

ON MOTIVIC VANISHING CYCLES OF CRITICAL LOCI

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Abstract

Let U be a smooth scheme over an algebraically closed field \mathbb{K} of characteristic zero and $f : U \rightarrow \mathbb{A}^1$ a regular function, and write $X = \text{Crit}(f)$, as a closed \mathbb{K} -subscheme of U . The *motivic vanishing cycle* $MF_{U,f}^{\text{mot},\phi}$ is an element of the $\hat{\mu}$ -equivariant motivic Grothendieck ring $\mathcal{M}_X^{\hat{\mu}}$, defined by Denef and Loeser [8, 9] and Looijenga [18], and used in Kontsevich and Soibelman’s theory of motivic Donaldson–Thomas invariants [16].

We prove three main results:

(a) $MF_{U,f}^{\text{mot},\phi}$ depends only on the third-order thickenings $U^{(3)}, f^{(3)}$ of U, f .

(b) If V is another smooth \mathbb{K} -scheme, $g : V \rightarrow \mathbb{A}^1$ is regular, $Y = \text{Crit}(g)$, and $\Phi : U \rightarrow V$ is an embedding with $f = g \circ \Phi$ and $\Phi|_X : X \rightarrow Y$ an isomorphism, then $\Phi|_X^*(MF_{V,g}^{\text{mot},\phi}) = MF_{U,f}^{\text{mot},\phi} \odot \Upsilon(P_\Phi)$ in a certain quotient ring $\bar{\mathcal{M}}_X^{\hat{\mu}}$ of $\mathcal{M}_X^{\hat{\mu}}$, where $P_\Phi \rightarrow X$ is a principal \mathbb{Z}_2 -bundle associated to Φ and $\Upsilon : \{\text{principal } \mathbb{Z}_2\text{-bundles on } X\} \rightarrow \bar{\mathcal{M}}_X^{\hat{\mu}}$ a natural morphism.

(c) If (X, s) is an *oriented algebraic d -critical locus* in the sense of Joyce [14], there is a natural motive $MF_{X,s} \in \bar{\mathcal{M}}_X^{\hat{\mu}}$, such that if (X, s) is locally modelled on $\text{Crit}(f : U \rightarrow \mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $MF_{U,f}^{\text{mot},\phi}$.

Using results of Pantev, Toën, Vezzosi and Vaquié [21], these imply the existence of natural motives on moduli schemes of coherent sheaves on a Calabi–Yau 3-fold equipped with ‘orientation data’, as required in Kontsevich and Soibelman’s motivic Donaldson–Thomas theory [16], and on intersections $L \cap M$ of oriented Lagrangians L, M in an algebraic symplectic manifold (S, ω) .

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1. Introduction

Brav, Bussi, Dupont, Joyce and Szendrői [3] proved some results on perverse sheaves, \mathcal{D} -modules and mixed Hodge modules of vanishing cycles on critical loci, and gave some applications to categorification of Donaldson–Thomas invariants of Calabi–Yau 3-folds, and to defining ‘Fukaya categories’ of complex or algebraic symplectic manifolds using perverse sheaves. This paper is a sequel to [3], in which we prove analogous results for motives and motivic vanishing cycles, with applications to motivic Donaldson–Thomas invariants.

Let \mathbb{K} be an algebraically closed field of characteristic zero, U a smooth \mathbb{K} -scheme, $f : U \rightarrow \mathbb{A}^1$ a regular function, and $U_0 = f^{-1}(0)$, $X = \text{Crit}(f)$ as closed \mathbb{K} -subschemes of U . Following Denef and Loeser [8, 9] and Looijenga [18], in §2 we will define the *motivic nearby cycle* $MF_{U,f}^{\text{mot}}$ in the monodromic Grothendieck group $K_0^{\hat{\mu}}(U_0)$ of $\hat{\mu}$ -equivariant motives on U_0 , and the *motivic vanishing cycle* $MF_{U,f}^{\text{mot},\phi}$ in the ring $\mathcal{M}_X^{\hat{\mu}} = K_0^{\hat{\mu}}(X)[\mathbb{L}^{-1}]$ with Tate motive $\mathbb{L} = [\mathbb{A}^1]$ inverted.

Here $MF_{U,f}^{\text{mot}}$ is the motivic analogue of the constructible complex of nearby cycles $\psi_f(\mathbb{Q}_U) \in \text{Perv}(U_0)$ in [3], and $MF_{U,f}^{\text{mot},\phi}$ the motivic analogue of the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^{\bullet} = \phi_f(\mathbb{Q}_U[\dim U - 1]) \in \text{Perv}(X)$ in [3] (at least when $X \subseteq U_0$). The fibre $MF_{U,f}^{\text{mot}}(x)$ of $MF_{U,f}^{\text{mot}}$ at $x \in U_0$ is the *motivic Milnor fibre* of f at x from [8, 9, 18], the algebraic analogue of the Milnor fibre $MF_f(x)$ at x of a holomorphic function $f : U \rightarrow \mathbb{C}$ on a complex manifold U .

We will prove three main results, Theorems 3.2, 4.4 and 5.10. The first, Theorem 3.2, says that $MF_{U,f}^{\text{mot},\phi} \in \mathcal{M}_X^{\hat{\mu}}$ depends only on the third-order thickenings $U^{(3)}, f^{(3)}$ of U, f at X , where $\mathcal{O}_{U^{(3)}} = \mathcal{O}_U/I_X^3$, for $I_X \subseteq \mathcal{O}_U$ the

ideal of functions $U \rightarrow \mathbb{A}^1$ vanishing on X , and $f^{(3)} = f|_{U^{(3)}}$. We also show by example that $U^{(2)}, f^{(2)}$ do not determine $MF_{U,f}^{\text{mot},\phi}$.

Our second and third main results involve *principal \mathbb{Z}_2 -bundles* $P \rightarrow X$ over a \mathbb{K} -scheme X . In §2.5 we define a natural motive $\Upsilon(P) \in \mathcal{M}_X^\mu$ for each principal \mathbb{Z}_2 -bundle $P \rightarrow X$. As in Denef and Loeser [9] and Looijenga [18], there is a (non-obvious) commutative, associative multiplication \odot on \mathcal{M}_X^μ which appears in the motivic Thom–Sebastiani Theorem [8, 9, 18].

In §4–§5 we need the $\Upsilon(P)$ to satisfy $\Upsilon(P \otimes_{\mathbb{Z}_2} Q) = \Upsilon(P) \odot \Upsilon(Q)$ for all principal \mathbb{Z}_2 -bundles $P, Q \rightarrow X$, but we cannot prove this in \mathcal{M}_X^μ . Our solution in §2.5 is to define a new ring of motives $\bar{\mathcal{M}}_Y^\mu$ for each \mathbb{K} -scheme Y to be the quotient of $(\mathcal{M}_Y^\mu, \odot)$ by the ideal generated by pushforwards $\phi_*(\Upsilon(P \otimes_{\mathbb{Z}_2} Q) - \Upsilon(P) \odot \Upsilon(Q))$ for all \mathbb{K} -scheme morphisms $\phi : X \rightarrow Y$ and principal \mathbb{Z}_2 -bundles $P, Q \rightarrow X$, and then $\Upsilon(P \otimes_{\mathbb{Z}_2} Q) = \Upsilon(P) \odot \Upsilon(Q)$ holds in $\bar{\mathcal{M}}_X^\mu$.

Essentially the same issue occurs in Kontsevich and Soibelman [16], which inspired this part of our paper. In defining the motivic rings $\bar{\mathcal{M}}^\mu(X)$ in which their motivic Donaldson–Thomas invariants take values, in [16, §4.5] they impose a complicated relation, which as in [16, §5.1] implies that the motivic vanishing cycle $MF_{E,q}^{\text{mot},\phi}$ of a nondegenerate quadratic form q on a vector bundle $E \rightarrow U$ depends only on the triple $(\text{rank } E, \Lambda^{\text{top}} E, \det q)$. As in §2.5, this implies our relation $\Upsilon(P \otimes_{\mathbb{Z}_2} Q) = \Upsilon(P) \odot \Upsilon(Q)$. So Kontsevich and Soibelman’s ring $\bar{\mathcal{M}}^\mu(X)$ is a quotient of our ring $\bar{\mathcal{M}}_X^\mu$.

Our second main result, Theorem 4.4, says that if U, V are smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ are regular, $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$, and $\Phi : U \rightarrow V$ is an embedding with $f = g \circ \Phi$ and $\Phi|_X : X \rightarrow Y$ an isomorphism, then $\Phi|_X^*(MF_{V,g}^{\text{mot},\phi}) = MF_{U,f}^{\text{mot},\phi} \odot \Upsilon(P_\Phi)$ in $\bar{\mathcal{M}}_X^\mu$, for $P_\Phi \rightarrow X$ a principal \mathbb{Z}_2 -bundle parametrizing orientations of the nondegenerate quadratic form $\text{Hess } g$ on $N_{UV}|_X$, with $N_{UV} \rightarrow U$ the normal bundle of $\Phi(U)$ in V . The analogous result [3, Th. 5.4] for perverse sheaves of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$ says that $\Phi|_X^*(\mathcal{PV}_{V,g}^\bullet) \cong \mathcal{PV}_{U,f}^\bullet \otimes_{\mathbb{Z}_2} P_\Phi$.

For U, V, f, g, Φ as above, [14, Prop. 2.23] shows that étale locally on V we have equivalences $V \sim U \times \mathbb{A}^n$ identifying $g \sim f \boxplus z_1^2 + \cdots + z_n^2$ and $\Phi \sim \text{id}_U \times 0$. So if we could work étale locally, we would have

$$\begin{aligned} \Phi|_X^*(MF_{V,g}^{\text{mot},\phi}) &\sim (\text{id}_X \times 0)^*(MF_{U \times \mathbb{A}^n, f \boxplus z_1^2 + \cdots + z_n^2}^{\text{mot},\phi}) \\ &= MF_{U,f}^{\text{mot},\phi} \boxtimes MF_{\mathbb{A}^n, z_1^2 + \cdots + z_n^2}^{\text{mot},\phi} = MF_{U,f}^{\text{mot},\phi} \boxtimes 1_{\{0\}} = MF_{U,f}^{\text{mot},\phi}, \end{aligned}$$

using the motivic Thom–Sebastiani theorem in the second step. However, for motives we must work Zariski locally, so we need a more complicated proof involving the (étale locally trivial) correction factor $\Upsilon(P_\Phi)$. In singularity

theory, passing from f to $f \boxplus z_1^2 + \cdots + z_n^2$ is known as *stabilization*, so Theorem 4.4 studies the behaviour of motivic vanishing cycles under stabilization.

Our third main result, Theorem 5.10, concerns a new class of geometric objects called *d-critical loci*, introduced in Joyce [14], and explained in §5.1. An (algebraic) d-critical locus (X, s) over \mathbb{K} is a \mathbb{K} -scheme X with a section s of a certain natural sheaf \mathcal{S}_X^0 on X . A d-critical locus (X, s) may be written Zariski locally as a critical locus $\text{Crit}(f : U \rightarrow \mathbb{A}^1)$ of a regular function f on a smooth \mathbb{K} -scheme U , and s records some information about U, f (in the notation above, s remembers $f^{(2)}$). There is also a complex analytic version.

Algebraic d-critical loci are classical truncations of the *derived critical loci* (more precisely, *-1-shifted symplectic derived schemes*) introduced in derived algebraic geometry by Pantev, Toën, Vaquié and Vezzosi [21]. Theorem 5.10 roughly says that if (X, s) is an algebraic d-critical locus over \mathbb{K} with an ‘orientation’, then we may define a natural motive $MF_{X,s}$ in $\overline{\mathcal{M}}_X^{\hat{\mu}}$, such that if (X, s) is locally modelled on $\text{Crit}(f : U \rightarrow \mathbb{A}^1)$ then $MF_{X,s}$ is locally modelled on $MF_{U,f}^{\text{mot}, \phi} \odot \Upsilon(P)$, where $P \rightarrow X$ is a principal \mathbb{Z}_2 -bundle relating the ‘orientations’ on (X, s) and $\text{Crit}(f)$. The proof uses Theorem 4.4.

Bussi, Brav and Joyce [4] prove Darboux-type theorems for the k -shifted symplectic derived schemes of Pantev et al. [21], and use them to construct a truncation functor from -1 -shifted symplectic derived schemes to algebraic d-critical loci. Combining this with results of [3, 14, 21] and this paper gives new results on categorifying Donaldson–Thomas invariants of Calabi–Yau 3-folds, and on motivic Donaldson–Thomas invariants. In particular, as we explain in §5.2, Kontsevich and Soibelman [16] wish to associate a motivic Milnor fibre to each point of the moduli \mathbb{K} -schemes $\mathcal{M}_{\text{st}}^{\alpha}(\tau)$ of τ -stable coherent sheaves on a Calabi–Yau 3-fold over \mathbb{K} . The issue of how these vary in families over the base $\mathcal{M}_{\text{st}}^{\alpha}(\tau)$ is not really addressed in [16]. Our paper answers this question.

In the rest of the paper, §2 introduces motivic Milnor fibres and motivic vanishing cycles, and §3–§5 state and prove Theorems 3.2, 4.4 and 5.10.

Ben-Bassat, Brav, Bussi and Joyce [2] will extend the results of [3, 4] and this paper from (derived) schemes to (derived) Artin stacks.

Conventions. Throughout we work over a base field \mathbb{K} which is algebraically closed and of characteristic zero, for instance $\mathbb{K} = \mathbb{C}$. All \mathbb{K} -schemes are assumed to be of finite type, unless we explicitly say otherwise. We discuss extending our results to \mathbb{K} -schemes locally of finite type in Remark 5.11.

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2. Background material

We begin by discussing rings of motives, motivic Milnor fibres, and motivic vanishing cycles. Sections 2.1–2.3 broadly follow Denef and Loeser [7–10] and Looijenga [18]. Sections 2.4–2.5 contain some new material, much of which is based on ideas in Kontsevich and Soibelman [16].

2.1. Rings of motives on a \mathbb{K} -scheme X . We define ($\hat{\mu}$ -equivariant) Grothendieck groups of schemes.

Definition 2.1. Let X be a \mathbb{K} -scheme (always assumed of finite type). By an X -scheme we mean a \mathbb{K} -scheme S together with a morphism $\Pi_S^X : S \rightarrow X$. The X -schemes form a category Sch_X , with morphisms $\alpha : S \rightarrow T$ satisfying $\Pi_S^X = \Pi_T^X \circ \alpha$.

Write $K_0(\text{Sch}_X)$ for the *Grothendieck group* of Sch_X . It is the abelian group generated by symbols $[S]$, for S an X -scheme, with relations $[S] = [T]$ if $S \cong T$ in Sch_X , and $[S] = [T] + [S \setminus T]$ if $T \subseteq S$ is a closed X -subscheme. There is a natural commutative ring structure on $K_0(\text{Sch}_X)$, with $[S] \cdot [T] = [S \times_X T]$.

Write \mathbb{A}_X^1 for the X -scheme $\pi_X : \mathbb{A}^1 \times X \rightarrow X$, and define $\mathbb{L} = [\mathbb{A}_X^1]$ in $K_0(\text{Sch}_X)$. We denote by $\mathcal{M}_X = K_0(\text{Sch}_X)[\mathbb{L}^{-1}]$ the ring obtained from $K_0(\text{Sch}_X)$ by inverting \mathbb{L} . When $X = \text{Spec } \mathbb{K}$ we write $K_0(\text{Sch}_{\mathbb{K}}), \mathcal{M}_{\mathbb{K}}$ instead of $K_0(\text{Sch}_X), \mathcal{M}_X$. If X, Y are \mathbb{K} -schemes there are natural *external products*

$$(2.1) \quad \begin{aligned} \boxtimes : K_0(\text{Sch}_X) \times K_0(\text{Sch}_Y) &\rightarrow K_0(\text{Sch}_{X \times Y}), \quad \boxtimes : \mathcal{M}_X \times \mathcal{M}_Y \rightarrow \mathcal{M}_{X \times Y} \\ \text{with } [\Pi_S^X : S \rightarrow X] \boxtimes [\Pi_T^Y : T \rightarrow Y] &= [\Pi_{S \times T}^X \times \Pi_T^Y : S \times T \rightarrow X \times Y]. \end{aligned}$$

If $f : X \rightarrow Y$ is a morphism of \mathbb{K} -schemes, we define *pushforwards* $f_* : K_0(\text{Sch}_X) \rightarrow K_0(\text{Sch}_Y)$, $f_* : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ and *pullbacks* $f^* : K_0(\text{Sch}_Y) \rightarrow K_0(\text{Sch}_X)$, $f^* : \mathcal{M}_Y \rightarrow \mathcal{M}_X$ by $f_*([\Pi_S^X : S \rightarrow X]) = [f \circ \Pi_S^X : S \rightarrow Y]$ and $f^*([\Pi_T^Y : T \rightarrow Y]) = [\pi_X : S \times_{\Pi_T^Y, Y, f} X \rightarrow X]$. They have the usual functorial properties.

Definition 2.2. For $n = 1, 2, \dots$, write μ_n for the group of all n^{th} roots of unity in \mathbb{K} , which is assumed algebraically closed of characteristic zero as in §1, so that $\mu_n \cong \mathbb{Z}_n$. Then μ_n is the \mathbb{K} -scheme $\text{Spec}(\mathbb{K}[x]/(x^n - 1))$. The μ_n form a projective system, with respect to the maps $\mu_{nd} \rightarrow \mu_n$ mapping $x \mapsto x^d$ for all $d, n = 1, 2, \dots$. Define the group $\hat{\mu}$ to be the projective limit of the μ_n . Note that $\hat{\mu}$ is not a \mathbb{K} -scheme, but is a pro-scheme.

