . Equivalently, p is a critical point of f if the differential df_p does not have full rank.

Definition 1.0.2. Given a compact, smooth manifold K , a map $\gamma : K \rightarrow C^\infty(M, N)$ is called **smooth** if the induced map $\Gamma : M \times K \rightarrow N$, given by $\Gamma(m, t) = \gamma(t)(m)$ for all $m \in M$ and $t \in K$, is smooth. We can thus identify the set of smooth maps $K \rightarrow C^\infty(M, N)$ with $C^\infty(M \times K, N)$.

A **smooth path** (respectively, **smooth homotopy**) in $C^\infty(M, N)$ is a smooth map $\gamma : K \rightarrow C^\infty(M, N)$ for $K = [0, 1]$ (respectively, $[0, 1] \times [0, 1]$).

Morse functions, Cerf paths, and Cerf homotopies are smooth functions, paths, and homotopies, respectively, in $C^\infty(M, \mathbb{R})$ with a certain concretely understood local behavior around their critical points. Moreover, they are **generic**:

Definition 1.0.3. A subspace U of a topological space X is called **generic** if it is an open and dense subset of X .

Before introducing Morse functions, Cerf paths, and Cerf homotopies in detail, we need to specify a topology on $C^\infty(M \times K, \mathbb{R})$ (for $K = *, [0, 1], [0, 1] \times [0, 1]$) with respect to which they will be generic and introduce a tool that allows us to prove genericity results. In more generality, we will specify a topology on $C^\infty(M, N)$ for any smooth manifolds M and N in the next section.

1.1 Whitney topologies

We will first review **jet bundles** following Chapter 1 of [Mic80].

Definition 1.1.1. Let M and N be two smooth manifolds. A **germ** of smooth functions between M and N based at $x \in M$ is an equivalence class of pairs (U_x, f) , where:

- U_x is a neighborhood of x in M ,
- f is a smooth map $U_x \rightarrow N$.

Two such pairs (U_x, f) and (U'_x, f') are equivalent if there exists some open set $V \subset U_x \cap U'_x$ such that $f|_V = f'|_V$.

Definition 1.1.2. A **k-jet** with source M and target N is an equivalence class of triples (x, σ, y) , where:

- $x \in M, y \in N$,
- σ is a germ of smooth functions between M and N based at $x \in M$, such that $\sigma(x) = y$.

Two such triples (x, σ, y) and (x', σ', y') are equivalent if:

- $x = x'$ and $y = y'$,
- all partial derivatives of σ and σ' up to order k at x agree.

Definition 1.1.3. We denote by $\mathbf{J}^k(\mathbf{M}, \mathbf{N})$ the set of all k -jets with source M and target N .

The set $J^k(M, N)$ admits a smooth manifold structure, and the projections

$$\begin{array}{ccc} J^k(M, N) & J^k(M, N) & J^k(M, N) \\ \downarrow \pi_M & \downarrow \pi_N & \downarrow \pi_{M \times N} \\ M & N & M \times N \end{array}$$

are fiber bundles. Moreover,

- $\pi_{M \times N} : J^1(M, N) \rightarrow M \times N$ and
- $\pi_M : J^k(M, \mathbb{R}^n) \rightarrow M$

are vector bundles. For more detailed statements and proofs of these facts, the reader can consult Chapter 1 of [Mic80].

We are now well-equipped to construct a topology on $C^\infty(M, N)$, namely the weak Whitney topology. More precisely, we will construct a topology on $C^r(M, N)$ for all r , following the exposition in [Kup19], which the reader can consult for more details.

Let us first recall the definition of the compact-open topology:

Definition 1.1.4. Let X and Y be two topological spaces. We define the compact-open topology on the space of continuous maps $\text{Maps}(X, Y)$ through the sub-basis given by the sets:

$$N(K, V) := \{f \in \text{Maps}(X, Y) \mid f(K) \subset V\},$$

where $K \subset X$ is compact and $V \subset Y$ is open.

Remark 1.1.1. The category of compactly generated, weakly Hausdorff spaces becomes an enriched category using the compact-open topology on mapping sets. Moreover, there is a tensor-hom adjunction

$$\text{Maps}(X \times Y, Z) \cong \text{Maps}(X, \text{Maps}(Y, Z)),$$

which justifies the choice of the compact-open topology on $\text{Maps}(X, Y)$ as a reasonable one.

A first attempt to define a topology on $C^r(M, N)$ could be to consider the subspace topology with respect to the evident inclusion map $C^r(M, N) \hookrightarrow \text{Maps}(M, N)$ and the compact-open topology on $\text{Maps}(M, N)$. However, this is inadequate as it does not keep track of differentiability. We will amend this issue by using jet bundles. Namely, every C^r -function $f : M \rightarrow N$ defines a section

$$\begin{array}{ccc} & J^r(M, N) & \\ j^r f \nearrow & \downarrow \pi_M & \\ & M & \end{array}$$

inducing a map (a priori of sets):

$$C^r(M, N) \xrightarrow{j^r} \Gamma(M, J^r(M, N)).$$

Here, $\Gamma(M, J^r(M, N))$ is the space of sections of π_M equipped with the subspace topology with respect to the inclusion

$$\Gamma(M, J^r(M, N)) \hookrightarrow \text{Maps}(M, J^r(M, N))$$

and the compact-open topology on $\text{Maps}(M, J^r(M, N))$. Moreover, there is a map

$$\Gamma(M, J^r(M, N)) \rightarrow C^r(M, N)$$

defined by

$$(s : M \rightarrow J^r(M, N)) \mapsto \pi_N \circ s,$$

which is a left inverse of j^r , thus showing that j^r is an inclusion.

Definition 1.1.5. The **weak Whitney C^r -topology** on $C^r(M, N)$ is the subspace topology with respect to the inclusion $j^r : C^r(M, N) \hookrightarrow \Gamma(M, J^r(M, N))$ and the above chosen topology on $\Gamma(M, J^r(M, N))$.



Figure 1.1: Two transverse and one non-transverse intersection between a circle and a line in $\mathbb{R}^2$.

Definition 1.1.6. For every $r \geq 0$, there exists an inclusion $C^\infty(M, N) \hookrightarrow C^r(M, N)$. We define the **weak Whitney C^∞ -topology** on $C^\infty(M, N)$ to be the coarsest topology with respect to which all these inclusions are continuous.

Remark 1.1.2. Another popular topology on $\text{Maps}(X, Y)$ is the fine topology defined by the sub-basis which is given by the collection:

$$N(U) := \{g \in \text{Maps}(X, Y) \mid (\text{id} \times g)(X) \subset U\},$$

where U is an open subset of $X \times Y$. Using the fine topology instead, the above construction lets us define the **strong Whitney C^r -topology** on $C^r(M, N)$. Since all mapping spaces we will consider have compact sources, the weak and strong Whitney topologies will coincide, so we can safely ignore this distinction from now on and merely write “ C^r -topology” to refer to the weak Whitney C^r -topology.

We have now specified a topology on $C^\infty(M, N)$ with respect to which we can study generic subspaces of smooth functions. To do so efficiently, it will help to have a tool to prove genericity results. This will be Thom’s jet transversality theorem, which we introduce in the next section.

1.2 Thom’s jet transversality theorem

We will first review transversality, which is a key notion in differential topology. It provides a criterion for the intersection between two submanifolds to be suitably generic. For example, the first two instances in Figure 1.1 feature a transverse intersection between a circle and a line in $\mathbb{R}^2$: perturbing the submanifolds by a small amount does not change the type of 0-manifold defined by their intersection. A useful criterion for capturing this behavior, which also constitutes the definition of transversality for submanifolds, is:

Definition 1.2.1. Let W be a smooth manifold and let $M, N \subset W$ be smooth submanifolds. We say that M and N **intersect transversally at** $w \in M \cap N$ (and

write $M \pitchfork N$ at w) if the map

$$T_w M \oplus T_w N \rightarrow T_w W.$$

induced by the respective inclusions is an isomorphism. We say that M and N **intersect transversally** (and write $M \pitchfork N$) if M intersects N transversally at all points $w \in M \cap N$.

More generally, there is a notion of transversality for maps:

Definition 1.2.2. Let W be a smooth manifold as above, let $f : M \rightarrow W$ and $g : N \rightarrow W$ be smooth maps, and let $m \in M, n \in N$ be such that $f(m) = g(n) = w$, i.e., (m, n) is an element of the pullback $M \times_W N$. We say that f intersects g transversally at (m, n) (and write $f \pitchfork g$ at (m, n)) if the map

$$df(T_m M) \oplus dg(T_n N) \rightarrow T_w W$$

induced by the respective inclusions is an isomorphism. We say that f and g intersect transversally if they intersect transversally at all $(m, n) \in M \times_W N$.

Remark 1.2.1. By taking the maps f and g in the above definition to be submanifold inclusions, we recover the definition of transversality for submanifolds. Similarly, by taking one of the maps to be a submanifold inclusion, we obtain the definition of a transversal intersection between a map and a submanifold.

One of the most important consequences of transversality is the following theorem characterizing transversal intersections:

Theorem 1.2.2. *Let W be a smooth manifold, $f : M \rightarrow W$ be a smooth map, and $N \subset W$ a smooth submanifold of W . Assume further that $f \pitchfork N$. Then, $f^{-1}(N)$ is a submanifold of M . Moreover,*

$$\text{codim}(f^{-1}(N)) = \text{codim}(N).$$

Remark 1.2.3. A useful consequence of the above theorem is the observation that if $\dim(M) < \text{codim}(N)$, then the only way f can be transverse to N is if $f^{-1}(N) = \emptyset$.

We will need to review some more preliminary definitions and facts before stating Thom's jet transversality theorem. These are standard and we will refer to [GG73] for more details and proofs. The following proposition will later help us conclude that the space of Morse functions is an open subset of $C^\infty(M, \mathbb{R})$:

Proposition 1.2.1 (Proposition 4.5 in [GG73]). Let M and N be two smooth manifolds with W a submanifold of N and consider

$$T_W = \{f \in C^\infty(M, N) \mid f \pitchfork W\}.$$

Then, T_W is an open subset of $C^\infty(M, N)$ (in the C^1 - and hence the C^∞ -topology) if W is a closed submanifold of N .

The following will later help us conclude that the space of Morse functions is a dense subset of $C^\infty(M, N)$:

Definition 1.2.3. A subspace X of a topological space Y is called **residual** if it is a countable intersection of open and dense subsets of Y .

Definition 1.2.4. A topological space Y is called a **Baire space** if every residual subset X of Y is dense.

Proposition 1.2.2 (Proposition 3.3 in [GG73]). Let M and N be two smooth manifolds. Then, $C^\infty(M, N)$ equipped with the C^∞ -topology is a Baire space.

We are now well equipped to state Thom's jet transversality theorem and an immediate corollary:

Theorem 1.2.4 (Thom's Jet Transversality Theorem (theorem 4.9 in [GG73])). *Let M and N be two smooth manifolds with W a smooth submanifold of $J^k(M, N)$ and consider*

$$T_W = \{f \in C^\infty(M, N) \mid j^k f \pitchfork W\}.$$

Then, T_W is a residual subset of $C^\infty(M, N)$.

Corollary 1.2.5. *Let $M, N, W \subset J^k(M, N)$ and T_W be as above. Then, T_W is a dense subset of $C^\infty(M, N)$. If, furthermore, W is a closed submanifold, T_W is also open.*

Proof. The first part of the corollary follows immediately from Thom's jet transversality theorem and the fact that $C^\infty(M, N)$ is a Baire space.

For the second part of the corollary, consider the maps

$$C^\infty(M, N) \xrightarrow{j^k} \Gamma(M, J^k(M, N)) \xleftarrow{i} C^\infty(M, J^k(M, N))$$

and let

$$\tilde{T}_W = \{f \in C^\infty(M, J^k(M, N)) \mid f \pitchfork W\}.$$

Then, $\tilde{T}_W$ is open if W is a closed submanifold of $J^k(M, N)$ by Proposition 1.2.1. Moreover, $T_W \cong (j^k \circ i)^{-1}(\tilde{T}_W)$, which completes the proof of the corollary. $\square$

We will, lastly, introduce a useful generalization of the above theorem, Thom's multijet transversality theorem.

Definition 1.2.5. Let X be a topological space. We will denote by X^s the s -fold Cartesian product of X by itself. We will denote by $\text{Conf}_s(X)$ the **configuration space** of s points in X . Namely,

$$\text{Conf}_s(X) = \{(x_1, \dots, x_s) \in X^s \mid x_i = x_j \Leftrightarrow i = j\}.$$

We can naturally identify $\text{Conf}_s(X)$ with an open subset of X . When M is a smooth manifold, we therefore conclude that $\text{Conf}_s(M)$ inherits a smooth manifold structure.

Definition 1.2.6. Given smooth manifolds M and N , let $\pi_M^s : J^k(M, N)^s \rightarrow M^s$ be the map induced by the first factor projection $\pi_M : J^k(M, N) \rightarrow M$. We define the **s -fold k -jet bundle of M and N** , denoted by $\mathbf{J}_s^k(\mathbf{M}, \mathbf{N})$, to be the pullback

$$\begin{array}{ccc} J_s^k(M, N) & \xhookrightarrow{\tilde{i}} & J^k(M, N)^s \\ \tilde{\pi}_M^s \downarrow & \lrcorner & \downarrow \pi_M^s \\ \text{Conf}_s(M) & \xhookrightarrow{i} & M^s \end{array}$$

In particular, $J_s^k(M, N)$ can be identified with $(\pi_M^s)^{-1}(i(\text{Conf}_s(M)))$, which is an open subset of $J^k(M, N)^s$, since $i(\text{Conf}_s(M))$ is an open subset of M^s . This lets us equip $J_s^k(M, N)$ with a smooth manifold structure.

Unwinding definitions, we see that any (smooth) section $s : M^s \rightarrow J^k(M, N)^s$ naturally induces a (smooth) section $\tilde{s} : \text{Conf}_s(M) \rightarrow J_s^k(M, N)$. We have seen that any smooth map $f : M \rightarrow N$ induces a smooth section $j^k f : M \rightarrow J^k(M, N)$, which in turn induces a smooth section

$$(j^k f)^s : M^s \rightarrow J^k(M, N)^s,$$

and hence a smooth section

$$j_s^k f : \text{Conf}_s(M) \rightarrow J_s^k(M, N).$$

We can now state Thom's multijet transversality theorem:

Theorem 1.2.6 (Thom's Multijet Transversality Theorem, theorem 4.13 in [GG73]). *Let M and N be two smooth manifolds with W a smooth submanifold of $J_s^k(M, N)$ and consider*

$$T_W = \{f \in C^\infty(M, N) \mid j_s^k f \pitchfork W\}.$$

Then, T_W is a residual, and hence dense, subset of $C^\infty(M, N)$. Moreover, if W is compact, then T_W is open.

We will now use these tools to introduce and study Morse functions.

1.3 Morse functions

1.3.1 Definition and basic properties

Thom's jet transversality theorem indicates that in order to find a generic subset of $C^\infty(M, \mathbb{R})$, we should perhaps look for a closed submanifold of $J^1(M, \mathbb{R})$. Recall that $J^1(M, \mathbb{R})$ is the total space of the vector bundle

$$\begin{array}{ccc} P^1(\mathbb{R}^m, \mathbb{R}) & \longrightarrow & J^1(M, \mathbb{R}) \\ & & \downarrow \pi_{M \times \mathbb{R}} \\ & & M \times \mathbb{R} \end{array}$$

whose fiber $P^1(\mathbb{R}^m, \mathbb{R})$ is the space of degree 1 polynomials $\mathbb{R}^m \rightarrow \mathbb{R}$ with no constant term, where $m = \dim(M)$. There is a natural stratification of $P^1(\mathbb{R}^m, \mathbb{R})$ with two strata: the open stratum which consists of all non-zero polynomials and the one-point closed stratum which consists of the zero polynomial. This induces a stratification of $J^1(M, \mathbb{R})$ with two strata:

- An open stratum S_0 whose points consist of equivalence classes of triples (x, σ, y) , where σ is a germ at x with non-trivial differential.
- A closed stratum S_1 whose points consist of equivalence classes of triples (x, σ, y) , where σ is a germ at x with trivial differential. We observe that $\text{codim}(S_1) = \dim(P^1(\mathbb{R}^m, \mathbb{R})) = \dim(M)$. Moreover, we have

$$p \text{ is a critical point of } f \in C^\infty(M, \mathbb{R}) \Leftrightarrow j^1 f(p) \in S_1.$$

We can now define non-degenerate or Morse critical points and Morse functions:

Definition 1.3.1. Let $f \in C^\infty(M, \mathbb{R})$ and let $p \in M$ be a critical point of f . We say that p is **non-degenerate**, or **Morse**, if $j^1 f \pitchfork S_1$ at p . We say that f is Morse if all of its critical points are non-degenerate, or, equivalently, if $j^1 f \pitchfork S_1$. We will denote the space of all Morse functions by $\text{Morse}(M)$.

Thom's jet transversality theorem immediately implies the following:

Theorem 1.3.1 (Genericity of Morse functions). *Let M be a smooth manifold. Then, the space of Morse functions $\text{Morse}(M)$ is an open and dense subset of $C^\infty(M, \mathbb{R})$.*

We will now collect some useful facts about Morse functions that will be later needed.

Proposition 1.3.1. The critical points of a Morse function $f \in \text{Morse}(M)$ are isolated.

Proof. The critical points of f are exactly the points in $(j^1 f)^{-1}(S_1)$. Since f is Morse, we have that $j^1 f \pitchfork S_1$ and Theorem 1.2.2 implies that $(j^1 f)^{-1}(S_1)$ is a submanifold of M whose codimension equals the dimension M . Therefore, it is a 0-dimensional submanifold of M and, hence, it consists of isolated points. $\square$

We can locally characterize Morse critical points without making any explicit reference to the jet space $J^1(M, \mathbb{R})$. To do so we will introduce the Hessian of a function $f \in C^\infty(M, \mathbb{R})$ at a critical point p .

Definition 1.3.2. Let p be a critical point of $f \in C^\infty(M, \mathbb{R})$. We define the **Hessian** of f at p

$$d^2 f_p : T_p M \times T_p M \rightarrow \mathbb{R}$$

as follows:

Given $v_1, v_2 \in T_p M$, choose $\gamma : \mathbb{R}^2 \rightarrow M$ with $\gamma(0) = p$ and $\frac{\partial \gamma}{\partial x_i}(0) = v_i \in T_p M$ for $i = 1, 2$. We now set

$$d^2 f_p(v_1, v_2) = \left. \frac{\partial^2 (f \circ \gamma)}{\partial x_1 \partial x_2} \right|_{x_1=x_2=0}.$$

Proposition 1.3.2. The Hessian is a well-defined, symmetric and bilinear. After choosing a local coordinate system around p , it takes the form $\frac{\partial^2 f}{\partial x_1 \partial x_2}(p)$.

Proof. It follows from the chain rule. For details, see Appendix A in [Cha18]. $\square$

Proposition 1.3.3 (Proposition 6.6 in [GG73]). Let $f \in C^\infty(M, \mathbb{R})$ and let $p \in M$ be a critical point of f . Then, the Hessian of f at p is non-degenerate if and only if p is a Morse critical point.

Definition 1.3.3. Let p be a Morse critical point of $f \in C^\infty(M, \mathbb{R})$. We define the **index** of p to be the index of the Hessian of f at p .

We can now use the non-degeneracy of the Hessian to obtain an explicit local coordinate description of Morse critical points:

Theorem 1.3.2 (Morse Lemma). *Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by*

$$g(x_1, \dots, x_n) = c - (x_1^2 + \dots + x_k^2) + x_{k+1}^2 + \dots + x_n^2,$$

for some constant c . Then,

- (a) g is a Morse function on $\mathbb{R}^n$ with a single critical point at 0 of index k .
- (b) Given any $f \in C^\infty(M, \mathbb{R})$ with a Morse critical point $p \in M$ of index k , there exists a neighborhood V of the origin in $\mathbb{R}^n$ and an embedding $\phi : (V, 0) \rightarrow (M, p)$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & M \\ g \downarrow & & \downarrow f \\ \mathbb{R} & \xlongequal{\quad} & \mathbb{R} \end{array}$$

We can now specialize to a slightly more restrictive class of functions:

Definition 1.3.4 (Excellent Morse functions). A Morse function $f : M \rightarrow \mathbb{R}$ is called **excellent** if any two of its critical points have distinct images. We will denote the space of all excellent Morse functions on M by $\mathcal{F}^0(M)$.

Thom's multijet transversality theorem implies the following:

Theorem 1.3.3 (Genericity of excellent Morse functions). *Let M be a smooth manifold. Then, the space of excellent Morse functions $\mathcal{F}^0(M)$ is an open and dense subset of $C^\infty(M, \mathbb{R})$.*

We have, so far, introduced Morse functions, explained why they are generic among all smooth real-valued functions and explicitly described their local behavior. Another useful property of Morse functions which we will now introduce is their stability. It will be useful for later applications to introduce stability in the context of smooth manifold triads:

Definition 1.3.5 (Definition 1.3 in [Mil63]). $(W; \partial_0 W, \partial_1 W)$ is a (compact) **smooth manifold triad** if W is a (compact) smooth n -manifold and ∂W is the disjoint union of two open and closed submanifolds $\partial_0 W$ and $\partial_1 W$.

A smooth map of smooth manifold triads

$$F : (W; \partial_0 W, \partial_1 W) \rightarrow (W'; \partial_0 W', \partial_1 W')$$

$F : W \rightarrow W'$ that restricts to smooth maps $F : \partial_0 W \rightarrow \partial_0 W'$ and $F : \partial_1 W \rightarrow \partial_1 W'$.

Example 1.3.4. A simple example of a smooth manifold triad is the interval $(I = [0, i]; \{0\}, \{i\})$, for some $i > 0$.

Definition 1.3.6. A Morse function f on a smooth manifold triad $(W; \partial_0 W, \partial_1 W)$ is a smooth map of smooth manifold triads with target $(I = [0, i]; \{0\}, \{i\})$ such that the only critical points of f are of Morse type and lie in the interior of W .

We will denote by $\mathcal{F}^0(W, I)$ the space of excellent Morse functions on a smooth manifold triad $(W; \partial_0 W, \partial_1 W)$ with target $(I = [0, i]; \{0\}, \{i\})$.

Remark 1.3.5. We will sometimes refer to a smooth manifold triad $(W; \partial_0 W, \partial_1 W)$ merely as W , when the rest of the data are either clear or unnecessary given the context.

Definition 1.3.7. A diffeomorphism D of smooth manifold triads W, W' is a diffeomorphism $D : W \rightarrow W'$ that restricts to diffeomorphisms $D : \partial_0 W \rightarrow \partial_0 W'$ and $D : \partial_1 W \rightarrow \partial_1 W'$. We will denote by $\text{Diff}(W; \partial_0 W, \partial_1 W)$ the group of diffeomorphisms of the smooth manifold triad W .

The group $\mathcal{G} := \text{Diff}(W; \partial_0 W, \partial_1 W) \times \text{Diff}(I; \{0\}, \{i\})$ acts on $\mathcal{F}^0(W, I)$ via

$$(g, g') \cdot f = g' \circ f \circ g^{-1}.$$

Given $f \in \text{Morse}(W, I)$, this action induces an orbit map

$$\begin{array}{c} \mathcal{G} \\ \downarrow (g, g') \mapsto g' \circ f \circ g^{-1} \\ \mathcal{F}^0(W, I) \end{array}$$

Theorem 1.3.6 (Stability of Morse functions, see p.205 in [Cer73]). *Given $f \in \mathcal{F}^0(W, I)$, there exists an open neighborhood U of f in $\mathcal{F}^0(W, I)$ that lies within the orbit of the above action. Moreover, the map*

$$\begin{array}{c} \mathcal{G} \\ \downarrow (g, g') \mapsto g' \circ f \circ g^{-1} \\ \text{Orb}(\mathcal{G}) \end{array}$$

is a fiber bundle.

Proof. See the Appendix in [Cer73]. □

Corollary 1.3.7. *Let $\mathcal{G}_e$ denote the identity component of the group $\mathcal{G}$ and let $\mathcal{F}^0(W, I; f)$ denote the path-connected component of f in $\mathcal{F}^0(W, I)$. Then, the map*

$$\begin{array}{c} \mathcal{G}_e \\ \downarrow (g, g') \mapsto g' \circ f \circ g^{-1} \\ \mathcal{F}^0(W, I; f) \end{array}$$

is a fiber bundle.

Proof. We will first show that the map in question is well-defined. Given $(g, g') \in \mathcal{G}_e$, there exists a path $\gamma : [0, 1] \rightarrow \mathcal{G}_e$ with $\gamma(0) = \{id\}$ and $\gamma(1) = (g, g')$. Then, $\gamma(t) \cdot f$ is a path in $\mathcal{F}^0(W, I)$ from f to $g' \circ f \circ g^{-1}$, which shows that $g' \circ f \circ g^{-1} \in \mathcal{F}^0(W, I; f)$. It now suffices to show that the map in question is surjective and the corollary will immediately follow from Theorem 1.3.6. Let $h \in \mathcal{F}^0(W, I; f)$ and let $\gamma : [0, 1] \rightarrow \mathcal{F}^0(W, I; f)$ be such that $\gamma(0) = f$ and $\gamma(1) = h$. We know by Theorem 1.3.6 that for any $k \in \mathcal{F}^0(W, I)$, there exists an open neighborhood U_k of k that lies within the orbit of $\mathcal{G}$. Consider the open cover of $\mathcal{F}^0(W, I)$ by the open sets $\{U_k\}_{k \in \mathcal{F}^0(W, I)}$. Then, $\{\gamma^{-1}(U_{\gamma(t)})\}_{t \in [0, 1]}$ is an open cover of $[0, 1]$. Let $\{\gamma^{-1}(U_{\gamma(t_i)})\}_{(t_0=0 \leq t_1 \leq \dots \leq t_{n-1} \leq t_n=1)}$ be an open subcover, which we know must exist since $[0, 1]$ is compact. Now consider the graph $G = (V, E)$, which has

- $n + 1$ vertices labeled by $\{t_0, \dots, t_n\}$,
- an edge between vertices labeled t_i and t_j exactly when $\gamma^{-1}(U_{t_i}) \cap \gamma^{-1}(U_{t_j})$ is non-empty.

The graph G must be connected, otherwise $\{\gamma^{-1}(U_{\gamma(t_i)})\}_{(t_0=0 \leq t_1 \leq \dots \leq t_{n-1} \leq t_n=1)}$ wouldn't be an open cover. Consider a path that connects vertices $t_0 = 0$ and $t_n = 1$ and let $\tilde{t}_0 = 0, \tilde{t}_1, \dots, \tilde{t}_l = 1$ be all the vertices it passes by in exactly that order. By construction, for each $0 \leq j \leq l - 1$, there exists some $D_j \in \mathcal{G}_e$ such that $D_j \cdot \gamma(\tilde{t}_j) = \gamma(\tilde{t}_{j+1})$. Let $D = D_{l-1} \circ D_{l-2} \circ \dots \circ D_0$. Then, $h = \gamma(1) = \gamma(\tilde{t}_l) = D \cdot f$, which completes the proof. $\square$

Morse functions will help us decompose manifolds into simple pieces that can be fully understood in terms of surgery. To achieve this, we need to equip a given manifold not only with a Morse function but also with some additional data. This will be the focus of the next two sections.

1.3.2 Morse data

Gradient-like vector fields

In this section, W will either denote a smooth, closed manifold or a compact smooth manifold triad $(W; \partial_0 W, \partial_1 W)$. Let's first recall the definition of (gradient) vector fields.

Definition 1.3.8 ((Gradient) vector field). A **vector field** ξ on W is a section of its tangent bundle:

$$\begin{array}{c} TW \\ \xi \nearrow \downarrow \pi \\ W \end{array}$$

Recall that the differential of a function $f \in C^\infty(W, \mathbb{R})$ is a section of the cotangent bundle of W :

$$\begin{array}{c} T^*W \\ df \nearrow \downarrow \pi^* \\ W \end{array}$$

Further recall that a Riemannian metric g on W defines an identification of the cotangent and the tangent bundle of W :

$$\begin{array}{ccc} TW & \xrightarrow{(u,w) \mapsto g_w(u,-)} & T^*W \\ & \searrow \pi \quad \swarrow \pi^* & \\ & W & \end{array}$$

Given a Riemannian metric g on W and a function $f \in C^\infty(W, \mathbb{R})$, the vector field obtained by pulling back the differential along this identification is called the **gradient vector field of f with respect to g** and is denoted by $\nabla_g f$:

$$\begin{array}{ccc} TW & \xrightarrow{(u,w) \mapsto g_w(u,-)} & T^*W \\ & \searrow \pi \quad \swarrow \pi^* & \\ & W & \\ \nabla_g f \nearrow & & \nwarrow df \end{array}$$

By the Morse lemma, if p is an index i Morse critical point of $f : W \rightarrow \mathbb{R}$, there exists a local coordinate chart (U, ϕ) on W centered at p , where f takes the form

$$f(x_1, \dots, x_n) = -x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots x_n^2.$$

Definition 1.3.9. Let f be a Morse function on W . A Riemannian metric g on W is **adapted to f** if around each critical point p of f , g agrees with the pullback of the standard Euclidean metric under a Morse coordinate chart centered at p .

Definition 1.3.10 (Gradient-like vector fields). Let f be a Morse function on W . A **gradient-like vector field for f** is a gradient vector field of f with respect to some Riemannian metric g on W adapted to f .

Remark 1.3.8. It is easy to show that there exist metrics adapted to f by a partition of units argument. Conversely, in general, there are metrics which are not adapted to f . See Chapter 6.2 in [Coh06] for an example. In particular, this implies that **not** every gradient vector field of f is a gradient-like vector field for f , though the inverse is true.

We will primarily use gradient - like vector fields through the flow lines they define. Hence, we are led to the following definition:

Definition 1.3.11 (Horizontal distribution). A horizontal distribution for a Morse function f is an equivalence class of gradient - like vector fields for f , wherein two gradient - like vector fields for f are equivalent if they induce the same flow lines.

Standard Morse models

We have already introduced a collection of standard Morse functions, namely all these functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$ that take the form

$$g(x_1, \dots, x_n) = c - (x_1^1 + \dots + x_k^2) + (x_{k+1}^2 + \dots + x_n^2).$$

Our goal now is to define a standard model for an n -dimensional cobordism equipped with a Morse function that has a single critical point of index k , which we will refer to as a **standard Morse model of index k in $\mathbb{R}^n$** and denote by $\overline{M}_k^n$. Recall the definition of a cobordism:

Given two smooth, closed $(n - 1)$ -dimensional manifolds Y_0 and Y_1 , a cobordism between them is defined by a 5-tuple $(W; \partial_0 W, \partial_1 W; \iota_0, \iota_1)$ ¹, where:

1. W is a compact n -dimensional manifold with boundary,
2. The boundary of W is decomposed as $\partial_0 W \sqcup \partial_1 W$,
3. There are boundary identifications:

$$\iota_0 : \partial_0 W \rightarrow Y_0$$

and

$$\iota_1 : \partial_1 W \rightarrow Y_1.$$

Following [Mil63], the underlying smooth manifold triad of $\overline{M}_k^n$, denoted by $(M_k^n; \partial_0 M_k^n, \partial_1 M_k^n)$ is defined as follows:

¹The first three pieces of data constitute a smooth manifold triad.

- M_k^n is the submanifold of $\mathbb{R}^n$ defined as

$$M_k^n := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid -1 \leq -|x|^2 + |y|^2 \leq 1 \text{ and } |x||y| < \sinh(1) \cosh(1)\}.$$

- The incoming boundary of M_k^n is

$$\partial_0 M_k^n := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid -|x|^2 + |y|^2 = -1\}.$$

- The outgoing boundary of M_k^n is

$$\partial_1 M_k^n := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid -|x|^2 + |y|^2 = 1\}.$$

$\overline{M}_k^n$ is a cobordism between $S^{k-1} \times D^{n-k}$ and $D^k \times S^{n-k-1}$ (D^r denotes the open disk of radius 1 in $\mathbb{R}^r$, centered at 0 and $\overline{D}^r$ denotes its closure) with boundary identifications defines as follows:

- The incoming boundary identification $\iota_0 : S^{k-1} \times D^{n-k} \rightarrow \partial_0 M_k^n$ is defined by

$$(u, \theta v) \mapsto (u \cosh(\theta), v \sinh(\theta)), \text{ where } 0 \leq \theta < 1.$$

- The outgoing boundary identification $\iota_1 : D^k \times S^{n-k-1} \rightarrow \partial_1 M_k^n$ is defined by

$$(\theta u, v) \mapsto (u \sinh(\theta), v \cosh(\theta)), \text{ where } 0 \leq \theta < 1.$$

The standard Morse model of index k in $\mathbb{R}^n$ comes naturally equipped with an excellent Morse function that has a single critical point of index k at the origin $0 \in M_k^n$, namely the function

$$h : (M_k^n; \partial_0 M_k^n, \partial_1 M_k^n) \rightarrow ([-1, 1]; \{-1\}, \{1\})$$

given by

$$\mathbb{R}^k \times \mathbb{R}^n \ni (x, y) \mapsto -|x|^2 + |y|^2.$$

It also comes naturally equipped with a gradient - like vector field, namely the gradient of h with respect to the standard Euclidean metric on $\mathbb{R}^n$. This vector field defines gradient flow lines in M_k^n which in turn define an identification

$$S^{k-1} \times (D^{n-k} \setminus \{0\}) \rightarrow (D^k \setminus \{0\}) \times S^{n-k-1}$$

given by

$$(u, \theta v) \mapsto (\theta u, v), \text{ where } 0 < \theta < 1.$$

Definition 1.3.12 (Standard Morse model embeddings). Let $f : W \rightarrow I$ be a function and let $p \in W$ be a Morse critical point of index k . A standard Morse model embedding adapted to f at p is an embedding $\phi : M_k^n \rightarrow W$ such that:

- $\phi(0) = p$,
- There exists an increasing embedding $\phi' : [-1, 1] \rightarrow I$ such that the following diagram commutes:

$$\begin{array}{ccc} M_k^n & \xrightarrow{\phi} & W \\ h \downarrow & & \downarrow f \\ [-1, 1] & \xrightarrow{\phi'} & I \end{array}$$

If ϕ' exists, it is uniquely determined by ϕ and f . We will sometimes denote a standard Morse model embedding by (ϕ, ϕ') , when we want to later refer to ϕ' explicitly.

Proposition 1.3.4. Given a function $f : W \rightarrow I$ with a Morse critical point of index k , there always exists a standard Morse model embedding adapted to f at p .

Proof. This follows from the Morse lemma. For a detailed proof, see [Mil63]. $\square$

Every standard Morse model embedding ϕ induces embeddings of framed surgery spheres:

$$\begin{aligned} \phi \circ \iota_0 : S^{k-1} \times D^{n-k} &\rightarrow W, \\ \phi \circ \iota_1 : D^k \times S^{n-k-1} &\rightarrow W, \end{aligned}$$

that lie within the level sets $f^{-1}(\phi'(-1))$ and $f^{-1}(\phi'(1))$, respectively. By abuse of notation, we will denote both of these embeddings by ϕ when the context is clear.

Definition 1.3.13. Let $f : W \rightarrow I$ be a Morse function with a Morse critical point $p \in W$ of index k and a standard Morse model embedding $\phi : M_k^n \rightarrow W$ adapted to f at p . Let ξ be a gradient - like vector field for f . We say that ξ is **compatible** with ϕ if in the image of ϕ we have

$$\xi = \nabla_g f,$$

where g is the push-forward of the standard Euclidean metric under ϕ .

A horizontal distribution for f is **compatible** with ϕ if it has a representative gradient - like vector field which is compatible with ϕ .

We will now state two lemmas regarding standard Morse model embeddings that will become useful later when constructing elementary birth-death paths. Their proofs are quite straightforward and will not be included.

Lemma 1.3.9. *Let $f : W \rightarrow I$ be a smooth function with a Morse critical point $p \in W$ of index k and a standard Morse model embedding $(\phi, \phi') : M_k^n \rightarrow W$ adapted to f at p . Let*

$$\gamma : (S^{k-1} \times D^{n-k}) \times [0, 1] \rightarrow W$$

be an isotopy such that

- $\gamma(-, 0) = \phi$,
- *the image of γ lies in the level set $f^{-1}(\phi'(-1))$.*

Then, there exists an isotopy $\Gamma : M_k^n \times [0, 1] \rightarrow W$ of standard Morse model embeddings adapted to f at p that induces γ when restricted to $S^{k-1} \times D^{n-k}$. The analogous statement for the dual framed attaching sphere also holds.

Lemma 1.3.10. *Let $f : W \rightarrow I$ be a Morse function with a Morse critical point $p \in W$ of index k and a standard Morse model embedding $\phi : M_k^n \rightarrow W$ adapted to f at p . Let g be a Riemannian metric on W adapted to f such that the gradient-like vector field $\nabla_g f$ is compatible with ϕ . Let $\tilde{g}$ be another Riemannian metric on W adapted to f that agrees with g in a neighborhood of p . Then, there exists a standard Morse model embedding $\tilde{\phi} : M_k^n \rightarrow W$ adapted to f at p , that agrees with ϕ in a neighborhood of p and is compatible with $\nabla_{\tilde{g}} f$.*

Regular level sets

Definition 1.3.14. Let $f : W \rightarrow I$ be an excellent Morse function, where I is an interval with endpoints i_0 and i_1 . A set of regular values $\{i_0 = a_0 < \dots < a_m = i_1\}$ will be called **complete** if for any $0 \leq i \leq m - 1$, there exists at most one critical value in the interval $[a_i, a_{i+1}]$.

Let p be a critical point of index k and $(\phi, \phi') : M_k^n \rightarrow W$ a standard Morse model embedding adapted to f at p . A complete set of regular values $\{a_0 < \dots < a_m\}$ is **compatible** with (ϕ, ϕ') if there exists $0 \leq i \leq m - 1$ such that $a_i = \phi'(-1)$ and $a_{i+1} = \phi'(1)$.

We are now ready to define the notion of Morse data associated with an excellent Morse function f :

Definition 1.3.15. Let $f : W \rightarrow I$ be an excellent Morse function and let $\text{Crit}(f)$ denote the set of all its critical points. A choice of **Morse data associated with f** consists of:

- For each $p \in \text{Crit}(f)$, a choice of standard Morse model embeddings $(\phi_p, \phi'_p) : M_{\text{ind}(p)}^n \rightarrow W$ adapted to f at p .
- A horizontal distribution τ compatible with every standard Morse model embedding (ϕ_p, ϕ'_p) .
- A complete set of regular values $\{a_0 < \dots < a_m\}$ compatible with every standard Morse model embedding (ϕ_p, ϕ'_p) .

For brevity, we will typically denote a choice of Morse data associated with f by $\mathcal{D}_f$. We will denote the set of all Morse data associated with f by $\mathcal{MD}(W; f)$ and the set of all pairs $(f, \mathcal{D}_f)$ by $\mathcal{MD}(W)$. As we have seen already, the Morse lemma and a partition of units argument shows that for any excellent Morse function f , $\mathcal{MD}(W; f)$ is non-empty.

It will be useful to also consider choices of *partial* Morse data associated with f . This is defined just like Morse data, but without the datum of the complete set of regular values. We denote a choice of partial Morse data associated with f by $\mathcal{PD}_f$, the set of all such choices by $\mathcal{PMD}(W; f)$, and the set of all pairs $(f, \mathcal{PD}_f)$ by $\mathcal{PMD}(W)$.

It is not hard to regard $\mathcal{PMD}(W)$ as a space. The collection of all standard Morse model embeddings forms a subset of the space of embeddings of standard Morse models in W , and hence inherits the subspace topology. Similarly, the collection of horizontal distributions compatible with these embeddings is a subset of the space of all horizontal distributions, which itself is a quotient of the space of gradient-like vector fields, a subspace of the space of vector fields. This equips $\mathcal{PMD}(W)$ with a natural topology.

It is also useful to view $\mathcal{MD}(W)$ as a space, but for our purposes it is more appropriate to restrict the allowed paths of regular values. Thus, by a mild abuse of notation, we describe $\mathcal{MD}(W)$ as a directed graph:

Definition 1.3.16. The directed graph $\mathcal{MD}(W)$ is defined as follows:

1. Its vertices are the elements of $\mathcal{MD}(W)$, i.e. a Morse function on W together with a choice of Morse data.
2. Its edges are of two kinds:

- (a) edges consisting of a smooth path of Morse functions, together with
 - a smooth isotopy of standard Morse model embeddings,
 - a compatible smooth isotopy of horizontal distributions, and
 - a compatible smooth isotopy of regular values.

In particular, the number of regular values remains constant throughout. We refer to these as *isotopy edges*;

- (b) edges corresponding to the birth or death of a single regular value. Here the Morse function and horizontal distribution are required to be the same at both vertices, while the standard Morse model embeddings at the source vertex canonically determine those at the target vertex by flowing along the horizontal distribution until the compatibility conditions are satisfied. We will refer to these as *regular value edges*.

A *path* in $\mathcal{MD}(W)$ is a finite directed path in this graph. Note that every path in $\mathcal{MD}(W)$ has an inverse. We similarly define the directed graph $\mathcal{MD}(W; f)$ by requiring the Morse function f to remain fixed throughout, and define paths in $\mathcal{MD}(W; f)$ accordingly.

Remark 1.3.11. Isotopy edges naturally encode paths $\gamma: [0, 1] \rightarrow \mathcal{MD}(W)$. Likewise, we may associate regular-value edges with paths $\gamma: [0, 1] \rightarrow \mathcal{MD}(W)$. Such paths are piecewise constant: they remain constant before and after the birth or death of a regular value. Throughout such a path, the horizontal distribution stays fixed, while the set of regular values—and possibly the standard Morse model embeddings—changes at the critical time.

We will often represent paths in the graph $\mathcal{MD}(W)$ as actual paths in the set $\mathcal{MD}(W)$ from this point of view. For example, we will typically denote an element of $\mathcal{MD}(W)$ by $(f, \mathcal{D}_f)$ and a path in $\mathcal{MD}(W)$ by $(f_s, \mathcal{D}_{f_s})$.

It will be useful to note that extensions of partial Morse data are connected:

Lemma 1.3.12. *Let $(f, \mathcal{PD}_f)$ be a choice of partial Morse data on W , and let $(f, \mathcal{D}_f)$ and $(f, \mathcal{D}'_f)$ be two extensions of $(f, \mathcal{PD}_f)$ to full Morse data. Then there exists a path in $\mathcal{MD}(W; f)$ between $(f, \mathcal{D}_f)$ and $(f, \mathcal{D}'_f)$ which fixes $(f, \mathcal{PD}_f)$ throughout.*

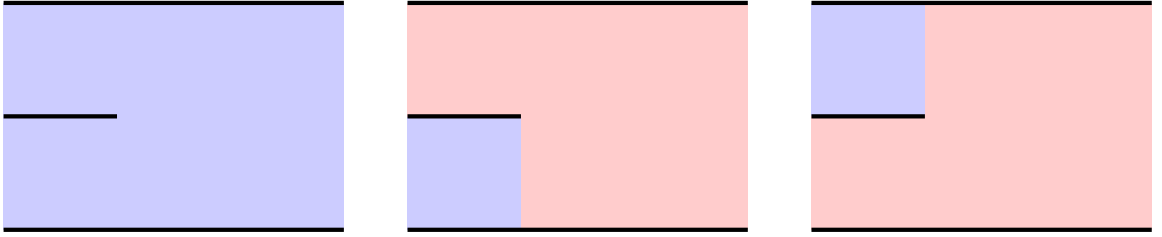
Proof. The key observation is that any partial Morse data admits a canonical minimal extension: namely, one obtained by taking the regular values determined by the endpoints of the target interval together with the standard Morse model embeddings.

Every extension of the partial Morse data necessarily contains these regular values by compatibility. Any other extension can be connected to this minimal extension by a path in $\mathcal{MD}(W; f)$. Since paths in $\mathcal{MD}(W; f)$ are invertible, the claim follows. $\square$

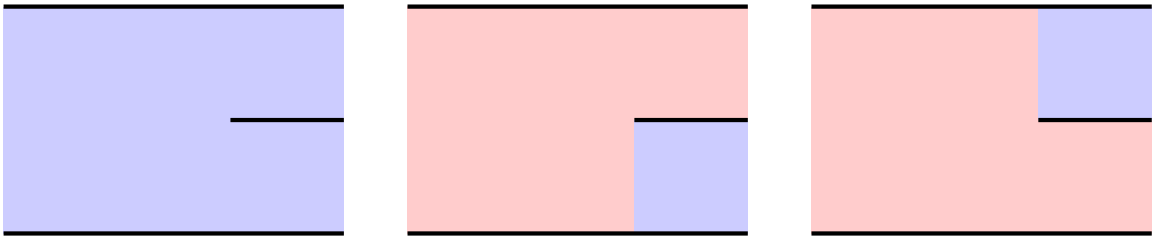
Corollary 1.3.13. *Let $(f, \mathcal{D}_f)$ and $(f', \mathcal{D}_{f'})$ be two choices of Morse data on W . If their induced partial Morse data are connected by a path of partial Morse data, then $(f, \mathcal{D}_f)$ and $(f', \mathcal{D}_{f'})$ are connected by a path in $\mathcal{MD}(W)$.*

We now introduce an equivalence relation on the set of all paths in $\mathcal{MD}(W; f)$. Before doing so, it is helpful to visualize *regular-value edges*. We color an interval between two regular values **blue** if it contains no critical values and **red** if it contains a single critical value. In the figures below, the horizontal axis represents the path parameter and the vertical axis the function value; black horizontal segments depict the regular values, which move in a piecewise constant fashion.

An edge recording the *death* of a regular value may be visualized in any of the following three ways:



Analogously, an edge recording the *birth* of a regular value can be visualized in any of the following three ways:

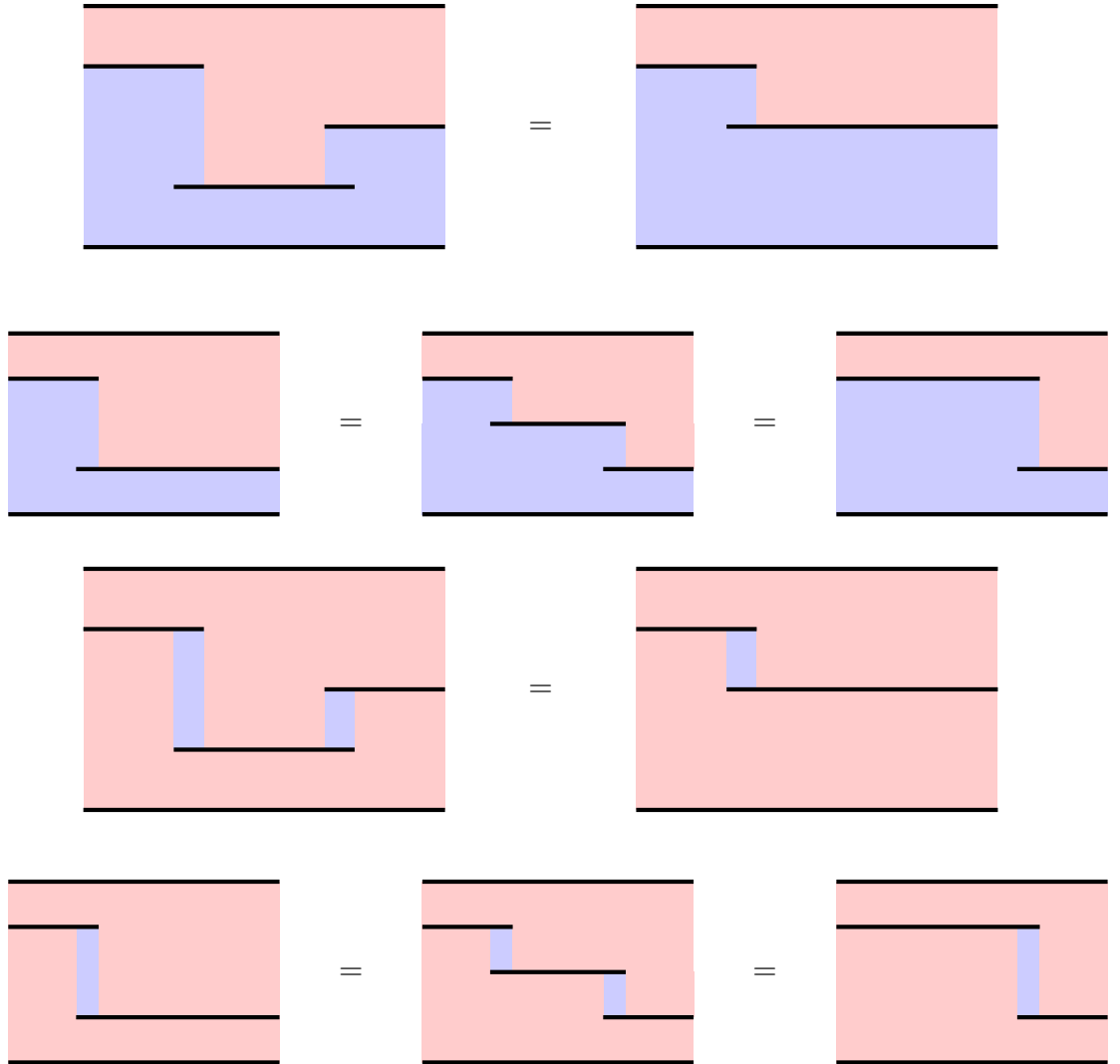


In all pictures, **blue** regions indicate intervals free of critical values, **red** regions indicate an interval containing a single critical value, and black segments depict the chosen regular values along the path parameter.

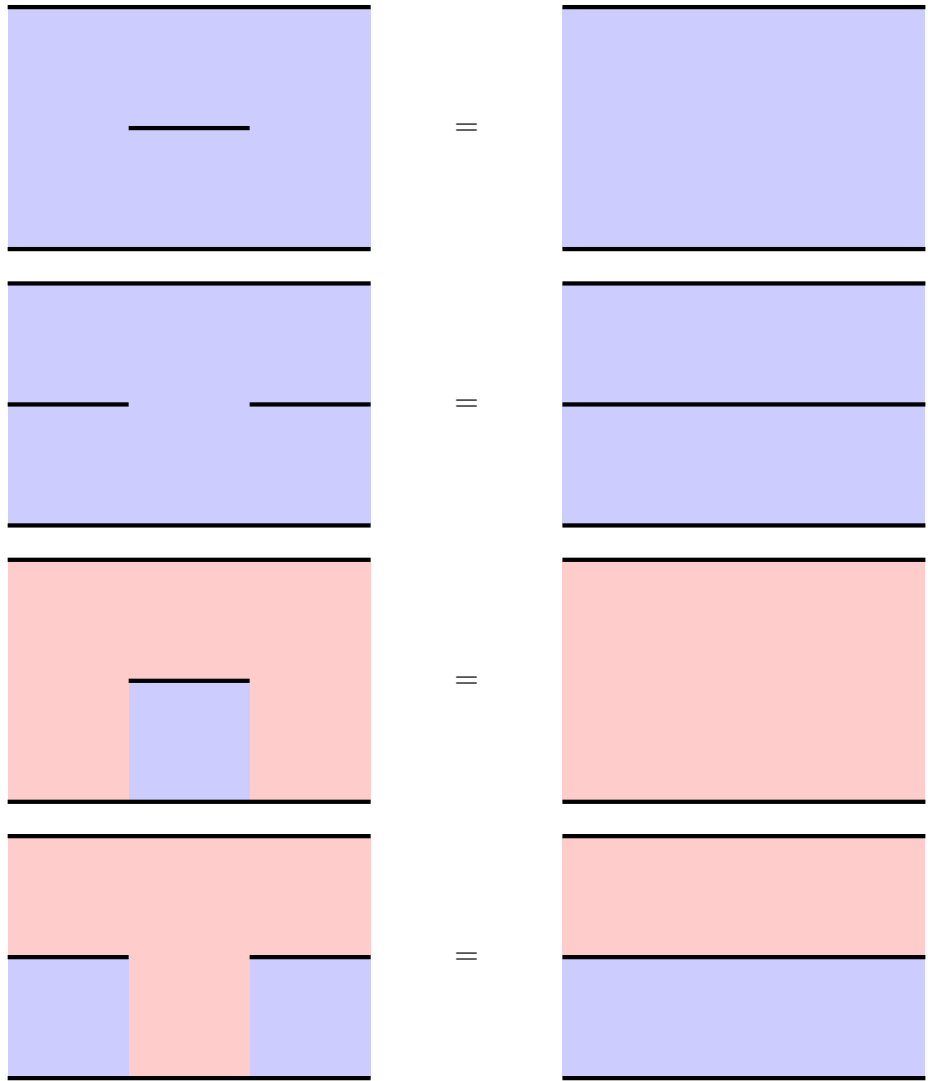
Definition 1.3.17. Two paths in $\mathcal{MD}(W)$ are said to be equivalent if they are related by the equivalence relation generated by the following moves:

1. **Isotopy moves:** paths of isotopy edges that define isotopic composite isotopies are equivalent.
2. **Commutation moves:** composition of an isotopy edge with a regular value edge commutes. In addition, edges recording data in disjoint intervals between regular values commute.
3. **Regular–value moves:** paths of regular value edges are equivalent if they can be transformed into one another by a finite sequence of the following graphical relations:

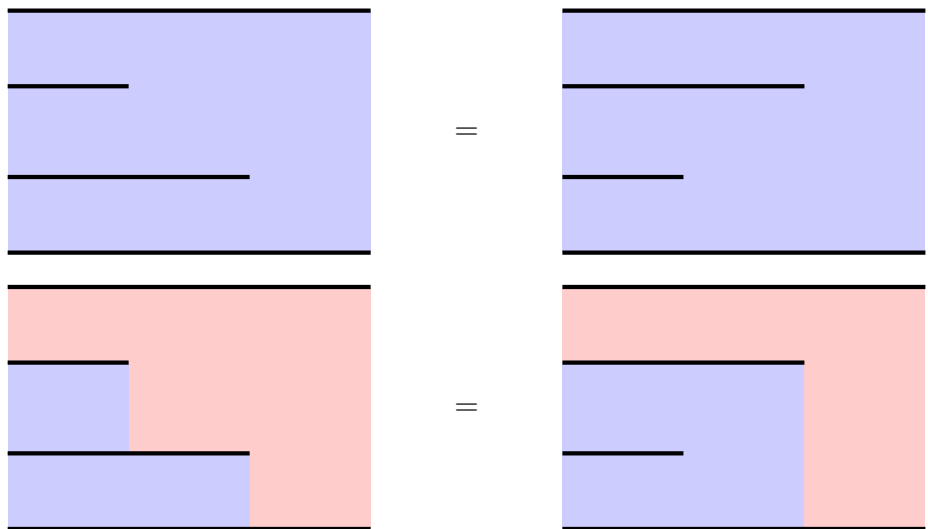
Sliding red region moves

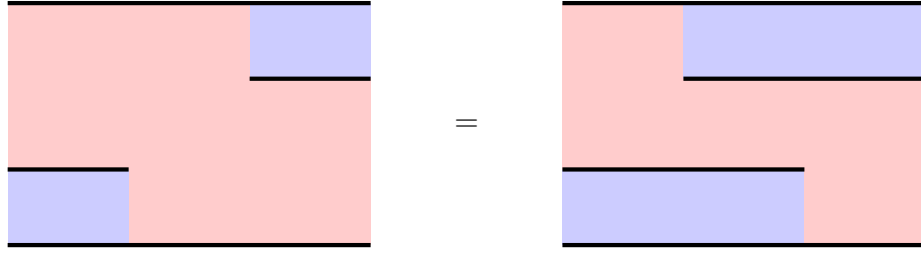


Inverse cancellation moves



Coincidence moves





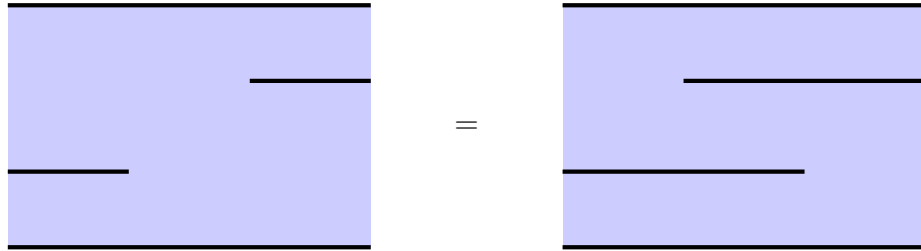
These graphical equalities are to be understood up to rotation about the x - and y -axes.

Equipped with this definition, we can now show the following:

Lemma 1.3.14. *Any two paths of regular value edges in $\mathcal{MD}(W; f)$ with the same endpoints are equivalent.*

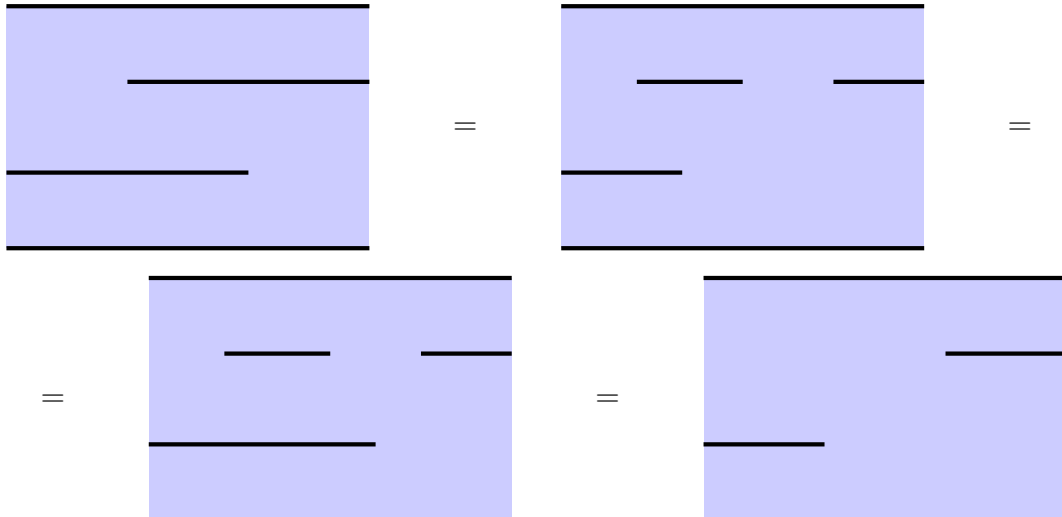
Proof. We argue by reduction to a normal form.

First, consider the case where the endpoint vertices correspond to Morse data whose underlying Morse functions have no critical points—equivalently, all intervals are labeled [blue](#). We first verify the Frobenius relation:



up to rotation about the x -axis.

This follows from a combination of inverse cancellation and coincidence moves as follows:

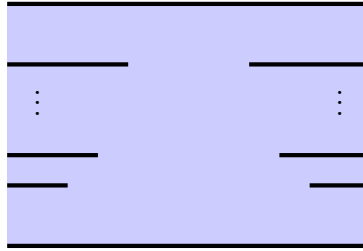


Let us begin with the special case where the endpoint vertices correspond to Morse data having only two regular values, namely the endpoints of the target interval of the Morse function. In this situation, we claim that the normal form of any path is given by



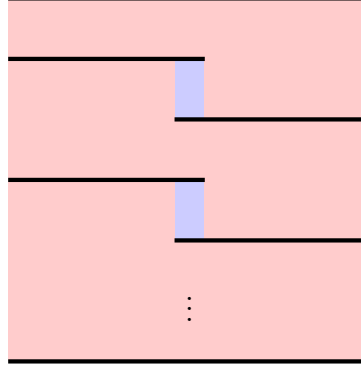
—that is, the trivial path with no additional regular values.

A general path may contain extra horizontal segments of regular values appearing inside the blue region. We show that the number of such segments can always be decreased, which yields the claim by induction. Let t_0 denote the time of the first birth of a regular value, and focus on the horizontal segment created at t_0 . We can shrink this segment forward in the time parameter t until we encounter the birth or death of another segment adjacent to it. At that point, we apply the Frobenius relation or the coincidence moves to continue shrinking past that time. Ultimately, the segment can be completely contracted using the inverse cancellation move. This shows that any path whose endpoints correspond to Morse data with exactly two regular values is equivalent to the canonical normal form above. Using inverse cancellation moves and this observation, we further conclude that any path whose endpoints correspond to Morse data whose underlying Morse functions have no critical points is equivalent to the normal form



Next, we consider the case where the endpoint vertices correspond to Morse data in which every interval between consecutive regular values contains exactly one critical value, i.e. every interval is labeled red. Using the sliding red region moves, together with the earlier observation that any floating interval of regular values inside a blue region can be eliminated, we conclude that any such path is equivalent to the normal

form



Finally, applying the inverse cancellation moves reduces the general case of a path whose endpoint vertices correspond to Morse data with critical points to the case above, thereby completing the proof. $\square$

Corollary 1.3.15. *Let γ_0, γ_1 be paths in $\mathcal{MD}(W)$ sharing the same endpoints. If their induced paths of partial Morse data are equivalent in $\mathcal{PM}\mathcal{D}(W)$, then γ_0 is equivalent to γ_1 .*

Proof. This is a consequence of Lemma 1.3.14 together with the commutation moves. $\square$

Finally, just as we have a notion of Morse function preserving diffeomorphisms, we will need a notion of Morse data preserving diffeomorphisms.

Definition 1.3.18. Let W and W' be smooth manifold triads equipped with Morse functions f, f' and associated Morse data $\mathcal{D}_f, \mathcal{D}_{f'}$, respectively. An equivalence $D: W \rightarrow W'$ of smooth manifold triads that preserves the Morse functions is said to **preserve the associated Morse data** if the following hold:

- D preserves the horizontal distributions;
- D induces an ordered bijection on the regular level sets;
- D preserves the standard Morse model embeddings up to the action of $G_{(n,k)}^{(\text{or})}$.

We will now explain the last point concretely. Let p be a critical point of f of index k , and consider the standard Morse model embedding

$$\begin{array}{ccc} M_k^n & \xrightarrow{\iota_p} & W \\ h \downarrow & & \downarrow f \\ [-1, 1] & \xrightarrow{\iota'_p} & I \end{array}$$

adapted to f at p , which is part of $\mathcal{D}_f$. Since D preserves the Morse functions, $p' = D(p)$ is a critical point of f' of the same index k . Let

$$\begin{array}{ccc} M_k^n & \xrightarrow{\tilde{\iota}_p} & W' \\ h \downarrow & & \downarrow f' \\ [-1, 1] & \xrightarrow{\tilde{\iota}'_p} & J \end{array}$$

be the standard Morse model embedding adapted to f' at p' , which is part of $\mathcal{D}_{f'}$.

We say that D preserves the standard Morse model embeddings ι_p and $\tilde{\iota}_p$ if there exists $Q \in G_{(n,k)}^{(\text{or})}$ such that the following diagram commutes:

$$\begin{array}{ccccc} W & \xrightarrow{D} & & & W' \\ & \searrow f & & & \swarrow f' \\ & & I & \xrightarrow{D'} & J \\ \iota_p \uparrow & & \uparrow \iota'_p & & \uparrow \tilde{\iota}'_p \\ & & [-1, 1] & = & [-1, 1] \\ & \nearrow h & & & \nwarrow h \\ M_k^n & \xrightarrow{Q} & & & M_k^n \\ & & & & \uparrow \tilde{\iota}_p \\ & & & & W' \end{array}$$

The inner square is included for completeness, although its commutativity already follows from the assumption that D preserves the Morse functions. The existence of Q uniquely determines it.

1.4 Cerf's fibration lemma

Morse data constitute the necessary tool for describing a cobordism in terms of elementary pieces. We will not only care about excellent Morse functions, however, but we will also have to consider Cerf paths, which, in particular, include families of excellent Morse functions. We will consequently need to talk about families of Morse data.

There is an evident projection map

$$\begin{array}{c} \mathcal{PMD}(W) \\ \downarrow \\ \mathcal{F}^0(W) \end{array}$$

whose fiber over $f \in \mathcal{F}^0(W)$ is $\mathcal{PMD}(W; f)$. In order to study families of Morse functions and associated families of Morse data, it would be very convenient if this

map were a fibration and if we further had an understanding of the homotopy type of its fiber. Thankfully, we find ourselves in this situation. Even more thankfully, Cerf found himself in a similar situation too and we can directly benefit from his insights without much extra work.

We will primarily use Cerf's fibration lemma and the stability of Morse functions.

Lemma 1.4.1 (Cerf's fibration lemma). *Let E and B be two topological spaces and let $p : E \rightarrow B$ be a continuous map. The following is a sufficient condition for p to be a fiber bundle:*

For each $x_0 \in B$, there exists a topological group G that acts from the left on E and B in such a way that the diagram

$$\begin{array}{ccc} G \times E & \longrightarrow & E \\ \text{Id}_G \times p \downarrow & & \downarrow p \\ G \times B & \longrightarrow & B \end{array}$$

commutes, and the map $G \rightarrow B$ given by $g \mapsto g \cdot x_0$ admits a continuous section on a neighborhood of x_0 .

Proof. This is Lemma 1, p. 115 in [Cer68]. □

Taking $G = \text{Diff}(W; \partial_0 W, \partial_1 W) \times \text{Diff}(I; \{0\}, \{i\})$ for $f \in \mathcal{F}^0(W, I)$, Cerf's fibration lemma and the stability of Morse functions imply that the projection map

$$\begin{array}{c} \mathcal{PMD}(W) \\ \downarrow \\ \mathcal{F}^0(W) \end{array}$$

is a fiber bundle. Following Cerf, let $\mathcal{P}(W)$ denote the space of pairs $(f, \mathcal{P}_f)$, where $\mathcal{P}_f$ denotes a choice of standard Morse model embeddings $\{(\phi_p, \phi'_p)\}_{p \in \text{Crit}(f)}$ adapted to f at its respective critical points. Cerf's fibration lemma and the stability of Morse functions apply in the same way to show that the projection map

$$\begin{array}{ccc} \mathcal{P}(W; f) & \longrightarrow & \mathcal{P}(W) \\ & & \downarrow \\ & & \mathcal{F}^0(W) \end{array}$$

is a fiber bundle. Moreover, Cerf explicitly determines the homotopy type of the fiber $\mathcal{P}(W; f)$:

- If f has no critical points, $\mathcal{P}(W; f)$ is empty and $\mathcal{PM}\mathcal{D}(W; f)$ is contractible.
- If f has a single critical point p of index k , let $J_0^1\mathcal{P}(W; f)$ denote the space of 1-jets at the origin of the standard Morse embeddings adapted to f at p . Cerf shows that the map

$$\begin{array}{c} \mathcal{P}(W; f) \\ \downarrow \phi \mapsto j^1\phi(0) \\ J_0^1\mathcal{P}(W; f) \end{array}$$

is a fiber bundle with contractible fiber. Moreover, $J_0^1\mathcal{P}(W; f) \cong O(k, n-k)$ in the unoriented case and $J_0^1\mathcal{P}(W; f) \cong SO(k, n-k)$ in the oriented case. Detailed proofs of these statements appear in the Appendix of [Cer73]. Their combination implies that $\mathcal{P}(W; f)$ is weakly equivalent to $O(k, n-k)$ (respectively $SO(k, n-k)$) in the unoriented (respectively oriented) case.

Let $G_{n,k} := O(k) \times O(n-k)$ and $G_{n,k}^{\text{or}} := (O(k) \times O(n-k)) \cap SO(n)$. It is known that $G_{(n,k)}^{(\text{or})}$ is homotopy equivalent to $(S)O(k, n-k)$. Moreover, $G_{(n,k)}^{(\text{or})}$ acts freely on $\mathcal{P}(W; f)$ and $J_0^1\mathcal{P}(W; f)$. The homeomorphism $J_0^1\mathcal{P}(W; f) \cong O(k, n-k)$ (respectively $J_0^1\mathcal{P}(W; f) \cong SO(k, n-k)$) is equivariant with respect to these actions. Moreover, the following diagram commutes:

$$\begin{array}{ccc} G_{(n,k)}^{(\text{or})} \times \mathcal{P}(W; f) & \longrightarrow & \mathcal{P}(W; f) \\ \text{Id} \times j_0^1 \downarrow & & \downarrow j_0^1 \\ G_{(n,k)}^{(\text{or})} \times J_0^1\mathcal{P}(W; f) & \longrightarrow & J_0^1\mathcal{P}(W; f) \end{array}$$

The combination of these facts implies that $\mathcal{P}(W; f)$ is weakly equivalent to $G_{(n,k)}^{(\text{or})}$.

Lemma 1.4.2. *$\mathcal{PM}\mathcal{D}(W; f)$ is weakly equivalent to $\mathcal{P}(W; f)$ and hence weakly equivalent to $G_{(n,k)}^{(\text{or})}$.*

Proof. The key observation is that the fiber of the projection map $\mathcal{PM}\mathcal{D}(W; f) \rightarrow \mathcal{P}(W; f)$ is convex, and therefore contractible. The result then follows once we establish that this projection is a Serre fibration— we will now sketch the proof of a slightly weaker but sufficient fact.

Suppose W is a manifold equipped with a Morse function f having a single critical point p , and let U be a Morse coordinate chart around p . Using f , we obtain a nested sequence of neighborhoods around p :

$$U \supset U_{1/2} \supset U_{1/4} \supset \cdots \supset U_{1/2^n} \supset \cdots .$$

Let $F(U)$ denote the space of admissible horizontal distributions on U . Restriction gives a tower of maps

$$F(U) \rightarrow F(U_{1/2}) \rightarrow \cdots \rightarrow F(U_{1/2^n}) \rightarrow \cdots ,$$

and we define $G(W, f)$ to be the filtered colimit of this system. Let $\mathcal{H}(W; f)$ denote the space of admissible horizontal distributions on all of W with respect to f . A standard partition of units argument shows that the natural restriction map

$$\mathcal{H}(W; f) \rightarrow G(W, f)$$

is a Serre fibration. Now consider the pullback diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{PM}\mathcal{D}}(W; f) & \longrightarrow & \mathcal{H}(W; f) \\ \downarrow & & \downarrow \\ \mathcal{P}(W; f) & \longrightarrow & G(W, f) \end{array}$$

The space $\widetilde{\mathcal{PM}\mathcal{D}}(W; f)$ is a weaker version of $\mathcal{PM}\mathcal{D}(W; f)$, where we only require the standard Morse model embedding and the admissible horizontal distribution to be compatible over some $U_{1/2^n}$. The map

$$\widetilde{\mathcal{PM}\mathcal{D}}(W; f) \rightarrow \mathcal{P}(W; f)$$

is a Serre fibration as the pullback of a Serre fibration, and its fibers are convex, hence contractible. It is, therefore, a homotopy equivalence. Finally, we observe that the natural map

$$\mathcal{PM}\mathcal{D}(W; f) \rightarrow \widetilde{\mathcal{PM}\mathcal{D}}(W; f)$$

is a homotopy equivalence, which implies that the original projection

$$\mathcal{PM}\mathcal{D}(W; f) \rightarrow \mathcal{P}(W; f)$$

is also a homotopy equivalence. □

1.5 Cerf paths

We have already seen that the space of excellent Morse functions is an open and dense subset of the space of all smooth real valued functions on a manifold. This makes it very easy to show that there exists an excellent Morse function on any given manifold. It does not, however, make it easy to choose one, since the space of excellent Morse functions is far from contractible and, in fact, far from connected. The reader unfamiliar with this statement is encouraged to think about two different Morse functions on the cylinder $f_0, f_1 : S^1 \times [0, 1] \rightarrow [0, 1]$ with a different number of critical points. By the stability of Morse functions, any two excellent Morse functions connected via a path of excellent Morse functions are in the orbit of the group $\text{Diff}(S^1 \times [0, 1]; S^1 \times \{0\}, S^1 \times \{1\}) \times \text{Diff}(I; \{0\}, \{1\})$ and hence must have the same number of critical points. On the other extreme, the space of all real valued functions on a manifold is contractible, hence any two excellent Morse functions are connected via a path of smooth real valued functions. Our goal in this section is to describe a subspace of the space of real valued functions on a manifold W , $\mathcal{F}^1(W)$, that is explicit and small enough so that we can concretely understand it and broad enough so that

$$\text{Maps}([0, 1], (\mathcal{F}^0(W) \cup \mathcal{F}^1(W), \mathcal{F}^0(W)))$$

is a generic subset of

$$\text{Maps}([0, 1], (C^\infty(W), \mathcal{F}^0(W))).$$

The space $\mathcal{F}^1(W)$ consists of functions that only fail to be excellent Morse by having either a single cusp critical point or two Morse critical points at the same critical level. We will now study the former case in some detail.

1.5.1 Birth - death paths

As before, let W denote a smooth manifold or a smooth manifold triad and let $f \in C^\infty(W; \mathbb{R})$ or $f \in C^\infty(W; I)$, where $I = ([0, i]; \{0\}, \{1\})$, respectively. For parts of this section, we will follow [Cha18]. We have previously introduced the Hessian of a critical point $p \in \text{Crit}(f)$ in section 1.3, as a symmetric bilinear map $d^2 f_p : T_p W \times T_p W \rightarrow \mathbb{R}$. The kernel of $d^2 f_p$ consists of all $v \in T_p W$ such that $d^2 f_p(w, v) = 0, \forall w \in T_p W$. We can now similarly define the intrinsic third derivative at a critical point $p \in \text{Crit}(f)$, following [Cha18], as a map

$$d^3 f_p : \ker d^2 f_p \times \ker d^2 f_p \times \ker d^2 f_p \rightarrow \mathbb{R},$$

constructed as follows:

Given $v_1, v_2, v_3 \in \ker d^2 f_p$, choose $\eta : \mathbb{R}^3 \rightarrow M$ such that $\eta(0) = p$ and $\frac{\partial \eta}{\partial x_i} = v_i$ for $i = 1, 2, 3$. We set

$$d^3 f_p(v_1, v_2, v_3) = \left. \frac{\partial^3 (f \circ \eta)}{\partial x_1 \partial x_2 \partial x_3} \right|_{x_1=x_2=x_3=0}.$$

Just as with the intrinsic Hessian above, we can show that $d^3 f_p$ is well-defined (i.e. independent of the choice of η) by an explicit calculation in a local chart, which can be found in [Cha18].

Definition 1.5.1 (Cusp critical points). Given $f \in C^\infty(W)$ as above, a point $p \in W$ is a **cusp critical point** of f if:

- (i) $p \in \text{Crit}(f)$
- (ii) $\dim_{\mathbb{R}}(\ker(d^2 f_p)) = 1$. Let $v \in T_p M$ be such that $\mathbb{R}v = \ker(d^2 f_p)$.
- (iii) $d^3 f_p(v, v, v) \neq 0$

We further define the **index** of p to be the index of the Hessian of f at p .

In analogy with Morse critical points, cusp critical points admit a standard local coordinate description:

Theorem 1.5.1 (Cusp lemma). *Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by*

$$g(x_1, \dots, x_n) = c - (x_1^1 + \dots + x_k^2) + x_{k+1}^2 + \dots + x_{n-1}^2 + x_n^3,$$

for some constant c . Then,

- (a) *g has a single cusp critical point at 0 of index k .*
- (b) *Given any $f \in C^\infty(W, \mathbb{R})$ with a cusp critical point $p \in W$ of index k , there exists a neighborhood V of the origin in $\mathbb{R}^n$ and embeddings $\phi : (V, 0) \rightarrow (W, p)$, $\phi' : \mathbb{R} \rightarrow \mathbb{R}$ such that the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ g \downarrow & & \downarrow f \\ \mathbb{R} & \xrightarrow{\phi'} & \mathbb{R} \end{array}$$

Remark 1.5.2. In contrast to the Morse lemma, we cannot, in general, take ϕ' to be the identity.

Definition 1.5.2. We denote by $\mathcal{F}_\alpha^1(W)$ the subspace of $C^\infty(W; \mathbb{R})$ consisting of functions f such that:

- f has a single cusp critical point,
- all other critical points of f are of Morse type,
- for any two critical points p_1, p_2 of f , we have $f(p_1) \neq f(p_2)$.

Definition 1.5.3 (Birth-death paths). A *birth-death* path on W is a smooth path $\gamma : [0, 1] \rightarrow \mathcal{F}^0(W) \cup \mathcal{F}_\alpha^1(W)$ that satisfies the three conditions listed below.

1. There exists a single time $t_0 \in (0, 1)$ when $\gamma(t_0) \in \mathcal{F}_\alpha^1$.
2. $\gamma(\frac{t_0}{2})$ and $\gamma(\frac{t_0+1}{2})$ lie in different connected components of $\mathcal{F}^0(W)$.
3. Let $p \in W$ be the cusp critical point of $\gamma(t_0)$ and let $v \in T_p W$ be such that $\mathbb{R}v = \ker(d^2 f_p)$. Consider

$$\Gamma : [0, 1] \times W \rightarrow \mathbb{R}$$

given by

$$(t, w) \mapsto \gamma(t)(w).$$

We require that

$$d\left(\frac{\partial \Gamma}{\partial t} \Big|_{t=t_0}\right)_p(v) \neq 0.$$

Note that the second condition can be derived from this transversality condition, but we list it here separately for emphasis.

In this case, we will also say that γ crosses $\mathcal{F}_\alpha^1(W)$ generically at a single point. If $\gamma(t_0)$ has a cusp critical point of index k , γ is called a birth - death path of index k . Moreover, if $\gamma(\frac{t_0}{2})$ has fewer Morse critical points than $\gamma(\frac{t_0+1}{2})$, then γ will be called a **birth path of index k** and otherwise γ will be called a **death path of index k** .

Remark 1.5.3. Not every generalized birth-death path is a birth-death path. Consider, for example, the birth path $\Gamma : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\Gamma(x_1, \dots, x_n) = c - (x_1^1 + \dots + x_k^2) + x_{k+1}^2 + \dots + x_{n-1}^2 - (2t - 1)^3 x_n + x_n^3.$$

Theorem 1.5.4 (Birth path lemma). Let $B : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be given by

$$B(x_1, \dots, x_n) = c - (x_1^1 + \dots + x_k^2) + x_{k+1}^2 + \dots + x_{n-1}^2 - (2t - 1)x_n + x_n^3,$$

for some constant c . Then,

- (a) B is a birth path and $B(1/2)$ has a single cusp critical point at the origin 0.
- (b) Given any birth path of index k , $\Gamma : [0, 1] \times W \rightarrow \mathbb{R}$, there exists a neighborhood V of the origin in $\mathbb{R}^n$ and embeddings $\phi : V \rightarrow W$, $\phi' : \mathbb{R} \rightarrow \mathbb{R}$ that respect the critical points / critical values, such that the following diagram commutes:

$$\begin{array}{ccc} [0, 1] \times V & \xrightarrow{\phi} & [0, 1] \times W \\ B \downarrow & & \downarrow \Gamma \\ \mathbb{R} & \xrightarrow{\phi'} & \mathbb{R} \end{array}$$

There is an analogous death path lemma obtained by substituting $t \mapsto -t$ in the above.

Proof. A proof can be found in Appendix A of [Cha18]. $\square$

1.5.2 Crossing paths

We will now define crossing paths and Cerf paths.

Definition 1.5.4. We denote by $\mathcal{F}_\beta^1(W)$ the subspace of $C^\infty(W; \mathbb{R})$ consisting of functions f such that:

- all critical points of f are of Morse type,
- there exist exactly two critical points $p, q \in W$ of f such that $f(p) = f(q)$.

Definition 1.5.5 (Crossing paths). A **crossing** path on W is a smooth path $\gamma : [0, 1] \rightarrow \mathcal{F}^0(W) \cup \mathcal{F}_\beta^1(W)$ such that:

- there exists a single time $t_0 \in (0, 1)$ when $\gamma(t_0) \in \mathcal{F}_\beta^1$,
- $\gamma(\frac{t_0}{2})$ and $\gamma(\frac{t_0+1}{2})$ lie in different connected components of $\mathcal{F}^0(W)$.

In this case, we will also say that γ crosses $\mathcal{F}_\beta^1(W)$ generically at a single point.

We now set $\mathcal{F}^1(W) := \mathcal{F}_\alpha^1(W) \cup \mathcal{F}_\beta^1(W)$.

Definition 1.5.6 (Cerf paths). A *Cerf path* is a smooth path

$$\gamma : [0, 1] \rightarrow (\mathcal{F}^0(W) \cup \mathcal{F}^1(W), \mathcal{F}^0(W))$$

for which there exists a partition $0 = a_0 < a_1 < \dots < a_{m-1} < a_m = 1$ such that for all $0 \leq i \leq m-1$, the path

$$\gamma|_{[a_i, a_{i+1}]}$$

is either a path of excellent Morse functions or a birth-death path or a crossing path. We will denote by $\text{Cerf}(W)$ the space of all Cerf paths on W and refer to Cerf paths that cross $\mathcal{F}^1(W)$ at most once as *simple*.

A useful tool for analyzing Cerf paths is their singularity graphic, also known as a Cerf graphic. Given a Cerf path γ , its Cerf graphic is the subset of $I \times \mathbb{R}$ consisting of the critical values of the functions along the path. Intuitively, it records how the critical points move and interact over time. Examples of Cerf graphics corresponding to death paths and crossing paths are shown in Figure 1.2.

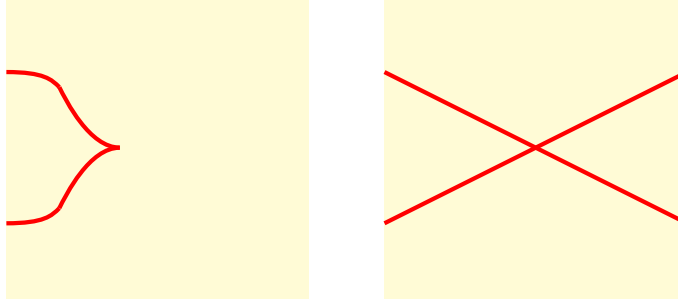


Figure 1.2: Cerf graphics corresponding to crossing paths and death paths.

Theorem 1.5.5 (Genericity of Cerf paths). *Cerf(W) is a generic subspace of the space of all smooth paths on $C^\infty(W, \mathbb{R})$.*

Proof. This theorem is another consequence of Thom’s (multi)jet transversality theorem. For a detailed proof, see [Cer68, Chapter II, Proposition 4, p. 29]. $\square$

It will be helpful to further examine the structure of the stratified space $\mathcal{F}^{\leq 1}(W) := \mathcal{F}^0(W) \cup \mathcal{F}^1(W)$. A connected component of this space is often depicted pictorially, as illustrated in Figure 1.3. One should, of course, be mindful of the usual limitations—and often the misleading nature—of representing infinite-dimensional spaces using two-dimensional pictures. In this case, however, Figure 1.3 suggests a structure that is, in fact, accurate: the space $\mathcal{F}^0(W) \cup \mathcal{F}^1(W)$ is a conical, codimension-one stratified space. Moreover, each point in $\mathcal{F}^1(W)$ admits a neighborhood of the form $U \times I$, where U is a neighborhood in $\mathcal{F}^1(W)$ and I is the cone on two points—that is, an interval. This result was established by Cerf in [Cer68, Chapter II, Proposition 6, p. 33].

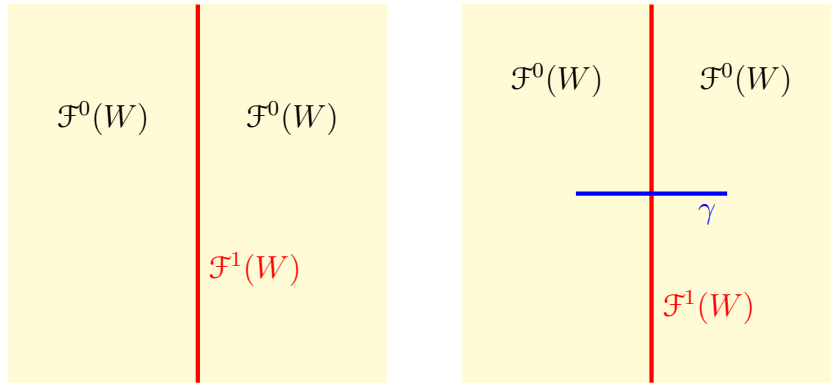


Figure 1.3: A connected component of $\mathcal{F}^0(W) \cup \mathcal{F}^1(W)$ and a simple Cerf path γ on it.

1.6 Elementary Cerf paths

Before delving into the details of elementary Cerf paths, it is helpful to begin with a conceptual overview of how various constructions in Morse and Cerf theory are organized and interrelated. To this end, we present a table of analogies, shown in Figure 1.4 below. This table includes notions that have already been introduced, as well as others that will be developed later in this chapter. The latter appear in shaded cells. We hope that the analogies drawn here will provide the reader with a rough guide to what lies ahead.

The constructions introduced in this section form the technical backbone of the first part of the thesis and will be used repeatedly in the arguments that follow. These include the following pairwise-related notions: standard birth–death models and elementary birth–death paths, as well as standard ascending/descending models and elementary crossing paths. Keeping the broad roadmap of Figure 1.4 in mind, we now begin by describing standard birth–death models and elementary birth–death paths in detail.

Morse theory	Cerf theory	
Functions with no critical points	Paths of excellent Morse functions	
Excellent Morse functions (that have critical points)	Birth–death paths	Crossing paths
Morse functions with a single critical point	Birth–death paths crossing $\mathcal{F}_\alpha^1$ at a single time t_0	Crossing paths crossing $\mathcal{F}_\beta^1$ at a single time t_0
Standard Morse function of index i on $\mathbb{R}^n$: $h_i(x) = -\sum_{j=1}^i x_j^2 + \sum_{j=i+1}^n x_j^2$	Standard birth–death path of index i on $\mathbb{R}^n$: $P_t^i(x) = h_i(x) + x_n^3 \pm (2t-1)x_n$	<i>No direct analogue</i>
Morse lemma: For every n -manifold M with Morse function f and critical point p of index i , there exists an embedding of $(\mathbb{R}^n, h_i)$ adapted to f at p	Cusp lemma: For every n -manifold M with birth–death path l and cusp critical point p of index i , there exists an embedding of $(\mathbb{R}^n \times \mathbb{R}, P_t^i)$ adapted to l at p	<i>No direct analogue</i>
Standard Morse model of index i : a model manifold triad with a single critical point of index i	Standard birth–death model of index i : a model manifold triad with a birth–death path that has a cusp critical point of index i	Standard ascending/ descending model of index i in $\mathbb{R}^n$
<i>No direct analogue</i>	Elementary birth–death paths: such birth–death paths are determined by the initial Morse function and a standard birth–death model embedding	Elementary crossing paths: such crossing paths are determined by the initial Morse function and standard ascending- descending model embeddings in crossing position

Figure 1.4: A table of analogies between notions in Morse and Cerf theory.

1.6.1 Standard birth-death models

(Preliminary) standard birth-death models

We will describe the standard birth-death model for a pair of critical points of index i and $i + 1$ in $\mathbb{R}^n$, closely following Chapter 3.1 of [Cer73].

Let i be an integer such that $0 \leq i \leq n - 1$, $\lambda \in [0, 1]$ and choose $\epsilon > 0$ (ϵ will soon be chosen to be small enough to satisfy a desired condition). Consider the path

$$L : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$$

given by

$$(\lambda, x) \mapsto l_\lambda(x) := -x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_{n-1}^2 + x_n^3 - (2\lambda - 1)\epsilon x_n.$$

This is just a small modification of the birth-death path that locally characterizes the generic birth-death paths of index i . The function l_λ has:

- no critical points for $\lambda < 1/2$,
- one cusp critical point at the origin for $\lambda = 1/2$,
- two Morse critical points for $\lambda > 1/2$, namely the points $(0, \dots, 0, \pm \sqrt{\frac{(2\lambda-1)\epsilon}{3}})$.

The critical points of l_1 are the points $(0, \dots, 0, \pm \sqrt{\frac{\epsilon}{3}})$. For all $\lambda \in (1/2, 1]$, the critical points of l_λ lie on the interval $[-\sqrt{\frac{\epsilon}{3}}, \sqrt{\frac{\epsilon}{3}}]$ in the n -th coordinate. The point $(0, \dots, 0, -\sqrt{\frac{(2\lambda-1)\epsilon}{3}})$ is an index $i + 1$ Morse critical point of l_λ , while the point $(0, \dots, 0, \sqrt{\frac{(2\lambda-1)\epsilon}{3}})$ is an index i Morse critical point of l_λ and the path L is a birth path of index i .

We will now modify the path (l_λ) in order to obtain a birth - death path in $\mathbb{R}^n$ that remains constant in time outside of a compact set in $\mathbb{R}^n$. We will call such paths **compactly supported**.

Let $\omega : \mathbb{R} \rightarrow [0, 1]$ be an even, smooth bump function that is identically 1 near 0, has support equal to $[-1, 1]$ and negative derivative on $\omega^{-1}((0, 1)) \cap \{t > 0\}$, and let $\bar{\omega} : \mathbb{R}^n \rightarrow [0, 1]$ be defined as $\bar{\omega}(x_1, \dots, x_n) = \omega(|x_1|^2 + \dots + |x_n|^2)$. This makes $\bar{\omega}$ a smooth bump function in $\mathbb{R}^n$ that has compact support around the origin and furthermore depends only on the Euclidean distance to the origin. We now modify the path (l_λ) by setting

$$l_{\bar{\omega};\lambda}(x) = -x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_{n-1}^2 + x_n^3 - (2\lambda\bar{\omega}(x) - 1)\epsilon x_n,$$

where $x = (x_1, \dots, x_n)$. This modified path $(l_{\bar{\omega};\lambda})$ has the following evident properties:

- On $\bar{\omega}^{-1}(0)$, the functions l_0 and $l_{\bar{\omega};\lambda}$ coincide for all values of $\lambda \in [0, 1]$.
- On $\bar{\omega}^{-1}(1)$, the paths (l_λ) and $(l_{\bar{\omega};\lambda})$ coincide.

Therefore, the modified path $(l_{\bar{\omega};\lambda})$ is constant in time outside of a compact set and coincides with the path (l_λ) in a neighborhood of the origin, namely $\bar{\omega}^{-1}(1)$. We need to make sure that everything interesting happens within that neighborhood of the origin. By choosing ϵ small enough, it is clear that we can ensure that all critical points of (l_λ) lie within $\bar{\omega}^{-1}(1)$. We also need to ensure that by modifying (l_λ) , we haven't introduced any new critical points on the compact set $\overline{\bar{\omega}^{-1}(0, 1)}$. This can also be ensured by choosing ϵ small enough:

Proposition 1.6.1. There exists $\epsilon' > 0$ such that for any $0 < \epsilon \leq \epsilon'$, the path $l_{\bar{\omega};\lambda}$ has no critical points on the compact set $\overline{\bar{\omega}^{-1}(0, 1)}$.

Proof. We will first prove that if $(y_1, \dots, y_n)$ is a critical point of $l_{\bar{\omega};\lambda}$ in $\overline{\bar{\omega}^{-1}(0, 1)}$, we must have that $y_n \neq 0$:

Suppose that $(y_1, \dots, y_{n-1}, 0)$ is a critical point of $l_{\bar{\omega};\lambda}$. Then,

$$\begin{aligned} \frac{\partial l_{\bar{\omega};\lambda}}{\partial x_k}(y_1, \dots, y_{n-1}, 0) &= 0 \Rightarrow \\ \left[\pm 2x_k + 2\lambda \frac{\partial \bar{\omega}(x)}{\partial x_k} \epsilon x_n \right]_{(y_1, \dots, y_{n-1}, 0)} &= 0 \Rightarrow y_k = 0 \quad \forall k, \end{aligned}$$

which implies that the only possible critical point of $l_{\bar{\omega};\lambda}$ whose last coordinate is zero is the origin and it's easy to see by considering $\frac{\partial l_{\bar{\omega};\lambda}}{\partial x_n}$ that this only happens when $\lambda = 0$. In particular, no critical point of $l_{\bar{\omega};\lambda}$ in $\overline{\bar{\omega}^{-1}(0, 1)}$ can have last coordinate zero.

