

# Numerical determination of the basin of attraction for exponentially asymptotically autonomous dynamical systems

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## Abstract

Numerical methods to determine the basin of attraction for autonomous equations focus on a bounded subset of the phase space. For non-autonomous systems, any relevant subset of the phase space, which now includes the time as one coordinate, is unbounded in  $t$ -direction. Hence, a numerical method would have to use infinitely many points.

To overcome this problem, we introduce a transformation of the phase space. Restricting ourselves to exponentially asymptotically autonomous systems, we can map the infinite time interval to a finite, compact one. The basin of attraction of a solution becomes the basin of attraction of an exponentially stable equilibrium. Now we are able to generalise numerical methods from the autonomous case. More precisely, we characterise a Lyapunov function as a solution of a suitable linear first-order partial differential equation and approximate it using Radial Basis Functions.

**Keywords:** basin of attraction, asymptotically autonomous differential equation, Lyapunov function, Radial Basis Function

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## 1 Introduction

Non-autonomous differential equations play an important role in modelling many processes in biology, industry and economics. We study a general equation of the form  $\dot{x} = f(t, x)$ , where  $f$  is smooth and  $x \in \mathbb{R}^n$ . Important questions are the stability of a special solution and its basin of attraction consisting of all solutions that approach the special solution.

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In the case of equilibria in autonomous differential equations, the basin of attraction can be determined using sublevel sets of Lyapunov functions. Methods for the construction of Lyapunov functions in autonomous systems include Zubov's equation [7] and linear programming [8]. A special Lyapunov function can be characterised as the solution of a first-order partial differential equation, and then be approximated using Meshless Collocation, in particular Radial Basis Functions [2, 5]. The method was extended to discrete [3], non-smooth [4] and time-periodic systems [6].

Turning back to non-autonomous systems, we can add the time as one additional variable. However, the basin of attraction of a solution is now an unbounded set, since the phase space includes the time as one coordinate. Hence, a numerical method would have to use infinitely many points.

To overcome this problem, we introduce a transformation of the phase space. Instead of the additional equation  $\dot{t} = 1$  we transform  $t$  into the new variable  $\tau$ , mapping the infinite time interval with respect to  $t$  to a finite, compact one with respect to  $\tau$ . The point  $t = \infty$  will be mapped to  $\tau = 0$ . Hence, we have to restrict ourselves to asymptotically autonomous systems, for which the limit  $f(t, x)$  for  $t \rightarrow \infty$  makes sense. More precisely, we assume that  $f(t, x)$  tends to  $g(x)$  as  $t \rightarrow \infty$  uniformly in  $x$ . The transformed differential equation reads

$$\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = F(\tau, x), \quad (1.1)$$

where the dot still denotes the derivative with respect to  $t$  and  $F \in \mathbb{R}^{n+1}$ . Note that (1.1) is an autonomous system in  $\mathbb{R}^{n+1}$ .

We assume that  $x_0(t) = 0$  is an exponentially stable solution. The basin of attraction consists of solutions  $x(t)$  such that  $\lim_{t \rightarrow \infty} x(t) = 0$ . In the new variable, the solutions  $(\tau(t), x(t))$  will converge to  $\lim_{t \rightarrow \infty} (\tau(t), x(t)) = (0, 0)$ . The basin of attraction of the zero solution is thus mapped to the basin of attraction of the equilibrium  $(0, 0)$  in the transformed system (1.1). Since (1.1) is autonomous, we can use numerical methods for the determination of equilibria in the autonomous case, which have been described above. More precisely, we characterise a Lyapunov function as a solution of a suitable linear first-order partial differential equation and approximate it using Radial Basis Functions.

There are further points to consider: first of all, the transformed system is only defined for  $\tau \leq 0$  or  $\tau \geq 0$  depending on how we transform the time  $t$  to  $\tau$ . We, however, need that the equilibrium  $(0, 0)$  is not on the boundary of the phase space. We overcome this problem by defining the transformation from  $t$  to  $\tau$  in the following way: we map  $t$  twice to  $\tau$ , once  $t \in (-\infty, \infty)$  is mapped to  $\tau \in (-\infty, 0)$ , and once  $t \in (-\infty, \infty)$  is mapped to  $\tau \in (0, \infty)$  with back ward orientation;  $t = \infty$  is mapped to  $\tau = 0$ , cf. Figure 1. Hence, the system (1.1) is symmetric about  $\tau = 0$ . We will make use of this symmetry by using a symmetric grid for the Radial Basis Function approximation, so that the collocation matrix can be reduced.

We also need that the equilibrium  $(0, 0)$  in the transformed system (1.1) is exponentially stable. This will be achieved by an exponential transformation from

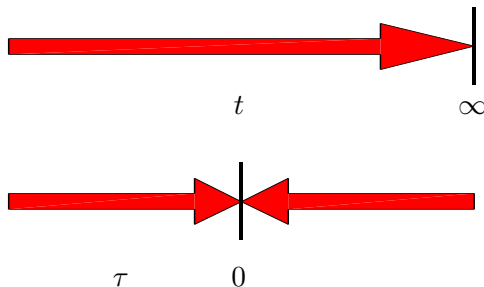


Figure 1: The time  $t$  is mapped twice to the new variable  $\tau$ , once with the same orientation to  $\tau < 0$ , once with backward orientation to  $\tau > 0$ . The point  $t = \infty$  is mapped to  $\tau = 0$ .

$t$  to  $\tau$ . This is a Fowler-type transformation [1] and is used, for example, in the study of ground states of elliptic equations [9]. Furthermore, we assume that the system  $\dot{x} = f(t, x)$  is exponentially asymptotically autonomous, i.e. it converges exponentially fast to the limiting autonomous system  $\dot{x} = g(x)$  as  $t \rightarrow \infty$ .

Note that the first main contribution of the paper is the transformation of the exponentially asymptotically autonomous system to an autonomous one, and the basin of attraction of the zero solution to the symmetric basin of attraction of the exponentially stable equilibrium  $(0, 0)$ . The basin of attraction of this new, transformed system can be studied with a variety of methods. The second main contribution is to present the study of the transformed basin of attraction with Radial Basis Functions, making use of the introduced symmetry of the problem.

Let us give an overview over the contents: In Section 2 we introduce exponentially asymptotically autonomous systems as well as the time transformation. Furthermore, we study the basin of attraction of the original and transformed system, and recall Lyapunov functions and their use to study basins of attraction. In Section 3 we recall Radial Basis Functions and their use to construct Lyapunov functions. Using a symmetric grid, we show how to reduce the collocation matrix. In Section 4, the method is applied to an example.

## 2 Time transformation

### 2.1 Exponentially asymptotically autonomous dynamical systems

In this paper, we consider the non-autonomous differential equation

$$\dot{x} = f(t, x), \tag{2.1}$$

where  $x \in \mathbb{R}^n$ . We denote by  $\varphi(t, t_0, x_0) := x(t)$  the solution  $x(t)$  of (2.1) with initial value  $x(t_0) = x_0$ . We assume that  $f(t, x)$  tends to a limiting function  $g(x)$  exponentially fast as  $t \rightarrow \infty$ . More precisely, we use the following definition. In this paper,  $\|\cdot\|$  denotes an arbitrary but fixed norm of  $\mathbb{R}^n$ .

