

Transonic Shock Waves and Free Boundary Problems for the Nonlinear Wave System

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Abstract

In this Thesis, we study the problem of Shock Reflection-Diffraction by a Wedge for the Nonlinear Wave System. Shock Reflection-Diffraction problems in Fluid Dynamics, and the corresponding wave patterns have been studied by many authors, via many different analytical, experimental and numerical methods.

The Nonlinear Wave System is obtained from the Full Euler System, by making the assumption that the inertial terms are small. It has many similar properties (such as the structure of the characteristics) in common with it, so is a useful model for understanding the Full Euler System, and also plays an important role in some operator splitting schemes in numerical analysis.

The solutions to the problem of Shock Reflection-Diffraction by a wedge that we seek are Self-Similar Regular Shock Reflection-Diffraction configurations. The first necessary ingredient in the structure of these solutions is the existence of local reflection at the point where the incident shock hits the wedge. We find that there exists sonic and detachment angles, depending only on the initial data, determining the existence of the local reflection, and the pattern which appears. With the local theory in hand, we use a similar approach to that used in [1, 26], by using a nonlinear method of continuity that relies on the Leray-Schauder degree theory from [87], to obtain existence in all cases where the local reflection exists.

There are three key difficulties in the approach. The first is the changing elliptic or degenerate elliptic structure as the wedge angle passes through the sonic angle, and the corresponding change in wave patterns. In particular, we find that for some subsonic wedge angles, we require a new technique to attain suitable regularity near the reflection point, from the limited structure estimates available. The second is the degenerate obliqueness of the free boundary condition where the shock wave meets the flat part of the wedge. Because obliqueness is a property that relies not just on the values of the coefficients, but on the shape of the boundary, this is handled via a new technique involving imposing a cutoff on the boundary itself. The third is ensuring that, in the iteration problem, we obtain a gain-in-regularity so that compactness of the iteration map holds. This involves careful analysis of the regularity of both the density and boundary.

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Part I

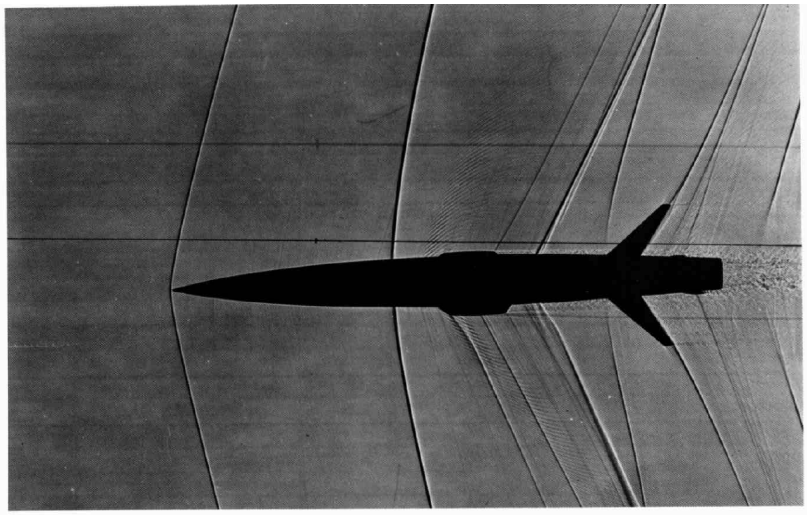
Part I - Introduction and Mathematical Formulation

Chapter 1

Introduction

1.1 Introduction

Shock waves are steep fronts within a compressible fluid, where there is a sharp change in the pressure and velocity. These can occur in nature due to physical processes such as aircraft travelling faster than the speed of sound, explosions, or solar wind. Fluid motion is modelled by the Euler equations; these were first formulated by Euler in [32, 33, 34], based on earlier work by Bernoulli in [5], and were amongst the first PDEs describing a physical situation to ever be written down. When a shock wave hits an object, shock

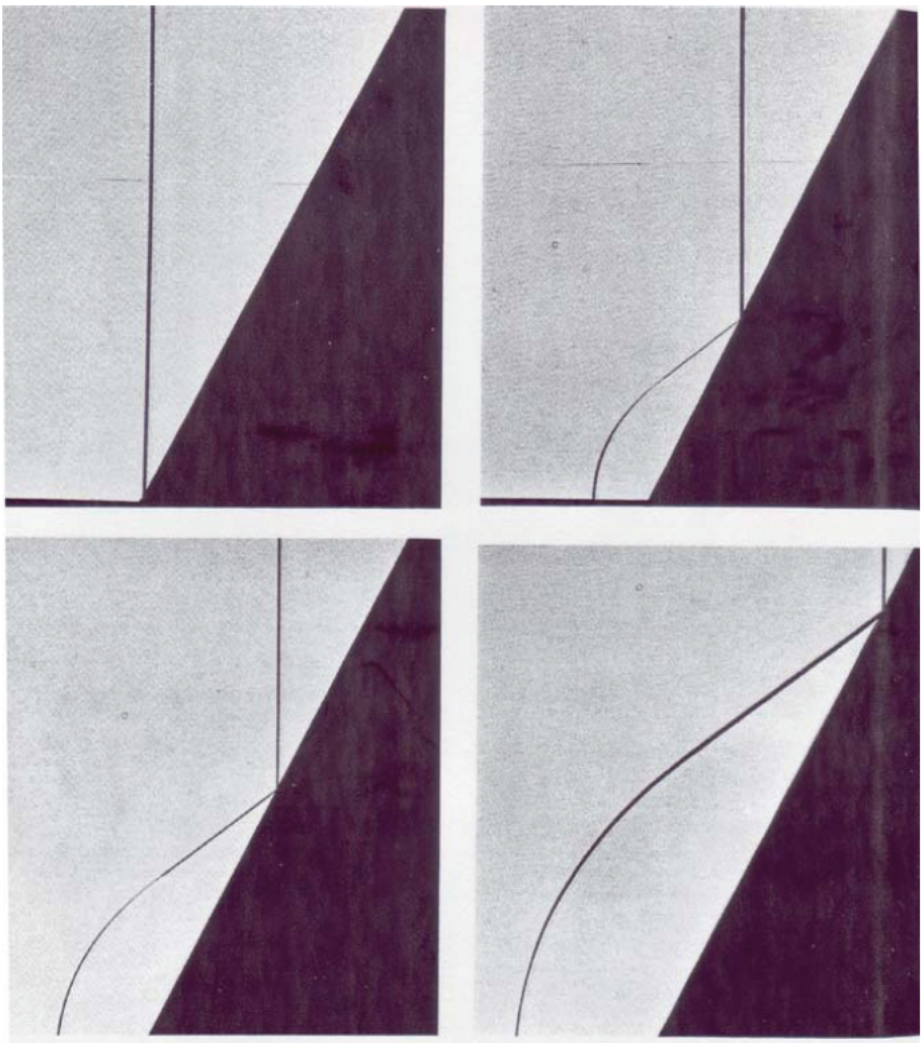


255. Airplane model in free flight at $M=1.1$. Shadowgraphs show a winged model launched into atmospheric air from a gun, as in figure 252. The bow wave is marginally attached at this slightly supersonic speed. In the plan view above, the wings are lifting, as shown by trailing vortices from the tips. In the side view below, the herringbone pattern is produced by pressure pulses from grooves in the wing that trip the boundary layer to make it turbulent over the rear half. *NASA photographs, courtesy of W. G. Vincenti*

Figure 1.1: A supersonic aircraft model [78][Fig. 255, pp. 157]

reflection-diffraction phenomena take place. One fundamental problem of this type is shock reflection-diffraction by a wedge. When an incident shock wave collides with a wedge, it experiences a shock reflection-diffraction process, and many authors have studied the wave patterns that occur; see [2, 3, 4, 61, 80, 81, 82] and the references within.

The compressible Euler system for isentropic flow in two physical dimensions consists



Regular reflection of a shock wave from a wedge. Shadowgraphs show a vertical plane shock wave striking a 60° wedge from the left. The wedge angle and shock-wave strength are such that the reflection is regular. The angles of the incident and reflected shock waves to the face of the wedge are not equal, because the phenomenon is non-linear. Insofar as viscosity is negligible, there is no reference length available, so the flow is self-similar: the wave pattern is seen to expand linearly in time from the instant of contact with the leading edge. *Schardin 1965, courtesy of H. Oertel, Sr.*

Figure 1.2: Regular Shock Reflection-Diffraction for a large-angled wedge; [78][pp.142]

of three conservation laws:

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0, \\ (\rho u)_t + (u^2\rho + p)_x + (uv\rho)_y &= 0, \\ (\rho v)_t + (uv\rho)_x + (v^2\rho + p)_y &= 0, \end{aligned} \tag{1.1.1}$$

where (p, ρ) are the pressure and density of the fluid and (u, v) are the components of the velocity. The pressure-density constitutive relation is given by

$$p(\rho) := \frac{\rho^\gamma}{\gamma}, \tag{1.1.2}$$

where $\gamma > 1$ is the abiatic index of the fluid. Also, the local speed of sound c is given by $c^2(\rho) := p'(\rho) = \rho^{\gamma-1}$.

In this thesis, we will consider the Nonlinear Wave System, which is obtained from (1.1.1) by making the assumption the inertial terms (i.e. the terms quadratic in the velocity) are small, and defining $(m, n) = (\rho u, \rho v)$ to be the pseudo-momenta:

$$\begin{aligned} \rho_t + m_x + n_y &= 0, \\ m_t + p_x &= 0, \\ n_t + p_y &= 0. \end{aligned} \tag{1.1.3}$$

Though it is unknown if this system represents physical situations, it is a useful model in understanding the full Euler system. For instance, its structures, in particular the characteristic structure, are more similar to the Full Euler System than some other related systems, such as the Unsteady Transonic Small Disturbance equations; see [79] for discussion of the characteristic structure and other basic properties of the Nonlinear Wave System and other related systems. The Nonlinear Wave System also appears within a numerical operator-splitting scheme for the Full Euler System, where the compressible Euler equations are split into the Nonlinear Wave System (the pressure system), and Gradient Flow (the pressure-less system); see [69, 71, 89, 90]. Some other, closely related transonic shock wave problems to the one considered in this thesis have been studied both analytically and numerically for the Nonlinear Wave System; for a sample of these, see [7, 13, 48, 50, 51, 75] and the references within.

The fundamental questions that arise in the topic of shock reflection-diffraction for a wedge include:

(i) The structure of the shock reflection-diffraction configurations,

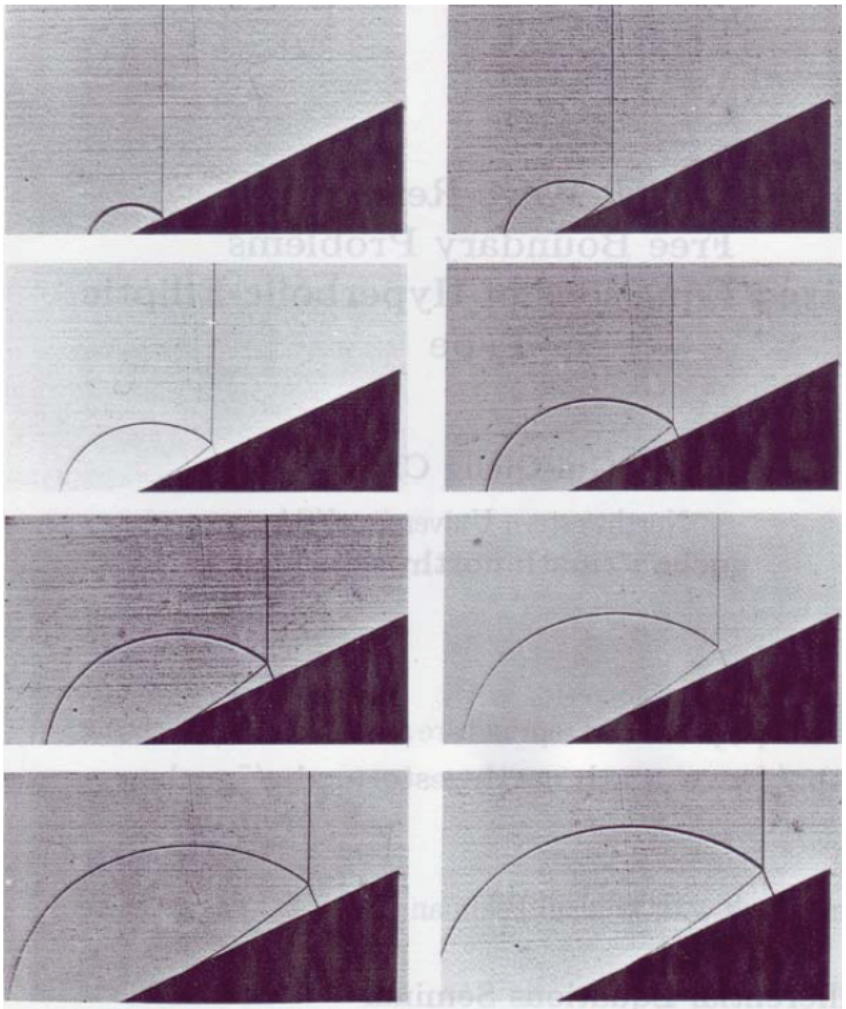
(ii) The transition criteria between different patterns of shock reflection-diffraction,

(iii) The dependence of the patterns upon physical parameters, such as the wedge angle, and the strength of the incident shock.

The complexity of shock reflection-diffraction configurations when a shock wave hits a wedge, was first noted by Mach [61], who observed two patterns; Regular Reflection (RR), and Simple Mach Reflection (SMR). The problem then remained dormant until the 1940s, until it was found there are even more complex shock patterns than Mach ever found, for example Double Mach Reflection (DMR), the von Neumann Reflection (vNR), and the Guderley Reflection (GR). For a sample of experimental and numerical work identifying these types of reflection, see [2, 3, 4, 28, 41, 42, 70, 74, 74, 75, 76, 78, 80, 81, 82, 83] and the references within.

Asymptotic analysis has also been made for various other reflection-diffraction configurations arising from shock waves interacting with objects; for a sample of these, see [2, 3, 4, 28, 36, 44, 45, 46, 49, 59, 60, 63, 73, 80, 81, 82, 83] and the references within. There has also been many numerical experiments; for a sample of these, see [2, 4, 21, 30, 31, 38, 39, 40, 41, 47, 64, 65, 67, 68, 75, 85, 86] and the references within.

However, many questions remain unanswered, particularly for global structure and transitions between different shock reflection-diffraction patterns. Often, when carrying out experiments (both numerical and physical), limitations in the experiments, such as the accuracy possible in numerical algorithms, and the precision of measuring equipment, lead to unclear results, and unsatisfactory answers to the fundamental questions we wish to answer. For example, it was observed in [2] that, for Shock Reflection-Diffraction by a Wedge, the



Mach reflection of a shock wave from a wedge. Shadowgraphs show a vertical plane shock wave striking a 25° wedge. The regular reflection shown on the opposite page is replaced for smaller wedge angles by this pattern of Mach reflection. A third shock wave, the Mach stem, runs normal to the surface and joins the incident and reflected shock waves at the triple point. The curve running down left from that point to meet the surface smoothly is the slip line, across which the entropy is discontinuous because the air on each side has been subjected to different shock waves. The time between photographs is 6×10^{-5} s. *Schardin 1965, courtesy of H. Oertel, Sr.*

Figure 1.3: Mach Reflection for a 25-degree angled wedge; [78][pp.142]

detachment and sonic angle are only a small fraction of a degree apart, so it was not possible to accurately determine transition criteria. As another example, the theory indicates that in Mach Reflection configurations, it is not normally possible to obtain a pure three-shock pattern and a fourth wave must be present [28, 66], but pure-three shock patterns appeared in experiments; see [78] and Figure 1.3. Therefore, a sensible approach to further the understanding of shock reflection-diffraction phenomena is to carry out rigorous mathematical analysis, i.e. establishing the global existence, regularity, and structural stability of shock reflection-diffraction configurations.