It, then, suffices to show that there exists $\epsilon' > 0$ such that $l_{\bar{\omega};\lambda}$ has no critical points in $\overline{\bar{\omega}^{-1}(0, 1)} \setminus \{x_n = 0\}$ for all $0 < \epsilon \leq \epsilon'$. We will show the stronger statement that there exists $\epsilon' > 0$ such that

$$\frac{\partial(x_n^3 + (2\lambda\bar{\omega}(x) - 1)\epsilon x_n)}{\partial x_n} \neq 0 \text{ in } \overline{\bar{\omega}^{-1}(0, 1)} \setminus \{x_n = 0\}, \quad \forall 0 < \epsilon \leq \epsilon'.$$

Expanding the partial derivative, we have

$$\frac{\partial(x_n^3 + (2\lambda\bar{\omega}(x) - 1)\epsilon x_n)}{\partial x_n} = 3x_n^2 + 2\lambda \frac{\partial \bar{\omega}(x)}{\partial x_n} \epsilon x_n + (2\lambda\bar{\omega}(x) - 1)\epsilon := P(\epsilon).$$

For every $\epsilon \in [0, 1]$ consider the statement: $P(\epsilon)$ has a zero in $\overline{\omega^{-1}(0, 1)} \setminus \{x_n = 0\}$. Since $[0, 1]$ is compact, this statement is either always false or there exists a minimum value of ϵ for which this statement becomes true. If the statement is always false in $[0, 1]$, let $\epsilon' = 1$. If the statement is not always false in $[0, 1]$, let $\epsilon' = \frac{1}{2} \min(\epsilon \in [0, 1] \mid P(\epsilon) \text{ is true})$. Note that in this case we can't have $\epsilon' = 0$, since $3x_n^2$ does not have a zero in $\overline{\omega^{-1}(0, 1)} \setminus \{x_n = 0\}$, which completes the proof. $\square$

To sum up, if ϵ is small enough, $l_{\bar{\omega};\lambda}$ and l_λ have the same critical points and coincide in a neighborhood of these critical points, $l_{\bar{\omega};0}$ and l_0 coincide in all of $\mathbb{R}^n$, while $l_{\bar{\omega};\lambda}$ and l_0 coincide in $\omega^{-1}(0)$.

Following Cerf, we will now modify the path $(l_{\bar{\omega};\lambda})$ in order to obtain a birth-death path in $\mathbb{R}^n$ that agrees with $(l_{\bar{\omega};\lambda})$ in a neighborhood of its critical points, and is the last coordinate projection map $x_n : \mathbb{R}^n \rightarrow \mathbb{R}$ away from a compact set. To do that, it suffices to find a diffeomorphism $\chi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that $l_0 \circ \chi(x_1, \dots, x_n) = x_n$ and then consider the family $l_{\bar{\omega};\lambda} \circ \chi$. Note that for fixed $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, the map $\mathbb{R} \rightarrow \mathbb{R}$ given by $x_n \mapsto l_0(x_1, \dots, x_{n-1}, x_n)$ is a diffeomorphism since l_0 has no critical points. Hence, the map

$$\chi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, l_0(x_1, \dots, x_n))$$

is a diffeomorphism. Moreover, $l_0 \circ \chi : \mathbb{R}^n \rightarrow \mathbb{R}$ is just the last coordinate projection map as desired.

Let $(b_\lambda) := (l_{\bar{\omega};\lambda} \circ \chi)$ be the new modified path. Since $(l_{\bar{\omega};\lambda})$ is compactly supported, the same is true for (b_λ) . We will denote by $B^k(r)$ the closed ball of radius r in $\mathbb{R}^k$, centered at 0. Let $B \times J$ be a cylinder of the form $B^{n-1}(\mu) \times B^1(\mu) \subset \mathbb{R}^n$. We choose μ large enough so that $B \times J$ contains the support of (b_λ) . The resulting cylinder $B \times J$ equipped with the path (b_λ) will be the n -dimensional **(preliminary) standard birth-death model** of index i . We will soon have to slightly modify it again, hence why it is preliminary, but not in any major way, hence why “preliminary” is parenthesized. Note that it has the following properties:

- $b_0(x) = x_n$, for all $x \in \mathbb{R}^n$,
- $b_\lambda(x) = x_n$, for all $x \in \mathbb{R}^n \setminus B \times J$ and for all $\lambda \in [0, 1]$.

Standard Morse data on b_0 and b_1

The (preliminary) standard birth-death model provides us with a model transition between two different connected components in the space of excellent Morse functions through a function containing a cusp singularity. This transition is quite natural and easy to explicitly understand at a level of functions. Indeed, if that was our sole goal, the construction of the (preliminary) standard birth-death model above would be unnecessarily complicated and the definition of birth-death paths would suffice. However, recall that for our intended application, we will consider excellent Morse functions equipped with Morse data. We would thus like to have a model transition between two different connected components in the space of excellent Morse functions equipped with Morse data through a function containing a cusp singularity. Describing such a transition is rather subtle. Informally, the difficulty lies in the fact that the Morse critical points which lie close to the cusp within a generic birth-death path admit two kinds of coordinate systems: a coordinate system that comes from the cusp lemma and a Morse coordinate system whose existence is implied by the Morse lemma. A priori, there is no natural compatibility condition between them. In order to address this issue, we will now make a choice of standard Morse data for the Morse functions b_0 and b_1 in $\mathbb{R}^n$, namely the endpoints of the birth-death path our (preliminary) standard birth-death model is equipped with. In particular, this entails a standard choice of Morse coordinate charts around the critical points of b_1 .

Morse data on b_0

The Morse function b_0 has no critical points. Therefore, in order to define Morse data associated with b_0 , it suffices to choose a gradient-like vector field on $\mathbb{R}^n$ for b_0 . Let $\mathcal{M}$ be the pull-back of the standard Euclidean metric on $\mathbb{R}^n$ by the diffeomorphism χ and let v be the gradient vector field of b_0 on $\mathbb{R}^n$ with respect to $\mathcal{M}$. The distribution defined by v constitutes the standard Morse data associated with b_0 in $\mathbb{R}^n$.

Morse data on b_1

We will now define standard Morse data associated with b_1 in $\mathbb{R}^n$. We will first define Morse data associated with l_1 , then modify them to get Morse data associated with $l_{\bar{\omega};1}$, and then pull those back via ψ to Morse data associated with b_1 in $\mathbb{R}^n$. We will start by choosing appropriate standard Morse model embeddings adapted to l_1 at c_1 and c_2 .

Recall that l_1 has two Morse critical points, the index $i + 1$ critical point $c_1 :=$

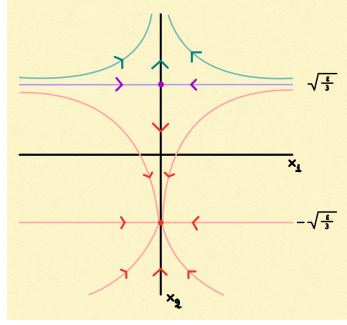


Figure 1.5: The critical points of $l_1(x_1, x_2) = -x_1^2 + x_2^3 - \epsilon x_2$ in $\mathbb{R}^2$ colored red and purple. Some flow lines are suggestively drawn and appropriately colored after the critical point they flow towards (with the exception of the x_2 -axis). The flow lines that do not flow towards a critical point are colored teal.

$(0, \dots, 0, -\sqrt{\frac{\epsilon}{3}})$ and the index i critical point $c_2 := (0, \dots, 0, \sqrt{\frac{\epsilon}{3}})$. Consider the gradient flow of l_1 with respect to the standard Euclidean metric in $\mathbb{R}^n$.

By inspecting the gradient flow equation, we see that the descending disk of c_1 (denoted by $D(c_1)$) lies in the hyperplane $\{x_{i+1} = \dots = x_{n-1} = 0\}$ and the ascending disk of c_2 (denoted by $A(c_2)$) lies in the hyperplane $\{x_1 = \dots = x_i = 0\}$. These hyperplanes intersect the 0-level set of l_1 (denoted by V_0) at the three points $(0, \dots, 0)$, $(0, \dots, \pm\sqrt{\epsilon})$ and by inspecting the direction of the gradient flow we see that only the origin $\bar{0} := (0, \dots, 0)$ further lies at the intersection of $D(c_1)$ and $A(c_2)$. V_0 is determined by the vanishing of a polynomial equation and we can readily calculate its tangent space at $\bar{0}$ to be the hyperplane $\{x_n = 0\}$. Moreover, in a neighborhood of $\bar{0}$, $D(c_1)$ and $A(c_2)$ coincide with the hyperplanes $\{x_{i+1} = \dots = x_{n-1} = 0\}$ and $\{x_1 = \dots = x_i = 0\}$, respectively. Hence, we conclude that the restrictions of $D(c_1)$ and $A(c_2)$ at V_0 intersect there transversally at a single point. We will now construct standard Morse model embeddings $k_1 : M_{i+1}^n \rightarrow \mathbb{R}^n$ and $k_2 : M_i^n \rightarrow \mathbb{R}^n$ adapted to l_1 at c_1 and c_2 respectively whose induced attaching spheres coincide with $D(c_1) \cap V_0$ and $A(c_2) \cap V_0$.

Let $k_1 : M_{i+1}^n \rightarrow \mathbb{R}^n$ and $k_2 : M_i^n \rightarrow \mathbb{R}^n$ be two standard Morse model embeddings adapted to l_1 at the critical points c_1 and c_2 , respectively. Let S_1 (respectively S_2) be the level set where $k_1(\partial_{\text{in}}(M_{i+1}^n))$ (respectively $k_2(\partial_{\text{out}}(M_i^n))$) lies and assume that k_1, k_2 are chosen so that S_1 lies above V_0 and S_2 lies below V_2 . The embeddings k_1 and k_2 push forward the standard Euclidean metric around the critical points c_1 and c_2 . Let $\mathcal{M}$ denote the Riemannian metric on $\mathbb{R}^n$ which agrees with the standard Euclidean metric outside of a neighborhood of the critical points c_1 and c_2 (we

choose the neighborhood to be contained within the standard Morse models without intersecting S_1 and S_2) and further require $\mathcal{M}$ to agree with the push forward of the standard Euclidean metric under the standard Morse model embeddings in a smaller neighborhood of the critical points. We know that such a metric always exists by a partition of unity argument. We now modify the standard Morse model embeddings k_1, k_2 to obtain embeddings r_1, r_2 respectively, that are compatible with the metric $\mathcal{M}$ and agree with k_1, k_2 on a neighborhood of the critical points. This is always possible by lemma 1.3.10. We further choose r_1, r_2 so that the embeddings $r_1(\partial_{\text{in}}(M_{i+1}^n))$ and $r_2(\partial_{\text{out}}(M_i^n))$ continue to lie on the level sets S_1 and S_2 , respectively, a requirement which uniquely specifies r_1 and r_2 . Consider the embedding

$$\tilde{r}_1 : S^i \xrightarrow{\text{id} \times \{0\}} S^i \times D^{n-i-1} \xrightarrow{r_1} S_1$$

induced by r_1 and note that $\tilde{r}_1(S^i)$ is the attaching sphere of c_1 on S_1 with respect to the Riemannian metric $\mathcal{M}$. Likewise, $\partial D(c_1)$ is the attaching sphere of c_1 on S_1 with respect to the Euclidean metric on $\mathbb{R}^n$. Since the space of Riemannian metrics is contractible, there is a path of Riemannian metrics that connects $\mathcal{M}$ with the standard Euclidean metric on $\mathbb{R}^n$. This path induces an isotopy γ of embeddings of S^i in S_1 from $\tilde{r}_1$ to an embedding whose images coincides with $D(c_1) \cap S_1$. Since S^i is compact and S_1 has no boundary, the isotopy extension lemma implies that there exists an isotopy $\tilde{\gamma}$ of S_1 that induces γ . Then, $\tilde{\gamma} \circ r_1$ is an isotopy of embeddings of the framed attaching sphere $S^i \times D^{n-i-1}$ in S_1 . By Lemma 1.3.9, there exists an isotopy Γ of standard Morse model embeddings adapted to l_1 at c_1

$$\begin{array}{ccc} M_{i+1}^n \times [0, 1] & \xrightarrow{\Gamma} & \mathbb{R}^n \\ h \times \text{id} \downarrow & & \downarrow l_1 \\ [-1, 1] \times [0, 1] & \xrightarrow{\Gamma'} & \mathbb{R} \end{array}$$

such that $\Gamma_1 = r_1$ and $\Gamma_t|_{\partial_{\text{in}}(M_{i+1}^n)} = \tilde{\gamma} \circ r_1|_{\partial_{\text{in}}(M_{i+1}^n)}$. We now define $\tilde{j}_1 := \Gamma_1$. By construction, $\tilde{j}_1$ is a standard Morse model embedding whose induced attaching sphere

$$S^i \xrightarrow{\text{id} \times \{0\}} S^i \times D^{n-i-1} \xrightarrow{\tilde{j}_1} S_1$$

coincides with $D(c_1) \cap S_1$. We modify r_2 in an entirely analogous way and obtain a standard Morse model embedding $\tilde{j}_2 : M_i^n \rightarrow \mathbb{R}^n$ whose induced attaching sphere

$$S^{n-i-1} \xrightarrow{\{0\} \times \text{id}} D^i \times S^{n-i-1} \xrightarrow{\tilde{j}_2} S_2$$

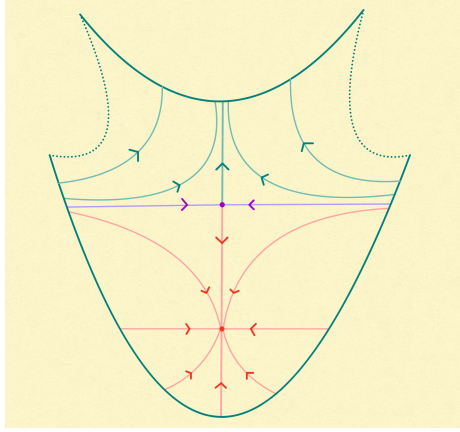


Figure 1.6: A smooth manifold triad in $\mathbb{R}^2$ which contains the critical points of l_1 and is bounded by submanifolds of level sets of l_1 . The boundary is colored teal and some gradient flow lines are pictured, appropriately colored after the target they flow towards. This is a useful mental picture for a first approximation to the standard birth-death model.

coincides with $A(c_2) \cap S_2$. We will now extend $\tilde{j}_1$ by using the gradient flow of l_1 with respect to $\mathcal{M}$ from V_0 to S_1 and define j_1 to be the induced standard Morse model embedding. Since $\mathcal{M}$ agrees with the standard Euclidean metric in the region of $\mathbb{R}^n$ bounded by S_1 and V_0 , the embedding

$$S^i \xrightarrow{\text{id} \times \{0\}} S^i \times D^{n-i-1} \xrightarrow{j_1} V_0$$

coincides with $D(c_1) \cap V_0$. We likewise extend $\tilde{j}_2$ to a standard Morse model embedding j_2 such that

$$S^{n-i-1} \xrightarrow{\{0\} \times \text{id}} D^i \times S^{n-i-1} \xrightarrow{j_2} V_0$$

coincides with $A(c_2) \cap V_0$. So far, j_1 and j_2 are two standard Morse model embeddings whose attaching spheres intersect transversally at a single point in the level set V_0 . We would like to have some tighter control on the intersection of $j_1(S^i \times D^{n-i-1})$ and $j_2(D^i \times S^{n-i-1})$ in V_0 , namely some tighter control on how the framings behave in this intersection. Intuitively, and as illustrated in Figure 1.7, we would like the intersection of $j_1(S^i \times D^{n-i-1})$ with $j_2(D^i \times S^{n-i-1})$ to be diffeomorphic to $D^i \times D^{n-i-1}$ in a way that the factor D^i (respectively D^{n-i-1}) pulls back to the fiber of S^{n-i-1} (respectively S^i). We will now formalize this intuitive picture.

Definition 1.6.1 (Critical points in cancelling position). Let W be a smooth n -dimensional manifold equipped with a Morse function $f : W \rightarrow \mathbb{R}$ that has two critical points c_1, c_2 of index $i+1$ and i , respectively. Let $\phi_1 : M_{i+1}^n \rightarrow W$ and

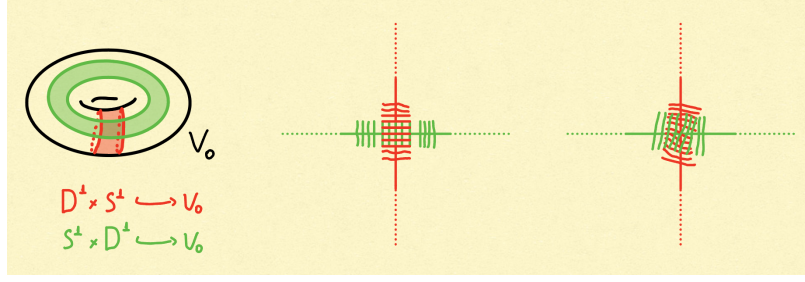


Figure 1.7: Framed attaching spheres in framed vs non-framed cancelling position.

$\phi_2 : M_i^n \rightarrow W$ be two standard Morse model embeddings adapted to f at c_1 and c_2 respectively. We say that c_1 and c_2 are in **cancelling position with respect to** ϕ_1, ϕ_2 if:

- The induced embeddings of framed attaching spheres $\phi_1|_{S^i \times D^{n-i-1}}$ and $\phi_2|_{D^i \times S^{n-i-1}}$ lie at the same level set V_0 .
- $\phi_1(S^i \times \{0\})$ and $\phi_2(\{0\} \times S^{n-i-1})$ intersect transversally at a single point in V_0 .

We say that c_1 and c_2 are in **cancelling position** if they are in cancelling position with respect to some standard Morse model embeddings ϕ_1 and ϕ_2 .

Definition 1.6.2 (Standard Morse model embeddings in cancelling position). Let $W, f, c_1, c_2, \phi_1, \phi_2$ be as above and assume that c_1, c_2 are in cancelling position with respect to ϕ_1, ϕ_2 . We say that the standard Morse model embeddings are in cancelling position if:

- there exist diffeomorphisms $r_1 \in O(n-i-1), r_2 \in O(i)$,
- an embedding $s_1 : D^i \rightarrow S^i$ that coincides with the northern hemisphere inclusion up to the right action by $O(i+1)$,
- an embedding $s_2 : D^{n-i-1} \rightarrow S^{n-i-1}$ that coincides with the northern hemisphere inclusion up to the right action by $O(n-i)$,
- such that the following diagram commutes:

$$\begin{array}{ccc}
 D^i \times D^{n-i-1} & \xrightarrow{s_1 \times r_1} & S^i \times D^{n-i-1} \\
 r_2 \times s_2 \downarrow & & \downarrow \phi_1 \\
 D^i \times S^{n-i-1} & \xrightarrow{\phi_2} & W
 \end{array}$$

Proposition 1.6.2. There is a choice of standard Morse model embeddings in cancelling position adapted to the critical points of l_1 in $\mathbb{R}^n$.

Proof. Here is a sketch of a proof. We have already constructed the standard Morse model embeddings j_1 and j_2 with respect to which the critical points of l_1 are in cancelling position. We need to modify j_1 and j_2 so that they are themselves in cancelling position. We can show more generally that this is always possible. Namely, let $W, f, c_1, c_2, \phi_1, \phi_2$ be as in definition 1.6.1, assume that c_1 and c_2 are in cancelling position with respect to ϕ_1, ϕ_2 and let V_0 be the level set where ϕ_1 and ϕ_2 meet. Then, we observe that $\phi_1|_{S^i \times D^{n-i-1}}$ and $\phi_2|_{D^i \times S^{n-i-1}}$ are isotopic in V_0 to embeddings ψ_1 and ψ_2 that make the diagram

$$\begin{array}{ccc} D^i \times D^{n-i-1} & \xrightarrow{s_1 \times r_1} & S^i \times D^{n-i-1} \\ r_2 \times s_2 \downarrow & & \downarrow \psi_1 \\ D^i \times S^{n-i-1} & \xrightarrow{\psi_2} & W \end{array}$$

commute, wherein s_1, s_2, r_1, r_2 are as in definition 1.6.2. To see this, let's assume that $\phi_1(S^i \times \{0\}) \cap \phi_2(\{0\} \times S^{n-i-1}) = \{p\} \in V_0$. Since this intersection is transversal, there is a local coordinate system around $p \in V_0$ that makes $\phi_1(S^i \times \{0\}) \cup \phi_2(\{0\} \times S^{n-i-1})$ locally look like the $\mathbb{R}^i \cup \mathbb{R}^{n-i-1}$ hyperplane union in $\mathbb{R}^{n-1}$. We can then isotope the framings within this chart to intersect as in the definition 1.6.2. Then, by lemma 1.3.9, we can isotope ϕ_1 and ϕ_2 so that they assume cancelling position. $\square$

(Final) standard birth-death models

The arguments presented above are not specific to the function l_1 in $\mathbb{R}^n$, and apply equally well to $l_{\bar{\omega},1}$ and, by composing everything with the diffeomorphism ψ , to b_1 . Let ϕ_1 and ϕ_2 be two such standard Morse model embeddings and let U_1 and U_2 be their corresponding images. We choose a Riemannian metric on $\mathbb{R}^n$ that is compatible with ϕ_1 and ϕ_2 . Let's suppose that the “double neighborhood” $U_1 \cup U_2$ lies between level sets L_1 and L_2 in $\mathbb{R}^n$. Using the gradient flow of the chosen Riemannian metric, we can construct a “saturated double neighborhood” U that lies between L_1 and L_2 characterized by the fact that all points of U either lie in $U_1 \cup U_2$ or on a gradient flow line that leads to $U_1 \cup U_2$. Furthermore, by Lemma 2 in Chapter 3.2.2 of [Cer73], we can choose ϕ_1 and ϕ_2 so that U contains $B \times J$. We choose such ϕ_1 and ϕ_2 once and for all. The corresponding U will be the standard saturated double neighborhood of index i in $\mathbb{R}^n$ and we will be denoting it by $U_{(n,i)}^{\text{std}}$ or simply by $U_{(n,i)}^{\text{std}}$ when (n, i) is implicit or irrelevant. This is naturally a smooth manifold triad.

Definition 1.6.3 (Standard birth-death model). The n -dimensional standard birth-death model of index i consists of the smooth manifold triad $U_{(n,i)}^{\text{std}}$ equipped with the birth-death path $(b_\lambda)_{\lambda \in [0,1]}$. It comes further equipped with **standard Morse data associated with b_0 and b_1** as follows:

- Standard Morse data associated with b_0 :
 1. b_0 has no critical points, and hence admits no standard Morse model embeddings.
 2. The associated horizontal distribution defined by the Riemannian metric we chose in the construction of $U_{(n,i)}^{\text{std}}$.
 3. The chosen regular level sets consist of the incoming and outgoing boundary of $U_{(n,i)}^{\text{std}}$.

Equipped with these data, $U_{(n,i)}^{\text{std}}$ becomes diffeomorphic to a cylinder $D^{n-1} \times \Xi$, where Ξ is the interval $\text{im}(b_1)$. We choose such a diffeomorphism once and for all (this amounts to just an identification of $\partial_{\text{in}} U_{(n,i)}^{\text{std}}$ with D^{n-1}).

- Standard Morse data associated with b_1 :
 1. The standard Morse model embeddings ϕ_1 and ϕ_2 that are in cancelling position as chosen above.
 2. The associated horizontal distribution defined by the Riemannian metric we chose in the construction of $U_{(n,i)}^{\text{std}}$.
 3. The chosen regular level sets consist of the incoming and outgoing boundary of $U_{(n,i)}^{\text{std}}$ and the 0-level set of b_1 .

1.6.2 Elementary birth-death paths

Elementary birth paths

Definition 1.6.4. Let $f : W \rightarrow I$ be an excellent Morse function. An embedding

$$\phi : U_{(n,i)}^{\text{std}} \rightarrow W$$

is **adapted to f with respect to b_λ** if there exists an orientation preserving embedding $\phi' : \Xi \rightarrow I$ such that the diagram

$$\begin{array}{ccc} U_{(n,i)}^{\text{std}} & \xrightarrow{\phi} & W \\ b_\lambda \downarrow & & \downarrow f \\ \Xi & \xrightarrow{\phi'} & I \end{array}$$

commutes.

A birth path $(f_\lambda)_{\lambda \in [0,1]}$ that starts at $f \in \mathcal{F}^0(W)$ is called an **elementary birth path of index i** if there exists an embedding $\phi : U_{(n,i)}^{\text{std}} \rightarrow W$ adapted to f with respect to b_0 such that for all $\lambda \in [0, 1]$ we have

1. $f_\lambda = f$ in the complement of the image of ϕ , and
2. the diagram

$$\begin{array}{ccc} U_{(n,i)}^{\text{std}} & \xrightarrow{\phi} & W \\ b_\lambda \downarrow & & \downarrow f_\lambda \\ \Xi & \xrightarrow{\phi'} & I \end{array}$$

commutes for all $\lambda \in [0, 1]$.

Elementary death paths

Definition 1.6.5. A death path $(f_\lambda)_{\lambda \in [0,1]}$ is called an **elementary death path of index i** if $(f_{1-\lambda})_{\lambda \in [0,1]}$ is an elementary birth path of index i .

Remark 1.6.1. Let $f \in \mathcal{F}^0(W)$ be an excellent Morse function with two cancellable critical points c_1 and c_2 . A choice of Morse data associated with f that includes two standard Morse model embeddings in cancelling position adapted to f at c_1 and c_2 , uniquely determines an embedding of the standard birthdeath model adapted to f and hence an elementary death path starting from f .

Elementary birth-death paths play a crucial role in the constructions that follow. In particular, it will be important to understand some homotopical information about the space of all birth - death paths relative to the space of all elementary birth - death paths. Thankfully, we only need to understand a modest amount that has mostly been covered in Cerf's work.

Lemma 1.6.2. *Let f be an excellent Morse function on W . Let $\mathbf{Birth}_f$ (respectively $\mathbf{Death}_f$) denote the space of all birth paths (respectively death paths) that start at f and let $\mathbf{EBirth}_f$ (respectively $\mathbf{EDeath}_f$) denote the space of all elementary birth paths (respectively elementary death paths) that start at f . Then,*

$$\pi_0(\mathbf{Death}_f, \mathbf{EDeath}_f) = 0$$

and

$$\pi_0(\mathbf{Birth}_f, \mathbf{EBirth}_f) = 0.$$

Proof. See Chapter 3 of [Cer73]. □

Lemma 1.6.3. *Let f be an excellent Morse function on W and let γ_0, γ_1 be elementary birth paths (respectively elementary death paths) starting at f induced by standard birth - death model embeddings ϕ_0 and ϕ_1 adapted to f with respect to b_0 (respectively b_1). If there exists a path of birth paths (respectively a path of death paths) between γ_0 and γ_1 , then there exists an isotopy of standard birth - death model embeddings adapted to f with respect to b_0 (respectively b_1) between ϕ_0 and ϕ_1 .*

Remark 1.6.4. We can interpret this lemma as a weak version of the statements

$$\pi_1(\mathfrak{Death}_f, \mathfrak{EDeath}_f) = 0$$

and

$$\pi_1(\mathfrak{Birth}_f, \mathfrak{EBirth}_f) = 0.$$

While I believe these statements are true, I don't currently have a complete proof of them. Thankfully, the weaker statement above is sufficient for our purposes and is substantially easier to prove.

Proof. We will address the case of birth paths. The case of death paths can be reduced to it using the stability of Morse functions.

Let g_0 and g_1 be the endpoint excellent Morse functions of γ_0 and γ_1 , respectively. Since there is a path of birth paths between γ_0 and γ_1 , there exists a path of excellent Morse functions between g_0 and g_1 . This can only happen if the embeddings ϕ_0 and ϕ_1 are such that $\text{im}(f \circ \phi_0)$ and $\text{im}(f \circ \phi_1)$ are not separated by any critical values. Let W' be a smooth manifold triad that is a submanifold of W containing $\text{im}(\phi_0)$ and $\text{im}(\phi_1)$ such that $f|_{W'}$ has no critical points. In order for γ_0 and γ_1 to be connected via a path of excellent Morse functions, we need to further have that the embeddings ϕ_0 and ϕ_1 lie in the same connected component of W' . Recall that the standard birth - death model equipped with b_0 is diffeomorphic to the cylinder $D \times \Xi$. The embeddings ϕ_0 and ϕ_1 are, then, determined up to isotopy and up to possibly a reflection (which doesn't effect the resulting elementary birth - death path) by the image of D . Any two embeddings of D in the same connected component of W' must be isotopic, which completes the proof. □

1.6.3 Standard descending models

We will now present the analogous constructions for crossing paths, closely following Chapter 2 of [Cer73].

Consider the n -dimensional standard Morse model of index k , M_k^n equipped with the standard Morse function $h : M_k^n \rightarrow [-1, 1]$. Choose a bump function $\omega : \mathbb{R} \rightarrow [0, 1]$, that satisfies the following conditions:

- $\omega(0) = 1$ and $\frac{d\omega}{dt}(0) = \frac{d^2\omega}{dt^2}(0) = 0$,
- the support of ω is $[-1, 1]$,
- ω is invariant under the involution $t \mapsto -t$, and
- $\frac{d\omega}{dt}$ is strictly negative on $(0, 1)$.

We now define the bump function $\bar{\omega} : \mathbb{R}^k \times \mathbb{R}^{n-k} \rightarrow [0, 1]$ given by

$$\bar{\omega}(x, y) = \omega(-|x|^2 + |y|^2) \cdot \omega\left(\frac{|x||y|}{\sinh(1) \cosh(1)}\right).$$

Note that the support of $\bar{\omega}$ is exactly the closure of M_k^n in $\mathbb{R}^n$. Choose $\epsilon > 0$ small enough such that for $x \in M_k^n$ and $\lambda \in [0, 1]$, the path (h_λ) defined by

$$h_\lambda(x) = h(x) - \lambda\epsilon\bar{\omega}(x)$$

has a single critical point throughout, namely the origin $0 \in M_k^n$ whose critical value traces the corresponding path $-\lambda\epsilon$. This is possible by an argument similar to the one used in the construction of the standard birth - death models above. We will refer to M_k^n equipped with the path (h_λ) as the **n -dimensional standard descending model of index k** .

1.6.4 Elementary crossing paths

Elementary descending paths

Definition 1.6.6. A path $(f_\lambda)_{\lambda \in [0, 1]}$ that starts at $f \in \mathcal{F}^0(W)$ is called an **elementary descending path of index k** if there exists a standard Morse model embedding

$$\begin{array}{ccc} M_k^n & \xrightarrow{\phi} & W \\ h \downarrow & & \downarrow f \\ \Xi & \xrightarrow{\phi'} & I \end{array}$$

such that for all $\lambda \in [0, 1]$ we have

1. $f_\lambda = f$ in the complement of the image of ϕ , and
2. the diagram

$$\begin{array}{ccc} M_k^n & \xrightarrow{\phi} & W \\ h_\lambda \downarrow & & \downarrow f_\lambda \\ \Xi & \xrightarrow{\phi'} & I \end{array}$$

commutes for all $\lambda \in [0, 1]$.

Elementary ascending paths

Definition 1.6.7. A path $(f_\lambda)_{\lambda \in [0,1]}$ is called an **elementary ascending path of index k** if $(f_{1-\lambda})_{\lambda \in [0,1]}$ is an elementary descending path of index k .

Remark 1.6.5. Given a collection of disjoint standard Morse model embeddings adapted to f at different critical points, it is straightforward now to define a path that is simultaneously ascending / descending with respect to them.

Elementary crossing paths

The following lemma plays a key role in justifying the definition of elementary crossing paths which follows.

Lemma 1.6.6 (Lemma 4, Chapter 2.1.2, p.42 in [Cer73]). *Let f be an excellent Morse function on W with a critical point c of index k . Let D be a descending disk of c with respect to some Riemannian metric and let δ be the lowest value in the interval $f(D)$. Then, there exists a standard Morse model embedding $\phi : M_k^n \rightarrow W$ adapted to f at c such that the following hold:*

1. *the descending disk defined by ϕ is D ,*
2. *the image of ϕ is contained in an arbitrary small neighborhood of D , and*
3. *the elementary descending path (f_λ) defined by ϕ is such that $f_1(c)$ is any given point in the open set $(\delta, f(c))$.*

Definition 1.6.8. Let f be an excellent Morse function on W that has two consecutive critical points c_0 and c_1 with $f(c_0) < f(c_1)$. A crossing path $(f_\lambda)_{\lambda \in [0,1]}$ starting at f such that $f_1(c_0) > f_1(c_1)$ is called an **elementary crossing path** if it is simultaneously an elementary ascending path for c_0 and an elementary descending path for c_1 with respect to two disjoint standard Morse model embeddings.

Remark 1.6.7. Cerf's definition of elementary crossing paths is slightly different, as it involves only an elementary descending path for c_1 . Here, we opt for a more symmetric definition. This is not a substantial difference, but rather a matter of style. It is fairly straightforward to see that the space of elementary crossing paths starting at f , as defined by Cerf, is homotopy equivalent to the space of elementary crossing paths starting at f as defined here.

As with elementary birth-death paths, elementary crossing paths play a crucial role in the constructions that follow. In particular, it will be important to understand some homotopical information about the space of all crossing paths relative to the space of all elementary crossing paths. Thankfully, we only need to understand a modest amount that has already been covered in Cerf's work. We will denote by $\mathbf{Cross}_f$ the space of all crossing paths starting at f and by $\mathbf{ECross}_f$ the space of all elementary crossing paths starting at f . We will further denote by $\overline{\mathbf{ECross}_f}$ the Kan complex whose points are the points in $\mathbf{ECross}_f$ and whose higher simplices are induced by n -parameter standard Morse model embeddings.

Lemma 1.6.8 ([Cer73, Proposition 2, p. 49]). *Let f be an excellent Morse function on W . The inclusion*

$$\mathbf{ECross}_f \hookrightarrow \mathbf{Cross}_f$$

induces isomorphisms

$$\pi_0(\overline{\mathbf{ECross}_f}) \xrightarrow{\cong} \pi_0(\mathbf{Cross}_f)$$

and, for all $\beta \in \mathbf{ECross}_f$ and all $j \geq 1$,

$$\pi_j(\overline{\mathbf{ECross}_f}; \beta) \xrightarrow{\cong} \pi_j(\mathbf{Cross}_f; \beta).$$

Remark 1.6.9. We defined the spaces $\mathbf{Birth}_f$, $\mathbf{Death}_f$, $\mathbf{EBirth}_f$, and $\mathbf{EDeath}_f$ in Lemma 1.6.2. Write $\overline{\mathbf{EBirth}_f}$ (resp. $\overline{\mathbf{EDeath}_f}$) for the Kan complex whose 0-simplices are the points of $\mathbf{EBirth}_f$ (resp. $\mathbf{EDeath}_f$) and whose higher simplices are induced by n -parameter standard birth (resp. death) model embeddings. In light of Lemma 1.6.3 and Lemma 1.6.8, one may ask whether, for all $\beta \in \mathbf{EBirth}_f$ and all $j \geq 1$,

$$\pi_j(\overline{\mathbf{EBirth}_f}; \beta) \xrightarrow{\cong} \pi_j(\mathbf{Birth}_f; \beta)$$

and similarly, for all $\beta \in \mathbf{EDeath}_f$ and all $j \geq 1$,

$$\pi_j(\overline{\mathbf{EDeath}_f}; \beta) \xrightarrow{\cong} \pi_j(\mathbf{Death}_f; \beta).$$

Following Cerf's proof of Lemma 1.6.8, one can approach this via Cerf's *lemme des chemins élémentaires* [Cer73, p. 200]. It implies that

$$\pi_j(\overline{\mathfrak{EDeath}}_f; \beta) \xrightarrow{\cong} \pi_j(\mathfrak{Death}_f; \beta)$$

if and only if $\pi_j(\overline{\mathfrak{EDeath}}_\beta; \beta)$ is trivial. Here $\mathfrak{EDeath}_\beta$ denotes the Kan complex whose points are the elementary death paths having the same intersection with $\mathcal{F}^1$ as β , and whose higher simplices are induced by n -parameter standard Morse model embeddings. The analogous statement holds for $\mathfrak{EBirth}_f$.

1.7 Cerf homotopies

In the previous section, we introduced the space $\mathcal{F}^1(W)$, which satisfied the property that any two excellent Morse functions on W are connected via a Cerf path in $\mathcal{F}^0(W) \cup \mathcal{F}^1(W)$. In this section, we will introduce the space $\mathcal{F}^2(W)$ and the notion of Cerf homotopies. These will, analogously, satisfy the property that any two Cerf paths in $\mathcal{F}^0(W) \cup \mathcal{F}^1(W)$ are connected via a Cerf homotopy in $\mathcal{F}^0(W) \cup \mathcal{F}^1(W) \cup \mathcal{F}^2(W)$. We will closely follow Chapter 1 of [Cer73] and Section 2 of [GK15].

Definition 1.7.1 (A_3 critical points). Let $p \in W$ be a critical point of some smooth function f . We say that p is an A_3 **critical point of index** i if there exists a local coordinate chart centered at p in which f takes the form:

$$-x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_{n-1}^2 \pm x_n^4.$$

Definition 1.7.2 (Codimension of critical points). Let $p \in W$ be a critical point of some smooth function f .

1. If p is a Morse critical point, we define its codimension to be 0.
2. If p is a cusp critical point, we define its codimension to be 1.
3. If p is an A_3 critical point, we define its codimension to be 2.

Definition 1.7.3 (Codimension of a smooth function). Let f be a smooth function on W that only admits Morse or cusp or A_3 critical points. Let

$$v_1(f) = \text{sum of the codimensions of the critical points of } f,$$

$$v_2(f) = \text{sum of the multiplicities of the critical values of } f.$$

We then define the **codimension of** f to be

$$\text{codim}(f) = v_1(f) + v_2(f).$$

Equipped with this definition, $\mathcal{F}^0(W)$ consists of the codimension 0 smooth functions on W and $\mathcal{F}^1(W)$ consists of the codimension 1 smooth functions on W .

Definition 1.7.4. $\mathcal{F}^2(W)$ consists of the codimension 2 smooth functions on W . Unwinding this definition, we have the following cases for $f \in \mathcal{F}^2(W)$:

- (a) $v_1(f) = 2$ and $v_2(f) = 0$, which in turn consists of two cases:
 - (i) f has a single A_3 critical point, and all other critical points of f are of Morse type. Moreover, no two critical points of f share the same critical value. We denote by $\mathcal{F}_\alpha^2$ the subset of $\mathcal{F}^2$ consisting of functions with this property.
 - (ii) f has two cusp critical points and all other critical points of f are of Morse type. Moreover, no two critical points of f share the same critical value.
- (b) $v_1(f) = v_2(f) = 1$, which in turn consists of two cases:
 - (i) f has a cusp critical point, and all other critical points of f are of Morse type. There is a Morse critical point that shares the same critical level with the cusp critical point and no other two critical points of f share the same critical level. We denote by $\mathcal{F}_\beta^2$ the subset of $\mathcal{F}^2$ consisting of functions with this property.
 - (ii) f has a cusp critical point and all other critical points of f are of Morse type. There are two Morse critical points of f that share the same critical level and no other two critical points of f share the same critical level.
- (c) $v_1(f) = 0$ and $v_2(f) = 2$, which in turn consists of two cases:
 - (i) f only has Morse critical points. There are three Morse critical points that share the same critical level and no other two critical points of f share the same critical level. We denote by $\mathcal{F}_\gamma^2$ the subset of $\mathcal{F}^2$ consisting of functions with this property.
 - (ii) f only has Morse critical points. There are two pairs of Morse critical points that share the same critical level and no other two critical points of f share the same critical level.

Informally, Cerf homotopies are smooth maps

$$\gamma : [0, 1] \times [0, 1] \rightarrow \mathcal{F}^0(W) \cup \mathcal{F}^1(W) \cup \mathcal{F}^2(W)$$

that intersect the codimension 1 and codimension 2 strata transversally. Formally, we use the following definition from [GK15]:

Definition 1.7.5 (Cerf homotopies). Let $\gamma_0(s)$ and $\gamma_1(s)$ be two Cerf paths in $\mathcal{F}^0(W) \cup \mathcal{F}^1(W)$. A **Cerf homotopy** $\Gamma(s, t)$ between them is a homotopy through

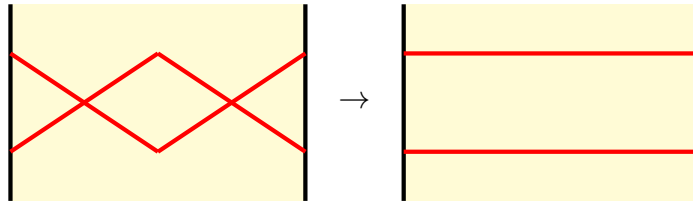
$$\mathcal{F}^0(W) \cup \mathcal{F}^1(W) \cup \mathcal{F}^2(W)$$

such that, for each fixed s , the map $t \mapsto \Gamma(s, t)$ is a Cerf path, except at finitely many values $s_0, \dots, s_m \in [0, 1]$. At each such exceptional value s_i , there exists a unique time $t_i \in [0, 1]$ such that:

- $\Gamma(s_i, t)|_{t \in [0, q]}$ is a Cerf path for all $q < t_i$, and
- $\Gamma(s_i, t)|_{t \in [q, 1]}$ is a Cerf path for all $q > t_i$.

At each such singular point (s_i, t_i) , one of the following events occurs (up to reversing the parameter s or t):

1. $\Gamma(s_i, t_i) \in \mathcal{F}_\beta^1$, but the 1-parameter family $t \mapsto \Gamma(s_i, t)$ fails to be a Cerf path because it does not cross $\mathcal{F}_\beta^1$ generically. Around (s_i, t_i) , the Cerf graphics of $\Gamma(s, t)$ change (in the s -parameter) via a *Reidemeister II* move, as shown:

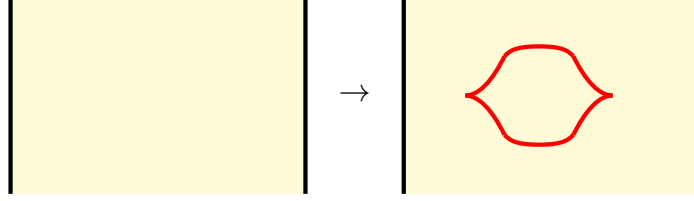


2. $\Gamma(s_i, t_i) \in \mathcal{F}_\alpha^1$, but the 1-parameter family $t \mapsto \Gamma(s_i, t)$ fails to be a Cerf path because it does not cross $\mathcal{F}_\alpha^1$ generically. Let $p \in W$ be the cusp critical point of $\Gamma(s_i, t_i)$. There are two cases to consider:

- (a) There exist $\delta, \epsilon > 0$ such that in a neighborhood of p and $\Gamma_{s_i, t_i}(p)$, there is a coordinate system $(x_1, \dots, x_n)$ for which, whenever $|s - s_i| < \delta$ and $|t - t_i| < \epsilon$, the map $\Gamma(s, t)$ is given by:

$$\Gamma_{s, t}(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + (t - t_i)^2 x_n - (s - s_i)^2 x_n + x_n^3.$$

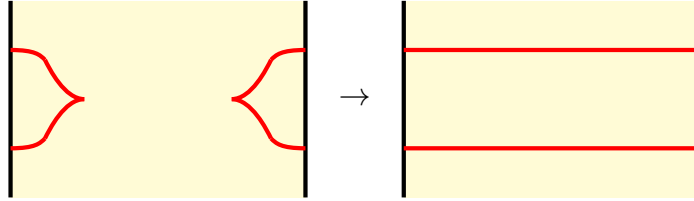
Moreover, there are no other singularities in the preimage of a small neighborhood of $\Gamma_{s_i, t_i}(p)$. Around (s_i, t_i) , the Cerf graphics of $\Gamma(s, t)$ change via an *eye birth-death* move, as shown:



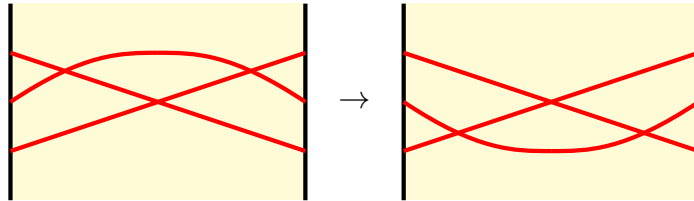
(b) Similarly, if the local model is instead:

$$\Gamma_{s,t}(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 - (t-t_i)^2 x_n - (s-s_i)^2 x_n + x_n^3,$$

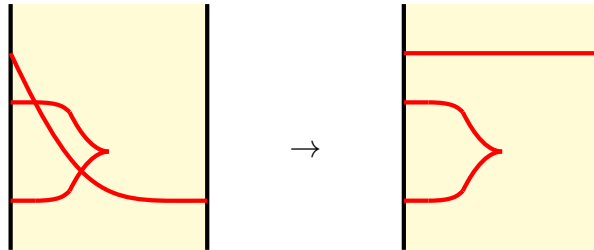
then around (s_i, t_i) , the Cerf graphics of $\Gamma(s, t)$ change via a *merge-unmerge* move, as shown:



3. $\Gamma(s_i, t_i) \in \mathcal{F}_\gamma^2$. Around (s_i, t_i) , the Cerf graphics of $\Gamma(s, t)$ change via a *Reidemeister III* move, as shown:



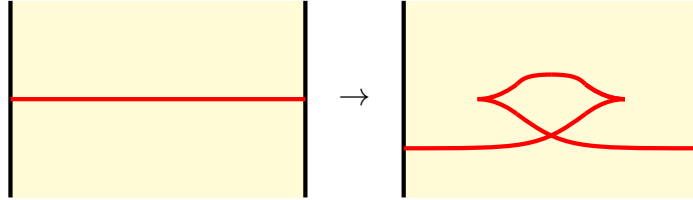
4. $\Gamma(s_i, t_i) \in \mathcal{F}_\beta^2$. Around (s_i, t_i) , the Cerf graphics of $\Gamma(s, t)$ change via a *beak* move, as shown:



5. $\Gamma(s_i, t_i) \in \mathcal{F}_\alpha^2$. Let $p \in W$ be the unique critical point of $\Gamma(s_i, t_i)$ of type A_3 . There exist $\delta, \epsilon > 0$ such that in a neighborhood of p and $\Gamma_{s_i, t_i}(p)$, there is a coordinate system $(x_1, \dots, x_n)$ for which, whenever $|s - s_i| < \delta$ and $|t - t_i| < \epsilon$, the map $\Gamma(s, t)$ is given by:

$$\Gamma_{s,t}(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + (t - t_i)^2 x_n - (s - s_i)^2 x_n \pm x_n^4.$$

Moreover, there are no other singularities in the preimage of a small neighborhood of $\Gamma_{s_i, t_i}(p)$. Around (s_i, t_i) , the Cerf graphics of $\Gamma(s, t)$ change via a *swallowtail* move, as shown:



This Cerf graphic move corresponds to the $+x_n^4$ case. Its upside-down reflection corresponds to the $-x_n^4$ case.

6. $\Gamma(s_i, t_i) \in \mathcal{F}^2 \setminus (\mathcal{F}_\alpha^2 \cup \mathcal{F}_\beta^2 \cup \mathcal{F}_\gamma^2)$. This occurs when we have two codimension 1 phenomena occurring at the same time. For example, $\Gamma(s_i, t_i)$ might have two cusp critical points. We require that around (s_i, t_i) , $\Gamma(s_i, t_i)$ crosses $\mathcal{F}^2$ generically in the obvious way.

The following result is widely regarded as standard:

Theorem 1.7.1. *The space of Cerf homotopies between two Cerf paths is a generic subspace of the space of all homotopies.*

This theorem appears as early as in the work of Hatcher–Wagoner [HW73], and more recently in that of Gay–Kirby [GK15]. However, a single, self-contained reference that explicitly provides a complete proof is not readily available in the literature. In this thesis, we only require a mild application of genericity—namely, the existence of a Cerf homotopy between any two given Cerf paths. For this purpose, it suffices to prove the following:

Theorem 1.7.2. *The space of Cerf homotopies between two Cerf paths is a dense subspace of the space of all homotopies.*

Proof. The proof proceeds in two main steps. The first step analyzes the local behavior of generic 2-parameter families using Thom's jet transversality theorem. The second step addresses the global structure by invoking Thom's multijet transversality theorem.

Step 1: Local structure via jet transversality.

We apply Thom's jet transversality theorem to the space of smooth maps $\phi : M \times I^2 \rightarrow \mathbb{R}$ to determine the local singularities that generically arise in 2-parameter families of functions. The theorem implies that the only such singularities are of types A_1 (Morse), A_2 (cusp), and A_3 (swallowtail). These correspond to codimension 0, 1, and 2 strata in the jet space. To fully understand the local behavior, we must also analyze how these singularities unfold within 2-parameter families—a notion we will make precise shortly. This leads us to conclude that the only local phenomena that occur in generic 2-parameter families correspond to the swallowtail, eye birth-death and merge-unmerge moves.

Consider the jet space $J^3(M \times I^2, \mathbb{R})$ and let $n = \dim(M)$. A point in this space can be explicitly described by the tuple:

$$\left(x, s, t; f(x, s, t); \frac{\partial f}{\partial x}, \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}; \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial s}, \frac{\partial^2 f}{\partial x \partial t}, \frac{\partial^2 f}{\partial s^2}, \frac{\partial^2 f}{\partial s \partial t}, \frac{\partial^2 f}{\partial t^2}; \right. \\ \left. \frac{\partial^3 f}{\partial x^3}, \frac{\partial^3 f}{\partial x^2 \partial s}, \frac{\partial^3 f}{\partial x^2 \partial t}, \frac{\partial^3 f}{\partial x \partial s^2}, \frac{\partial^3 f}{\partial x \partial s \partial t}, \frac{\partial^3 f}{\partial x \partial t^2}, \frac{\partial^3 f}{\partial s^3}, \frac{\partial^3 f}{\partial s^2 \partial t}, \frac{\partial^3 f}{\partial s \partial t^2}, \frac{\partial^3 f}{\partial t^3} \right)$$

Here, $x = (x_1, \dots, x_n)$ denotes a point in M , s and t are the two coordinates on I^2 , and f is a germ defined around the point (x, s, t) . We will often write $f(x, s, t) := f_{s,t}(x)$. The partial derivative $\frac{\partial f}{\partial x}$ denotes the n -tuple $\left(\frac{\partial f}{\partial x_i} \right)_{1 \leq i \leq n}$, and similar conventions apply for the higher derivatives. In particular, $\frac{\partial^2 f}{\partial x^2}$ is the n^2 -tuple consisting of the entries of the Hessian of $f_{s,t}$ at x , and $\frac{\partial^3 f}{\partial x^3}$ is the n^3 -tuple consisting of the entries of the third derivative tensor of $f_{s,t}$ at x .

We now define the following submanifolds of $J^3(M \times I^2, \mathbb{R})$:

- $\Sigma^{1,0,0}$: consists of all points in $J^3(M \times I^2, \mathbb{R})$ for which $\frac{\partial f}{\partial x} = 0$ and $\det(\text{Hess}(f_{s,t}(x))) \neq 0$. This imposes n independent conditions, so $\Sigma^{1,0,0}$ is a codimension n submanifold of $J^3(M \times I^2, \mathbb{R})$.
- $\Sigma^{1,1,0}$: consists of all points for which $\frac{\partial f}{\partial x} = 0$ and the Hessian $\text{Hess}(f_{s,t}(x))$ has a 1-dimensional kernel. Let $v \in T_x M$ be a nonzero vector spanning this kernel. We require, in addition, that the third derivative tensor evaluated on (v, v, v) is

nonzero. This imposes $n + 1$ independent conditions, so $\Sigma^{1,1,0}$ is a codimension $n + 1$ submanifold.

- $\Sigma^{1,1,1}$: consists of all points for which $\frac{\partial f}{\partial x} = 0$ and $\text{Hess}(f_{s,t}(x))$ has a 1-dimensional kernel. Let $v \in T_x M$ again span this kernel. We require that the third derivative tensor evaluated on (v, v, v) vanishes. This imposes $n + 2$ independent conditions, so $\Sigma^{1,1,1}$ is a codimension $n + 2$ submanifold.

(If the Hessian has kernel of dimension > 1 one obtains strata of strictly larger codimension.)

By Thom's jet transversality theorem a generic F satisfies

$$j^3 F \pitchfork \Sigma^{1,0,0}, \quad j^3 F \pitchfork \Sigma^{1,1,0}, \quad j^3 F \pitchfork \Sigma^{1,1,1}.$$

Hence

- $(j^3 F)^{-1}(\Sigma^{1,0,0}) \subset M \times I^2$ is a submanifold of codimension n , so it has dimension $(n + 2) - n = 2$. These are the A_1 (Morse) singularities.
- $(j^3 F)^{-1}(\Sigma^{1,1,0})$ has codimension $n + 1$, hence dimension $(n + 2) - (n + 1) = 1$. These are the A_2 (cusp) singularities.
- $(j^3 F)^{-1}(\Sigma^{1,1,1})$ has codimension $n + 2$, hence dimension $(n + 2) - (n + 2) = 0$. These are the A_3 (swallowtail) singularities.

In the A_3 case, transversality at the 3-jet level forces the third derivative in the kernel direction to vanish; transversality lifted to the 4-jet level then forces the fourth derivative in that direction to be nonzero. (One can make this explicit by considering the natural strata in J^4 .)

To make the consequences of the jet transversality argument more concrete, we look at the local algebraic models, or *normal forms*, for the singularities that arise. A celebrated classification theorem due to Arnold states that an A_k singularity has local normal form

$$\pm x_1^{k+1} + Q(x_2, \dots, x_n),$$

where Q is a non-degenerate quadratic form. In particular:

- A_1 : $f(x) = \pm x_1^2 \pm \dots \pm x_n^2$.
- A_2 : $f(x) = x_n^3 + Q(x_1, \dots, x_{n-1})$.
- A_3 : $f(x) = \pm x_n^4 + Q(x_1, \dots, x_{n-1})$.

The derivation of these normal forms is standard and proceeds by the Splitting Lemma (Relative Morse Lemma) followed by a one-dimensional analysis of the remaining degenerate germ.

Step 1.1: Applying the Splitting Lemma

The Splitting Lemma provides a canonical way to decompose a function germ near a degenerate critical point.

Lemma 1.7.3 (Splitting Lemma, stated as Relative Morse Lemma in p. 364 in [Gol78]). *Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a smooth function germ with $f(0) = 0$ and $Df(0) = 0$. Suppose the Hessian matrix $\text{Hess}_x(f)|_0$ has rank $n - m$ (i.e., the corank is m). Then there exists a local coordinate system $(y_1, \dots, y_m, z_1, \dots, z_{n-m})$ around the origin such that f takes the form:*

$$f(y, z) = g(y_1, \dots, y_m) + Q(z_1, \dots, z_{n-m})$$

where Q is a non-degenerate quadratic form (i.e., $Q(z) = \sum_{i=1}^{n-m} \epsilon_i z_i^2$ with $\epsilon_i = \pm 1$), and the germ $g : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}, 0)$ has the property that its 1-jet and 2-jet both vanish at the origin, i.e., $g(0) = 0$, $Dg(0) = 0$, and $\text{Hess}_y(g)|_0 = 0$.

We apply this lemma to our singularities of interest:

- **For an A_1 (Morse) singularity:** The Hessian is non-degenerate by definition. Its rank is n , so the corank is $m = 0$. The Splitting Lemma implies that the function is equivalent to a non-degenerate quadratic form in all n variables. This is precisely the Morse Lemma, which establishes the normal form $f(x) \sim \sum \pm x_i^2$.
- **For an A_2 (Cusp) or A_3 (Swallowtail) singularity:** By definition, the Hessian has a one-dimensional kernel. Its rank is $n - 1$, so the corank is $m = 1$. The Splitting Lemma guarantees the existence of local coordinates $(y_1, z_1, \dots, z_{n-1})$ such that:

$$f(y_1, z) = g(y_1) + Q(z_1, \dots, z_{n-1})$$

where g is a germ of a single variable satisfying $g(0) = g'(0) = g''(0) = 0$.

The problem of finding the normal form for f in n variables is now reduced to finding the normal form for the 1-dimensional germ $g(y_1)$.

Step 1.2: Analysis of the 1D Degenerate Germ

We now classify the germ $g(y_1)$ based on its first non-vanishing higher derivative, which is the defining characteristic of the A_k singularity types.

A₂ Singularity (Cusp)

For an A_2 singularity, the third derivative of f in the direction of the Hessian's kernel is non-zero. This translates to the condition that $g'''(0) \neq 0$. Since $g(0) = g'(0) = g''(0) = 0$, the Taylor expansion of g at the origin is:

$$g(y_1) = \frac{g'''(0)}{3!}y_1^3 + \frac{g^{(4)}(0)}{4!}y_1^4 + \dots = y_1^3 \underbrace{\left(\frac{g'''(0)}{6} + \frac{g^{(4)}(0)}{24}y_1 + \dots \right)}_{\phi(y_1)}$$

The function $\phi(y_1)$ is smooth, and importantly, $\phi(0) = g'''(0)/6 \neq 0$. We can therefore define a new smooth function $\psi(y_1) = \sqrt[3]{\phi(y_1)}$, which is well-defined near the origin. This allows us to write:

$$g(y_1) = (y_1 \cdot \psi(y_1))^3$$

Let's define a new coordinate $u_1 = y_1 \cdot \psi(y_1)$. This is a valid local change of coordinates because its derivative at the origin is non-zero:

$$\left. \frac{du_1}{dy_1} \right|_{y_1=0} = (\psi(y_1) + y_1\psi'(y_1)) \Big|_{y_1=0} = \psi(0) = \sqrt[3]{\phi(0)} \neq 0$$

By the Inverse Function Theorem, the map $y_1 \mapsto u_1$ is a local diffeomorphism. In this new coordinate system, the germ g takes the form $g(u_1) = u_1^3$. Combining this with the Splitting Lemma, we find that the original function f is locally equivalent to $u_1^3 + Q(z_1, \dots, z_{n-1})$, which is the normal form for an A_2 singularity.

A₃ Singularity (Swallowtail)

The argument is identical. For an A_3 singularity, we have $g(0) = g'(0) = g''(0) = g'''(0) = 0$, but $g^{(4)}(0) \neq 0$. The Taylor expansion is:

$$g(y_1) = \frac{g^{(4)}(0)}{4!}y_1^4 + \frac{g^{(5)}(0)}{5!}y_1^5 + \dots = y_1^4 \underbrace{\left(\frac{g^{(4)}(0)}{24} + \frac{g^{(5)}(0)}{120}y_1 + \dots \right)}_{\phi(y_1)}$$

where $\phi(0) = g^{(4)}(0)/24 \neq 0$. We can define a smooth function $\psi(y_1) = \sqrt[4]{|\phi(y_1)|}$. Then we can write:

$$g(y_1) = \text{sgn}(\phi(0)) \cdot (y_1 \cdot \psi(y_1))^4$$

We define the new coordinate $u_1 = y_1 \cdot \psi(y_1)$. As before, this is a valid coordinate change since $\left. \frac{du_1}{dy_1} \right|_0 = \psi(0) \neq 0$. In this coordinate, $g(u_1) = \pm u_1^4$, where the sign is

determined by the sign of $g^{(4)}(0)$. Combining this with the Splitting Lemma, we find that the original function f is locally equivalent to $\pm u_1^4 + Q(z_1, \dots, z_{n-1})$, which is the normal form for an A_3 singularity.

These normal forms describe the singularity at a single point in the family. However, we are interested in how these singularities behave as we vary the parameters (s, t) . This is captured by the concept of an *unfolding* of a singularity. The definitions here follow [Gol78].

Definition 1.7.6. Let $f : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}$ be a germ of a function $\mathbb{R}^n \rightarrow \mathbb{R}$ at 0. A *k-parameter unfolding* of f is a germ $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow \mathbb{R}$, where $F(0, x) = f(x)$ for $x \in \mathbb{R}^n$. We will denote $F(t, x)$ by $F_t(x)$, for $t \in \mathbb{R}^k$.

Definition 1.7.7 (Definition 3, p.365 in [Gol78]). Let $F_t, G_s : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}$, where $s, t \in \mathbb{R}^k$, be two *k*-parameter unfoldings of f . These unfoldings are (*right*) *equivalent* if there is a *k*-parameter family of diffeomorphisms $H : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow \mathbb{R}^n$, a smooth invertible change of parameters $t = t(s)$ and a smooth germ $K : (\mathbb{R}^k, 0) \rightarrow \mathbb{R}$ such that

$$H_0(x) = x \quad \text{and} \quad t(0) = 0,$$

$$G_s(x) = F_{t(s)}(H_s(x)) + K_s$$

Definition 1.7.8 (Definition 3, p.365 in [Gol78]). A *k*-parameter unfolding $F_t : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}$ of f is (*locally*) *stable* (at $t = 0$) if every nearby *k*-parameter unfolding G_s of f is equivalent to F_t .

Theorem 4.1 in [Gol78] asserts that stable unfoldings are generic in the space of all unfoldings when the parameter space is low-dimensional (≤ 5). This theorem implies that in order to understand the local behavior of generic 2-parameter families, it now suffices to understand the stable unfoldings of the A_1 (Morse), A_2 (cusp) and A_3 (swallowtail) singularities. Morse critical points are stable so their 2-parameter unfoldings are just 2-parameter families of Morse functions. Cusp and swallowtail critical points are more interesting. Theorems A.3 and A.4 in [Lek09] imply that their stable 2-parameter unfoldings are given by Table 1.1.

These, respectively, correspond to a path of Cerf paths, eye birth-death, merge-unmerge and swallowtail moves, as desired.

Step 2: Global structure via multijet transversality.

We now consider how the singularities identified in Step 1 may interact globally. For example, we are interested in whether multiple singularities (possibly of the same or different types) can occur simultaneously—at the same time slice and at the same

Germ	Stable 2-parameter unfoldings
$x_n^3 + Q(x_1, \dots, x_{n-1})$	$x_n^3 + tx_n + Q(x_1, \dots, x_{n-1})$
$x_n^3 + Q(x_1, \dots, x_{n-1})$	$x_n^3 + t^2x_n + sx_n + Q(x_1, \dots, x_{n-1})$
$x_n^3 + Q(x_1, \dots, x_{n-1})$	$x_n^3 - t^2x_n + sx_n + Q(x_1, \dots, x_{n-1})$
$\pm x_n^4 + Q(x_1, \dots, x_{n-1})$	$\pm x_n^4 + sx_n^2 + tx_n + Q(x_1, \dots, x_{n-1})$

Table 1.1: The stable 2-parameter unfoldings of A_2 and A_3 critical points. Here, Q is a non-degenerate quadratic form.

critical level. Thom's multijet transversality theorem helps us identify which such interactions occur densely in the space of 2-parameter families. The theorem implies that the only such interactions we need to consider correspond to the Reidemeister II, Reidemeister III, beak and coincidence moves.

We need to consider the following cases:

- Multiple Morse critical points share same critical levels at the same time
- Multiple cusp critical points share same critical levels at the same time
- Multiple Morse and cusp critical points share same critical levels at the same time
- Multijet phenomena that include swallowtail critical points

Let π_{I^2} denote the composite

$$J^3(M \times I^2, \mathbb{R}) \xrightarrow{\pi_{M \times I^2}} M \times I^2 \longrightarrow I^2,$$

$\pi_{\mathbb{R}}$ denote the projection

$$J^3(M \times I^2, \mathbb{R}) \rightarrow \mathbb{R},$$

ι_r denote the inclusion

$$J_r^3(M \times I^2, \mathbb{R}) \rightarrow J^3(M \times I^2, \mathbb{R})^r,$$

and Δ_{X^k} denote the diagonal of X^k . We have:

1. Multiple Morse critical points share same critical levels at the same time:

Let N_M denote the submanifold

$$\iota_r^{-1}((\Sigma^{1,0,0})^r) \times \pi_{I^2}^{-1}(\Delta_{(I^2)^r}) \times \pi_{\mathbb{R}}^{-1}(\Delta_{\mathbb{R}^{r_1}} \times \Delta_{\mathbb{R}^{r_2}} \times \dots \times \Delta_{\mathbb{R}^{r_k}}) \subset J_r^3(M \times I^2, \mathbb{R}).$$

Let F be our 2-parameter family and let's suppose that for some time s_0, t_0 , F_{s_0, t_0} is a Morse function with r Morse critical points, such that r_i of them are at

the same critical level c_i , for $1 \leq i \leq k$. This implies that $j_r^3(F)$ intersects N_M . By Thom's multijet transversality, for a dense subset of 2-parameter families, $j_r^3(F)$ intersects N_M transversally. Let's compute the codimension of N_M in $J_r^3(M \times I^2, \mathbb{R})$:

- Being a non-degenerate critical point imposes n independent conditions, hence the codimension of $\iota_r^{-1}((\Sigma^{1,0,0})^r)$ is $n \cdot r$.
- Fixing the time parameter for r points imposes $2 \cdot (r - 1)$ independent conditions, hence the codimension of $\pi_{I^2}^{-1}(\Delta_{(I^2)^r})$ is $2 \cdot (r - 1)$.
- Having r_i critical points at the same critical level imposes $r_i - 1$ independent conditions, hence the codimension of $\pi_{\mathbb{R}}^{-1}(\Delta_{\mathbb{R}^{r_1}} \times \Delta_{\mathbb{R}^{r_2}} \times \dots \times \Delta_{\mathbb{R}^{r_k}})$ is $(\sum_1^k r_i) - k$.

Therefore, the codimension of N_M in $J_r^3(M \times I^2, \mathbb{R})$ is $n \cdot r + 2 \cdot (r - 1) + (\sum_1^k r_i) - k$. Recall that source of $j_r^3(F)$ is $\text{Conf}_r(M \times I^2)$, which has dimension $r \cdot (n + 2)$. If $j_r^3(F)$ intersects N_M transversally, we must have

$$n \cdot r + 2 \cdot (r - 1) + (\sum_1^k r_i) - k \leq r \cdot (n + 2) \Leftrightarrow (\sum_1^k r_i) - 2 \leq k$$

Now, we make the following observation: if the function F_{s_0, t_0} has a Morse critical point at a different critical level with the rest, that would contribute equally to the codimension of N_M and the dimension of $\text{Conf}_r(M \times I^2)$. We can then assume that

- $r_i \geq 2$, for all $1 \leq i \leq k$,
- $\sum_1^k r_i = r$,
- $r \geq 2$ and $k \geq 1$.

This implies that

$$r = \sum_1^k r_i \geq \sum_1^k 2 = 2k \Leftrightarrow k \leq \frac{r}{2}.$$

Combined with the inequality established above, we have

$$r - 2 \leq k \leq \frac{r}{2}, \quad r \geq 2, \quad \text{and } k \geq 1.$$

This implies that $2 \leq r \leq 4$, which leads to the following cases:

- $r = 2$:

We must, then, have $k = 1$. This corresponds to a 2-parameter family that passes by functions that have two Morse critical points at the same critical level. Transversality implies that there exists a 1-dimensional submanifold M_2 of $(M \times I^2)^2$ consisting of points $((p_0, s, t), (p_1, s, t))$ such that $f(p_0) = f(p_1)$. We now consider the map $M_2 \rightarrow I^2$ that encodes the s - and t -coordinates of the points in M_2 . This is the *bifurcation diagram* of F . Generically, a map from a 1-manifold to a 2-manifold is an immersion, hence we may assume that the bifurcation diagram is an immersed submanifold of I^2 . If we further consider the projection map

$$I^2 \xrightarrow{(s,t) \mapsto s} I$$

restricted to the bifurcation diagram, it will have a finite number of critical points. These correspond to the Reidemeister II move.

- $r = 3$:

We must, then, have $k = 1$. This corresponds to a 2-parameter family that passes by a function that have three Morse critical points at the same critical level. Unpacking the implications of transversality in this case, we see that this occurs at isolated times and corresponds to the Reidemeister III move.

- $r = 4$:

We must, then, have $k = 2$, and $r_1 = r_2 = 2$. This occurs at isolated times and corresponds to a coincidence move, where we have at time (s_0, t_0) a Morse function with two pairs of Morse critical points (a_1, b_1) and (a_2, b_2) such that (a_i, b_i) lie at the same critical level.

2. Multiple cusp critical points share same critical levels at the same time:

A similar analysis to the one above shows that this can only correspond to a coincidence move, where we have at time (s_0, t_0) a generalized Morse function with two pairs of cusp critical points (a_1, b_1) and (a_2, b_2) such that (a_i, b_i) lie at the same critical level.

3. Multiple Morse and cusp critical points share same critical levels at the same time:

A similar analysis to the one above shows that this can only correspond to the following two moves:

- A beak move. This occurs at isolated times where the 2-parameter family F passes by a generalized Morse function that has a cusp critical point and a Morse critical point that lie at the same critical level.
 - A coincidence move. This occurs at isolated times where the 2-parameter family F passes by a generalized Morse function that has a cusp critical point c and a pair of Morse critical points (a_1, b_1) such that $f(a_1) = f(b_1) \neq f(c)$.
4. Multijet phenomena that include swallowtail critical points: It is straightforward to see that Thom's multijet transversality implies that there aren't any multijet phenomena that include swallowtail critical points. Namely, if F passes by a function that has a swallowtail critical point, all other critical points of that function must be Morse and distinct critical points must lie at distinct critical levels.

This completes the proof establishing the density of the space of Cerf homotopies in the space of all homotopies. □

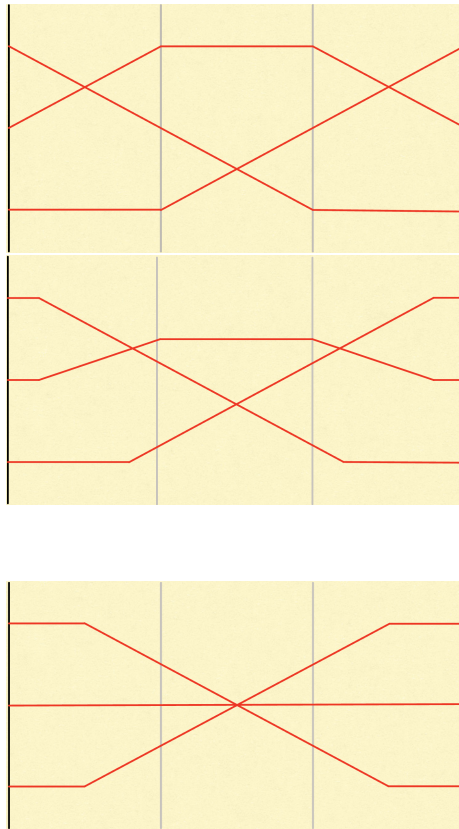
Remark 1.7.4. Showing that the space of Cerf homotopies is also open—and hence generic—is not difficult. Intuitively, it follows from the fact that transversality is an open condition. Indeed, if we were only concerned with the *local structure* of the singularities that appear (rather than their *global interactions*), openness would follow immediately from Thom's jet transversality theorem. However, this conclusion does not directly follow from Thom's multijet transversality theorem, and must instead be verified separately. One reason for this is that Thom's multijet transversality involves maps defined on configuration spaces, which are not compact. The proof of openness is entirely analogous to Cerf's proof that the space of Cerf paths forms an open subset of the space of all paths; see Proposition 4 on p. 29 of [Cer68]. The interested reader is encouraged to adapt Cerf's proof to the setting of Cerf homotopies. Since this is not required for our purposes, we omit the details.

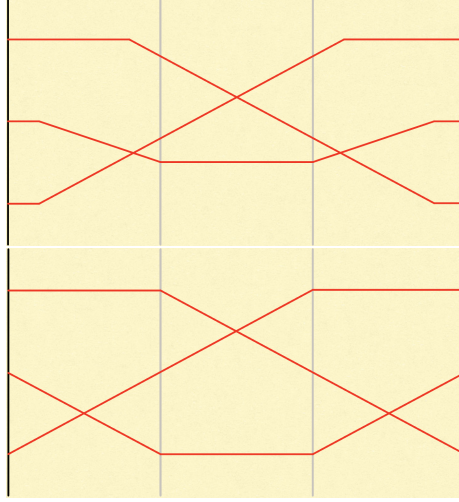
1.8 Elementary Reidemeister III homotopies

Just as there is a notion of elementary Cerf paths, it is natural to consider an analogous notion of elementary Cerf homotopies. This idea is not developed in [Cer73], and we will likewise not pursue it in full generality here, as it is not required

for the main results of this thesis. However, a version of this notion will be needed in the case of Reidemeister III homotopies. These will closely mirror the elementary crossing paths introduced earlier. We now outline some basic ideas and properties of these elementary Reidemeister III homotopies, which will become useful in what follows.

As before, we begin by fixing standard Morse model embeddings for the three critical points involved in the Reidemeister III move, and then consider a two-parameter family of elementary ascending and descending paths for them. While such a family can be chosen explicitly, the particular choice will not matter for our purposes and offers limited additional insight. Instead, we illustrate the idea through a sequence of Cerf graphics, which encode a representative movie of the homotopy and allow one to directly extract the desired family of elementary ascending and descending paths for the critical points involved.





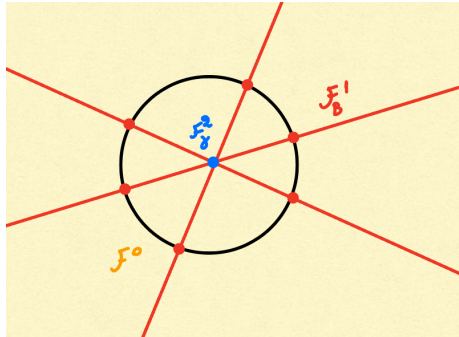
Let Reid denote the space of Reidemeister III homotopies, and let $\text{EReid} \subset \text{Reid}$ denote the subspace of elementary Reidemeister III homotopies. Let $f \in \mathcal{F}^0(W)$, and define Reid_f and EReid_f to be the subspaces of Reid and EReid , respectively, consisting of homotopies that are based at f —that is, they satisfy $\Gamma(0,0) = f$. Similarly, for $g \in \mathcal{F}_\gamma^2(W)$, define ${}_g\text{Reid}$ and ${}_g\text{EReid}$ to denote the subspaces of homotopies that pass through g .

We will need the following lemma:

Lemma 1.8.1. *Let f be an excellent Morse function on W . Then*

$$\pi_0(\text{Reid}_f, \text{EReid}_f) = 0.$$

Proof. This follows by adapting Cerf’s argument for Lemma 1.6.8. The key point is to understand the local structure of the stratified space $\mathcal{F}^{\leq 2}$ around the $\mathcal{F}_\gamma^2$ stratum. The relevant geometry is depicted in the following diagram:



This figure suggests that the space

$$\mathcal{F}^0(W) \cup \mathcal{F}_\beta^1(W) \cup \mathcal{F}_\gamma^2(W)$$

is a codimension-2 conical stratified space. More precisely, each point in $\mathcal{F}_\gamma^2(W)$ admits a neighborhood of the form $U \times B$, where U is an open neighborhood in $\mathcal{F}_\gamma^2(W)$ and B is a cone over a codimension-1 stratified space—namely, a circle with six distinguished points corresponding to the codimension-1 strata. This conical structure follows from an argument similar to the one used by Cerf to describe the conical structure of $\mathcal{F}^{\leq 1}$ in [Cer68, Chapter II, Proposition 6, p. 33].

With this local model in hand, we invoke Cerf’s elementary paths lemma. Consider the following diagram of maps:

$$\begin{array}{ccc} \text{Reid} & & \text{EReid} \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{F}^0(W) & & \mathcal{F}^0(W) \end{array}$$

where ι records the initial map, i.e., $\Gamma \mapsto \Gamma(0, 0)$. By Cerf’s fibration lemma and the stability of excellent Morse functions, both maps are locally trivial fiber bundles. It follows that

$$\pi_k(\text{Reid}_f, \text{EReid}_f) \cong \pi_k(\text{Reid}, \text{EReid}) \quad \text{for all } k \geq 0.$$

Next, consider the maps

$$\begin{array}{ccc} \text{Reid} & & \text{EReid} \\ \downarrow \gamma & & \downarrow \gamma \\ \mathcal{F}_\gamma^2(W) & & \mathcal{F}_\gamma^2(W) \end{array}$$

which record the point at which the homotopy crosses the $\mathcal{F}_\gamma^2$ stratum. Although we lack a stability theorem for $\mathcal{F}_\gamma^2(W)$, these maps are surjective, which suffices to conclude that

$$\pi_0({}_g\text{Reid}, {}_g\text{EReid}) = 0 \quad \Rightarrow \quad \pi_0(\text{Reid}, \text{EReid}) = 0.$$

It remains to analyze the space ${}_g\text{Reid}$. By the product structure around the $\mathcal{F}_\gamma^2$ stratum, it has six connected components, corresponding to the six distinct directions in which a Reidemeister III homotopy may unfold, and each component is contractible. For each such direction, there exists an elementary Reidemeister III homotopy. Hence, $\pi_0({}_g\text{Reid}, {}_g\text{EReid}) = 0$, which completes the proof. $\square$

Chapter 2

Surgery presentations of bordism categories

The purpose of this chapter is to provide a detailed overview of Juhasz’s classification of TQFTs via surgery ([Juh18]), simultaneously presenting both the oriented and unoriented versions. Our goal is to express Juhasz’s proof of this classification result in a language that will make its categorification more accessible. In particular, we claim no originality in the proof itself, only in its exposition. While this chapter is not logically necessary for obtaining our main results, we hope that it will help clarify the structure of the argument we are subsequently presenting. All categories appearing in this chapter are 1-categories.

Section 2.1 serves as a warm-up. It introduces the presentation of linear diffeomorphism diagrams, $\text{Diff}_{(d-1; \sim)}^{(\text{or})}$, and shows that the associated category $F\text{Diff}_{(d-1; \sim)}^{(\text{or})}$ is equivalent to the groupoid of smooth manifolds and isotopy classes of diffeomorphisms, $\text{Man}_{(d-1; \sim)}^{(\text{or})}$.

Section 2.2 introduces the presentation of linear surgery diagrams, $\text{Sur}_{(d-1; d)}^{(\text{or})}$. Notably, $F\text{Sur}_{(d-1; d)}^{(\text{or})}$ corresponds to the category $\mathcal{F}(\mathcal{G}_n)/\mathcal{R}$ discussed in [Juh18]. A key advantage of $F\text{Sur}_{(d-1; d)}^{(\text{or})}$ is that all of its 0- and 1-cells are diagrams labeled by $(d-1)$ -dimensional data, whereas the 1-cells of $\text{Bord}_{(d-1; d)}^{(\text{or})}$ are a priori characterized by d -dimensional data.

Section 2.3 introduces the category of smooth manifolds and equivalence classes of d -dimensional cobordisms equipped with Morse functions, $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$, and shows that it is equivalent to $\text{Bord}_{(d-1; d)}^{(\text{or})}$.

Then, section 2.4 introduces the category of smooth manifolds and equivalence classes of d -dimensional cobordisms equipped with Morse data, $\text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or})}$,

and shows that it is equivalent to $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$, thus establishing the equivalences

$$\text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or})} \xrightarrow{F^{\text{data}}} \text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})} \xrightarrow{F^{\text{Morse}}} \text{Bord}_{(d-1; d)}^{(\text{or})}.$$

Morse data allow us to associate cobordisms with linear surgery diagrams. We describe this in detail in section 2.5, where we construct the functors

$$F \text{Sur}_{(d-1; d)}^{(\text{or})} \xrightleftharpoons[S]{B} \text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or})}$$

and show that they are equivalences, thus establishing the equivalence between $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{(\text{or})}$.

2.1 Linear diffeomorphism diagrams

This section serves as a warm-up. Rather than analyzing the full bordism category, we focus on the groupoid $\text{Man}_{(d-1; \sim)}^{(\text{or})}$, whose 0-cells are closed, (oriented) $(d-1)$ -manifolds and whose 1-cells are isotopy classes of orientation-preserving diffeomorphisms. We introduce a generators-and-relations presentation of $\text{Man}_{(d-1; \sim)}^{(\text{or})}$ as a symmetric monoidal category, using *linear diffeomorphism diagrams*, which we denote by $\text{Diff}_{(d-1; \sim)}^{(\text{or})}$.

2.1.1 The presentation of linear diffeomorphism diagrams

$$\text{Diff}_{(d-1; \sim)}^{(\text{or})}$$

The presentation $\text{Diff}_{(d-1; \sim)}^{(\text{or})}$ is defined as follows:

Generating 0-cells

Closed, connected, (oriented) $(d-1)$ -dimensional manifolds.

Generating 1-cells

Diagrams of the form



where $\phi : X \rightarrow Y$ is a(n orientation preserving) diffeomorphism. Composition of 1-cells is via vertical concatenation.

Generating relations

Isotopy relation:

$$\begin{array}{c} Y \\ | \\ \phi \bullet \\ | \\ X \end{array} = \begin{array}{c} Y \\ | \\ \bullet \psi \\ | \\ X \end{array}$$

if ϕ and ψ are isotopic diffeomorphisms.

Unit relation:

$$\begin{array}{c} | \\ | \\ | \\ | \\ X \end{array} = \begin{array}{c} X \\ | \\ \bullet \text{Id}_X \\ | \\ X \end{array}$$

The purpose of the unit relation is to relate the unit in $F\text{Diff}_{(d-1; \sim)}^{(\text{or})}$, which is the diagram decorated with no diffeomorphisms, to the unit in $\text{Man}_{(d-1; \sim)}^{(\text{or})}$, which is the identity diffeomorphism.

Composition relation:

$$\begin{array}{c} Z \\ | \\ \psi \bullet \\ | \\ Y \\ | \\ \phi \bullet \\ | \\ X \end{array} = \begin{array}{c} Z \\ | \\ \bullet \psi \circ \phi \\ | \\ X \end{array}$$

The purpose of the composition relation is to relate the composition in $F\text{Diff}_{(d-1; \sim)}^{(\text{or})}$, which is given by concatenating diagrams, to the composition in $\text{Man}_{(d-1; \sim)}^{(\text{or})}$, which is given by composing diffeomorphisms.

Braid relation:

$$\begin{array}{c} B \sqcup A \\ (12) \text{ } \bullet \\ A \sqcup B \end{array} = \begin{array}{c} B \sqcup A \\ \bullet (12) \\ A \sqcup B \end{array}$$

Here, the LHS denotes braid 1-cell that exists in $F \text{Diff}_{(d-1; \sim)}^{(\text{or})}$ due to $\text{Diff}_{(d-1; \sim)}^{(\text{or})}$ being a symmetric monoidal presentation. The RHS denotes the generating 1-cell of $\text{Diff}_{(d-1; \sim)}^{(\text{or})}$ labeled by the swap diffeomorphism.

2.1.2 The equivalence between $F \text{Diff}_{(d-1; \sim)}^{(\text{or})}$ and $\text{Man}_{(d-1; \sim)}^{(\text{or})}$

Given the above definitions, the equivalence between $F \text{Diff}_{(d-1; \sim)}^{(\text{or})}$ and $\text{Man}_{(d-1; \sim)}^{(\text{or})}$ becomes almost tautological. The functor

$$\text{Man}_{(d-1; \sim)}^{(\text{or})} \xrightarrow{D} F \text{Diff}_{(d-1; \sim)}^{(\text{or})}$$

is the identity on 0-cells and sends a diffeomorphism $\phi : X \rightarrow Y$ to the linear surgery diagram

$$\begin{array}{c} Y \\ \phi \text{ } \bullet \\ X \end{array}$$

The functor

$$F \text{Diff}_{(d-1; \sim)}^{(\text{or})} \xrightarrow{M} \text{Man}_{(d-1; \sim)}^{(\text{or})}$$

is also the identity on 0-cells, sends the unit diagram

$$\begin{array}{c} \\ \\ X \end{array}$$

to the identity diffeomorphism $\text{Id}_X : X \rightarrow X$ and the generating 1-cells



to the diffeomorphism $\phi : X \rightarrow Y$. The unit and composition relations imply that these functors are inverses of each other on the nose, hence establishing the equivalence between $F \text{Diff}_{(d-1; \sim)}^{(\text{or})}$ and $\text{Man}_{(d-1; \sim)}^{(\text{or})}$.

2.2 Linear surgery diagrams

The purpose of this section is to introduce the presentation of linear surgery diagrams, $\text{Sur}_{(d-1; d)}^{(\text{or})}$. We will later demonstrate in section 2.5 that $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ is equivalent to the d -dimensional bordism category $\text{Bord}_{(d-1; d)}^{(\text{or})}$. To more accurately mimic our strategy for obtaining surgery presentations for bordism bicategories, we will introduce $\text{Sur}_{(d-1; d)}^{(\text{or})}$ in two stages. First, we will present $\text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$, which initially omits some necessary relations. We will then introduce the additional relations needed to fully define $\text{Sur}_{(d-1; d)}^{\text{or}}$.

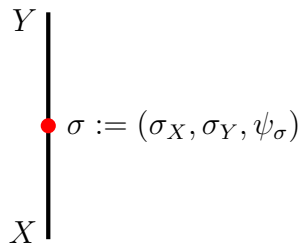
2.2.1 The presentation of linear surgery diagrams $\text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$

We define $\text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$ by enhancing the presentation of $\text{Diff}_{(d-1; \sim)}^{(\text{or})}$ with additional generating 1-cells and relations. Those inherited from $\text{Diff}_{(d-1; \sim)}^{(\text{or})}$ are referred to as *diffeomorphism type* generating 1-cells and relations. The new generating 1-cells and relations specific to $\text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$ are as follows:

Additional generating 1-cells

Surgery type generating 1-cells:

Diagrams of the form



where $(\sigma_X, \sigma_Y, \psi_\sigma)$ is a symmetric surgery triple. Namely,

$$\begin{aligned}\sigma_X &: S^{k-1} \times D^{n-k} \hookrightarrow X, \\ \sigma_Y &: D^k \times S^{n-k-1} \hookrightarrow Y, \\ \psi_\sigma &: X \setminus (\sigma_X(S^{k-1} \times \{0\})) \xrightarrow{\cong} Y \setminus \sigma_Y(\{0\} \times S^{n-k-1}).\end{aligned}$$

In this case, we will say that σ has index k . The embeddings σ_X, σ_Y and the diffeomorphism ϕ_σ are assumed to be orientation preserving when we define the oriented version of the category of linear surgery diagrams. Moreover, ψ_σ is required to be compatible with the framings of the spheres, i.e., the following diagram is commutative:

$$\begin{array}{ccc} S^{k-1} \times (D^{n-k} \setminus \{0\}) & \xrightarrow{c} & (D^k \setminus \{0\}) \times S^{n-k-1} \\ \sigma \downarrow & & \downarrow \sigma' \\ X \setminus \sigma_X(S^{k-1} \times \{0\}) & \xrightarrow{\psi_\sigma} & Y \setminus \sigma_Y(\{0\} \times S^{n-k-1}) \end{array}$$

where c is the diffeomorphism given by

$$(u, \lambda v) \mapsto (v, \lambda u), \quad 0 < \lambda < 1.$$

Additional generating relations

Naturality type relations:

$$\begin{array}{ccc} \begin{array}{c} Z \\ \phi \bullet \\ Y \\ \sigma := (\sigma_X, \sigma_Y, \psi_\sigma) \bullet \\ X \end{array} & = & \begin{array}{c} Z \\ \bullet \phi^* \sigma := (\sigma_X, \phi \circ \sigma_Y, \phi \circ \psi_\sigma) \\ X \end{array} \\ \\ \begin{array}{c} Z \\ \sigma := (\sigma_Y, \sigma_Z, \psi_\sigma) \bullet \\ Y \\ \phi \bullet \\ X \end{array} & = & \begin{array}{c} Z \\ \bullet \phi_* \sigma := (\phi^{-1} \circ \sigma_Y, \sigma_Z, \psi_\sigma \circ \phi) \\ X \end{array} \end{array}$$

Frame change type relations:

$$\sigma := (\sigma_X, \sigma_Y, \psi_\sigma) \begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} = \begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} Q \cdot \sigma := (\sigma_X \circ Q^{-1}, \sigma_Y \circ Q^{-1}, \psi_\sigma)$$

where $Q \in G_{(n,k)}^{(\text{or})}$. We define:

$$G_{(n,k)} := O(k) \times O(n-k), \quad G_{(n,k)}^{(\text{or})} := G_{(n,k)} \cap SO(n).$$

Here, Q (and hence Q^{-1}) induces diffeomorphisms:

$$S^{k-1} \times D^{n-k} \longrightarrow S^{k-1} \times D^{n-k}, \quad D^k \times S^{n-k-1} \longrightarrow D^k \times S^{n-k-1}.$$

The group $Q \in G_{(n,k)}^{(\text{or})}$ has two connected components unless $k = 0$ or $k = n$. If we are only interested in the 1-categorical setting, it suffices to include a single representative Q from the connected component that does not contain the identity. In [Juh18], Juhász considers the *framing flip*, which induces the diffeomorphism

$$S^{k-1} \times D^{n-k} \rightarrow S^{k-1} \times D^{n-k},$$

given by

$$(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \mapsto (-x_1, \dots, x_k, -y_{k+1}, \dots, y_n)$$

and the diffeomorphism

$$D^k \times S^{n-k-1} \rightarrow D^k \times S^{n-k-1}$$

given by

$$(y_1, \dots, y_k, x_{k+1}, \dots, x_n) \mapsto (-y_1, \dots, y_k, -x_{k+1}, \dots, x_n).$$

Here, we allow for a general Q in $Q \in G_{(n,k)}^{(\text{or})}$ in preparation for obtaining surgery presentations of bordism bicategories.

2.2.2 The presentation of linear surgery diagrams $\text{Sur}_{(d-1; d)}^{(\text{or})}$

We will now define $\text{Sur}_{(d-1; d)}^{(\text{or})}$ by enhancing the presentation $\text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$ with some further generating relations. Namely:

Additional generating relations

Surgery type relations:

Cusp relations

$$\begin{array}{ccc}
 \begin{array}{c} Z \\ \rho := (\rho_Y, \rho_Z, \psi_\rho) \\ Y \\ \sigma := (\sigma_X, \sigma_Y, \psi_\sigma) \\ X \end{array} & = & \begin{array}{c} Z \\ \phi \\ X \end{array}
 \end{array}$$

Here, the following conditions are in place:

- The surgery triples σ and ρ have index k and $k + 1$, respectively.
- The attaching spheres $\sigma_Y(\{0\} \times S^{n-k-1})$ and $\rho_Y(S^k \times \{0\})$ intersect transversally in Y at a single point. We will refer to such a pair of attaching spheres as being in cancelling position. Following Cerf (see section 1.6 for details), this identifies a path-connected component of $\text{Diff}(X, Z)$.
- The diffeomorphism ϕ is an element of this path-connected component, hence it is well-defined up to isotopy, which is enough given the isotopy relation.

Remark 2.2.1. Later, we will need to upgrade this relation to a generating 2-cell in a 2-category. Specifying ϕ up to isotopy will then not be sufficient. Thankfully, by working a bit harder, we can define what it means for a pair of **framed** attaching spheres to be in a canceling position, strengthening the condition above to also consider the framings. This will allow us to construct an explicit canceling diffeomorphism ϕ , given a pair of surgery triples whose induced framed attaching spheres are in canceling position.

