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Matrix-Weighted Networks for Modeling Multidimensional Dynamics: Theoretical Foundations and Applications to Network Coherence

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Networks are powerful tools for modeling interactions in complex systems. While traditional networks use scalar edge weights, many real-world systems involve multidimensional interactions. For example, in social networks, individuals often have multiple interconnected opinions that can affect different opinions of other individuals, which can be better characterized by matrices. We propose a general framework for modeling such multidimensional interacting dynamics: matrix-weighted networks (MWNs). We present the mathematical foundations of MWNs and examine consensus dynamics and random walks within this context. Our results reveal that the coherence of MWNs gives rise to nontrivial steady states that generalize the notions of communities and structural balance in traditional networks.

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Networks have emerged as a powerful model for studying interacting systems across a wide range of disciplines, enhancing our understanding of complex systems such as the human brain, the Internet, and human societies [1]. Networks are constructed from pairwise interactions between entities, which, combined together, allow signals to propagate in connected systems. Much effort has been made to understand the intricate interplay between dynamical systems on networks and underlying network structure [2–5] with two main foci: examining the network properties that influence dynamics on networks, and utilizing dynamical processes to extract valuable information about underlying network structure. For example, modular structure in networks can characterize timescale separation of dynamical systems [5,6] and, in the presence of negative links, structural balance can help separate distinct dynamical behaviors [7–9].

How to characterize pairwise interaction is a key modeling decision that significantly impacts the analysis of network dynamics. Widespread choices include binary values indicating the presence of an edge between two nodes, a positive real number to indicate connection strength, and a real value whose sign represents consolidation or contradiction [1]. Recent studies have expanded the concept of edge weights to complex values [10,11] to represent the "state" of connections.

Multidimensional interactions are another prevalent form of interaction in numerous complex systems [12–16]. In social networks, individuals hold interconnected opinions on various issues that are intrinsically interrelated and change collectively [16–22]. Similar situations arise in global trade networks, where countries influence each other through multiple dimensions such as currency exchange, export composition, and world market shares. We can also think of multispecies metapopulation dynamics with interrelated interactions between species [23,24], many-body quantum systems with linear operators characterizing the connections [25], and multidimensional synchronization on networks [26,27]. Such multidimensional interactions—where a node has multiple channels to interact with other nodes—have been commonly characterized by multiplex and multilayer networks [28–32].

However, the details of such nontrivial and united multidimensional interactions *between nodes* are not fully reflected in traditional network models. Edges in existing models with multidimensional node states are typically assumed to simply transmit a vector from one node to another, and/or to compare two vector states and react according to their distances, e.g., [16], in a simple manner

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without considering interactions among its components. These assumptions would be too limited to capture more complex multidimensional interactions identified in various real-world network systems. This observation has led us to study multidimensional *interaction* models. Models for multidimensional interactions have been previously studied, particularly in opinion dynamics [33,34]. In these models, agents possess opinion vectors on various topics that can change in interdependent ways through communication, depending on how agents interpret their neighbors' opinions. While these models have found applications in machine learning [35–38], their translation to physical systems and complex network dynamics remains largely unexplored, leaving opportunities to enhance our understanding of multifaceted interactions in complex networks.

In this Letter, we introduce a framework specifically designed for multidimensional interactions on networks: "matrix-weighted networks" (MWNs). This is a generalization of conventional scalar- or complex-weighted network models, in which one can represent a linear transformation of state vectors between a pair of nodes connected by an edge. In most general settings, the weights can be any matrices of any size (even rectangular ones may be allowed if the dimensions of node states differ from node to node) and self-loops are permitted. Then, not only multidimensional interactions of node states but also any network with any partitions can be analyzed from the angle of MWNs. MWNs offer a new perspective on modeling multidimensional interactions by directly encoding collective transformations across state dimensions. For example, in ecological systems, predation may simultaneously reduce a prey's population size and alter its reproduction rate—effects that MWNs capture through a single transformation matrix. Such coupled changes are obscured in flattened representations by scalar-weighted and multilayer network representations. In this Letter, we start with the cases of square matrix weights and directed edge reciprocity that, as we argue, can be seen as the generalization of classical undirected networks to the matrix-weighted setting. More specifically, we consider the cases when edges characterize reciprocal interactions, where the effect of one characteristic dimension of node states on another dimension from one node to another is the same as the one when the orders of both nodes and dimensions are reversed. We start by providing mathematical foundations for MWNs, and then characterize the dynamical behavior through two key dynamics, consensus dynamics and random walks that are reformulated with this framework, both theoretically and numerically. Our main results concern the definition of the coherence of MWNs, which determines the existence of nontrivial steady states for the dynamics and is associated with large-scale patterns generalizing the notions of communities and structural balance [39]. Finally, we discuss opportunities for future research and connections with existing network models.