**Definition 2.1** Let  $\sigma \in \mathbb{N}$ . The system of differential equations  $\dot{x} = f(t, x)$ ,  $f \in C^\sigma(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  is called **exponentially asymptotically autonomous**, if there exists a function  $g \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$  and  $\alpha > 0$  such that for each compact set  $K \subset \mathbb{R}^n$  and each  $\epsilon > 0$  there is a  $T$  with

$$\|\partial^\gamma f(t, x) - \partial^\gamma g(x)\| e^{\alpha t} < \epsilon \text{ for all } \gamma \text{ with } |\gamma| \leq \sigma, \quad (2.2)$$

for all  $x \in K$  and all  $t > T$ . We use the notation  $\partial^\gamma f(t, x) = \frac{\partial^{\gamma_0}}{\partial t^{\gamma_0}} \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \dots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}} f(t, x)$ .

Note that the assumptions on the derivatives do not follow from the assumption on  $f$ , e.g. consider  $f(t) = e^{-t} \sin(e^{2t})$ . The definition is related to the definition of asymptotically autonomous systems, where one assumes that (2.2) holds for  $\gamma = 0$  and  $\alpha = 0$ .

## 2.2 Time transformation

The non-autonomous system (2.1) can easily be extended to an autonomous system by adding the time as an additional variable. We will, however, add a transformed time variable  $\tau$ . We use an exponential map to transform the time of the exponentially asymptotically autonomous system into the new variable  $\tau$ ; this is a Fowler-type transformation [1]. The transformed system will have  $(\tau, x) \in \mathbb{R} \times \mathbb{R}^n$  as variables, and be of the form

$$\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = F(\tau, x), \quad (2.3)$$

where the dot still denotes the derivative with respect to the time  $t$  and  $F_i: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 0, 1, \dots, n$ . The new variable  $\tau$  represents the old time in the following way:  $\tau \in (-\infty, 0)$  corresponds to  $t \in (-\infty, \infty)$ , the point  $\tau = 0$  corresponds to  $t = \infty$  and  $\tau \in (0, \infty)$  corresponds again to  $t \in (-\infty, \infty)$ , but with time running backwards. The main contribution of the following theorem is the smoothness of the function  $F$ .

**Theorem 2.2** Let  $\dot{x} = f(t, x)$  be an exponentially asymptotically autonomous system with limiting function  $g$  and exponent  $\alpha > 0$ .

Let  $0 < \beta \leq \frac{\alpha}{\sigma}$ . Define the new variable  $\tau \in \mathbb{R}$  by

$$\tau = -e^{-\beta t} \text{ for } \tau < 0, \quad (2.4)$$

$$\tau = e^{-\beta t} \text{ for } \tau > 0. \quad (2.5)$$

The inverse function is in both cases  $t = -\frac{1}{\beta} \ln |\tau|$ .

Define  $F_0(\tau, x) := -\beta\tau$  and for  $i = 1, \dots, n$

$$F_i(\tau, x) := \begin{cases} f_i\left(-\frac{1}{\beta} \ln |\tau|, x\right) & \text{for } \tau \neq 0, \\ g_i(x) & \text{for } \tau = 0. \end{cases}$$

Then  $F \in C^\sigma(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$  and  $DF(0, x) = \begin{pmatrix} -\beta & 0 \\ 0 & Dg(x) \end{pmatrix}$ .

Consider the new system

$$\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = F(\tau, x). \quad (2.6)$$

- We have the following relation between solution of (2.6) and (2.1)

$$S_t(\theta, \xi) = \left( \theta e^{-\beta t}, \varphi \left( t - \frac{1}{\beta} \ln |\theta|, -\frac{1}{\beta} \ln |\theta|, \xi \right) \right),$$

where  $S_t(\theta, \xi) := \begin{pmatrix} \tau(t) \\ x(t) \end{pmatrix}$  is the solution of (2.6) with initial values  $\tau(0) = \theta$  and  $x(0) = \xi$ .

- If  $(\tau(t), x(t))$  with  $\tau(t_0) \neq 0$  for a  $t_0 \in \mathbb{R}$  is a solution of (2.6), then  $x(t)$  is a solution of  $\dot{x} = f(t, x)$ .
- If  $(\tau(t), x(t))$  with  $\tau(t_0) = 0$  for a  $t_0 \in \mathbb{R}$  is a solution of (2.6), then  $x(t)$  is a solution of  $\dot{x} = g(x)$ .

PROOF: We first show the relation between solutions of the old and the new system (2.1) and (2.6). The solution of  $\dot{\tau} = -\beta\tau$  with initial value  $\tau(0) = \theta$  is clearly  $\theta e^{-\beta t}$ . The initial point  $\tau = \theta$  corresponds to the initial time  $t_0 = -\frac{1}{\beta} \ln |\theta|$ , which shows the relation. The other two statements are direct consequences.

Before we prove the result about the smoothness, we show an auxiliary lemma.

**Lemma 2.3** Let  $i \in \{1, \dots, n\}$  and let  $\tilde{\gamma} \in \mathbb{N}_0^{n+1}$  with  $\tilde{\gamma}_0 = 0$ .

Then for  $k \in \mathbb{N}$  with  $|\tilde{\gamma}| + k \leq \sigma$  there are constants  $c_{i,k,1}, \dots, c_{i,k,k} \in \mathbb{R}$  such that for all  $\tau \neq 0$  we have

$$\partial_\tau^k \partial^{\tilde{\gamma}} F_i(\tau, x) = \frac{1}{\tau^k} \sum_{j=1}^k c_{i,k,j} \partial_t^j \partial^{\tilde{\gamma}} f_i(t, x), \quad (2.7)$$

where  $t = -\frac{1}{\beta} \ln |\tau|$ .

PROOF: It is enough to show the lemma for  $\tilde{\gamma} = 0$ . We use induction with respect to  $k$ . For  $k = 1$  we have

$$\begin{aligned} \partial_\tau F_i(\tau, x) &= \partial_t f_i \left( -\frac{1}{\beta} \ln |\tau|, x \right) \frac{\partial t}{\partial \tau} \\ &= -\frac{\partial_t f_i \left( -\frac{1}{\beta} \ln |\tau|, x \right)}{\beta \tau} \end{aligned}$$

since  $\frac{\partial t}{\partial \tau} = -\frac{1}{\beta \tau}$ . This shows (2.7) with  $c_{i,1,1} = -\frac{1}{\beta}$  for all  $i$ .

Now assume that the statement is true up to  $k - 1$ . Then

$$\begin{aligned}
\partial_\tau^k F_i(\tau, x) &= \partial_\tau \partial_\tau^{k-1} F_i(\tau, x) \\
&= \partial_\tau \left( \frac{1}{\tau^{k-1}} \sum_{j=1}^{k-1} c_{i,k-1,j} \partial_t^j f_i(t, x) \right) \\
&= -(k-1) \frac{1}{\tau^k} \sum_{j=1}^{k-1} c_{i,k-1,j} \partial_t^j f_i(t, x) \\
&\quad + \frac{1}{\tau^{k-1}} \sum_{j=1}^{k-1} c_{i,k-1,j} \partial_t^{j+1} f_i(t, x) \cdot \left( -\frac{1}{\beta \tau} \right)
\end{aligned}$$

which shows (2.7) with suitable constants.  $\square$

Now we show that  $F \in C^\sigma(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ . For  $F_0$  this is clear. Also, for any point  $(\tau, x)$  with  $\tau \neq 0$ ,  $F$  is obviously in  $C^\sigma$ . Now let  $\tau = 0$  and fix  $i \in \{1, \dots, n\}$ .

For the derivatives we distinguish between the cases  $\gamma_0 = 0$  and  $\gamma_0 \neq 0$ . For  $\gamma_0 = 0$ , due to the smoothness of  $f$  and  $g$ , we only have to show that  $\lim_{t \rightarrow \infty} \partial^\gamma f(t, x) = \partial^\gamma g(x)$ . This is clear by (2.2).

