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Ruled special Lagrangian 3-folds in $\mathbb{C}^3$

Dominic Joyce,
Lincoln College, Oxford

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1 Introduction

This is the fourth in a series of papers [8, 9, 10] constructing explicit examples of special Lagrangian submanifolds (SL m-folds) in $\mathbb{C}^m$. This paper focusses on *ruled* special Lagrangian 3-folds in $\mathbb{C}^3$, that is, SL 3-folds N in $\mathbb{C}^3$ admitting a fibration $\pi : N \to \Sigma$ for some 2-manifold Σ, such that each fibre $\pi^{-1}(\sigma)$ is a real affine straight line in $\mathbb{C}^3$.

Quite a lot is already known about ruled SL 3-folds in $\mathbb{C}^3$. In particular, given a minimal surface X in $\mathbb{R}^3$, Harvey and Lawson [6, §III.3.C] constructed a ruled SL 3-fold in $\mathbb{C}^3$ from the *normal bundle* $\nu(X)$ of X in $\mathbb{R}^3$. Borisenko [1, §3] showed how to generalize this using a harmonic function ρ on X, to define special Lagrangian *twisted normal bundles* in $\mathbb{C}^3$.

An important family of ruled special Lagrangian 3-folds are the *special Lagrangian cones* N_0 in $\mathbb{C}^3$. Bryant [3, Ex. 4] showed how to generalize these to the *twisted special Lagrangian cones* in $\mathbb{C}^3$, using a function ρ on $\Sigma = N_0 \cap \mathcal{S}^5$ which is an eigenfunction of the Laplacian Δ with eigenvalue 2. Bryant's construction is similar to Borisenko's, but not a generalization. Bryant also proved other results on ruled SL 3-folds [3, §7].

We shall (locally) write each ruled 3-fold N in $\mathbb{C}^3$ in the form

$$N = \bigl\{r\,\phi(\sigma) + \psi(\sigma) : r \in \mathbb{R}, \quad \sigma \in \Sigma\bigr\},$$

where Σ is a surface, $\phi : \Sigma \to \mathcal{S}^5$ and $\psi : \Sigma \to \mathbb{C}^3$ are smooth maps, and $\mathcal{S}^5$ is the unit sphere in $\mathbb{C}^3$. To each ruled 3-fold N we associate an *asymptotic cone*

$$N_0 = \bigl\{r\,\phi(\sigma) : r \in \mathbb{R}, \quad \sigma \in \Sigma\bigr\}$$

in $\mathbb{C}^3$, to which N is asymptotic at infinity (in a fairly weak sense).

In this paper we shall study the set of ruled SL 3-folds N asymptotic to a fixed SL cone N_0. We begin in §2 and §3 by introducing special Lagrangian geometry, ruled submanifolds and cones, and §4 reviews the work of Harvey and Lawson, Borisenko and Bryant referred to above. The new material begins in §5, where we study the equations on ϕ and ψ for N to be special Lagrangian.

It turns out that if N is special Lagrangian then N_0 is, and ϕ satisfies a certain nonlinear equation. Taking N_0 to be special Lagrangian and ϕ to be

fixed, our first main result, Theorem 5.5, is that provided N is not locally isomorphic to some $\mathbb{R}^3$ in $\mathbb{C}^3$, it is special Lagrangian if and only if ψ satisfies a *linear* equation. Therefore, the set of ruled SL 3-folds N asymptotic to a fixed SL cone N_0 in $\mathbb{C}^3$ has the structure of a vector space.

Our second main result, Theorem 6.1, is similar to Borisenko's construction of twisted SL normal bundles, and Bryant's construction of twisted SL cones. We show that if N_0 is a special Lagrangian cone in $\mathbb{C}^3$ then Σ has the structure of a Riemann surface, and that for every holomorphic vector field w on Σ we can construct a ruled SL 3-fold N asymptotic to N_0. Borisenko's result can be regarded as a special case of this. Bryant's result is different, and can be combined with it to give a larger family of ruled SL 3-folds.

The rest of the paper gives applications of Theorem 6.1. In §6 we show that if N_0 is a special Lagrangian cone on T^2 then there is a 2-parameter family of ruled SL 3-folds N asymptotic to N_0, and diffeomorphic to $T^2 \times \mathbb{R}$. There is also a variant of this yielding a 1-parameter family of ruled SL 3-folds diffeomorphic to a nontrivial real line bundle over the Klein bottle.

Section 7 gives explicit examples of ruled SL 3-folds. Using a $\mathrm{U}(1)^2$-invariant T^2-cone due to Harvey and Lawson, we write down two explicit families of ruled SL 3-folds in $\mathbb{C}^3$ depending on a holomorphic function on $\mathbb{C}$, which include new kinds of singularities of SL 3-folds. We also use explicit formulae for a family of SL T^2-cones in [8] to give an explicit family of ruled SL 3-folds diffeomorphic to $T^2 \times \mathbb{R}$.

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2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [6].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [6, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [6, §III].

Definition 2.2 Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g, a real 2-form ω and a complex m-form Ω on $\mathbb{C}^m$ by

$$g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \quad \text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \tag{1}$$

Then $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m, and let $\theta \in [0, 2\pi)$. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$, with *phase* $\mathrm{e}^{i\theta}$, if L is calibrated with respect to $\cos\theta\,\mathrm{Re}\,\Omega + \sin\theta\,\mathrm{Im}\,\Omega$, in the sense of Definition 2.1.

We will often abbreviate 'special Lagrangian' by 'SL', and 'm-dimensional submanifold' by 'm-fold', so that we shall talk about SL m-folds in $\mathbb{C}^m$. Usually we take $\theta = 0$, so that L has phase 1, and is calibrated with respect to $\mathrm{Re}\,\Omega$. When we discuss special Lagrangian submanifolds without specifying a phase, we mean them to have phase 1.

Harvey and Lawson [6, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

Proposition 2.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ with phase $\mathrm{e}^{i\theta}$ if and only if $\omega|_L \equiv 0$ and $(\sin\theta\,\mathrm{Re}\,\Omega - \cos\theta\,\mathrm{Im}\,\Omega)|_L \equiv 0$.*

Note that an m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega|_L \equiv 0$. Thus special Lagrangian submanifolds with phase $\mathrm{e}^{i\theta}$ are Lagrangian submanifolds satisfying the extra condition that $(\sin\theta\,\mathrm{Re}\,\Omega - \cos\theta\,\mathrm{Im}\,\Omega)|_L \equiv 0$, which is how they get their name.

3 Ruled submanifolds of $\mathbb{C}^m$ and cones

We now set up some notation for discussing *ruled* submanifolds in $\mathbb{C}^m$, which will be used in the rest of the paper.

Definition 3.1 Let N be a real k-dimensional submanifold in $\mathbb{C}^m$. A *ruling* (Σ, π) of N is a $(k-1)$-dimensional manifold Σ and a smooth map $\pi : N \to \Sigma$, such that for each $\sigma \in \Sigma$ the fibre $\pi^{-1}(\sigma)$ is a real affine straight line in $\mathbb{C}^m$. A *ruled submanifold* is a triple (N, Σ, π), where N is a submanifold of $\mathbb{C}^m$ and (Σ, π) a ruling of N.

Usually we will refer to the ruled submanifold as N, taking Σ, π to be given. An *r-orientation* for (Σ, π) is a choice of orientation for the real line $\pi^{-1}(\sigma)$ for each $\sigma \in \Sigma$, which varies continuously with σ. A ruled submanifold (N, Σ, π) with an r-orientation is called an *r-oriented ruled submanifold*.

Let (N, Σ, π) be an r-oriented ruled submanifold of N, and let $\mathcal{S}^{2m-1}$ be the unit sphere in $\mathbb{C}^m$. Define a map $\phi : \Sigma \to \mathcal{S}^{2m-1}$ such that $\phi(\sigma)$ is the unique unit vector parallel to $\pi^{-1}(\sigma)$ and in the positive direction with respect to the orientation on $\pi^{-1}(\sigma)$, for each $\sigma \in \Sigma$. It is easy to see that ϕ is a smooth map.

Define a map $\psi : \Sigma \to \mathbb{C}^m$ such that $\psi(\sigma)$ is the unique vector in $\pi^{-1}(\sigma)$ orthogonal to $\phi(\sigma)$, for each $\sigma \in \Sigma$. Then ψ is smooth, and we have

$$N = \big\{ r\,\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R} \big\}. \tag{2}$$

Note that some submanifolds N of $\mathbb{C}^m$, such as vector subspaces $\mathbb{R}^k$ for $k \geqslant 2$, may admit more than one distinct ruling (Σ, π), (Σ', π'). In this case we consider (N, Σ, π) and (N, Σ', π') to be different ruled submanifolds. Though we generally refer to a ruled submanifold as N, dropping Σ, π, we always have a particular ruling in mind. Also, whether a ruled submanifold admits an r-orientation is essentially independent of whether it admits an orientation.

Some explanation of what we mean by 'submanifold' is called for here. Sometimes we will treat submanifolds as *embedded*, and regard them as subsets of $\mathbb{C}^m$. But mostly we allow N to be an *immersed* submanifold. That is, N is a real k-dimensional manifold together with an immersion $\iota : N \to \mathbb{C}^m$, which need not be an embedding. We generally suppress the immersion ι.

However, we do not identify N with its image in $\mathbb{C}^m$. In particular, two points $p, q \in N$ with $\iota(p) = \iota(q)$ can have $\pi(p) \neq \pi(q)$, so that the domain of π is N, not its image in $\mathbb{C}^m$. Also, locally we can always choose an r-orientation for a ruling (Σ, π), even though globally an r-orientation may not exist, as we will see in examples later.

Next we discuss *cones* in $\mathbb{C}^m$.

Definition 3.2 A (singular) submanifold N in $\mathbb{C}^m$ is called a *cone*, with vertex 0, if whenever $p \in N$ then $r\,p \in N$ for all $r \geqslant 0$. Let N be a cone in $\mathbb{C}^m$. We call N *two-sided* if $N = -N$, or equivalently if whenever $p \in N$ then $r\,p \in N$ for all $r \in \mathbb{R}$. If N is not two-sided, we call it *one-sided*.

Cones N in $\mathbb{C}^m$, other than vector subspaces $\mathbb{R}^k$, are *singular* at their vertex 0. So we will have to deal with singular submanifolds. In discussing singularities, there is a tension between the embedded and immersed points of view. When dealing with embedded submanifolds we regard N as a subset of $\mathbb{C}^m$, and a singular point as a point in $\mathbb{C}^m$ where N does not satisfy the usual submanifold conditions.

In the immersed case, in general we should regard N as a singular manifold together with an immersion $\iota : N \to \mathbb{C}^m$. However, there is a class of singularities of immersed submanifolds which arise when N is a *nonsingular* manifold, and $\iota : N \to \mathbb{C}^m$ is a smooth map which is not an immersion at every point. A singular point is then the image $q = \iota(p)$ in $\mathbb{C}^m$ of a point $p \in N$ where ι is not an immersion. Note that $\iota^{-1}(q)$ may have positive dimension.

We can use this point of view to interpret two-sided cones as examples of ruled submanifolds.

Definition 3.3 Let N_0 be a k-dimensional, two-sided cone in $\mathbb{C}^m$, with an isolated singularity at 0, and regarded for the moment simply as a subset of $\mathbb{C}^m$. Then $N_0 \cap \mathcal{S}^{2m-1}$ is a nonsingular $(k-1)$-dimensional submanifold of

$\mathcal{S}^{2m-1}$, closed under the action of $-1 : \mathcal{S}^{2m-1} \to \mathcal{S}^{2m-1}$. Define $\Sigma = (N_0 \cap \mathcal{S}^{2m-1})/\{\pm 1\}$. Then Σ is a nonsingular $(k-1)$-manifold.