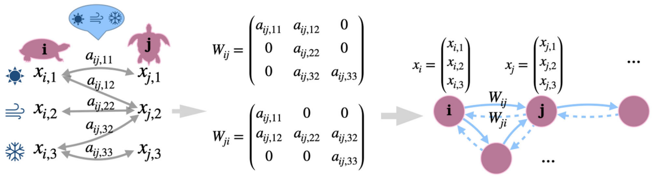

FIG. 1. Illustration of an MWN, where each node is equipped with three state variables. In an opinion dynamics setting, each variable can be interpreted as an opinion on a specific topic. Here, we illustrate how the state vector of node v_i affects the state vector of v_j and vice versa.

Matrix-weighted networks (MWNs)—We consider networks in the form of connected, weighted, and directed graphs $G = (V, E, \mathcal{W})$, where $V = \{v_1, v_2, \dots, v_n\}$ is the node set and $E \subset V \times V$ is the edge set. We use a matrix $\mathbf{W}_{ij} \in \mathbb{R}^{n_d \times n_d}$ to characterize the relationship between nodes v_i and v_j, where n_d is the characteristic dimension. $\mathbf{W}_{ij}$ is the all-zero matrix if and only if there is no directed edge from v_i to v_j. Otherwise, the matrix encodes how an n_d-dimensional signal on node v_j impacts the signal on node v_i. See Fig. 1 for an example. We define the $(nn_d) \times (nn_d)$ supraweight matrix $\mathcal{W}$ to have block structure, with the i, j block being $\mathbf{W}_{ij}$, the matrix weight between nodes v_i and v_j. We use "supra" to indicate matrix representations that act on the full concatenated state vector across all nodes (e.g., $\mathcal{W}, \mathcal{L}$). Furthermore, we decompose each matrix weight into two parts, one accounting for the magnitude $w_{ij} = \|\mathbf{W}_{ij}\|_2$, and the other for the transformation $\mathbf{R}_{ij}$ with $\|\mathbf{R}_{ij}\|_2 = 1$ (for $w_{ij} > 0$), such that $\mathbf{W}_{ij} = w_{ij}\mathbf{R}_{ij}$. In this Letter, we assume that $\mathbf{W}_{ij} = \mathbf{W}_{ji}^T$, i.e., the interaction between the same pair of dimensions is reciprocal between the same pair of nodes [40]. Then $w_{ij} = w_{ji}$, $\mathbf{R}_{ij} = \mathbf{R}_{ji}^T$, and finally, $\mathcal{W}^T = \mathcal{W}$. This choice ensures that the eigenvalues of $\mathcal{W}$ are real and can be understood as an extension of undirected networks to the matrix-weighted setting.

We define the degree (or strength) of a node v_i as the sum of the matrix norm of incident edges,

$$d_i = \sum_j \|\mathbf{W}_{ij}\|_2 = \sum_j w_{ij},$$

and the supradegree matrix as

$$\mathcal{D} = \mathbf{D} \otimes \mathbf{I},$$

where $\mathbf{D}$ is the diagonal matrix with $\mathbf{d} = (d_i)$ on its diagonal, $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{n_d \times n_d}$, and $\otimes$ denotes Kronecker product. We define the supra-Laplacian matrix as

$$\mathcal{L} = \mathcal{D} - \mathcal{W}.$$

We note that there are other types of Laplacians in the multidimensional setting, such as the one based on node-edge interactions [34], but ours is more consistent with existing network models, thus more appropriate to adapt network-based methods. If we consider a vector $\mathbf{x} = (\mathbf{x}_1; \mathbf{x}_2; \ldots; \mathbf{x}_n)$ with $\mathbf{x}_i \in \mathbb{R}^{n_d}$ for each i, then

$$\mathbf{x}^T \mathcal{L} \mathbf{x} = \sum_{(v_i,v_j),(v_j,v_i)\in E} w_{ij} \left\| \mathbf{x}_i - \mathbf{R}_{ij}\mathbf{x}_j \right\|_2^2 \geq 0. \tag{1}$$