Now we show that for  $\gamma$  with  $|\gamma| \leq \sigma$  and  $\gamma_0 \geq 1$  we have  $\partial^\gamma F_i(0, x) = 0$  and that the derivatives are continuous at  $\tau = 0$ . We use induction with respect to  $\gamma_0$ . For  $\gamma_0 = 1$  we set  $\gamma = (1, 0, \dots, 0) + \tilde{\gamma}$  so that  $\partial^\gamma = \partial_\tau \partial^{\tilde{\gamma}}$  and  $\tilde{\gamma}_0 = 0$ . We calculate, using the transformation from  $\tau$  back to  $t$

$$\begin{aligned}
|\partial_\tau \partial^{\tilde{\gamma}} F_i(0, x)| &= \lim_{\tau \rightarrow 0} \left| \frac{\partial^{\tilde{\gamma}} f_i \left( -\frac{1}{\beta} \ln |\tau|, x \right) - \partial^{\tilde{\gamma}} g_i(x)}{\tau} \right| \\
&= \lim_{t \rightarrow \infty} |\partial^{\tilde{\gamma}} f_i(t, x) - \partial^{\tilde{\gamma}} g_i(x)| e^{\beta t} \\
&= 0
\end{aligned}$$

by (2.2) as  $\beta \leq \alpha$ .

Note that  $\frac{\partial t}{\partial \tau} = -\frac{1}{\beta \tau}$ . We show that the derivative is continuous since for  $\tau \neq 0$  we have with Lemma 2.3

$$\begin{aligned}
|\partial_\tau \partial^{\tilde{\gamma}} F_i(\tau, x)| &= \frac{|\partial_t \partial^{\tilde{\gamma}} f_i \left( -\frac{1}{\beta} \ln |\tau|, x \right)|}{\beta |\tau|} \\
\text{and } \lim_{\tau \rightarrow 0} |\partial_\tau \partial^{\tilde{\gamma}} F_i(\tau, x)| &= \lim_{t \rightarrow \infty} \frac{1}{\beta} |\partial_t \partial^{\tilde{\gamma}} f_i(t, x)| e^{\beta t} = 0
\end{aligned}$$

by (2.2) as  $\beta \leq \alpha$ .

Now we assume that the statement is true for all  $\gamma_0$  up to  $k - 1 \geq 1$  and we show it for  $\gamma_0 = k$ . Note that  $k \leq |\gamma| \leq \sigma$  so that  $2 \leq k \leq \sigma$ . We set  $\gamma = (k, 0, \dots, 0) + \tilde{\gamma}$  so that  $\tilde{\gamma}_0 = 0$ . Using Lemma 2.3, we have

$$|\partial_\tau^k \partial^{\tilde{\gamma}} F_i(0, x)| = \lim_{\tau \rightarrow 0} \left| \frac{\partial_\tau^{k-1} \partial^{\tilde{\gamma}} F_i(\tau, x) - 0}{\tau} \right|$$

$$\begin{aligned}
&= \lim_{\tau \rightarrow 0} \left| \frac{1}{\tau^{k-1}} \sum_{j=1}^{k-1} c_{i,k-1,j} \partial_t^j \partial^{\tilde{\gamma}} f_i(t, x) \right| \\
&= \lim_{t \rightarrow \infty} \left| \sum_{j=1}^{k-1} c_{i,k-1,j} \partial_t^j \partial^{\tilde{\gamma}} f_i(t, x) \right| e^{(k-1)\beta t} \\
&= 0
\end{aligned}$$

by (2.2) as  $\beta \leq \frac{1}{\sigma}\alpha \leq \frac{1}{k-1}\alpha$ .

Finally, we show that the derivative is continuous since for  $\tau \neq 0$ . We have with Lemma 2.3

$$\begin{aligned}
|\partial_\tau^k \partial^{\tilde{\gamma}} F_i(\tau, x)| &\leq \frac{1}{|\tau|^k} \sum_{j=1}^k |c_{i,k,j}| \cdot \left| \partial_t^j \partial^{\tilde{\gamma}} f_i \left( -\frac{1}{\beta} \ln |\tau|, x \right) \right| \\
\text{and } \lim_{\tau \rightarrow 0} |\partial_\tau^k \partial^{\tilde{\gamma}} F_i(\tau, x)| &\leq \lim_{t \rightarrow \infty} \sum_{j=1}^k |c_{i,k,j}| \cdot \left| \partial_t^j \partial^{\tilde{\gamma}} f_i(t, x) \right| e^{\beta k t} = 0
\end{aligned}$$

by (2.2) as  $\beta \leq \frac{1}{\sigma}\alpha \leq \frac{1}{k}\alpha$ . □

### 2.3 Basin of attraction

In the following we will consider the three systems:

- Old system (2.1):  $\dot{x} = f(t, x)$ , where  $x \in \mathbb{R}^n$  (non-autonomous).
- Limiting system:  $\dot{x} = g(x)$ , where  $x \in \mathbb{R}^n$  (autonomous).
- New system (2.3):  $\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = F(\tau, x)$  where  $(\tau, x) \in \mathbb{R}^{n+1}$  (autonomous).

We assume that  $x(t) = 0$  is a solution of  $\dot{x} = f(t, x)$ , i.e.  $f(t, 0) = 0$  for all  $t \in \mathbb{R}$ , moreover we assume that 0 is an exponentially asymptotically stable equilibrium of  $\dot{x} = g(x)$ . More precisely, we make the following assumptions throughout the rest of the paper.

#### Assumptions 2.4

- $f(t, 0) = 0$  for all  $t \in \mathbb{R}$ .
- All eigenvalues of  $Dg(0)$  have negative real part.

Under these assumptions, we have the following results on the equilibria and their stability in the new and limiting system. We will consider the old non-autonomous system (2.1) later in Proposition 2.8.

**Proposition 2.5** *Let the Assumptions 2.4 hold. Then*

1.  $0 \in \mathbb{R}^n$  is exponentially stable equilibrium of  $\dot{x} = g(x)$ .
2.  $(0, 0) \in \mathbb{R}^{n+1}$  is exponentially stable equilibrium of the system  $\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = F(\tau, x)$ .

PROOF: For the first statement, we note that  $g(0) = 0$  since  $f$  is exponentially asymptotically autonomous. The stability follows directly from the assumptions.

Now we turn to the second statement. We can conclude that  $(0, 0)$  is an equilibrium of this system, since  $F(0, 0) = (0, g(0)) = (0, 0)$  by Theorem 2.2. By the same theorem,  $DF(0, 0) = \begin{pmatrix} -\beta & 0 \\ 0 & Dg(0) \end{pmatrix}$  and thus  $(0, 0)$  is exponentially stable.  $\square$

In the new (autonomous) system, we can thus define the basin of attraction  $A_F(0, 0)$  of  $(0, 0)$  in the usual way.

**Definition 2.6** *Let the Assumptions 2.4 hold.*

*The basin of attraction  $A_F(0, 0)$  of  $(0, 0)$  with respect to the new system  $\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = F(\tau, x)$  is defined by*

$$A_F(0, 0) := \{(\theta, \xi) \in \mathbb{R}^{n+1} \mid \lim_{t \rightarrow \infty} S_t(\theta, \xi) = (0, 0)\},$$

where  $S_t(\theta, \xi)$  denotes the solution of (2.3).

Similarly the basin of attraction of 0 with respect to the (autonomous) limiting system  $\dot{x} = g(x)$  is defined by

$$A_g(0) := \{\xi \in \mathbb{R}^n \mid \lim_{t \rightarrow \infty} S_t^g \xi = 0\},$$

where  $S_t^g \xi := x(t)$  denotes the solution  $x(t)$  of  $\dot{x} = g(x)$  with initial values  $x(0) = \xi$ .