Shock Wave problems also fit into the structure of Global Entropy Solutions for Hyperbolic Conservation Laws. Riemann solutions (i.e. solutions for piecewise constant initial data) are often used as building blocks to determine solutions for more complicated problems, such as fluid dynamics problems. See [10, 29, 41, 52, 53, 54, 62] for an overview of

this area.

We now discuss some of the technical issues involved in the rigorous mathematical analysis for the problem tackled in this thesis. We will seek Regular Self-Similar Shock Reflection-Diffraction configurations for the Nonlinear Wave System, aiming to prove von Neumann's detachment conjecture.

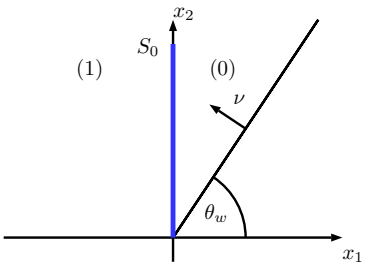


Figure 1.4: Initial data at $t = 0$

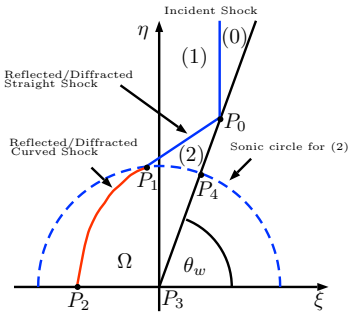


Figure 1.5: Self-similar supersonic regular shock reflection-diffraction configuration

Firstly, we must carry out local analysis near the point where the incident shock meets the wedge, to set up our boundary conditions. Motivated by the shape of wave patterns attained experimentally, and invariance of the problem under the transformation $(t, \mathbf{x}) \mapsto (\alpha t, \alpha x)$, we consider self-similar solutions, using the change of variables $\boldsymbol{\xi} = \frac{\mathbf{x}}{t}$. The initial data at $t = 0$ then becomes an asymptotic boundary condition as $|\boldsymbol{\xi}| \to \infty$. We find that, using the Rankine-Hugoniot conditions, the incident shock moves to the right, to a fixed location $\xi = \xi_1 > 0$ in the self-similar coordinates, depending only on the data. We then turn to the local theory at the point $P_0 = (\xi_1, \xi_1 \tan\theta_w)$ where the incident shock meets the wedge, and find that there exists a sonic angle θ_{sonic} and detachment angle θ_d satisfying $0 < \theta_d < \theta_{sonic} < \frac{\pi}{2}$, such that when the wedge angle θ_w satisfies:

- $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$, then local regular reflection exists and is supersonic;
- $\theta_w \in (\theta_d, \theta_{sonic})$ then local regular reflection exists and is subsonic,
- $\theta_w < \theta_d$, then local regular reflection does not exist.

In fact, we actually find that, for each wedge angle $\theta_w \in (\theta_d, \frac{\pi}{2})$, there exists two possible values for the local reflection, the weak state and the strong state. There is some debate as to which is more physical. We choose the weak state and discard the strong state, mainly as this is the one that agrees continuously with the normal reflection solution, i.e. the solution in the case where $\theta_w = \frac{\pi}{2}$, and the wedge is a vertical wall. It is also the one that tends to appear during physical experiments. This will be discussed further in §2.3 and §3.7.

We will formulate our problem as a mixed-type problem. Most of our analysis takes place in an elliptic region Ω, which is separated from the hyperbolic region by the shock wave, which is formulated as a free boundary, where an oblique boundary condition will be imposed, and the sonic arc, where the location is known, but the ellipticity degenerates. In particular, we note that within the elliptic region, we can reduce the three conservation laws that make up the Nonlinear Wave System into a single mixed-type divergence-form equation for the density, and the Rankine-Hugoniot conditions into an oblique boundary

condition for the density. We can then recover the velocity components, using the second two conservation laws, taking advantage of the structure of the region, and the regularity of the solution. In particular, the $C^{1,\alpha}$ regularity of the density, up to the origin, is required to show that the momenta are not multi-valued.

To deal with the free boundary, we first observe the conditions imposed, which consist of an oblique boundary condition, and an ODE describing the shape of the boundary, are enough to fully determine both the position of the boundary, and the value of the density on it. In our problem, the obliqueness of the boundary condition, and the ellipticity within the elliptic region rely heavily on the strength of the shock wave being strictly positive. As such, we must first attain *a priori* estimates on the strength of the shock, before we can make any regularity estimates. A property that turns out to be useful in proving the positive strength of the shock, is that in common with closely-related problems for the Nonlinear Wave System (for example [7, 13]), the density is monotone on the free boundary and the elliptic region Ω is convex. The convexity is important in the creation of the barrier functions that allow us to prove the strength of the shock across the free boundary is strictly positive. Free boundaries also arise in many other physical problems where there is a sharp transition in the variables involved, and first originated in the study of the melting of ice. For a recent overview of the state of research in free boundary problems, see [22] and the references within.

The other part of the transonic boundary, the sonic arc, is not a free boundary, but we instead have the issue that the ellipticity of the PDE degenerates as we approach it. To deal with this issue, we take advantage of the specific form of the elliptic degeneracy to obtain estimates near the sonic arc in appropriately weighted and scaled $C^{1,\alpha}$ spaces.

Another issue we must tackle is that at the point where the our shock wave reaches the flat part of the wedge, the obliqueness of the boundary condition degenerates. Careful analysis is required to impose a one-point Dirichlet condition at the point where the obliqueness degenerates. Using the precise structure of our problem, it is possible to obtain algebraic growth bounds, which allow us to both impose the one-point Dirichlet condition, and obtain suitable local regularity.

To obtain the existence of solutions, we use a nonlinear method of continuity, taking advantage of Leray-Schauder degree theory from [87]. Previously to this project, this type of method has been used in the case of potential flow, for the cases of Prandtl-Meyer reflection in [1], and Shock Reflection-Diffraction by a Wedge in [26]. There are three key difficulties in the approach used here. The first is the different treatment of the changing elliptic or degenerate elliptic structure as the wedge angle passes through the sonic arc. In particular, we find that, for some subsonic wedge angles, we require a new technique to attain suitable regularity near the reflection point, from the limited structure estimates available. The second is the degenerate obliqueness of the free boundary condition where the shock wave meets the flat part of the wedge that was mentioned above. Because obliqueness is a property that relies not just on the values of the coefficients, but on the shape of the boundary, this is handled via a new technique involving imposing a cutoff on the boundary itself. The third is the iteration problem now being in terms of both the density, and the boundary; so careful analysis involving the specific form of our problem is required to obtain a gain-in-regularity, so the required compactness for the degree theory holds.

This type of rigorous mathematical analysis, involving questions such as existence, uniqueness, and stability, has also been carried out for a wide variety of other shock wave problems, corresponding to a large range of physical problems; see [6, 8, 9, 11, 12, 14, 15, 16, 18, 19, 20, 23, 24, 25, 27, 35, 63, 72, 88, 91] and the references within for a sample of

these. The assumptions made in analytical study of shock wave problems are often rather restrictive, such as being restricted to a small perturbation of a simple problem, the solution only being valid in a small neighbourhood of some point (such as a reflection point), or analysis only being possible for steady equations. It is likely that in the future, the methods used here will be used to remove these types of assumption in other problems.

We also remark that the mathematical issues present here and in other transonic shock wave problems are also present in many other areas of research that use mathematical analysis, such as solid mechanics, materials science, and differential geometry. Thus any progress for shock wave problems is likely to contribute towards progress in other fields of research and lead to new research opportunities.

We now outline the structure of the thesis. In Part I, we mathematically formulate the problem we study in this thesis, Self-Similar Regular Shock Reflection-Diffraction by a Wedge for the Nonlinear Wave System, as a free boundary problem. In particular, we carry out the asymptotic analysis near the reflection point. We also prove some basic formulas and properties for the Nonlinear Wave System, which will be used throughout the thesis. We find it is most convenient to formulate the problem in terms of just the density component ρ; we can eliminate (m, n) and obtain a second order divergence form equation, along with mixed boundary conditions for ρ.

In Part II, we prove von Neumann's sonic conjecture, for the Nonlinear Wave System. We choose to first prove the sonic conjecture, for the benefit of the reader, as the analysis involved in dealing with the transition between supersonic and subsonic reflection-diffraction configurations is rather complex.

In Part III, we prove von Neumann's detachment conjecture. The main point here will be to add in the changes required to analyse the transition between supersonic and subsonic reflection configurations. We also describe here how to recover the components (m, n).

1.2 Shock reflection-diffraction by a wedge for the Nonlinear Wave System

The Nonlinear Wave System in two spatial dimensions consists of three conservation laws:

$$\begin{cases} \rho_t + m_{x_1} + n_{x_2} = 0, \\ m_t + p_{x_1} = 0, \\ n_t + p_{x_2} = 0, \end{cases} \tag{1.2.1}$$

for $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^2_+$, where ρ is the density, p is the pressure and (m, n) are the conserved momenta. The pressure-density constitutive relation is

$$p(\rho) = \rho^\gamma/\gamma, \qquad \gamma > 1, \tag{1.2.2}$$

where γ is called the abiatic index. The sonic speed $c = c(\rho) > 0$ is determined by

$$c^2(\rho) := p'(\rho) = \rho^{\gamma-1}. \tag{1.2.3}$$

Let $\mathcal{S}_0$ denote the vertical planar shock in $(t, \mathbf{x})$ coordinates, with left constant state $(1) = (\rho_1, m_1, 0)$ and right state $(2) = (\rho_0, 0, 0)$, satisfying

$$m_1 = \sqrt{(p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0)}, \qquad \rho_1 > \rho_0. \tag{1.2.4}$$

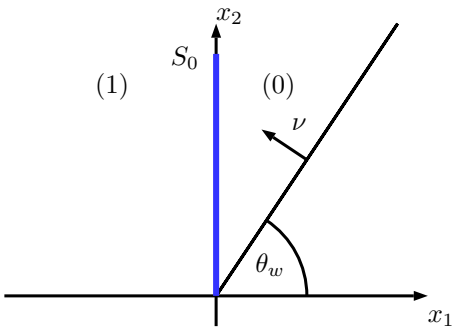


Figure 1.6: Initial Data at $t = 0$

When $\mathcal{S}_0$ hits a wedge with wedge angle $\theta_w \in (0, \frac{\pi}{2})$,

$$W := \{\mathbf{x} = (x_1, x_2) \mid x_1 \geq 0, x_1 = x_2 \cot\theta_w\} \cup \{\mathbf{x} = (x_1, x_2) \mid x_1 \leq 0, x_2 = 0\}, \tag{1.2.5}$$

as in Figure 1.6, it experiences a reflection-diffraction process, which we can formulate as an initial boundary value problem as follows:

Problem 1.2.1. *Seek a solution of* (1.2.1) *with the initial condition at* $t = 0$*;*

$$(\rho, m, n)\,|_{t=0} = \begin{cases} (\rho_0, 0, 0) & \text{in } \{0 \leq x_1 \leq \cot\theta_w\, x_2\} \\ (\rho_1, m_1, 0) & \text{in } \{x_1 < 0, x_2 > 0\}, \end{cases}$$

with the slip boundary condition

$$(m, n) \cdot \boldsymbol{\nu} = 0 \qquad \text{on } \partial W, \tag{1.2.6}$$

where $\boldsymbol{\nu}$ *is the outer unit normal to* ∂W*.*

Chapter 2

Mathematical Formulation and Main Theorems

In this Chapter, we shall mathematically formulate the Self-Similar Shock Reflection-Diffraction Problem, and state our main theorems.

2.1 Self-similar problem

We observe that Problem 1.2.1 is invariant under the self-similar scaling

$$(\mathbf{x}, t) \mapsto (\alpha\mathbf{x}, \alpha t), \quad (\rho, m, n) \mapsto (\rho, m, n) \qquad \text{for } \alpha > 0. \tag{2.1.1}$$

Thus we seek self-similar solutions of the form

$$(\rho, m, n)(t, \mathbf{x}) = (\rho, m, n)(\boldsymbol{\xi}) \qquad \text{for } \boldsymbol{\xi} = \frac{\mathbf{x}}{t}, \tag{2.1.2}$$

defined on the self-similar domain

$$\Lambda \equiv \Lambda^{(\theta_w)} := \{\boldsymbol{\xi} = (\xi, \eta) \in \mathbb{R}^2_+ \mid \eta > \max\{0, \xi \tan\theta_w\}\}.$$

Substituting the self-similar coordinates $\boldsymbol{\xi} = (\xi, \eta)$ into the system (1.2.1) leads to the following system of conservation laws;

$$\begin{aligned} (m - \xi\rho)_\xi + (n - \eta\rho)_\eta + 2\rho &= 0, \\ (p(\rho) - \xi m)_\xi - (\eta m)_\eta + 2m &= 0, \\ (\xi n)_\xi - (p(\rho) - \eta n)_\eta - 2n &= 0. \end{aligned} \tag{2.1.3}$$

Whenever $(\rho, m, n) \in C^2$, we can eliminate (m, n), and obtain the second-order divergence-form equation

$$((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta + \xi\rho)_\xi + ((c^2 - \eta^2)\rho_\eta - \xi\eta\rho\xi + \eta\rho)_\eta - 2\rho = 0. \tag{2.1.4}$$

We observe that once ρ is known within a region, the final two equations in (2.1.3) can be used to obtain (m, n), provided (m, n) are known on a suitable boundary portion, and the region is of suitable structure. We will see in Chapter 11 that this is possible for Regular Shock Reflection-Diffraction configurations, thus throughout this thesis, we shall mainly consider ρ and (2.1.4). The form of (2.1.4) shows that (2.1.3) elliptic (subsonic) if

$$c^2 - |\boldsymbol{\xi}|^2 > 0 \tag{2.1.5}$$

and hyperbolic (supersonic) if

$$c^2 - |\boldsymbol{\xi}|^2 < 0. \tag{2.1.6}$$

Also, motivated by (2.1.5) and (2.1.6), for a constant state ρ, we call the circle $\{|\boldsymbol{\xi}| = c(\rho)\}$ the sonic circle for that state; i.e. where the constant state ρ transitions between being elliptic and hyperbolic. This also means that it is often convenient to consider the polar form of (2.1.3) and (2.1.4); taking the polar coordinates (r, θ), where $r = |\boldsymbol{\xi}|$, we obtain

$$\begin{aligned}
(r\rho - \cos\theta m - \sin\theta n)_r + \left(\frac{\sin\theta}{r}m - \frac{\cos\theta}{r}n\right)_\theta &= \rho + \frac{\cos\theta}{r}m + \frac{\sin\theta}{r}n,\\
(rm - \cos\theta\, p(\rho))_r + \left(\frac{\sin\theta}{r}p(\rho)\right)_\theta &= m + \frac{\cos\theta}{r}p(\rho),\\
(rn - \sin\theta\, p(\rho))_r + \left(-\frac{\cos\theta}{r}p(\rho)\right)_\theta &= n + \frac{\sin\theta}{r}p(\rho).
\end{aligned} \tag{2.1.7}$$

and

$$\left((c^2 - r^2)\rho_r\right)_r + \frac{c^2}{r}\rho_r + \left(\frac{c^2}{r^2}\rho_\theta\right)_\theta = 0. \tag{2.1.8}$$

2.2 The Rankine-Hugoniot shock condition

In this section, we study the Rankine-Hugoniot condition on shock waves. In particular, we derive the location of the incident shock in self-similar coordinates, and formulate the Rankine-Hugoniot condition as an oblique boundary condition.