Hence, the supra-Laplacian matrix maintains the property of the graph Laplacian to be positive semidefinite. We then define the supra-random-walk Laplacian matrix as

$$\mathcal{L}_{\mathrm{rw}} = \mathcal{I} - \mathcal{P},$$

where $\mathcal{I} \in \mathbb{R}^{nn_d \times nn_d}$ is the identity matrix and $\mathcal{P} = \mathcal{D}^{-1}\mathcal{W}$ is the supratransition matrix. If $n_d = 1$, we retrieve the classic definitions. We will show later that the two Laplacians are closely related to the consensus dynamics and the random walks in the multidimensional case.

Coherence—We define the “transformation” of a directed walk P [41] with composing edges $e_1, e_2 \ldots, e_k$ as

$$\mathbf{R}(P) = \mathbf{R}(e_1)\mathbf{R}(e_2)\ldots\mathbf{R}(e_k),$$

where $\mathbf{R}(e)$ returns the transformation of edge e. This notion is essential to understand how signals propagate along walks in the graph. As we will see, in certain graphs, different walks between the same pair of nodes will be associated with different transformations, and hence to conflicting signals, while in other graphs, walks between the same pair of nodes will always result in the same transformation and thus to reinforcement akin to resonance. The absence of conflicts along the walks is described by the notion of “coherence,” which we define now, and determines the asymptotic of dynamics studied in this Letter, similar to structural balance in the study of signed networks [42].

We define a matrix-weighted network G to be “coherent” if the transformation of every directed cycle is $\mathbf{I}$. An immediate consequence for a graph to be coherent is that all matrix weights are orthogonal under the assumptions adopted in this Letter, since for each edge $(v_i, v_j) \in E$, we have $(v_j, v_i) \in E$, and for the directed cycle of the two reciprocal edges to have transformation $\mathbf{I}$, we have $\mathbf{R}_{ij}\mathbf{R}_{ji} = \mathbf{R}_{ij}\mathbf{R}_{ij}^T = \mathbf{I}$. Note that this definition, which naturally describes the geometrical problems considered in this Letter, is strict; hence we also quantify the *level of coherence* by how far the network is from being coherent, e.g., through the portion of cycles that do not have transformation $\mathbf{I}$. Furthermore, the notion can be relaxed in other contexts, as in Supplemental Material (SM) [43].

The notion of coherence also characterizes particular block structures in the MWN. Suppose there is a partition of G such that (i) any edges within each block have transformation $\mathbf{I}$, and (ii) any edges between the same pair of blocks have the same transformation; see Fig. 2. Then if (iii) any directed cycles (of such blocks) formed by between-block edges have transformation $\mathbf{I}$, the MWN is coherent, since any directed cycles in the network, apart from edges inside each block, form directed cycles of these blocks (if not empty). We also show in SM (Sec. II.B) that such block structure always exists in coherent MWN, hence uniquely defining the coherence.

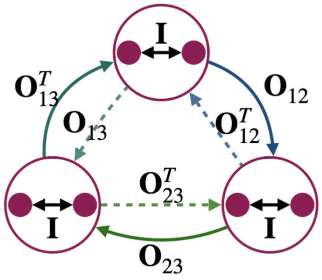

FIG. 2. Example of the block structure of a coherent MWN, where the label indicates the orthogonal matrix weights and $\mathbf{O}_{13} = \mathbf{O}_{12}\mathbf{O}_{23}$.