Crossing relations

$$\begin{array}{ccc}
 \begin{array}{c} Z \\ \rho := (\rho_Y, \rho_Z, \psi_\rho) \\ Y \\ \sigma := (\sigma_X, \sigma_Y, \psi_\sigma) \\ X \end{array} & = & \begin{array}{c} Z \\ \tilde{\sigma} := (\tilde{\sigma}_{Y'}, \tilde{\sigma}_Z, \psi_{\tilde{\sigma}}) \\ Y' \\ \tilde{\rho} := (\tilde{\rho}_X, \tilde{\rho}_{Y'}, \psi_{\tilde{\rho}}) \\ X \end{array}
 \end{array}$$

Here, the following conditions are in place:

- $\text{Im}(\sigma_Y) \subset Y$ is disjoint from $\text{Im}(\rho_Y) \subset Y$.
- $\text{Im}(\tilde{\rho}_{Y'}) \subset Y'$ is disjoint from $\text{Im}(\tilde{\sigma}_{Y'}) \subset Y'$.
- $\psi_\rho \circ \psi_\sigma = \psi_{\tilde{\sigma}} \circ \psi_{\tilde{\rho}}$, wherever both sides are defined.
- $\psi_\sigma \circ \tilde{\rho}_X = \rho_Y$ and $\psi_{\tilde{\rho}} \circ \sigma_X = \tilde{\sigma}_{Y'}$.
- $\psi_\rho \circ \sigma_Y = \tilde{\sigma}_Z$ and $\psi_{\tilde{\sigma}} \circ \tilde{\rho}_{Y'} = \rho_Z$.

Remark 2.2.2. The above conditions imply that the left hand side 1-cell determines the right hand side 1-cell on the nose up to a diffeomorphism $Y' \rightarrow Y''$.

This completes the definition of $\text{Sur}_{(d-1; d)}^{(\text{or})}$. Figure 2.1 collects all the pieces of this definition in one place.

Generating 0-cells

Closed, connected, (oriented) $(d - 1)$ -dimensional manifolds
--

Generating 1-cells

Diffeomorphism type	Surgery type
$\begin{array}{c} Y \\ \\ \phi \bullet \\ \\ X \end{array}$	$\begin{array}{c} Y \\ \\ \sigma = (\sigma_X, \sigma_Y, \phi_\sigma) \bullet \\ \\ X \end{array}$

Generating relations

Diffeomorphism type	Naturality type	Surgery type	Frame change type
<u>Isotopy</u> $\begin{array}{c} Y \\ \\ \phi \bullet \\ \\ X \end{array} = \begin{array}{c} Y \\ \\ \psi \bullet \\ \\ X \end{array}$ if ϕ and ψ are isotopic <u>Unit</u> $\begin{array}{c} \\ \\ X \end{array} = \begin{array}{c} \\ \\ \bullet \text{Id} \\ \\ X \end{array}$ <u>Composition</u> $\begin{array}{c} Z \\ \\ \psi \bullet \\ \\ Y \\ \\ \phi \bullet \\ \\ X \end{array} = \begin{array}{c} Z \\ \\ \bullet \psi \circ \phi \\ \\ X \end{array}$ <u>Braid</u> $\begin{array}{c} B \sqcup A \\ \\ (12) \bullet \\ \\ A \sqcup B \end{array} = \begin{array}{c} B \sqcup A \\ \\ \bullet (12) \\ \\ A \sqcup B \end{array}$	$\begin{array}{c} Z \\ \\ \sigma \bullet \\ \\ Y \\ \\ \phi \bullet \\ \\ X \end{array} = \begin{array}{c} Z \\ \\ \bullet \phi_* \sigma \\ \\ X \end{array}$ $\begin{array}{c} Z \\ \\ \sigma \bullet \\ \\ Y \\ \\ \phi \bullet \\ \\ X \end{array} = \begin{array}{c} Z \\ \\ \bullet \phi^* \sigma \\ \\ X \end{array}$	<u>Cusp</u> $\begin{array}{c} Z \\ \\ \tau \bullet \\ \\ Y \\ \\ \sigma \bullet \\ \\ X \end{array} = \begin{array}{c} Z \\ \\ \bullet \phi \\ \\ X \end{array}$ when the attaching spheres σ and τ are in cancelling position, and ϕ is a compatible cancelling diffeomorphism <u>Crossing</u> $\begin{array}{c} Z \\ \\ \rho \bullet \\ \\ Y \\ \\ \sigma \bullet \\ \\ X \end{array} = \begin{array}{c} Z \\ \\ \bullet \sigma' \\ \\ Y' \\ \\ \bullet \rho' \\ \\ X \end{array}$ when (σ, ρ) and (σ', ρ') are in compatible crossing positions	$\begin{array}{c} Y \\ \\ \sigma \bullet \\ \\ X \end{array} = \begin{array}{c} Y \\ \\ \bullet Q \cdot \sigma \\ \\ X \end{array}$ when σ is of index k and $Q \in G_{(n,k)}^{(\text{or})}$

Figure 2.1: The presentation $\text{Sur}_{(d-1; d)}^{(\text{or})}$

2.3 Bordism categories equipped with Morse functions

We begin by recalling the definition of the (oriented) d -dimensional bordism category $\text{Bord}_{(d-1; d)}^{(\text{or})}$ in some detail. It has:

- **0-cells:** Closed, (oriented) $(d-1)$ -dimensional manifolds.
- **1-cells:** Equivalence classes of (oriented) d -dimensional cobordisms between them.

Here, given two cobordisms

$$(W; \partial_0 W, \partial_1 W; \iota_0, \iota_1) \quad \text{and} \quad (W'; \partial_0 W', \partial_1 W'; \iota'_0, \iota'_1)$$

between Y_0 and Y_1 , an *equivalence* between them is an orientation-preserving diffeomorphism

$$D : W \rightarrow W'$$

that preserves the boundary identifications, meaning

$$\iota_0 = \iota'_0 \circ D \quad \text{and} \quad \iota_1 = \iota'_1 \circ D.$$

Our goal in this chapter is to show that $\text{Bord}_{(d-1; d)}^{(\text{or})}$ is equivalent to the category of linear surgery diagrams $F\text{Sur}_{(d-1; d)}^{(\text{or})}$. While one could establish this equivalence directly by constructing a functor

$$F\text{Sur}_{(d-1; d)}^{(\text{or})} \rightarrow \text{Bord}_{(d-1; d)}^{(\text{or})}$$

and verifying that it is fully faithful and essentially surjective, a more instructive approach is to break the proof into intermediate steps. This becomes especially important in the 2-categorical setting, where checking all the details at once would be intractable.

The purpose of this section is to describe the first such intermediate step, by introducing the category $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$ and showing that the forgetful functor

$$\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})} \xrightarrow{F^{\text{Morse}}} \text{Bord}_{(d-1; d)}^{(\text{or})}$$

is an equivalence.

Mimicking the construction of $\text{Sur}_{(d-1; d)}^{(\text{or})}$, we introduce $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$ in two stages. We begin by defining the subcategory $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or}), \mathcal{F}^0}$.

2.3.1 The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or}), \mathcal{F}^0}$

The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or}), \mathcal{F}^0}$ is defined as follows:

0-cells

Closed, (oriented) $(d - 1)$ -dimensional manifolds.

1-cells

A 1-cell between 0-cells Y_0 and Y_1 is (up to soon to be specified equivalence) a d -dimensional cobordism $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1)$ equipped with an excellent Morse function $f : (W; \partial_0 W, \partial_1 W) \rightarrow (I; \{0\}, \{i\})$, where I is the interval $[0, i]$, for some $i > 0$. Composition is by gluing of cobordisms and Morse functions. This requires a choice of collar, but any two choices will result in equivalent 1-cells.

Generating relations

There are two kinds of relations we quotient out by:

1. Two 1-cells $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; f : W \rightarrow I)$ and $(W; \partial_0 W, \partial_1 W; \iota'_0 : \partial_0 W \rightarrow Y_0, \iota'_1 : \partial_1 W \rightarrow Y_1; f' : W \rightarrow I)$ between 0-cells Y_0 and Y_1 with the same underlying smooth manifold triad and isotopic boundary identifications are equivalent if there exists a **path of excellent Morse functions** on $(W; \partial_0 W, \partial_1 W;)$ connecting f and f' that fixes a collar of the boundary of W throughout.
2. Two 1-cells $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; f : W \rightarrow I)$ and $(W'; \partial_0 W', \partial_1 W'; \iota'_0 : \partial_0 W' \rightarrow Y_0, \iota'_1 : \partial_1 W' \rightarrow Y_1; f' : W' \rightarrow J)$ between 0-cells Y_0 and Y_1 are equivalent if there exists a(n orientation preserving) diffeomorphism

$$D : (W; \partial_0 W, \partial_1 W) \rightarrow (W'; \partial_0 W', \partial_1 W')$$

of smooth manifold triads that respects the boundary identifications and the Morse functions. Namely, the diagrams

$$\begin{array}{ccc} \partial_0 W & \xrightarrow{D} & \partial_0 W' \\ \iota_0 \downarrow & & \downarrow \iota'_0 \\ Y_0 & \xlongequal{\quad} & Y_0 \end{array}$$

$$\begin{array}{ccc}
\partial_1 W & \xrightarrow{D} & \partial_1 W' \\
\downarrow \iota_1 & & \downarrow \iota'_1 \\
Y_1 & \xlongequal{\quad} & Y_1
\end{array}$$

commute, i.e. D is an equivalence of cobordisms, and there exists a diffeomorphism of smooth manifold triads $D' : (I; \{0\}, \{i\}) \rightarrow (J; \{0\}, \{j\})$ such that the following diagram commutes:

$$\begin{array}{ccc}
W & \xrightarrow{D} & W' \\
f \downarrow & & \downarrow f' \\
I & \xrightarrow{D'} & J
\end{array}$$

Here, the existence of D' uniquely determines it.

There is an evident forgetful functor

$$F_0^{\text{Morse}} : \text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or}), \mathcal{F}^0} \rightarrow \text{Bord}_{(d-1; d)}^{(\text{or})},$$

that simply forgets the Morse functions. It is the identity, and hence essentially surjective, on 0-cells. It is full on 1-cells, since any manifold admits a Morse function, as a consequence of the genericity of Morse functions. It is not, however, faithful on 1-cells, since the space of excellent Morse functions is not connected. We will fix this by introducing some further relations, which will complete the definition of $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$.

2.3.2 The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$

The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$ is obtained as the quotient of $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or}), \mathcal{F}^0}$ by the following generating relations:

1. Two 1-cells

$$(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; f) \quad \text{and}$$

$$(W; \partial_0 W, \partial_1 W; \iota'_0 : \partial_0 W \rightarrow Y_0, \iota'_1 : \partial_1 W \rightarrow Y_1; f')$$

between 0-cells Y_0 and Y_1 with the same underlying smooth manifold triad and isotopic boundary identifications are equivalent if there exists a **birth–death path** on $(W; \partial_0 W, \partial_1 W)$ connecting f and f' that fixes a collar neighborhood of ∂W throughout.

2. Two 1-cells

$$(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; f) \quad \text{and}$$

$$(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; f')$$

between 0-cells Y_0 and Y_1 with the same underlying smooth manifold triad and isotopic boundary identifications are equivalent if there exists a **crossing path** on $(W; \partial_0 W, \partial_1 W)$ connecting f and f' that also fixes a collar neighborhood of ∂W throughout.

Therefore, two 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$ that only differ by a choice of Morse function will be equivalent as any two Morse functions are connected via a Cerf path. The functor F_0^{Morse} above induces a functor

$$F^{\text{Morse}} : \text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})} \rightarrow \text{Bord}_{(d-1; d)}^{(\text{or})},$$

which, again, simply forgets the Morse functions.

Proposition 2.3.1. The functor F^{Morse} is an equivalence of categories

Proof. F^{Morse} is essentially surjective on 0-cells and full on 1-cells for the same reasons as F_0^{Morse} . It, thus, suffices to show that it is faithful on 1-cells, which is a consequence of the genericity of Cerf paths. Namely, let $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; f)$ and $(W'; \partial_0 W', \partial_1 W'; \iota'_0 : \partial_0 W' \rightarrow Y_0, \iota'_1 : \partial_1 W' \rightarrow Y_1; f')$ be two 1-cells between the 0-cells Y_0 and Y_1 such that the induced 1-cells $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1)$ and $(W'; \partial_0 W', \partial_1 W'; \iota'_0 : \partial_0 W' \rightarrow Y_0, \iota'_1 : \partial_1 W' \rightarrow Y_1)$ are equivalent in $\text{Bord}_{(d-1; d)}^{(\text{or})}$. This implies that there exists a(n orientation preserving) diffeomorphism

$$D : (W; \partial_0 W, \partial_1 W) \rightarrow (W'; \partial_0 W', \partial_1 W')$$

of smooth manifold triads that respects the boundary identifications. Namely, the diagrams

$$\begin{array}{ccc} \partial_0 W & \xrightarrow{D} & \partial_0 W' \\ \iota_0 \downarrow & & \downarrow \iota'_0 \\ Y_0 & \xlongequal{\quad} & Y_0 \end{array}$$

$$\begin{array}{ccc} \partial_1 W & \xrightarrow{D} & \partial_1 W' \\ \iota_1 \downarrow & & \downarrow \iota'_1 \\ Y_1 & \xlongequal{\quad} & Y_1 \end{array}$$

commute. The diffeomorphism D , however, need not be Morse function preserving with respect to f and f' . Let D^*f' be the pullback of the Morse function f' on the smooth manifold triad $(W; \partial_0 W, \partial_1 W)$. Then, the 1-cells $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; D^*f')$ and $(W'; \partial_0 W', \partial_1 W'; \iota'_0 : \partial_0 W' \rightarrow Y_0, \iota'_1 : \partial_1 W' \rightarrow Y_1; f')$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$ since the diffeomorphism D now preserves the Morse function by construction. The 1-cells $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; f)$ and $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1; D^*f')$ are also equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or})}$ as they only differ by the choice of Morse function and any two Morse functions are connected via a Cerf path, as a consequence of the genericity of Cerf paths. $\square$

Remark 2.3.1. The functor F^{Morse} is actually an isomorphism of categories.

2.4 Bordism categories equipped with Morse data

The purpose of this section is to introduce the second intermediate step in relating $\text{Bord}_{(d-1; d)}^{(\text{or})}$ to $F\text{Sur}_{(d-1; d)}^{(\text{or})}$, namely the d -dimensional bordism category equipped with Morse data, denoted $\text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or})}$. In analogy with Section 2.3, we will construct this category in two stages, beginning with the subcategory $\text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or}), \mathcal{F}^0}$.

2.4.1 The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or}), \mathcal{F}^0}$

The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or}), \mathcal{F}^0}$ is defined as follows:

0-cells

Closed, (oriented) $(d-1)$ -dimensional manifolds.

1-cells

A 1-cell between 0-cells Y_0 and Y_1 is (up to soon to be specified equivalence) a d -dimensional cobordism $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1)$ equipped with an excellent Morse function $f : (W; \partial_0 W, \partial_1 W) \rightarrow (I; \{0\}, \{i\})$, where I is the interval $[0, i]$ and associated Morse data, which we denote by $\mathcal{D}_f$. Recall that $\mathcal{D}_f$ consists of a choice of standard Morse model embeddings adapted at the critical points of f , a choice of regular level sets compatible with the standard Morse model embeddings and including the boundary level sets, and a choice of horizontal distribution for f compatible with vector field induced by the standard Morse embeddings in their image. Composition is by gluing of cobordisms, Morse functions and Morse data. This

requires a choice of collar, but any two choices will result in equivalent 1-cells. For brevity, we will denote a cobordism $(W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1)$ by $\overline{W}$, when we don't need to refer to it explicitly.

Generating relations

There are two kinds of relations we quotient out by:

1. Two 1-cells $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ between 0-cells Y_0 and Y_1 , with the same underlying smooth manifold triad and isotopic boundary identifications, are declared equivalent if there exists a path of Morse data in $\mathcal{MD}(W)$ connecting $(f, \mathcal{D}_f)$ and $(f', \mathcal{D}_{f'})$, as in Definition 1.3.16.
2. Two 1-cells $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W'}; f'; \mathcal{D}_{f'})$ between 0-cells Y_0 and Y_1 if there exists a(n orientation preserving) equivalence of cobordisms

$$D : \overline{W} \rightarrow \overline{W'}$$

that preserves the Morse functions and the associated Morse data, as in Definition 1.3.18.

There is an evident forgetful functor

$$F_0^{\text{data}} : \text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0} \rightarrow \text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0},$$

that simply forgets the Morse data.

Proposition 2.4.1. The functor F_0^{data} is an equivalence of categories.

Proof. The functor F_0^{data} is the identity (and hence essentially surjective) on 0-cells. By the Morse lemma, every smooth manifold triad equipped with a Morse function admits compatible Morse data, which implies that F_0^{data} is full on 1-cells. Therefore, it suffices to show that F_0^{data} is faithful on 1-cells, which follows from the stability of Morse functions. In particular, we need to verify that both types of relations in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0}$ induce corresponding relations in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$.

1. Let $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ such that the underlying 1-cells $(\overline{W}; f)$ and $(\overline{W}; f')$ are related in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0}$ via a path of excellent Morse functions from f to f' . We must show that $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$.

For simplicity, suppose that f (and hence f') has at most one critical point of index k . Since the general case follows by decomposing $(W; \partial_0 W, \partial_1 W)$ into submanifolds satisfying this assumption, we restrict to this case for clarity. Consider the fiber bundle

$$\begin{array}{ccc} \mathcal{PMD}(W; f) & \longrightarrow & \mathcal{PMD}(W) \\ & & \downarrow \\ & & \mathcal{F}^0(W) \end{array}$$

we introduced in section 1.3.2. Let

$$\gamma : [0, 1] \rightarrow \mathcal{F}^0(W)$$

be a path of excellent Morse functions that realizes the equivalence of $(\overline{W}; f)$ and $(\overline{W}; f')$ in $\text{Bord}_{(d-1; d)}^{\text{Morse}, (\text{or}), \mathcal{F}^0}$. By the path lifting property for fiber bundles, there exists a lift

$$\begin{array}{ccc} & & \mathcal{PMD}(W) \\ & \nearrow \tilde{\gamma} & \downarrow \\ [0, 1] & \xrightarrow{\gamma} & \mathcal{F}^0(W) \end{array}$$

such that $\tilde{\gamma}(0) = (f; \mathcal{PD}_f)$ and $\tilde{\gamma}(1) = (f'; \tilde{\mathcal{PD}}_{f'})$, where $\tilde{\mathcal{PD}}_{f'}$ is not necessarily the same as $\mathcal{PD}_{f'}$. If f (and hence also f') has no critical points, the fiber over f' , $\mathcal{PMD}(W; f')$, is contractible and the result follows immediately. If f (and hence also f') has a single critical point of index k , then the fiber over f' , $\mathcal{PMD}(W; f')$, admits a free action of $G_{(n,k)}^{(\text{or})}$ and is weakly equivalent to $G_{(n,k)}^{(\text{or})}$. We can, thus, choose the lift $\tilde{\gamma}$ so that $\tilde{\mathcal{PD}}_{f'}$ and $\mathcal{PD}_{f'}$ are in the same $G_{(n,k)}^{(\text{or})}$ -orbit. This implies that there exists $Q \in G_{(n,k)}^{(\text{or})}$ such that

$$\begin{array}{ccc} W & \xrightarrow{\text{Id}} & W \\ \uparrow \iota_p & & \uparrow \tilde{\iota}_p \\ M_k^n & \xrightarrow{Q} & M_k^n \end{array}$$

and the horizontal distribution in $\tilde{\mathcal{PD}}_{f'}$ agrees with the horizontal distribution in $\mathcal{PD}_{f'}$.

Therefore, $\tilde{\gamma}$ is a path of partial Morse data from $(\overline{W}; f; \mathcal{PD}_f)$ to $(\overline{W}; f'; \tilde{\mathcal{PD}}_{f'})$. We now extend $(\overline{W}; f'; \tilde{\mathcal{PD}}_{f'})$ to full Morse data $(\overline{W}; f'; \tilde{\mathcal{D}}_{f'})$ in such a way that the identity on W is Morse-data-preserving from $(\overline{W}; f'; \tilde{\mathcal{PD}}_{f'})$ to $(\overline{W}; f; \mathcal{PD}_f)$. By Corollary 1.3.13, the path $\tilde{\gamma}$ then

extends to a path from $(\overline{W}; f; \mathcal{D}_f)$ to $(\overline{W}; f'; \tilde{\mathcal{D}}_{f'})$. Moreover, the identity diffeomorphism $W \rightarrow W$ exhibits the equivalence of the 1-cells $(\overline{W}; f'; \tilde{\mathcal{D}}_{f'})$ and $(\overline{W}; f; \mathcal{D}_f)$. This completes the first part of the proof.

2. Let $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ such that the corresponding 1-cells $(\overline{W}; f)$ and $(\overline{W}'; f')$ are related in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0}$ due to a Morse function preserving equivalence of cobordisms $D : \overline{W} \rightarrow \overline{W}'$. We will show that $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$. For simplicity, let's assume again without loss of generality that f (and hence also f') have at most a single critical point of index k . Let $D^*(\mathcal{D}_{f'})$ be the pullback of the Morse data associated with f' to Morse data associated with $D^*(f') = f$ in $\overline{W}$. The 1-cells $(\overline{W}; f; D^*(\mathcal{D}_{f'}))$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$, by construction. If f has no critical points, then $\mathcal{PM}\mathcal{D}(W; f)$ is contractible and hence there is a path of Morse data connecting $D^*(\mathcal{D}_{f'})$ and $\mathcal{D}_f$, which implies that $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f; D^*(\mathcal{D}_{f'}))$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$. If f has a single critical point of index k , then there is a path of partial Morse data connecting $\mathcal{PD}_f$ and the orbit of $D^*(\mathcal{PD}_{f'})$ under the action of $G_{(n,k)}^{\text{or}}$. We conclude once again that $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f; D^*(\mathcal{D}_{f'}))$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$, thus completing the proof.

□

Remark 2.4.1. The functor F_0^{data} is actually an isomorphism of categories.

2.4.2 The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$

The bordism category $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$ is obtained as the quotient of $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ by the generating relations:

1. Two 1-cells $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ between 0-cells Y_0 and Y_1 with the same underlying smooth manifold triad and isotopic boundary identifications are equivalent if:
 - The Morse function f has two cancellable critical points p and q of index k and $k+1$, respectively.
 - The ascending attaching sphere of p and the descending attaching sphere of q , as determined by the Morse data $\mathcal{D}_f$ lie in the same level set, where

they intersect transversally at a single point. Let $A(p)$ be the ascending disk of p and $D(q)$ the descending disk of q , as determined by the Morse data $\mathcal{D}_f$.

- The Morse functions f and f' are connected via an **elementary death path** induced by an embedding of the standard double neighborhood on W adapted to f at p and q that preserves $A(p)$ and $D(q)$. We further assume that this standard double neighborhood embedding does not intersect any of the standard Morse model embeddings in $\mathcal{D}_f$, other than those adapted to f at p and q .
 - The functions f and f' only differ in the image of the standard double neighborhood embedding from the previous step. We ask that all standard Morse model embeddings in $\mathcal{D}_{f'}$ agree with the corresponding ones in $\mathcal{D}_f$. Note that, since the space of horizontal distributions compatible with given standard Morse model embeddings is contractible, this requirement determines $\mathcal{D}_{f'}$ up to equivalence.
2. Two 1-cells $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ between 0-cells Y_0 and Y_1 with the same underlying smooth manifold triad and isotopic boundary identifications are equivalent if they are related by an **elementary crossing path**.

The functor F_0^{data} above induces a functor

$$F^{\text{data}} : \text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}} \rightarrow \text{Bord}_{(d-1; d)}^{\text{Morse, (or)}}.$$

Proposition 2.4.2. The functor F^{data} is an equivalence of categories.

Proof. F^{data} is essentially surjective on 0-cells and full on 1-cells for the same reasons as F_0^{data} . It, thus, suffices to show that it is faithful on 1-cells. Let $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$ between 0-cells Y_0 and Y_1 , such that the corresponding 1-cells $(\overline{W}; f)$ and $(\overline{W}'; f')$ in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or)}}$ are equivalent. We have to show that $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$. It suffices to show that in the special case when $(\overline{W}; f)$ and $(\overline{W}'; f')$ are related via a generating relation.

We have already seen in the proof that F_0^{data} is an equivalence that if $(\overline{W}; f)$ and $(\overline{W}'; f')$ are related via a generating relation in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0}$, then $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ are related in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ and hence will be related in

$\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$.

Let $(\overline{W}; f)$ and $(\overline{W}; f')$ be related via a death path relation in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or)}}$, i.e. there is a death path $\gamma \in \mathfrak{Death}_f$ that kills the critical points p and q of f , such that $\gamma(1) = f'$. Since $\pi_0(\mathfrak{EDeath}_f, \mathfrak{Death}_f) = 0$, there is an isotopy $\Gamma : [0, 1] \rightarrow \mathfrak{Death}_f$ such that $\Gamma(0) = \gamma$ and $\Gamma(1) = \tilde{\gamma} \in \mathfrak{EDeath}_f$. Let $\tilde{f}'$ be the endpoint Morse function of the elementary death path $\tilde{\gamma}$. Let $\tilde{\mathcal{D}}_f$ be new Morse data adapted to f that consist of the same standard Morse model embeddings as $\mathcal{D}_f$ apart from those adapted to f at p and q . The standard Morse model embeddings at p and q will come from a standard double neighborhood embedding that induces the elementary death path $\tilde{\gamma}$. We further assume that the partition into level sets in $\tilde{\mathcal{D}}_f$ agrees with that in $\mathcal{D}_f$ in the complement of $[f(p), f(q)]$. Furthermore, the horizontal distribution in $\tilde{\mathcal{D}}_f$ is chosen to be compatible with that induced by the standard double neighborhood embedding. A choice of $\tilde{\mathcal{D}}_f$ as above uniquely specifies a choice of Morse data $\tilde{\mathcal{D}}_{\tilde{f}'}$ for $\tilde{f}'$. The 1-cells $(\overline{W}; f; \tilde{\mathcal{D}}_f)$ and $(\overline{W}; \tilde{f}'; \tilde{\mathcal{D}}_{\tilde{f}'})$ are, then, equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$ as they are related by an elementary death path. The 1-cells $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f; \tilde{\mathcal{D}}_f)$ are related in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ and, hence, in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$ as they live over the same 1-cell $(\overline{W}; f)$ in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0}$ and we have already seen that F_0^{data} is faithful. Lastly, there is a path of excellent Morse functions connecting f' and $\tilde{f}'$, given by $\Gamma(t)(1)$. Hence, the 1-cells $(\overline{W}; \tilde{f}')$ and $(\overline{W}; f')$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0}$ and, thus, the 1-cells $(\overline{W}; \tilde{f}'; \tilde{\mathcal{D}}_{\tilde{f}'})$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$, which implies that they are also equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$. This completes the proof that the 1-cells $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$, as desired.

Let's assume that $(\overline{W}; f)$ and $(\overline{W}; f')$ are related via a crossing path relation in $\text{Bord}_{(d-1; d)}^{\text{Morse, (or)}}$, i.e. there is a crossing path $\gamma \in \mathfrak{Cross}_f$ along which the critical level of the critical point p essentially crosses the critical level of the critical point q exactly once and no other critical levels cross, such that $\gamma(1) = f'$. Using the fact that $\pi_0(\mathfrak{ECross}_f, \mathfrak{Cross}_f) = 0$, the same argument as in the case of death paths applies and shows that the 1-cells $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ are equivalent in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$, which completes the proof. $\square$

Remark 2.4.2. The functor F^{Morse} is actually an isomorphism of categories.

2.5 The equivalence between $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{(\text{or})}$

We have, so far, constructed the equivalences

$$\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}} \xrightarrow{F^{\text{data}}} \text{Bord}_{(d-1; d)}^{\text{Morse, (or)}} \xrightarrow{F^{\text{Morse}}} \text{Bord}_{(d-1; d)}^{(\text{or})}.$$

The purpose of this section is to construct the functors

$$F \text{Sur}_{(d-1; d)}^{(\text{or})} \xrightleftharpoons[S]{B} \text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$$

and show that they are equivalences, thus completing the comparison between $\text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{(\text{or})}$. We will first construct the functors

$$F \text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0} \xrightleftharpoons[S_0]{B_0} \text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$$

and show that they are equivalences.

2.5.1 The equivalence between $F \text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$ and $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$

We will first construct the functor

$$B_0 : F \text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0} \rightarrow \text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}.$$

To do so, we need to specify where it maps the 0-cells and the generating 1-cells, and verify that the proposed assignment satisfies the generating relations.

0-cells

B_0 is the identity on 0-cells.

Generating 1-cells

Diffeomorphism type generating 1-cells

B_0 sends the 1-cell



to the 1-cell in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ with source X and target Y consisting of:

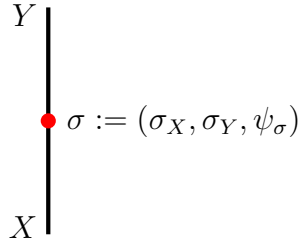
1. the mapping cylinder cobordism

$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X : X \times \{0\} \rightarrow X, \phi : X \times \{1\} \rightarrow Y),$$

2. the Morse function $f : X \times [0, 1] \rightarrow [0, 1]$, which is simply the second factor projection map,
3. the horizontal distribution $T_{f(z)}([0, 1]) \subset T_z(X \times [0, 1])$.
4. the complete set of regular values $\{0, 1\}$.

Surgery type generating 1-cells

B_0 sends the 1-cell



to the 1-cell in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ with source X and target Y consisting of:

1. The symmetric trace of the surgery cobordism $\overline{W(X, Y, \sigma)}$. The underlying manifold triad of $\overline{W(X, Y, \sigma)}$ is given by Milnor's trace of the surgery $W(X, \sigma_X)$, as constructed in [Mil63]. The bottom boundary identification is the identity, while the top boundary identification is determined by σ_Y and ψ_σ .
2. The Morse function that $\overline{W(X, Y, \sigma)}$ naturally carries shifted by 1 so its target is the $[0, 2]$ interval.
3. The Morse data that $\overline{W(X, Y, \sigma)}$ is naturally equipped with. Note that the chosen complete set of regular values is $\{0, 2\}$.

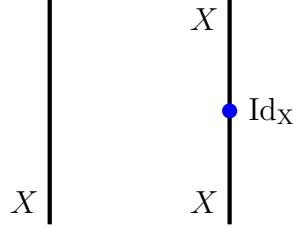
We now need to verify that B_0 is well-defined, i.e. that it respects all of the generating relations.

Generating relations

Diffeomorphism type relations

Unit.

The images of the 1-cells



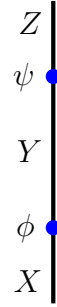
under B_0 are both the identity mapping cylinder on X , namely

$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X : X \times \{0\} \rightarrow X, \text{Id}_X : X \times \{1\} \rightarrow X),$$

equipped with the standard Morse data. The identity diffeomorphism witnesses their equivalence in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$. Therefore, this relation is satisfied by B_0 .

Composition.

The image of the 1-cell



under B_0 has underlying smooth manifold triad given by the gluing

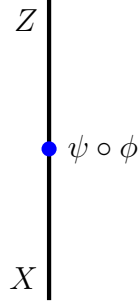
$$W := X \times [0, 1] \bigcup_{\phi: X \times \{1\} \rightarrow Y \times \{0\}} Y \times [0, 1]$$

with boundary identifications

$$\text{Id}_X : X \times \{0\} \rightarrow X, \quad \psi : Y \times \{1\} \rightarrow Z.$$

It is equipped with a Morse function $f: W \rightarrow [0, 2]$ obtained by gluing the projections $X \times [0, 1] \rightarrow [0, 1]$ and $Y \times [0, 1] \rightarrow [0, 1]$. The horizontal distribution is $T_{f(w)}([0, 1]) \subset T_w(W \times [0, 1])$, and the regular values are $\{0, 1, 2\}$.

The image of the 1-cell



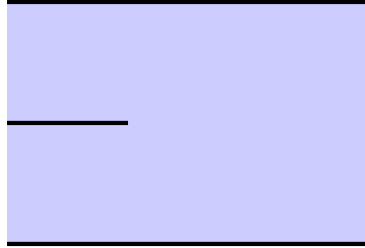
under B_0 is the mapping cylinder cobordism

$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X : X \times \{0\} \rightarrow X, \psi \circ \phi : X \times \{1\} \rightarrow Z),$$

equipped with the projection Morse function $g : X \times [0, 1] \rightarrow [0, 1]$. Here the horizontal distribution is $T_{g(z)}([0, 1]) \subset T_z(X \times [0, 1])$, and the regular values are $\{0, 1\}$.

These two bordisms with Morse data are related in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ as follows:

1. There exists a path of regular value edges in $\mathcal{MD}(W; f)$ that eliminates the regular value 1. Recall that this can be depicted as:



2. Using the horizontal distribution, we obtain a Morse-data-preserving diffeomorphism from the resulting bordism to

$$X \times [0, 2],$$

with boundary identifications

$$\text{Id}_X : X \times \{0\} \rightarrow X, \quad \psi \circ \phi : X \times \{2\} \rightarrow Z,$$

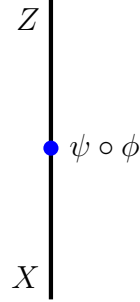
and Morse data given by:

- Morse function: projection $h : X \times [0, 2] \rightarrow [0, 2]$;
- horizontal distribution: $T_{h(z)}([0, 2]) \subset T_z(X \times [0, 2])$;
- regular values: $\{0, 2\}$.

3. The diffeomorphism

$$X \times [0, 2] \longrightarrow X \times [0, 1], \quad (x, t) \mapsto (x, t/2),$$

identifies this bordism with the image of the 1-cell

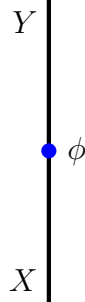


under B_0 , preserving the Morse data.

Therefore, this relation is satisfied by B_0 .

Isotopy.

The image of the 1-cell

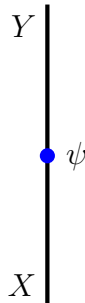


under B_0 is the mapping cylinder cobordism

$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X : X \times \{0\} \rightarrow X, \phi : X \times \{1\} \rightarrow Y),$$

equipped with the projection Morse function $f : X \times [0, 1] \rightarrow [0, 1]$. The horizontal distribution is $T_{f(z)}([0, 1]) \subset T_z(X \times [0, 1])$, and the regular values are $\{0, 1\}$.

Similarly, the image of the 1-cell



under B_0 is the mapping cylinder cobordism

$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X : X \times \{0\} \rightarrow X, \psi : X \times \{1\} \rightarrow Y),$$

with the same Morse data (projection Morse function, horizontal distribution $T_{f(z)}([0, 1])$, and regular values $\{0, 1\}$).

Now suppose there exists an isotopy θ from ϕ to ψ . This isotopy induces a path of Morse data on

$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X : X \times \{0\} \rightarrow X, \phi : X \times \{1\} \rightarrow Y),$$

that fixes the Morse function and regular values while isotoping the horizontal distribution. The final horizontal distribution then determines a Morse-data-preserving diffeomorphism from the bordism at the end of this path to

$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X : X \times \{0\} \rightarrow X, \psi : X \times \{1\} \rightarrow Y),$$

with the Morse data as described above. Therefore, this relation is satisfied by B_0 .

Naturality-type relations

The image of the 1-cell

$$\begin{array}{c} Z \\ \phi \bullet \\ Y \\ \sigma := (\sigma_X, \sigma_Y, \psi_\sigma) \bullet \\ X \end{array}$$

under B_0 has underlying smooth manifold triad given by the gluing

$$W := W(X, \sigma_X) \bigcup_{\psi: \chi(X, \sigma_X) \rightarrow Y \times \{0\}} Y \times [0, 1],$$

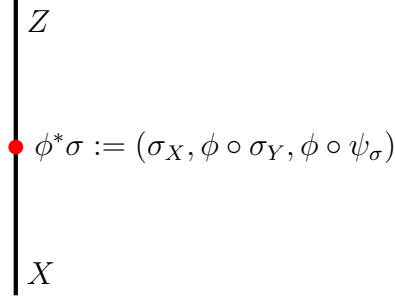
with boundary identifications

$$\begin{aligned} \text{Id}_X : X \setminus (\sigma_X(S^{k-1} \times \{0\})) &\bigcup_{\sigma_X: S^{k-1} \times (D^{n-k} \setminus \{0\}) \rightarrow X \setminus (\sigma_X(S^{k-1} \times \{0\}))} S^{k-1} \times D^{n-k} \rightarrow X, \\ \phi : Y \times \{1\} &\rightarrow Z. \end{aligned}$$

It comes equipped with a Morse function $f : W \rightarrow [0, 3]$ obtained by gluing the standard Morse function on $W(X, \sigma_X)$ (shifted by 1) with the projection

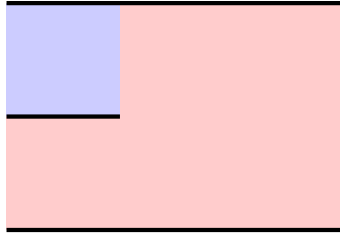
$Y \times [0, 1] \rightarrow [0, 1]$. The standard Morse model embedding is the one naturally associated to $W(X, \sigma_X)$, and the horizontal distribution is obtained by gluing the standard horizontal distributions on $W(X, \sigma_X)$ and $Y \times [0, 1]$. The regular values are $\{0, 1, 2\}$.

The image of the 1-cell



under B_0 is the cobordism $\overline{W(X, Z, \phi^* \sigma)}$. It comes with the Morse function $g : W \rightarrow [0, 2]$ induced by the standard Morse function on $W(X, \sigma_X)$, together with its natural Morse model embedding and horizontal distribution. The regular values are $\{0, 2\}$.

These two bordisms are related in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ as follows. First, there is a path of regular value edges in $\mathcal{MD}(W; f)$ that eliminates the regular value 2. Recall that this can be depicted as:



At the end of this path, we obtain a Morse-data-preserving diffeomorphism from W , equipped with its modified Morse data, to $W(X, \sigma_X)$ with its standard Morse data.

Any such diffeomorphism is determined by a map $[0, 3] \rightarrow [0, 2]$ sending $0 \mapsto 0$, $3 \mapsto 2$, and $1 \mapsto 1$ (preserving the critical value). Moreover, any two such maps are isotopic, and hence the resulting Morse-data-preserving diffeomorphism is unique up to isotopy.

Therefore, the naturality-type relation

$$\begin{array}{c} Z \\ \phi \\ Y \\ \sigma := (\sigma_X, \sigma_Y, \psi_\sigma) \\ X \end{array} \quad = \quad \begin{array}{c} Z \\ \bullet \\ \phi^* \sigma := (\sigma_X, \phi \circ \sigma_Y, \phi \circ \psi_\sigma) \\ X \end{array}$$

is satisfied by B_0 .

A completely analogous argument shows that B_0 also respects the naturality relation

$$\begin{array}{c} Z \\ \sigma := (\sigma_Y, \sigma_Z, \psi_\sigma) \\ Y \\ \phi \\ X \end{array} \quad = \quad \begin{array}{c} Z \\ \bullet \\ \phi_* \sigma := (\phi^{-1} \circ \sigma_Y, \sigma_Z, \psi_\sigma \circ \phi) \\ X \end{array}$$

Frame change type relations

The image of the 1-cell

$$\begin{array}{c} Y \\ \sigma := (\sigma_X, \sigma_Y, \psi_\sigma) \\ X \end{array}$$

under B_0 consists of the smooth manifold triad

$$W(X, \sigma_X) := X \setminus (S^{k-1} \times \{0\}) \times [-1, 1] \bigcup_{\sigma_X, \text{ flow lines}} M_k^n.$$

The image of the 1-cell

$$\begin{array}{c} Y \\ \bullet \\ Q \cdot \sigma := (\sigma_X \circ Q^{-1}, \sigma_Y \circ Q^{-1}, \psi_\sigma) \\ X \end{array}$$

under B_0 consists of the smooth manifold triad

$$W(X, \sigma_X \circ Q^{-1}) := X \setminus (S^{k-1} \times \{0\}) \times [-1, 1] \bigcup_{\sigma_X \circ Q^{-1}, \text{ flow lines}} M_k^n.$$

Here, recall that $Q \in G_{(n,k)}^{(\text{or})}$, which induces a diffeomorphism $M_k^n \rightarrow M_k^n$ that respects the Morse function and flow lines of M_k^n , which we also refer to as Q by abuse of notation. The diffeomorphisms

$$\text{Id} : X \setminus (S^{k-1} \times \{0\}) \times [-1, 1] \rightarrow X \setminus (S^{k-1} \times \{0\}) \times [-1, 1]$$

and

$$Q : M_k^n \rightarrow M_k^n$$

glue to produce a diffeomorphism

$$D : W(X, \sigma_X) \rightarrow W(X, \sigma_X \circ Q^{-1}).$$

Both $W(X, \sigma_X)$ and $W(X, \sigma_X \circ Q^{-1})$ come naturally equipped with standard Morse model embeddings, given by the evident inclusions, and the following diagram is commutative:

$$\begin{array}{ccc} W(X, \sigma_X) & \xrightarrow{D} & W(X, \sigma_X \circ Q^{-1}) \\ \uparrow \iota & & \uparrow \iota' \\ M_k^n & \xrightarrow{Q} & M_k^n \end{array}$$

Moreover, it is straightforward to see that D preserves the boundary identifications and the Morse data. It is, thus, an equivalence of 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$, as desired. This completes the proof that the functor B_0 is well defined.

We will now construct the functor

$$S_0 : \text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0} \rightarrow F\text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0},$$

which will end up being inverse to B_0 .

Our choice of Morse data was mainly motivated by making the definition of S_0 as easy to state as possible. As part of the Morse data, we have a choice of regular level sets, which is compatible with the choice of standard Morse model embeddings. This means that between two regular level sets there exists at most one critical point. In order to construct S_0 , we will specify where it sends 1-cells with only two regular level sets in their Morse data, as any 1-cell can be obtained as a composite of those.

If we only have two regular level sets X and Y , and no critical points between them, the horizontal distribution which is part of the Morse data induces a diffeomorphism $\phi : X \rightarrow Y$ and S_0 will send this 1-cell to the diffeomorphism type generating 1-cell labeled by ϕ .

If we only have two regular level sets X and Y , and a single critical point between them, the standard Morse model embedding and the horizontal distribution which are part of the Morse data will induce a surgery triple datum σ on (X, Y) , since the regular level sets are assumed to be compatible with the standard Morse model embeddings. S_0 will then send this 1-cell to the surgery type generating 1-cell labeled by σ .

Just like with B_0 , we have to show that S_0 is well-defined. This will be equivalent to showing that B_0 is faithful on 1-cells. In particular, we need to verify that S_0 respects the generating relations in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$.

Morse data preserving diffeomorphisms

Let $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ related by a Morse-data-preserving diffeomorphism. By definition, such a diffeomorphism preserves the Morse function, the horizontal distributions, and all regular level sets. It also preserves the standard Morse model embeddings up to an action of $G_{(n,k)}^{(\text{or})}$. It follows that the induced images of $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ under S_0 differ by frame change and naturality type relations. Therefore, S_0 respects Morse-data-preserving diffeomorphisms.

Paths of regular value edges

Let $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ that are related by a path of regular value edges in $\mathcal{MD}(W; f)$. Then, their images under S_0 will be related via composition and naturality type relations.

Isotopies of Morse data with fixed regular level sets

Let $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}'; f'; \mathcal{D}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0}$ that are related by a path of Morse data, with a fixed choice of regular level sets.. We will show that their images under S_0 are related in $F\text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$. There are three cases to consider:

1. $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f'; \mathcal{D}_{f'})$ are related by a path of excellent Morse functions and a compatible path of Morse data. Let $\gamma : [0, 1] \rightarrow \mathcal{F}^0(W)$ be a path of excellent Morse functions with $\gamma(0) = f$ and $\gamma(1) = f'$. By the stability of Morse functions and corollary 1.3.7, there exists an isotopy of diffeomorphisms in W that induces the isotopy of Morse functions γ . By pulling back the isotopy of Morse data via this isotopy of diffeomorphisms, we get an isotopy of Morse data on W that keeps the Morse function fixed throughout. Thus, this is the only case we need to consider.
2. Let's suppose that $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f; \mathcal{D}'_f)$ are related via an isotopy of horizontal distributions. It suffices to consider 1-cells with just two regular level sets, as a general 1-cell will be a composite of those. In this case, we can directly see that the images of the 1-cells under S_0 will be related via a combination of composition, naturality and isotopy relations. Note that we arrive at the same observation if we also allow a compatible isotopy of standard Morse model embeddings with fixed framed attaching spheres.
3. Let's suppose that $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f; \mathcal{D}'_f)$ are related via an isotopy of standard Morse model embeddings, ϕ_t , and a compatible isotopy of horizontal distributions, τ_t . Let's suppose without loss of generality that we have chosen exactly two regular level sets on $\overline{W}$, X and Y . We will show that ϕ_t and τ_t are induced by ambient isotopies of X and Y .

By considering the closure of the standard Morse model embeddings, we obtain an isotopy $\overline{\phi}_t^X$ of $S^{k-1} \times \overline{D^{n-k}}$ on X and an isotopy $\overline{\phi}_t^Y$ of $\overline{D^k} \times S^{n-k-1}$ on Y . Let $\iota_X : X \times [0, 1] \rightarrow X$ and $\iota_Y : Y \times [0, 1] \rightarrow Y$ be ambient isotopies extending $\overline{\phi}_t^X$ and $\overline{\phi}_t^Y$, respectively. Their existence is guaranteed by the ambient isotopy lemma. Consider the isotopy of Morse data induced by ϕ_t , τ_t composed with the ambient isotopies ι_X^{-1} and ι_Y^{-1} . This produces an isotopy of Morse data that fixes the framed attaching spheres. By the previous point, this induces a new isotopy of Y , which we denote by θ_Y . Then, the ambient isotopies ι_X and $\theta_Y \circ \iota_Y$ induce the original isotopy of standard Morse model embeddings. This implies that the images of $(\overline{W}; f; \mathcal{D}_f)$ and $(\overline{W}; f; \mathcal{D}'_f)$ under S_0 are related via naturality and isotopy relations.

This completes the proof that the functor S_0 is well-defined. It is now easy to check that the functors B_0 , S_0 are inverses of each other, which implies that the categories $F \text{Sur}_{(d-1; d)}^{(\text{or}), \mathcal{F}^0}$ and $\text{Bord}_{(d-1; d)}^{\text{Morse data}, (\text{or}), \mathcal{F}^0}$ are equivalent, and, in fact, isomorphic.

2.5.2 The equivalence between $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$

We will now show that the functors B_0, S_0 induce functors

$$F \text{Sur}_{(d-1; d)}^{(\text{or})} \xrightleftharpoons[S]{B} \text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$$

which will allow us to conclude that $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$ are equivalent, and, in fact, isomorphic. The relations in $\text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{\text{Morse data, (or)}}$ have been chosen in order to make this step essentially tautological:

1. Two 1-cells in $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ are related via a cusp relation if and only if their images under B_0 are related via an elementary birth - death path and vice versa.
2. Two 1-cells in $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ are related via a crossing relation if and only if their images under B_0 are related via an elementary crossing path and vice versa.

This completes the proof of the equivalence between $F \text{Sur}_{(d-1; d)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d)}^{(\text{or})}$.

Chapter 3

Surgery presentations of bordism bicategories

In this chapter, we categorify the results of Chapter 2 to obtain a surgery presentation of the symmetric monoidal $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$. This $(2, 1)$ -category has as objects closed, (oriented) $(d-1)$ -dimensional manifolds, as 1-morphisms (oriented) d -dimensional cobordisms, and as 2-morphisms isotopy classes of orientation-preserving diffeomorphisms.

Section 3.1 recalls from [SP14] the notion of *unbiased semistrict symmetric monoidal 2-categories* and their presentations via generators and relations, which we refer to as *unbiased semistrict symmetric monoidal 2-presentations*. Since these are the only kinds of 2-categorical presentations considered in this thesis, we will often refer to them simply as *2-presentations*.

Section 3.2 serves as a warm-up. It introduces the 2-presentation of planar diffeomorphism diagrams, $\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$, and shows that the associated 2-category $F\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ is equivalent to the 2-groupoid of smooth manifolds, diffeomorphisms, and isotopy classes of isotopies, denoted $\text{Man}_{(d-1; \sim_2)}^{(\text{or})}$. Section 3.3 then introduces the 2-presentation of planar surgery diagrams, $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$, which categorifies $\text{Sur}_{(d-1; d)}^{(\text{or})}$.

Section 3.4 introduces the symmetric monoidal $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$, whose objects are closed, (oriented) $(d-1)$ -manifolds, whose 1-morphisms are (oriented) d -dimensional cobordisms equipped with Morse functions, and whose 2-morphisms are equivalence classes of orientation-preserving mapping cylinders equipped with Cerf paths. As a direct application of the genericity of Morse functions, Cerf paths, and Cerf homotopies, we show that the forgetful functor

$$F^{\text{Cerf}} : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$$

is an equivalence.

Section 3.5 introduces the symmetric monoidal $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or)}}$, whose 1-morphisms are (oriented) cobordisms equipped with Morse functions and compatible Morse data, and whose 2-morphisms are mapping cylinders equipped with Cerf paths and compatible Cerf data. We show that the forgetful functor

$$F^{\text{data}} : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or)}} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or)}}$$

is an equivalence, thereby establishing the chain of equivalences

$$\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or)}} \xrightarrow{F^{\text{data}}} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or)}} \xrightarrow{F^{\text{Cerf}}} \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}.$$

Cerf data allow us to associate diffeomorphisms of cobordisms with planar surgery diagrams. We describe this in detail in section 3.6, where we construct the functors

$$F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})} \xrightleftharpoons[S]{B} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or)}}$$

and show that they are equivalences, thus establishing the equivalence between $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$.

3.1 2-presentations

Our goal in this chapter is to introduce a generators-and-relations presentation of the symmetric monoidal $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$. To even begin such a task, we must first clarify what is meant by a presentation of a symmetric monoidal bicategory via generators and relations—a notion that is far from straightforward. In fact, even defining symmetric monoidal bicategories is a formidable undertaking. Fortunately, this has been addressed in detail by Schommer-Pries in [SP14], and we will adopt the technical framework developed there.

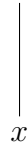
To simplify matters, we work with *unbiased semistrict symmetric monoidal 2-categories*. These are stricter versions of symmetric monoidal bicategories in which many coherence 2-cells are taken to be trivial. Crucially, every symmetric monoidal bicategory is equivalent to an unbiased semistrict symmetric monoidal 2-category. We begin by recalling the definition of these categories and then describe their presentations. Our aim is not to reproduce the full theory from scratch, but to provide sufficient background for the unfamiliar reader to follow the upcoming developments. For further detail, we refer to [SP14].

3.1.1 Unbiased semistrict symmetric monoidal 2-categories

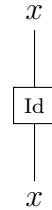
Wire diagrams

We will use the wire diagram calculus for 2-categories described by Bartlett in [Bar14]. We briefly recall the basics, following [Bar14] almost verbatim. Given a 2-category $\mathcal{C}$, we denote:

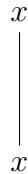
- An object x of $\mathcal{C}$ is represented by the diagram



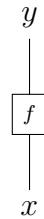
- The identity 1-morphism on an object x of $\mathcal{C}$ is drawn as



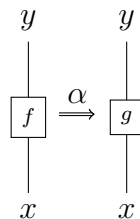
When it is clear from context that we are not simply referring to the object x , we will often abbreviate this by omitting the label:



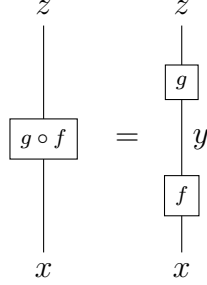
- A 1-morphism $f : x \rightarrow y$ of $\mathcal{C}$ by the diagram



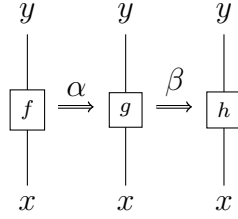
- A 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$ by the diagram



Composition of 1-morphisms in $\mathcal{C}$ is represented by vertical stacking of diagrams:

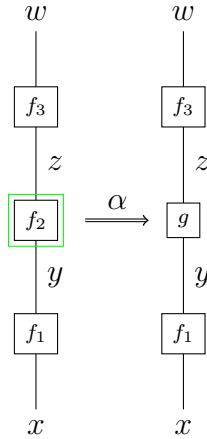


Vertical composition of 2-morphisms is (somewhat confusingly) represented by *horizontal stacking* of wire diagrams:



We will simply refer to this as *composition of wire diagrams*.

Now suppose f_1, f_2, f_3 are composable 1-morphisms and $\alpha : f_2 \Rightarrow g$ is a 2-morphism. Then the *whiskering* of α by the identity 2-morphisms on f_1 and f_3 is represented as follows:



Finally, horizontal composition in $\mathcal{C}$ is expressed through a combination of whiskering and wire diagram composition.

For more details, the reader is encouraged to consult [\[Bar14\]](#).

Definitions

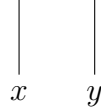
Equipped with the wire diagram calculus, we can now define unbiased semistrict symmetric monoidal 2-categories. We begin by recalling the definition of a *stringent monoidal 2-category*, following [Bar14].

Definition 3.1.1. A **stringent monoidal 2-category** $M = (M, 1, \otimes, \phi_{f,g})$ consists of:

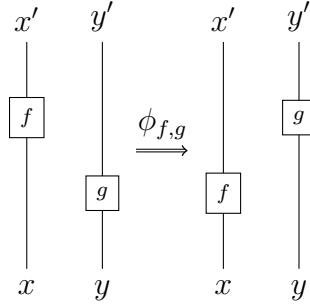
- A strict 2-category M ,
- An object $1 \in M$, drawn as the invisible wire:



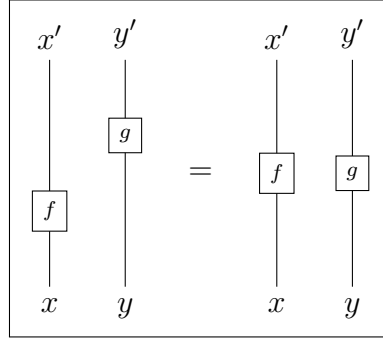
- Strict left- and right-tensor functors $x \otimes -$ and $- \otimes x$ from M to itself for every object $x \in M$. For any two objects $x, y \in M$, we draw $x \otimes y$ as



- Interchangor 2-isomorphisms $\phi_{f,g}: (f \otimes \text{Id}) \circ (\text{Id} \otimes g) \Rightarrow (\text{Id} \otimes g) \circ (f \otimes \text{Id})$, drawn as



This data is subject to relations, which we omit here and refer the reader to [Bar14] for a complete list. These relations ensure that the interchangor 2-cells are natural and behave coherently with respect to composition of 1-cells, tensoring, and units. We highlight here a particularly important identity, known as *nudging*:

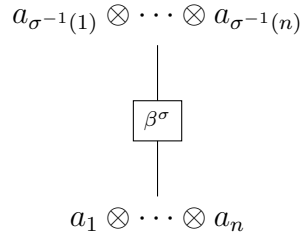


Definition 3.1.2. An **unbiased semistrict symmetric monoidal 2-category** is a stringent monoidal 2-category $(M, 1, \otimes, \phi_{f,g})$ equipped with:

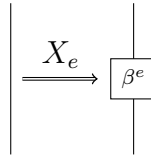
- For every n -tuple of objects $a_1, \dots, a_n \in M$ and every $\sigma \in \Sigma_n$, a 1-cell

$$\beta^\sigma : a_1 \otimes \dots \otimes a_n \rightarrow a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(n)},$$

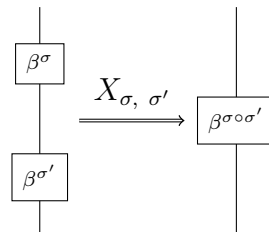
drawn as



- The following structure 2-cells and their inverses (omitting labels for readability):
 - For the identity permutation $e \in \Sigma_n$, a 2-isomorphism



- For all $\sigma, \sigma' \in \Sigma_n$, a 2-isomorphism



- For every n -tuple of objects $a_1, \dots, a_n \in M$, every $\sigma \in \Sigma_n$ with $\sigma(i) = j$, and every 1-morphism $f_i : a_i \rightarrow b_i$, define:

$$\begin{aligned}\tilde{f}_i &:= \text{Id} \otimes \dots \otimes f_i \otimes \dots \otimes \text{Id} : a_1 \otimes \dots \otimes a_i \otimes \dots \otimes a_n \rightarrow a_1 \otimes \dots \otimes b_i \otimes \dots \otimes a_n, \\ \tilde{f}_i^\sigma &:= \text{Id} \otimes \dots \otimes f_{\sigma^{-1}(j)} \otimes \dots \otimes \text{Id} : a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(j)} \otimes \dots \otimes a_{\sigma^{-1}(n)} \\ &\rightarrow a_{\sigma^{-1}(1)} \otimes \dots \otimes b_{\sigma^{-1}(j)} \otimes \dots \otimes a_{\sigma^{-1}(n)}.\end{aligned}$$

Then, there exists a coherence 2-cell

This data is subject to relations, which we omit here. We refer the reader to [SP14, Section 2.6] for a complete list. These relations express the associativity and unitality coherence laws for X_e and $X_{\sigma, \sigma'}$, and ensure compatibility with the operadic composition in symmetric groups.

3.1.2 Presentations of unbiased semistrict symmetric monoidal 2-categories

We describe presentations of unbiased semistrict symmetric monoidal 2-categories in five stages. To keep things concise, we refer the reader to [SP14, Sections 2.10, 2.12] for full details.

Definition 3.1.3. An **unbiased semistrict symmetric monoidal 0-computad** consists of a set G_0 . A **word** in G^0 is defined recursively as follows:

- The empty word is a word in G^0 ;
- If $a \in G^0$, then a is a word in G^0 ;
- If u and v are words in G^0 , then $u \otimes v$ is a word in G^0 .

We denote the set of words in G^0 by $W(G^0)$.

Definition 3.1.4. An **unbiased semistrict symmetric monoidal 1-computad** consists of a set G^0 , a set G^1 , and a pair of maps $s, t : G^1 \rightarrow W(G^0)$. We define **elementary sentences** in G^1 as follows:

- The empty sentence is a sentence in G^1 with source and target the empty word.
- For every word $w \in W(G^0)$, the wire diagram

$$\begin{array}{c} w \\ | \\ \boxed{\text{Id}_w} \\ | \\ w \end{array} = \begin{array}{c} w \\ | \\ w \end{array}$$

is an elementary sentence in G^1 .

- For every word $w = (w_i)$ in $W(G^0)$ of length n and every permutation $\sigma \in \Sigma_n$, the wire diagram

$$\begin{array}{c} w' \\ | \\ \boxed{\beta^\sigma} \\ | \\ w \end{array}$$

where $w' = (w_{\sigma^{-1}(i)})$ is the result of applying σ to w , is an elementary sentence in G^1 .

- For any words $w, w' \in W(G^0)$ and $f \in G^1$, the wire diagram

$$\begin{array}{c} w \otimes t(f) \otimes w' \\ | \\ \boxed{\text{Id}_w \otimes f \otimes \text{Id}_{w'}} \\ | \\ w \otimes s(f) \otimes w' \end{array}$$

is an elementary sentence in G^1 . If w or w' is empty, this is modified in the obvious way.

If γ denotes a wire diagram, we define $s(\gamma)$ to be its bottom label and $t(\gamma)$ to be its top label.

A **sentence** in G^1 is defined recursively as follows:

- If f is an elementary sentence in G^1 , then f is a sentence in G_1 .
- If f and g are sentences in G^1 with $t(f) = s(g)$, then the composition $f \circ g$ is a sentence in G^1 with source $s(f)$ and target $t(g)$.

- If f and g are sentences in G^1 , then $f \otimes g$ is a sentence in G^1 with source $s(f) \otimes s(g)$ and target $t(f) \otimes t(g)$.

By definition, sentences in G^1 are represented by wire diagrams, with composition and tensoring of wire diagrams defined as above. We denote the set of such sentences by $S(G^0, G^1, s, t)$.

Definition 3.1.5. An **unbiased semistrict symmetric monoidal 2-computad** consists of an unbiased semistrict symmetric monoidal 1-computad (G^0, G^1, s, t) , a set G_2 , and a pair of maps $s, t : G_2 \rightarrow S(G^0, G^1, s, t)$. These maps are required to satisfy

$$s(s(e)) = s(t(e)) \quad \text{and} \quad t(s(e)) = t(t(e))$$

for all $e \in G_2$.

We define **elementary paragraphs** in G_2 as follows:

- For every sentence $f \in S(G^0, G^1, s, t)$, the identity on f is an elementary paragraph in G_2 with source and target f .
- The structure 2-cells X_e, X_σ, σ' , the coherence 2-cells $\beta_{(f_i)}^\sigma$, and their inverses (for all $\sigma, \sigma' \in \Sigma_n$ and $f \in S(G^0, G^1, s, t)$) are elementary paragraphs in G_2 with their natural sources and targets.
- The interchangor 2-cells $\phi_{f,g}$ (for all $f, g \in S(G^0, G^1, s, t)$) are elementary paragraphs in G_2 with their natural sources and targets.
- For any sentences $f, g \in S(G^0, G^1, s, t)$ and $u \in G^2$, the tensor $\text{Id}_f \otimes u \otimes \text{Id}_g$ is an elementary paragraph in G_2 with source $f \otimes s(u) \otimes g$ and target $f \otimes t(u) \otimes g$.

A **paragraph** in G^2 is defined recursively as follows:

- If u is an elementary paragraph in G^2 , then u is a paragraph in G^2 .
- If u and v are paragraphs in G^2 with $t(u) = s(v)$, then the composition (concatenation) of the wire diagrams representing u and v is a paragraph in G_2 with source $s(u)$ and target $t(v)$.
- If u and v are paragraphs in G^2 with $t(s(u)) = s(s(v))$, then the horizontal composition of u and v is a paragraph in G^2 with source $s(u) \circ s(v)$ and target $t(u) \circ t(v)$.

- If u and v are paragraphs in G^2 , then $u \otimes v$ is a paragraph in G^2 with source $s(u) \otimes s(v)$ and target $t(u) \otimes t(v)$.

Definition 3.1.6. Let $G = (G^0, G^1, G^2)$ be an unbiased semistrict symmetric monoidal 2-computad. We define $F(G)$ as the unbiased semistrict symmetric monoidal 2-category constructed as follows:

- Objects: words in G^0 ;
- 1-morphisms: sentences in G^1 ;
- 2-morphisms: equivalence classes of paragraphs in G^2 , where the equivalence relation is the finest one generated by the relations imposed by the structure of an unbiased semistrict symmetric monoidal 2-category.

Definition 3.1.7. Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let R be a set of pairs of 2-morphisms (α, β) in $\mathcal{C}$ such that α and β have the same source and target. We call R a set of *generating relations*. It determines an equivalence relation—also denoted by R by abuse of notation—on the 2-morphisms of $\mathcal{C}$, defined as the finest equivalence relation containing R that is compatible with the symmetric monoidal bicategory structure of $\mathcal{C}$.

Remark 3.1.1. In this work, we only consider relations at the level of 2-morphisms.

Definition 3.1.8. An **unbiased semistrict symmetric monoidal 2-presentation**, or simply a **2-presentation**, $\mathcal{P}$ consists of an unbiased semistrict symmetric monoidal 2-computad G together with a set of generating relations R for the unbiased semistrict symmetric monoidal 2-category $F(G)$. We define

$$F(\mathcal{P}) := F(G)/R$$

to be the unbiased semistrict symmetric monoidal 2-category presented by $\mathcal{P}$.

3.2 Planar diffeomorphism diagrams

This section serves as a warm-up. Instead of studying the full bordism $(2, 1)$ -category, we will focus on the bigroupoid $\text{Man}_{(d-1; \sim_2)}^{(\text{or})}$ that has:

- 0-cells: closed, (oriented) $(d - 1)$ -dimensional manifolds,
- 1-cells: (orientation preserving) diffeomorphisms

- 2-cells: isotopy classes of isotopies.

We will graphically encode this structure using planar diffeomorphism diagrams, which we now introduce.

3.2.1 The 2-presentation of planar diffeomorphism diagrams

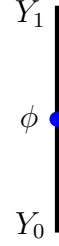
$$\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$$

The 2-presentation $\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ is defined as follows:

- **Generating 0-cells**

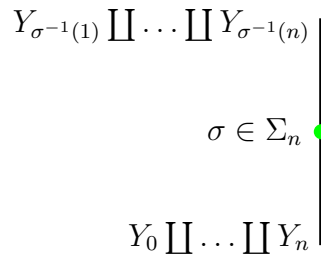
Closed, connected, (oriented) $(d - 1)$ -dimensional manifolds.

- **Generating 1-cells**

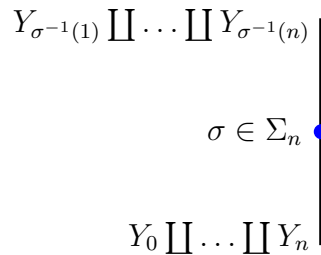


where $\phi : Y_0 \rightarrow Y_1$ is a(n orientation preserving) diffeomorphism.

Remark 3.2.1. We will denote the structure 1-cell β^σ of the generated unbiased semistrict symmetric monoidal 2-category by



This is distinct from the generating 1-cell

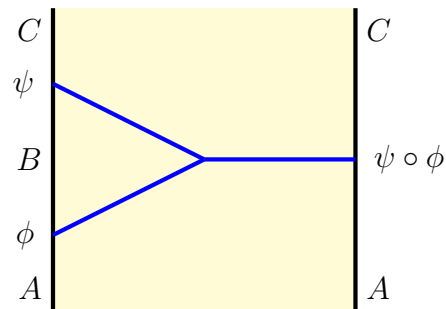


corresponding to the diffeomorphism induced by σ . The reader is cautioned as this seemingly small distinction has caused the author quite some headache.

- **Generating 2-cells**

Compositor 2-cells

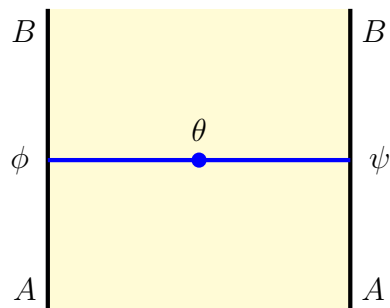
Planar diagrams of the form



alongside their mirror image.

Isotopy 2-cells

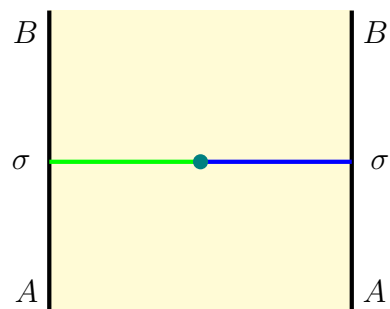
Planar diagrams of the form



where θ is an isotopy from ϕ to ψ .

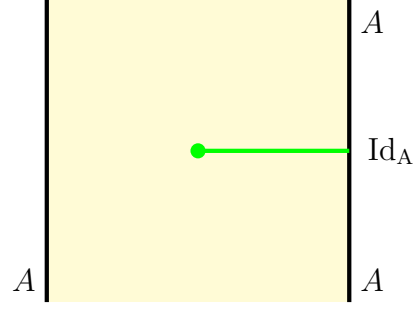
Symmetry 2-cells

Planar diagrams of the form

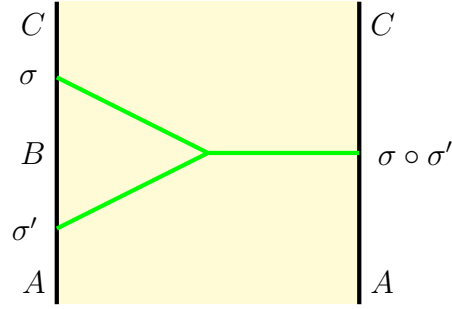


alongside their inverses. These 2-cells express the compatibility between the structure 1-cells β^σ and the generating 1-cells arising from diffeomorphisms induced by elements of the symmetric group.

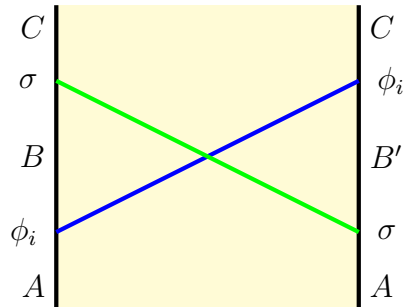
Remark 3.2.2. We will denote the structure 2-cells X_e by



We will omit the label Id_A from now on, wherever it is required and can be easily deduced from context. We will denote the structure 2-cells $X_{\sigma, \sigma'}$ by

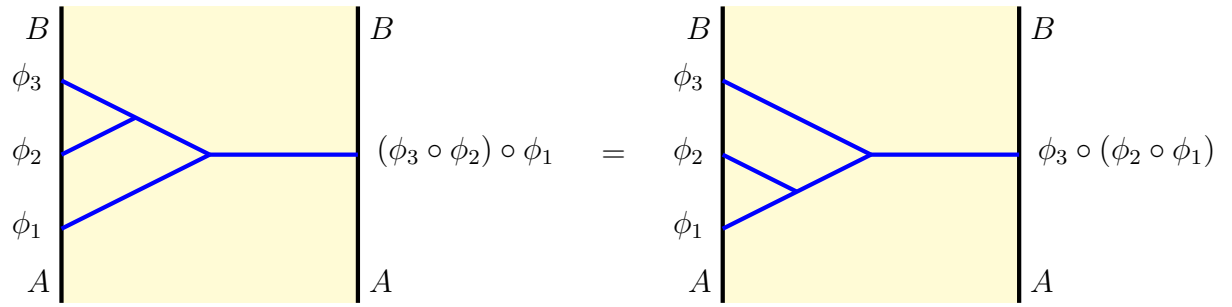


We will denote the structure 2-cells $\beta_{(\phi_i)}^\sigma$ by



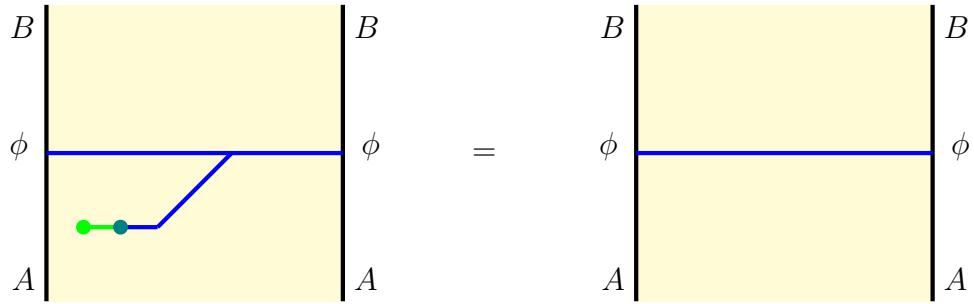
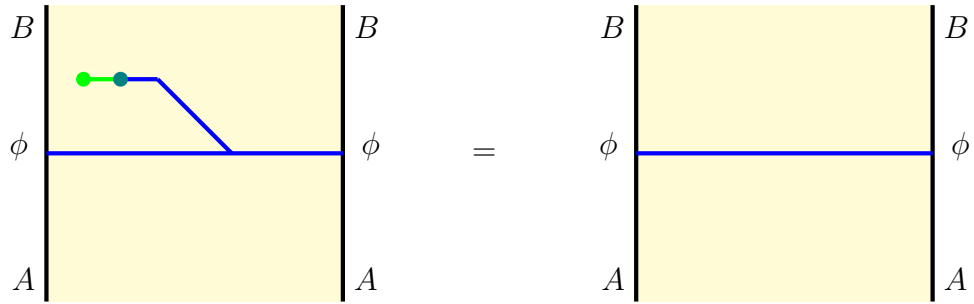
- **Relations**

1. Associativity relations



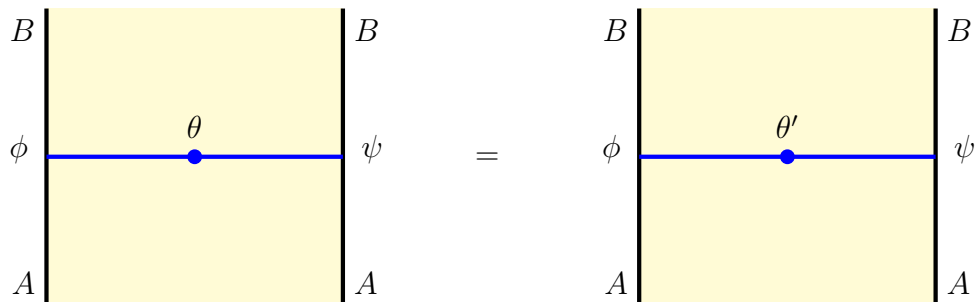
alongside their mirror image.

2. Unitality relations

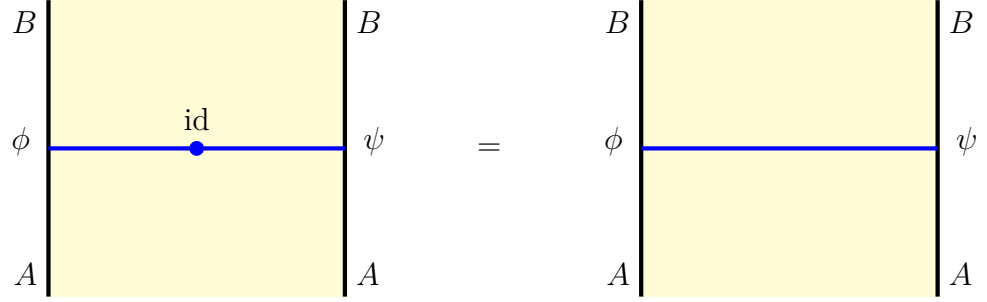


alongside their mirror images.

3. Isotopy invariance:

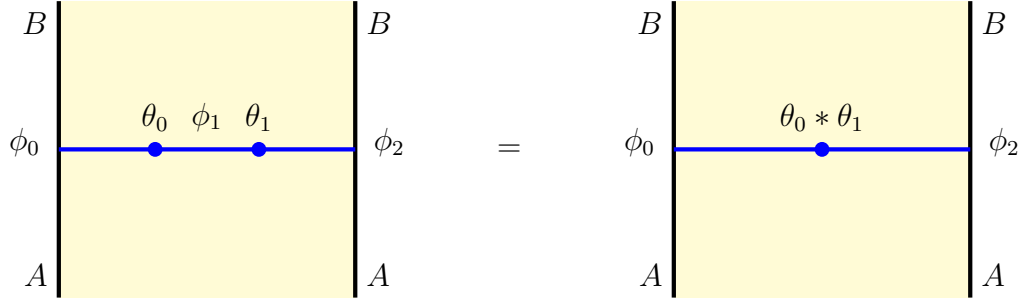


if θ and θ' are isotopic isotopies. Moreover,



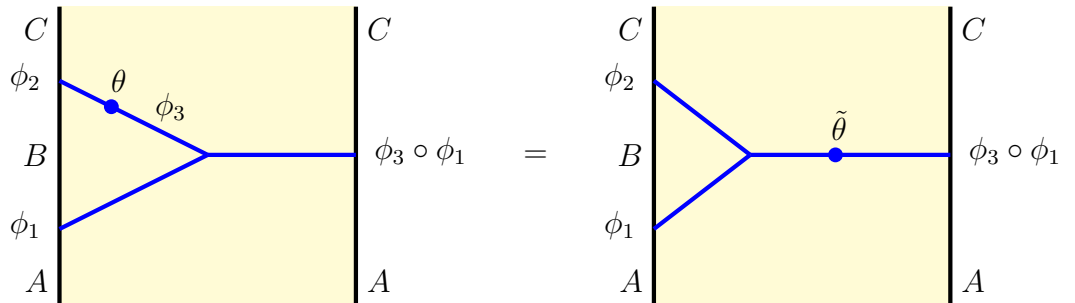
where id denotes the constant isotopy $A \times [0, 1] \rightarrow B$ given by $(a, t) \mapsto \phi(a)$.

4. Concatenation of isotopies:

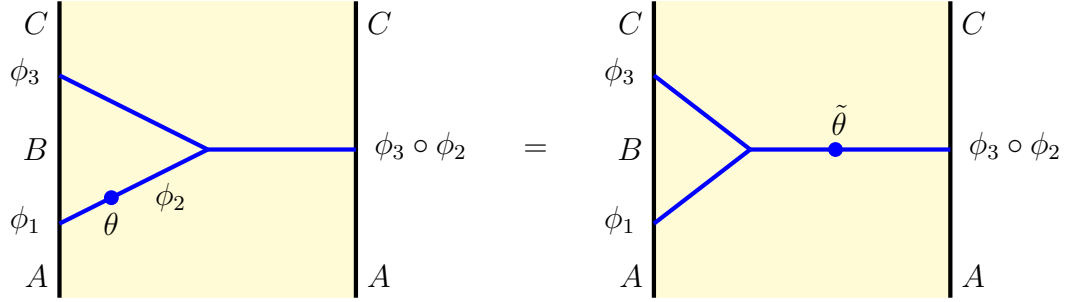


Here, $\theta_0 * \theta_1$ is the concatenation of the isotopies θ_0 and θ_1 . Any two ways of concatenating two isotopies are isotopic, hence we don't need to make a specific choice given the isotopy invariance relation.

5. Isotopy - compositor crossing:



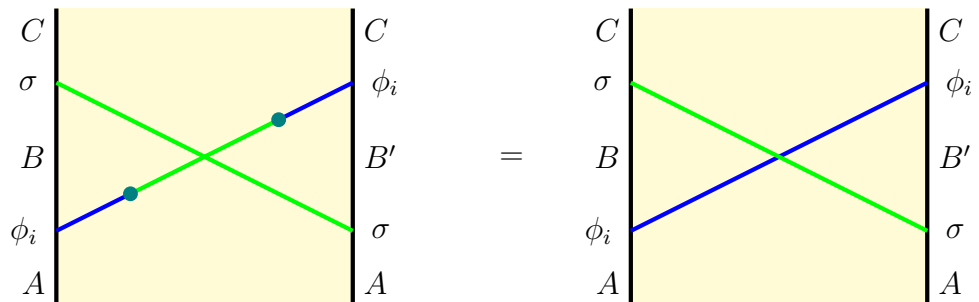
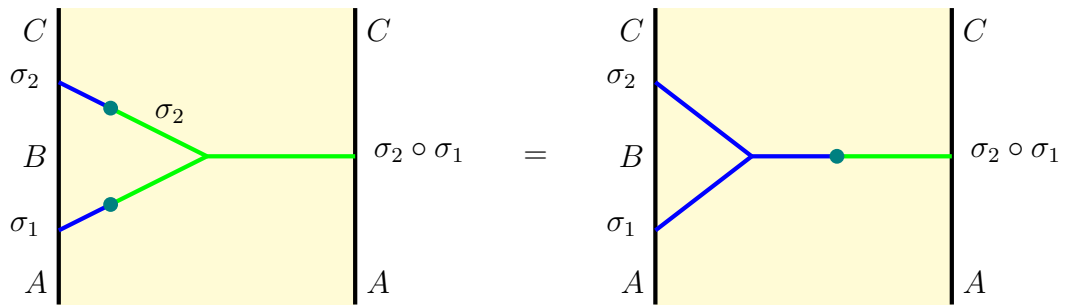
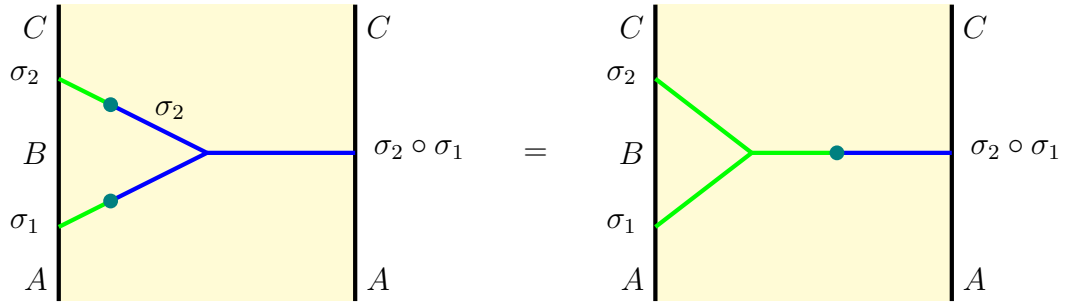
Here, $\tilde{\theta}$ is the induced isotopy from $\phi_2 \circ \phi_1$ to $\phi_3 \circ \phi_1$, given by level-wise precomposing θ by ϕ_1 .



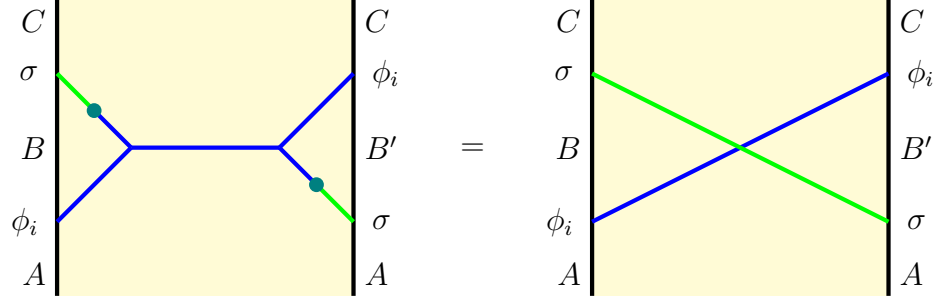
Here, $\tilde{\theta}$ is the induced isotopy from $\phi_2 \circ \phi_1$ to $\phi_3 \circ \phi_1$, given by level-wise post-composing θ by ϕ_3 .

The isotopy-compositor crossing relations further include all the mirror images of the above.

6. Symmetry - compositor crossing:

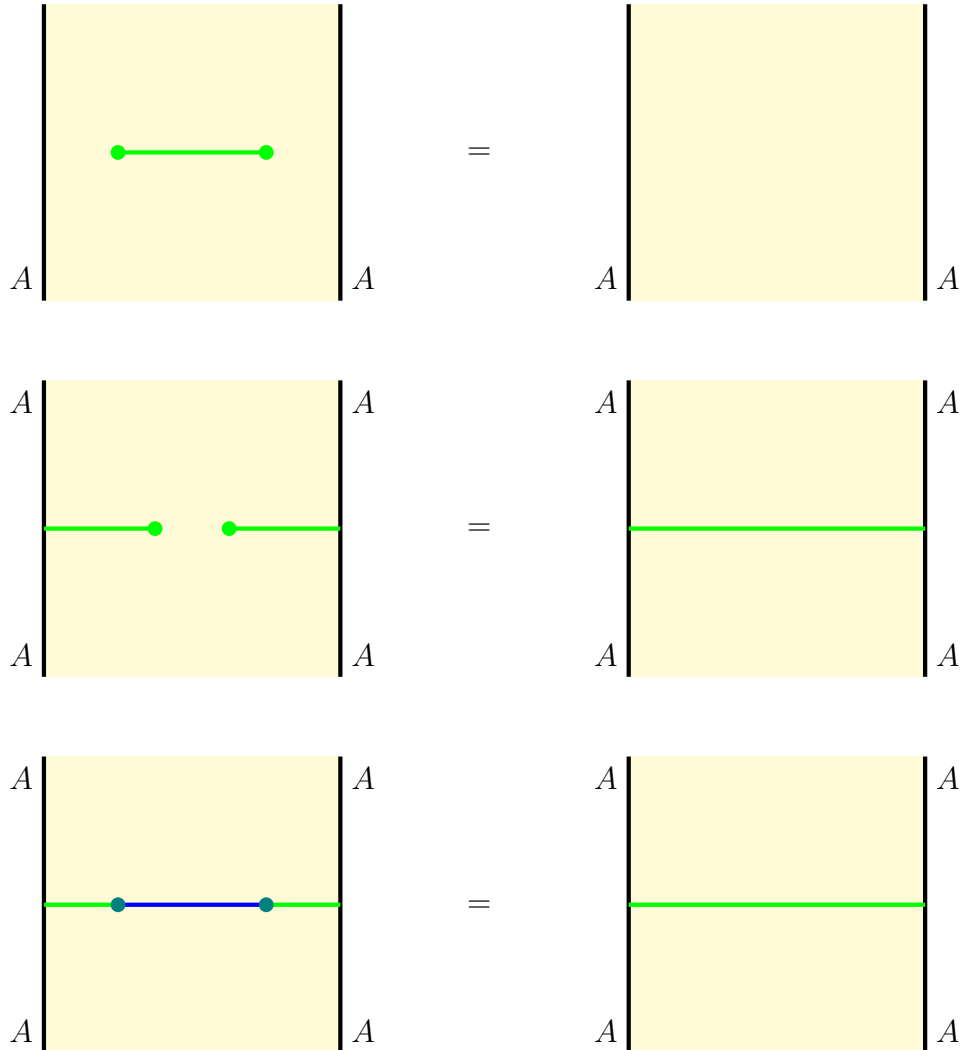


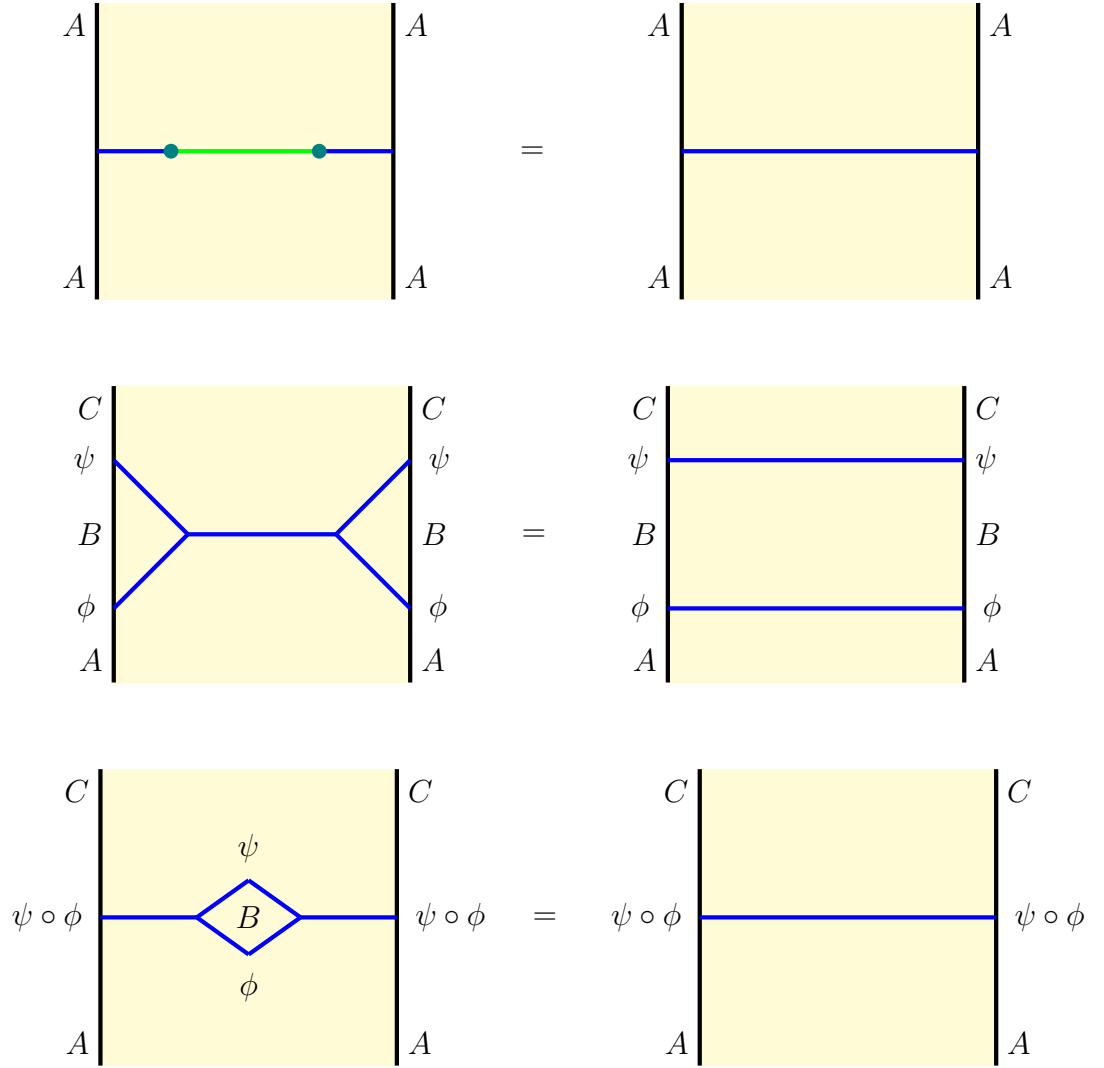
alongside their mirror images.



alongside its mirror image, whenever σ and ϕ_i satisfy the necessary conditions in the definition of the structure 2-cell $\beta_{(\phi_i)}^\sigma$.

6. Invertibility relations:





This completes the definition of the unbiased semistrict symmetric monoidal 2-presentation $\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$.

3.2.2 The equivalence between $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ and $\text{Man}_{(d-1; \sim_2)}^{(\text{or})}$

There is an evident functor $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})} \rightarrow \text{Man}_{(d-1; \sim_2)}^{\text{or}}$ which sends the unit 2-cell to the identity isotopy from the identity diffeomorphism to itself, the compositor 2-cell to the identity isotopy from $\psi \circ \phi$ to itself and the isotopies of diffeomorphisms 2-cells to the underlying isotopies. We can directly check that this functor respects the relations, i.e. it is well-defined.

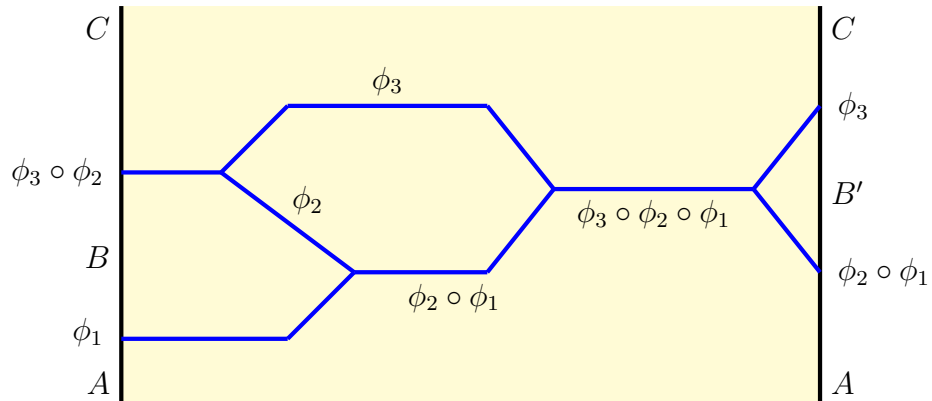
In this subsection we will show that this functor $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})} \rightarrow \text{Man}_{(d-1; \sim_2)}^{\text{or}}$ is an equivalence using the Whitehead theorem for symmetric monoidal bicategories,

proved in [SP14]. Namely, we will show that it is essentially surjective on objects, essentially full on 1-morphisms and fully faithful on 2-morphisms. To do so, we will prove a normal form lemma for the 2-cells in $F\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$. We begin with the following lemma:

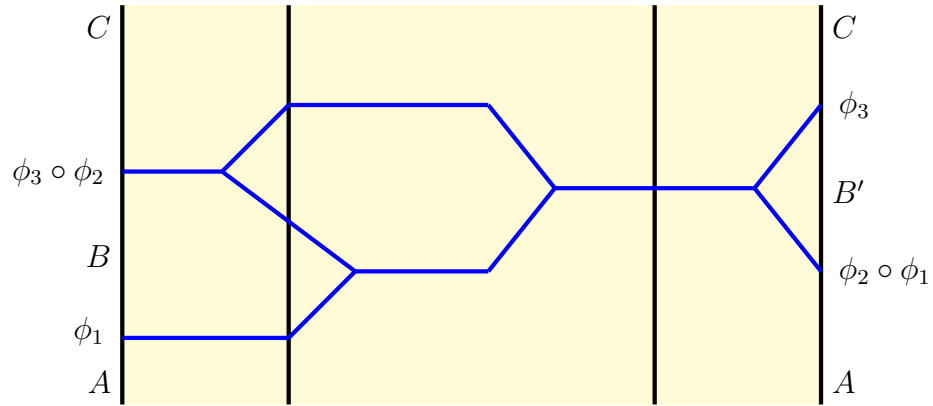
Lemma 3.2.3. *The 2-cells in $F\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ satisfy the Frobenius relation:*

The diagram illustrates the Frobenius relation for 2-cells. It consists of three parts, each showing an equality between two 2-cell diagrams. The diagrams are represented as rectangular regions with vertical lines on the left and right, and horizontal lines at the top and bottom. The regions are filled with a light yellow color. The labels C , B , B' , and A are placed at the top, middle, and bottom of the vertical lines. The labels ϕ_1 , ϕ_2 , ϕ_3 , and $\phi_3 \circ \phi_2$ are placed near the horizontal lines. The labels ϕ_2^{-1} and $\phi_3 \circ \phi_2 \circ \phi_1$ are placed near the diagonal lines. The first equality shows a 2-cell with a blue line from A to B' that splits into two paths: one through B and another through B , both labeled with ϕ_2 . The second equality shows a similar 2-cell with a blue line from A to B' that splits into two paths: one through B and another through B , both labeled with ϕ_2^{-1} . The third equality shows a similar 2-cell with a blue line from A to B' that splits into two paths: one through B and another through B , both labeled with $\phi_3 \circ \phi_2 \circ \phi_1$.