We will show that the Assumptions 2.4 guarantee that the zero solution in the original, non-autonomous system is exponentially stable and we can show relations between the basins of attraction of the three systems. Let us first define the exponential stability of the zero solution of the non-autonomous system.

**Definition 2.7**  $x(t) = 0$  is exponentially stable for the non-autonomous system (2.1) if

- it is stable, i.e. for all  $\epsilon > 0$ ,  $T \in \mathbb{R}$  there is a  $\delta > 0$  such that

$$\|\varphi(t, t_0, x_0)\| < \epsilon \text{ for all } t \geq t_0, t_0 > T \text{ and } \|x_0\| < \delta.$$

- it is exponentially attractive, i.e. for all  $T \in \mathbb{R}$  there are  $C, \nu, \delta > 0$  such that

$$\|\varphi(t, t_0, x_0)\| < Ce^{-\nu t} \text{ for all } t \geq t_0, t_0 > T \text{ and } \|x_0\| < \delta.$$



Its basin of attraction is defined by

$$A_f(0) = \{(t_0, x_0) \in \mathbb{R}^{n+1} \mid \varphi(t, t_0, x_0) \xrightarrow{t \rightarrow \infty} 0\},$$

where  $\varphi(t, t_0, x_0)$  denotes the solution of (2.1).

**Proposition 2.8** *Let the Assumptions 2.4 hold.*

*Then the zero solution of (2.1) is exponentially stable in the sense of Definition 2.7. The basins of attraction  $A_F(0, 0)$ ,  $A_f(0)$  and  $A_g(0)$  satisfy*

- $(\tau, 0) \in A_F(0, 0)$  for all  $\tau \in \mathbb{R}$
- $(\tau, x) \in A_F(0, 0) \iff (-\tau, x) \in A_F(0, 0)$  (symmetry)
- $(0, x) \in A_F(0, 0) \iff x \in A_g(0)$
- $(\tau, x) \in A_F(0, 0) \setminus (\{0\} \times A_g(0)) \iff \left(-\frac{1}{\beta} \ln |\tau|, x\right) \in A_f(0)$

PROOF: By 2. of Proposition 2.5,  $(0, 0)$  is exponentially stable equilibrium of (2.3). We show now that the zero solution of (2.1) is stable in the sense of Definition 2.7. Let  $T \in \mathbb{R}$  and choose  $\epsilon > 0$ . Since  $(0, 0)$  is stable for (2.3), for the above  $\epsilon > 0$  there is a  $\delta > 0$  such that  $\|S_t(\theta, \xi)\| < \epsilon$  holds for all  $\|(\theta, \xi)\| < \delta$  and  $t \geq 0$ . Set  $\tilde{\delta} = \delta/\sqrt{2}$  and  $T = -\frac{1}{\beta} \ln \tilde{\delta}$ . Then we have

$$\begin{aligned} \|(\pm e^{-\beta t_0}, x_0)\|^2 &= e^{-2\beta t_0} + \|x_0\|^2 \\ &< e^{-2\beta T} + \tilde{\delta}^2 \\ &= \delta^2 \end{aligned}$$

such that  $\|S_{t-t_0}(\pm e^{-\beta t_0}, x_0)\| < \epsilon$  for all  $t \geq t_0, t_0 > T$  and  $\|x_0\| < \tilde{\delta}$ .

Using the relation  $S_t(\theta, \xi) = \left(\theta e^{-\beta t}, \varphi\left(t - \frac{1}{\beta} \ln |\theta|, -\ln |\theta|, \xi\right)\right)$  of Theorem 2.2 we conclude

$$\|\varphi(t, t_0, x_0)\|^2 = \|S_{t-t_0}(\pm e^{-\beta t_0}, x_0)\|^2 - e^{-2\beta t} < \epsilon^2$$

The exponential attraction is proved in a similar way. All other statements are clear by the relations between the old, new and limiting system.  $\square$

As  $A_F(0, 0)$  is the basin of attraction of an exponentially stable equilibrium in an autonomous system, we can use standard methods to determine it. We will use Lyapunov functions to determine the basin of attraction, and Radial Basis Functions to construct Lyapunov functions. Once we have determined  $A_F(0, 0)$ , we can recover the basin of attraction  $A_f(0)$  with respect to the old system  $\dot{x} = f(t, x)$  by Proposition 2.8.

## 2.4 Lyapunov function

In this section we consider the autonomous system (2.3)

$$\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = F(\tau, x)$$

and assume that  $(0, 0) \in \mathbb{R} \times \mathbb{R}^n$  is an exponentially stable equilibrium with basin of attraction  $A_F(0, 0)$ .

Lyapunov functions can be used to determine the basin of attraction. We cite the following classical results: Theorem 2.9 explains the use of Lyapunov functions for the determination of the basin of attraction. A Lyapunov function provides information on the basin of attraction through its sublevel sets. There is no straightforward analytic way to construct Lyapunov functions for general systems except for linear ones.

The main condition on a Lyapunov function is that its orbital derivative is negative. We will later construct a Lyapunov function using Radial Basis Functions, however, it turns out that the orbital derivative locally near the equilibrium is not negative. Therefore, the following theorem allows for an exceptional set  $E$  near the equilibrium, where the orbital derivative may be positive. To use the theorem we have to show that  $E$  is a subset of the basin of attraction; this will be done using a local Lyapunov function (see Lemma 2.10).

**Theorem 2.9 ([2, Theorem 2.26])** *Let  $(0, 0) \in B \subset \mathbb{R}^{n+1}$  be an open set,  $K \subset B$  a compact set and  $(0, 0) \in E$  an open set. Let  $s \in C^1(B, \mathbb{R})$  and*

1.  $(0, 0) \in \overset{\circ}{K}$ ,
2.  $s'(\tau, x) < 0$  for all  $(\tau, x) \in K \setminus E$ , i.e.  $s$  is decreasing along solutions in  $K \setminus E$ ,
3.  $K = \{(\tau, x) \in B \mid s(\tau, x) \leq R\}$  with an  $R \in \mathbb{R}$ , i.e.  $K$  is a sublevel set of  $s$ ,
4.  $E \subset A_F(0, 0)$ .

*Then  $K \subset A_F(0, 0)$ .*

Note that the orbital derivative  $s'(x)$  is the derivative along a solution of (2.3) and is given by the chain rule:

$$\frac{d}{dt}s(\tau(t), x(t)) = \partial_{\tau}s(\tau(t), x(t))F_0(\tau(t), x(t)) + \sum_{i=1}^n \partial_{x_i}s(\tau(t), x(t))F_i(\tau(t), x(t)).$$

The main condition 2. is that the orbital derivative is negative in a sublevel set  $K$  with the exception of the set  $E$ . One can use the Lyapunov function for the linearised system at the equilibrium point as a local Lyapunov function for the nonlinear system. Local means that the area where the orbital derivative is negative,

is a (usually small) neighborhood of the equilibrium. This will enable us to find a subset  $E$  of the basin of attraction which can be used in Theorem 2.9.

The linearised system at the equilibrium point  $(0,0)$  is given by  $\begin{pmatrix} \dot{\tau} \\ \dot{x} \end{pmatrix} = DF(0,0) \begin{pmatrix} \tau \\ x \end{pmatrix}$ . This is a linear system and, thus, one can easily calculate a Lyapunov function  $L$  which is not only a Lyapunov function for the linearised system, but also for the nonlinear system in a neighborhood of  $(0,0)$ , for details cf. [2]. We can find a subset  $E$  of the basin of attraction using Lemma 2.10, which can then be used in Theorem 2.9.