Let S be an X -scheme. A *good μ_n -action on S* is a group action $\sigma_n : \mu_n \times S \rightarrow S$ which is a morphism of X -schemes, such that each orbit is contained in an affine subscheme of S . This last condition is automatically satisfied when S is quasi-projective. A *good $\hat{\mu}$ -action on S* is a group action $\hat{\sigma} : \hat{\mu} \times S \rightarrow S$ which factors through a good μ_n -action, for some n . We will write $\hat{\iota} : \hat{\mu} \times S \rightarrow S$ for the trivial $\hat{\mu}$ -action on S , for any S , which is automatically good.

The *monodromic Grothendieck group* $K_0^{\hat{\mu}}(\text{Sch}_X)$ is the abelian group generated by symbols $[S, \hat{\sigma}]$, for S an X -scheme and $\hat{\sigma} : \hat{\mu} \times S \rightarrow S$ a good $\hat{\mu}$ -action, with the relations:

- (i) $[S, \hat{\sigma}] = [T, \hat{\tau}]$ if S, T are isomorphic as X -schemes with $\hat{\mu}$ -actions;
- (ii) $[S, \hat{\sigma}] = [T, \hat{\sigma}|_T] + [S \setminus T, \hat{\sigma}|_{S \setminus T}]$ if $T \subseteq S$ is a closed, $\hat{\mu}$ -invariant X -subscheme of S ; and
- (iii) $[S \times \mathbb{A}^n, \hat{\sigma} \times \hat{\tau}_1] = [S \times \mathbb{A}^n, \hat{\sigma} \times \hat{\tau}_2]$ for any linear $\hat{\mu}$ -actions $\hat{\tau}_1, \hat{\tau}_2$ on \mathbb{A}^n .

There is a natural commutative ring structure on $K_0^{\hat{\mu}}(\text{Sch}_X)$ with multiplication ‘ \cdot ’ defined by $[S, \hat{\sigma}] \cdot [T, \hat{\tau}] = [S \times_X T, \hat{\sigma} \times \hat{\tau}]$. Write $\mathbb{L} = [\mathbb{A}_X^1, \hat{\iota}]$ in $K_0^{\hat{\mu}}(\text{Sch}_X)$. Define $\mathcal{M}_X^{\hat{\mu}} = K_0^{\hat{\mu}}(\text{Sch}_X)[\mathbb{L}^{-1}]$ to be the ring obtained from $K_0^{\hat{\mu}}(\text{Sch}_X)$ (with multiplication ‘ \cdot ’) by inverting \mathbb{L} . When $X = \text{Spec } \mathbb{K}$ we write $K_0^{\hat{\mu}}(\text{Sch}_{\mathbb{K}}), \mathcal{M}_{\mathbb{K}}^{\hat{\mu}}$ instead of $K_0^{\hat{\mu}}(\text{Sch}_X), \mathcal{M}_X^{\hat{\mu}}$.

If X, Y are \mathbb{K} -schemes there are natural *external products*

$$(2.2) \quad \begin{aligned} \boxtimes : K_0^{\hat{\mu}}(\text{Sch}_X) \times K_0^{\hat{\mu}}(\text{Sch}_Y) &\rightarrow K_0^{\hat{\mu}}(\text{Sch}_{X \times Y}), \quad \boxtimes : \mathcal{M}_X^{\hat{\mu}} \times \mathcal{M}_Y^{\hat{\mu}} \rightarrow \mathcal{M}_{X \times Y}^{\hat{\mu}} \\ \text{with } [S \rightarrow X, \hat{\sigma}] \boxtimes [T \rightarrow Y, \hat{\tau}] &= [S \times T \rightarrow X \times Y, \hat{\sigma} \times \hat{\tau}]. \end{aligned}$$

Pushforwards and pullbacks work for the rings $\mathcal{M}_X^{\hat{\mu}}$ in the obvious way.

There are also natural morphisms of commutative rings

$$\begin{aligned} i_X : K_0^{\hat{\mu}}(\text{Sch}_X) &\longrightarrow K_0^{\hat{\mu}}(\text{Sch}_X), \quad i_X : \mathcal{M}_X \longrightarrow \mathcal{M}_X^{\hat{\mu}}, \quad i_X : [S] \longmapsto [S, \hat{\iota}], \\ \Pi_X : K_0^{\hat{\mu}}(\text{Sch}_X) &\longrightarrow K_0(\text{Sch}_X), \quad \Pi_X : \mathcal{M}_X^{\hat{\mu}} \longrightarrow \mathcal{M}_X, \quad \Pi_X : [S, \hat{\sigma}] \longmapsto [S]. \end{aligned}$$

Following Looijenga [18, §7] and Denef and Loeser [9, §5], we introduce a second commutative, associative multiplication ‘ \odot ’ on $K_0^{\hat{\mu}}(\text{Sch}_X), \mathcal{M}_X^{\hat{\mu}}$ (written ‘ $*$ ’ in [9, 18]).

Definition 2.3. Let X be a \mathbb{K} -scheme, and $[S, \hat{\sigma}], [T, \hat{\tau}]$ generators of $K_0^{\hat{\mu}}(\text{Sch}_X)$ or $\mathcal{M}_X^{\hat{\mu}}$. Then there exists $n \geq 1$ such that the $\hat{\mu}$ -actions $\hat{\sigma}, \hat{\tau}$ on S, T factor through μ_n -actions σ_n, τ_n . Define J_n to be the Fermat curve

$$J_n = \{(t, u) \in (\mathbb{A}^1 \setminus \{0\})^2 : t^n + u^n = 1\}.$$

Let $\mu_n \times \mu_n$ act on $J_n \times (S \times_X T)$ by

$$(\alpha, \alpha') \cdot ((t, u), (v, w)) = ((\alpha \cdot t, \alpha' \cdot u), (\sigma_n(\alpha)(v), \tau_n(\alpha')(w))).$$

Write $J_n(S, T) = (J_n \times (S \times_X T)) / (\mu_n \times \mu_n)$ for the quotient \mathbb{K} -scheme, and define a μ_n -action v_n on $J_n(S, T)$ by

$$v_n(\alpha)((t, u), v, w) = ((\alpha \cdot t, \alpha \cdot u), v, w)(\mu_n \times \mu_n).$$

Let \hat{v} be the induced good $\hat{\mu}$ -action on $J_n(S, T)$, and set

$$(2.3) \quad [S, \hat{\sigma}] \odot [T, \hat{\tau}] = (\mathbb{L} - 1) \cdot [(S \times_X T) / \mu_n, \hat{\iota}] - [J_n(S, T), \hat{v}]$$

in $K_0^{\hat{\mu}}(\text{Sch}_X)$ or $\mathcal{M}_X^{\hat{\mu}}$. This turns out to be independent of n , and defines commutative, associative products \odot on $K_0^{\hat{\mu}}(\text{Sch}_X), \mathcal{M}_X^{\hat{\mu}}$.

Now let X, Y be \mathbb{K} -schemes. As for (2.1)–(2.2), we define products

$$\square : K_0^{\hat{\mu}}(\text{Sch}_X) \times K_0^{\hat{\mu}}(\text{Sch}_Y) \rightarrow K_0^{\hat{\mu}}(\text{Sch}_{X \times Y}), \quad \square : \mathcal{M}_X^{\hat{\mu}} \times \mathcal{M}_Y^{\hat{\mu}} \rightarrow \mathcal{M}_{X \times Y}^{\hat{\mu}}$$

by following the definition above for $[S, \hat{\sigma}] \in K_0^{\hat{\mu}}(\text{Sch}_X)$ and $[T, \hat{\tau}] \in K_0^{\hat{\mu}}(\text{Sch}_Y)$, but taking products $S \times T$ rather than fibre products $S \times_X T$. These \square are also commutative and associative in the appropriate sense.

Taking $Y = \text{Spec } \mathbb{K}$ and using $X \times \text{Spec } \mathbb{K} \cong X$, we see that \square makes $K_0^{\hat{\mu}}(\text{Sch}_X), \mathcal{M}_X^{\hat{\mu}}$ into modules over $K_0^{\hat{\mu}}(\text{Sch}_{\mathbb{K}}), \mathcal{M}_{\mathbb{K}}^{\hat{\mu}}$.

For generators $[S, \hat{\sigma}]$ and $[T, \hat{\iota}] = i_X([T])$ in $K_0^{\hat{\mu}}(\text{Sch}_X)$ or $\mathcal{M}_X^{\hat{\mu}}$ where $[T, \hat{\iota}]$ has trivial $\hat{\mu}$ -action $\hat{\iota}$, one can show that $[S, \hat{\sigma}] \odot [T, \hat{\iota}] = [S, \hat{\sigma}] \cdot [T, \hat{\iota}]$. Thus i_X is a ring morphism $(K_0(\text{Sch}_X), \cdot) \rightarrow (K_0^{\hat{\mu}}(\text{Sch}_X), \odot)$. However, Π_X is not a ring morphism $(K_0^{\hat{\mu}}(\text{Sch}_X), \odot) \rightarrow (K_0(\text{Sch}_X), \cdot)$. Since $\mathbb{L} = [\mathbb{A}_X^1, \hat{\iota}]$ this implies that $M \cdot \mathbb{L} = M \odot \mathbb{L}$ for all M in $K_0^{\hat{\mu}}(\text{Sch}_X)$ or $\mathcal{M}_X^{\hat{\mu}}$.

Remark 2.4. Our principal references for this section [7–9, 18] work in terms not of \mathbb{K} -schemes but \mathbb{K} -varieties, by which they mean reduced, separated \mathbb{K} -schemes of finite type, not necessarily irreducible.

We assume that all our schemes are of finite type unless we explicitly say otherwise. The reduced and separated conditions in [7–9, 18] are not actually needed, but are included as an aesthetic choice, as mathematicians in some areas prefer varieties to schemes. However, our results (Theorem 3.2 for instance) concern non-reduced schemes, and would be false if we replaced schemes by varieties, so we have fixed on working with finite type schemes.

To see that dropping the reduced and separated conditions changes nothing, note that if S is an X -scheme then the reduced X -subscheme $S^{\text{red}} \subseteq S$ is closed with $S \setminus S^{\text{red}} = \emptyset$, so $[S] = [S^{\text{red}}]$ in $K_0(\text{Sch}_X)$. Also, any non-separated scheme can be cut into finitely many separated schemes using the relation $[S] = [T] + [S \setminus T]$ for $T \subseteq S$ closed. Therefore $K_0(\text{Var}_X) \cong K_0(\text{Sch}_X)$.

Example 2.5. Define elements $\mathbb{L}^{1/2}$ in $K_0^{\hat{\mu}}(\text{Sch}_X)$ and $\mathcal{M}_X^{\hat{\mu}}$ by

$$(2.4) \quad \mathbb{L}^{1/2} = [X, \hat{\iota}] - [X \times \mu_2, \hat{\rho}],$$

where $[X, \hat{\iota}]$ with trivial $\hat{\mu}$ -action $\hat{\iota}$ is the identity in $K_0^{\hat{\mu}}(\text{Sch}_X), \mathcal{M}_X^{\hat{\mu}}$, and $X \times \mu_2 = X \times \{1, -1\}$ is two copies of X with nontrivial $\hat{\mu}$ -action $\hat{\rho}$ induced by the left action of μ_2 on itself, exchanging the two copies of X . Applying (2.3) with $n = 2$, we can show that $\mathbb{L}^{1/2} \odot \mathbb{L}^{1/2} = \mathbb{L}$. Thus, $\mathbb{L}^{1/2}$ in (2.4) is a square root for \mathbb{L} in the rings $(K_0^{\hat{\mu}}(\text{Sch}_X), \odot)$ and $(\mathcal{M}_X^{\hat{\mu}}, \odot)$. Note that $\mathbb{L}^{1/2} \cdot \mathbb{L}^{1/2} \neq \mathbb{L}$.

We can now define unique elements $\mathbb{L}^{n/2}$ in $K_0^{\hat{\mu}}(\text{Sch}_X)$ for all $n = 0, 1, 2, \dots$ and $\mathbb{L}^{n/2} \in \mathcal{M}_X^{\hat{\mu}} = K_0^{\hat{\mu}}(\text{Sch}_X)[\mathbb{L}^{-1}]$ for all $n \in \mathbb{Z}$ in the obvious way, such that $\mathbb{L}^{m/2} \odot \mathbb{L}^{n/2} = \mathbb{L}^{(m+n)/2}$ for all m, n .

2.2. Arc spaces and the motivic zeta function.

Definition 2.6. Let U be a \mathbb{K} -scheme. For each $n \in \mathbb{N} = \{0, 1, \dots\}$ we consider the space $\mathcal{L}_n(U)$ of arcs modulo t^{n+1} on U . This is a \mathbb{K} -scheme, whose K -points, for any field K containing \mathbb{K} , are the $K[t]/t^{n+1}K[t]$ -points of U . For $n \leq m$ there are projections $\pi_m^n : \mathcal{L}_m(U) \rightarrow \mathcal{L}_n(U)$ mapping $\alpha \mapsto \alpha \bmod t^{m+1}$. Note that $\mathcal{L}_0(U) = U$ and $\mathcal{L}_1(U)$ is the tangent sheaf of U .

Example 2.7. If $U \subseteq \mathbb{A}^m$ is a \mathbb{K} -scheme defined by $f_k(x_1, \dots, x_m) = 0$ for $k = 1, \dots, l$, then $\mathcal{L}_n(U) \subseteq \mathbb{A}^{m(n+1)}$ is given in the variables x_i^j for $i = 1, \dots, m$ and $j = 0, \dots, n$ by the equations

$$f_k(x_1^0 + x_1^1 t + \dots + x_1^n t^n, \dots, x_m^0 + x_m^1 t + \dots + x_m^n t^n) \equiv 0 \bmod t^{n+1}, \quad k = 1, \dots, l.$$

We now recall the motivic zeta function from Denef and Loeser [9, §3.2].

Definition 2.8. Let U be a smooth \mathbb{K} -scheme and $f : U \rightarrow \mathbb{A}^1$ a non-constant regular function. Then f induces morphisms $\mathcal{L}_n(f) : \mathcal{L}_n(U) \rightarrow \mathcal{L}_n(\mathbb{A}^1)$ for $n \geq 1$. Any point β of $\mathcal{L}_n(\mathbb{A}^1)$ yields a power series $\beta(t) \in K[[t]]/t^{n+1}$, for some field K containing \mathbb{K} . So we define maps

$$\text{ord}_t : \mathcal{L}_n(\mathbb{A}^1) \rightarrow \{0, 1, \dots, n, \infty\},$$

with $\text{ord}_t \beta$ the largest m such that t^m divides $\beta(t)$. Set

$$\mathfrak{U}_n := \{\alpha \in \mathcal{L}_n(U) : \text{ord}_t(\mathcal{L}_n(f)(\alpha)) = n\}.$$

This is a locally closed subscheme of $\mathcal{L}_n(U)$. Note that \mathfrak{U}_n is actually a U_0 -scheme, through the morphism $\pi_0^n : \mathcal{L}_n(U) \rightarrow U$, where U_0 denotes the locus of $f = 0$ in U . Indeed $\pi_0^n(\mathfrak{U}_n) \subset U_0$, since $n \geq 1$. We consider the morphism

$$\bar{f}_n : \mathfrak{U}_n \rightarrow \mathbb{G}_m := \mathbb{A}^1 \setminus \{0\},$$

sending a point α in \mathfrak{U}_n to the coefficient of t^n in $\mathcal{L}_n(f)(\alpha)$. There is a natural action of \mathbb{G}_m on \mathfrak{U}_n given by $a \cdot \alpha(t) = \alpha(at)$, where $\alpha(t)$ is the vector of power series corresponding to α in some local coordinate system. Since $\bar{f}_n(a \cdot \alpha) = a^n \bar{f}_n(\alpha)$ it follows that \bar{f}_n is an étale locally trivial fibration.

We denote by $\mathfrak{U}_{n,1}$ the fibre $\bar{f}_n^{-1}(1)$. Note that the action of \mathbb{G}_m on \mathfrak{U}_n induces a good action ρ_n of μ_n (and hence a good action $\hat{\rho}$ of $\hat{\mu}$) on $\mathfrak{U}_{n,1}$. Since \bar{f}_n is a locally trivial fibration, the U_0 -scheme $\mathfrak{U}_{n,1}$ and the action of μ_n on it, completely determines both the scheme \mathfrak{U}_n and the morphism

$$(\bar{f}_n, \pi_0^n) : \mathfrak{U}_n \longrightarrow \mathbb{G}_m \times U_0.$$

Indeed it is easy to verify that \mathfrak{U}_n , as a $(\mathbb{G}_m \times U_0)$ -scheme, is isomorphic to the quotient of $\mathfrak{U}_{n,1} \times \mathbb{G}_m$ under the μ_n -action defined by $a(\alpha, b) = (a\alpha, a^{-1}b)$.

The *motivic zeta function* $Z_f(T) \in \mathcal{M}_{U_0}^{\hat{\mu}}[[T]]$ of $f : U \rightarrow \mathbb{A}^1$ is the power series in T over $\mathcal{M}_{U_0}^{\hat{\mu}}$ defined by

$$(2.5) \quad Z_f(T) := \sum_{n \geq 1} [\mathfrak{U}_{n,1} \rightarrow U_0, \hat{\rho}] \mathbb{L}^{-n \dim U} T^n.$$

We will recall a formula for $Z_f(T)$ in terms of resolution of singularities.