Define a subset $\tilde{N}_0$ of $\Sigma \times \mathbb{C}^m$ by $\tilde{N}_0 = \bigl\{(\{\pm p\}, rp) : p \in N_0 \cap \mathcal{S}^{2m-1},\, r \in \mathbb{R}\bigr\}$. Then $\tilde{N}_0$ is a nonsingular k-manifold. Define $\iota : \tilde{N}_0 \to \mathbb{C}^m$ by $\iota : (\{\pm p\}, rp) \mapsto rp$ and $\pi : \tilde{N}_0 \to \Sigma$ by $\iota : (\{\pm p\}, rp) \mapsto \{\pm p\}$. Then $\iota(\tilde{N}_0) = N_0$. Also, ι is an immersion except on $\iota^{-1}(0) \cong \Sigma$. Therefore, we may consider $\tilde{N}_0$ to be a singular immersed submanifold of $\mathbb{C}^m$, with immersion ι.

Clearly, $(\tilde{N}_0, \Sigma, \pi)$ is a *ruled submanifold*, with image N_0 in $\mathbb{C}^m$. Thus, every two-sided cone N_0 in $\mathbb{C}^m$ may be regarded as a ruled submanifold. We will generally suppress the notation $\tilde{N}_0$. We shall call N_0 an *r-oriented two-sided cone* if we are given an r-orientation for $(\tilde{N}_0, \Sigma, \pi)$.

Let N_0 be an r-oriented two-sided cone. Then as in Definition 3.1 we can define maps $\phi : \Sigma \to \mathcal{S}^{2m-1}$ and $\psi : \Sigma \to \mathbb{C}^m$ such that N_0 is written in the form (2). From the definition we see that $\psi \equiv 0$. Conversely, any ruled submanifold of the form (2) with $\psi \equiv 0$ is obviously an r-oriented two-sided cone.

Definition 3.4 Let (N, Σ, π) be a ruled k-dimensional submanifold in $\mathbb{C}^m$. Define

$$N_0 = \bigl\{\mathbf{v} \in \mathbb{C}^m : \mathbf{v} \text{ is parallel to } \pi^{-1}(\sigma) \text{ for some } \sigma \in \Sigma\bigr\}. \tag{3}$$

Then N_0 is usually a k-dimensional two-sided cone in $\mathbb{C}^m$. We call it the *asymptotic cone* of N.

The statement that N_0 is a k-dimensional cone calls for caution here. For instance, it may be that $k > 1$, but that all the fibres of $\pi : N \to \Sigma$ are parallel, so that N_0 is just a straight line in $\mathbb{C}^m$, of dimension 1.

What we can say is that if N is an immersed submanifold with immersion $\iota : N \to \mathbb{C}^m$, then N_0 is the image of a smooth map $\iota_0 : N \to \mathbb{C}^m$, but ι_0 may not be an immersion at any point. In particular, if N is r-oriented then we may identify N with $\Sigma \times \mathbb{R}$ as a manifold and define $\iota : N \to \mathbb{C}^m$ by $\iota : (\sigma, r) \to r\,\phi(\sigma) + \psi(\sigma)$, as in (2), and $\iota_0 : N \to \mathbb{C}^m$ is given by $\iota_0 : (\sigma, r) \to r\,\phi(\sigma)$.

To explain in what sense a ruled submanifold N is asymptotic to its asymptotic cone N_0, we make the following definition.

Definition 3.5 Let N_0 be a closed cone in $\mathbb{C}^m$, nonsingular except at 0, and let N be a closed, nonsingular submanifold in $\mathbb{C}^m$. We say that N is *asymptotic to N_0 with order $O(r^\alpha)$* for some $\alpha < 1$ if there exists a compact subset K in N, a constant $R > 0$ and a diffeomorphism $\Phi : N_0 \setminus \overline{B}_R(0) \to N \setminus K$ such that

$$\begin{aligned} &\bigl|\Phi(\mathbf{x}) - \mathbf{x}\bigr| = O(r^\alpha), \quad \bigl|\nabla\Phi - I\bigr| = O(r^{\alpha-1}) \quad \text{and} \\ &\bigl|\nabla^k \Phi\bigr| = O(r^{\alpha-k}) \quad \text{for } k = 2, 3, \ldots, \text{ as } r \to \infty. \end{aligned} \tag{4}$$

Here $\overline{B}_R(0)$ is the closed ball of radius R in $\mathbb{C}^m$, r is the radius function on $\mathbb{C}^m$, and I is the identity map on $\mathbb{C}^m$. If N is asymptotic to some cone N_0 with order $O(r^\alpha)$ for some $\alpha < 1$ then we say N is *asymptotically conical* with order $O(r^\alpha)$.

For a ruled submanifold N with asymptotic cone N_0 written in the form

$$N = \bigl\{r\,\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R}\bigr\} \quad \text{and} \quad N_0 = \bigl\{r\,\phi(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R}\bigr\},$$

we may define Φ by $\Phi : r\,\phi(\sigma) \mapsto r\,\phi(\sigma) + \psi(\sigma)$ for $\sigma \in \Sigma$ and $|r| > R$. If Σ is compact, so that ψ is bounded, it is not difficult to show that N is asymptotic to N_0 with order $O(1)$.

This is quite a weak form of convergence, as it says only that N stays within a fixed distance of N_0 near infinity in $\mathbb{C}^m$, rather than converging to N_0. However, some ruled submanifolds converge more strongly than this, and in §6.1 we will describe a class of ruled SL 3-folds which are asymptotic to their asymptotic cones with order $O(r^{-1})$.

Here is an elementary result on ruled special Lagrangian m-folds in $\mathbb{C}^m$. The proof is easy, and we omit it.

Proposition 3.6 *Let N be a ruled special Lagrangian m-fold in $\mathbb{C}^m$, and let N_0 be the asymptotic cone of N, as in Definition 3.4. Then N_0 is a special Lagrangian cone in $\mathbb{C}^m$, provided it is m-dimensional.*

The reason for requiring N_0 to be m-dimensional here is that there do exist ruled SL m-folds N whose tangent cones have $\dim N_0 < m$, and so cannot be regarded as special Lagrangian except in a rather singular sense. For instance, if $N = \Sigma \times \mathbb{R}$ in $\mathbb{C}^{m-1} \times \mathbb{C}$, where Σ is an SL $(m-1)$-fold in $\mathbb{C}^{m-1}$, then all the lines in the obvious ruling of N are parallel, and so N_0 is just $\mathbb{R}$ in $\mathbb{C}^m$.

Motivated by this proposition, the general point of view we will take is to fix a special Lagrangian cone N_0 in $\mathbb{C}^3$, and study the ruled special Lagrangian 3-folds N in $\mathbb{C}^3$ asymptotic to N_0. In the notation of Definition 2.1, N_0 determines the map $\phi : \Sigma \to \mathcal{S}^5$, and we shall look for maps $\psi : \Sigma \to \mathbb{C}^3$ such that N defined by (1) is special Lagrangian.

4 Review of previous work

Before beginning our new material, we briefly review three previous papers that have contributed to the theory of ruled special Lagrangian 3-folds. These are Harvey and Lawson [6], Borisenko [1] and Bryant [3].

4.1 Harvey and Lawson's SL normal bundles in $\mathbb{C}^m$

Harvey and Lawson [6, §III.3.C] gave the following construction of SL m-folds in $\mathbb{C}^m$. Let $\mathbb{R}^m$ have coordinates $(x_1, \ldots, x_m)$ and Euclidean metric $\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_m^2$, let X be a submanifold of $\mathbb{R}^m$, and let $\nu(X)$ be the normal bundle of X. That is, for each $x \in X$ the fibre ν_x of $\nu(X)$ is the orthogonal complement $T_xX^\perp$ of T_xX in $\mathbb{R}^m$.

Write $\nu(X)$ as a subset of $\mathbb{R}^m \oplus \mathbb{R}^m$ by

$$\nu = \bigl\{(x, y) : x \in X, \quad y \in \nu_x = T_xX^\perp \subset \mathbb{R}^n\bigr\}.$$

Then it is a classical fact that $\nu(X)$ is Lagrangian with respect to the symplectic structure $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_m \wedge \mathrm{d}y_m$, where $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ are the obvious coordinates on $\mathbb{R}^m \oplus \mathbb{R}^m$. Identify $\mathbb{R}^m \oplus \mathbb{R}^m$ with $\mathbb{C}^m$ using the complex coordinates $(z_1, \ldots, z_m)$, where $z_j = x_j + iy_j$. Then ω is the usual Kähler form on $\mathbb{C}^m$. Thus, for any submanifold X of $\mathbb{R}^m$, the normal bundle $\nu(X)$ is a Lagrangian submanifold of $\mathbb{C}^m$.

Harvey and Lawson were interested in the conditions on X for $\nu(X)$ to be special Lagrangian. Their answer is given in the following definition [6, Def. III.3.15] and theorem [6, Th. III.3.11].

Definition 4.1 Let X be a k-dimensional submanifold of $\mathbb{R}^m$, and A the second fundamental form of X. Then A_x lies in $\nu_x \otimes S^2 T_x^* X$ for each $x \in X$. We call X *austere* if for all $x \in X$ and $y \in \nu_x$, the invariants of odd order of the quadratic form $A_x \cdot y$ in $S^2 T_x^* X$ vanish. Equivalently, for all $x \in X$ and $y \in \nu_x$, the collection of eigenvalues $\lambda_1, \ldots, \lambda_k$ of $A_x \cdot y$, with multiplicity, should be invariant under multiplication by -1.

Austere submanifolds of $\mathbb{R}^n$ are studied by Bryant [2].

Theorem 4.2 *Let X be a k-dimensional submanifold of $\mathbb{R}^m$. Then the normal bundle $\nu(X)$ is special Lagrangian in $\mathbb{C}^m$, with phase i^{m-k}, if and only if X is austere.*

As the sign of the phase does not matter, we can take $\nu(X)$ to have phase 1 if $m - k$ is even, and phase i if $m - k$ is odd.

Now the condition for X to be *minimal* in $\mathbb{R}^n$ is that $\mathrm{Tr}(A_x \cdot y) = 0$ for all $x \in X$ and $y \in \nu_x$, as $\mathrm{Tr}(A)$ is the mean curvature of X. But $\mathrm{Tr}(A_x \cdot y)$ is an odd-order invariant of $A_x \cdot y$, and so vanishes when X is austere. Thus all austere submanifolds are minimal. Furthermore, when X is 2-dimensional, $\mathrm{Tr}(A_x \cdot y)$ is the *only* odd-order invariant of $A_x \cdot y$, and so X is austere if and only if it is minimal. So we prove:

Corollary 4.3 *Let X be a minimal surface in $\mathbb{R}^m$. Then the normal bundle $\nu(X)$ is special Lagrangian in $\mathbb{C}^m$, with phase i^{m-2}.*

We shall be concerned only with the case $m = 3$. When $k = \dim X$ is 0,1 or 3, the construction above yields only affine special Lagrangian 3-planes $\mathbb{R}^3$ in $\mathbb{C}^3$. But when $k = 2$, by considering minimal surfaces in $\mathbb{R}^3$ it yields nontrivial examples of special Lagrangian 3-folds $\nu(X)$ in $\mathbb{C}^3$, with phase i. Note that in this case $\nu(X)$ is automatically a *ruled* special Lagrangian 3-fold, with ruling (X, π), where $\pi : \nu(X) \to X$ is the natural projection, whose fibres are real straight lines in $\mathbb{C}^3$.