**Lemma 2.10 (Local Lyapunov function)** *Let  $(0,0)$  be an equilibrium of  $(\dot{\tau}, \dot{x}) = F(\tau, x)$ , such that all eigenvalues of  $DF(0,0)$  have negative real part. Denote by  $C \in \mathbb{R}^{(n+1) \times (n+1)}$  the unique solution of the matrix equation  $DF(0,0)^T C + C DF(0,0) = -I$  and define the local Lyapunov function*

$$L(\tau, x) = (\tau, x) C \begin{pmatrix} \tau \\ x \end{pmatrix}.$$

*Then, there is a compact set  $E$  with a neighborhood  $B$  such that  $(0,0) \in \overset{\circ}{E}$ . Moreover,  $L'(\tau, x) < 0$  holds for all  $(\tau, x) \in E \setminus \{(0,0)\}$  and  $E = \{(\tau, x) \in B \mid L(\tau, x) \leq R^*\}$  with some  $R^* > 0$ .*

Finally, Theorem 2.11 proves the existence of a special Lyapunov function which satisfies an equation for its orbital derivative. This equation will be used in the next chapter to approximate  $V$  using Radial Basis Functions.

**Theorem 2.11 ([2, Theorem 2.38 and Theorem 2.46])** *Consider (2.3) with  $F \in C^\sigma(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$  and let  $(0,0)$  be an equilibrium such that all eigenvalues of  $DF(0,0)$  have negative real part.*

*Then, there exists a Lyapunov function  $V \in C^\sigma(A_F(0,0), \mathbb{R})$  such that  $V(0,0) = 0$  and*

$$V'(\tau, x) = -\tau^2 - \|x\|^2 \text{ for all } (\tau, x) \in A_F(0,0).$$

### 3 Approximation with Radial Basis Functions

We use Radial Basis Functions to approximate the function  $V$  of Theorem 2.11. Radial Basis Functions are a powerful method to interpolate scattered data, and also to solve linear partial differential equations, cf. for example [10]. As  $V$  satisfies the first-order linear PDE

$$V'(\tau, x) = -\tau^2 - \|x\|^2$$

and is a  $C^\sigma$  function, we can use Radial Basis Functions to approximate  $V$  by the approximant  $v_f$ .

For the method, we choose a Radial Basis Function  $\Phi(\tilde{x}) := \psi(\|\tilde{x}\|)$ , where  $\tilde{x} = (\tau, x)$ . In this paper, we use the family of Wendland's compactly supported Radial Basis Functions where  $k$  denotes their smoothness.

Furthermore, we choose a finite set of data points  $X \subset \mathbb{R} \times \mathbb{R}^n$ . The approximant is given by  $v_f$  as in Lemma 3.1, where the coefficients are determined by solving a system of linear equations. Note that this system has a unique solution for any choice of pairwise distinct data points which are no equilibria of (2.3) since the Fourier transform of  $\psi$  is positive and the linear operators are not singular, cf. [2, 5] for more details.

In this special case, we use a symmetric set of points  $X = X^+ \cup X^- \cup X^0$ , where  $X^0 = \{(0, y_1), \dots, (0, y_{N^0})\}$ ,  $X^- = \{(-\tau_1, x_1), \dots, (-\tau_N, x_N)\}$  and  $X^+ = \{(\tau_1, x_1), \dots, (\tau_N, x_N)\}$ , where  $\tau_k > 0$  for all  $k = 1, \dots, N$ , for an example cf. Figure 3. Since the linear operator and the data are symmetric we can use a smaller collocation matrix.

In the following we denote by  $D$  the linear operator of the orbital derivative, i.e.

$$Dv(\tau, x) := v'(\tau, x) = \langle \nabla_{(\tau, x)} v(\tau, x), F(\tau, x) \rangle.$$

The scalar product  $\langle \cdot, \cdot \rangle$  is in  $\mathbb{R}^{n+1}$ , i.e.  $\langle \tilde{x}, \tilde{y} \rangle := \sum_{i=0}^n x_i y_i$ , where  $\tilde{x} = (\tau, x) \in \mathbb{R}^{n+1}$ . The superscript  $\tilde{y}$  in Lemma 3.1 denotes the application of the operator with respect to  $\tilde{y}$ , and  $\delta_{\tilde{x}_0}$  denotes Dirac's delta operator, i.e.  $\delta_{\tilde{x}_0} v(\tilde{x}) := v(\tilde{x}_0)$ .

**Lemma 3.1** *Let  $v_f$  be the unique solution of the full collocation problem, i.e.*

$$\begin{aligned} v_f(\tau, x) &= \sum_{k=1}^{N^0} \tilde{a}_k (\delta_{(0, y_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}) \\ &\quad + \sum_{k=1}^N \tilde{b}_k^- (\delta_{(-\tau_k, x_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}) \\ &\quad + \sum_{k=1}^N \tilde{b}_k^+ (\delta_{(\tau_k, x_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}), \end{aligned}$$

where  $\tilde{x} = (\tau, x) \in \mathbb{R}^{n+1}$  and  $(\tilde{a}, \tilde{b}^-, \tilde{b}^+) \in \mathbb{R}^{N^0+2N}$  is the solution of

$$\begin{pmatrix} \tilde{A} & \tilde{D} & \tilde{E} \\ \tilde{D}^T & \tilde{B} & \tilde{F} \\ \tilde{E}^T & \tilde{F}^T & \tilde{C} \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b}^- \\ \tilde{b}^+ \end{pmatrix} = \begin{pmatrix} c \\ d \\ d \end{pmatrix},$$

$c_k = -\|y_k\|^2$ ,  $d_k = -\tau_k^2 - \|x_k\|^2$  and

$$\begin{aligned} \tilde{A}_{ik} &= (\delta_{(0, y_i)} \circ D)^{\tilde{x}} (\delta_{(0, y_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}), \\ \tilde{B}_{ik} &= (\delta_{(-\tau_i, x_i)} \circ D)^{\tilde{x}} (\delta_{(-\tau_k, x_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}), \\ \tilde{C}_{ik} &= (\delta_{(\tau_i, x_i)} \circ D)^{\tilde{x}} (\delta_{(\tau_k, x_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}), \end{aligned}$$

$$\begin{aligned}
\tilde{D}_{ik} &= (\delta_{(0,y_i)} \circ D)^{\tilde{x}} (\delta_{(-\tau_k, x_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}), \\
\tilde{E}_{ik} &= (\delta_{(0,y_i)} \circ D)^{\tilde{x}} (\delta_{(\tau_k, x_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}), \\
\tilde{F}_{ik} &= (\delta_{(-\tau_i, x_i)} \circ D)^{\tilde{x}} (\delta_{(\tau_k, x_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}).
\end{aligned}$$

Let  $v$  be the solution of the **reduced collocation problem**, i.e.

$$\begin{aligned}
v(\tau, x) &= \sum_{k=1}^{N^0} a_k (\delta_{(0,y_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}) \\
&\quad + \sum_{k=1}^N b_k ([\delta_{(-\tau_k, x_k)} + \delta_{(\tau_k, x_k)}] \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}),
\end{aligned}$$

where  $\tilde{x} = (\tau, x) \in \mathbb{R}^{n+1}$  and  $(a, b) \in \mathbb{R}^{N^0+N}$  is the solution of

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ 2d \end{pmatrix}, \quad (3.1)$$

$c_k = -\|y_k\|^2$ ,  $d_k = -\tau_k^2 - \|x_k\|^2$  and

$$\begin{aligned}
A_{ik} &= (\delta_{(0,y_i)} \circ D)^{\tilde{x}} (\delta_{(0,y_k)} \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}) \\
B_{ik} &= (\delta_{(0,y_i)} \circ D)^{\tilde{x}} ([\delta_{(-\tau_k, x_k)} + \delta_{(\tau_k, x_k)}] \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y}) \\
D_{ik} &= ([\delta_{(-\tau_i, x_i)} + \delta_{(\tau_i, x_i)}] \circ D)^{\tilde{x}} ([\delta_{(-\tau_k, x_k)} + \delta_{(\tau_k, x_k)}] \circ D)^{\tilde{y}} \Phi(\tilde{x} - \tilde{y})
\end{aligned}$$

Then (3.1) has a unique solution and  $v = v_f$ .