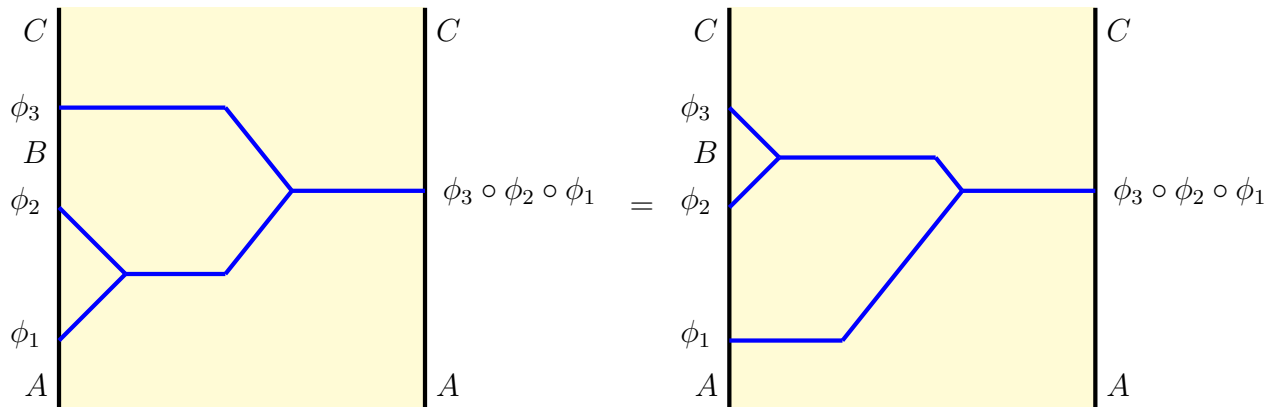
Proof. We will show that the first equality holds, the second following by a similar argument. Using the third type of invertibility relations, the LHS of the first equation is equivalent to the 2-cell



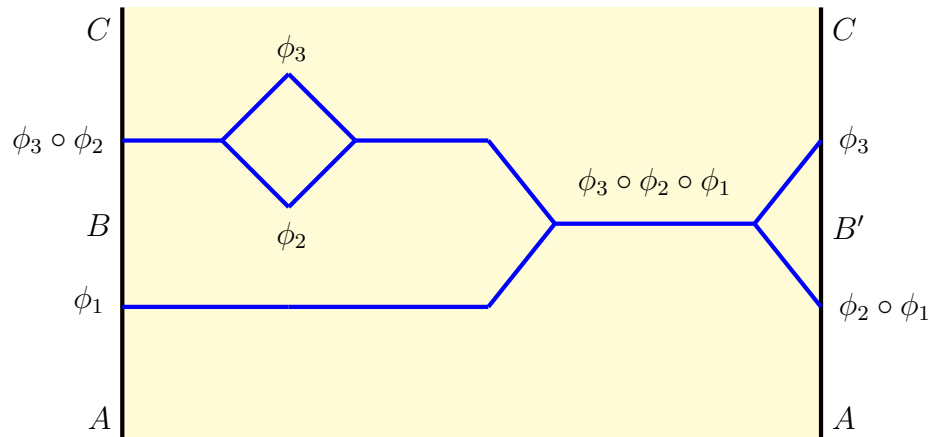
We now express this 2-cell as a composition of 2-cells:



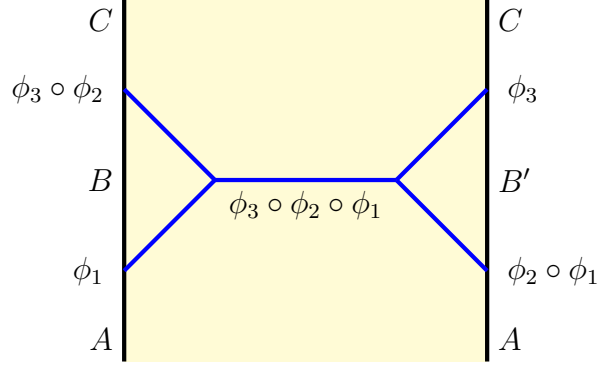
Using the associator relation in the middle 2-cell, we have



which implies that the 2-cell we started with is equivalent to the 2-cell



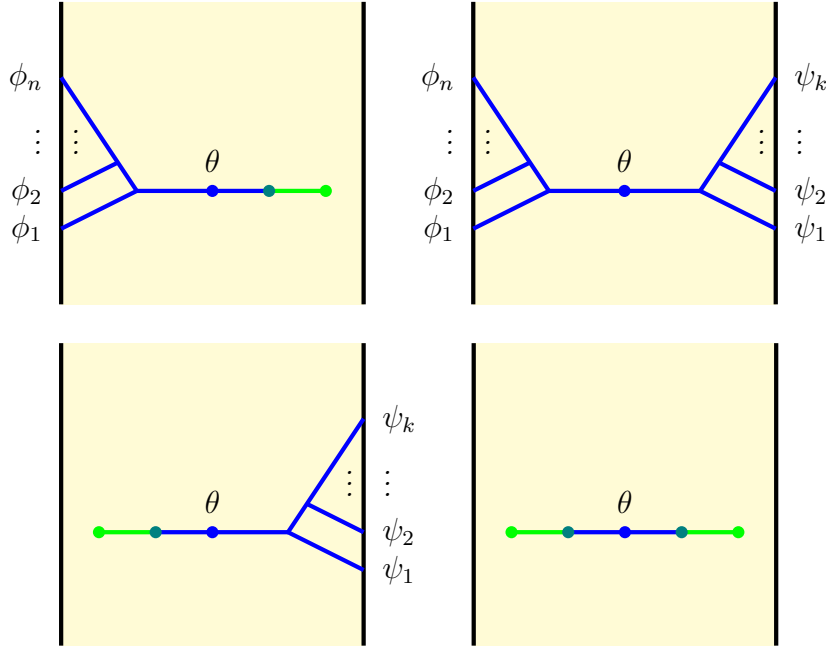
which is in turn, using the fourth type of invertibility relations, equivalent to



thus completing the proof. $\square$

We will use Lemma 3.2.3 to prove the following normal form lemma:

Lemma 3.2.4. *Any 2-cell in $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ is equivalent to either the empty diagram 2-cell or a 2-cell having one of the following four forms:*

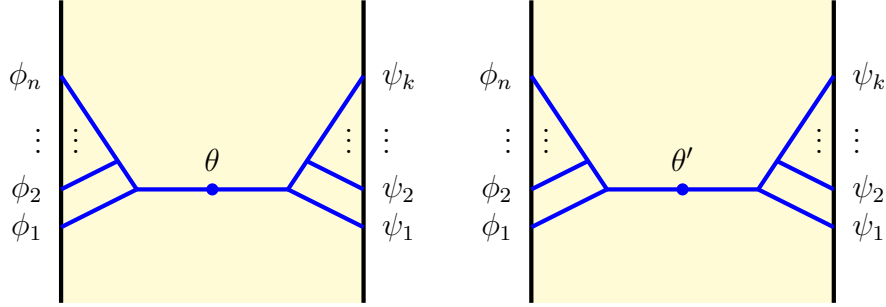


Before proving lemma 3.2.4, we first show how it implies the theorem:

Theorem 3.2.5. *The functor $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})} \rightarrow \text{Man}_{(d-1; \sim_2)}^{\text{or}}$ is an equivalence of symmetric monoidal 2-categories.*

Proof. We will apply Whitehead's theorem for bicategories. The functor in question is the identity (hence essentially surjective) on 0-cells, essentially full (in fact, full) on 1-cells and full on 2-cells. The only non-trivial step is to show that it is faithful on

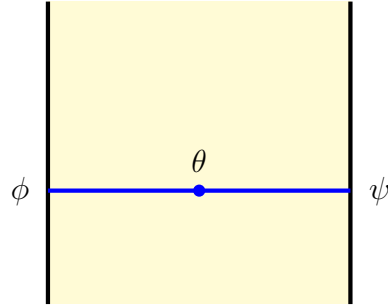
2-cells. Suppose that we are given two 2-cells in $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ that map to the same 2-cell in $\text{Man}_{(d-1; \sim_2)}^{\text{or}}$. By lemma 3.2.4, there are three cases to consider. Suppose that the two 2-cells are



These 2-cells map to the same 2-cell in $\text{Man}_{(d-1; \sim_2)}^{\text{or}}$ if and only if θ and θ' are isotopic, and by the isotopy invariance relation we conclude that the two 2-cells are equivalent in $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$. The same argument works for the other two cases, thus showing that the functor is faithful in 2-cells and hence an equivalence by Whitehead's theorem for symmetric monoidal bicategories. $\square$

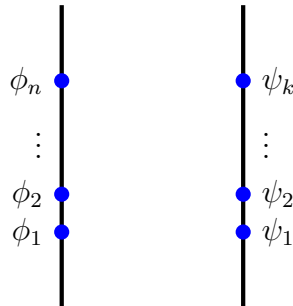
We will now prove lemma 3.2.4:

Proof. Step 1. We first reduce the lemma to the following claim: Any 2-cell in $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ between generating 1-cells is equivalent to a 2-cell having the following form:



Let's see how this claim proves lemma 3.2.4:

Suppose that we are given a 2-cell C in $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ between the 1-cells



with $n, k > 0$.

By iteratively using the third invertibility relation, we get the equivalences

$$\begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array} = \begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array}$$

and

$$\begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} = \begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} .$$

Consider the 2-cell

$$C' = \begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array} \circ C \circ \begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} .$$

C' is equivalent to C , since the 2-cells we pre- and post- composed with are equivalent to the identity. But we can also express C' as a composition of the three 2-cells

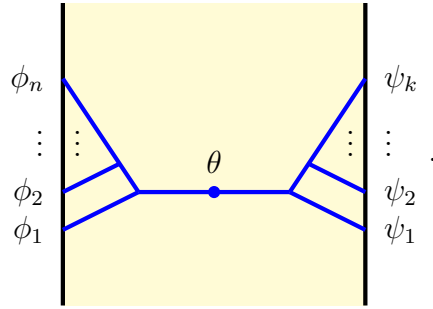
$$\begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \phi_n \\ \vdots \\ \phi_2 \\ \phi_1 \end{array} \circ \tilde{C} \circ \begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \psi_k \\ \vdots \\ \psi_2 \\ \psi_1 \end{array} ,$$

where

$$\tilde{C} = \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) \circ C \circ \left(\begin{array}{c} \text{Diagram 2} \end{array} \right).$$

The diagram on the left shows a vertical yellow rectangle with two black vertical lines. Blue lines connect the left black line to the right black line. On the right black line, there are labels $\phi_n, \dots, \phi_2, \phi_1$ from top to bottom. The diagram on the right is similar, with labels $\psi_k, \dots, \psi_2, \psi_1$ on the right black line.

Since $\tilde{C}$ is a 2-cell between elementary 1-cells, the above claim now implies that C is equivalent to



The cases $n = 0$ or $k = 0$ follow identically, which shows that lemma 3.2.4 follows from the claim above.

Step 2. We will now prove the claim made in Step 1, which focuses on 2-cells between two generating 1-cells. This is the most involved part of the proof of theorem 3.2.5.

First, note that any such 2-cell in $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ consists of an underlying labeled graph with a marked source (respectively target) vertex which lies in the source (respectively target) linear diffeomorphism diagram. This graph only has vertices of valence 1 (corresponding to unitors), 2 (corresponding to isotopy 2-cells and symmetry 2-cells) and 3 (corresponding to compositors). Using the isotopy - compositor crossing and the unitality relations, we can eliminate all univalent vertices apart from the source and target vertices, and thus assume that our given graph only has vertices of valence 2 and 3. It suffices to show that any such graph is equivalent, through the relations in $F \text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ to a bivalent tree, as the result will then follow by using the concatenation of isotopies and invertibility of symmetry 2-cells relations. In turn, it suffices to show that the graph is equivalent to a tree without introducing any univalent vertices along the way, since then we wouldn't be able to have a trivalent

vertex without having a cycle.

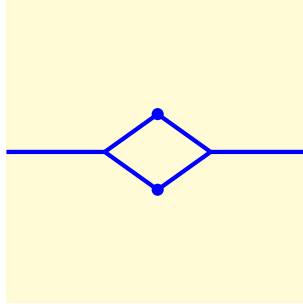
We will use an induction argument on the number of elementary cycles, ensuring that no univalent vertices are introduced along the way.

Definition 3.2.1 (Elementary cycles and graphs). Let s, t label the starting and ending vertices respectively. Each cycle in our graph divides the plane in two regions. We call the region that contains s, t exterior and its complement interior with respect to the cycle. An **elementary cycle** is a cycle whose interior is empty. An **elementary graph** is a graph every cycle of which is either elementary or contains an elementary cycle in the closure of its interior.

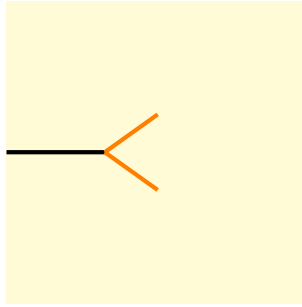
A given labeled graph can only fail to be elementary if there exist cycles with valence 1 vertices in their interior. Since we assume that our given graph has no univalent vertices, we can safely assume that it is elementary. In the base case and induction step we will show how to eliminate a given elementary cycle using again an induction argument on its length, while being careful to not introduce any new cycles along the way.

Base case

Let's assume that the underlying labeled graph which defines a given 2-cell only has a single elementary cycle. We will use an induction argument to eliminate it. Let $n \geq 2$ be the length of the cycle. If $n = 2$, the cycle must look like

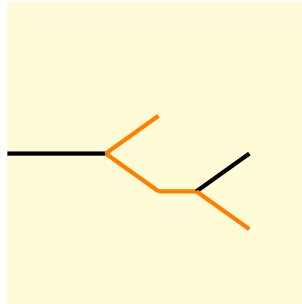


Using isotopy - compositor crossing and the invertible relations, we can eliminate it. If $n > 2$, we will show that the cycle is equivalent to a cycle of length $n - 1$, without introducing any new cycles. We will now color the part of the graph that is part of the cycle orange, and the part of the graph which is not part of the cycle black. Then, there must exist a trivalent vertex of the graph that looks like



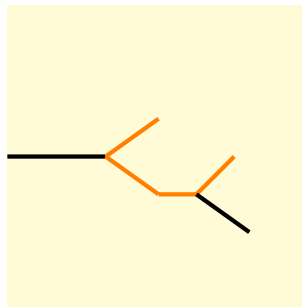
otherwise we would not be able to choose a direction within the cycle. Let's consider now the various possibilities for what the graph may look like if we try to complete its lower right side:

1.



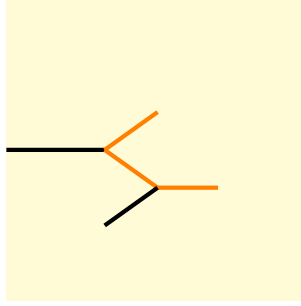
This case is not allowed since the corresponding cycle is not elementary.

2.



In this case, we can use the associativity relations to obtain an elementary cycle of length $n - 1$ without introducing any new cycles along the way.

3.



In this case, we can use the Frobenius relations to obtain an elementary cycle of length $n - 1$ without introducing any new cycles along the way.

This completes the proof of the base case. Namely, an elementary graph of the form we consider with a single elementary cycle is equivalent to a bivalent tree.

Induction step

Let's suppose that the given graph has k elementary cycles. The inductive argument we used in the base case applies verbatim here and shows that the graph is equivalent to one with $k - 1$ elementary cycles. This completes the proof of Step 2 and hence of theorem [3.2.5](#).

□

3.3 2-presentation of planar surgery diagrams

The goal of this section is to introduce the unbiased semistrict symmetric monoidal 2-presentation of planar surgery diagrams in a sequence of increasing complexity. In particular, we will construct the tower of 2-presentations

$$\begin{array}{c}
 \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0} \\
 \downarrow \\
 \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1} \\
 \downarrow \\
 \text{Sur}_{(d-1; d; \sim)}^{(\text{or})}
 \end{array}$$

In subsequent sections, we will construct corresponding towers of bordism bicategories decorated with data that come from Morse - Cerf theory and show that all categories at the same level are equivalent. These towers help us to slowly and steadily build

our way towards a surgery presentation of the bordism bicategory, one manageable step at a time.

3.3.1 The 2-category of planar surgery diagrams $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$

We will define $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ by enhancing the 2-presentation $\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ with some further generating 1-cells, 2-cells and relations. We will denote the generating 1-cells, 2-cells and relations coming from $\text{Diff}_{(d-1; \sim_2)}^{(\text{or})}$ as diffeomorphism type generating 1-cells, 2-cells and relations, respectively. The additional generating 1-cells, 2-cells and relations of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ are as follows:

Additional generating 1-cells

Surgery type generating 1-cells:

These are exactly the surgery type generating 1-cells in $\text{Sur}_{(d-1; d)}^{(\text{or})}$. Recall that these 1-cells are diagrams of the form

$$\begin{array}{c} Y \\ | \\ \bullet \sigma = (\sigma_X, \sigma_Y, \psi_\sigma) \\ | \\ X \end{array}$$

where $(\sigma_X, \sigma_Y, \psi_\sigma)$ is a symmetric surgery triple, as defined in 2.2.

Additional generating 2-cells

Surgery type generating 2-cells:

Isotopies of Morse data

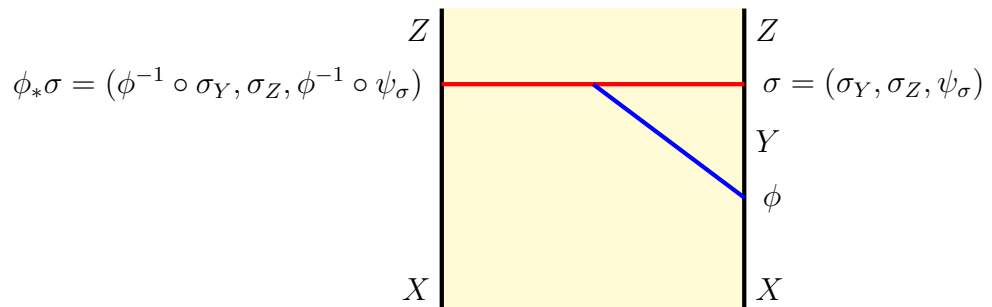
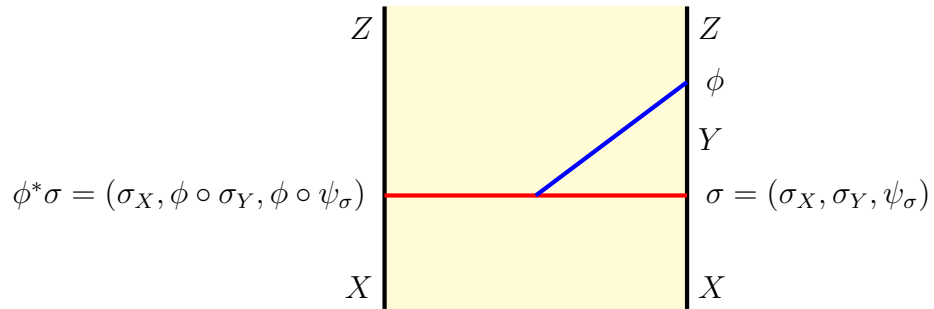
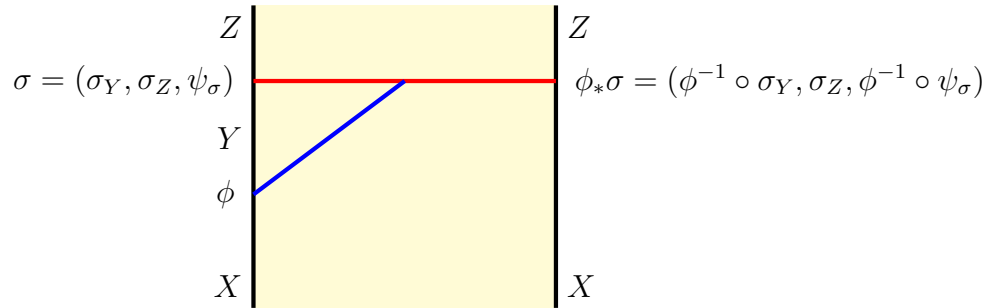
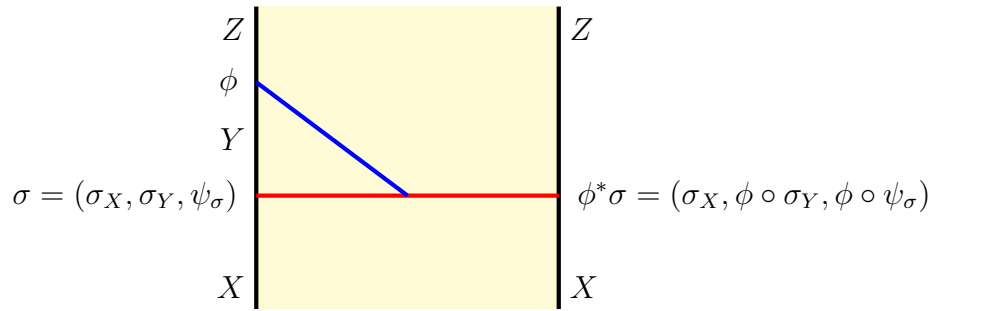
$$\begin{array}{ccc} Y & \begin{array}{|c|} \hline \text{Yellow Box} \\ \hline \end{array} & Y \\ \sigma = (\sigma_X, \sigma_Y, \psi_\sigma) & \begin{array}{c} \text{---} \theta^M \text{---} \\ \bullet \end{array} & \sigma' = (\sigma'_X, \sigma'_Y, \psi_{\sigma'}) \\ X & \begin{array}{|c|} \hline \text{Yellow Box} \\ \hline \end{array} & X \end{array}$$

where θ^M is an isotopy of Morse data and boundary identifications on $W(X, Y, \sigma)$ from the natural Morse data that $W(X, Y, \sigma)$ is equipped with to the Morse data

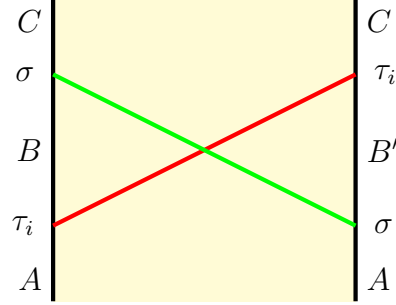
that induce the surgery triple σ' .

Naturality type generating 2-cells:

These are the planar surgery diagrams which encode the composition of a symmetric surgery triple with a diffeomorphism and take the form



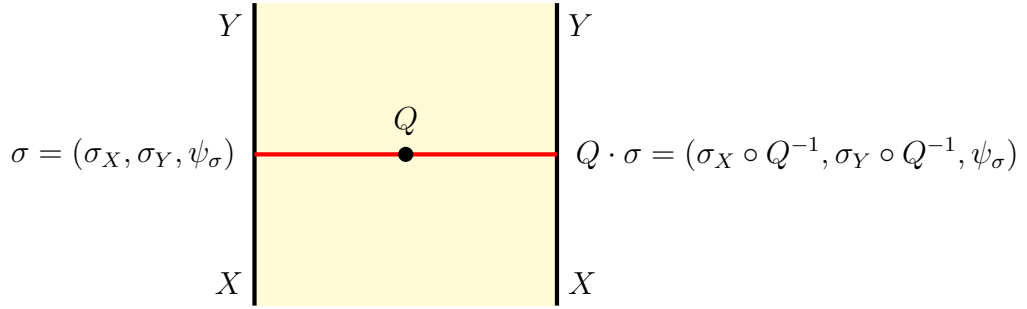
Remark 3.3.1. We will denote the structure 2-cells $\beta_{(\tau_i)}^\sigma$ by



Remark 3.3.2. For brevity, we may sometimes suppress an explicit green-to-blue color change when writing a naturality type 2-cell. In such cases, the reader should interpret the 2-cell as including first a green-to-blue color change, followed by a blue-red crossing.

Frame change type generating 2-cells:

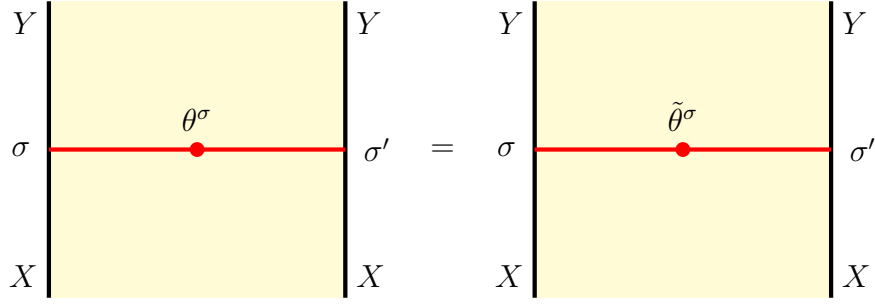
For each symmetric surgery triple of index k and each $Q \in G_{n,k}^{(\text{or})}$, we have generating 2-cells that take the form



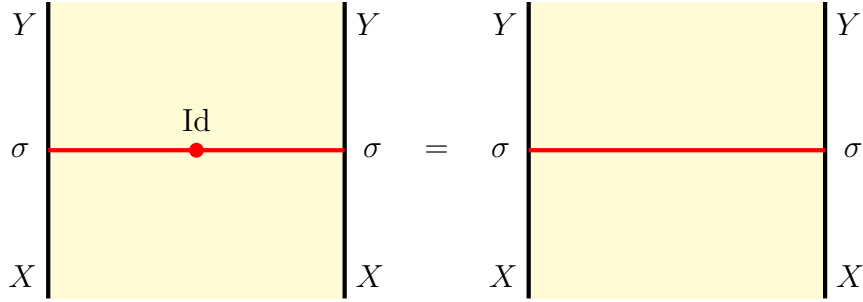
Additional generating relations

Surgery type relations:

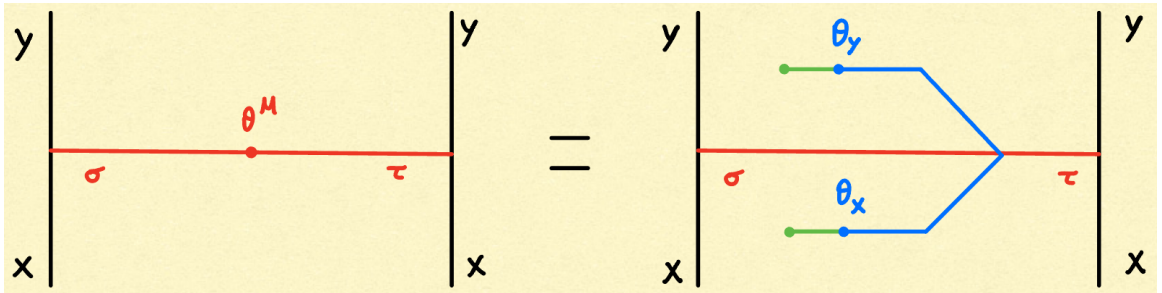
Isotopy of Morse data invariance



where θ^σ and $\tilde{\theta}^\sigma$ are isotopic isotopies of Morse data. Moreover, we have



Compatibility of isotopies



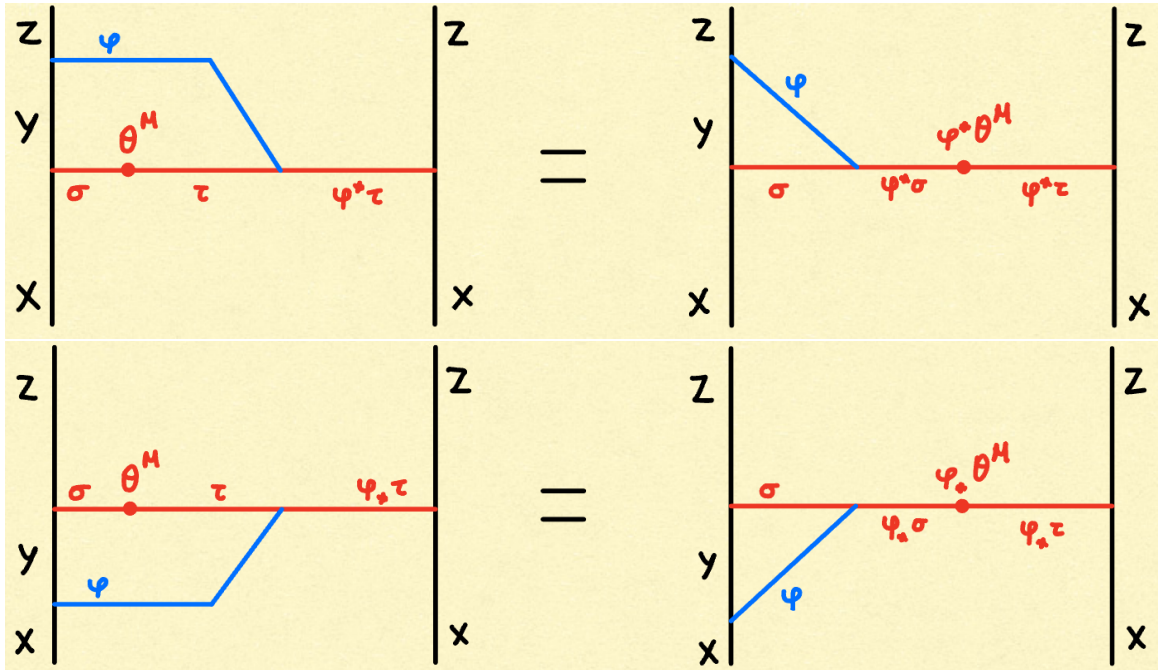
This relation holds under certain conditions on θ_M , θ_X and θ_Y . Namely, θ^M denotes an isotopy of Morse data and boundary identifications on $W(X, Y, \sigma)$, while θ_X and θ_Y induce an isotopy of Morse data and boundary identifications on $W(X, Y, \sigma)$. The above relation holds when the corresponding isotopies of Morse data and boundary identifications are equal.

Remark 3.3.3. In section 2.5 we saw how the isotopy extension lemma implies that, given θ^M , we can always find compatible θ_X and θ_Y . This makes the isotopy of Morse data 2-cells obsolete. Indeed, for the same reason, they did not appear as relations at the 1-categorical case. However, they are still useful, as they make the definition of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ a bit more natural, and, as we will see later, make the functor

$$S_0 : \text{Bord}_{(d-1; d; d+1)}^{\text{Cerf data, (or), } \mathcal{F}^0} \rightarrow \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$$

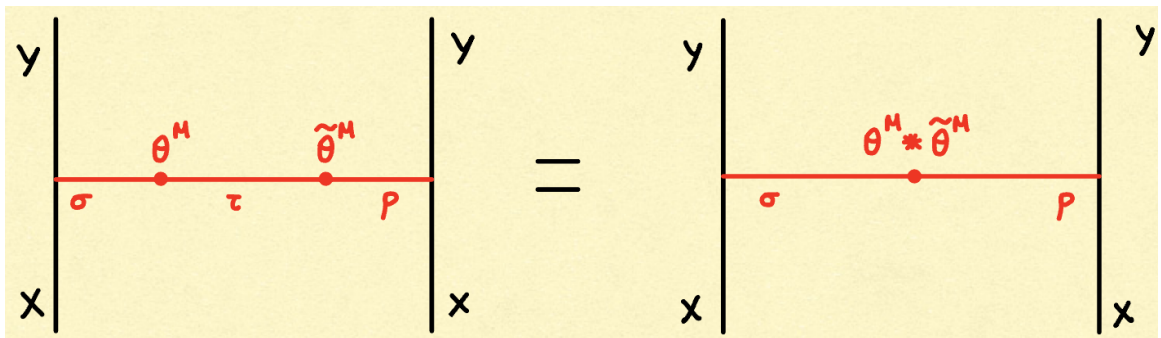
a bit easier to define.

Isotopy of Morse data - naturality crossing

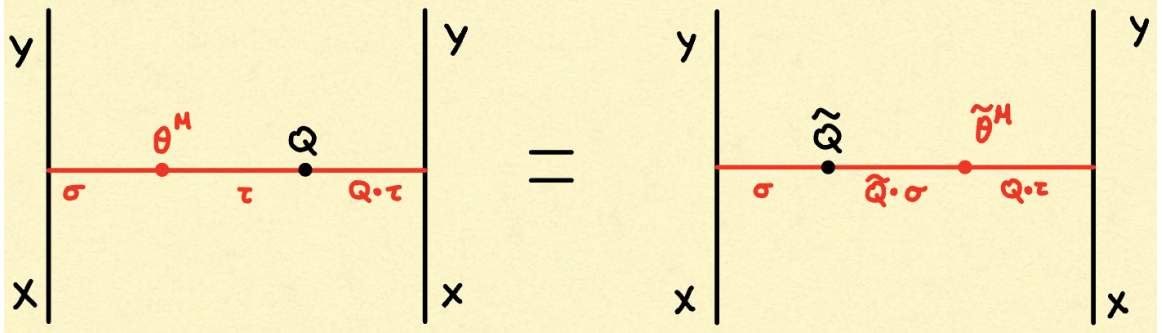


alongside their mirror images.

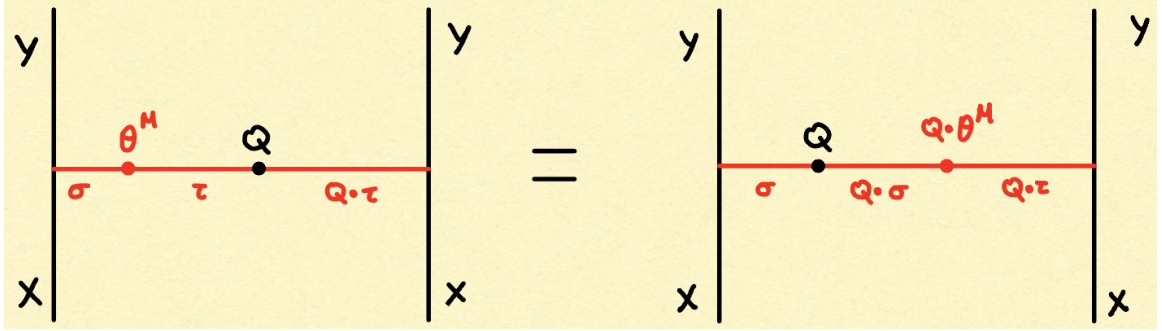
Concatenation of isotopies of Morse data



Isotopy of Morse data - frame change crossing

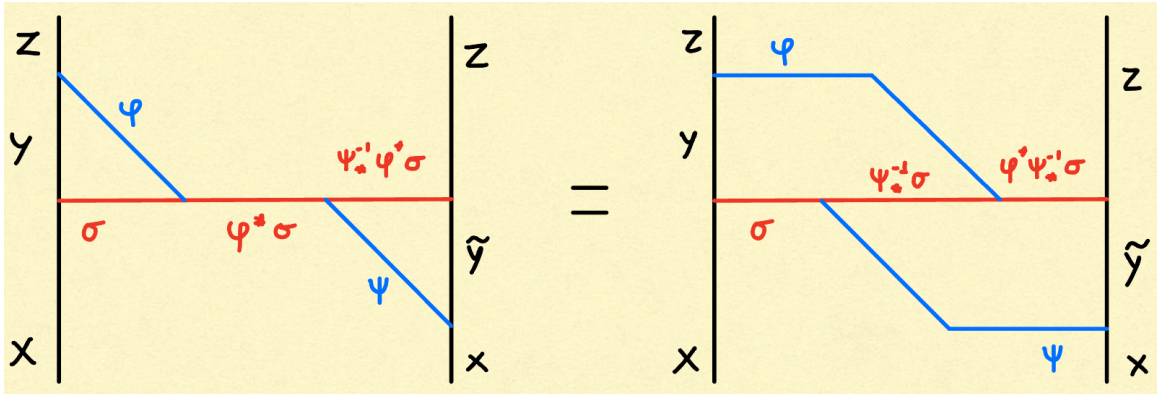


if the isotopies of Morse data θ^M and $\tilde{\theta}^M$ differ by a path $\gamma : [0, 1] \rightarrow G_{(n,k)}^{(\text{or})}$ from $\tilde{Q}$ to Q . Note that this relation trivializes all isotopies of Morse data that come from loops in $G_{(n,k)}^{(\text{or})}$. Moreover, in the special case of the constant path at Q , we get the relation:

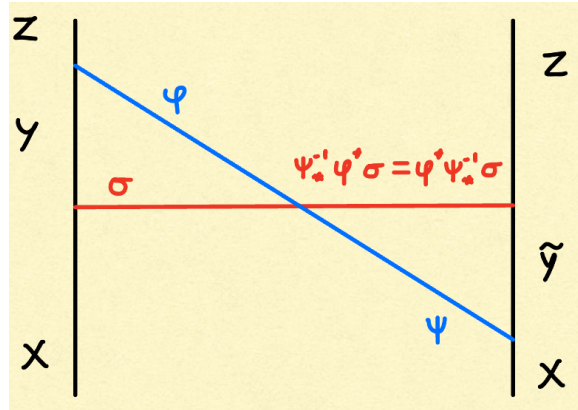


Naturality type relations:

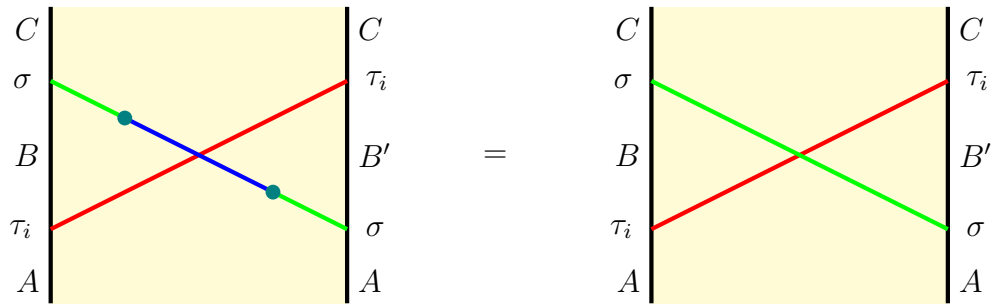
Naturality - 4-valence



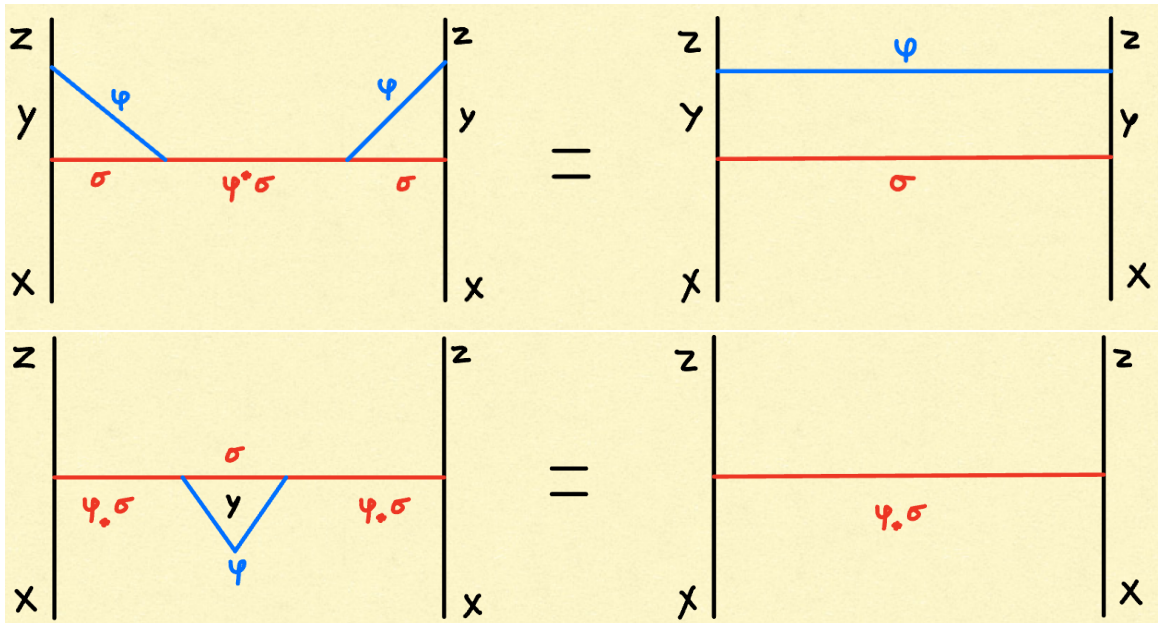
alongside their mirror image. We further now introduce surgery diagrams of the form

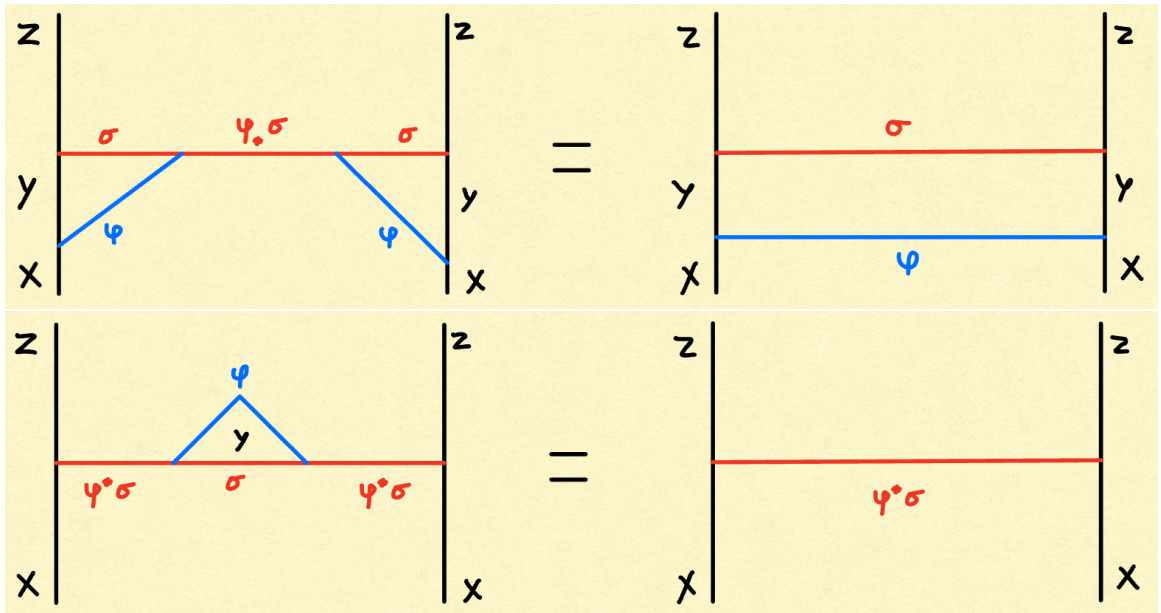


alongside their mirror image, representing the same 2-cell as the surgery diagrams in the above relation. Moreover, we have the following relation:

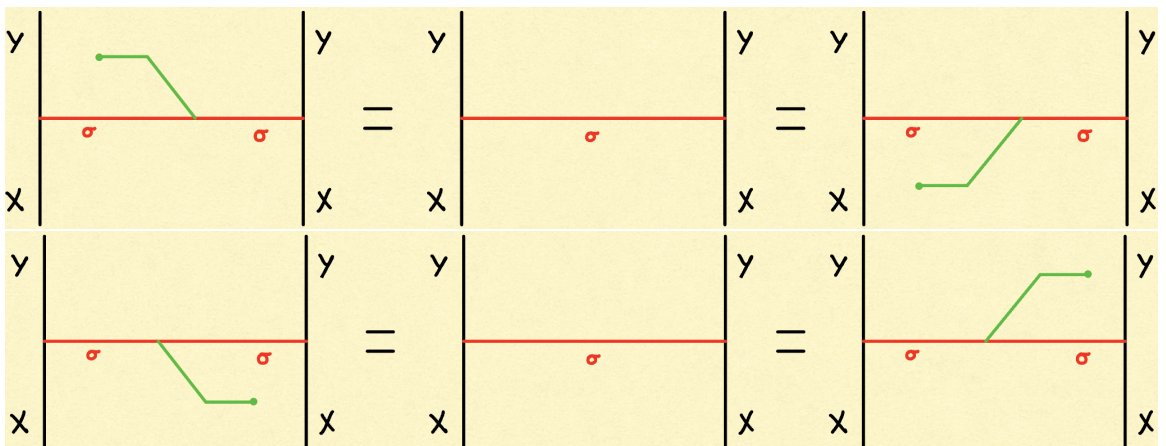


Naturality - Reidemeister II

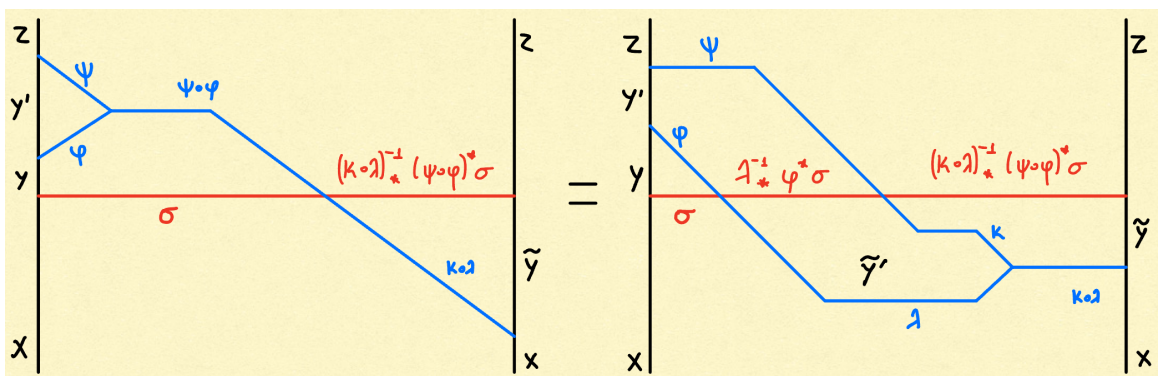




Naturality - Unitality



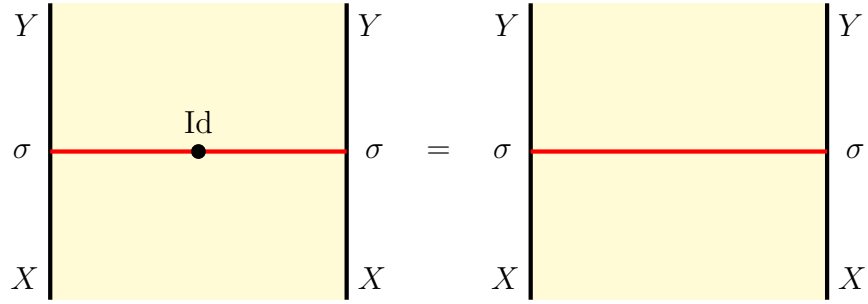
Naturality - Compositor



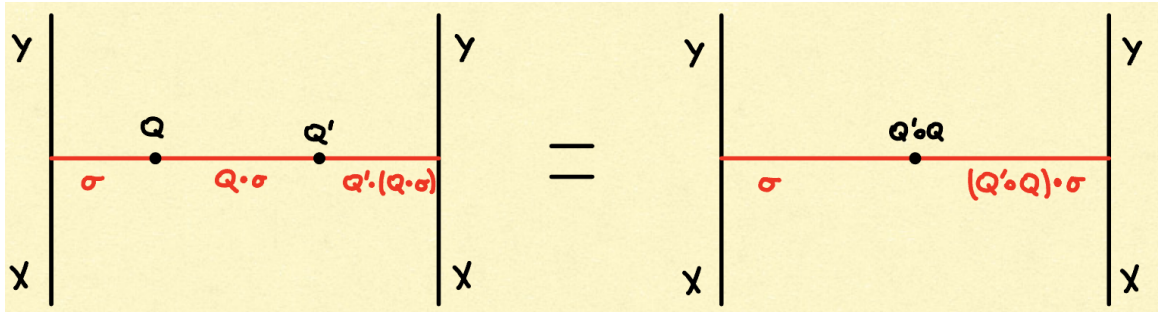
alongside their mirror image.

Frame change type relations:

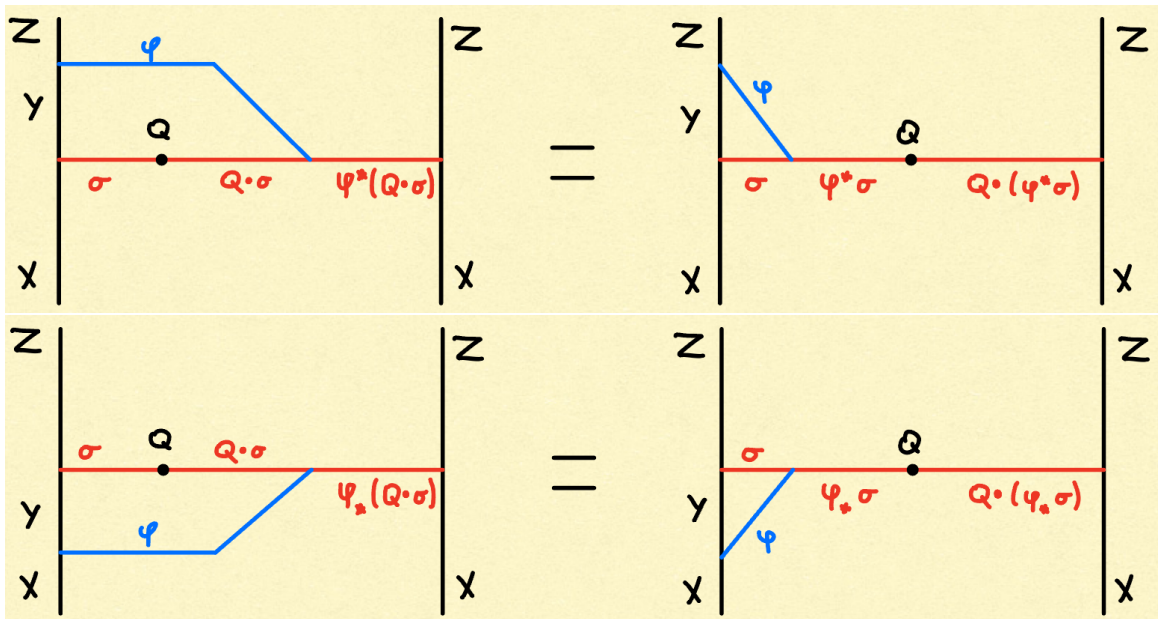
Frame no change



Frame change composition



Frame change - naturality crossing



alongside their mirror images.

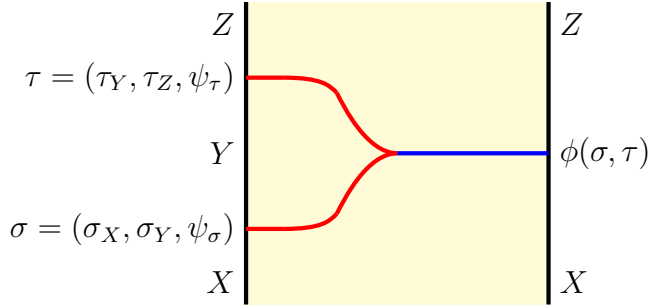
3.3.2 The 2-presentation of planar surgery diagrams $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$

We will define $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ by enhancing the presentation $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ with some further generating 2-cells and generating relations.

Additional generating 2-cells

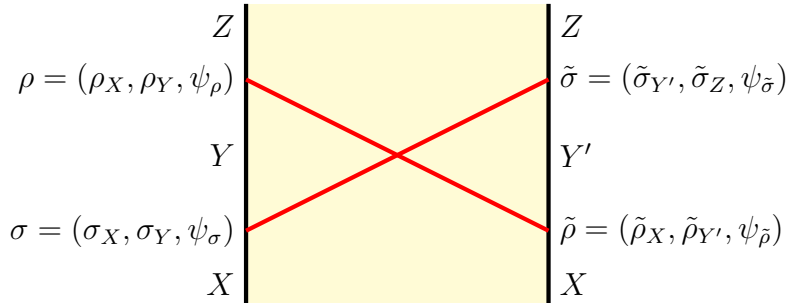
Surgery type generating 2-cells:

Cusp



alongside its mirror image. The framed attaching spheres σ_Y and τ_Y are in cancelling position in Y . This specifies an embedding of the standard double neighborhood on the bordism obtained by gluing $W(X, Y, \sigma)$ and $W(Y, Z, \tau)$. This embedding, in turn, determines an elementary death path which specifies a cancelling diffeomorphism $\phi(\sigma, \tau)$ on the nose.

Crossing



Here, as in the crossing relations of $\text{Sur}_{(d-1; d)}^{(\text{or})}$, the following conditions are in place:

- $\text{Im}(\sigma_Y) \subset Y$ is disjoint from $\text{Im}(\rho_Y) \subset Y$.

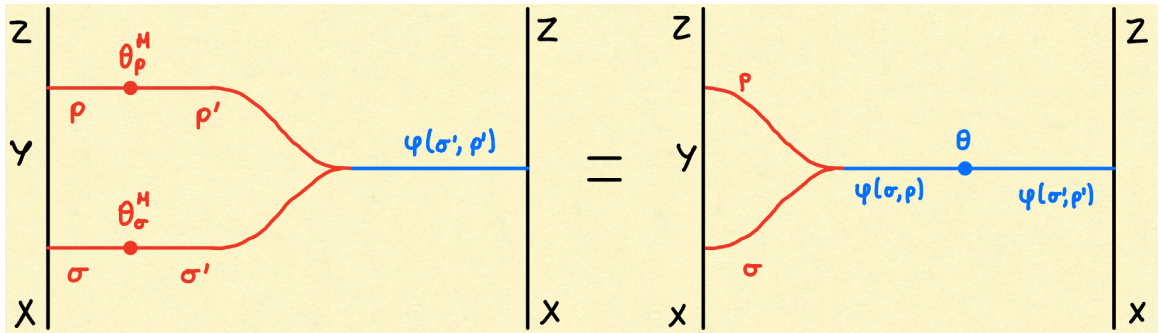
- $\text{Im}(\tilde{\rho}_{Y'}) \subset Y'$ is disjoint from $\text{Im}(\tilde{\sigma}_{Y'}) \subset Y'$.
- $\psi_\rho \circ \psi_\sigma = \psi_{\tilde{\sigma}} \circ \psi_{\tilde{\rho}}$, wherever both sides are defined.
- $\psi_\sigma \circ \tilde{\rho}_X = \rho_Y$ and $\psi_{\tilde{\rho}} \circ \sigma_X = \tilde{\sigma}_{Y'}$.
- $\psi_\rho \circ \sigma_Y = \tilde{\sigma}_Z$ and $\psi_{\tilde{\sigma}} \circ \tilde{\rho}_{Y'} = \rho_Z$.

Remark 3.3.4. The above conditions imply that the source 1-cell determines the target 1-cell on the nose up to a diffeomorphism $Y' \rightarrow Y''$.

Additional generating relations

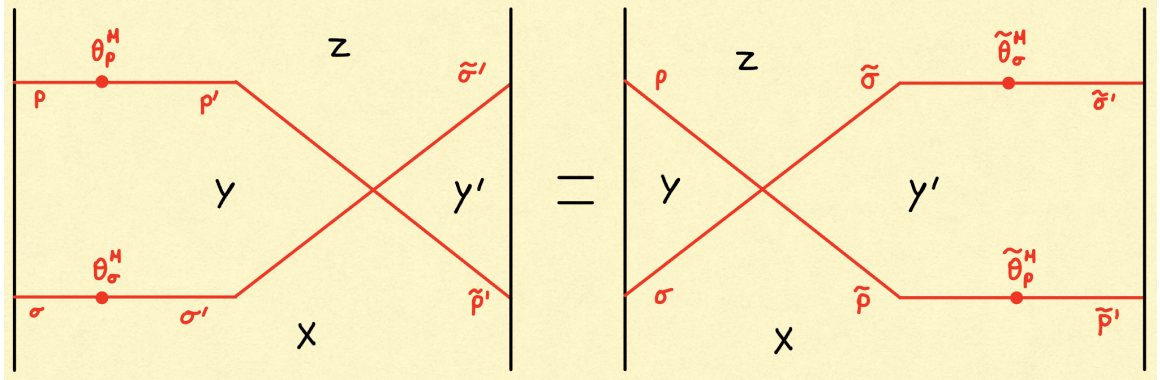
Surgery type relations:

Cusp - isotopy relations



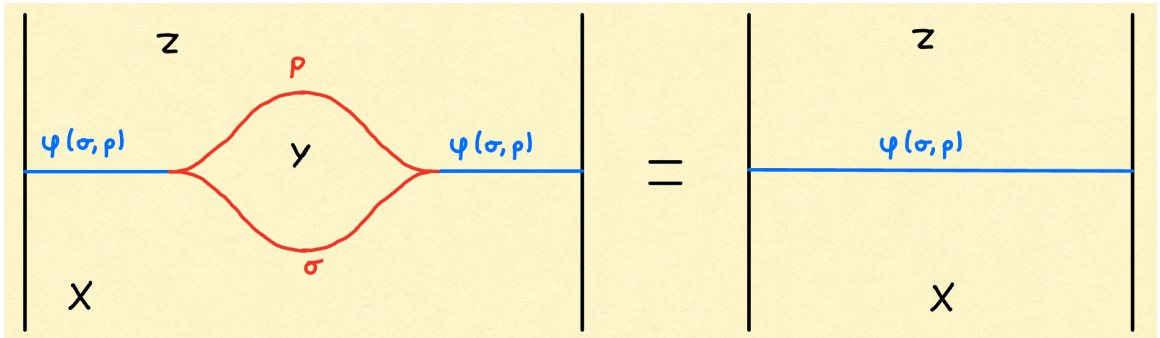
alongside their mirror image. Here, θ_σ^M and θ_ρ^M are isotopies of Morse data that remain in cancelling position throughout. They induce an isotopy of elementary death paths which in turn determines an isotopy of cancelling diffeomorphisms, θ . Note that in this case, the data in the left hand side of the relation completely determine the data on the right hand side of the relation.

Crossing - isotopy relations

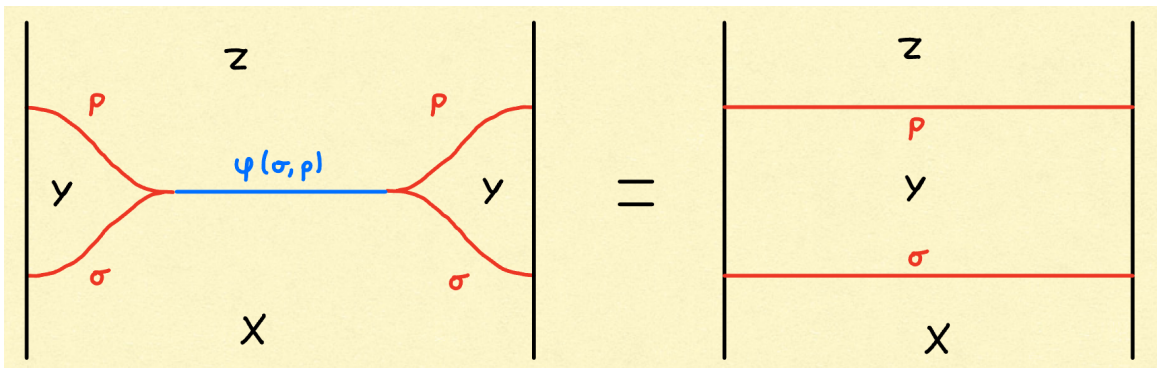


Here, θ_σ^M and θ_p^M are isotopies of Morse data that remain in crossing position throughout. This induces an isotopy of elementary crossing paths which completely specifies $\tilde{\theta}_\sigma^M$ and $\tilde{\theta}_p^M$ up to an isotopy $Y' \times [0, 1] \rightarrow Y'$ that is the identity at time 0. Note that in this case, the data on the left hand side of the relation determine the data on the right hand side up to some ambiguity, but the space of ambiguities is contractible.

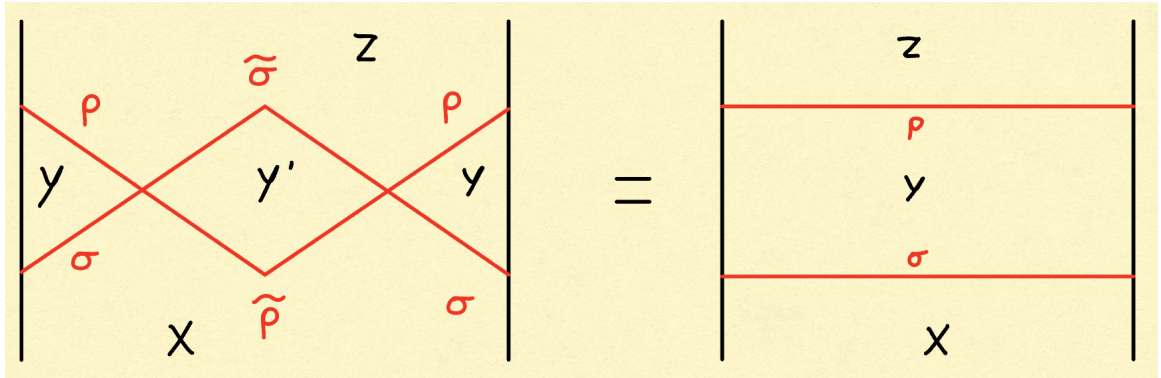
Eye



Merge - unmerge



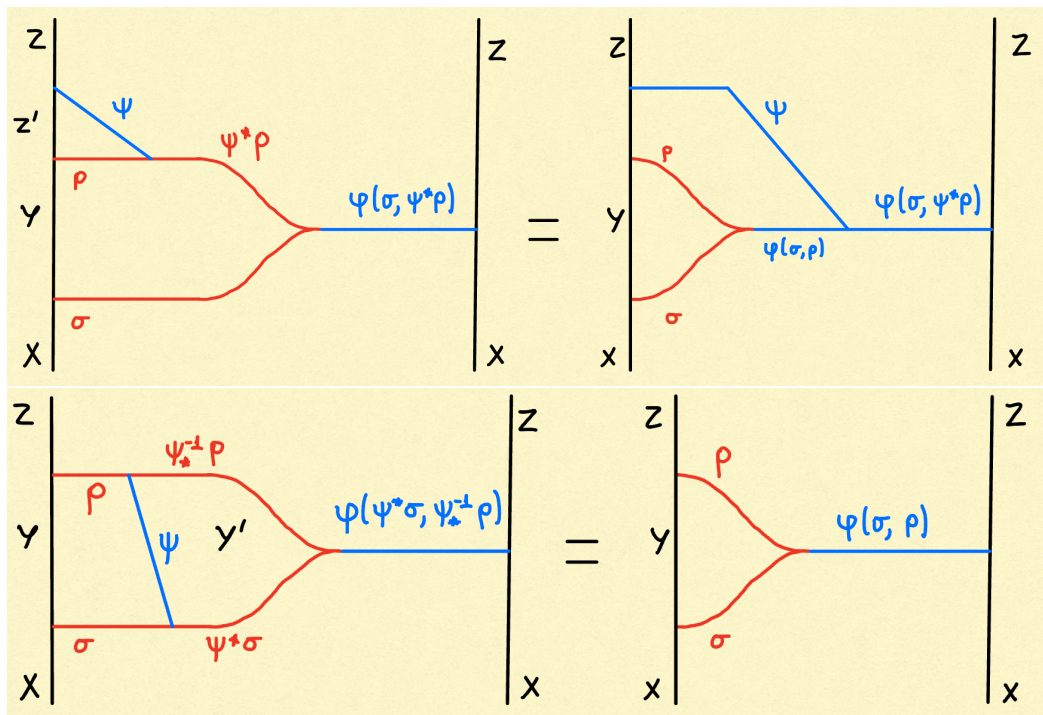
Reidemeister II

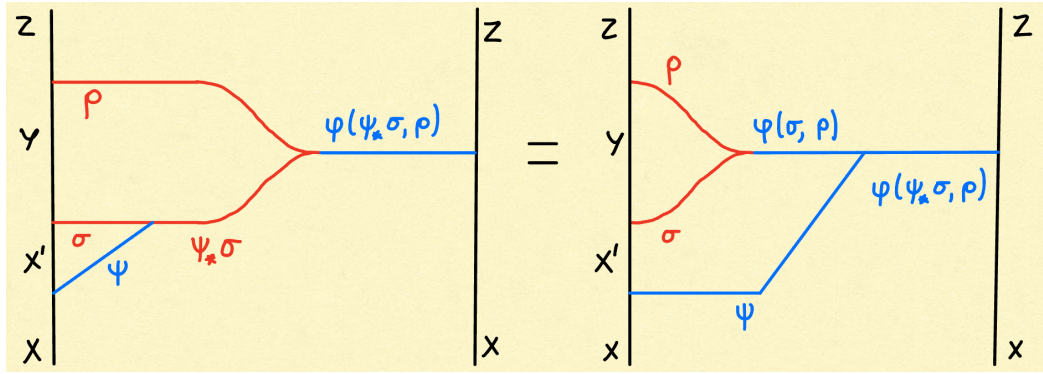


Remark 3.3.5. The eye, merge - unmerge and Reidemeister II relations encode the invertibility of the cusp and crossing 2-cells.

Naturality type relations:

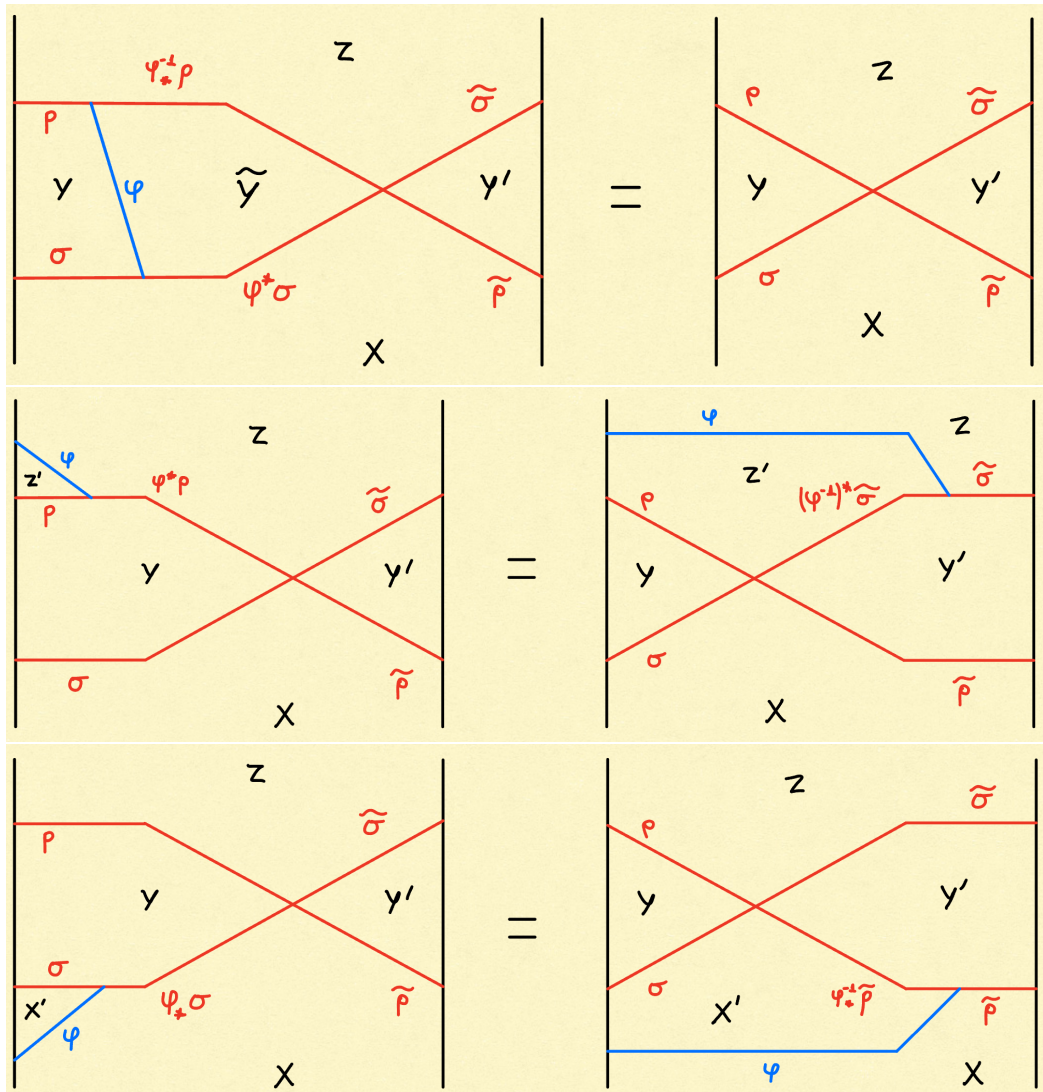
Naturality - beak





alongside their mirror images.

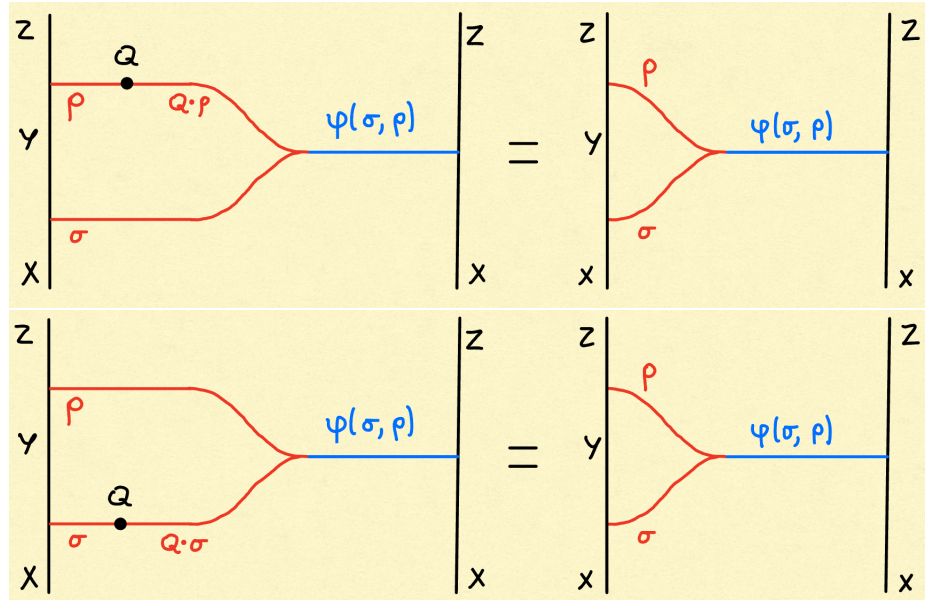
Naturality - Reidemeister III



alongside their mirror images.

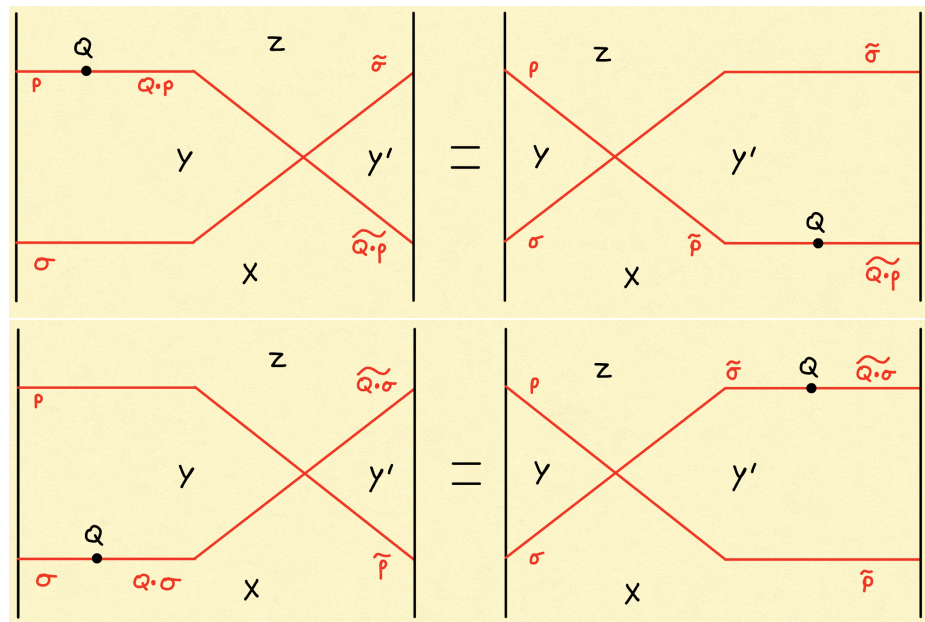
Frame change type relations:

Frame change - cusp



alongside their mirror images.

Frame change - crossing



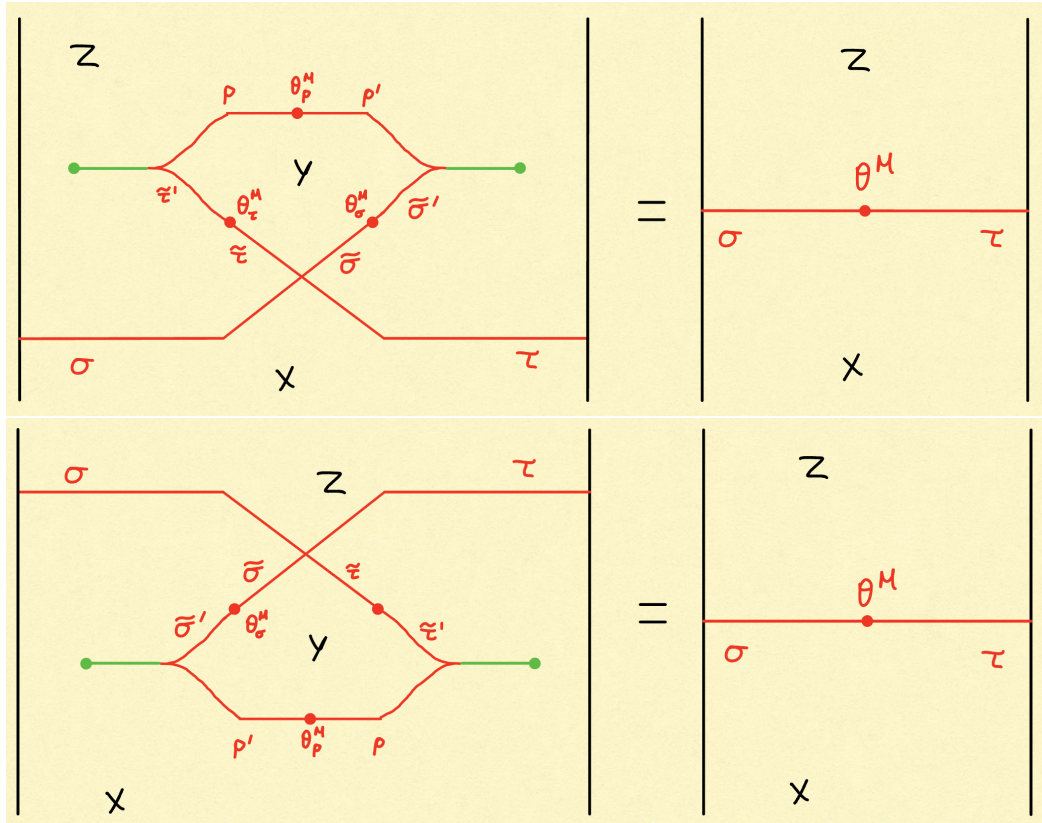
3.3.3 The 2-category of planar surgery diagrams $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$

We define $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ by enhancing the presentation $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ with additional generating relations, all of *surgery type*. These new relations encode Cerf homotopies that cross the codimension-2 strata. We obtain these relations by pulling them back from $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$ to $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})}$, and finally to $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$. Consequently, the functors F_1^{data} and B_1 , defined in Sections 3.5 and 3.6 respectively, will play a role here.

We acknowledge that this entails referring the reader to a later section in order to fully understand a construction appearing here; nonetheless, we believe it is valuable to provide a complete definition of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ at this stage. In future work, we intend to replace this with a smaller set of relations given by an intrinsic description in terms of surgery data and to prove that these suffice to account for all codimension-2 relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$. Achieving this requires a stability theorem for Cerf paths—a standard fact, though one not readily found in the literature.

Additional generating relations of surgery type

Swallowtail

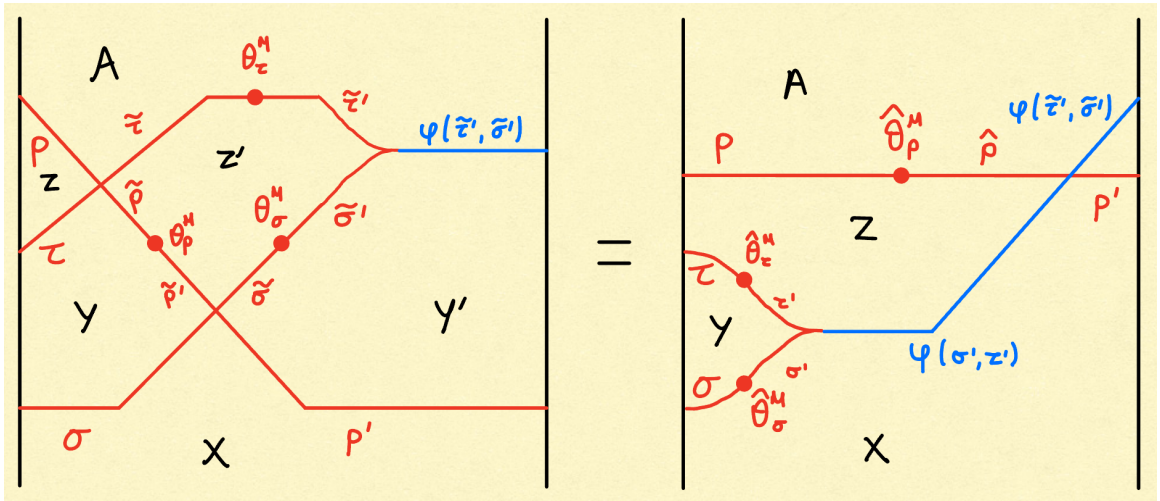


Here, the surgery data on left hand side 2-cells define a Cerf path on $W(X, Z, \sigma)$ using $F_1^{\text{data}} \circ B_1$, which we will denote by $\{f_s\}$. The conditions for the swallowtail relation are:

1. There is a two parameter swallowtail homotopy $F_{s,t}$ such that $F_{s,0} = f_s$, and $g_s := F_{s,1}$ is a path of excellent Morse functions.
2. We can equip the path of excellent Morse functions g_s with an isotopy of Morse data connecting σ and τ . Note that we can always equip g_s with Cerf data that consist of an isotopy of Morse data connecting σ and τ up to applying a frame change on τ . The extra assumption we make here is that the embeddings σ and τ are chosen to be isotopic and, thus, no framing flip is necessary to connect them. θ^M is an isotopy of Morse data we obtain this way and any two choices for it will be equivalent due to the faithfulness of the functor F_1^{data} .

Remark 3.3.6. In future work, we will show that it suffices to only include the swallowtail relations for which the isotopies of Morse data $\theta_\rho^M, \theta_\sigma^M$ and θ_τ^M are all identities. This is especially easy to see in low dimensions. If X, Y, Z are 0- or 1-dimensional, it is easy to see that any isotopies of Morse data that start in cancelling position have to remain in cancelling position throughout, hence we can use the cusp - isotopy relations to eliminate $\theta_\rho^M, \theta_\sigma^M$ and θ_τ^M . Albeit a bit less clear a priori, we can reach the same conclusion when X, Y, Z are 2-dimensional using the results in Appendix C of Behrens' thesis [Beh14].

Beak

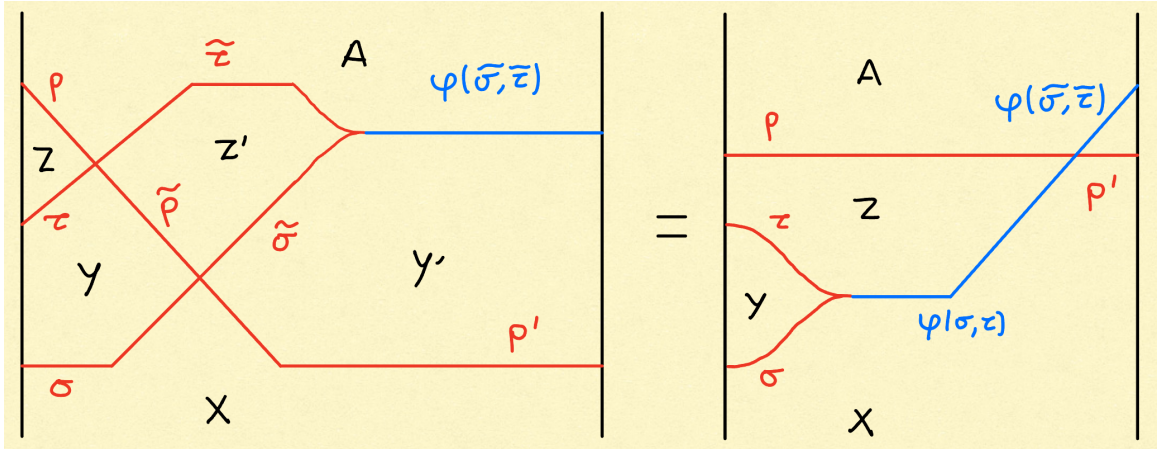


alongside their reflections on the x - and y - axes.

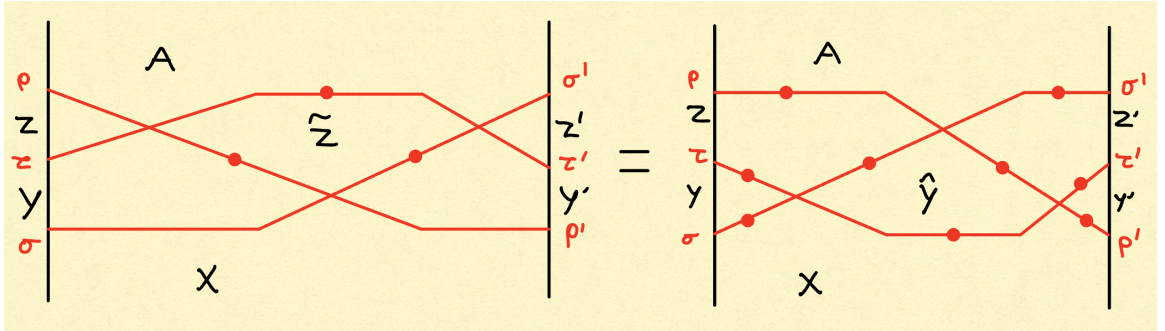
Here, the surgery data on left hand side 2-cells define a Cerf path on the bordism represented by the source 1-cell using $F_1^{\text{data}} \circ B_1$, which we will denote by $\{f_s\}$. The conditions for the beak relation are:

1. There is a two parameter beak homotopy $F_{s,t}$ such that $F_{s,0} = f_s$, and $g_s := F_{s,1}$ has the Cerf graphic suggested by the surgery diagram on the right hand side..
2. The isotopies of Morse data $\hat{\theta}_\sigma^M$, $\hat{\theta}_\tau^M$ and $\hat{\theta}_\rho^M$ come from a choice of Cerf data for $\{g_s\}$ and any two choices will lead to equivalent surgery diagrams due to the faithfulness of the functor F_1^{data} . Note that a choice of Cerf data resulting in the surgery diagram always exists since ρ and $\hat{\rho}$ must be isotopic embeddings by spelling out all the compatibility conditions for the surgery data in the left hand side diagram.

Remark 3.3.7. In future work, we will show that it suffices to only include the beak relations for which all the isotopies of Morse data are identities, namely the relations:



Reidemeister III



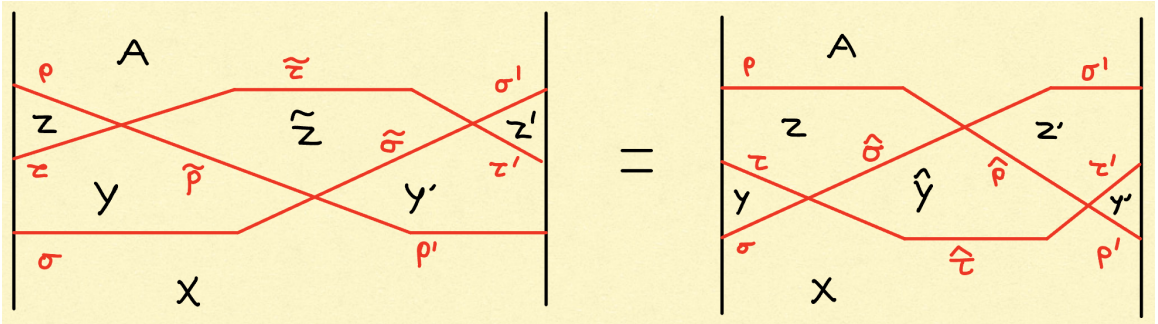
Here, we have omitted some labels for readability (and because the Reidemeister III relation will eventually be simplified and we won't have to worry about them anyway). The same principle as per the two previous relations applies: the surgery data on left hand side 2-cells define a Cerf path on the bordism represented by the source 1-cell using $F_1^{\text{data}} \circ B_1$, which we will denote by $\{f_s\}$. The conditions for the Reidemeister III relation are:

1. There is a two parameter Reidemeister III homotopy $F_{s,t}$ such that $F_{s,0} = f_s$, and $g_s := F_{s,1}$ has the Cerf graphic suggested by the surgery diagram on the right hand side..
2. The isotopies of Morse data on the right hand side 2-cell (whose labels are omitted) come from a choice of Cerf data for $\{g_s\}$ and any two choices will lead to equivalent surgery diagrams due to the faithfulness of the functor F_1^{data} . By spelling out all the compatibility conditions for the surgery data in the left hand side diagram, we see that there won't be a need for frame change type 2-cells on the right hand side surgery diagram.

Remark 3.3.8. We have seen in Lemma 1.8.1 that

$$\pi_0(\text{Reid}_f, \text{ElReid}_f) = 0,$$

which implies that it suffices to solely include the following Reidemeister III relations:



3.4 Bordism $(2, 1)$ -categories equipped with Cerf paths

Recall that our goal in this chapter is to show that the d -dimensional bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$ is equivalent to the $(2, 1)$ -category of planar surgery diagrams $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$. While in principle it should be possible to do this directly by constructing a functor

$$\text{Sur}_{(d-1; d; \sim)}^{(\text{or})} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$$

and verifying the Whitehead criteria, directly checking all the details is overwhelming, if not utterly unmanageable. It is more instructive to break down the proof of the equivalence into intermediate steps. The purpose of this section is to describe the first such intermediate step by introducing the $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$ and showing that the forgetful functor

$$\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})} \xrightarrow{F^{\text{Cerf}}} \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$$

is an equivalence. Mimicking the construction of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$, we will introduce $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$ in three stages, constructing a tower of $(2, 1)$ -categories

$$\begin{array}{c} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0} \\ \downarrow \\ \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^1} \\ \downarrow \\ \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})} \end{array}$$

We begin now by introducing $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$.

Remark 3.4.1. Verifying all of the coherence data required to define a symmetric monoidal bicategory is a formidable task. Fortunately, there is a convenient framework for constructing symmetric monoidal bicategories due to Shulman and Hansen [SH19], which works exceptionally well for bordism bicategories and applies to $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$ as well as all other bordism $(2, 1)$ -categories we will consider. This approach is explained in detail in Chapter 3 of [SP14], and we encourage the interested reader to consult it for more background. In this thesis, we will define all necessary bordism $(2, 1)$ -categories somewhat informally by specifying

their morphisms and their composition, while implicitly relying on the framework of Shulman and Hansen to guarantee that the resulting structures indeed form symmetric monoidal bicategories.

3.4.1 The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$

The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$ is defined as follows:

0-cells

Closed, (oriented) $(d - 1)$ -dimensional manifolds.

1-cells

A 1-cell between 0-cells Y_0 and Y_1 is a d -dimensional cobordism

$$\overline{W} = (W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1)$$

equipped with a collar and an excellent Morse function

$$f : (W; \partial_0 W, \partial_1 W) \rightarrow (I; \{0\}, \{i\}),$$

where I is the interval $[0, i]$. Composition is by gluing of cobordisms and Morse functions, with the help of the chosen collars.

2-cells

We will first need a preliminary definition:

Definition 3.4.1 (Cylindrical 2-cobordisms). Let $\overline{W}_0$ and $\overline{W}_1$ be two cobordisms from Y_0 to Y_1 . A **cylindrical 2-cobordism** $\overline{Z} \times \overline{I}$ between them consists of the following data:

1. A smooth manifold with corners $(Z; \partial_0 Z, \partial_1 Z) \times I$ obtained as the product of a smooth manifold triad with $I = [0, i]$.
2. Isotopies of diffeomorphisms $j_0 : Y_0 \times I \rightarrow \partial_0 Z \times I$ and $j_1 : Y_1 \times I \rightarrow \partial_1 Z \times I$.
3. For each $t \in I$, we have an induced cobordism

$$\overline{Z}_t := (Z; \partial_0 Z, \partial_1 Z; j_0(-, t) : Y_0 \rightarrow \partial_0 Z, j_1(-, t) : Y_1 \rightarrow \partial_1 Z).$$

Part of the data of the cylindrical 2-cobordism $\overline{Z} \times \overline{I}$ are equivalences of cobordisms $h_0 : \overline{W}_0 \rightarrow \overline{Z}_0$ and $h_1 : \overline{W}_1 \rightarrow \overline{Z}_1$.