Definition 2.9. Let U be a smooth \mathbb{K} -scheme and $f : U \rightarrow \mathbb{A}^1$ a non-constant regular function. By Hironaka's Theorem [11] we can choose a *resolution* (V, π) of f . That is, V is a smooth \mathbb{K} -scheme and $\pi : V \rightarrow U$ a proper morphism, such that $\pi|_{V \setminus \pi^{-1}(U_0)} : V \setminus \pi^{-1}(U_0) \rightarrow U \setminus U_0$ is an isomorphism for $U_0 = f^{-1}(0)$, and $\pi^{-1}(U_0)$ has only normal crossings as a \mathbb{K} -subscheme of V .

Write $E_i, i \in J$ for the irreducible components of $\pi^{-1}(U_0)$. For each $i \in J$, denote by N_i the multiplicity of E_i in the divisor of $f \circ \pi$ on V , and by $\nu_i - 1$ the multiplicity of E_i in the divisor of $\pi^*(dx)$, where dx is a local non vanishing volume form at any point of $\pi(E_i)$. For $I \subset J$, we consider the smooth \mathbb{K} -scheme $E_I^\circ = (\bigcap_{i \in I} E_i) \setminus (\bigcup_{j \in J \setminus I} E_j)$.

Let $m_I = \gcd(N_i)_{i \in I}$. We introduce an unramified Galois cover \tilde{E}_I° of E_I° , with Galois group μ_{m_I} , as follows. Let V' be an affine Zariski open subset of V , such that, on V' , $f \circ \pi = uv^{m_I}$, with $u : V' \rightarrow \mathbb{A}^1 \setminus \{0\}$ and $v : V' \rightarrow \mathbb{A}^1$. Then the restriction of \tilde{E}_I° above $E_I^\circ \cap V'$, denoted by $\tilde{E}_I^\circ \cap V'$, is defined as

$$\tilde{E}_I^\circ \cap V' = \{(z, w) \in \mathbb{A}^1 \times (E_I^\circ \cap V') : z^{m_I} = u(w)^{-1}\}.$$

Note that E_I° can be covered by such affine open subsets V' of V . Gluing together the covers $\tilde{E}_I^\circ \cap V'$ in the obvious way, we obtain the cover \tilde{E}_I° of E_I° which has a natural μ_{m_I} -action ρ_I , obtained by multiplying the z -coordinate with the elements of μ_{m_I} . This μ_{m_I} -action on \tilde{E}_I° induces a $\hat{\mu}$ -action $\hat{\rho}_I$ on \tilde{E}_I° in the obvious way.

Denef and Loeser [10] and Looijenga [18] prove that $Z_f(T)$ is rational:

Theorem 2.10. *In the situation of Definition 2.9, in $\mathcal{M}_{U_0}^{\hat{\mu}}[[T]]$ we have:*

$$(2.6) \quad Z_f(T) = \sum_{\emptyset \neq I \subset J} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ \rightarrow U_0, \hat{\rho}_I] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

2.3. Motivic nearby and vanishing cycles. Following Denef and Loeser [7, 9, 10], we state the main definition of the section:

Definition 2.11. Let U be a smooth \mathbb{K} -scheme and $f : U \rightarrow \mathbb{A}^1$ a regular function. If f is non-constant, expanding the rational function $Z_f(T)$ as a power series in T^{-1} and taking minus its constant term, yields a well defined

element of $\mathcal{M}_{U_0}^{\hat{\mu}}$, which we call *motivic nearby cycle* of f . Namely,

$$(2.7) \quad MF_{U,f}^{\text{mot}} := - \lim_{T \rightarrow \infty} Z_f(T) = \sum_{\emptyset \neq I \subseteq J} (1 - \mathbb{L})^{|I|-1} [\tilde{E}_I^\circ \rightarrow U_0, \hat{\rho}_I].$$

If f is constant we set $MF_{U,f}^{\text{mot}} = 0$. We write

$$MF_{U,f}^{\text{mot}}(x) := \text{Fibre}_x(MF_{U,f}^{\text{mot}}) \in \mathcal{M}_{\mathbb{K}}^{\hat{\mu}}$$

for each $x \in U_0$, which we call the *motivic Milnor fibre* of f at x .

Now let $X = \text{Crit}(f) \subseteq U$, as a closed \mathbb{K} -subscheme of U , and $X_0 = X \cap U_0$. Consider the restriction $MF_{U,f}^{\text{mot}}|_{U_0 \setminus X_0}$ in $\mathcal{M}_{U_0 \setminus X_0}^{\hat{\mu}}$. In Definition 2.9 we can choose (V, π) with $\pi|_{V \setminus \pi^{-1}(X_0)} : V \setminus \pi^{-1}(X_0) \rightarrow U \setminus X_0$ an isomorphism. Write D_1, \dots, D_k for the irreducible components of $\pi^{-1}(U_0 \setminus X_0) \cong U_0 \setminus X_0$. They are disjoint as $\pi^{-1}(U_0 \setminus X_0)$ is nonsingular. The closures $\overline{D}_1, \dots, \overline{D}_k$ (which need not be disjoint) are among the divisors E_i , so we write $\overline{D}_a = E_{i_a}$ for $a = 1, \dots, k$, with $\{i_1, \dots, i_k\} \subseteq I$. Clearly $N_{i_a} = \nu_{i_a} = 1$ for $a = 1, \dots, k$.

Then in (2.7) the only nonzero contributions to $MF_{U,f}^{\text{mot}}|_{U_0 \setminus X_0}$ are from $I = \{i_a\}$ for $a = 1, \dots, k$, with $\tilde{E}_{\{i_a\}}^\circ \cong E_{\{i_a\}}^\circ \cong D_a$, and the $\hat{\mu}$ -action on $\tilde{E}_{\{i_a\}}^\circ$ is trivial as it factors through the action of $\mu_1 = \{1\}$. Hence

$$MF_{U,f}^{\text{mot}}|_{U_0 \setminus X_0} = \sum_{a=1}^k [\tilde{E}_{\{i_a\}}^\circ, \hat{\ell}] = \sum_{a=1}^k [D_a, \hat{\ell}] = [U_0 \setminus X_0, \hat{\ell}].$$

Therefore $[U_0, \hat{\ell}] - MF_{U,f}^{\text{mot}}$ is supported on $X_0 \subseteq U_0$, and by restricting to X_0 we regard it as an element of $\mathcal{M}_{X_0}^{\hat{\mu}}$.

As $f|_X : X \rightarrow \mathbb{A}^1$ is locally constant on $X^{\text{red}} = \text{Crit}(f)^{\text{red}}$, $f(X)$ is finite, and $X = \coprod_{c \in f(X)} X_c$ with $X_c = X \cap U_c$, where $U_c = f^{-1}(c) \subset U$. Define $MF_{U,f}^{\text{mot}, \phi} \in \mathcal{M}_X^{\hat{\mu}}$ by

$$(2.8) \quad MF_{U,f}^{\text{mot}, \phi}|_{X_c} = \mathbb{L}^{-\dim U/2} \odot ([U_c, \hat{\ell}] - MF_{U,f-c}^{\text{mot}})|_{X_c} \in \mathcal{M}_{X_c}^{\hat{\mu}}$$

for each $c \in f(X)$, where $\mathbb{L}^{-\dim U/2} \in \mathcal{M}_{X_c}^{\hat{\mu}}$ is defined in Example 2.5, and the product \odot in Definition 2.3. We call $MF_{U,f}^{\text{mot}, \phi}$ the *motivic vanishing cycle* of f .

Sometimes it is convenient to regard $MF_{U,f}^{\text{mot}, \phi}$ as an element of $\mathcal{M}_U^{\hat{\mu}}$ supported on X , via the inclusion $\mathcal{M}_X^{\hat{\mu}} \hookrightarrow \mathcal{M}_U^{\hat{\mu}}$. Also, for each $x \in X$ we set

$$(2.9) \quad MF_{U,f}^{\text{mot}, \phi}(x) := \mathbb{L}^{-\dim U/2} \odot (1 - MF_{U,f}^{\text{mot}}(x)) \in \mathcal{M}_{\mathbb{K}}^{\hat{\mu}}.$$

In §3–§4 we will use the fact that if U, V are smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ are regular, $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$, and $\Phi : U \rightarrow V$ is étale with $f = g \circ \Phi$, so that $\Phi|_X : X \rightarrow Y$ is étale, then $\Phi|_X^*(MF_{V,g}^{\text{mot}, \phi}) = MF_{U,f}^{\text{mot}, \phi}$.

Remark 2.12. (a) Because of (2.6) the right hand side of (2.7) is independent of the choice of resolution (V, π) , although a priori this is not obvious.

(b) As in [9, §3.5], one should regard $MF_{U,f}^{\text{mot}}(x)$ as the correct motivic incarnation of the *Milnor fibre* $MF_{U,f}(x)$ of f at x when $\mathbb{K} = \mathbb{C}$, which is in itself not at all motivic. This is indeed true for the Hodge realization [9, Th. 3.10]. Moreover, using the perverse sheaf notation of Brav et al. [3, §2], $MF_{U,f}^{\text{mot}}$ is the virtual motivic incarnation of the *complex of nearby cycles* $\psi_f(\mathbb{Q}_U)$ of U, f in $D_c^b(U_0)$, and $MF_{U,f}^{\text{mot}, \phi}$ as the virtual motivic incarnation of the *perverse sheaf of vanishing cycles* $\mathcal{PV}_{U,f}^\bullet \in \text{Perv}(X)$ of U, f , defined by

$$(2.10) \quad \mathcal{PV}_{U,f}^\bullet|_{X_c} = \phi_{f-c}[-1](\mathbb{Q}_U[\dim U]) \quad \text{for all } c \in f(X).$$

Note the analogy between (2.8) and (2.10).

(c) Our formulae (2.8)–(2.9) are based on Denef and Loeser [9, Not. 3.9], but with a different normalization, since Denef and Loeser have $(-1)^{\dim U}$ rather than $\mathbb{L}^{-\dim U/2}$. As in Examples 2.13 and 2.15 below, our normalization ensures that $MF_{V,q}^{\text{mot}, \phi} = 1$ whenever q is a nondegenerate quadratic form on a finite-dimensional \mathbb{K} -vector space V . The normalizing factor $\mathbb{L}^{-\dim U/2}$ also appears in Kontsevich and Soibelman [16, §5.1].

Example 2.13. Define $f : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ by $f(z) = z^2$. In Definition 2.9 we may take $U = V = \mathbb{A}^1$ and $\pi = \text{id}_{\mathbb{A}^1}$. Then $\pi^{-1}(0) = \{0\}$ is one divisor $E_0 = \{0\}$, with $N_0 = 2$ and $\nu_0 = 1$. In (2.6) the only nonzero term is $I = \{0\}$, and $\tilde{E}_{\{0\}}^\circ = \mu_2 = \{1, -1\}$ is two points with $\hat{\mu}$ -action $\hat{\rho}$ induced by the left action of μ_2 on itself. Hence $Z_f(T) = [\mu_2, \hat{\rho}] \cdot (\mathbb{L}^{-1}T^2)/(1 - \mathbb{L}^{-1}T^2)$. Taking the limit $T \rightarrow \infty$, equation (2.7) yields $MF_{\mathbb{A}^1, z^2}^{\text{mot}} = [\mu_2, \hat{\rho}]$. Thus (2.4) and (2.8) give

$$(2.11) \quad MF_{\mathbb{A}^1, z^2}^{\text{mot}, \phi} = \mathbb{L}^{-1/2} \odot (1 - [\mu_2, \hat{\rho}]) = \mathbb{L}^{-1/2} \odot \mathbb{L}^{1/2} = 1.$$

2.4. The motivic Thom–Sebastiani Theorem. Here is the motivic Thom–Sebastiani Theorem of Denef–Loeser and Looijenga [8, 9, 18], stated using the notation of §2.1–§2.3.

Theorem 2.14. *Let U, V be smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ regular functions, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$. Write $f \boxplus g : U \times V \rightarrow \mathbb{A}^1$ for the regular function mapping $f \boxplus g : (u, v) \mapsto f(u) + g(v)$. Then $MF_{U \times V, f \boxplus g}^{\text{mot}, \phi} = MF_{U, f}^{\text{mot}, \phi} \boxtimes MF_{V, g}^{\text{mot}, \phi}$ in $\mathcal{M}_{X \times Y}^{\hat{\mu}}$.*

Example 2.15. Define $f : \mathbb{A}^n \rightarrow \mathbb{A}^1$ by $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$ for $n \geq 1$. Then using Theorem 2.14, induction on n , and (2.11) shows that

$$(2.12) \quad MF_{\mathbb{A}^n, z_1^2 + \dots + z_n^2}^{\text{mot}, \phi} = MF_{\mathbb{A}^1, z^2}^{\text{mot}, \phi} \boxtimes \dots \boxtimes MF_{\mathbb{A}^1, z^2}^{\text{mot}, \phi} = 1 \boxtimes \dots \boxtimes 1 = 1.$$

If V is a finite-dimensional \mathbb{K} -vector space and q a nondegenerate quadratic form on V , then $(V, q) \cong (\mathbb{A}^n, z_1^2 + \cdots + z_n^2)$ for $n = \dim V$, so $MF_{V,q}^{\text{mot},\phi} = 1$. The purpose of the factors $\mathbb{L}^{-\dim U/2}$ in (2.8)–(2.9) was to achieve this.

Our next result shows how motivic vanishing cycles change under stabilization by a nondegenerate quadratic form. The term $\mathbb{L}^{\dim U/2} \odot MF_{E,q}^{\text{mot},\phi}$ in (2.13) may be regarded as the *relative motivic vanishing cycle* of (E, q) relative to U .

Theorem 2.16. *Let U be a smooth \mathbb{K} -scheme, $\pi : E \rightarrow U$ a vector bundle over U , $f : U \rightarrow \mathbb{A}^1$ a regular function, q a nondegenerate quadratic form on E , and $X = \text{Crit}(f)$. Regard (the total space of) E as a smooth \mathbb{K} -scheme and $q, f \circ \pi : E \rightarrow \mathbb{A}^1$ as regular functions on E , so that $f \circ \pi + q : E \rightarrow \mathbb{A}^1$ is also a regular function. Identify U with the zero section in E , so that $X \subseteq U \subseteq E$, and we have $\mathcal{M}_X^\mu \subseteq \mathcal{M}_U^\mu \subseteq \mathcal{M}_E^\mu$. Then in \mathcal{M}_E^μ we have*

$$(2.13) \quad MF_{E, f \circ \pi + q}^{\text{mot},\phi} = MF_{U, f}^{\text{mot},\phi} \odot (\mathbb{L}^{\dim U/2} \odot MF_{E, q}^{\text{mot},\phi}).$$

Proof. Consider the function $f \boxplus q : U \times E \rightarrow \mathbb{A}^1$ and morphism $\pi \times \text{id} : E \rightarrow U \times E$. We have $(f \boxplus q) \circ (\pi \times \text{id}) = f \circ \pi + q : E \rightarrow \mathbb{A}^1$. We first claim that

$$(2.14) \quad (\pi \times \text{id})^*(Z_{f \boxplus q}(T)) = Z_{f \circ \pi + q}(T) \in \mathcal{M}_E^\mu[[T]].$$

To see this, note that étale locally on $U \times E$, there exist isomorphisms $U \times E \cong E \times \mathbb{A}^m$, where $m = \dim U$, making the following two diagrams equivalent:

$$(2.15) \quad \begin{array}{ccc} E & \xrightarrow{f \circ \pi + q} & \mathbb{A}^1 \\ \downarrow \pi \times \text{id} & \searrow & \\ U \times E & \xrightarrow{f \boxplus q} & \mathbb{A}^1 \end{array}, \quad \begin{array}{ccc} E & \xrightarrow{f \circ \pi + q} & \mathbb{A}^1 \\ \downarrow \text{id} \times 0 & \searrow & \\ E \times \mathbb{A}^m & \xrightarrow{(f \circ \pi + q) \boxplus 0} & \mathbb{A}^1. \end{array}$$

It follows that in Definition 2.8, we have étale local isomorphisms

$$(2.16) \quad \begin{aligned} (\pi \times \text{id})^*((\mathfrak{U} \times \mathfrak{E})_n) &\cong \mathfrak{E}_n \times \mathcal{L}_n(\mathbb{A}^m)_0, \\ (\pi \times \text{id})^*((\mathfrak{U} \times \mathfrak{E})_{n,1}) &\cong \mathfrak{E}_{n,1} \times \mathcal{L}_n(\mathbb{A}^m)_0, \end{aligned}$$

where $\mathfrak{E}_n, \mathfrak{E}_{n,1}$ and $(\mathfrak{U} \times \mathfrak{E})_n, (\mathfrak{U} \times \mathfrak{E})_{n,1}$ are $\mathfrak{U}_n, \mathfrak{U}_{n,1}$ in Definition 2.8 for $f \circ \pi + q : E \rightarrow \mathbb{A}^1$ and $f \boxplus q : U \times E \rightarrow \mathbb{A}^1$, and $\mathcal{L}_n(\mathbb{A}^m)_0$ is the subspace of arcs in $\mathcal{L}_n(\mathbb{A}^m)$ based at 0. The second equation also holds with $\hat{\mu}$ -actions, where $\mathcal{L}_n(\mathbb{A}^m)_0$ has the trivial $\hat{\mu}$ -action $\hat{\iota}$.