2.2.1 The incident shock

If we study a piecewise smooth weak solution of (2.1.3) with a jump in (ρ, m, n) across a curve $\mathcal{S}$ with unit normal $\boldsymbol{\nu} = (\nu_1, \nu_2)$, then the definition of weak solution yields the Rankine-Hugoniot jump condition across $\mathcal{S}$ as in physics:

$$\begin{aligned}
[(m, n)] \cdot \boldsymbol{\nu} - [\rho]\boldsymbol{\xi} \cdot \boldsymbol{\nu} &= 0,\\
[p]\nu_1 - [m]\boldsymbol{\xi} \cdot \boldsymbol{\nu} &= 0,\\
[p]\nu_2 - [n]\boldsymbol{\xi} \cdot \boldsymbol{\nu} &= 0,
\end{aligned} \tag{2.2.1}$$

where the bracket $[w]$ denotes the jump of the quantity w across $\mathcal{S}$. A discontinuity of (ρ, m, n) on a curve $\mathcal{S}$ is called a shock wave if it further satisfies the *physical entropy condition*;

$$\rho \text{ increases across } \mathcal{S}, \text{ where } \boldsymbol{\nu} \text{ is oriented so that } \boldsymbol{\xi} \cdot \boldsymbol{\nu} < 0, \tag{2.2.2}$$

I.e. ρ increases in the direction of increasing time. The plane incident shock in $(t, \mathbf{x})$ coordinates with the left and right constant states (ρ, m, n)

$$(0) = (\rho_0, 0, 0), \qquad (1) = (\rho_1, m_1, 0)$$

corresponds to a continuous weak solution of (2.1.3) in the self-similar coordinates $\boldsymbol{\xi}$ of the following form:

$$(\rho, m, n) = \begin{cases} (\rho_0, 0, 0) & \text{for } \xi > \xi_1,\\ (\rho_1, m_1, 0) & \text{for } \xi < \xi_1, \end{cases} \tag{2.2.3}$$

where from the Rankine-Hugoniot conditions (2.2.1),

$$\xi_1 = \overline{c}(\rho_1, \rho_0), \tag{2.2.4}$$

$$m_1 = \sqrt{(p_1 - p_0)(\rho_1 - \rho_0)}, \tag{2.2.5}$$

and

$$\overline{c}(a, b) := \sqrt{\frac{p(b) - p(a)}{a - b}}. \tag{2.2.6}$$

We note that due to the strict convexity of $\rho \mapsto p(\rho)$, $\overline{c}$ is strictly monotone in each variable (s, t). This leads to an asymptotic boundary condition for ρ:

$$\rho \to \overline{\rho} := \begin{cases} \rho_0 & \text{for } \xi > \xi_1, \eta > \xi \tan\theta_w, \\ \rho_1 & \text{for } \xi < \xi_1, \eta > 0 \end{cases} \qquad \text{as } |\boldsymbol{\xi}| \to \infty, \tag{2.2.7}$$

in the sense that $\lim_{R\to\infty} \|\rho - \overline{\rho}\|_{0,\Lambda\setminus B_R(0)} = 0$. We also note, using the second and third equations in (2.1.3), the slip boundary condition in Problem 1.2.1 is equivalent to the oblique boundary condition

$$\nabla\rho \cdot \boldsymbol{\nu} = 0 \qquad \text{on } \{\eta = 0\} \cup \{\eta = \xi \tan\theta_w\}. \tag{2.2.8}$$

Now, having turned the initial boundary condition and slip boundary condition in the space-time coordinates into an asymptotic boundary condition and an oblique boundary condition in the self-similar coordinates, we can state the following reformulation of our problem:

Problem 2.2.1. *Seek a solution ρ of equation* (2.1.3) *in the self-similar domain Λ with the slip boundary condition* (2.2.8) *and asymptotic boundary condition* (2.2.7).

2.2.2 The Rankine-Hugoniot condition in polar coordinates

For a curved shock, is it more convenient to consider the Rankine-Hugoniot conditions in polar coordinates (r, θ) centred at the origin in the self-similar coordinates. Due to the entropy condition (2.2.2), any C^1 shock wave $\mathcal{S}$ can be written as $\mathcal{S} = \{r = r(\theta)\}$ in polar coordinates (r, θ) with respect to the origin in the $\boldsymbol{\xi}$ coordinates, and the region $\{r < r(\theta)\}$ is the region where ρ is larger.

We now assume that across a shock wave, (ρ, m, n) jumps from a uniform state (ρ_-, m_-, n_-) in a region Ω_- to a variable state (ρ, m, n), with $\rho > \rho_-$ in a region Ω. Using (2.1.7), we find that,

$$\begin{aligned} r[\rho] - \cos\theta[m] - \sin\theta[n] - r'(\theta)\sin\theta[m] + r'(\theta)\cos\theta[n] = 0, \\ r[m] - \cos\theta[p] - r'(\theta)\sin\theta[p] = 0, \\ r[n] - \sin\theta[p] + r'(\theta)\cos\theta[p] = 0. \end{aligned} \tag{2.2.9}$$

By first eliminating $[m]$ and $[n]$ to obtain an expression for r', and then eliminating r' and taking the tangential derivative (see for example [50][§1.3] for full calculations), we can write (2.2.9) in the following way:

$$\begin{aligned} \beta \cdot \nabla\rho = \beta_1\rho_r + \beta_2\rho_\theta = 0, \\ r'(\theta) = g(r, \rho) \end{aligned} \tag{2.2.10}$$

where $\beta = (\beta_1, \beta_2)$ and $g(r, \rho)$ satisfy

$$\begin{aligned} \beta_1 &= r'(\theta)\big(3\overline{c}^2(c^2 - r^2) - c^2(r^2 - \overline{c}^2)\big), \\ \beta_2 &= \overline{c}^2(c^2 - r^2) - 3c^2(r^2 - \overline{c}^2), \\ g(r, \rho) &= -\frac{r}{\overline{c}}\sqrt{r^2 - \overline{c}^2}, \end{aligned} \tag{2.2.11}$$

where $\overline{c} = \overline{c}(\rho, \rho_-)$. Thus, as by (2.2.2) the inner unit normal to Ω in polar coordinates points in direction $(-1, r'(\theta))$, the obliqueness of (2.2.11) as a boundary condition on $\mathcal{S} \cap \partial\Omega$ is equivalent to

$$\beta \cdot (-1, r'(\theta)) = -r'(\theta) r^2 (c^2 - \overline{c}^2) \geq 0, \tag{2.2.12}$$

where, as $\rho > \rho_-$, the above is only equal to zero if $r'(\theta) = 0$, i.e. $r^2 = \overline{c}^2$. In the case where the obliqueness fails,

$$\begin{aligned} \beta_1 &= 0, \text{ and} \\ \beta_2 = \overline{c}^2(c^2 - r^2) &= \overline{c}^2(c^2 - \overline{c}^2) > 0. \end{aligned} \tag{2.2.13}$$

2.3 Structure of solutions: Self-similar regular shock reflection-diffraction configurations

We now discuss the structure of solutions ρ of Problem 2.2.1 corresponding to shock reflection-diffraction. As $(\rho_1, m_1, 0)$ does not satisfy the slip boundary condition (1.2.6) on ∂W, the solution must differ from $(\rho_1, m_1, 0)$ in $\{\xi < \xi_1\} \cap \Lambda$. The solutions we seek are Self-Similar Regular Shock Reflection-Diffraction configurations; our mathematical formulation of these will be similar to that used where the same problem was tackled for the Potential Flow system in [26][§2.4]. A necessary condition for these configurations to exist

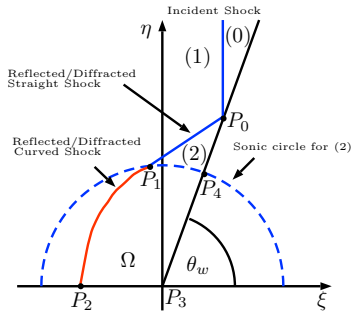


Figure 2.1: Self-Similar Supersonic Regular Shock Reflection-Diffraction configuration

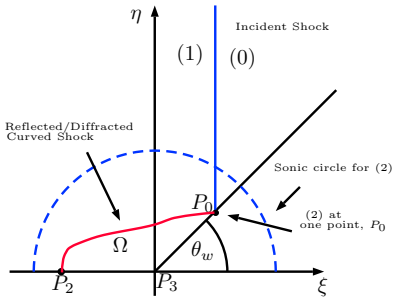


Figure 2.2: Self-Similar Subsonic Regular Shock Reflection-Diffraction configuration

is for there to exist a constant state (2) which satisfies the Rankine-Hugoniot conditions (2.2.1) and slip boundary condition (1.2.6) at the reflection point $P_0 = (\xi_1, \xi_1 \tan\theta_w)$ where the incident shock meets the wedge. Supersonic reflection corresponds to when the point P_0 where the incident shock intersects the wedge is outside the sonic circle for state (2); subsonic reflection corresponds to when P_0 is either on the boundary or the interior of the sonic circle for state (2).

In the case where the state (2) is supersonic at P_0, the supersonic reflection configuration from Figure 2.1 consists of three uniform states (0),(1),(2), and a non-uniform state in domain $\Omega = P_1P_2P_3P_4$. The solution is equal to state (0) and (1) ahead of and behind the incident shock $\mathcal{S}_0$, away from $P_0P_2P_3$. The solution is equal to state (2) in subregion $P_0P_1P_4$, noting that (2) is supersonic in $P_0P_1P_4$. The non-uniform state in Ω is subsonic; i.e. (2.1.3) for ρ is elliptic in Ω. We denote the boundary parts of Ω by

$$\Gamma_{shock} := P_1P_2, \quad \Gamma_{sym} := P_2P_3, \quad \Gamma_{wedge} = P_3P_4, \quad \Gamma_{sonic} = P_1P_4.$$

In the subsonic case, ρ matches with state (2) just at the single point P_0, where it is elliptic, and ρ is elliptic in Ω.

In Chapter 3, we will find that for fixed initial data $0 < \rho_0 < \rho_1$ and $\gamma > 1$, there exists a detachment angle and a sonic angle, $0 < \theta_d < \theta_{sonic}$ The local regular reflection exists for $\theta_w \geq \theta_d$, does not exist for $\theta_w < \theta_d$, is supersonic for $\frac{\pi}{2} \geq \theta_w > \theta_{sonic}$, is sonic for $\theta_w = \theta_{sonic}$ and is subsonic for $\theta_d < \theta_w < \theta_{sonic}$. In actual fact, for $\theta_d < \theta_w < \frac{\pi}{2}$ there exists exactly two local reflections; the weak state and the strong state. There is long debate as to which is physical for local theory; see for example [2, 3, 4, 28]. We choose the weak state, and discard the strong state, as the weak state agrees continuously with the normal reflection solution as $\theta_w \uparrow \frac{\pi}{2}$, which implies a stability condition with the unique normal reflection; see Theorem 3.2.1 and §3.7 for further discussion. Once the local reflection is found, finding a regular shock reflection-diffraction solution is a matter of solving a boundary value problem in Ω, which we describe in the following Section.

2.4 von Neumann conjecture and formulation as a free boundary problem

This project is motivated by the von Neumann Detachment Conjecture in [80, 81], which for our purposes can be formulated as

The von Neumann Detachment Conjecture Whenever $\theta_w > \theta_d$, there exists a supersonic regular reflection-diffraction solution to Problem 2.2.1, as in Figure 2.1.

To solve the von Neumann Detachment Conjecture, we first formulate shock reflection-diffraction configurations as Free Boundary Problems. The technicalities involved in continuously linking different cases as θ_w crosses the sonic angle turn out to be rather complex; thus, to simplify our analysis, we first deal with the Supersonic case in Part II, and then extend to the detachment angle in Part III.

We thus formulate the following two free boundary problems, based on Figures 2.1, 2.2.

Problem 2.4.1 (Supersonic Regular Shock Reflection-Diffraction as a Free Boundary Problem)**.** *For a supersonic wedge angle $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$, find a shock wave Γ_{shock}, and a function ρ, defined in a region Ω as shown in Figure 2.1 such that*

(i) *ρ satisfies* (2.1.3) *in Ω,*

(ii) (2.1.3) *is strictly elliptic in Ω;*

$$c^2 > |\boldsymbol{\xi}|^2 \qquad \text{in } \Omega,$$

(iii) *Γ_{shock} can be written as $\{r = r(\theta)\}$, and the Rankine-Hugoniot conditions* (2.2.10) *and the entropy condition* (2.2.2) *are satisfied,*

(iv) $\rho = \rho_2$ *on* Γ_{sonic},

(v) $\nabla\rho \cdot \boldsymbol{\nu} = 0$ *on* $\Gamma_{wedge} \cup \Gamma_{sym}$,

where $\boldsymbol{\nu}$ *is the interior unit normal to* Ω.

Problem 2.4.2 (Subsonic Regular Shock Reflection-Diffraction as a Free Boundary Problem)**.** *For a subsonic or sonic wedge angle* $\theta_w \in (\theta_d, \theta_{sonic}]$*, find a shock wave* Γ_{shock}*, and a function* ρ*, defined in a region* Ω *as shown in Figure 2.2 such that*

(i) ρ *satisfies 2.1.3 in* Ω,

(ii) (2.1.3) *is strictly elliptic in* Ω;

$$c^2 > |\boldsymbol{\xi}|^2 \qquad \text{in } \Omega,$$

(iii) Γ_{shock} *can be written as* $\{r = r(\theta)\}$*, and the Rankine-Hugoniot conditions* (2.2.10) *and the entropy condition* (2.2.2) *are satisfied,*

(iv) $\rho = \rho_2$ *at* $\overline{\Gamma_{sonic}} := \{P_0\}$,

(v) $\nabla\rho \cdot \boldsymbol{\nu} = 0$ *on* $\Gamma_{wedge} \cup \Gamma_{sym}$,

where $\boldsymbol{\nu}$ *is the interior unit normal to* Ω.

2.5 Leray-Schauder fixed point degree and nonlinear method of continuity

We now discuss a relevant result in degree theory from [87], that we will use in our problem. We also given an outline of the nonlinear method of continuity that will be used.

For our purposes, we require a very general result from [87][Section 11, Problem 13.6a].

Definition 2.5.1. Let X be a Banach space, $K \subset X$ and $f : K \to X$ be an operator.

- f is compact if it is continuous, and maps bounded sets into relatively compact sets.
- f is i-compact if it is continuous, and the image $f(K)$ is relatively compact.

We note that if K is bounded, i-compactness is equivalent to compactness. If K is unbounded, i-compactness is stronger than compactness.

Definition 2.5.2. A mapping $f : \overline{G} \subset M \to M$ is called admissible if;

(i) M is a closed convex subset of some Banach space X,

(ii) G is a relatively open set in M,

(iii) f is i-compact and has no fixed points on ∂G, where ∂G is determined by the subspace topology on M.

From the above definition, we see that when we are defining our iteration set, we can impose Banach space conditions and convex conditions (via X and M) that must be preserved by the iteration map, and open conditions (via G) that do not need to be preserved. This provides much more freedom than say, the Schauder Fixed Point Theorem, where only convex conditions can be imposed, and they must be preserved by the mapping.