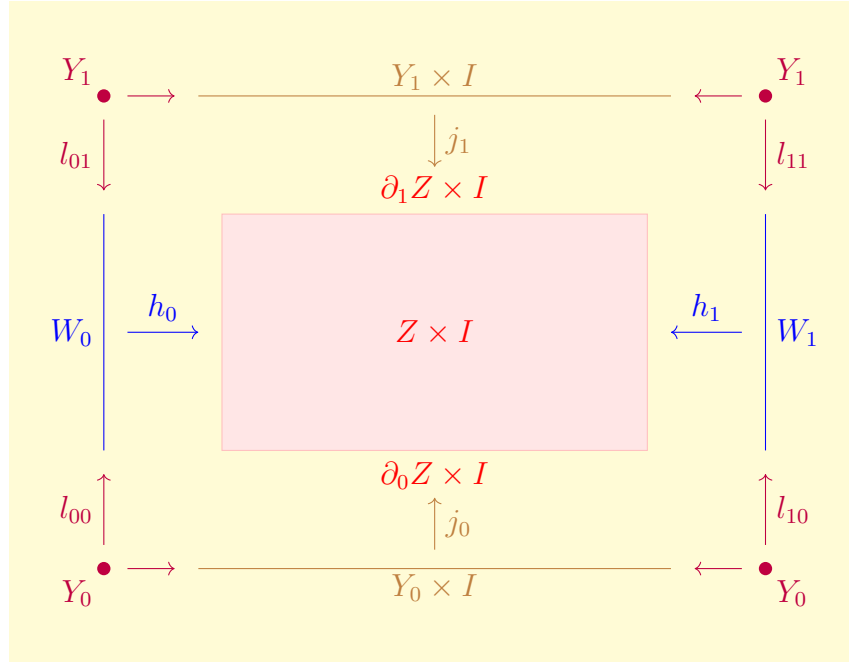
This data is required to satisfy compatibility conditions on the corners, namely:

$$\begin{array}{ccc} W_0 & \xrightarrow{h_0} & Z \times \{0\} \\ \iota_{00} \uparrow & & \uparrow j_0 \\ Y_0 & \xrightarrow{\text{Id} \times \{0\}} & Y_0 \times \{0\} \end{array} \quad \begin{array}{ccc} Y_1 & \xrightarrow{\text{Id} \times \{0\}} & Y_1 \times \{0\} \\ \iota_{01} \downarrow & & \downarrow j_1 \\ W_0 & \xrightarrow{h_0} & Z \times \{0\} \end{array}$$

and

$$\begin{array}{ccc} W_1 & \xrightarrow{h_1} & Z \times \{1\} \\ \iota_{10} \uparrow & & \uparrow j_0 \\ Y_0 & \xrightarrow{\text{Id} \times \{1\}} & Y_0 \times \{1\} \end{array} \quad \begin{array}{ccc} Y_1 & \xrightarrow{\text{Id} \times \{1\}} & Y_1 \times \{1\} \\ \iota_{11} \downarrow & & \downarrow j_1 \\ W_1 & \xrightarrow{h_1} & Z \times \{1\} \end{array}$$

Pictorially, we can organize the data of a cylindrical 2-cobordism as follows:



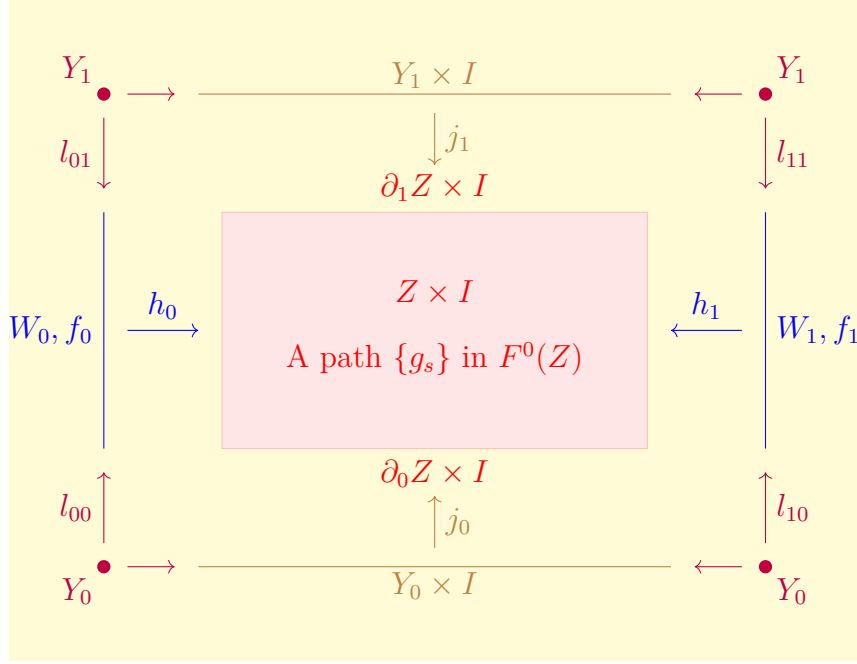
We will refer to $Z \times \{0\} \cup Z \times \{1\}$ as the vertical boundary and $\partial_0 Z \times I \cup \partial_1 Z \times I$ as the horizontal boundary of $(Z; \partial_0 Z, \partial_1 Z) \times I$.

We can now describe the 2-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$:

Let $(\overline{W_0}, f_0)$ and $(\overline{W_1}, f_1)$ be two 1-cells from Y_0 to Y_1 . A 2-cell between them is (up to soon to be specified equivalence) a cylindrical 2-cobordism $\overline{Z} \times I$ between $\overline{W_0}$ and $\overline{W_1}$ equipped with a path of excellent Morse functions $\{g_s\}$ such that the boundary equivalences of cobordisms h_0 and h_1 are Morse function preserving diffeomorphisms

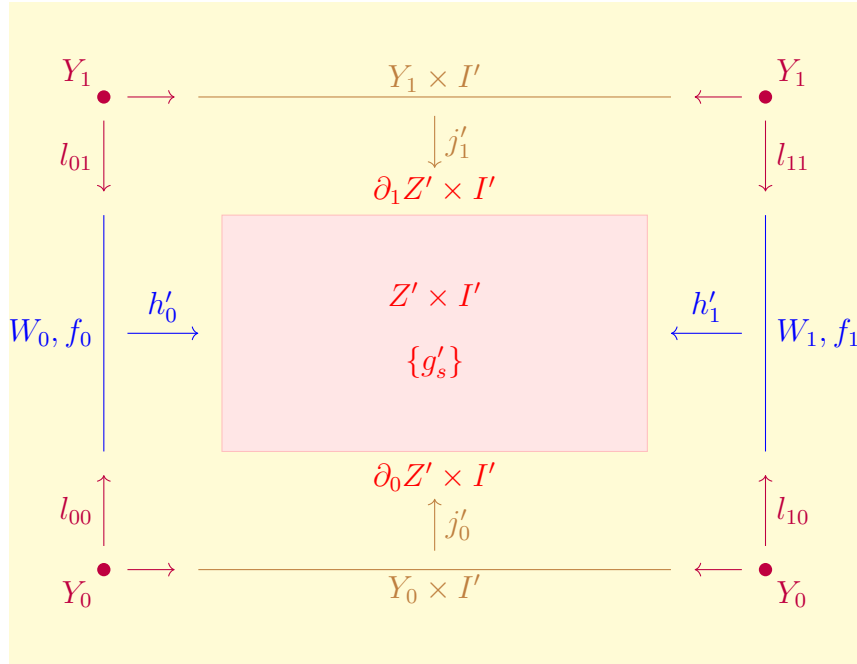
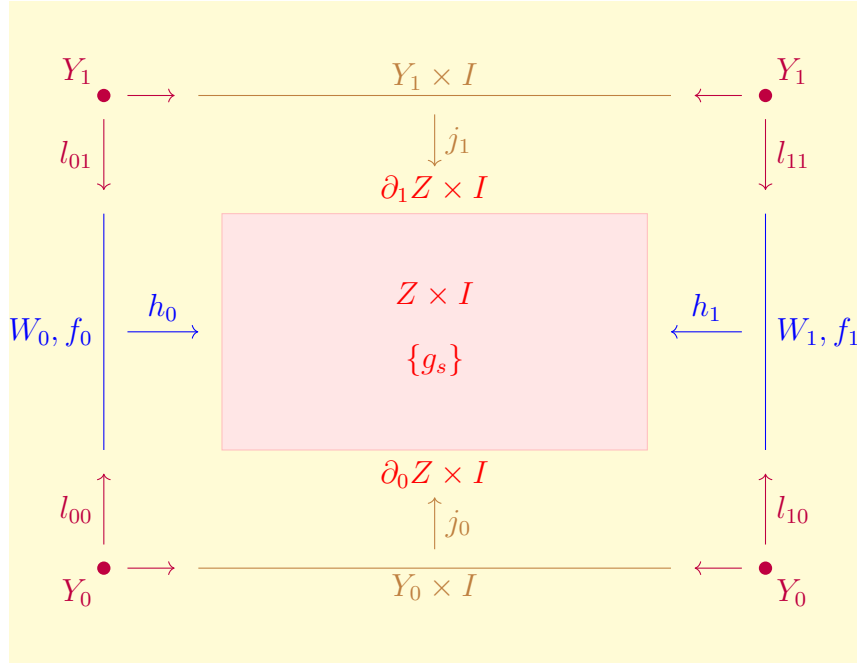
$$h_k : (\overline{W_k}, f_k) \rightarrow (\overline{Z_k}, g_k), \text{ for } k = 0, 1.$$

Moreover, the 1-cells $(\overline{W_0}, f_0)$ and $(\overline{W_1}, f_1)$ are cobordisms equipped with the germ of a collar and the 2-cobordism $\overline{Z \times I}$ comes equipped with a compatible germ of a collar both in its horizontal and vertical boundary, with the Cerf path remaining constant throughout the vertical and horizontal collars. Pictorially, we visualize a 2-cell as follows:



Equivalence relation

Let $(\overline{W_0}, f_0)$ and $(\overline{W_1}, f_1)$ be two 1-cells from Y_0 to Y_1 and let $(\overline{Z \times I}, \{g_s\})$, $(\overline{Z' \times I'}, \{g'_s\})$ be two 2-cells between them. Instead of spelling out all of the data symbolically, let's make them explicit through the following two pictures:



There are three kinds of relations we quotient out by:

1. Reparameterization of the interval I . Namely, consider any interval $I' = [0, i']$ and a diffeomorphism $\phi : I \rightarrow I'$ such that $\phi(0) = 0$ and $\phi(i) = i'$. This induces a diffeomorphism

$$\Phi = \text{Id}_Z \times \phi : Z \times I \rightarrow Z \times I'.$$

We then get an equivalent 2-cell by taking

- $Z' = Z$
- $h'_0 = h_0$ and $h'_1 = h_1$
- $j'_0 = \Phi \circ j_0$ and $j'_1 = \Phi \circ j_1$
- $g'_s = g_{\phi^{-1}(s)}$

2. Change of the underlying smooth manifold triad. Namely, consider another smooth manifold triad $(Z'; \partial_0 Z', \partial_1 Z')$ and a diffeomorphism of smooth manifold triads $d : (Z; \partial_0 Z, \partial_1 Z) \rightarrow (Z'; \partial_0 Z', \partial_1 Z')$. This induces a diffeomorphism

$$D = d \times \text{Id}_I : Z \times I \rightarrow Z' \times I.$$

We then get an equivalent 2-cell by taking

- $I' = I$
- $h'_0 = D \circ h_0 = d \circ h_0$ and $h'_1 = D \circ h_1 = d \circ h_1$
- $j'_0 = D \circ j_0$ and $j'_1 = D \circ j_1$
- $g'_s = g_s \circ d^{-1}$

3. Homotopy of excellent Morse functions and compatible isotopy of boundary identifications. Namely, let $G_{s,t}$ be a homotopy of excellent Morse functions that remains constant through the germ of a collar of $Z \times I$ such that $G_{s,0} = g_s$. Let $H_{0,t}$ (respectively $H_{1,t}$) be an isotopy of diffeomorphisms between cobordisms $\overline{W}_0 \rightarrow \overline{Z}_0$ (respectively $\overline{W}_0 \rightarrow \overline{Z}_0$) such that

$$H_{0,t} : (\overline{W}_0, f_0) \rightarrow (\overline{Z}_0, G_{0,t})$$

and, respectively,

$$H_{1,t} : (\overline{W}_1, f_1) \rightarrow (\overline{Z}_1, G_{1,t})$$

are Morse function preserving. Lastly, let $J_{0,t}$ (respectively $J_{1,t}$) be an isotopy of isotopies between $Y_0 \times I$ and $\partial_0 Z \times I$ (respectively $Y_1 \times I$ and $\partial_1 Z \times I$), compatible with $H_{0,t}$ (respectively $H_{1,t}$). The compatibility conditions spelled out are:

$$\begin{array}{ccc} W_0 & \xrightarrow{H_{0,t}} & Z \times \{0\} \\ \iota_{00} \uparrow & & \uparrow J_{0,t} \\ Y_0 & \xrightarrow{\text{Id} \times \{0\}} & Y_0 \times \{0\} \end{array} \quad \begin{array}{ccc} Y_1 & \xrightarrow{\text{Id} \times \{0\}} & Y_1 \times \{0\} \\ \iota_{01} \downarrow & & \downarrow J_{1,t} \\ W_0 & \xrightarrow{H_{0,t}} & Z \times \{0\} \end{array}$$

and, respectively,

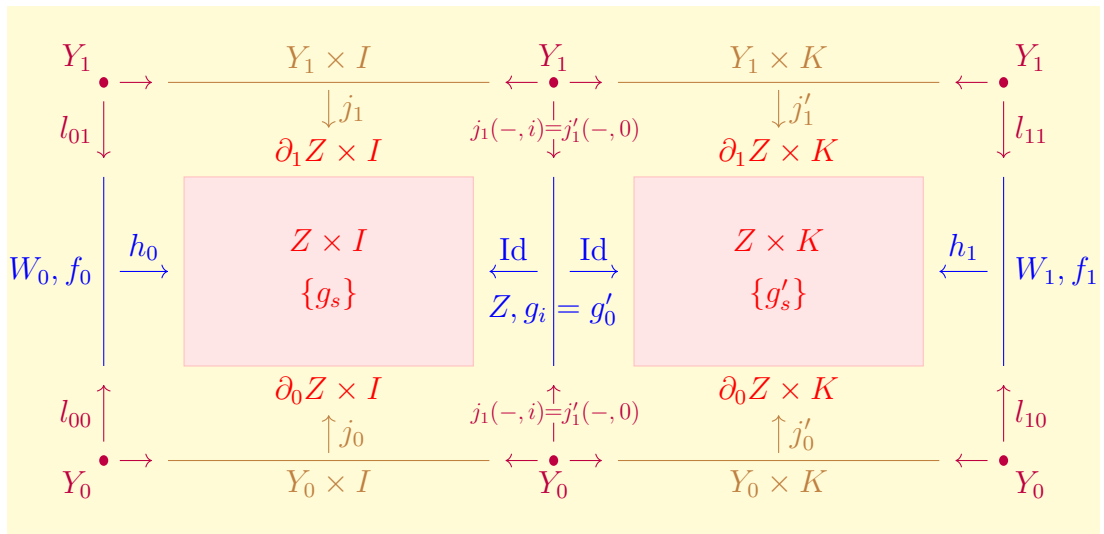
$$\begin{array}{ccc}
 W_1 & \xrightarrow{H_{1,t}} & Z \times \{1\} \\
 \uparrow \iota_{10} & & \uparrow J_{0,t} \\
 Y_0 & \xrightarrow{\text{Id} \times \{1\}} & Y_0 \times \{1\}
 \end{array}
 \quad
 \begin{array}{ccc}
 Y_1 & \xrightarrow{\text{Id} \times \{1\}} & Y_1 \times \{1\} \\
 \downarrow \iota_{11} & & \downarrow J_{1,t} \\
 W_1 & \xrightarrow{H_{1,t}} & Z \times \{1\}
 \end{array}$$

We then get an equivalent 2-cell by taking

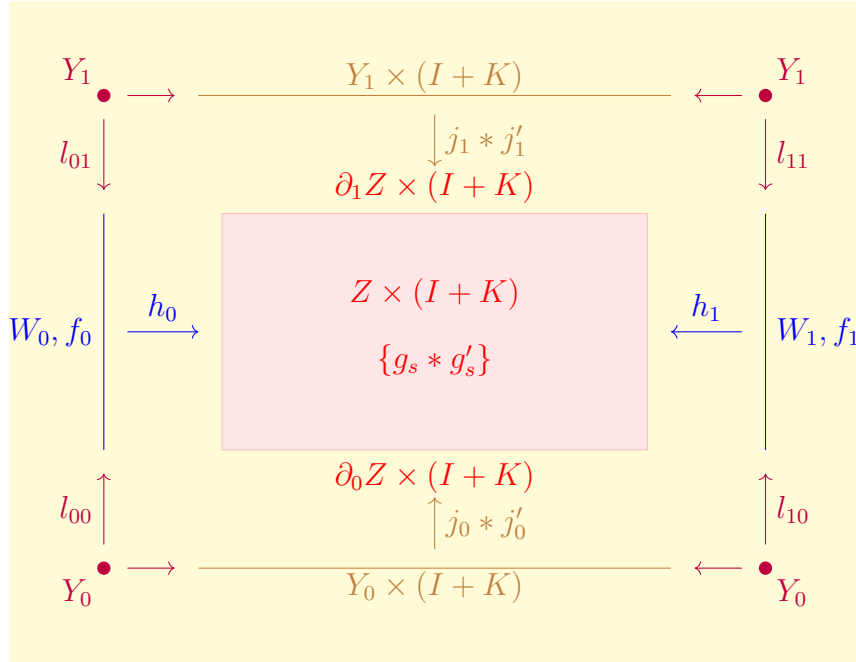
- $Z' = Z$ and $I = I'$
- $h'_0 = H_{0,1}$ and $h'_1 = H_{1,1}$
- $j'_0 = J_{0,1}$ and $j'_1 = J_{1,1}$
- $g'_s = G_{s,1}$

This equivalence relation completes the definition of the 2-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^0}$.

The horizontal composition of 2-cells (in our diagrams this refers to composition going up along the page) is given by reparameterizing and gluing with the help of germs of collars. There is no canonical way of doing that, but any two choices will lead to equivalent 2-cells. The vertical composition (in our diagrams this refers to composition going from left to right along the page) is determined by the relations above and by defining the composition



to equal the 2-cell



This completes the definition of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$.

3.4.2 The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^1}$

The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^1}$ is defined as follows:

0-cells

Closed, (oriented) $(d - 1)$ -dimensional manifolds.

1-cells

A 1-cell between 0-cells Y_0 and Y_1 is a d -dimensional cobordism

$$\overline{W} = (W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1)$$

equipped with a collar and an excellent Morse function

$$f : (W; \partial_0 W, \partial_1 W) \rightarrow (I; \{0\}, \{i\}),$$

where I is the interval $[0, i]$. Composition is by gluing of cobordisms and Morse functions, with the help of the chosen collars.

2-cells

The 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^1}$ are defined exactly like the 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$, but the path (g_s) of functions can now be any Cerf path and is not restricted to be a

path of excellent Morse functions. A 2-cell is further equipped with a partition into cut-times $(c_0 < \dots < c_m)$ such that $(g_s)_{s \in [a_i, a_{i+1}]}$ is either a path of excellent Morse functions, or a birth-death path or a crossing path.

Equivalence relation

The equivalence relation on the 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$ also apply here. Furthermore, we have:

1. Homotopy of birth-death paths and compatible isotopy of boundary identifications. This relation applies if the path (g_s) is a birth-death path and the partition into cut-times is just (a_0, a_1) . Its form is then exactly the same as the homotopy of excellent Morse functions relation.
2. Homotopy of crossing paths and compatible isotopy of boundary identifications. This is exactly analogous to the relation above.
3. Isotopies of partitions. Birth of new partition times. Removal of a partition time if the resulting partition is admissible. For example, if $(g_s)_{s \in [a_{i-1}, a_i]}$ is a birth-death path and $(g_s)_{s \in [a_i, a_{i+1}]}$ is a crossing path, we cannot remove time a_i from the partition. These guarantee that any two partitions into cut-times are equivalent.
4. Eye birth-death, merge-unmerge and Reidemeister II homotopies of Cerf paths and compatible isotopies of boundary identifications.
5. Naive Cerf homotopies and compatible isotopy of boundary identifications. These are the Cerf homotopies that witness a codimension-2 phenomenon in the simplest possible way, namely by passing through functions that have two cusp critical points or two pairs of Morse critical points at the same critical level or one of each.

Composition in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^1}$ is defined just like in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$.

3.4.3 The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$

This is simply the quotient of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^1}$ by the Cerf homotopies that cross $\mathcal{F}^2$ (swallowtail, beak and Reidemeister III) and compatible isotopies of boundary identifications.

Proposition 3.4.1. The forgetful functor

$$F^{\text{Cerf}} : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$$

is an equivalence of bicategories.

Proof. This is an immediate consequence of the genericity of Cerf paths and density of Cerf homotopies. $\square$

3.5 Bordism $(2, 1)$ -categories equipped with Cerf data

The purpose of this section is to introduce the second intermediate step in relating $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$ with $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$, namely the d -dimensional bordism $(2, 1)$ -category equipped with Cerf data $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$. In analogy with section 2.3, we will construct $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$ in three stages, building a tower of $(2, 1)$ -categories

$$\begin{array}{c} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0} \\ \downarrow \\ \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1} \\ \downarrow \\ \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})} \end{array}$$

We begin by introducing a convenient refinement of Morse data.

3.5.1 Colored Morse data

Recall from Definition 1.3.15 that Morse data for an excellent Morse function f consist of a collection of standard Morse model embeddings, a horizontal distribution, and a choice of regular level sets, all subject to the specified compatibility conditions. In addition, we label the intervals between consecutive regular values: *red* if the interval contains a single critical value, and *blue* otherwise. To more closely parallel the definition of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$, it is convenient to refine this structure by allowing four possible labels for such intervals: red, blue, green, and transparent. The admissible labelings are defined as follows.

Definition 3.5.1 (Admissible labelings). Let a_i and a_{i+1} be two consecutive regular values chosen as part of the Morse data of f . An *admissible labeling* of the interval $[a_i, a_{i+1}]$ is given by:

1. If there exists a critical value in $[a_i, a_{i+1}]$, then the interval is labeled **red**.
2. If there is no critical value in $[a_i, a_{i+1}]$, then:
 - If the horizontal distribution induces a diffeomorphism $f^{-1}(a_i) \rightarrow f^{-1}(a_{i+1})$ that is not merely a permutation induced by a symmetric group action on the connected components, then the interval is labeled **blue**.
 - If the induced diffeomorphism is given by a nontrivial symmetric group action, then the interval may be labeled either **blue** or **green**.
 - If the induced diffeomorphism is the identity, then the interval may be labeled either **blue**, **green**, or **transparent**.

Combining this with Definition 1.3.15, we obtain:

Definition 3.5.2 (Colored Morse data). Let $f : W \rightarrow I$ be an excellent Morse function, and let $\text{Crit}(f)$ denote its set of critical points. A choice of **colored Morse data associated with f** consists of:

- for each $p \in \text{Crit}(f)$, a pair of standard Morse model embeddings $(\phi_p, \phi'_p) : M_{\text{ind}(p)}^n \rightarrow W$ adapted to f at p ;
- a horizontal distribution τ compatible with every standard Morse model embedding (ϕ_p, ϕ'_p) ;
- a complete set of regular values $\{a_0 < \dots < a_m\}$ compatible with every standard Morse model embedding (ϕ_p, ϕ'_p) ;
- an admissible labeling of the intervals $[a_i, a_{i+1}]$ for all $0 \leq i \leq m-1$.

For brevity, we denote a choice of colored Morse data for f by $\mathcal{CD}_f$. The set of all such choices is written $\mathcal{CMD}(W; f)$, and the set of all pairs $(f, \mathcal{CD}_f)$ is denoted $\mathcal{CMD}(W)$.

We now define paths of colored Morse data. As in the case of ordinary Morse data, this is done by introducing the graph $\mathcal{CMD}(W)$, adapting Definition 1.3.16.

Definition 3.5.3. The directed graph $\mathcal{CMD}(W)$ is defined as follows:

1. **Vertices.** The vertices are the elements of $\mathcal{CMD}(W)$, i.e. a Morse function on W together with a choice of colored Morse data.
2. **Edges.** There are three kinds of edges:
 - (a) *Isotopy edges:* given by a path of Morse functions, together with a smooth isotopy of standard Morse model embeddings and a compatible isotopy of horizontal distributions, keeping the set of regular values fixed. These occur only between vertices whose colored Morse data use only red and blue labels.
 - (b) *Color-changing edges:* changing labels from transparent to green or from green to blue, and their inverses.
 - (c) *Regular-value edges:* corresponding to the birth or death of a regular value. Here the Morse function and horizontal distribution remain fixed, while the standard Morse model embeddings at the source vertex canonically determine those at the target vertex by flowing along the horizontal distribution until the compatibility conditions are satisfied. For a death move, i.e. merging two intervals, the labels of the source vertex determine the labels at the target as follows:

$$\begin{aligned}
\text{transparent} + \text{anything} &\mapsto \text{anything}, \\
\text{red} + \text{blue} &\mapsto \text{red}, \\
\text{red} + \text{green} &\mapsto \text{not allowed}, \\
\text{blue} + \text{blue} &\mapsto \text{blue}, \\
\text{green} + \text{green} &\mapsto \text{green}, \\
\text{green} + \text{blue} &\mapsto \text{not allowed}.
\end{aligned}$$

A birth move is defined as the inverse of a death move.

A *path* in $\mathcal{CMD}(W)$ is a finite directed path in this graph. Every path in $\mathcal{CMD}(W)$ has an inverse. Similarly, we define the directed graph $\mathcal{CMD}(W; f)$ by requiring the Morse function f to remain fixed throughout, and we define paths in $\mathcal{CMD}(W; f)$ accordingly.

Remark 3.5.1. This is the analogue of Remark 1.3.11 in the colored setting. Isotopy edges naturally encode paths $\gamma: [0, 1] \rightarrow \mathcal{CMD}(W)$. Likewise, we may associate regular-value edges with paths $\gamma: [0, 1] \rightarrow \mathcal{CMD}(W)$. Such paths are piecewise constant: they remain constant before and after the birth or death of a regular value. Throughout such a path, the horizontal distribution stays fixed, while one or more

of the set of regular values, the standard Morse model embeddings, and the interval labels may change at the critical time.

We will often represent paths in the graph $\mathcal{CMD}(W)$ as actual paths in the set $\mathcal{CMD}(W)$, from this point of view. For example, we will typically denote an element of $\mathcal{CMD}(W)$ by $(f, \mathcal{CD}_f)$ and a path in $\mathcal{CMD}(W)$ by $(f_s, \mathcal{CD}_{f_s})$.

Definition 3.5.4. Two paths in $\mathcal{CMD}(W)$ are said to be *equivalent* if they are related by the equivalence relation generated by the following moves:

1. The isotopy, commutation, and regular-value moves in $\mathcal{MD}(W)$, as defined in Definition 1.3.17.
2. Additional regular-value moves:
 - (a) Suppose two consecutive intervals can be merged by the death of a regular value, producing a final interval labeled f_0 . If the labels of the initial intervals can be changed so that their merge instead yields a final interval labeled f_1 , then the following two paths are equivalent:
 - (i) perform the merge and then change the label $f_0 \rightarrow f_1$ (if $f_0 \neq f_1$);
 - (ii) first change the labels of the initial intervals and then perform the merge.

For example, if two consecutive intervals are labeled green, then performing the death that merges them—yielding a green interval—and subsequently changing its label to blue is equivalent to first changing the two green intervals to blue and then performing the death.

- (b) If the inverses of two paths are related by move (a), then the paths themselves are also related.
- (c) Any two sequences of label changes with the same endpoints are equivalent. In particular, the composite of a label-change path with its inverse is equivalent to the constant path.

Equipped with this definition, we can now show the following:

Lemma 3.5.2. *Any two paths of regular-value edges in $\mathcal{CMD}(W; f)$ with the same endpoints are equivalent.*

Proof. The argument proceeds by reduction to a normal form, just as in Lemma 1.3.14. We distinguish two cases.

Case 1: The endpoints lie in $\mathcal{MD}(W; f)$. In this case every interval between consecutive regular values is labeled either red or blue. We show that any such path in $\mathcal{CMD}(W; f)$ is equivalent to a path entirely contained in $\mathcal{MD}(W; f)$.

If the path consists exclusively of label changes (and hence no births or deaths of regular values), then the claim follows directly from move (c) of Definition 3.5.4, which relates any two sequences of label changes with the same endpoints.

Suppose instead that the path contains births or deaths of regular values. We first show that it is equivalent to a path with the following property: whenever two intervals are merged or unmerged during a birth or death, both are labeled red or blue. To achieve this, consider any merge or unmerge that fails this property. Apply move (c) to rewrite a participating interval labeled c as $c \rightarrow \text{blue} \rightarrow c$, and then use move (a) or (b) of Definition 3.5.4 to adjust accordingly. This produces a path with the desired property.

Thus every interval labeled green or transparent arises only through a label change, and such intervals disappear (via a label change) before they interact with any birth or death of regular values. Consequently, the path reduces to one consisting only of red and blue intervals, and the proof of Lemma 1.3.14 applies to yield a normal form.

Case 2: The endpoints lie outside $\mathcal{MD}(W; f)$. In this case, we use move (c) of Definition 3.5.4 to pre- and post-compose the given path with paths equivalent to the constant path, which rewrite any interval labeled c (transparent or green) as $c \rightarrow \text{blue} \rightarrow c$. This reduces the situation to Case 1 and completes the proof. $\square$

Corollary 3.5.3. *Let γ_0, γ_1 be paths in $\mathcal{CMD}(W)$ sharing the same endpoints. If their induced paths of partial Morse data are equivalent in $\mathcal{PMD}(W)$, then γ_0 is equivalent to γ_1 .*

Proof. This is a consequence of Lemma 3.5.2 together with the commutation moves. $\square$

Definition 3.5.5. A *colored Morse data preserving diffeomorphism* is a Morse data preserving diffeomorphism in the sense of Definition 1.3.18, considered between 1-cells equipped with colored Morse data. In particular, we do *not* require it to preserve the labels of the corresponding intervals.

We can now introduce $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

3.5.2 The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$

The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ is defined as follows:

0-cells

Closed, (oriented) $(d - 1)$ -dimensional manifolds.

1-cells

A 1-cell between 0-cells Y_0 and Y_1 is a d -dimensional cobordism

$$\overline{W} = (W; \partial_0 W, \partial_1 W; \iota_0 : \partial_0 W \rightarrow Y_0, \iota_1 : \partial_1 W \rightarrow Y_1)$$

equipped with an excellent Morse function

$$f : (W; \partial_0 W, \partial_1 W) \rightarrow (I; \{0\}, \{i\}),$$

where $I = [0, i]$, together with associated colored Morse data, denoted $\mathcal{CD}_f$.

Recall from Definition 3.5.2 that $\mathcal{CD}_f$ consists of:

- standard Morse model embeddings adapted at the critical points of f ;
- a horizontal distribution for f , compatible with the vector fields induced by the standard embeddings on their images;
- a choice of regular level sets, compatible with the embeddings and including the boundary levels;
- an admissible labeling of the intervals between consecutive regular values.

Composition of 1-cells is given by gluing cobordisms and gluing Morse functions, using the collar germs induced by the horizontal distributions that are part of the Morse data.

2-cells

Let $(\overline{W}_0, f_0, \mathcal{CD}_{f_0})$ and $(\overline{W}_1, f_1, \mathcal{CD}_{f_1})$ be two 1-cells from Y_0 to Y_1 . A 2-cell between them is, up to a soon-to-be-specified equivalence, a cylindrical 2-cobordism

$$\overline{Z \times I}$$

between $\overline{W}_0$ and $\overline{W}_1$, equipped with:

- a path of excellent Morse functions $\{g_s\}$, and

- an associated path of colored Morse data $\mathcal{CD}_{g_s}$, as in Definition 3.5.3.

We will view such a path of colored Morse data as a map $\gamma: [0, 1] \rightarrow \mathcal{CMD}(W)$, following Remark 3.5.1. It comes equipped with cut times $0 = c_0 < c_1 < \dots < c_k = 1$ such that each restriction $\gamma|_{[c_i, c_{i+1}]}$ represents an edge in the graph $\mathcal{CMD}(W)$: either a smooth isotopy (labeling an isotopy edge) or a piecewise constant path (labeling a regular-value edge), as in Definition 3.5.3.

The boundary identifications are required to be colored Morse data preserving diffeomorphisms:

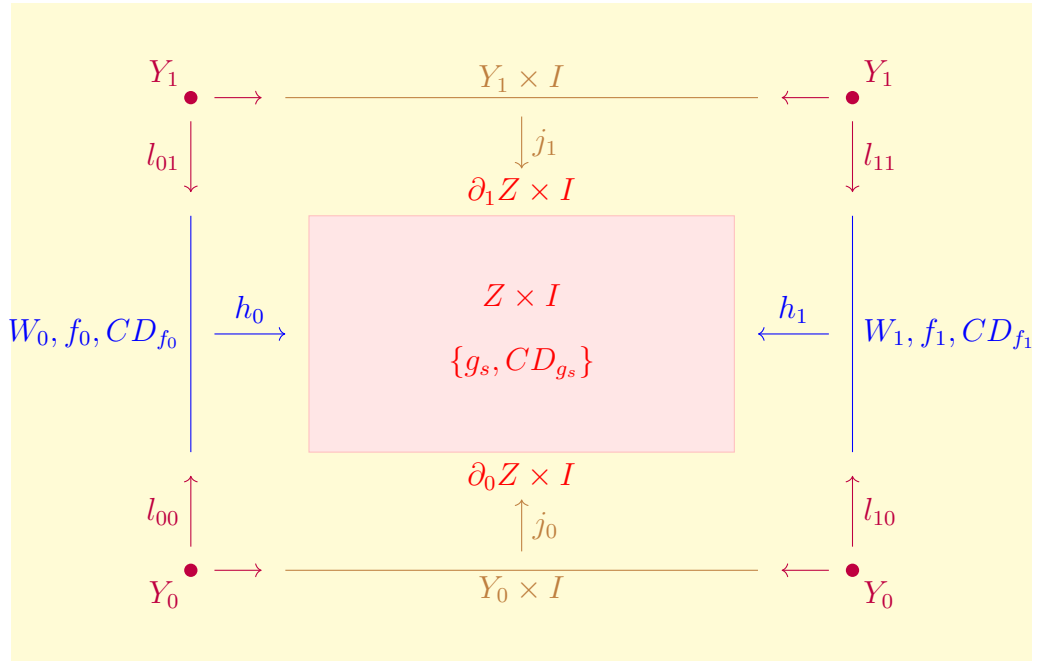
$$h_0 : (\overline{W}_0, f_0, \mathcal{CD}_{f_0}) \rightarrow (\overline{Z}_0, g_0, \mathcal{CD}_{g_0}),$$

$$h_1 : (\overline{W}_1, f_1, \mathcal{CD}_{f_1}) \rightarrow (\overline{Z}_i, g_i, \mathcal{CD}_{g_i}).$$

The Morse data $\mathcal{CD}_{f_0}$ and $\mathcal{CD}_{f_1}$ induce collar germs on the boundaries of the 1-cells. Similarly, the path $\mathcal{CD}_{g_s}$ induces a collar germ on the vertical boundary of $\overline{Z} \times \overline{I}$. We require that the boundary identifications h_0, h_1 respect these collar germs, since they are Morse data preserving.

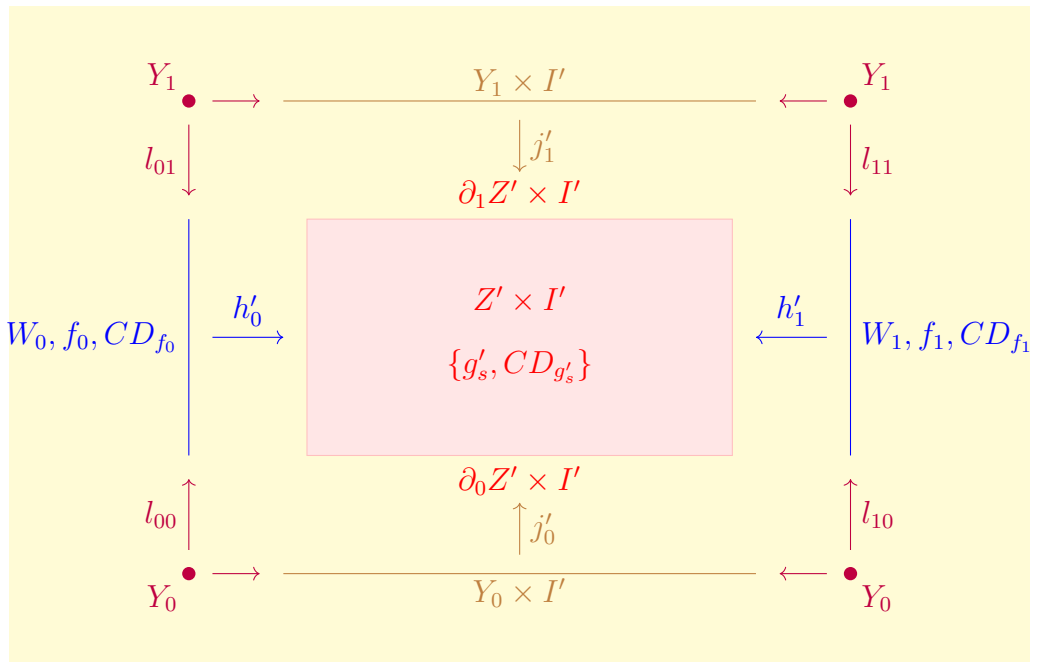
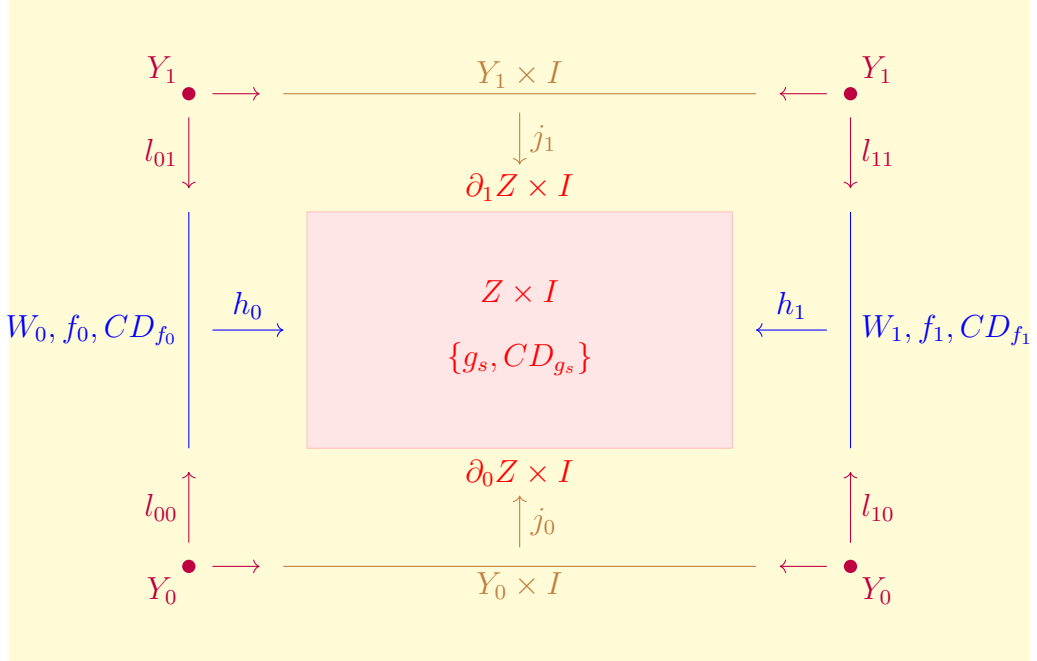
Finally, we further equip $\overline{Z} \times \overline{I}$ with a collar germ along its horizontal boundary and require that the Cerf path and the associated colored Morse data remain constant along this horizontal collar.

Pictorially, we visualize a 2-cell as follows:



Equivalence relation

Let $(\overline{W}_0, f_0, \mathcal{CD}_{f_0})$ and $(\overline{W}_1, f_1, \mathcal{CD}_{f_1})$ be two 1-cells from Y_0 to Y_1 and let $(\overline{Z} \times \overline{I}, \{g_s, \mathcal{CD}_{g_s}\})$, $(\overline{Z'} \times \overline{I'}, \{g'_s, \mathcal{CD}_{g'_s}\})$ be two 2-cells between them. Instead of spelling out all of the data symbolically, let's make them explicit through the following two pictures:



There are four kinds of relations we quotient out by:

1. **Reparameterization of the interval I .** Namely, consider any interval $I' = [0, i']$ and a diffeomorphism $\phi : I \rightarrow I'$ such that $\phi(0) = 0$ and $\phi(i) = i'$. This induces a diffeomorphism

$$\Phi = \text{Id}_Z \times \phi : Z \times I \rightarrow Z \times I'.$$

We then get an equivalent 2-cell by taking

- $Z' = Z$
- $h'_0 = h_0$ and $h'_1 = h_1$
- $c'_i = \phi(c_i)$
- $j'_0 = \Phi \circ j_0$ and $j'_1 = \Phi \circ j_1$
- $g'_s = g_{\phi^{-1}(s)}$ and $\mathcal{D}_{g'_s} = \mathcal{D}_{g_{\phi^{-1}(s)}}$

2. **Change of the underlying smooth manifold triad.** Namely, consider another smooth manifold triad $(Z'; \partial_0 Z', \partial_1 Z')$ and a diffeomorphism of smooth manifold triads $d : (Z; \partial_0 Z, \partial_1 Z) \rightarrow (Z'; \partial_0 Z', \partial_1 Z')$. This induces a diffeomorphism

$$D = d \times \text{Id}_I : Z \times I \rightarrow Z' \times I.$$

We then get an equivalent 2-cell by taking

- $I' = I$
- $c'_i = c_i$
- $h'_0 = D \circ h_0 = d \circ h_0$ and $h'_1 = D \circ h_1 = d \circ h_1$
- $j'_0 = D \circ j_0$ and $j'_1 = D \circ j_1$
- $g'_s = g_s \circ d^{-1}$ and $\mathcal{D}_{g'_s} = d \circ \mathcal{D}_{g_s}$

3. **Equivalence of paths of colored Morse data and compatible isotopies of boundary identifications.** This relation is given by an equivalence between two paths of colored Morse data $\mathcal{CD}_{G_t}$ and $\mathcal{CD}_{G'_t}$ as in Definition 3.5.4. We do not require this equivalence to fix the endpoints of the paths. However, we do assume that the initial and final endpoints are related by isotopies of colored Morse data (i.e. the number of regular values remains constant).

More explicitly, suppose we are given isotopies $\mathcal{CD}_{0,t}$ and $\mathcal{CD}_{1,t}$ of colored Morse data such that:

$$\mathcal{CD}_{0,t}: \mathcal{CD}_{G_0} \rightsquigarrow \mathcal{CD}_{G'_0},$$

$$\mathcal{CD}_{1,t}: \mathcal{CD}_{G_j} \rightsquigarrow \mathcal{CD}_{G'_j},$$

where the paths $\mathcal{CD}_{G_t}$ and $\mathcal{CD}_{G'_t}$ are parameterized by $[0, j]$.

Our compatibility condition for the boundary identifications is as follows. There exist isotopies

$$H_{0,t}: \overline{W_0} \rightarrow \overline{Z}, \quad H_{1,t}: \overline{W_1} \rightarrow \overline{Z}$$

such that

$$H_{0,t}: (\overline{W_0}, \mathcal{CD}_{f_0}) \rightarrow (\overline{Z}, \mathcal{CD}_{0,t}), \quad H_{1,t}: (\overline{W_1}, \mathcal{CD}_{f_1}) \rightarrow (\overline{Z}, \mathcal{CD}_{1,t})$$

are colored Morse data preserving diffeomorphisms for all t . In addition, there exist isotopies of isotopies

$$J_{0,t}: Y_0 \times I \rightarrow \partial_0 Z \times I, \quad J_{1,t}: Y_1 \times I \rightarrow \partial_1 Z \times I$$

compatible with $H_{0,t}$ and $H_{1,t}$, in the sense that the following diagrams commute for all t :

$$\begin{array}{ccc} W_0 & \xrightarrow{H_{0,t}} & Z \times \{0\} \\ \iota_{00} \uparrow & & \uparrow J_{0,t} \\ Y_0 & \xrightarrow{\text{Id} \times \{0\}} & Y_0 \times \{0\} \end{array} \quad \begin{array}{ccc} Y_1 & \xrightarrow{\text{Id} \times \{0\}} & Y_1 \times \{0\} \\ \iota_{01} \downarrow & & \downarrow J_{1,t} \\ W_0 & \xrightarrow{H_{0,t}} & Z \times \{0\} \end{array}$$

$$\begin{array}{ccc} W_1 & \xrightarrow{H_{1,t}} & Z \times \{1\} \\ \iota_{10} \uparrow & & \uparrow J_{0,t} \\ Y_0 & \xrightarrow{\text{Id} \times \{1\}} & Y_0 \times \{1\} \end{array} \quad \begin{array}{ccc} Y_1 & \xrightarrow{\text{Id} \times \{1\}} & Y_1 \times \{1\} \\ \iota_{11} \downarrow & & \downarrow J_{1,t} \\ W_1 & \xrightarrow{H_{1,t}} & Z \times \{1\} \end{array}$$

We then obtain an equivalent 2-cell by setting:

- $Z' = Z$,
- the final path of colored Morse data on Z to be $\mathcal{CD}_{G'_t}$,
- $h'_0 = H_{0,1}$ and $h'_1 = H_{1,1}$,
- $j'_0 = J_{0,1}$ and $j'_1 = J_{1,1}$.

4. **Frame change.** Let $\{\phi_s : M_k^n \rightarrow Z\}$ be an isotopy of standard Morse model embeddings that is part of $\mathcal{D}_{g_s}$, and let $\{Q_s\}$ be a path in $G_{(n,k)}^{(\text{or})}$. Consider the isotopy of Morse data $\mathcal{D}'_{g_s}$ obtained by substituting $\{\phi_s : M_k^n \rightarrow Z\}$ in $\mathcal{D}_{g_s}$ with $\{Q_s \cdot \phi_s : M_k^n \rightarrow Z\}$ and keeping all other elements of $\mathcal{D}_{g_s}$ the same. We then get an equivalent 2-cell by taking

- $Z' = Z$, $I' = I$, and $c'_i = c_i$
- $h'_0 = h_0$ and $h'_1 = h_1$
- $j'_0 = j_0$ and $j'_1 = j_1$
- $g'_s = g_s$ and $\mathcal{D}_{g'_s} = \mathcal{D}'_{g_s}$

This completes the description of the relations. Finally, the composition in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ is defined in direct analogy with the composition in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^0}$, which completes the definition of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

Consider the forgetful functor

$$F_0^{\text{data}} : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^0},$$

that simply forgets the Cerf data.

Proposition 3.5.1. The functor F_0^{data} is an equivalence of symmetric monoidal bicategories.

Proof. We will apply the Whitehead criteria:

- Being essentially surjective on 0-cells: F_0^{data} is the identity (and hence essentially surjective on 0-cells).
- Being essentially full on 1-cells: According to the Morse lemma, every smooth manifold triad that comes equipped with a Morse function admits compatible Morse data, which implies that F_0^{data} is full (and hence essentially full) on 1-cells.
- Being full on 2-cells: Let $(\overline{W}_0, f_0, \mathcal{CD}_{f_0})$ and $(\overline{W}_1, f_1, \mathcal{CD}_{f_1})$ be two 1-cells from Y_0 to Y_1 . Let $(\overline{Z \times I}, \{g_s\})$ be a 2-cell between $(\overline{W}_0, f_0)$ and $(\overline{W}_1, f_1)$ in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^0}$, with vertical boundary identifications h_0, h_1 and horizontal boundary identifications j_0, j_1 . We must show that there exists a path of colored Morse data $\{\mathcal{CD}_{g_s}\}$ associated with $\{g_s\}$ on $Z \times I$, and that h_0 (respectively h_1)

is colored Morse data preserving with respect to $\mathcal{CD}_{f_0}$ and $\mathcal{CD}_{g_0}$ (respectively $\mathcal{CD}_{f_1}$ and $\mathcal{CD}_{g_i}$).

The proof of the faithfulness of the functor

$$F_0^{\text{data}}: \text{Bord}_{(d-1; d)}^{\text{Morse data, (or), } \mathcal{F}^0} \longrightarrow \text{Bord}_{(d-1; d)}^{\text{Morse, (or), } \mathcal{F}^0}$$

given in Proposition 2.4.1 implies that there exists a path of partial Morse data $\{\mathcal{PD}_{g_s}\}$ associated with $\{g_s\}$ on $Z \times I$, and that h_0 (respectively h_1) is partial Morse data preserving with respect to $\mathcal{PD}_{f_0}$ and $\mathcal{PD}_{g_0}$ (respectively $\mathcal{PD}_{f_1}$ and $\mathcal{PD}_{g_i}$).

Such a path of partial Morse data canonically determines a minimal path of colored Morse data, where:

- all intervals are labeled red or blue, and
- all regular values are determined by the standard Morse model embeddings.

The colored Morse data at the endpoints of this path may not agree with $\mathcal{CD}_{f_0}$ and $\mathcal{CD}_{f_1}$. In that case, we adjust the colored Morse data at the endpoints using the colored version of Lemma 1.3.12.

This completes the proof that a path of colored Morse data $\{\mathcal{CD}_{g_s}\}$ with the desired properties exists.

- Being faithful on 2-cells: Let $(\overline{W_0}, f_0, \mathcal{CD}_{f_0})$ and $(\overline{W_1}, f_1, \mathcal{CD}_{f_1})$ be two 1-cells from Y_0 to Y_1 . Let $(\overline{Z \times I}, \{g_s, \mathcal{CD}_{g_s}\})$ and $(\overline{Z' \times I'}, \{g'_s, \mathcal{CD}_{g'_s}\})$ be two 2-cells between them and let's assume that the 2-cells $(\overline{Z \times I}, \{g_s\})$ and $(\overline{Z' \times I'}, \{g'_s\})$ are equivalent in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^0}$. We will then show that $(\overline{Z \times I}, \{g_s, \mathcal{CD}_{g_s}\})$ and $(\overline{Z' \times I'}, \{g'_s, \mathcal{CD}_{g'_s}\})$ are equivalent in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. After the proof of the fullness of F_0^{data} , it suffices to only consider the case when $(\overline{Z \times I}, \{g_s\})$ and $(\overline{Z' \times I'}, \{g'_s\})$ are related via a generating relation in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^0}$. Moreover, using Lemma 3.5.2, it suffices to show that there exists an equivalence of the underlying partial Morse data. We will start with some special cases and we will then reduce all other cases to those:

1. First, we will address the case $(g_s)_{s \in I} = (g'_s)_{s \in I}$ and then we will reduce all other cases to this one. In this case, $\mathcal{PD}_{g_s}$ and $\mathcal{PD}'_{g_s}$ will be two paths of partial Morse data on the same path of excellent Morse functions g_s .

Moreover, let's assume that the identity $Z \rightarrow Z$ is colored Morse data preserving for $(\mathcal{CD}_{g_0}, \mathcal{CD}'_{g_0})$ and $(\mathcal{CD}_{g_i}, \mathcal{CD}'_{g_i})$. Let's suppose without loss of generality that g_0 has a single critical point of index k (the case when g_0 has no critical points is easy and if it has more than one critical points, we can decompose it into elementary pieces that don't).

Let $\mathcal{F}^0(Z)_{g_0}$ (respectively $\mathcal{MD}(Z)_{g_0}$) denote the connected component of g_0 in $\mathcal{F}^0(Z)$ (respectively $\mathcal{MD}(Z)$). Cerf's fibration lemma implies that we have a fiber bundle

$$\begin{array}{c} \mathcal{MD}(Z)_{g_0} \\ \downarrow \\ \mathcal{F}^0(Z)_{g_0} \end{array}$$

There is a free action of $G_{(n,k)}^{\text{or}}$ on $\mathcal{MD}(Z)_{g_0}$ which induces a fiber bundle

$$\begin{array}{c} \mathcal{MD}(Z)_{g_0} / G_{(n,k)}^{\text{or}} \\ \downarrow \\ \mathcal{F}^0(Z)_{g_0} \end{array}$$

whose fiber, $\mathcal{MD}(Z; g_0) / G_{(n,k)}^{\text{or}}$ is contractible. Since we assume that the identity on Z is Morse data preserving for $(\mathcal{PD}_{g_0}, \mathcal{PD}'_{g_0})$ and $(\mathcal{PD}_{g_i}, \mathcal{PD}'_{g_i})$, the images of $\mathcal{PD}_{g_s}$ and $\mathcal{PD}_{g'_s}$ in $\mathcal{MD}(Z)_{g_0} / G_{(n,k)}^{\text{or}}$ are two paths with the same endpoints over the same path in $\mathcal{F}^0(Z)_{g_0}$. The long exact sequence in homotopy groups then implies that $\mathcal{PD}_{g_s}$ and $\mathcal{PD}_{g'_s}$ are connected in $\mathcal{MD}(Z)_{g_0} / G_{(n,k)}^{\text{or}}$ through an endpoint preserving homotopy H . Thus, there exists an endpoint preserving homotopy in $\mathcal{MD}(Z)_{g_0}$ from $\mathcal{PD}_{g'_s}$ to a path of Morse data that coincides with $\mathcal{PD}_{g_s}$ up to a path in $G_{(n,k)}^{\text{or}}$, which implies that the original two 2-cells we considered are equivalent in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

2. We will now assume that $(g_s)_{s \in I}$ is isotopic to $(g'_s)_{s \in I}$ rel boundary and again assume that the identity $Z \rightarrow Z$ is colored Morse data preserving for $(\mathcal{CD}_{g_0}, \mathcal{CD}'_{g_0})$ and $(\mathcal{CD}_{g_i}, \mathcal{CD}'_{g_i})$. The same proof as above will apply in this case.
3. Second, we will consider the same case as above but now we will not assume that the identity on Z is colored Morse data preserving for $(\mathcal{CD}_{g_0}, \mathcal{CD}'_{g_0})$ and $(\mathcal{CD}_{g_i}, \mathcal{CD}'_{g_i})$, but rather two, potentially different, diffeomorphisms

isotopic to the identity are. This reduces to the previous case. We have colored Morse data preserving diffeomorphisms h_0, h'_0 and an isotopy H between them. This induces an isotopy on $Z \times I$ that agrees with H on $Z \times \{0\}$ and is constantly the identity on $Z \times \{i\}$. This in turn induces an isotopy of Morse functions and colored Morse data on $Z \times I$ ending with a path of colored Morse data $\mathcal{CD}''_{g_i}$ such that the identity on Z is colored Morse data preserving for $(\mathcal{CD}_{g_0}, \mathcal{CD}''_{g_0})$ and, moreover, $\mathcal{CD}''_{g_i} = \mathcal{CD}'_{g_i}$. We then apply the same argument on the other end and recover the conditions of the previous case.

4. Third, we will consider the general case when (g_s) and (g'_s) are connected via path of excellent Morse functions and there is also a compatible isotopy of boundary identifications. This can now be reduced to the above two.
5. The other generating relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^0}$ translate verbatim to relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$.

□

3.5.3 The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1}$

This is now defined in analogy with $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or}), \mathcal{F}^1}$. Namely, the bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1}$ is defined as follows:

0-cells

Closed, (oriented) $(d - 1)$ -dimensional manifolds.

1-cells

Exactly the same 1-cells as $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$

2-cells

The 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1}$ are defined exactly as in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$, except that the path (g_s) of functions and the associated Cerf data may now be any elementary Cerf path with associated Cerf data, and are no longer required to form a path of excellent Morse functions.

A 2-cell is further equipped with a partition into cut times $c_0 < \dots < c_k$ that refines the cut times used in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$. For each interval $[c_i, c_{i+1}]$, the restriction $(g_s)_{s \in [c_i, c_{i+1}]}$ may now also be:

- an elementary birth–death path (with associated Cerf data, i.e. Morse data at its endpoints that determine the path), or
- an elementary crossing path (again with associated Cerf data given by Morse data at its endpoints).

Moreover, the Morse data must be compatible at all cut times.

Equivalence relation

All of the relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ also apply here. Furthermore, we have:

1. Homotopy of elementary birth–death paths and compatible isotopy of boundary identifications. Here we only regard homotopies that come from isotopies of standard birth–death model embeddings, or equivalently from isotopies of standard Morse model embeddings in cancelling position. This relation applies if the path (g_s) is an elementary birth–death path and the partition into cut-times is just (a_0, a_1) . Its form is then exactly the same as the homotopy of excellent Morse functions with associated Morse data relation.
2. Homotopy of elementary crossing paths and compatible isotopy of boundary identifications. Here we only regard homotopies that come from isotopies of standard Morse model embeddings in crossing position. This is exactly analogous to the relation above.
3. Eye birth–death, merge–unmerge, and Reidemeister II Cerf homotopies. Here, two 2-cells are equivalent if their underlying Cerf paths are equivalent via one of these three kinds of homotopies.
4. Naive Cerf homotopies alongside compatible homotopies of Cerf data and compatible isotopy of boundary identifications. Recall that these are the Cerf homotopies that witness a codimension-2 phenomenon in the simplest possible way, namely by passing through functions that have two cusp critical points or two pairs of Morse critical points at the same critical level or one of each.

Composition in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^1}$ is defined just like in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

Consider now the forgetful functor

$$F_1^{\text{data}} : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^1} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^1},$$

that simply forgets the Cerf data.

Proposition 3.5.2. The functor F_1^{data} is an equivalence of symmetric monoidal bicategories.

Proof. Applying the Whitehead criteria in this case includes everything we covered in the proof of proposition 3.5.1. The extra things we need to check are:

- For fullness on 2-cells:
 - Every birth-death path is isotopic to an elementary birth-death path conjugated by isotopies of Morse data fixing the starting point. This follows from lemma 1.6.2.
 - Every crossing path is isotopic to an elementary crossing path conjugated by isotopies of Morse data fixing the starting point. This follows from Lemma 1.6.8.
- For faithfulness on 2-cells:
 - If two elementary birth-death paths are connected via a path of birth-death paths, then they are connected via a path of elementary birth-death paths induced by an isotopy of standard birth-death model embeddings. This is lemma 1.6.3.
 - Every homotopy of crossing paths between two elementary crossing paths is homotopic to a homotopy of elementary crossing paths induced by an isotopy of standard Morse model embeddings in crossing position. This follows again from lemma 1.6.8.

□

3.5.4 The bordism $(2, 1)$ -category $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or)}}$

Earlier, we constructed $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or)}}$ as the quotient of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or), } \mathcal{F}^1}$ by a collection of generating relations $\mathcal{R}$ consisting of Cerf homotopies—namely, the Cerf homotopies that cross the codimension-2 stratum $\mathcal{F}^2$ (swallowtail, beak, and Reidemeister III).

We now construct $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or)}}$ as the quotient of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^1}$ by the pullback of $\mathcal{R}$ along F_1^{data} . Since we are pulling back precisely the new relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or)}}$ along F_1^{data} , which we have already shown to be an equivalence, it follows that the induced forgetful functor

$$F^{\text{data}}: \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or)}} \longrightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf, (or)}}$$

is well-defined and is an equivalence of bicategories.

3.6 The equivalence between $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ and $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$

The purpose of this section is to prove that

$$F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})} \simeq \text{Bord}_{(d-1; d; \sim)}^{(\text{or})}.$$

To this end, we compare the towers

$$\begin{array}{ccc} F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0} & & \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0} \\ \downarrow & & \downarrow \\ F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1} & \text{and} & \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1} \\ \downarrow & & \downarrow \\ F \text{Sur}_{(d-1; d; \sim)}^{(\text{or})} & & \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})} \end{array}$$

and show that they are levelwise equivalent. Together with the fact that $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or})}$ is equivalent to $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf}, (\text{or})}$, which in turn is equivalent to $\text{Bord}_{(d-1; d; \sim)}^{(\text{or})}$, this establishes the desired equivalence.

3.6.1 The equivalence between $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ and $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$

The functor $B_0 : F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$

To construct the functor

$$B_0 : F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0},$$

we must first specify what it assigns to the generating 0-, 1- and 2-cells of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ and then verify that this assignment respects the generating relations. We have essentially already described the assignment B_0 on the generating 0-, 1- and 2-cells in Section 2.5, where we compared $\text{Sur}_{(d-1; d)}^{(\text{or})}$ with $\text{Bord}_{(d-1; d)}^{\text{Cerf data}, (\text{or})}$. For completeness, we now recall these constructions:

0-cells

B_0 is the identity on 0-cells.

Generating 1-cells

Diffeomorphism-type generating 1-cells

The functor B_0 sends the generating 1-cell



to the 1-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ with source X and target Y , consisting of:

1. the mapping cylinder cobordism

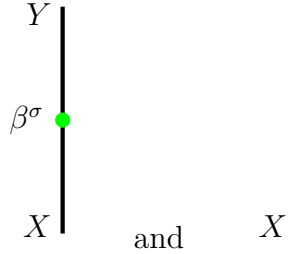
$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X: X \times \{0\} \rightarrow X, \phi: X \times \{1\} \rightarrow Y),$$

2. the Morse function $f: X \times [0, 1] \rightarrow [0, 1]$, given by projection,

3. the horizontal distribution $T_{f(z)}([0, 1]) \subset T_z(X \times [0, 1])$,

4. the complete set of regular values $\{0, 1\}$, with the interval $[0, 1]$ labeled blue.

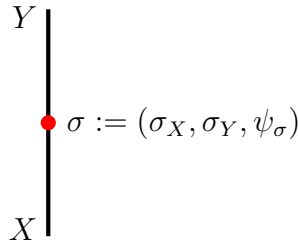
Remark 3.6.1. The images of the generating 1-cells



are defined in the same way, except that the interval between the regular values is labeled green and transparent, respectively.

Surgery-type generating 1-cells

The functor B_0 sends the generating 1-cell



to the 1-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ with source X and target Y , consisting of:

1. the symmetric trace of the surgery cobordism $\overline{W(X, Y, \sigma)}$. Its underlying manifold triad is given by Milnor's trace of the surgery $W(X, \sigma_X)$, as constructed in [Mil63]. The bottom boundary identification is the identity, while the top boundary identification is determined by σ_Y and ψ_σ ;
2. the Morse function that $\overline{W(X, Y, \sigma)}$ naturally carries, shifted by 1 so that its target is the interval $[0, 2]$;
3. the Morse data that $\overline{W(X, Y, \sigma)}$ is naturally equipped with. In particular, the chosen complete set of regular values is $\{0, 2\}$, and the interval $[0, 2]$ is labeled red.

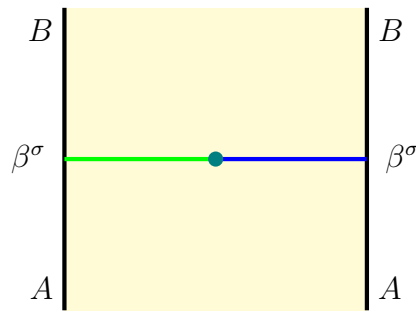
Remark 3.6.2. Let $(\overline{W}, f, \mathcal{CD}_f)$ be a 1-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. Using the colored Morse data on W , we can construct an equivalent cobordism W' which is a composite of mapping cylinders and symmetric surgery traces, each equipped with their standard Morse data and with interval colorings inherited from those on W . Moreover, using the colored Morse data on W and on W' , one can find a colored Morse data preserving diffeomorphism between them, which is unique up to isotopy. This shows that B_0 is essentially full on 1-cells.

Generating 2-cells

Diffeomorphism type generating 2-cells

Symmetry 2-cells

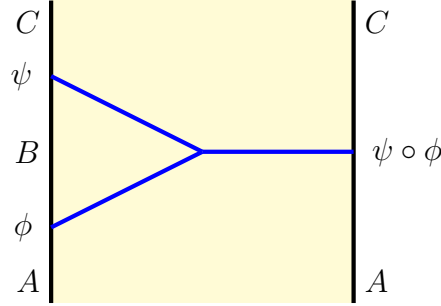
The functor B_0 sends the generating 2-cells



to the label-swap 2-cell that changes the label of the interval $[0, 1]$ from green to blue.

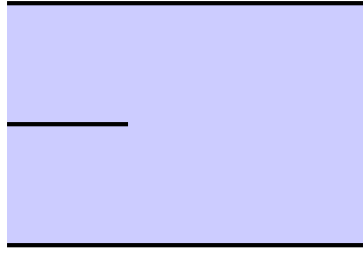
Compositor 2-cells

The functor B_0 sends the generating 2-cells



to the 2-cells defined by the following composite in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$:

1. Start with a path of regular-value edges in $\mathcal{MD}(W; f)$ (where (W, f) denotes the underlying cobordism and Morse function of the source) that eliminates the regular value 1. This is depicted schematically as:



2. Using the horizontal distribution, obtain a Morse data-preserving diffeomorphism from the resulting bordism to

$$X \times [0, 2],$$

with boundary identifications

$$\text{Id}_X : X \times \{0\} \rightarrow X, \quad \psi \circ \phi : X \times \{2\} \rightarrow Z,$$

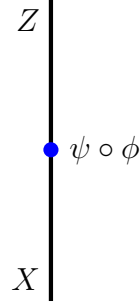
and equipped with the following Morse data:

- Morse function: projection $h : X \times [0, 2] \rightarrow [0, 2]$;
- horizontal distribution: $T_{h(z)}([0, 2]) \subset T_z(X \times [0, 2])$;
- regular values: $\{0, 2\}$.

3. The diffeomorphism

$$X \times [0, 2] \longrightarrow X \times [0, 1], \quad (x, t) \mapsto (x, t/2),$$

identifies this bordism with the image of the 1-cell

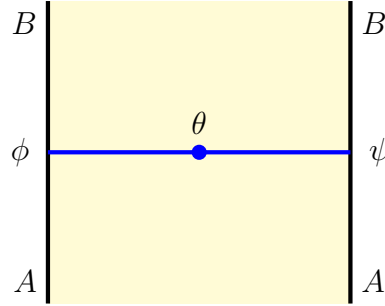


under B_0 , preserving the Morse data.

All relevant intervals are labeled blue throughout.

Isotopy.

The functor B_0 sends the generating 2-cells

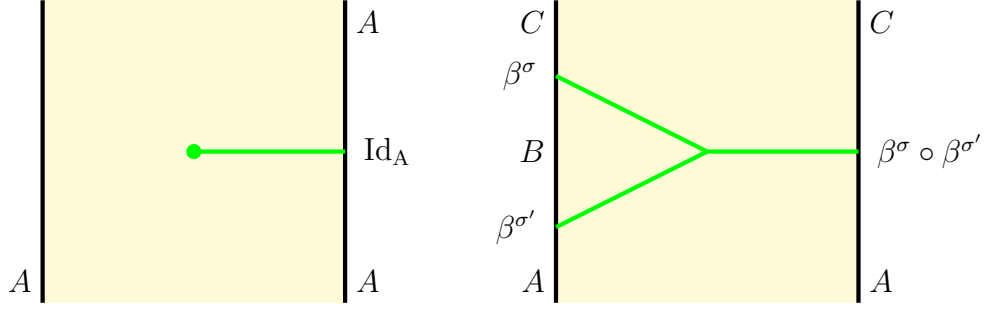


to the following 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. The isotopy θ from ϕ to ψ induces an isotopy $\phi^{-1} \circ \theta$ from the identity to $\phi^{-1} \circ \psi$. This, in turn, induces an isotopy of horizontal distributions on the mapping cylinder cobordism

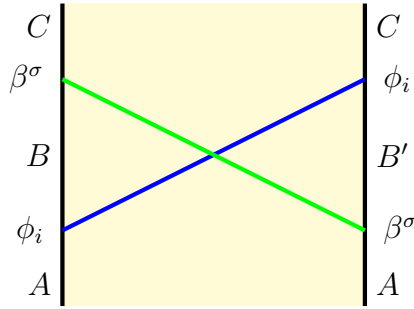
$$(X \times [0, 1]; X \times \{0\}, X \times \{1\}; \text{Id}_X: X \times \{0\} \rightarrow X, \phi: X \times \{1\} \rightarrow Y),$$

which is the image under B_0 of the source of the isotopy 2-cell. At the end of this path, the cobordism is equipped with Morse data whose horizontal distribution integrates to the diffeomorphism $\phi^{-1} \circ \psi$. This horizontal distribution then defines a Morse data preserving diffeomorphism whose target is the image under B_0 of the target of the isotopy 2-cell. This construction defines a 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. All relevant intervals are labeled blue throughout.

Remark 3.6.3. The images under B_0 of the 2-cells



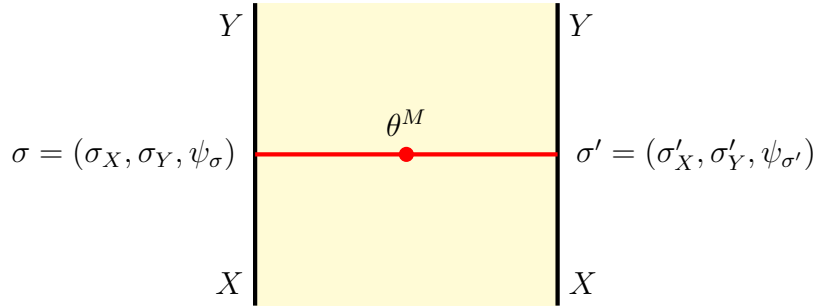
and



are defined in the same way as above, with the addition of the appropriate label swaps indicated by the diagrams.

Surgery type generating 2-cells

Recall that the only kind of surgery type generating 2-cell is an isotopy of Morse data:

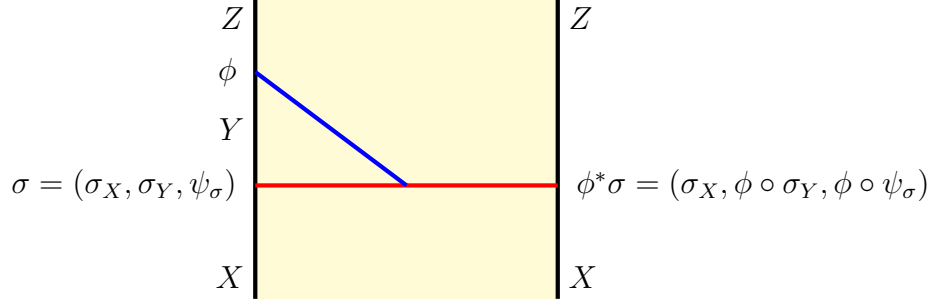


Here, θ^M is an isotopy of Morse data (and boundary identifications) on $W(X, Y, \sigma)$, from the natural Morse data that $W(X, Y, \sigma)$ is equipped with to Morse data that induce the surgery triple σ' .

Therefore, the decorations in this 2-cell naturally determine its image under B_0 : it is the isotopy-of-Morse-data 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ on $W(X, Y, \sigma)$, followed by the Morse data-preserving diffeomorphism uniquely determined by the Morse data on $W(X, Y, \sigma)$ at the end of the isotopy and the Morse data on $W(X, Y, \sigma')$.

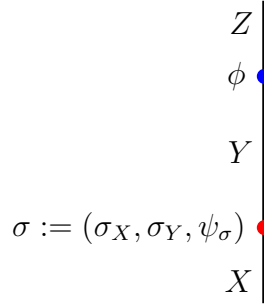
Naturality type generating 2-cells

The functor B_0 sends the generating 2-cells



to the following 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

First, observe that the image under B_0 of the source 1-cell



has underlying smooth manifold triad

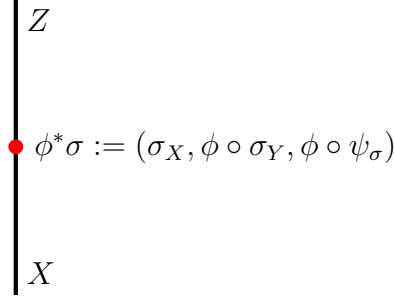
$$W := W(X, \sigma_X) \bigcup_{\psi: \chi(X, \sigma_X) \rightarrow Y \times \{0\}} Y \times [0, 1],$$

with boundary identifications

$$\text{Id}_X: X \setminus \sigma_X(S^{k-1} \times \{0\}) \cup_{\sigma_X} S^{k-1} \times D^{n-k} \longrightarrow X, \quad \phi: Y \times \{1\} \rightarrow Z.$$

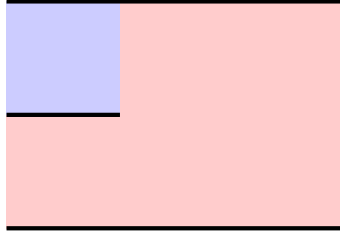
It comes equipped with a Morse function $f: W \rightarrow [0, 3]$ obtained by gluing the standard Morse function on $W(X, \sigma_X)$ (shifted by 1) with the projection $Y \times [0, 1] \rightarrow [0, 1]$. The standard Morse model embedding is the one naturally associated to $W(X, \sigma_X)$, and the horizontal distribution is obtained by gluing the standard horizontal distributions on $W(X, \sigma_X)$ and $Y \times [0, 1]$. The regular values are $\{0, 1, 2\}$.

The image under B_0 of the target 1-cell



is the cobordism $\overline{W(X, Z, \phi^* \sigma)}$. It carries the Morse function $g: W \rightarrow [0, 2]$ induced by the standard Morse function on $W(X, \sigma_X)$, together with its natural Morse model embedding and horizontal distribution. The regular values are $\{0, 2\}$.

These two bordisms are connected in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ by a 2-cell defined as follows. First, there is a path of regular-value edges in $\mathcal{MD}(W; f)$ that eliminates the regular value 2:

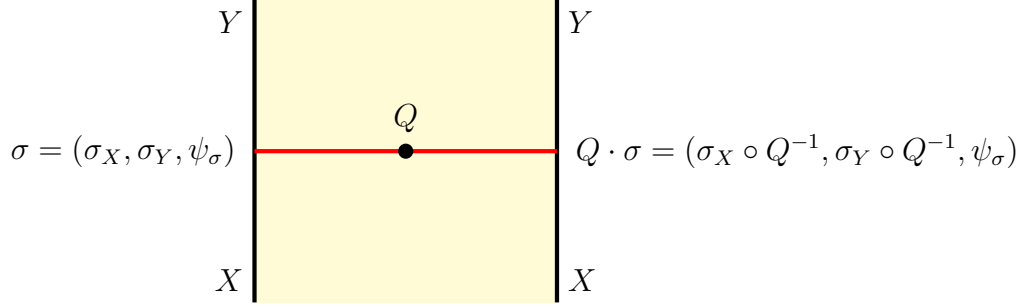


At the end of this path we obtain a Morse data-preserving diffeomorphism from W , equipped with its modified Morse data, to $W(X, \sigma_X)$ with its standard Morse data. Any such diffeomorphism is determined by a map $[0, 3] \rightarrow [0, 2]$ sending $0 \mapsto 0$, $3 \mapsto 2$, and $1 \mapsto 1$ (preserving the critical value). Moreover, any two such maps are isotopic, so the resulting Morse data-preserving diffeomorphism is unique up to isotopy.

Hence this determines a well-defined 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. Throughout, the labels of all intervals are red and blue exactly as indicated in the diagrams. The image of the other three kinds of naturality-type 2-cells under B_0 is defined analogously.

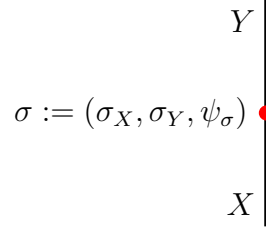
Frame change type generating 2-cells

The functor B_0 sends the generating 2-cells



to the following 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

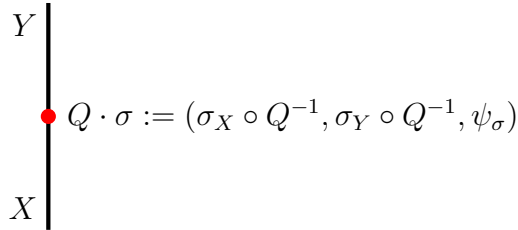
First, observe that the image of the source 1-cell



under B_0 is the smooth manifold triad

$$W(X, \sigma_X) := X \setminus (S^{k-1} \times \{0\}) \times [-1, 1] \bigcup_{\sigma_X, \text{ flow lines}} M_k^n.$$

The image of the target 1-cell



under B_0 is the smooth manifold triad

$$W(X, \sigma_X \circ Q^{-1}) := X \setminus (S^{k-1} \times \{0\}) \times [-1, 1] \bigcup_{\sigma_X \circ Q^{-1}, \text{ flow lines}} M_k^n.$$

Here $Q \in G_{(n,k)}^{(\text{or})}$, which induces a diffeomorphism $Q : M_k^n \rightarrow M_k^n$ preserving both the Morse function and the flow lines of M_k^n . By abuse of notation, we denote this diffeomorphism also by Q . Gluing the diffeomorphisms

$$\text{Id} : X \setminus (S^{k-1} \times \{0\}) \times [-1, 1] \longrightarrow X \setminus (S^{k-1} \times \{0\}) \times [-1, 1]$$

and

$$Q : M_k^n \longrightarrow M_k^n$$

produces a diffeomorphism

$$D : W(X, \sigma_X) \longrightarrow W(X, \sigma_X \circ Q^{-1}).$$

Both $W(X, \sigma_X)$ and $W(X, \sigma_X \circ Q^{-1})$ come naturally equipped with standard Morse model embeddings, given by the evident inclusions, and the following diagram commutes:

$$\begin{array}{ccc} W(X, \sigma_X) & \xrightarrow{D} & W(X, \sigma_X \circ Q^{-1}) \\ \uparrow \iota & & \uparrow \iota' \\ M_k^n & \xrightarrow{Q} & M_k^n \end{array}$$

Moreover, D preserves both the boundary identifications and the Morse data. It is therefore a 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$, as desired. All intervals are labeled red throughout, as indicated in the diagrams.

Proposition 3.6.1. The assignment B_0 described above induces a well-defined symmetric monoidal functor

$$B_0 : F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0} \longrightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}.$$

Proof. We must check that B_0 respects all generating relations in $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$. For each family of generating relations, we indicate the corresponding relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

1. Diffeomorphism type relations.

- *Associativity* is realized by the (all-blue) coincidence regular-value moves between Morse data; see Definition 1.3.17.
- *Unitality* and *symmetry-compositor crossing* are realized by the first two additional regular-value moves for colored Morse data in Definition 3.5.4.
- *Isotopy invariance* and *concatenation of isotopies* are realized by the isotopy moves between Morse data in Definition 1.3.17.
- *Isotopy-compositor crossing* is realized by the commutation moves in Definition 1.3.17.
- *Invertibility* is realized by the inverse-cancellation regular value moves in Definition 1.3.17 together with the third additional regular-value move in Definition 3.5.4.

2. Surgery type relations.

- *Isotopy of Morse data* (invariance, compatibility, and concatenation) is realized by the isotopy moves in Definition 1.3.17.
- *Isotopy of Morse data–naturality crossings* and *isotopy of Morse data–frame change crossings* are realized by the commutation moves in Definition 1.3.17.

3. Naturality type relations.

- *Naturality 4-valence* and *naturality–compositor* relations are realized by the red–blue coincidence regular-value moves and the sliding-red-region moves in Definition 1.3.17.
- *Naturality–Reidemeister II* is realized by the red–blue inverse-cancellation regular-value moves in Definition 1.3.17.
- *Naturality–unitality* is realized by the first two additional regular-value moves in Definition 3.5.4.

4. Frame change type relations.

- *No-change* (trivial frame change): the two sides have identical images under B_0 .
- *Composition of frame changes*: the left-hand side maps to the vertical composite of the two Morse-data–preserving diffeomorphisms, while the right-hand side maps to the single diffeomorphism given by their composition. This equality follows directly from how vertical composition of 2-cells is defined in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.
- *Frame change–naturality crossing* is realized by the relation that changes the underlying smooth manifold triad of a 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$.

These cover all relations in $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$, so B_0 is well defined. $\square$

The functor $S_0 : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0} \rightarrow F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$

We now construct the functor

$$S_0 : \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0} \longrightarrow F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0} .$$

We will not verify that S_0 is symmetric monoidal—although we expect this to be the case—since this is unnecessary for our purposes. By contrast, the symmetric monoidal structure of B_0 was automatic from its computadic definition. Our ultimate goal is to show that B_0 is a symmetric monoidal equivalence. To do this, it suffices to prove that B_0 and S_0 are inverse as plain functors, which is precisely what we will establish after constructing S_0 .

To construct S_0 , we must first specify what it assigns to 0-, 1- and 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ and then show that this assignment respects all relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. We have essentially already described the assignment S_0 on the 0-, 1- and 2-cells in Section 2.5, where we compared $\text{Sur}_{(d-1; d)}^{(\text{or})}$ with $\text{Bord}_{(d-1; d)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. For completeness, we now recall and expand on these constructions:

0-cells

S_0 is the identity on 0-cells.

1-cells

Our choice of Morse data was designed to make the definition of S_0 as straightforward as possible. Recall that part of the Morse data is a choice of regular level sets, compatible with the chosen standard Morse model embeddings. This ensures that between any two consecutive regular level sets there is at most one critical point. To define S_0 , it therefore suffices to specify its value on 1-cells whose Morse data contain only two regular level sets, since any 1-cell can be expressed as a composite of such pieces.

Suppose we have two regular level sets X and Y with no critical points between them. The horizontal distribution (which is part of the Morse data), together with the boundary identifications (if non-trivial), induces a diffeomorphism $\phi: X \rightarrow Y$, and S_0 sends this 1-cell to the diffeomorphism-type 1-cell labeled by ϕ . The label of the interval $[f(X), f(Y)]$ (blue, green, or transparent) determines the color of the dot—or its absence—recording the diffeomorphism in the diffeomorphism type 1-cell: if it is blue, we obtain a generating 1-cell with a blue dot; if it is green, we obtain the structure 1-cell β^σ with a single green dot; and if it is transparent, we obtain the empty 1-cell, which is the unit in $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$.

Suppose instead that we have two regular level sets X and Y with a single critical point between them. The standard Morse model embedding, horizontal distribution

(which are part of the Morse data), together with the boundary identifications (if non-trivial), then induce a surgery triple σ on (X, Y) , since the regular level sets are chosen to be compatible with the standard Morse model embeddings. In this case, S_0 sends the 1-cell to the surgery-type generating 1-cell labeled by σ .

This fully specifies S_0 on the 1-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$. Moreover, $S_0 \circ B_0$ acts as the identity on the 1-cells of $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$, which can be verified directly on the generating 1-cells of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$. In particular, this implies that S_0 is full on 1-cells.

2-cells

Recall that a 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ consists of three components: a colored Morse data preserving diffeomorphism, followed by a path of colored Morse data, followed by another colored Morse data preserving diffeomorphism. Moreover, the path of colored Morse data $\gamma: [0, 1] \rightarrow \mathcal{CMD}(W)$ is equipped with cut times $0 = c_0 < c_1 < \dots < c_k = 1$ such that each restriction $\gamma|_{[c_i, c_{i+1}]}$ represents a single edge in the graph $\mathcal{CMD}(W)$: either a smooth isotopy (labeling an isotopy edge) or a piecewise constant path (labeling a regular-value edge), as in Definition 3.5.3. Therefore, to define the assignment of S_0 on 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$, it suffices to specify it on colored Morse data preserving diffeomorphisms, on smooth isotopies of colored Morse data (isotopy edges), and on piecewise constant paths of colored Morse data (regular-value edges).

Morse data preserving diffeomorphisms

Let $(\overline{W}, f, \mathcal{CD}_f)$ and $(\overline{W}', f', \mathcal{CD}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$, and let $D: W \rightarrow W'$ be a colored Morse data preserving diffeomorphism. By definition, such a diffeomorphism preserves the Morse function, the horizontal distributions, and all chosen regular level sets. It also preserves the standard Morse model embeddings up to an action of $G_{(n,k)}^{(\text{or})}$. These constraints are extremely rigid: colored Morse data preserving diffeomorphisms are unique up to isotopy through such diffeomorphisms, and they are entirely determined by a diffeomorphism of the target intervals of f and f' that respects all chosen regular and critical values.

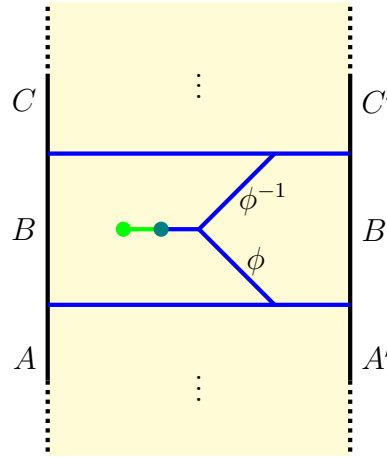
In order to associate a surgery diagram to such a diffeomorphism D , we proceed as follows. First, use the Morse data on the source and target to replace each cobordism by its associated *standard cobordism*—that is, a cobordism lying in the image of B_0 , decomposed as a composite of mapping cylinders and traces of surgeries. The

diffeomorphism D then induces a diffeomorphism between these standard cobordisms. This induced diffeomorphism is completely determined by two pieces of data:

- the diffeomorphisms that D induces on the chosen regular level sets, and
- the actions of $G_{(n,k)}^{(\text{or})}$ that D induces on the standard Morse model embeddings.

Together, these determine a surgery diagram representing the image of D under S_0 .

Concretely, suppose D induces a diffeomorphism ϕ from a chosen regular level set B of (W, f) to the corresponding regular level set B' of (W', f') . Assume the intervals with endpoints at the corresponding regular values are labeled blue. Then, the local part of the surgery diagram around these regular level sets is:



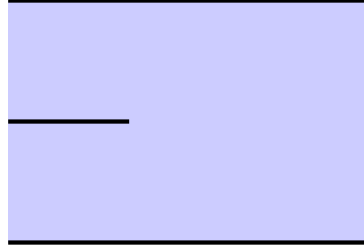
If instead one of the corresponding intervals is labeled red, we obtain the same local picture except that the relevant horizontal lines are red rather than blue. Moreover, such red lines may be further decorated with a black dot recording a frame change, labeled by the corresponding element of $G_{(n,k)}^{(\text{or})}$ uniquely determined by D .

Finally, if the relevant intervals are labeled green or transparent, we first insert a color-swap surgery diagram (symmetry 2-cell or unitor) to turn them blue, then apply one of the diagrams above, and then postcompose with another color-swap diagram to restore the target labels. This completes the description of the surgery diagram associated to a colored Morse data preserving diffeomorphism.

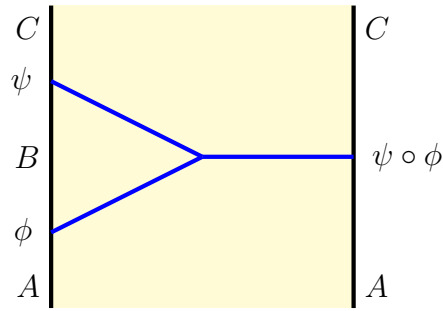
Paths of regular value edges

Let $(\overline{W}, f, \mathcal{CD}_f)$ and $(\overline{W}, f, \mathcal{CD}'_f)$ be two 1-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$ connected by a 2-cell that records a regular value edge in $\mathcal{CMD}(W)$ (i.e. the birth or death of a regular value or label swap). We assign such a 2-cell to a 2-cell in $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ that is either a compositor or a naturality type 2-cell.

For example, suppose the regular value edge corresponds to the picture

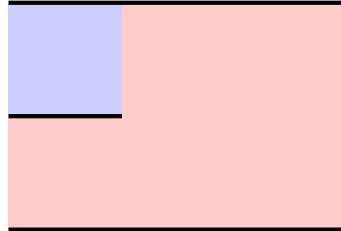


Then, the associated surgery diagram is:

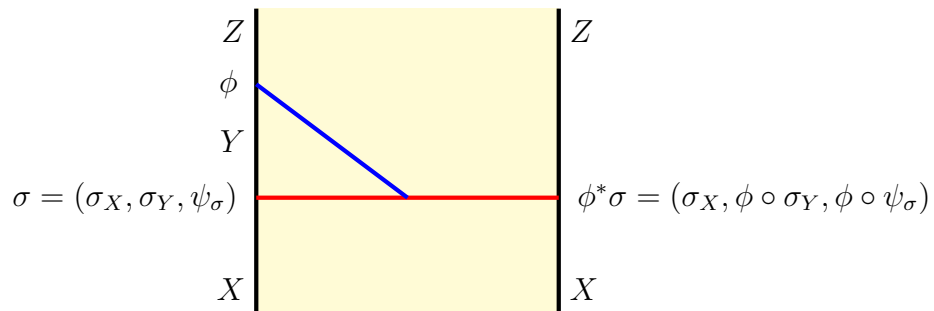


Here, the labels are induced by the underlying horizontal distribution, which stays fixed throughout the path.

Similarly, if the regular value edge corresponds to the picture



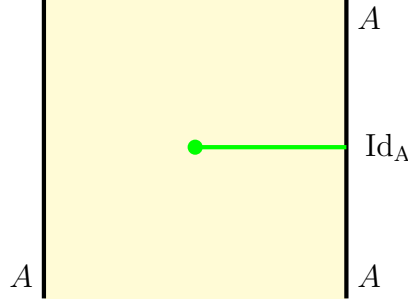
then the associated surgery diagram is:



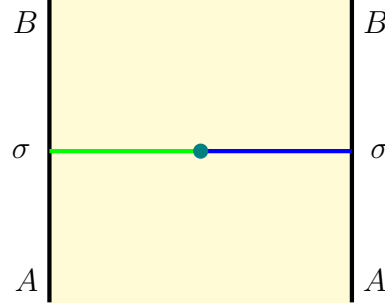
Here, the labels are determined by the underlying horizontal distribution (which again remains fixed) together with the standard Morse model embeddings in the

source and target. The remaining cases of birth/death of regular values are all analogous to the two examples above.

Regarding label swaps, a transparent-to-green label swap is assigned to the unitor 2-cell



and the green-to-blue label swap is assigned to the symmetry 2-cell



This completes the description of the surgery diagrams associated to paths of regular-value edges.

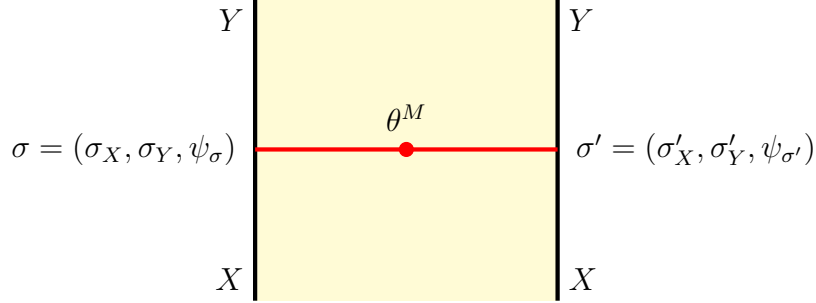
Isotopies of Morse data

Let $(\overline{W}; f; \mathcal{CD}_f)$ and $(\overline{W}; f'; \mathcal{CD}_{f'})$ be two 1-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ connected by an isotopy of Morse data.

We first note that it suffices to consider the case where the underlying path of excellent Morse functions remains constant. Indeed, suppose this is not the case. Let $\gamma: [0, 1] \rightarrow \mathcal{F}^0(W)$ be a path of excellent Morse functions with $\gamma(0) = f$ and $\gamma(1) = f'$. By the stability of Morse functions and Corollary 1.3.7, there exists an isotopy of diffeomorphisms of W inducing the isotopy of Morse functions γ . Pulling back the given isotopy of Morse data along this isotopy of diffeomorphisms produces an isotopy of Morse data on W that keeps the Morse function fixed throughout.

Thus, any isotopy of Morse data is equivalent to an isotopy of Morse data with fixed Morse function, followed by a Morse data preserving diffeomorphism. It therefore suffices to treat the case where the underlying Morse function stays constant.

In this situation, the isotopy of Morse data directly corresponds to the isotopy-of-Morse-data 2-cell in $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$, namely:



(This shows the case with a single critical point; in general, one obtains composites of such 2-cells.) This completes the description of the surgery diagrams associated to isotopies of Morse data, and hence the description of the surgery diagrams that S_0 assigns to the 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$.

Proposition 3.6.2. The assignment S_0 described above induces a well-defined functor

$$S_0: \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0} \longrightarrow F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$$

Proof. We must check that S_0 respects all relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$. For each family of relations, we indicate the corresponding relations in $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$.