4.2 Borisenko's twisted SL normal bundles in $\mathbb{C}^3$

For each minimal surface X in $\mathbb{R}^3$, Harvey and Lawson's construction yields a normal bundle $\nu(X)$ in $\mathbb{C}^3$, which is a ruled SL 3-fold. In [1, §3], Borisenko generalized this construction to generate a family of ruled SL 3-folds in $\mathbb{C}^3$, which

we will call *twisted special Lagrangian normal bundles*, depending on a minimal surface X in $\mathbb{R}^3$ and a harmonic function $\rho : X \to \mathbb{R}$. Here is Borisenko's result [1, Th. 1].

Theorem 4.4 *Let X be an oriented, regular minimal surface in $\mathbb{R}^3$ and $\rho : X \to \mathbb{R}$ a harmonic function. Let (s,t) be local coordinates on X compatible with the orientation, and write the immersion $X \to \mathbb{R}^3$ as $(s,t) \mapsto \mathbf{x}(s,t)$. Define vector-valued functions $\mathbf{n}, \mathbf{p} : X \to \mathbb{R}^3$ by*

$$\mathbf{n} = \frac{\mathbf{x}_s \times \mathbf{x}_t}{|\mathbf{x}_s \times \mathbf{x}_t|} \quad \text{and} \quad \mathbf{p} = \frac{\rho_s \mathbf{x}_t - \rho_t \mathbf{x}_s}{|\mathbf{x}_s \times \mathbf{x}_t|} \times \mathbf{n}, \tag{5}$$

where $\mathbf{x}_s = \frac{\partial \mathbf{x}}{\partial s}$, and so on. Then $\mathbf{n}, \mathbf{p}$ are well-defined and independent of the choice of coordinates (s,t). Define

$$N = \bigl\{\mathbf{x} + i\bigl(\mathbf{p}(\mathbf{x}) + r\,\mathbf{n}(\mathbf{x})\bigr) : \mathbf{x} \in X, \quad r \in \mathbb{R}\bigr\}. \tag{6}$$

Then N is a ruled special Lagrangian 3-fold in $\mathbb{C}^3 = \mathbb{R}^3 \oplus i\,\mathbb{R}^3$ with phase i.

Here '$\times$' is the usual cross product on $\mathbb{R}^3$. It is easy to see that $\mathbf{n}$ is the positive unit normal to X, defined using the orientations on X and $\mathbb{R}^3$, and $\mathbf{p}$ is the gradient vector of ρ, that is $\mathbf{p}^a = g^{ab}(\mathrm{d}\rho)_b$ in index notation, where g is the metric on X induced from the Euclidean metric on $\mathbb{R}^3$.

If we reverse the orientation on X then $\mathbf{n}$ changes sign and $\mathbf{p}$ is fixed. Reversing the sign of t in (6), we see that N is unchanged by reversing the orientation of X. In fact the construction works for non-orientable X as well.

When ρ is constant, Borisenko's construction reduces to Harvey and Lawson's special Lagrangian normal bundle $\nu(X)$. If N is one of Borisenko's twisted normal bundles, then the *asymptotic cone* N_0 of N is a subset of the special Lagrangian 3-plane $i\,\mathbb{R}^3$ in $\mathbb{C}^3$.

4.3 Bryant's results on ruled SL 3-folds

In [3], Bryant proves a number of results on ruled special Lagrangian 3-folds. Here is one, given in [3, Ex. 4] and based on work in [2, §4], which is similar to Borisenko's result.

Theorem 4.5 *Let N_0 be an r-oriented, two-sided special Lagrangian cone in $\mathbb{C}^3$, with ruling (Σ, π). Then Σ is an oriented Riemannian 2-manifold with an isometric immersion $\phi : \Sigma \to \mathcal{S}^5$. Suppose Σ is simply-connected. Let $\rho : \Sigma \to \mathbb{R}$ be any solution of the second-order, linear elliptic equation $*\mathrm{d}(*\mathrm{d}\rho) + 2\rho = 0$.*

Define a $\mathbb{C}^3$-valued 1-form β on Σ by $\beta = \phi * \mathrm{d}\rho - \rho * \mathrm{d}\phi$. Then β is closed, so there exists $\mathbf{b} : \Sigma \to \mathbb{C}^3$ with $\mathrm{d}\mathbf{b} = \beta$. Define $N = \bigl\{r\,\phi(\sigma) + \mathbf{b}(\sigma) : \sigma \in \Sigma, r \in \mathbb{R}\bigr\}$. Then N is a ruled special Lagrangian 3-fold in $\mathbb{C}^3$, asymptotic to N_0.

When $\rho \equiv 0$ we have $\beta = 0$ and $\mathbf{b}$ is constant, so N is a translation of N_0 in $\mathbb{C}^3$. Thus these examples generalize special Lagrangian cones in $\mathbb{C}^3$, and Bryant

calls them *twisted special Lagrangian cones*. They do not agree with Borisenko's twisted special Lagrangian normal bundles.

Next we summarize the results of [3, §3.7], which considers the family of all ruled SL 3-folds in $\mathbb{C}^3$. Let Λ be the set of oriented lines in $\mathbb{C}^3$. Then Λ is a 10-dimensional manifold, which fibres over $\mathcal{S}^5$ with fibre $\mathbb{R}^5$. Bryant shows that Λ carries a *real analytic, Levi-flat almost CR structure* (E, J), which is a subbundle E of $T\Lambda$ with fibre $\mathbb{R}^8$ (in this case), and an almost complex structure J on the fibres of E, satisfying some conditions.

A ruled 3-fold N in $\mathbb{C}^3$ may be regarded as the total space of a 2-dimensional family of (oriented) real lines, and thus as a 2-manifold Σ in Λ. Bryant shows that N is special Lagrangian if and only if Σ is *E-holomorphic*, that is, the tangent spaces of Σ lie in E and are closed under J.

Bryant also uses the theory of exterior differential systems to measure how 'big' the various families of ruled SL 3-folds are. In the sense of exterior differential systems, Harvey and Lawson's family of special Lagrangian normal bundles in $\mathbb{C}^3$, and the family of special Lagrangian cones in $\mathbb{C}^3$, both depend on two functions of one variable.

Similarly, Borisenko's family of twisted special Lagrangian normal bundles in $\mathbb{C}^3$, and Bryant's family of twisted special Lagrangian normal bundles in $\mathbb{C}^3$, both depend on four functions of one variable. But the family of all ruled special Lagrangian 3-folds in $\mathbb{C}^3$ depends on *six* functions of one variable. So generic ruled SL 3-folds do not come from Theorems 4.2–4.5.

5 General results on ruled SL 3-folds

In this section we shall parametrize an r-oriented ruled special Lagrangian 3-fold N in $\mathbb{C}^3$ using maps $\phi : \Sigma \to \mathcal{S}^5$ and $\psi : \Sigma \to \mathbb{C}^3$ as in Definition 3.1, and determine the conditions on ϕ and ψ for N to be special Lagrangian.

The following notation will be used throughout this section. Let Σ be a 2-dimensional, connected, real analytic manifold. Let $\phi : \Sigma \to \mathcal{S}^5$ be a real analytic immersion, where $\mathcal{S}^5$ is the unit sphere in $\mathbb{C}^3$. Let $\psi : \Sigma \to \mathbb{C}^3$ be a real analytic map. Define

$$N = \bigl\{ r\,\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R} \bigr\}. \tag{7}$$

Then N is an r-oriented ruled 3-fold in $\mathbb{C}^3$. We suppose N is special Lagrangian.

Now N is the image of the real analytic map $\Phi : \Sigma \times \mathbb{R} \to \mathbb{C}^3$ given by

$$\Phi(\sigma, r) = r\,\phi(\sigma) + \psi(\sigma). \tag{8}$$

As ϕ is an immersion, Φ is an immersion almost everywhere in $\Sigma \times \mathbb{R}$. The images of points where Φ is not an immersion are generally singular points of N. Regarding N as an immersed copy of $\Sigma \times \mathbb{R}$ with (possibly singular) immersion Φ, we may define $\pi : N \to \Sigma$ by $\pi : (\sigma, r) \to \sigma$, and then (Σ, π) is a ruling of N.

The asymptotic cone N_0 of N is

$$N_0 = \big\{ r\,\phi(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R} \big\}, \tag{9}$$

which is the image of $\Phi_0 : \Sigma \times \mathbb{R} \to \mathbb{C}^3$ given by

$$\Phi_0(\sigma, r) = r\,\phi(\sigma). \tag{10}$$

As we have assumed ϕ is an immersion, Φ_0 is an immersion except when $r = 0$, so that N_0 is nonsingular as an immersed submanifold except at 0. Note that N_0 is special Lagrangian, by Proposition 3.6.

As $\phi : \Sigma \to \mathcal{S}^5$ is an immersion, the pull-back $\phi^*(g)$ of the round metric on $\mathcal{S}^5$ makes Σ into a *Riemannian 2-manifold*. Also, as N_0 is special Lagrangian it is oriented, since $\operatorname{Re}\Omega$ is a nonvanishing 3-form on N_0. We use this to define a natural *orientation* on Σ, such that the non-vanishing 2-form $\phi^*(\phi \cdot \operatorname{Re}\Omega)$ on Σ is positive. Equivalently, local coordinates (s,t) on Σ are oriented if

$$\operatorname{Re}\Omega\big(\phi, \tfrac{\partial\phi}{\partial s}, \tfrac{\partial\phi}{\partial t}\big) > 0. \tag{11}$$

Thus Σ is an oriented Riemannian 2-manifold. Therefore it has a natural *complex structure* J. Near any point $\sigma \in \Sigma$ we can choose a holomorphic coordinate $z = s + it$. The corresponding real coordinates (s,t) on Σ have the property that $J\big(\frac{\partial}{\partial s}\big) \equiv \frac{\partial}{\partial t}$. We shall call local coordinates (s,t) on Σ with this property *oriented conformal coordinates*.

Section 5.1 analyzes the conditions on ϕ and ψ for N to be special Lagrangian, and §5.2 studies ruled SL 3-folds using an 'evolution equation' approach. These ideas are combined in §5.3 to prove our first main result, Theorem 5.5, which will be applied in the rest of the paper.

5.1 The special Lagrangian equations on ϕ and ψ

We shall find the conditions on ϕ, ψ for N to be special Lagrangian at each nonsingular point. Suppose Φ is an immersion at (σ, r) in $\Sigma \times \mathbb{R}$, and let $p = \Phi(\sigma, r)$. Choose oriented conformal coordinates (s,t) on Σ near σ. Then $T_pN = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle_{\mathbb{R}}$, where

$$\begin{aligned} \mathbf{v}_1 = \frac{\partial\Phi}{\partial r}(\sigma, r) = \phi(\sigma), \qquad \mathbf{v}_2 = \frac{\partial\Phi}{\partial s}(\sigma, r) = r\frac{\partial\phi}{\partial s}(\sigma) + \frac{\partial\psi}{\partial s}(\sigma) \\ \text{and} \qquad \mathbf{v}_3 = \frac{\partial\Phi}{\partial t}(\sigma, r) = r\frac{\partial\phi}{\partial t}(\sigma) + \frac{\partial\psi}{\partial t}(\sigma). \end{aligned} \tag{12}$$

We need T_pN to be a special Lagrangian 3-plane $\mathbb{R}^3$ in $\mathbb{C}^3$, with phase 1. By Proposition 2.3, the condition for this is that $\omega|_{T_pN} = \operatorname{Im}\Omega|_{T_pN} = 0$, which is equivalent to

$$\omega(\mathbf{v}_1, \mathbf{v}_2) = \omega(\mathbf{v}_1, \mathbf{v}_3) = \omega(\mathbf{v}_2, \mathbf{v}_3) = \operatorname{Im}\Omega(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 0. \tag{13}$$

Substituting in for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ using (12) gives equations upon ϕ and ψ and their derivatives, which are linear or quadratic polynomials in r. As the equations should hold for all $r \in \mathbb{R}$, the coefficient of each power of r should vanish.