We now state the existence of the Leray-Schauder degree for admissible maps;

Proposition 2.5.3. *There exists exactly one integer valued index for admissible maps f defined by Definition 2.5.2, which satisfies the following axioms;*

A1 - Normalization *If $f : \overline{G} \to M$ is a constant map whose image point lies in G, then $i(f, G) = 1$.*

A2 - Existence Principle *If $i(f, G) \neq 0$, then f has a fixed point on G.*

A3 - Additivity *If $G_1, \ldots, G_m$ are pairwise disjoint open subsets of G, and the complement $\overline{G} \setminus \cup_{j=1}^{m} G_j$ contains no fixed points of f, then*

$$i(f, G) = \sum_{j=1}^{m} i(f, G_j). \tag{2.5.1}$$

A4- Homotopy Invariance *Let J be a compact interval in $\mathbb{R}$ and let U be a relatively open set in $M \times J$. If $H : \overline{U} \to M$ is i-compact and if $H(x, t) \neq x$ on ∂U, then*

$$i(H(\cdot, t), U_t) = \textit{constant} \qquad \textit{for all } t \in J, \tag{2.5.2}$$

where $U_t = \{x \mid (x, t) \in U\}$.

After we have obtained our estimates, the main points of the existence argument will be:

- Creating an invertible coordinate mapping that maps regions with similar properties to those of the elliptic Ω region (in particular, the shape of Γ_{shock}), to the unit square $(0, 1)^2$. Then, using this mapping to construct an invertible correspondence between functions ρ and shock curves Γ_{shock} that are close to Regular Shock Reflection-Diffraction configurations, and pairs of functions (u, v) where u is defined on the unit square, and v is defined on the unit interval.

- Creating an iteration map $\mathcal{I}$ on $\mathcal{K}$, mapping wedge angles and possible solutions to the solution of an approximate problem. The approximate problem will contain cutoffs to deal with the degenerate ellipticity and obliqueness, which must later be removed to obtain solutions.

- Using continuous dependence of the problem that defines $\mathcal{I}$, and uniqueness of it's solutions, to prove appropriate continuity and compactness properties for Proposition 2.5.3 to apply. In particular, proving a gain-in-regularity is required for compactness to hold; this will follow by careful analysis of where the Hölder exponents in our estimates arise.

- Proving that fixed points of this iteration map correspond to admissible solutions; in particular, we must remove the cutoffs by using barrier functions.

- Using the fact that in the case $\theta_w = \frac{\pi}{2}$, i.e. the case where wedge is a vertical wall, the unique solution to Problem 2.4.1 is normal reflection, we find the unique fixed point for $\theta_w = \frac{\pi}{2}$ is the one corresponding to normal reflection. We then calculate the fixed point degree at $\theta_w = \frac{\pi}{2}$ to show that it is non-zero, and use **(A4)** to deduce existence of fixed points for all wedge angles.

Chapter 3

Local Theory & Von Neumann Conjecture for the Nonlinear Wave System

In this Chapter, we shall study local reflection near the point P_0, as well as derive some basic properties of the Nonlinear Wave System. As mentioned in Section 2.3, we will establish existence of local reflection for all wedge angles and establish that there exists a detachment angle and sonic angle $0 < \theta_d < \theta_{sonic}$, depending only on the shock strength via ρ_1/ρ_0 and the abiatic index γ, such that the local regular reflection is supersonic for wedge angles $\theta_w \in (\theta_{sonic}, \frac{\pi}{2}]$, subsonic for $\theta_w \in [\theta_d, \theta_{sonic})$, and does not exist for $\theta_w < \theta_d$. We also find that there always exists exactly two local reflections at point P_0, the strong state and the weak state. We discuss at the end of the chapter why we choose to discard the strong one, and choose the weak one to form our Regular Shock Reflection-Diffraction configurations.

3.1 Normal reflection and its uniqueness

We first consider the shock reflection in the case $\theta_w = \frac{\pi}{2}$; i.e. where the wedge is a vertical wall. The incident shock reflects normally and the reflected shock is a plane at $\xi = \overline{\xi}_1 < 0$, which is defined below. We seek $\overline{\xi}_1 < 0$ and a uniform state $(2) = (\rho_n, m_n, n_n)$ which satisfies the slip boundary condition $(m_n, n_n) \cdot \nu = 0$ on $\Gamma_{wedge} \cup \Gamma_{sym}$ and the Rankine-Hugoniot conditions (2.2.1), jumping to $(\rho_1, m_1, 0)$ across $\mathcal{S}_1 = \{\xi = \overline{\xi}_1\}$. From (2.2.1) on $\mathcal{S}_1$, we have

$$n_n = n_1 = 0. \tag{3.1.1}$$

Also, from the slip boundary condition (1.2.6) on $\Gamma_{wedge} = \{\xi = 0\}$,

$$m_n = 0. \tag{3.1.2}$$

As (3.1.1) implies implies the slip boundary condition is true on $\Gamma_{sym} = \{\eta = 0\}$, so normal reflection is a global solution. We shall see later that in the case $\theta_w < \frac{\pi}{2}$, we always have $n_2 \neq 0$ so that local reflection is never a global solution.

Now the Rankine-Hugoniot conditions (2.2.1) on $\{\xi = \overline{\xi}_1\}$, and the formula (3.4.1) imply that

$$\overline{\xi}_1 = -\overline{c}(\rho_n, \rho_1) < 0, \tag{3.1.3}$$

and

$$(p(\rho_n) - p(\rho_1))(\rho_n - \rho_1) = (p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0) = m_1^2. \tag{3.1.4}$$

Now we see there is a unique solution $\rho_n > \rho_1$ of (3.1.4); indeed, if we define

$$F(s) := (p(s) - p(\rho_1))(s - \rho_1), \tag{3.1.5}$$

then

$$\begin{aligned} F(\rho_1) = 0, \qquad \lim_{s\uparrow\infty} F(s) = \infty, \text{ and} \\ F'(s) = c^2(s)(s - \rho_1) + p(s) - p(\rho_1) > 0 \qquad \text{for } s > \rho_1. \end{aligned} \tag{3.1.6}$$

Therefore we conclude the following:

Theorem 3.1.1. *There exists a unique normal reflection state (2), which is a global solution to the Problem 2.4.1 in the case $\theta_w = \frac{\pi}{2}$, with velocity $(m_n, n_n) = \mathbf{0}$, and density $\rho_n \in (\rho_1, \infty)$. The location of the reflected shock is*

$$\overline{\xi}_1 = -\overline{c}(\rho_n, \rho_1) \in (-c_n, -c_1), \tag{3.1.7}$$

where $\overline{c}(\cdot,\cdot)$ is defined in (2.2.6)*, and $c_n = \rho_n^{\gamma-1}$ is the sonic speed for state (2).*

3.2 Local regular reflection and state (2)

We recall from §2.1 that the incident shock $\mathcal{S}_0 = \{\xi = \xi_1\}$ separates the uniform state $(0) = (\rho_0, 0, 0)$ from the downstream uniform state $(1) = (\rho_1, m_1, 0)$. The incident shock $\mathcal{S}_0$ intersects the wedge boundary $\{\eta = \xi \tan\theta_w\}$ at point

$$P_0 := (\xi_1, \xi_1 \tan\theta_w), \tag{3.2.1}$$

for ξ_1 as in (2.2.4). We study the local reflection at point P_0; i.e. we seek a straight shock $\mathcal{S}_1$ passing through P_0, which separates the upstream state $(1) = (\rho_1, m_1, 0)$ from a downstream constant state $(2) = (\rho_2, m_2, n_2)$ satisfying the slip boundary condition (2.2.8),the Rankine-Hugoniot conditions (2.2.1), and the entropy condition (2.2.2) are satisfied on $\mathcal{S}_1$.

We now state the main result for existence of local regular reflection.

Theorem 3.2.1. *Let $\rho_1 > \rho_0 > 0$. Then there exists*

$$0 < \theta_d < \theta_{sonic} < \frac{\pi}{2}, \tag{3.2.2}$$

depending only on ρ_1/ρ_0 and γ, called the detachment and sonic angles respectively, such that

(i) *For each $\theta_w \in (\theta_d, \frac{\pi}{2})$, there exists exactly two states (2) satisfying* (2.2.1)*,* (2.2.8) *and* (2.2.2)*, the weak reflection state $(\rho_2^{w,\theta_w}, m_2^{w,\theta_w}, n_2^{w,\theta_w})$ and the strong reflection state $(\rho_2^{s,\theta_w}, m_2^{s,\theta_w}, n_2^{s,\theta_w})$, such that*

$$\rho_1 < \rho_n < \rho_2^{w,\theta_w} < \rho_2^{s,\theta_w}, \tag{3.2.3}$$

where ρ_n is the density of the normal reflection state (2) and

$$\lim_{\theta_w\downarrow\theta_d} (\rho_2^{w,\theta_w}, m_2^{w,\theta_w}, n_2^{w,\theta_w}) = \lim_{\theta_w\downarrow\theta_d} (\rho_2^{s,\theta_w}, m_2^{s,\theta_w}, n_2^{s,\theta_w}).$$

(ii) *For $\theta_w = \theta_d$, there exists a unique state (2), satisfying the slip boundary condition* (2.2.8)*, the entropy condition* (2.2.2) *and the Rankine-Hugoniot conditions* (2.2.1) *which is equal to the limit in* **(i)**.

(iii) *The weak state ρ_2^{w,θ_w} depends continuously on $\theta_w \in [\theta_d, \frac{\pi}{2}]$ and C^∞-smoothly on $\theta_w \in (\theta_d, \frac{\pi}{2}]$.*

The strong state ρ_2^{s,θ_w} depends continuously on $\theta_w \in (\theta_d, \frac{\pi}{2})$ and C^∞-smoothly on $\theta_w \in (\theta_d, \frac{\pi}{2})$.

(iv)

$$\lim_{\theta_w \uparrow \frac{\pi}{2}} (\rho_2^{w,\theta_w}, m_2^{w,\theta_w}, n_2^{w,\theta_w}) = (\rho_n, 0, 0),$$

$$\lim_{\theta_w \uparrow \frac{\pi}{2}} (\rho_2^{s,\theta_w}, m_2^{s,\theta_w}, n_2^{s,\theta_w}) = (\infty, 0, \infty),$$

where ρ_n is the normal reflection density from (3.1.7).

(v) *State (1) is always strictly hyperbolic at P_0:*

$$|\boldsymbol{\xi}_{P_0}| > c_1 \qquad \text{for all } \theta_w \in [\theta_d, \frac{\pi}{2}] \tag{3.2.4}$$

(vi) *The ellipticity of state (2) at P_0 is as follows:*

$$|\boldsymbol{\xi}_{P_0}| > c^{s,\theta_w} \text{for } \theta_w \in (\theta_d, \frac{\pi}{2}), \tag{3.2.5}$$

$$|\boldsymbol{\xi}_{P_0}| < c^{w,\theta_w} \text{for } \theta_w \in (\theta_{sonic}, \frac{\pi}{2}], \tag{3.2.6}$$

$$|\boldsymbol{\xi}_{P_0}| = c^{w,\theta_w} \text{for } \theta_w = \theta_{sonic}, \tag{3.2.7}$$

$$|\boldsymbol{\xi}_{P_0}| > c^{w,\theta_w} \text{for } \theta_w \in [\theta_{detach}, \theta_{sonic}). \tag{3.2.8}$$

(vii) *For the weak (strong) reflection,*

$$|\boldsymbol{\xi}_{P_0}|^2 > (<) \frac{2c_2^2\bar{c}_2^2}{c_2^2 + \bar{c}_2^2}. \tag{3.2.9}$$

(viii) *Let $\mathcal{S}_1$ denote the shock wave that separates state (1) and (2). Then denoting the unit normal to $\mathcal{S}_1$ by $(\sin\theta_s, -\cos\theta_s)$ for an angle θ_s,*

$$0 < \theta_w - \theta_s < \frac{\pi}{2}, \text{ when } \theta_w \in (\theta_d, \frac{\pi}{2}), \tag{3.2.10}$$

i.e $\mathcal{S}_1$ meets Γ_{wedge} at an angle $\theta_{meet} \in (0, \frac{\pi}{2})$, where the angle is taken to point from Γ_{wedge} to $\mathcal{S}_1$. In the case $\theta_w = \frac{\pi}{2}$, $\theta_s = \frac{\pi}{2}$, and the normal reflection shock wave is parallel to Γ_{wedge}.

3.3 Shock polar for the Nonlinear Wave System

In this Section, we explicitly construct the weak and strong states (2), to prove Theorem 3.2.1. We express the unit normal to the shock wave pointing from state (1) to state (2), and the unit tangent to the shock wave pointing towards the reflection point P_0, as

$\boldsymbol{\nu} = (\sin\theta_s, -\cos\theta_s)$, $\boldsymbol{\tau} = (\cos\theta_s, \sin\theta_s)$ respectively, for some θ_s. We now try to find (ρ_2, m_2, n_2) which satisfy the Rankine-Hugoniot conditions (2.2.1), the entropy condition (2.2.2) and the slip boundary condition (2.2.8), by formulating equations for $(\rho_2, m_2, n_2, \theta_s)$ in terms of the variable θ_w, and depending on the initial data (ρ_0, ρ_1, γ). By (2.2.1),

$$(\rho_2 - \rho_1)\xi_1 \sec\theta_w \sin(\theta_s - \theta_w) = m_2 \sin\theta_s - n_2 \cos\theta_s, \tag{3.3.1}$$

$$(m_2 - m_1)\cos\theta_s + n_2 \sin\theta_s = 0. \tag{3.3.2}$$

The entropy condition (2.2.2) becomes $\sin(\theta_w - \theta_s) > 0$, i.e.

$$0 < \theta_w - \theta_s < \pi. \tag{3.3.3}$$

The slip boundary condition (2.2.8) becomes

$$-m_2 \sin\theta_w + n_2 \cos\theta_w = 0. \tag{3.3.4}$$

From (3.3.1)-(3.3.4), we can obtain the equivalent system of equations

$$\cos^2\theta_w = \xi_1^2 \frac{[p][\rho] - m_1^2}{[p]^2 - m_1^2\xi_1^2} := f(\rho), \tag{3.3.5}$$

$$\cos(\theta_w - \theta_s) = \frac{m_1 \sin\theta_w}{\sqrt{[p][\rho]}}, \tag{3.3.6}$$

$$m_2 = m_1 - \sqrt{[p][\rho]}\sin\theta_s, \tag{3.3.7}$$

$$n_2 = \sqrt{[p][\rho]}\cos\theta_s, . \tag{3.3.8}$$

From here, we can use (3.3.5), in particular studying the shape of the graph of f, to determine exactly for which θ_w a suitable ρ exists. We then see that existence of a suitable ρ for a given θ_w is both a necessary and sufficient condition for local reflection to exist, as once ρ is obtained, we can then always obtain suitable θ_s, m_2, n_2 using (3.3.6)–(3.3.8) and the constraint (3.3.3). To be more precise, to obtain (m_2, n_2, θ_s) from (ρ, θ_w), we first obtain θ_s from (3.3.6) and the constraint (3.3.3) followed by, then m_2 and n_2 from (3.3.7) and (3.3.8). We also note that (3.3.6) strengthens the constraint (3.3.3) to the condition $0 < \theta_w - \theta_s < \frac{\pi}{2}$.