1. Reparameterization of the interval I .

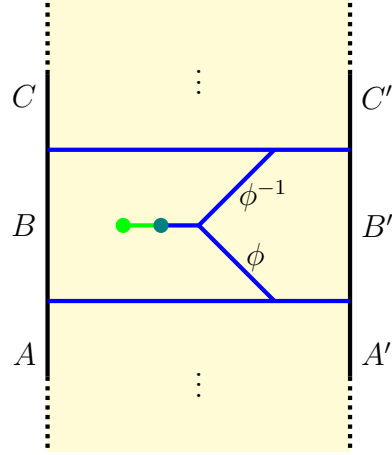
The exact interval is irrelevant in the definition of S_0 , hence this relation is automatically respected.

2. Change of the underlying smooth manifold triad.

Every 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$ is equivalent to one whose first boundary identification is the identity. Let α be such a 2-cell, and let β be an equivalent 2-cell whose underlying smooth manifold triad is different. It suffices to show that the 2-cells assigned by S_0 to α and β are equivalent in $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$.

The surgery diagram for β begins with a diagram encoding the Morse data preserving diffeomorphism appearing as its first boundary identification.

Locally, such a diagram looks like:



When surgery type 1-cells are involved, the diagram takes an analogous form, potentially including frame changes. Using the normal form for diffeomorphism-type 2-cells, together with the isotopy of Morse data - naturality crossing, isotopy of Morse data - frame change crossing, naturality - 4-valence, naturality Reidemeister II, naturality compositor, and frame change naturality crossing relations, one can slide the compositors and frame changes encoding this initial Morse data preserving diffeomorphism across the surgery diagram and compose them with the diagram for the Morse data preserving diffeomorphism given by the final boundary identification. This recovers the surgery diagram for α and shows that this relation is respected.

3. Equivalence of paths of colored Morse data and compatible isotopies of boundary identifications.

- *Isotopy moves between Morse data* as in Definition 1.3.17 are realized by the isotopy invariance, concatenation of isotopies, and isotopy of Morse data (invariance, compatibility, and concatenation) relations
- *Commutation moves* as in Definition 1.3.17 are realized by the isotopy - compositor crossing, isotopy of Morse data - naturality crossing, and isotopy of Morse data-frame change crossing.
- *Regular value moves* as in Definition 1.3.17 are realized as follows:
 - *Sliding-red-region moves* are realized by the naturality 4-valence, and naturality compositor relations.

- *Coincide moves* are realized by the associativity, naturality 4-valence, and naturality - compositor moves.
- *Inverse-cancellation moves* are realized by the invertibility diffeomorphism type, and naturality - Reidemeister II relations.
- *Additional regular value moves* as in Definition 3.5.4 are realized by the unitality, symmetry-compositor crossing, invertibility, and naturality-unitality moves.

4. Frame change.

This is realized by the isotopy of Morse data - frame change crossing relations.

These cover all relations in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$, so B_0 is well defined. □

Proving the equivalence

Proposition 3.6.3. The functors B_0 and S_0 constructed above are equivalences and inverses of one another (as plain functors).

Proof. As observed already in their definitions, these functors are both the identity on 0-cells. Moreover, we have seen that B_0 is essentially full on 1-cells, S_0 is full on 1-cells, and that the composite

$$S_0 \circ B_0$$

is the identity (on the nose) on 1-cells.

It therefore remains to show that B_0 and S_0 are fully faithful on 2-cells and, moreover, that the composite

$$S_0 \circ B_0$$

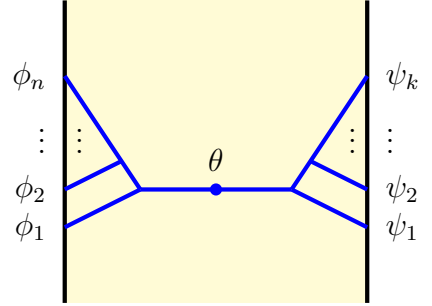
is the identity (on the nose) on 2-cells. By the Whitehead criteria, this will then imply that B_0 and S_0 are inverse equivalences of categories. To establish these facts, it will help to describe a normal form for the 2-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ and $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ so that we can more easily compare them.

Let's begin by describing a normal form for the 2-cells in $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$. There are three cases to consider:

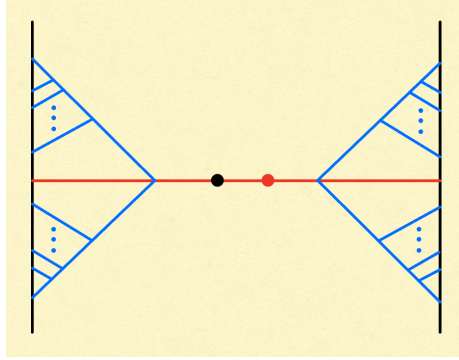
1. *The source and target 1-cells contain no surgery type generating 1-cells.*

We have already seen that in this case a 2-cell between them admits the normal

form



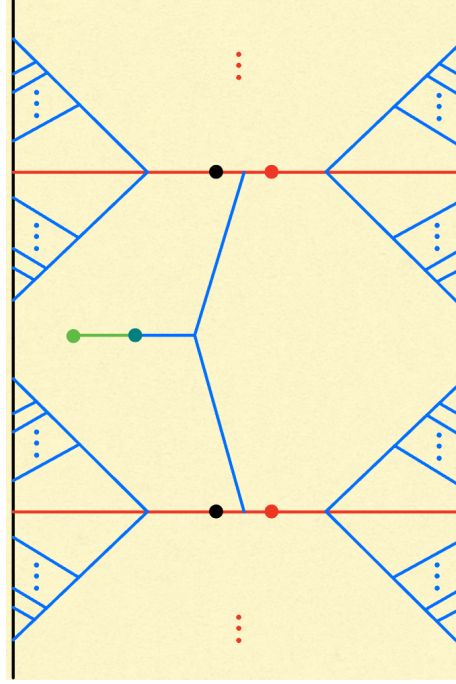
2. *The source and target 1-cells contain a single surgery type generating 1-cell.*
 In this case a 2-cell between them has the normal form



which can be derived using the relations in $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ as follows. First, use the isotopy - compositor crossing relations to arrange that all isotopies occur only in blue segments that connect directly to the red line. Then, apply the compatibility of isotopies relation to rewrite all such blue isotopies as isotopies of Morse data. As in the diffeomorphism type case, use relations to eliminate cycles consisting of blue segments. Next, apply the naturality compositor relations to break any blue line that crosses the diagram without meeting the red line into two pieces that intersect the red line. Use the isotopy of Morse data – naturality crossings and the frame change – naturality crossings to slide blue segments past red and black dots. Apply the naturality – compositor relations again to bring the blue segments into the form shown above. Finally, use the isotopy of Morse data – frame change crossings, concatenation of isotopies of Morse data, and frame change composition relations to obtain the claimed normal form.

3. *The source and target 1-cells each contain a single surgery-type generating 1-cell.*

In this case a 2-cell between them admits the normal form



The argument is analogous to the previous case of a single surgery type generator. The only new feature here is the presence of blue segments connecting two distinct red lines. Using the naturality – compositor relations, together with the isotopy of Morse data – naturality crossings and the frame change – naturality crossings, we can slide and collect these blue segments into the configuration displayed in the normal form above.

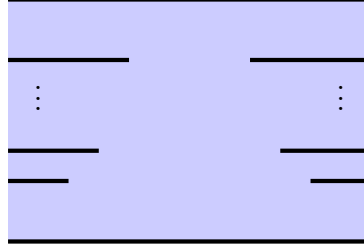
We now describe a normal form for the 2-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$. As noted in the construction of the functor S_0 , the stability of Morse functions implies that every 2-cell in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$ is equivalent to one whose underlying path of Morse functions remains constant. We will therefore assume this throughout when describing normal forms.

There are three cases to consider, directly analogous to the normal forms for 2-cells in $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$:

1. *The source and target 1-cells are cobordisms equipped with Morse functions, each with no critical points.*

Recall that a path of regular value edges with such endpoints is equivalent to

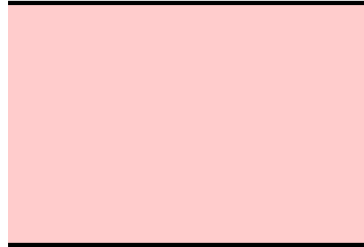
the normal form (see the proof of Lemma 1.3.14)



Using the commutation and isotopy moves of Definition 1.3.17, we may assume that the path of Morse data has a single isotopy edge inserted in the middle of this normal form. The image of the resulting 2-cell under S_0 is exactly the first case of the normal forms for 2-cells in $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$, namely the case where the source and target 1-cells contain no surgery type generating 1-cells.

2. *The source and target 1-cells are cobordisms equipped with Morse functions, each with a single critical point.*

Recall that a path of regular value edges whose endpoints have Morse data consisting of a single red interval and no blue intervals is equivalent to the normal form (see the proof of Lemma 1.3.14):



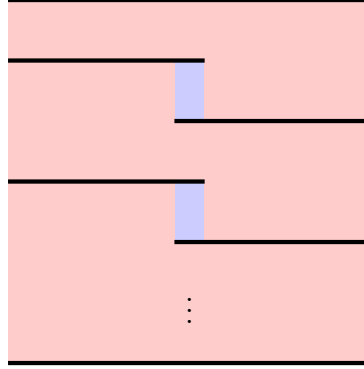
Using the commutation and isotopy moves of Definition 1.3.17, we may assume that our path of Morse data has a single isotopy edge inserted in the middle of this normal form. Using the change of underlying smooth manifold triad relation, we may also insert a potential frame change in the middle.

More generally, the endpoints may also contain blue intervals. In this case, applying the inverse cancellation moves, we obtain the following normal form: first a path of regular values eliminating all blue intervals in the source, then the red normal form described above, and finally a path of regular values creating all blue intervals in the target.

The image of the resulting 2-cell under S_0 is exactly the second case of the normal forms for 2-cells in $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$, namely the case where the source and target 1-cells each contain a single surgery type generating 1-cell.

3. *The source and target 1-cells are cobordisms equipped with Morse functions, each with multiple critical points.*

Recall that a path of regular value edges whose endpoints have Morse data consisting of multiple red intervals and no blue intervals is equivalent to the normal form (see the proof of Lemma 1.3.14):



Using the commutation and isotopy moves of Definition 1.3.17, we may assume that our path of Morse data has a single isotopy edge inserted in the middle of this normal form. Using the change of underlying smooth manifold triad relation, we may also insert a potential frame change in the middle.

More generally, the endpoints may also contain blue intervals. In this case, applying the inverse cancellation moves, we obtain the following normal form: first a path of regular values eliminating all blue intervals in the source, then the red normal form described above, and finally a path of regular values creating all blue intervals in the target.

The image of the resulting 2-cell under S_0 is exactly the third case of the normal forms for 2-cells in $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$, namely the case where the source and target 1-cells each contain multiple surgery type generating 1-cells.

Having described these normal forms, we see that $S_0 \circ B_0$ acts as the identity on 2-cells of $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ in normal form, and hence on all 2-cells. We have already seen that $S_0 \circ B_0$ is the identity on 0- and 1-cells, so altogether we obtain $S_0 \circ B_0 = \text{Id}$.

We now show that the functors S_0 and B_0 are fully faithful on 2-cells:

1. *Fullness:*

The fullness of S_0 follows immediately from the fact that $S_0 \circ B_0 = \text{Id}$. For the fullness of B_0 , let W_0 and W_1 be two 1-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$ lying in the image of S_0 , and let α be a 2-cell between them in normal form. The Morse data of W_0 and W_1 determine the path of regular values that forms part of

α . Thus, the remaining data of α consist of the isotopy edges, the Morse data preserving diffeomorphisms (possibly contributing a frame flip in the middle), and any blue squares preceding the isotopies. Each of these three pieces can be directly encoded in a surgery diagram in normal form, whose image under B_0 recovers α on the nose by definition.

2. Faithfulness:

The faithfulness of B_0 follows immediately from the fact that $S_0 \circ B_0 = \text{Id}$. For the faithfulness of S_0 , let α and β be two 2-cells of $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ in normal form with the same source and target. The Morse data on the source and target will determine the path of regular values that forms part of α and β . The rest of the data are the isotopy edges, the Morse data preserving diffeomorphisms (possibly contributing a frame flip in the middle), and any blue squares preceding the isotopies. If the rest of the data define 2-cells in normal form $S_0(\alpha)$ and $S_0(\beta)$ which are equivalent in $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$, they must be related via an isotopy of Morse data - frame change crossing, which induces a relation in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$, hence α and β are equivalent.

This completes the proof that the functors B_0 and S_0 are equivalences and inverses of one another (as plain functors). □

Corollary 3.6.4. *The functor $B_0 : F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0} \rightarrow \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^0}$ is a symmetric monoidal equivalence.*

3.6.2 The equivalence between $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ and $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^1}$

The functors S_0 and B_0 extend to functors

$$F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1} \xrightleftharpoons[S_1]{B_1} \text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^1}$$

This follows from the observation that the cusp generating 2-cells in $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ encode precisely the elementary birth-death paths, while the crossing generating 2-cells encode precisely the elementary crossing paths. In particular, this specifies the assignments of B_1 and S_1 on the new 2-cells appearing in $F\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ and $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data, (or), } \mathcal{F}^1}$.

Proposition 3.6.4. *The assignments B_1 and S_1 induce well-defined functors.*

Proof. We must check that B_1 and S_1 respect the additional relations used to pass from $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^0}$ to $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ and from $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^0}$ to $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1}$.

The two corresponding sets of extra relations were chosen so as to match directly:

1. cusp–isotopy relations correspond to homotopies of elementary birth–death paths,
2. crossing–isotopy relations correspond to homotopies of elementary crossing paths,
3. eye, merge–unmerge, and Reidemeister II relations correspond to the respective homotopies,
4. naturality–beak, naturality–Reidemeister III, frame change–cusp, and frame change–crossing relations correspond to changes of the underlying smooth manifold triad in an elementary birth–death path or crossing path,
5. naive Cerf homotopies correspond to relations that hold automatically in any unbiased semistrict symmetric monoidal 2-category with respect to the order of vertical composition.

This establishes that B_1 and S_1 respect all additional relations and hence are well defined. $\square$

Proposition 3.6.5. The functors B_1 and S_1 constructed above are equivalences and inverses of one another (as plain functors).

Proof. By construction, the functor S_1 is full on cusp and crossing generating 2-cells, while B_1 is full on elementary birth–death and crossing paths. Together with the fullness of S_0 and B_0 , this implies that S_1 and B_1 are full on all 2-cells.

Moreover, $S_1 \circ B_1$ acts as the identity on cusp and crossing generating 2-cells. Since we already know that $S_0 \circ B_0 = \text{Id}$, it follows that $S_1 \circ B_1 = \text{Id}$.

Faithfulness of S_1 and B_1 on 2-cells follows from a normal form argument, entirely analogous to the one used for S_0 and B_0 . In particular, the computadic definition of $F \text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ provides a normal form for its 2-cells, which is compatible with the normal form for 2-cells in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1}$, where 2-cells decompose into composites of paths of excellent Morse functions, elementary birth–death paths, and elementary crossing paths, all equipped with their associated Cerf data.

Finally, the precise correspondence between the additional relations introduced in $\text{Sur}_{(d-1; d; \sim)}^{(\text{or}), \mathcal{F}^1}$ and those introduced in $\text{Bord}_{(d-1; d; \sim)}^{\text{Cerf data}, (\text{or}), \mathcal{F}^1}$, as described in Proposition 3.6.4, together with these normal forms, completes the proof. $\square$

Corollary 3.6.5. *The functor $B_1 : F\mathrm{Sur}_{(d-1; d; \sim)}^{(\mathrm{or}), \mathcal{F}^1} \rightarrow \mathrm{Bord}_{(d-1; d; \sim)}^{\mathrm{Cerf\ data, (or), \mathcal{F}^1}}$ is a symmetric monoidal equivalence.*

3.6.3 The equivalence between $F\mathrm{Sur}_{(d-1; d; \sim)}^{(\mathrm{or})}$ and $\mathrm{Bord}_{(d-1; d; \sim)}^{\mathrm{Cerf\ data, (or)}}$

$\mathrm{Bord}_{(d-1; d; \sim)}^{\mathrm{Cerf\ data, (or)}}$ is obtained as the quotient of $\mathrm{Bord}_{(d-1; d; \sim)}^{\mathrm{Cerf\ data, (or), \mathcal{F}^1}}$ by a set of relations $\mathcal{R}$ corresponding to the Cerf homotopies that cross the codimension-2 strata exactly once trasversally (beak, Reidemeister III, and swallowtail homotopies).

$F\mathrm{Sur}_{(d-1; d; \sim)}^{(\mathrm{or})}$ is obtained as the quotient of $F\mathrm{Sur}_{(d-1; d; \sim)}^{(\mathrm{or}), \mathcal{F}^1}$ by the set of relations that constitute the pullback of $\mathcal{R}$ by B_1 (or equivalently the push-forward by S_1).

We immediately conclude that the functors S_1 and B_1 can be extended to functors

$$F\mathrm{Sur}_{(d-1; d; \sim)}^{(\mathrm{or})} \xrightleftharpoons[S]{B} \mathrm{Bord}_{(d-1; d; \sim)}^{\mathrm{Cerf\ data, (or)}}$$

which are equivalences and, in fact, inverses of each other.

This concludes the proof of our main result:

Theorem 3.6.6. *The symmetric monoidal bicategories $F\mathrm{Sur}_{(d-1; d; \sim)}^{(\mathrm{or})}$ and $\mathrm{Bord}_{(d-1; d; \sim)}^{(\mathrm{or})}$ are equivalent.*

Part II

From surgery presentations to finite presentations

Chapter 4

Skeletal 2-presentations

The purpose of this chapter is to define what it means for a presentation of an unbiased semistrict symmetric monoidal 2-category to be *skeletal*, and to describe a general procedure for skeletalizing such presentations. We begin by recalling and establishing some foundational definitions.

4.1 Definition

We first start with some preliminary definitions. We will use the wire diagram calculus introduced in Section 3.1 throughout.

Definition 4.1.1 (Invertible 2-cells). A 2-cell

$$\begin{array}{ccc} y & & y \\ | & & | \\ \boxed{f} & \xRightarrow{\alpha} & \boxed{g} \\ | & & | \\ x & & x \end{array}$$

in a 2-category $\mathcal{C}$ is called **invertible** if there exists a 2-cell

$$\begin{array}{ccc} y & & y \\ | & & | \\ \boxed{g} & \xRightarrow{\alpha^{-1}} & \boxed{f} \\ | & & | \\ x & & x \end{array}$$

in $\mathcal{C}$ such that $\alpha \circ \alpha^{-1} = \text{id}_g$ and $\alpha^{-1} \circ \alpha = \text{id}_f$. Such a 2-cell α^{-1} will be called an **inverse** of α .

Definition 4.1.2 (Invertible 1-cells). A 1-cell $f : x \rightarrow y$ in a 2-category $\mathcal{C}$ is called **invertible** if there exists a 1-cell $f^{-1} : y \rightarrow x$ in $\mathcal{C}$ and invertible 2-cells $\iota_f : f^{-1} \circ f \rightarrow \text{id}_x$, $\tilde{\iota}_f : f \circ f^{-1} \rightarrow \text{id}_y$.

We will now define some equivalence relations on the collection of 1-cells of a 2-category $\mathcal{C}$.

Definition 4.1.3. Given two 1-cells $f, g : x \Rightarrow y$ in $\mathcal{C}$, we say that f is **isomorphic** to g ($f \sim g$) if there exists an invertible 2-cell $\alpha : f \Rightarrow g$.

A more general equivalence relation is given by weak conjugation:

Definition 4.1.4 (Weak conjugation). Let V be a collection of invertible 1-cells in a 2-category $\mathcal{C}$, closed under composites and inverses up to isomorphism, that is:

- for each $v \in V$, there exists an inverse v^{-1} of v in V ,
- for each composable $v_0, v_1 \in V$, there exists $v \in V$ such that $v \sim v_0 \circ v_1$.

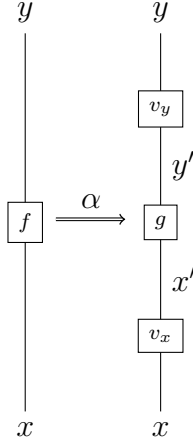
Given two 1-cells

$$\begin{array}{c} y \\ | \\ \boxed{f} \\ | \\ x \end{array} \quad , \quad \begin{array}{c} y' \\ | \\ \boxed{g} \\ | \\ x' \end{array}$$

in $\mathcal{C}$, we say that f is **weakly conjugate to g via V** ($f \sim_V g$) if there exist 1-cells

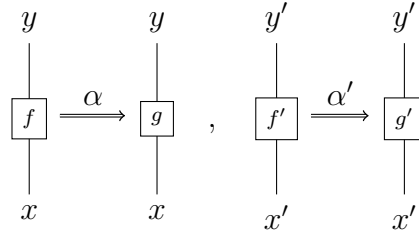
$$\begin{array}{c} x' \\ | \\ \boxed{v_x} \\ | \\ x \end{array} \quad , \quad \begin{array}{c} y \\ | \\ \boxed{v_y} \\ | \\ y' \end{array}$$

in V , and an invertible 2-cell



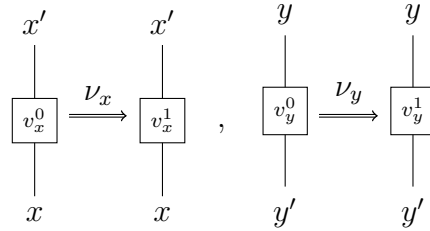
We leave it as an exercise to show that weak conjugation is indeed an equivalence relation. We will now define the analogous notion for 2-cells:

Definition 4.1.5 (Biconjugation). Let $\mathcal{V}$ and $\mathcal{H}$ be two collections of invertible 2-cells in a 2-category $\mathcal{C}$, closed under composites and inverses. Given two 2-cells



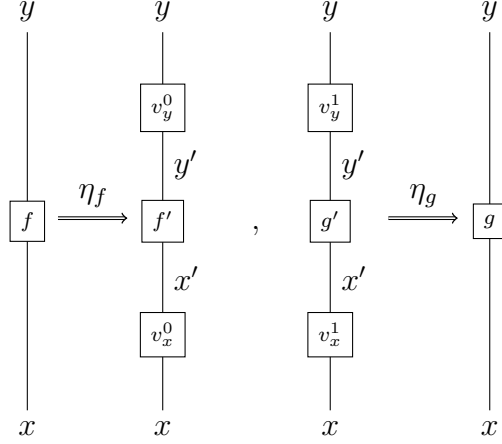
in $\mathcal{C}$, we say that α is *biconjugate* to α' via $(\mathcal{V}, \mathcal{H})$ ($\alpha \sim_{\mathcal{H}} \alpha'$) if there exist:

- invertible 2-cells between invertible 1-cells



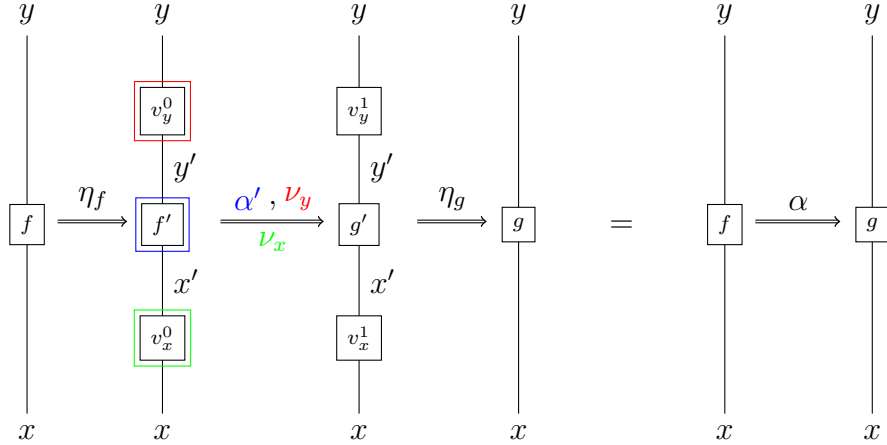
in $\mathcal{H}$,

- invertible 2-cells



in $\mathcal{V}$,

such that



We once again leave it as an exercise to show that biconjugation is indeed an equivalence relation. We will next explain what we mean by skeletal presentations of unbiased semistrict symmetric monoidal 2-categories after first recalling some “elementary” definitions mostly appearing in Section 3.1:

Definition 4.1.6. Let $\mathcal{P} = (G^0, G^1, G^2; R)$ be a presentation of an unbiased semistrict symmetric monoidal 2-category.

The *elementary* 1-cells in $F\mathcal{P}$ are of two kinds:

- 1-cells of the form $\beta^\sigma : w \mapsto w'$. Here, $w = (w_i)$ is a word in G^0 of length n , $\sigma \in \Sigma_n$ is a permutation, and $w' = (w_{\sigma^{-1}(i)})$ is the result of applying this permutation to w . We let Σ denote the collection of all elementary 1-cells of this kind.

- 1-cells of the form $\text{id}_w \otimes f \otimes \text{id}_{w'}$, where $f \in G^1$ and w, w' are words in G^0 (i.e., objects in $F\mathcal{P}$). We will say that such an elementary 1-cell is $\otimes$ -generated by f .

Remark 4.1.1. The 1-cells in Σ depend only on G^0 , and not on G^1 .

The *elementary 2-cells* in $F\mathcal{P}$ are of three kinds:

- The structure 2-cells X_e and $X_{\sigma, \sigma'}$ (for all $\sigma, \sigma' \in \Sigma_n$), together with their inverses. We let $\mathcal{S}$ denote the collection of all elementary 2-cells of this kind.
- The coherence 2-cells $\beta_{(f_i)}^\sigma$ (for all $\sigma \in \Sigma_n$ and $f_i \in S(G^0, G^1, s, t)$), the interchangor 2-cells $\phi_{f,g}$ (for all $f, g \in S(G^0, G^1, s, t)$), and their inverses. We let $\mathcal{J}$ denote the collection of all elementary 2-cells of this kind.
- 2-cells of the form $\text{id}_f \circ (\text{id}_{\text{id}_w} \otimes \alpha \otimes \text{id}_{\text{id}_{w'}}) \circ \text{id}_g$, where $\alpha \in G^2$ and f, g are 1-cells in $F\mathcal{P}$. We will say that such an elementary 2-cell is $\otimes$ -generated by α .

Remark 4.1.2. The 2-cells in $\mathcal{S}$ depend only on G^0 , and not on G^1 or G^2 . The 2-cells in $\mathcal{J}$ depend only on G^0 and G^1 , and not on G^2 .

Definition 4.1.7 (0-skeletal presentations). A presentation $\mathcal{P} = (G^0, G^1, G^2; R)$ of an unbiased semistrict symmetric monoidal 2-category is called **0-skeletal** if it satisfies the following condition: for every pair of *distinct generating* 0-cells x, y in G^0 , there exist no invertible 1-cells $f : x \rightarrow y$ in $F\mathcal{P}$.

Remark 4.1.3. Note that generating 0-cells are still allowed to have non-trivial automorphisms in $F\mathcal{P}$.

Example 4.1.4. Consider $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$. This presentation is *not* 0-skeletal, as we can find two distinct diffeomorphic $(d-1)$ -dimensional manifolds with invertible 1-cells between them. However, we can construct a 0-skeletal presentation as follows:

- Choose a representative $(d-1)$ -dimensional manifold from each diffeomorphism class (using the axiom of choice)
- Construct the full sub-presentation spanned by these chosen manifolds, namely the sub-presentation that has:
 0. Generating 0-cells: the chosen representative $(d-1)$ -dimensional manifolds
 1. Generating 1-cells: all the generating 1-cells of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ with source and target generated by the generating 0-cells chosen in the previous step

2. Generating 2-cells: all the generating 2-cells of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ with source and target generated by the generating 1-cells chosen in the previous step
3. Generating relations: all the generating relations of $\text{Sur}_{(d-1; d; \sim)}^{(\text{or})}$ with source and target generated by the generating 2-cells chosen in the previous step

This sub-presentation is 0-skeletal and generates an equivalent unbiased semistrict symmetric monoidal 2-category. This equivalence relies crucially on our choice to work with symmetric surgery data; in general, the process of 0-skeletalization requires more careful consideration.

Remark 4.1.5. The unbiased semistrict symmetric monoidal 2-category $F\mathcal{P}$ generated by a 0-skeletal presentation $\mathcal{P}$ is not, generally, 0-skeletal itself.

To define 1-skeletal presentations, we first establish a preorder on the set of initial generating 1-cells.

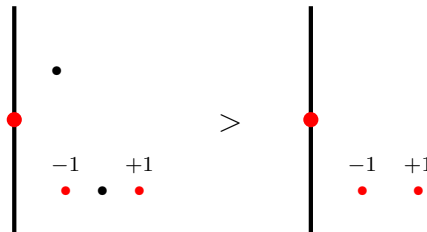
Definition 4.1.8. Let f, g be initial generating 1-cells in G_i^1 . We say $f > g$ if there exists a 1-cell h in $F\mathcal{P}_i$ such that:

- h is $\otimes$ -generated by g , and
- h is weakly conjugate to f via Σ .

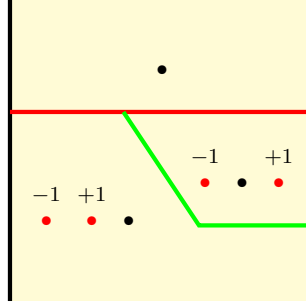
In this case, we say that f is **weakly permutogenerated** by g . If h is already equivalent to f without any conjugation via Σ , we say that f is **weakly $\otimes$ -generated** by g .

Definition 4.1.9 (1-skeletal presentations). A presentation $\mathcal{P} = (G^0, G^1, G^2; R)$ of an unbiased semistrict symmetric monoidal 2-category is called **1-skeletal** if it satisfies the following condition: there is no pair of distinct generating 1-cells $f, g \in G^1$ such that f is weakly permutogenerated by g .

Example 4.1.6. The presentation $\text{Sur}_{01\sim}$ is not 1-skeletal, since



as witnessed by the existence of the following invertible 2-cell:



4.2 Skeletalization procedure

Given a 0-skeletal presentation, our aim in this section is to construct an equivalent 1-skeletal presentation through a sequence of well-defined, though non-canonical, steps. While there is a useful analogy between skeletal categories and skeletal presentations, the reader should be aware of the following caveats:

Warning. It is not always possible to construct a skeletalization of an arbitrary presentation without imposing finiteness assumptions. For example, suppose a presentation has generating 1-cells $f_1, f_2, \dots$ such that for all i , $f_i > f_{i+1}$, and $f_i \not\asymp f_j$ for any $j < i$. In this situation, one cannot find a subset of generating 1-cells containing no pair g, h with $g > h$ and also satisfying the condition that every generating 1-cell r is weakly permutogenerated by some g in the subset. Both conditions are necessary for producing a 1-skeletal presentation that still generates an equivalent category. Thus, the procedure we describe cannot always be carried out. However, whenever it can be carried out, it yields a 1-skeletal presentation that generates a category equivalent to the one generated by the original presentation. In the examples we consider, the procedure is applicable, and is, therefore, sufficient for our purposes.

Moreover, the analogy with skeletal categories has an important limitation. Skeletalizations of categories always exist and are unique up to isomorphism. By contrast, skeletalizations of presentations, when they exist, are not unique up to isomorphism. By definition, however, they always generate equivalent (though not necessarily isomorphic) categories.

Let $\mathcal{P}_i = (G_i^0, G_i^1, G_i^2; R_i)$ be a 0-skeletal presentation, which we call the *initial presentation*. Here:

- (G_i^0, G_i^1, G_i^2) forms an unbiased semistrict symmetric monoidal 2-computad, denoted G_i .
- G_i^k denotes the collection of all *initial generating k -cells*. The k -cells of $\mathcal{F}(\mathcal{P}_i)$ will be referred to as *initial k -cells*.
- R_i denotes the collection of all *initial generating relations*. The relations generated by R_i will be referred to as *initial relations*.

We will construct, if possible, a 1-skeletal presentation $\mathcal{P}_f = (G_f^0, G_f^1, G_f^2; R_f)$, called the *final presentation*, such that $F\mathcal{P}_i$ and $F\mathcal{P}_f$ are equivalent as unbiased semistrict symmetric monoidal 2-categories. Similarly, we denote by G_f the computad (G_f^0, G_f^1, G_f^2) , and by G_f^k and R_f the collections of final generating k -cells and relations respectively. Final k -cells and relations are defined analogously.

The skeletalization procedure outlined below is general but non-canonical: different choices made along the way may yield distinct—yet equivalent—final presentations. For concrete understanding, the reader is encouraged to follow this procedure alongside its application in Section 5.1.1, where we apply it to $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$, the simplest presentation considered in this thesis.

Our construction proceeds in four major stages:

0. Construction of the underlying 0-computad (G_f^0)
1. Construction of the underlying 1-computad (G_f^1)
2. Construction of the underlying 2-computad (G_f^2)
3. Construction of the final relations (R_f)

4.2.1 Stage 0: Final generating 0-cells

As the initial presentation $\mathcal{P}_i$ is already 0-skeletal, we simply set $G_f^0 := G_i^0$.

4.2.2 Stage 1: Final generating 1-cells

We choose $G_f^1 \subseteq G_i^1$ to satisfy the following two conditions:

1. **(Completeness)** Every initial generating 1-cell in G_i^1 is weakly permutogenerated by some element of $G_f^1 \cup \Sigma$.
2. **(Minimality)** No two distinct elements $f, g \in G_f^1$ satisfy $f > g$.

The subset G_f^1 thus forms a collection of minimal elements (“roots”) with respect to the preorder defined above, and will serve as the chosen set of final generating 1-cells. For a concrete example of Stage 1, see Subsection 5.1.1.

We emphasize again that such a choice is not always possible; if it fails, the procedure terminates at this stage. For the examples we consider, however, the choice can be made, and we then proceed to the next stage.

4.2.3 Stage 2: Final generating 2-cells

The selection of final generating 2-cells G_f^2 is more intricate than the previous steps and requires careful consideration. We proceed in three phases: first choosing auxiliary “pointing” 2-cells, then using these to define two types of final generating 2-cells that together will form G_f^2 .

- Phase 2.1: Selection of pointing 2-cells

For each initial generating 1-cell $f: x \Rightarrow y$ in G_i^1 , we select a collection of invertible 2-cells

$$\begin{array}{ccc} y & & y \\ | & & | \\ \boxed{f} & \xRightarrow{\eta_j^f} & \boxed{g_j} \\ | & & | \\ x & & x \end{array}$$

in $F\mathcal{P}_i$, where each $g_j: x \Rightarrow y$ is a final (*not necessarily generating*) 1-cell in $F\mathcal{P}_f$. In the special case where f is already a final generating 1-cell, we include the identity 2-cell on f among the η_j^f . We refer to all such η_j^f as *pointing 2-cells*, and denote their collection by $\vec{P}$.

Remark 4.2.1. Although the presentation $\mathcal{P}_f$ has not yet been fully specified, our prior choices of the final generating 0- and 1-cells (G_f^0 and G_f^1) already determine all 1-cells in $F\mathcal{P}_f$, independently of the choices still to be made for G_f^2 and R_f . We may therefore refer unambiguously to these final 1-cells already.

For each chosen η_j^f , we also select its inverse $(\eta_j^f)^{-1}$. Let $\overleftrightarrow{P}$ denote the collection of all such 2-cells in $F\mathcal{P}_i$.

We then define the set $\mathcal{V}$ to be the closure of $\overleftrightarrow{P} \cup (\mathcal{S} \cup \mathcal{I}_i)$ under horizontal and vertical composition and under $\otimes$ -generation. Here, $\mathcal{I}_i$ denotes the collection of elementary 2-cells of the second kind in $F\mathcal{P}_i$, as per Definition 4.1.6.

Hence, $\mathcal{V}$ consists of all paragraphs in $\overleftrightarrow{P}$, according to Definition 3.1.5. We refer to any 2-cell in $\mathcal{V}$ as a *generalized pointing 2-cell*.

Remark 4.2.2. While pointing 2-cells do not appear directly in the final skeletal presentation, they serve a crucial role in the selection of both final generating 2-cells and relations. Their importance lies in their ability to connect initial and final 1-cells in a controlled manner.

Remark 4.2.3. From this point onward, we will almost exclusively work with generalized pointing 2-cells. For brevity, we will write simply “pointing 2-cells” to mean *generalized* pointing 2-cells. Whenever we wish to refer to pointing 2-cells that are *not* generalized pointing 2-cells, we will explicitly call them (non-generalized) pointing 2-cells.

We now introduce a definition that will be useful in the next two phases of this stage of the skeletalization procedure:

Definition 4.2.1. A 2-cell in $F\mathcal{P}_i$ is called **1-final** if both its source and target are final 1-cells.

Consider an initial 2-cell of the form

$$\begin{array}{ccc} y & & y \\ | & & | \\ \boxed{f} & \xRightarrow{\alpha} & \boxed{g} \\ | & & | \\ x & & x \end{array}$$

where f and g are not necessarily final 1-cells.

Our choice of pointing 2-cells ensures that there exist such 2-cells γ and δ for which the composite 2-cell

$$\begin{array}{ccccccc} y & & y & & y & & y \\ | & & | & & | & & | \\ \boxed{\tilde{f}} & \xRightarrow{\gamma^{-1}} & \boxed{f} & \xRightarrow{\alpha} & \boxed{g} & \xRightarrow{\delta} & \boxed{\tilde{g}} \\ | & & | & & | & & | \\ x & & x & & x & & x \end{array}$$

is 1-final.

Any such 1-final 2-cell obtained from α is called a **1-finalization** of α .

- Phase 2.2: Selection of type I final generating 2-cells

We now select a set of final generating 2-cells of type I, based on the collection of pointing 2-cells $\mathcal{V}$. Consider the subset $\mathcal{V}_1 \subset \mathcal{V}$ consisting of all 1-final pointing 2-cells. We select a subset ${}_I G_f^2 \subset \mathcal{V}_1$ satisfying the following condition:

Every 2-cell in $\mathcal{V}_1$ can be expressed, up to R_i , as a composite of 2-cells $\otimes$ -generated by elements of ${}_I G_f^2 \cup \mathcal{S} \cup \mathcal{J}_f$.

Here, $\mathcal{J}_f$ denotes the collection of elementary 2-cells of the second kind in $F\mathcal{P}_f$, as per Definition 4.1.6. Since $\mathcal{J}_f$ depends only on the underlying 1-computad of $\mathcal{P}_f$, we can already refer to it unambiguously at this stage.

The 2-cells in ${}_I G_f^2$ constitute our chosen *final generating 2-cells of type I*. The composites of 2-cells $\otimes$ -generated by elements of ${}_I G_f^2 \cup \mathcal{S} \cup \mathcal{J}_f$ —that is, the paragraphs in ${}_I G_f^2$, according to Definition 3.1.5—will be referred to as the *final 2-cells of type I*.

Remark 4.2.4. We do not impose the stronger condition that no two 2-cells in ${}_I G_f^2$ be equivalent via R_i . This flexibility may prove useful in practical applications.

- Phase 2.3: Selection of type II final generating 2-cells

We now select a set of final generating 2-cells of type II, based on the collection of initial generating 2-cells that are *not* in $\mathcal{V}$. To do so, we first impose a preorder on G_i^2 using our chosen pointing 2-cells.

Definition 4.2.2. Let α, β be initial generating 2-cells in G_i^2 . We write $\alpha > \beta$ if there exists a 2-cell γ in $F\mathcal{P}_i$ such that:

- γ is $\otimes$ -generated by β , and
- γ is biconjugate to α via $(\mathcal{V}, \mathcal{S})$.

When $\alpha > \beta$, we say that α is $(\mathcal{V}, \mathcal{S})$ -**generated** by β .

We select a subset $A_f^2 \subset G_i^2$, excluding the initial pointing 2-cells, such that A_f^2 forms a collection of *roots* for this preorder. That is, every initial generating non-pointing 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated by some 2-cell in $A_f^2 \cup \mathcal{J}_f$, up to R_i . We do not require that distinct $\alpha, \beta \in A_f^2$ satisfy $\alpha \not> \beta$.

Since our final generating 2-cells must have both source and target being final 1-cells, we next select a 1-finalization for each 2-cell in A_f^2 . We denote by ${}_{II} G_f^2$

the resulting collection of 1-finalizations, which constitutes our chosen set of final generating 2-cells of type II.

We complete the construction of the underlying final 2-computad by setting $G_f^2 := {}_I G_f^2 \cup {}_{II} G_f^2$. For a concrete example of Stage 2, see Subsection 5.1.1.

Remark 4.2.5. Although one might initially hope that ${}_{II} G_f^2$ alone would suffice as the complete set of final generating 2-cells, this is generally not the case. In particular, we must account for final generating 2-cells that arise from pointing 2-cells. For example, if all initial generating 2-cells were invertible and chosen as pointing 2-cells, then ${}_{II} G_f^2$ would be empty, yielding a final presentation that is not generally equivalent to the initial one.

One might attempt to resolve this by imposing additional restrictions on the choice of pointing 2-cells—e.g., requiring uniqueness of a pointing 2-cell for each initial generating 1-cell. While such a restriction could in principle allow ${}_{II} G_f^2$ to suffice, it is often impractical to implement in concrete examples. Our choice to explicitly include final generating 2-cells of type I provides a simple and robust alternative.

4.2.4 Stage 3: Final generating relations

The selection of final generating relations is the most complicated to describe and requires a carefully structured approach that builds upon our previous work with 2-cells. We proceed in three phases. In Phase 3.1, we select pointing relations, which play the same conceptual role for relations as pointing 2-cells did for 2-cells. These pointing relations will allow us to express (up to equivalence) arbitrary 2-cells in terms of final generating 2-cells of type I or type II.

- Phase 3.1: Selection of pointing relations

We begin by choosing specific collections of initial and final relations that will serve as pointing relations, analogous to the pointing 2-cells introduced in Stage 2 above. We distinguish two types of pointing relations:

1. Initial and final pointing relations of type I

Our choice of final generating 2-cells of type I ensures that every initial 1-final pointing 2-cell is equivalent, up to R_i , to some final 2-cell of type I. We would like this final 2-cell to be determined in a suitably unique way. Based on this, we select two collections of relations:

- ${}_IT_i \subset R_i$: initial pointing relations of type I. These are relations between pointing 2-cells, which include all invertibility relations between pointing 2-cells
- ${}_IT_f \subset {}_IT_i$: final pointing relations of type I between final 2-cells of type I

These collections must satisfy the following two properties:

- Every 1-final pointing 2-cell in $F\mathcal{P}_i$ is equivalent, up to ${}_IT_i$, to a final 2-cell of type I.
- Given any two final 2-cells α and β of type I, if α is equivalent to β up to ${}_IT_i$, then α is equivalent to β up to ${}_IT_f$.

2. Initial and final pointing relations of type II

Our selection of final generating 2-cells of type II ensures that each initial generating non-pointing 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated by a final generating 2-cell of type II or a 2-cell in $\mathcal{J}_f$, up to R_i . We now select:

- ${}_{II}T_i \subset R_i$: initial pointing relations of type II. These are relations between 2-cells that are $(\mathcal{V}, \mathcal{S})$ -generated by initial generating non-pointing 2-cells
- ${}_{II}T_f \subset {}_{II}T_i$: final pointing relations of type II between final 2-cells

These must satisfy the following defining properties:

- Every initial generating non-pointing 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated, up to $T_i := {}_IT_i \cup {}_{II}T_i$, by a final generating 2-cell of type II or a 2-cell in $\mathcal{J}_f$.
- Given any two final 2-cells α and β , which are $(\mathcal{V}, \mathcal{S})$ -generated by some final generating 2-cell of type II or a 2-cell in $\mathcal{J}_f$, if α and β are equivalent up to T_i , then α and β are equivalent up to $T_f := {}_IT_f \cup {}_{II}T_f$.

• Phase 3.2: Selection of good 1-finalizations

For each initial generating 2-cell α , we now select a 1-finalization β that is $\otimes$ -generated by a final generating 2-cell or a 2-cell in $\mathcal{J}_f$, up to conjugation by $\mathcal{S}$ and up to T_i . For each chosen 1-finalization, we also select a corresponding such final 2-cell. Moreover, we require that the collection of all these chosen final 2-cells includes all final generating 2-cells.

Our choices in stage 2 ensure that this selection is possible. We refer to these chosen 1-finalizations as *good*.

Let

$$\mathcal{P}_i^T := (G_i^0, G_i^1, G_i^2; T_i),$$

$$\mathcal{P}_f^T := (G_f^0, G_f^1, G_f^2; T_f)$$

and

$$\mathcal{P}_{\text{good}} := (G_{\text{good}}^0, G_{\text{good}}^1, G_{\text{good}}^2; R_{\text{good}}),$$

where:

- $G_{\text{good}}^0 = G_i^0 = G_f^0$,
- $G_{\text{good}}^1 = G_f^1$,
- G_{good}^2 consists of all final 2-cells that are equivalent, up to T_i to a good 1-finalization,
- $R_{\text{good}} = T_f$

Proposition 4.2.1. The inclusion functor

$$I_i^g : F\mathcal{P}_{\text{good}} \rightarrow F\mathcal{P}_i^T$$

is an equivalence of unbiased semistrict symmetric monoidal 2-categories.

Proof. We will apply the Whitehead criteria:

- I_i^g is the identity, and hence essentially surjective, on 0-cells
- We chose the final generating 1-cells so that I_i^g is full on 1-cells
- Let

$$\left| \begin{array}{c} \xrightarrow{\alpha} \end{array} \right|$$

be an initial generating 2-cell in $F\mathcal{P}_i^T$ with final source and target. Here, we have omitted the labels from the 1-cells for readability. We will show that α is in the image of I_i^g . Since α is an initial generating 2-cell, there exists a good 1-finalization β given by

$$\left| \begin{array}{c} \xrightarrow{\gamma^{-1}} \end{array} \right| \left| \begin{array}{c} \xrightarrow{\alpha} \end{array} \right| \left| \begin{array}{c} \xrightarrow{\delta} \end{array} \right|$$

where γ and δ are pointing 2-cells. Recall that ${}_IT_i$ includes all invertibility relations for pointing 2-cells, hence α is equivalent, up to T_i , to:

$$\left| \begin{array}{c} \xrightarrow{\gamma} \end{array} \right| \left| \begin{array}{c} \xrightarrow{\gamma^{-1}} \end{array} \right| \left| \begin{array}{c} \xrightarrow{\alpha} \end{array} \right| \left| \begin{array}{c} \xrightarrow{\delta} \end{array} \right| \left| \begin{array}{c} \xrightarrow{\delta^{-1}} \end{array} \right|$$

Here, γ is a pointing 2-cell with final source and target, hence it is equivalent, up to T_i , to a final 2-cell of type I. Since good 1-finalizations include all final generating 2-cells of type I, γ is in the image of I_i^g . The same holds true for δ^{-1} . Lastly, $\gamma^{-1} \circ \alpha \circ \delta$ is a good 1-finalization, hence it lies in the image of I_i^g . We thus conclude that α is in the image of I_i^g . A straightforward generalization of this argument shows that I_i^g is full on 2-cells.

- The defining conditions of the initial and final pointing relations are chosen to imply that I_i^g is faithful on 2-cells.

□

Let $I_i^f : F\mathcal{P}_f^T \rightarrow F\mathcal{P}_i^T$ denote the inclusion functor. It is clear that I_i^g factors through $F\mathcal{P}_f^T$. Namely, the functor

$$I_f^g : F\mathcal{P}_{\text{good}} \rightarrow F\mathcal{P}_f^T$$

induced by our choice of final 2-cell for each good 1-finalization makes the following diagram commute

$$\begin{array}{ccc} & & F\mathcal{P}_f^T \\ & \nearrow I_f^g & \downarrow I_i^f \\ F\mathcal{P}_{\text{good}} & \xrightarrow{I_i^g} & F\mathcal{P}_i^T \end{array}$$

due to the conditions imposed on the initial and final pointing relations.

Proposition 4.2.2. The functor I_f^g is an equivalence of unbiased semistrict symmetric monoidal 2-categories.

Proof. Applying the Whitehead criteria in this case is quite straightforward. □

Corollary 4.2.6. *The functor $I_i^f : F\mathcal{P}_f^T \rightarrow F\mathcal{P}_i^T$ is an equivalence of unbiased semistrict symmetric monoidal 2-categories. Namely, the 2-presentations $\mathcal{P}_i^T$ and $\mathcal{P}_f^T$ are equivalent.*

- Phase 3.3: Selection of the remaining final generating relations

We begin by extending our notion of 1-finality to relations:

Definition 4.2.3. A relation $\alpha \sim \beta$ between 2-cells in $F\mathcal{P}_i$ is called **1-final** if both α and β are 1-final 2-cells.

Consider an initial relation of the form

$$\begin{array}{c} y \\ | \\ \boxed{f} \\ | \\ x \end{array} \xRightarrow{\alpha} \begin{array}{c} y \\ | \\ \boxed{g} \\ | \\ x \end{array} \sim \begin{array}{c} y \\ | \\ \boxed{f} \\ | \\ x \end{array} \xRightarrow{\beta} \begin{array}{c} y \\ | \\ \boxed{g} \\ | \\ x \end{array}$$

where α and β are not necessarily 1-final. Our selection of pointing 2-cells ensures the existence of pointing 2-cells γ and δ such that we can form the 1-final relation:

$$\begin{array}{c} y \\ | \\ \boxed{\tilde{f}} \\ | \\ x \end{array} \xRightarrow{\gamma^{-1}} \begin{array}{c} y \\ | \\ \boxed{f} \\ | \\ x \end{array} \xRightarrow{\alpha} \begin{array}{c} y \\ | \\ \boxed{g} \\ | \\ x \end{array} \xRightarrow{\delta} \begin{array}{c} y \\ | \\ \boxed{\tilde{g}} \\ | \\ x \end{array} \sim \begin{array}{c} y \\ | \\ \boxed{\tilde{f}} \\ | \\ x \end{array} \xRightarrow{\gamma^{-1}} \begin{array}{c} y \\ | \\ \boxed{f} \\ | \\ x \end{array} \xRightarrow{\beta} \begin{array}{c} y \\ | \\ \boxed{g} \\ | \\ x \end{array} \xRightarrow{\delta} \begin{array}{c} y \\ | \\ \boxed{\tilde{g}} \\ | \\ x \end{array}$$

Any such 1-final relation constructed from $\alpha \sim \beta$ is called a *1-finalization* of $\alpha \sim \beta$.

To select the remaining final generating relations, we proceed as follows:

1. Consider the set $R_i \setminus T_i$ of remaining initial generating relations, i.e. the set of all initial generating relations in R_i not induced by T_i .
2. Define an equivalence relation on $R_i \setminus T_i$ induced by T_i : two generating relations r_1, r_2 in $R_i \setminus T_i$ are equivalent if either implies the other through relations generated by T_i .
3. Select a set of representatives $[R_i \setminus T_i]$ for each equivalence class.

4. For each relation in $[R_i \setminus T_i]$, choose a 1-finalization. This 1-finalization corresponds uniquely, via the process used to show fullness of the functor I_i^g , to a relation between final 2-cells.
5. Let $R_f \setminus T_f$ denote the collection of all relations between final 2-cells obtained through this process.

We complete the construction by setting

$$R_f := (R_f \setminus T_f) \cup T_f, \quad \mathcal{P}_f := (G_f^0, G_f^1, G_f^2; R_f).$$

We now have:

Proposition 4.2.3. The presentations $\mathcal{P}_f$ and $\mathcal{P}_i$ are equivalent. In particular, whenever the skeletalization $\mathcal{P}_f$ exists, it generates an unbiased semistrict symmetric monoidal 2-category equivalent to the one generated by the initial presentation $\mathcal{P}_i$.

Proof. We first observe that the initial set of relations is equivalent to the set of relations given by $(R_f \setminus T_f) \cup T_i$. Indeed, by construction, every relation in $(R_f \setminus T_f) \cup T_i$ is generated by a relation in $R_i = (R_i \setminus T_i) \cup T_i$. Conversely, let $r \in (R_i \setminus T_i) \cup T_i$. If $r \in T_i$, then $r \in (R_f \setminus T_f) \cup T_i$ as well. If instead $r \in (R_i \setminus T_i)$, then using the relations in T_i we can rewrite r as a relation in $(R_f \setminus T_f)$, by the construction of $(R_f \setminus T_f)$.

Therefore, the initial presentation

$$\mathcal{P}_i := (G_i^0, G_i^1, G_i^2; R_i)$$

is equivalent to $(G_i^0, G_i^1, G_i^2; (R_f \setminus T_f) \cup T_i)$. By Corollary 4.2.6, the presentations $\mathcal{P}_i^T := (G_i^0, G_i^1, G_i^2; T_i)$ and $\mathcal{P}_f^T := (G_f^0, G_f^1, G_f^2; T_f)$ are equivalent. It follows that $(G_i^0, G_i^1, G_i^2; (R_f \setminus T_f) \cup T_i)$ is equivalent to $(G_f^0, G_f^1, G_f^2; (R_f \setminus T_f) \cup T_f)$, which is just $\mathcal{P}_f := (G_f^0, G_f^1, G_f^2; R_f)$.

To summarize, we obtain the chain of equivalences

$$\begin{aligned} \mathcal{P}_i &:= (G_i^0, G_i^1, G_i^2; R_i) = (G_i^0, G_i^1, G_i^2; (R_i \setminus T_i) \cup T_i) \cong (G_i^0, G_i^1, G_i^2; (R_f \setminus T_f) \cup T_i) \\ &\cong (G_f^0, G_f^1, G_f^2; (R_f \setminus T_f) \cup T_f) = (G_f^0, G_f^1, G_f^2; T_f) =: \mathcal{P}_f^T \end{aligned}$$

which completes the proof. $\square$

For a concrete example of Stage 3, see Subsection 5.1.1.

Chapter 5

Constructing a finite presentation of $\text{Bord}_{01\sim}^{(\text{or})}$

This chapter focuses on the skeletalization of two key presentations: $\text{Sur}_{01\sim}^{\text{or}}$ and $\text{Sur}_{01\sim}$. We will construct skeletal versions of these presentations and compare them with the invertible part of the presentations of $\text{Bord}_{012}^{\text{or}}$ and Bord_{012} established by Schommer-Pries in [SP14]. Following Example 4.1.4, we will assume that all 2-categories of surgery diagrams under consideration are already 0-skeletal.

Moreover, we will work with slightly simplified versions of $\text{Sur}_{01\sim}^{\text{or}}$ and $\text{Sur}_{01\sim}$. Since all diffeomorphisms between 0-manifolds arise from symmetric group actions, we will discard all diffeomorphism-type generating 1-cells, 2-cells, and relations. Likewise, any generating 2-cells and relations involving isotopies will be omitted. In all other cases where diffeomorphisms appear in the presentations, they will be indicated in green, following the conventions established in Chapter 3.

5.1 Skeletalization of $\text{Sur}_{01\sim}^{\text{or}}$

In this section, we provide a detailed implementation of the skeletalization procedure developed in Section 4.2, applying it to obtain a skeletal presentation for $\text{Sur}_{01\sim}^{\text{or}}$. For clarity and pedagogical purposes, we divide this construction into three steps, beginning with the skeletalization of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$.

5.1.1 Skeletalization of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$

We first apply our skeletalization procedure to $\mathcal{P}_i = \text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$, which serves as the foundational example for understanding the general process. This presentation, while simple, exhibits many key features that will appear in more complex cases.

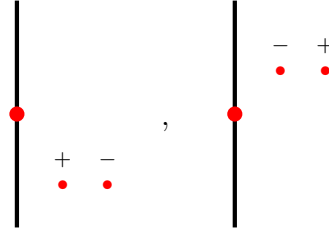
Stage 0: Final generating 0-cells

Since the initial presentation is already 0-skeletal, the final generating 0-cells remain unchanged from $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$, consisting of two elements:

1. $\bullet^+$ (positively oriented point)
2. $\bullet^-$ (negatively oriented point)

Stage 1: Final generating 1-cells

To recall the corresponding step in the skeletalization procedure, see Subsection 4.2.2. We select G_f^1 to consist of two final generating 1-cells:



These correspond to the generating 1-cells of $\text{Bord}_{012}^{\text{or}}$, described in [SP14]:

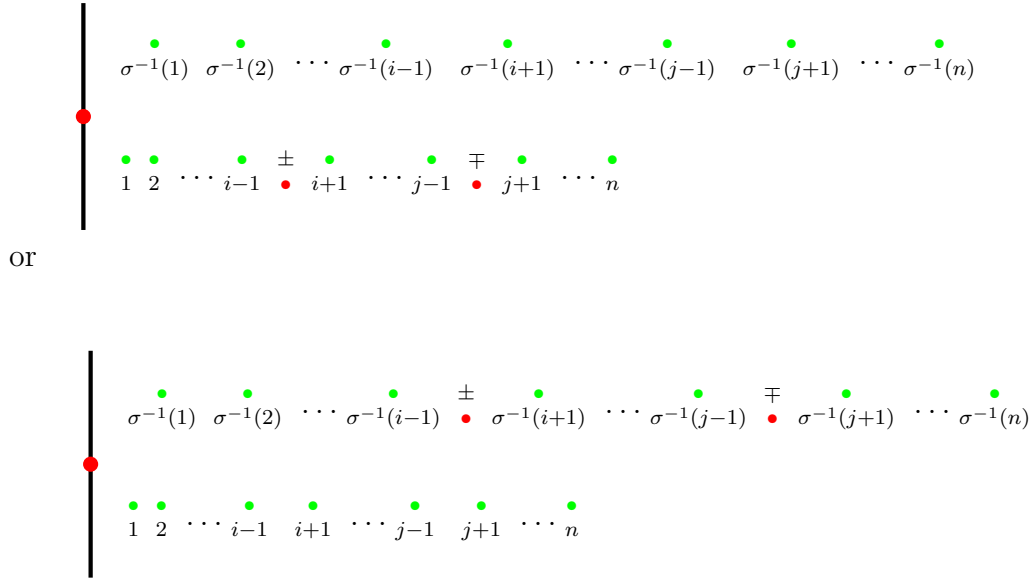


Remark 5.1.1 (Notational differences). Two important differences from [SP14] should be noted as they can potentially cause some confusion:

- Our 1-cells are read from bottom to top (versus left to right in [SP14])
- Our presentation explicitly shows boundary identifications (which are implicit in [SP14])

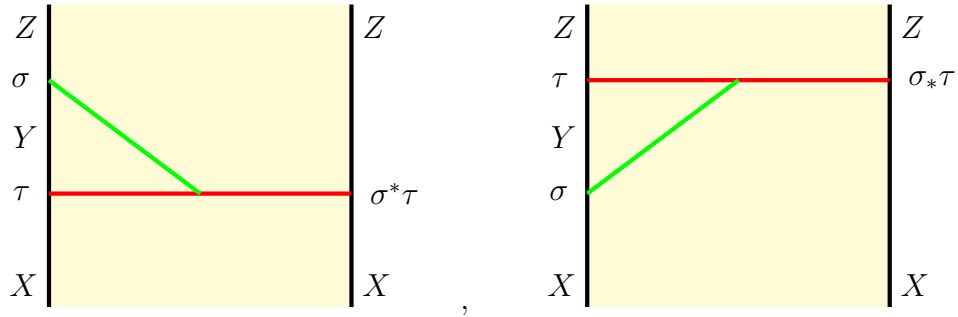
Verification that G_f^1 satisfies the desired conditions:

To show that this choice of G_f^1 satisfies the conditions required by our skeletalization procedure from Subsection 4.2.2, we must verify that every initial generating 1-cell is weakly permutogenerated by one of our chosen final generators (completeness) and that no final generating 1-cell is weakly permutogenerated by the other (minimality). Initial generating 1-cells in $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$ take one of two forms:



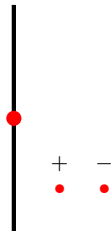
Remark 5.1.2. We omit the signs of the green points in these diagrams for readability.

Using the invertible 2-cells of naturality type:



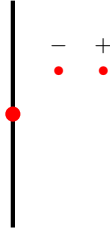
we can show that:

1. Any generating 1-cell of the first form is weakly permutogenerated by our first final generator:



2. Any generating 1-cell of the second form is weakly permutogenerated by our

second final generator:



Moreover, neither of these final generating 1-cells can be permutogenerated by the other, ensuring our choice satisfies the minimality condition of the skeletalization procedure.

Stage 2: Final generating 2-cells

To recall the corresponding step in the skeletalization procedure, see Subsection 4.2.4.

Phase 2.1: Selection of pointing 2-cells

For each initial generating 1-cell $f: x \Rightarrow y$, we must select invertible 2-cells in $F \text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$ of the form

$$\begin{array}{ccc} & y & y \\ & | & | \\ \boxed{f} & \xRightarrow{\eta_j^f} & \boxed{g_j} \\ & | & | \\ & x & x \end{array}$$

where each g_j is a final 1-cell in $F\mathcal{P}_f$.

Our choice here is maximal: for each initial generating 1-cell f , we select *all* invertible 2-cells in $F \text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$ satisfying this condition.

Proposition 5.1.1. With this choice, the pointing 2-cells in $F \text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$ are precisely the composites of:

- Initial generating 2-cells of naturality type
- 2-cells in $\mathcal{S} \cup \mathcal{J}_i$

Phase 2.2: Selection of type I final generating 2-cells

We can show that every 1-final pointing 2-cell decomposes as follows:

Proposition 5.1.2. Every 1-final pointing 2-cell can be expressed as a composite of:

- Elementary 2-cells in $\mathcal{S}$

- Elementary 2-cells in $\mathcal{J}_f$

up to the equivalence relation induced by R_i .

Corollary 5.1.3. *There are no final generating 2-cells of type I.*

Phase 2.3: Selection of type II final generating 2-cells

In this case, we find that $A_f^2 = \emptyset$ because all initial generating 2-cells are pointing 2-cells. Consequently:

Proposition 5.1.3. There are no final generating 2-cells of type II.

Remark 5.1.4. This remarkably simple outcome - having no final generating 2-cells of either type - reflects the special nature of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$ where all essential 2-cell structure is captured by symmetries and interchangors.

Stage 3: Final generating relations

To recall the corresponding step in the skeletalization procedure, see Subsection 4.2.4. Since there are no final generating 2-cells, we do not expect the final presentation to admit any generating relations. Even though it is easy to directly check that in this example, it is still useful to carefully go through the skeletalization procedure for future reference.

Phase 3.1: Selection of pointing relations

It is easy to check that the following choices are compatible with the skeletalization procedure:

- ${}_IT_i$ consists of all naturality type relations
- ${}_{II}T_i = \emptyset$
- ${}_IT_f = {}_{II}T_f = \emptyset$

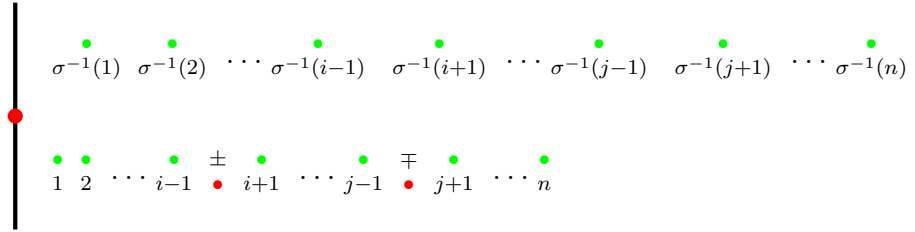
Remark 5.1.5. In this case, we have $T_i = R_i$.

Phase 3.2: Selection of good 1-finalizations

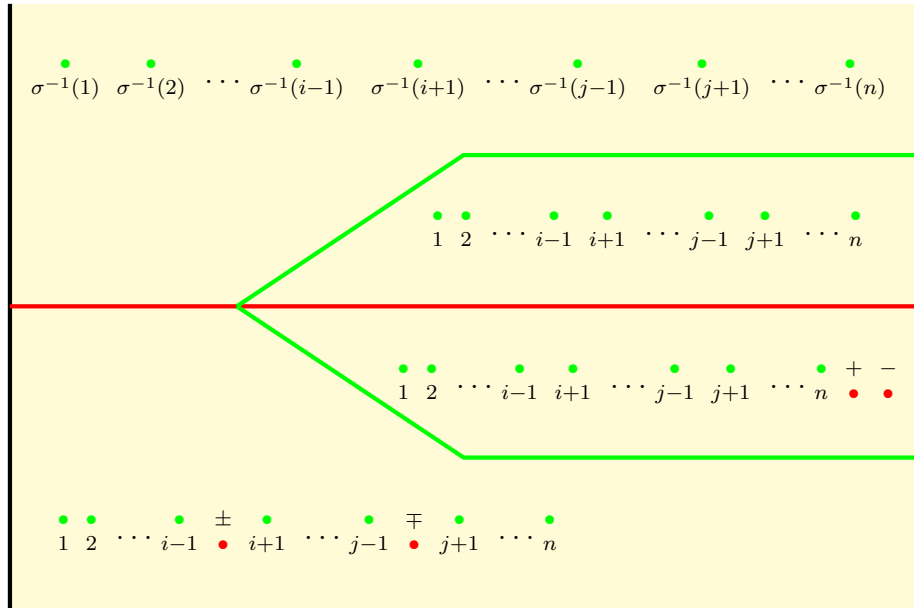
Remark 5.1.6. Since $T_f = \emptyset$ and $R_i \setminus T_i = \emptyset$, this step is not necessary, but we will nonetheless address it.

We will describe explicit 1-finalizations for each initial generating 2-cell, by specifying a pointing 2-cell with source any given initial generating 1-cell:

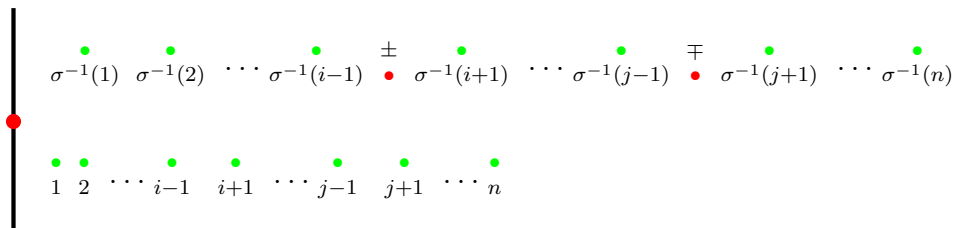
1. If the given initial generating 1-cell is already final, the chosen pointing 2-cell will be the identity.
2. If the given non-final initial generating 1-cell is of the form



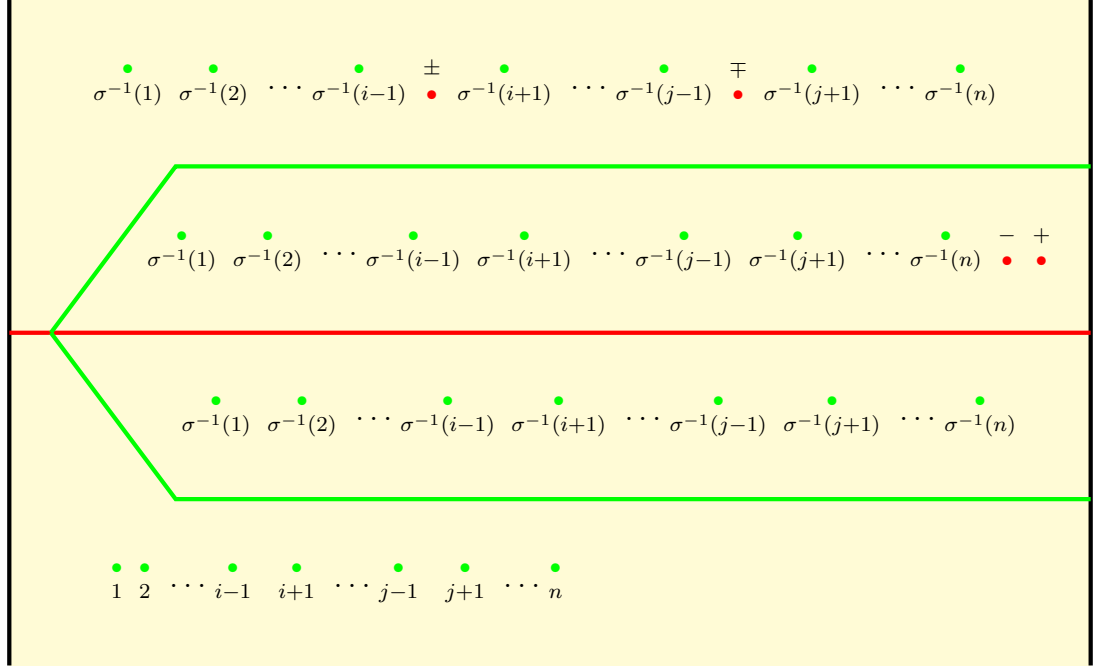
then the chosen pointing 2-cell will be the naturality type 2-cell:



3. If the given non-final initial generating 1-cell is of the form



then the chosen pointing 2-cell will be the naturality type 2-cell:



Phase 3.3: Selection of the remaining final generating relations

Since $T_i = R_i$ in this case, there are no initial generating relations remaining to study. We, therefore, conclude that $T_f = \emptyset$.

Remark 5.1.7. This completes our skeletalization of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$. The resulting skeletal presentation consists only of two 0-cells and two 1-cells, with all higher structure captured implicitly through the symmetric monoidal bicategory structure.

5.1.2 Skeletalization of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$

We now proceed to skeletalize $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$, building upon our previous results. This case introduces new complexities while maintaining some structure we have already identified.

Stages 0 and 1: Final generating 0- and 1-cells

The fundamental structure at the lowest categorical levels remains unchanged from $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$:

- The final generating 0-cells are again $\bullet^+$ and $\bullet^-$.
- The final generating 1-cells remain the same two generators we identified previously

Stage 2: Final generating 2-cells

To recall the corresponding step in the skeletalization procedure, see Subsection 4.2.4.

Phase 2.1: Selection of pointing 2-cells

The pointing 2-cells in $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$ are identical to those in $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$. Let $\mathcal{V}$ be the collection of all pointing 2-cells in $F \text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$.

Phase 2.2: Selection of type I final generating 2-cells

The pointing 2-cells of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$ are the same as those of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$, therefore $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$ will also have no type I final generating 2-cells, i.e. ${}_I G_f^2 := \emptyset$.

Phase 2.3: Selection of type II final generating 2-cells

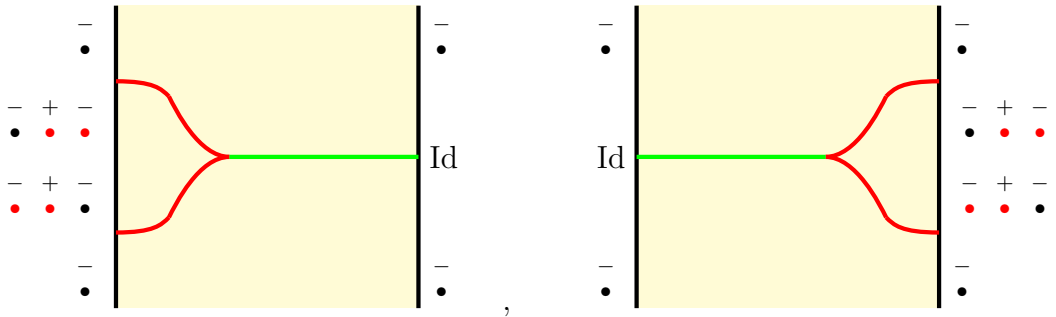
We first need to select A_f^2 . The non-pointing initial generating 2-cells consist of two types:

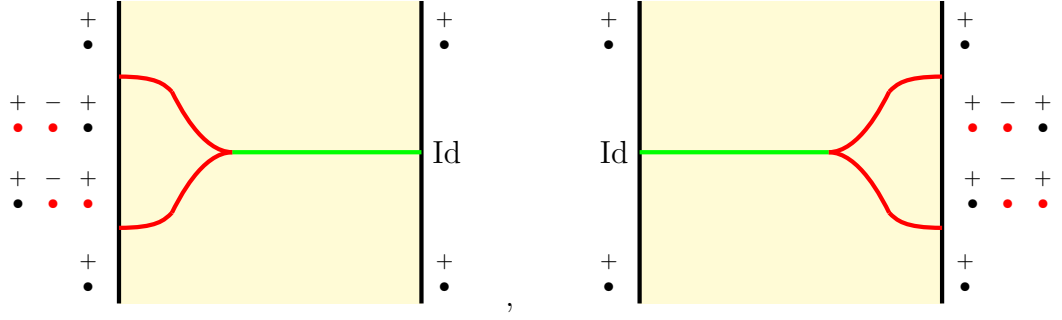
- Cusps
- Crossings

We can make the following simplifications:

Lemma 5.1.8. 1. Any crossing 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated by an interchangor 2-cell

2. Any cusp initial generating 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated by one of the following four cusps:

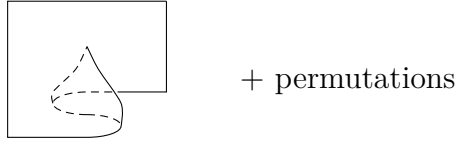




Proof. This follows immediately from our choice of good 1-finalizations, as outlined below in Phase 3.2. of the procedure. $\square$

We now choose A_f^2 to consist precisely of these four initial generating 2-cells. Since these cells are already 1-final, we have ${}_{II}G_f^2 := A_f^2$.

Remark 5.1.9. These cells correspond to the following generating 2-cells of $\text{Bord}_{012}^{\text{or}}$ described in [SP14]:



Stage 3: Final generating relations

To recall the corresponding step in the skeletalization procedure, see Subsection 4.2.4.

Phase 3.1: Selection of pointing relations

It is easy to check that the following choices are compatible with the skeletalization procedure:

1. ${}_IT_i$ consists of all initial generating relations between pointing 2-cells
2. ${}_{II}T_i$ consists of:
 - (a) All naturality-cusp relations
 - (b) All naturality-crossing relations
3. ${}_IT_f = {}_{II}T_f = \emptyset$

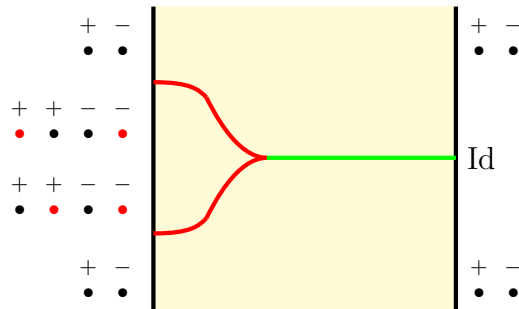
Remark 5.1.10. In this case, we have $T_i = R_i$.

Phase 3.2: Selection of good 1-finalizations

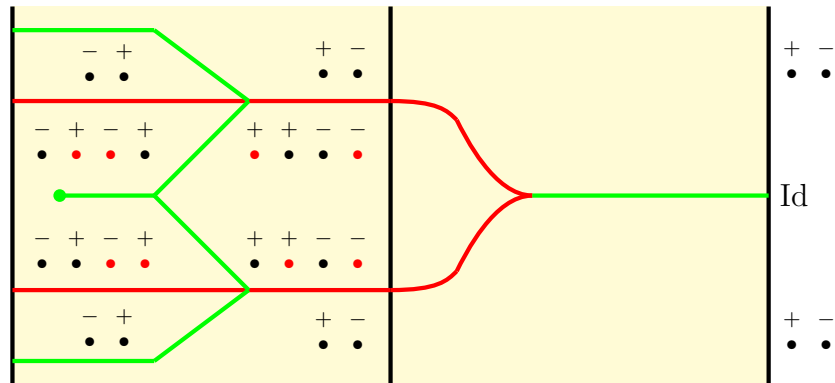
Remark 5.1.11. As in the skeletalization of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$ this step is not necessary here, since $T_f = \emptyset$ and $R_i \setminus T_i = \emptyset$, but we will nonetheless address it.

We will describe explicit 1-finalizations for each initial generating 2-cell:

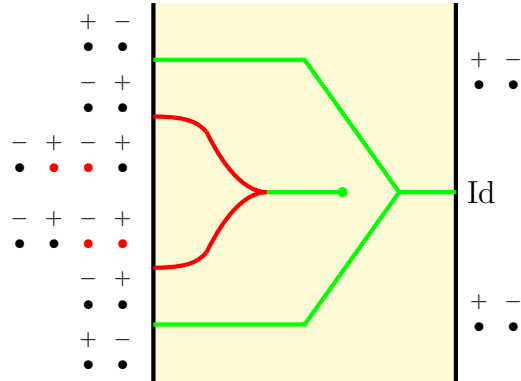
1. If the given initial generating 2-cell is already 1-final, its good 1-finalization will be itself.
2. If the given initial generating 2-cell is pointing, its good 1-finalization is that described in the skeletalization of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$.
3. If the given initial generating 2-cell is a cusp, we will describe its good 1-finalization through an example that is easy to generalize. Namely, we choose the good 1-finalization of the cusp 2-cell



to be

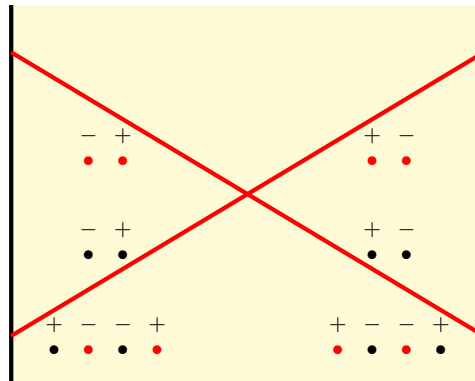


which, in turn, is equivalent to

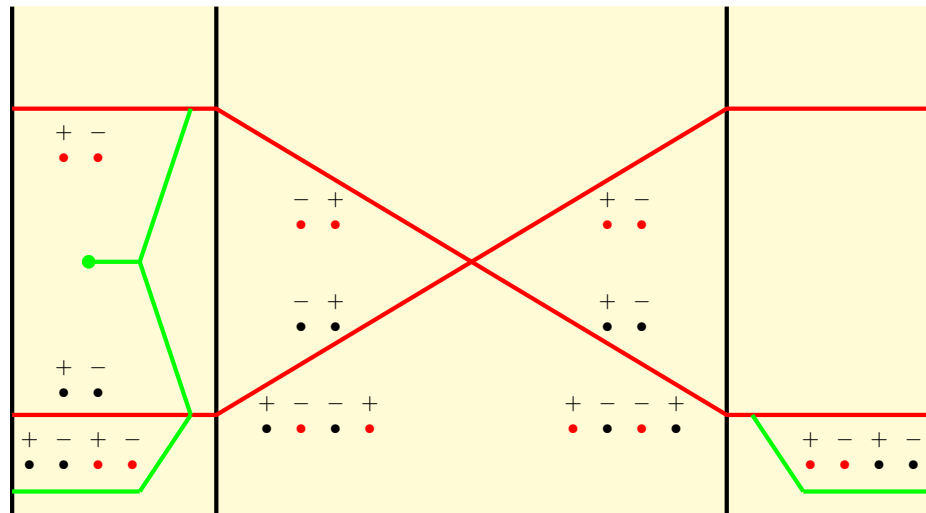


via the relation generated by R_i .

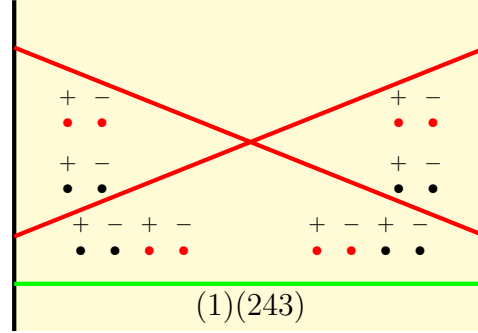
4. If the given initial generating 2-cell is a crossing, we will describe its good 1-finalization through an example that is easy to generalize. Namely, we choose the good 1-finalization of the crossing 2-cell



to be



which, in turn, is equivalent to

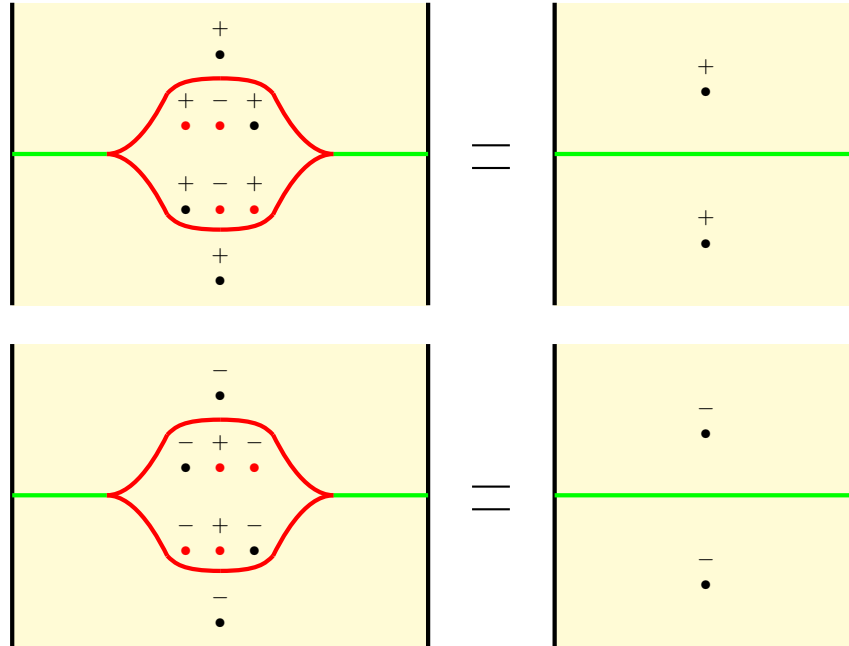


via the relation generated by R_i .

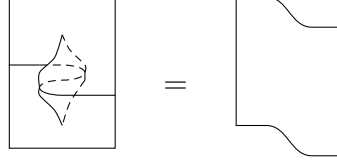
Phase 3.3: Selection of the remaining final generating relations

There are three families of remaining initial generating relations that we need to consider: eye birth-death, merge-unmerge, and Reidemeister II. Relations in the last family are equivalent, up to T_i , to relations that automatically hold for interchangor 2-cells, so we don't need to consider them any further. We will now examine the relations in the first two families up to T_i .

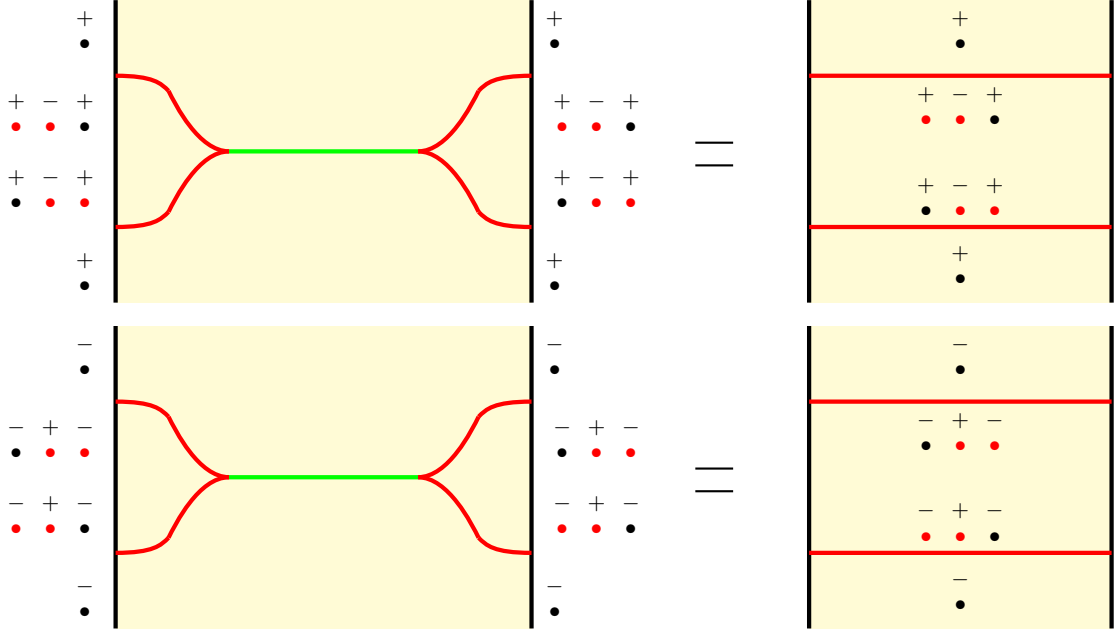
Eye birth - death: Up to relations induced by T_i , all eye birth - death relations are induced by the following two relations:



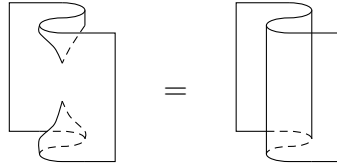
They correspond to the following relations of $\text{Bord}_{012}^{\text{or}}$, described in [SP14]:



Merge - unmerge: Up to relations generated by T_i , all merge - unmerge relations are induced by the following two relations:



They correspond to the following relations of $\text{Bord}_{012}^{\text{or}}$, described in [SP14]:



Remark 5.1.12. This completes our skeletalization of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$. The resulting skeletal presentation consists of two 0-cells, two 1-cells, four 2-cells and four relations.

5.1.3 Skeletalization of $\text{Sur}_{01\sim}^{\text{or}}$

We now proceed to skeletalize $\text{Sur}_{01\sim}^{\text{or}}$ building upon our previous results.

Stages 0, 1 and 2: Final generating 0-, 1- and 2-cells

The final generating 0-, 1- and 2- cells are the same as in $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$. The same holds true for the pointing 2-cells.

Stage 3: Final generating relations

To recall the corresponding step in the skeletalization procedure, see Subsection 4.2.4.

Phase 3.1: Selection of pointing relations

The pointing relations are also the same as in $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$. Namely,

1. ${}_IT_i$ consists of all initial generating relations between pointing 2-cells
2. ${}_{II}T_i$ consists of all naturality-cusp and naturality-crossing relations
3. ${}_IT_f = {}_{II}T_f = \emptyset$

Remark 5.1.13. In this case, it is *no longer* true that all relations in R_i are induced by the relations in T_i .

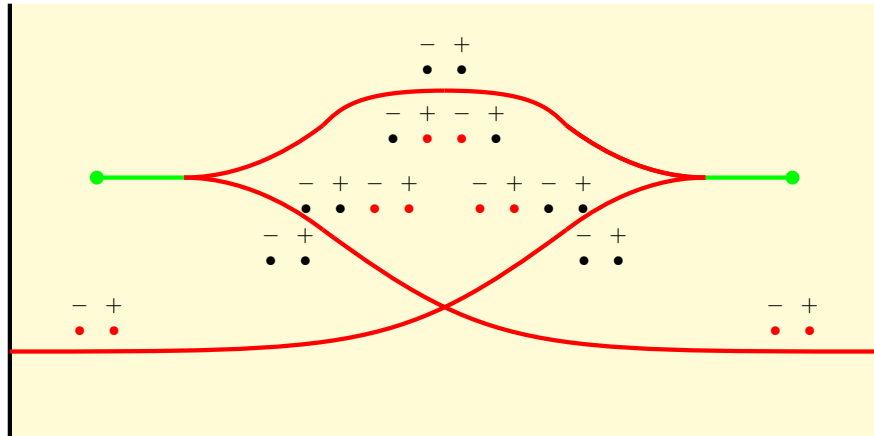
Phase 3.2: Selection of good 1-finalizations

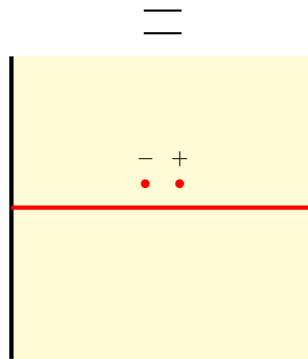
The good 1-finalizations are the same as in $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^1}$.

Phase 3.3: Selection of the remaining final generating relations

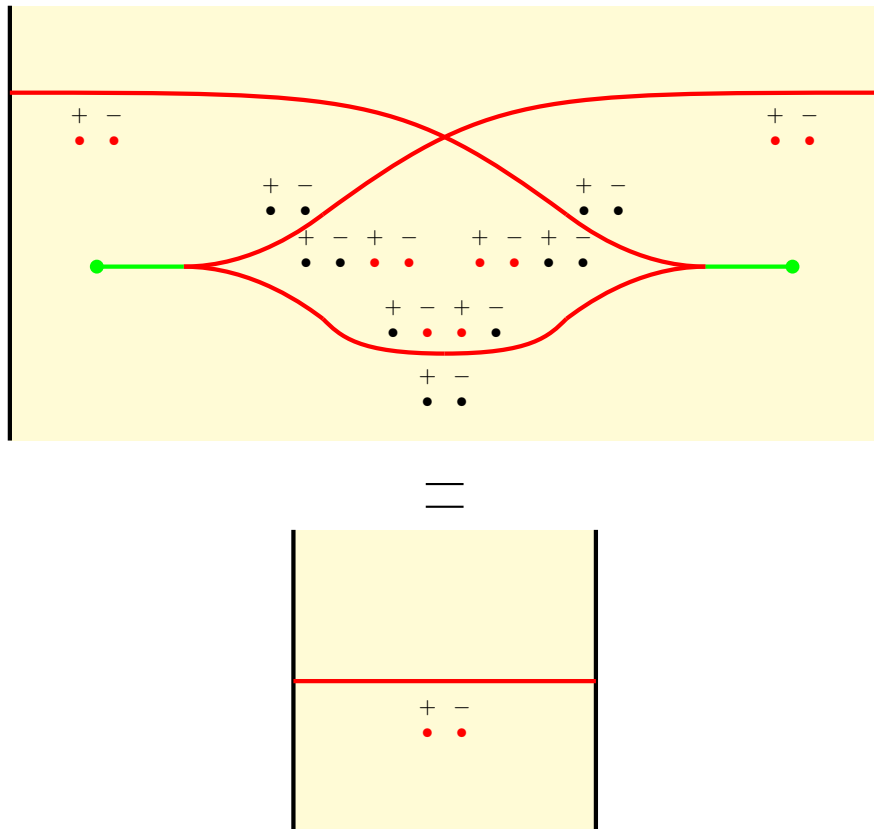
There are three families of remaining initial generating relations that we need to consider: swallowtail, beak, and Reidemeister III. Relations in the last two families are equivalent, up to T_i , to relations that automatically hold for interchangor 2-cells, so we don't need to consider them any further. We will now examine the relations in the first family up to T_i .

Swallowtail: Up to relations generated by T_i , all swallowtail relations are induced by the following two relations:

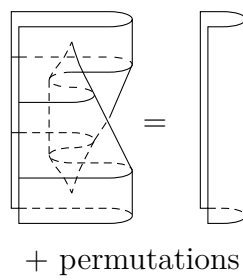




and



They correspond to the following relations of $\text{Bord}_{012}^{\text{or}}$, described in [SP14]:



Remark 5.1.14. This completes our skeletalization of $\text{Sur}_{01\sim}^{\text{or}}$. The resulting skeletal presentation consists of two 0-cells, two 1-cells, four 2-cells and six relations.

5.2 Skeletalization of $\text{Sur}_{01\sim}$

In this section, we provide a (not so detailed) implementation of the skeletalization procedure of section 5.1.1, applying it to obtain a skeletal presentation for $\text{Sur}_{01\sim}$. Just like in the oriented case, we divide this construction into three steps, beginning with the skeletalization of $\text{Sur}_{01\sim}^{\mathcal{F}^0}$.

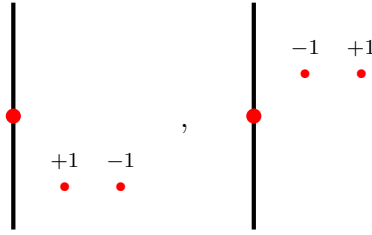
5.2.1 Skeletalization of $\text{Sur}_{01\sim}^{\mathcal{F}^0}$

Stage 0: Final generating 0-cells

The same as the initial generating 0-cell of $\text{Sur}_{01\sim}^{\mathcal{F}^0}$, namely $\bullet$.