So we find that (13) holding for all r is equivalent to the equations

$$\omega\bigl(\phi,\tfrac{\partial\phi}{\partial s}\bigr)=\omega\bigl(\phi,\tfrac{\partial\phi}{\partial t}\bigr)=\omega\bigl(\tfrac{\partial\phi}{\partial s},\tfrac{\partial\phi}{\partial t}\bigr)=\operatorname{Im}\Omega\bigl(\phi,\tfrac{\partial\phi}{\partial s},\tfrac{\partial\phi}{\partial t}\bigr)=0, \tag{14}$$

$$\begin{aligned}\omega\bigl(\phi,\tfrac{\partial\psi}{\partial s}\bigr)=\omega\bigl(\phi,\tfrac{\partial\psi}{\partial t}\bigr)=\omega\bigl(\tfrac{\partial\phi}{\partial s},\tfrac{\partial\psi}{\partial t}\bigr)+\omega\bigl(\tfrac{\partial\psi}{\partial s},\tfrac{\partial\phi}{\partial t}\bigr)=0,\\ \operatorname{Im}\Omega\bigl(\phi,\tfrac{\partial\phi}{\partial s},\tfrac{\partial\psi}{\partial t}\bigr)+\operatorname{Im}\Omega\bigl(\phi,\tfrac{\partial\psi}{\partial s},\tfrac{\partial\phi}{\partial t}\bigr)=0,\end{aligned} \tag{15}$$

$$\text{and}\qquad \omega\bigl(\tfrac{\partial\psi}{\partial s},\tfrac{\partial\psi}{\partial t}\bigr)=\operatorname{Im}\Omega\bigl(\phi,\tfrac{\partial\psi}{\partial s},\tfrac{\partial\psi}{\partial t}\bigr)=0. \tag{16}$$

Here we have arranged the equations so that (14) does not involve ψ at all, (15) is linear in ψ, and (16) is quadratic in ψ. Note that (14) implies that $\langle\phi,\frac{\partial\phi}{\partial s},\frac{\partial\phi}{\partial t}\rangle_{\mathbb{R}}$ is an SL 3-plane in $\mathbb{R}^3$, which is the condition for N_0 to be special Lagrangian.

In the next proposition we shall show that (15) and (16) are equivalent to one of two linear equations on ψ. We will need the following notation, adapted from [10, §5] with an extra factor of 2. Define an anti-bilinear cross product $\times:\mathbb{C}^3\times\mathbb{C}^3\to\mathbb{C}^3$ by

$$(\mathbf{u}\times\mathbf{v})^b=2\mathbf{u}^{a_1}\mathbf{v}^{a_2}(\operatorname{Re}\Omega)_{a_1a_2a_3}g^{a_3b}, \tag{17}$$

using the index notation for (real) tensors on $\mathbb{C}^3$. Calculation using (1) shows that in coordinates '$\times$' is given by

$$(u_1,u_2,u_3)\times(v_1,v_2,v_3)=(\bar u_2\bar v_3-\bar u_3\bar v_2,\bar u_3\bar v_1-\bar u_1\bar v_3,\bar u_1\bar v_2-\bar u_2\bar v_1). \tag{18}$$

It is equivariant under the SU(3)-action, as $\operatorname{Re}\Omega$ and g are SU(3)-invariant.

Proposition 5.1 *In the situation above, N is special Lagrangian if and only if*

$$\omega\Bigl(\phi,\frac{\partial\phi}{\partial s}\Bigr)\equiv 0\quad\text{and}\quad\frac{\partial\phi}{\partial t}\equiv\phi\times\frac{\partial\phi}{\partial s}, \tag{19}$$

and ψ satisfies either

(i) $\omega\bigl(\phi,\frac{\partial\psi}{\partial s}\bigr)\equiv 0$ *and* $\frac{\partial\psi}{\partial t}\equiv\phi\times\frac{\partial\psi}{\partial s}+f\phi$, *where '$\times$' is defined in* (18) *and $f:\Sigma\to\mathbb{R}$ is some real function; or*

(ii) $\frac{\partial\psi}{\partial s}(\sigma)$ *and* $\frac{\partial\psi}{\partial t}(\sigma)$ *lie in* $\bigl\langle\phi(\sigma),\frac{\partial\phi}{\partial s}(\sigma),\frac{\partial\phi}{\partial t}(\sigma)\bigr\rangle_{\mathbb{R}}$ *for all* $\sigma\in\Sigma$.

Proof. Above we showed that N is special Lagrangian if and only if (14)–(16) hold. We will show that (19) is equivalent to (14), and (15)–(16) are equivalent to (i) or (ii). Fix $\sigma\in\Sigma$, and let $C=\bigl|\frac{\partial}{\partial s}(\sigma)\bigr|$. Then $C>0$. As (s,t) are oriented conformal coordinates and $|\phi|\equiv 1$, one can show that if (14) holds then $\phi(\sigma),C^{-1}\frac{\partial\phi}{\partial s}(\sigma),C^{-1}\frac{\partial\phi}{\partial t}(\sigma)$ are an oriented orthonormal basis of a special Lagrangian 3-plane in $\mathbb{C}^3$.

Let (w_1,w_2,w_3) be the unique complex coordinate system on $\mathbb{C}^3$ in which we have $\phi(\sigma)=(1,0,0)$, $C^{-1}\frac{\partial\phi}{\partial s}(\sigma)=(0,1,0)$ and $C^{-1}\frac{\partial\phi}{\partial t}(\sigma)=(0,0,1)$. Then

(w_1, w_2, w_3) are related to the usual coordinates (z_1, z_2, z_3) on $\mathbb{C}^3$ by an SU(3) transformation. Thus, as g, ω and Ω are SU(3)-invariant, by (1) we have

$$g = |\mathrm{d}w_1|^2 + \cdots + |\mathrm{d}w_3|^2, \quad \omega = \frac{i}{2}(\mathrm{d}w_1 \wedge \mathrm{d}\bar{w}_1 + \cdots + \mathrm{d}w_3 \wedge \mathrm{d}\bar{w}_3), \\ \text{and} \quad \Omega = \mathrm{d}w_1 \wedge \mathrm{d}w_2 \wedge \mathrm{d}z_3. \tag{20}$$

In the coordinates (w_1, w_2, w_3), write

$$\phi(\sigma) = (1, 0, 0), \qquad \tfrac{\partial\phi}{\partial s}(\sigma) = (0, C, 0), \qquad \tfrac{\partial\phi}{\partial t}(\sigma) = (0, 0, C), \tag{21}$$

$$\tfrac{\partial\psi}{\partial s}(\sigma) = (a_1 + ib_1, a_2 + ib_2, a_3 + ib_3), \\ \text{and} \qquad \tfrac{\partial\psi}{\partial t}(\sigma) = (c_1 + id_1, c_2 + id_2, c_3 + id_3), \tag{22}$$

for $a_j, b_j, c_j, d_j \in \mathbb{R}$. Using (20)–(22), equations (15) and (16) become

$$b_1 = d_1 = d_2 - b_3 = d_3 + b_2 = 0, \tag{23}$$

$$a_1d_1 + a_2d_2 + a_3d_3 - b_1c_1 - b_2c_2 - b_3c_3 = 0, \tag{24}$$

$$\text{and} \qquad a_2d_3 + b_2c_3 - a_3d_2 - b_3c_2 = 0, \tag{25}$$

cancelling factors of C.

Setting $b_1 = d_1 = 0$ and substituting $d_2 = b_3$ and $d_3 = -b_2$ by (23), equations (24) and (25) give

$$-b_2(a_3 + c_2) + b_3(a_2 - c_3) = 0 \quad \text{and} \quad -b_2(a_2 - c_3) - b_3(a_3 + c_2) = 0,$$

which is equivalent to

$$\begin{pmatrix} -b_2 & b_3 \\ -b_3 & -b_2 \end{pmatrix} \begin{pmatrix} a_3 + c_2 \\ a_2 - c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{26}$$

The 2×2 matrix appearing here has determinant $b_2^2 + b_3^2$. If this is nonzero then the matrix is invertible, so the column matrix must be zero. If the determinant is zero then $b_2 = b_3 = 0$. So (26) holds if and only if either

(a) $a_3 + c_2 = a_2 - c_3 = 0$, or

(b) $b_2 = b_3 = 0$.

These two possibilities correspond to parts (i) and (ii) of the proposition.

As the transformation from the (z_1, z_2, z_3) coordinates to the (w_1, w_2, w_3) coordinates lies in SU(3), and '$\times$' is SU(3)-equivariant, the formula for '$\times$' in the (w_1, w_2, w_3) coordinates is the same as (18) in the (z_1, z_2, z_3) coordinates. Thus, combining (18) and (21) we see that $\frac{\partial\phi}{\partial t}(\sigma) = \phi(\sigma) \times \frac{\partial\phi}{\partial s}(\sigma)$. Since this holds for all $\sigma \in \Sigma$, equation (14) implies (19). Conversely, it is easy to show that (19) implies (14).

Suppose (a) holds, so that $c_2 = -a_3$ and $c_3 = a_2$. Then by (23) we have

$$\tfrac{\partial\psi}{\partial s}(\sigma) = (a_1, a_2 + ib_2, a_3 + ib_3) \quad \text{and} \quad \tfrac{\partial\psi}{\partial t}(\sigma) = (c_1, -a_3 + ib_3, a_2 - ib_2). \tag{27}$$

Clearly $\omega\bigl(\phi(\sigma),\frac{\partial\psi}{\partial s}(\sigma)\bigr)=0$ as $b_1=0$. Equations (18), (21) and (27) give

$$\tfrac{\partial\psi}{\partial t}(\sigma)=\phi(\sigma)\times\tfrac{\partial\psi}{\partial s}(\sigma)+c_1\phi(\sigma).$$

Thus part (i) of the proposition holds at σ, with $f(\sigma)=c_1$. Conversely, if (i) holds at σ then (27) holds, so (23)–(25) hold, and thus (15) and (16) hold at σ.

Now suppose (b) holds, so that $b_2=b_3=0$. Then (23) gives

$$\tfrac{\partial\psi}{\partial s}(\sigma)=(a_1,a_2,a_3)\quad\text{and}\quad\tfrac{\partial\psi}{\partial t}(\sigma)=(c_1,c_2,c_3).$$

That is, $\frac{\partial\psi}{\partial s}(\sigma)$ and $\frac{\partial\psi}{\partial t}(\sigma)$ are real vectors in the (w_1,w_2,w_3) coordinates. It follows that part (ii) of the proposition holds at σ. Conversely, if part (ii) holds at σ then (23)–(25) hold, and so (15) and (16) hold at σ.

We have shown that (19) is equivalent to (14), and (15) and (16) hold if and only if (i) or (ii) holds at every point σ in Σ. These are not exclusive options; both (i) and (ii) may hold at some points. But we need one of (i) and (ii) to hold at *every* point in Σ, rather than (i) at some points and (ii) at others.