Proposition 3.3.1. *Let $f : (\rho_1, \infty) \to \mathbb{R}$ be defined as in* (3.3.5)*. Then:*

(i) *$0 < f(\rho) < 1$ for $\rho > \rho_1$, exactly when $\rho > \rho_n$, where ρ_n is the density of the normal reflection state*

(ii) *There exists $\rho_n < \rho_d < \infty$ where*

$$\begin{aligned} &f'(\rho_d) = 0, \quad f' > 0 \text{ on } [\rho_n, \rho_d), \text{ and} \\ &f' < 0 \text{ on } (\rho_d, \infty). \end{aligned} \tag{3.3.9}$$

(iii) *There exists $\rho_n < \rho_s < \rho_d < \infty$ where the state (2) determined by* (3.3.5)*-*(3.3.8) *for a fixed ρ is degenerate at P_0 if $\rho = \rho_s$, hyperbolic if $\rho \in (\rho_s, \infty))$ and elliptic if $\rho \in [\overline{\rho}_2, \rho_s)$.*

Proof. We first prove **(i)**. We firstly note that from (3.1.6), that $[p][\rho] - m_1^2 \geq 0$ for $\rho > \rho_1$ exactly when $\rho \geq \rho_n$, and the expression is strictly monotone in ρ. Next, we note that

$$\begin{aligned} [p]^2 - m_1^2\xi_1^2 &= [p][\rho]\overline{c}(\rho,\rho_1) - m_1^2\xi_1^2 \\ &> \xi_1^2([p][\rho] - m_1^2), \end{aligned} \tag{3.3.10}$$

because $\rho > \rho_1 > \rho_0$, and the strict monotonicity of $\overline{c}(\cdot,\rho_1)$ due to (2.2.6).

Next, we prove **(ii)**. We find that $f'(\rho)$ satisfies

$$f'(\rho)([p]^2 - m_1^2\xi_1^2) = 2c^2[p]\bigg(\frac{[\rho]}{[p]} + \frac{1}{c^2} - 2f(\rho)\bigg), \tag{3.3.11}$$

so that

$$f'(\rho) > (<)0 \text{ if and only if } f(\rho) < (>)\frac{\xi_1^2}{2}\bigg(\frac{1}{\overline{c}^2(\rho,\rho_1)} + \frac{1}{c^2}\bigg). \tag{3.3.12}$$

We next note that $f(\rho_n) = 0$ by (3.1.4), and $\lim_{\rho\to\infty} f(\rho) = 0$ as the numerator is of order $\rho^{\gamma+1}$ and the denominator of order $\rho^{2\gamma}$. Thus, as f is positive and not identically zero, there exists a global maximum point $\rho_d \in (0,\infty)$ such that $f'(\rho_d) = 0$. We now claim that f is strictly increasing for $\rho_n < \rho < \rho_d$, and strictly decreasing for $\rho_d < \rho < \rho_n$. Suppose not, then by a preliminary argument there exists $s < t$ with $f(s) = f(t)$, $f'(s) \leq 0$, $f'(t) \geq 0$ and $f(s) = f(t)$. But as the expression on the right hand side of (3.3.12) is strictly monotone decreasing in ρ, it can be used to show this implies $f(s) > f(t)$, a contradiction.

Finally, we prove **(iii)**. As $r(P_1)^2 = \xi_1^2\sec^2\theta_w = \frac{\xi_1^2}{f(\rho)}$, we note that

$$c^2 > (<)r^2 \text{ if and only if } f(\rho) < (>)\frac{\xi_1^2}{c^2}. \tag{3.3.13}$$

We first note that for $\rho > \rho_d$, $f(\rho) > \frac{\xi_1^2}{c^2}$ by (3.3.12), so strong reflections are always supersonic. For $\rho_n < \rho < \rho_d$, f is strictly increasing, and $\frac{\xi_1^2}{c^2}$ is strictly decreasing, so as ρ_n is supersonic and ρ_d is subsonic, there exists a unique ρ_s which satisfies the conditions required. □

Now, using Proposition 3.3.1, can conclude Theorem 3.2.1 by choosing

$$\begin{aligned} \rho^{w,\theta_w} &:= \big(f\mid_{[\rho_n,\rho_d]}\big)^{-1}(\cos^2(\theta_w)), \\ \rho^{s,\theta_w} &:= \big(f\mid_{[\rho_d,\infty)}\big)^{-1}(\cos^2(\theta_w)), \\ \theta_d &= \cos^{-1}\Big(\sqrt{f(\rho_d)}\Big), \text{ and} \\ \theta_{sonic} &= \cos^{-1}\Big(\sqrt{f(\rho_s)}\Big), \end{aligned}$$

and using (3.3.3) and (3.3.5) to obtain (3.2.10), and (3.3.12) to obtain (3.2.9).

We also further note that $\frac{\rho_2}{\rho_1}$, θ_d and θ_{sonic} are dependent only on $\frac{\rho_1}{\rho_0}$ and γ, as substituting in the expressions for ξ_1, m_1 in terms of ρ_0, ρ_1, γ,

$$\cos^2\theta_w = f(\rho) = c^2(1,\frac{\rho_0}{\rho_1})\frac{(p(\frac{\rho_2}{\rho_1}) - p(1))(\frac{\rho_2}{\rho_1} - 1) - (p(1) - p(\frac{\rho_0}{\rho_1}))(1 - \frac{\rho_0}{\rho_1})}{(p(\frac{\rho_2}{\rho_1}) - p(1))^2 - (p(1) - p(\frac{\rho_0}{\rho_1}))^2}.$$

3.4 Basic properties of the weak state (2), and the definition of supersonic & subsonic wedge angles

In this Section, we fix a wedge angle $\theta_w \in (\theta_d, \frac{\pi}{2})$, we consider the weak state (2) associated with θ_w, and denote by (ρ_2, m_2, n_2) the density and momenta associated with with state (2).

Motivated by (2.1.5)-(2.1.6) and Theorem 3.2.1 **(vi)**,

Definition 3.4.1. We call a wedge angle $\theta_w \in (\theta_d, \frac{\pi}{2}]$

- Supersonic if $\theta_w \in (\theta_{sonic}, \frac{\pi}{2}]$,
- Sonic if $\theta_w = \theta_{sonic}$,
- Subsonic if $\theta_w \in (\theta_d, \theta_{sonic})$.

We shall now define mathematically the points and lines from Figure 2.1. First, we note that eliminating $[m]$ and $[n]$ in (2.2.1), and using the entropy condition (2.2.2),

$$\boldsymbol{\xi} \cdot \boldsymbol{\nu} = -\overline{c}(\rho_2, \rho_1) \in (-c_2, -c_1) \tag{3.4.1}$$

for $\overline{c}(s,t)$ from (2.2.6), where the inequality for $\overline{c}$ follows by monotonicity in both variables, and (3.2.3). This expression implies that the straight shock $\mathcal{S}_1$ between states (1) and (2) meets the sonic circle for state (2) at two points. We now show that in the supersonic case, the nearest intersection to P_0 lies within Λ. Using (3.4.1), $\mathcal{S}_1$ can be written in polar coordinates (r, θ) as

$$r = \csc(\theta - \theta_s) c(\rho_2, \rho_1).$$

In the supersonic case, $r(\theta_w) > c_2$. Also r is strictly decreasing on $(\theta_w, \theta_s + \frac{\pi}{2}) \subset (\theta_s, \theta_s + \frac{\pi}{2})$ with $r(\theta_s + \frac{\pi}{2}) = c(\rho_2, \rho_1)$. Thus we can set

$$\theta_1(\theta_w) := \theta_s + \sin^{-1}\left(\frac{c(\rho_2, \rho_1)}{c(\rho_2)}\right) \in (\theta_w, \pi) \tag{3.4.2}$$

and define point P_1 by

$$P_1 := c_2(\cos\theta_1, \sin\theta_1). \tag{3.4.3}$$

Definition 3.4.2. Let $\theta_w \in (\theta_d, \frac{\pi}{2}]$. Then the following lines, points and segments are defined for the wedge angle θ_w corresponding to the data and the parameters of state (2):

- Line $\mathcal{S}_1$ to be the straight shock separating states (1) and (2), and $\boldsymbol{\nu}_{\mathcal{S}_1} = (-\sin\theta_s, \cos\theta_s)$ to be the unit normal to $\mathcal{S}_1$ oriented so that;

  $$\mathcal{S}_1 = \{\boldsymbol{\xi} \cdot \boldsymbol{\nu}_{\mathcal{S}_1} = -\overline{c}(\rho_2, \rho_1)\}, \tag{3.4.4}$$

  where we recall that from (3.2.10), the angle θ_{meet} between Γ_{wedge} and $\mathcal{S}_1$ satisfies $\theta_{meet} \in (0, \frac{\pi}{2})$ whenever $\theta_w \in (\theta_d, \frac{\pi}{2})$.
- Point P_1, using (3.4.2) for supersonic wedge angles, and $P_1 = P_0$ for sonic/subsonic wedge angles.
- $P_3 = \mathbf{0}$.

3.5. The self-similar nonlinear wave system in the (x, y) coordinates flattening the sonic circle for state (2)

- For supersonic wedge angles, $P_4 = c_2(\cos\theta_w, \sin\theta_w)$; i.e the intersection in Λ of the sonic circle for state (2) with the wedge.
- For subsonic/sonic wedge angles, $P_4 = P_0$.
- $\Gamma_{wedge} := P_3P_4$.
- For supersonic wedge angles, Γ_{sonic} is the upper arc P_1P_4 of the sonic circle of state (2). For sonic/subsonic wedge angles, $P_1 = P_0$.

Remark 3.4.3. From the explicit formulae defining them, P_0 depends continuously on $\theta_w \in [\theta_d, \frac{\pi}{2})$. P_1, P_4 depend continuously on $\theta_w \in [\theta_d, \frac{\pi}{2}]$.

3.5 The self-similar nonlinear wave system in the (x, y) coordinates flattening the sonic circle for state (2)

Motivated by (2.1.5), and the polar coordinate expression (2.1.7), as well as condition **(iv)** in Problems 2.4.1 and 2.4.2, we now look at the Self-Similar Nonlinear Wave System in the transformed polar coordinates which flatten the sonic circle for state (2). Let (ρ, r) be a C^2 solution of (2.1.3) in a domain $\mathcal{D} \subset \mathbb{R}^2$. We will assume that for some $M \geq 0$,

$$0 < \rho \leq M \qquad \text{in } \mathcal{D}. \tag{3.5.1}$$

Throughout this section, C will be a large constant that depends on M, and may get larger at each occurrence. Let $\rho_2 > 0$ be a constant density. We introduce the function

$$\psi = c_2^2 - c^2 \qquad \text{in } \mathcal{D}, \tag{3.5.2}$$

where we note that when $\rho = \rho_2$, $\psi = 0$; so ψ is expected to be small near Γ_{sonic} in supersonic cases.

We now derive our equation in terms of ψ. Estimates for ρ are equivalent to estimates for ψ, provided ρ is bounded away from zero. Let (r, θ) be the polar coordinates for $\boldsymbol{\xi}$ centred at $\mathbf{0}$, and then introduce the (x, y) coordinates on $\mathcal{D} \setminus \{\mathbf{0}\}$,

$$(x, y) = (c_2 - r, \theta). \tag{3.5.3}$$

We note by (2.1.5)-(2.1.6), the sonic circle for state (2) is $\{x = 0\}$, (2) is subsonic on $\{0 < x < c_2\}$ and supersonic on $\{x < 0\}$. Writing (2.1.7) in non-divergence form, ψ satisfies the following equation in $\mathcal{D} \setminus \{\mathbf{0}\}$;

$$\begin{aligned} &\sum_{i,j=1}^{2} \hat{A}_{ij}(x,\psi)\partial_i j\psi + \sum_{i=1}^{2} \hat{A}_i(x,\psi)\partial_i\psi + \sum_{i,j=1}^{2} \hat{B}_{ij}(x,\psi)\partial_i\psi\partial_j\psi := \\ &\quad (2c_2x - \psi + O_1(x,\psi))\psi_{xx} + (c_2 + O_2(x,\psi))\psi_x - (1 + O_3(x,\psi))\psi_x^2 \\ &\qquad + (1 + O_4(x,\psi))\psi_{yy} - \left(\frac{1}{(\gamma-1)c_2^2} + O_5(x,\psi)\right)\psi_y^2 = 0, \end{aligned} \tag{3.5.4}$$

where

$$\begin{aligned} &O_1(x,z) = -x^2, && O_2(x,z) = -3x + \frac{z}{c_2}, \\ &O_3(x,z) = -\frac{\gamma-2}{\gamma-1}\big(2c_2x - z - x^2\big), && O_4(x,z) = \frac{c_2^2 - z}{(c_2-x)^2} - 1, \\ &O_5(x,z) = \frac{1}{(c_2-x)^2} - \frac{1}{c_2^2}, \end{aligned} \tag{3.5.5}$$

3.5. The self-similar nonlinear wave system in the (x, y) coordinates flattening the sonic circle for state (2)

so that for $i = 2, 3, 4, 5$,

$$\frac{|O_1(x,z)|}{x^2} + \frac{|DO_1(x,z)|}{x} + \frac{|O_i(x,z)|}{|x|+|z|} + |DO_i(x,z)| \leq C. \tag{3.5.6}$$

We now assume that ρ satisfies the oblique boundary condition (2.2.11), jumping discontinuously from state (1), to a state ρ, across a shock curve Γ_{shock}. We assume introduce the additional assumption that for $M > 0$ from (3.5.1),

$$\rho \geq \rho_1 + \frac{1}{M}, \tag{3.5.7}$$

where we note that $\rho_2 \geq \rho_1 + \frac{1}{M}$ for M depending only on the initial data, from (3.2.3) and continuous dependence of ρ_2 on the parameters. Firstly, we find that provided $r^2 - \overline{c}^2 > 0$ (which is the case when ψ, x are small),

$$r'(\theta) = -\frac{c_2}{\overline{c}_2}\sqrt{c_2^2 - \overline{c}_2^2} + O_6(x, \psi), \tag{3.5.8}$$

where

$$|O_6(x,z)| \leq C(|x| + |z|). \tag{3.5.9}$$

Thus we can write, whenever x, ψ are sufficiently small,

$$\begin{aligned} \Gamma_{shock} \cap \{0 < x < \epsilon\} &= \{y = f(x)\}, \\ \Omega \cap \{0 < x < \epsilon\} &= \{\theta_w < y < f(x) \mid x \in (0, \epsilon)\}, \end{aligned} \tag{3.5.10}$$

where

$$f'(x) = \frac{-1}{r'(y)} = \frac{\overline{c}}{r\sqrt{r^2 - \overline{c}^2}} = \frac{\overline{c_2}}{c_2\sqrt{c_2^2 - \overline{c}_2^2}} + O_7(x, z), \tag{3.5.11}$$

with

$$\begin{aligned} |O_7(x,z)| &\leq C(|x| + |z|), \\ |DO_7(x,z)| &\leq C. \end{aligned} \tag{3.5.12}$$

Thus the boundary condition (2.2.11) is equivalent to

$$\hat{M}\psi := \hat{\beta}_1\psi_x + \hat{\beta}_2\psi_y = 0, \tag{3.5.13}$$

where

$$\begin{aligned} \hat{\beta}_1 &= 3\overline{c}^2(c^2 - r^2) - c^2(r^2 - \overline{c}^2) = -c_2^2(c_2^2 - \overline{c}_2^2) + O_8(x, \psi), \\ \hat{\beta}_2 &= f'(x)\big(\overline{c}^2(c^2 - r^2) - 3c^2(r^2 - \overline{c}^2)\big) = -3c_2\overline{c}_2\sqrt{c^2 - \overline{c}^2} + O_9(x, \psi), \end{aligned} \tag{3.5.14}$$

and for $i = 8, 9$,

$$\begin{aligned} |O_i(x,z)| &\leq C(|x| + |z|), \\ |DO_i(x,z)| &\leq C. \end{aligned} \tag{3.5.15}$$

Therefore, for $\hat{\boldsymbol{\nu}} = (f'(x), -1)$,

$$\hat{\beta} \cdot \hat{\boldsymbol{\nu}} = 2r^2 f'(x)(c^2 - \overline{c}^2) = 2\overline{c}_2 c_2\sqrt{c_2^2 - \overline{c}_2^2} + O_{10}(x, z), \tag{3.5.16}$$

where

$$\begin{aligned} |O_{10}(x,z)| &\leq C(|x| + |z|), \\ |DO_{10}(x,z)| &\leq C. \end{aligned} \tag{3.5.17}$$

3.6 Ellipticity and obliqueness principles for the Nonlinear Wave System

As mentioned when formulating our Regular Self-Similar Shock Reflection-Diffraction configurations, we expect the ellipticity to degenerate at the sonic arc in supersonic cases. We will also find later that the obliqueness degenerates at P_2 where the shock wave meets the flat part of the wedge. Therefore, we will now derive ellipticity and obliqueness principles, which will allow us to control these degeneracies.