With the characteristic block structure, we can connect all coherent networks with the “identity-transformed MWN” where the transformation of each edge is $\mathbf{I}$. The identity-transformed MWN is a typical coherent MWN, and has supra-Laplacian,

$$\bar{\mathcal{L}} = \bar{\mathbf{L}} \otimes \mathbf{I},$$

where $\bar{\mathbf{L}} = \mathbf{D} - \bar{\mathbf{W}}$ and $\bar{\mathbf{W}} = (w_{ij})$ only maintains the magnitude of each edge weight; thus $\bar{\mathbf{L}}$ is the graph Laplacian for positive scalar weights. For a coherent MWN, we encode its intrinsic block structure into a block diagonal matrix, $\mathcal{S}$: its (i, i) block returns the transformation of any directed paths from nodes in the first block to node v_i’s block. Then,

$$\mathcal{L} = \mathcal{S}^T \bar{\mathcal{L}} \mathcal{S}.$$

Hence, the supra-Laplacian also has eigenvalue 0, and the associated eigenvectors span the subspace

$$\mathbf{X} = \mathcal{S}^T(\mathbf{1} \otimes \mathbf{I}). \tag{2}$$

We also show in SM (Sec. II.C) that the existence of 0 as an eigenvalue always leads to coherence, hence intrinsic to the coherent MWN. Additionally, we show in SM that coherent MWNs always have such a block structure. Note that nonidentity intrablock interactions can still yield coherence when an alternative partition of nodes into blocks satisfies the conditions.

Consensus dynamics—Consensus has been one of the most important dynamics on networks, with various applications [44–47]. Note that matrix relationship (or “logic constraints”) has been considered in the belief formation [16], but we incorporate transformation in the propagation process, as we will see later. Now, let us endow each node with a vector state variable, $\mathbf{y}_i \in \mathbb{R}^{n_d}$, and the interdependence between the dimensions are encoded in the matrix weights $\{\mathbf{W}_{ij}\}$. The "consensus dynamics" can be defined as

$$\dot{\mathbf{y}}_i = \sum_j w_{ij}(\mathbf{R}_{ij}\mathbf{y}_j - \mathbf{y}_i),$$

i.e., each node adjusts its state such that the difference to its neighbors, after the characteristic transformation, is reduced, with information from all dimensions integrated. Summarizing the states of all nodes into a vector $\mathbf{y} = (\mathbf{y}_1; \ldots; \mathbf{y}_n)$, we have $\dot{\mathbf{y}} = -\mathcal{L}\mathbf{y}$.

The node state vectors then exhibit distinct behaviors in the multidimensional space, depending on the underlying MWN. Indeed, we can solve the system where

$$\mathbf{y}(t) = \exp(-\mathcal{L}t)\mathbf{y}(0),$$

and thus the behavior depends on the eigenproperties of the supra-Laplacian. From the eigenvectors associated with eigenvalue 0 in Eq. (2), we can see that multiconsensus can be achieved at the steady state for coherent MWNs, where nodes in different blocks in the corresponding partition (e.g., Fig. 2) are characterized by different vectors (see Sec. II.D in SM for more detail). In all other networks, the consensus dynamics will reach the steady state of all-zero vector for all nodes, i.e., global consensus, in a sufficiently long time, with the relaxation time depending on the level of coherence. We show later that the consensus dynamics on such networks can exhibit timescale separation, where the states of nodes approach multiconsensus in a short time while converging to the all-zero state in the long run.

Random walks—Random walks are another important dynamics on networks, aiming at modeling how an entity randomly explores the underlying structure, with applications in various fields [48–55]. Now, let us consider a random walker that can randomly move on the network while changing its states in a high-dimensional space $\mathbb{R}^{n_d}$ that characterizes its features. Specifically, for a random walker on node v_i at time t, it will choose one of v_i's neighbors randomly at $t+1$, with probability w_{ij}/d_i for each neighbor v_j, while the characteristic vector will also transform according to the edge connecting them, through $\mathbf{R}_{ij}$. Hence, if we average the characteristic vectors at each node by the corresponding probability, denoted by $\mathbf{y}_i$ for each node v_i, the (discrete-time) random walks on the network can be described as

$$\mathbf{y}_j(t+1) = \sum_i \frac{w_{ij}}{d_i}\mathbf{R}_{ij}^T\mathbf{y}_i(t) = \sum_i \frac{\mathbf{W}_{ij}^T\mathbf{y}_i(t)}{d_i}.$$

Furthermore, if we consider $\mathbf{y} = (\mathbf{y}_1; \ldots; \mathbf{y}_n)$, then $\mathbf{y}(t+1) = \mathcal{P}^T\mathbf{y}(t) = (\mathcal{P}^T)^{t+1}\mathbf{y}(0)$, where $\mathcal{P}$ is the supra-transition matrix.

