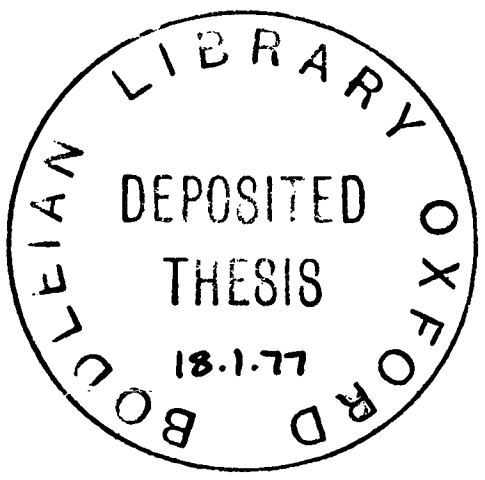
STUDY OF REACTIONS BETWEEN HEAVY NUCLEI

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ABSTRACT

A semiclassical approach has been used to study those reactions between heavy nuclei (or "heavy ions" as they are more commonly referred to) in which a single nucleon or a single cluster of nucleons is transferred from one nucleus to the other. A method has been developed for evaluating transfer amplitudes for the transfer of a neutron using the semiclassical theory of Brink and Pixton. As this leads to a tedious three dimensional integral, we have developed another method for the same calculation. It is found that if we work in momentum space and assume a straight line trajectory for the relative motion, the expressions simplify considerably and we are left with a straightforward single integral for the transfer amplitude. An added advantage is that the formulae are exactly the same in the post and prior representations. The theory has then been generalised for the transfer of a charged particle. Some numerical calculations have been performed for the reaction $^{208}Pb(^{11}B, ^{10}B)^{209}Pb$ at 114 MeV laboratory energy. A comparison of the transfer amplitudes calculated in the post and prior representations using the first method has been made. The agreement is reasonably good between these two results as well as between these and the results using the second method.

We go on to set up a semiclassical theory for evaluating angular distributions. The starting point is the expression of the transition matrix in DWBA. A partial wave expansion is made for the distorted waves and the WKB approximation used for the radial wave functions. Further simplifications are made resulting in a formula for the transition amplitude which is a partial wave sum containing a term recognisable as the semiclassical transfer amplitude evaluated earlier.

The method has been used to calculate angular distributions for the reactions $^{26}Mg(^{11}B, ^{10}B)^{27}Mg$ and $^{26}Mg(^{11}B, ^{10}Be)^{27}Al$ at 114 MeV laboratory energy. A detailed numerical study of the formula shows some interesting features which enable us to approximate the transfer amplitude by a simple parametrised exponential formula. The resulting angular distributions are compared with experiment showing good agreement. A parametrisation of the elastic scattering phase shifts as well make it possible to obtain a closed expression for the angular distribution. A numerical comparison of the angular distribution obtained from this formula is made with former results. The agreement is good.

We also study the theoretical reaction $^{16}O(^{17}O^*, ^{16}O)^{17}O^*$ for a range of energies from 212.5 MeV to 37.1875 MeV and compare our results with that using the quantum mechanical code LOLA of De Vries (DeVries, 1973). The elastic scattering phase shifts are calculated in a

potential with an energy dependent imaginary part. The agreement is good till 53.125 MeV. At 37.1875 MeV there is disagreement. This is attributed to the breakdown of our formalism for weakly absorbing potentials.

Some calculations have been performed for the α-particle transfer reaction $^{16}O(^{16}O, ^{12}C)^{20}Ne$. Comparison with LOLA shows disagreement which we have not been able to explain.

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CHAPTER I

INTRODUCTION

The subject of heavy ion reactions has become such a complex one that it is not possible in this work to mention the many aspects of this interesting field. We shall restrict ourselves to the study of those reactions between two nuclei in which a single nucleon or a single cluster of nucleons is transferred from one nucleus to the other as they come within the interaction fields of one another. We shall try to understand the reaction mechanism and to elicit some information on nuclear structure.

Several theories have been proposed to study transfer reactions (Fleury and Alexander, 1974 and references therein; Ehlers et. al., 1974). Perhaps the most successful quantum mechanical one has been the distorted wave Born approximation (DWBA) theory. As this leads to a six dimensional numerical integral for the transition amplitude, attempts have been made to simplify the theory. In its earliest applications this was accomplished by introducing the so-called 'no recoil approximation' (Buttle and Goldfarb, 1966; Buttle and Goldfarb, 1968; Schmittroth et. al. 1970). Although this considerably reduced computing time it soon became clear the terms left out by this approximation were by no means negligible. Since then attempts have been made to include recoil in an approximate way (Dodd and Greider, 1969; Buttle and

Goldfarb, 1971; Nagarajan, 1972). Other approximations have also been made by various authors to make the evaluation of the DWBA matrix element easier (Satchler, 1964; Austern et. al., 1964; Kammuri and Yoshida, 1969; Glendenning and Nagarjan, 1974; and references above). Presently, it is possible to compute this matrix element without making approximations as faster computers are now available and programs have been developed (De Vries, 1973; Tamura and Low, 1974; Tamura, 1974) which enable one to do so (e.g. De Vries and Kubo, 1973; Tamura and Low, 1973). However, these programs are expensive to run and there is still need for a quicker way to obtain the results.

Although the problem is essentially a quantum mechanical one, at high energies one may use semiclassical concepts to formulate a theory for transfer reactions. Such theories are often more of an aid to the understanding of physics than a means of performing systematic calculations. Even this aspect makes them of considerable value in view of the complexity and cost of exact quantum calculations. Consequently, a number of authors have devoted themselves to a semiclassical study of these reactions and extracted some interesting information (Dar, 1965; Scott, 1974; and references therein; Glendenning, 1975). We shall briefly look at some of these theories.

As a first step to obtaining angular distributions it is useful to have a formula for the transfer amplitude. Broglia and Winther (Broglia and Winther, 1972a; Broglia and Winther, 1972b) arrive at one by solving the quantum

mechanical equations of motion for the reaction $c_2(a,c_1)b$ ($a = c_1 + x$, $b = c_2 + x$) while at the same time assuming a distinct classical path for the relative motion. The total wave function is written as a linear combination of the wave functions in the initial and final channels, the expansion coefficients $c_{if}(t)$ at $t = \infty$ corresponding to the transfer amplitude. (The subscripts i, f stand for the quantum numbers of the initial and final states respectively). Substitution of this wave function into the Schrodinger equation gives coupled equations for c_{if}. After going through a considerable deal of mathematics and simplifications, one arrives at the formula:

$$c_{if}(\infty) = \frac{1}{i\hbar} \int_{-\infty}^{+\infty} \left(\psi^{c_1}(\xi_{c_1})\,\psi^{b}(\xi_b), (V_{c_2 a} - U_{c_2 a})\, e^{\,i \frac{m_x}{M} \underline{v}\cdot(m_b \underline{r}_{xc_1} + m_{c_2} \underline{r}_{xa})} \right.$$
$$\left. \times\, e^{\frac{i}{\hbar}[(E_{c_1} + E_b - E_{c_2} - E_a)t' - \gamma(t')]}\, \psi^{A}(\xi_A)\psi^{b}(\xi_b)\right) dt' \qquad (I.1.1)$$

where ξ_i are the intrinsic coordinates, ψ^i the time-independent eigenfunctions of the intrinsic Hamiltonians for the nuclei i, E_i the corresponding eigenvalues and m_i the masses. $\underline{v}$ is velocity of the transferred particle x at the point of transfer, m_x its mass and $\underline{r}_{xc_1}$, $\underline{r}_{xa}$ its position with respect to the centres of mass of nuclei c_1 and a respectively. $(V_{c_2 a} - U_{c_2 a})$ is the interaction potential and $\gamma(t')$ is a time dependent phase which takes into account the effect of the difference of optical potentials and reduced masses in the initial and final channels.

A parallel formula to eq. (I.1.1) for the transfer amplitude has been obtained by Brink and Pixton (Pixton, 1973; Anayas-Weiss et. al., 1974) in a semiclassical theory derived by them. They assume that the particle x (which is taken to be neutral) starts out at $t = -\infty$ in a bound state of the potential V_1 provided by the moving core c_1. The probability that the particle transfers to a bound state of the potential V_2 of the core c_2 at $t = \infty$ is given by:

$$A_{if} = \frac{1}{\hbar} \int \langle \Psi_2, V \Psi_1 \rangle \, dt \, . \qquad \text{(I.1.2)}$$

where Ψ_1, Ψ_2 refer to bound states of the particle in the moving potentials V_1, V_2 respectively. V is equal to V_2 in the prior representation and V_1 in the post representation. Writing the wave functions Ψ_1 and Ψ_2 in terms of states ψ_1, ψ_2 in fixed potentials we get:

$$\langle \Psi_2, V \Psi_1 \rangle = U_{if}(t) \exp\left[-\frac{it}{\hbar}\left(Q + \tfrac{1}{2} m_x \dot{\underline{s}}^2\right)\right] \qquad \text{(I.1.3)}$$

where,

$$U_{if}(t) = \int e^{\frac{i}{\hbar} m_x \dot{\underline{s}} \cdot \underline{r}_2} \psi_2(\underline{r}_2) V_{1,2}(\underline{r}_2, \underline{s}) \psi_1(\underline{r}_2 - \underline{s}) \, d^3 \underline{r}_2 \qquad \text{(I.1.4)}$$

Here Q is the reaction Q value; $\underline{s}(t)$ is the relative separation of the cores and $\underline{r}_2$ is the coordinate of x relative to the core c_2. It has also been shown that the transfer amplitudes are equal in the post and prior representations. Pixton has suggested some modifications to simplify the three dimensional integral (I.1.4). In

the prior representation, we approximate the radial part of $\psi_1(\underline{r}_2 - \underline{s})$ by a Hankel function. This enables us to reduce (I.1.4) to a sum of one dimensional integrals. Pixtonused his formulae to perform some numerical calculations. He obtained transfer amplitudes in the post and prior representations for the reaction $^{208}Pb(^{11}B, ^{10}B)^{209}Pb$ performed at 114 MeV laboratory energy. The semiclassical path of the projectile relative to the target was approximated by a straight line. A comparison of the results showed that the amplitudes in the post and prior representations did not agree, suggesting that perhaps the Hankel function approximation is not a good one.

We would like to mention that the transfer amplitudes A_{if} depend on the magnetic quantum numbers λ_1, λ_2 of the initial and final states ψ_1 and ψ_2, and hence on the coordinate system. The total transfer probability which is the sum of squares of the transfer amplitudes is, however, rotationally invariant.

Eqns. (I.1.2) - (I.1.4) are essentially the same as eq. (I.1.1), A_{if} of eq. (I.1.2) corresponding to c_{if} of eq. (I.1.1). However, the derivation of Brink and Pixton is more straightforward because they assume a simple potential model for the initial and final nuclear states. A highly simplified version of this method has been given by Buck (Buck, 1976) from which one can see very quickly the various approximations and assumptions which go into it without lo-sing the semiclassical picture. It is

worth remarking at this stage that recoil corrections are taken into account in both the above theories in the sense of Dodd and Greider (Dodd and Greider, 1969) through an exponential factor ($exp\left(\frac{i}{\hbar} m_x \dot{\underline{s}} \cdot \underline{r}_2\right)$ in eq. (I.1.4)) and a similar factor obtained by simplifying the exponentials in eq. (I.1.1). The equivalence has been shown by Anayas-Weiss et. al. (Anayas-Weiss et. al., 1974). Both the above procedures give only a transfer amplitude and a transfer probability corresponding to a particular orbit $\underline{s}(t)$ of relative motion.

There is a problem of relating the transfer amplitude A_{if}(or c_{if}) to the scattering cross section. In a very simplistic way this relation is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{i\to f} \propto |A|^2 \sqrt{\frac{d\sigma_i}{d\Omega}\frac{d\sigma_f}{d\Omega}} \qquad \text{(I.1.5)}$$

where $$\frac{d\sigma_f}{d\Omega} = \frac{b\, db}{\sin\theta_f\, d\theta_f} \qquad \text{(I.1.6)}$$

b corresponding to the classical impact parameter. Here it has been assumed that only one trajectory contributes and A_{if} is evaluated for that particular one. Thus the contribution of other trajectories is neglected and interference and deflection effects are not accounted for. Further, the effect of absorption into other channels is also neglected. Now there are more sophisticated semi-classical theories available for evaluating cross sections in transfer reactions, which correct for the above

defects. A common denominator in these theories is a partial wave summation formula for the reaction amplitude $f(\theta,\varphi)$ (where θ, φ is the scattering angle) of the form:

$$f(\theta,\varphi) \propto \sum_{l} \sqrt{2l+1}\ \eta^{\tau}(l)\ Y_{lm}(\theta,\varphi) \qquad \text{(I.1.7)}$$

where $\eta^{\tau}(l)$ represents the transfer amplitude combined with a damping factor which takes into account absorption into other channels. m is the z-component of the transferred angular momentum. τ stands for the various other quantities $\eta^{\tau}(l)$ depends on.

The expression (I.1.7) can be arrived at in a variety of ways the essential difference being the exact meaning of $\eta^{\tau}(l)$ in each case. This largely depends on the aim of the particular calculation. The final form of $\eta^{\tau}(l)$ reflects the approximations and assumptions that have gone in its derivation and contains all the physics. For example, Strutinsky (Strutinsky, 1973) derives a simple model which takes into account the dynamic as well as quantal dispersion of the wave packet. To this end, he assumes that only a relatively narrow region of l-values contributes to the reaction taking place at the nuclear surface, and therefore approximates $\eta^{\tau}(l)$ by:

$$\eta^{\tau}(l) = \eta(l_0) \exp\left\{-(l-l_0)^2/(\Delta l)^2\right\} \exp(2i\delta_l) \qquad \text{(I.1.8)}$$

where $l_0 \gg \Delta l \gg 1$. The phase shift

$$\delta_l = \frac{1}{2}\left[\Theta_c(l-l_0) - \kappa\cdot(l-l_0)^2 + \cdots\right] \quad ,$$

where $\Theta_c = (\frac{2d\delta_\ell}{d\ell})_{\ell=\ell_0}$ is the scattering angle at the central value of $\ell = \ell_0$ in the incident wave packet. The parameter κ determines the dynamic angular dispersion of the packet due to the nuclear interaction. The resulting angular distributions usually gave a good fit to experiment. The small values of $\Delta\ell$ obtained for some cases of multi-nucleon transfer was attributed to the dynamic dispersion. Strutinsky was also able to obtain an estimate of the reaction time with the help of his formula.

An interesting study of the formula (I.1.7) and its comparison with the crude semiclassical formula (I.1.5) as well as with quantum mechanical calculations has been made by Broglia et. al. (Broglia et. al., 1974). Though they have discussed elastic and inelastic scattering, their method can be readily generalised to transfer reactions. They have tested different approximate methods to calculate the scattering amplitude. The reaction mechanism is in all cases described by the classical reaction amplitudes while the geometry is dealt with in different degrees of approximation. First of all they derive a formula for the scattering amplitude similar to eq. (I.1.7), with $\eta^T(\ell)$ given by:

$$\eta^T(\ell) \sim c_{if}(\ell,\infty)\, e^{i(\delta_\ell^{(i)} + \delta_\ell^{(f)})} \qquad \text{(I.1.9)}$$

Here $c_{if}(\ell,\infty)$ is given by eq. (I.1.1) (we have included the ℓ dependence of c_{if} in the notation). $\delta_\ell^{(i)}, \delta_\ell^{(f)}$ are the elastic scattering phase shifts in the initial and final channels respectively. For inelastic scattering

these are obtained from complex potentials. Broglia et al. were able to obtain eq. (1.1.5) as a limiting case of eq. (1.1.7). They then used their expressions to calculate cross sections for three different types of potentials for the scattering of $^{16}O + ^{56}Ni$: (I) weak nuclear attraction so that the deflection function is approximately the Coulomb deflection function (II) nuclear attraction strong enough to produce a maximum in the deflection function (III) nuclear attraction so strong that orbiting occurs. A comparison was made with quantum mechanical cross sections. The results from the crude formula (I.1.5) were seen to agree with the quantal results for case (I) for elastic scattering, and for inelastic scattering at low energies. At high energies there were oscillations in the quantal cross sections about the semiclassical distribution. The formula fails for cases (II) and (III) as it gives a singularity in the cross section. Using eq. (I.1.7) they obtained good agreement for small scattering angles in all three cases. For case (I) agreement at larger angles can be improved by calculating the elastic scattering phase shifts more accurately. The discrepancy at large angles in cases (II) and (III) between the semiclassical and quantal results is believed to be due to the different ways in which absorption is taken into account in the two theories. This was tested by repeating the calculation in a real potential and seeing that the disagreement still existed. As a conclusion they point out that it is still not very clear as to how to include the reflective effects associated with imaginary potentials in the calculation

of the classical trajectory. The diffractive and interference phenomena are taken into account by use of eq. (I.1.7).

The same results as above have been obtained by Koeling and Malfliet (Koeling and Malfliet, 1975) using a different formulation. They use a Feynman path integral formalism to derive their results. Their approach is more flexible and allows for generalisations. It is especially suited for treating complex interactions in the small wavelength approximations and reproduce other methods as limiting cases.

Recently, Landowne et. al. have used semiclassical methods to derive a formula for angular distributions in transfer reactions. They start with the expression of the T-matrix in DWBA, with the form factor F written as a linear combination of $F_{\lambda\mu}$, where $\lambda\mu$ is the transferred angular momentum. The distorted waves are then expanded in a partial wave summation. The introduction of an intrinsic coordinate system causes some simplifications to be made. As a result one can perform the angular integration and we are left with radial integrals $I_{\ell_f \ell_i}$. The radial wave functions are now approximated by WKB wave functions. By using the analytic properties of these functions and going into the complex plane the integrals $I_{\ell_f \ell_i}$ can be evaluated. The integration is performed by choosing a contour which circumvents both turning points. The resulting quantity corresponds to the semiclassical reaction amplitude. The equation for the transition amplitude is similar to eq.(I.1.5) being given by:

$$T_{if} = \sum_{l_f l_i} i^{l_i - l_f} (l_f\, m\, \lambda - m | l_i\, 0) \sqrt{2l_f+1}\; e^{i(\delta_{l_i} + \delta_{l_f})} \times \times I_{l_f l_i} Y_{l_f m}(\vartheta)$$

(I.1.10)

$(l_f\ m\lambda - m | l_i 0)$ is a Clebsch Gordan coefficient. Under the assumption of the initial and final orbits being well matched so that the relative motion can be approximated by an average trajectory, the integrals $I_{l_f l_i}$ became identical to the reaction amplitude of Broglia and Winther (eq. (I.1.1)). The formula (I.1.10) has been used to evaluate angular distributions for inelastic scattering and a one particle transfer reaction. The results gave good agreement with DWBA.

A theory for transfer reactions based on the DWBA has been developed by Crowley (Crowley, 1976). Following a discussion of some of its general properties, the transfer t-matrix is derived in the DWBA and reduced to a form suitable for expressing the integrals in momentum space representation. The use of eikonal distorted waves then leads to an expression of the t-matrix which is improved by the introduction of an ansatz. Further substantiation is provided by means of a semiclassical approach. The theory describes distorted-wave effects in terms of elastic scattering phase shifts, and the JWKB approximation proposed as a suitable method for their calculation at high energies. The theory has been applied to the study of some reactions. The results have been compared with conventional DWBA calculations and in some cases with experiment.

We shall now go on to briefly describe what this work is about.

In this work we shall use semiclassical concepts to set up a simple thoery to evaluate transfer amplitudes and angular distributions for transfer reactions between heavy ions. The starting point is the semiclassical theory of Brink and Pixton. Chapter II is divided into five sections. The first section contains a brief outline of the above theory for neutron transfer and the relevant formulae for calculating transfer amplitudes. The need to use a better wave function than the Hankel function approximation is pointed out. This point is investigated in the section that follow. In section II.2 we develop a method whereby the coordinate space integral (I.1.4) can be somewhat simplified. No approximation is made for the wave functions which are taken to be the solutions of Schrodinger equations. The resulting formula, though comparatively simpler is still fairly tedious to compute as we have to perform a three dimensional integration. We therefore present another formulation in section II.3 by means of which the four dimensional integral (I.1.2) is reduced to a sum of one dimensional integrals. This is achieved by assuming a straight line orbit for the relative motion and carrying out calculations in momentum space rather than in coordinate space as before. The formula we obtain provides a very quick and easy way to compute transfer amplitudes. This formula has the additional advantage of having exactly the same form in

the post and prior representation. To test the various formulae and to see if the post-prior discrepancy disappears by using the correct bound state wave functions rather than the Hankel function, we repeated the calculations of Pixton for the reaction $^{208}Pb(^{11}B, ^{10}B)^{209}Pb$. The results are presented in section II.4. Indeed the discrepancy is reduced when calculations are carried out in coordinate space and there is reasonable agreement between these results and those carried out in momentum space. In section II.5 we re-formulate theory for proton transfer. In this case one has to account for the effect of the long range Coulomb force of the final nucleus while the particle is still bound to the initial nucleus. This causes an additional time dependent phase factor to appear in eq. (I.1.3) which modifies the Q-value to an effective Q-value - Q_{eff}. We also have to subtract a term from the interaction potential V to compensate the long range part of V. The need to modify the Q-value has already been pointed out by Buttle and Goldfarb (Buttle and Goldfarb, 1972), while the correction to V is discussed by De Vries et. al. (De Vries et. al., 1974).

In Chapter III we go on to derive a formula for determining angular distributions. Our approach is similar to that of Landowne et. al. (Landowne et. al., 1976) in some respects. Like them we start with the expression of the transition matrix in DWBA, expand the distorted waves in a partial wave sum and use WKB approximation for the radial wave functions. However, our handling of

the problem is different. By using an average orbit we are able to perform one of the partial wave sums, ending up with a formula of the type (I.1.7). $\eta^T(\ell)$ is given as a product of $e^{i(\delta_i^{(i)} + \delta_i^{(f)})}$ and the semiclassical transfer amplitudes A_{if}. We discuss some applications of this formula in Chapter IV.

In chapter IV we have applied the above formula to study reaction between ^{11}B and ^{26}Mg at 114 MeV laboratory energy. Section IV.1 gives numerical details. In section IV.2 we have first looked at $|\eta^T(\ell)|$ for some of the reactions and notice that it is peaked about a certain ℓ-value. Thereafter the fall is exponential. This encouraged us to write a simple parametrised formula for A_{if} in the form of a decaying exponential. The parameters were fitted by computing A_{if} from eq. (I.1.2) at two points. A comparison of A_{if} using eq. (I.1.2) and the parametrised formula is shown graphically. In the relevant region, nowhere is the disagreement more than 10%. The same is true of $|\eta^T(\ell)|$. In section IV.3 we show a comparison of angular distributions using the parametrised form of A_{if} with experiment. The result is good. A comparison also is made of spectrosscopic factors obtained by us and the DWBA code LOLA of De Vries (De Vries, 1973). We go on to parametrise the phase shifts as well in section IV.4. This enables us to obtain a closed formula for angular distributions. We use this formula to calculate the angular distributions for the reaction $^{26}Mg(^{11}B, ^{10}B)^{27}Mg$ at 114 MeV and compare it with the previous result. The agreement is good.

The theoretical reaction $^{16}O(^{17}O^*, ^{16}O)^{17}O^*$ is studied in chapter V for five different energies in the range 212.5 MeV to 37.1875 MeV. The elastic scattering phase shifts are calculated in a potential with an energy dependent imaginary part. The results show good agreement with LOLA at high energies (till 53.125 MeV). There is disagreement at 37.1875 MeV, which is attributed to the breakdown of our formalism for weakly absorbing potentials. The results of Crowley are also presented for comparison. These agree in the shape of the angular distributions but there is some energy dependent disagreement in magnitude.

We have performed some calculations for the α-transfer reaction $^{16}O(^{16}O, ^{12}C)^{20}Ne$ for the transfer of an α-particle from the 0^+ ground state in ^{16}O to the 0^+ ground state in ^{20}Ne. Calculations were performed by calculating various parameters occuring in the equation in the initial and final channels. While the shape of the angular distributions in the two cases agreed there was some difference in the magnitude. A comparison with LOLA showed that for small angles there was a shift in the curves by about $.5^o$ but they soon become out of phase. There is also a disagreement in the magnitude. A calculation has also been made of the angular distribution for the transfer of the α-particle to the 4.25 (4^+) state of ^{20}Ne. No comparison is made with LOLA for this case.

CHAPTER II

CALCULATION OF TRANSFER AMPLITUDES

II.1 Semiclassical Theory for Neutron Transfer

In this section a semiclassical theory for neutron transfer reaction developed by Brink and Pixton (Pixton, 1972; Anayas Weiss et al, 1974) is presented. A detailed derivation of the formulae obtained for the transfer amplitudes (eqs. (II.1.2) - (II.1.5)) has been made by Pixton (Pixton, 1972), and only the final results are quoted here. The same procedure has been followed to obtain corresponding expressions for charged particle transfer in section II.5 of this chapter where the method is outlined. In the case of charged particle transfer one has to take into account the Coulomb interaction between the transferred particle and the interacting nuclei, whereas for neutron transfer this extra term in the interaction potential is absent.

We shall first consider the neutron transfer reaction:

$$a_1 + c_2 \rightarrow a_2 + c_1$$

$$(c_1 + n) + c_2 \rightarrow (c_2 + n) + c_1, \qquad \text{(II.1.1)}$$

in which a neutron is transferred from a bound state in nucleus a_1 to a bound state in nucleus a_2.

In what follows it is assumed that the initial and final nuclei follow well defined classical trajectories. The theory is semiclassical in that the transfer amplitude is calculated quantum mechanically whereas the orbits which the ions describe are classical.

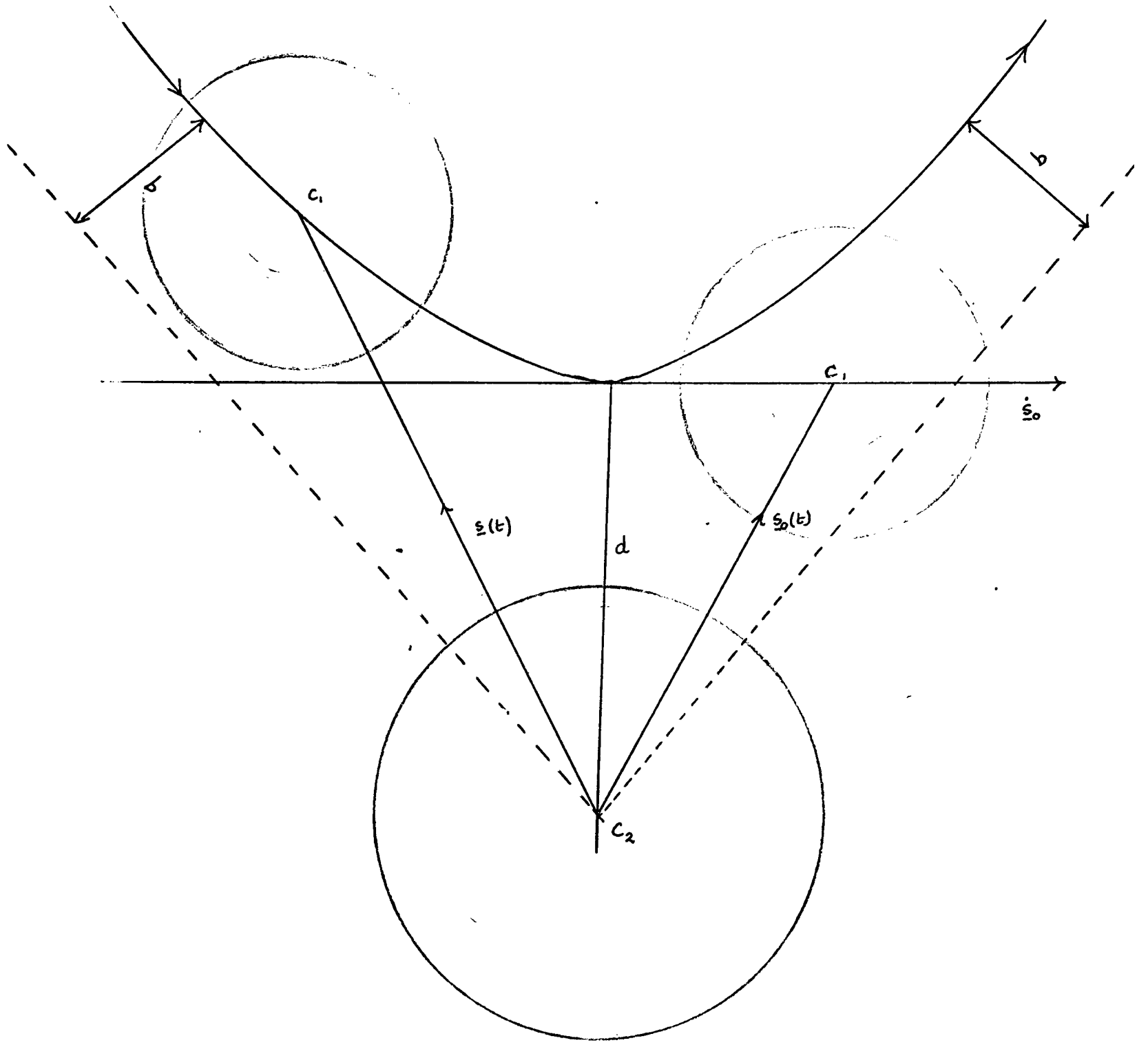


Fig. 1

Fig. 1 illustrates the orbit $\underline{s}(t)$ of relative motion of c_1 and c_2. The impact parameter is b and the distance of closest approach is d. In later work we approximate the actual orbit $\underline{s}(t)$ by a straight line orbit $\underline{s}_o(t)$ with constant velocity $\dot{\underline{s}}_o$ equal to the tangential velocity $\dot{\underline{s}}(t)$ at the point of closest approach.

The nucleon transfer is described by a Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V_1(\underline{r}_1 - \underline{s}_1(t)) + V_2(\underline{r}_2 - \underline{s}_2(t)),$$

where V_1 and V_2 are the potentials of the cores c_1 and c_2 respectively, m is the mass of the transferred nucleon and $\underline{s}_1(t)$ and $\underline{s}_2(t)$ are the classical trajectories of the centres of the cores c_1 and c_2. At $t = -\infty$ the neutron is assumed to be in a bound state ψ_1 in the potential V_1 while after transfer it is in a bound state ψ_2 in the potential V_2.

Making this assumption the quantal transfer probability can be evaluated by integrating over the orbit of relative motion of the nuclei. The expression obtained for the transfer probability (Pixton, 1972; Anayas-Weiss et. al., 1974) is:

$$P(\lambda_1, \lambda_2) = |A(\lambda_1, \lambda_2)|^2 \qquad \text{(II.1.2)}$$

where

$$A(\lambda_1, \lambda_2) = \frac{1}{\hbar}\int_{-\infty}^{+\infty} U_{\lambda_1\lambda_2}(t) \exp\left[-it\left(Q + \tfrac{1}{2}m\dot{s}^2\right)\right] dt \qquad \text{(II.1.3)}$$

with $\underline{s}(t) = \underline{s}_1(t) - \underline{s}_2(t)$.

and $U_{\lambda_1\lambda_2}(t)$ given in the prior representation by:

$$U_{\lambda_1\lambda_2}(t) = \int \psi_2^*(\underline{r}_2)\, V_2(\underline{r}_2)\, \psi_1(\underline{r}_2 - \underline{s}(t)) \exp\left(+\frac{i}{\hbar} m\dot{\underline{s}}\cdot\underline{r}_2\right) d^3\underline{r}_2 \qquad \text{(II.1.4)}$$

and in the post representation by:

$$U_{\lambda_1\lambda_2}(t) = \int \psi_2^*(\underline{r}_2)\, V_1(|\underline{r}_2 - \underline{s}(t)|)\, \psi_1(\underline{r}_2 - \underline{s}(t))\, \exp\left(+\frac{i}{\hbar} m \dot{\underline{s}} \cdot \underline{r}_2\right) d^3\underline{r}_2 \qquad \text{(II.1.5)}$$

Here $\psi_1(\underline{r}_1)$ and $\psi_2(\underline{r}_2)$ are given by:

$$\psi_1(\underline{r}_1) = u_1(r_1)\, Y_{\ell_1\lambda_1}(\theta_1\varphi_1), \qquad \text{(II.1.6)}$$

and
$$\psi_2(\underline{r}_2) = u_2(r_2)\, Y_{\ell_2\lambda_2}(\theta_2\varphi_2), \qquad \text{(II.1.7)}$$

where $\underline{r}_2$ is the coordinate of the transferred nucleon n, measured relative to the core c_2, and $\underline{r}_1 = \underline{r}_2 - s(t)$ is the coordinate of n relative to c_1: $\ell_1\lambda_1$, $\ell_2\lambda_2$ are the angular momentum quantum numbers. These wave functions satisfy the equations:

$$(T_1 + V_1)\, \psi_1(\underline{r}_1) = \epsilon_1\, \psi_1(\underline{r}_1) \qquad \text{(II.1.8)}$$

and
$$(T_2 + V_2)\, \psi_2(\underline{r}_2) = \epsilon_2\, \psi_2(\underline{r}_2) \qquad \text{(II.1.9)}$$

$Q = \epsilon_1 - \epsilon_2$ is the difference of the bound state energies. This formalism takes into account recoil effects through the phases appearing in eq. (II.1.3) and (II.1.4) (or eq. (II.1.5)) in the sense of Dodd and Greider (Dodd and Greider, 1969). The relationship has been shown explicitly by Anayas-Weiss et. al. (Anayas-Weiss et. al., 1974).

The total transfer probability for the transfer of a neutron from a state with angular momentum j_1 to a state with angular momentum j_2 can be obtained by coupling the neutron spin with the orbital angular momentum using appropriate Clebsch Gordan coefficients and summing over final states and averaging over the initial ones to give:

$$P_{21} = \frac{1}{2j_1+1} \sum_{n_1 n_2} \Big| \sum_{\substack{\lambda_1\lambda_2 \\ m_s}} \langle \ell_1\lambda_1, \Delta m_s | j_1 n_1 \rangle \langle \ell_2\lambda_2, \Delta m_s | j_2 n_2 \rangle\, A(\lambda_1,\lambda_2) \Big|^2 \qquad \text{(II.1.10)}$$

The cores c_1 and c_2 have been assumed to be spinless. The quantity $U_{\lambda_1\lambda_2}(t)$ appearing in $A(\lambda_1,\lambda_2)$ (eq. (II.1.3)) can be calculated either in the prior representation (eq. (II.1.4)) or in the post representation (II.1.5) as Pixton has shown that the two representations are equal. Thus:

$$\left[A(\lambda_1, \lambda_2)\right]_{\text{prior}} = \left[A(\lambda_1, \lambda_2)\right]_{\text{post}}$$

An exact evaluation of the transfer amplitude $A(\lambda_1,\lambda_2)$ involves a four dimensional integration over $\underline{r}_2$ and t, which requires a knowledge of the bound state wave functions $\psi_1(\underline{r}_1)$ and $\psi_2(\underline{r}_2)$. These can, of course, be obtained numerically by solving the corresponding Schrodinger equations. In order to simplify the calculation, however, Pixton made some approximations. In the prior representation he assumed that only the tail of the wave function $\psi_1(r_1)$ contributed to the integral (II.1.4) and that this tail could be approximated by a Hankel function. With this assumption he was able to simplify eq. (II.1.4) to a sum of one dimensional integrals which had to be computed numerically. The trajectory of the projectile relative to the target was assumed to be a straight line.

He studied the reaction $^{208}Pb(^{11}B, ^{10}B)^{209}Pb$ performed at 114 MeV laboratory energy. The transfer amplitudes $A(\lambda_1,\lambda_2)$ were evaluated for the transfer of a neutron from the 1 $p_{3/2}$ state in ^{11}B to the 2 $g_{9/2}$ state in ^{209}Pb for all possible combinations of λ_1, λ_2, using the Hankel

function approximation in the post and prior representations. A comparison of the results showed that there was not a good agreement between them. The ratio of the values of the post and prior representations was almost 1.5. This suggested that perhaps the Hankel function approximation was not a good one.

In the following section (II.2. and II.3) we show two ways in which $U_{\lambda_1\lambda_2}(t)$ can be evaluated without making the Hankel function approximation.

II.2 Calculation of Transfer Amplitudes in the Coordinate Representation

Here a method is developed to evaluate the transfer amplitude $A(\lambda_1,\lambda_2)$ without using an approximate form for $\Psi_1(\underline{r}_1)$. We shall assume that $\Psi_1(\underline{r}_1)$ and $\Psi_2(\underline{r}_2)$ are the exact solution of (II.1.8) and (II.1.9) respectively. By a rotation of axes such that the z-axis lies along $\underline{s}$ (Fig. 3), the integration over the azimuthal angle φ can be performed. Instead of the four dimensional integral (II.1.3), we are now left with a sum of three dimensional integrals which can be computed numerically. In section II.4 the results obtained from this calculation for the reaction $^{208}Pb(^{11}B, ^{10}B)^{209}Pb$ are presented and the results for the post and prior representations compared.

Two coordinate systems are used for the calculations described in this section.* The first is illustrated in fig. 2. The actual orbit of relative motion $\underline{s}(t)$ is replaced by the straight line orbit $\underline{s}_0(t)$ (Fig. 1) and D is the point of closest approach. At any instant of time t, $\underline{s}_0$ makes an angle α with C_2D. The x axis is chosen along C_2D and the y axis as the direction of $\dot{\underline{s}}_0$, while the z-axis is taken perpendicular to the reaction plane as shown in fig. 2. C_1, C_2 denote the centres of mass of the cores c_1, c_2 respectively, while n denotes the position of the neutron.

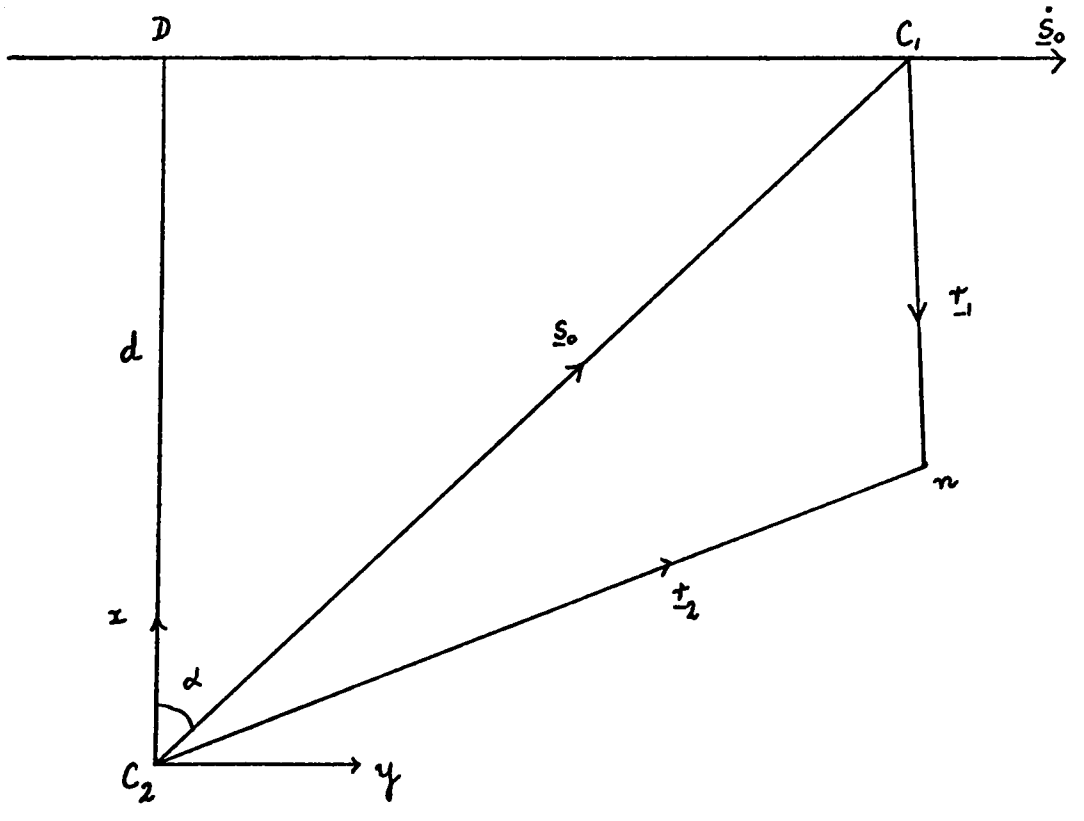


Fig. 2.