PROOF: Note that the matrix elements  $M$  of the matrices  $\tilde{A}$  to  $\tilde{F}$  are of the form

$$\begin{aligned}
M(\tilde{x}_0, \tilde{y}_0) &:= (\delta_{\tilde{x}_0} \circ D)^{\tilde{x}} (\delta_{\tilde{y}_0} \circ D)^{\tilde{y}} \psi(\|\tilde{x} - \tilde{y}\|) \\
&= (\delta_{\tilde{x}_0} \circ D)^{\tilde{x}} [\psi_1(\|\tilde{x} - \tilde{y}_0\|) \langle \tilde{y}_0 - \tilde{x}, F(\tilde{y}_0) \rangle] \\
&= -\psi_2(\|\tilde{x}_0 - \tilde{y}_0\|) \langle \tilde{x}_0 - \tilde{y}_0, F(\tilde{y}_0) \rangle \\
&\quad \times \langle \tilde{x}_0 - \tilde{y}_0, F(\tilde{x}_0) \rangle \\
&\quad - \psi_1(\|\tilde{x}_0 - \tilde{y}_0\|) \langle F(\tilde{x}_0), F(\tilde{y}_0) \rangle
\end{aligned}$$

where  $\psi_1(r) := \frac{1}{r} \frac{\partial \psi(r)}{\partial r}$  (note that  $\psi_1$  is defined for all  $r \geq 0$ ) and  $\psi_2(r) := \begin{cases} \frac{1}{r} \frac{\partial \psi_1(r)}{\partial r} & \text{for } r > 0 \\ 0 & \text{otherwise} \end{cases}$ , see also [2].

We show that for any  $\tau_1, \tau_2 \in \mathbb{R}$  and  $x, y \in \mathbb{R}^n$  we have

$$M((-\tau_1, x), (-\tau_2, y)) = M((\tau_1, x), (\tau_2, y)).$$

Indeed, since  $F_i(\tau, x) = F_i(-\tau, x)$  for  $i = 1, \dots, n$  and  $F_0(\tau, x) = -F_0(-\tau, x)$ , cf. Theorem 2.2, we have

$$M((-\tau_1, x), (-\tau_2, y))$$

$$\begin{aligned}
&= -\psi_2 \left( \sqrt{(-\tau_1 + \tau_2)^2 + \|x - y\|^2} \right) \langle (-\tau_1 + \tau_2, x - y), F(-\tau_1, x) \rangle \\
&\quad \times \langle (-\tau_1 + \tau_2, x - y), F(-\tau_2, y) \rangle \\
&\quad - \psi_1 \left( \sqrt{(-\tau_1 + \tau_2)^2 + \|x - y\|^2} \right) \langle F(-\tau_1, x), F(-\tau_2, y) \rangle \\
&= -\psi_2 \left( \sqrt{(\tau_1 - \tau_2)^2 + \|x - y\|^2} \right) \langle (\tau_1 - \tau_2, x - y), F(\tau_1, x) \rangle \\
&\quad \times \langle (\tau_1 - \tau_2, x - y), F(\tau_2, y) \rangle \\
&\quad - \psi_1 \left( \sqrt{(\tau_1 - \tau_2)^2 + \|x - y\|^2} \right) \langle F(\tau_1, x), F(\tau_2, y) \rangle \\
&= M((\tau_1, x), (\tau_2, y)).
\end{aligned}$$

Hence,  $\tilde{D} = \tilde{E}$ ,  $\tilde{B} = \tilde{C}$  and  $\tilde{F} = \tilde{F}^T$ . This shows

$$\begin{aligned}
c &= \tilde{A}\tilde{a} + \tilde{D}\tilde{b}^- + \tilde{E}\tilde{b}^+ = \tilde{A}\tilde{a} + \tilde{E}\tilde{b}^- + \tilde{D}\tilde{b}^+, \\
d &= \tilde{D}^T\tilde{a} + \tilde{B}\tilde{b}^- + \tilde{F}\tilde{b}^+ = \tilde{E}^T\tilde{a} + \tilde{C}\tilde{b}^- + \tilde{F}^T\tilde{b}^+, \\
d &= \tilde{E}^T\tilde{a} + \tilde{F}^T\tilde{b}^- + \tilde{C}\tilde{b}^+ = \tilde{D}^T\tilde{a} + \tilde{F}\tilde{b}^- + \tilde{B}\tilde{b}^+.
\end{aligned}$$

Hence,

$$\begin{pmatrix} \tilde{A} & \tilde{D} & \tilde{E} \\ \tilde{D}^T & \tilde{B} & \tilde{F} \\ \tilde{E}^T & \tilde{F}^T & \tilde{C} \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b}^+ \\ \tilde{b}^- \end{pmatrix} = \begin{pmatrix} c \\ d \\ d \end{pmatrix},$$

i.e.  $\begin{pmatrix} \tilde{a} \\ \tilde{b}^+ \\ \tilde{b}^- \end{pmatrix}$  is a solution of the full collocation problem. Since this problem has a

unique solution, and  $\begin{pmatrix} \tilde{a} \\ \tilde{b}^- \\ \tilde{b}^+ \end{pmatrix}$  is also a solution, we have  $\tilde{b}^+ = \tilde{b}^-$ .

Note that the reduced system has a unique solution. Now if  $(\tilde{a}, \tilde{b}^+, \tilde{b}^-)$  is a solution of the first system, then  $a := \tilde{a}$  and  $b := \tilde{b}^- = \tilde{b}^+$  is a solution of the second system; indeed, the ‘first row’ is the ‘first row’ of the old system and the ‘second row’ is the sum of the ‘second and third row’ of the old system.

Now let  $(a, b)$  be a solution of the reduced system. Then define  $a =: \tilde{a}$  and  $b =: \tilde{b}^- =: \tilde{b}^+$ . Then again the ‘first rows’ are the same. Since  $\tilde{D} = \tilde{E}$ ,  $\tilde{B} = \tilde{C}$  and  $\tilde{F} = \tilde{F}^T$ , as we have shown above, we can divide the ‘second row’ of the new system by 2 and obtain the ‘second and third row’ (which are in fact equal) of the full collocation problem.  $\square$