Let σ be a generic point in Σ. Then if (i) or (ii) holds at σ, it also holds in a small neighbourhood of σ in Σ. But Σ is connected, ϕ,ψ are real analytic, and (i), (ii) are closed conditions, so if one of (i) or (ii) holds in an open set in Σ then it holds in all of Σ. This completes the proof. □

Observe that parts (i) and (ii) are both *linear* restrictions on ψ. So ψ satisfies one of two linear equations. The proposition suggests that there are really two different kinds of ruled special Lagrangian 3-folds. However, we will show later that the only ruled SL 3-folds admitting rulings satisfying (ii) but not (i) are SL 3-planes $\mathbb{R}^3$ in $\mathbb{C}^3$, so that all interesting ruled SL 3-folds satisfy part (i).

Note also that N is unchanged by transformations of the form

$$\phi\mapsto\phi,\qquad\psi\mapsto\psi+\alpha\phi,\tag{28}$$

where $\alpha:\Sigma\to\mathbb{R}$ is a real analytic function. This can be regarded as a kind of *gauge transformation* of (ϕ,ψ). We can fix α and ψ uniquely by requiring that $g(\phi,\psi)\equiv 0$, as we did in Definition 3.1, but we will not always do this. Alternatively, we can use α to ensure that the function f in part (i) of the proposition is zero.

5.2 Evolution equations for ruled SL 3-folds

We will now study ruled special Lagrangian 3-folds using the 'evolution equation' approach developed by the author in [8, 9, 10]. This depends on the following result, proved in [8, Th. 3.3].

Theorem 5.2 *Let P be a compact, orientable, real analytic $(m-1)$-manifold, χ a real analytic, nonvanishing section of $\Lambda^{m-1}TP$, and $\Psi:P\to\mathbb{C}^m$ a real analytic immersion such that $\Psi^*(\omega)\equiv 0$ on P. Then there exists $\epsilon>0$ and a*

unique real analytic family $\bigl\{\Psi_t : t \in (-\epsilon,\epsilon)\bigr\}$ *of real analytic maps* $\Psi_t : P \to \mathbb{C}^m$ *with* $\Psi_0 = \Psi$*, satisfying the equation*

$$\Bigl(\frac{\mathrm{d}\Psi_t}{\mathrm{d}t}\Bigr)^b = (\Psi_t)_*(\chi)^{a_1\ldots a_{m-1}}(\mathrm{Re}\,\Omega)_{a_1\ldots a_{m-1}a_m}g^{a_m b}, \tag{29}$$

using the index notation for (real) tensors on $\mathbb{C}^m$*. Define* $\Phi : P \times (-\epsilon,\epsilon) \to \mathbb{C}^m$ *by* $\Phi(p,t) = \Psi_t(p)$*. Then* $N = \mathrm{Im}\,\Phi$ *is a nonsingular immersed special Lagrangian submanifold of* $\mathbb{C}^m$*.*

Here the assumption that P is compact is often unnecessary. The theorem constructs SL m-folds in $\mathbb{C}^m$ by evolving arbitrary real analytic $(m-1)$-submanifolds P in $\mathbb{C}^m$ with $\omega|_P \equiv 0$. The trouble with this is that as the set of such submanifolds is infinite-dimensional, the theorem is really an infinite-dimensional evolution problem, and so difficult to solve explicitly.

To get round this, in [8, 9, 10] the author found various special classes $\mathcal{C}$ of real analytic $(m-1)$-submanifolds P in $\mathbb{C}^m$ with $\omega|_P \equiv 0$, such that the evolution (29) stayed within the class $\mathcal{C}$. Usually $\mathcal{C}$ was only finite-dimensional, so that (29) reduced to an o.d.e. in finitely many variables, which might even be solved explicitly.

We will use the same idea to study ruled SL 3-folds in $\mathbb{C}^3$ in our next proposition. We shall see that if $\mathcal{C}$ is the class of ruled 2-manifolds P in $\mathbb{C}^3$ with $\omega|_P \equiv 0$, then the evolution (29) stays within $\mathcal{C}$, and the resulting SL 3-folds N are ruled. In what follows, we say that a function defined on a compact interval S in $\mathbb{R}$ is *real analytic* if it extends to a real analytic function on an open neighbourhood of S in $\mathbb{R}$.

Proposition 5.3 *Let* S *be the circle* $\mathbb{R}/\mathbb{Z}$*, or a compact interval in* $\mathbb{R}$*, and let* s *be a coordinate on* S*, taking values in* $\mathbb{R}/\mathbb{Z}$ *or* $\mathbb{R}$*. Suppose* $\phi_0 : S \to \mathcal{S}^5$ *and* $\psi_0 : S \to \mathbb{C}^3$ *are real analytic maps satisfying*

$$\omega\Bigl(\phi_0, \frac{\partial\phi_0}{\partial s}\Bigr) \equiv \omega\Bigl(\phi_0, \frac{\partial\psi_0}{\partial s}\Bigr) \equiv 0 \quad \text{in } S. \tag{30}$$

Then there exists $\epsilon > 0$ *and unique real analytic maps* $\phi : S \times (-\epsilon,\epsilon) \to \mathcal{S}^5$ *and* $\psi : S \times (-\epsilon,\epsilon) \to \mathbb{C}^3$ *with* $\phi(s,0) = \phi_0(s)$ *and* $\psi(s,0) = \psi_0(s)$ *for all* $s \in S$*, satisfying*

$$\frac{\partial\phi}{\partial t} = \phi \times \frac{\partial\phi}{\partial s}, \qquad \frac{\partial\psi}{\partial t} = \phi \times \frac{\partial\psi}{\partial s} \tag{31}$$

$$\text{and} \quad \omega\Bigl(\phi, \frac{\partial\phi}{\partial s}\Bigr) \equiv \omega\Bigl(\phi, \frac{\partial\psi}{\partial s}\Bigr) \equiv 0 \quad \text{in } S \times (-\epsilon,\epsilon). \tag{32}$$

Define $N = \bigl\{\Phi(r,s,t) : r \in \mathbb{R},\ s \in S,\ t \in (-\epsilon,\epsilon)\bigr\}$*, where* $\Phi : (r,s,t) \mapsto r\,\phi(s,t) + \psi(s,t)$*. Then* N *is an* r*-oriented ruled special Lagrangian* 3*-fold in* $\mathbb{C}^3$*.*

Proof. We shall apply Theorem 5.2. Define P to be $\mathbb{R} \times S$, with coordinates (r,s), and let $\chi = 2\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial s}$. Then χ is a real analytic, nonvanishing section of $\Lambda^2 TP$, as in the theorem.

Consider a family of maps $\Psi_t : P \to \mathbb{C}^3$ depending smoothly on $t \in (-\epsilon, \epsilon)$, of the form $\Psi_t(r,s) = r\,\phi(s,t) + \psi(s,t)$, where $\phi, \psi : S \times (-\epsilon,\epsilon) \to \mathbb{C}^3$ are real analytic maps with $\phi(s,0) \equiv \phi_0(s)$ and $\psi(s,0) \equiv \psi_0(s)$. We do not yet assume that ϕ maps to $\mathcal{S}^5$ in $\mathbb{C}^3$, but we will prove this later.

Then $\Psi_t^*\bigl(\frac{\partial}{\partial r}\bigr) = \phi$ and $\Psi_t^*\bigl(\frac{\partial}{\partial s}\bigr) = r\frac{\partial\phi}{\partial s} + \frac{\partial\psi}{\partial s}$, so (29) becomes

$$\begin{aligned} r\frac{\partial\phi^b}{\partial t} + \frac{\partial\psi^b}{\partial t} &= 2\phi^{a_1}\Bigl(r\frac{\partial\phi^{a_2}}{\partial s} + \frac{\partial\psi^{a_2}}{\partial s}\Bigr)(\mathrm{Re}\,\Omega)_{a_1a_2a_3}g^{a_3b} \\ &= \Bigl(r\,\phi\times\frac{\partial\phi}{\partial s} + \phi\times\frac{\partial\psi}{\partial s}\Bigr)^b, \end{aligned}$$

by the definition of '$\times$' in (17). Equating linear and constant terms in r on each side, we see that (29) is equivalent to (31).

The only problem in applying Theorem 5.2 in this situation is that P is not compact. However, as the Ψ_t are defined using maps ϕ, ψ on S, which is compact, this does not matter, and the theorem applies. This can be checked by examining its proof in [8, §3].

Thus, by Theorem 5.2, there exists $\epsilon > 0$ and unique real analytic maps $\phi, \psi : S \times (-\epsilon,\epsilon) \to \mathbb{C}^3$ with $\phi(s,0) \equiv \phi_0(s)$ and $\psi(s,0) \equiv \psi_0(s)$, satisfying (31), and such that N is special Lagrangian. It remains only to prove (32), and that ϕ maps into $\mathcal{S}^5$.

Equation (32) follows from the equation $\Phi^*(\omega)\bigl(\frac{\partial}{\partial r}, \frac{\partial}{\partial s}\bigr) = 0$, which holds as N is Lagrangian. And $\frac{\partial}{\partial t}\bigl(|\phi|^2\bigr) = 2g\bigl(\phi, \frac{\partial\phi}{\partial t}\bigr) = 2g\bigl(\phi, \phi\times\frac{\partial\phi}{\partial s}\bigr) = 0$, by (31) and the definition of '$\times$'. Thus $|\phi|$ is independent of t. But $\bigl|\phi(s,0)\bigr| = 1$ as $\phi(s,0) = \phi_0(s)$ lies in $\mathcal{S}^5$, so $|\phi| \equiv 1$, and ϕ maps $S \times (-\epsilon,\epsilon) \to \mathcal{S}^5$. □

Observe the similarity between equation (19) and part (i) of Proposition 5.1, and equations (31) and (32) of Proposition 5.3. The only real difference is that part (i) of Proposition 5.1 allows $\frac{\partial\psi}{\partial t} \equiv \phi\times\frac{\partial\psi}{\partial s} + f\phi$ for some $f : \Sigma \to \mathbb{R}$, whereas (31) requires f to be zero. We reconcile these by noting, as above, that we can set f to be zero using a transformation of the form (28) depending on $\alpha : \Sigma \to \mathbb{R}$. The 'evolution' construction has the effect of fixing α.

Remark. One consequence of the proposition is that if P is a real analytic ruled 2-manifold in $\mathbb{C}^3$ with $\omega|_P = 0$, then P extends locally to a unique ruled special Lagrangian 3-fold N in $\mathbb{C}^3$. This fact is implicit in Bryant's Cartan–Kähler theory calculations [3, §3.7], and so is not new.

5.3 Main results

The next proposition deals with ruled SL 3-folds satisfying part (ii) but not part (i) of Proposition 5.1.

Proposition 5.4 *Every ruled special Lagrangian* 3*-fold* N *in* $\mathbb{C}^3$ *locally admits an r-oriented ruling* (Σ,π) *satisfying part* (i) *of Proposition 5.1. An r-oriented ruled special Lagrangian* 3*-fold* (N,Σ,π) *in* $\mathbb{C}^3$ *satisfying part* (ii) *but not part* (i) *of Proposition 5.1 is locally isomorphic to an affine special Lagrangian* 3*-plane* $\mathbb{R}^3$ *in* $\mathbb{C}^3$.

Proof. Suppose N is a ruled SL 3-fold in $\mathbb{C}^3$, with r-oriented ruling (Σ,π) satisfying part (ii) but not part (i) of Proposition 5.1. As N is real analytic wherever is is nonsingular by [6, Th. III.2.7], we may take (Σ,π) to be real analytic, at least locally.