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* Note that the value of $A(\lambda_1, \lambda_2)$ depends on the coordinate system used. In the rest of this chapter we take $A(\lambda_1, \lambda_2)$ to refer to its value in the coordinate system illustrated in fig. 2.

Our first step in the evaluation of $A(\lambda_1,\lambda_2)$ is to determine the quantity $U_{\lambda_1\lambda_2}(t)$ given in the prior representation by (II.1.3). We shall therefore set about simplifying this equation as much as we can. To attain this we begin by separating $\psi_1(\underline{r}_1)$ and $\psi_2(\underline{r}_2)$ into radial and angular parts according to eqs. (II.1.6) and (II.1.7) respectively. Substituting these into (II.1.3) we obtain:

$$U_{\lambda_1\lambda_2}(t) = \int u_2^*(r_2)\, V_2(t_2)\, u_1(r_1)\, Y^*_{\lambda_2\lambda_2}(\theta_2\varphi_2)\, Y_{\lambda_1\lambda_1}(\theta_1\varphi_1)\, e^{i\underline{p}\cdot\underline{r}_2}\, d^3\underline{r}_2 \qquad \text{(II.2.1)}$$

where

$$\underline{p} = \frac{m\dot{\underline{s}}_0}{\hbar}$$

and

$$\underline{r}_1 = \underline{r}_2 - \underline{s}_0(t)$$

The coordinate system illustrated in fig. 2 is very suitable for calculating the time integral eq. (II.1.3) but is not so convenient for calculating $U_{\lambda_1\lambda_2}(t)$. For this purpose we introduce another coordinate system illustrated in fig. 3 with the z-axis chosen along $\underline{s}_0(t)$. This corresponds to a rotation of axes through Euler angles $(\alpha, \frac{\Pi}{2}, 0)$. The definition of Brink and Satchler is used for the Euler angles and the rotation matrices (which follow) (Brink and Satchler, 1968).

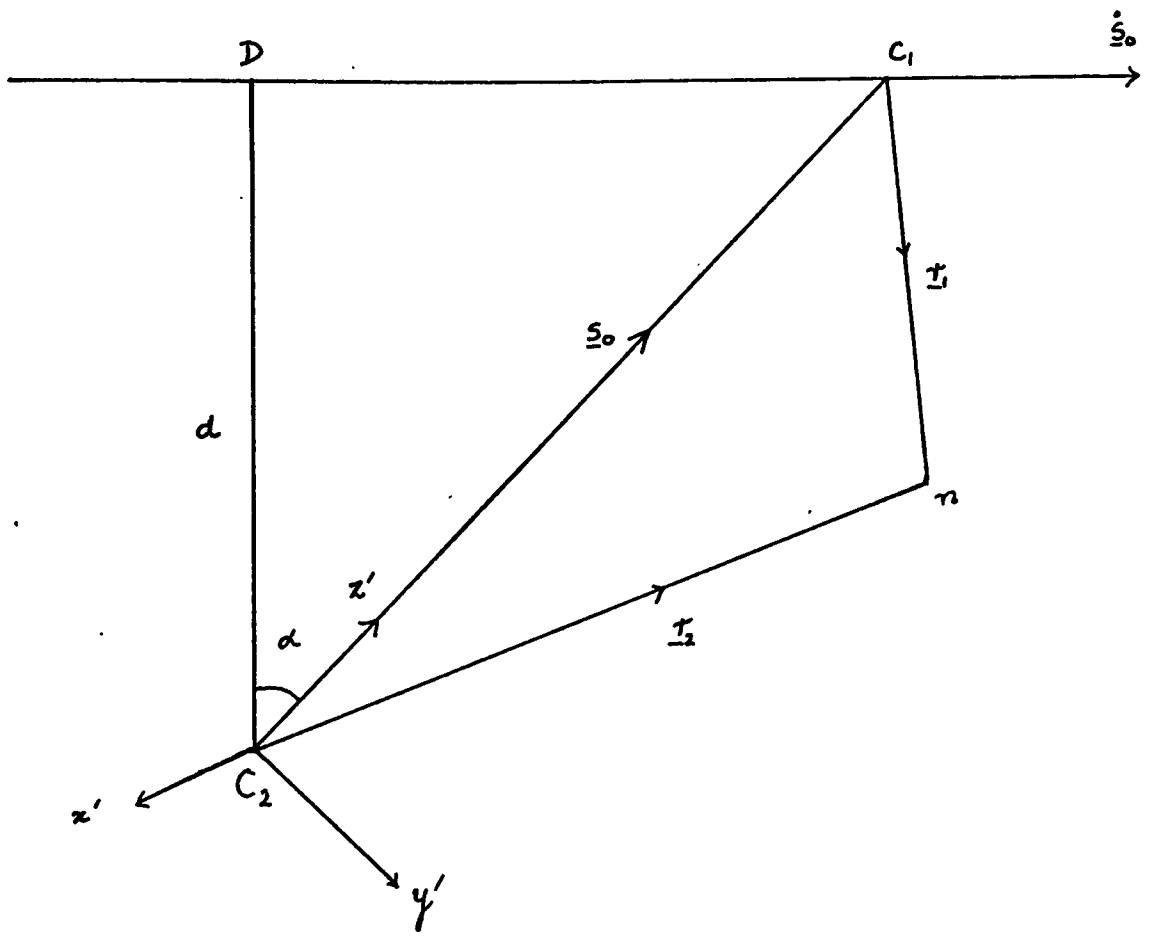


Fig. 3.

Under the rotation, the spherical harmonics in the wave functions will be transformed in the primed frame to:

$$Y_{l_1\lambda_1'}(\theta_1'\varphi_1') = \sum_{\lambda_1} D^{l_1}_{\lambda_1\lambda_1'}(\alpha, \tfrac{\pi}{2}, 0)\, Y_{l_1\lambda_1}(\theta_1\varphi_1) \qquad \text{(II.2.2)}$$

$$Y_{l_2\lambda_2'}(\theta_2'\varphi_2') = \sum_{\lambda_2} D^{l_2}_{\lambda_2\lambda_2'}(\alpha, \tfrac{\pi}{2}, 0)\, Y_{l_2\lambda_2}(\theta_2\varphi_2) \qquad \text{(II.2.3)}$$

where the $D^{\ell}_{\lambda\lambda'}$ are the rotation matrices. Multiplying (II.2.2) and (II.2.3) on the right by $(D^{\ell_1}_{\lambda_1\lambda_1'})^*$ and $(D^{\ell_2}_{\lambda_2\lambda_2'})^*$ respectively and summing over λ_1' and λ_2' we get the inverse relations:

$$Y_{l_1\lambda_1}(\theta_1\varphi_1) = \sum_{\lambda_1'} \mathcal{D}^{l_1}_{\lambda_1'\lambda_1}(0, -\tfrac{\pi}{2}, -\alpha)\, Y_{l_1\lambda_1'}(\theta_1'\varphi_1')$$

and $$Y_{l_2\lambda_2}(\theta_2\varphi_2) = \sum_{\lambda_2'} \mathcal{D}^{l_2}_{\lambda_2'\lambda_2}(0, -\tfrac{\pi}{2}, -\alpha)\, Y_{l_2\lambda_2'}(\theta_2'\varphi_2')$$

where we have used the property:

$$(D^{\ell}_{\lambda\lambda}, (\alpha, \beta, \gamma))^* = D^{\ell}_{\lambda',\lambda}(-\gamma, -\beta, -\alpha)$$

Substituting these expressions for the $Y_{\ell\lambda}$'s into (II.2.1) we obtain a relation between $U_{\lambda_1\lambda_2}(t)$ and $U'_{\lambda_1',\lambda_2'}(t)$:

$$U_{\lambda_1\lambda_2}(t) = \sum_{\lambda_1'\lambda_2'} \mathcal{D}^{l_1}_{\lambda_1'\lambda_1}(0, -\tfrac{\pi}{2}, -\alpha)\, \mathcal{D}^{l_2}_{\lambda_2'\lambda_2}(0, -\tfrac{\pi}{2}, -\alpha)\, U'_{\lambda_1'\lambda_2'}(t)$$

where

$$U'_{\lambda_1'\lambda_2'}(t) = \int u_2^*(r_2)\, V_2(r_2)\, u_1(r_1)\, Y_{l_1\lambda_1'}(\theta_1'\varphi')\, Y^*_{l_2\lambda_2'}(\theta_2'\varphi') \times$$

$$\times\, e^{i(p_y' \cos\theta_2' + p_y' \sin\theta_2' \sin\varphi')\, r_2}\, r_2^2\, dr_2\, d(\cos\theta_2')\, d\varphi' \qquad \text{(II.2.4)},$$

where $\varphi_1' = \varphi_2' = \varphi'$

and

$$\underline{p}\cdot\underline{r}_2 = (p_z' \cos\theta_2' + p_y' \sin\theta_2' \sin\varphi')\, r_2$$

p_z' and p_y' being the components of $\underset{\sim}{p}$ along z' and y' respectively.

Using the definition of spherical harmonics:

$$Y_{lm}(\theta\varphi) = N_{lm}\, P_l^m(\cos\theta)\, e^{im\varphi}$$

where $N_{\ell m}$ is a normalisation constant and $P_\ell^m(\cos\theta)$ is an associated Legendre polynomial, we have:

$$Y^*_{l_2\lambda_2'}(\theta_2'\varphi_2')\, Y_{l_1\lambda_1'}(\theta_1'\varphi') = N^*_{l_2\lambda_2}\, N_{l_1\lambda_1}\, P_{l_2}^{\lambda_2'}(\cos\theta_2')\, P_{l_1}^{\lambda_1'}(\cos\theta_1')\, e^{i(\lambda_1'-\lambda_2')\varphi'}$$

Substituting this into (II.2.4) and separating out the integral over φ' we have

$$I_{(\varphi')} = \int_0^{2\pi} e^{i p_y' \sin\theta_2' \sin\varphi' r_2}\, e^{i(\lambda_1'-\lambda_2')\varphi'}\, d\varphi'$$

Let $\qquad \lambda_2' - \lambda_1' = \Delta\lambda' \quad , \quad p_y' \sin\theta_2' r_2 = \beta$

Then (Jahnke and Emde, 1945) :

$$I_{(\varphi')}(\lambda,\beta) = \int_0^{2\pi} e^{-i(\Delta\lambda'\varphi' - \beta\sin\varphi')}\, d\varphi'$$

$$= 2\pi\, J_{\Delta\lambda'}(\beta)$$

$$= 2\pi\, J_{\Delta\lambda'}(p_y' \sin\theta_2' r_2)$$

Hence eq. (II.2.4) becomes:

$$U'_{\lambda_1'\lambda_2'}(t) = 2\pi\, N_{\lambda_1'\lambda_2'} \int u_2^*(t_2)\, V_2(t_2)\, u_1(t_1)\, P_{l_2}^{\lambda_2'}(\cos\theta_2')\, P_{l_1}^{\lambda_1'}(\cos\theta_1') \times$$

$$\times\, J_{\Delta\lambda'}(p_y' \sin\theta_2' t_2)\, e^{i p_z' \cos\theta_2' t_2} \times$$

$$\times\, t_2^2\, dt_2\, d(\cos\theta_2')\ ,$$

where $N_{\lambda_1'\lambda_2'} = N_{l_1\lambda_1}\ N^*_{l_2\lambda_2'}$

The D-matrices can be written in terms of the d-matrices:

$$D^{l}_{\lambda'\lambda}(\alpha\beta\gamma) = e^{-i(\alpha\lambda'+\beta\lambda')} d^{l}_{\lambda'\lambda}(\beta),$$

to yield:

$$U_{\lambda_1\lambda_2}(t) = e^{i\alpha(\lambda_1-\lambda_2)} \sum_{\lambda_1'\lambda_2'} d^{l_1}_{\lambda_1'\lambda_1}(-\tfrac{\pi}{2})\, d^{l_2}_{\lambda_2'\lambda_2}(-\tfrac{\pi}{2})\, U'_{\lambda_1'\lambda_2'}(t) \qquad \text{(II.2.5)}$$

The three dimensional integral (II.2.1) has thus been reduced to a two dimensional integral which has to be computed numerically. The calculation of the transfer amplitude (II.1.3) hence involves a three dimensional integral to be evaluated by numerical means.

The actual number of distinct integrals appearing in the summation in eq. (II.2.5) is less than it may seem. This is because there is a symmetry between the terms dependent on λ_1', λ_2' and between those dependent on $-\lambda_1'$, $-\lambda_2'$.

This can be seen as follows. From the symmetry properties of associated Legendre polynomials and Bessel functions we know:

$$P_{\ell\lambda}(x) = (-1)^{\lambda} P_{\ell_1 -\lambda}(x)$$

and $$J_{\lambda}(y) = (-1)^{\lambda} J_{-\lambda}(y)$$

Thus:

$$P_{\ell_2}^{-\lambda_2'}(\cos\theta_2')\, P_{\ell_1}^{-\lambda_1'}(\cos\theta_1')\, J_{-\Delta\lambda'}(p_y' \sin\theta_2' r_2)$$

$$= (-1)^{\lambda_2'} P_{\ell_2}^{\lambda_2'}(\cos\theta_2')\, (-1)^{\lambda_1'} P_{\ell_1}^{\lambda_1'}(\cos\theta_1') \times$$

$$(-1)^{\lambda_2' - \lambda_1'} J_{\Delta\lambda}(p_y' \sin\theta_2' r_2)$$

$$= P_{\ell_2}^{\lambda_2'}(\cos\theta_2')\, P_{\ell_1}^{\lambda_1'}(\cos\theta_1')\, J_{\Delta\lambda}(p_y' \sin\theta_2' r_2)$$

The number of integrals is therefore reduced from $(2\ell_1 + 1)(2\ell_2 + 1)$ to $\left[(2\ell_1 + 1)(2\ell_2 + 1) + 1\right]/2$.

Further, it has been shown by Pixton (Pixton, 1972) by considering the symmetry properties of $A(\lambda_1, \lambda_2)$ under reflection in the x - y plane, that it is non-vanishing only for those values of λ_1, λ_2 for which:

$$\ell_1 + \ell_2 + \lambda_1 + \lambda_2 \text{ is even.}$$

Because of this, we need calculate the sum (II.2.5) just for those values of λ_1, λ_2 which satisfy the above condition.

Under complex conjugation $U_{\lambda_1\lambda_2}(t)$ behaves as:

$$U_{\lambda_1\lambda_2}(\underline{p}\ ,\ -\alpha) \;=\; U^{*}_{\lambda_1\lambda_2}(\underline{p}\ ,\ \alpha).$$

Therefore, the integrand of eq. (II.1.3) which we shall denote by $W_{\lambda_1\lambda_2}(t)$ has the property:

$$W^{*}_{\lambda_1\lambda_2}(t) \;=\; W_{\lambda_1\lambda_2}(-t),$$

where explicitly:

$$W_{\lambda_1\lambda_2}(t) = U_{\lambda_1\lambda_2}(t)\ \exp\left[-\frac{it}{\hbar}\left(Q + \tfrac{1}{2} m \dot{s}^2\right)\right]$$

$$\therefore \qquad P_{21} \;=\; \frac{1}{\hbar^2}\left|\int_{-\infty}^{+\infty} W_{\lambda_1\lambda_2}(t)\,dt\right|^2$$

$$= \frac{4}{\hbar^2}\left|\int_{0}^{\infty} \mathrm{Re}\, W_{\lambda_1\lambda_2}(t)\,dt\right|^2$$

This property helps to simplify the computing of the integral.

The formulae obtained in this section have been used to compute transfer amplitudes for the $^{208}Pb(^{11}B,\ ^{10}B)^{209}Pb$ reaction studied by Pixton. The results, both for the post and prior represnetations are presented in section 4 of this chapter.

II.3 Calculation of Transfer Amplitudes in Momentum Representation

Actual calculations show that the evaluation of the integral (II.1.3) giving the transfer amplitude, is a lengthy and time consuming process even after using the simplifications of the previous section. If, however, we write eq. (II.1.3) in the momentum representation the expressions simplify considerably and we end up with a one dimensional integral which is much easier to compute. We first obtain an expression for the transfer amplitude by working in the post representation and later show that it is identical with the one obtained in the prior representation.

Post Representation

As before we assume a straight line orbit for the projectile (fig. 1). The expression for the transfer amplitude is:

$$A(\lambda_1,\lambda_2) = \frac{1}{i\hbar}\int_{-\infty}^{+\infty} U_{\lambda_1\lambda_2}(t)\,\exp\left[-it\left(Q+\tfrac{1}{2}m\dot{s}_0^2\right)\right]dt\,, \tag{II.1.3}$$

with $$U_{\lambda_1\lambda_2}(t) = \int \psi_2^*(\underline{r}_2)\,V_1(r_1)\,\psi_1(\underline{r}_2-\underline{s}_0)\,e^{\frac{i}{\hbar}m\dot{\underline{s}}_0\cdot\underline{r}_2} \tag{II.1.5}$$

Remembering that:

$$(T_1 + V_1)\,\Psi_1(\underline{r}_1) = \epsilon_1\Psi_1(\underline{r}_1) \tag{II.1.8}$$

we get:

$$U_{\lambda_1\lambda_2}(t) = \int \psi_2^*(\underline{r}_2)\,(\varepsilon_1 - T_1)\,\psi_1(\underline{r}_1)\, e^{\frac{i}{\hbar} m \dot{\underline{s}}_0 \cdot \underline{r}_2}\, d^3\underline{r}_2 \qquad \text{(II.3.1)}$$

Changing to momentum space by substituting:

$$\psi_2^*(\underline{r}_2) = \frac{1}{(2\pi)^{3/2}} \int \tilde{\psi}_2^*(\underline{k}_2)\, e^{-i\underline{k}_2 \cdot \underline{r}_2}\, d^3\underline{k}_2 \qquad \text{(II.3.2)}$$

$$\psi_1(\underline{r}_1) = \frac{1}{(2\pi)^{3/2}} \int \tilde{\psi}_1(\underline{k}_1)\, e^{i\underline{k}_1 \cdot \underline{r}_1}\, d^3\underline{k}_1 \qquad \text{(II.3.3)}$$

we get:

$$\begin{aligned} U_{\lambda_1\lambda_2}(t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} & \tilde{\psi}_2^*(\underline{k}_2)\,(\varepsilon_1 - \tfrac{\hbar^2 k_1^2}{2m})\,\tilde{\psi}_1(\underline{k}_1) \times \\ & \times e^{i(\underline{k}_1 - \underline{k}_2 + \frac{m\dot{\underline{s}}_0}{\hbar})\cdot \underline{r}_2}\, e^{-i\underline{k}_1 \cdot \underline{s}_0}\, d^3\underline{k}_1\, d^3\underline{k}_2\, d^3\underline{r}_2 \end{aligned}$$

$$= \int_{-\infty}^{+\infty} \tilde{\psi}_2^*(\underline{k}_2)\,(\varepsilon_1 - \tfrac{\hbar^2 k_1^2}{2m})\,\tilde{\psi}_1(\underline{k}_1)\, e^{-i\underline{k}_1 \cdot \underline{s}_0}\, \delta^3(\underline{k}_1 - \underline{k}_2 + \tfrac{m\dot{\underline{s}}_0}{\hbar})\, d^3\underline{k}_1\, d^3\underline{k}_2$$

$$= \int_{-\infty}^{+\infty} \tilde{\psi}_2^*(\underline{k}_1 + \tfrac{m\dot{\underline{s}}_0}{\hbar})\,(\varepsilon_1 - \tfrac{\hbar^2 k_1^2}{2m})\,\tilde{\psi}_1(\underline{k}_1)\, e^{-i\underline{k}_1 \cdot \underline{s}_0}\, d^3\underline{k}_1$$

$$= \int_{-\infty}^{+\infty} \tilde{\psi}_2^*(\underline{k}_1 + \tfrac{m\dot{\underline{s}}_0}{\hbar})\,(\varepsilon_1 - \tfrac{\hbar^2 k_1^2}{2m})\,\tilde{\psi}_1(\underline{k}_1)\, e^{-i\underline{k}_1 \cdot \underline{d}}\, e^{-i\underline{k}_1 \cdot \dot{\underline{s}}_0 t}\, d^3\underline{k}_1 \,, \qquad \text{(II.3.4)}$$

where we have substituted

$$\underline{s}_o = \underline{d} + \dot{\underline{s}}_o t,$$

d being the distance of closest approach and $\dot{\underline{s}}_o$ the velocity at that point (see fig. 1).

When evaluating the transfer amplitude from eq. (II.3.4) it is convenient to choose a coordinate system with z-axis along the direction of $\dot{\underline{s}}_o$ as shown in fig. 4. It is obtainable from the system in fig. 2 by a rotation through Euler angles $(\frac{\Pi}{2}, \frac{\Pi}{2}, 0)$.

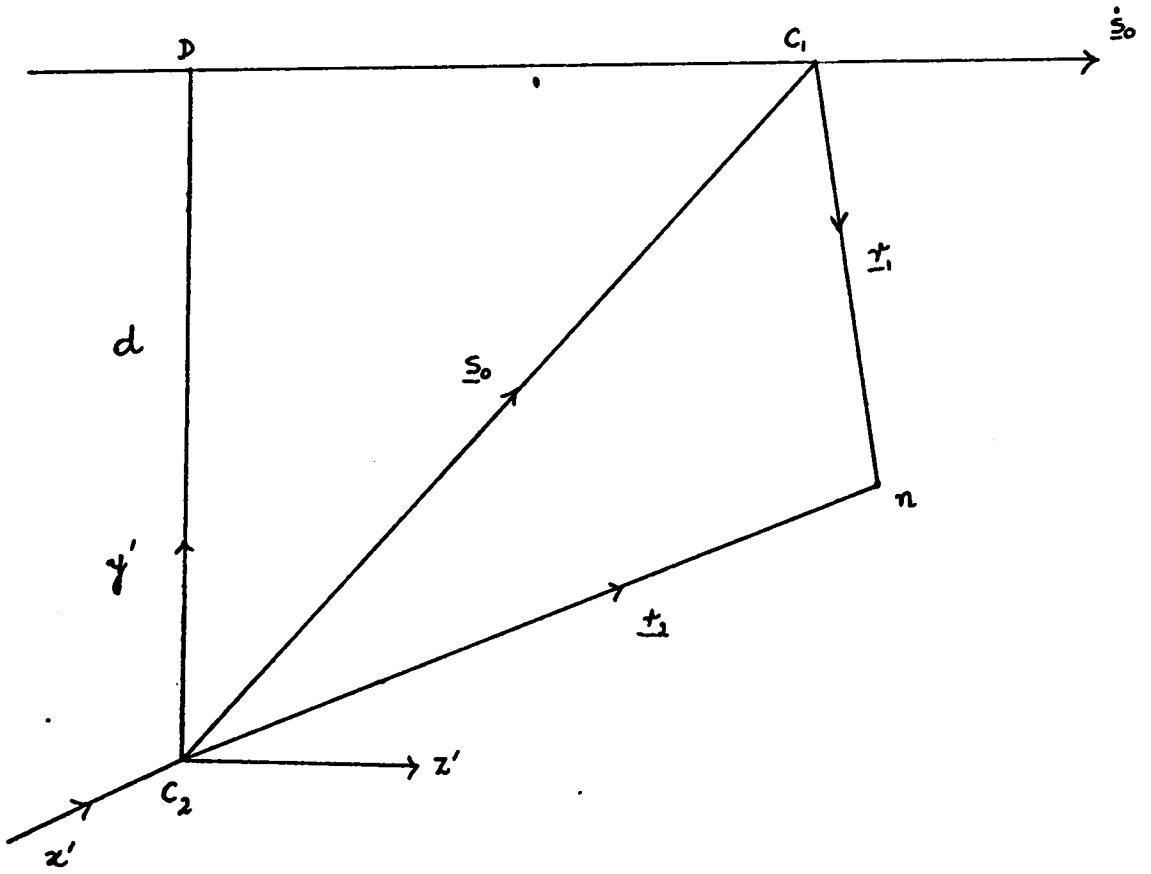


Fig. 4

An expression for the transfer amplitude $A'(\lambda_1', \lambda_2')$ in this frame of reference is obtained by substituting eq. (II.3.4) into eq. (II.1.3) and remembering that now:

$$\underline{k}_1' \cdot \dot{\underline{s}}_0 = k_{1z}' |\dot{\underline{s}}_0|$$

Thus,

$$A'(\lambda_1', \lambda_2') = \frac{1}{\hbar} \int_{-\infty}^{+\infty} \tilde{\psi}_2^* \left(\underline{k}_1' + \frac{m \dot{\underline{s}}_0}{\hbar} \right) \left(\varepsilon_1 - \frac{\hbar^2 k_1^2}{2m} \right) \tilde{\psi}_1 (\underline{k}_1') \, e^{-i \underline{k}_1' \cdot \underline{d}} \times$$

$$\times e^{-\frac{i}{\hbar} \left(Q + \frac{1}{2} m \dot{s}_0^2 + \hbar^2 k_{1z}' |\dot{\underline{s}}_0| \right) t} \, d^3 \underline{k}_1' \, dt$$

$$= \frac{2\pi}{\hbar} \int \tilde{\psi}_2^* \left(\underline{k}_1' + \frac{m \dot{\underline{s}}_0}{\hbar} \right) \left(\varepsilon_1 - \frac{\hbar^2 k_1^2}{2m} \right) \tilde{\psi}_1 (\underline{k}_1') \, e^{-i \underline{k}_1' \cdot \underline{d}} \times$$

$$\times \delta \left(k_{1z}' |\dot{\underline{s}}_0| + \frac{Q + \frac{1}{2} m \dot{s}_0^2}{\hbar} \right) d^3 \underline{k}_1$$

Writing $\tilde{\psi}(\underline{k})$ as:

$$\tilde{\psi}(\underline{k}) = \tilde{u}(k) \, Y_{\ell m}(\theta_k, \varphi_k)$$

and integrating over k_{1z}' we have:

$$A'(\lambda_1', \lambda_2') = \frac{2\pi}{\hbar |\dot{\underline{s}}_0|} \int \tilde{u}_2^*(k_2) \, \tilde{u}_1(k_1) \, Y_{\ell_2 \lambda_2'}^*(\theta_{k_2}', \varphi_{k_2}') \, Y_{\ell_1 \lambda_1}(\theta_{k_1'}, \varphi_{k_1'}) \times$$

$$\times \left(\varepsilon_1 - \frac{\hbar^2 k_1^2}{2m} \right) e^{-i \underline{k}_1' \cdot \underline{d}} \, dk_{1x}' \, dk_{1y}' \qquad \text{(II.3.5)}$$

where $\underline{k}_2 = \underline{k}_1 + \frac{m \dot{\underline{s}}_0}{\hbar}$,

$$k'_{1\gamma} = K = -\frac{\left(Q + \frac{1}{2} m \dot{s}_0^2\right)}{\hbar |\dot{s}_0|}$$

$$d'_x = d'_z = 0, \quad d'_y = d,$$

and $$\varphi'_{k_1} = \varphi'_{k_2} = \varphi_{k_1} \quad \text{(say)}$$

Various quantities in eq. (II.3.5) are illustrated in fig. 5. The integral in eq. (II.3.5) is taken over the plane through PB perpendicular to O_2O_1B. Note that the position of this plane depends on the Q-value of the reaction.

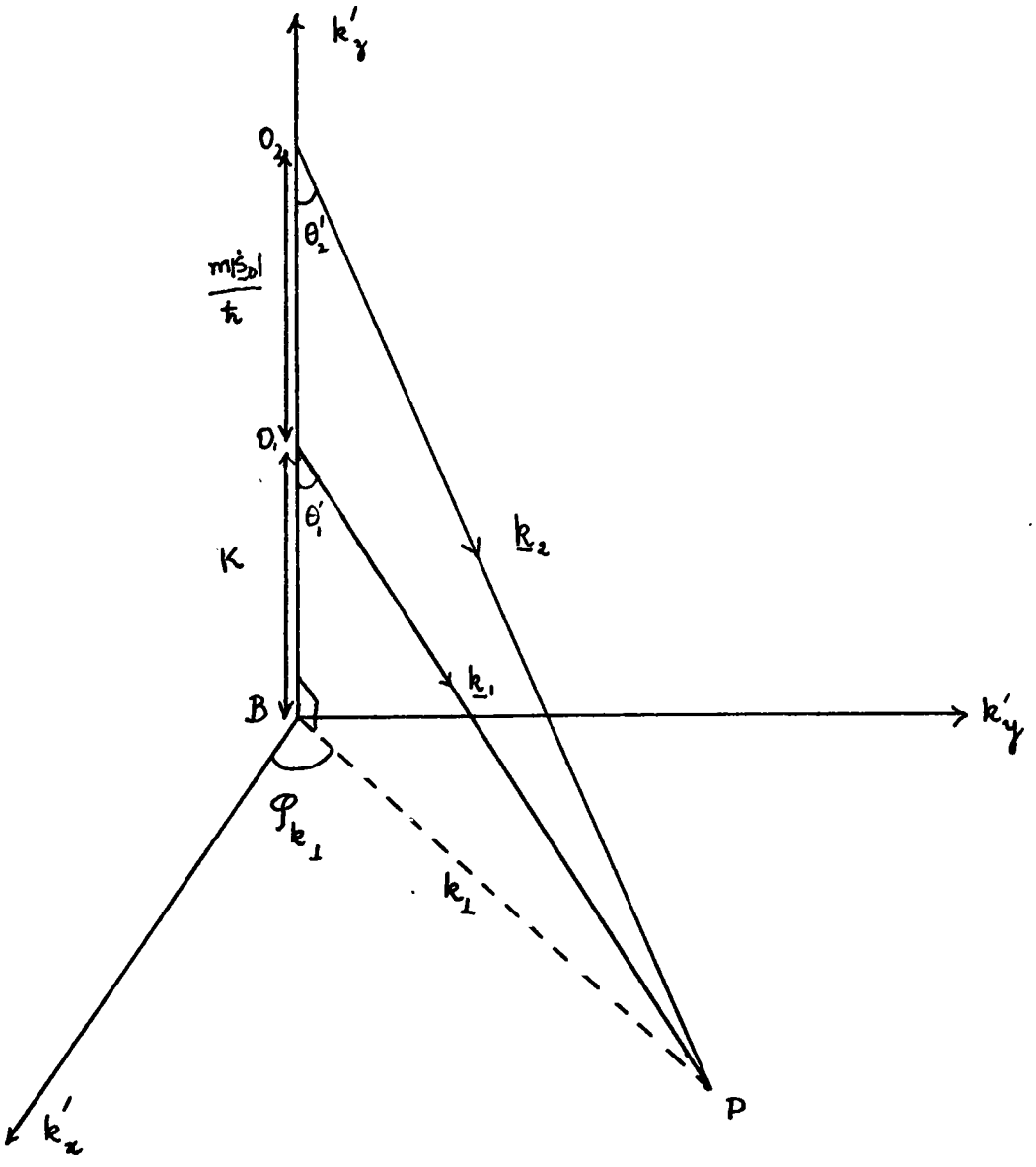


Fig. 5

To evaluate the integral (II.3.5) we make a change of variables

$$k'_{1x} = k_{\underline{1}} \cos \varphi_{k_{\underline{1}}}$$

$$k'_{1y} = k_{\underline{1}} \sin \varphi_{k_{\underline{1}}}$$

the relation between $\underline{k}_{\underline{1}}$ and $\underline{k}_1$ and $\underline{k}_2$ being indicated in Fig. 5.

The expression for $A'(\lambda'_1, \lambda'_2)$ now becomes:

$$A'(\lambda'_1,\lambda'_2) = \frac{2\pi}{\hbar|\dot{\underline{s}}_0|} \int_0^{2\pi}\int_0^{\infty} \tilde{u}_2^*(k_2)\,\tilde{u}_1(k_1)\, P_{l_2}^{\lambda'_2}(\cos\theta'_2)\, P_{l_1}^{\lambda'_1}(\cos\theta'_1)\, e^{-i(\lambda'_2-\lambda'_1)\varphi_{k_\perp}} \times \left(\varepsilon_1 - \frac{\hbar^2 k_1^2}{2m}\right) e^{-ik_\perp d\sin\varphi_{k_\perp}} k_\perp\, dk_\perp\, d\varphi_{k_\perp}$$

Integrating over φ_{k_1} we get:

$$A'(\lambda'_1,\lambda'_2) = \frac{(2\pi)^2}{\hbar|\dot{\underline{s}}_0|} \int_0^{\infty} \tilde{u}_2^*(k_2)\,\tilde{u}_1(k_1)\, P_{l_2}^{\lambda'_2}(\cos\theta'_2)\, P_{l_1}^{\lambda'_1}(\cos\theta'_1) \times \left(\varepsilon_1 - \frac{\hbar^2 k_1^2}{2m}\right) J_{\lambda'_1-\lambda'_2}(k_\perp d)\, k_\perp\, dk_\perp \qquad \text{(II.3.6)}$$

The four dimensional integral (II.1.3) has thus been reduced to a one dimensional integral eq. (II.3.6) which is fairly easy to compute and provides a simple way to calculate the transfer amplitude.

To obtain this expression in the original frame of reference (fig. 2), we perform an inverse rotation through the Euler angles, $(0, \frac{\Pi}{2}, \frac{\Pi}{2})$ so that the expressions for $A(\lambda_1, \lambda_2)$ in the two frames are related by:

$$A(\lambda_1,\lambda_2) = \sum_{\lambda_1'\lambda_2'} D^{j_2*}_{\lambda_2'\lambda_2}(0,\tfrac{\pi}{2},\tfrac{\pi}{2})\; D^{j_1}_{\lambda_1'\lambda_1}(0,\tfrac{\pi}{2},\tfrac{\pi}{2})\; A'(\lambda_1',\lambda_2')\,.$$

$$= e^{-i(\lambda_1-\lambda_2)\pi/2} \sum_{\lambda_1'\lambda_2'} d^{j_1}_{\lambda_1'\lambda_1}(\tfrac{\pi}{2})\; d^{j_2}_{\lambda_2'\lambda_2}(\tfrac{\pi}{2})\; A'(\lambda_1',\lambda_2') \qquad \text{(II.3.7)}$$

Prior Representation

Instead of using the expression given by eq. (II.1.5) for $U_{\lambda_1\lambda_2}(t)$, if we begin with the expression (II.1.4) in the prior representation we see that the formula obtained for $A(\lambda_1, \lambda_2)$ (eq. (II.3.6) and (II.3.7)) is exactly the same in both cases:

In the prior representation we have:

$$U_{\lambda_1\lambda_2}(t) = \int \Psi_2^*(\underline{r}_2)\, V_2(r_2)\, \Psi_1(\underline{r}_2 - \underline{s}_0)\, e^{\frac{i}{\hbar} m \dot{\underline{s}}_0 \cdot \underline{r}_2} \qquad \text{(II.3.8)}$$

As before:

$$(T_2 + V_2)\; \Psi_2(\underline{r}_2) \;=\; \epsilon_2\; \Psi_2(\underline{r}_2)$$

or $$\Psi_2^*(r_2)\; (T_2 + V_2) \;=\; \Psi_2^*(\underline{r}_2)\epsilon_2,$$

so that:

$$U_{\lambda_1\lambda_2}(t) = \int \Psi_2^*(\underline{r}_2)\, (\epsilon_2 - T_2)\, \Psi_1(\underline{r}_1)\, e^{im\dot{\underline{s}}_0\cdot\underline{r}_2/\hbar}\, d^3\underline{r}_2$$

Changing to momentum space and simplifying:

$$U_{\lambda_1\lambda_2}(t) = \int_{-\infty}^{+\infty} \tilde{\Psi}_2^{*}\left(\underline{k}_1 + \frac{m\dot{\underline{s}}_0}{\hbar}\right)\left(\varepsilon_2 - \frac{\hbar^2 k_2^2}{2m}\right)\tilde{\Psi}_1(\underline{k}_1)\, e^{-i\underline{k}_1\cdot(\underline{d}+\dot{\underline{s}}_0 t)}\, d^3\underline{k}_1$$

Following exactly the same procedure as before we get:

$$A'(\lambda_2',\lambda_1') = \frac{2\pi}{\hbar|\dot{\underline{s}}_0|}\int \tilde{u}_2^{*}(k_2)\,\tilde{u}_1(k_1)\,P_{L_2}^{\lambda_2'}(\cos\theta_2')\,P_{L_1}^{\lambda_1'}(\cos\theta_1') \times \left(\varepsilon_2 - \frac{\hbar^2 k_2^2}{2m}\right) J_{\lambda_1'-\lambda_2'}(k_\perp d)\, k_\perp\, dk_\perp \qquad \text{(II.3.9)}$$

where all the quantities appearing in the equation have the same meaning as before.

Remembering that:

$$\underline{k}_2 = \underline{k}_1 + \frac{m\,\dot{\underline{s}}_0}{\hbar}$$

$$k_{1z}' = -\frac{Q + \frac{1}{2}m\dot{s}_0^2}{\hbar|\dot{\underline{s}}_0|},$$

and that $\dot{\underline{s}}_0$ lies along the z-axis, we have:

$$k_2^2 = k_1^2 + \frac{m^2\dot{s}_0^2}{\hbar^2} + \frac{2m}{\hbar}k_{1z}'|\dot{\underline{s}}_0|$$

$$= k_1^2 + \frac{m^2\dot{s}_0^2}{\hbar^2} + \frac{2m}{\hbar}|\dot{\underline{s}}_0|\left(-\frac{Q+\frac{1}{2}m\dot{s}_0^2}{\hbar|\dot{\underline{s}}_0|}\right)$$

or $$\frac{\hbar^2 k_2^2}{2m} = \frac{\hbar^2 k_1^2}{2m} - Q$$

$$\therefore \quad \varepsilon_2 - \frac{\hbar^2 k_2^2}{2m} = -\frac{\hbar^2 k_1^2}{2m} + (\varepsilon_2 + Q) = -\frac{\hbar^2 k_1^2}{2m} + \varepsilon_1$$

Substituting in (II.3.9), we get:

$$A'(\lambda_1', \lambda_2') = \frac{2\pi i}{\hbar |\dot{s}_0|} \int \tilde{u}_2^*(k_2)\, \tilde{u}_1(k_1)\, P_{l_2}^{\lambda_2'}(\cos\theta_2')\, P_{l_1}^{\lambda_1'}(\cos\theta_1') \times \left(\varepsilon_1 - \frac{\hbar^2 k_1^2}{2m}\right) J_{\lambda_1' - \lambda_2'}(k_\perp d)\, k_\perp\, dk_\perp$$

which is identical to eq. (II.3.6).

This method thus has the additional advantage that we do not have to perform two separate calculations for the post and prior representations and we are in no way prejudiced to use one representation in preference to the other.

In the following section, numerical results obtained using eq. (II.3.7) for the transfer of a neutron from the $1p_{3/2}$ state in ^{11}B to the $2g_{9/2}$ state in ^{209}Pb in the ^{208}Pb (^{11}B, ^{10}B)^{209}Pb reaction at 114 MeV laboratory energy are tabulated together with the corresponding results in the coordinate representation for the post and prior representations. The transfer probability P_{21} (eq. (II.1.10)) is also calculated for this case as well as for transfer to excited states in ^{209}Pb. The results are compared with experimental differential cross sections at the grazing angle as well as with Pixton's results.