**Remark 3.2** *Due to the symmetry we have the following formulae for the matrix elements by similar calculations as in the proof of Lemma 3.1:*

$$\begin{aligned}
A_{ik} &= -\psi_2(\|y_i - y_k\|) \langle (0, y_i - y_k), F(0, y_i) \rangle \langle (0, y_i - y_k), F(0, y_k) \rangle \\
&\quad - \psi_1(\|y_i - y_k\|) \langle F(0, y_i), F(0, y_k) \rangle,
\end{aligned}$$

$$\begin{aligned}
B_{ik} &= -2\psi_2 \left( \sqrt{\tau_k^2 + \|y_i - x_k\|^2} \right) \langle (\tau_k, y_i - x_k), F(0, y_i) \rangle \langle (\tau_k, y_i - x_k), F(\tau_k, x_k) \rangle \\
&\quad - 2\psi_1 \left( \sqrt{\tau_k^2 + \|y_i - x_k\|^2} \right) \langle F(0, y_i), F(\tau_k, x_k) \rangle, \\
D_{ik} &= -2\psi_2 \left( \sqrt{(\tau_i - \tau_k)^2 + \|x_i - x_k\|^2} \right) \langle (\tau_i - \tau_k, x_i - x_k), F(\tau_i, x_i) \rangle \\
&\quad \times \langle (\tau_i - \tau_k, x_i - x_k), F(\tau_k, x_k) \rangle \\
&\quad - 2\psi_1 \left( \sqrt{(\tau_i - \tau_k)^2 + \|x_i - x_k\|^2} \right) \langle F(\tau_i, x_i), F(\tau_k, x_k) \rangle \\
&\quad - 2\psi_2 \left( \sqrt{(\tau_i + \tau_k)^2 + \|x_i - x_k\|^2} \right) \langle (\tau_i + \tau_k, x_i - x_k), F(\tau_i, x_i) \rangle \\
&\quad \times \langle (\tau_i + \tau_k, x_i - x_k), F(-\tau_k, x_k) \rangle \\
&\quad - 2\psi_1 \left( \sqrt{(\tau_i + \tau_k)^2 + \|x_i - x_k\|^2} \right) \langle F(\tau_i, x_i), F(-\tau_k, x_k) \rangle.
\end{aligned}$$

We obtain the following error estimate using Wendland's compactly supported functions  $\Phi = \psi_{\ell,k}(c\|\cdot\|) \in C^{2k}(\mathbb{R}^{n+1})$ , where  $k$  is the smoothness index of the compactly supported functions. The proof makes use of Theorem 2.11, ensuring that  $V \in W_2^{k+1+n/2}(\Omega)$ .

**Theorem 3.3 ([5, Corollary 4.11])** *Let  $\Omega \subset A_F(0,0) \subset \mathbb{R}^{n+1}$  be a domain with smooth boundary. Denote by  $k$  the smoothness index of the compactly supported Wendland function. Let  $k > \frac{1}{2}$  if  $n$  is even or  $k > 1$  if  $n$  is odd. Set  $\tau := k+1+n/2$ ,  $\sigma := \lceil \tau \rceil$  and assume  $F \in C^\sigma(\Omega)$ .*

*Then, employing this basis function yields*

$$\|V' - v'\|_{L^\infty(\Omega)} \leq Ch_X^{k-\frac{1}{2}} \|V\|_{W_2^{k+1+n/2}(\Omega)}.$$

Here the mesh norm  $h_X := \sup_{\tilde{x} \in \Omega} \min_{\tilde{x}_j \in X} \|\tilde{x} - \tilde{x}_j\|$  measures the maximum distance a point in  $\Omega$  can have from any point in  $X$ .

For given  $\epsilon > 0$ , the error between  $V'$  and  $v'$  can be made smaller than  $\epsilon$  by choosing the grid points  $X$  dense enough in  $\Omega$ , i.e. by making  $h_X$  small enough. Note that  $V'(\tau, x) = -\tau^2 - \|x\|^2$  and thus

$$v'(\tau, x) \leq V'(\tau, x) + \epsilon = -\tau^2 - \|x\|^2 + \epsilon < 0$$

for  $(\tau, x) \notin \overline{B_\epsilon(0,0)}$ . Hence, locally near the equilibrium point the orbital derivative  $v'$  may be positive, cf. the example in Section 4, Figure 5. This is why we need a local Lyapunov function to obtain information about the local basin of attraction.

## 4 Example

As an example we consider the exponentially asymptotically autonomous, one-dimensional ODE

$$\dot{x} = -x + x^3 + e^{-2t}x^2. \quad (4.1)$$

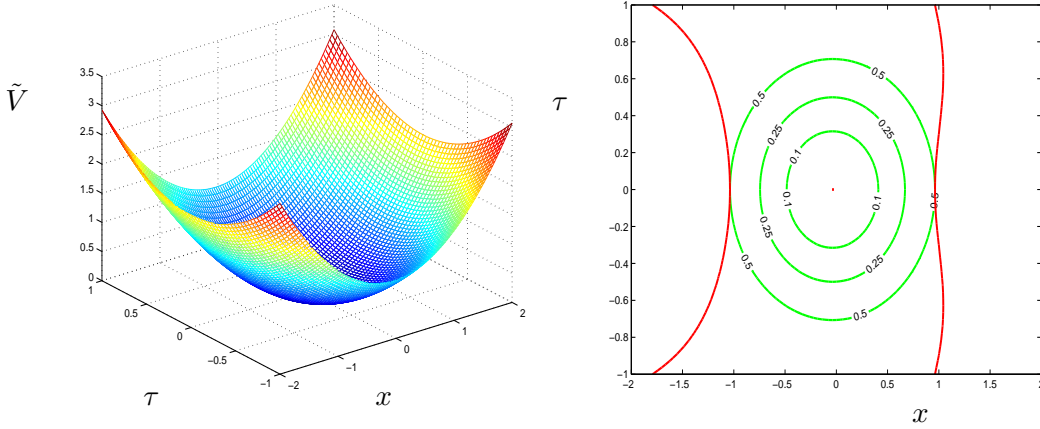


Figure 2: Left: The local Lyapunov function  $L(\tau, x) = \tau^2 + \frac{1}{2}x^2$ ; Right: Level sets of the local Lyapunov function  $L$  (green) and the zero level set of  $L'(\tau, x)$  (red). Hence, the set  $\{(\tau, x) \in \mathbb{R}^2 \mid L(\tau, x) < \frac{1}{2}\}$  is a sublevel set of the basin of attraction  $A_F(0, 0)$ .

This system verifies the smoothness assumptions with any  $\sigma \in \mathbb{N}$ ,  $g(x) = -x + x^3$  and  $\alpha < 2$ .

We choose  $k = 2$  and need  $F \in C^4(\mathbb{R}^2)$  for the error estimate. Following Theorem 2.2 we need to choose  $\beta \leq \frac{\alpha}{\sigma} < \frac{1}{2}$ .

However, if we choose  $\beta = \frac{1}{2}$ , then a direct calculation shows that after the transformation  $\tau = \pm e^{-t/2}$  or  $t = -\ln \tau^2$ , the ODE (4.1) becomes

$$\begin{aligned}\dot{\tau} &= -\frac{1}{2}\tau, \\ \dot{x} &= -x + x^3 + \tau^4 x^2.\end{aligned}$$

It is immediately clear, that in this case we even have  $F \in C^\infty$ . We have  $DF(0, 0) = \text{diag}(-\frac{1}{2}, -1)$ .

A local Lyapunov function is calculated using  $DF(0, 0)^T C + C DF(0, 0) = -I$  to be  $L(\tau, x) = \tau^2 + \frac{1}{2}x^2$ . Then

$$L'(\tau, x) = -\tau^2 - x^2 + x^4 + \tau^4 x^3,$$

which shows that  $K := \{(\tau, x) \in \mathbb{R}^2 \mid L(\tau, x) < \frac{1}{2}\}$  is a subset of the basin of attraction, cf. Figure 2.