Stage 1: Final generating 1-cells

We choose G_f^1 to consist of two final generating 1-cells:



These correspond to the generating 1-cells of $\text{Bord}_{(0; 1; 2)}^{\text{or}}$, described in [SP14]:



We can verify that any initial generating 1-cell in $\text{Sur}_{01\sim}^{\mathcal{F}^0}$ is weakly permutogenerated by one of these chosen final generating 1-cell via essentially the same argument as in the oriented case.

Stage 2: Final generating 2-cells

Phase 2.1: Selection of pointing 2-cells

The pointing 2-cells in $F\text{Sur}_{01\sim}$ are all composites of

- Initial generating 2-cells of naturality type
- 2-cells in $\mathcal{S} \cup \mathcal{J}_i$

We let $\mathcal{V}$ denote the collection of all pointing 2-cells in $F\text{Sur}_{01\sim}$.

Phase 2.2: Selection of type I final generating 2-cells

We can show that every 1-final pointing 2-cell decomposes as follows:

Proposition 5.2.1. Every 1-final pointing 2-cell can be expressed as a composite of:

- Elementary 2-cells in $\mathcal{S}$
- Elementary 2-cells in $\mathcal{J}_f$

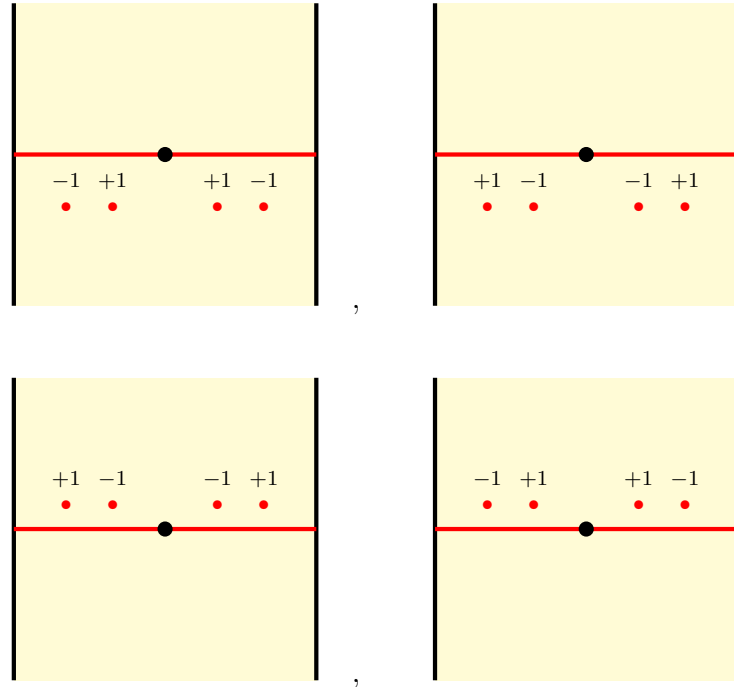
up to the equivalence relation induced by R_i .

Corollary 5.2.1. *There are no final generating 2-cells of type I.*

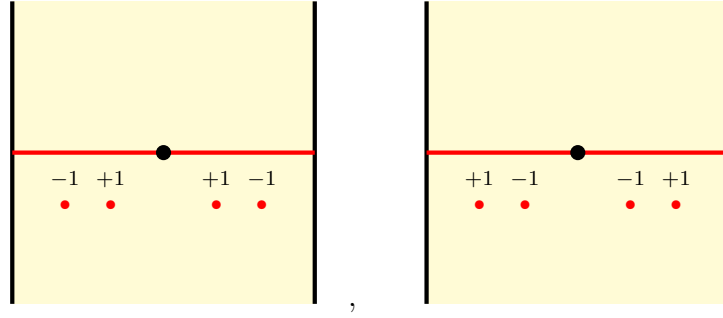
Phase 2.3: Selection of type II final generating 2-cells

We first need to select A_f^2 . The non-pointing initial generating 2-cells consist of frame change type 2-cells. We can make the following simplification:

Lemma 5.2.2. *Any frame change type initial generating 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated by one of the following four:*

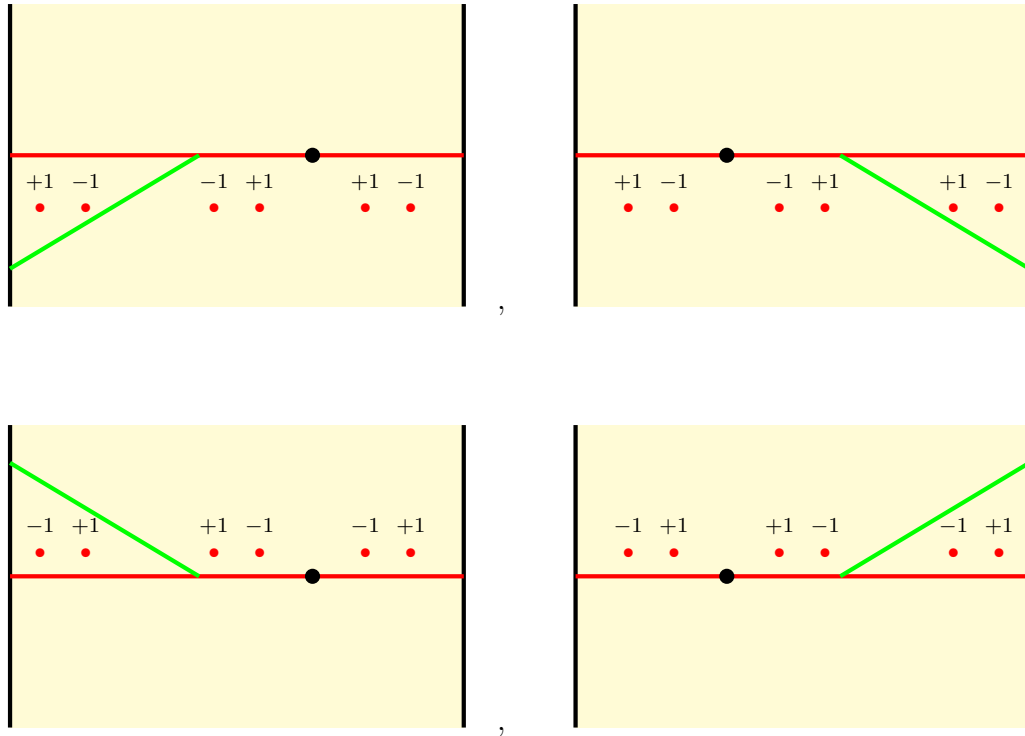


Remark 5.2.3. Note that the two 2-cells

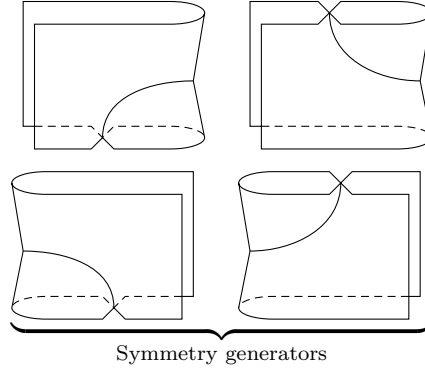


are $(\mathcal{V}, \mathcal{S})$ -generated by each other, and the same holds true for the other two, so we could have chosen B_f^2 to only consist of two 2-cells if we so wanted. The interested reader is encouraged to carry out the skeletalization procedure for this more economical choice of B_f^2 and verify that it leads to a presentation equivalent to that obtained here.

We now choose a 1-finalization for each 2-cell in A_f^2 , as follows:



These four 2-cells constitute ${}_{II}G_f^2 = G_f^2$ and correspond to the following generating 2-cells of $\text{Bord}_{01\sim}$ described in [SP14]:



Stage 3: Final generating relations

Phase 3.1: Selection of pointing relations

The pointing relations of $\text{Sur}_{01\sim}$ are as follows:

1. ${}_IT_i$ consists of naturality type relations between pointing 2-cells
2. ${}_{II}T_i$ consists of frame change - naturality type relations that involve symmetries which do not affect the relative position of the two points in the image of the 0-dimensional attaching sphere
3. ${}_IT_f = {}_{II}T_f = \emptyset$

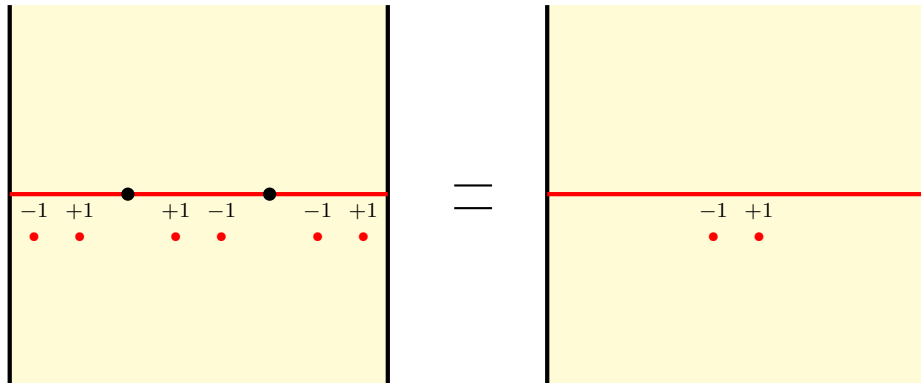
Phase 3.2: Selection of good 1-finalizations

This is mostly the same as in the oriented case.

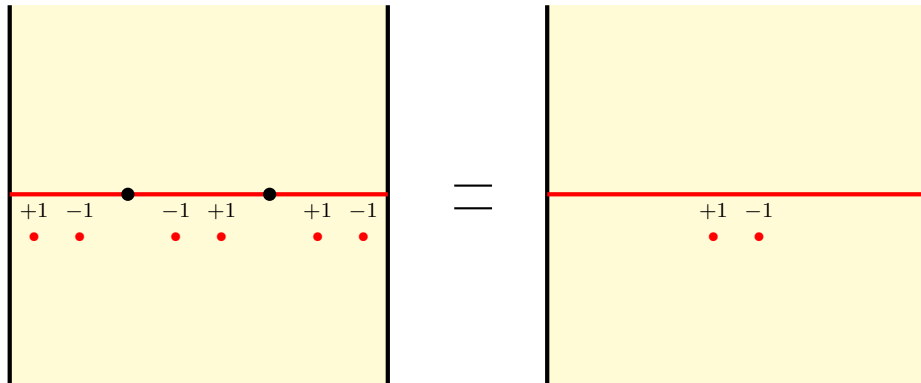
Phase 3.3: Selection of the remaining final generating relations

There are two families of remaining initial generating relations we need to consider: composition of frame changes and frame change - naturality relations.

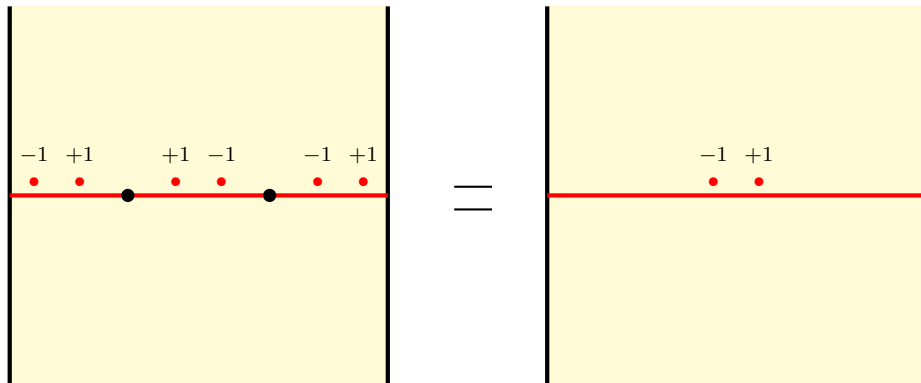
Composition of frame changes: Up to relations induced by T_i , all relations coming from compositions of frame changes are induced by the following four:



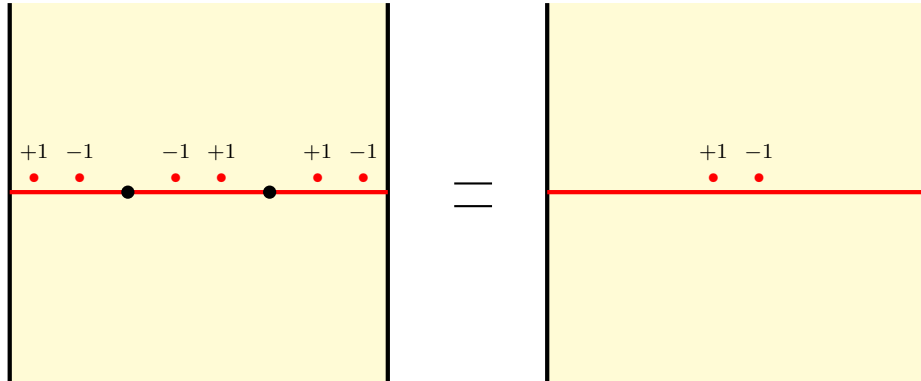
—
—



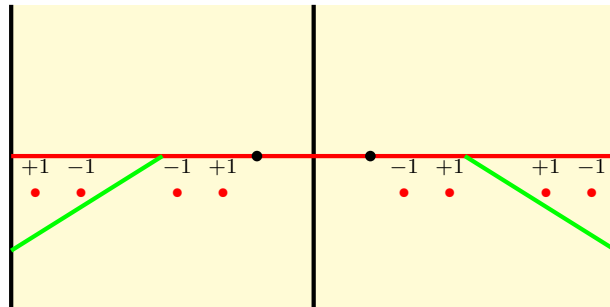
—
—



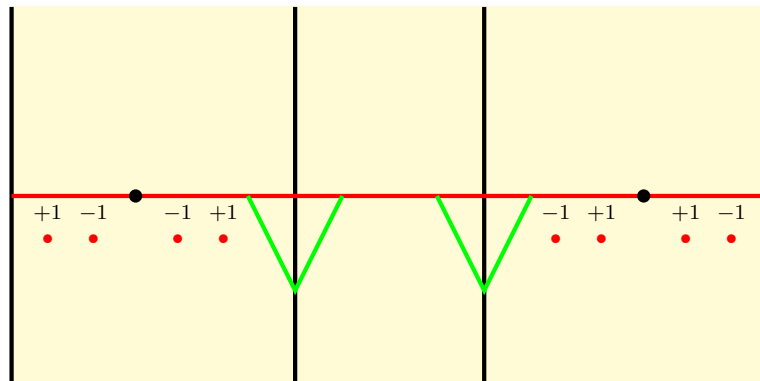
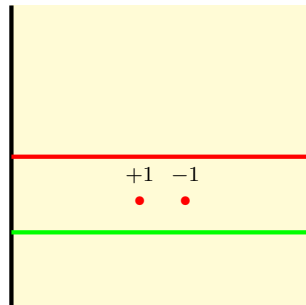
—
—

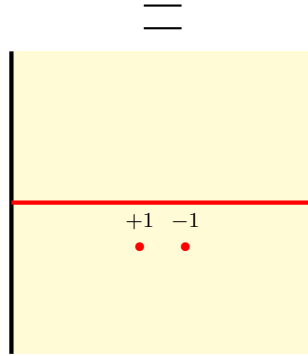


Each of them will correspond to a final relation after 1-finalizing according to the skeletalization procedure. We explicitly see this in two examples below:

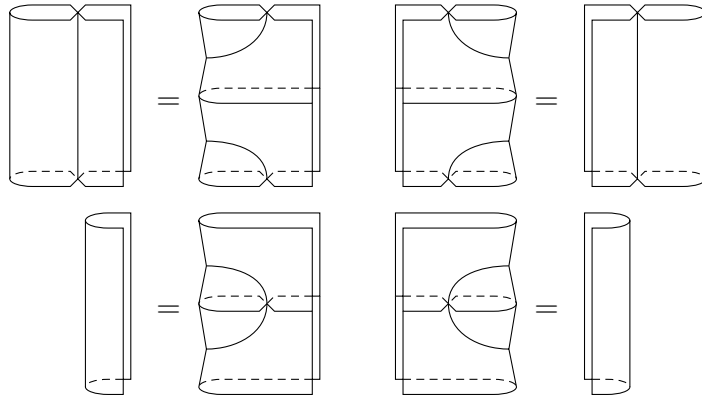


=

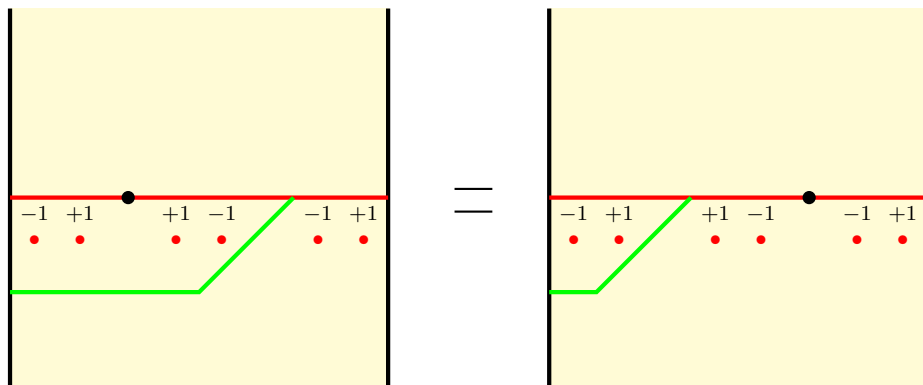


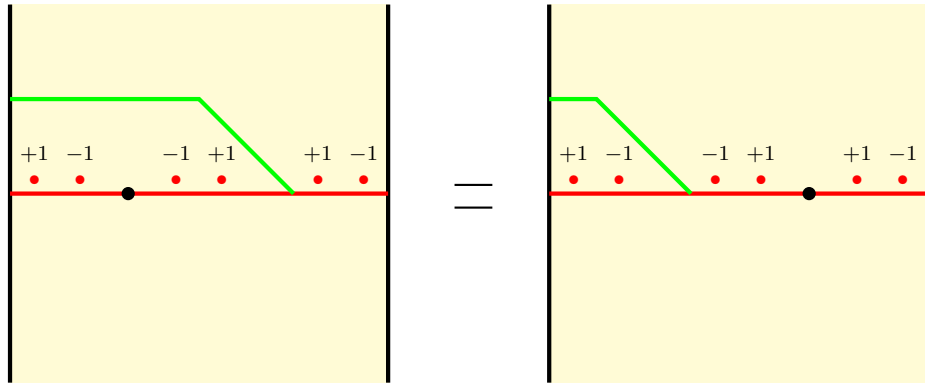


These four relations correspond to the following relations of $\text{Bord}_{01\sim}$ described in [SP14]:

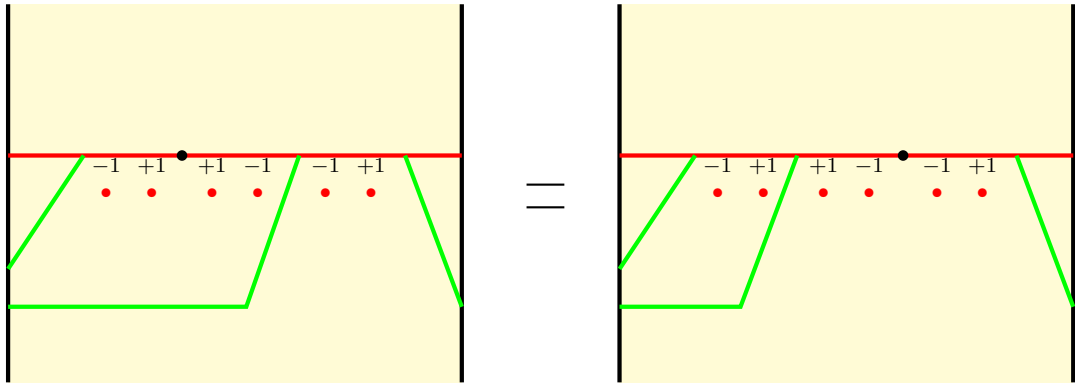


Frame change - naturality relations: Up to relations induced by T_i , all frame change - naturality relations are induced by the following two:

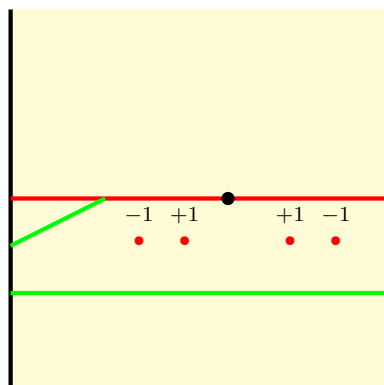




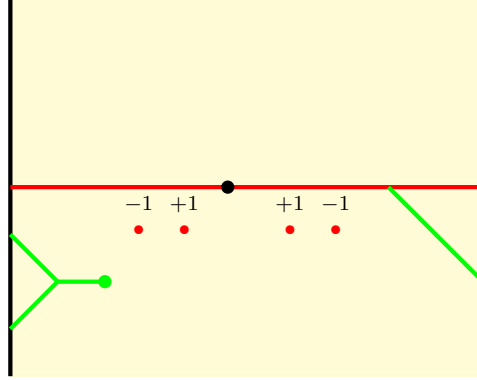
Each of them will correspond to a final relation after 1-finalizing according to the skeletalization procedure. We explicitly see this in the example below:



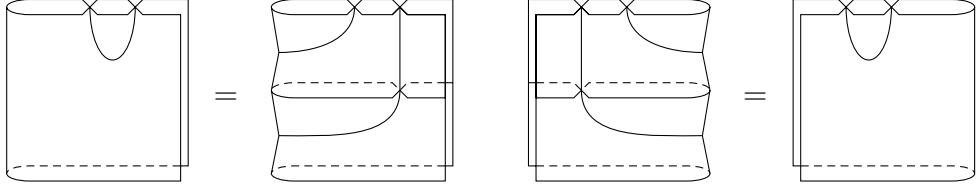
The left hand side is equivalent, through pointing relations, to the final 2-cell



and the right hand side is equivalent, through pointing relations, to the final 2-cell



These two relations correspond, after rearranging and using the already established relations, to the following relations of $\text{Bord}_{01\sim}$ described in [SP14]:



Remark 5.2.4. This completes our skeletalization of $\text{Sur}_{01\sim}^{\mathcal{F}^0}$. The resulting skeletal presentation consists of one generating 0-cells, two generating 1-cells, four generating 2-cells, and six generating relations.

5.2.2 Skeletalization of $\text{Sur}_{01\sim}^{\mathcal{F}^1}$

We now proceed to skeletalize $\text{Sur}_{01\sim}^{\mathcal{F}^1}$, building upon our previous results.

Stages 0 and 1: Final generating 0- and 1-cells

The fundamental structure at the lowest categorical levels remains unchanged from $\text{Sur}_{01\sim}^{\mathcal{F}^0}$:

- The final generating 0-cells is again a single point $\bullet$.
- The final generating 1-cells remain the same two generators we identified previously

Stage 2: Final generating 2-cells

Phase 2.1: Selection of pointing 2-cells

The pointing 2-cells in $F\text{Sur}_{01\sim}^{\mathcal{F}^1}$ are all 2-cells in $F\text{Sur}_{01\sim}^{\mathcal{F}^0}$. Let $\mathcal{V}$ denote their collection.

Phase 2.2: Selection of type I final generating 2-cells

${}_I G_f^2$ consists of all the final generating 2-cells of $\text{Sur}_{01\sim}^{\mathcal{F}^0}$, namely the four symmetry generators. Note that according to the choices we made, these are now type I final generating 2-cells, while they were type II final generating 2-cells for $\text{Sur}_{01\sim}^{\mathcal{F}^0}$.

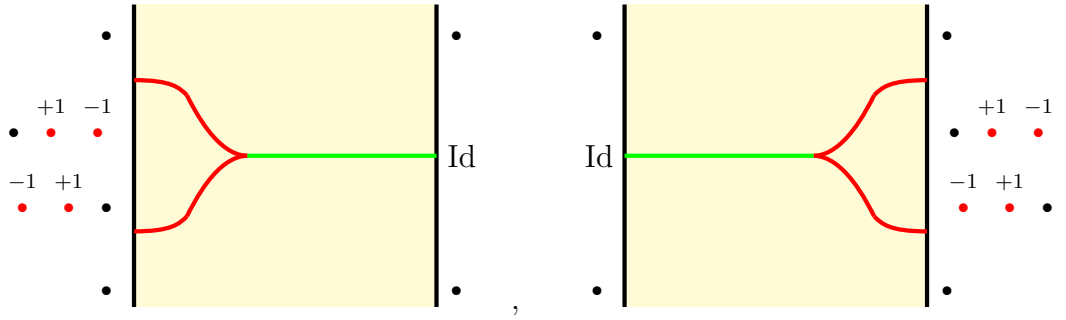
Phase 2.3: Selection of type II final generating 2-cells

We first need to select A_f^2 . The non-pointing initial generating 2-cells consist of two types:

- Cusps
- Crossings

We can make the following simplifications:

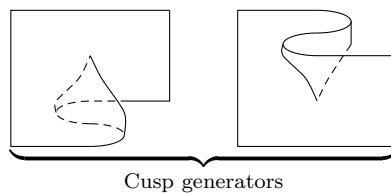
- Lemma 5.2.5.** 1. *Any crossing 2-cell is permutogenerated by an interchangor 2-cell*
2. *Any cusp initial generating 2-cell is permutogenerated by one of the following two cusps:*



Proof. This is similar to the oriented case. □

We now choose A_f^2 to consist precisely of these two initial generating 2-cells. Since these cells are already 1-final, we have ${}_{II} G_f^2 := A_f^2$.

Remark 5.2.6. These cells correspond to the following generating 2-cells of Bord_{012} described in [SP14]:



Stage 3: Final generating relations

Phase 3.1: Selection of pointing relations

The pointing relations of $\text{Sur}_{01\sim}^{\mathcal{F}^1}$ are as follows:

1. ${}_IT_i$ consists of all relations in $\text{Sur}_{01\sim}^{\mathcal{F}^0}$
2. ${}_{II}T_i$ consists of:
 - Naturality-cusp and naturality-crossing relations
 - Frame change-cusp and frame change-crossing relations
3. ${}_IT_f$ consists of all final relations in $\text{Sur}_{01\sim}^{\mathcal{F}^0}$
4. ${}_{II}T_f = \emptyset$

Phase 3.2: Selection of good 1-finalizations

This is mostly the same as in the oriented case.

Phase 3.3: Selection of the remaining final generating relations

There are three families of remaining initial generating relations that we need to consider: eye birth-death, merge-unmerge, and Reidemeister II. Relations in the last family (Reidemeister II) are equivalent, up to T_i , to relations that automatically hold for interchangor 2-cells, so we do not need to consider them further. The first two families—eye birth-death, and merge-unmerge—are treated in essentially the same way as in the oriented case, so we omit the details here. In the end, we obtain one merge-unmerge final relation, and one eye birth-death final relation.

Remark 5.2.7. This completes our skeletalization of $\text{Sur}_{01\sim}^{\mathcal{F}^0}$. The resulting skeletal presentation consists of one generating 0-cells, two generating 1-cells, six generating 2-cells, and eight generating relations.

5.2.3 Skeletalization of $\text{Sur}_{01\sim}$

We now proceed to skeletalize $\text{Sur}_{01\sim}$ building upon our previous results.

Stages 0, 1 and 2: Final generating 0-, 1- and 2-cells

The final generating 0-, 1- and 2- cells are the same as in $\text{Sur}_{01\sim}^{\mathcal{F}^1}$. The same holds true for the pointing 2-cells.

Stage 3: Final generating relations

Phase 3.1: Selection of pointing relations

The pointing relations are also the same as in $\text{Sur}_{01\sim}^{\mathcal{F}^1}$.

Phase 3.2: Selection of good 1-finalizations

The good 1-finalizations are the same as in $\text{Sur}_{01\sim}^{\mathcal{F}^1}$.

Phase 3.3: Selection of the remaining final generating relations

There are three families of remaining initial generating relations that we need to consider: swallowtail, beak, and Reidemeister III. Relations in the last two families (beak, and Reidemeister III) are equivalent, up to T_i , to relations that automatically hold for interchangor 2-cells, so we do not need to consider them further. The first family—swallowtail—are treated in essentially the same way as in the oriented case, so we omit the details here. In the end, we obtain two swallowtail final relations.

Remark 5.2.8. This completes our skeletalization of $\text{Sur}_{01\sim}$. The resulting skeletal presentation consists of one generating 0-cells, two generating 1-cells, six generating 2-cells, and ten generating relations.

Chapter 6

Constructing a finite presentation of $\text{Bord}_{12\sim}^{\text{or}}$

This chapter focuses on the skeletalization of $\text{Sur}_{12\sim}^{\text{or}}$, a key result of this thesis. We implement the skeletalization procedure developed in Section 4.2 to produce a finite presentation equivalent to $\text{Sur}_{12\sim}^{\text{or}}$. Up to a well-understood ambiguity—which we make explicit and are currently working to fully resolve—this finite presentation coincides with the $\mathcal{F}^2$ -presentation described in Appendix ??, and is almost identical to the invertible part of the “Morse presentation” of $\text{Bord}_{12\sim}^{\text{or}}$ conjectured in [BDSPV14].

Before beginning the skeletalization procedure, we slightly simplify $\text{Sur}_{12\sim}^{\text{or}}$ to streamline the construction. Following Example 4.1.4, we assume that all 2-categories of surgery diagrams under consideration are already 0-skeletal, with a single generating 0-cell: a once-and-for-all chosen positively oriented circle. Moreover, we omit the “Isotopy of Morse data” 2-cells from the list of generating 2-cells by using the “Compatibility of isotopies” relation and the ambient isotopy lemma, as explained in Remark 3.3.3. All relations involving isotopy of Morse data 2-cells are adapted accordingly.

To enhance clarity, we organize the construction into three steps, beginning with the skeletalization of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$.

6.1 Skeletalization of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$

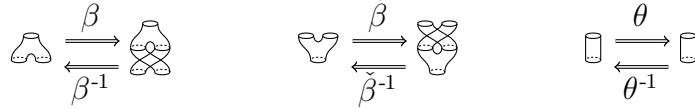
The purpose of this section is to show that $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$ is equivalent to a skeletal presentation, which we refer to as the $\mathcal{F}^0$ presentation. The $\mathcal{F}^0$ presentation has 1 generating 0-cell, 4 generating 1-cells, 6 generating 2-cells, and 7 generating relations, and is defined as follows:

- Generating 0-cell: $\varnothing$

- Generating 1-cells:



- Generating 2-cells:



- Generating relations:

1. **Inverses.**

Each of the generating 2-cells ω satisfies $\omega\omega^{-1} = \text{id}$ and $\omega^{-1}\omega = \text{id}$.

2. **Naturality - Frame flip.**

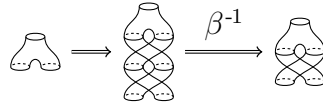
$$\text{pair of pants} \xrightarrow{\beta^2} \text{pair of pants with dot} \xRightarrow{\quad} \text{pair of pants} = \text{pair of pants with colored dot} \xrightarrow[\theta^{-1}, \theta^{-1}]{\theta} \text{pair of pants} \quad \text{N-F } (x)$$

3. **Isotopy of Morse data invariance.**

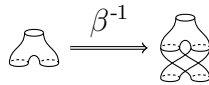
$$\text{cap with green dot} \xrightarrow{\theta} \text{cup} = \text{cup} \xrightarrow{\text{id}} \text{cup} \quad \text{IMD } (x)$$

Remark 6.1.1. For many of the relations we introduce, a rotated version along the x -axis and/or the z -axis also holds. For brevity, we include only one representative form of each such relation and indicate the existence of its rotated counterparts by annotating it with “ (x) ”, “ (z) ”, or “ (xz) ”. Our conventions regarding rotation of relations follow those established in [BDSPV14].

Remark 6.1.2. We will often denote the composite 2-cell



simply by



The analogous convention holds for $\check{\beta}^{-1}$.

We will now begin the skeletalization procedure for $\mathcal{P}_i = \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$.

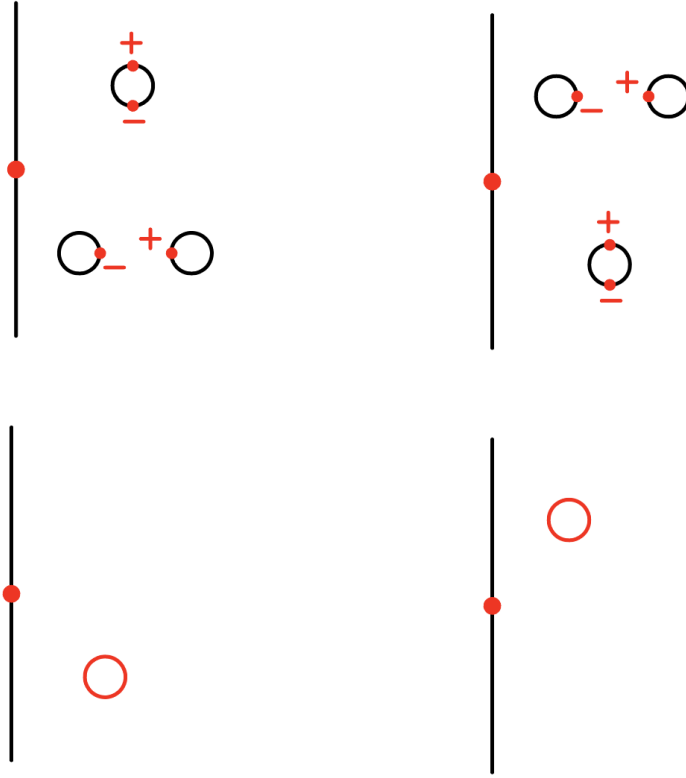
Stage 0: Final generating 0-cells

Since the initial presentation is already 0-skeletal, the final generating 0-cells remain unchanged from $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$, consisting solely of the chosen positively oriented circle:



Stage 1: Final generating 1-cells

We select G_f^1 to consist of four final generating 1-cells:



Remark 6.1.3. The pictures above only capture the embeddings of the attaching spheres. The rest of the surgery data are then uniquely determined up to isotopy and will not generally be shown for readability.

These correspond to the generating 1-cells of the $\mathcal{F}^0$ presentation:



Verification that G_f^1 satisfies the desired conditions:

To show that this choice of G_f^1 satisfies the conditions required by our skeletalization procedure from Section 4.2, we must verify that every initial generating 1-cell is weakly permutogenerated by one of our chosen final generators (completeness) and that no final generating 1-cell is permutogenerated by the other (minimality).

Minimality condition. Given a 1-cell f , we consider

$$\#(\text{Connected components of } f\text{'s target}) - \#(\text{Connected components of } f\text{'s source}).$$

This number remains the same for all 1-cells permutogenerated by f . Moreover, every 2-cell α in $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$ is invertible and satisfies the following condition:

$$\#(\text{Connected components of } \alpha\text{'s source}) = \#(\text{Connected components of } \alpha\text{'s target}).$$

Equipped with these two observations, we can easily verify that no two 1-cells in G_f^1 are permutogenerated by one another.

Completeness. This closely parallels the completeness argument used when selecting the final generating 1-cells of $\text{Sur}_{01\sim}^{\text{or}, \mathcal{F}^0}$. The only difference is that here we need not only naturality-type 2-cells to encode swaps of distinct connected components, but also isotopies to bring the surgery spheres into the “correct” position.

Stage 2: Final generating 2-cells

Phase 2.1: Selection of pointing 2-cells

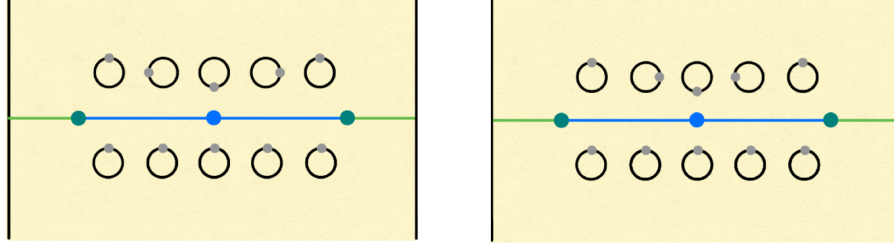
The pointing 2-cells in $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$ are chosen to be all composites of

- Initial generating 2-cells of diffeomorphism type
- Initial generating 2-cells of naturality type
- 2-cells in $\mathcal{S} \cup \mathcal{J}_i$

We let $\mathcal{V}$ denote the collection of pointing 2-cells in $F \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$.

Phase 2.2: Selection of type I final generating 2-cells

We choose the final generating 2-cells of type I to be the two 1-final 2-cells:



These correspond to the following generating 2-cells of the Morse presentation described in [BDSPV14]:

$$\begin{array}{ccc} \text{Cylinder} & \xrightleftharpoons[\theta^{-1}]{\theta} & \text{Cylinder} \end{array}$$

We will refer to them as the right- and left-handed Dehn twists, respectively. Using the standard form of 2-cells in $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$, we can now verify that this choice of IG_f^2 is compatible with the skeletalization procedure, namely:

Lemma 6.1.4. *Every 1-final pointing 2-cell is, up to the equivalence relation generated by R_i , a composite of 2-cells $\otimes$ -generated by*

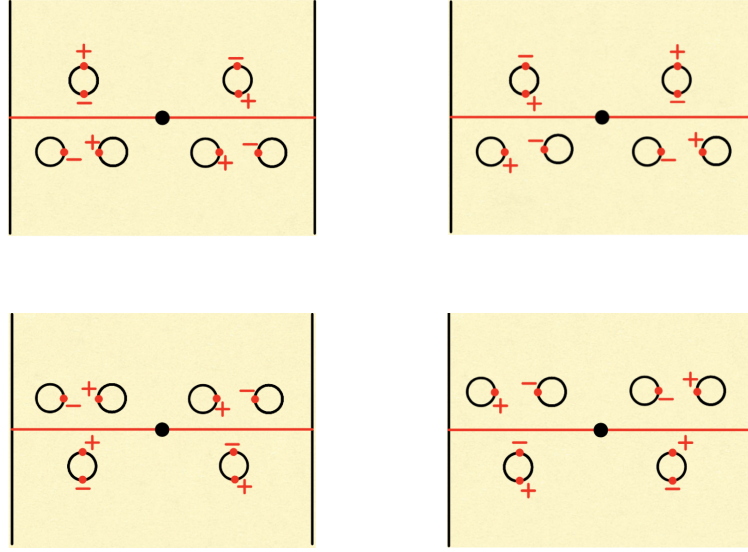
- *Dehn twists*
- *2-cells in $\mathcal{S} \cup \mathcal{I}_f$*
- *Identity 2-cells*

Remark 6.1.5. We henceforth denote 2π by θ .

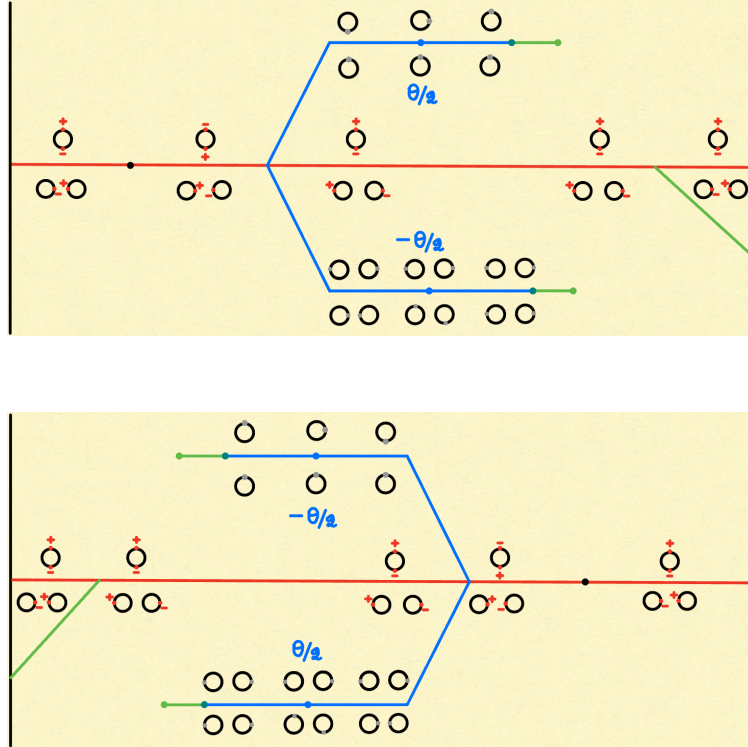
Phase 2.3: Selection of type II final generating 2-cells

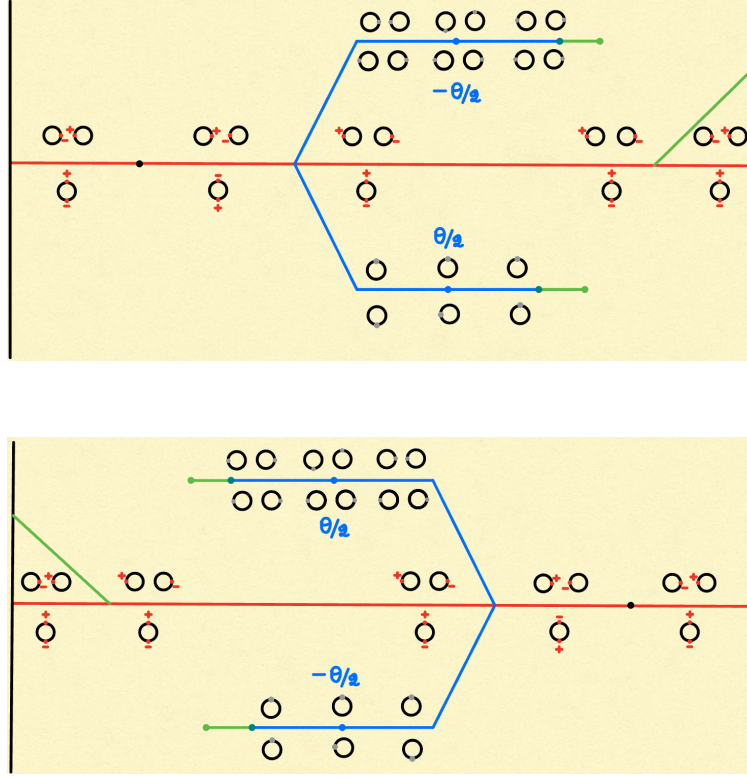
We first need to select A_f^2 . The non-pointing generating 2-cells are precisely the initial generating 2-cells of frame change type. The following lemma then follows easily:

Lemma 6.1.6. *Any initial generating 2-cell of frame change type is $(\mathcal{V}, \mathcal{S})$ -generated by one of the following four:*



We now choose A_f^2 to consist precisely of these four initial generating 2-cells. To choose ${}_IG_f^2$, we select a 1-finalization for each of the four 2-cells that make up A_f^2 as follows:





These correspond to the following generating 2-cells of the $\mathcal{F}^0$ presentation:

$$\begin{array}{ccc}
 \text{Diagram 1} & \xrightleftharpoons[\beta^{-1}]{\beta} & \text{Diagram 2} \\
 \text{Diagram 3} & \xrightleftharpoons[\check{\beta}^{-1}]{\check{\beta}} & \text{Diagram 4}
 \end{array}$$

Remark 6.1.7. Unless explicitly stated otherwise, our convention is that if a single isotopy label is used for multiple connected components, then the same isotopy is applied to each of them.

Stage 3: Final generating relations

Selection of pointing relations

The pointing relations of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$ are as follows:

1. ${}_IT_i$ consists of:
 - Diffeomorphism type relations
 - Naturality type relations
2. ${}_{II}T_i$ consists of:

- Frame change–naturality relations involving symmetries that preserve the relative position of the two points in the image of the 0-dimensional attaching sphere.
3. ${}_IT_f$ consists of the relation witnessing the fact that the right- and left-handed Dehn twists are inverses of one another, namely

$$\begin{array}{c} \text{cylinder} \end{array} \xrightarrow{\theta} \begin{array}{c} \text{cylinder} \end{array} \xrightarrow{\theta^{-1}} \begin{array}{c} \text{cylinder} \end{array} = \text{id}$$

and

$$\begin{array}{c} \text{cylinder} \end{array} \xrightarrow{\theta^{-1}} \begin{array}{c} \text{cylinder} \end{array} \xrightarrow{\theta} \begin{array}{c} \text{cylinder} \end{array} = \text{id}$$

4. ${}_{II}T_f$ is empty

It is straightforward to check that these choices are compatible with the skeletalization procedure, namely:

Lemma 6.1.8. 1. *Every 1-final pointing 2-cell is equivalent, up to ${}_IT_i$, to an identity 2-cell or a composite of 2-cells $\otimes$ -generated by:*

- (a) *Dehn twists*
- (b) *2-cells in $\mathcal{S} \cup \mathcal{J}_f$*

These are exactly the final 2-cells of type I.

- 2. *Given any two final 2-cells of type I α and β , if α is equivalent to β up to ${}_IT_i$, then α is equivalent to β up to ${}_IT_f$.*
- 3. *The only non-pointing initial generating 2-cells are those of frame change type. Any such 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated, up to T_i , by a final generating 2-cell of type II.*
- 4. *Given two final 2-cells α and β , which are $(\mathcal{V}, \mathcal{S})$ -generated by some final generating 2-cell of type II, if α and β are equivalent up to T_i , then they are equivalent up to T_f .*

Phase 3.2: Selection of good 1-finalizations

It suffices to select a good 1-finalization for each initial generating 2-cell. We distinguish the following cases:

1. α is a pointing initial generating 2-cell:

If α involves index 1 surgeries whose 0-dimensional attaching spheres lie in disconnected components that are not adjacent, we begin by pre- and post-composing α with swaps to bring the component containing $+1$ to the immediate right of the component containing -1 , without otherwise modifying the configuration. We then further pre- and post-compose with counterclockwise isotopies to place the surgery spheres in standard position.

2. α is an initial generating 2-cell in A_f^2 :

In this case, the desired 1-finalization is the one already selected as part of the definition of the final generating 2-cells.

3. α is a non-pointing initial generating 2-cell not in A_f^2 :

We first pre- and post-compose α with swaps and counterclockwise isotopies to obtain a 2-cell equivalent (up to T_i) to one of the four elements of A_f^2 . We then use the corresponding 1-finalization selected in the previous case.

Phase 3.3: Selection of the remaining final generating relations

There are three families of remaining initial generating relations we need to consider:

1. Frame change composition
2. Frame change - naturality crossings that involve symmetries which change the relative position of the two points in the image of the 0-dimensional attaching sphere
3. Isotopy of Morse data invariance

Frame change composition

Up to the equivalence relation induced by T_i , all relations coming from frame change composition are induced by the following four:

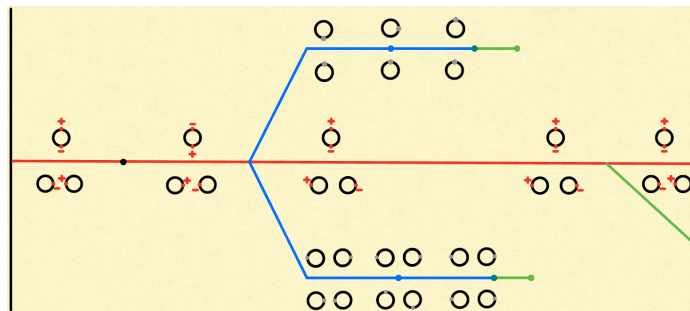
$$\left| \begin{array}{ccc} \textcircled{+} & \textcircled{-} & \textcircled{+} \\ \hline \textcircled{-} \textcircled{+} & \textcircled{+} \textcircled{-} & \textcircled{-} \textcircled{+} \end{array} \right| = \left| \begin{array}{c} \textcircled{+} \\ \hline \textcircled{-} \textcircled{+} \end{array} \right|$$

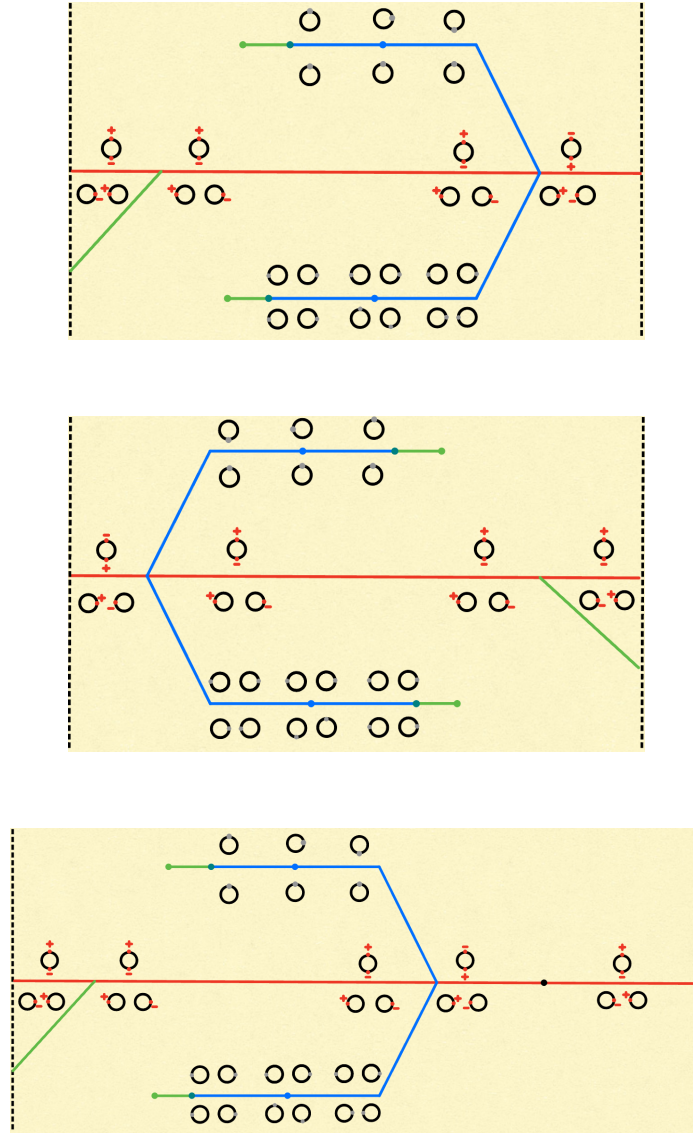
$$\left| \begin{array}{ccc} \textcircled{-} & \textcircled{+} & \textcircled{-} \\ \hline \textcircled{+} \textcircled{-} & \textcircled{-} \textcircled{+} & \textcircled{+} \textcircled{-} \end{array} \right| = \left| \begin{array}{c} \textcircled{-} \\ \hline \textcircled{+} \textcircled{-} \end{array} \right|$$

$$\left| \begin{array}{ccc} \textcircled{-} \textcircled{+} & \textcircled{+} \textcircled{-} & \textcircled{-} \textcircled{+} \\ \hline \textcircled{+} & \textcircled{-} & \textcircled{+} \end{array} \right| = \left| \begin{array}{c} \textcircled{-} \textcircled{+} \\ \hline \textcircled{+} \end{array} \right|$$

$$\left| \begin{array}{ccc} \textcircled{+} \textcircled{-} & \textcircled{-} \textcircled{+} & \textcircled{+} \textcircled{-} \\ \hline \textcircled{-} & \textcircled{+} & \textcircled{-} \end{array} \right| = \left| \begin{array}{c} \textcircled{+} \textcircled{-} \\ \hline \textcircled{-} \end{array} \right|$$

These four relations correspond to the four braiding invertibility relations of the Morse presentation in [BDSPV14]. For example, let's focus on the first of these relations, with the other three admitting a similar treatment. This happens to be a relation between 1-final 2-cells. Using the good 1-finalizations described in the previous phase, and their induced functor from Proposition 4.2.2, we see that the left-hand side is equivalent to the final 2-cell:





Remark 6.1.9. Here, the 2-cell does not fit in a single line, so we are denoting it as a composite of four 2-cells, each occupying a single line. This will happen often from now on, and we will use the dotted side as an indication.

The composite of the two middle 2-cells is a 1-final pointing 2-cell that is equivalent to an identity 2-cell, while the first and last 2-cells are final generating 2-cells. Therefore, the left hand side of the first frame change composition relation is equivalent to the final 2-cell

$$\text{Diagram 1} \xrightarrow{\beta} \text{Diagram 2} \xrightarrow{\beta^{-1}} \text{Diagram 3}$$

and the corresponding final relation is the braiding invertibility relation

$$\text{Y-shape} \xrightarrow{\beta} \text{Crossed Y-shape} \xrightarrow{\beta^{-1}} \text{Y-shape} = \text{id}$$

Similarly, the remaining three frame change composition relations correspond to the other three braiding invertibility relations:

$$\text{Y-shape} \xrightarrow{\beta^{-1}} \text{Crossed Y-shape} \xrightarrow{\beta} \text{Y-shape} = \text{id}$$

$$\text{Y-shape} \xrightarrow{\check{\beta}} \text{Crossed Y-shape} \xrightarrow{\check{\beta}^{-1}} \text{Y-shape} = \text{id}$$

$$\text{Y-shape} \xrightarrow{\check{\beta}^{-1}} \text{Crossed Y-shape} \xrightarrow{\check{\beta}} \text{Y-shape} = \text{id}$$

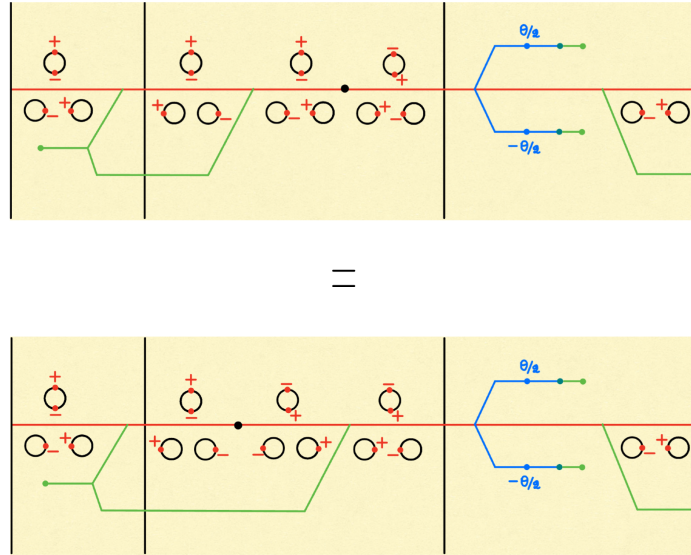
Frame change - naturality crossings

Up to the equivalence relation generated by T_i , all frame change–naturality crossing relations are consequences of the following two:

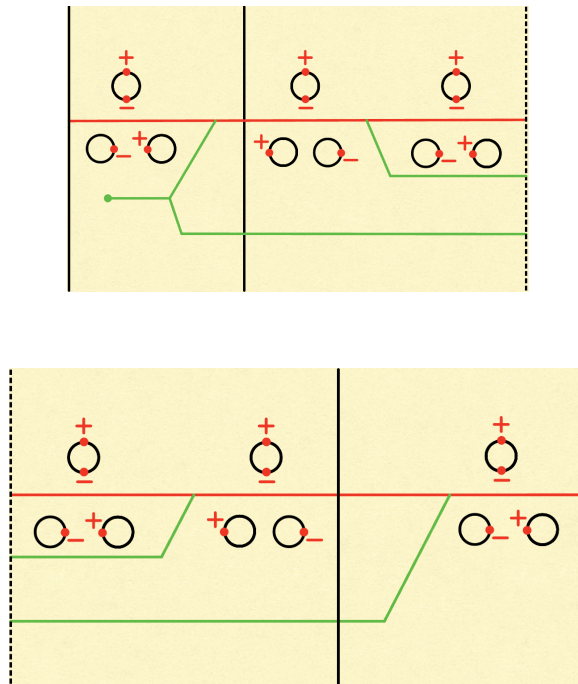
We will see that these correspond to the “N-F (x)” relations of $\mathcal{F}^0$ presentation:

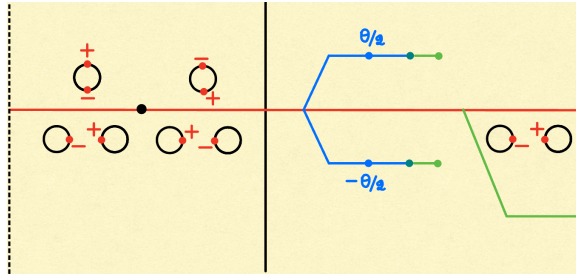
$$\begin{aligned} \text{Y-shape} \xrightarrow{\beta^2} \text{Crossed Y-shape} \xrightarrow{\beta^{-2}} \text{Y-shape} &= \text{Y-shape} \xrightarrow[\theta^{-1}, \theta^{-1}]{\theta} \text{Y-shape} \\ \text{Y-shape} \xrightarrow{\check{\beta}^2} \text{Crossed Y-shape} \xrightarrow{\check{\beta}^{-2}} \text{Y-shape} &= \text{Y-shape} \xrightarrow[\theta]{\theta^{-1}, \theta^{-1}} \text{Y-shape} \end{aligned}$$

For example, let's focus on the first of these relations, with the other admitting a similar treatment. The two 2-cells involved in this relation are not 1-final, so we first need to make an arbitrary 1-finalization. We choose the 1-finalization which leads to the relation:

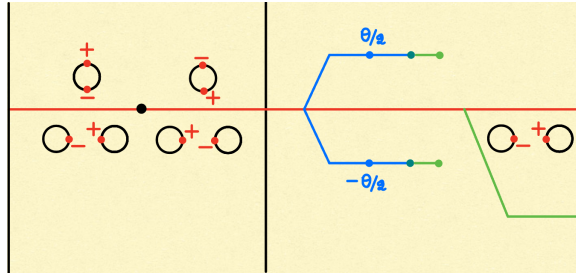


Using the good 1-finalizations, we see that the left-hand side is equivalent to the final 2-cell

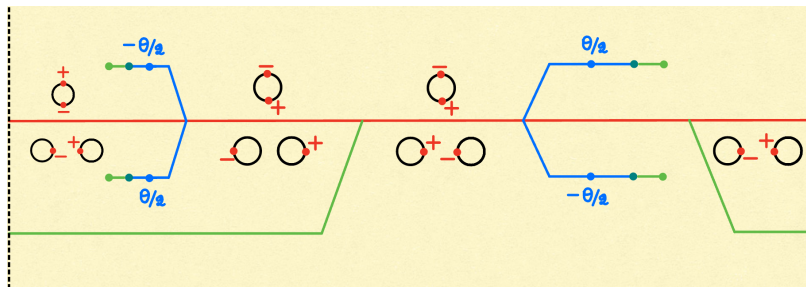
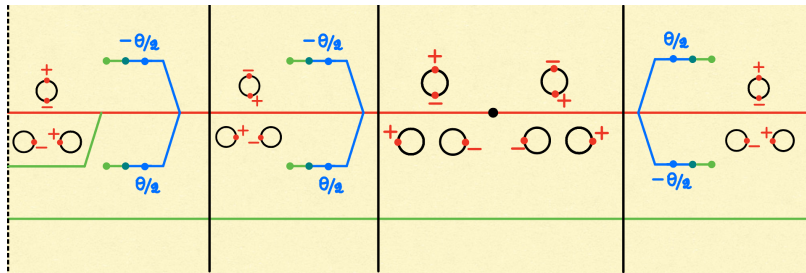
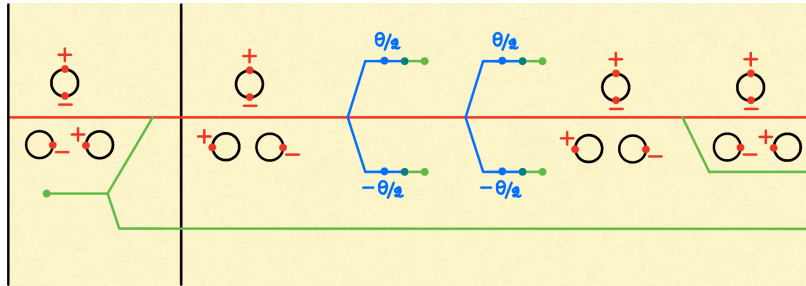




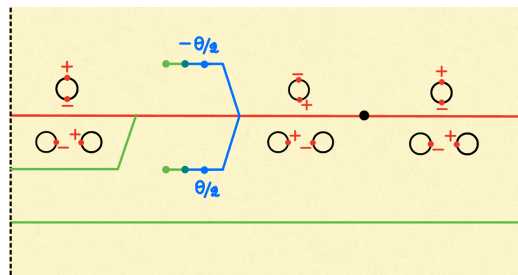
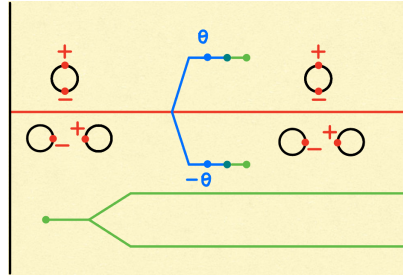
which is equivalent, up to ${}_IT_i$, to the final generating 2-cell



Likewise, we see that the right-hand side is equivalent to the final 2-cell



which is equivalent, up to T_i , to the final 2-cell



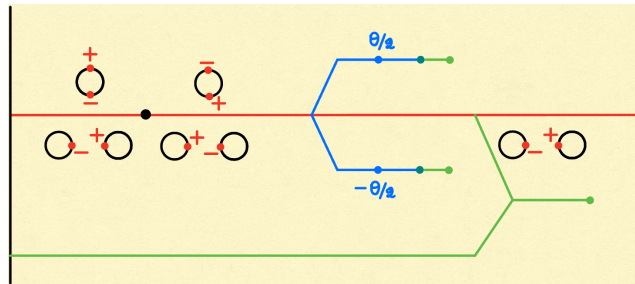
In summary, the initial generating relation

$$\begin{array}{|c|} \hline \begin{array}{c} \text{Diagram 1} \end{array} \\ \hline \end{array} = \begin{array}{|c|} \hline \begin{array}{c} \text{Diagram 2} \end{array} \\ \hline \end{array}$$

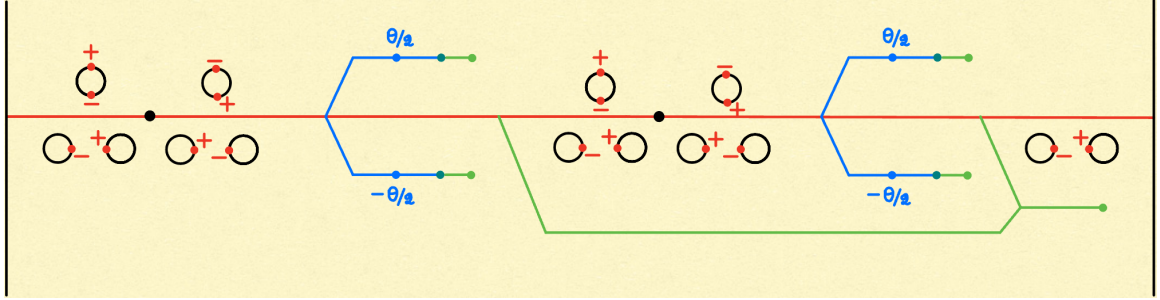
corresponds, via the skeletalization procedure, to the relation

The diagram illustrates an equality between two 2-cell configurations. On the left, a red horizontal line contains a black dot. Above the line are two 1-cells (circles with $+$ and $-$ signs). Below the line are two pairs of 1-cells. A blue 2-cell with labels $\theta/2$ and $-\theta/2$ connects the two pairs. A green line branches off to the right. On the right, the configuration is rearranged, with the blue 2-cell now labeled θ and $-\theta$, and the green line branching off to the left. The two sides are separated by an equals sign.

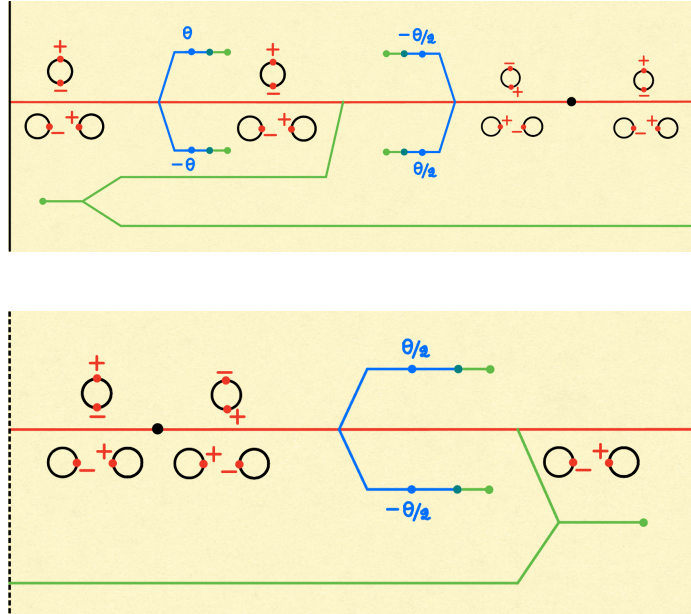
We can now compose both sides of this relation with the invertible final 2-cell



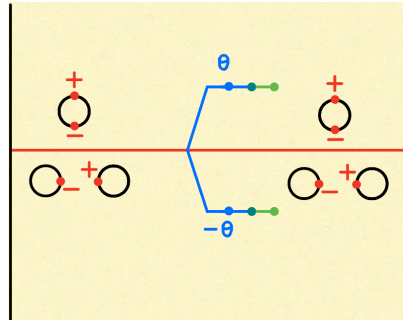
The left-hand side becomes



The right-hand side becomes



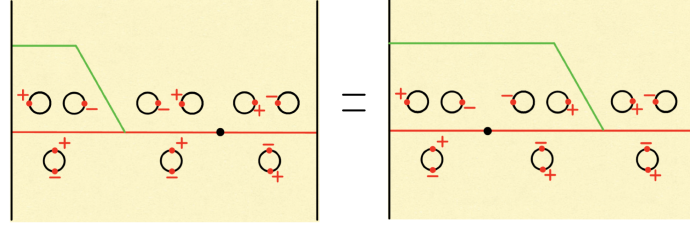
which is equivalent, up to T_i and the braiding invertibility relations, to the final 2-cell



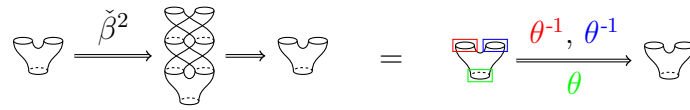
This corresponds to the N-F relation of the $\mathcal{F}^0$ presentation:

$$\begin{array}{c} \text{cup} \end{array} \xrightarrow{\beta^2} \begin{array}{c} \text{cup with crossings} \end{array} \Rightarrow \begin{array}{c} \text{cup} \end{array} = \begin{array}{c} \text{cup with colored boxes} \end{array} \xrightarrow[\theta^{-1}, \theta^{-1}]{\theta} \begin{array}{c} \text{cup} \end{array}$$

Similarly, the initial generating relation

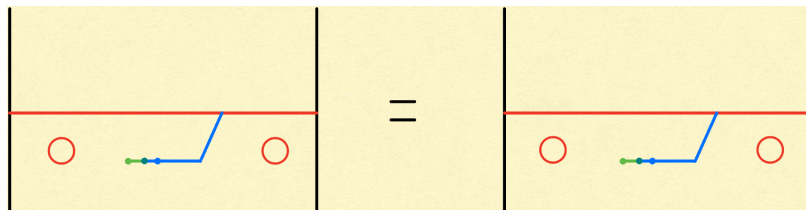
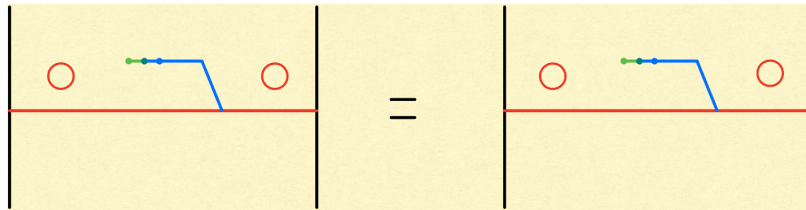
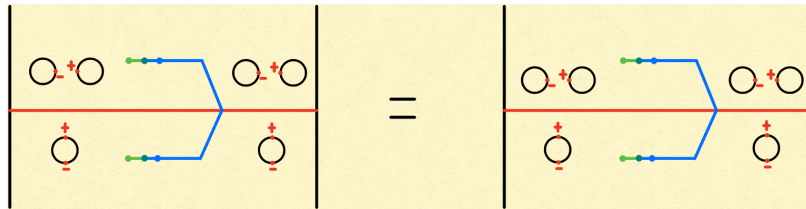
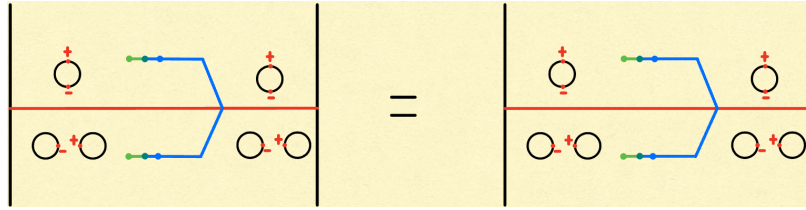


corresponds to the the N-F x relation



Isotopy of Morse data invariance

Up to the equivalence relation generated by T_i , all isotopy of Morse data invariance relations are consequences of the following four families:



We have omitted isotopy labels in the diagrams to improve readability. Each relation compares diagrams with identical underlying shapes but differing isotopy labels that, nonetheless, represent isotopic isotopies of Morse data. Basic considerations from the mapping class group show that the first two families of relations do not yield any new final relations. The third family corresponds to the IMD relation in the $\mathcal{F}^0$ presentation:

$$\begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightarrow{\theta} \text{⌞} = \text{⌞} \xrightarrow{\text{id}} \text{⌞}$$

The fourth family similarly corresponds to the IMD x relation in the $\mathcal{F}^0$ presentation. The above discussion establishes the following result:

Theorem 6.1.10. *The surgery presentation $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$ is equivalent to the $\mathcal{F}^0$ presentation: a skeletal presentation with one generating 0-cell, four generating 1-cells, six generating 2-cells, and seven generating relations, as defined at the beginning of this section.*

6.2 Skeletalization of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$

The purpose of this section is to show that $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$ is equivalent to a skeletal presentation, which we refer to as the $\mathcal{F}^1$ presentation. This presentation builds on the $\mathcal{F}^0$ presentation by adding 22 additional generating 2-cells and 23 additional generating relations:

- Further generating 2-cells:

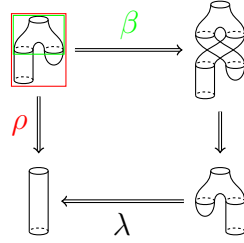
$$\begin{array}{ccc} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\rho^{-1}]{\rho} \text{⌞} \xrightleftharpoons[\lambda]{\lambda^{-1}} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} & \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\check{\rho}^{-1}]{\check{\rho}} \text{⌞} \xrightleftharpoons[\check{\lambda}]{\check{\lambda}^{-1}} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} & \\ \\ \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\alpha^{-1}]{\alpha} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} & \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\check{\alpha}^{-1}]{\check{\alpha}} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} & \\ \\ \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\phi_1^{-1}]{\phi_1} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\phi_2^{-1}]{\phi_2} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} & & \\ \\ \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\text{II}^{-1}]{\text{II}} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} & \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\text{III}^{-1}]{\text{III}} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} & \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \xrightleftharpoons[\text{III}'^{-1}]{\text{III}'} \begin{array}{c} \text{⌞} \\ \text{⌞} \end{array} \end{array}$$

- Further generating relations:

1. **Inverses.**

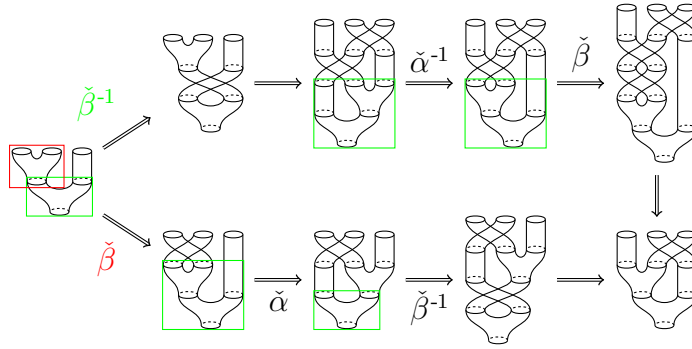
Each of the further generating 2-cells ω satisfies $\omega\omega^{-1} = \text{id}$ and $\omega^{-1}\omega = \text{id}$.

2. **Naturality - Frame flip - Cusp.**

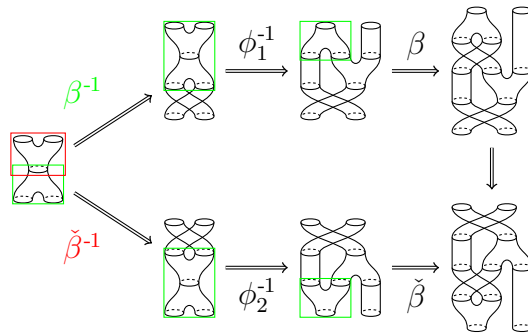


N-F-Cusp (xz)

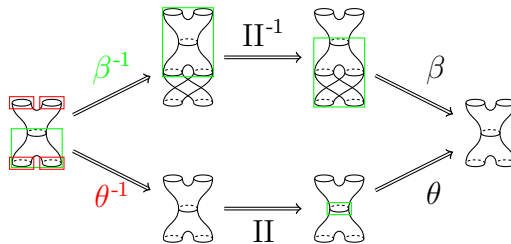
3. **Naturality - Frame flip - Crossing.**



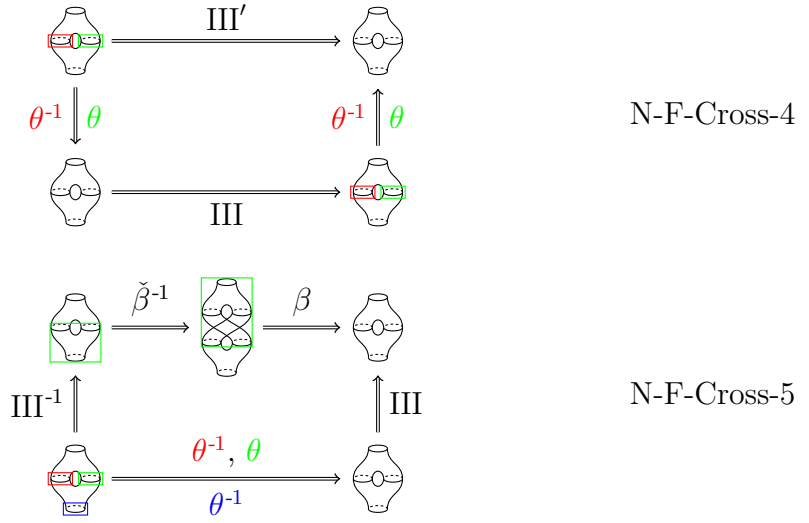
N-F-Cross-1 (x)



N-F-Cross-2 (z)



N-F-Cross-3 (x)



Remark 6.2.1. It is instructive to compare the $\mathcal{F}^1$ presentation with the Morse presentation in [BDSPV14]. Notably, the N-F-Cross (x) relations do not appear explicitly in the Morse presentation. While they can be derived from the other relations given there, our current derivation relies on relations involving non-invertible 2-cells, and it is unclear whether such a derivation exists using only invertible 2-cells.

Moreover, the Morse presentation includes a single relation labeled (FFG2) in place of the N-F-Cross-3 and N-F-Cross-3 (x) relations found here. Our two relations together imply (FFG2), but we are not currently aware of a way to deduce them from (FFG2) from the other relations in the Morse presentation.

Lastly, we can show that the N-F-Cross-4 and N-F-Cross-5 relations introduced here are equivalent to the (FFG3) and (FFG4) z relations in the Morse presentation.

We will now begin the skeletalization procedure for $\mathcal{P}_i = \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$.

Stages 0 and 1: Final generating 0- and 1-cells

The final generating 0- and 1-cells of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$ are identical to those of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$.

Stage 2: Final generating 2-cells

Phase 2.1: Selection of pointing 2-cells

The pointing 2-cells in $F \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$ are all the 2-cells in $F \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$, namely all composites of the following:

- Initial generating 2-cells of diffeomorphism type
- Initial generating 2-cells of naturality type

- Initial generating 2-cells of frame-change type
- 2-cells in $\mathcal{S} \cup \mathcal{J}_i$

We let $\mathcal{V}$ denote the collection of pointing 2-cells in $F \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$.

Phase 2.2: Selection of type I final generating 2-cells

Since the pointing 2-cells in $F \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$ are exactly the 2-cells in $F \text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$, and since the generating relations set of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$ is strictly contained in that of $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$, it follows that the type I final generating 2-cells in $\mathcal{F}^1$ are precisely the final generating 2-cells already selected in $\mathcal{F}^0$. These consist of the right- and left-handed Dehn twists and the four geometric braidings.

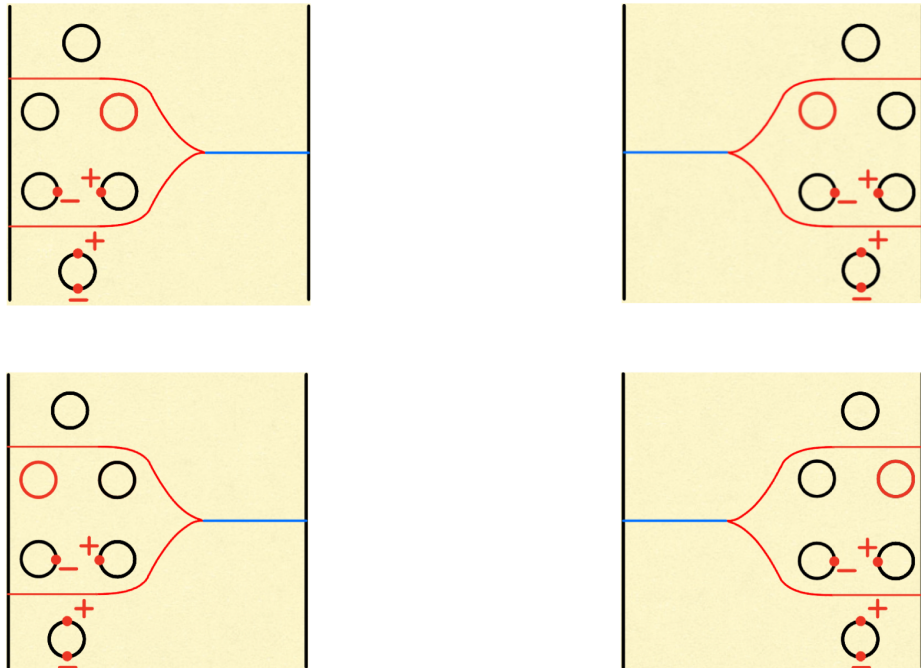
Phase 2.3: Selection of type II final generating 2-cells

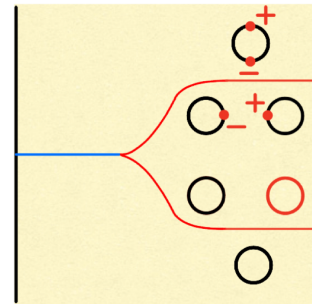
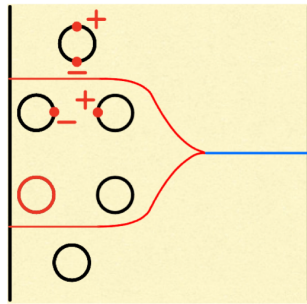
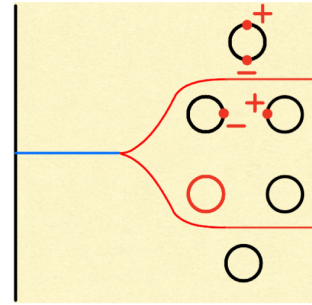
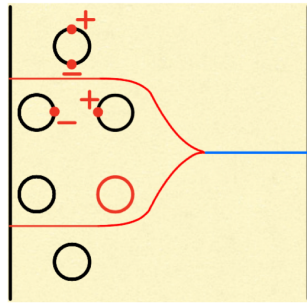
We first need to select A_j^2 . The non-pointing initial generating 2-cells are of the following two types:

- Cusps
- Crossings

It is easy to check that we can make the following simplifications:

Lemma 6.2.2. *Any cusp initial generating 2-cell is $(\mathcal{V}, \mathcal{S})$ -generated by one of the following 8 cusps:*

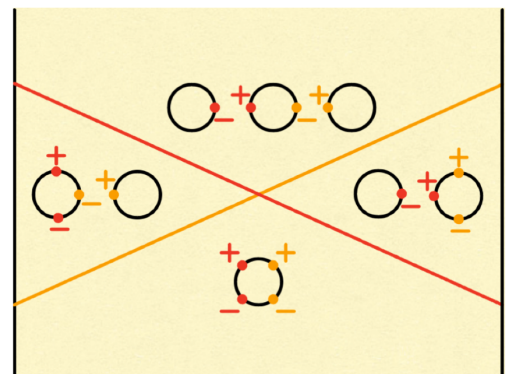
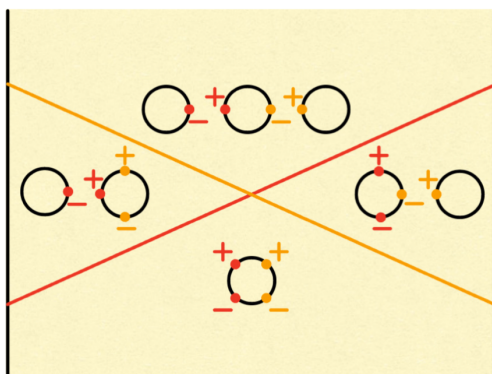




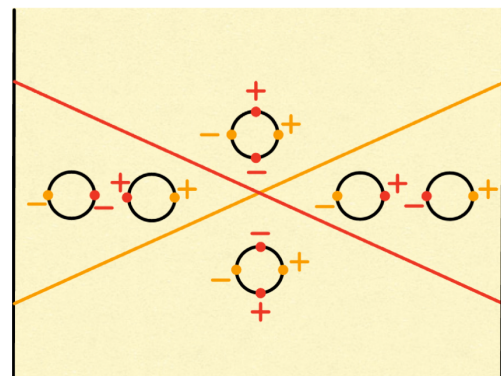
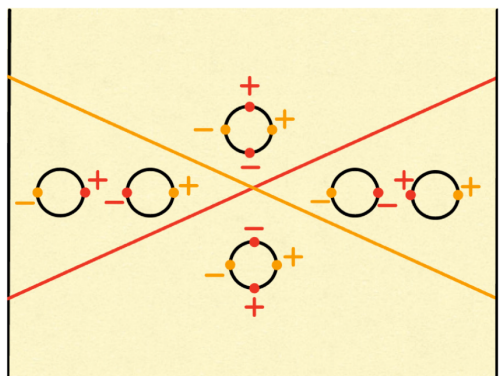
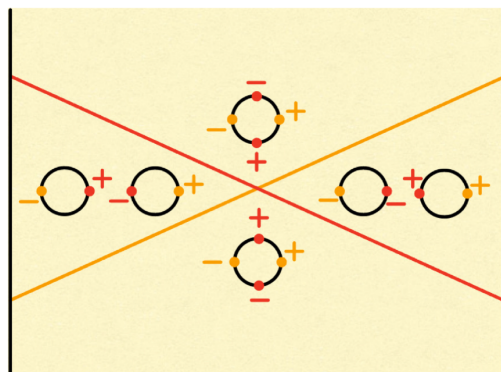
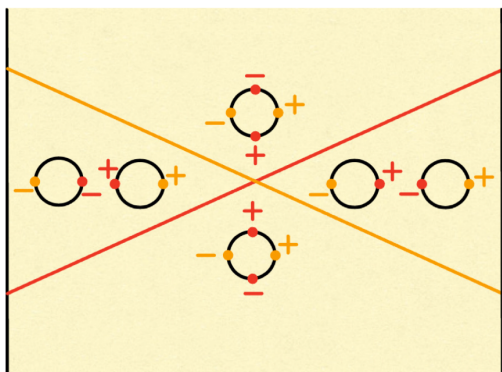
Lemma 6.2.3. *Any crossing initial generating 2-cell is permutogenerated by one of the following 14 crossings:*

Case 1: 

Type 1A: 

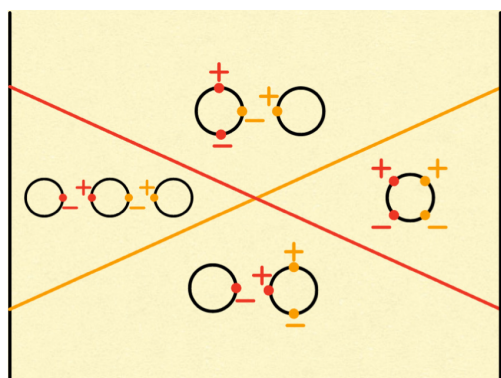
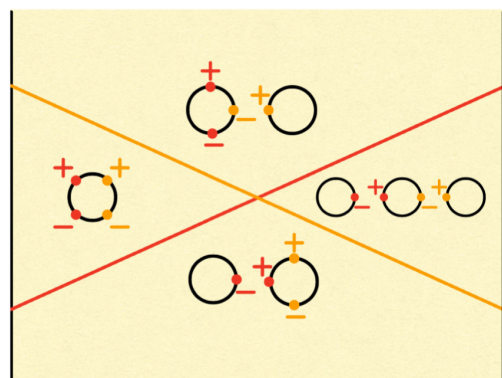


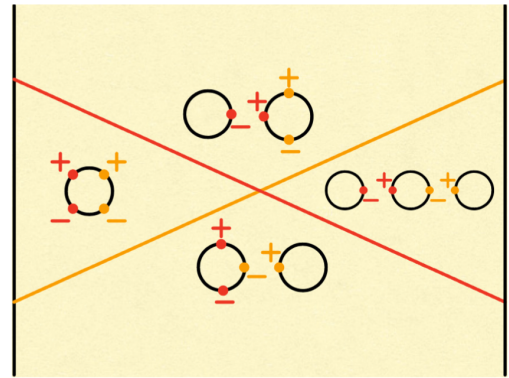
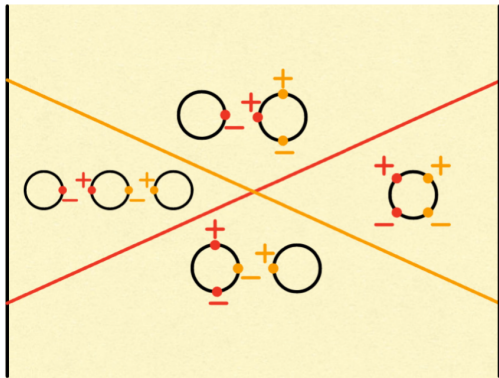
Type 1B: 

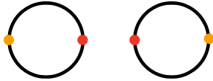


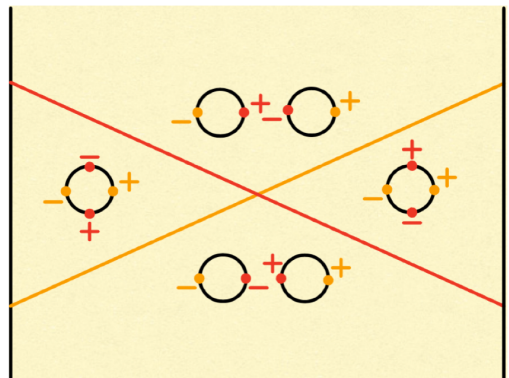
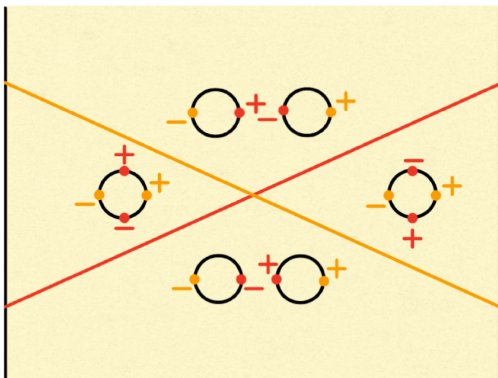
Case 2: 

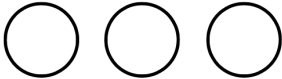
Type 2A:   /   /   /  



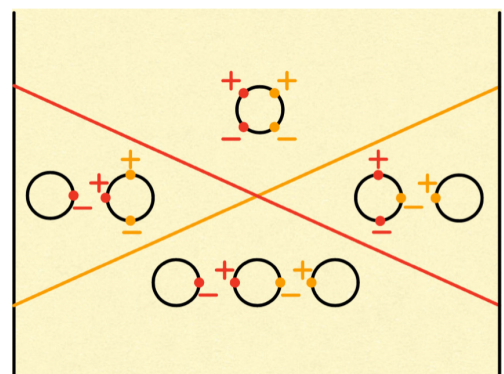
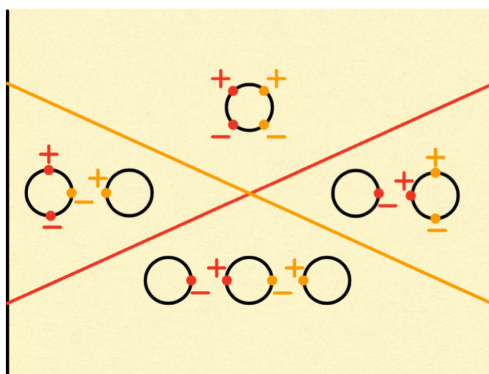


Type 2B: 



Case 3: 

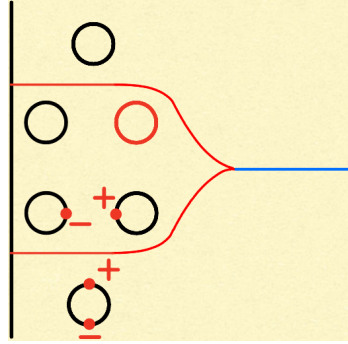




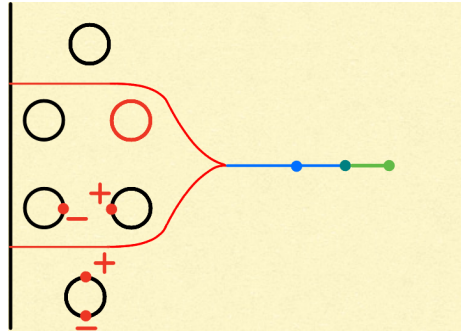
We now define A_f^2 to consist of the 8 cusp 2-cells and 14 crossing 2-cells selected above. These 22 2-cells account for the additional generating 2-cells in the $\mathcal{F}^1$ presentation. To construct ${}_{II}G_f^2$, we choose a 1-finalization for each of these 22 2-cells as follows:

1. Cusps

The chosen 1-finalization for



is



Here, we choose an isotopy from the canceling diffeomorphism to the identity, which keeps the west pole of the circle fixed throughout - any two such isotopies are isotopic rel boundary and no further specification is thus needed. This final generating 2-cell corresponds to the generating 2-cell of the $\mathcal{F}^1$ presentation:

$$\text{Diagram of a cusp} \xrightarrow{\check{\rho}} \text{Diagram of a cylinder}$$

We choose the 1-finalizations of the remaining 7 initial cusps in A_f^2 in an entirely analogous manner. The final generating 2-cells thus obtained correspond to the generating 2-cells of the $\mathcal{F}^1$ presentation:

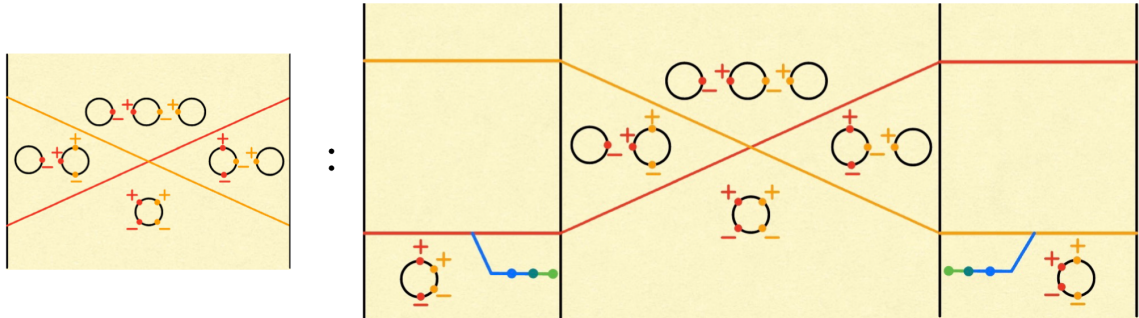
$$\begin{array}{c}
\begin{array}{ccccc}
\text{Y-shape} & \xrightleftharpoons{\check{\rho}} & \text{Cylinder} & \xrightleftharpoons[\check{\lambda}]{\check{\lambda}^{-1}} & \text{Y-shape} \\
& \xleftarrow{\check{\rho}^{-1}} & & \xleftarrow{\check{\lambda}} &
\end{array} \\
\\
\begin{array}{ccccc}
\text{Y-shape} & \xrightleftharpoons{\rho} & \text{Cylinder} & \xrightleftharpoons[\lambda]{\lambda^{-1}} & \text{Y-shape} \\
& \xleftarrow{\rho^{-1}} & & \xleftarrow{\lambda} &
\end{array}
\end{array}$$

2. Crossings

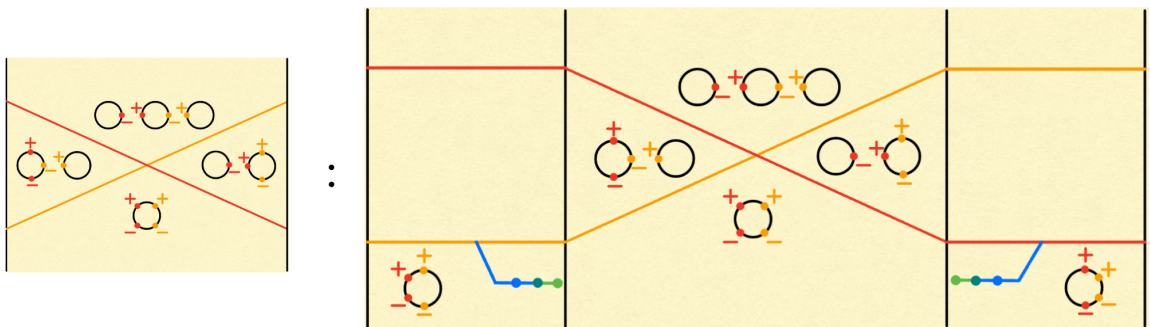
The chosen 1-finalizations for the 14 crossing 2-cells in A_f^2 are grouped below by type, together with their corresponding final 2-cells in the $\mathcal{F}^1$ presentation.