II.4 A Numerical Comparison of the Two Methods:

The neutron transfer reaction:

$$^{208}Pb(^{11}B,\ ^{10}B)\ ^{209}Pb$$

at 114 MeV laboratory energy has been considered and the transfer amplitude calculated using both eq. (II.2.5) and eq. (II.3.7).

The potentials V_1 and V_2 have been taken to be Woods-Saxon wells, the parameters of which are chosen so as to be typical of the parameters used by other authors. The radius parameter is set equal to $1.25\ A^{\frac{1}{3}}$ and the diffuseness parameter to 0.65. No spin orbit coupling is introduced. The bound state wave functions have been calculated by a computer programme which automatically fixes the depth of the potential so as to give the correct binding energy.

To test the programmes and to compare various methods for calculating transfer amplitudes, a typical value of the distance of closest approach, d was taken from the formula:

$$d = r_o(A_1^{\frac{1}{3}} + A_2^{\frac{1}{3}}), \qquad \text{(II.4.1)}$$

with $r_o = 1.25$. The slowing down of the projectile owing to the Coulomb repulsion of the cores has been taken into account in the calculation of $|\dot{\underline{s}}_o|$, the velocity appropriate to a grazing collision (fig. 1). It has been obtained from the formula:

$$\dot{\underline{s}}_o^2 = v^2 - 2\frac{Z_1 Z_2}{d}\frac{e^2}{\mu}, \qquad \text{(II.4.2)}$$

where v is the asymptotic velocity of the projectile given by:

$$\tfrac{1}{2} m_{a_1} v^2 = E_{lab}, \qquad \text{(II.4.3)}$$

or equivalently, $\tfrac{1}{2}\mu v^2 = E_{cm}$,

m_{a_1} being the mass of the projectile and μ the reduced mass of the projectile and target. $Z_1 e$, $Z_2 e$ are the charges of the two nuclei.

The transfer amplitude was first calculated using eq. (II.1.3) where $U_{\lambda_1\lambda_2}(t)$ was computed according to eq. (II.2.5). As we can see, this involves a double integration over r_2 and θ_2. To perform this integration it was found convenient to transform the coordinates to a new set of variables ξ_1, ξ_2 related to the old set as shown in fig. 6, and integrating over ξ_1 and ξ_2 numerically. The details of the transformation are discussed in appendix (II.A).

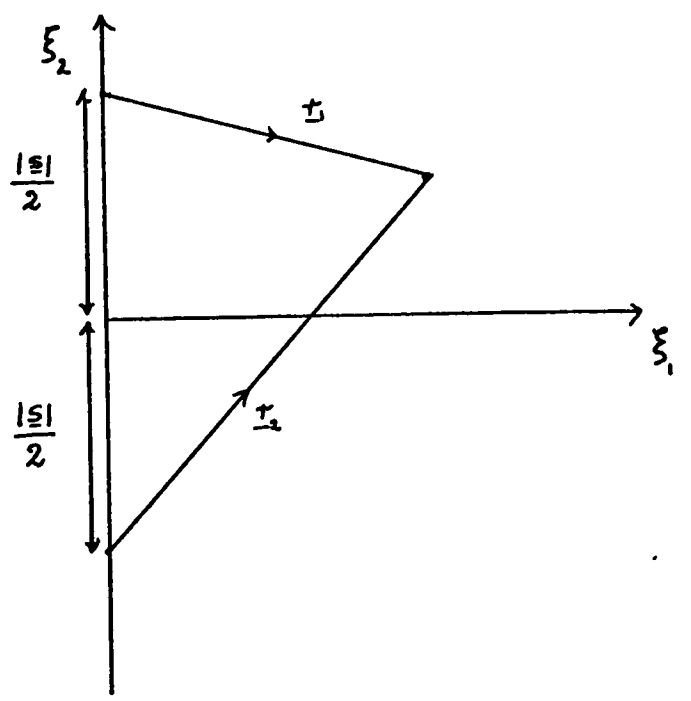


Fig. 6

The integration routine used is such that it automatically determines whether it should use Simpson's rule, the 5-point rule or the 9-point quadrature for integration, depending on the accuracy demanded. Amongst the input parameters to the subroutine one has to specify the limits of the integration, the initial number of parts into which the integration interval is divided and the absolute error demanded. Care is needed when choosing these parameters, as the results seem unduly sensitive to the choice made. Consequently, a number of calculations were performed using different starting values till fairly stable results were obtained. The time integration was performed using Simpson's Rule.

We present a table (I) of the results in which the limits of integration have been varied, to give some idea of the sort of convergence achieved. The first line in each set of results corresponds to the calculation being performed in the prior representation while the second line to that in the post representation. The results are relatively stable, but the integrals have not converged completely. As these calculations were fairly tedious, we did not make an attempt to get more stability.

We can see from table I that the agreement between the post and prior representations is reasonably good. The discrepancy between the two results seems to be largely numerical rather than actual. Thus it appears that a replacement of the Hankel function by the actual

bound state wave function removes the former disagreement between the post and prior representation results, suggesting that that approximation was perhaps an over-simplification of the problem.

The present results are further supported by the numbers obtained by calculating the transfer amplitude in the momentum representation, i.e. using eqs. (II.3.6) and (II.3.7). The quantities d, $|\underline{s}_o|$, and the bound state wave functions $u_i(r_i)$ are calculated as before. The Fourier transforms $\tilde{u}_i(k_i)$ (Appendix II.B) are computed using a Simpson's rule subroutine, and the final integration over k_1 is also done by Simpson's rule. The results are presented in Table II together with the third set of results from Table I for comparison. The agreement seems good. It has already been shown earlier that the expressions for $A(\lambda_1,\lambda_2)$ in the post and prior representations are exactly the same for momentum space calculations. Hence there is only one set of numbers in this case compared to a pair of sets in the case of coordinate space results.

Fortunately, in this case the numbers are only marginally sensitive to the limits of integration. This is to be expected, as we have only one numerical integration here compared to three earlier. Consequently, we expect these results to be more accurate than the previous ones. An additional advantage is that it takes far less computer time to obtain lines 3, 6 and 9 of Table II than the corresponding ones in coordinate space results.

$t_{min} = 0$ $t_{max} = 4$ $\xi_{1min} = 0$

ξ_{1max}	ξ_{2min}	ξ_{2max}	λ_1 \ λ_2	−4	−3	−2	−1	0	+1	+2	+3	+4
6.1	−5	6.1	−1	+.010 +.003		−.009 −.010		−.032 −.023		+.234 +.258		−.305 −.951
7	−6	7		+.006 +.007		−.009 −.005		−.029 −.032		+.226 +.241		−.309 −.392
7	−7	8		+.006 +.012		−.009 −.007		−.029 −.038		+.226 +.246		−.309 −.376
6.1	−5	6.1	0		−.001 −.001		−.014 −.015		+.097 +.107		−.258 −.287	
7	−6	7			−.001 −.006		−.013 −.016		+.094 +.102		−.262 −.283	
7	−7	8			−.001 −.000		−.013 −.018		+.094 +.100		−.262 −.279	
6.1	−5	6.1	+1	−.010 −.005		+.012 +.007		+.017 +.017		−.158 −.162		+.548 +.606
7	−6	7		−.005 −.011		+.008 +.007		+.016 +.025		−.152 −.165		+.542 +.581
7	−7	8		−.005 −.013		+.008 +.009		+.016 +.026		−.151 −.169		+.541 +.584

TABLE I: A comparison of transfer amplitudes using the method of section II.2 for various limits of integration on ξ_1, ξ_2, t. The first line in each set corresponds to results in the prior representation while the second line to the post representation.

Represent-ation	λ_1 \ λ_2	-4	-3	-2	-1	0	+1	+2	+3	+4
Prior		+.006		-.009		-.029		+.226		-.809
Post	-1	+.012		-.007		-.038		+.246		-.876
Mom.Space		+.012		-.001		-.033		+.219		-.799
Prior			-.001		-.013		+.094		-.262	
Post	0		.000		-.018		+.100		-.279	
Mom.Space			.000		-.015		+.089		-.259	
Prior		-.005		+.008		+.016		-.151		+.541
Post	+1	-.013		+.009		+.026		-.169		+.584
Mom.Space		-.010		+.008		+.021		-.149		+.529

Table II

A comparison of the transfer amplitudes $A(\lambda_1,\lambda_2)$ calculated using the methods described in sections 2 and 3.

Calculation of Transfer Probabilities:

The transfer probabilities P_{21} (eq. II.1.10) for the transfer of a neutron from the $1p_{3/2}$ state in ^{11}B to the ground and excited states in ^{209}Pb are calculated with the help of eqs. (II.3.6) and (II.3.7). The details of the calculation are the same as before.

The results are presented in table III together with experimental differential cross section at the grazing angle (Anayas-Weiss et. al., 1973) and with Pixton's results.

Final State	Experimental Cross Section at grazing angle (mb/sr)	Calculated Transfer Probabilities	
		Present Work	Pixton's Results
$2g_{9/2}$	3.20	0.616	.592
$1i_{11/2}$	1.35	1.08	.731
$1j_{15/2}$	0.89	2.60	.479
$3d_{5/2}$		.019	.052
$4s_{\frac{1}{2}}$	-	.000	.000
$2g_{7/2}$	2.80	.425	.439
$3d_{3/2}$		.006	.028

Table III

We see from the table that the results of the present calculation do not agree well with Pixton's results, lending support to the supposition that the Hankel function approximation was not a very good one.

II.5 Modification for Proton Transfer

The theory for proton transfer is parallel to that for neutron transfer except that in this case one has to take into account the effect of the long range Coulomb force on the proton wave function, which is affected even before transfer. This effect manifests itself mainly in the phase of the wave function.

Keeping this in mind we shall follow the same steps as Pixton (Pixton, 1972) to obtain an expression for the transfer amplitude $A(\lambda_1,\lambda_2)$ for proton transfer. As in the case of neutron transfer we consider the reaction:

$$a_1 + c_2 = a_2 + c_1$$

$$(c_1 + p) + c_2 = (c_2 + p) + c_1, \qquad \text{(II.5.1)}$$

where this equation has the same mea-ning as (II.1.1) with the exception that we read 'proton' for 'neutron'. We again assume that the two ions move in distinct classical orbit $\underline{s}_1(t)$ and $\underline{s}_2(t)$ respectively, and calculate the transfer probability quantum mechanically. At the moment of transfer the relative velocity of the ions is assumed constant, so that any relative acceleration between them is neglected.

Let us consider the transfer of the proton from a state $\psi_1(\underline{r}, t)$ in nucleus a_1 to the state $\psi_2(\underline{r}, t)$ in nucleus a_2, where $\psi_1(\underline{r}, t)$ and $\psi_2(\underline{r}, t)$ are stationary state wave functions satisfying the Schrodinger equations.

$$i\hbar \frac{\partial \Psi_1(\underline{r},t)}{\partial t} = (T + V_1 + \Delta\varepsilon_2(t))\,\Psi_1(\underline{r},t) \qquad \text{(II.5.2)}$$

$$i\hbar \frac{\partial \Psi_2(\underline{r},t)}{\partial t} = (T + V_2 + \Delta\varepsilon_1(t))\,\Psi_2(\underline{r},t) \qquad \text{(II.5.3)}$$

V_1 and V_2 are the potentials of the isolated cores c_1 and c_2 respectively and consist of a short range nuclear part V_n and a long range Coulomb part V_{coul}.

Thus: $V_{1,2} = V_{n(1,2)} + V_{coul(1,2)}$

$\Delta\varepsilon_2(t)$ represents the average effect of the Coulomb potential of nucleus a_2 on the proton when it is bound in nucleus a_1, and similarly for $\Delta\varepsilon_1(t)$. We take these to be numbers given by:

$$\Delta\varepsilon_1(t) = \frac{Z_{c_1} Z_p e^2}{|\underline{s}(t)|} \qquad \text{(II.5.4)}$$

$$\Delta\varepsilon_2(t) = \frac{Z_{c_2} Z_p e^2}{|\underline{s}(t)|} \qquad \text{(II.5.5)}$$

where $|\underline{s}|$ is the separation of the cores (Fig. 1). $Z_{c_1}e$, $Z_{c_2}e$, $Z_p e$ are the charges of c_1, c_2 and p respectively.

The wave function Φ of the proton interacting with both nuclei is given by the Schrodinger equation:

$$i\hbar \frac{\partial \Phi(\underline{r},t)}{\partial t} = (T + V_1 + V_2)\,\Phi(\underline{r},t) \qquad \text{(II.5.6)}$$

with the boundary condition that:

$$\Phi(\underline{r},\ t) \to \Psi_1(\underline{r},\ t) \text{ as } t \to -\infty$$

We write

$$\Phi(\underline{r},\ t) = \Phi_1 + \Phi_2$$

where

$$i\hbar \frac{\partial \Phi_1}{\partial t} = (T + V_1 + \Delta\varepsilon_2(t))\,\Phi_1 + (V_1 - \Delta\varepsilon_1(t))\,\Phi_2 \qquad \text{(II.5.7)}$$

$$i\hbar \frac{\partial \Phi_2}{\partial t} = (T + V_2 + \Delta\varepsilon_1(t))\,\Phi_2 + (V_2 - \Delta\varepsilon_2(t))\,\Phi_1\ , \qquad \text{(II.5.8)}$$

and impose the boundary conditions.

$$\left.\begin{aligned} \Phi_1(\underline{r},\ t) &\to \Psi_1(\underline{r},\ t) \\ \Phi_2(\underline{r},\ t) &\to 0 \end{aligned}\right\} \text{ as } t \to -\infty$$

on Φ_1 and Φ_2.

Solving eq. (II.5.8) approximately by iteration taking for a first approximation:

$$\Phi_1(\underline{r},\ t) \sim \Psi_1(\underline{r},\ t),$$

we get:

$$i\hbar \frac{\partial \Phi_2(\underline{r},t)}{\partial t} - (T + V_2 + \Delta\varepsilon_1(t))\,\Phi_2 \approx (V_2 - \Delta\varepsilon_2(t))\,\Psi_1 \qquad \text{(II.5.9)}$$

As $\Delta\varepsilon_2(t)$ compensates V_2 when the nuclei are far apart, we expect $(V_2 - \Delta\varepsilon_2(t))\ \Psi_1$ to be small at all points except the point of transfer. The transfer probability is given by:

$$P_{21} = |\langle \Psi_2(\underline{r},t), \Phi_2(\underline{r},t)\rangle|^2 , \qquad t \to \infty$$

To obtain an expression for P_{21} we multiply eq.(II.5.9) on the left by $\Psi_2^*(\underline{r},\ t)$ and integrate over $\underline{r}$ to get:

$$\langle \Psi_2(\underline{r},t),\ (V_2 - \Delta\varepsilon_2(t))\Psi_1(\underline{r},t)\rangle = i\hbar\langle \Psi_2(\underline{r},t), \frac{\partial \Phi_2(\underline{r},t)}{\partial t}\rangle -$$

$$\langle \Psi_2(\underline{r},t),\ (T + V_2 + \Delta\varepsilon_1(t))\Phi_2(\underline{r},t)\rangle \qquad \text{(II.5.10)}$$

Taking the complex conjugate of eq. (II.5.3) and substituting in eq. (II.5.10) we obtain:

$$\langle \bar{\Psi}_2(\pm,t), (V_2 - \Delta\varepsilon_2(t))\bar{\Psi}_1(\pm,t)\rangle = i\hbar \left\langle \bar{\Psi}_2(\pm,t), \frac{\partial \bar{\Phi}_2(\pm,t)}{\partial t}\right\rangle +$$

$$i\hbar \left\langle \frac{\partial \bar{\Psi}_2(\pm,t)}{\partial t}, \bar{\Phi}_2(\pm,t)\right\rangle$$

$$= i\hbar \frac{\partial}{\partial t} \langle \bar{\Psi}_2(\pm,t), \bar{\Phi}_2(\pm,t)\rangle \qquad \text{(II.5.11)}$$

Integrating both sides over t and squaring, the expression for P_{21} becomes:

$$P_{21} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{+\infty} \langle \bar{\Psi}_2, (V_2 - \Delta\varepsilon_2(t))\bar{\Psi}_1\rangle \, dt \right|^2 \qquad \text{(II.5.12)}$$

This is the expression in the prior representation. The corresponding equation in the post representation obtained by solving eq. (II.5.7) by iteration and following through the same steps is:

$$P_{21} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{+\infty} \langle \bar{\Psi}_2, (V_1 - \Delta\varepsilon_1(t))\bar{\Psi}_1\rangle \, dt \right|^2 \qquad \text{(II.5.13)}$$

We shall now prove the post - prior equality. Using eqs. (II.5.2) and (II.5.3) we have:

$$i\hbar \frac{\partial}{\partial t} \langle \Psi_2(\pm,t), \Psi_1(\pm,t) \rangle = \langle \Psi_2, i\hbar \frac{\partial \Psi_1}{\partial t} \rangle - \langle i\hbar \frac{\partial \Psi_1}{\partial t}, \Psi_1 \rangle$$

$$= \langle \Psi_2, (T + V_1 + \Delta\varepsilon_2(t)) \Psi_1 \rangle - \langle (T + V_2 + \Delta\varepsilon_1(t)) \Psi_2, \Psi_1 \rangle$$

$$= \langle \Psi_2, \{ (V_1 - \Delta\varepsilon_1(t)) - (V_2 - \Delta\varepsilon_2(t)) \} \Psi_1 \rangle$$

Integrating both sides of this equation over time:

$$\int_{-\infty}^{+\infty} \langle \Psi_2, (V_1 - \Delta\varepsilon_1(t)) \Psi_1 \rangle - \int_{-\infty}^{+\infty} \langle \Psi_2, (V_2 - \Delta\varepsilon_2(t)) \Psi_1 \rangle$$

$$= i\hbar \left[\langle \Psi_2, \Psi_1 \rangle \right]_{-\infty}^{+\infty}$$

$$\simeq 0 \ ,$$

since there is no overlap between the nucleon states in the two nuclei when they are separated by large distances (i.e. at large times). We now have

$$P_{21} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{+\infty} \langle \Psi_2, (V_1 - \Delta\varepsilon_1(t)) \Psi_1 \rangle \, dt \right|^2$$

$$= \frac{1}{\hbar^2} \left| \int_{-\infty}^{+\infty} \langle \Psi_2, (V_2 - \Delta\varepsilon_2(t)) \Psi_1 \rangle \, dt \right|^2 \ ,$$

which is the post - prior equality.

The corresponding equation for neutron transfer were (Pixton, 1972):

$$P_{21} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{+\infty} \langle \tilde{\Psi}_2, V_1 \tilde{\Psi}_1 \rangle \, dt\right|^2 \qquad \text{(II.5.15)}$$

$$= \frac{1}{\hbar^2}\left|\int_{-\infty}^{+\infty} \langle \tilde{\Psi}_2, V_2 \tilde{\Psi}_1 \rangle \, dt\right|^2 \qquad \text{(II.5.16)},$$

where $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are solutions of the equations:

$$i\hbar \frac{\partial \tilde{\Psi}_1}{\partial t} = (T + V_1)\,\tilde{\Psi}_1 \qquad \text{(II.5.17)}$$

$$i\hbar \frac{\partial \tilde{\Psi}_2}{\partial t} = (T + V_2)\,\tilde{\Psi}_2 \qquad \text{(II.5.18)}$$

Comparing eqs. (II.5.2), (II.5.3) and eqs. (II.5.17), (II.5.18) we see that Ψ_1, Ψ_2 are related to $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ respectively through the relations:

$$\Psi_1(\underline{r}, t) = \tilde{\Psi}_1(\underline{r}, t)\, e^{-i\phi_2(t)} \qquad \text{(II.5.19)}$$

$$\Psi_2(\underline{r}, t) = \tilde{\Psi}_2(\underline{r}, t)\, e^{-i\phi_1(t)} \qquad \text{(II.5.20)},$$

where
$$\phi_2(t) = \int_0^t \Delta\varepsilon_2(t)\, dt \qquad \text{(II.5.21)}$$

$$\phi_1(t) = \int_0^t \Delta\varepsilon_1(t)\, dt, \qquad \text{(II.5.22)}$$

and $t = 0$ corresponds to the point of closest approach.

Thus,

$$P_{21} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{+\infty} \langle \tilde{\Psi}_2, \tilde{V}_2 \tilde{\Psi}_1 \rangle \, e^{i(\varphi_1(t) - \varphi_2(t))} \, dt \right|^2 , \tag{II.5.23}$$

where $$\tilde{V}_2 = V_2 - \Delta\varepsilon_2(t) \tag{II.5.24}$$

in the prior representation, with a corresponding equation in the post representation. Comparing eq. (II.5.23) with eq. (II.5.16), we see that the differences between the two is that V_2 in the latter is replaced by $\tilde{V}_2$ and there is an additional phase factor. The necessity of making a correction to V_2 due to the Coulomb interaction potentials has also been pointed out by De Vries et al. (De Vries et. al., 1974).

Near the point of closest approach, $\Delta\varepsilon_1(t)$ and $\Delta\varepsilon_2(t)$ can be approximated by:

$$\Delta\varepsilon_1(t) \simeq \frac{Z_{c_1} Z_p e^2}{d} \tag{II.5.25}$$

$$\Delta\varepsilon_2(t) \simeq \frac{Z_{c_2} Z_p e^2}{d} \tag{II.5.26}$$

so that

$$\varphi_1(t) - \varphi_2(t) = \frac{(Z_{c_1} - Z_{c_2}) Z_p e^2}{d} \, t$$

The effect of these changes on $A(\lambda_1,\lambda_2)$ and $U_{\lambda_1\lambda_2}(t)$ of eqs. (II.1.3) and (II.1.4) is that they now transform to:

$$A(\lambda_1,\lambda_2)=\frac{1}{\hbar}\int_{-\infty}^{+\infty} U_{\lambda_1\lambda_2}(t)\exp\left[-\frac{it}{\hbar}\left(Q_{eff}+\frac{1}{2}m\dot{s}^2\right)\right]dt \qquad \text{(II.5.27)}$$

with $$U_{\lambda_1\lambda_2}(t)=\int \psi_2^*(\underline{r}_2)\tilde{V}_2(\underline{r}_2)\psi_1(\underline{r}_2-\underline{s})\exp\left(+\frac{i}{\hbar}m\dot{\underline{s}}\cdot\underline{r}_2\right)d^3\underline{r}_2 \qquad \text{(II.5.28)}$$

and

$$Q_{eff} = Q + \frac{(Z_{c_2}-Z_{c_1})Z_p e^2}{d} \qquad \text{(II.5.29)}$$

and all the other symbols have the same meaning as before. The importance of the quantity Q_{eff} has been stressed by other authors (Buttle and Godlfarb, 1972; Broglia and Winther, 1972).

The evaluation of $A(\lambda_1,\lambda_2)$ in momentum space follows through exactly as before, giving finally for $A'(\lambda_1',\lambda_2')$ of eq. (II.3.6):

$$A'(\lambda_1',\lambda_2')=\frac{(2\pi)^2}{\hbar|\dot{s}_0|}\int_0^\infty \tilde{u}_2^*(k_2)\tilde{u}_1(k_1)P_{L_2}^{\lambda_2'}(\cos\theta_2')\,P_{L_1}^{\lambda_1'}(\cos\theta_1')\times$$
$$\times\left(\tilde{\varepsilon}_1-\frac{\hbar^2k_1^2}{2m}\right)J_{\lambda_1'-\lambda_2'}(k_\perp d)\,k_\perp\,dk_\perp \qquad \text{(II.5.30)}$$

where $$\tilde{\varepsilon}_1=\varepsilon_1-\frac{Z_{c_1}Z_p e^2}{d}$$

and all other symbols have the same meaning as before.

CHAPTER III

CALCULATION OF ANGULAR DISTRIBUTIONS IN TRANSFER REACTIONS

III.1 A Formula for the Calculation of Angular Distributions

To make comparisons with experimental data it is necessary to calculate angular distributions of transfer reactions. In this chapter we start from the distorted wave Born approximation (DWBA) theory and obtain an expression for the angular distribution as a partial wave sum. The transfer amplitudes in the sum turn out to be the same as the semiclassical amplitudes evaluated in chapter II. The formula for angular distributions is of the same form as that of other authors (Strutinsky, 1973; Koeling and Malfliet, 1975; Landowne et. al., 1976). The method developed here resembles the one used by Landowne et. al. only in that the distorted waves are expanded as a partial wave sum and that the radial wave functions are approximated by WKB wave functions. The actual handling of the problem is different and somewhat simpler.

The procedure is broadly divided into the following steps:

1. We begin with the expression of the T-matrix in DWBA and first separate out phase factors containing recoil corrections. The remaining distorted waves are then expanded in partial waves.

2. The radial wave functions appearing in the DWBA matrix elements which occur in the partial wave summations are approximated by WKB wave functions.

3. Further simplifications enable us to perform one of the partial wave summations leaving us with a familiar looking expression for the scattering amplitude which consists of a partial wave summation containing a quantity identifiable as the transfer amplitude.

Let us now discuss these points in detail.

In the distorted wave Born approximation theory, the transition amplitude for the transfer reaction:

$$(c_1 + x) + c_2 \longrightarrow c_1 + (c_2 + x)$$

$$a_1 + c_2 \longrightarrow c_1 + a_2$$

(where c_1, c_2, a_1, a_2 have the same meaning as in chapter II, and x is the transferred particle or cluster of particles) is given by: (e.g. Dodd and Greider, 1969; Austern et. al., 1964).

$$T_{if} = \int \chi_f^{(-)*}(\underline{k}_f, \underline{r}_f)\, \psi_f^*(\underline{r}_2)\, \Delta V\, \psi_i(\underline{r}_2 - \underline{s})\, \chi_i^{(+)}(\underline{k}_i, \underline{r}_i)\, d^3\underline{r}_i\, d^3\underline{r}_f \qquad \text{(III.1.1)}$$

where ψ_f, ψ_i are the bound state wave functions of the transferred cluster x in the isolated nuclei a_2 and a_1 respectively and correspond to ψ_2, ψ_1 of chapter II.

$\chi_i^{(+)}$, $\chi_f^{(-)}$ are the distorted waves. They are elastic scattering wave functions which describe the relative motion of the pair a_1, c_2 before the collision and the pair c_1, a_2 after the collision. ΔV is the interaction responsible for the transition. The system of coordinates is illustrated in fig. 7. $\underline{r}_1$, $\underline{r}_2$ are the positions of x relative to the centres of mass of the cores c_1 and c_2 respectively. $\underline{r}_i$ represents the position of c_2 relative to the centre of mass, A_1, of c_1 and x, while $\underline{r}_f$ represents that of c_1 relative to the centre of mass, A_2, of c_2 and x. $\underline{s}$ is the coordinate of c_1 with respect to c_2. The

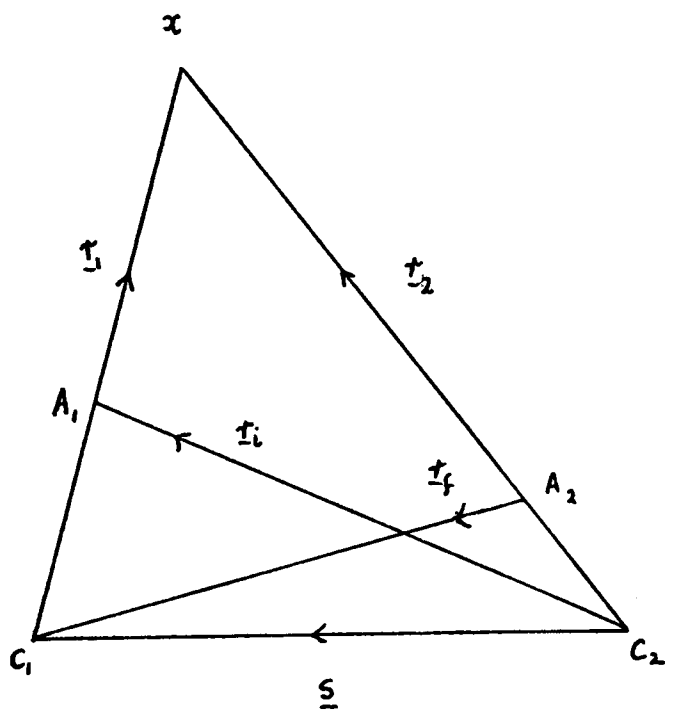


Fig. 7.

various coordinates are related by the equations:

$$\underline{r}_f = \underline{s} - \mu_{xc_2}\underline{r}_2 \qquad \text{(III.1.2)}$$

$$\underline{r}_i = (1-\mu_{c_1x})\underline{s} + \mu_{c_1x}\underline{r}_2 \quad , \qquad \text{(III.1.3)}$$

where $$\mu_{c_2x} = \frac{m_x}{m_x + m_{c_2}} \quad , \qquad \text{(III.1.4)}$$

$$\mu_{c_1x} = \frac{m_x}{m_x + m_{c_1}} \quad , \qquad \text{(III.1.5)}$$

m_x, m_{c_1} and m_{c_2} being the masses of the transferred particle x and the cores c_1 and c_2 respectively.

Before proceeding with the derivation we shall look at the interaction potential ΔV in some detail (c.f. eq. (III.1.1)). It consists of a nuclear part ΔV^n and a Coulomb part ΔV^{coul} given in the post representation by:

$$\Delta V^n = V^n_{c_1x}(\underline{r}_i) + V^n_{c_1c_2}(\underline{s}) - U^{opt,n}_{c_1a_2}(\underline{r}_f) \qquad \text{(III.1.6)}$$

$$\Delta V^{coul} = V^{coul}_{c_1x}(\underline{r}_i) + V^{coul}_{c_1c_2}(\underline{s}) - U^{opt,coul}_{c_1a_2}(\underline{r}_f) \qquad \text{(III.1.7)}$$

and in the prior representation by:

$$\Delta V^{n} = V^{n}_{c_2 x}(\underline{r}_2) + V^{n}_{c_1 c_2}(\underline{s}) - U^{opt,n}_{a_1 c_2}(\underline{r}_1) \tag{III.1.8}$$

$$\Delta V^{coul} = V^{coul}_{c_2 x}(\underline{r}_2) + V^{coul}_{c_1 c_2}(\underline{s}) - U^{opt,coul}_{a_1 c_2}(\underline{r}_1) \tag{III.1.9}$$

where $V_{c_1 x}$, $V_{c_2 x}$ are the bound state potentials (corresponding to V_1, V_2 of chapter II) and $V_{c_1 c_2}$ is the core-core potential, while U^{opt} is the optical potential for the respective channels. It is expected to compensate the major contribution of the core-core potential, so that there is a tendency among those doing DWBA calculations to approximate ΔV by $V^{n}_{1,2}$ only. However, it has been pointed out by De Vries et al. (De Vries et. al., 1974) that while it does not appreciably affect the results to omit the last two terms in eqs. (III.1.6) and (III.1.8), the contribution due to ΔV^{coul} must be included, we thus take ΔV to be:

$$\Delta V(\underline{r}_{1,2}, \underline{s}) = V^{n}_{c_{1,2} x}(\underline{r}_{1,2}) + \Delta V^{coul}(\underline{r}_{1,2}, \underline{s}), \quad (\text{where, } \underline{r}_1 = \underline{r}_2 - \underline{s}) \tag{III.1.6}$$

and consider an approximate form for $\Delta V^{coul}(\underline{r}_{1,2}, \underline{s})$ later in the chapter. We substitute ΔV in eq. (III.1.1) by $\Delta V(\underline{r}_2, \underline{s})$ of eq. (III.1.6) and continue our discussion with this expression of T_{if}.

W.K.B. Approximation for $\chi_{i,f}$

At high energies and if the reduced mass is sufficiently large the relative motion has a small wavelength. The heavy ion scattering which we study here is one of such cases. It is known that the semiclassical J.W.K.B. is a good approximation in such cases. Hence we may use the W.K.B. method to approximate the wave functions of relative motion of the projectile and target. In the W.K.B. approximation the distorted waves can be written as (Schiff, 1968; Glauber, 1964):

$$\chi_f^{(-)}(\pm_f) = A_f(\pm_f)\, e^{\frac{i}{\hbar} S_f(\pm_f)} \qquad \text{(III.1.10)}$$

where $S_f(\underline{r}_f)$ is the appropriate solution of the Hamiltonian Jacobi equation:

$$\frac{1}{2\mu_f}\left(\frac{\partial S_f}{\partial \pm}\right)^2 + U_f(+) = E_f \quad ,$$

μ_f, U_f, E_f being the reduced mass, potential and energy respectively in the final channel. It can be computed as an action integral along the classical trajectory that passes through the point $\underline{r}_f$ and has asymptotic momentum $\underline{k}_f$ as $t \to +\infty$ (c.f. Austern, 1969). $A_f^2(\underline{r}_f)$ corresponds to the classical density function, as it satisfies the same equation of continuity :

$$\operatorname{div}\left(\frac{A_f^2}{\mu_f}\,\frac{\partial S}{\partial \underline{r}}\right) = 0 \quad .$$

In the semiclassical limit ($\hbar \to 0$) the most rapidly varying term in eq.(III.1.10) is $\frac{1}{\hbar} S_f(\underline{r}_f)$. We expand this as a Taylor series in $(\underline{r}_f - \underline{s})$ about $\underline{s}$ up to first order in $(\underline{r}_f - \underline{s})$:

$$S_f(\underline{r}_f) \simeq S_f(\underline{s}) + (\underline{r}_f - \underline{s}) \cdot \left[\frac{\partial S_f(\underline{r}_f)}{\partial \underline{r}_f}\right]_{\underline{r}_f = \underline{s}}$$

$$= S_f(\underline{s}) + (\underline{r}_f - \underline{s}) \cdot \underline{p}_f(\underline{s}) \quad ,$$

where $\underline{p}_f(\underline{s})$ is the local momentum of the projectile on the classical orbit passing through $\underline{s}$ and has the direction of the final momentum $\underline{k}_f$ for large t. Hence, $\chi_f^{(-)}$ becomes:

$$\chi_f^{(-)}(\underline{r}_f) \simeq \chi_f^{(-)}(\underline{s})\, e^{\frac{i}{\hbar}(\underline{r}_f - \underline{s}) \cdot \underline{p}_f} \quad , \qquad \text{(III.1.11)}$$

where $$\chi_f^{(-)}(\underline{s}) = A_f(\underline{s})\, e^{\frac{i}{\hbar} S_f(\underline{s})}$$

We have assumed:

$$A_f(\underline{r}_f) \simeq A_f(\underline{s})$$

The analogous expression for $\chi_i^{(+)}$ is:

$$\chi_i^{(+)}(\underline{r}_i) \simeq \chi_i^{(+)}(\underline{s})\, e^{\frac{i}{\hbar}(\underline{r}_i - \underline{s})\cdot \underline{p}_i(\underline{s})} \qquad \text{(III.1.12)}$$

where $\underline{p}_i(\underline{s})$ is evaluated on a classical orbit with asymptotic momentum $\underline{k}_i$ as $t \to -\infty$.

Substituting eq. (III.1.11) and (III.1.12) into (III.1.1) and changing the variable of integration from $\underline{r}_i$, $\underline{r}_f$ to $\underline{s}$, $\underline{r}_2$, the six dimensional integral separates to give (c.f. Dodd and Greider, 1969)

$$T_{if} = \int \chi_f^{(-)*}(\underline{s})\, G(\underline{s})\, \chi_i^{(+)}(\underline{s})\, d^3\underline{s} \ , \qquad \text{(III.1.13)}$$

where

$$G(\underline{s}) = e^{-\frac{i}{\hbar}\mu_{c_1 x}\underline{p}_i\cdot\underline{s}} \int \psi_f^*(\underline{r}_2)\, \Delta V(\underline{r}_2, \underline{s})\, \psi_i(\underline{r}_2 - \underline{s})\, e^{\frac{i}{\hbar}\underline{p}\cdot\underline{r}_2}\, d^3\underline{r}_2 \ , \qquad \text{(III.1.14)}$$

with

$$\underline{p} = \mu_{c_1 x}\underline{p}_i + \mu_{c_2 x}\underline{p}_f$$

$$= m_x\, \underline{v}(\underline{s}) \ ,$$

where

$$\underline{v}(\underline{s}) = \frac{m_{c_1}\underline{v}_f(\underline{s}) + m_{c_2}\underline{v}_i(\underline{s})}{m_{c_1} + m_{c_2} + m_x}$$

$\underline{v}_i$, $\underline{v}_f$ are the velocities at the point $\underline{s}$ in the initial and final channels respectively.

Partial Wave Expansion of $\chi_{i,f}(\underline{s})$

To evaluate eq. (III.1.13) we make a partial wave expansion of the amplitude functions $\chi_i^{(+)}(\underline{s})$ and $\chi_f^{(-)*}(\underline{s})$. Choosing a coordinate frame so the $\underline{k}_i$ defines the z-axis and (ϑ, Φ) gives the direction of $\underline{k}_f$, we obtain:

$$\chi_i^{(+)}(\underline{s}) = \frac{(4\pi)^{1/2}}{k_i s} \sum_{l_i} (2l_i+1)^{1/2} i^{l_i} e^{i\delta_{l_i}^{(i)}} f_{l_i}(s)\, Y_{l_i 0}(\hat{\underline{s}})$$

$$\chi_f^{(-)*}(\underline{s}) = \frac{4\pi}{k_f s} \sum_{l_f m_f} i^{-l_f} e^{i\delta_{l_f}^{(f)}} f_{l_f}(s)\, Y^*_{l_f m_f}(\hat{\underline{s}})\, Y_{l_f m_f}(\vartheta, \Phi) \qquad \text{(III.1.15)}$$

where $\delta_l^{(i,f)} = \delta_l^{n(i,f)} + \delta_l^{c(i,f)}$ is the sum of the nuclear and Coulomb elastic scattering phase shifts in the initial and final channels respectively, and ϑ is the scattering angle in the reaction plane. $f_{l_i, l_f}(s)$ are radial wave functions.

A substitution of eq. (III.1.15) into eq. (III.1.13) yields:

$$T_{if} = \frac{(4\pi)^{3/2}}{\sqrt{k_i k_f}} \sum_{l_i l_f m_f} (2l_i+1)^{1/2} e^{i(\delta_{l_i}^{(i)} + \delta_{l_f}^{(f)})} i^{l_i - l_f} Y_{l_f m_f}(\vartheta, \Phi)\, I_{l_i l_f m_f}\,, \qquad \text{(III.1.16)}$$

where $$I_{l_i l_f m_f} = \frac{1}{\sqrt{k_i k_f}} \int_0^\infty f_{l_f}(s)\, G_{l_i l_f}^{m_f}(s)\, f_{l_i}(s)\, ds\,, \qquad \text{(III.1.17)}$$

with

$$G^{m_f}_{\ell_i \ell_f} = \int Y^*_{\ell_f m_f}(\hat{\underline{s}})\, G(s,\theta,\varphi)\, Y_{\ell_i 0}(\hat{\underline{s}})\, \sin\theta\, d\theta\, d\varphi \qquad \text{(III.1.18)}$$

Semiclassical Approximation for $Y_{\ell m}$

Our next step is to simplify eq. (III.1.18). As we expect a dominant contribution only from the partial waves with large values of ℓ_f, ℓ_i we may substitute the asymptotic form of the $Y_{\ell m}$'s in eq. (III.1.18), i.e., we substitute (Abromowitz and Stegun, 1965):

$$Y_{\ell m}(\theta,\varphi) \simeq \frac{B_{\ell m} \cos\left[(\ell+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right]}{\sqrt{\sin\theta}}$$

where

$$B_{\ell m} = \frac{1}{\pi}(-1)^m \left[\frac{2\ell+1}{2} \frac{\Gamma(\ell-m+1)\,\Gamma(\ell+m+1)}{(\Gamma(\ell+3/2))^2}\right]^{1/2}$$

In eq. (III.1.16) m_f is the z-component of angular momentum transfer, while ℓ_i, ℓ_f are the angular momenta of relative motion of the nuclei in the initial and final channels respectively. Hence we expect $\ell_i \gg 0$, $\ell_f \gg m_f$, and may use the asymptotic form of the Γ-function for large argument to simplify $B_{\ell m}$ (Appendix III.A) obtaining:

$$B_{\ell m} \simeq \frac{1}{\pi}(-1)^m$$

$$\therefore\ Y_{\ell m}(\theta,\varphi) \simeq \frac{e^{im\varphi}}{\pi}(-1)^m \frac{\cos\left[(\ell+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right]}{\sqrt{\sin\theta}} \qquad \text{(III.1.19)}$$

If we replace the $Y_{\ell m}$'s in eq. (III.1.18) by the asymptotic form (III.1.19) we get:

$$G^{m_f}_{l_i l_f}(s) = \frac{(-1)^{m_f}}{\pi^2}\int e^{-im_f\varphi} G(s,\theta,\varphi)\cos\left[(l_f+\tfrac{1}{2})\theta - \tfrac{\pi}{4} + \tfrac{m_f\pi}{2}\right]\times$$

$$\times \cos\left[(l_i+\tfrac{1}{2})\theta - \tfrac{\pi}{4}\right] d\theta\, d\varphi$$

$$= \frac{(-1)^{m_f}}{2\pi^2}\int e^{-im_f\varphi} G(s,\theta,\varphi)\left[\cos((l_f-l_i)\theta + \tfrac{m_f\pi}{2}) - \right.$$

$$\left.\cos((l_i+l_f+1)\theta + \tfrac{\pi}{2}(m_f-1))\right] d\theta\, d\varphi$$

As the second term in the integrand is rapidly oscillating we expect a very small contribution from it and therefore neglect it. We thus obtain:

$$G^{m_f}_{l_i l_f}(s) \simeq \frac{(-1)^{m_f}}{2\pi^2}\int e^{-im_f\varphi} G(s,\theta,\varphi)\cos\left[(l_f-l_i)\theta + \tfrac{m_f\pi}{2}\right] d\theta\, d\varphi \qquad \text{(III.1.20)}$$

W.K.B. Approximation for the Radial Matrix Element

The next step is to make a semiclassical evaluation of the radial matrix element (III.1.17). We use a method which given for a real potential by Landau and Lifshitz (Landau and Lifshitz, 1965) and has been generalised to a complex potential by Landowne et. al. (Landowne et. al., 1976).