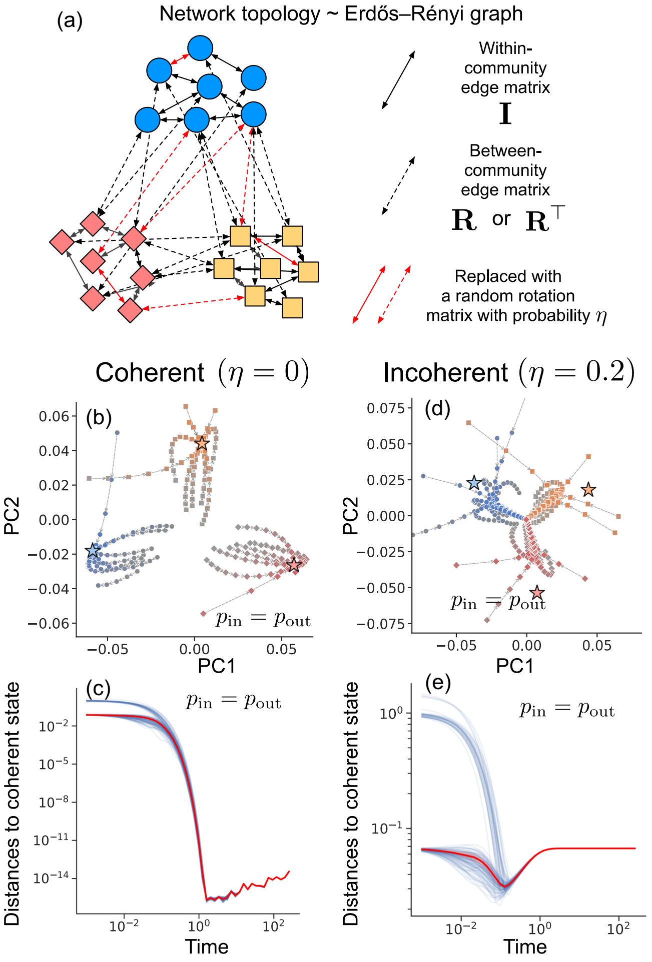

FIG. 3. Consensus dynamics in a three-block matrix-weighted stochastic block model with 10 dimensional state space. (a) Schematic of the network and the edge matrix weights, with the block structure solely encoded in the edge rotation matrices ($p_{\rm in} = p_{\rm out} = 0.3$). (b), (d) Principal component analysis projections of node state vectors onto the first two principal components (PC1, PC2); stars show theoretical steady states; each symbol represents a node with hue indicating its block membership and increasing saturation over time. (c), (e) Euclidean distances to steady states. The blue lines show individual node distances and red lines indicate average distances, with semitransparent color bands showing 95% confidence intervals derived from bootstrap methods.

With multidimensional interaction incorporated, the random walkers can explore the multidimensional sphere and exhibit averaged behavior that varies according to the structure of the MWN. If the network is coherent (and not bipartite), the averaged vector at the steady state will have different directions for nodes in different blocks associated with the coherence (see Sec. II.D in SM for more detail). In all other networks, either there will be no steady state, or the averaged vector at each node will be all-zero at steady state, i.e., there is no particular direction that is preferred by the random walkers. The relaxation time depends on the level of coherence.

Numerical results—We validate our theoretical results using a matrix-weighted stochastic block model. We generate networks with 120 nodes and three equal-sized communities, with edge probability $p_{\mathrm{in}} = 0.3$ within communities and $p_{\mathrm{out}} = 0.3$ between communities [Fig. 3(a)]. The matrix weights are constructed as follows. First, we create a base rotation matrix $\mathbf{R}$ by applying random 2D rotations between pairs of dimensions, with angles drawn uniformly from $(-\pi, \pi]$. For nodes v_i and v_j in blocks k and ℓ, respectively, we set $\mathbf{R}_{ij} = \mathbf{R}^{\ell-k}$, ensuring coherence by construction. To study the effects of incoherence, we introduce controlled perturbations: with probability η, each matrix weight is randomly rotated by angles drawn from $\mathcal{N}(0, \sigma^2)$.