We claim that the isomorphisms (2.16) also hold Zariski locally, and thus on the level of motives. To see this, note that we may cover U by Zariski open neighbourhoods $U' \subseteq U$, such that on U' we can choose étale coordinates (z_1, \dots, z_m) , and writing $E' = E|_{U'}$ and $q' = q|_{U'}$, we can choose an algebraic connection ∇ on E' which preserves q' . Since (2.16) over E' concerns formal arcs starting in the images $(\pi \times \text{id})(E')$ and $E' \times 0$ in $U' \times E'$

and $E' \times \mathbb{A}^m$, it depends only on the formal completions $(\widehat{U' \times E'})_{(\pi \times \text{id})(E')}$ and $(\widehat{E' \times \mathbb{A}^m})_{E' \times 0}$.

Informally, we think of points of $(\widehat{U' \times E'})_{(\pi \times \text{id})(E')}$ as pairs $(u_0, (u_1, e_1))$ where $u_0, u_1 \in U'$, and $e_1 \in E'|_{u_1}$, and u_0, u_1 are ‘infinitesimally close’ in U' , and $f \boxplus q$ maps $(u_0, (u_1, e_1)) \mapsto f(u_0) + q'|_{u_1}(e_1)$. Similarly, we think of points of $(\widehat{E' \times \mathbb{A}^m})_{E' \times 0}$ as pairs $((u, e), x)$, where $u \in U'$, and $e \in E'|_u$, and $x \in \mathbb{A}^m$ is ‘infinitesimally close’ to 0, and $(f \circ \pi + q) \boxplus 0$ maps $((u, e), x) \mapsto f(u) + q'|_u(e)$.

We can now define a unique isomorphism of formal schemes

$$\Phi : (\widehat{U' \times E'})_{(\pi \times \text{id})(E')} \xrightarrow{\cong} (\widehat{E' \times \mathbb{A}^m})_{E' \times 0}$$

which (informally) on points maps $\Phi : (u_0, (u_1, e_1)) \mapsto ((u, e), x)$, where $u = u_0$, and $x = (z_1(u_1) - z_1(u_0), \dots, z_m(u_1) - z_m(u_0))$, and $e \in E'|_{u_0}$ is the unique point such that if $\phi : (\widehat{\mathbb{A}^m})_0 \rightarrow U'$ is the unique morphism of formal schemes with $\phi(0) = u_0$, $\phi(x) = u_1$ and $z_i \circ \phi(y_1, \dots, y_m) = z_i(u_0) + y_i$ for $i = 1, \dots, m$, then parallel translation from $t = 0$ to $t = 1$ along the formal path $t \mapsto \phi(t \cdot x)$ in U' for $t \in \mathbb{A}^1$ in the vector bundle $E' \rightarrow U'$ using the connection ∇ maps $e \in E'|_{u_0}$ at $t = 0$ to $e_1 \in E'|_{u_1}$ at $t = 1$. As ∇ preserves q' we have $q'|_{u_0}(e) = q'|_{u_1}(e_1)$, so that $f(u_0) + q'|_{u_1}(e_1) = f(u) + q'|_u(e)$. Thus Φ is compatible with the commutative triangles (2.15), and induces Zariski local isomorphisms (2.16) over U' , proving the claim.

Since $[\mathcal{L}_n(\mathbb{A}^m)_0, \hat{l}] = \mathbb{L}^{mn}$, equation (2.16) at the level of motives gives

$$(\pi \times \text{id})^*[(\mathcal{U} \times \mathfrak{E})_{n,1}, \hat{\rho}] = [\mathfrak{E}_{n,1}, \hat{\rho}] \cdot \mathbb{L}^{mn}.$$

Multiplying this by $\mathbb{L}^{-n(\dim E + m)} T^n$, summing over all $n \geq 1$, and using (2.5), proves (2.14). Taking the limit $T \rightarrow \infty$ in (2.14) and using (2.7) yields

$$(2.17) \quad (\pi \times \text{id})^*(MF_{U \times E, f \boxplus q}^{\text{mot}}) = MF_{E, f \circ \pi + q}^{\text{mot}}.$$

Now both sides of (2.13) are supported on $X \subseteq U \subseteq E$, and as in Definition 2.11 $X = \coprod_{c \in f(X)} X_c$ with $X_c = X \cap f^{-1}(c)$. For each $c \in f(X)$ we have

$$\begin{aligned} MF_{E, f \circ \pi + q}^{\text{mot}, \phi}|_{X_c} &= \mathbb{L}^{-\dim E/2} \odot ([E_c, \hat{l}] - MF_{E, (f-c) \circ \pi + q}^{\text{mot}})|_{X_c} \\ &= \mathbb{L}^{\dim U/2} \odot (\pi \times \text{id})|_{X_c}^* [\mathbb{L}^{-\dim U/2 - \dim E/2} \odot \\ &\quad ([(U \times E)_c, \hat{l}] - MF_{U \times E, (f-c) \boxplus q}^{\text{mot}})|_{X_c \times U}] \\ &= \mathbb{L}^{\dim U/2} \odot (\pi \times \text{id})|_{X_c}^* [MF_{U \times E, f \boxplus q}^{\text{mot}, \phi}|_{X_c \times U}] \\ &= \mathbb{L}^{\dim U/2} \odot (\pi \times \text{id})|_{X_c}^* [MF_{U, f}^{\text{mot}, \phi} \boxtimes MF_{E, q}^{\text{mot}, \phi}|_{X_c \times U}] \\ &= MF_{U, f}^{\text{mot}, \phi} \odot (\mathbb{L}^{\dim U/2} \odot MF_{E, q}^{\text{mot}, \phi})|_{X_c}, \end{aligned}$$

using (2.8) in the first and third steps, (2.17) with $f - c$ in place of f in the second, Theorem 2.14 in the fourth, and comparing the definitions of \odot and

\square in Definition 2.3 in the fifth: $MF_{U,f}^{\text{mot},\phi} \odot MF_{E,q}^{\text{mot},\phi}$ involves a fibre product (over X or U or E , all have the same effect), but $MF_{U,f}^{\text{mot},\phi} \square MF_{E,q}^{\text{mot},\phi}$ has no fibre product, and the effect of $(\pi \times \text{id})^*$ is to take the fibre product. This proves the restriction of (2.13) to X_c for each $c \in f(X)$, and the theorem follows. \square

2.5. Motives of principal \mathbb{Z}_2 -bundles. We define principal \mathbb{Z}_2 -bundles $P \rightarrow X$, associated motives $\Upsilon(P)$, and a quotient ring of motives $\bar{\mathcal{M}}_X^{\hat{\mu}}$ in which $\Upsilon(P \otimes_{\mathbb{Z}_2} Q) = \Upsilon(P) \odot \Upsilon(Q)$ for all P, Q .

Definition 2.17. Let X be a \mathbb{K} -scheme. A *principal \mathbb{Z}_2 -bundle* $P \rightarrow X$ is a proper, surjective, étale morphism of \mathbb{K} -schemes $\pi : P \rightarrow X$ together with a free involution $\sigma : P \rightarrow P$, such that the orbits of $\mathbb{Z}_2 = \{1, \sigma\}$ are the fibres of π . The *trivial \mathbb{Z}_2 -bundle* is $\pi_X : X \times \mathbb{Z}_2 \rightarrow X$. We will use the ideas of *isomorphism* of principal bundles $\iota : P \rightarrow Q$, *section* $s : X \rightarrow P$, *tensor product* $P \otimes_{\mathbb{Z}_2} Q$, and *pullback* $f^*(P) \rightarrow W$ under a morphism of \mathbb{K} -schemes $f : W \rightarrow X$, all of which are defined in the obvious ways.

Write $\mathbb{Z}_2(X)$ for the abelian group of isomorphism classes $[P]$ of principal \mathbb{Z}_2 -bundles $P \rightarrow X$, with multiplication $[P] \cdot [Q] = [P \otimes_{\mathbb{Z}_2} Q]$ and identity $[X \times \mathbb{Z}_2]$. Since $P \otimes_{\mathbb{Z}_2} P \cong X \times \mathbb{Z}_2$ for each $P \rightarrow X$, each element of $\mathbb{Z}_2(X)$ is self-inverse, and has order 1 or 2.

If $P \rightarrow X$ is a principal \mathbb{Z}_2 -bundle over X , define a motive

$$(2.18) \quad \Upsilon(P) = \mathbb{L}^{-1/2} \odot ([X, \hat{\iota}] - [P, \hat{\rho}]) \in \mathcal{M}_X^{\hat{\mu}},$$

where $\hat{\rho}$ is the $\hat{\mu}$ -action on P induced by the μ_2 -action on P from the principal \mathbb{Z}_2 -bundle structure, as $\mu_2 \cong \mathbb{Z}_2$. If $P = X \times \mathbb{Z}_2$ is the trivial \mathbb{Z}_2 -bundle then

$$\Upsilon(X \times \mathbb{Z}_2) = \mathbb{L}^{-1/2} \odot ([X, \hat{\iota}] - [X \times \mathbb{Z}_2, \hat{\rho}]) = \mathbb{L}^{-1/2} \odot \mathbb{L}^{1/2} \odot [X, \hat{\iota}] = [X, \hat{\iota}],$$

using (2.4). Note that $[X, \hat{\iota}]$ is the identity in the ring $\mathcal{M}_X^{\hat{\mu}}$.

As $\Upsilon(P)$ only depends on P up to isomorphism, Υ factors through $\mathbb{Z}_2(X)$, and we may consider Υ as a map $\mathbb{Z}_2(X) \rightarrow \mathcal{M}_X^{\hat{\mu}}$.

For our applications in §4–§5 we want $\Upsilon : \mathbb{Z}_2(X) \rightarrow \mathcal{M}_X^{\hat{\mu}}$ to be a group morphism with respect to the multiplication \odot on $\mathcal{M}_X^{\hat{\mu}}$, but we cannot prove that it is. Our (somewhat crude) solution is to pass to a quotient ring $\bar{\mathcal{M}}_X^{\hat{\mu}}$ of $\mathcal{M}_X^{\hat{\mu}}$ such that the induced map $\Upsilon : \mathbb{Z}_2(X) \rightarrow \bar{\mathcal{M}}_X^{\hat{\mu}}$ is a group morphism.

In fact we want more than this: if we simply define $\bar{\mathcal{M}}_X^{\hat{\mu}}$ to be the quotient ring of $\mathcal{M}_X^{\hat{\mu}}$ by the relations $\Upsilon(P \otimes_{\mathbb{Z}_2} Q) - \Upsilon(P) \odot \Upsilon(Q) = 0$ for all $[P], [Q]$ in $\mathbb{Z}_2(X)$ then pushforwards $\phi_* : \bar{\mathcal{M}}_X^{\hat{\mu}} \rightarrow \bar{\mathcal{M}}_Y^{\hat{\mu}}$ will not be defined for general morphisms $\phi : X \rightarrow Y$. The proof of Theorem 5.10 below implicitly uses pushforwards ϕ_* for Zariski open inclusions $\phi : R \hookrightarrow X$, and for the generalization to stacks [2, Prop. 5.8, Th. 5.14] we will need pushforwards ϕ_* for smooth $\phi : X \rightarrow Y$. We will define $\bar{\mathcal{M}}_X^{\hat{\mu}}$ so that all pushforwards exist.

So, for each \mathbb{K} -scheme Y , define I_Y^μ to be the ideal in the commutative ring $(\mathcal{M}_Y^\mu, \odot)$ generated by elements $\phi_*(\Upsilon(P \otimes_{\mathbb{Z}_2} Q) - \Upsilon(P) \odot \Upsilon(Q))$ for all \mathbb{K} -scheme morphisms $\phi : X \rightarrow Y$ and principal \mathbb{Z}_2 -bundles $P, Q \rightarrow X$, and define $\bar{\mathcal{M}}_Y^\mu = \mathcal{M}_Y^\mu / I_Y^\mu$ to be the quotient, as a commutative ring with multiplication ‘ \odot ’, with projection $\Pi_Y^\mu : \mathcal{M}_Y^\mu \rightarrow \bar{\mathcal{M}}_Y^\mu$.

Note that in $\bar{\mathcal{M}}_Y^\mu$ we do not have the second multiplication ‘ \cdot ’, since we do not require I_Y^μ to be an ideal in $(\mathcal{M}_Y^\mu, \cdot)$. Also the external product \boxtimes and projection $\Pi_Y : \mathcal{M}_Y^\mu \rightarrow \mathcal{M}_Y$ on \mathcal{M}_Y^μ do not descend to $\bar{\mathcal{M}}_Y^\mu$. Apart from these, all of §2.1–§2.4, in particular the operations \odot, \boxtimes , pushforwards ϕ_* and pullbacks ϕ^* , elements $\mathbb{L}, \mathbb{L}^{1/2}, MF_{U,f}^{\text{mot}}, MF_{U,f}^{\text{mot}, \phi}, \Upsilon(P)$, and Theorems 2.10, 2.14 and 2.16, make sense in $\bar{\mathcal{M}}_Y^\mu$ rather than \mathcal{M}_Y^μ by applying Π_Y^μ . We will use the same notation in $\bar{\mathcal{M}}_Y^\mu$ as in \mathcal{M}_Y^μ .

Taking $Y = X$ and $\phi = \text{id}_X$, we see that $\bar{\mathcal{M}}_X^\mu$ has the property that

$$(2.19) \quad \Upsilon(P \otimes_{\mathbb{Z}_2} Q) = \Upsilon(P) \odot \Upsilon(Q) \quad \text{in } \bar{\mathcal{M}}_X^\mu$$

for all principal \mathbb{Z}_2 -bundles $P, Q \rightarrow X$.

Remark 2.18. (a) When we define a ring R by generators and relations, such as $K_0(\text{Sch}_X), K_0^\mu(\text{Sch}_X), \mathcal{M}_X^\mu, \bar{\mathcal{M}}_X^\mu$, and we impose an apparently arbitrary relation, such as Definition 2.2(iii), or the quotient by I_X^μ in Definition 2.17, then there is a risk that R may be small or even zero. If so, theorems we prove in R will be of little or no value. Thus, when we make such a definition, we should justify that R is ‘reasonably large’, for instance by producing morphisms from R to other interesting rings. We do this in **(b), (c)**.

(b) Our definition of $\bar{\mathcal{M}}_X^\mu$ is based on the motivic rings of Kontsevich and Soibelman [16]. In [16, §4.3] they discuss motivic rings $\text{Mot}^\mu(X)$ which coincide with our $K_0^\mu(\text{Sch}_X)$, as in Denef and Loeser [9]. Then in [16, §4.5] they define motivic rings $\bar{\mathcal{M}}^\mu(X)$ as the quotient of $\text{Mot}^\mu(X)$ by a complicated relation involving cohomological equivalence of algebraic cycles, and inverting \mathbb{L} . As in [16, §5.1], this equivalence relation implies that the analogue of Theorem 2.20 holds in their $\bar{\mathcal{M}}^\mu(X)[\mathbb{L}^{-1}]$, which implies that (2.19) also holds in their $\bar{\mathcal{M}}^\mu(X)[\mathbb{L}^{-1}]$. It follows that their $\bar{\mathcal{M}}^\mu(X)[\mathbb{L}^{-1}]$ is a quotient of our $\bar{\mathcal{M}}_X^\mu$.

(c) As in Kontsevich and Soibelman [16, 17], a major goal in Donaldson–Thomas theory is *categorification*, i.e. the replacement of $K_0(\text{Sch}_X), K_0^\mu(\text{Sch}_X)$ by the Grothendieck groups of suitable categories of motives. To each \mathbb{K} -scheme X one should associate triangulated \mathbb{Q} -linear tensor categories $\text{DM}_X^\mathbb{Q}, \text{DM}_X^{\mu, \mathbb{Q}}$ with functors $M_X : \text{Sch}_X \rightarrow \text{DM}_X^\mathbb{Q}, M_X^\mu : \text{Sch}_X^\mu \rightarrow \text{DM}_X^{\mu, \mathbb{Q}}$ satisfying a package of properties we will not discuss. Brav et al. [3] prove categorified versions of the results of this paper, in the contexts of \mathbb{Q} -linear perverse sheaves,

\mathcal{D} -modules, and mixed Hodge modules. A kind of universal categorification may be given by Voevodsky's category of motives [5, 23].

Using similar arguments to [16, §4.5, §5.1], involving the triangulated \mathbb{Q} -linear properties of $\mathrm{DM}_X^{\hat{\mu}, \mathbb{Q}}$ in an essential way, one can show that for any such categorification with the right properties, an analogue of Theorem 2.20 below holds in $K_0(\mathrm{DM}_X^{\hat{\mu}, \mathbb{Q}})[\mathbb{L}^{-1}]$, so $K_0(M_X^{\hat{\mu}}) : K_0(\mathrm{Sch}_X^{\hat{\mu}})[\mathbb{L}^{-1}] \rightarrow K_0(\mathrm{DM}_X^{\hat{\mu}, \mathbb{Q}})[\mathbb{L}^{-1}]$ factors via our ring $\bar{\mathcal{M}}_X^{\hat{\mu}}$, giving a morphism $\bar{\mathcal{M}}_X^{\hat{\mu}} \rightarrow K_0(\mathrm{DM}_X^{\hat{\mu}, \mathbb{Q}})[\mathbb{L}^{-1}]$.