We used  $T = 2$ , and  $c = 2$  for the parameter of the Wendland function  $\psi_{4,2}$ , which means  $\psi(r) = (1 - cr)^6[35c^2r^2 + 18cr + 3]$  for  $r \leq \frac{1}{c}$  and  $\psi(r) = 0$  otherwise. The points of the grid  $X$  were  $x \in \{-0.66, -0.6, -0.54, \dots, 0, 0.06, 0.12, \dots, 0.6, 0.66\}$  and  $t \in \{-2, -1.95, -1.9, \dots, 2\}$ . This results in  $N_0 = 22$  at  $t = 0$  and  $N = 902$ . For a plot of the grid points, cf. Figure 3.

Now we approximate the global Lyapunov function  $V$  satisfying  $V'(\tau, x) = -\tau^2 - x^2$  and obtain a subset  $S$  of the basin of attraction  $A_F(0, 0)$ , cf. Figure 4. Note that



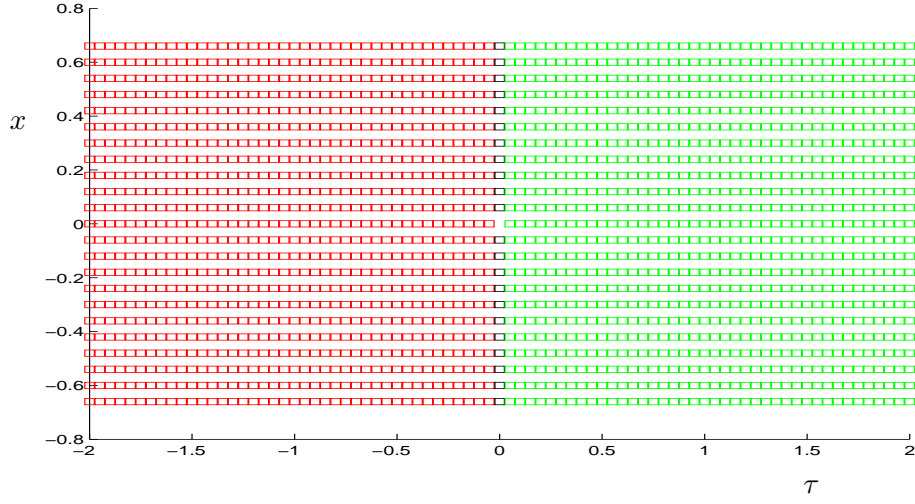


Figure 3: This figure shows the grid points used in the calculations. The set  $X^-$  with negative  $\tau$  values is shown in red, the symmetric image of  $X^-$ , the set  $X^+$  with positive  $\tau$  values, is shown in green, and the points with  $\tau = 0$  are shown in black, note that the equilibrium  $(0,0)$  is excluded.

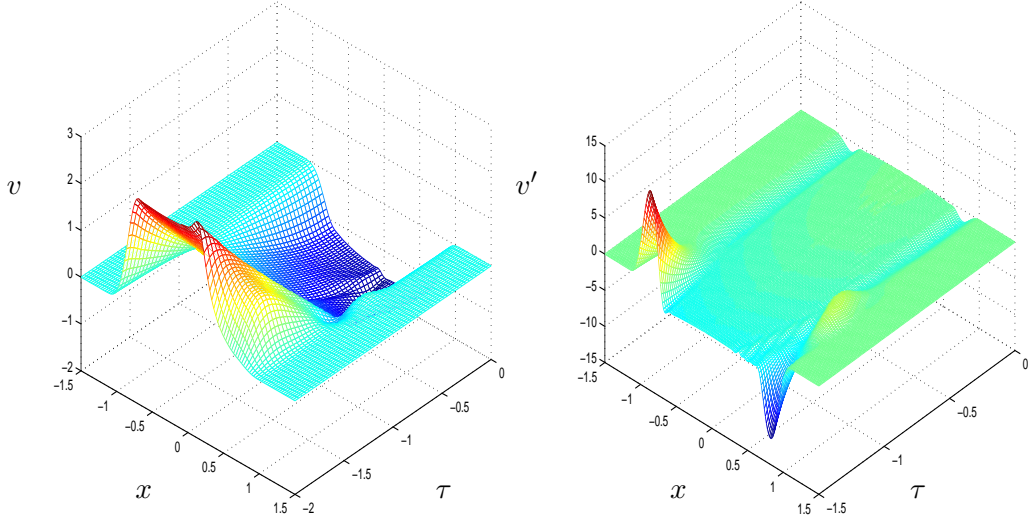


Figure 4: Left: The Lyapunov function  $v(\tau, x)$ , calculated by Radial Basis Function approximation of  $V$  satisfying  $V'(\tau, x) = -\tau^2 - x^2$ . Right: The orbital derivative  $v'(\tau, x)$ ; note that the two peaks are outside the area where the grid points lie.

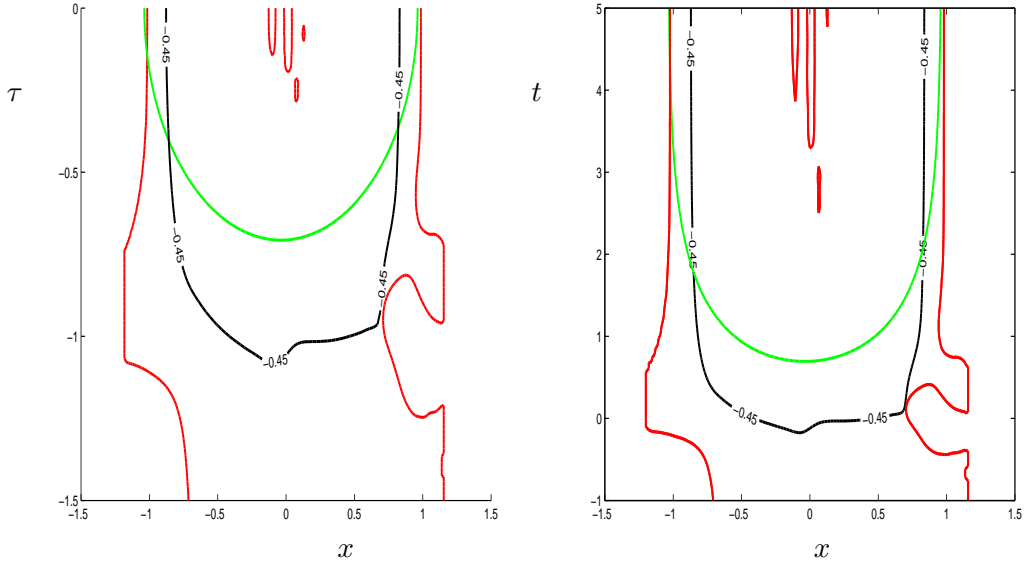


Figure 5: Both figures show the zero level set of  $v'(\tau, x)$  (red) which separates the regions of negative and positive sign of  $v'$ , the sublevel set  $\{(\tau, x) \in \mathbb{R}^2 \mid L(\tau, x) < \frac{1}{2}\}$  of the local Lyapunov function (green) which includes areas with positive orbital derivative  $v'$  and the sublevel set  $\{(\tau, x) \in \mathbb{R}^2 \mid v(\tau, x) \leq -0.45\}$  (black). Both the green and black sets are subsets of the basin of attraction. Left: The figure displays the  $(\tau, x)$  plane and shows subsets of the basin of attraction  $A_F(0, 0)$  of the system (2.3); Right: The sets are transformed back to the  $(t, x)$  plane and the figure shows subsets of the basin of attraction  $A_f(0)$  of the zero solution.

in this case  $A_g(0) = (-1, 1)$ . Using the relation  $t = -\ln \tau^2$ , we can map  $S$  back to the original system in  $(t, x)$  and obtain a subset of the basin of attraction  $A_f$ , cf. Figure 5.

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