3.6.1 Ellipticity principle

Using similar calculations to [50][Lemma 3.4], we derive an ellipticity principle for solutions of (2.1.4). We start with an interior ellipticity principle.

Proposition 3.6.1. *Let $\mathcal{D} \subset \mathbb{R}^2$ be an open, bounded region, and $\rho \in C^2(\mathcal{D})$ is a solution to (2.1.4) with $c^2 - r^2 > 0$ in $\mathcal{D}$, and $\frac{1}{M} \leq \rho \leq M$. Then there exists $\sigma > 0$, depending only on M, such that if $b \in C^2(\mathcal{D})$ and if $b \in C^2(\mathcal{D} \cup \Gamma)$ and*

$$|b| + |Db| + |D^2 b| \leq \sigma, \tag{3.6.1}$$

then $c^2 - r^2 - b$ cannot attain a minimum in $\mathcal{D}$.

Proof. Firstly, we find that if $r^2 \leq \frac{1}{2M^{\gamma-1}}$, then $c^2 - r^2 \geq \frac{1}{2M^{\gamma-1}}$. So, if σ is small depending on M, we may use the polar form (2.1.7). We find that $u = c^2$ satisfies

$$0 = Qu := (c^2 - r^2)u_{rr} + \frac{c^2}{r^2}u_{\theta\theta} + \left(\frac{1}{\gamma - 1}\left(1 - \frac{r^2}{c^2}\right) + \frac{r^2}{c^2}\right)u_r^2 + \frac{1}{\gamma - 1}\frac{1}{r^2}u_\theta^2 + \left(\frac{c^2}{r} - 2r\right)u_r. \tag{3.6.2}$$

Treating u as a quasilinear operator with coefficients evaluated at u, we find that

$$Qr^2 \geq 2c^2 \geq \frac{2}{M^{\gamma-1}}, \tag{3.6.3}$$

Thus if $|Db|$ and $|D^2 b|$ are sufficiently small, depending only on M, $Q(c^2 - r^2 - b) < 0$, and so the result follows by the comparison principle. □

We now turn to the shock boundary.

Proposition 3.6.2. *Let $\mathcal{D} \subset \mathbb{R}^2$ be an open, bounded region, and $\Gamma \subset \partial\mathcal{D}$. Suppose that $\rho \in C^2(\mathcal{D}) \cap C^1(\mathcal{D} \cup \Gamma)$ satisfies (2.1.4) in $\mathcal{D}$, (2.2.11) on Γ, and $\frac{1}{M} + \rho_1 \leq \rho \leq M$ in $\mathcal{D} \cup \Gamma$, $c^2 - r^2 > 0$ in $\mathcal{D} \cup \Gamma$ and $r^2 - \bar{c}^2 > 0$ on Γ. Then there exists $\sigma > 0$ depending only on M such that if $b \in C^(\mathcal{D} \cup \Gamma)$ and*

$$|b| \leq \sigma, \quad \partial_r b \geq -2r, \quad \partial_\theta b \leq 0, \ \text{on } \Gamma \tag{3.6.4}$$

then $c^2 - r^2 - b$ cannot attain a minimum over $\mathcal{D} \cup \Gamma$ on Γ.

Proof. Firstly, as $r^2 > \bar{c}^2 > c_1^2$ on Γ, we may use the polar coordinate expressions. At a minimum for $c^2 - r^2 - b$ on $\Omega \cup \Gamma$ located on Γ, $c^2 - r^2 \leq \sigma$ by (3.6.4). Also, due to the strict monotonicity of $\bar{c}$ in both variables, and the condition $\rho \geq \frac{1}{M} + \rho_1$, $c^2 - \bar{c}^2 \geq \frac{1}{C(M)}$, so that

$$r^2 - \bar{c}^2 \geq c^2 - \bar{c}^2 - \delta \geq \frac{1}{C(M)} - \delta. \tag{3.6.5}$$

Thus, for δ sufficiently small depending on M,

$$\beta_2 = \overline{c}^2(c^2 - r^2) - 3c^2(r^2 - \overline{c}^2) \leq -\frac{1}{C(M)}, \tag{3.6.6}$$

and

$$\beta_1 = r'(\theta)(3\overline{c}^2(c^2 - r^2) - c^2(r^2 - \overline{c}^2) \geq \frac{1}{C(M)}, \tag{3.6.7}$$

so that

$$\beta \cdot \nabla(c^2 - r^2 - b) = -\beta_1(2r + \partial_r b) - \beta_2 \partial_\theta b < 0, \tag{3.6.8}$$

therefore result follows by the Hopf maximum principle. □

We now turn to to the wedge boundary.

Proposition 3.6.3. *Let $\mathcal{D} \subset \{\theta > \theta_0\} \subset \mathbb{R}^2$ be an open, bounded region, and that $\Gamma \subset \partial\mathcal{D}$ is a ray $\{\theta = \theta_0\}$ emanating from the origin. Suppose that $\rho \in C^2(\mathcal{D} \cup \Gamma)$ satisfies (2.1.4) in $\mathcal{D}$, the slip boundary condition on (2.2.8) on Γ, $\frac{1}{M} \leq \rho \leq M$ in $\mathcal{D} \cup \Gamma$, and $c^2 - r^2 > 0$ in $\mathcal{D} \cup \Gamma$. Then there exists $\sigma > 0$, depending only on M, such that if*

$$|b| + |Db| + |D^2 b| \leq \sigma, \tag{3.6.9}$$

then $c^2 - r^2 - b$ cannot attain a minimum on $\mathcal{D} \cup \Gamma$.

Proof. As in the proof of Proposition 3.6.1, we may use the polar form (2.1.7). We then note that we can extend ρ by even reflection:

$$\begin{aligned} \mathcal{D}^{ext} &= \mathcal{D} \cup \{\mathbf{x} \in \mathbb{R}^2 \mid (r, \theta_0 - \theta) \in \mathcal{D}\}, \\ \rho^{ext}(r, \theta) &= \rho(r, \theta_0 - \theta). \end{aligned} \tag{3.6.10}$$

As (2.2.8) is equivalent to $\rho_\theta = 0$, $\rho^{ext} \in C^2(\mathcal{D}^{ext})$. Also, as (2.1.7) is invariant under the transformation $(r, \theta) \mapsto (r, \theta_0 - \theta)$, ρ^{ext} satisfies (2.1.7) in $\mathcal{D}^{ext}$, and the result follows in the same way as the interior case. □

3.6.2 Obliqueness principle

Here, we derive an obliqueness principle for (2.1.4) and (2.2.11).

Proposition 3.6.4. *Suppose that $\mathcal{D} \subset \mathbb{R}^2$ is an open bounded region, $\Gamma \subset \partial\mathcal{D}$, and and that $\rho \in C^2(\mathcal{D}) \cap C^{1,\alpha}(\mathcal{D} \cup \Gamma)$ satisfies (2.1.4) in $\mathcal{D}$, and (2.2.11) on Γ, and $\rho_1 + \frac{1}{M} \leq \rho \leq M$ in $\mathcal{D} \cup \Gamma$. Then there exists σ only depending on M such that if $b \in C^2(\mathcal{D}) \cap C^{1,\alpha}(\mathcal{D} \cup \Gamma)$ satisfies*

$$\partial_r b \leq 2r, \qquad \partial_\theta b \leq 0, \qquad |b| \leq \sigma, \tag{3.6.11}$$

then $r^2 - \overline{c}^2 - b$ cannot attain it's minimum over Γ, in the interior of Γ.

Proof. This follows by the same saddle point argument as originally in [50][Lemma 3.2], where b is constant. The key point of the argument is that at a minimum point for $r^2 - \overline{c}^2 - b$,

$$r^2 - \overline{c}^2 \leq \delta, \tag{3.6.12}$$

so that

$$\begin{aligned} c^2 - r^2 &\geq c^2 - \overline{c}^2 - \delta \\ &\geq C(\tau) - \delta \\ &\geq C(\tau), \end{aligned} \tag{3.6.13}$$

provided that δ is sufficiently small; thus

$$\beta_1 < 0 \qquad \text{and } \beta_2 > 0, \tag{3.6.14}$$

so that

$$M(r^2 - \bar{c}^2 - b) = \beta_1(2r - \partial_r b) + \beta_2 \partial_\theta b \leq 0. \tag{3.6.15}$$

A minimum point $P \in \Gamma$ for $r^2 - \bar{c}^2 - b$ is either a minimum within $\mathcal{D} \cup \Gamma$, in which case the result follows by the Hopf maximum principle, or a saddle point. In the saddle point case, the $C^{1,\alpha}$ regularity can be exploited to construct a modified function which attains a minimum over Γ at P, leading to a contradiction; see [50][Lemma 3.2] for details. □

3.7 von Neumann's sonic and detachment conjectures

Here, we continue the discussion from §2.4. We have seen that local theory indicates that in the case $\theta_w < \frac{\pi}{2}$, there are two possible choices for state (2). There has been a long debate as to which is more physical; see [2, 28] and the other references referred to in the introduction.

In deciding which state (2) to use, we note that shock reflection-diffraction is not a local problem; so the choice of state (2) should depend on global properties of solutions. We take the same point of view as [17, 26], which that the selection of state (2) should depend on the the stability of the configuration with respect to θ_w; so we take the weak state (2), as it converges to normal reflection as $\theta_w \uparrow \frac{\pi}{2}$, which is the unique solution in the case $\theta_w = \frac{\pi}{2}$. Also, it has been observed that in experiments, the weak state (2) is typically the one which appears; see in particular [3, 4] and the references within.

Stability/Criterion Criterion to Select the Correct State (2): Since the normal reflection solution is unique when the wedge angle $\theta_w = \frac{\pi}{2}$, the global regular reflection-diffraction configuration should be stable and converge to the unique normal reflection solution when $\theta_w \uparrow \frac{\pi}{2}$, provided that such a global configuration can be constructed.

We also remark here that the difference between the wedge angles θ_{sonic} and θ_d is normally very small, i.e. θ_{sonic} and θ_d are only small fractions of degree apart; see Figures 3.1 and 3.2.

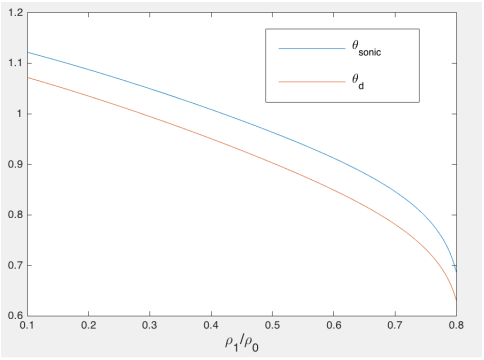


Figure 3.1: ρ_1/ρ_0 against θ_d, θ_{sonic} when $\gamma = 2.5$

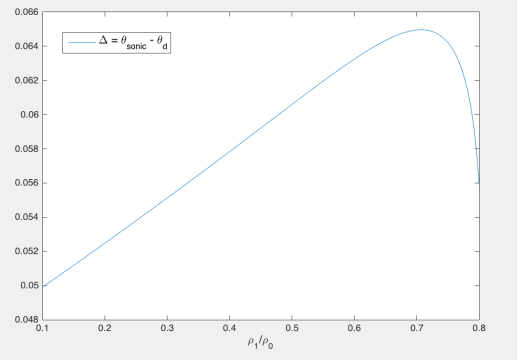


Figure 3.2: ρ_1/ρ_0 against $\Delta = \theta_{sonic} - \theta_d$ when $\gamma = 2.5$

Part II

Part II - Existence up to the Sonic Angle

Chapter 4

Admissible Solutions and Basic Estimates for Problem 2.4.1

4.1 Main theorem

In this Part, we shall prove existence of solutions to Problem 2.4.1:

Theorem 4.1.1. *Let $\theta_w \in (\theta_{sonic}, \frac{\pi}{2}]$. Then there exists a region Ω of the form described by Figure 2.1, and functions $\rho \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \Gamma_{sonic}) \cap C(\overline{\Omega})$, $r \in C^2((\theta_1, \pi)) \cap C^1([\theta_1, \pi])$, that solve Problem 2.4.1.*

4.2 Definition of admissible solutions

Let $\gamma > 1$, $\rho_1 > \rho_0 > 0$ be fixed. We use the terminology and notation introduced in Chapter 3. We will now define admissible solutions for Problem 2.4.1, motivated by the structure of Regular Supersonic Shock Reflection-Diffraction configuration, described in Section 2.3.

Definition 4.2.1. Fix $\theta_w \in [\theta_{sonic}, \frac{\pi}{2})$. A solution (ρ, r) to Problem 2.4.1 is called an admissible solution if it satisfies the following:

(i) The shock $\Gamma_{shock} = \{r = r(\theta) \mid \theta \in (\theta_1, \pi)\}$ satisfies

$$\Gamma_{shock} \subset (\Lambda \setminus B_{c_1}), \tag{4.2.1}$$

and Γ_{shock} is C^2 in its relative interior. Moreover, Γ^{ext}_{shock}, the extension of Γ_{shock} by reflecting in $\{\eta = 0\}$ and adding P_2 is C^1 in its relative interior, including P_2.

(ii) For some $\alpha \in (0, 1)$, ρ and r satisfy

$$\begin{aligned} &\rho \in C^{0,\alpha}(\overline{\Omega}) \cap C^{0,1}(\overline{\Omega} \setminus \{P_2\}) \cap C^{1,\alpha}(\overline{\Omega} \setminus \Gamma_{sonic}) \cap C^2(\Omega), \\ &r \in C^{1,\alpha}([\theta_1, \pi]) \cap C^2((\theta_1, \pi)) \cap C^{1,1}([\theta_1, \pi)) \end{aligned} \tag{4.2.2}$$

(iii) Equation (2.1.3) is strictly elliptic in $\overline{\Omega} \setminus \overline{\Gamma_{sonic}}$:

$$c^2 > |\boldsymbol{\xi}|^2 \qquad \text{in } \overline{\Omega} \setminus \overline{\Gamma_{sonic}}. \tag{4.2.3}$$

(iv) The boundary condition (2.2.11) is strictly oblique on the interior of Γ_{shock};

$$r^2 - \bar{c}^2 > 0 \qquad \text{on } \Gamma_{shock} \tag{4.2.4}$$

Remark 4.2.2. Due to condition **(i)**, $r'(\pi) = 0$, so that by (2.2.12), the obliqueness of the boundary condition (2.2.11) on Γ_{shock} degenerates at P_2. Thus by (2.2.13), (ρ, r) satisfy the the following one-point Dirichlet condition at P_2:

$$\bar{c}(\rho, \rho_1) = r(\pi) \qquad \text{at } P_2. \tag{4.2.5}$$

In the following Sections, we will now derive some basic estimates for admissible solutions. We always assume that (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1. We aim to obtain some basic pointwise estimates for admissible solutions. Some of these will be uniform in the wedge angle $\theta_w \in (\theta_{sonic}, \frac{\pi}{2}]$; for others, we fix $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$, and the estimates will be uniform in $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$.