Now there is a natural 1-1 correspondence

$$(2.20) \quad \begin{aligned} \mathbb{Z}_2(X) &= \{ \text{isomorphism classes } [P] \text{ of principal } \mathbb{Z}_2\text{-bundles } P \rightarrow X \} \\ &\longleftrightarrow \{ \text{isomorphism classes } [L, \iota] \text{ of pairs } (L, \iota), \text{ where } L \rightarrow X \text{ is} \\ &\quad \text{a line bundle and } \iota : L \otimes_{\mathcal{O}_X} L \rightarrow \mathcal{O}_X \text{ an isomorphism} \}, \end{aligned}$$

defined as follows: to each principal \mathbb{Z}_2 -bundle $\pi : P \rightarrow X$ we associate the line bundle $L = (P \times \mathbb{A}^1)/\mathbb{Z}_2$ over X (identifying line bundles with their total spaces), where \mathbb{Z}_2 acts in the given way on P and as $z \mapsto -z$ on \mathbb{A}^1 . The isomorphism $\iota : L \otimes_{\mathcal{O}_X} L \rightarrow \mathcal{O}_X$ acts on points by $\iota((p, z)\mathbb{Z}_2) \otimes ((p, z')\mathbb{Z}_2) = (x, zz') \in X \times \mathbb{A}^1 = \mathcal{O}_X$, where $p \in P$ with $\pi(p) = x$ and $z, z' \in \mathbb{A}^1$. Conversely, to each line bundle $\lambda : L \rightarrow X$ with isomorphism $\iota : L \otimes_{\mathcal{O}_X} L \rightarrow \mathcal{O}_X$, we define P to be the \mathbb{K} -subscheme of points $l \in L$ with $\iota(l \otimes l) = (x, 1)$ for $x = \lambda(l)$, with projection $\pi = \lambda|_P : P \rightarrow X$ and \mathbb{Z}_2 -action $\sigma : l \mapsto -1 \cdot l$.

For smooth U , we can express $\Upsilon(P)$ in (2.18) in terms of the motivic vanishing cycle associated to the corresponding (L, ι) in (2.20).

Lemma 2.19. *Let U be a smooth \mathbb{K} -scheme, $P \rightarrow U$ a principal \mathbb{Z}_2 -bundle, and (L, ι) correspond to P under the 1-1 correspondence (2.20). Define a regular function $q : L \rightarrow \mathbb{A}^1$ on the total space of L by $\iota(l \otimes l) = (x, q(l)) \in \mathcal{O}_X \cong X \times \mathbb{A}^1$ for $l \in L$, so that q is a nondegenerate quadratic form on the fibres of $L \rightarrow X$, and $\mathrm{Crit}(q) \subset L$ is the zero section of L , which we identify with U . Then*

$$(2.21) \quad \Upsilon(P) = \mathbb{L}^{\dim U/2} \odot MF_{L,q}^{\mathrm{mot}, \phi} \quad \text{in both } \mathcal{M}_U^{\hat{\mu}} \text{ and } \bar{\mathcal{M}}_U^{\hat{\mu}}.$$

Proof. By a very similar proof to Example 2.13, we may show that $MF_{L,q}^{\mathrm{mot}} = [P, \hat{\rho}]$, for $\hat{\rho}$ as in (2.18). Equation (2.21) then follows from (2.8), (2.18) and $\dim L = \dim U + 1$. \square

We can generalize (2.21) to an expression (2.22) for motivic vanishing cycles $MF_{E,q}^{\mathrm{mot}, \phi}$ of nondegenerate quadratic forms on vector bundles. The proof uses (2.19), and so holds only in $\bar{\mathcal{M}}_U^{\hat{\mu}}$ rather than in $\mathcal{M}_U^{\hat{\mu}}$. Note that $\mathbb{L}^{\dim U/2} \odot MF_{E,q}^{\mathrm{mot}, \phi}$ in (2.22) also occurs in equation (2.13) of Theorem 2.16.

Theorem 2.20. *Let U be a smooth \mathbb{K} -scheme, $E \rightarrow U$ a vector bundle of rank r , and $q \in H^0(S^2 E^*)$ a nondegenerate quadratic form on the fibres of E . Regard $q : E \rightarrow \mathbb{A}^1$ as a regular function on the total space of E , which*

is a nondegenerate homogeneous quadratic polynomial on each fibre E_u of E , so that $\text{Crit}(q) \subseteq E$ is the zero section of E , which we identify with U .

Then $\Lambda^r E \rightarrow U$ is a line bundle, and the determinant $\det(q)$ is a non-vanishing section of $(\Lambda^r E^*)^{\otimes 2}$, or equivalently an isomorphism $(\Lambda^r E) \otimes_{\mathcal{O}_U} (\Lambda^r E) \rightarrow \mathcal{O}_U$. Thus there is a principal \mathbb{Z}_2 -bundle $P \rightarrow U$, unique up to isomorphism, corresponding to $(\Lambda^r E, \det(q))$ under the 1-1 correspondence (2.20). We have

$$(2.22) \quad \Upsilon(P) = \mathbb{L}^{\dim U/2} \odot MF_{E,q}^{\text{mot},\phi} \quad \text{in } \bar{\mathcal{M}}_U^{\hat{\mu}}.$$

Proof. We will prove the theorem by induction on $r = \text{rank } E$. The first step, with $r = 1$, follows from Lemma 2.19. For the inductive step, suppose the theorem holds for all U, E, q with $\text{rank } E = r \leq n$ for $n \geq 1$, and let U, E, q be as in the theorem with $\text{rank } E = n + 1$. It is enough to prove (2.22) Zariski locally on U . For each $x \in U$, we can choose a Zariski open neighbourhood U' of x in U and a section $s \in H^0(E|_{U'})$ such that $q(s, s)$ is nonvanishing on U' .

Write $E|_{U'} = F \oplus L$, where $L = \langle s \rangle$ is the line subbundle of E spanned by s , and $F = L^\perp$ the orthogonal vector subbundle of L in E with respect to q . This makes sense as $q(s, s) \neq 0$ on U' , so $L^\perp \cap L = \{0\}$. Then $q|_{U'} = q_F \oplus q_L$, for $q_F \in H^0(S^2 F^*)$ and $q_L \in H^0(S^2 L^*)$ nondegenerate quadratic forms on F, L .

Using the projection $E|_{U'} = F \oplus L \rightarrow F$, we can regard the total space of $E|_{U'}$ as a line bundle over the total space of F , with $E|_{U'} \cong \pi^*(L)$ for $\pi : F \rightarrow U'$ the projection. Then Theorem 2.16 with $F, E|_{U'}, q_F, q_L$ in place of U, E, f, q gives

$$(2.23) \quad MF_{E|_{U'}, q|_{U'}}^{\text{mot},\phi} = MF_{F, q_F}^{\text{mot},\phi} \odot (\mathbb{L}^{\dim F/2} \odot MF_{E|_{U'}, q_L}^{\text{mot},\phi}).$$

As $\text{rank } F = n$, the inductive hypothesis gives

$$(2.24) \quad \Upsilon(Q) = \mathbb{L}^{\dim U/2} \odot MF_{F, q_F}^{\text{mot},\phi},$$

where $Q \rightarrow U'$ is the principal \mathbb{Z}_2 -bundle corresponding to $(\Lambda^r F, \det(q_F))$ under (2.20). Also Lemma 2.19 gives

$$(2.25) \quad \Upsilon(R) = \mathbb{L}^{\dim F/2} \odot MF_{E|_{U'}, q_L}^{\text{mot},\phi},$$

with $R \rightarrow F$ the principal \mathbb{Z}_2 -bundle corresponding to $(E|_{U'} \cong \pi^*(L), q_L)$.

We now have

$$(2.26) \quad \begin{aligned} MF_{E|_{U'}, q|_{U'}}^{\text{mot},\phi} &= (\mathbb{L}^{-\dim U/2} \odot \Upsilon(Q)) \odot \Upsilon(R) \\ &= \mathbb{L}^{-\dim U/2} \odot (\Upsilon(Q) \odot \Upsilon(R|_{U'})) = \mathbb{L}^{-\dim U/2} \odot \Upsilon(P|_{U'}), \end{aligned}$$

where we consider that $U' \subseteq F \subseteq E|_{U'}$, and regard (2.23)–(2.26) as equations in $\bar{\mathcal{M}}_{E|_{U'}}^{\hat{\mu}} \supseteq \bar{\mathcal{M}}_F^{\hat{\mu}} \supseteq \bar{\mathcal{M}}_{U'}^{\hat{\mu}}$, with $MF_{E|_{U'}, q|_{U'}}^{\text{mot},\phi}, MF_{F, q_F}^{\text{mot},\phi}, \Upsilon(Q)$ supported on U' and $MF_{F, q_F}^{\text{mot},\phi}, \Upsilon(R)$ supported on F . Here the first step of (2.26) combines (2.23)–(2.25), the second uses $\Upsilon(Q)$ supported on U' so that $\Upsilon(Q) \odot \Upsilon(R) =$

$\Upsilon(Q) \odot (\Upsilon(R)|_{U'}) = \Upsilon(Q) \odot \Upsilon(R|_{U'})$, and the third uses $P \cong Q \otimes_{\mathbb{Z}_2} R|_{U'}$ since $(E|_{U'}, q) = (F, q_F) \oplus (L, q_L)$ and (2.19). Equation (2.26) is equivalent to the restriction of (2.22) to $U' \subseteq U$. As we can cover U by such Zariski open U' , equation (2.22) holds. This proves the inductive step, and the theorem. \square

Remark 2.21. (a) the combination of equations (2.13) and (2.22) in Theorems 2.16 and 2.20 will be important in §4–§5. The main reason for passing to the rings $\overline{\mathcal{M}}_X^{\hat{\mu}}$ is so that (2.22) holds.

(b) Theorem 2.20 is more-or-less equivalent to material in Kontsevich and Soibelman [16, §5.1]. It implies that $MF_{E,q}^{\text{mot},\phi}$ depends only on $U, r, \Lambda^r E, \det(q)$, which is important in their definition of motivic Donaldson–Thomas invariants. As for our $\overline{\mathcal{M}}_X^{\hat{\mu}}$, Kontsevich and Soibelman [16, §4.5] also introduce an extra relation in their ring of motives to make the analogue of Theorem 2.20 true.

(c) In equations (2.13), (2.21) and (2.22), we can regard $\mathbb{L}^{\dim U/2} \odot MF_{E,q}^{\text{mot},\phi}$ as the *relative motivic vanishing cycle* $MF_{E \rightarrow U, q}^{\text{mot},\phi, \text{rel}}$ of (E, q) relative to U .

In fact, as in Davison and Meinhardt [6, §3.3], one can define such a relative vanishing cycle $MF_{E \rightarrow U, q}^{\text{mot},\phi, \text{rel}}$ without assuming U is smooth, only using that $E \rightarrow U$ is a smooth morphism. Then versions of Lemma 2.19 and Theorem 2.20 hold without assuming U is smooth. But we will not need this.

(d) We defined the ring $\overline{\mathcal{M}}_Y^{\hat{\mu}}$ by imposing the pushforward ϕ_* of relation (2.19) in $\mathcal{M}_X^{\hat{\mu}}$ under all morphisms $\phi : X \rightarrow Y$. As in (c), we may rewrite (2.22) as

$$(2.27) \quad \Upsilon(P) = MF_{E \rightarrow U, q}^{\text{mot},\phi, \text{rel}},$$

for $P \rightarrow U$ the principal \mathbb{Z}_2 -bundle corresponding to $(\Lambda^r E, \det(q))$ under (2.20), and where now both sides of (2.27) make sense for U singular as well as for U smooth, by [6, §3.3]. We may regard (2.27) as a relation in $\mathcal{M}_U^{\hat{\mu}}$, which is equivalent to (2.19). Thus, an alternative definition of the rings $\overline{\mathcal{M}}_Y^{\hat{\mu}}$, closer in spirit to Kontsevich and Soibelman [16, §4.5 & §5.1], is to impose the relation in $\mathcal{M}_Y^{\hat{\mu}}$ that for all \mathbb{K} -scheme morphisms $\phi : U \rightarrow Y$, rank r vector bundles $E \rightarrow U$, and nondegenerate quadratic forms $q \in H^0(S^2 E^*)$, the pushforward $\phi_*(MF_{E \rightarrow U, q}^{\text{mot},\phi, \text{rel}})$ depends only on U, ϕ and $(\Lambda^r E, \det(q))$.

3. Dependence of $MF_{U,f}^{\text{mot},\phi}$ on f

We will use the following notation, [3, §4]:

Definition 3.1. Let U be a smooth \mathbb{K} -scheme, $f : U \rightarrow \mathbb{A}^1$ a regular function, and $X = \text{Crit}(f)$ as a closed \mathbb{K} -subscheme of U . Write $I_X \subseteq \mathcal{O}_U$

for the sheaf of ideals of regular functions $U \rightarrow \mathbb{A}^1$ vanishing on X , so that $I_X = I_{df}$. For each $k = 1, 2, \dots$, write $X^{(k)}$ for the k^{th} order thickening of X in U , that is, $X^{(k)}$ is the closed \mathbb{K} -subscheme of U defined by the vanishing of the sheaf of ideals I_X^k in \mathcal{O}_U . Also write X^{red} for the reduced \mathbb{K} -subscheme of U , and $X^{(\infty)}$ or \hat{U} for the formal completion of U along X .

Then we have a chain of inclusions of closed \mathbb{K} -subschemes of U

$$(3.1) \quad X^{\text{red}} \subseteq X = X^{(1)} \subseteq X^{(2)} \subseteq X^{(3)} \subseteq \dots \subseteq X^{(\infty)} = \hat{U} \subseteq U,$$

although technically $X^{(\infty)} = \hat{U}$ is not a scheme, but a formal scheme.

Write $f^{(k)} := f|_{X^{(k)}} : X^{(k)} \rightarrow \mathbb{K}$, and $f^{\text{red}} := f|_{X^{\text{red}}} : X^{\text{red}} \rightarrow \mathbb{A}^1$, and $f^{(\infty)}$ or $\hat{f} := f|_{\hat{U}} : \hat{U} \rightarrow \mathbb{A}^1$, so that $f^{(k)}, f^{\text{red}}$ are regular functions on the \mathbb{K} -schemes $X^{(k)}, X^{\text{red}}$, and $f^{(\infty)} = \hat{f}$ a formal function on the formal \mathbb{K} -scheme $X^{(\infty)} = \hat{U}$. Note that $f^{\text{red}} : X^{\text{red}} \rightarrow \mathbb{A}^1$ is locally constant, since $X = \text{Crit}(f)$.

As for perverse sheaves of vanishing cycles in Brav et al. [3, §4], we can ask: how much of the sequence (3.1) does $MF_{U,f}^{\text{mot},\phi}$ depend on? That is, is $MF_{U,f}^{\text{mot},\phi}$ determined by $(X^{\text{red}}, f^{\text{red}})$, or by $(X^{(k)}, f^{(k)})$ for some $k = 1, 2, \dots$, or by (\hat{U}, \hat{f}) , as well as by (U, f) ? Our next theorem shows that $MF_{U,f}^{\text{mot},\phi}$ is determined by $(X^{(3)}, f^{(3)})$, and hence a fortiori also by $(X^{(k)}, f^{(k)})$ for $k > 3$ and by (\hat{U}, \hat{f}) :

Theorem 3.2. *Let U, V be smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ be regular functions, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$ as closed \mathbb{K} -subschemes of U, V , so that the motivic vanishing cycles $MF_{U,f}^{\text{mot},\phi}$, $MF_{V,g}^{\text{mot},\phi}$ are defined on X, Y . Define $X^{(3)}, f^{(3)}$ and $Y^{(3)}, g^{(3)}$ as in Definition 3.1, and suppose $\Phi : X^{(3)} \rightarrow Y^{(3)}$ is an isomorphism with $g^{(3)} \circ \Phi = f^{(3)}$, so that $\Phi|_X : X \rightarrow Y \subseteq Y^{(3)}$ is an isomorphism. Then*

$$(3.2) \quad MF_{U,f}^{\text{mot},\phi} = \Phi|_X^*(MF_{V,g}^{\text{mot},\phi}) \quad \text{in } \mathcal{M}_X^{\hat{\mu}} \text{ and } \bar{\mathcal{M}}_X^{\hat{\mu}}.$$

Proof. The proof of Theorem 3.2 is based on the following [3, Prop. 4.3]:

Proposition 3.3. *Let U, V be smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ be regular functions, and $X = \text{Crit}(f) \subseteq U$, $Y = \text{Crit}(g) \subseteq V$. Using the notation of Definition 3.1, suppose $\Phi : X^{(k+1)} \rightarrow Y^{(k+1)}$ is an isomorphism with $g^{(k+1)} \circ \Phi = f^{(k+1)}$ for some $k \geq 2$. Then for each $x \in X$ we can choose a smooth \mathbb{K} -scheme T and étale morphisms $\pi_U : T \rightarrow U$, $\pi_V : T \rightarrow V$ such that*

- (a) $e := f \circ \pi_U = g \circ \pi_V : T \rightarrow \mathbb{A}^1$;
- (b) setting $Q = \text{Crit}(e)$, then $\pi_U|_{Q^{(k)}} : Q^{(k)} \rightarrow X^{(k)} \subseteq U$ is an isomorphism with a Zariski open neighbourhood $\tilde{X}^{(k)}$ of x in $X^{(k)}$; and
- (c) $\Phi \circ \pi_U|_{Q^{(k)}} = \pi_V|_{Q^{(k)}} : Q^{(k)} \rightarrow Y^{(k)}$.