Choose any embedded real analytic curve $\gamma:[0,1]\rightarrow\Sigma$. Define $S=[0,1]$ and $\phi_0:S\rightarrow\mathcal{S}^5$, $\psi_0:S\rightarrow\mathbb{C}^3$ by $\phi_0(s)=\phi(\gamma(s))$ and $\psi_0(s)=\psi(\gamma(s))$. Apply Proposition 5.3. This gives real analytic maps $\phi':S\times(-\epsilon,\epsilon)$ and $\psi':S\times(-\epsilon,\epsilon)\rightarrow\mathbb{C}^3$, and constructs a ruled SL 3-fold N' in $\mathbb{C}^3$ from them, with a ruling (Σ',π') that satisfies part (i) of Proposition 5.1.

Now N and N' intersect in the ruled surface $\pi^{-1}(\operatorname{Im}\gamma)$ in $\mathbb{C}^3$. Therefore, by [6, Th. III.5.5], N and N' coincide locally. Thus, locally N admits a ruling (Σ',π') satisfying part (i) of Proposition 5.1. Since by assumption (Σ,π) does not satisfy (i), (Σ,π) and (Σ',π') must be different.

It is not difficult to show that different, nearby curves γ in Σ will yield different rulings (Σ',π') of N. Thus N admits not just two, but infinitely many different rulings. But Bryant [3, Th. 6] shows that any SL 3-fold with more than two distinct rulings is planar, that is, locally isomorphic to $\mathbb{R}^3$ in $\mathbb{C}^3$. $\square$

Here is our main result, which follows from Propositions 5.1–5.4.

Theorem 5.5 *Let (N,Σ,π) be a non-planar, r-oriented, ruled SL 3-fold in $\mathbb{C}^3$. Then there exist real analytic maps $\phi:\Sigma\rightarrow\mathcal{S}^5$ and $\psi:\Sigma\rightarrow\mathbb{C}^3$ such that*

$$N=\bigl\{r\,\phi(\sigma)+\psi(\sigma):\sigma\in\Sigma,\quad r\in\mathbb{R}\bigr\}. \tag{33}$$

If (s,t) are oriented conformal coordinates on $U\subset\Sigma$, then ϕ,ψ satisfy

$$\omega\Bigl(\phi,\frac{\partial\phi}{\partial s}\Bigr)\equiv\omega\Bigl(\phi,\frac{\partial\psi}{\partial s}\Bigr)\equiv 0, \tag{34}$$

$$\frac{\partial\phi}{\partial t}=\phi\times\frac{\partial\phi}{\partial s}\quad\text{and}\quad\frac{\partial\psi}{\partial t}\equiv\phi\times\frac{\partial\psi}{\partial s}+f\phi \tag{35}$$

for some real analytic function $f:U\rightarrow\mathbb{R}$. Conversely, if ϕ,ψ satisfy these equations then N is special Lagrangian wherever it is nonsingular.

Again, note that (34) and (35) are *linear* in ψ, regarding ϕ as fixed and f as linear in ψ. This means that the family of ruled special Lagrangian 3-folds N with a fixed asymptotic cone N_0 has the structure of a vector space.

6 Holomorphic vector fields and ruled SL 3-folds

In Theorems 4.4 and 4.5 we described constructions of ruled SL 3-folds in $\mathbb{C}^3$ by Borisenko and Bryant. Borisenko's result involved a harmonic function ρ on a minimal surface X in $\mathbb{R}^3$, and Bryant's involved an SL cone N_0 in $\mathbb{C}^3$ and a function ρ on $\Sigma=N_0\cap\mathcal{S}^5$ satisfying $*\mathrm{d}(*\mathrm{d}\rho)+2\rho=0$.

We shall now present a construction of ruled special Lagrangian 3-folds in $\mathbb{C}^3$ which is similar to both Borisenko's and Bryant's constructions, but not the same as either. The data we use is an SL cone N_0 in $\mathbb{C}^3$ and a holomorphic vector field w on the Riemann surface $\Sigma=N_0\cap\mathcal{S}^5$. Here is our result.

Theorem 6.1 *Let N_0 be an r-oriented, two-sided special Lagrangian cone in $\mathbb{C}^3$. Then as in §5 we can write*

$$N_0 = \big\{ r\,\phi(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R} \big\}, \tag{36}$$

where Σ is a Riemann surface and $\phi : \Sigma \to \mathcal{S}^5$ a real analytic map, such that if (s,t) are oriented conformal coordinates on an open set $U \subset \Sigma$ then ϕ satisfies

$$\omega\Big(\phi, \frac{\partial \phi}{\partial s}\Big) \equiv 0 \quad \textit{and} \quad \frac{\partial \phi}{\partial t} = \phi \times \frac{\partial \phi}{\partial s}. \tag{37}$$

Let w be a holomorphic vector field on Σ and define $\psi : \Sigma \to \mathbb{C}^3$ by $\psi = \mathcal{L}_w \phi$, where $\mathcal{L}_w$ is the Lie derivative. Define

$$N = \big\{ r\,\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R} \big\}. \tag{38}$$

Then N is an r-oriented ruled special Lagrangian 3-fold in $\mathbb{C}^3$.

Proof. The first part of the theorem follows from the material of §5. If $w \equiv 0$ then the second part is trivial. So suppose $w \not\equiv 0$, so that w has only isolated zeros. Let $\sigma \in \Sigma$ be a point where w is nonzero. Then it is easy to show that as w is a holomorphic vector field on Σ, there exist oriented conformal coordinates (s,t) on an open neighbourhood U of σ in Σ such that $w = \frac{\partial}{\partial s}$ in U. Therefore $\psi = \frac{\partial \phi}{\partial s}$ in U.

Taking $\frac{\partial}{\partial s}$ of the first equation of (37), we find

$$\omega\Big(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial s}\Big) + \omega\Big(\phi, \frac{\partial^2 \phi}{(\partial s)^2}\Big) \equiv \omega\Big(\phi, \frac{\partial \psi}{\partial s}\Big) \equiv 0, \tag{39}$$

as ω is antisymmetric and $\psi = \frac{\partial \phi}{\partial s}$. Similarly, taking $\frac{\partial}{\partial s}$ of the second equation of (37) gives

$$\frac{\partial^2 \phi}{\partial s \partial t} = \frac{\partial \phi}{\partial s} \times \frac{\partial \phi}{\partial s} + \phi \times \frac{\partial^2 \phi}{(\partial s)^2} = \phi \times \frac{\partial^2 \phi}{(\partial s)^2},$$

as '$\times$' is antisymmetric. Using $\frac{\partial^2}{\partial s \partial t} = \frac{\partial^2}{\partial t \partial s}$ and $\psi = \frac{\partial \phi}{\partial s}$, this becomes

$$\frac{\partial \psi}{\partial t} = \phi \times \frac{\partial \psi}{\partial s}. \tag{40}$$

Comparing (37), (39) and (40) with equations (34) and (35), we see by Theorem 5.5 that the subset $\pi^{-1}(U)$ of N is a ruled SL 3-fold. Now every point $\sigma \in \Sigma$ has such a neighbourhood U, except for the isolated zeros of w. Thus N is an r-oriented ruled SL 3-fold, except possibly along a discrete set of lines. But to be special Lagrangian is a closed condition on the nonsingular part of N, so N is special Lagrangian. □

Here is a sketch of how the theorem relates to Borisenko's construction in Theorem 4.4. Let X be an oriented minimal surface in $\mathbb{R}^3$, with unit normal

$\mathbf{n}: X \to \mathcal{S}^2$. Then $\mathbf{n}$ is the *Gauss map* of X. Suppose for simplicity that $\mathbf{n}$ is a diffeomorphism between X and an open set Σ of $\mathcal{S}^2$.

Note that $\mathbf{n}$ is a conformal map, though it is some sense orientation-reversing. Thus the harmonic function $\rho: X \to \mathbb{R}$ pushes forward to a harmonic function $\rho': \Sigma \to \mathbb{R}$ under $\mathbf{n}$, since X and Σ are conformal. Define $\alpha = \mathrm{d}\rho' - iJ\mathrm{d}\rho'$. Then α is a holomorphic (1,0)-form on Σ.

Now using the second fundamental form of X one can construct a natural holomorphic section β of $T^{(1,0)}\Sigma \otimes_{\mathbb{C}} T^{(1,0)}\Sigma$. The contraction $\alpha \cdot \beta$ is a holomorphic section of $T^{(1,0)}\Sigma$. Let w be the real part of $\alpha \cdot \beta$. Then w is a holomorphic vector field on Σ. The map $\phi: \Sigma \to \mathcal{S}^5$ is just multiplication by i, and $\psi = \mathcal{L}_w \phi$ in Theorem 6.1 corresponds to Borisenko's term $i\mathbf{p}: \Sigma \to \mathbb{C}^3$ in (6).

Thus ψ in Theorem 6.1 corresponds to the 'twisting' term $\mathbf{p}$ in Theorem 4.4. However, our result works for all SL cones in $\mathbb{C}^3$, whereas Borisenko's construction works only when the asymptotic cone is $i\,\mathbb{R}^3$. To make up for this, Borisenko also includes the extra term $\mathbf{x}$ in (6), which does not appear in our construction.

Bryant's construction in Theorem 4.5 does not coincide with our construction above, even though both begin with a special Lagrangian cone. In fact, because of the linearity in ψ noted after Theorem 5.5 we can combine the two constructions to get:

Theorem 6.2 *Let N_0, Σ and ϕ be as above, and suppose Σ is simply-connected. Let w be a holomorphic vector field on Σ and define $\psi: \Sigma \to \mathbb{C}^3$ by $\psi = \mathcal{L}_w \phi$, where $\mathcal{L}_w$ is the Lie derivative. Let $\rho: \Sigma \to \mathbb{R}$ be any solution of the second-order, linear elliptic equation $*\mathrm{d}(*\mathrm{d}\rho) + 2\rho = 0$. Define a $\mathbb{C}^3$-valued 1-form β on Σ by $\beta = \phi * \mathrm{d}\rho - \rho * \mathrm{d}\phi$. Then β is closed, so there exists $\mathbf{b}: \Sigma \to \mathbb{C}^3$ with $\mathrm{d}\mathbf{b} = \beta$. Define*

$$N = \bigl\{ r\,\phi(\sigma) + \psi(\sigma) + \mathbf{b}(\sigma) : \sigma \in \Sigma, \quad r \in \mathbb{R} \bigr\}. \tag{41}$$

Then N is an r-oriented ruled special Lagrangian 3-fold in $\mathbb{C}^3$.

To see that the terms ψ and $\mathbf{b}$ here represent genuinely different perturbations of N_0, observe that in oriented conformal coordinates (s,t) we have

$$\begin{aligned} &\tfrac{\partial \psi}{\partial s} = u\tfrac{\partial^2 \phi}{(\partial s)^2} + \tfrac{\partial u}{\partial s}\tfrac{\partial \phi}{\partial s} + v\tfrac{\partial^2 \phi}{\partial s \partial t} + \tfrac{\partial v}{\partial s}\tfrac{\partial \phi}{\partial t}, \quad \tfrac{\partial \psi}{\partial t} = u\tfrac{\partial^2 \phi}{\partial s \partial t} + \tfrac{\partial u}{\partial t}\tfrac{\partial \phi}{\partial s} + v\tfrac{\partial^2 \phi}{(\partial t)^2} + \tfrac{\partial v}{\partial t}\tfrac{\partial \phi}{\partial t}, \\ &\tfrac{\partial \mathbf{b}}{\partial s} = -\tfrac{\partial \rho}{\partial t}\phi + \rho\tfrac{\partial \phi}{\partial t} \qquad \text{and} \qquad \tfrac{\partial \mathbf{b}}{\partial t} = \tfrac{\partial \rho}{\partial s}\phi - \rho\tfrac{\partial \phi}{\partial s}, \end{aligned}$$

where $w = u\frac{\partial}{\partial s} + v\frac{\partial}{\partial t}$.