4.3 Monotonicity of ρ on Γ_{shock} and convexity of Ω

In this Section, we establish monotonicity of ρ on Γ_{shock}, as well well as convexity of the elliptic region Ω.

Proposition 4.3.1. *Let (ρ, r) be an admissible solution of Problem 2.4.1 in the sense of Definition 4.2.1. Then ρ is monotone on Γ_{shock}.*

Proof. This follows by the same argument as originally in [7], and later adapted in [13] to require only uniform $C^{0,\alpha}$ regularity. The same argument used for convex-cornered wedges (i.e. Shock Reflection-Diffraction down a step) holds here with no change; see [13][Appendix] for details of the proof. □

Next, we prove that ρ is increasing in the direction from P_1 to P_2, and that Ω is convex.

Proposition 4.3.2. *Let (ρ, r) be an admissible solution of Problem 2.4.1 in the sense of Definition 4.2.1. Then ρ is monotone increasing on Γ_{shock} from P_1 to P_2, and the elliptic region Ω is convex.*

Proof. We calculate the curvature of Γ_{shock}. Let $\rho_{\boldsymbol{\tau}}$ denote the tangential derivative of ρ, pointing from P_1 to P_2, i.e. the direction of increasing θ. We find, by explicit calculation, using the formula for r' in (2.2.11), that the curvature is equal to

$$r^2 + 2(r')^2 - rr'' = \frac{r^3}{\bar{c}^2\sqrt{r^2 - \bar{c}^2}}(\bar{c}^2)'\rho_{\boldsymbol{\tau}}, \tag{4.3.1}$$

so that either ρ is monotone decreasing, and the curve Γ_{shock} is convex, or ρ is monotone increasing, and Γ_{shock} is concave. As Γ_{shock} is vertical at P_2 by Definition 4.2.1 **(i)**, and is tangent to $\mathcal{S}_1$ at P_1 (and as $\theta_s \leq \theta_w \leq \frac{\pi}{2}$), we conclude that ρ is monotone increasing, and Γ_{shock} is concave, so Ω is convex.

We also note here that this argument implies that when $\theta_w = \frac{\pi}{2}$, the shock must be a vertical line, and ρ must be constant; thus equal to the normal reflection solution, due to the Dirichlet boundary condition on Γ_{sonic}. □

The convexity of Ω leads to the following Corollary:

Corollary 4.3.3. *Let $\boldsymbol{\nu}_w = (-\sin\theta_w, \cos\theta_w)$ denote the outer unit normal to Γ_{wedge}. Then for any admissible solution to Problem 2.4.1 in the sense of Definition 4.2.1, denoting by $\boldsymbol{\nu}_{sh}$ the inner unit normal to Γ_{shock},*

$$\boldsymbol{\nu}_{sh} \cdot \boldsymbol{\nu}_w \leq -\delta, \tag{4.3.2}$$

where δ depends only on the initial data.

Proof. At P_1,

$$\boldsymbol{\nu}_{sh} \cdot \boldsymbol{\nu}_w = \boldsymbol{\nu}_{\mathcal{S}_1} \cdot \boldsymbol{\nu}_w = \sin(\theta_s - \theta_w) < -\delta, \tag{4.3.3}$$

with δ depending only on the initial data, due to (3.2.10), and continuous dependence of θ_s on the initial data and wedge angle. At P_2, $\boldsymbol{\nu}_{sh} = (1, 0)$, so that

$$\boldsymbol{\nu}_{sh} \cdot \boldsymbol{\nu}_w = -\sin\theta_w < -\sin\theta_{sonic} < 0, \tag{4.3.4}$$

and hence (4.3.2) holds by convexity. □

4.4 Bounds for the elliptic region Ω and admissible solutions ρ within Ω

We first note that by the comparison principle, and the ellipticity (4.2.3) and the obliqueness (4.2.4),

Lemma 4.4.1. *For an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, ρ can only attain its global maximum and minimum on $\Gamma_{shock} \cup \Gamma_{sonic}$:*

$$\sup_{\overline{\Omega}} \rho = \sup_{\Gamma_{sonic} \cup \Gamma_{shock}} \rho, \qquad \inf_{\overline{\Omega}} \rho = \inf_{\Gamma_{sonic} \cup \Gamma_{shock}} \rho. \tag{4.4.1}$$

We next note the following properties of ρ and the region Ω:

Proposition 4.4.2. *For an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$:*

$$\Omega \subset B_{c_2}, \tag{4.4.2}$$

$$\rho_1 < \rho \leq C \qquad \text{in } \overline{\Omega}. \tag{4.4.3}$$

Proof. Firstly, (4.4.2) holds by the structure of the region described in Problem 2.4.1, as by (2.2.11), $r' \leq 0$, so $r \leq r(\theta_1) = c_2$ on Γ_{shock}.

We now turn to (4.4.3); by (4.4.1), we only need to bound ρ on Γ_{shock}. By ellipticity and obliqueness from Definition 4.2.1, and as Γ_{shock} lies outside the sonic circle for state (1) by (4.2.1),

$$\overline{c}^2 \leq r^2 \leq c_2^2,$$

and

$$c^2 \geq r^2 > c_1^2,$$

so (4.4.3) holds. □

We now note that by the explicit formula (2.2.11) for r', we can obtain a uniform Lipschitz bound for Γ_{shock};

Proposition 4.4.3. *There exists C depending only on the initial data such that*

$$\mathrm{Lip}(r) \leq C. \tag{4.4.4}$$

We now turn to uniform separations of the components of $\partial\Omega$. This uniform separation implies that when obtaining boundary estimates, we can consider each part of the boundary separately, and piece local estimates together to obtain a global one.

Proposition 4.4.4. *Let $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$. There exists $C > 0$ depending only on (ρ_0, ρ_1, γ), such that for any admissible solution with wedge angle $\theta_w \in (\theta_w^*, \frac{\pi}{2})$,*

$$\mathrm{dist}(\Gamma_{sonic}, \Gamma_{sym}) \geq \frac{1}{C}, \tag{4.4.5}$$

$$\mathrm{dist}(\Gamma_{shock}, \Gamma_{wedge}) \geq \frac{1}{C}. \tag{4.4.6}$$

Proof. Firstly, (4.4.5) follows by first noting that in the polar coordinates, $\Gamma_{sym} = \{\theta = \pi\}$, and $\Gamma_{sonic} = \{r = c_2 \mid \theta \in (\theta_w, \theta_1)\}$. Then, we note that by (3.4.2), $\theta_1 < \pi + \frac{1}{C}$ where C only depends on the initial data, by continuous dependence on parameters. (4.4.6) follows similarly, as $\Gamma_{shock} = \{(r, \theta) \mid r = r(\theta), \theta \in (\theta_1, \pi)\}$, $r > c_1$ by (4.2.1) and $\theta_1 > \theta_w + \frac{1}{C}$ by (3.4.2) and continuous dependence on parameters. □

4.5 A uniform positive lower bound for $\rho - \rho_1$

In this section, we derive a lower bound for $\rho - \rho_1$. We note that by the obliqueness $r^2 \geq \overline{c}^2$, implies a lower bound for $r - c_1$; hence this shows a lower bound for the hyperbolicity of the Nonlinear Wave System in the region where Shock Reflection-Diffraction configurations are equal to state (1). This estimate will be critical for deriving ellipticity and obliqueness estimates on Ω.

Proposition 4.5.1. *Let $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$. There exists τ, depending only on the initial data and θ_w^*, such that for any admissible solution (ρ, r) of Problem 2.4.1 in the sense of Definition 4.2.1 with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$,*

$$\rho > \rho_1 + \tau. \tag{4.5.1}$$

Proof. By Proposition 4.3.2 and Lemma 4.4.1 it suffices to prove $\rho > \rho_1 + \tau$ at P_2, with τ depending only on the initial data and θ_w^*. We use a similar argument to that used in [13][Chapter 4], which was used to show $\rho > \rho_1$ on the boundary for their approximate solutions, with a fixed wedge angle; the same argument can be used as the structure of their shock wave is similar to ours. Here, the difference is that we are varying the wedge angle, and taking a limit of exact solutions, so the contradiction obtained from the limiting function leads to uniform lower bound for $\rho - \rho_1$.

To outline the argument - we first suppose that the proposition is false; then there exists a sequence of admissible solutions (ρ_k, r_k) with wedge angles $\theta_w^{(k)}$ and points $\boldsymbol{\xi}_k \in \Gamma_{shock}^{(k)} := \{r = r_k(\theta) \mid \theta \in (\theta_1, \pi)\}$ such that $\rho_k(P_2) \to \rho_1$. By the same argument as the one used in [13][Chapter 4], using local obliqueness and ellipticity arising from the results of Section 3.6, we can show that r converges in $C^{1,\alpha}$ to some limiting r, and $\theta_w^{(k)}$ converges to some θ_w, and ρ_k converges to some ρ, that satisfy (2.1.3) and (2.2.11). The barrier functions from [13][Lemma 4.2,Lemma 4.5], whose construction relies on the convexity of ω, can then be used to derive a contradiction. □

Using the obliqueness (4.2.4), we can then conclude

Corollary 4.5.2. *Let $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$. There exists τ, depending only on the initial data, such that for any admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$,*

$$\begin{aligned} &\rho_1 + \tau < \rho \leq \rho_2 \qquad \text{in } \overline{\Omega}, \\ &r > c_1 + \tau \qquad \text{on } \Gamma_{shock}. \end{aligned} \tag{4.5.2}$$

4.6 Uniform obliqueness & ellipticity estimates

The global lower bound for $\rho - \rho_1$, combined with the ellipticity and obliqueness principles in §3.6 leads us to the following global ellipticity & obliqueness estimates for admissible solutions:

Proposition 4.6.1. *Let $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$. Then there exists $\lambda > 0$, $\delta > 0$, depending only on the initial data and θ_w^*, such that for any admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$,*

$$\begin{aligned} &c^2 - r^2 \geq \lambda \text{dist}(\boldsymbol{\xi}, \Gamma_{sonic}) \qquad \text{in } \overline{\Omega}, \\ &r^2 - \overline{c}^2 \geq \delta(\pi - \theta) \qquad \text{on } \overline{\Gamma_{shock}}. \end{aligned} \tag{4.6.1}$$

Proof. For the first inequality, we first note that at P_2, due to Corollary 4.5.2,

$$c^2 - r^2 = c^2 - \overline{c}^2 \geq \frac{1}{C}, \tag{4.6.2}$$

as $\overline{c}$ defined by (2.2.6) is continuous, and strictly monotone in both variables, and C depends only on τ and the initial data. As τ depends on θ_w^* and the initial data, C depends on θ_w^* and the initial data. Because $\boldsymbol{\xi} \mapsto \text{dist}(\boldsymbol{\xi}, \Gamma_{sonic})$ is merely Lipschitz, we must use a smooth approximation to be able to apply the ellipticity principle in §3.6. Due to the similarities in the structure of Ω, the construction in [26][Lemma 9.6.4] can be used, and thus the proof follows in the same way. To prove the second inequality, we may use Proposition 3.6.4, as at P_1,

$$r^2 - \overline{c}^2 = c_2^2 - \overline{c}_2^2 \geq \frac{1}{C},$$

where C only depends on the initial data, again as $\rho_2 > \rho_1$ and $\overline{c}$ is continuous and monotone in both variables. The proof then follows with δ depending on C and τ, and the uniform positive lower bound for $\pi - \theta_1(\theta_w)$; thus only on the data and θ_w^*. □

Chapter 5

Regularity away from the Sonic Arc

In this Chapter, we will obtain uniform estimates for admissible solutions away from the sonic arc. The main points of interest here will be the wedge vertex P_3, and the point P_2 where the obliqueness degenerates, as noted in Remark 4.2.2. We define

$$\Omega_\epsilon = \{\boldsymbol{\xi} \in \Omega \mid \text{dist}(\boldsymbol{\xi}, \Gamma_{sonic}) < \epsilon\}, \tag{5.0.1}$$

and find estimates in $\Omega \setminus \Omega_\epsilon$, for $\epsilon > 0$ small.

For this Chapter, we fix $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$, and obtain uniform estimates for admissible solutions with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$.

5.1 Regularity away from $P_2 \cup \Gamma_{sonic}$

We now look at the regularity of (ρ, r) away from $P_2 \cup \Gamma_{sonic}$. By Proposition 4.6.1, we find that for any $\epsilon, \delta > 0$, (2.1.4) is strictly elliptic in $\Omega \setminus \Omega_\epsilon$ with constant depending on ϵ. We also find, from (2.2.11), (4.5.2) and (4.6.1) that there exists C depending only on the data and θ_w^* such that

$$r'(\theta) \leq -\frac{1}{C}(\pi - \theta)^{1/2}, \text{ and } \beta \cdot \boldsymbol{\nu} \geq \frac{1}{C}(\pi - \theta)^{1/2}, \tag{5.1.1}$$

so that the boundary condition (2.2.11) is strictly oblique on $\Gamma_{shock} \setminus B_\delta(P_2)$ with obliqueness constant depending on δ. Also, by the formula for r', on $\Gamma_{shock} \setminus B_\delta(P_2)$ r' is a Lipschitz function of ρ and r, with Lipschitz constant depending on δ and the initial data. Thus, away from P_3, we can use Propositions B.0.2, B.0.4, and B.0.6 to $C^{1,\alpha}$ estimates, depending on the data and θ_w^*. We then use [56][Corollary 2.3] to obtain a $C^{0,\alpha}$ estimate near P_3, followed by [57][Proposition 4.7.8] combined with [77][Lemma 1] to obtain a $C^{1,\alpha}$ estimate.

We remark that the exponent α in the $C^{1,\alpha}$ estimate near P_3 does not depend just on the exponent in a Hölder estimate for A_{ij}, but the value of the $C^{0,\alpha}$; so we will only use the $C^{0,\alpha}$ estimate in our iteration problem.

Theorem 5.1.1. *For any $\epsilon, \delta > 0$, there exists $\alpha \in (0,1)$, $C > 0$ depending on δ, ϵ, the initial data and θ_w^* such that if (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$,*

$$\begin{aligned} &\|\rho\|_{1,\alpha,\Omega\setminus(\Omega_\epsilon \cup B_\delta(P_2))} \leq C, \\ &\|r\|_{2,\alpha,[\theta_1+\epsilon,\pi-\delta])} \leq C. \end{aligned} \tag{5.1.2}$$

5.2 Regularity near P_2

In this section, we now turn to regularity near P_2, where the obliqueness degenerates. We start by first estimating the degeneracy of the obliqueness at P_2.

Proposition 5.2.1. *There exists C depending only on the data such that on Γ_{shock}, such that*

$$\begin{aligned} r^2 - \bar{c}^2 &\leq C(\pi - \theta) \qquad \text{on } \Gamma_{shock}, \\ -C(\pi-\theta)^{1/2} \leq r'(\theta) &\leq 0 \qquad \text{for } \theta \in [\theta_1, \pi] \end{aligned} \tag{5.2.1}$$

Proof. As ρ is monotone on Γ_{shock}, and $\theta \mapsto r(\theta)$ is Lipschitz with bound depending only on the data given by (4.4.4),

$$r^2 - \bar{c}^2 \leq r^2(P_2) + C(\pi - \theta) - c^2(P_2) = C(\pi - \theta), \qquad \text{for all } (r(\theta), \theta) \in \Gamma_{shock},$$

due to the one-point Dirichlet condition (4.2.5) at P_2. The explicit formula in (2.2.11), combined with (8.2.6) and (8.2.7) then imply the second inequality. □

We now use the fact that the oblique boundary condition points towards Γ_{sym} to prove an algebraic growth estimate on Γ_{shock}.