Let U, V, f, g, X, Y, Φ be as in Theorem 3.2, and $x \in X$. Apply Proposition 3.3 with $k = 2$ to get $T, \pi_U, \pi_V, e, Q, \tilde{X}^{(2)}$ satisfying (a)–(c). As π_U, π_V are

étale with $e = f \circ \pi_U = g \circ \pi_V$, we see that

$$\pi_U|_Q^*(MF_{U,f}^{\text{mot},\phi}) = MF_{T,e}^{\text{mot},\phi} = \pi_V|_Q^*(MF_{V,g}^{\text{mot},\phi}).$$

Part (c) gives $\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}}$, so restricting to $Q \subset Q^{(2)}$ yields $\Phi|_{\tilde{X}} \circ \pi_U|_Q = \pi_V|_Q$ for $\tilde{X} = X \cap \tilde{X}^{(2)}$. Thus we see that

$$\pi_U|_Q^*(MF_{U,f}^{\text{mot},\phi}) = \pi_U|_Q^* \circ \Phi|_{\tilde{X}}^*(MF_{V,g}^{\text{mot},\phi}).$$

But $\pi_U|_Q : Q \rightarrow \tilde{X}$ is an isomorphism by (b), so we may omit $\pi_U|_Q^*$, yielding

$$MF_{U,f}^{\text{mot},\phi}|_{\tilde{X}} = \Phi|_{\tilde{X}}^*(MF_{V,g}^{\text{mot},\phi}).$$

This is the restriction of (3.2) to $\tilde{X} \subseteq X$. Since we can cover X by such Zariski open \tilde{X} , Theorem 3.2 follows. \square

Remark 3.4. We can define motivic vanishing cycles $MF_{\hat{U},\hat{f}}^{\text{mot},\phi}$ for a class of formal functions \hat{f} on formal schemes \hat{U} using Theorem 3.2. Let U be a smooth \mathbb{K} -scheme, $X \subseteq U$ a closed \mathbb{K} -subscheme, and \hat{U} the formal completion of U along X . Suppose $f : \hat{U} \rightarrow \mathbb{A}^1$ is a formal function with $\text{Crit}(f) = X \subseteq \hat{U}$. Then there is a unique $MF_{\hat{U},f}^{\text{mot},\phi}$ in $\mathcal{M}_X^{\hat{\mu}}$ or $\bar{\mathcal{M}}_X^{\hat{\mu}}$ with the property that if $U' \subseteq U$ is Zariski open with $X' = X \cap U'$ and $g : U' \rightarrow \mathbb{A}^1$ is regular with $g + I_{X'}^3 = \hat{f}|_{U'} + I_X^3$ in $H^0(\mathcal{O}_{U'}/I_{X'}^3)$ then $MF_{\hat{U},f}^{\text{mot},\phi}|_{X'} = MF_{U',g}^{\text{mot},\phi}|_{X'}$. Theorem 3.2 shows that $MF_{U',g}^{\text{mot},\phi}|_{X'}$ is independent of the choice of g with $g + I_{X'}^3 = \hat{f}|_{U'} + I_X^3$, so $MF_{\hat{U},f}^{\text{mot},\phi}$ is well-defined. Motivic Milnor fibres for formal functions were also defined by Nicaise and Sebag [20] in the context of formal geometry.

In Brav et al. [3, Th. 4.2] we proved an analogue of Theorem 3.2 for perverse sheaves of vanishing cycles. The next example shows that Theorem 3.2 with $X^{(2)}, Y^{(2)}, f^{(2)}, g^{(2)}$ in place of $X^{(3)}, Y^{(3)}, f^{(3)}, g^{(3)}$ is false, so we cannot do better than $(X^{(3)}, f^{(3)})$ in Theorem 3.2.

Example 3.5. Let $U = V = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$ as smooth \mathbb{K} -schemes, define regular $f : U \rightarrow \mathbb{A}^1$ and $g : V \rightarrow \mathbb{A}^1$ by $f(x, y) = x^2$ and $g(x, y) = x^2 y$. Then $X = \text{Crit}(f) = \{(x, y) \in V : x = 0\} = \text{Crit}(g) = Y$, as $y \neq 0$. Hence $X^{(2)} = \{(x, y) \in V : x^2 = 0\} = Y^{(2)}$, so $f^{(2)} = f|_{X^{(2)}} = 0 = g|_{Y^{(2)}} = g^{(2)}$. Thus $\Phi = \text{id}_{X^{(2)}} : X^{(2)} \rightarrow Y^{(2)}$ is an isomorphism with $g^{(2)} \circ \Phi = f^{(2)}$. However, using Lemma 2.19 it is easy to show that

$$MF_{U,f}^{\text{mot},\phi} = \mathbb{L}^{-1/2} \odot \Upsilon(X \times \mathbb{Z}_2) \neq \mathbb{L}^{-1/2} \odot \Upsilon(P) = MF_{V,g}^{\text{mot},\phi} = \Phi|_X^*(MF_{V,g}^{\text{mot},\phi}),$$

where $X \times \mathbb{Z}_2 \rightarrow X$ is the trivial principal \mathbb{Z}_2 -bundle and $P \rightarrow Y$ the nontrivial principal \mathbb{Z}_2 -bundle over $X = Y = \mathbb{A}^1 \setminus \{0\}$.

The next theorem is an immediate corollary of Theorem 3.2, in which we take $U = V$, $X = Y$ and $\Phi = \text{id}_{X^{(3)}}$.

Theorem 3.6. *Let U be a smooth \mathbb{K} -scheme and $f, g : U \rightarrow \mathbb{A}^1$ regular functions. Suppose $X := \text{Crit}(f) = \text{Crit}(g)$ and $f^{(3)} = g^{(3)}$, that is, $f + I_X^3 = g + I_X^3$ in $H^0(\mathcal{O}_U/I_X^3)$, where $I_X \subseteq \mathcal{O}_U$ is the ideal of regular functions vanishing on X . Then $MF_{U,f}^{\text{mot},\phi} = MF_{U,g}^{\text{mot},\phi}$ in $\mathcal{M}_X^{\hat{\mu}}$ and $\bar{\mathcal{M}}_X^{\hat{\mu}}$.*

4. Stabilizing motivic vanishing cycles

To set up notation for our main result, which is Theorem 4.4 below, we need the following theorem, which is proved in Joyce [14, Prop.s 2.24 & 2.25(c)].

Theorem 4.1 (Joyce [14]). *Let U, V be smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ be regular, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$ as \mathbb{K} -subschemes of U, V . Let $\Phi : U \hookrightarrow V$ be a closed embedding of \mathbb{K} -schemes with $f = g \circ \Phi : U \rightarrow \mathbb{A}^1$, and suppose $\Phi|_X : X \rightarrow V \supseteq Y$ is an isomorphism $\Phi|_X : X \rightarrow Y$. Then:*

(i) *For each $x \in X \subseteq U$ there exist a Zariski open neighbourhood U' of x in U , a smooth \mathbb{K} -scheme V' , and morphisms $j : V' \rightarrow V$, $\Phi' : U' \rightarrow V'$, $\alpha : V' \rightarrow U'$, $\beta : V' \rightarrow \mathbb{A}^n$, and $q_1, \dots, q_n : U' \rightarrow \mathbb{A}^1 \setminus \{0\}$, where $n = \dim V - \dim U$, such that $j : V' \rightarrow V$ and $\alpha \times \beta : V' \rightarrow U' \times \mathbb{A}^n$ are étale, $\Phi|_{U'} = j \circ \Phi'$, $\alpha \circ \Phi' = \text{id}_{U'}$, $\beta \circ \Phi' = 0$, and*

$$(4.1) \quad g \circ j = f \circ \alpha + (q_1 \circ \alpha) \cdot (z_1^2 \circ \beta) + \dots + (q_n \circ \alpha) \cdot (z_n^2 \circ \beta).$$

Thus, setting $f' = f|_{U'}$, $g' = g \circ j$, $X' = \text{Crit}(f') = X \cap U'$, and $Y' = \text{Crit}(g')$, then $f' = g' \circ \Phi'$, and $\Phi'|_{X'} : X' \rightarrow Y'$, $j|_{Y'} : Y' \rightarrow Y$, $\alpha|_{Y'} : Y' \rightarrow X$ are étale. We also require that $\Phi \circ \alpha|_{Y'} = j|_{Y'} : Y' \rightarrow Y$.

(ii) *Write N_{UV} for the normal bundle of $\Phi(U)$ in V , regarded as a vector bundle on U in the exact sequence*

$$(4.2) \quad 0 \longrightarrow TU \xrightarrow{d\Phi} \Phi^*(TV) \xrightarrow{\Pi_{UV}} N_{UV} \longrightarrow 0,$$

so that $N_{UV}|_X$ is a vector bundle on X . Then there exists a unique $q_{UV} \in H^0(S^2 N_{UV}|_X^)$ which is a nondegenerate quadratic form on $N_{UV}|_X$, such that whenever $x, U', V', j, \Phi', \alpha, \beta, n, q_a$ are as in (i), then there is an isomorphism $\hat{\beta} : \langle dz_1, \dots, dz_n \rangle_{U'} \rightarrow N_{UV}^*|_{U'}$ making the following commute:*

$$\begin{array}{ccc} N_{UV}^*|_{U'} & \xrightarrow{\Pi_{UV}^*|_{U'}} & \Phi|_{U'}^*(T^*V) = \Phi'^* \circ j^*(T^*V) \\ \uparrow \hat{\beta} & & \downarrow \Phi'^*(dj^*) \\ \langle dz_1, \dots, dz_n \rangle_{U'} & \xrightarrow{\Phi'^*(d\beta^*)} & \Phi'^*(T^*V'), \end{array}$$

and if $X' = X \cap U'$, then

$$q_{UV}|_{X'} = [q_1 \cdot S^2 \hat{\beta}(dz_1 \otimes dz_1) + \dots + q_n \cdot S^2 \hat{\beta}(dz_n \otimes dz_n)]|_{X'}.$$

Following [14, Def.s 2.26 & 2.34], we define:

Definition 4.2. Let U, V be smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ be regular, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$ as \mathbb{K} -subschemes of U, V . Suppose $\Phi : U \hookrightarrow V$ is an embedding of \mathbb{K} -schemes with $f = g \circ \Phi : U \rightarrow \mathbb{A}^1$ and $\Phi|_X : X \rightarrow Y$ an isomorphism. Then Theorem 4.1(ii) defines the normal bundle N_{UV} of U in V , a vector bundle on U of rank $n = \dim V - \dim U$, and a nondegenerate quadratic form $q_{UV} \in H^0(S^2 N_{UV}^*|_X)$. Taking top exterior powers in the dual of (4.2) gives an isomorphism of line bundles on U

$$\rho_{UV} : K_U \otimes \Lambda^n N_{UV}^* \xrightarrow{\cong} \Phi^*(K_V),$$

where K_U, K_V are the canonical bundles of U, V .

Write X^{red} for the reduced \mathbb{K} -subscheme of X . As q_{UV} is a nondegenerate quadratic form on $N_{UV}|_X$, its determinant $\det(q_{UV})$ is a nonzero section of $(\Lambda^n N_{UV}^*|_X)^{\otimes 2}$. Define an isomorphism of line bundles on X^{red} :

$$(4.3) \quad J_\Phi = \rho_{UV}^{\otimes 2} \circ (\text{id}_{K_U^2|_{X^{\text{red}}}} \otimes \det(q_{UV})|_{X^{\text{red}}}) : K_U^{\otimes 2}|_{X^{\text{red}}} \xrightarrow{\cong} \Phi|_{X^{\text{red}}}^*(K_V^{\otimes 2}).$$

Since principal \mathbb{Z}_2 -bundles $\pi : P \rightarrow X$ are a topological notion, and X^{red} and X have the same topological space, principal \mathbb{Z}_2 -bundles on X^{red} and on X are equivalent. Define $\pi_\Phi : P_\Phi \rightarrow X$ to be the principal \mathbb{Z}_2 -bundle which parametrizes square roots of J_Φ on X^{red} . That is, local sections $s_\alpha : X \rightarrow P_\Phi$ of P_Φ correspond to local isomorphisms $\alpha : K_U|_{X^{\text{red}}} \rightarrow \Phi|_{X^{\text{red}}}^*(K_V)$ on X^{red} with $\alpha \otimes \alpha = J_\Phi$. Equivalently, P_Φ is the principal \mathbb{Z}_2 -bundle corresponding to $(\Lambda^n N_{UV}|_X, \det(q_{UV}))$ under the 1-1 correspondence (2.20).

The reason for restricting to X^{red} above is the next result [14, Prop. 2.27], whose proof uses the fact that X^{red} is reduced in an essential way.

Lemma 4.3. *In Definition 4.2, the isomorphism J_Φ in (4.3) and the principal \mathbb{Z}_2 -bundle $\pi_\Phi : P_\Phi \rightarrow X$ depend only on U, V, X, Y, f, g and $\Phi|_X : X \rightarrow Y$. That is, they do not depend on $\Phi : U \rightarrow V$ apart from $\Phi|_X : X \rightarrow Y$.*

Using the notation of Definition 4.2, we can state our main result:

Theorem 4.4. *Let U, V be smooth \mathbb{K} -schemes, $f : U \rightarrow \mathbb{A}^1$, $g : V \rightarrow \mathbb{A}^1$ be regular, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$ as \mathbb{K} -subschemes of U, V . Let $\Phi : U \hookrightarrow V$ be an embedding of \mathbb{K} -schemes with $f = g \circ \Phi : U \rightarrow \mathbb{A}^1$, and suppose $\Phi|_X : X \rightarrow Y \supseteq Y$ is an isomorphism $\Phi|_X : X \rightarrow Y$. Then*

$$(4.4) \quad \Phi|_X^*(MF_{V,g}^{\text{mot},\phi}) = MF_{U,f}^{\text{mot},\phi} \odot \Upsilon(P_\Phi) \quad \text{in } \overline{\mathcal{M}}_X^\beta,$$

where P_Φ is as in Definition 4.2, and Υ is defined in (2.18).

In Brav et al. [3, Th. 5.4] we proved an analogue of Theorem 4.4 for perverse sheaves of vanishing cycles.

4.1. Proof of Theorem 4.4. Suppose U, V, f, g, X, Y, Φ are as in Theorem 4.4, and use the notation N_{UV}, q_{UV} from Theorem 4.1(ii) and J_Φ, P_Φ from Definition 4.2. Let $x, U', V', j, \Phi', \alpha, \beta, q_1, \dots, q_n, f', g', X', Y'$ be as in Theorem 4.1(i). Then in $\bar{\mathcal{M}}_{X'}^\mu$ we have

$$\begin{aligned}
(4.5) \quad & \Phi|_X^* (MF_{V,g}^{\text{mot},\phi})|_{X'} = \Phi'|_X^* \circ j|_{Y'}^* (MF_{V,g}^{\text{mot},\phi}) = \Phi'|_X^* (MF_{V',g'}^{\text{mot},\phi}) \\
& = \Phi'|_X^* \circ (\alpha \times \beta)^* (MF_{U' \times \mathbb{A}^n, f \circ \pi_{U'} + \sum_{i=1}^n (q_i \circ \pi_{U'}) \cdot (z_i^2 \circ \pi_{\mathbb{A}^n})}^{\text{mot},\phi}) \\
& = (\text{id}_{X'} \times 0)^* (MF_{U',f'}^{\text{mot},\phi} \odot \mathbb{L}^{\dim U/2} \odot MF_{U' \times \mathbb{A}^n, \sum_{i=1}^n (q_i \circ \pi_{U'}) \cdot (z_i^2 \circ \pi_{\mathbb{A}^n})}^{\text{mot},\phi}) \\
& = (\text{id}_{X'} \times 0)^* (MF_{U',f'}^{\text{mot},\phi} \odot \Upsilon(P_{q_1 \dots q_n})) = MF_{U',f'}^{\text{mot},\phi} \odot \Upsilon(P_{q_1 \dots q_n}|_{X'}) \\
& = MF_{U',f'}^{\text{mot},\phi} \odot \Upsilon(P_\Phi|_{X'}) = (MF_{U,f}^{\text{mot},\phi} \odot \Upsilon(P_\Phi))|_{X'},
\end{aligned}$$

using $\Phi|_{U'} = j \circ \Phi'$ in the first step, $j : V' \rightarrow V$ étale with $g' = g \circ j$ in the second, $\alpha \times \beta : V' \rightarrow U' \times \mathbb{A}^n$ étale and (4.1) in the third, and $\alpha \circ \Phi' = \text{id}_{U'}$, $\beta \circ \Phi' = 0$, and Theorem 2.16 in the fourth.

In the fifth step of (4.5), we apply Theorem 2.20 to the vector bundle $U' \times \mathbb{A}^n \rightarrow U'$ and nondegenerate quadratic form $\sum_{i=1}^n (q_i \circ \pi_{U'}) \cdot (z_i^2 \circ \pi_{\mathbb{A}^n})$, and we write $P_{q_1 \dots q_n} \rightarrow U'$ for the principal \mathbb{Z}_2 -bundle corresponding to $(\mathcal{O}_{U'}, q_1 \dots q_n)$ under (2.20). The sixth step uses $MF_{U',f'}^{\text{mot},\phi}$ supported on $X' \cong X' \times \{0\}$, the seventh that $P_{q_1 \dots q_n}|_{X'} \cong P_\Phi|_{X'}$ since Theorem 4.1(ii) implies an identification between $q_1 \dots q_n$ and $\det(q_{UV})$ on X' and $P_{q_1 \dots q_n}|_{X'}, P_\Phi|_{X'}$ parametrize square roots of $q_1 \dots q_n$ and $\det(q_{UV})$ on X' , and the eighth that $U' \subseteq U$ is open with $f' = f|_{U'}$. Equation (4.5) proves the restriction of (4.4) to the Zariski open set $X' \subseteq X$. Since we can cover X by such open X' , Theorem 4.4 follows.