Generically five of $\phi, \frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t}, \frac{\partial^2 \phi}{(\partial s)^2}, \frac{\partial^2 \phi}{\partial s \partial t}$ and $\frac{\partial^2 \phi}{(\partial t)^2}$ will be linearly independent, and so the ψ and $\mathbf{b}$ terms cannot agree, since if they did then only four would be linearly dependent. In fact, in the generic case the author expects that locally Theorem 6.2 gives *all* the ruled special Lagrangian 3-folds asymptotic to N_0. But this will not in general be true at points where ϕ and its first and second derivatives are too linearly dependent.

6.1 Ruled SL 3-folds over compact Riemann surfaces

We shall now apply Theorem 6.1 in the case when Σ is a *compact* Riemann surface, without boundary. If Σ is a compact, connected Riemann surface of genus g then the vector space of holomorphic vector fields on Σ has dimension 6 when $g = 0$, dimension 2 when $g = 1$, and dimension 0 when $g \geqslant 2$. So to get nontrivial holomorphic vector fields we should take Σ to be $\mathcal{S}^2$ or T^2.

However, it follows from well known facts in minimal surface theory that any special Lagrangian cone on $\mathcal{S}^2$ in $\mathbb{C}^3$ must be an SL 3-plane $\mathbb{R}^3$. The author first learnt this from Robert Bryant, and a proof can be found in Haskins [7, Th. B]. When applied to $N_0 = \mathbb{R}^3$, Theorem 6.2 just gives back a different ruling of the same $\mathbb{R}^3$.

Therefore the only interesting case is $\Sigma \cong T^2$. Then we can prove:

Theorem 6.3 *Let N_0 be an r-oriented two-sided special Lagrangian cone on T^2. That is, N_0 may be defined as in* (36)*, where $\Sigma \cong T^2$ is a Riemann surface and $\phi : \Sigma \to \mathcal{S}^5$ a real analytic immersion satisfying* (37) *in oriented conformal coordinates. Then there exists a 2-dimensional family of distinct, r-oriented, ruled special Lagrangian 3-folds N with asymptotic cone N_0, which are asymptotic to N_0 with order $O(r^{-1})$ in the sense of Definition 3.5.*

Proof. Any Riemann surface $\Sigma \cong T^2$ may be written as $\mathbb{R}^2/\Lambda$, where $\Lambda \cong \mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$, and the coordinates (s, t) on $\mathbb{R}^2$ are oriented conformal coordinates. Write Σ in this way. Then the holomorphic vector fields w on Σ are of the form $u\frac{\partial}{\partial s} + v\frac{\partial}{\partial t}$ for $u, v \in \mathbb{R}$.

Lift ϕ to a Λ-invariant map $\phi : \mathbb{R}^2 \to \mathcal{S}^5$. For each $u, v \in \mathbb{R}$, define

$$N_{u,v} = \bigl\{r\,\phi(s,t) + u\tfrac{\partial\phi}{\partial s}(s,t) + v\tfrac{\partial\phi}{\partial t}(s,t) : r, s, t \in \mathbb{R}\bigr\}. \tag{42}$$

Then $N_{u,v}$ is a ruled special Lagrangian 3-fold in $\mathbb{C}^3$ wherever it is nonsingular, by Theorem 6.1, and it clearly has asymptotic cone N_0.

It remains to prove that $N_{u,v}$ is asymptotic to N_0 with order $O(r^{-1})$, in the sense of Definition 3.5. Let $R > 0$, and define $\Phi : N_0 \setminus \overline{B}_R(0) \to N_{u,v}$ by

$$\begin{aligned}\Phi : r\,\phi(s,t) \longmapsto r\,\phi\bigl(s - \tfrac{u}{r}, t - \tfrac{v}{r}\bigr) + u\tfrac{\partial\phi}{\partial s}\bigl(s - \tfrac{u}{r}, t - \tfrac{v}{r}\bigr)\\ + v\tfrac{\partial\phi}{\partial t}\bigl(s - \tfrac{u}{r}, t - \tfrac{v}{r}\bigr)\end{aligned} \tag{43}$$

for $|r| > R$ and $s, t \in \mathbb{R}$. Then Φ is well-defined, and using the expansions

$$\begin{gathered}\tfrac{\partial\phi}{\partial s}\bigl(s - \tfrac{u}{r}, t - \tfrac{v}{r}\bigr) = \tfrac{\partial\phi}{\partial s}(s,t) + O(r^{-1}), \quad \tfrac{\partial\phi}{\partial t}\bigl(s - \tfrac{u}{r}, t - \tfrac{v}{r}\bigr) = \tfrac{\partial\phi}{\partial t}(s,t) + O(r^{-1})\\ \text{and} \quad \phi\bigl(s - \tfrac{u}{r}, t - \tfrac{v}{r}\bigr) = \phi(s,t) - \tfrac{u}{r}\tfrac{\partial\phi}{\partial s}(s,t) - \tfrac{v}{r}\tfrac{\partial\phi}{\partial t}(s,t) + O(r^{-2})\end{gathered}$$

for large r, one can show that

$$\Phi\bigl(r\,\phi(s,t)\bigr) = r\,\phi(s,t) + O(r^{-1})$$

for large r, which is the first equation of (4), with $\alpha = -1$. The other equations of (4) may be proved in the same way. Therefore $N_{u,v}$ is asymptotic to N_0 with order $O(r^{-1})$. $\square$

In §3 we saw that ruled submanifolds are asymptotic to their asymptotic cones with order $O(1)$. But the ruled SL 3-folds in the theorem are asymptotic with order $O(r^{-1})$, which is stronger. The reason for this is that each line $\pi^{-1}(\sigma)$ in N_0 is translated by $\psi(\sigma)$ to make N, and $\psi(\sigma)$ is tangent to N_0 along $\pi^{-1}(\sigma)$. Thus the distance between N and N_0 is roughly proportional to the curvature of N_0, which is $O(r^{-1})$.

One can show by the same method that any ruled SL 3-fold constructed in Theorem 6.1 is asymptotic to N_0 with order $O(r^{-1})$ over compact subsets of Σ. The author believes that all ruled SL 3-folds asymptotic with order $O(r^{-1})$ to their asymptotic cones arise from the construction of Theorem 6.1.

In Theorem 6.3 we took N_0 to be fibred by a T^2 family of lines. Thus $N_0 \cap \mathcal{S}^5$ is actually two copies of T^2, as each line intersects $\mathcal{S}^5$ in two points, and the two T^2 are swapped by the action of -1 on $\mathcal{S}^5$. So N_0 is two opposite T^2-cones meeting at their common vertex 0, and for generic u, v we expect $N_{u,v}$ to be a nonsingular immersed 3-submanifold diffeomorphic to $T^2 \times \mathbb{R}$, with two T^2 ends at infinity.

However, there is another kind of special Lagrangian T^2-cone, in which N_0 is one T^2-cone rather than two, invariant under ± 1. Suppose this is the case. Let $\tilde{\Sigma} = N_0 \cap \mathcal{S}^5$, so that $\tilde{\Sigma}$ is a Riemann surface isomorphic to T^2. Then the action of -1 on $\mathcal{S}^5$ restricts to a free, orientation-reversing involution on $\tilde{\Sigma}$, and $\Sigma = \tilde{\Sigma}/\{\pm 1\}$ is the *Klein bottle*.

Now (N_0, Σ, π) is a ruled 3-fold as in Definition 3.3. However, N_0 is *not* r-orientable, so that we cannot define $\phi : \Sigma \to \mathcal{S}^5$ because of sign problems. Instead, we define $\phi : \tilde{\Sigma} \to \mathcal{S}^5$ to be the identity map on the double cover $\tilde{\Sigma}$ of Σ. This gives a corresponding immersed, r-oriented SL T^2-cone $\tilde{N}_0$, isomorphic as a (singular) immersed manifold to $\tilde{\Sigma} \times \mathbb{R}$ with immersion $\iota : (\tilde{\sigma}, r) \mapsto r\,\phi(\tilde{\sigma})$, which is the double cover of N_0.

We may then apply Theorem 6.3 to $\tilde{N}_0$ to get a 2-parameter family of ruled SL 3-folds $\tilde{N}_{u,v}$ asymptotic to $\tilde{N}_0$. It turns out that $\tilde{N}_{u,v}$ is the double cover of a ruled SL 3-fold $N_{u,v}$ asymptotic to N_0, fibred over the Klein bottle, if and only if the holomorphic vector field $w = u\frac{\partial}{\partial s} + v\frac{\partial}{\partial t}$ changes sign under $-1 : \tilde{\Sigma} \to \tilde{\Sigma}$. The vector space of such w is 1-dimensional. Thus we prove:

Theorem 6.4 *Let N_0 be a two-sided special Lagrangian cone on the Klein bottle. Then there exists a 1-dimensional family of distinct, non r-orientable, ruled special Lagrangian 3-folds N with asymptotic cone N_0, which are asymptotic to N_0 with order $O(r^{-1})$ in the sense of Definition 3.5.*

When the 3-folds N in the theorem are nonsingular, they are immersed 3-submanifolds diffeomorphic to the total space of a nontrivial real line bundle over the Klein bottle. They have one end at infinity, which is asymptotically a T^2-cone.

Finally we note that we can generate interesting ruled SL 3-folds N from SL cones N_0 over Riemann surfaces Σ with any genus $g \geqslant 1$, by applying Theorem 6.1 to nontrivial *meromorphic* vector fields w on Σ with poles at $\sigma_1, \ldots, \sigma_k$ in Σ. Then N is asymptotic to the union of N_0 and k SL 3-planes $\Pi_1, \ldots, \Pi_k$, which are tangent to N_0 along the lines $\pi^{-1}(\sigma_1), \ldots, \pi^{-1}(\sigma_k)$ respectively.

7 Explicit examples of ruled SL 3-folds

We now apply Theorem 6.1 to give some explicit examples of ruled special Lagrangian 3-folds. We start by considering the $\mathrm{U}(1)^2$-invariant SL T^2-cone found by Harvey and Lawson [6, §III.3.A]. Define $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ by

$$\phi : (s,t) \longmapsto \tfrac{1}{\sqrt{3}}\bigl(\mathrm{e}^{is}, \mathrm{e}^{-\frac{i}{2}s-\frac{i\sqrt{3}}{2}t}, \mathrm{e}^{-\frac{i}{2}s+\frac{i\sqrt{3}}{2}t}\bigr), \tag{44}$$

and write $N_0 = \bigl\{r\,\phi(s,t) : r,s,t \in \mathbb{R}\bigr\}$. Then N_0 is a special Lagrangian T^2-cone, and (s,t) are oriented conformal coordinates on $\Sigma = \mathbb{R}^2$, considered as a Riemann surface.

Note that ϕ is invariant under the lattice $\Lambda \cong \mathbb{Z}^2$ in $\mathbb{R}^2$ generated by $(2\pi, 2\pi/\sqrt{3}\,)$ and $(0, 4\pi/\sqrt{3}\,)$, but we shall not pass to the quotient $\mathbb{R}^2/\Lambda$. Let $u(s,t)+iv(s,t)$ be a holomorphic function of $s+it$. Then $w = u(s,t)\frac{\partial}{\partial s} + v(s,t)\frac{\partial}{\partial t}$ is a holomorphic vector field on Σ. Applying Theorem 6.1 gives:

Theorem 7.1 *Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be functions such that $u(s,t) + iv(s,t)$ is a holomorphic function of $s+it$. Define*

$$\begin{aligned} N = \Bigl\{ \tfrac{1}{\sqrt{3}}\Bigl(\mathrm{e}^{is}\bigl(r + iu(s,t)\bigr), \mathrm{e}^{-\frac{i}{2}s-\frac{i\sqrt{3}}{2}t}\bigl(r - \tfrac{i}{2}u(s,t) - \tfrac{i\sqrt{3}}{2}v(s,t)\bigr), \\ \mathrm{e}^{-\frac{i}{2}s+\frac{i\sqrt{3}}{2}t}\bigl(r - \tfrac{i}{2}u(s,t) + \tfrac{i\sqrt{3}}{2}v(s,t)\bigr)\Bigr) : r,s,t \in \mathbb{R}\Bigr\}. \end{aligned} \tag{45}$$

Then N is a ruled special Lagrangian 3-fold in $\mathbb{C}^3$.