The method consists in first of all approximating the wave functions $f_\ell(s)$ in the regions on both sides

of a classical turning point s_0, but not in its immediate neighbourhood by the W.K.B. solutions:

$$f_l(s) = \begin{cases} \sqrt{\frac{k}{\gamma_k(s)}} \sin\left(\int_{s_0}^{s} \gamma_k(s')\,ds' + \frac{\pi}{4}\right), & s > s_0 \\ \frac{1}{2}\sqrt{\frac{k}{K(s)}} \exp\left(\int_{s_0}^{s} K(s')\,ds'\right), & s < s_0 \end{cases} \qquad \text{(III.1.21)}$$

where $\gamma_k(s)$ and $K(s)$ are given by:

$$\gamma_k(s) = \frac{1}{\hbar}\sqrt{2\mu(E-V)}\,,$$

$$K(s) = \frac{1}{\hbar}\sqrt{2\mu(V-E)}\,,$$

with μ the reduced mass, E the centre of mass energy and V the (real) potential:

$$V = \frac{\hbar^2(l+\frac{1}{2})^2}{2\mu s^2} + U(s) \qquad \text{(III.1.22)}$$

In what follows we assume that $E_f > E_i$ and $s_{of} < s_{oi}$.

Classical turning points correspond to zeros of $(V(s) - E)$. In the following discussion we assume $V(s)$ has the shape shown in fig. 8 and that only one turning point s_0 is important in each case. On the figure are also indicated the positions of E_f, E_i and the corresponding turning points s_{of}, s_{oi}. If U(s) is an attractive

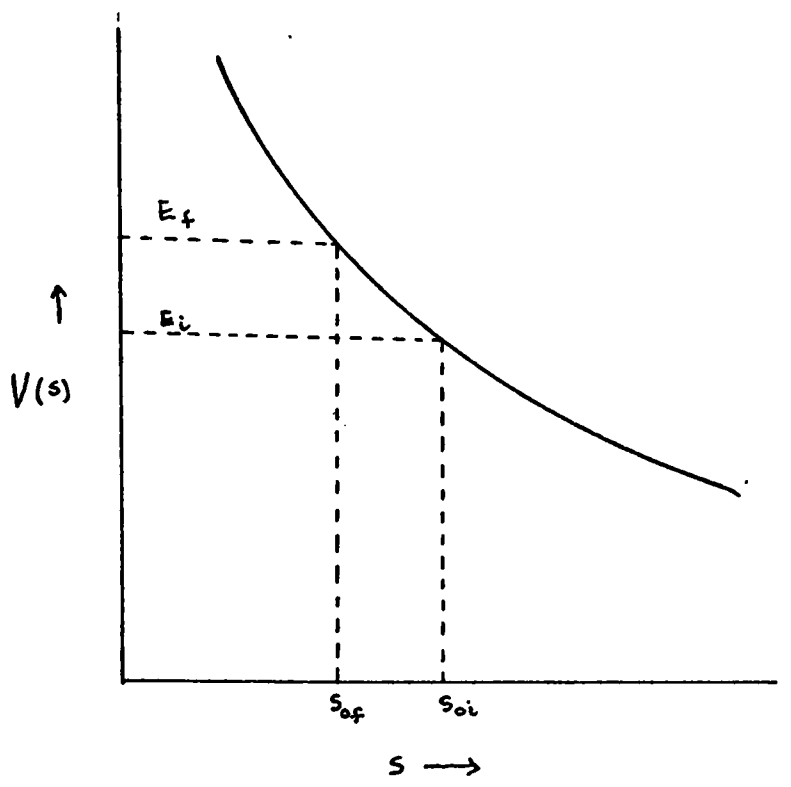


Fig. 8

potential there is often more than one turning point. This question will be discussed later in this section.

Following Landau and Lifshitz, we break up $f_{\ell_f}(s)$ in the region $s > s_{of}$ into a sum of two complex conjugate parts $f^{(+)}_{\ell_f}$ and $f^{(-)}_{\ell_f}$, expressing the sine as a sum of two exponentials. Thus $f^{(+)}_{\ell_f}(s)$ is defined as:

$$f^{(+)}_{\ell_f}(s) = \begin{cases} -\frac{i}{2}\sqrt{\frac{k_f}{K_f(s)}}\ \exp\left[-\int_{s_{of}}^{s} K(s')\,ds'\right], & s < s_{of} \\ \frac{1}{2}\sqrt{\frac{k_f}{ik_f(s)}}\ \exp\left[i\int_{s_{of}}^{s} ik_f(s')\,ds' + \frac{i\pi}{4}\right], & s > s_{of} \end{cases}$$

The integral (III.1.17), which we shall denote symbolically as:

$$q_{12} = \int_0^\infty f_{l_f}(s)\, G(s)\, f_{l_i}(s)\, ds \qquad \text{(III.1.23)}$$

now breaks up into a sum of two complex conjugate parts $g_{12} = g_{12}^{(+)} + g_{12}^{(-)}$ with

$$q_{12}^{(+)} = \int_0^\infty f_{l_f}^{(+)}(s)\, G(s)\, f_{l_i}(s)\, ds$$

To evaluate this integral we regard the coordinate s as a complex variable, and displace the path of integration off the real axis into the upper half-plane. The displaced path of integration does not pass through the points $s = s_{oi}$, s_{of} on the real axis, near which the semiclassical approximation is inapplicable. Hence we can use for f_{l_i} and $f_{l_f}^{(+)}$, over the whole path, the functions which are their asymptotic expressions in the upper half-plane. These are:

$$f_{l_i}(s) = \frac{1}{2}\sqrt{\frac{k_i}{K_i(s)}}\, \exp\left[\int_{s_{oi}}^{s} K_i(s')\, ds'\right],$$

$$f_{l_f}^{(+)}(s) = -\frac{i}{2}\sqrt{\frac{k_f}{K_f(s)}}\, \exp\left[-\int_{s_{of}}^{s} K_f(s')\, ds'\right],$$

where κ_i, κ_f are positive on the real axis for $s < s_{of}$. Thus,

$$q_{12}^{(+)} = \frac{-i\sqrt{k_i k_f}}{4} \int_{C_1} \exp\left[\int_{s_{oi}}^{s} K_i(s')ds' - \int_{s_{of}}^{s} K_f(s')ds'\right] \frac{G(s)}{\sqrt{K_i(s)K_f(s)}}\, ds \qquad \text{(III.1.24)}$$

where C_1 is the displaced contour in the upper half-plane (fig. 9a).

Similarly,

$$q_{12}^{(-)} = \frac{i\sqrt{k_i k_f}}{4} \int_{C_2} \exp\left[-\int_{s_{oi}}^{s} K_i(s')\,ds' + \int_{s_{of}}^{s} K_f(s')\,ds'\right] \frac{G(s)}{\sqrt{K_i(s)K_f(s)}}\, ds \qquad \text{(III.1.25)}$$

where C_2 is the displaced contour in the lower half-plane (fig. 9a). Remembering that:

$$f_{\ell_f}^{(+)}(s) = -f_{\ell_f}^{(-)}(s) \qquad \text{for} \quad s < s_{of},$$

and that $f_\ell^{(\pm)}(s)$ for $s > s_{of}$ are analytic continuations of $f_\ell^{(\pm)}(s)$ for $s < s_{of}$, we may combine eqs. (III.1.24) and (III.1.25) to give:

$$q_{12} \simeq \frac{i\sqrt{k_i k_f}}{4} \int_{C} \exp\left[\int_{s_{of}}^{s} K_f(s')ds' - \int_{s_{oi}}^{s} K_i(s')ds'\right] \frac{G(s)}{\sqrt{K_i(s)K_f(s)}}\, ds, \qquad \text{(III.1.26)}$$

C being the contour which circumvents s_{oi}, s_{of} as shown in fig. 9b.

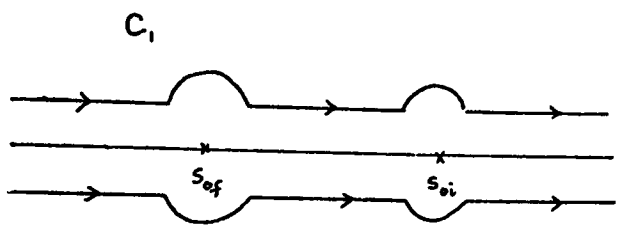


Fig. 9a

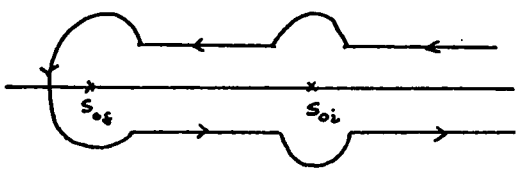


Fig. 9b

Let us now consider the case when U(s) appearing in eq. (III.1.22) represents a real scattering potential. V(s) will then be of the form shown in fig. 10, and there will be three turning points. For energies below the Coulomb barrier only one is real while the other two become imaginary and complex conjugates of one another. When the scattering potential is complex all the classical turning points in the W.K.B. approximation move into the

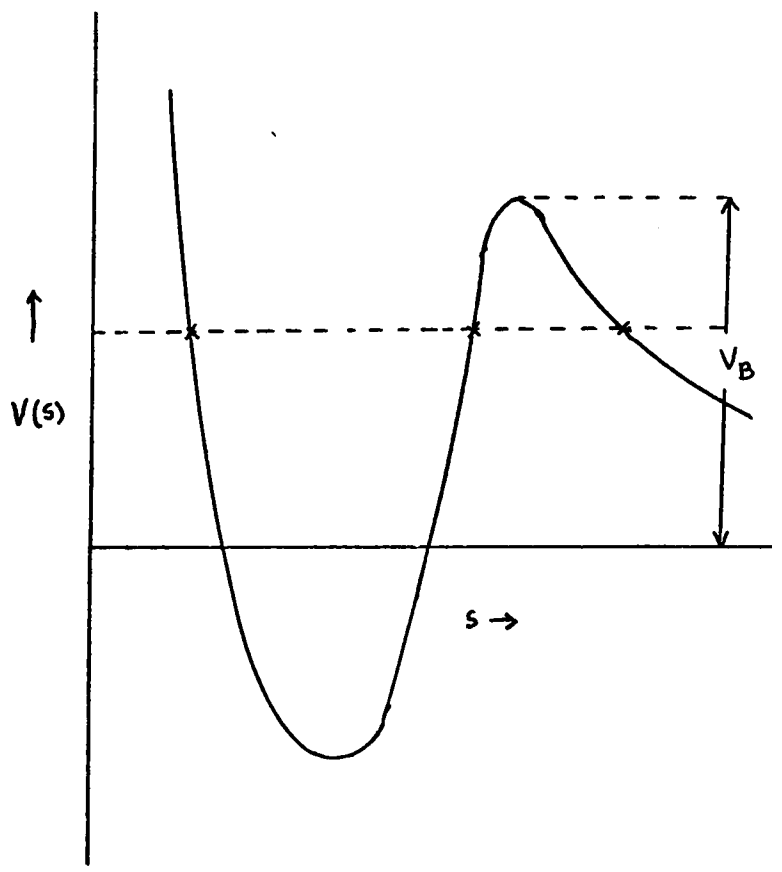


Fig. 10

complex s-plane. Malfliet et. al. (Malfliet and Koeling, 1975; Malfliet et. al., 1973) and Knoll and Schaeffer (Knoll and Schaeffer, 1975) have shown that the optical model scattering phases are well approximated by the W.K.B. phases calculated with respect to the outermost complex turning point when the absorption is strong.

Using the wave functions (III.1.21) corresponding to this outermost complex turning point, Landowne et. al. (Landowne, 1976; Landowne et. al., 1976) have generalised the procedure outlined above to evaluate integrals of the type g_{12} (eq. (III.1.21)) in the form of eq. (III.1.26). Proceeding on the same lines as before they have shown that eq. (III.1.17) becomes:

$$I_{l_i l_f m_f} = \frac{1}{4} \int_C ds \; G^{m_f}_{l_i l_f}(s) \frac{\exp\left[i \int_{s_{of}}^{s} \gamma k_f(s')ds' - \int_{s_{oi}}^{s} \gamma k_i(s')ds'\right]}{\sqrt{\gamma k_f(s)\, \gamma k_i(s)}} \qquad \text{(III.1.27)}$$

where C is the contour which circumvents the two turning points s_{of}, s_{oi} which cause a singularity in the integrand (fig. 11). Eq. (III.1.27) is essentially the same as eq. (III.1.26), γk and κ being related by

$$\gamma k = -i\kappa$$

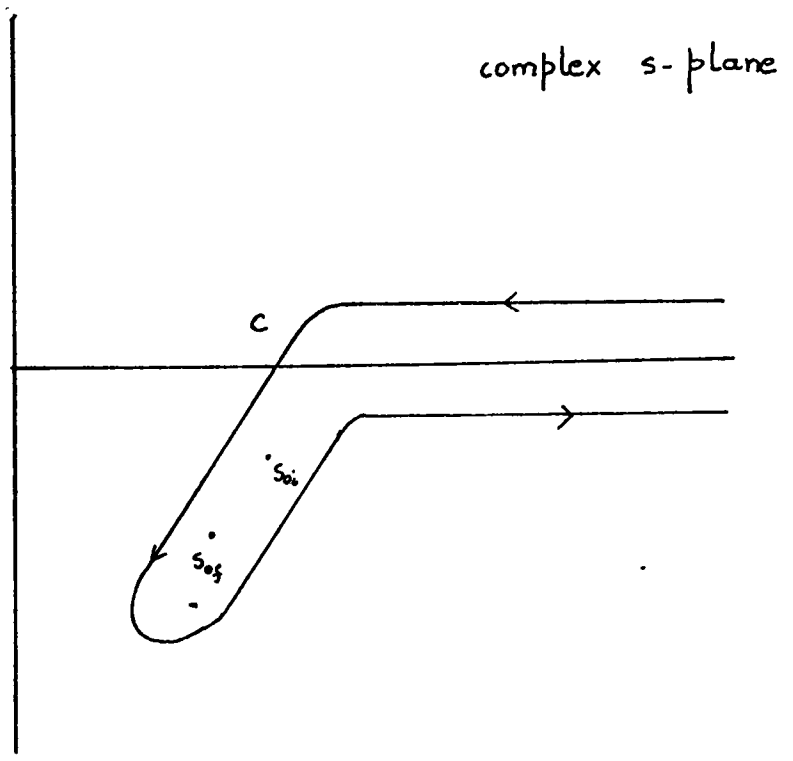


Fig. 11

Expansion about the Final Orbit

Each of the quantities $\nu k_i(s)$, $\nu k_f(s)$ in eq. (III.1.27) correspond to a classical orbit with particular values of E, ℓ, U(s). If these classical orbits are not too different in the initial and final channels, we may expand about the final orbit. The possibility of expanding about some average orbit is discussed later.

Under the above assumption of well-matched orbits the turning points s_{of}, s_{oi} will be close together. We can expand the exponent:

$$\Delta\phi(s) = \int_{s_{of}}^{s} \nu k_f(s')\, ds' - \int_{s_{oi}}^{s} \nu k_i(s')\, ds' \qquad \text{(III.1.28)}$$

appearing in eq. (III.1.27) to first order in the energy difference $E_f - E_i$, the angular momentum difference $\ell_f - \ell_i$, the difference in reduced mass $\mu_f - \mu_i$ and the difference in optical potentials, while simultaneously defining a time differential as:

$$\frac{ds}{k} = \frac{\hbar}{\mu} \frac{ds}{\dot{s}} = \frac{\hbar}{\mu} dt$$

$$\text{or} \qquad t = \frac{\mu}{\hbar} \int_{s_0}^{s} \frac{ds}{k(s)}$$

(thus $s = s_0$ corresponds to $t=0$),

obtaining (appendix III.B)

$$\Delta\phi(s) = \Delta\phi_0(s) + (\ell_f - \ell_i)\,\bar{\theta}(s) \,, \tag{III.1.29}$$

with

$$\Delta\phi_0(s) = \frac{1}{\hbar}(E_f - E_i)t + \frac{1}{\hbar}\int_0^t \Big[\Big[U_i(s(t')) - U_f(s(t'))\Big] + \frac{1}{2}(\mu_f - \mu_i)(\dot{\underline{s}}(t'))^2\Big]\,dt' \tag{III.1.30}$$

Here $\dot{\underline{s}}$ is the relative velocity corresponding to the position vector $\underline{s}$ of the projectile at a time t, and $\bar{\theta}(s)$ is the azimuthal angle in the scattering plane measured from its symmetry axis. In eqs. (III.1.29) and (III.1.30) $\bar{\theta}(s)$ and $s(t)$ correspond to the final orbit.

Substituting eq. (III.1.29) into eq. (III.1.27) we get:

$$I_{l_i l_f m_f} = \frac{1}{4} \int_C ds \quad \frac{G^{m_f}_{l_i l_f} \; e^{i\Delta\phi(s)}}{\sqrt{2k_f \, 2k_i}} \tag{III.1.31}$$

Using eq. (III.1.20) this becomes:

$$I_{l_i l_f m_f} = \int_0^{\pi} d\theta \int_C \frac{G^{m_f}(s,\theta)}{\sqrt{2k_f \, 2k_i}} \cos\left[(l_f - l_i)\theta + \frac{m_f \pi}{2}\right] e^{i\Delta\phi(s)} \, ds \, ; \tag{III.1.32}$$

where

$$G^{m_f}(s,\theta) = \frac{(-1)^{m_f}}{8\pi^2} \int_0^{2\pi} e^{-i m_f \varphi} \, G(s,\theta,\varphi) \, d\varphi \tag{III.1.33}$$

We can evaluate $I_{l_i l_f m_f}$ by writing it as a sum of two integrals $I^{(+)}_{l_i l_f m_f}$, $I^{(-)}_{l_i l_f m_f}$ corresponding to the two branches of the contour C.

Thus:

$$I^{(\pm)}_{l_i l_f m_f} = \int_0^{\pi} d\theta \int_{\delta_0}^{\infty} \frac{G^{m_f}(s,\theta)}{\sqrt{2k_f \, 2k_i}} \cos\left[(l_f - l_i)\theta + \frac{m_f \pi}{2}\right] e^{\pm i\Delta\phi(s)} ds \tag{III.1.34}$$

Our next step is to expand $\delta^{(i)}(l_i)$ about l_f in Taylor's series and retain terms up to first order in $(l_f - l_i)$:

$$\delta^{(i)}(l_i) \simeq \delta^{(i)}(l_f) + \tfrac{1}{2}(l_i - l_f)\,\Theta^{(i)}_{l_f} \quad ,$$

where $\Theta^{(i)}_{l_f} = \left[2\,\frac{d\,\delta^{(i)}(l)}{dl} \right]_{l=l_f}$ corresponds to the classical deflection function for the initial orbit evaluated at $l = l_f$. Also, in the first order we may approximate $\Theta^{(i)}_{l_f}$ by $\Theta^{(f)}_{l_f}$ which corresponds to the classical deflection function for the final orbit.

Further

$$(2l_i+1)^{1/2} = (2l_f + 1 + 2(l_i - l_f))^{1/2}$$

$$= (2l_f+1)^{1/2} \left[1 + \frac{2(l_i-l_f)}{2l_f+1} \right]^{1/2}$$

$$\simeq (2l_f+1)^{1/2}$$

$$\therefore \quad T_{if} = \frac{(4\pi)^{3/2}}{\sqrt{k_i k_f}} \sum_{l_i l_f m_f} (2l_f+1)^{1/2}\, e^{i(\delta^{(i)}_{l_i} + \delta^{(f)}_{l_f})}\, Y_{l_f m_f}(\theta,\Phi) \times$$

$$\times\, e^{i(l_i - l_f)\left(\frac{\pi}{2} + \frac{\Theta^{(f)}_{l_f}}{2}\right)} \left(I^{(+)}_{l_i l_f m_f} + I^{(-)}_{l_i l_f m_f} \right)$$

$$= A \sum_{l_f m_f} \mathcal{F}_{m_f}(l_f)\, \mathcal{I}_{m_f}(l_f) \quad , \qquad \text{(III.1.35)}$$

where

$$A = \frac{(4\pi)^{3/2}}{\sqrt{k_i k_f}} \quad , \qquad \text{(III.1.36)}$$

$$\mathcal{Y}_{m_f}(l_f) = (2l_f+1)^{1/2}\, e^{i(\delta_{l_f}^{(i)} + \delta_{l_f}^{(f)})}\, Y_{l_f m_f}(\vartheta,\Phi)\ ,$$

$$\mathcal{I}_{m_f}(l_f) = \sum_{l_i} e^{i(l_i - l_f)(\frac{\pi}{2} + \frac{\Theta_{l_f}^{(f)}}{2})} \left(I^{(+)}_{l_i l_f m_f} + I^{(-)}_{l_i l_f m_f} \right)$$

(III.1.37)

Now we may write with the help of eqs. (III.1.32, 29):

$$\sum_{l_i} e^{i(l_i - l_f)(\frac{\pi}{2} + \frac{\Theta_{l_f}^{(f)}}{2})}\, I^{(+)}_{l_i l_f m_f} = \frac{1}{2}\int_{s_0}\int_{0}^{\infty}\int^{\pi} \frac{G^{m_f}(s,\theta)}{\sqrt{2k_i\, 2k_f}}\, e^{i\Delta\phi_0(s)} \times$$

$$\times\, e^{\frac{-i m_f \pi}{2}} \left(\sum_{l_i=0}^{\infty} \left[e^{i(l_i - l_f)(\theta_s(l_f) - \theta)} + e^{i(l_i - l_f)(\theta_s(l_f) + \theta)}\, e^{i m_f \pi} \right] \right) ds\, d\theta$$

(III.1.38)

where $\theta_s(l_f)$ is defined by:

$$\theta_s(l_f) = \bar{\theta}(s) + \frac{\Theta_{l_f}^{(f)}(s)}{2} + \frac{\pi}{2}$$

(III.1.39)

Putting $l_i = l_f + l$ and assuming that l is large we can extend the limit of the l-summation from $-l_f$ to $-\infty$.

$$\therefore \sum_{l_i=-\infty}^{\infty} \left[e^{i(l_i - l_f)(\theta_s(l_f) - \theta)} + e^{i(l_i - l_f)(\theta_s(l_f) + \theta)}\, e^{i m_f \pi} \right]$$

$$= 2\pi \left[\delta(\theta_s(l_f) - \theta) + \delta(\theta_s(l_f) + \theta)\, e^{i m_f \pi} \right]$$

(III.1.40)

Since θ lies between 0 and Π and $\theta_s(\ell_f)$ is positive (it being the angle which $\underline{s}$ makes with the z-axis at any time t, t being positive(c.f. fig. (12)) we do not expect a contribution to the integral from the second δ -function.

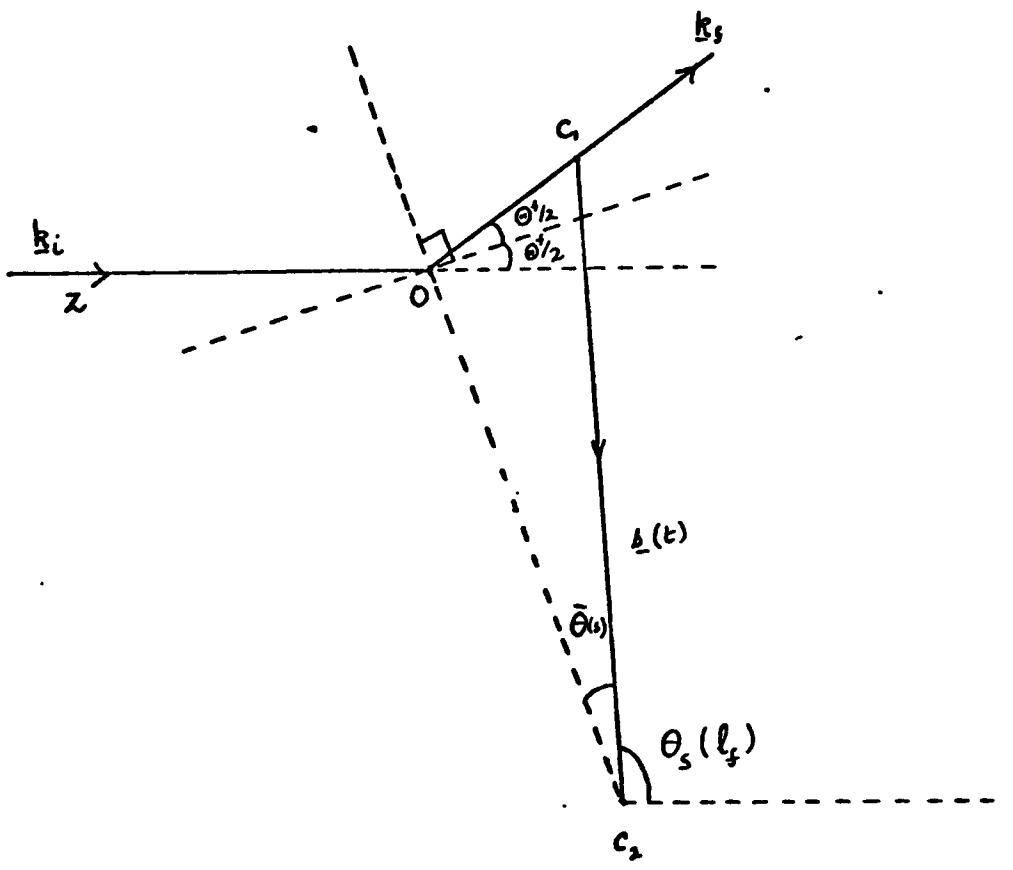


Fig. 12

This figure illustrates the relation (III.1.39) between tthe angles $\theta_s(\ell_f)$, $\bar{\theta}(s)$ and $\Theta^{(f)}_{\ell_f}$. C_1 gives the position of the projectile at time $t > 0$. O denotes the position of the projectile at $t = 0$. The z-axis is along $\hat{\underline{k}}_i$, while OC_2 gives the direction of the symmetry axis.

We therefore have from eqs. (III.1.40, 39):

$$\sum_{l_i} e^{i(l_i-l_f)\left(\frac{\pi}{2}+\frac{\Theta^{(+)}}{2}l_f\right)} I^{(+)}_{l_i l_f m_f} = \frac{\hbar\pi}{\mu_f} e^{-i m_f \frac{\pi}{2}} \int_{s_0}^{\infty}\int_{0}^{2\pi} G^{m_f}(s,\theta)\, e^{i\Delta\phi_0(s)} \times \delta(\theta_s(l_f)-\theta)\, d\theta \frac{ds}{\dot{s}}$$

$$= \frac{\hbar\pi}{\mu_f} e^{-i m_f \frac{\pi}{2}} \int_{s_0}^{\infty} G^{m_f}(s,\theta_s)\, e^{i\Delta\phi_0(s)} \frac{ds}{\dot{s}} \qquad \text{(III.1.41)}$$

where we have assumed:

$$k_i \simeq k_f = \frac{\mu_f \dot{s}}{\hbar}$$

Using the relation

$$ds = \dot{s}\, dt$$

and changing variables from s to t, with $s = s_0$ corresponding to $t = 0$, we may re-write eq. (III.1.41) as:

$$\sum_{l_i} e^{i(l_i-l_f)\left(\frac{\pi}{2}+\frac{\Theta^{(+)}}{2}l_f\right)} I^{(+)}_{l_i l_f m_f} = \frac{\hbar\pi}{\mu_f} e^{-i m_f \frac{\pi}{2}} \int_{0}^{\infty} G^{m_f}(s(t),\theta_s(t)) \times e^{i\Delta\phi_0(t)}\, dt$$

Changing variable from t to -t and using the time reversal property of $G^{m_f}(t)$ we get:

$$\sum_{l_i} e^{i(l_i-l_f)\left(\frac{\pi}{2}+\frac{\Theta^{(+)}}{2}l_f\right)} I^{(+)}_{l_i l_f m_f} = \frac{\hbar\pi}{\mu_f} e^{-i m_f \frac{\pi}{2}} \int_{-\infty}^{0} G^{m_f}(s(t),\theta_s(l_f(t))\, e^{-i\Delta\phi_0(t)}\, dt \qquad \text{(III.1.42)}$$

Similarly we get:

$$\sum_{\ell_i} e^{i(\ell_i-\ell_f)\left(\frac{\pi}{2}+\frac{\Theta^{(f)}_{\ell_f}}{2}\right)} I^{(-)}_{\ell_i\ell_f m_f} = \frac{\hbar\pi}{\mu_f} e^{-\frac{i m_f \pi}{2}} \int_0^\infty G^{m_f}(s(t),\theta_s(\ell_f,t)) \times e^{-i\Delta\phi_0(t)}\, dt \qquad \text{(III.1.43)}$$

(The ℓ_f dependence of θ_s has been introduced in the notation)

Adding eqs. (II.1.42) and (III.1.43) we get:

$$\mathcal{I}_{m_f}(\ell_f) = \sum_{\ell_i} e^{i(\ell_i-\ell_f)\left(\frac{\pi}{2}+\frac{\Theta^{(f)}_{\ell_f}}{2}\right)} \left(I^{(+)}_{\ell_i\ell_f m_f} + I^{(-)}_{\ell_i\ell_f m_f}\right)$$

$$= \frac{\hbar\pi}{\mu_f} e^{-\frac{i m_f \pi}{2}} \int_{-\infty}^{+\infty} G^{m_f}(s(t),\theta_s(\ell_f,t))\, e^{-i\Delta\phi_0(t)}\, dt \qquad \text{(III.1.44)}$$

In eq. (III.1.44) for $\mathcal{I}_{m_f}(\ell_f)$, $s(t)$, $\theta_s(\ell_f, t)$ and $\Phi_0(t)$ correspond to their values calculated along the final orbit defined by the energy, E_f, reduced mass, μ_f and angular momentum ℓ_f. Instead of using the final orbit for our derivation we could have chosen an orbit $\underline{s}_0$ which is some average of the initial and final orbits. In that case we would get equations analogous to (III.1.35) and (III.1.44):

Thus: $$T_{if} = A \sum_{\ell m} (2\ell+1)^{1/2} e^{i(\delta^{(i)}_\ell + \delta^{(f)}_\ell)} Y_{\ell m}(\vartheta,\Phi)\, \mathcal{I}_m(\ell) \qquad \text{(III.1.45)}$$

with

$$\mathcal{I}_m(\ell) = \frac{\hbar\pi}{\mu} e^{-\frac{i m \pi}{2}} \int_{-\infty}^{+\infty} G^m(s_0(t),\theta_{s_0}(\ell,t))\, e^{-i\Delta\bar{\phi}_0(t)}\, dt \qquad \text{(III.1.46)}$$

(μ is some average of μ_f, μ_i)

where $\overline{\Delta\Phi_0}$ is now calculated over an average orbit. It may be separated into two parts

$$\overline{\Delta\phi_0}(t) = \Delta\phi_0'(t) + \frac{1}{2\hbar}(\mu_f - \mu_i)\int_0^t (\dot{s}_0(t'))^2 \, dt' , \tag{III.1.47}$$

with

$$\Delta\phi_0'(t) = \frac{1}{\hbar}(E_f - E_i)t - \frac{1}{\hbar}\int_0^t \left[U_f(s_0(t')) - U_i(s_0(t'))\right] dt' \tag{III.1.48}$$

Identification of $\mathcal{I}_m(\ell)$ as the Semiclassical Amplitude of Chapter II:

First of all let us look at the component $\Delta\Phi_0'$ (eq. (III.1.48)) of the time dependent phase $\overline{\Delta\Phi_0}(t)$ appearing in eq. (III.1.45). We shall assume that the most important contribution of $(U_f - U_i)$ is that of the Coulomb terms and comes from parts of the orbit $s_0(t)$ near the point of closest approach

$$\text{i.e.} \quad U_f(s_0(t)) - U_i(s_0(t)) \simeq \frac{(Z_{c_1} - Z_{c_2})Z_x e^2}{d}$$

where Z_{c_1} e, Z_{c_2} e, Z_xe are the changes on the cores c_1, c_2 and the transferred particle x respectively. d is the distance of closest approach. $\Delta\Phi_0'$ now becomes:

$$\Delta\phi_0' = \frac{1}{\hbar}(E_f - E_i)t + \frac{1}{\hbar}\frac{(Z_{c_1} - Z_{c_2})Z_x e^2}{d}\, t$$

$$= \frac{1}{\hbar} Q_{eff}\, t$$

where Q_{eff} is the same as the modified Q - value defined in eq. (II.5.29)

We shall further assume:

$$(\dot{\underline{s}}_o(t))^2 \simeq (\dot{\underline{s}}_o)^2_{t=0}$$

in the second term of eqs. (III.1.47).

With these approximations eq. (III.1.45) becomes:

$$\mathcal{I}_m(\ell) = \frac{\hbar\Gamma_i}{\mu} e^{-i\frac{m\pi}{2}} \int_{-\infty}^{+\infty} G^m(s_o(t), \theta_{s_o}(\ell,t)) e^{-\frac{1}{2\hbar}(\mu_f-\mu_i)\dot{s}_o^2 t - \frac{i}{\hbar}Q_{eff}t} dt$$

(III.1.49)

We shall now write $G^m(s_o(t), \theta_{s_o}(\ell,t))$ explicitly with the help of eqs. (III.1.33) and (III.1.14). Thus,

$$G^m(s_o(t), \theta_{s_o}(\ell,t)) = \frac{(-1)^m}{8\pi^2}\int_0^{2\pi}\int e^{-im\varphi} \psi_2^*(\underline{r}_2)\, \Delta V(\underline{r}_2, \underline{s}_o)\, \psi_1(\underline{r}_2 - \underline{s}_o) \times e^{\frac{i}{\hbar}\underline{p}\cdot\underline{r}_2} e^{-\frac{i}{\hbar}\underline{p}_i\cdot\underline{s}_o\mu_{ix}} d^3\underline{r}_2\, d\varphi$$

(III.1.50)

where ψ_1, ψ_2 are the bound state wave functions defined in chapter II. The next step is to obtain an approximation for the term $\Delta V^{coul}(\underline{r}_2, \underline{s}_o)$ appearing in $\Delta V(\underline{r}_2, \underline{s}_o)$ (c.f. eq. (III.1.16)). We shall use the form:

$$\Delta V^{coul}(\underline{r}_2, \underline{s}_o) = V_{1,2}^{coul}(\underline{r}_{1,2}) + U_{a_{2,1}c_{1,2}}^{opt,coul}(\underline{r}_{i,f}) - V_{c_1c_2}^{coul}(\underline{s})$$

$$\simeq V_{1,2}^{coul}(\underline{r}_{1,2}) - \Delta\varepsilon_{1,2}$$

(III.1.7,9)

where $$\Delta\varepsilon_{1,2} = \frac{Z_{c_{1,2}} Z_x e^2}{d}$$

Hence in the post representation we have (assuming a central nuclear potential):

$$\Delta V = V_1(|\underline{t}_2 - \underline{s}_0|) - \Delta\varepsilon_1 \qquad \text{(III.1.51)}$$

while in the prior representation we get:

$$\Delta V = V_2(t_2) - \Delta\varepsilon_2 \qquad \text{(III.1.52)}$$

The approximation of replacing $(U^{opt,coul}_{a_{1,2}c_{2,1}}(\underline{t}_{i,f}) - V^{coul}_{c_1c_2}(\underline{s}_0))$ by $\Delta\varepsilon_{2,1}$ can be justified by the following considerations. As the distance of closest approach is greater than the sum of the radii of the two nuclei, the Coulomb interactions between them may be considered as that between point charges

$$\text{i.e. } V^{coul}_{c_1c_2}(\underline{s}_0) - U^{opt,coul}_{a_{1,2}c_{2,1}}(\underline{t}_{i,f}) \simeq \frac{Z_{c_1}Z_{c_2}e^2}{|\underline{s}_0|} - \frac{(Z_{c_{1,2}} + Z_x)Z_{c_{2,1}}}{|\underline{t}_{i,f}|}$$

If the mass of the transferred particle is much less than that of c_1 and c_2 we may replace $|\underline{r}_{i,f}|$ in the above equation by $|\underline{s}_0|$. Also, as the reaction is expected to be a peripheral one the most important contribution to this term is expected to come from $|\underline{s}_0| \simeq d$. Thus we replace:

$$V^{coul}_{c_1c_2}(\underline{s}) - U^{opt,coul}_{a_{1,2}c_{2,1}} \simeq \frac{Z_{c_{2,1}}Z_x}{d} = \Delta\varepsilon_{2,1}$$

Comparing eq. (III.1.52) with eq. (II.5.24) we see that ΔV in the prior representation is exactly the same as $\tilde{V}_2(r_2)$, and similarly for the post representation.

Making this substitution for ΔV in eq. (III.1.50) and assuming the $\underline{p}$ depends only on $s_0(t)$ and lies in the plane of the orbit $\underline{s}_0(t)$, we get:

$$G^m(s_0(t), \theta_s(\ell,t)) = \frac{(-1)^m}{8\pi^2} \int_0^{2\pi} e^{-im\varphi} U_{\lambda_1\lambda_2}(t)\, e^{-\frac{i}{\hbar} p_i \cdot s_0 \mu_{\alpha\alpha}} d\varphi \qquad \text{(III.1.53)}$$

where,

$$U_{\lambda_1\lambda_2}(t) = \int \psi_2^*(\underline{r}_2)\, \tilde{V}_2(r_2)\, \psi_1(\underline{r}_2 - \underline{s}_0)\, e^{\frac{i}{\hbar} p \cdot \underline{r}_2}\, d^3\underline{r}_2$$

may be identified as the quantity defined by eq. (II.5.28) (or eq. (II.1.4) for neutron transfer). $\ell_1\lambda_1$, $\ell_2\lambda_2$ are the angular momentum quantum numbers of the transferred cluster in the initial and final states respectively (c.f. eqs. (II.1.6), (II.1.7)).