5. Motivic vanishing cycles on d-critical loci

In §5.1 we introduce *d-critical loci* from Joyce [14]. Our main result Theorem 5.10, which associates a motive $MF_{X,s} \in \bar{\mathcal{M}}_X^\mu$ to each oriented d-critical locus (X, s) , is stated and discussed in §5.2, and proved in §5.4. Section 5.3 describes a torus localization formula for $MF_{X,s}$ due to Maulik [19].

In §5.1 we only need our \mathbb{K} -schemes to be locally of finite type, not of finite type. In §5.2–§5.4 we take \mathbb{K} -schemes to be finite type, but discuss extension to the locally of finite type case in Remark 5.11.

5.1. Background material on d-critical loci. Here are the main definitions and results on d-critical loci, from Joyce [14, Th.s 2.1, 2.20, 2.28 & Def.s 2.5, 2.18, 2.31 & Ex. 2.38]. In fact [14] develops two versions of the

theory, *algebraic d -critical loci* on \mathbb{K} -schemes and *complex analytic d -critical loci* on complex analytic spaces, but we discuss only the former.

Theorem 5.1. *Let X be a \mathbb{K} -scheme, which must be locally of finite type, but not necessarily of finite type. Then there exists a sheaf \mathcal{S}_X of \mathbb{K} -vector spaces on X , unique up to canonical isomorphism, which is uniquely characterized by the following two properties:*

- (i) *Suppose $R \subseteq X$ is Zariski open, U is a smooth \mathbb{K} -scheme, and $i : R \hookrightarrow U$ is a closed embedding. Then we have an exact sequence of sheaves of \mathbb{K} -vector spaces on R :*

$$0 \longrightarrow I_{R,U} \longrightarrow i^{-1}(\mathcal{O}_U) \xrightarrow{i^\#} \mathcal{O}_X|_R \longrightarrow 0,$$

where $\mathcal{O}_X, \mathcal{O}_U$ are the sheaves of regular functions on X, U , and $i^\#$ is the morphism of sheaves of \mathbb{K} -algebras on R induced by i .

There is an exact sequence of sheaves of \mathbb{K} -vector spaces on R :

$$0 \longrightarrow \mathcal{S}_X|_R \xrightarrow{\iota_{R,U}} \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \xrightarrow{d} \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)},$$

where d maps $f + I_{R,U}^2 \mapsto df + I_{R,U} \cdot i^{-1}(T^*U)$.

- (ii) *Let $R \subseteq S \subseteq X$ be Zariski open, U, V be smooth \mathbb{K} -schemes, $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ closed embeddings, and $\Phi : U \rightarrow V$ a morphism with $\Phi \circ i = j|_R : R \rightarrow V$. Then the following diagram of sheaves on R commutes:*

$$(5.1) \quad \begin{array}{ccccc} 0 \rightarrow \mathcal{S}_X|_R & \xrightarrow{\iota_{S,V}|_R} & \frac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}|_R & \xrightarrow{d} & \frac{j^{-1}(T^*V)}{I_{S,V} \cdot j^{-1}(T^*V)}|_R \\ \downarrow \text{id} & & \downarrow i^{-1}(\Phi^\#) & & \downarrow i^{-1}(d\Phi) \\ 0 \rightarrow \mathcal{S}_X|_R & \xrightarrow{\iota_{R,U}} & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} & \xrightarrow{d} & \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}. \end{array}$$

Here $\Phi : U \rightarrow V$ induces $\Phi^\# : \Phi^{-1}(\mathcal{O}_V) \rightarrow \mathcal{O}_U$ on U , so we have

$$(5.2) \quad i^{-1}(\Phi^\#) : j^{-1}(\mathcal{O}_V)|_R = i^{-1} \circ \Phi^{-1}(\mathcal{O}_V) \longrightarrow i^{-1}(\mathcal{O}_U),$$

a morphism of sheaves of \mathbb{K} -algebras on R . As $\Phi \circ i = j|_R$, equation (5.2) maps $I_{S,V}|_R \rightarrow I_{R,U}$, and so maps $I_{S,V}^2|_R \rightarrow I_{R,U}^2$. Thus (5.2) induces the morphism in the second column of (5.1). Similarly, $d\Phi : \Phi^{-1}(T^*V) \rightarrow T^*U$ induces the third column of (5.1).

There is a natural decomposition $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where \mathbb{K}_X is the constant sheaf on X with fibre \mathbb{K} , and $\mathcal{S}_X^0 \subset \mathcal{S}_X$ is the kernel of the composition

$$\mathcal{S}_X \longrightarrow \mathcal{O}_X \xrightarrow{i_X^\#} \mathcal{O}_{X^{\text{red}}},$$

with X^{red} the reduced \mathbb{K} -subscheme of X , and $i_X : X^{\text{red}} \hookrightarrow X$ the inclusion.

Definition 5.2. An algebraic d -critical locus over a field \mathbb{K} is a pair (X, s) , where X is a \mathbb{K} -scheme, locally of finite type, but not necessarily of finite type, and $s \in H^0(\mathcal{S}_X^0)$ for \mathcal{S}_X^0 as in Theorem 5.1, such that for each $x \in X$, there exists a Zariski open neighbourhood R of x in X , a smooth \mathbb{K} -scheme U , a regular function $f : U \rightarrow \mathbb{A}^1 = \mathbb{K}$, and a closed embedding $i : R \hookrightarrow U$, such that $i(R) = \text{Crit}(f)$ as \mathbb{K} -subschemes of U , and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$. We call the quadruple (R, U, f, i) a *critical chart* on (X, s) .

Let (X, s) be an algebraic d -critical locus, and (R, U, f, i) a critical chart on (X, s) . Let $U' \subseteq U$ be Zariski open, and set $R' = i^{-1}(U') \subseteq R$, $i' = i|_{R'} : R' \hookrightarrow U'$, and $f' = f|_{U'}$. Then (R', U', f', i') is a critical chart on (X, s) , and we call it a *subchart* of (R, U, f, i) . As a shorthand we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s) , with $R \subseteq S \subseteq X$. An *embedding* of (R, U, f, i) in (S, V, g, j) is a locally closed embedding $\Phi : U \hookrightarrow V$ such that $\Phi \circ i = j|_R$ and $f = g \circ \Phi$. As a shorthand we write $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$. If $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ and $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ are embeddings, then $\Psi \circ \Phi : (R, U, f, i) \hookrightarrow (T, W, h, k)$ is also an embedding.

Theorem 5.3. Let (X, s) be an algebraic d -critical locus, and $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s) . Then for each $x \in R \cap S \subseteq X$ there exist subcharts $(R', U', f', i') \subseteq (R, U, f, i)$, $(S', V', g', j') \subseteq (S, V, g, j)$ with $x \in R' \cap S' \subseteq X$, a critical chart (T, W, h, k) on (X, s) , and embeddings $\Phi : (R', U', f', i') \hookrightarrow (T, W, h, k)$, $\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k)$.

Theorem 5.4. Let (X, s) be an algebraic d -critical locus, and $X^{\text{red}} \subseteq X$ the associated reduced \mathbb{K} -scheme. Then there exists a line bundle $K_{X,s}$ on X^{red} which we call the **canonical bundle** of (X, s) , which is natural up to canonical isomorphism, and is characterized by the following properties:

- (i) If (R, U, f, i) is a critical chart on (X, s) , there is a natural isomorphism

$$(5.3) \quad \iota_{R,U,f,i} : K_{X,s}|_{R^{\text{red}}} \longrightarrow i^*(K_U^{\otimes 2})|_{R^{\text{red}}},$$

where $K_U = \Lambda^{\dim U} T^*U$ is the canonical bundle of U in the usual sense.

- (ii) Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s) . Then (4.3) defines an isomorphism of line bundles on $\text{Crit}(f)^{\text{red}}$:

$$J_\Phi : K_U^{\otimes 2}|_{\text{Crit}(f)^{\text{red}}} \xrightarrow{\cong} \Phi^*|_{\text{Crit}(f)^{\text{red}}}(K_V^{\otimes 2}).$$

Since $i : R \rightarrow \text{Crit}(f)$ is an isomorphism with $\Phi \circ i = j|_R$, this gives

$$i|_{R^{\text{red}}}^*(J_\Phi) : i^*(K_U^{\otimes 2})|_{R^{\text{red}}} \xrightarrow{\cong} j^*(K_V^{\otimes 2})|_{R^{\text{red}}},$$

and we must have

$$(5.4) \quad \iota_{S,V,g,j}|_{R^{\text{red}}} = i|_{R^{\text{red}}}^*(J_{\Phi}) \circ \iota_{R,U,f,i} : K_{X,s}|_{R^{\text{red}}} \longrightarrow j^*(K_V^{\otimes 2})|_{R^{\text{red}}}.$$

Definition 5.5. Let (X, s) be an algebraic d-critical locus, and $K_{X,s}$ its canonical bundle from Theorem 5.4. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red} . That is, an orientation is a line bundle L on X^{red} , together with an isomorphism $L^{\otimes 2} = L \otimes L \cong K_{X,s}$. A d-critical locus with an orientation will be called an *oriented d-critical locus*.

Example 5.6. Let X be a smooth \mathbb{K} -scheme. Then in Theorem 5.1 we have $\mathcal{S}_X \cong \mathbb{K}_X$ and $\mathcal{S}_X^0 \cong 0$. The section $s = 0 \in H^0(\mathcal{S}_X^0)$ makes $(X, 0)$ into an algebraic d-critical locus, covered by the critical chart $(R, U, f, i) = (X, X, 0, \text{id}_X)$. Theorem 5.4(i) for this chart shows that $K_{X,0} \cong K_X^{\otimes 2}$, where K_X is the usual canonical bundle of X . Thus, $(X, 0)$ has a natural orientation $K_{X,0}^{1/2} = K_X$.

As we call $K_{X,0}$ the canonical bundle of $(X, 0)$, one might have expected $K_{X,0} \cong K_X$. The explanation is that as a derived scheme, $\text{Crit}(0 : X \rightarrow \mathbb{A}^1)$ is not X , but the shifted cotangent bundle $T^*X[1]$, and the degree -1 fibres of the projection $T^*X[1] \rightarrow X$ include an extra factor of K_X in $K_{X,0}$.

In [4, Th. 6.6] we show that algebraic d-critical loci are classical truncations of objects in derived algebraic geometry known as *-1-shifted symplectic derived schemes*, introduced by Pantev, Toën, Vaquié and Vezzosi [21].

Theorem 5.7 (Bussi, Brav and Joyce [4]). *Suppose (\mathbf{X}, ω) is a -1-shifted symplectic derived scheme over a field \mathbb{K} in the sense of Pantev et al. [21], and let $X = t_0(\mathbf{X})$ be the associated classical \mathbb{K} -scheme of \mathbf{X} . Then X extends naturally to an algebraic d-critical locus (X, s) . The canonical bundle $K_{X,s}$ from Theorem 5.4 is naturally isomorphic to the determinant line bundle $\det(\mathbb{L}_{\mathbf{X}})|_{X^{\text{red}}}$ of the cotangent complex $\mathbb{L}_{\mathbf{X}}$ of \mathbf{X} .*

Pantev et al. [21] show that derived moduli schemes of coherent sheaves on a Calabi–Yau 3-fold have -1 -shifted symplectic structures, giving [4, Cor. 6.7]:

Corollary 5.8. *Suppose Y is a Calabi–Yau 3-fold over \mathbb{K} , and \mathcal{M} is a classical moduli \mathbb{K} -scheme of simple coherent sheaves in $\text{coh}(Y)$, or simple complexes of coherent sheaves in $D^b \text{coh}(Y)$, with perfect obstruction theory $\phi : \mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{M}}$ as in Thomas [22] or Huybrechts and Thomas [12]. Then \mathcal{M} extends naturally to an algebraic d-critical locus (\mathcal{M}, s) . The canonical bundle $K_{\mathcal{M},s}$ from Theorem 5.4 is naturally isomorphic to $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\text{red}}}$.*

Here we call $F \in \text{coh}(Y)$ *simple* if $\text{Hom}(F, F) = \mathbb{K}$, and F^\bullet in $D^b \text{coh}(Y)$ *simple* if $\text{Hom}(F^\bullet, F^\bullet) = \mathbb{K}$ and $\text{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Thus, d-critical loci will have applications in Donaldson–Thomas theory for Calabi–Yau 3-folds [15–17, 22]. Orientations on (\mathcal{M}, s) are closely related to *orientation data* in the work of Kontsevich and Soibelman [16, 17].

Pantev et al. [21] also show that derived intersections $L \cap M$ of (derived) algebraic Lagrangians L, M in an algebraic symplectic manifold (S, ω) have -1 -shifted symplectic structures, so that Theorem 5.7 gives them the structure of algebraic d-critical loci. Thus we may deduce [4, Cor. 6.8]:

Corollary 5.9. *Suppose (S, ω) is an algebraic symplectic manifold over \mathbb{K} , and L, M are algebraic Lagrangians in S . Then the intersection $X = L \cap M$, as a \mathbb{K} -subscheme of S , extends naturally to an algebraic d-critical locus (X, s) . The canonical bundle $K_{X,s}$ from Theorem 5.4 is isomorphic to $K_L|_{X^{\text{red}}} \otimes K_M|_{X^{\text{red}}}$.*

5.2. The main result, and applications. Here is our main result, which will be proved in §5.4.

Theorem 5.10. *Let (X, s) be a finite type algebraic d-critical locus with a choice of orientation $K_{X,s}^{1/2}$. There exists a unique motive $MF_{X,s} \in \bar{\mathcal{M}}_X^\mu$ with the property that if (R, U, f, i) is a critical chart on (X, s) , then*

$$(5.5) \quad MF_{X,s}|_R = i^*(MF_{U,f}^{\text{mot}, \phi}) \odot \Upsilon(Q_{R,U,f,i}) \quad \text{in } \bar{\mathcal{M}}_R^\mu,$$

where $Q_{R,U,f,i} \rightarrow R$ is the principal \mathbb{Z}_2 -bundle parametrizing local isomorphisms $\alpha : K_{X,s}^{1/2}|_{R^{\text{red}}} \rightarrow i^*(K_U)|_{R^{\text{red}}}$ with $\alpha \otimes \alpha = \iota_{R,U,f,i}$, for $\iota_{R,U,f,i}$ as in (5.3).

Remark 5.11. The theory of algebraic d-critical loci (X, s) in [14] is developed for \mathbb{K} -schemes X locally of finite type, but not necessarily of finite type. The construction of $MF_{X,s}$ in Theorem 5.10 is local in X . Thus, we can easily extend Theorem 5.10 to X only locally of finite type, provided we have suitable generalizations of the motivic rings $K_0(\text{Sch}_X)$, \mathcal{M}_X , $K_0^\mu(\text{Sch}_X)$, \mathcal{M}_X^μ , $\bar{\mathcal{M}}_X^\mu$ of §2 to \mathbb{K} -schemes X only locally of finite type.

Here are two natural ways to extend the groups $K_0(\text{Sch}_X)$ to X locally of finite type, based on the ‘stack functions’ $\text{SF}(X)$ and ‘local stack functions’ $\text{SF}(X)$ of Joyce [13, Def.s 3.1 & 3.9]. We can also generalize \mathcal{M}_X , $K_0^\mu(\text{Sch}_X)$, \mathcal{M}_X^μ and $\bar{\mathcal{M}}_X^\mu$ in the same ways.

- (i) We could define $K_0(\text{Sch}_X)$ to be generated by symbols $[S]$ for S a finite type \mathbb{K} -scheme and $\Pi_S^X : S \rightarrow X$ a morphism, with relation $[S] = [T] + [S \setminus T]$ if $T \subseteq S$ is a closed \mathbb{K} -subscheme. Note that S must be finite type as a \mathbb{K} -scheme, *not* as an X -scheme.

If X is not of finite type then $K_0(\text{Sch}_X)$ is a ring without identity, as $[X]$ is not an element of $K_0(\text{Sch}_X)$.

If X, Y are locally of finite type and $\phi : X \rightarrow Y$ a morphism, then pushforwards $\phi_* : K_0(\text{Sch}_X) \rightarrow K_0(\text{Sch}_Y)$ are defined for arbitrary ϕ , but pullbacks $\phi^* : K_0(\text{Sch}_Y) \rightarrow K_0(\text{Sch}_X)$ are defined only for ϕ of finite type.

- (ii) We could define elements of $K_0(\text{Sch}_X)$ to be \sim -equivalence classes of sums $\sum_{i \in I} c_i [S_i]$, where I is a possibly infinite indexing set, $\Pi_{S_i}^X : S_i \rightarrow X$ is a finite type morphism of \mathbb{K} -schemes for each $i \in I$ (so that S_i is locally of finite type as a \mathbb{K} -scheme), and $c_i \in \mathbb{Z}$ for $i \in I$, such that for any finite type \mathbb{K} -subscheme $X' \subseteq X$, we have $S_i \times_X X' \neq \emptyset$ for only finitely many $i \in I$. Define an equivalence relation \sim on such sums by $\sum_{i \in I} c_i [S_i] \sim \sum_{j \in J} d_j [T_j]$ if for all finite type \mathbb{K} -subschemes $X' \subseteq X$, we have $\sum_{i \in I} c_i [S_i \times_X X'] = \sum_{j \in J} d_j [T_j \times_X X']$ in $K_0(\text{Sch}_{X'})$, where $K_0(\text{Sch}_{X'})$ is as in §2.1 as X' is of finite type, and the sums make sense as there are only finitely many nonzero terms.