The good thing about this theorem is that it defines a large family of ruled SL 3-folds very explicitly. Therefore we can use it as a laboratory for studying the *singularities* of ruled SL 3-folds. For instance, if we put $u(s,t)+iv(s,t) = (s+it)^k$ for $k = 2, 3, \ldots$, then we generate a series of ruled SL 3-folds N_k with an isolated singularity at 0 in $\mathbb{C}^3$.

We can also exchange the dependent and independent variables in Theorem 7.1, and regard $s+it$ as a holomorphic function of $u+iv$. This yields:

Theorem 7.2 *Let $s, t : \mathbb{R}^2 \to \mathbb{R}$ be functions such that $s(u,v) + it(u,v)$ is a holomorphic function of $u+iv$. Define*

$$\begin{aligned} N = \Bigl\{ \tfrac{1}{\sqrt{3}}\Bigl(\mathrm{e}^{is(u,v)}\bigl(r + iu\bigr), \mathrm{e}^{-\frac{i}{2}s(u,v)-\frac{i\sqrt{3}}{2}t(u,v)}\bigl(r - \tfrac{i}{2}u - \tfrac{i\sqrt{3}}{2}v\bigr), \\ \mathrm{e}^{-\frac{i}{2}s(u,v)+\frac{i\sqrt{3}}{2}t(u,v)}\bigl(r - \tfrac{i}{2}u + \tfrac{i\sqrt{3}}{2}v\bigr)\Bigr) : r,u,v \in \mathbb{R}\Bigr\}. \end{aligned} \tag{46}$$

Then N is a ruled special Lagrangian 3-fold in $\mathbb{C}^3$.

This is less useful for modelling singularities, as N is always nonsingular near 0 in $\mathbb{C}^3$. But one can instead use it for modelling 'branched' asymptotic behaviour of ruled SL 3-folds. Putting $s(u,v)+it(u,v)=(u+iv)^k$ for $k=2,3,\ldots$, we find that near the line $\bigl\{(x,x,x):x\in\mathbb{R}\bigr\}$, N is a non-singular ruled SL 3-fold that is asymptotic to a k-fold branched cover of the Harvey–Lawson T^2-cone N_0 defined above.

Next we shall generalize a family of U(1)-invariant special Lagrangian T^2-cones in $\mathbb{C}^3$ given in [8, §8] to a family of ruled SL 3-folds, as illustrations of Theorems 6.3 and 6.4. They are written in terms of the *Jacobi elliptic functions*, which we now briefly introduce. The following material can be found in Chandrasekharan [5, Ch. VII].

For each $k\in[0,1]$, the Jacobi elliptic functions $\mathrm{sn}(t,k)$, $\mathrm{cn}(t,k)$ and $\mathrm{dn}(t,k)$ with modulus k are the unique solutions to the o.d.e.s

$$\begin{aligned}\bigl(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(t,k)\bigr)^2&=\bigl(1-\mathrm{sn}^2(t,k)\bigr)\bigl(1-k^2\mathrm{sn}^2(t,k)\bigr),\\ \bigl(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(t,k)\bigr)^2&=\bigl(1-\mathrm{cn}^2(t,k)\bigr)\bigl(1-k^2+k^2\mathrm{cn}^2(t,k)\bigr),\\ \bigl(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(t,k)\bigr)^2&=-\bigl(1-\mathrm{dn}^2(t,k)\bigr)\bigl(1-k^2-\mathrm{dn}^2(t,k)\bigr),\end{aligned}$$

with initial conditions

$$\begin{aligned}\mathrm{sn}(0,k)=0,\quad \mathrm{cn}(0,k)=1,\quad \mathrm{dn}(0,k)=1,\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(0,k)=1,\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(0,k)=0,\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(0,k)=0.\end{aligned}$$

They satisfy the identities

$$\mathrm{sn}^2(t,k)+\mathrm{cn}^2(t,k)=1\ \text{ and }\ k^2\mathrm{sn}^2(t,k)+\mathrm{dn}^2(t,k)=1,$$

and the differential equations

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(t,k)=\mathrm{cn}(t,k)\mathrm{dn}(t,k),\qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(t,k)=-\mathrm{sn}(t,k)\mathrm{dn}(t,k)\\ \text{and}\qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(t,k)=-k^2\mathrm{sn}(t,k)\mathrm{cn}(t,k).\end{aligned}\tag{47}$$

When $k=0$ or 1 they reduce to trigonometric functions:

$$\begin{aligned}\mathrm{sn}(t,0)=\sin t,\quad \mathrm{cn}(t,0)=\cos t,\quad \mathrm{dn}(t,0)=1,\\ \mathrm{sn}(t,1)=\tanh t,\quad \mathrm{cn}(t,1)=\mathrm{dn}(t,1)=\mathrm{sech}t.\end{aligned}$$

For $k\in[0,1)$ the Jacobi elliptic functions $\mathrm{sn}(t,k),\mathrm{cn}(t,k)$ and $\mathrm{dn}(t,k)$ are *periodic* in t, with a common period.

Using this notation we have the following result, adapted from [8, Th. 8.7].

Theorem 7.3 *Let b_1,b_2,b_3 be coprime integers satisfying $b_2\geqslant b_3>0>b_1$ and $b_1+b_2+b_3=0$. Define $a>0$ and $b\in[0,1)$ by*

$$a^2=b_2(b_3-b_1)\quad\textit{and}\quad b^2=\frac{b_1(b_2-b_3)}{b_2(b_1-b_3)}.\tag{48}$$

Define $\phi:\mathbb{R}^2\rightarrow\mathcal{S}^5$ *by*

$$\begin{aligned}\phi:(s,t)\mapsto\Big(&ie^{ib_1s}\big(\tfrac{b_2}{b_2-b_1}\big)^{1/2}\mathrm{dn}(at,b),ie^{ib_2s}\big(\tfrac{b_1}{b_1-b_2}\big)^{1/2}\mathrm{cn}(at,b),\\ &ie^{ib_3s}\big(\tfrac{b_1}{b_1-b_3}\big)^{1/2}\mathrm{sn}(at,b)\Big),\end{aligned}\tag{49}$$

and let $N_0=\big\{r\,\phi(s,t):r,s,t\in\mathbb{R}\big\}$. *Then* N_0 *is a special Lagrangian cone in* $\mathbb{C}^3$ *and* ϕ *is an oriented conformal map, so that* (s,t) *are oriented conformal coordinates on* $N_0\cap\mathcal{S}^5$. *Furthermore,* ϕ *is doubly periodic in* $\mathbb{R}^2$, *so that* N_0 *is a two-sided* T^2*-cone.*

Applying Theorem 6.1 to this example with $w=u\frac{\partial}{\partial s}+v\frac{\partial}{\partial t}$ for $u,v\in\mathbb{R}$, and using (47) to calculate $\psi=u\frac{\partial\phi}{\partial s}+v\frac{\partial\phi}{\partial t}$, yields:

Theorem 7.4 *Let* b_1,b_2,b_3 *be coprime integers satisfying* $b_2\geqslant b_3>0>b_1$ *and* $b_1+b_2+b_3=0$. *Define* $a>0$ *and* $b\in[0,1)$ *by* (48). *Let* $u,v\in\mathbb{R}$ *and define* $N_{u,v}$ *to be*

$$\begin{aligned}\Big\{\Big(&\big(\tfrac{b_2}{b_2-b_1}\big)^{1/2}e^{ib_1s}\big((ir-ub_1)\mathrm{dn}(at,b)-iva b^2\,\mathrm{sn}(at,b)\mathrm{cn}(at,b)\big),\\ &\big(\tfrac{b_1}{b_1-b_2}\big)^{1/2}e^{ib_2s}\big((ir-ub_2)\mathrm{cn}(at,b)-iva\,\mathrm{sn}(at,b)\mathrm{dn}(at,b)\big),\\ &\big(\tfrac{b_1}{b_1-b_3}\big)^{1/2}e^{ib_3s}\big((ir-ub_3)\mathrm{sn}(at,b)+iva\,\mathrm{cn}(at,b)\mathrm{dn}(at,b)\big)\Big):r,s,t\in\mathbb{R}\Big\}.\end{aligned}$$

Then $N_{u,v}$ *is a ruled special Lagrangian* 3*-fold in* $\mathbb{C}^3$.

It can be shown that when b_1 is even the 3-folds $N_{u,v}$ result from Theorem 6.3, and are generically nonsingular and diffeomorphic to $T^2\times\mathbb{R}$ as immersed submanifolds, and when b_1 is odd and $u=0$ the 3-folds $N_{0,v}$ result from Theorem 6.4, and are generically nonsingular and diffeomorphic to the total space of a real line bundle over the Klein bottle as immersed submanifolds.

We gave the cones of Theorem 7.3 as examples because one can write them down in a very explicit way. But these are only the simplest cases of two much larger explicit families of special Lagrangian T^2-cones in $\mathbb{C}^3$, which were constructed by the author in [8, §8] and [9, §6], and which intersect in the examples of Theorem 7.3.

The first of these families was also studied by Haskins [7, §3–§5], and in terms of minimal Lagrangian tori in $\mathbb{CP}^2$ by Castro and Urbano [4], and the second is related to examples due to Lawlor and Harvey, and was also studied by Bryant [3, §3.5] from a different point of view. By applying Theorem 6.1 to these families we can obtain many more explicit examples of ruled SL 3-folds diffeomorphic to $T^2\times\mathbb{R}$ or a real line bundle over the Klein bottle.

The constructions of [9, §6] also included a 1-parameter family of ruled SL 3-folds asymptotic to each T^2-cone, and results analogous to Theorems 6.3 and 6.4 are given for them in [9, Th.s 6.3 & 6.4]. This 1-parameter family is part of the 2-parameter family of Theorem 6.3, those with $u=0$.

Finally, we briefly discuss U(1)-invariant ruled SL 3-folds in $\mathbb{C}^3$. Let $G \subset \mathrm{SU}(3)$ be a Lie subgroup isomorphic to U(1). What can we say about G-invariant ruled SL 3-folds? Calculations by the author show the following. There is a 2-dimensional family of G-invariant SL cones, written down explicitly by Haskins [7, §3–§5] and the author [8, §8]. Applying Theorem 6.1, we can enlarge this to a 4-dimensional family of explicit G-invariant ruled SL 3-folds.

However, the family of all G-invariant ruled SL 3-folds is 6-dimensional. In the notation of §5, ϕ is already explicitly known by work of Haskins and the author, and we seek G-invariant solutions to the linear equations on ψ. These can be reduced to a linear first-order o.d.e. in 4 variables.

The coefficients of this o.d.e. involve the Jacobi elliptic functions, as these enter the explicit form of ϕ. Two solutions to this o.d.e. are known from Theorem 6.1, but the author has not been able to find the other two solutions, and so find an explicit form for general G-invariant ruled SL 3-folds.

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