Note that $U_{\lambda_1\lambda_2}(t)$ depends on φ through the coordinate $\underline{s}_0$ appearing in its expression. It also depends on ℓ through $\theta_{s_0}(\ell,t)$. Considering the rotational properties of $\psi_1(\underline{r}_2 - \underline{s}_0)$, $\psi_2(\underline{r}_2)$, we may write the φ dependence of $U_{\lambda_1\lambda_2}(t)$ simply as

$$U_{\lambda_1\lambda_2}(t) = U^{\ell,0}_{\lambda_1\lambda_2}(t)\, e^{i(\lambda_1-\lambda_2)\varphi}$$

where the superscript 0 indicates that $U_{\lambda_1\lambda_2}(t)$ has been calculated with $\underline{s}_0$ lying in the x - z plane; the superscript ℓ indicates its ℓ-dependence.

$$\therefore G^m(s_0(t), \xi_{s_0}(l,t)) = \frac{(-1)^m}{8\pi^2} e^{-\frac{i}{\hbar}\mu_i \underline{p}_i \cdot \underline{s}_0} \int_0^{2\pi} e^{-i(m+\lambda_2-\lambda_1)\varphi} U^{l,0}_{\lambda_1\lambda_2}(t) \times d\varphi$$

$$= \frac{(-1)^m}{8\pi^2} e^{-\frac{i}{\hbar}\underline{p}_i \cdot \underline{s}_0 \mu_{ix}} U^{l,0}_{\lambda_1\lambda_2}(t)\, \delta(m-\lambda_1+\lambda_2)$$

(III.1.54)

Substituting eq. (III.1.54) into eq. (III.1.49) we get:

$$\mathcal{I}_m(l) = \frac{\hbar\pi}{\mu} \frac{e^{im\pi/2}}{8\pi^2} \delta(m+\lambda_2-\lambda_1) \int_{-\infty}^{+\infty} U^{l,0}_{\lambda_1\lambda_2}(t)\, e^{-\frac{i}{\hbar}\underline{p}_i \cdot \underline{s}_0 \mu_{ix}} \times e^{-\frac{1}{2\hbar}(\mu_f-\mu_i)\dot{\underline{s}}_0^2 t - \frac{i}{\hbar} Q_{eff} t} dt$$

(III.1.55)

Now let:

$$\underline{p}_i(s_0(t)) \simeq \mu_i \frac{\dot{\underline{s}}_0}{\hbar}$$

where μ_i is the reduced mass in the initial channel.

$$\therefore \underline{p}_i \cdot \underline{s}_0 = \underline{p}_i \cdot \dot{\underline{s}}_0 t$$

$$\simeq \frac{\mu_i}{\hbar} \dot{\underline{s}}_0^2 t$$

(III.1.56)

Substituting eq. (III.1.56) into eq. (III.1.55), the time dependent phase appearing in eq. (III.1.55) can be simplified to yield:

$$Q_{eff} + \frac{1}{2}\left(\mu_f - \mu_i(1-2\mu_{c_1x})\right)\dot{s}_0^2 = Q_{eff} + \frac{m_x(m_{c_1}+m_{c_2})}{2(m_{c_1}+m_{c_2}+m_x)}\dot{s}_0^2$$

$$\simeq Q_{eff} + \frac{1}{2}m_x\dot{s}_0^2 \quad (\text{if } m_x << m_{c_1}+m_{c_2}+m_x)$$

$$\therefore \mathcal{I}_m(l) = \frac{\hbar^2\pi}{\mu} e^{i m\pi/2} \frac{1}{\hbar}\delta(m-\lambda_1+\lambda_2)\int_{-\infty}^{+\infty} U^{l,0}_{\lambda_1\lambda_2}(t)\, e^{-\frac{i}{\hbar}(Q_{eff}+\frac{1}{2}m_x\dot{s}_0^2)t}\, dt$$

$$= \frac{\hbar^2\pi}{\mu} e^{i m\pi/2} A^0_{\lambda_1\lambda_2}(l)\,\delta(m+\lambda_1-\lambda_2)$$

(III.1.57)

where $A^0_{\lambda_1\lambda_2}(l)$ can be recognised as corresponding to the semiclassical transfer amplitude $A(\lambda_1,\lambda_2)$ (eq. (II.5.27) or (II.1.3) for $Z_x = 0$) calculated in a frame of reference in which $\underline{s}_0$ lies in the x - z plane.

Approximation of Average Semiclassical Path to a Straight Line:

As in chapter II we shall approximate the average path by a straight line trajectory (fig. 1)

$$\underline{s}_0 = \underline{d} + \dot{\underline{s}}_0 t$$

(where d is the distance of closest approach) which is the average of the initial and final directions $\underline{k}_i$ and $\underline{k}_f$ respectively and is inclined at an angle $\vartheta/2$ to the z-axis. $\underline{s}_0$ will be taken to be in the x - z plane (fig. 13)

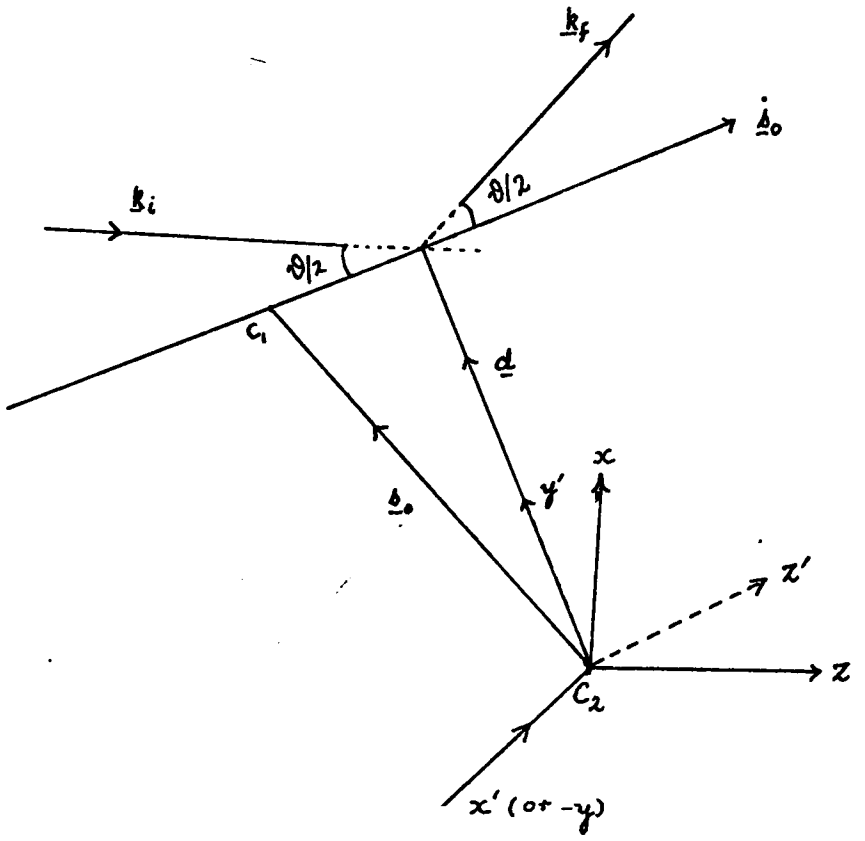


Fig. 13

It is now convenient to perform a rotation of axes (fig.13) through the Euler angles $(0, \frac{\vartheta}{2}, \frac{-\Pi}{2})$, such that the z' axis lies along $\underline{s}_o$ and the y' axis along $\underline{d}$. The primed system here is the same as that one pictured in fig. 4 and $A^{o'}_{\lambda_1\lambda_2'}(\ell)$ is the same as $A'(\lambda_1', \lambda_2')$ of eq. (II.5.30). The relation between the transfer amplitudes in the two frames of reference is given by:

$$A^{o}_{\lambda_1\lambda_2}(\ell) = \sum_{\lambda_1'\lambda_2'} \mathcal{D}^{\ell_1}_{\lambda_1'\lambda_1}\left(\frac{\pi}{2}, -\frac{\vartheta}{2}, 0\right) \mathcal{D}^{\ell_2 *}_{\lambda_2'\lambda_2}\left(\frac{\pi}{2}, -\frac{\vartheta}{2}, 0\right) A^{o'}_{\lambda_1'\lambda_2'}(\ell)$$

$$= \sum_{\lambda_1'\lambda_2'} e^{-i(\lambda_1'-\lambda_2')\pi/2}\, d^{\ell_1}_{\lambda_1'\lambda_1}\left(-\frac{\vartheta}{2}\right) d^{\ell_2}_{\lambda_2'\lambda_2}\left(-\frac{\vartheta}{2}\right) A^{o'}_{\lambda_1'\lambda_2'}(\ell)$$

(III.1.58)

$A^{o'}_{\lambda_1',\lambda_2'}(\ell)$ in these equations is given by eq. (II.5.17) which to recapitulate is:

$$A^{o'}_{\lambda_1'\lambda_2'}(\ell) = \frac{(2\pi)^2}{k|\dot{\underline{s}}_0|}\int_0^\infty \tilde{u}_2^*(k_2)\,\tilde{u}_1(k_1)\,P_{\ell_2}^{\lambda_2'}(\cos\theta_2')\,P_{\ell_1}^{\lambda_1'}(\cos\theta_1')\times$$
$$\times\left(\tilde{\varepsilon}_1-\frac{\hbar^2k_1^2}{2m}\right)J_{\lambda_1'-\lambda_2'}(k_\perp d)\,k_\perp\,dk_\perp$$
$$\left(d\simeq(\ell+\nu_2)\hbar/\mu|\dot{\underline{s}}_0|\right) \qquad \text{(II.5.17)}$$

the various quantities being the same as before except that they now refer to the transfer of a charged cluster rather than to a single proton.

Substituting eq. (III.1.58) into eq. (III.1.57) which is put back into (III.1.45) we get for the transition amplitude (λ_1,λ_2 dependence of T_{if} is now introduced in notation)

$$T_{if}^{\lambda_1\lambda_2}(\theta,\Phi) = \frac{(4\pi)^{3/2}}{\sqrt{k_ik_f}}\frac{1}{8\pi^2}\frac{k^2\pi}{\sqrt{\mu_i\mu_f}}\,e^{i(\lambda_1-\lambda_2)\bar{\pi}/2}\times$$
$$\times\sum_\ell(2\ell+1)^{\nu_2}\,e^{i(\delta_\ell^{(i)}+\delta_\ell^{(f)})/2}\,A^o_{\lambda_1\lambda_2}(\ell)\,Y_{\ell,\lambda_1-\lambda_2}(\theta,\Phi)$$
(III.1.59)

(where we have approximated μ by $\sqrt{\mu_i\mu_f}$)

For small angle scattering we may approximate the spherical harmonic by:

$$Y_{\ell,\lambda_1-\lambda_2}(\theta,\Phi)\simeq e^{i(\lambda_1-\lambda_2)\Phi}\sqrt{\frac{2\ell+1}{4\pi}}\;J_{\lambda_1-\lambda_2}\left((2\ell+1)\sin\tfrac{\theta}{2}\right)$$

$$\therefore\; T_{if}^{\lambda_1\lambda_2}(\theta,\Phi)\simeq\frac{k^2}{2\sqrt{\mu_ik_i\mu_fk_f}}\,e^{i(\lambda_1-\lambda_2)(\frac{\bar{\pi}}{2}+\Phi)}\sum_\ell(2\ell+1)\,e^{i(\delta_\ell^{(i)}+\delta_\ell^{(f)})/2}\,A^o_{\lambda_1\lambda_2}(\ell)\times$$
$$\times J_{\lambda_1-\lambda_2}\left((2\ell+1)\sin\tfrac{\theta}{2}\right)$$
(III.1.60)

The scattering amplitude $f_{\lambda_1\lambda_2}(\theta)$ is given by (in scattering plane):

$$f_{\lambda_1\lambda_2}(\theta) = -\sqrt{\frac{\mu_i\mu_f}{2\pi\hbar^2}}\; T_{if}^{\lambda_1\lambda_2}(\theta)$$

The amplitude $f(j_1\ n_1,\ j_2\ n_2)$ for transfer from a definite state $(j_1\ n_1)$ to a state $(j_2\ n_2)$ is obtained by combining the various $f_{\lambda_1\lambda_2}(\theta)$ with appropriate Clebsch Gordan coefficients and summing over the spin states:

$$f(j_1 n_1, j_2 n_2) = \sum_{\lambda_1\lambda_2 m_s} \langle l_1\lambda_1, s\, m_s | j_1 n_1\rangle \langle l_2\lambda_2, s\, m_s | j_2 n_2\rangle\, f_{\lambda_1\lambda_2}(\theta)$$

(III.1.61)

where $\langle l_i\lambda_i, s\, m_s | j_i n_i\rangle$ are Clebsch Gordan coefficients.
The differential cross section is then given by squaring the scattering amplitudes, summing over final spin projections and averaging over the initial ones to obtain:

$$\frac{d\sigma}{d\Omega} = \frac{2J_{a_2}+1}{(2J_{c_2}+1)(2j_1+1)(2j_2+1)}\ \frac{k_f}{k_i} \sum_{n_1 n_2} |f(j_1 n_1, j_2 n_2)|^2$$

(III.1.62)

where J_{a_2} and J_{c_2} are the spins of the final nucleus and the core c_2 respectively.

In successive chapters, we shall use the formulae obtained here to determine cross-sections for specific reactions and make some comparisons with experiment.

CHAPTER IV

CALCULATION OF ANGULAR DISTRIBUTIONS FOR $^{26}Mg + {}^{11}B$ REACTIONS

IV.1 Details of Calculations

In this chapter we have studied the reactions:

$$^{26}Mg(^{11}B, {}^{10}B)^{27}Mg, \qquad \text{(IV.1.1)}$$

$$^{26}Mg(^{11}B, {}^{10}Be)^{27}Al \qquad \text{(IV.1.2)}$$

performed at 114 MeV laboratory energy. We have first looked at the formula for the transition amplitude:

$$T_{if}^{\lambda_1\lambda_2}(\vartheta) = \frac{\hbar^2}{2\sqrt{\mu_i k_i \mu_f k_f}} e^{i(\lambda_1-\lambda_2)\frac{\pi}{2}} \sum_l (2l+1) e^{i(\delta_l^{(i)} + \delta_l^{(f)})} A^0_{\lambda_1,\lambda_2}(l) J_{\lambda_1-\lambda_2}((2l+1)\sin\tfrac{\vartheta}{2}) \qquad \text{(III.1.60)}$$

derived in chapter III, in some detail and then evaluated angular distributions. Before going on with the discussion we present some numerical details.

The bound state wave functions appearing in the expression*(eq. (II.3.6)) for $A^{0'}_{\lambda_1'\lambda_2'}(l)$ were calculated as in section II.4 in a Woods-Saxon potential with a radius parameter $R = r_0 A^{\frac{1}{3}}$ ($r_0 = 1.20$) and diffusivity $a = 0.65$. The elastic scattering phase shifts in this chapter were calculated using the W.K.B. approximation with the computer programme of B.J.B. Crowley (Crowley, 1975). The phase shifts obtained here agree with with those calculated quantum mechanically. The optical model parameters for the complex potential† were taken to be:

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* C.f. Eq.(III.1.38) for relation between $A^0_{\lambda_1,\lambda_2}(l)$ and $A^0_{\lambda_1',\lambda_2'}(l)$.

† $V^n + iW^n = -V_R/(1+\exp((r-R_R)/a_R)) - iV_I/(1+\exp((r-R_I)/a_I))$

$V_R = 35$ MeV $V_I = 25$ MeV

$r_{oR} = 1.066$ $r_{oI} = 1.216$ $R_i = r_{oi}(A_P^{\frac{1}{3}}+A_T^{\frac{1}{3}})$

$a_R = 0.8$ $a_I = 0.62$

The grazing velocity $|\dot{\underline{s}}_0|$ was calculated as a function of ℓ according to the formula:

$$|\dot{s}_0| = v\,\frac{\eta}{(l+\frac{1}{2})}\left[-1+\left\{1+\frac{(l+\frac{1}{2})^2}{\eta^2}\right\}^{\frac{1}{2}}\right], \qquad \text{(IV.1.3)}$$

where

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v}, \qquad \text{(IV.1.4)}$$

v being the asymptotic velocity of the projectile and $Z_1 e$, $Z_2 e$ the charges on the interacting nuclei. This expression was obtained as a solution of the simultaneous equations:

$$\frac{1}{2}\mu\,|\dot{\underline{s}}_0|^2 = \frac{1}{2}\mu v^2 - \frac{Z_1 Z_2 e^2}{d},$$

and

$$d = (\ell+\tfrac{1}{2})\hbar\,/\,\mu\,|\dot{\underline{s}}_0|$$

μ is the reduced mass and d the distance of closest approach. All these quantities were calculated in the initial channel.

IV.2. Numerical Study of Eq. (III.1.60):

Let us now look at the behaviour of some of the terms in eq. (III.1.60) in detail and see if we can find simplifications which can be exploited to our advantage. We study the ℓ-dependence of the nuclear phase, the transfer amplitude $A^{\circ}_{\lambda_1\lambda_2}(\ell)$, and the product of these two quantities.

Fig. 14 shows a plot of $e^{-2\delta^{n}_{im}(\ell)}$ (where $2\delta^{n}_{im}(\ell) = Im(\delta^{(i)n}_{\ell} + \delta^{(f)n}_{\ell})$ is the imaginary part of the nuclear phase shift) together with $|A^{o'}_{\lambda'_1\lambda'_2}(\ell)|$ calculated according to eq. (II.3.6), and their product $e^{-2\delta^{(n)}_{im}(\ell)} \times |A^{o'}_{\lambda'_1\lambda'_2}(\ell)|$ for the transfer of a neutron from the $1p_{3/2}$ state in ^{11}B to the $2s_{\frac{1}{2}}$ state in ^{27}Mg for the reaction (IV.1.1) for $\lambda'_1 = -1$, $\lambda'_2 = 0$. We can see from the figure that the imaginary part of the phase shift causes the contribution from lower values of ℓ to be damped out, thus providing a lower cut off to the summation in eq. (III.1.60). For large values of ℓ, $e^{-2\delta^{n}_{im}(\ell)}$ tends to unity. The upper cut off is provided by $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ which decays to zero as ℓ increases. In fact, the part of $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ not cut by the imaginary phase shift looks like an exponential curve. To show this we plotted $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ on a logarithmic scale (fig. 15a) for values of λ'_1, λ'_2 equal to (-1,0) and (0,0). In both cases they approximate quite well to a straight line indicating the exponential behaviour of the transfer amplitudes. The product $e^{-2\delta^{(n)}_{im}(\ell)} \times |A^{o'}_{\lambda'_1\lambda'_2}(\ell)|$ (which is really the more important quantity, as it is this which appears in the partial wave summation) is peaked around $\ell = 39$, for which value of ℓ, $e^{-2\delta^{(n)}_{im}(\ell)}$ is 62% of its maximum value. This function has been plotted again in fig. 15b on a larger scale, where the peak is more noticeable.

We make a similar study of $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ for the transfer of a neutron to the $1d_{3/2}$ excited state in ^{27}Mg, as well as for the transfer of a proton* from the $1p_{3/2}$ state in

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*Using the modified equation (II.5.30) instead of (II.3.6)

^{11}B to the $1d_{5/2}$ state in ^{27}Al for the reaction (IV.1.2). A logarithmic plot of $|A^{o'}_{\lambda'_1\lambda'_2}(\ell)|$ for some combinations of (λ'_1,λ'_2) is shown in figs. 15c and 16a. In each case the exponential behaviour of $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ is evident. Further, it is seen that for a specific transfer the straight line $\ln\left[A^{o'}_{\lambda'_1\lambda'_2}(\ell)\right]$ for various combinations of λ'_1,λ'_2 are parallel suggesting that the amplitudes bear simple ratios to one another.

These properties of $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ tempt us to approximate them by an exponential function rather than calculate them for each value of ℓ from eq. (II.3.6). In the following section we show how this may be done.

IV.3. Exponential Approximation for $A^{o}_{\lambda_1\lambda_2}(\ell)$

On the strength of the above arguments we shall write $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ as:

$$A^{o'}_{\lambda'_1\lambda'_2}(\ell) = C'_{\lambda'_1\lambda'_2}\, e^{\frac{\Lambda_0-\ell}{\Delta}} \qquad \text{(IV.3.1)}$$

(or $A^{o}_{\lambda_1\lambda_2}(\ell) = C_{\lambda_1\lambda_2} e^{\frac{\Lambda_0-\ell}{\Delta}}$, with $C_{\lambda_1\lambda_2} = \sum_{\lambda'_1\lambda'_2} \theta^{\ell_1}_{\lambda'_1\lambda_1}\, \theta^{\ell_2 *}_{\lambda'_2}\left(\frac{\pi}{2},-\frac{\pi}{2},0\right) C'_{\lambda'_1\lambda'_2}$ (from eq. (III.1.58))

where $C_{\lambda_1\lambda_2}$ contains the λ_1,λ_2 dependence and is independent of ℓ, and Λ_0, Δ are parameters to be fitted:

To determine Λ_0, Δ we calculate $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ for a particular pair of λ'_1,λ'_2 from eq. (II.3.6) at two different values of ℓ:- ℓ_a and ℓ_b. Λ_0 and Δ are then given by:

$$\Delta = \frac{\ell_b - \ell_a}{\ln A^{o'}_{\lambda'_1\lambda'_2}(\ell_b) - \ln A^{o'}_{\lambda'_1\lambda'_2}(\ell_a)} \qquad \text{(IV.3.2)}$$

$$\Lambda_0 = \ell_b + \Delta \ln A^{o'}_{\lambda'_1\lambda'_2}(\ell_a) \qquad \text{(IV.3.3)}$$

$C'_{-\ell_1,-\ell_2}$ (i.e. $\lambda'_1 = -\ell_1$, $\lambda'_2 = -\ell_2$) is taken to be unity.

To calculate $C'_{\lambda'_1\lambda'_2}$ for other values of λ'_1,λ'_2 we calculate $A^{o'}_{\lambda'_1\lambda'_2}(\ell_a)$, $A^{o'}_{\lambda'_1\lambda'_2}(\ell_b)$ from eq. (II.3.6)

Then:
$$C'_{\lambda'_1\lambda'_2} \simeq \frac{1}{2}\left[\frac{A^{o'}_{\lambda'_1\lambda'_2}(\ell_a)}{A^{o'}_{-\ell_1,-\ell_2}(\ell_a)} + \frac{A^{o'}_{\lambda'_1\lambda'_2}(\ell_b)}{A^{o'}_{-\ell_1,-\ell_2}(\ell_b)}\right] \qquad \text{(IV.3.4)}$$

ℓ_a is chosen to be roughly the point where the product $e^{-2\delta_{im}} \cdot A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ has its peak, while ℓ_b is taken somewhat lower down the curve.

A numerical comparison of $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ using eqs. (II.3.6) and (IV.3.1) is made for the reactions (IV.1.1) and (IV.1.2). The results of the transfer of a neutron from $1p_{3/2}$ state in ^{11}B to the $2s_{\frac{1}{2}}$ state as well as to the $1d_{3/2}$ are shown in figs. 15a, b for some combinations of (λ'_1,λ'_2). A similar comparison is shown in figs. 16a, b for the transfer of a proton from the $1p_{3/2}$ state in ^{11}B to the $1d_{5/2}$ state in ^{27}Al. It can be seen from the figures that the results obtained from the two formulae agree very well in each case, showing that the exponential formula is a good approximation for the transfer amplitudes.

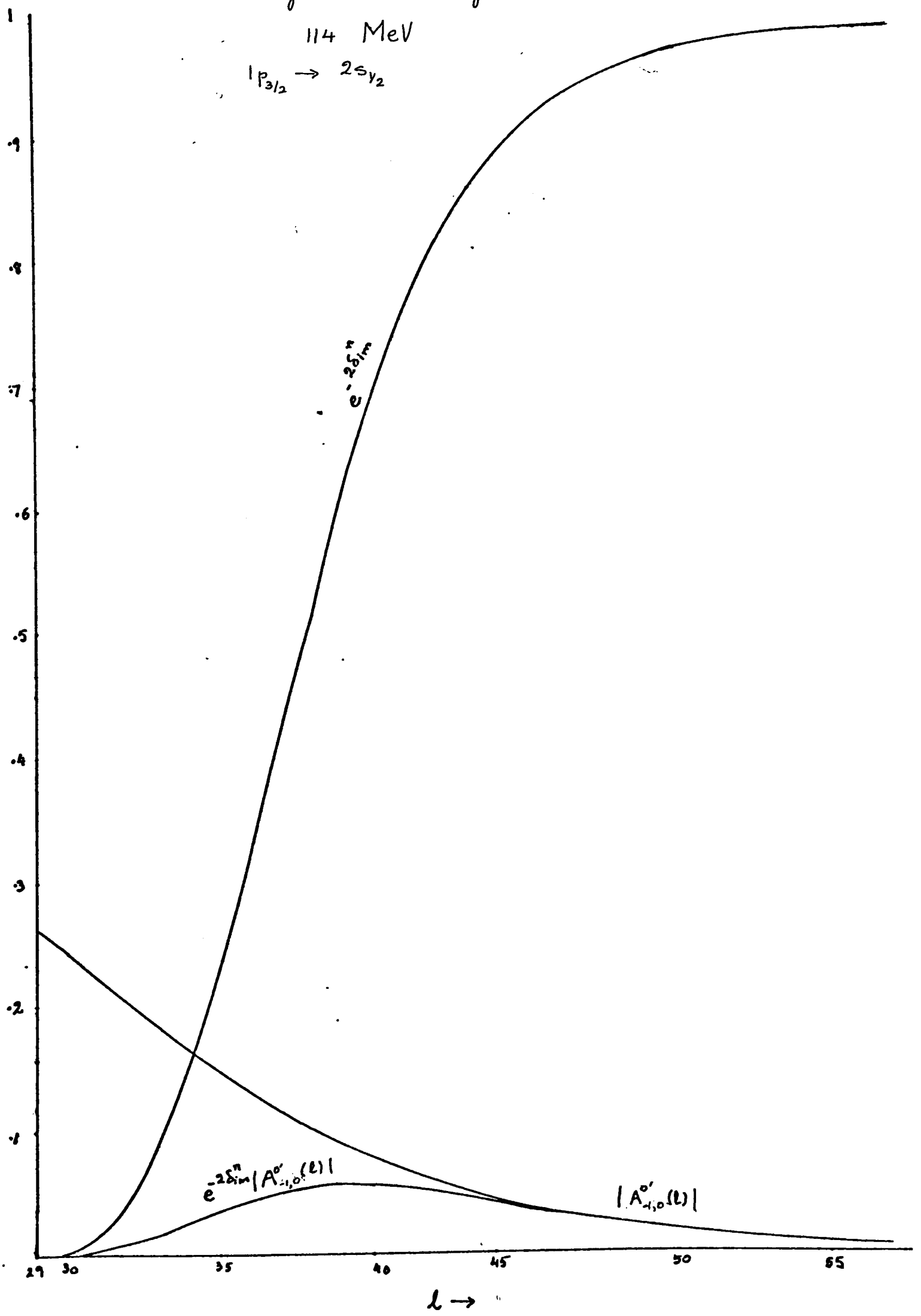

$^{26}Mg\,(^{11}B, ^{10}B)\,^{27}Mg$
114 MeV
$1p_{3/2} \rightarrow 2s_{1/2}$
1
·9
·8
·7
·6
·5
·4
·3
·2
·1
$e^{-2\delta_{im}^{n}}|A^{0'}_{-1,0}(l)|$
$|A^{0'}_{-1,0}(l)|$
29
30
35
40
45
50
55
$l \rightarrow$

Fig. 14

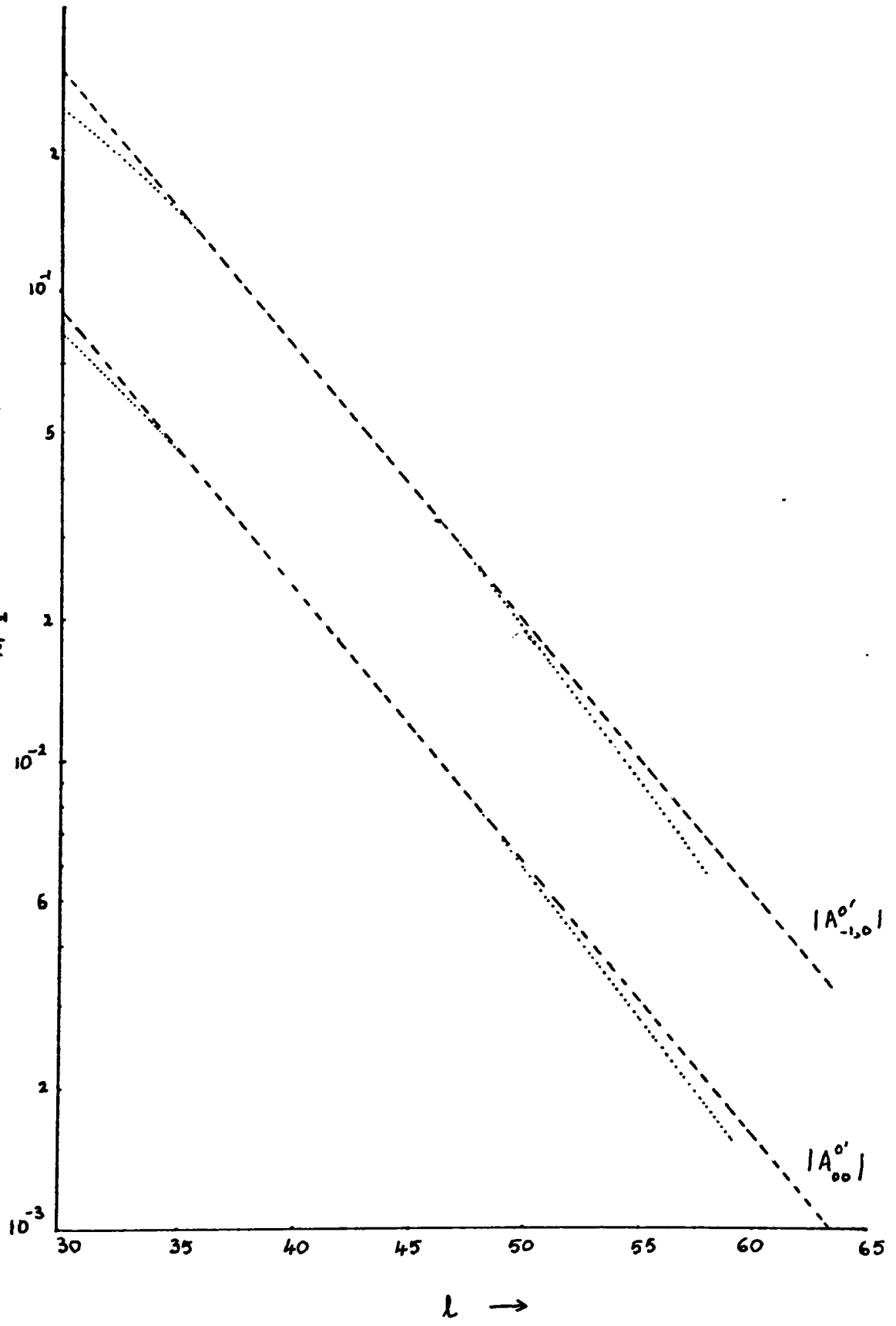
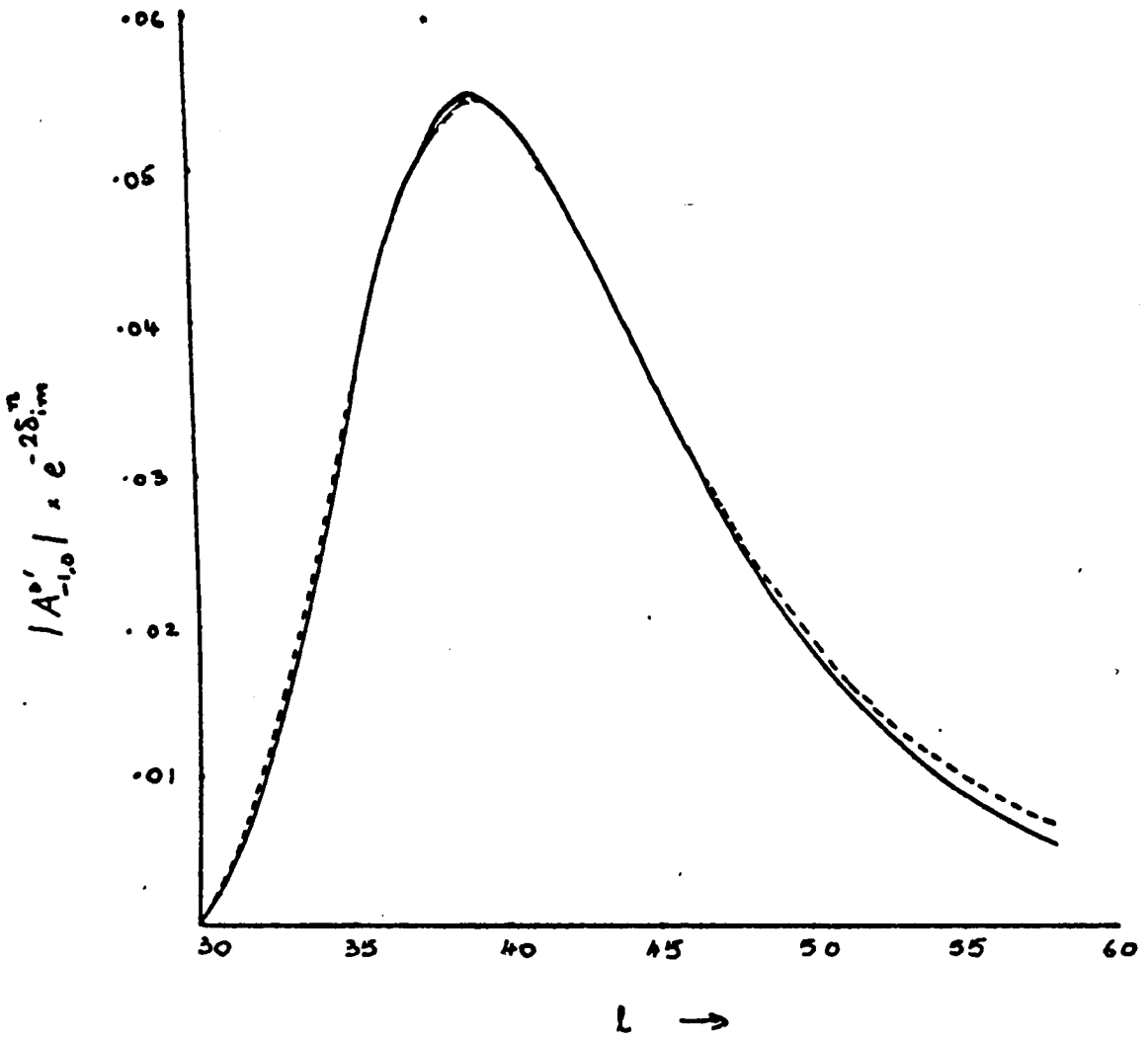


Fig. 15a

A comparison of $|A^{o'}_{\lambda_1',\lambda_2'}|$ calculated from eqs.(II.3.6) and (IV.3.1) for transfer of a neutron from a 1 $p_{3/2}$ state in ^{11}B to a 2 $s_{\frac{1}{2}}$ state in ^{27}Mg for (λ_1',λ_2') equal to (-1,0) and (0,0). The dotted curve in each case corresponds to the calculation using eq. (II.5.30) while the dashed curve to the parametrised formula eq.(IV.3.1). The values of the parameters are: Λ_0=20.9,Δ =7.44, $|C'_{-1,0}|$, $|C'_{oo}|$ = .268.



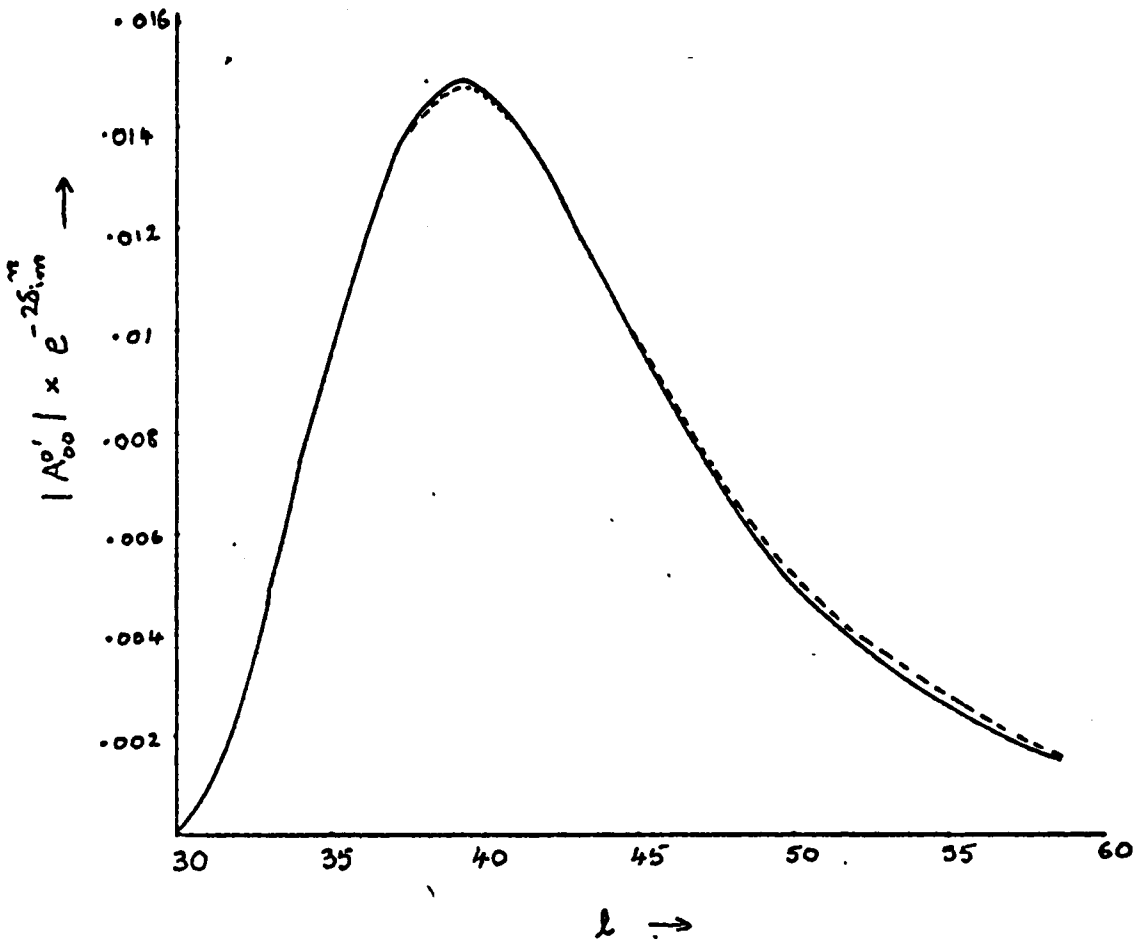


Fig. 15b

A comparison of $e^{-2\delta^n_{i,m}} \times |A^{o'}_{\lambda_1',\lambda_2'}|$ calculated using eqs. (II.3.6) and (IV.3.1) for the transfer of a neutron from a 1 $p_{3/2}$ state in ^{11}B to 2 $s_{\frac{1}{2}}$ state in ^{27}Mg for (λ_1',λ_2') equal to (-1,0) and (0,0). The solid curve in each case corresponds to the calculation using eq. (II.3.6) while the dashed curve to the parametrised formula (IV.3.1). Values of parameter were: $\Lambda_o = 20.9$, $\Delta = 7.44$, $|C'_{-1,o}| = 1$, $|C'_{o,o}| = .268$.