Then $K_0(\text{Sch}_X)$ is a ring with identity $[X]$, since $\text{id}_X : X \rightarrow X$ is a finite type morphism. Pullbacks $\phi^* : K_0(\text{Sch}_Y) \rightarrow K_0(\text{Sch}_X)$ are defined for arbitrary $\phi : X \rightarrow Y$, but pushforwards $\phi_* : K_0(\text{Sch}_X) \rightarrow K_0(\text{Sch}_Y)$ are defined only for ϕ of finite type.

For the purposes of generalizing Theorem 5.10, (i) does not work – for X only locally of finite type, $MF_{X,s}$ would in general not be supported on a finite type subscheme of X , and so could not be an element of $\bar{\mathcal{M}}_X^{\hat{\mu}}$ defined as in (i). But (ii) does work, and defining $\bar{\mathcal{M}}_X^{\hat{\mu}}$ using the method of (ii), Theorem 5.10 extends to X locally of finite type in a straightforward way. Note that we cannot push $MF_{X,s}$ forward to $\bar{\mathcal{M}}_{\mathbb{K}}$ if X is not of finite type, since $\pi : X \rightarrow \text{Spec } \mathbb{K}$ is not a finite type morphism, and $\pi_* : \bar{\mathcal{M}}_X \rightarrow \bar{\mathcal{M}}_{\mathbb{K}}$ is not defined.

In Brav et al. [3, Th. 6.9] we proved an analogue of Theorem 5.10 for perverse sheaves of vanishing cycles. Theorem 5.7 and Corollaries 5.8 and 5.9 imply:

Corollary 5.12. *Let (X, ω) be a -1 -shifted symplectic derived scheme over \mathbb{K} in the sense of Pantev et al. [21], and $X = t_0(X)$ the associated classical \mathbb{K} -scheme, assumed of finite type. Suppose we are given a square root $\det(\mathbb{L}_X)^{1/2}$ for $\det(\mathbb{L}_X)|_X$. Then we may define a natural motive $MF_{X,\omega}$ in $\bar{\mathcal{M}}_X^{\hat{\mu}}$.*

Corollary 5.13. *Suppose Y is a Calabi–Yau 3-fold over \mathbb{K} , and \mathcal{M} is a finite type moduli \mathbb{K} -scheme of simple coherent sheaves in $\text{coh}(Y)$, or simple complexes of coherent sheaves in $D^b \text{coh}(Y)$, with obstruction theory $\phi : \mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{M}}$ as in Thomas [22] or Huybrechts and Thomas [12]. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we may define a natural motive $MF_{\mathcal{M}} \in \bar{\mathcal{M}}_{\mathcal{M}}^{\hat{\mu}}$.*

Corollary 5.14. *Let (S, ω) be an algebraic symplectic manifold and L, M finite type algebraic Lagrangian submanifolds in S , and write $X = L \cap M$, as a subscheme of S . Suppose we are given square roots $K_L^{1/2}, K_M^{1/2}$ for K_L, K_M . Then we may define a natural motive $MF_{L,M} \in \bar{\mathcal{M}}_X^{\hat{\mu}}$.*

Corollary 5.13 has applications to Donaldson–Thomas theory. If Y is a Calabi–Yau 3-fold over \mathbb{C} and τ a suitable stability condition on coherent sheaves on M , the *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ are integers which ‘count’ the moduli schemes $\mathcal{M}_{\text{st}}^\alpha(\tau)$ of τ -stable coherent sheaves on Y with Chern character $\alpha \in H^{\text{even}}(Y; \mathbb{Q})$, provided there are no strictly τ -semistable sheaves in class α on Y . They were defined by Thomas [22].

Behrend [1] showed that $DT^\alpha(\tau)$ may be written as a weighted Euler characteristic $\chi(\mathcal{M}_{\text{st}}^\alpha(\tau), \nu)$, where $\nu : \mathcal{M}_{\text{st}}^\alpha(\tau) \rightarrow \mathbb{Z}$ is a certain constructible function called the *Behrend function*. Joyce and Song [15] extended the definition of $DT^\alpha(\tau)$ to classes α including τ -semistable sheaves (with $DT^\alpha(\tau) \in \mathbb{Q}$), and proved a wall-crossing formula for $DT^\alpha(\tau)$ for change of stability condition τ . Kontsevich and Soibelman [16] gave a (partly conjectural) motivic generalization of Donaldson–Thomas invariants, for Calabi–Yau 3-folds over \mathbb{K} , also with a wall-crossing formula.

Kontsevich and Soibelman define a motive over $\mathcal{M}_{\text{st}}^\alpha(\tau)$, by associating a formal power series to each (not necessarily closed) point, and taking its motivic Milnor fibre. The question of how these formal power series and motivic Milnor fibres vary in families over the base $\mathcal{M}_{\text{st}}^\alpha(\tau)$ is not really addressed in [16]. Corollary 5.13 answers this question, showing that Zariski locally in $\mathcal{M}_{\text{st}}^\alpha(\tau)$ we can take the formal power series and motivic Milnor fibres to all come from a regular function $f : U \rightarrow \mathbb{A}^1$ on a smooth \mathbb{K} -scheme U .

The square root $\det(\mathcal{E}^\bullet)^{1/2}$ required in Corollary 5.13 corresponds roughly to *orientation data* in Kontsevich and Soibelman [16, §5], [17].

5.3. Torus localization of motives. We will explain a torus localization formula for the motives $MF_{X,s}$ of Theorem 5.10, due to Daves Maulik [19]. We need the following notation:

Definition 5.15. Suppose (X, s) is a finite type algebraic d -critical locus over \mathbb{K} . Let the multiplicative group $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$ act on the \mathbb{K} -scheme X , and write the action as $\rho(\lambda) : X \rightarrow X$ for $\lambda \in \mathbb{G}_m$. Then

- (i) We say that s is \mathbb{G}_m -invariant, and (X, s) is a \mathbb{G}_m -invariant d -critical locus, if $\rho(\lambda)^*(s) = s$ for all $\lambda \in \mathbb{G}_m$, where the pullback $\rho(\lambda)^*$ is as in Joyce [14, Prop. 2.3].
- (ii) We say the \mathbb{G}_m -action ρ on X is *good* if X may be covered by Zariski open, affine, \mathbb{G}_m -invariant \mathbb{K} -subschemes $U \subseteq X$.
- (iii) We say the \mathbb{G}_m -action on X is *circle-compact* if for all $x \in X$ the limit $\lim_{\lambda \rightarrow 0} \rho(\lambda)x$ exists in X , where $\lambda \rightarrow 0$ in $\mathbb{G}_m = \mathbb{K} \setminus \{0\} \subset \mathbb{K}$. If X is proper, then any \mathbb{G}_m -action is circle-compact.

Suppose s is \mathbb{G}_m -invariant, and the \mathbb{G}_m -action ρ is good. Then Joyce [14, Prop.s 2.43 & 2.44] shows that (X, s) admits a cover by \mathbb{G}_m -equivariant critical

charts, and that two such charts can be compared on their overlap by \mathbb{G}_m -equivariant embeddings into a third \mathbb{G}_m -equivariant critical chart.

Write $X^{\mathbb{G}_m}$ for the \mathbb{G}_m -fixed subscheme of X , so that $X^{\mathbb{G}_m} \subseteq X$ is a finite type closed \mathbb{K} -subscheme, with inclusion $\iota : X^{\mathbb{G}_m} \hookrightarrow X$. Write $s^{\mathbb{G}_m}$ for the pullback $\iota^*(s) \in H^0(\mathcal{S}_{X^{\mathbb{G}_m}}^0)$ of s to $X^{\mathbb{G}_m}$. By [14, Cor. 2.45] $(X^{\mathbb{G}_m}, s^{\mathbb{G}_m})$ is an algebraic d-critical locus, and one can show that each \mathbb{G}_m -equivariant orientation $K_{X,s}^{1/2}$ for (X, s) induces a natural orientation $K_{X^{\mathbb{G}_m}, s^{\mathbb{G}_m}}^{1/2}$ for $(X^{\mathbb{G}_m}, s^{\mathbb{G}_m})$.

Write $X^{\mathbb{G}_m} = \coprod_{i \in I} X_i^{\mathbb{G}_m}$ for the decomposition of $X^{\mathbb{G}_m}$ into connected components, where I is a finite indexing set, and set $s_i^{\mathbb{G}_m} = s^{\mathbb{G}_m}|_{X_i^{\mathbb{G}_m}}$, so that $(X_i^{\mathbb{G}_m}, s_i^{\mathbb{G}_m})$ is a connected algebraic d-critical locus.

Following Maulik [19, §4], define the *virtual index* $\text{ind}^{\text{vir}}(X_i^{\mathbb{G}_m}, X)$ of $X_i^{\mathbb{G}_m}$ as follows: if $x \in X_i^{\mathbb{G}_m}$ is a \mathbb{K} -point then $T_x X$ is a \mathbb{K} -vector space with a \mathbb{G}_m -action induced by ρ , so we can split $T_x X = (T_x X)_0 \oplus (T_x X)_+ \oplus (T_x X)_-$, where $(T_x X)_0, (T_x X)_+, (T_x X)_-$ are the direct sums of eigenspaces of the \mathbb{G}_m action on which \mathbb{G}_m acts with zero weight, or positive weight, or negative weight, respectively. Then $(T_x X)_0 = T_x X_i^{\mathbb{G}_m}$. We define $\text{ind}^{\text{vir}}(X_i^{\mathbb{G}_m}, X) = \dim(T_x X)_+ - \dim(T_x X)_-$. This is independent of the choice of $x \in X_i^{\mathbb{G}_m}$.

Theorem 5.16 (Maulik [19]). *Suppose (X, s) is an oriented d-critical locus, and ρ is a good, circle-compact \mathbb{G}_m -action on X which fixes s and preserves the orientation $K_{X,s}^{1/2}$. Then as above, the \mathbb{G}_m -invariant subscheme $X^{\mathbb{G}_m}$ extends to an oriented d-critical locus $(X^{\mathbb{G}_m}, s^{\mathbb{G}_m}) = \coprod_{i \in I} (X_i^{\mathbb{G}_m}, s_i^{\mathbb{G}_m})$.*

Theorem 5.10 gives relative motives $MF_{X,s} \in \bar{\mathcal{M}}_X^{\hat{\mu}}$, $MF_{X_i^{\mathbb{G}_m}, s_i^{\mathbb{G}_m}} \in \bar{\mathcal{M}}_{X_i^{\mathbb{G}_m}}^{\hat{\mu}}$. Writing $\pi : X \rightarrow * = \text{Spec } \mathbb{K}$ and $\pi : X_i^{\mathbb{G}_m} \rightarrow *$ for the projections, we have absolute motives $\pi_*(MF_{X,s}), \pi_*(MF_{X_i^{\mathbb{G}_m}, s_i^{\mathbb{G}_m}}) \in \bar{\mathcal{M}}_{\mathbb{K}}^{\hat{\mu}}$. These are related by

$$(5.6) \quad \pi_*(MF_{X,s}) = \sum_{i \in I} \mathbb{L}^{-\text{ind}^{\text{vir}}(X_i^{\mathbb{G}_m}, X)/2} \odot \pi_*(MF_{X_i^{\mathbb{G}_m}, s_i^{\mathbb{G}_m}}).$$

In the special case in which $X^{\mathbb{G}_m}$ consists (as a scheme) only of finitely many isolated points, equation (5.6) reduces to

$$(5.7) \quad \pi_*(MF_{X,s}) = \sum_{x \in X^{\mathbb{G}_m}} \mathbb{L}^{-\text{ind}^{\text{vir}}(\{x\}, X)/2}.$$

As in [19], equations (5.6)–(5.7) are powerful tools for computing the absolute motives $\pi_*(MF_{X,s}) \in \bar{\mathcal{M}}_{\mathbb{K}}^{\hat{\mu}}$ in examples. Maulik first proves a torus localization formula for \mathbb{G}_m -equivariant motivic vanishing cycles, and then deduces Theorem 5.16 using results of [14, §2.6]. It seems likely that Theorem 5.16 also holds without the assumption that the \mathbb{G}_m -action ρ is good.

5.4. Proof of Theorem 5.10. Let (X, s) be an algebraic d-critical locus with orientation $K_{X,s}^{1/2}$. We must construct $MF_{X,s} \in \bar{\mathcal{M}}_X^{\hat{\mu}}$ satisfying (5.5) for each critical chart (R, U, f, i) . Since such $R \subseteq X$ form a Zariski open cover of X , and (5.5) determines $MF_{X,s}|_R$, there exists a unique $MF_{X,s}$ satisfying

(5.5) for all (R, U, f, i) if and only if the prescribed values $MF_{X,s}|_R$ agree on overlaps between critical charts. That is, we must prove that if (R, U, f, i) and (S, V, g, j) are critical charts, then

$$(5.8) \quad [i^*(MF_{U,f}^{\text{mot},\phi}) \odot \Upsilon(Q_{R,U,f,i})]|_{R \cap S} = [j^*(MF_{V,g}^{\text{mot},\phi}) \odot \Upsilon(Q_{S,V,g,j})]|_{R \cap S}.$$

Fix $x \in R \cap S \subseteq X$, and let $(R', U', f', i'), (S', V', g', j'), (T, W, h, k), \Phi, \Psi$ be as in Theorem 5.3. Then as in [3, Th. 6.9(ii)] there is a natural isomorphism of principal \mathbb{Z}_2 -bundles on R'

$$(5.9) \quad \Lambda_\Phi : Q_{T,W,h,k}|_{R'} \xrightarrow{\cong} i|_{R'}^*(P_\Phi) \otimes_{\mathbb{Z}_2} Q_{R,U,f,i}|_{R'},$$

for $P_\Phi \rightarrow \text{Crit}(f')$ the principal \mathbb{Z}_2 -bundle of orientations of $(N_{U'W}|_{\text{Crit}(f')}, q_{U'W})$ as in Definition 4.2, defined as follows: local isomorphisms

$$\alpha : K_{X,s}^{1/2}|_{R'^{\text{red}}} \longrightarrow i^*(K_U)|_{R'^{\text{red}}}, \quad \beta : K_{X,s}^{1/2}|_{R'^{\text{red}}} \longrightarrow k^*(K_W)|_{R'^{\text{red}}},$$

and $\gamma : i^*(K_U)|_{R'^{\text{red}}} \longrightarrow k^*(K_W)|_{R'^{\text{red}}}$

with $\alpha \otimes \alpha = \iota_{R,U,f,i}|_{R'^{\text{red}}}$, $\beta \otimes \beta = \iota_{T,W,h,k}|_{R'^{\text{red}}}$, $\gamma \otimes \gamma = i|_{R'^{\text{red}}}^*(J_\Phi)$ correspond to local sections $s_\alpha : R' \rightarrow Q_{R,U,f,i}|_{R'}$, $s_\beta : R' \rightarrow Q_{T,W,h,k}|_{R'}$, $s_\gamma : R' \rightarrow i|_{R'}^*(P_\Phi)$. Equation (5.4) shows that $\beta = \gamma \circ \alpha$ is a possible solution for β , and we define Λ_Φ in (5.9) such that $\Lambda_\Phi(s_\beta) = s_\gamma \otimes_{\mathbb{Z}_2} s_\alpha$ if and only if $\beta = \gamma \circ \alpha$.

We now have

$$\begin{aligned} & [k^*(MF_{W,h}^{\text{mot},\phi}) \odot \Upsilon(Q_{T,W,h,k})]|_{R'} \\ &= i'^*[\Phi|_{\text{Crit}(f')}^*(MF_{W,h}^{\text{mot},\phi})] \odot \Upsilon(Q_{T,W,h,k})|_{R'} \\ &= i'^*[MF_{U',f'}^{\text{mot},\phi} \odot \Upsilon(P_\Phi)] \odot \Upsilon(Q_{T,W,h,k})|_{R'} \\ &= i|_{R'}^*[MF_{U,f}^{\text{mot},\phi}] \odot \Upsilon(i|_{R'}^*(P_\Phi)) \odot \Upsilon(Q_{T,W,h,k}|_{R'}) \\ &= i|_{R'}^*[MF_{U,f}^{\text{mot},\phi}] \odot \Upsilon(i|_{R'}^*(P_\Phi) \otimes_{\mathbb{Z}_2} Q_{T,W,h,k}|_{R'}) \\ &= i|_{R'}^*[MF_{U,f}^{\text{mot},\phi}] \odot \Upsilon(Q_{R,U,f,i}|_{R'}) \\ (5.10) \quad &= [i^*(MF_{U,f}^{\text{mot},\phi}) \odot \Upsilon(Q_{R,U,f,i})]|_{R'}, \end{aligned}$$

using $\Phi|_{\text{Crit}(f')} \circ i' = k|_{R'}$ in the first step, Theorem 4.4 for $\Phi : (U', f') \rightarrow (W, h)$ in the second, $U' \subseteq U$, $f' = f|_{U'}$ and functoriality of Υ in the third, (2.19) in the fourth, and (5.9) in the fifth.

Similarly, from $\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k)$ we obtain

$$(5.11) \quad [k^*(MF_{W,h}^{\text{mot},\phi}) \odot \Upsilon(Q_{T,W,h,k})]|_{S'} = [j^*(MF_{V,g}^{\text{mot},\phi}) \odot \Upsilon(Q_{S,V,g,j})]|_{S'}.$$

Combining the restrictions of (5.10)–(5.11) to $R' \cap S'$ proves the restriction of (5.8) to $R' \cap S'$. Since we can cover $R \cap S$ by such Zariski open $R' \cap S' \subseteq R \cap S$, this proves (5.8), and hence Theorem 5.10.

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