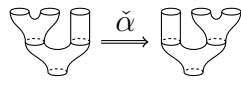
Case 1: 

Type 1A:  / 

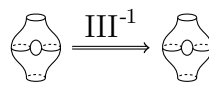
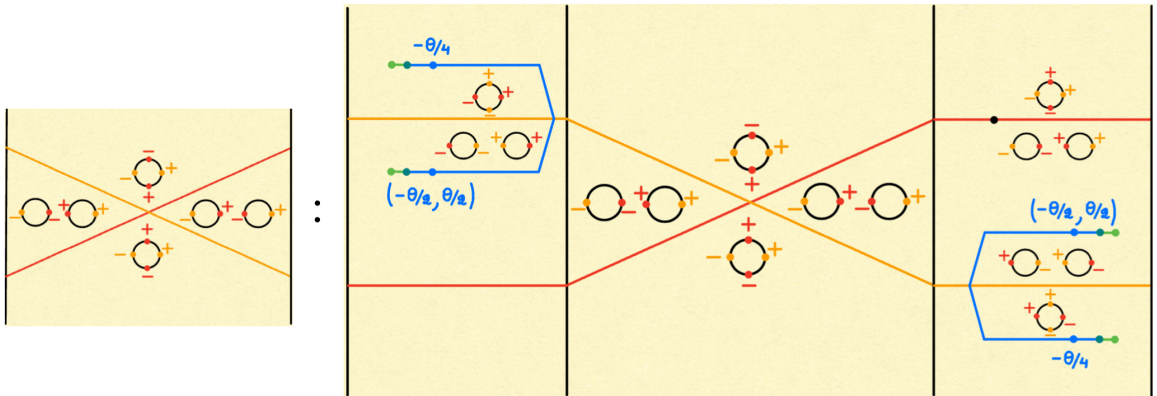
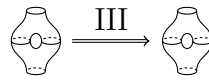
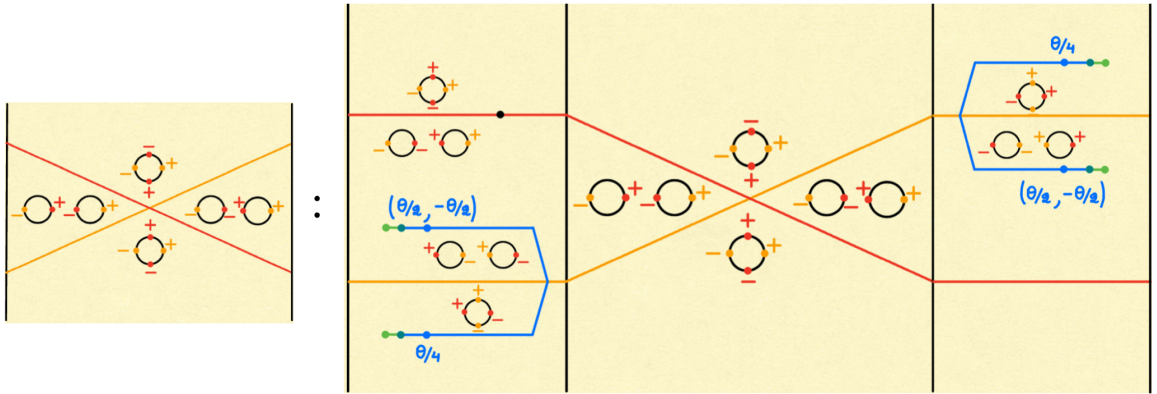


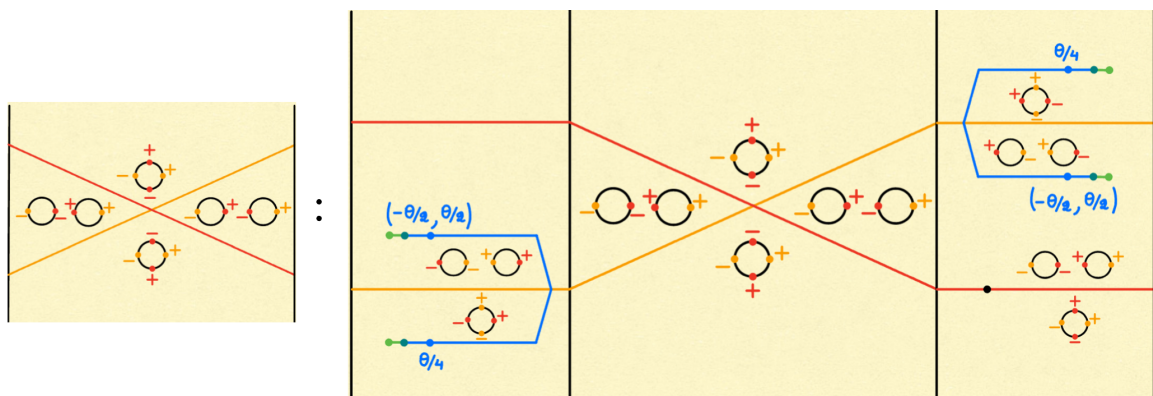
$$\begin{array}{ccc}
\text{Y-shape} & \xrightarrow{\check{\alpha}^{-1}} & \text{Y-shape}
\end{array}$$



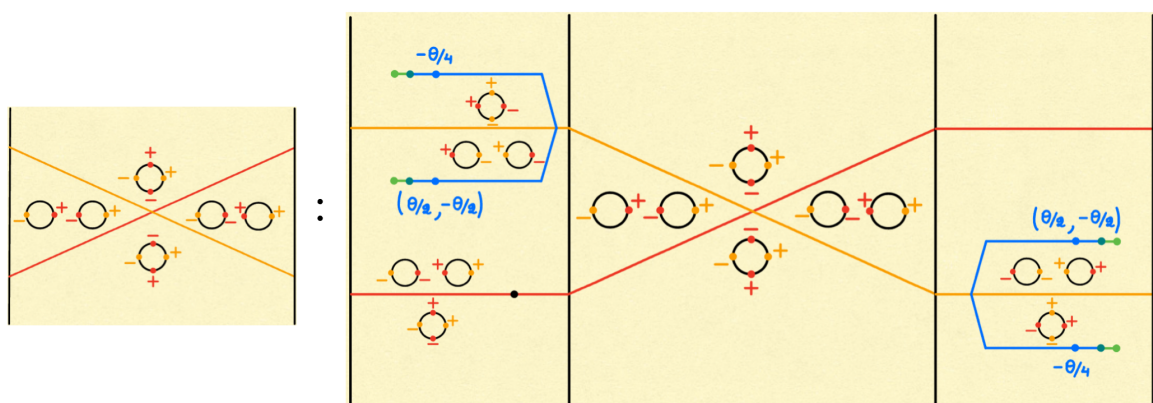


Type 1B: 





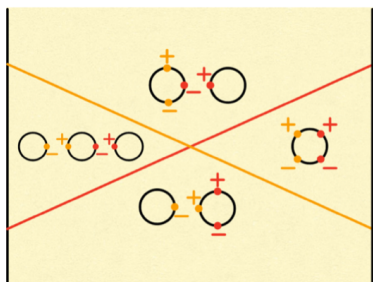
$$\text{Diagram} \xrightarrow{\text{III}'} \text{Diagram}$$



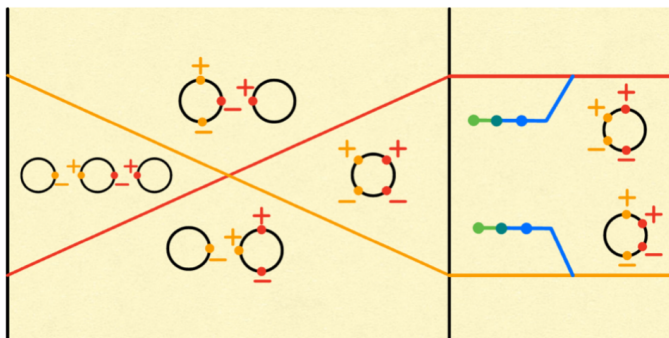
$$\text{Diagram} \xrightarrow{\text{III}'^{-1}} \text{Diagram}$$

Case 2:

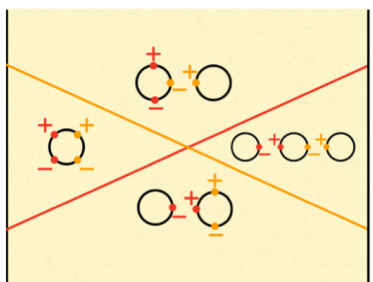
Type 2A:



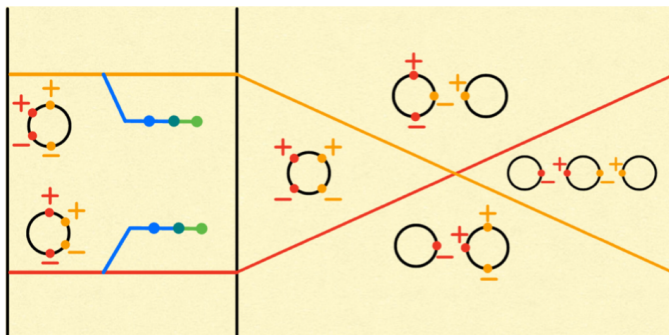
:



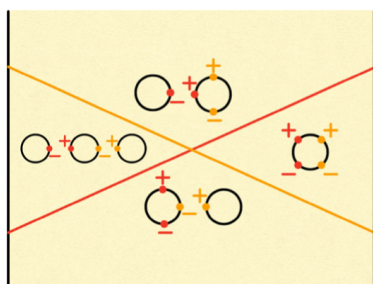
$$\text{Diagram} \xrightarrow{\phi_1} \text{Diagram}$$



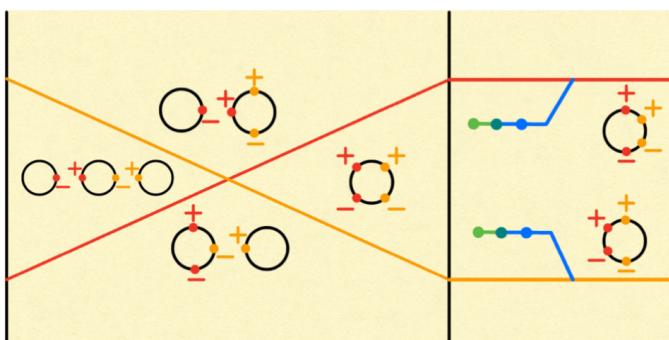
:



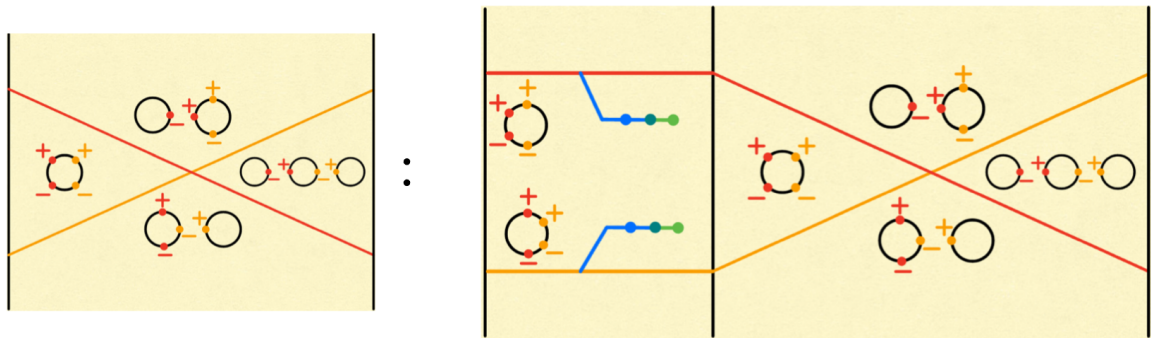
$$\text{Diagram} \xrightarrow{\phi_1^{-1}} \text{Diagram}$$



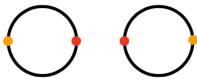
:

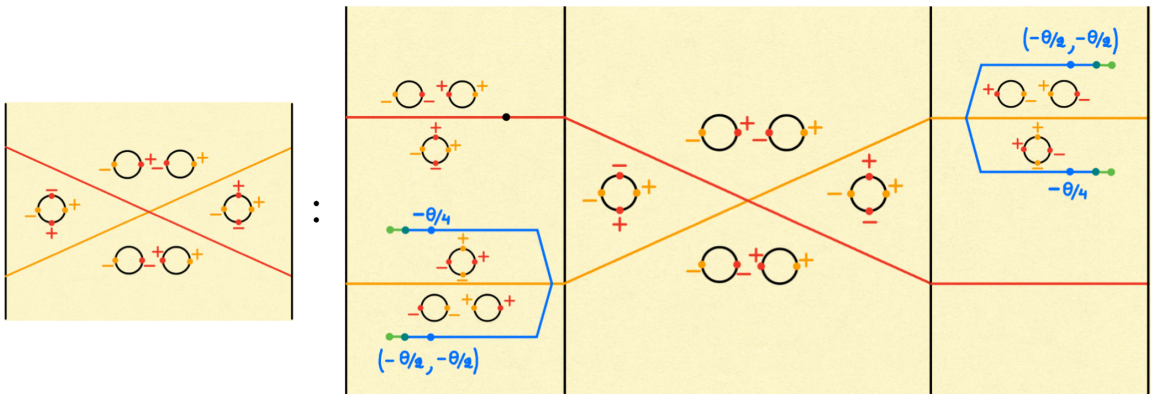


$$\text{Diagram} \xrightarrow{\phi_2} \text{Diagram}$$

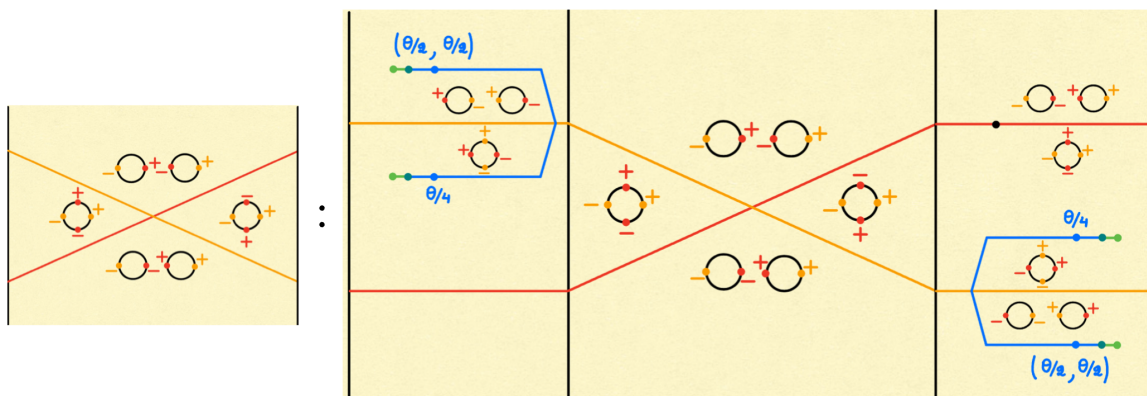


$$\text{X} \xrightarrow{\phi_1^{-1}} \text{Y}$$

Type 2B: 

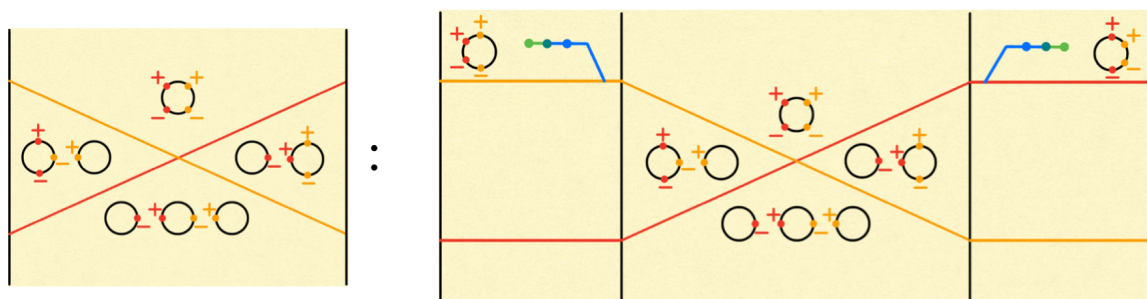


$$\text{X} \xrightarrow{\Pi} \text{Y}$$

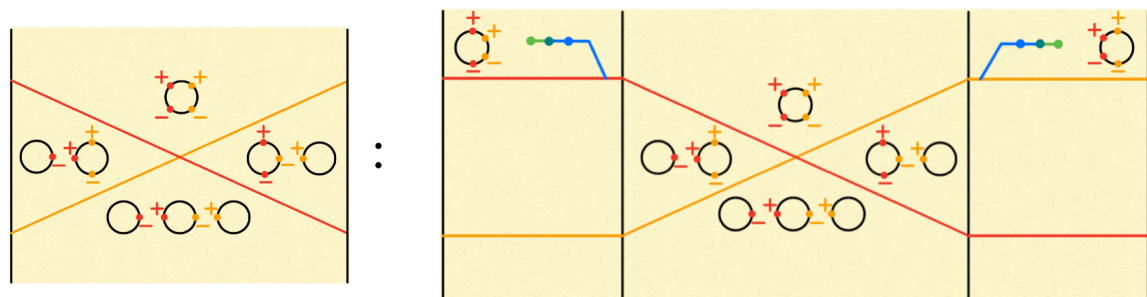


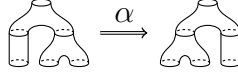
$$\text{Diagram} \xrightarrow{\Pi^{-1}} \text{Diagram}$$

Case 3: ○ ○ ○



$$\text{Diagram} \xrightarrow{\alpha^{-1}} \text{Diagram}$$





Stage 3: Final generating relations

Phase 3.1: Selection of pointing relations

The type I pointing relations in $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$ are chosen as follows:

- ${}_IT_i$ consists of all relations in $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$
- ${}_IT_f$ consists of all final relations in $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^0}$

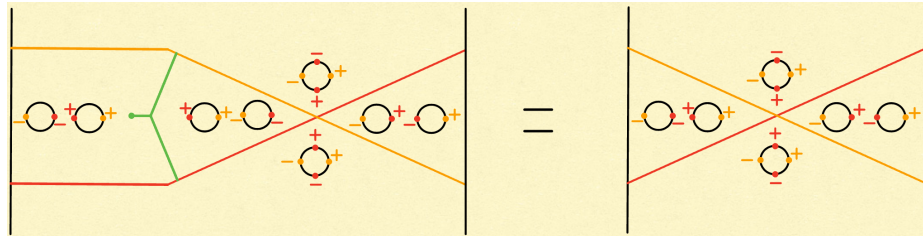
Lemma 6.2.4. *This choice of type I pointing relations is consistent with the skeletalization procedure.*

Proof. Follows immediately from Section 3.1. □

Choosing the type II pointing relations is a bit more intricate, though verifying that our choice is compatible with the skeletalization procedure should be relatively straightforward. Let's begin with choosing ${}_{II}T_i$ - it consists of:

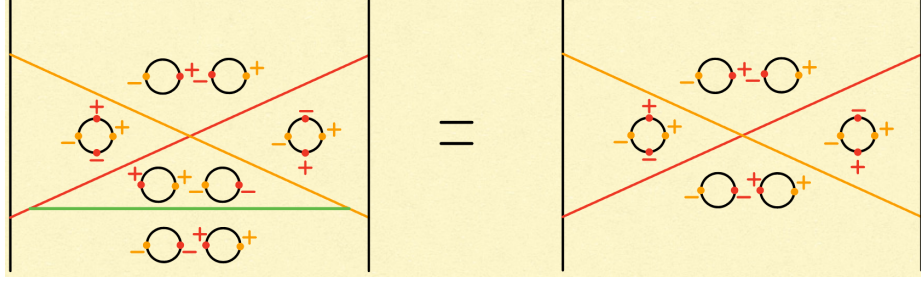
- All naturality - beak relations
- All naturality - Reidemeister III relations for Type 1A, Type 2A and Type 3 crossings. For Type 1B and 2B crossings we include all naturality - Reidemeister III relations that involve diffeomorphisms which do not permute past each other the circles where the image of the attaching surgery spheres lie. Moreover, the following naturality - Reidemeister III relations are also chosen to be in ${}_{II}T_i$:

– For Type 1B crossings:



and the analogous relations for the remaining three type 1B crossing 2-cells in A_f^2 .

- For Type 2B crossings:



and the analogous relations for the remaining type 2B crossing 2-cell in A_f^2 .

- The following frame flip-crossing relations:
 - **Type 1A crossings:** Relations that encode the movement of the frame flip dot across the descending diagonal in the Cerf graphic of the underlying surgery diagram.
 - **Type 2A crossings:** There are four such 2-cells in A_f^2 , each relating 2-cells whose source manifold is a disjoint union of two circles. The two bottom surgery data induce two embeddings of surgery spheres: one entirely contained within a single connected component, and one intersecting both. We consider frame flip-crossing relations across the diagonal corresponding to the embedding contained in a single component.
 - **Type 2B crossings:** Relations encoding the movement of the frame flip dot across the descending diagonal in the underlying Cerf graphic.
 - **Type 3 crossings:** Relations encoding the movement of the frame flip dot across the ascending diagonal in the underlying Cerf graphic.

Furthermore, we choose $_{II}T_f$ to be empty. The following lemma holds by a direct, albeit tedious, unwinding of the definitions:

Lemma 6.2.5. *This choice of type II pointing relations is consistent with the skeletalization procedure.*

Phase 3.2: Selection of good 1-finalization

This is analogous to the selection of good 1-finalizations for $\text{Sur}_{12\sim}^{\text{or}}$.

Selection of the remaining final generating relations

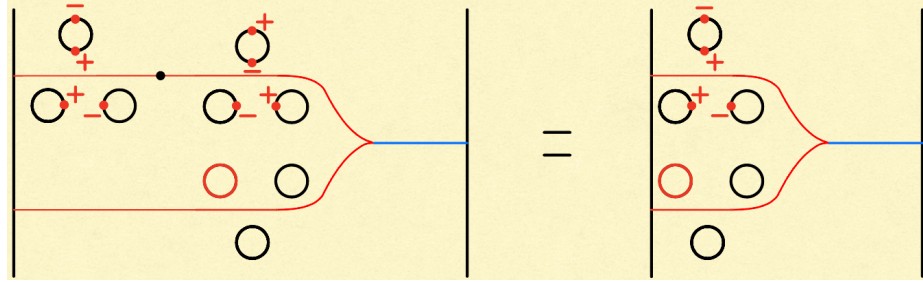
We now turn to the remaining initial generating relations that are not of pointing type. These include all eye birth-death, merge-unmerge, Reidemeister II, frame flip-cusp relations, as well as the non-pointing frame flip-crossing and naturality-Reidemeister III relations. For each of these families, we will select representative relations that generate all others up to T_i , and specify the corresponding final relation associated with each representative.

Eye birth-death, merge-unmerge and Reidemeister II

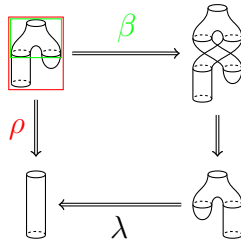
These correspond to the generating relations in the $\mathcal{F}^1$ presentation labeled *Inverses*.

Frame flip-cusp relations

Up to T_i , it suffices to consider the following frame flip-cusp relation and its analogues for the remaining seven cusp 2-cells in A_f^2 :



The corresponding relation in the $\mathcal{F}^1$ presentation is N-F-Cusp:



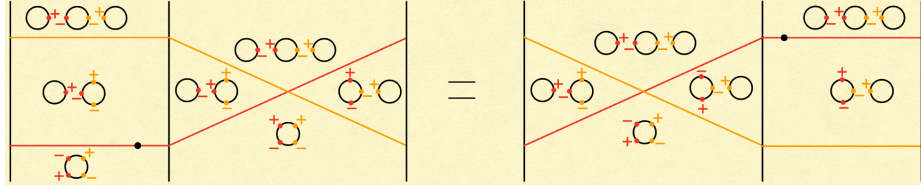
The corresponding relations for the other three cusp 2-cells representing death paths are N-F-Cusp x , N-F-Cusp z and N-F-Cusp xz . For cusp 2-cells representing birth paths, the relevant frame flip-cusp relations correspond to final relations equivalent to those already obtained for the cusp 2-cells representing death paths.

Relations involving crossings

1. Frame flip-crossing relations

- **Type 1A crossings:**

Up to the equivalence relation T_i , it suffices to consider the following:



This corresponds to the final relation N-F-Cross-1.

- **Type 1B crossings:**

Up to T_i , we distinguish two cases:

- For the crossings in A_f^2 corresponding to III and III⁻¹:
 - (a) The frame flip-crossing along the descending diagonal corresponds to N-F-Cross-4.
 - (b) The one along the ascending diagonal corresponds to N-F-Cross-5.
- For the crossings corresponding to III' and III'⁻¹: These can be deduced from the above two using relations in T_i .

- **Type 2A crossings:**

Up to T_i , it suffices to consider the frame flip-crossing across the diagonal associated with the surgery sphere embedding that intersects both components of the source manifold, for the crossings in A_f^2 corresponding to ϕ_1 and ϕ_2 . These give rise to the relations N-F-Cross-2 and N-F-Cross-2 z .

- **Type 2B crossings:**

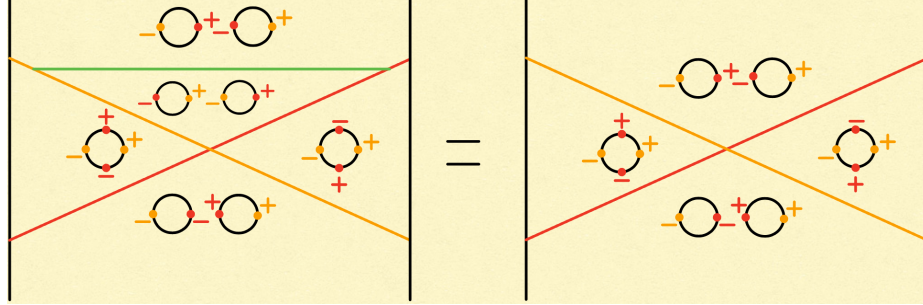
Up to T_i , it suffices to consider the frame flip-crossing along the ascending diagonal of the Type 2B crossing in A_f^2 corresponding to the II. This gives the relation N-F-Cross-3.

- **Type 3 crossings:**

Up to T_i , consider the frame flip-crossing along the descending diagonal of the Type 3 crossing in A_f^2 corresponding to α . This corresponds to N-F-Cross-1 x .

2. Naturality-Reidemeister II

Up to the equivalence relation T_i , it suffices to consider the following:



This corresponds to the final relation N-F-Cross-3 x .

The above discussion establishes the following result:

Theorem 6.2.6. *The surgery presentation $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$ is equivalent to the $\mathcal{F}^1$ presentation: a skeletal presentation with one generating 0-cell, four generating 1-cells, twenty-eight generating 2-cells, and thirty generating relations, as defined at the beginning of this section.*

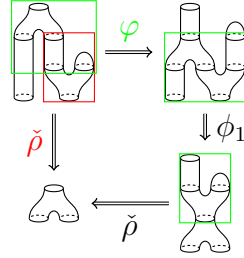
6.3 Skeletalization of $\text{Sur}_{12\sim}^{\text{or}}$

The purpose of this section is to describe the current status of our efforts to establish that Sur^{or} is equivalent to a skeletal presentation, which we refer to as the $\mathcal{F}^2$ presentation. This presentation extends the $\mathcal{F}^1$ presentation by incorporating 36 additional generating relations, organized as follows:

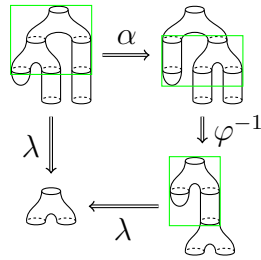
1. Swallowtail.

$$\begin{aligned}
 & \text{Diagram 1} \xrightarrow{\check{\lambda}^{-1}} \text{Diagram 2} \xrightarrow{\varphi^{-1}} \text{Diagram 3} \xrightarrow{\check{\rho}} \text{Diagram 4} = \text{id} & \text{ST1 } (x) \\
 & \text{Diagram 5} \xrightarrow{\rho^{-1}} \text{Diagram 6} \xrightarrow{\alpha} \text{Diagram 7} \xrightarrow{\lambda} \text{Diagram 8} = \text{id} & \text{ST2 } (x) \\
 & \text{Diagram 9} \xrightarrow{\rho^{-1}} \text{Diagram 10} \xrightarrow{\Pi} \text{Diagram 11} \xrightarrow{\rho} \text{Diagram 12} \xrightarrow{\theta^{-1}} \text{Diagram 13} = \text{id} & \text{ST3 } (xz)
 \end{aligned}$$

2. Beak.

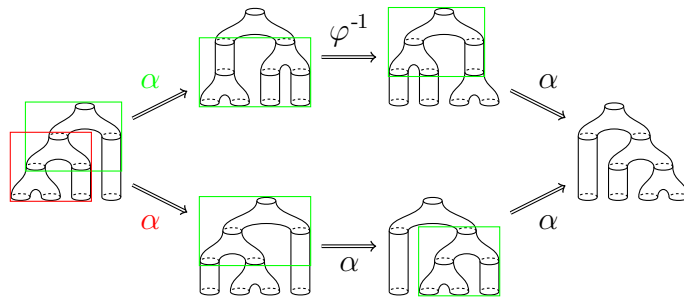


B1 (xz)

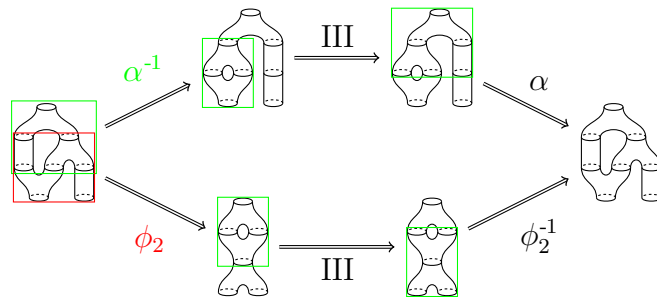


B2 (xz)

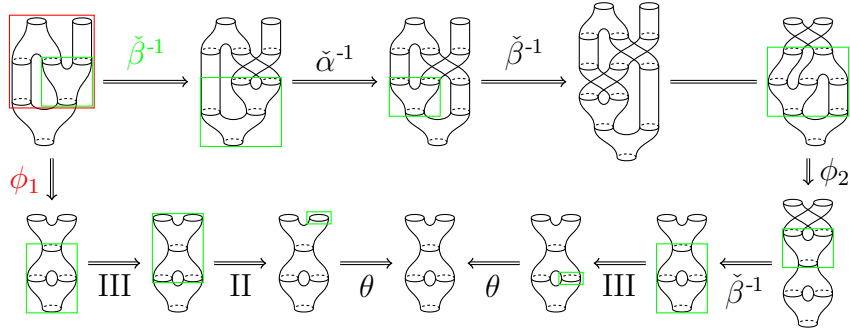
3. Reidemeister III.



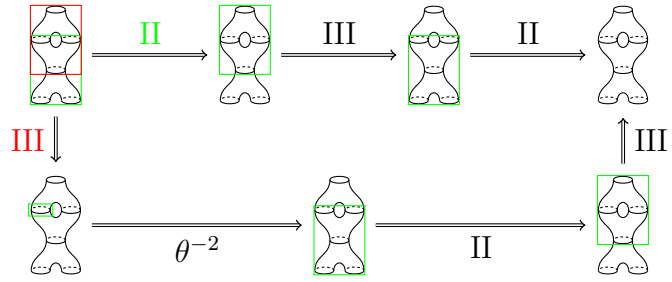
T1 (x)



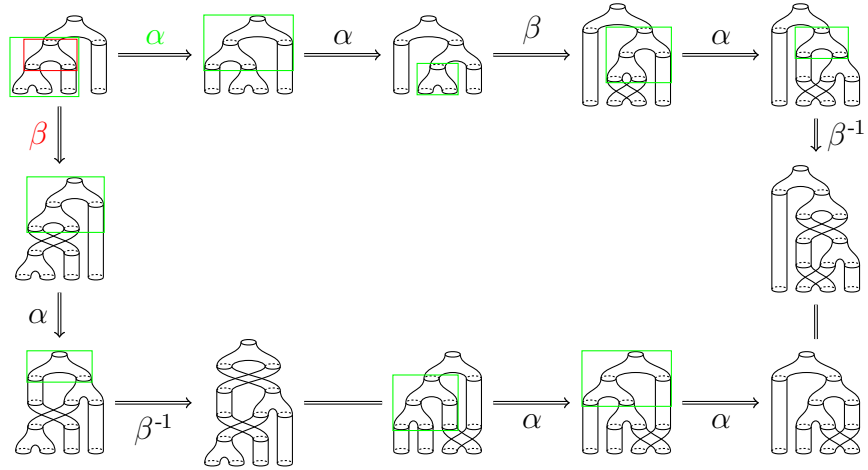
T2 (x)



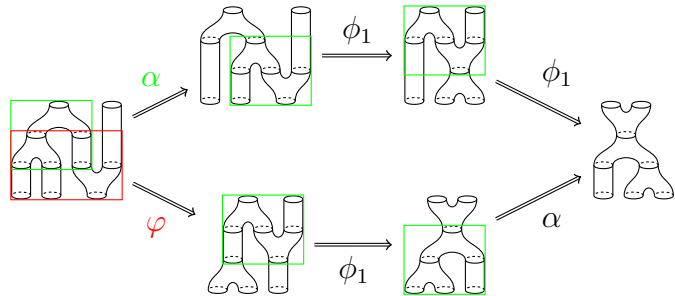
T3 (x)



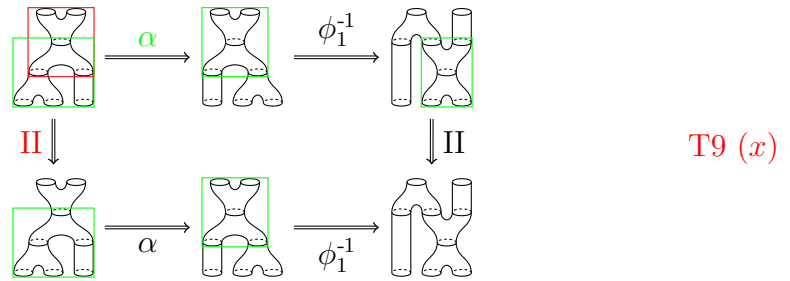
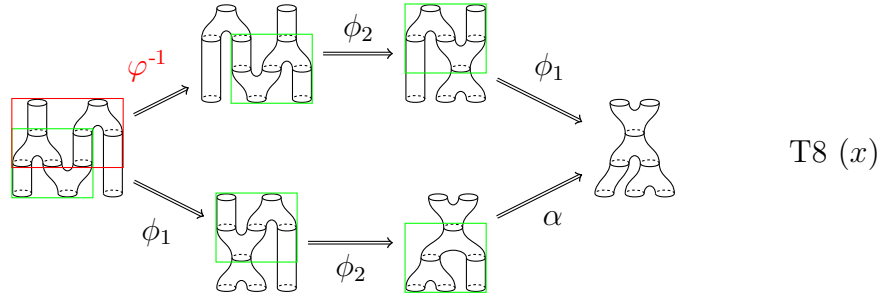
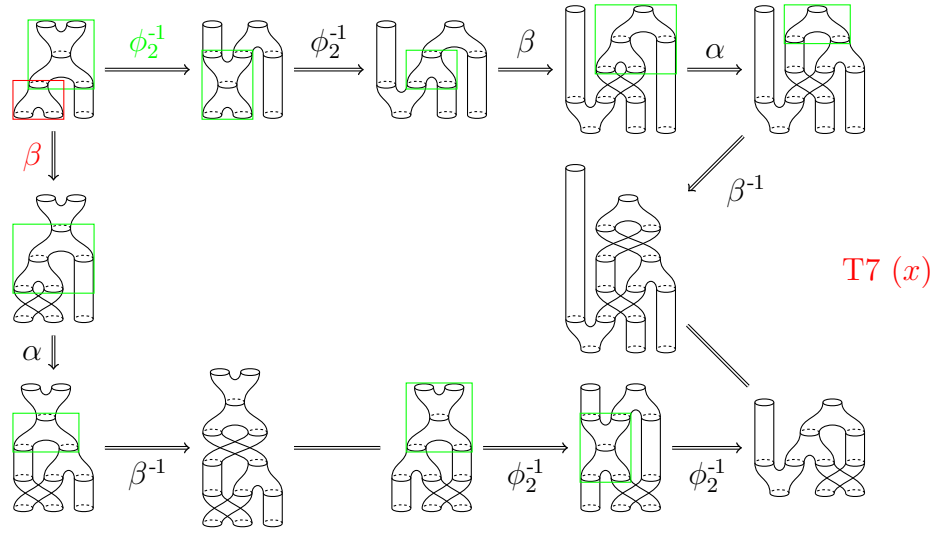
T4 (x)

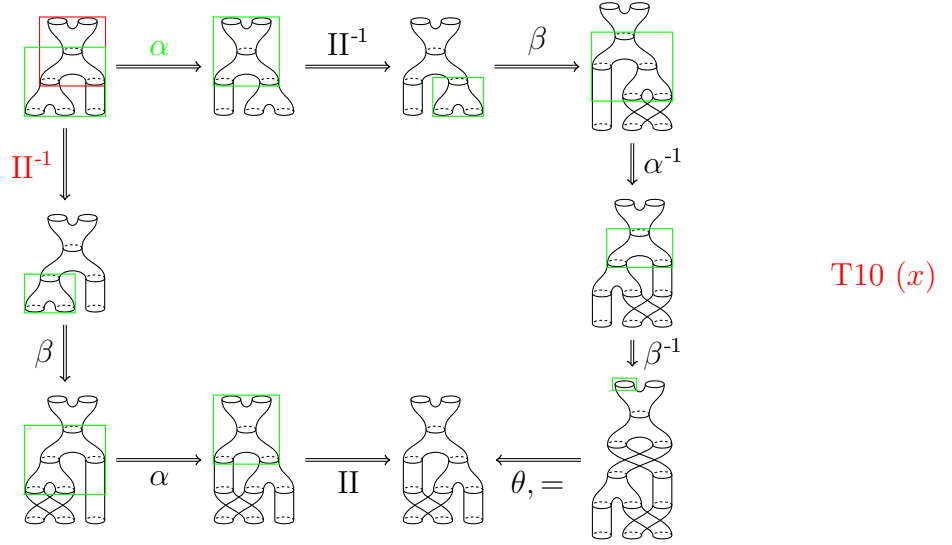


T5 (x)



T6 (x)





We will explain why exactly thirty-six initial generating relations remain to be accounted for and describe which additional relations in the $\mathcal{F}^2$ presentation correspond to them. However, the reader should note that we have not yet explicitly carried out the skeletalization for the Reidemeister III relations marked in red—namely: $T2(x)$, $T3(x)$, $T4(x)$, $T5(x)$, $T7(x)$, $T9(x)$, and $T10(x)$. While we do identify the initial generating relations to which these correspond, the precise form they take in the $\mathcal{F}^2$ presentation should for now be regarded as conjectural. We are currently working to complete the necessary computations and expect to resolve these remaining details shortly.

Remark 6.3.1. The Morse presentation in [BDSPV14] includes z -rotated versions of certain Reidemeister III relations. However, as we demonstrate below, these are redundant: each such relation follows from the others up to equivalence.

We now proceed to skeletalize $\text{Sur}_{12\sim}^{\text{or}}$, building upon the results established in the previous sections.

Stages 0, 1, and 2: Final generating 0-, 1-, and 2-cells

The final generating 0-, 1-, and 2-cells are identical to those of $\text{Sur}_{12\sim}^{\text{or}}$. The same holds for the pointing 2-cells.

Stage 3: Final generating relations

Phase 3.1: Selection of pointing relations

The set of initial pointing relations T_i includes all relations from $\text{Sur}_{12\sim}^{\text{or}, \mathcal{F}^1}$, while the set T_f of final pointing relations consists of all relations appearing in the $\mathcal{F}^1$ presentation.

Phase 3.2: Selection of good 1-finalizations

We retain the same choices of good 1-finalizations as in $\text{Sur}_{12\sim}^{\text{or}}$.

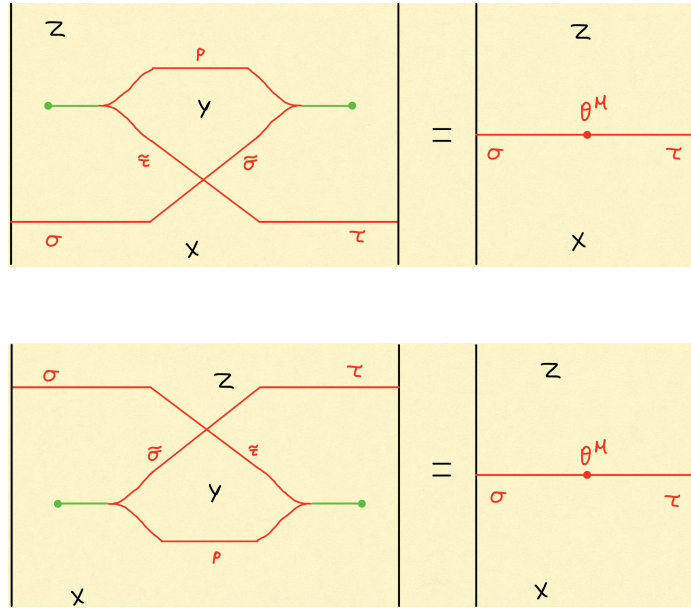
Phase 3.3: Selection of the remaining final generating relations

Three families of remaining initial generating relations require analysis: swallowtail, beak, and Reidemeister III. As expected, representatives of these families correspond to the swallowtail, beak, and Reidemeister III relations in the $\mathcal{F}^2$ presentation, respectively.

The swallowtail and beak relations can be treated relatively directly (up to T_i), with only a minor subtlety—which we will address via an explicit example. In contrast, the Reidemeister III relations remain one of the most challenging components of the project. We now examine each of these families in more detail.

Swallowtail

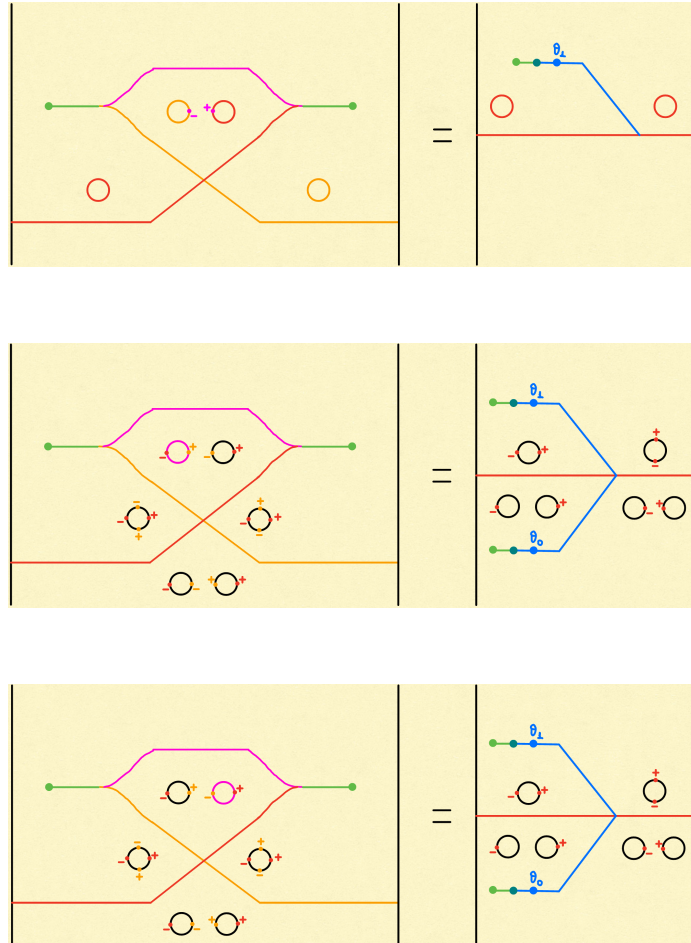
We begin by recalling that the initial swallowtail generating relations take one of the following two forms:

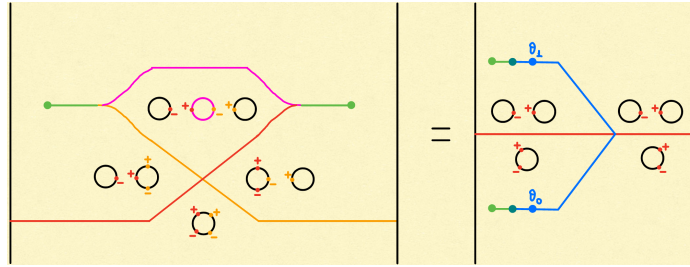


We focus on swallowtails of the first form, as the treatment of the second is entirely analogous. Up to T_i , all such relations are determined by the triple $(Y, \rho_Y, \tilde{\tau}_Y, \tilde{\sigma}_Y)$, where the surgery sphere ρ_Y intersects both $\tilde{\tau}_Y$ and $\tilde{\sigma}_Y$ transversely at a single point in Y , and where $\tilde{\tau}_Y$ and $\tilde{\sigma}_Y$ are disjoint. This geometric condition significantly constrains the possible configurations. There are two main cases to consider:

1. ρ is an index 1 surgery. Then $\tilde{\tau}$ and $\tilde{\sigma}$ are disjoint index 0 surgeries, and Y has two connected components.
2. ρ is an index 2 surgery. Then $\tilde{\tau}$ and $\tilde{\sigma}$ are disjoint index 1 surgeries, and there are two subcases:
 - (a) Y has two connected components, each appearing in the image of both $\tilde{\tau}$ and $\tilde{\sigma}$.
 - (b) Y has three connected components, only one of which lies in the image of both $\tilde{\tau}$ and $\tilde{\sigma}$.

Each of these cases is represented by one of the following diagrams. Up to T_i , any initial swallowtail generating relation of the first form is equivalent to one of these:





These correspond respectively to the relations ST1 x , ST3 xz , ST3 x , and ST2 x in the $\mathcal{F}^2$ presentation.

One subtlety remains in establishing this correspondence. In the initial swallowtail relations, the isotopy of Morse data on the right-hand side is not explicitly specified. Fortunately, in dimension 2, this ambiguity can be resolved directly. For example, to verify the relation ST3 (x), it suffices to determine the isotopy class of the right-hand side by tracing the image of a collection of generic curves in the source 1-cell through the sequence of 2-cells on the left-hand side. This allows us to determine any additional isotopies that must be included in the right-hand side. The same method applies to all swallowtail relations.

Finally, using an entirely analogous argument, we obtain the x -rotated counterparts of the swallowtail relations discussed above.

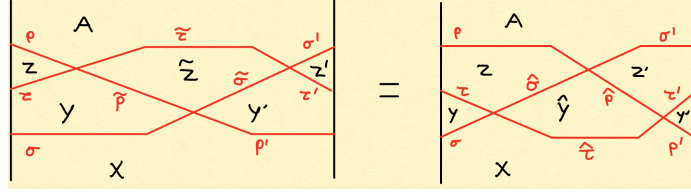
Beak

The analysis of the beak relations proceeds in a manner entirely analogous to that of the swallowtail relations, and so we omit the details here. The only subtlety arises from the same type of ambiguity encountered in the swallowtail case—namely, the specification of the isotopy on the right-hand side of the relation. As before, this can be resolved by examining the images of generic curves under the composite of 2-cells in the finalized form of the left-hand side, thereby determining the necessary isotopy directly.

Reidemeister III

The analysis of the Reidemeister III relations is the most intricate part of the skeletalization of $\text{Sur}_{12\sim}^{\text{or}}$. In what follows, we outline the main ideas underlying this part of the construction.

We begin by recalling that the initial Reidemeister III generating relations take the following form:

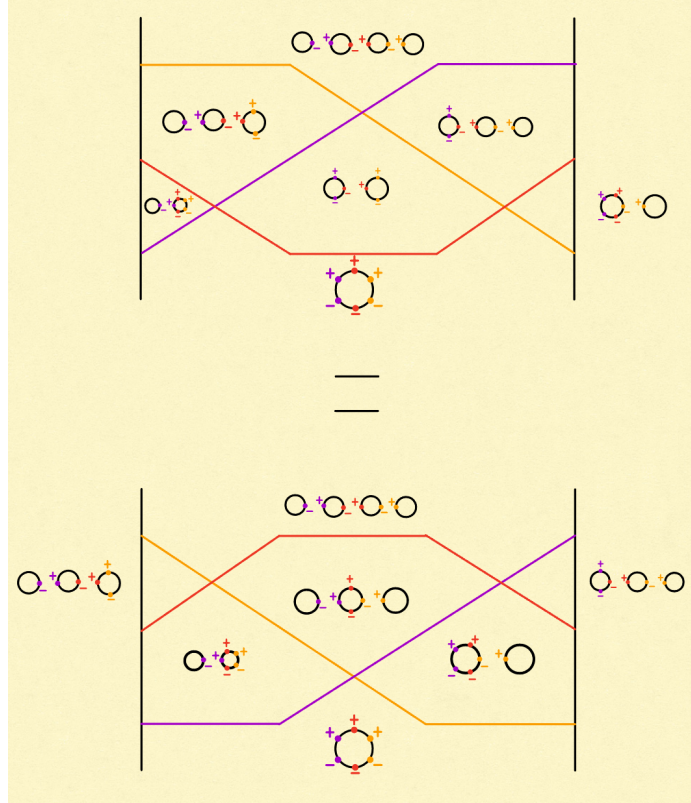


The first key observation is that a Reidemeister III relation is determined, up to T_i , by the following data: the source manifold X and the embeddings of surgery spheres

$$\sigma_X, \quad \psi_\sigma^{-1} \circ \tau_Y, \quad \text{and} \quad (\psi_\tau \circ \psi_\sigma)^{-1} \circ \rho_Z$$

into X .

As a concrete example, consider the following Reidemeister III relation in $\text{Sur}_{12\sim}^{\text{or}}$:



Here, we color the three strands of the underlying Cerf graphic to keep track of which corresponds to which embedding of surgery spheres, and we record all such embeddings in each manifold appearing in the diagram. In the example above, the source manifold is a single circle, and the surgery sphere embeddings yield a configuration of six points in it, grouped into three pairs. This configuration—considered up to rotation of the circle—uniquely determines the Reidemeister III relation up to T_i .

Furthermore, flipping the signs of both points in any one pair (i.e., for a fixed color) results in an equivalent Reidemeister III relation up to T_i : one simply pre- and post-composes with frame flips and invokes the frame flip–crossing relations studied earlier.

It is also worth noting that the source manifold is always a disjoint union of at most four circles. If there are more components, the entire cobordism can be decomposed into disjoint pieces.

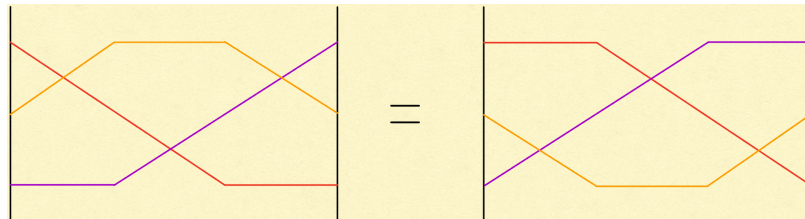
In summary, equivalence classes of Reidemeister III relations up to T_i are classified by a closed 1-manifold with at most four components and a configuration of three pairs of points in it, labeled (and colored) purple, red, and orange. This yields a significant simplification: although there are a priori infinitely many initial Reidemeister III relations, we have now reduced the problem to a finite—though still large—set of configurations, each corresponding to a specific arrangement of six labeled points. This brings us to our second key observation, which will help us refine this classification further.

Lemma 6.3.2. *Let a configuration of three pairs of points in a closed 1-manifold correspond to a Reidemeister III relation. If two colors in the configuration are swapped—for example, if all red points are relabeled purple and all purple points red—the resulting configuration determines a Reidemeister III relation that is equivalent to the original one up to T_i .*

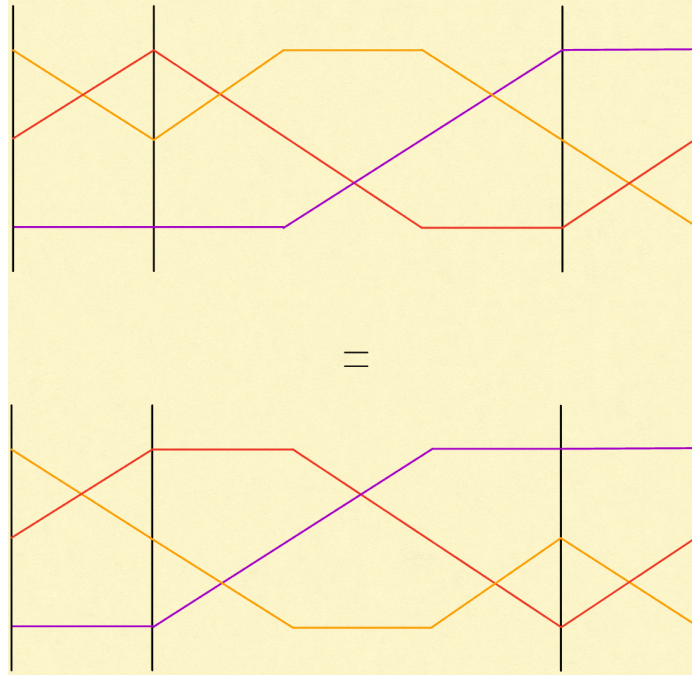
Proof. The lemma follows from one simple observation and a small trick.

The key observation is that there are two ways to realize a swap of two colors in a configuration. One is to swap the colors assigned to the points in the configuration while keeping the colors of the strands in the Reidemeister III diagram fixed. The other is to swap the colors of the strands in the diagram while keeping the configuration fixed. Both result in the same underlying diagram up to relabeling. Therefore, it suffices to show that swapping two colors of the strands in the diagram results in an equivalent Reidemeister III relation.

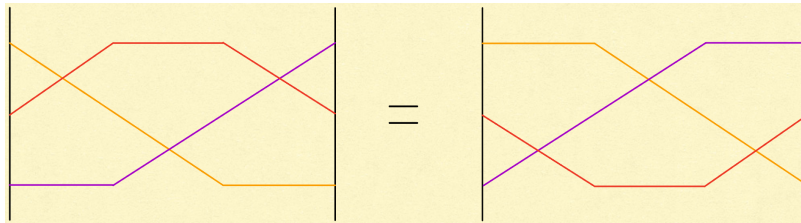
This leads us to the following trick. We start with a Reidemeister III relation as shown:



Our goal is to use relations in T_i to deduce the Reidemeister III relation obtained from swapping the colors red and orange in the strands of the underlying Cerf graphic. We pre- and post-compose with crossing 2-cells as shown:



Then, using the Reidemeister II relations, we obtain



A similar argument shows that we can also deduce the purple–orange color swap on the initial Reidemeister III relation up to T_i . Since any color swap can be written as a composition of such transpositions, it follows that we can deduce all color swaps. This completes the proof. □

This color swap symmetry significantly reduces the number of configurations we need to consider. Further constraints arise from the choice of source manifold, which imposes additional restrictions on the allowable configurations, as illustrated below. With these simplifications in place, we are now in a position to list all

relevant configurations that must be checked—each corresponding to a distinct final Reidemeister III relation.

1. The source manifold has one connected component

In this case, final Reidemeister III relations are represented by cyclic configurations of three pairs of points on a circle, colored purple, red, and orange, considered up to color swaps. We show that there are exactly five such configurations up to equivalence.¹

Definition 6.3.1. Given a configuration as above, define the *red distance* to be the smallest number of other points that a path must cross to go from one red point to the other (traversing the circle in either direction). Define the *purple distance* and *orange distance* analogously. Let $\min$ and $\max$ denote the minimum and maximum, respectively, of these three distances.

For any valid configuration, we must have $\min \geq 0$, $\max \leq 2$, and $\min \leq \max$. This leads to the following cases:

- $(\max, \min) = (0, 0)$: The only possible configuration is RROOPP. This corresponds to the final relation T5 x .
- $(\max, \min) = (1, 0)$: The only possible configuration is ROROPP. This corresponds to T2 x .
- $(\max, \min) = (1, 1)$: No such configuration is possible.
- $(\max, \min) = (2, 0)$: The only possible configuration is RROPPO. This corresponds to T1 x .
- $(\max, \min) = (2, 1)$: The only possible configuration is ROPRPO. This corresponds to T3 x .
- $(\max, \min) = (2, 2)$: The only possible configuration is ROPROP. This corresponds to T4 x .

These cases account for all Reidemeister III relations in the $\mathcal{F}^2$ presentation whose source has a single connected component.

¹Special credit to Christoph Weis for coming up with this argument on the spot during a longer-than-expected train trip!

2. The source manifold has two connected components

In this case, there is an additional constraint and an additional symmetry to account for. The constraint is that no connected component may contain only two points of the same color, as such a configuration would correspond to a disconnected cobordism. The additional symmetry is the interchange of the two connected components. Using arguments similar to those in the one-component case, we find that there are exactly eight admissible configurations up to equivalence:

- R ROPPO: corresponds to T6 x .
- R ROOPP: corresponds to T7 x .
- R ROPOP: corresponds to T2.
- PR ROOP: corresponds to T9 x .
- PR ROPO: corresponds to T3.
- PPR ROO: corresponds to T8 x .
- POR RPO: corresponds to T10 x .
- POR ROP: corresponds to T4.

These configurations account for all Reidemeister III relations in the $\mathcal{F}^2$ presentation with source manifolds consisting of two connected components.

3. The source manifold has three connected components

As in the previous case, we consider configurations of three pairs of points up to cyclic permutation, color swaps, and interchange of connected components, excluding those that lead to disconnected cobordisms. We find that there are exactly five admissible configurations up to equivalence:

- PPR RO O: corresponds to T6.
- POPR R O: corresponds to T8.
- PPOR R O: corresponds to T7.
- PR RPO O: corresponds to T9.
- PR RO OP: corresponds to T10.

These configurations account for all Reidemeister III relations in the $\mathcal{F}^2$ presentation with source manifolds consisting of three connected components.

4. The source manifold has four connected components

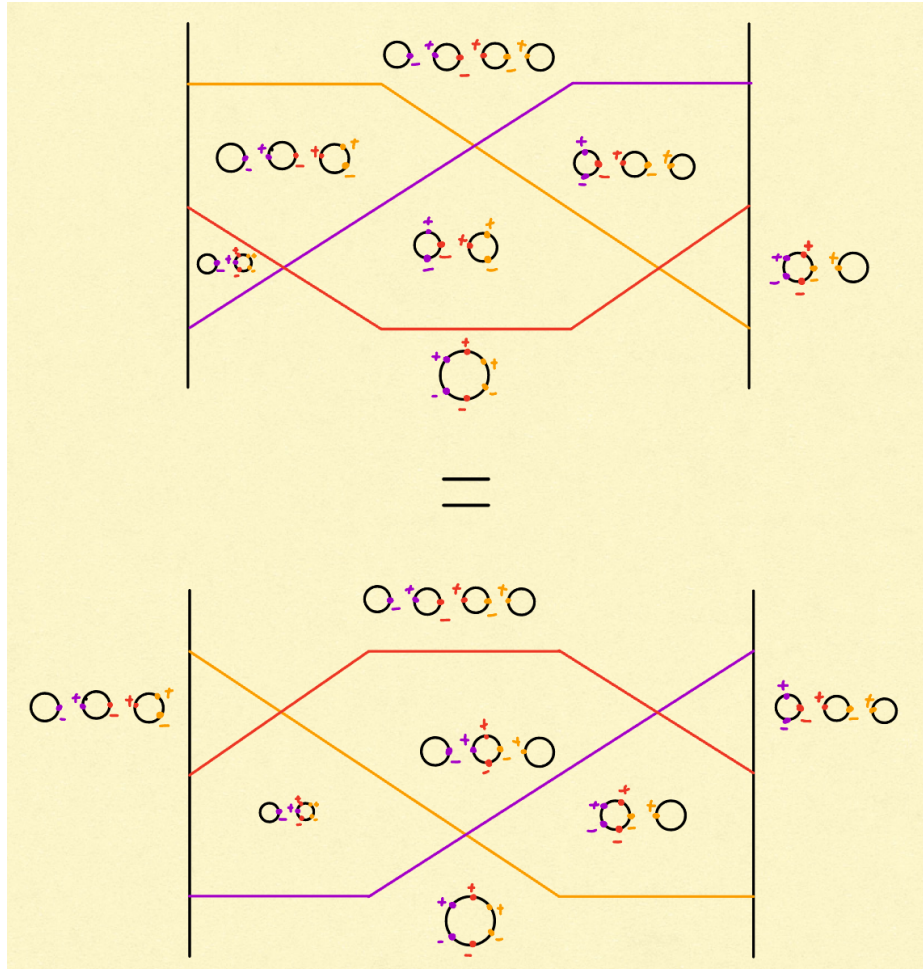
We find that there are exactly two admissible configurations up to equivalence:

- R RO OP P: corresponds to T1.
- POR R P O: corresponds to T5.

These configurations account for all Reidemeister III relations in the $\mathcal{F}^2$ presentation with source manifolds consisting of four connected components.

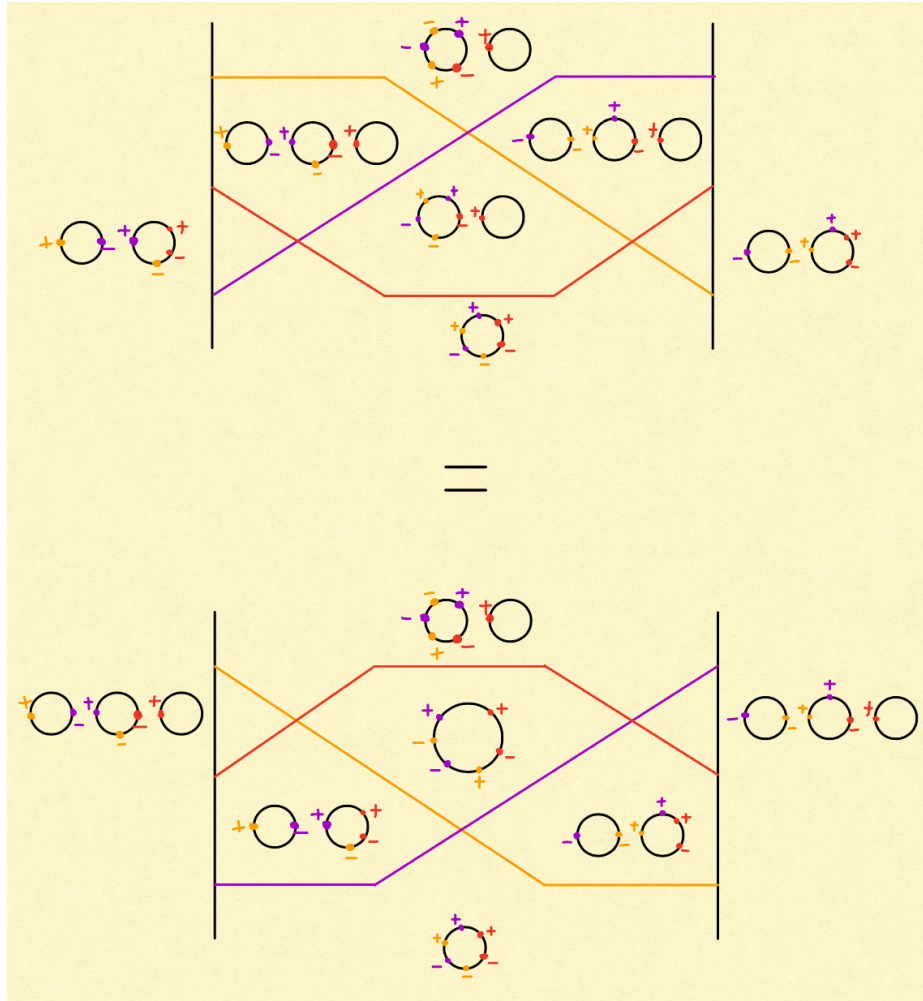
This analysis shows that it suffices to consider exactly twenty Reidemeister III relations in the final presentation, each corresponding to one of the configurations listed above. Moreover, we now have all the tools required to explicitly construct representatives for each of these relations. The procedure is as follows: given a configuration, we assign signs to the points, identify the resulting initial Reidemeister III relation, and then apply the skeletalization procedure to extract a relation between final 2-cells.

For some configurations, this process is especially straightforward. For example, T1 (x) corresponds to the configuration RROPPO, which gives rise to the initial Reidemeister III relation:



This relation can be directly translated into the T1 (x) relation in the $\mathcal{F}^2$ presentation.

On the other hand, T2 (x) corresponds to the configuration ROROPP, which yields the following initial Reidemeister III relation:



To obtain the corresponding final relation, we must carry out the 1-finalization according to the skeletalization procedure. While this involves no conceptual difficulties, it is somewhat tedious. For instance, applying the procedure to the right-hand side of the above relation produces the surgery diagram shown in Figure 6.1, which indeed matches the lower side of the T2 (x) relation in the $\mathcal{F}^2$ presentation. We have verified the relations corresponding to configurations T1 (x), T6 (x), and T8 (x). Applying the skeletalization procedure to the remaining Reidemeister III relations is ongoing work.

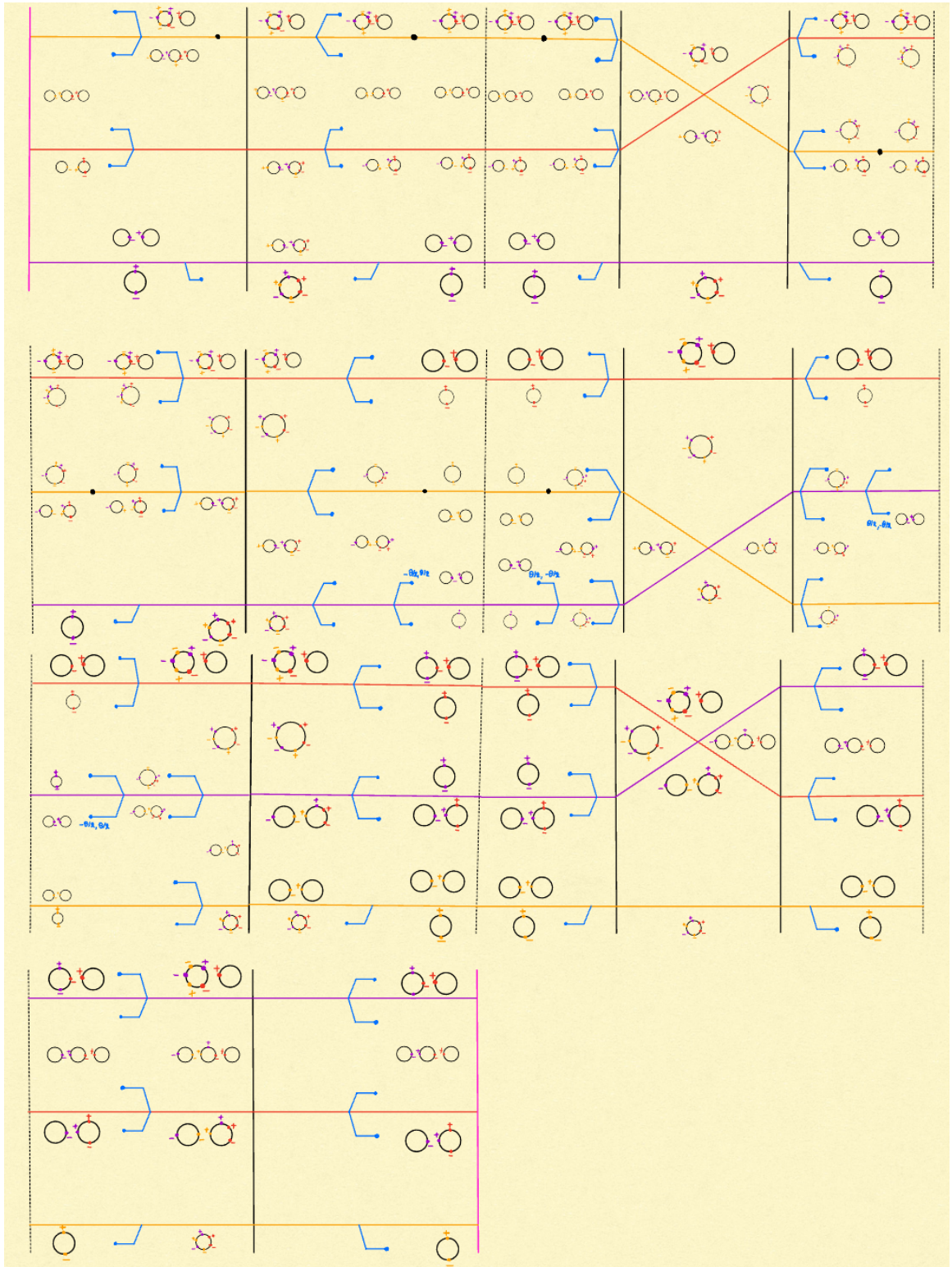


Figure 6.1: The 1-finalization corresponding to the RHS of the T2 x relation.

Appendix A

The $\mathcal{F}^2$ presentation of $\text{Bord}_{12\sim}^{\text{or}}$

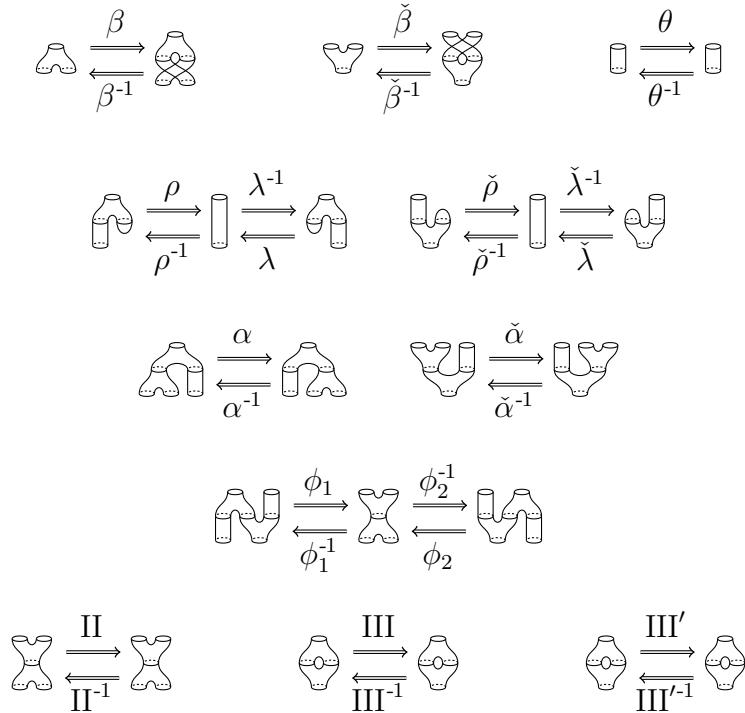
Although the $\mathcal{F}^2$ presentation has appeared in fragmented form throughout Chapter 6, we include it here in full due to its central importance. It is a presentation of an unbiased semistrict symmetric monoidal 2-category with 1 generating 0-cell, 4 generating 1-cells, 28 generating 2-cells and 66 generating relations defined as follows:

- Generating 0-cell: $\circ$

- Generating 1-cells:



- Generating 2-cells:



- Generating relations:

1. **Inverses.**

Each of the generating 2-cells ω satisfies $\omega\omega^{-1} = \text{id}$ and $\omega^{-1}\omega = \text{id}$.

2. **Naturality - Frame flip.**

$$\text{Diagram} \xrightarrow{\beta^2} \text{Diagram} \rightarrow \text{Diagram} = \text{Diagram} \xrightarrow[\theta^{-1}, \theta^{-1}]{\theta} \text{Diagram} \quad (\text{N-F}) \quad x$$

3. **Isotopy of Morse data invariance.**

$$\text{Diagram} \xrightarrow{\theta} \text{Diagram} = \text{Diagram} \xrightarrow{\text{id}} \text{Diagram} \quad (\text{IMD}) \quad x$$

4. **Naturality - Frame flip - Cusp.**

$$\begin{array}{ccc} \text{Diagram} & \xrightarrow{\beta} & \text{Diagram} \\ \downarrow \rho & & \downarrow \\ \text{Diagram} & \xleftarrow{\lambda} & \text{Diagram} \end{array} \quad (\text{N-F-Cusp}) \quad xz$$

5. **Naturality - Frame flip - Crossing.**

$$\begin{array}{ccccccc} & & \text{Diagram} & \rightarrow & \text{Diagram} & \xrightarrow{\check{\alpha}^{-1}} & \text{Diagram} & \xrightarrow{\check{\beta}} & \text{Diagram} \\ & \nearrow \check{\beta}^{-1} & & & & & & & \\ \text{Diagram} & & & & & & & & \\ & \searrow \check{\beta} & & & & & & & \\ & & \text{Diagram} & \xrightarrow{\check{\alpha}} & \text{Diagram} & \xrightarrow{\check{\beta}^{-1}} & \text{Diagram} & \rightarrow & \text{Diagram} \end{array} \quad (\text{N-F-Cross-1}) \quad x$$

$$\begin{array}{ccccccc} & & \text{Diagram} & \xrightarrow{\phi_1^{-1}} & \text{Diagram} & \xrightarrow{\beta} & \text{Diagram} \\ & \nearrow \beta^{-1} & & & & & \\ \text{Diagram} & & & & & & \\ & \searrow \check{\beta}^{-1} & & & & & \\ & & \text{Diagram} & \xrightarrow{\phi_2^{-1}} & \text{Diagram} & \xrightarrow{\check{\beta}} & \text{Diagram} \end{array} \quad (\text{N-F-Cross-2}) \quad z$$

(N-F-Cross-3) x

(N-F-Cross-4)

(N-F-Cross-5)

6. Swallowtail.

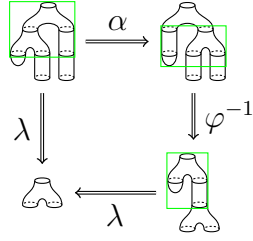
(ST1) x

(ST2) x

(ST3) xz

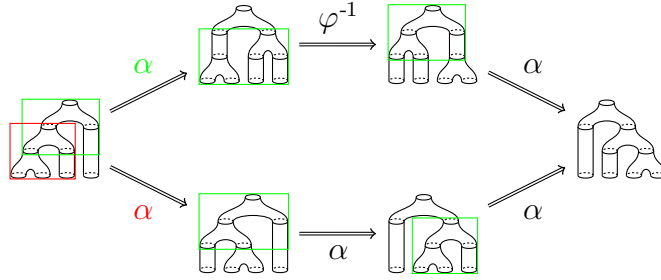
7. Beak.

(B1) xz

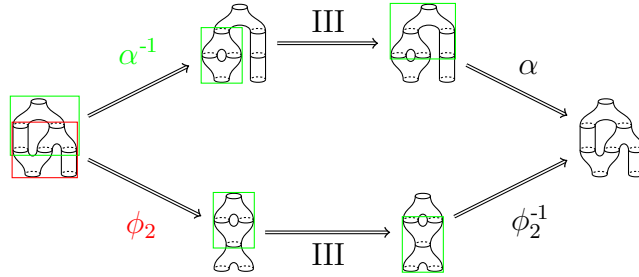


(B2) xz

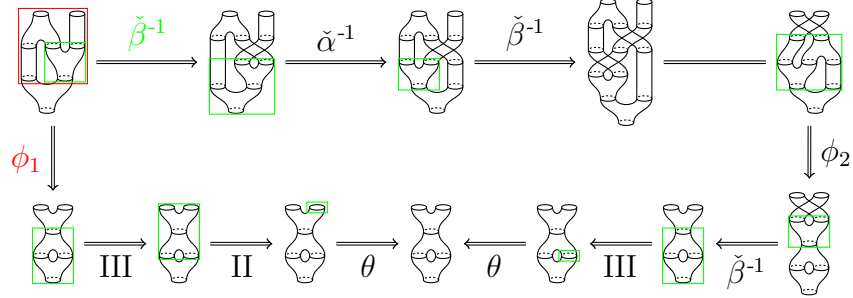
8. Triple point.



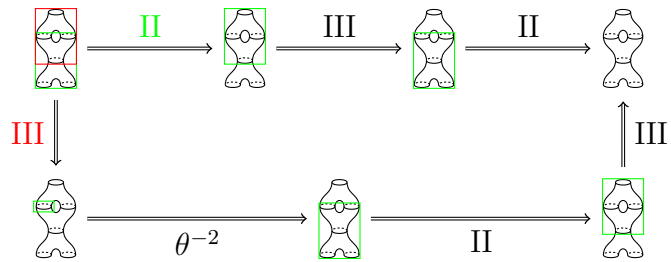
(T1) x



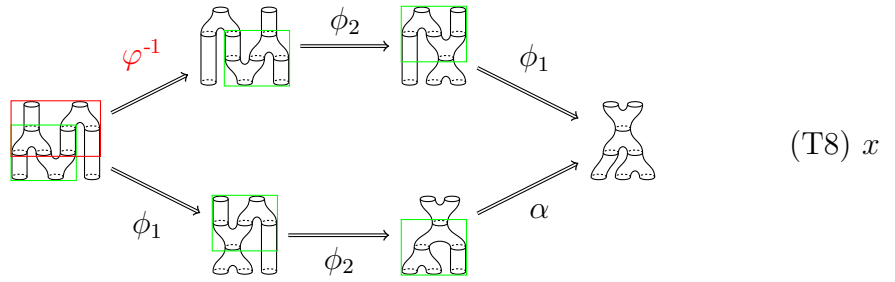
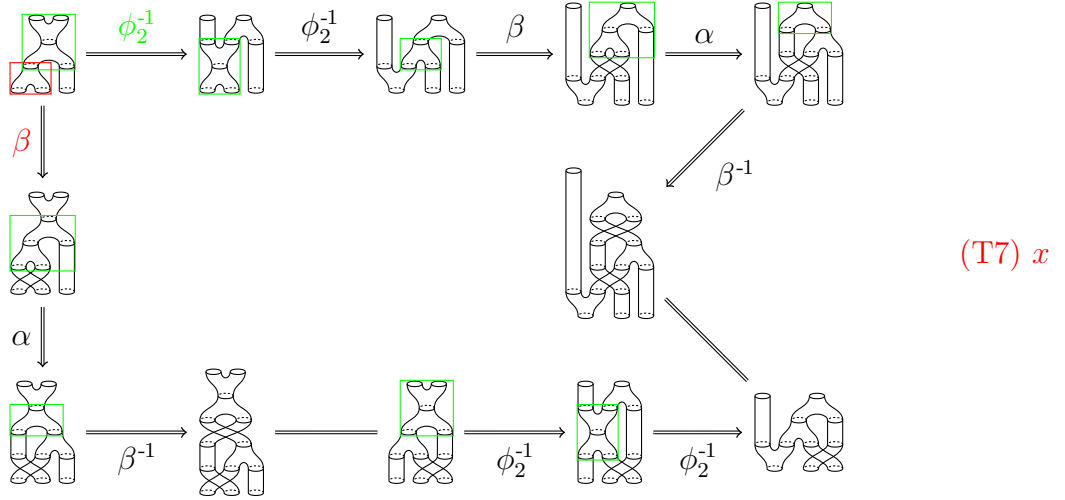
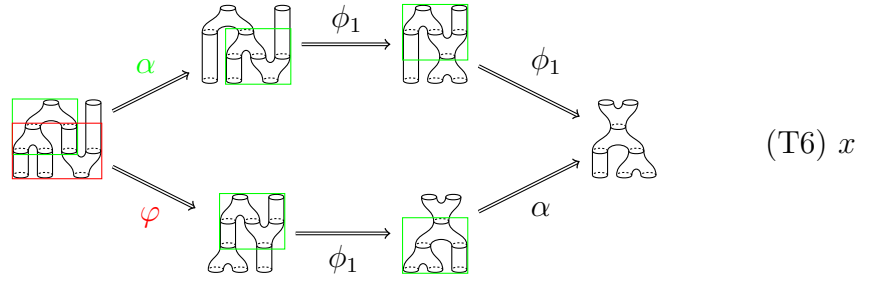
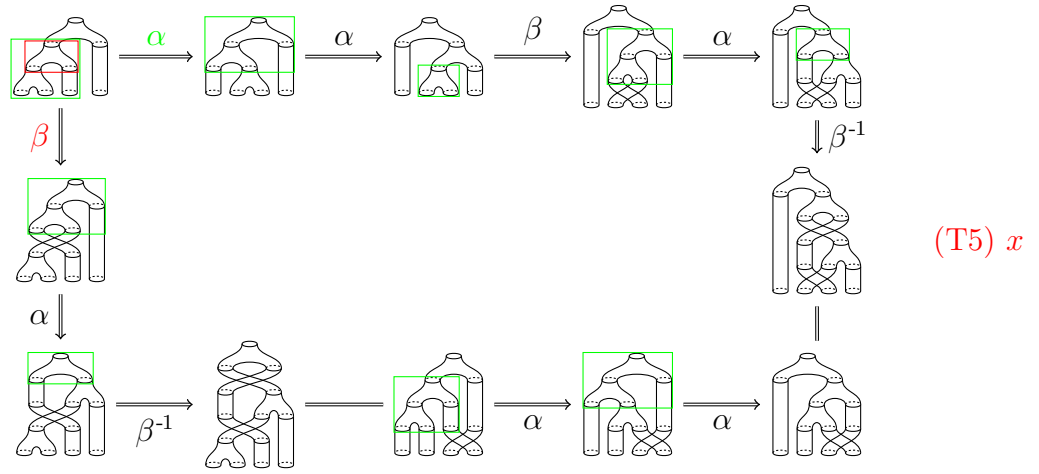
(T2) x

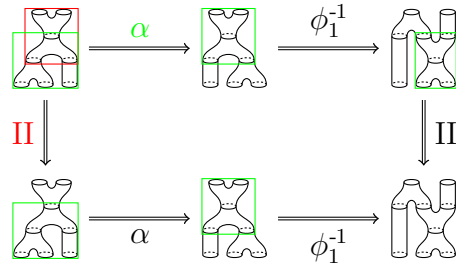


(T3) x

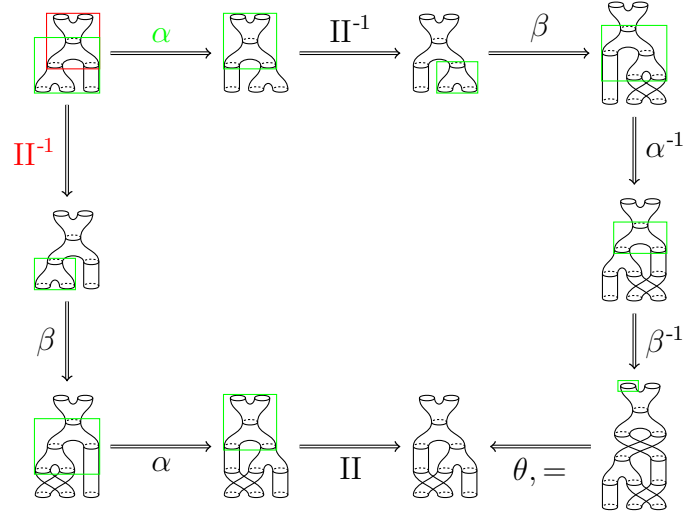


(T4) x





(T9) x



(T10) x

References

- [Ati88] Michael Atiyah. Topological quantum field theory. *Publications Mathématiques de l’IHÉS.* 68 (68): 175–186, 1988.
- [Bar14] Bruce Bartlett. Quasistrict symmetric monoidal 2-categories via wire diagrams. *arXiv:1409.2148v1*, 2014.
- [BD95] John Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.* 36, 6073–6105, 1995.
- [BDSPV14] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Extended 3-dimensional bordism as the theory of modular objects. *arXiv:1411.0945*, 2014.
- [BDSPV15] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Modular categories as representations of the 3-dimensional bordism 2-category. *arXiv:1509.06811*, 2015.
- [Beh14] Stefan Behrens. Smooth 4-manifolds and surface diagrams. *University of Bonn Doctoral Dissertation*, 2014.
- [BK01] Bojko Bakalov and Alexander Kirillov. *Lectures on Tensor Categories and Modular Functors*. American Mathematical Society, 2001.
- [BL03] John Baez and Laurel Langford. Higher-dimensional algebra iv: 2-tangles. *Adv. Math.* 180 (2003), 705–764, 2003.
- [Cer68] Jean Cerf. Sur les difféomorphismes de la sphère de dimension trois. *Springer Verlag, Lecture Notes in Mathematics, volume 53*, 1968.
- [Cer73] Jean Cerf. La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. *Publications Mathématiques de l’IHÉS, Volume 39 (1970), pp. 5-173*, 1973.

- [Cha18] Antony Charel. Gradient trajectories near real and complex a2-singularities. *ETH Doctoral Thesis*, 2018.
- [Coh06] Ralph Cohen. *Topics in Morse theory: lecture notes*. 2006.
- [CRS97] Scott Carter, Joachim Rieger, and Masahico Saito. A combinatorial description of knotted surfaces and their isotopies. *Advances in Mathematics Volume 127, Issue 1, Pages 1-51*, 1997.
- [Fre94] Dan Freed. Higher algebraic structures and quantization. *Commun.Math.Phys.* 159 (1994) 343-398, 1994.
- [GG73] Martin Golubitsky and Victor Guillemin. *Stable Mappings and Their Singularities*. Springer Verlag, 1973.
- [GK15] David Gay and Robion Kirby. Indefinite morse 2-functions; broken fibrations and generalizations. *Geom. Topol.* 19 (2015) 2465-2534, 2015.
- [Gol78] Martin Golubitsky. An introduction to catastrophe theory and its applications. *SIAM Review*, vol. 20, no. 2, 1978, pp. 352-87. *JSTOR*, 1978.
- [Hen96] M. Hennings. Invariants of links and 3-manifolds obtained from hopf algebras. *J. of London Math. Soc.* 54 (1996), 594-624, 1996.
- [HW73] Allen Hatcher and John Wagoner. Pseudo-isotopies of compact manifolds. *Astérisque*, tome 6, 1973.
- [Igu87] Kiyoshi Igusa. The space of framed functions. *Transactions of the American Mathematical Society Vol. 301, No. 2 (Jun., 1987), pp. 431-477*, 1987.
- [Juh18] András Juhász. Defining and classifying tqfts via surgery. *Quantum Topol.* 9 (2018), no. 2, 229-321, 2018.
- [KL01] T. Kerler and V. Lyubashenko. Non-semisimple topological quantum field theories for 3-manifolds with corners. *Lecture Notes in Math.*, 1765. Springer-Verlag, Berlin, 2001.
- [Kup19] Alexander Kupers. *Lectures on diffeomorphism groups of manifolds*. 2019.

- [Law93] Ruth Elke Lawrence. Triangulations, categories and extended topological field theories. *Quantum Topology*, Eds. R. Baadhio and L.H. Kauffman, *World Scientific (1993)* 191-208, 1993.
- [Lek09] Yanki Lekili. Wrinkled fibrations on near-symplectic manifolds. *Geom. Topol.* 13 (2009) 277-318, 2009.
- [Lur09] Jacob Lurie. On the classification of topological field theories. *Current Developments in Mathematics Volume 2008* , 129-280, 2009.
- [Lyu95] V. Lyubashenko. Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity. *Comm. Math. Phys.* 172 (1995), 467–516, 1995.
- [Mic80] Peter W Michor. Manifolds of differentiable mappings. *Shiva Publishing Limited*, 1980.
- [Mil63] John Milnor. *Morse Theory*. Princeton University Press, 1963.
- [Mil22] Kyle Austin Miller. Singularity theory for extended cobordism categories and an application to graph theory. *UC Berkeley*, 2022.
- [Pon38a] Lev Pontrjagin. Classification of continuous maps of a complex into a sphere, communication i. *Doklady Akademii Nauk SSSR* 19(3) (1938), 147-149, 1938.
- [Pon38b] Lev Pontrjagin. Classification of continuous maps of a complex into a sphere, communication ii. *Doklady Akademii Nauk SSSR* 19(5) (1938), 361-363, 1938.
- [Pst14] Piotr Pstragowski. On dualizable objects in monoidal bicategories, framed surfaces and the cobordism hypothesis. *Master's thesis, Universität Bonn, Germany*, 2014.
- [RT91] Nikolai Reshetikin and Vladimir Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inventiones Mathematicae*, 103(1):547–597, 1991.
- [SH19] Michael Shulman and Linde Wester Hansen. Constructing symmetric monoidal bicategories functorially. *arXiv:1910.09240v2*, 2019.

- [SP14] Christopher J. Schommer-Pries. The classification of two-dimensional extended topological field theories. *arXiv:1112.1000v2*, 2014.
- [Tho54] René Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.* 28 (1954) 17–86, 1954.
- [Til98] Ulrike Tillmann. S-structures for k-linear categories and the definition of a modular functor. *Journal of the London Mathematical Society*, 58:208–228, 1998.
- [Tur94] Vladimir Turaev. Quantum invariants of knots and 3-manifolds. *volume 18 of de Gruyter Studies in Mathematics*. Walter de Gruyter, Berlin, 1994.
- [Wal91] Kevin Walker. On witten’s 3-manifold invariants. <https://canyon23.net/math/1991TQFTNotes.pdf>, 1991.