Figures 3(b)–(e) compare dynamics on coherent versus incoherent networks in 10-dimensional state space. When the MWN is coherent ($\eta = 0$), nodes converge precisely to their theoretical steady states, with distinct vector directions for each block [Figs. 3(b) and (c)]. In contrast, incoherent MWNs ($\eta = 0.2$, $\sigma^2 = 0.1$) exhibit a two-phase behavior: nodes initially approach community-specific directions similar to the coherent case, but then slowly decay toward the origin [Figs. 3(d) and (e)]. This demonstrates how (small) perturbations to coherence can dramatically alter long-term dynamics while preserving short-term community structure. Random walk dynamics also exhibit analogous behavior (Sec. III in SM).

These results remain robust across different parameter regimes (see SM). Networks with stronger block structure ($p_{\mathrm{out}} = 0.1$) show qualitatively similar behavior but slower mixing in the incoherent case (Fig. 2 in SM). The behavior also persists with different state space dimensions ($n_d = 2$, 3, 5) and numbers of communities (Figs. 3–6 in SM), highlighting the generality of our framework. The key distinction between coherence and incoherence—preservation versus decay of community-specific directions in the steady states—appears universal across these variations.

Discussions—In this Letter, we introduced a novel framework of MWNs, laying the mathematical groundwork for analyzing both their structural and dynamical properties. This approach opens new avenues for research in network science. For instance, random walks underpin numerous techniques for extracting insights from network data, such as determining node importance through centrality measures [56] and developing efficient community detection algorithms [6,57]. Our framework provides an opportunity to extend these methods to datasets characterized by high-dimensional, interacting features. Our Letter also opens several fundamental questions for future research, e.g., to define relaxed versions of coherence, as we discuss in SM.

This Letter participates in the recent efforts to enrich the standard network paradigm to "higher orders" [58]. MWNs could provide a new angle to analyze and generalize models of multilayer networks [28] and temporal networks [59], explicitly representing transformations between multidimensional states and preserving the functional integrity of these transformations. Take a system where each node is copied in n_d layers. Its connections via interlayer and intralayer connections to other nodes can alternatively be encoded in a matrix, making it possible to represent the system as an MWN. The fundamental difference between multilayer networks and MWNs is that the former separates relationships into distinct layers, while the latter stacks different dimensions together and encodes linear transformations in edges, leading to collective interactions. In addition, our definition of the matrix of a path, as the product of matrices of its edges, is a natural generalization of switching networks [60], a popular model for temporal networks, where the system is modeled as a random sequence of matrices, instead of a sequence determined by a walk on a graph as we consider here. In fact, any network with any node groupings can be modeled as an MWN, by defining the state of a node group that represents a larger-scale entity as a vector and the interaction between groups as a matrix. While coherence is not a generic property of MWNs, its theoretical significance parallels that of connected components in network analysis: although most real-world networks are connected, the concept of disconnected components underlies foundational results like the Cheeger inequality and informs the analysis of timescale separation and community structure. Likewise, coherence serves as a critical conceptual anchor for modeling collective, structured multidimensional interactions, which in turn informs the design of empirical tools for analyzing empirical MWNs. We hope that MWNs not only deepen our understanding of complex systems but also provide a flexible framework for modeling a wide range of systems that have multidimensional dynamics at multiple scales.

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[39] The code and documentation to reproduce all numerical results can be found at our GitHub repository: https://github.com/skojaku/matrix-weight-net.
[40] We note that each matrix weight may not be symmetric.
[41] We note that for a directed walk P with composing edges $e_1, e_2 \ldots, e_k$, if we represent $e_l = (u_l, v_l) (l = 1, \ldots, k)$, then the only requirement is that $u_l = v_{l-1}$ for all $l = 2, \ldots, k$. A directed walk becomes a directed path if there is no other repetition of nodes in all composing edges, and a directed walk becomes a directed cycle if the former is true apart from the starting and end nodes being the same.
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