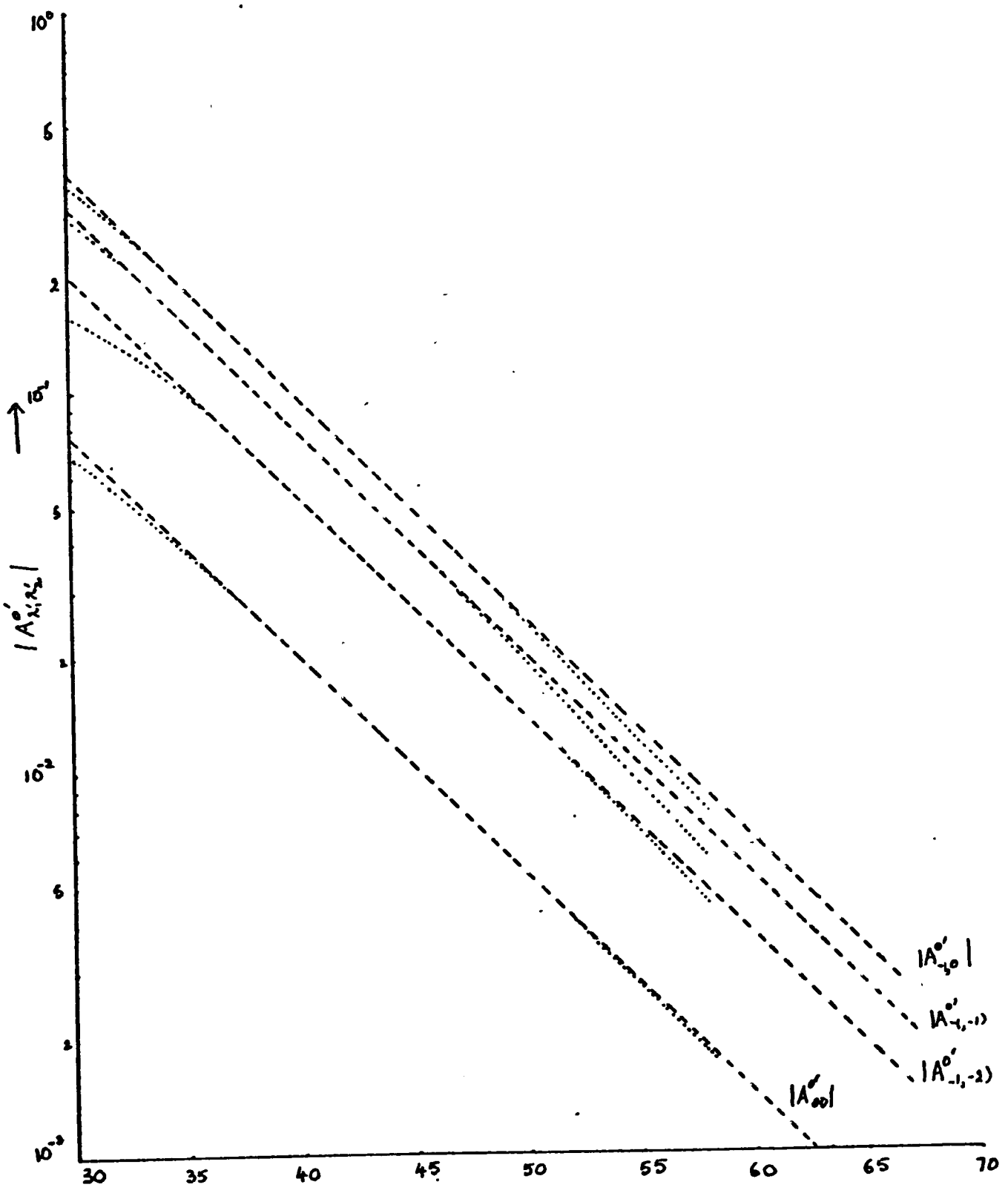


Fig. 15c

A comparison of $|A^{o'}_{\lambda_1',\lambda_2'}|$ calculated from eqs. (II.3.6) and (IV.3.1) for the transfer of a neutron from a 1 $p_{3/2}$ state in ^{11}B to a 1 $d_{3/2}$ state in ^{27}Mg for (λ_1', λ_2') equal to (0,0), (-1,-2) (-1,-1), (-1,0). The dotted curves correspond to calculations using eq. (II.3.6) while the dashed curve to the parametrised formula (IV.3.1). The values of the parameters are Λ_o=18.11, Δ=7.47, $|C'_{-1,-2}|$=1, $|C'_{-1,-1}|$=1.43 , $|C'_{-1,0}|$=1.77,

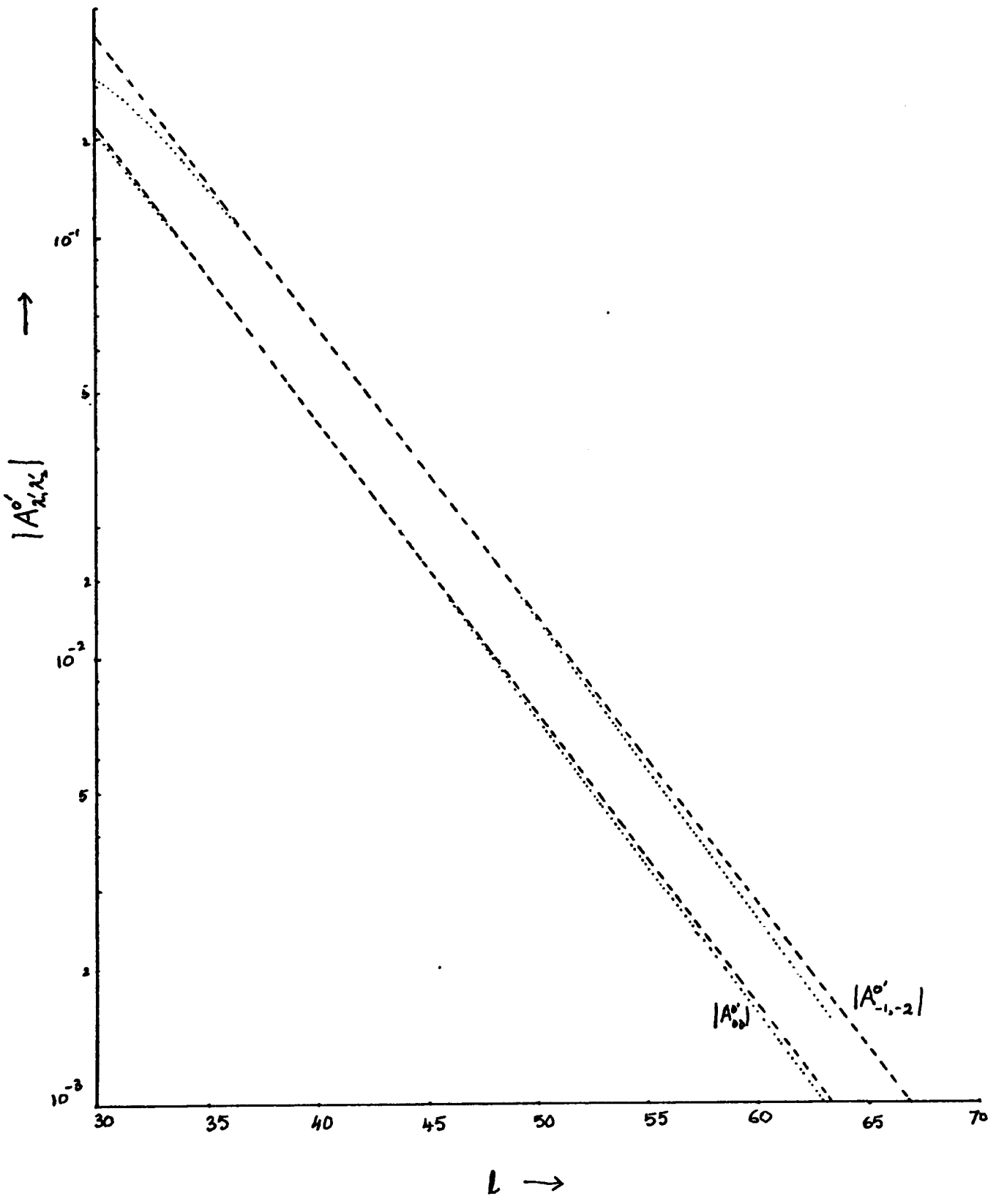


Fig. 16a

A comparison of $|A^{o'}_{\lambda_1',\lambda_2'}|$ calculated from eqs. (II.3.6) and (IV.3.1) for transfer of a proton from a 1 $p_{3/2}$ state in ^{11}B to a 1 $d_{5/2}$ state in ^{27}Al for values of (λ_1', λ_2') equal to (−1, −2) and (0, 0). The dotted curves correspond to calculations using eq. (II.5.30) while the dashed curves to the parametrised formula (IV.3.1). The values of the parameters are: $\Lambda_o = 21.18$, $\Delta = 6.60$, $|C'_{-1,-2}|=1$, $|C'_{oo}|= .590$.

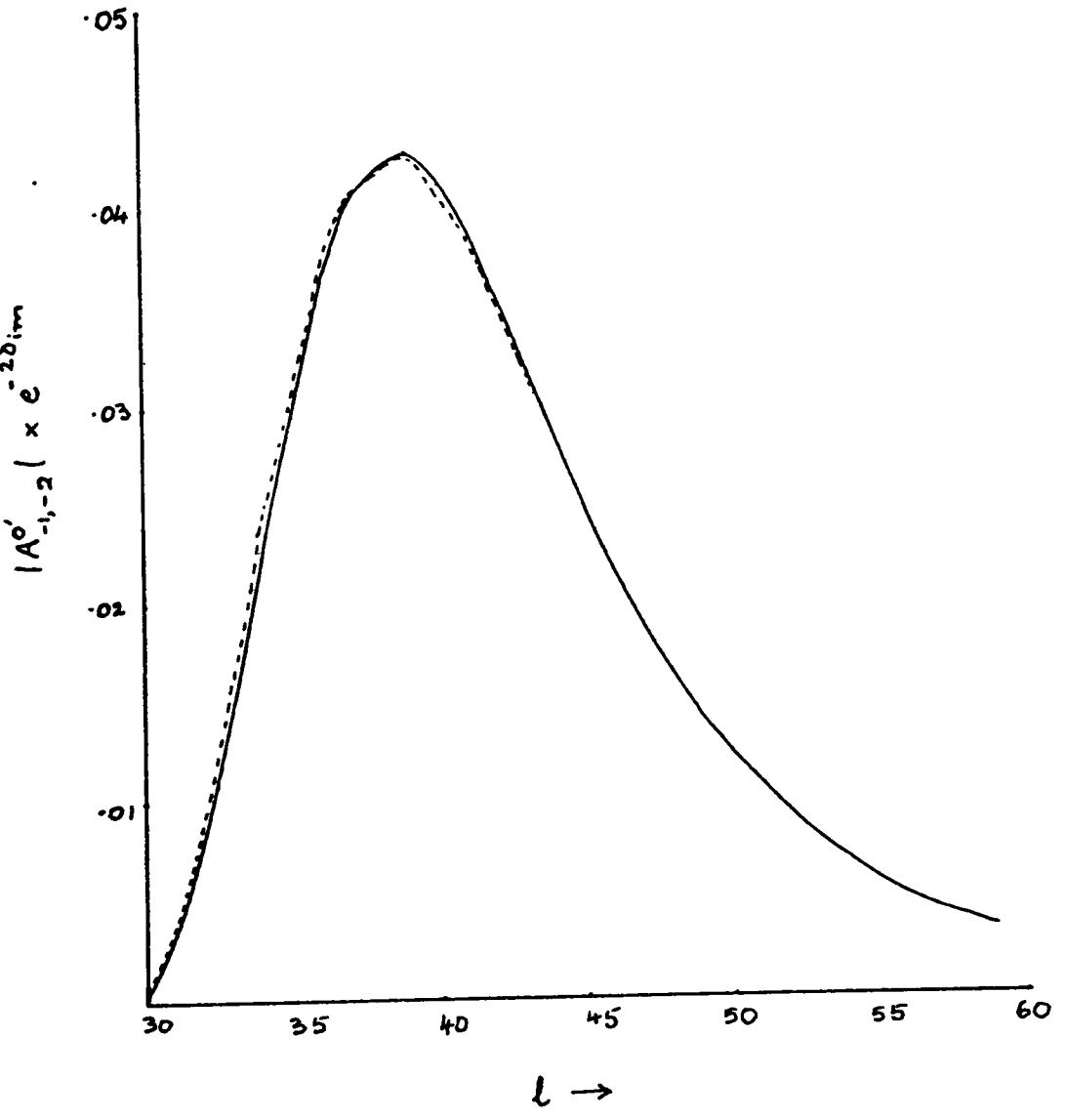


Fig. 16b

A comparison of $e^{-2\delta^n_{im}} \times |A^{o'}_{\lambda_1',\lambda_2'}|$ calculated using eqs. (II.3.6) and (IV.3.1) for the transfer of a proton from a 1 $p_{3/2}$ state in ^{11}B to a 1 $d_{5/2}$ state in ^{27}Al for (λ_1', λ_2') equal to (-1, -2). The solid curve corresponds to calculations using eq. (II.5.30) while the dashed curve to the parametrised formula (IV.3.1). The values of the parameters are: $\Lambda_o = 21.18$, $\Delta = 6.60$, $|C'_{-1,-2}| = 1$.

IV.4. Calculation of Angular Distributions and Comparison with Experiment:

Using the above approximation we have calculated with some success, angular distributions for the reactions (IV.1.1.) and (IV.1.2). The results are presented in figs. 17 and 18. In each case the cross sections are normalised by the corresponding spectroscopic factors obtained by comparison with experiment and plotted on the same scale as the experimental results (Paschopoulos et. al., 1975). The fit is generally seen to be good. Some angular distributions were also obtained with $A^{o'}_{\lambda'_1\lambda'_2}(\ell)$ calculated from eq. (II.3.6) for all values of ℓ instead of using the exponential approximation eq. (IV.3.1). The results are not plotted separately as there was an almost complete overlap between them and those shown in figs. 17 and 18.

A comparison of the spectroscopic factors obtained from the present calculations is made with those obtained from the exact finite range code LOLA of De Vries (De Vries, 1973) in table IV. The agreement is good.

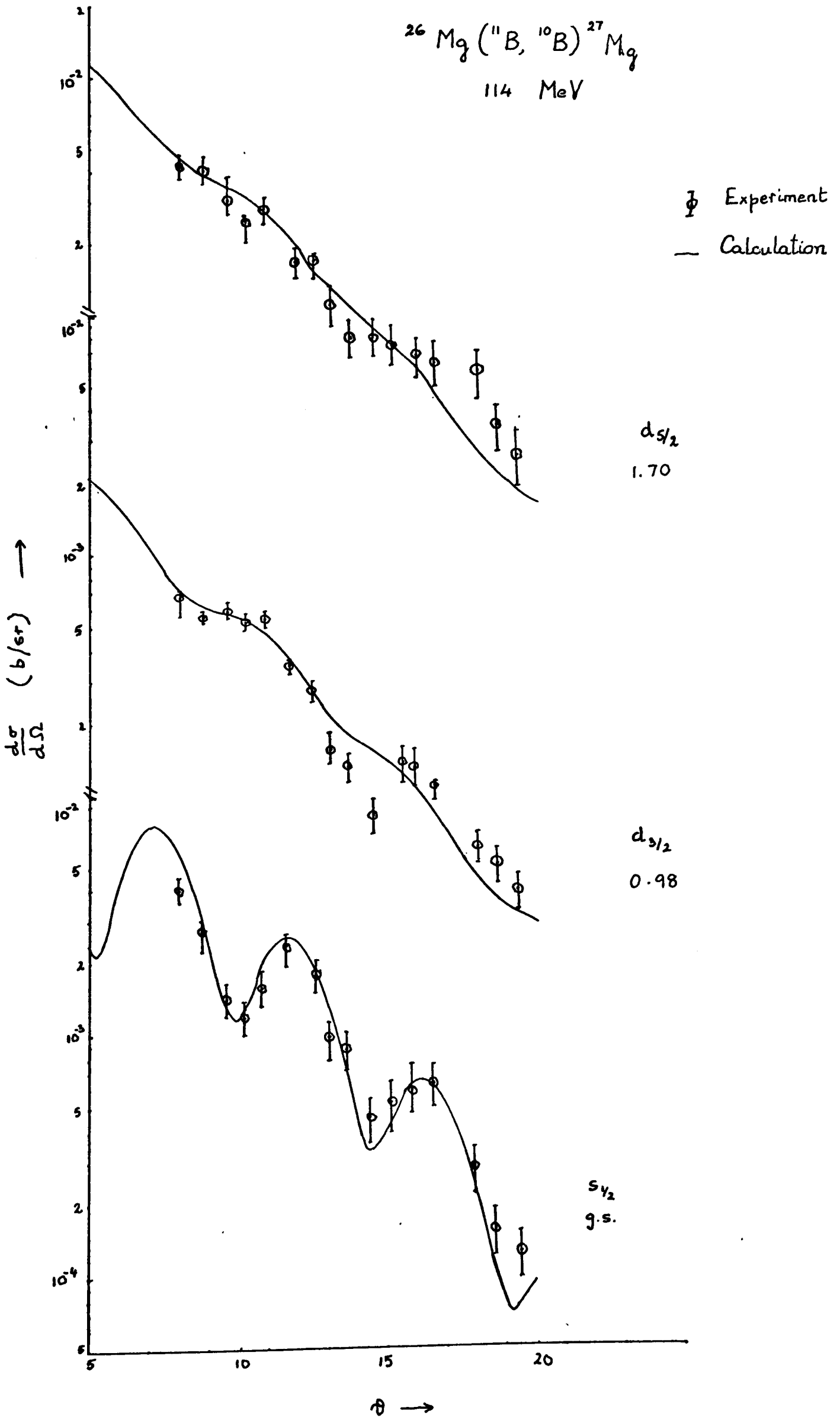

$^{26}Mg\,(^{11}B,\,^{10}B)\,^{27}Mg$
114 MeV
Experiment
Calculation
$d_{5/2}$
1.70
$d_{3/2}$
0.98
$s_{1/2}$
g.s.
$\frac{d\sigma}{d\Omega}$ (b/sr) →
θ →
10^{-2}
10^{-3}
10^{-4}
5
10
15
20

Figure 17

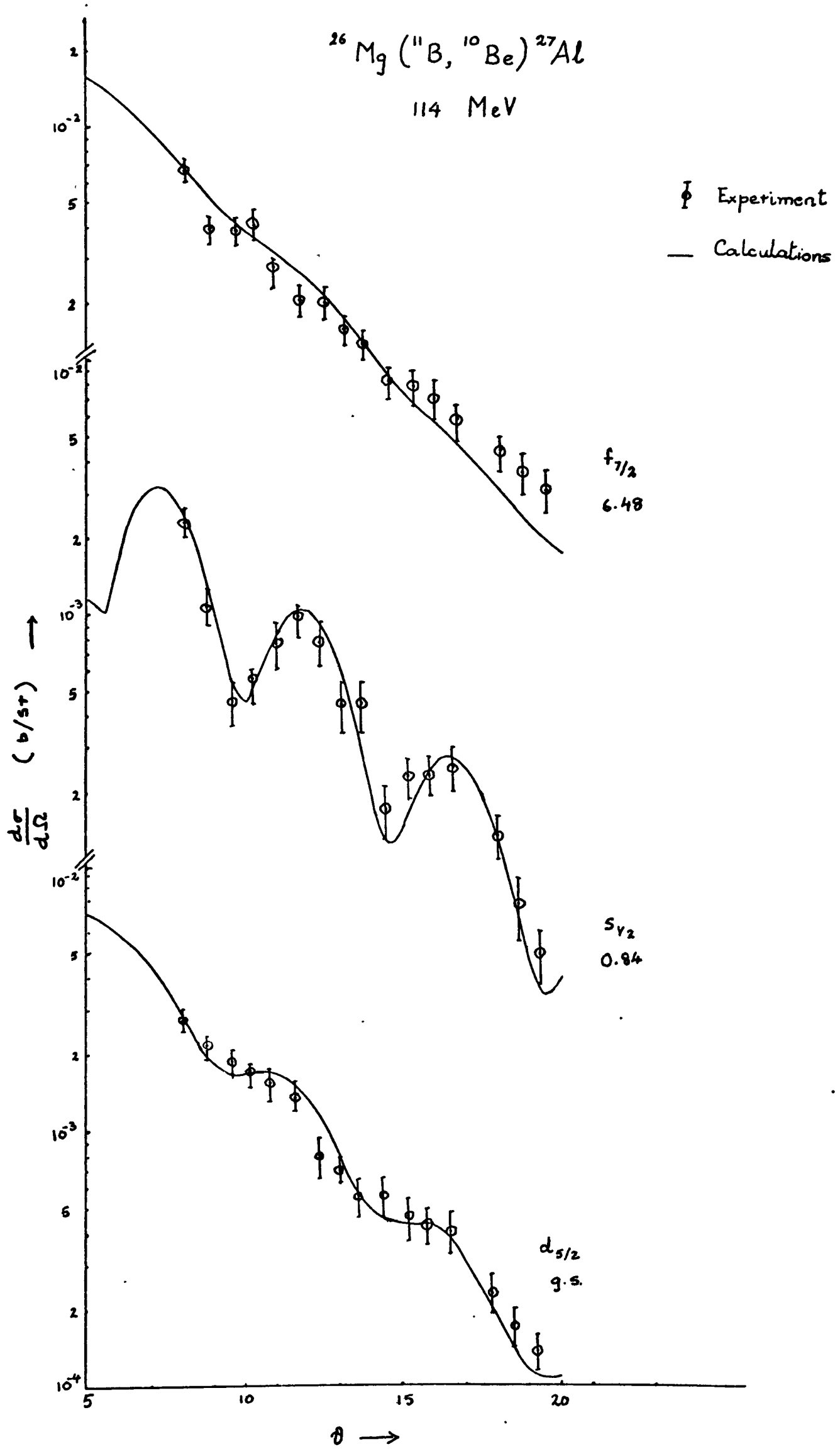

$^{26}Mg\ (^{11}B, ^{10}Be)\ ^{27}Al$
114 MeV
Experiment
Calculations
$f_{7/2}$
6.48
$s_{1/2}$
0.84
$d_{5/2}$
g.s.
$\frac{d\sigma}{d\Omega}$ (b/sr) →
θ →
2
10^{-2}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-2}
5
2
10^{-3}
6
2
10^{-4}
5
10
15
20

Figure 18

Reaction	E	J	Spectroscopic Factor	
			LOLA	Present Calculations
$^{26}Mg(^{11}B,^{10}Be)^{27}Al$	0	$5/2^+$	0.15	0.19
	0.84	$1/2^+$	0.20	0.18
	6.48	$7/2^-$	0.20	0.22
$^{26}Mg(^{11}B,^{10}B)^{27}Mg$	0.0	$1/2^+$	0.62	0.62
	0.98	$3/2^+$	0.50	0.37
	1.70	$5/2^+$	0.20	0.22

Table IV

In the following section we attempt to obtain a closed formula for $T_{if}^{\lambda_1\lambda_2}(\vartheta)$ by parametrising the nuclear phases as well.

IV.5. Closed Expression for Angular Distributions

The expression for $T_{if}^{\lambda_1\lambda_2}(\vartheta)$ (Eq. (III.1.60)) involves an evaluation of the amplitudes $A^{\circ}_{\lambda_1\lambda_2}(\ell)$ and the phase shifts δ_ℓ at a number of points till the summation converges. However, if we look at the product $e^{-2\delta_{im}^{n}(\ell)} \times A^{\circ}_{\lambda_1\lambda_2}$ (e.g. figs. 15b, 16b) we see that its absolute value is peaked around a certain ℓ-value. This suggests that we can approximate it by a relatively simple parametrised formula. We have already seen that $A^{\circ}_{\lambda_1\lambda_2}(\ell)$ can be replaced by:

$$A^{0}_{\lambda_1\lambda_2}(\ell) = C_{\lambda_1\lambda_2}\, e^{\frac{\Lambda_0-\ell}{\Delta}} \qquad \text{(IV.2.1)}$$

A formula that is seen to work fairly well for the nulcear phase shifts is (putting $\delta(\ell) = \delta^{(i)}(\ell) + \delta^{(f)}(\ell)$):

$$e^{2i\delta^{n}(\ell)} = \frac{1}{1 + e^{(\Lambda_0' + i\Lambda_1 - \ell)/\Delta'}} \qquad \text{(IV.5.1)}$$

Substituting eqs. (IV.2.1) and (IV.5.1) into (III.1.60) we get (in the reaction plane):

$$T_{if}^{\lambda_1\lambda_2}(\theta) = \frac{\hbar^2}{2\sqrt{\mu_i k_i \mu_f k_f}}\, e^{i(\lambda_1-\lambda_2)\frac{\pi}{2}} \sum_{\ell} (2\ell+1)\, e^{2i\delta_\ell^{c}} C_{\lambda_1\lambda_2}\, e^{(\Lambda_0-\Lambda_0')/\Delta} \times \frac{e^{\frac{\Lambda_0'-\ell}{\Delta}}}{1 + e^{\frac{\Lambda_0' + i\Lambda_1 - \ell}{\Delta'}}} \times J_{\lambda_1-\lambda_2}\!\left((2\ell+1)\sin\tfrac{\theta}{2}\right) \qquad \text{(IV.5.2)}$$

We can convert the sum to an integral by taking ℓ to be a continuous variable λ such that $\lambda = \ell + \frac{1}{2}$. We get:

$$T_{if}^{\lambda_1\lambda_2}(\theta) = B_{\lambda_1\lambda_2} \int_0^{\infty} \lambda\, e^{2i\delta^{c}(\lambda)} \frac{e^{\frac{\bar{\Lambda}_0-\lambda}{\Delta}}}{1 + e^{\frac{\bar{\Lambda}_0 + i\Lambda_1 - \lambda}{\Delta'}}} J_{\lambda_1-\lambda_2}(\lambda\theta)\, d\lambda, \qquad \text{(IV.5.3)}$$

where $$B_{\lambda_1\lambda_2} = \frac{\hbar^2}{2\sqrt{\mu_i k_i \mu_f k_f}}\, e^{i(\lambda_1-\lambda_2)\frac{\pi}{2}}\, e^{(\Lambda_0-\Lambda_0')/\Delta}\, C_{\lambda_1\lambda_2}$$

$$\bar{\Lambda}_0 = \Lambda_0 + \tfrac{1}{2}$$

and $\sin\theta/2 \simeq \theta/2$, which is valid for small angle scattering.

Expanding the Coulomb phase shift $\delta_\lambda^{\,c}$ about the point Λ around which the integral gets its maximum contribution and retaining terms up to first order in λ we get:

$$2\delta^c(\lambda) \approx 2\delta^c(\Lambda) + (\lambda-\Lambda)\,\Theta_\Lambda^c \quad ,$$

where Θ_Λ^c is the "critical angle" defined as the value of the quantal deflection function $\Theta^c(\lambda) = \frac{d2\delta(\lambda)}{d\lambda}$ at $\lambda = \Lambda$.

$$\Theta_\Lambda^c = 2\tan^{-1}\left(\frac{1}{\Lambda}\right)$$

where η is defined in eq. (IV.1.4).

Eq. (IV.5.3) now becomes:

$$T_{if}^{\lambda_1\lambda_2}(\vartheta) = B_{\lambda_1\lambda_2} e^{2i\delta^c(\Lambda)} \Lambda \int_0^\infty \frac{e^{\frac{\bar{\lambda}_0-\lambda}{\Delta}}}{1+e^{\frac{\bar{\lambda}_0+i\Lambda_1-\lambda}{\Delta'}}} e^{i(\lambda-\Lambda)\Theta_\Lambda^c} J_{\lambda_1-\lambda_2}(\lambda\vartheta)\,d\lambda \qquad \text{(IV.5.4)}$$

To evaluate this integral we shall employ Frahn's approach in the calculation of a similar integral (Frahn, 1973). First of all the Bessel function is replaced by its integral representation:

$$J_n(x) = \frac{\left(\frac{1}{2}x\right)^n}{\pi^{1/2}\,\Gamma\left(n+\frac{1}{2}\right)} \int_{-1}^{+1} (1-t^2)^{n-\frac{1}{2}}\, e^{itx}\, dt \qquad \text{(IV.5.5)}$$

giving:

$$T_{if}^{\lambda_1\lambda_2}(\vartheta) = \frac{\frac{1}{2}(\vartheta\lambda)^{\lambda_1-\lambda_2}}{\pi^{1/2}\,\Gamma(\lambda_1-\lambda_2+\frac{1}{2})} B_{\lambda_1\lambda_2} \Lambda e^{2i\delta_\Lambda^c} \int_{-1}^{+1} dt\,(1-t^2)^{\lambda_1-\lambda_2-\frac{1}{2}} \times$$

$$\times \int_0^\infty e^{it\lambda\vartheta}\, e^{i(\lambda-\Lambda)\Theta_\Lambda^c} \frac{e^{\frac{\bar{\lambda}_0-\lambda}{\Delta}}}{1+e^{\frac{\bar{\lambda}_0+i\Lambda_1-\lambda}{\Delta'}}} d\lambda$$

$$= B_{\lambda_1\lambda_2} \frac{\Lambda \left(\frac{1}{2}\vartheta\Lambda\right)^{\lambda_1-\lambda_2}}{\pi^{1/2}\,\Gamma(\lambda_1-\lambda_2+\frac{1}{2})} e^{2i\delta^c(\Lambda)} \int_{-1}^{+1} dt\,(1-t^2)^{\lambda_1-\lambda_2-\frac{1}{2}} e^{i\vartheta\Lambda t} \times F\left[(\Theta^c_\Lambda + \vartheta t)\Delta'\right]$$

(IV.5.6)

where

$$F\left[(\Theta^{c'}_\Lambda + \vartheta t)\Delta'\right] = \int_0^\infty d\lambda \frac{e^{\frac{\bar{\lambda}_0-\lambda}{\Delta}}}{1+e^{\frac{\bar{\lambda}_0+i\Lambda_1-\lambda}{\Delta'}}} e^{i(\lambda-\Lambda)(\Theta_\Lambda+\vartheta t)}$$

$$\equiv F(t)$$

(IV.5.7)

Let us now write $F_1(t)$ and $F_2(t)$ as even functions of t such that:

$$F_1(t) = \frac{1}{2}\left[F(t) + F(-t)\right]$$

$$F_2(t) = \frac{1}{2\vartheta\Delta' t}\left[F(t) - F(-t)\right]$$

and

$$F(t) = F_1(t) + \vartheta\Delta' t\, F_2(t)$$

(IV.5.8)

We can expand $F_{1,2}(t)$ around $t^2 = 1$ in a Taylor's series to give:

$$F_{1,2}(t) = \sum_n \left(\frac{\vartheta\Delta'}{n!}\right)^{2n} F^{(n)}_{1,2}\,(t^2-1)^n \quad ,$$

(IV.5.9)

where

$$F_{1,2}^{(n)} = \left(\partial^n F_{1,2} \,/\, \partial [(\vartheta\Delta' t)^2]^n \right)_{t^2=1}$$

Substituting eq. (IV.5.8) and (IV.5.9) into (IV.5.6) we get:

$$T_{if}^{\lambda_1\lambda_2}(\vartheta) = B_{\lambda_1\lambda_2} \Lambda \left(\tfrac{1}{2}\Lambda\vartheta\right)^{\lambda_1-\lambda_2} e^{2i\delta^c(\Lambda)} \times$$

$$\times \pi^{1/2}\, \Gamma(\lambda_1-\lambda_2+\tfrac{1}{2}) \times$$

$$\times \sum_n \int_{-1}^{+1} dt\, (1-t^2)^{\lambda_1-\lambda_2+\frac{1}{2}} e^{i\vartheta\Lambda t} \left(\frac{\vartheta\Delta'}{t}\right)^{2n} (t^2-1)^n \left[F_1^{(n)} + \vartheta\Delta' F_2^{(n)}\right]$$

(IV.5.10)

Simplifying the above expression and again using the integral representation (IV.5.5) of the Bessel function we get:

$$T_{if}^{\lambda_1\lambda_2}(\vartheta) = B_{\lambda_1\lambda_2} \Lambda\, e^{2i\delta^c(\Lambda)} \sum_{n=0}^{\infty} \left(\frac{-\Delta'}{\Lambda}\right)^n \frac{\Gamma(\lambda_1-\lambda_2+n+\frac{1}{2})}{\Gamma(\lambda_1-\lambda_2+\frac{1}{2})\, n!} \left[(2\Delta'\vartheta)^n F_1^{(n)} J_{\lambda_1-\lambda_2+n}(\vartheta\Lambda) + \tfrac{1}{2} i\, (2\Lambda\vartheta)^{n+1} F_2^{(n)} J_{\lambda_1-\lambda_2+n+1}(\vartheta\Lambda)\right]$$

Under conditions of strong absorption $\frac{\Delta'}{\Lambda}$ is small and we may retain the term only with $n = 0$. We get

$$T_{if}^{\lambda_1\lambda_2}(\vartheta) = B_{\lambda_1\lambda_2} \Lambda\, e^{2i\delta^c(\Lambda)} \left[F_1 J_{\lambda_1-\lambda_2}(\Lambda\vartheta) + i\vartheta\Delta' F_2 J_{\lambda_1-\lambda_2+1}(\vartheta\Lambda)\right]$$

(IV.5.11)

where F_1 and F_2 now are:

$$F_1 = \left[\tfrac{1}{2}\left(F(t) + F(-t)\right)\right]_{t=1}$$

$$F_2 = \left[\frac{1}{2\vartheta\Delta' t}\left(F(t) - F(-t)\right)\right]_{t=1}$$

Calculation of F(t):

$$F(\pm t) = \int d\lambda \frac{e^{\frac{\bar{\Lambda}_0 - \lambda}{\Delta}}}{1 + e^{\frac{\bar{\Lambda}_0 + i\Lambda_1 - \lambda}{\Delta}}} e^{i(\lambda-\Lambda)(\Theta_\Lambda \pm \vartheta t)}$$

From now on we shall denote $\Theta_\Lambda \pm \vartheta t$ by x for convenience. To evaluate the integral we perform a contourintegration in the complex λ-plane. For positive x we close the contour by a semi-circle in the upper half plane, while for negative x we close it by a semi-circle in the lower half plane (fig. 19)

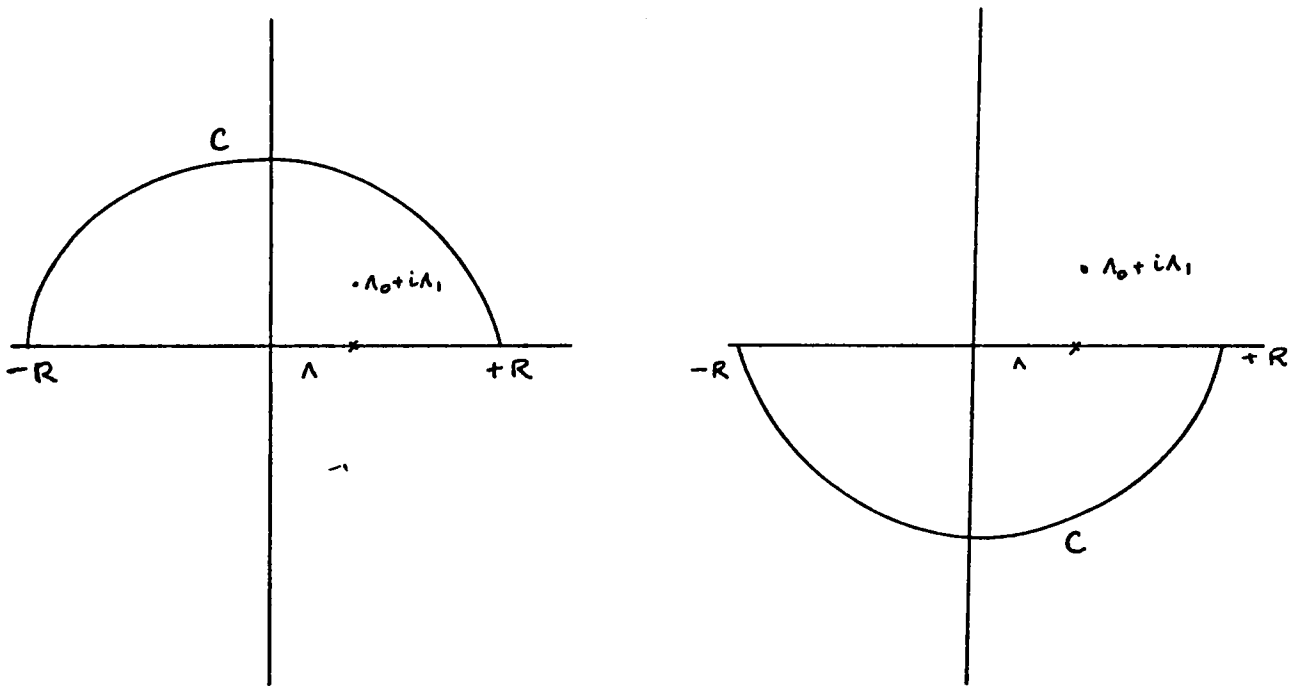


Fig. 19

It is easy to see that the contribution along the semi-circle C vanishes in either case.

The value of the integral, therefore is:

$$F = 2\Pi i \sum (\text{residues})^{\pm}$$

The poles of the integrand are at:

$$\frac{\bar{\Lambda}_0 + i\Lambda_1 - \lambda}{\Delta'} = \pm (2n+1)\pi i \qquad n = 0, 1, 2, \ldots$$

Since the residues are symmetric in the upper and lower plane, the value of the integral will be the same for positive and negative x.

We shall evaluate for positive x, i.e. we shall calculate the residues in the upper half plane.

$$\therefore F = 2\pi i \sum_{n=0}^{\infty} e^{i(\lambda_n - \lambda)x} e^{\frac{\bar{\Lambda}_0 - \lambda_n}{\Delta}},$$

where, $\lambda_n = \bar{\Lambda}_0 + i\Lambda_1 + (2n+1)\pi\Delta' i$

$$\therefore F = 2\pi i \Delta' \sum_n e^{-i[\Lambda_1 + (2n+1)\pi\Delta']/\Delta} e^{i(\bar{\Lambda}_0 + i\Lambda_1 + (2n+1)\pi\Delta i - \lambda)x}$$

$$= \pi i \Delta' e^{-i\left(\frac{\Delta_1}{\Delta} - (\bar{\Lambda}_0 - \lambda)x - i\Lambda_1 x\right)} \cdot 2 \sum_n e^{-(2n+1)\pi\Delta'\left(\frac{i}{\Delta} + x\right)}$$

$$= i\Delta'\pi\, e^{-i\left(\frac{\Delta_1}{\Delta} - (\bar{\Lambda}_0 - \lambda)x\right)} e^{-\Lambda_1 x} \frac{1}{\sinh\left[\pi\Delta'\left(\frac{i}{\Delta} + x\right)\right]}$$

$$\therefore F_1 = \frac{i\Delta'\pi}{2} e^{-i\left[\frac{\Delta_1}{\Delta} - (\bar{\Lambda}_0 - \lambda)(\Theta_\lambda + \vartheta)\right]} e^{-\Lambda_1(\Theta_\lambda + \vartheta)} \times$$

$$\times \left[\frac{1}{\sinh\left[\pi\Delta'\left(\frac{i}{\Delta} + \Theta_\lambda + \vartheta\right)\right]} + \frac{e^{-2[i(\bar{\Lambda}_0 - \lambda)\vartheta - \Lambda_1\vartheta]}}{\sinh\left[\Delta'\pi\left(\frac{i}{\Delta} + \Theta - \vartheta\right)\right]}\right]$$

$$F_2 = \frac{i\Delta'\pi}{2\vartheta\Delta'} e^{-i\left[\frac{\Delta_1}{\Delta} - (\bar{\Lambda}_0 - \lambda)(\Theta_\lambda + \vartheta)\right]} e^{-\Lambda_1(\Theta_\lambda + \vartheta)} \times$$

$$\times \left[\frac{1}{\sinh\left[\pi\Delta'\left(\frac{i}{\Delta} + \Theta + \vartheta\right)\right]} - \frac{e^{-2[i(\bar{\Lambda}_0 - \lambda)\vartheta - \Lambda_1\vartheta]}}{\sinh\left[\pi\Delta'\left(\frac{i}{\Delta} + \Theta - \vartheta\right)\right]}\right]$$

IV.6 Calculation of $^{26}Mg(^{11}B, ^{10}B)^{27}Mg$ Angular Distribution with the Closed Formula

To evaluate $T^{\lambda_1\lambda_2}(\theta)$ from eq. (IV.5.11) we have to first of all obtain values for the parameters Λ_o', Δ' and Λ_1 used to parametrise the nuclear phase shift, eq. (IV.5.1). This we do by fitting the phase shifts at two points ℓ_a and ℓ_b and simplifying , yielding for Δ' and Λ_o':

$$\Delta' = \frac{2(\ell_b - \ell_a)}{\ln\left[\frac{1 + e^{4\delta^n_{im}(\ell_a)} - 2e^{2\delta^n_{im}(\ell_a)}\cos 2\delta^n_r(\ell_a)}{1 + e^{4\delta^n_{im}(\ell_b)} - 2e^{2\delta^n_{im}(\ell_b)}\cos 2\delta^n_r(\ell_b)}\right]} \qquad \text{(IV.6.1)}$$

$$\text{and } \Lambda_o' = \ell_a + \frac{\Delta'}{2}\ln\left[1 + e^{4\delta^n_{im}(\ell_a)} - 2e^{2\delta^n_{im}(\ell_a)}\cos 2\delta^n_r(\ell_a)\right] \qquad \text{(IV.6.2)}$$

ℓ_a and ℓ_b are chosen near the points where $e^{-2\delta^n_{im}}$ bends. They are adjusted to give the best fit.

Λ_1 is determined by the formula:

$$\tan\frac{\Lambda_1}{\Delta} = -\frac{\sin\delta^n_r(\ell)}{\cos 2\delta^n_r(\ell) - e^{-2\delta^n_{im}(\ell)}} \qquad \text{(IV.6.3)}$$

This is determined at a point ℓ which is roughly midway between ℓ_a and ℓ_b. $\frac{\Lambda_1}{\Delta}$ is found to be fairly constant around this point.

We have evaluated angular distributions for the transfer of a neutron from the 1 $p_{3/2}$ state in ^{11}B to the 2 $s_{\frac{1}{2}}$ state in ^{27}Mg for the reaction (IV.1.1). The values of Λ_0', Δ', Λ_1 in this case were: $\Lambda_0' = 38.733$, $\Delta' = 2.455$, $\Lambda_1 = -.948$, obtained by fitting the imaginary phase shifts as $\ell = 33$ and $\ell_b = 40$. We take $\Lambda = \bar{\Lambda}_0 = 39.233$. The results plotted together with those obtained from eq. (III.1.60), eq. (IV.3.1) (fig. 19) show that there is reasonable agreement between the two.

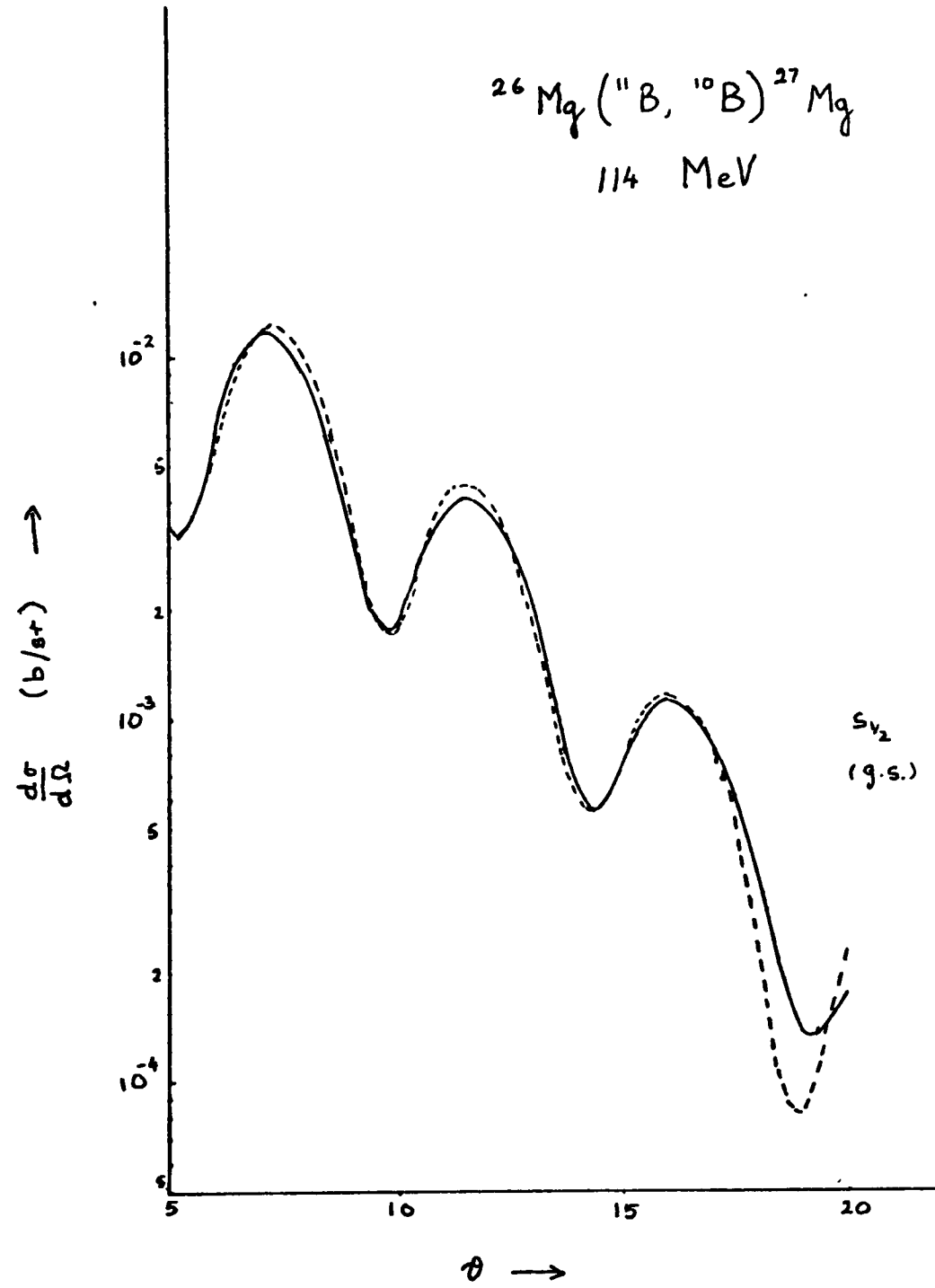


Figure 19

Comparison of angular distributions obtained from closed formula and full summation, for the reaction $^{26}Mg(^{11}B, ^{10}B)^{27}Mg$. The solid line denotes the calculation with the full sum while the dashed line the closed formula.

CHAPTER V

CALCULATION OF ANGULAR DISTRIBUTIONS FOR THE REACTION $^{16}O(^{17}O^*, ^{16}O)^{17}O^*$

V.I. Introduction

The instigation for carrying out the calculations described in this chapter came from a curious result quoted by Crowley (Crowley,1976a). He studied the theoretical reaction $^{17}O^*(^{16}O, ^{17}O^*)^{16}O$ to test his semiclassical theory for transfer reactions. Angular distributions were calculated using the semiclassical theory as well as the quantum mechanical DWBA code LOLA for a range of energies from 35 MeV to 200 MeV. A comparison of the results showed that although the shape of the angular distribution curves agreed, there was a systematic change in relative magnitudes which could be described quite well by the formula:

$$\frac{\sigma_W}{\sigma_L} = \exp\left\{\left(\frac{26.3}{E_i}\right)^{3/2}\right\} \qquad (V.1.1)$$

where σ_W , σ_L are the cross sections for the semiclassical theory and LOLA calculations respectively, and E_i is the centre of mass energy. No explanation was given for the result.

We decided to repeat the above calculations using our theory and to see if we got a similar disagreement. In the following sections we present our calculations and results.

V.2. Details of Calculation

We study the reaction $^{16}O(^{17}O^*, O^{16})^{17}O^*$ in which a single neutron is elastically transferred between $\frac{1}{2}^+$, 0.87 MeV first excited states of $^{17}O^*$. The choice of a stripping reaction rather than the corresponding pick up reaction $^{17}O^*(^{16}O, ^{17}O^*)^{16}O$ studied by Crowley is purely of convenience, as the kinematics in our computer program are for stripping rather than for pick up reactions. Calculations were perfromed for laboratory energies $E_{lab}(^{17}O^*)$ = 212.5, 106.25, 83.86, 53.125, 37.1875 MeV. These correspond to the laboratory energies $E_{lab}(^{16}O)$ = 200, 100, 80, 50, 35 MeV for the pick up reaction studied by Crowley. The bound state wave functions were calculated in a Woods Saxon well with radius parameter $R = r_o A^{\frac{1}{3}} (r_o = 1.35)$ and diffuseness a = 0.53. The elastic scattering phase shifts were obtained quantum mechanically from the computer program of Rae (Rae, 1976). The optical potential used was taken to have a pure Woods-Saxon form for the nuclear part:

[1]

$$V_n + iW_n = \frac{-V_R}{1 + e^{(t-R_R)/a_R}} - i\frac{V_I}{1 + e^{(t-R_I)/a_I}}$$

with the values of the parameters given by:

$$V_R = 30 \qquad V_I = 4.12558 + .25783E_i - .00100716E_i^2$$
$$r_{OR} = 1.3362 \qquad r_{OI} = 1.3362$$
$$a_R = 0.49 \qquad a_I = 0.3$$
$$R_R = 6.802 \qquad R_I = 6.802, \qquad (V.2.1)$$

where E_i is the centre of mass energy. A Coulomb potential derived from a uniform charge distribution of radius $R_c = 1.25(A_P^{\frac{1}{3}} + A_T^{\frac{1}{3}})$ was included (A_P, A_T are the masses of the projectile and target repsectively).

V.3. Discussion of Results

Figs. 20, 21, 22, 23, show a comparison of our results with LOLA for four different energies. (The results of Crowley (Crowley, 1976b) are also plotted for reference). The agreement generally seems to be good both in the shape and magnitude, though it does seem to be better as the energy increases. The shapes of Crowley's angular distributions, however, agree better with LOLA than do ours.

At 37.1875 MeV we get disagreement with LOLA in the shape of the angular distribution (fig. 24). This appears to be caused by the imaginary part of the optical potential which becomes quite shallow at this energy ($V_I = 8.45$ MeV). Our formula (eq. (III.I.60)) is rather sensitive to this depth, as it is the optical potential which accounts for the absorption and removes contributions from the interior. However, if it is not strong enought to do so, there are large contributions coming to the partial wave sum from

small values of ℓ. Indeed there is some contribution coming from the interior even at 53.125 MeV though it is quite small.

This effect is illustrated in figs. 25a, 25b for two different energies using the energy dependent optical potential (V.2.1). We have plotted $e^{-2\delta_{im}^{n}}$ and the product $e^{-2\delta_{im}^{n}} \times A_{oo}^{o'}$ (c.f. eqs. (III.1.60), (II.3.6)). For E_{lab} = 212.5 MeV (V_I = 20 MeV), this product is cut off for low ℓ-values as expected. However, for low energies (E_{lab} = 37.1875 MeV, V_I = 8.45 MeV) there is a sizeable contribution coming from the interior. This seems a possible reason why our formula does not give good results at low energies.

We should not really be surprised by the above observations as the assumption of a strongly absorbing potential is implicit in our derivation of eq. (III.1.60). In evaluating the radial integrals $I_{\ell_i \ell_f m_f}$ (c.f. chapter III) we have made the assumption that only the outermost turning point contributes to the integral. This condition holds when the imaginary part of the optical potential is strong. If it is too shallow, we get contributions from the interior turning points and our formula breaks down.

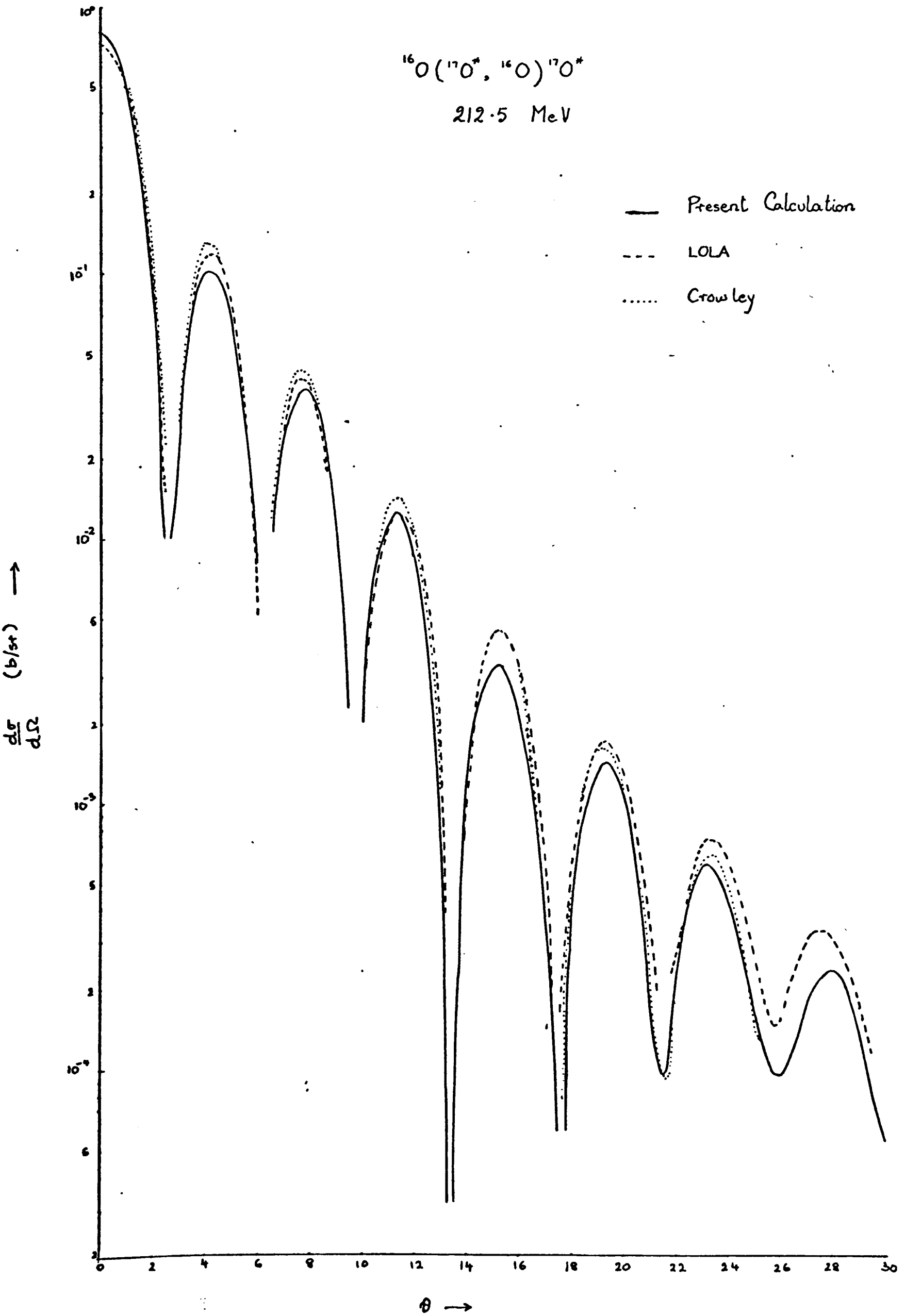

$^{16}O(^{17}O^{*}, ^{16}O)^{17}O^{*}$
212·5 MeV
Present Calculation
LOLA
Crowley
$\frac{d\sigma}{d\Omega}$ (b/sr)
$\theta \longrightarrow$

Figure 20

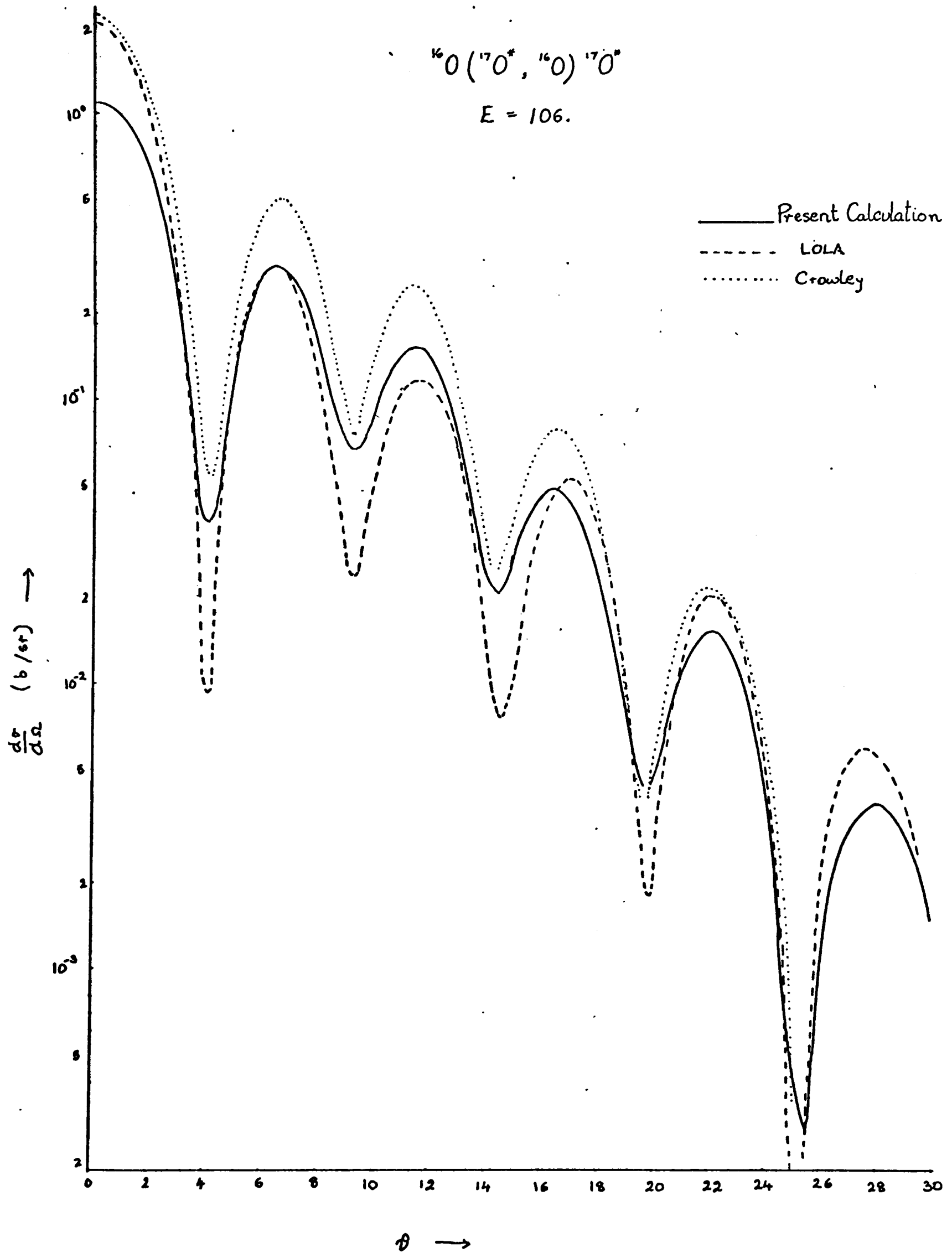

$^{16}O(^{17}O^{*}, ^{16}O)^{17}O^{*}$
E = 106.
Present Calculation
LOLA
Crowley
$\frac{d\sigma}{d\Omega}$ (b/sr)
$\vartheta \longrightarrow$
2
10^{0}
5
2
10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30

Figure 21

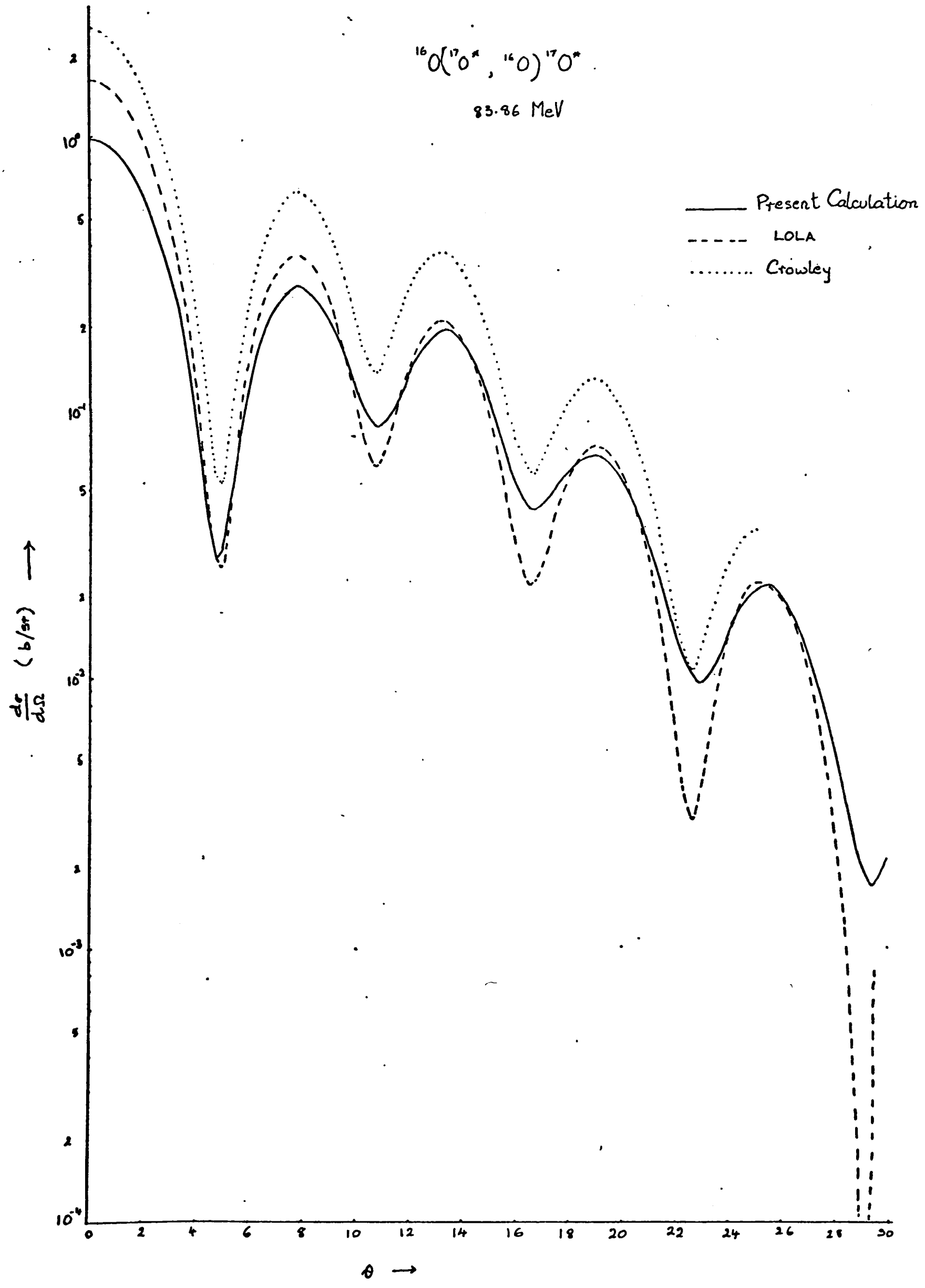

$^{16}O(^{17}O^*, ^{16}O)^{17}O^*$
83.86 MeV
Present Calculation
LOLA
Crowley
$\frac{d\sigma}{d\Omega}$ (b/sr) ⟶
2
10^0
5
2
10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-4}
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30
θ ⟶

Figure 22

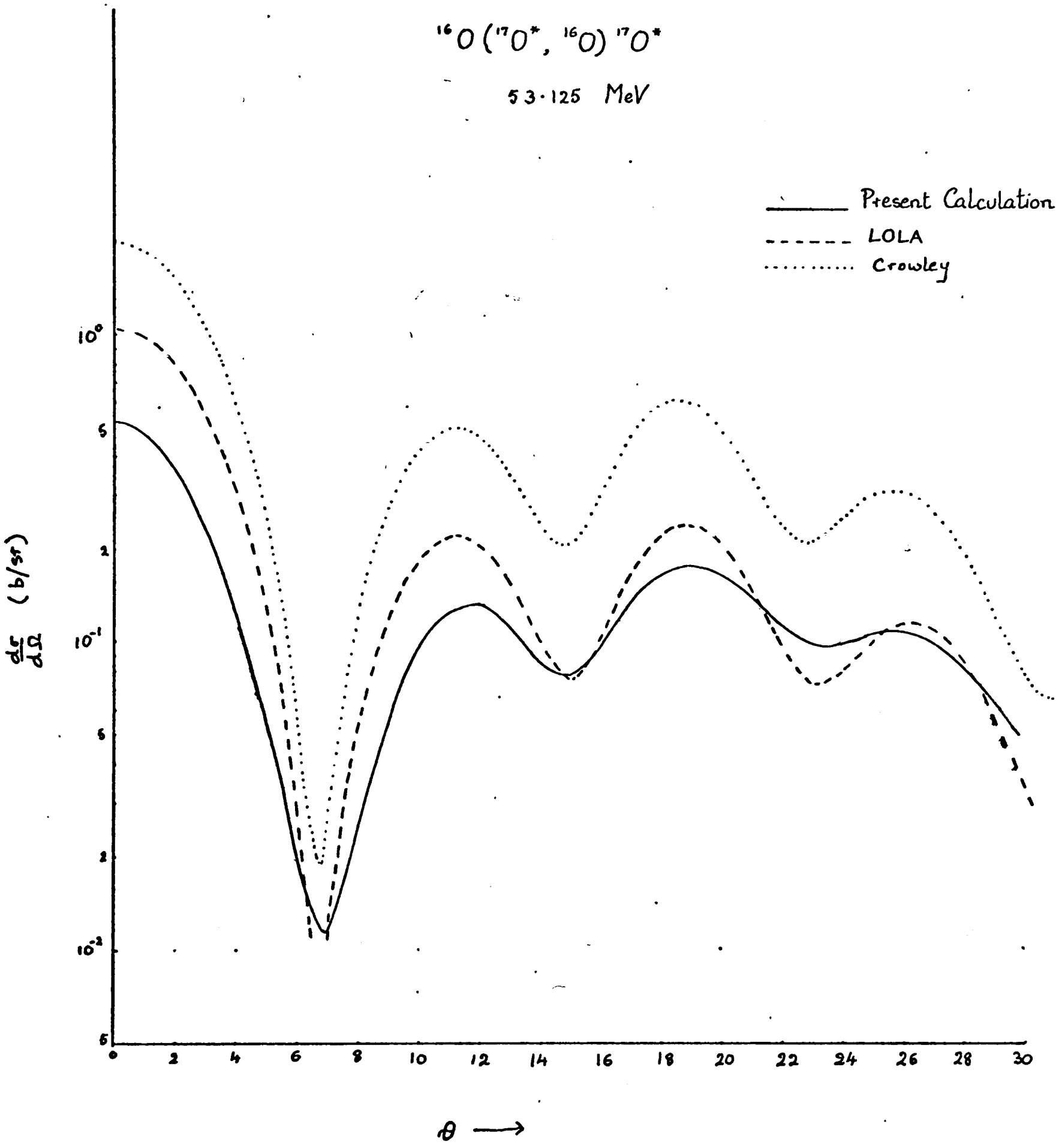

$^{16}O(^{17}O^*, ^{16}O)^{17}O^*$
53·125 MeV
Present Calculation
LOLA
Crowley
$\frac{d\sigma}{d\Omega}$ (b/sr)
10^0
5
2
10^{-1}
5
2
10^{-2}
5
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
$\theta \longrightarrow$

Figure 23

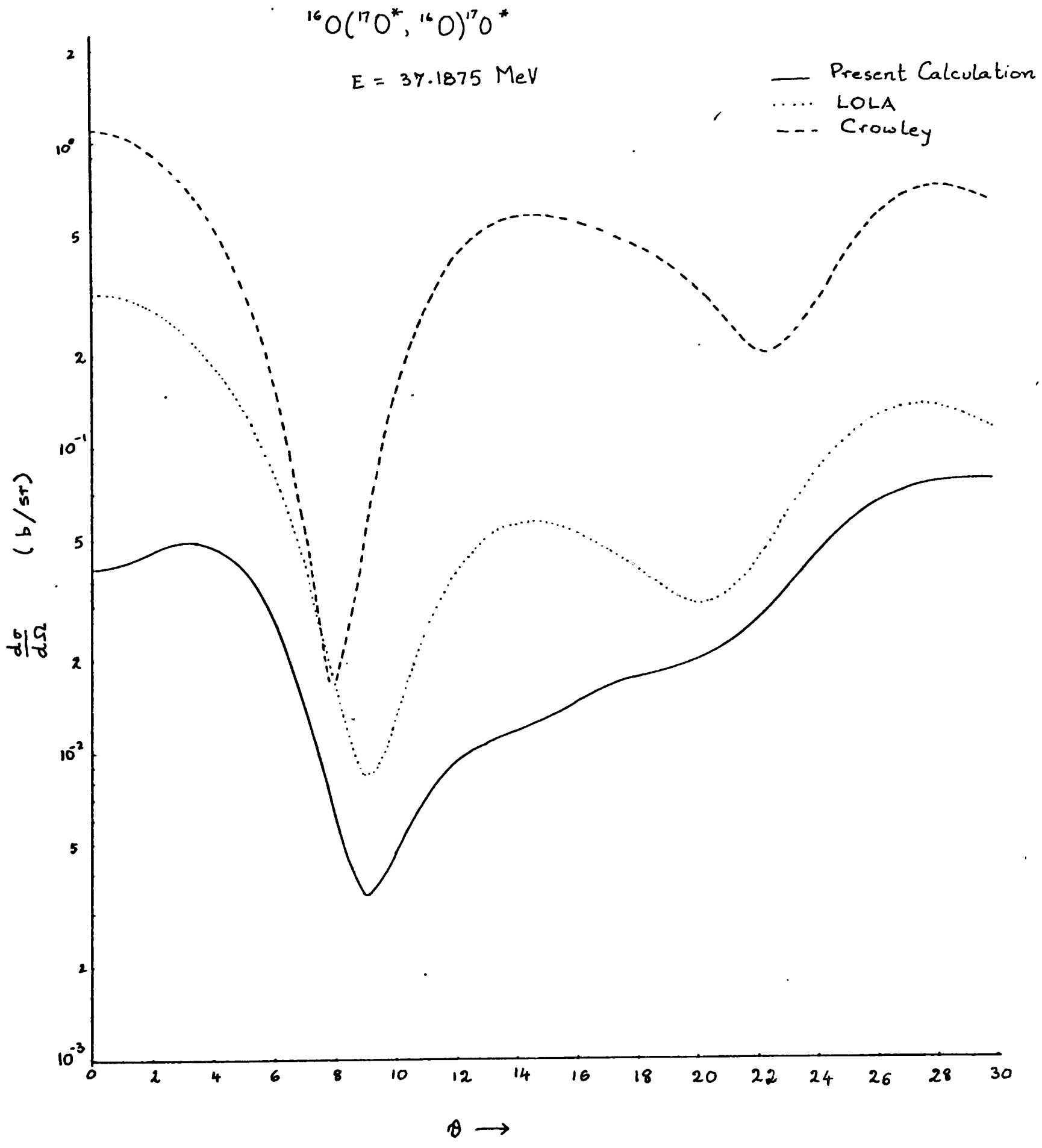

$^{16}O(^{17}O^*, ^{16}O)^{17}O^*$
E = 37.1875 MeV
Present Calculation
LOLA
Crowley
$\frac{d\sigma}{d\Omega}$ (b/sr)
2
10^0
5
2
10^{-1}
5
2
10^{-2}
5
2
10^{-3}
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
$\theta \longrightarrow$

Figure 24

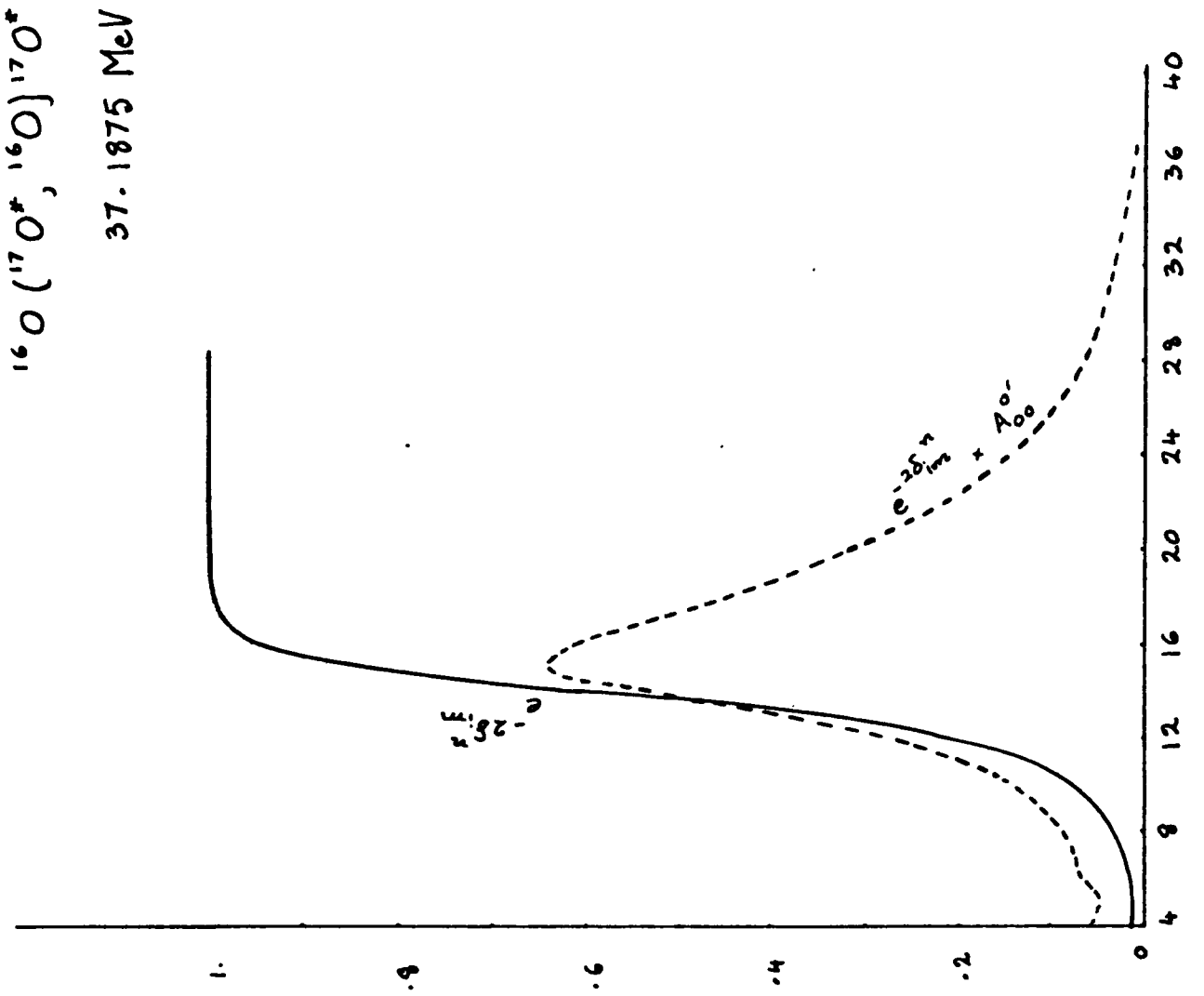


Figure 25b

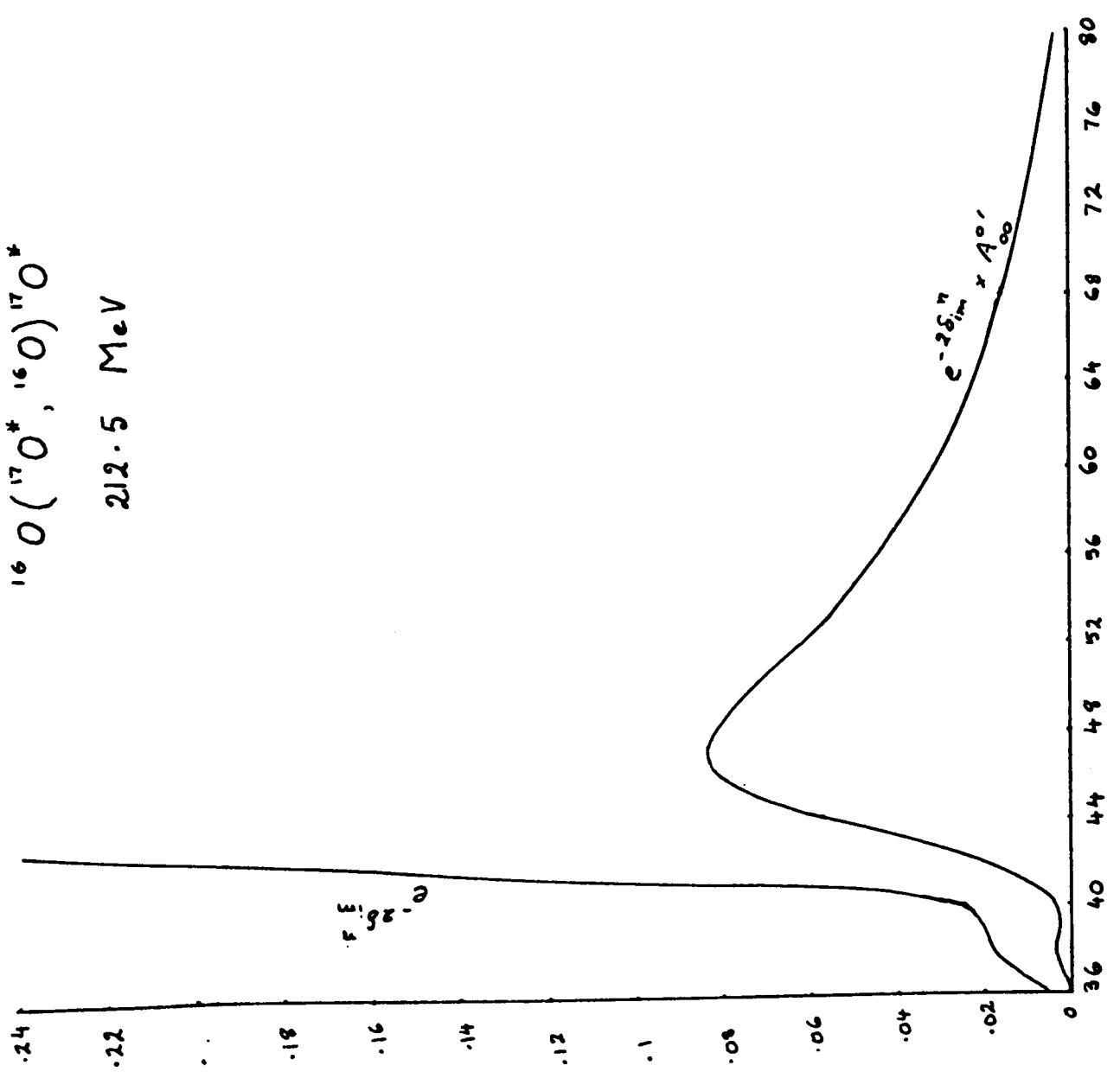


Figure 25a

CHAPTER IV

APPLICATION OF SEMICLASSICAL THEORY TO α-TRANSFER

IV.1 Introduction

The calculations presented in this chapter were started with a view to calculate $^{12}C - ^{16}O$ angular correlations measured in recent experiments at Orsay (Roussel et. al., 1976). They measure $^{12}C - ^{16}O$ correlation functions in the sequential reaction $^{16}O(^{16}O, ^{12}C)^{20}Ne^* \rightarrow \alpha + ^{16}O$ at 68 MeV laboratory energy. The correlation has been measured for the decay of a number of excited states of $^{20}Ne^*$, all of which are unbound.

We may look at the reaction as the transfer of an α-particle from one ^{16}O nucleus to the other giving $^{20}Ne^*$, which then decays again to ^{16}O and α. In order to test our theory for α-transfer before applying it to experiment, we carried out some preliminary calculations for the transfer of an α-particle from the 0^+ ground state in ^{16}O to the 0^+ ground state in ^{20}Ne, and compared our angular distributions with that obtained using the quantum mechanical code LOLA. A number of difficulties as well as interesting features appeared in this calculation which we present in succeeding sections. We then calculated the angular distribution for the 4^+ excited state of $^{20}Ne^*$ studied experimentally, but did not perform calculations for any of the unbound states or correlation functions.

VI.2 Details of Calculation

1. α-particle Wave Function

The first question which presents itself is what should the α-particle wave function be? To answer this question we shall look at the theory of Buck et al. (Buck et al., 1975). They have shown that the ^{16}O and ^{20}Ne nuclei can be quite well described in the cluster model in which the α-cluster is viewed as a single particle in a certain orbit with principal quantum number N and orbital quantum number L, circulating around our inert core (^{12}C in the case of ^{16}O, and ^{16}O in the case of ^{20}Ne). The values of N and L are related to the n_i and ℓ_i of the particles which make up the cluster by:

$$2N+L = \sum_{i=1} (2n_i+\ell_i)$$

Thus the α-particle ground state in ^{16}O should be 0^+ state with 2 nodes, while that in ^{20}Ne is the 0^+ state with 4 nodes. However, the α-particle ground state in ^{16}O is considered to be an admixture of more than one state. We take the 0^+ state with 3 nodes for our calculations.

Buck et al. have shown that these states may be calculated using a folded potential. We have used a Woods-Saxon potential, the parameters of which have been varied so as to approximate the folded potential as closely as possible. We calculate our wave functions in this well, the radius parameter of which is given by

$R = r_o A^{\frac{1}{3}}$ ($r_o = 0.9325$, A is the core mass) and diffuseness a = 0.89. The depth of the potential was adjusted to give the experimental bending energy. For ^{16}O this depth is 151.1 MeV while for ^{20}Ne it is 178.8 MeV.

The wave functions obtained are illustrated in fig 26. A cursory look at the ^{20}Ne wave function obtained here and that of Buck et al. shows that the positions and magnitudes of the maxima and minima agree in the two cases. The wave function of the 4^+ state of ^{20}Ne is also illustrated. This was also calculated in a Woods-Saxon well with the same radius and diffuseness as before. The well depth was slightly changed to 177.6 MeV to give the correct binding. In this case the positions of the first maximum and minimum seem to agree with those of Buck et al. but the second maximum is shifted by about .3 fm.

2. Choice of Optical Potential

The complex optical potential for calculating the elastic scattering phase shifts was taken to be of Woods-Saxon form for the nuclear part. The parameters for the real and imaginary parts of the potential were taken to be (Halbert et al., 1975):

$$V_R = 16. \qquad V_I = 10.$$
$$r_{oR} = 1.349 \qquad r_{oI} = 1.349$$
$$r_{oR} = 0.49 \qquad r_{oi} = 0.49$$
$$R = r_o(A_P^{\frac{1}{3}} + A_T^{\frac{1}{3}})$$

where E_i is the centre of mass energy and A_P and A_T are the projectile and target masses respectively.

VI.3 Discussion of Results

Angular distributions were calculated for the transfer of an α-particle from the 0^+ ground state in ^{16}O to the 0^+ ground state in ^{20}Ne (using eqs. (III.1.60), (II.5.30)) under four different approximate conditions:

1) The values of the velocities, reduced mass, momentum, appearing in eq. (II.5.30) for the transfer amplitude $A^{O'}_{OO}(\ell)$ were calculated in the initial channel.

2) These quantities calculated in the final channel.

3) Same as (1) but m_x in the definition of $A^O_{\lambda_1\lambda_2}$ (eq. (III.1.57)) replaced by $m_x(m_{c_1} + m_{c_2})/(m_{c_1}+m_{c_2}+m_x)$ where m_{c_1}, m_{c_2}, m_x now are the masses of ^{12}C, ^{16}O, α-particle respectively (c.f. equation above (III.1.57)).

4) Same as (2) but m_x replaced by $m_x(m_{c_1}+m_{c_2})/(m_{c_1}+m_{c_2}+m_x)$

The results are presented in fig. 27 together with those of LOLA for comparison. We see that cases (1) and (2) differ in magnitude by approximately an overall factor of 2.16, but are otherwise similar. Case (3)

is similar in shape to cases (1) and (2) till about 20^{o}. It produces a very sharp minimum at 26^{o} which is not present in the other cases. There is a difference in magnitude of the order of 1.9. Case (4) looks like case (3) except that the sharp minimum at 26^{o} is absent. An overall difference of magnitude or the order of 2.8 exists.

A comparison with LOLA shows that cases (1) and (2) agree better in shape than cases (3) and (4). However, the maxima and minima are shifted by about $.5^{o}$ with respect to LOLA at small angles. At larger angles our angular distributions become out of phase with LOLA. Further, case (1) disagrees with LOLA in magnitude by about a factor of 2.2, and the other cases by larger factors.

The disagreement in magnitudes between cases (1) and (2) (or (3) and (4)) is most likely due to the mismatch between the initial and final orbits. As the α-particle is heavy compared to a single nucleon, this mismatch is appreciable and seems to manifest itself in this way. The disagreement in magnitude between our results and LOLA is of the same order as that between cases (1) and (2), and (3) and (4) in our calculations. The reason for the disagreement in phase between LOLA and our results is not very clear.

Another curious feature is exhibited by the quantity $e^{-2\delta_{im}^{n}} \times A_{oo}^{o'}(l)$ (c.f. eq. (III.1.60)) which is illustrated in fig. 28 for case (1). Unlike the case of single nucleon

transfer, this quantity changes sign. For small values of ℓ it is positive. It has two small kinks and then goes to a negative value. The change of sign occurs at $\ell = 21$ which corresponds to a distance of closest approach of 7.446 fm. On the negative side of the graph $e^{-2\delta_{im}^{n}} \times A_{oo}^{o'}(\ell)$ has the expected shape, i.e. it peaks around a certain ℓ-value and decays (approximately exponentially) to zero.

The angular distributions obtained for the transfer of the α-particle to the 4.25 (4^+) excited state of $^{20}Ne^*$ is shown in fig. 29. Calculations were carried out for cases (1) and (2). The three experimental points obtained by Rousell et. al. are also marked. They are normalised so that the point at 36.8^o falls on curve 1. The agreement seems encouraging, though we really need more data to be able to say anything more conclusive.

Because of the difficulties encountered here, it was decided not to perform calculations for angular correlations till these could be better understood.

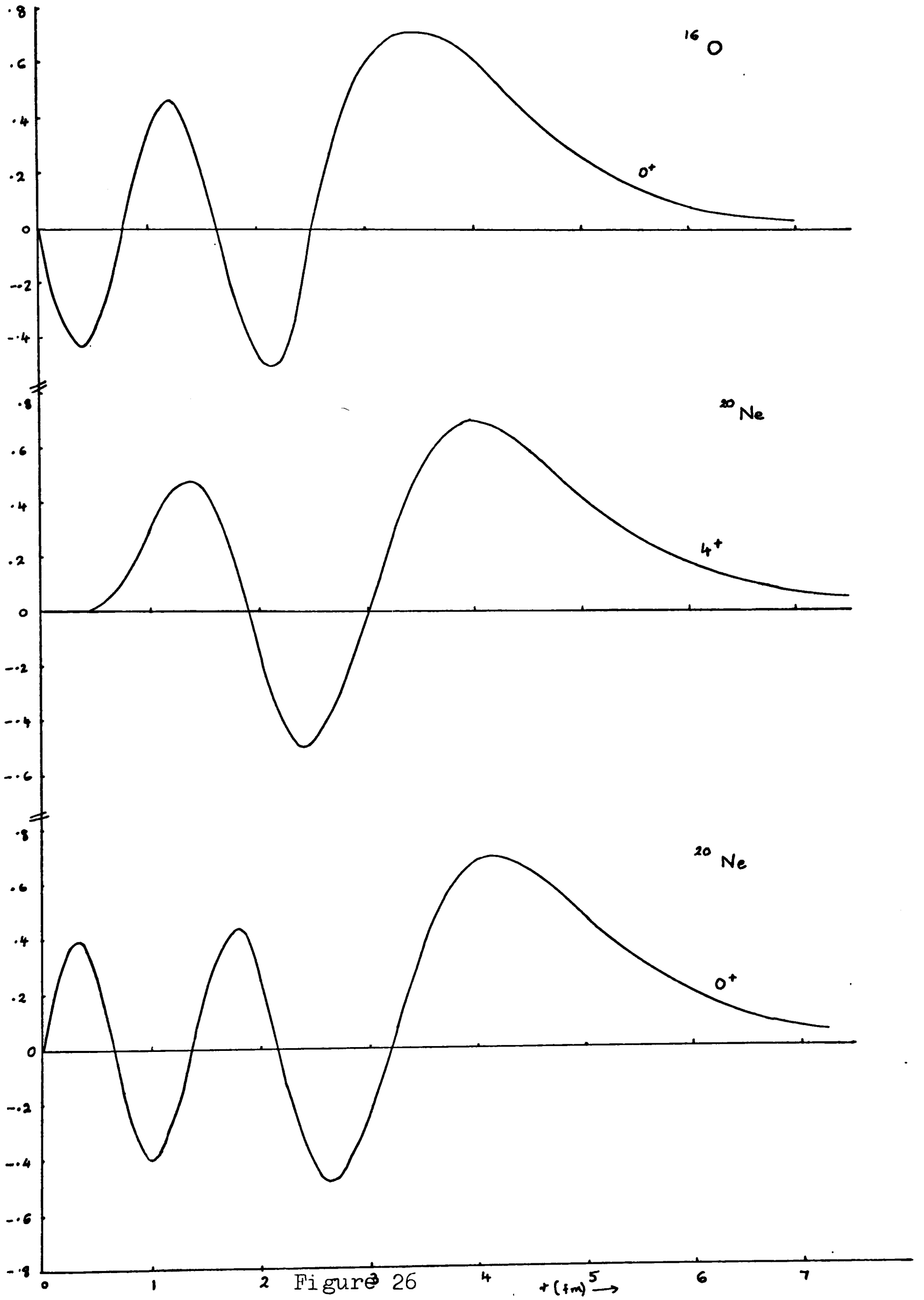


Figure 26

α-particle wave functions in the 0^+ ground state of ^{16}O and the 0^+ and 4^+ states in ^{20}Ne.

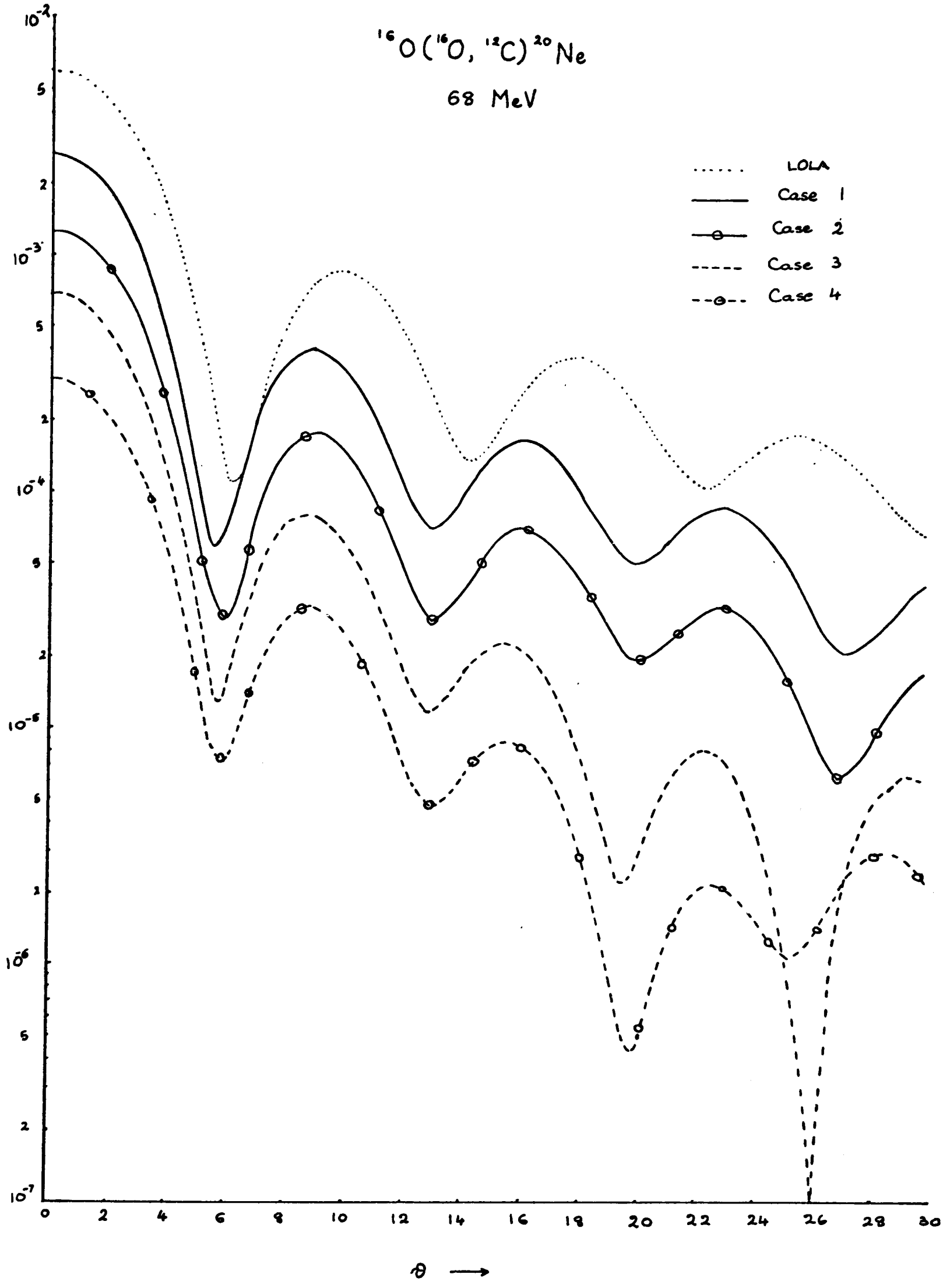


Figure 27

Angular distributions for the transfer of an α-particle from the 0^+ ground state in ^{16}O to the 0^+ ground state in ^{20}Ne.

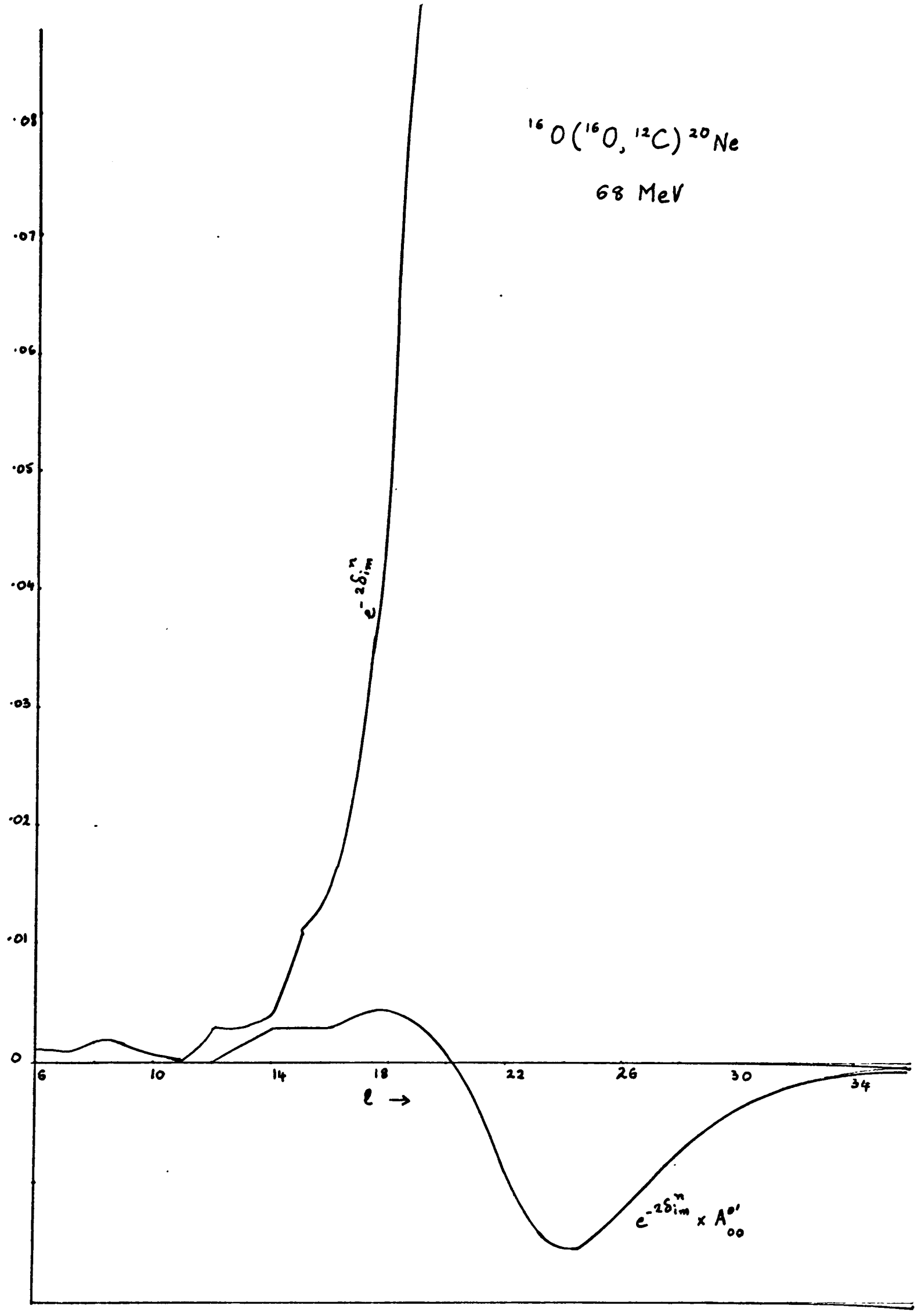

$^{16}O(^{16}O, ^{12}C)^{20}Ne$
68 MeV
$e^{-2\delta_{im}^{n}}$
$e^{-2\delta_{im}^{n}} \times A_{00}^{n'}$
·08
·07
·06
·05
·04
·03
·02
·01
0
6
10
14
18
22
26
30
34
ℓ →

Figure 28.

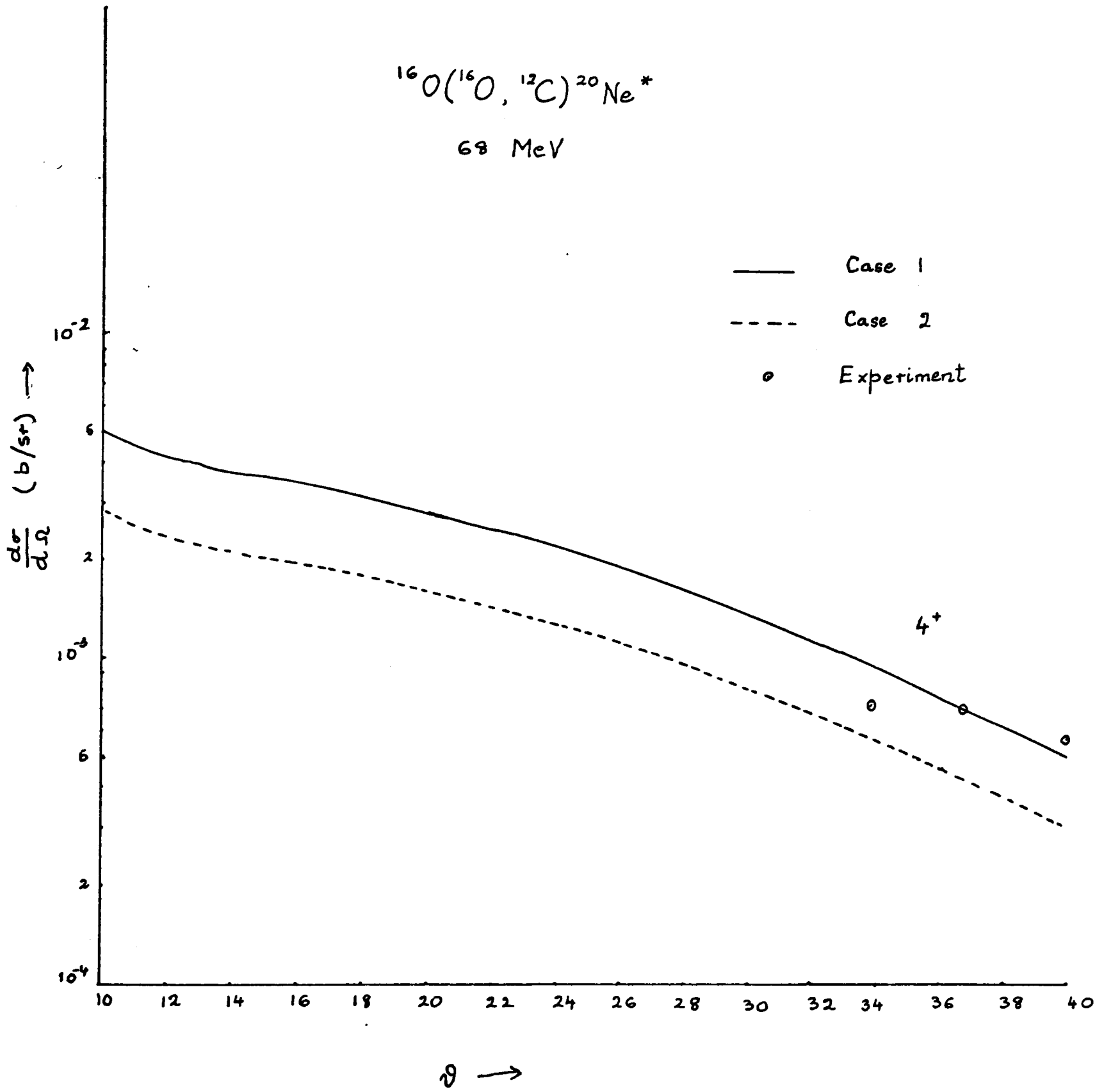


Figure 29

Angular distributions for the transfer of an α-particle from the 0^+ ground state in ^{16}O to the 4.25 (4^+) state in $^{20}Ne^*$.

CHAPTER VII

CONCLUSIONS

The simple semiclassical theory studied in this work provides a quick and easy way to calculate transfer amplitudes and angular distributions for transdfer reactions at high energies. The formula for evaluating transfer amplitudes includes in an approximate way recoil, as well as correction to the Q-value and interaction potential for charged particle transfer. It has the additional advantage of being identical in the post and prior forms. Being a straightforward single dimensional integral it is relatively simple to compute and is therefore practical to use in the determination of angular distributions.

Our formula for angular distributions is derived from the expression of the transition matrix in DWBA. An expansion of the distorted waves into partial waves sums gives us a double summation. By keeping our semiclassical picture for the reaction we are able to perform one of these summations analytically. In the formula of Landowne et. al. (Landowne et. al., 1976) the double summation exists till the end.

From the calculations we have performed it appears that our formula may be applied successfully to single nucleon transfer reactions at high energies. The number

of approximations that go into it, however, do restrict its use somewhat. As we have seen, the neglect of the contribution of the two inner turning points (when approximating the radial distorted wave functions by WKB solutions) requires a strong complex potential to describe the elastic scattering. If the imaginary part of this potential is weak our formula breaks down. This question has not been tested numerically, but it may be worthwhile doing so.

The approximation of the transfer amplitudes by a parametrised formula seems a very very promising one as it greatly reduces the complexity of the calculation. The conditions of validity of this also need a more thorough investigation. The closed formula we obtained may also prove to be a useful one for quick calculations. A slight disadvantage in its use is that we need a good parametrisation of the elastic scattering phases. This is, however, not a serious drawback and is fairly easily overcome.

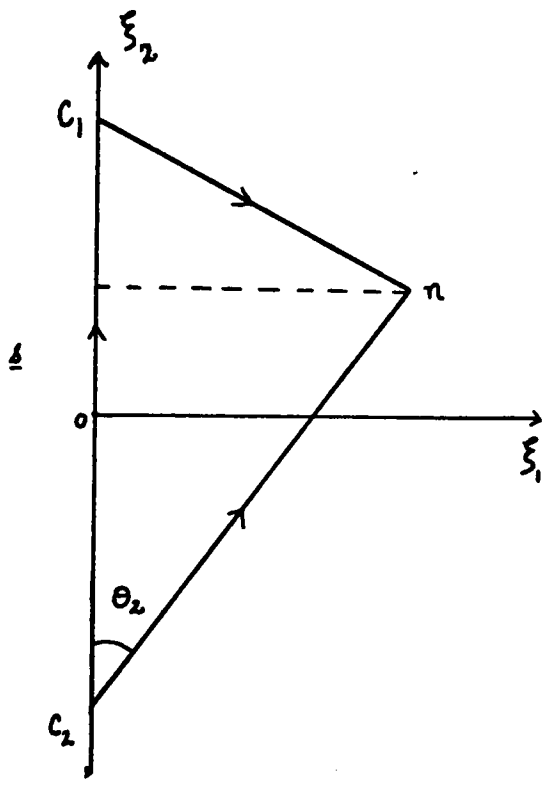
The comparisons we made for angular distributions calculated with our formula with those of Crowley's and LOLA show that we get good agreement with the other two calculations at high energies and for single neutron transfer. No direct comparisons have been made for proton transfer. However, a comparison of spectroscopic factors obtained for reactions between ^{26}Mg and ^{11}B at 114 MeV agree with with those of LOLA.

As regards the application of this formula to multinucleon transfer, we are still not in a position to say anything concrete. There is nothing in the theory which prevents us from using it for such reactions, provided they are one step processes and the wave functions of the transferred cluster is well known. However, the α-particle transfer reaction we studied show that there is disagreement in magnitude between the calculation made using initial channel parameters and final channel parameters. This leads us to believe that the theory may not work so well if the transferred cluster is heavy and there is mismatch between the initial and final channels.

APPENDIX

II.A Transformation of variables

We transform from variable r_2, θ_2 to variables ξ_1, ξ_2 according to the diagram:



$$\underline{r}_2 - \underline{r}_1 = \underline{s}$$

The origin is taken at a distance $|s|/2$ from c_1 and c_2 along the line joining them. This line is taken as the ξ_2 axis and a perpendicular through the origin gives the ξ_1 axis. The old and new coordinates are then related through the equations:

$$r_2 = \sqrt{\xi_1^2 + \left(\xi_2 + \frac{|\underline{s}|}{2}\right)^2}$$

$$\cos\theta_2 = \frac{\xi_2 + |\underline{s}|/2}{\sqrt{\xi_1^2 + \left(\xi_2 + |\underline{s}|/2\right)^2}}$$

r_1, θ_1 are given in terms of ξ_1, ξ_2 by the relations:

$$r_1 = \sqrt{\xi_1^2 + (|\underline{\xi}|/2 - \xi_1)^2}$$

$$\cos\theta_1 = \frac{\frac{|\underline{\xi}|}{2} - \xi_1}{\sqrt{\xi_1^2 + (|\underline{\xi}|/2 - \xi_1)^2}}$$

The limits of integration on ξ_1 are from 0 to ∞ and on ξ_2 from $-\infty$ to $+\infty$.

II.B Momentum Space Wave Functions

To obtain the wave functions in momentum space we take the Fourier transform of the wave function $\psi(\underline{r})$ in coordinate space:

$$\tilde{\psi}(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\underline{k}\cdot\underline{r}}\, \psi(r)\, d^3\underline{r} \qquad \text{(B1)}$$

Now, $$e^{i\underline{k}\cdot\underline{r}} = \sum_{lm} i^l j_l(kr)\, Y^*_{lm}(\theta_k \varphi_k)\, Y_{lm}(\theta_r \varphi_r) \qquad \text{(B2)}$$

Substituting into (B1) and simplifying we get:

$$\tilde{\psi}(\underline{k}) = \frac{4\pi}{(2\pi)^{3/2}} \sum_{l'm'} (-1)^{l'} \int i^{l'} j_{l'}(kr)\, Y^{*}_{l'm'}(\theta_r \varphi_r)\, Y^{*}_{l'm'}(\theta_k \varphi_k) \times$$

$$\times u_l(r)\, Y_{lm}(\theta_r \varphi_r)\, d(\cos\theta_r)\, d\varphi_r\, r^2 dr$$

$$= \sqrt{\frac{2}{\pi}} \sum_{l'm'} (-1)^{l'} i^{l'} Y_{l'm'}(\theta_k \varphi_k) \int j_{l'}(kr)\, u_{l'}(r)\, r^2 dr \times$$

$$\times \delta_{ll'}\, \delta_{mm'}$$

$$= \sqrt{\frac{2}{\pi}}\, Y_{lm}(\theta_k \varphi_k)\, i^{l} \int j_l(kr)\, u_l(r)\, r^2 dr$$

$$= \tilde{u}_l(k)\, Y_{lm}(\theta_k \varphi_k)$$

$$\therefore \tilde{u}_l(k) = (-i)^{l} \sqrt{\frac{2}{\pi}} \int j_l(kr)\, u_l(r)\, r^2 dr \qquad \text{(B3)}$$

This integral is evaluated numerically by using Simpson's Rule. To check the accuracy of the $\tilde{u}_l(k)$ calculated, we re-evaluate the $u_l(r)$ by using the inverse relation to B3:

$$u_l(r) = \sqrt{\frac{2}{\pi}}\, (-i)^{l} \int j_l(kr)\, \tilde{u}_l(kr)\, k^2\, dk \qquad \text{(B4)}$$

The numerical difficulties encountered in calculating (B3) and (B4) are fixing the limits of integration and the integration interval. However, these problems are fairly easily overcome. A look at the wave functions

$u_\ell(r)$ gives a good idea as to how big the integration limit should be for (B3). The step length is then varied till convergence is obtained.

Fixing the limits of the k-integration (B4) is not as straightforward. We can make a fairly accurate estimation by assuming that $u_\ell(r)$ is similar to an oscillator wave function. Let us write the time independent Schrodinger equation assuming an oscillator potential as:

$$\frac{\hbar^2 k^2}{2\mu} + \frac{1}{2}\mu\omega^2 r^2 = \hbar\omega\left(n + \frac{3}{2}\right) \qquad \text{(B5)}$$

where $\hbar\omega$ is the oscillator frequency and $n(=2n_o + \ell)$ is the number of oscillator quanta, n_o being the number of nodes in the wave function. μ is the reduced mass of the particle and k its momentum. Let x_o be the point at which $k = 0$, and k_o the momentum when $r = 0$. Then, from (B5) we have:

$$\frac{1}{2}\mu\omega^2 x_o^2 = \hbar\omega\left(n + \frac{3}{2}\right)$$

and
$$\frac{\hbar^2}{2\mu} k_o^2 = \hbar\omega\left(n + \frac{3}{2}\right)$$

$$\therefore \quad x_o = \left(\frac{\hbar}{\mu\omega}\right)(2n + 3) \qquad \text{(B6)}$$

and
$$k_o = \frac{\mu\omega}{\hbar}(2n+3) \qquad \text{(B7)}$$

Suppose x_m is a point at which $u_\ell(r)$ decays towards zero, and k_m is the point at which $\tilde{u}_\ell(k)$ decays to zero. Let us define α such that:

$$x_m = \alpha\, x_o \qquad \text{(B8)}$$

$$\text{and} \quad k_m \simeq \alpha\, k_o \qquad \text{(B9)}$$

x_o is known approximately from (B6) and x_m by a look at the behaviour of $u_\ell(r)$. α is then calculated from (B8) and k_m determined from (B7) and (B9).

This method of course gives only an estimate of k_m, and we still have to vary it somewhat to obtain convergence. The step length, Δk, is fixed automatically by taking the same number of points N for the k-integration as were used for the r-integration (B3).

Then, $$\Delta k = \frac{k_m}{N}$$

The value of Δk and k_m obtained in this way are used in the evaluation of the integral (II.3.6). Typical values for α range from about 2.5 to 4.

The $u_\ell(r)$ obtained from (B4) in this way agree with the original ones up to six places of decimal.

Appendix

III.A. Asymptotic Expansion of $B_{\ell m}$:

$$B_{lm} = \frac{1}{\pi}(-1)^m \left[\frac{2l+1}{2} \frac{\Gamma(l-m+1)\Gamma(l+m+1)}{(\Gamma(l+3/2))^2} \right]^{1/2}$$

For $\ell >> m$ we may use the asymptotic expansion of $\Gamma(z)$ for large arguments:

$$\ln \Gamma(z) \sim (z-\tfrac{1}{2}) \ln z - z + \tfrac{1}{2} \ln 2\pi$$

$$\ln \left[\frac{\pi}{(-1)^m} \times B_{lm} \right] \simeq \ln(l+\tfrac{1}{2}) + (l-m+\tfrac{1}{2}) \ln(l-m+1) -$$

$$-(l-m+1) + \tfrac{1}{2} \ln 2\pi + (l+m+\tfrac{1}{2}) \times$$

$$\times \ln(l+m+1) - (l+m+1) + \tfrac{1}{2} \ln 2\pi -$$

$$2(l+1) \ln(l+\tfrac{3}{2}) - (2l+3) - \ln 2\pi$$

$$= (l+\tfrac{1}{2}) \ln\left(1 - \frac{m+\frac{1}{2}}{l+3/2}\right) +$$

$$(l+\tfrac{1}{2}) \ln\left(1 + \frac{m-\frac{1}{2}}{l+\frac{3}{2}}\right) + m \ln\left(1 + \frac{2m}{l-m+1}\right)$$

$$+ \ln\left(1 - \frac{1}{l+\frac{3}{2}}\right) + 1$$

Remembering that ℓ is large and $\ell >> m$, we may expand the logarithims into a series and retain terms of the order $\frac{1}{\ell}$

$$\therefore \quad \ln\left[\frac{\pi}{(-1)^m} B_{\ell m}\right] \simeq (\ell+\tfrac{1}{2})\left(-\frac{m+\frac{1}{2}}{\ell+3/2}\right) +$$

$$+ (\ell+\tfrac{1}{2})\left(\frac{m-\frac{1}{2}}{\ell+3/2}\right) - \frac{1}{\ell+3/2} + 1$$

$$= -\frac{\ell+\frac{1}{2}}{\ell+\frac{3}{2}} - \frac{1}{\ell+3/2} + 1$$

$$= 0$$

$$\therefore \quad B_{\ell m} \simeq \frac{1}{\pi} (-1)^m$$

III.B. Expansion of $\Delta\phi(s)$:

$$\Delta\phi(s) = \frac{1}{\hbar}\int_{s_{of}}^{s} k_f(s')\,ds' - \frac{1}{\hbar}\int_{s_{oi}}^{s} k_i(s')\,ds'$$

$$= \frac{1}{\hbar}\int_{s_{of}}^{s}\sqrt{2\mu_f(E_f-V_f)}\,ds' - \frac{1}{\hbar}\int_{s_{oi}}^{s}\sqrt{2\mu_i(E_i-V_i)}\,ds'$$

$$= \frac{1}{\hbar}\int_{s_{of}}^{s}\sqrt{2\mu_f(E_f-V_f)}\,ds' - \frac{1}{\hbar}\int_{s_{oi}}^{s}\sqrt{2(\mu_f-\delta\mu)(E_f-V_f-\delta E+\delta V)}\,ds',$$

where $\delta\mu = \mu_f - \mu_i$, $\delta E = E_f - E_i$, $\delta V = V_f - V_i$

Expanding the square root to first order in $\delta\mu$, δE, δV we get:

$$\Delta\phi(s) \simeq \frac{1}{\hbar}\int_{s_{of}}^{s}\sqrt{2\mu_f(E_f-V_f(s'))}\,ds' - \frac{1}{\hbar}\int_{s_{oi}}^{s}\sqrt{2\mu_f}\left(1-\frac{\delta\mu_f}{2\mu_f}\right)\times$$
$$\times\sqrt{E_f-V_f(s')}\left(1-\frac{\delta E-\delta V}{2(E_f-V_f)}\right)ds'$$

Assuming that $s_{of} \simeq s_{oi} = s_o$ we get:

$$\Delta\phi(s) \simeq \frac{1}{\hbar}\int_{s_o}^{s}\frac{\delta E-\delta V}{\sqrt{E_f-V_f}}\sqrt{\frac{\mu_f}{2}}\,ds' + \frac{\delta\mu}{\hbar}\int_{s_o}^{s}\frac{1}{\sqrt{2\mu_f}}\sqrt{E_f-V_f(s)}\,ds'$$

Now $$\sqrt{E_f - V_f(s')} = \sqrt{\tfrac{1}{2}\mu}\;\dot{s}'$$

$$\therefore \hbar\,\Delta\phi(s) = \delta E\int_{s_o}^{s}\frac{dt}{ds'}\,ds' + \frac{1}{2}\,\delta\mu\int_{s_o}^{s}\dot{s}'\,ds' - \int_{s_o}^{s}\delta V\,\frac{dt}{ds'}\,ds'$$

We shall now covert the s' integral to a time integral remembering that $s' = s_0$ corresponds to $t' = 0$, and $s' = s$ to $t' = t$

$$\therefore \hbar\,\Delta\phi(t) = \delta E \int_0^t dt' + \frac{1}{2}\delta\mu \int_0^t \dot{s}'^2 dt - \int_0^t \delta V\, dt$$

Now $$V(s) = \frac{\hbar^2 (l+\frac{1}{2})^2}{2\mu s^2} + U(s)$$

$$\therefore \delta V = \frac{\hbar^2 (l_f+\frac{1}{2})}{\mu_f s^2}\,\delta l - \frac{\hbar^2 (l_f+\frac{1}{2})^2}{2\mu_f^2 s^2}\,\delta\mu + \delta U$$

Also: $$\frac{\hbar (l_f+\frac{1}{2})}{s^2} = \mu_f \dot{\theta}$$

where $\dot{\theta}$ is the angular velocity

$$\therefore \delta V = \hbar\,\dot{\theta}\,\delta l - \tfrac{1}{2} s^2 \dot{\theta}^2 \delta\mu + \delta U$$

$$\therefore \int \delta V\, dt' = \delta l \int_0^t \hbar\,\dot{\theta}\, dt - \delta\mu \int_0^t s'^2 \dot{\theta}^2 dt' + \int_0^t \delta U\, dt'$$

$$= \delta l\, \hbar\, \bar{\theta}(t) - \tfrac{1}{2}\delta\mu \int_0^t s'^2 \dot{\theta}^2 dt' + \int_0^t \delta U\, dt' \qquad (s' \equiv s(t'))$$

$$\therefore \hbar\,\Delta\phi(s) = \delta E\,.\,t + \tfrac{1}{2}\delta\mu \int_0^t (s'^2\theta^2 + \dot{s}'^2)\, dt - \int_0^t \delta U dt'$$

$$- \hbar\,\delta l\,\bar{\theta}(s)$$

$$\therefore \Delta\phi(s) = \frac{1}{\hbar}(E_f - E_i)t - (l_f - l_i)\bar{\theta}(s) + \frac{1}{\hbar}\int_0^t \Big[(U_i - U_f) + \tfrac{1}{2}(\mu_f - \mu_i)(\dot{s}')^2\Big] dt'$$

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