Lemma 5.2.2. *Let (ρ, r) be an admissible solution. Then there exists $R > 0$, depending only on the data and θ_w^*, such that for any integer $m \geq 1$, and $M > 0$*

$$\rho - \rho(P_2) - (+)M(\pi-\theta)^m \qquad \text{does not attain a maximum (minimum) on } \Gamma_{shock} \cap B_R(P_2). \tag{5.2.2}$$

Proof. We find, using (4.6.1) and (5.2.1), taking advantage of the uniform separation of P_2 and Γ_{sonic} from (4.4.6), that by the explicit formula (2.2.11),

$$\beta_2 \geq \frac{1}{C} - CR \geq \frac{1}{C}, \tag{5.2.3}$$

for C depending only on the data and θ_w^*. Therefore, we find that

$$\beta \cdot \nabla(\rho - \rho(P_2) + (-)(\pi - \theta)^m = +(-)\beta_2 m(\pi - \theta)^{m-1} > (<)0, \tag{5.2.4}$$

so that by the same argument used to derive the obliqueness principle Proposition 3.6.4, we conclude (5.2.2). □

Therefore, using (4.4.3) and Lemma 5.2.2, noting that by property **(iii)** in Problem 2.4.1, $\partial B_R(P_2) \cap \Gamma_{shock}$ will consist of P_2 and one point from the interior of Γ_{shock},

Lemma 5.2.3. *Let (ρ, r) be an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$. Then for R from Lemma 5.2.2, for any $m \geq 1$, there exists C depending on R, θ_w^* and the data such that*

$$|\rho - \rho(P_2)| \leq C(\pi - \theta)^m \qquad \text{on } \Gamma_{shock} \cap B_R(P_2). \tag{5.2.5}$$

We can now deduce the following:

Proposition 5.2.4. *Let (ρ, r) be an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$. Then there exists $\alpha \in (0,1)$ depending on the initial data and θ_w^* such that for $\epsilon > 0$ and $\beta \in (0,1)$, there exists C depending on β, ϵ, the initial data and θ_w^* such that if (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, then*

$$\|\rho\|^{(-\beta),\{P_2\}}_{1,\alpha,\Omega\setminus\Omega_\epsilon} \leq C, \tag{5.2.6}$$

$$\|r\|^{(-1-\beta/2),\{\pi\}}_{2,\alpha,[\theta_1+\epsilon,\pi]} \leq C. \tag{5.2.7}$$

Proof. Firstly, we focus on (5.2.6). Firstly, by using Lemma 5.2.3 to obtain the algebraic growth bound on Γ_{shock}. We then use the structure of Ω, in particular the property $r' \leq 0$ to deduce that for any $a > 0$, the ball $B_a(\xi_{P_2} + a, 0)$ is an exterior ball to Ω at P_2, so we may use [43][Theorem 1.1.15] to deduce that for C depending only on M from Lemma 5.2.3, and the ellipticity & boundedness of the coefficients in $B_R(P_2)$ (so only on the data), that

$$|\rho - \rho(P_2)| \leq C|\boldsymbol{\xi} - \boldsymbol{\xi}_{P_2}|^{\frac{m}{m+1}} \leq C|\boldsymbol{\xi} - \boldsymbol{\xi}_{P_2}|^{\beta}, \qquad \text{in } B_R(P_2) \cap \Omega, \tag{5.2.8}$$

for R small depending on the data and θ_w^*, choosing m large depending only on β. We also find that for any $\sigma > 0$, possibly reducing R depending only on the data and σ, using (2.2.11), (4.4.6), (3.6.4) and (5.2.1) imply

$$|\beta_1| \leq \sigma, \qquad \beta_2 > \frac{1}{C}, \qquad \text{on } B_R(P_2) \cap \Gamma_{shock}, \tag{5.2.9}$$

for C depending only on the data, uniform in R small enough. Thus by a standard argument with rescaled functions, we can write the boundary condition in the form required for Proposition B.0.7 to deduce that there exists α, depending only on the ellipticity in $B_R(P_2)$, such that for any $\beta \in (0,1)$, there exists $C(\beta)$ depending only on β, θ_w^* and the data such that

$$\|\rho\|^{(-\beta),\{P_2\}}_{1,\alpha,B_R(P_2)\cap\Omega} \leq C(\beta). \tag{5.2.10}$$

and we can combine with (5.1.2) to obtain (5.2.6).

We now turn to (5.2.7). We use the explicit formula for r' from (2.2.10), alongside (5.2.1) and the estimate (5.2.6). Consider the function

$$u(r,\theta) = r^2 \frac{r^2 - \bar{c}^2}{\bar{c}^2}. \tag{5.2.11}$$

From the lower bound for ρ in (4.4.3), u is a smooth function of ρ and r, thus (5.2.6) and (5.2.10) imply that

$$\|u\|^{(-\beta),\{P_2\}}_{1,\alpha,\Omega\setminus(\Omega_\epsilon\cup B_{c_1/2}(P_3))} \leq C(\beta). \tag{5.2.12}$$

Also, by formula (2.2.11),

$$\|r\|_{1,\beta/2,[\theta_1+\epsilon,\pi]} \leq C(\beta). \tag{5.2.13}$$

Thus, providing we choose $\beta/2 > \alpha$ so that $1 + \beta/2 > 1 + \alpha$, we can deduce that

$$U(\theta) := u(r,\theta) \text{ satisfies } \|U\|^{(-\beta)}_{1,\alpha,[\theta_1+\epsilon,\pi]} \leq C(\beta). \tag{5.2.14}$$

Next, we note that $r'(\theta) = \sqrt{U(\theta)}$. The main point here in obtaining a weighted $C^{2,\alpha}$ bound is to estimate the difference quotient term, taking advantage of the lower bound for

U arising from the obliqueness. We note that from the obliqueness in (4.6.1) and the lower bound in (5.2.6), that for C depending only on the data and θ_w^*,

$$U(\theta) \geq \frac{1}{C}(\pi - \theta), \tag{5.2.15}$$

so we find firstly that

$$r''(\theta) = -\frac{U'(\theta)}{2\sqrt{U(\theta)}}, \tag{5.2.16}$$

so that

$$|r''(\theta)| \leq C(\pi - \theta)^{\beta - \frac{3}{2}} \|U\|^{(-\beta)}_{1,0,[\theta_1+\epsilon,\pi]}, \tag{5.2.17}$$

and for $\theta \leq \tilde{\theta}$, we find that

$$\begin{aligned} |r''(\theta) - r''(\tilde{\theta})| &\leq \left| \frac{U'(\theta) - U'(\tilde{\theta})}{2\sqrt{U(\theta)}} \right| + \left| U'(\theta) \left(\frac{1}{2\sqrt{U(\theta)}} - \frac{1}{2\sqrt{U(\tilde{\theta})}} \right) \right| \\ &\leq C(\pi - \theta)^{\beta - 1 - \alpha - \frac{1}{2}} \|U\|^{(-\beta)}_{1,\alpha,[\theta_1+\epsilon,\pi]} |\theta - \tilde{\theta}|^{\alpha} \\ &\qquad + C(\pi - \theta)^{\beta - 1 - \frac{3}{2} + \beta - \alpha} |\theta - \tilde{\theta}|^{\alpha} \|U\|^{(-\beta)}_{0,\alpha,[\theta_1+\epsilon,\pi]}. \end{aligned} \tag{5.2.18}$$

Because the second term is the one of larger magnitude, we conclude that

$$\|r\|^{(-(2\beta - \frac{1}{2}))}_{2,\alpha,[\theta_1+\epsilon,\pi]} \leq C(\beta), \tag{5.2.19}$$

so that as $\beta \in (0, 1)$ is arbitrary, we deduce (5.2.7). □

Chapter 6

Regularity of Admissible Solutions near the Sonic Arc

In this Chapter, we obtain regularity results near the sonic arc, valid for supersonic wedge angles. Similarly to the methods in [1, 26], we exploit the degenerate elliptic structure (4.6.1) to obtain estimates in the weighted-and-scaled $C^{(par)}_{1,\alpha}$ norm defined by (A.1.8) for

$$\psi := c_2^2 - c^2(\rho) \tag{6.0.1}$$

The estimates we obtain here will depend on the length of the sonic arc being strictly positive; i.e. they are not are not uniform in $\theta_w \in [\theta_{sonic}, \frac{\pi}{2}]$. We will remove this assumption in Part III, by using a different scaling.

Here, we will assume that $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$ and obtain estimates depending on θ_w^* and the data, valid for wedge angles $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$.

6.1 Equation in (x, y) coordinates near Γ_{sonic}

We will now use the expressions derived in §3.5 for our analysis near the sonic arc. Let (r, θ) be the polar coordinates for $\boldsymbol{\xi}$, and

$$(x, y) = (c_2 - r, \theta).$$

Then we recall that $\psi = c_2^2 - c^2$ satisfies an equation of the form

$$\sum_{i,j=1}^{2} A_{ij}(x, \psi)\psi_{ii} + \sum_{i=1}^{2} A_i(x, \psi)\psi_i + \sum_{i,j=1}^{2} B_{ij}(x, \psi)\psi_i\psi_j = 0 \qquad \text{on } \Omega_\epsilon, \tag{6.1.1}$$

with expressions and estimates given by (3.5.4) and (3.5.5). We also see, from the definition of (x, y), that the slip boundary condition (2.2.8) becomes

$$\partial_y \psi = 0 \qquad \text{on } \Gamma_{wedge} = \{y = \theta_w\}. \tag{6.1.2}$$

Also, noting that by (4.5.2), the assumption (3.5.7) holds with M depending on the data and θ_w^*, we can write

$$\begin{aligned} \Gamma_{shock} \cap \partial\Omega_\epsilon &:= \{y = f(x)\}, \\ \Omega_\epsilon &= \{0 < x < \epsilon, \theta_w < y < f(x)\}, \end{aligned} \tag{6.1.3}$$

with expression for $f'(x)$ given by (3.5.11), and the Rankine-Hugoniot condition on Γ_{shock} can be written as

$$\hat{\beta} \cdot \nabla\psi = \hat{\beta}_1(x,\psi)\psi_x + \hat{\beta}_2(x,\psi)\psi_y = 0 \qquad \text{on } \partial\Omega_\epsilon \cap \Gamma_{shock}, \tag{6.1.4}$$

with expressions given by (3.5.14).

The following Lemma, relating smallness of ψ to ellipticity of (6.1.1) follows directly from the expressions in Section 3.5.

Lemma 6.1.1. *For any $0 < \delta < c_2$, $M > 0$, there exists $\epsilon > 0$ such that if $0 < x < \epsilon$ and $|z| \le (2c_2 - \delta)x$, then*

$$\frac{\delta}{2}|\zeta|^2 \le \frac{\hat{A}_{11}(x,z)}{x}\zeta_1^2 + \hat{A}_{22}(x,z)\zeta_2^2 \le \frac{2}{\delta}|\zeta|^2 \qquad \text{for } \zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^2. \tag{6.1.5}$$

6.2 Pointwise estimates of ρ, and structure of Γ_{shock} near Γ_{sonic}

In order to obtain the estimates near Γ_{sonic}, we now analyse the structure of Ω near Γ_{sonic}, as well as attain pointwise estimates for ψ.

Definition 6.2.1. We define the following points:

- $\overline{P}_1$ the second point of intersection of $\mathcal{S}_1$ with ∂B_{c_2}, which exists by the computation where P_1 was found in (3.4.2). Then $|P_0\overline{P}_1| > |P_0P_1|$.
- Q the midpoint of the arc $P_1\overline{P}_1$
- Q' the midpoint of the line segment $P_1\overline{P}_1$.

From how we defined $P_1, \overline{P}_1$ and Q,

$$\theta_{\overline{P}_1} = \pi + 2\theta_s - \theta_{P_1}$$

and

$$r_Q = c_2, \quad \theta_Q = \theta_s + \frac{\pi}{2} = \frac{\theta_{P_1} + \theta_{\overline{P}_1}}{2}.$$

We also note that using (3.4.4), the straight shock $\mathcal{S}_1$ between states (1) and (2) may be written as

$$y = \hat{f}_0(x) = \theta_s + \sin^{-1}\left(\frac{\overline{c}_2}{c_2 - x}\right) \qquad \text{for } x \in (0, c_2), \tag{6.2.1}$$

which satisfies

$$\hat{f}_0'(x) = \frac{\overline{c}_2}{(c_2 - x)\sqrt{(c_2 - x)^2 - \overline{c}_2^2}}. \tag{6.2.2}$$

We use these expressions to make the following estimates for the straight shock $\mathcal{S}_1$:

Proposition 6.2.2. *There exists $\epsilon_0 > 0$, $\omega > 0$ and $M \ge 1$ such that*

$$2\omega < \hat{f}_0'(x) \le M \text{ for } x \in (-\epsilon_0, \epsilon_0), \tag{6.2.3}$$

and

$$\|\hat{f}_0'\|_{C^3([-\epsilon_0,\epsilon_0])} \le M. \tag{6.2.4}$$

Proof. This follows directly by the explicit expression (6.2.1). More precisely, ϵ_0 can be taken as $c_2 - \bar{c}_2$, which by (3.2.3) can be bounded below depending only on the data. □

We now turn to a Lipschitz bound for ψ up to Γ_{sonic}, which follows from the degenerate ellipticity:

Lemma 6.2.3. *There exists $\epsilon_1 > 0$ and $\delta > 0$ depending only on $(\rho_0, \rho_1, \gamma, \theta_w^*)$, such that if (ρ, r) is an admissible solution with $\theta_w \in [\theta_w^*, \frac{\pi}{2})$, then*

$$0 \leq \psi \leq (2c_2 - \delta)x \qquad \text{for } x \in (0, \epsilon_1) \tag{6.2.5}$$

Proof. $0 \leq \psi$ follows as $\rho \leq \rho_2$ by (4.5.2). The other side of the inequality follows by (4.6.1); because $\operatorname{dist}(\boldsymbol{\xi}, \Gamma_{sonic}) \geq x$,

$$\lambda x \leq c^2 - (c_2 - x)^2 = -\psi + 2c_2 x - x^2, \tag{6.2.6}$$

and taking $\epsilon_1 < \lambda/2$, $\delta = \lambda/2$. □

This implies that (6.1.3) and (6.1.4) hold, and we can use the expressions from §3.5 to make the following structure estimates:

Proposition 6.2.4. *Let ϵ_0, ϵ_1 be the constants defined in Proposition 6.2.2 and Lemma 6.2.3. Then there exists $C > 0$, such that for any admissible solution with $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, and any $\epsilon \in (0, \epsilon_1)$,*

(i) *Ω_ϵ can be written in the form*

$$\Omega_\epsilon = \{0 < x < \epsilon, \theta_w < y < f(x)\}, \tag{6.2.7}$$

where $f'(x)$ is given by the formula (3.5.11)*, and*

$$\partial\Omega_\epsilon \cap \Gamma_{shock} = \{0 < x < \epsilon, y = f(x)\} \tag{6.2.8}$$

(ii) *The boundary condition on $\Gamma_{shock} \cap \{0 < x < \epsilon\}$ can be written in the form*

$$\hat{\beta} \cdot \nabla\psi = \hat{\beta}_1 \psi_x + \hat{\beta}_2 \psi_y = 0, \tag{6.2.9}$$

with the expressions as in (3.5.14)*.*

(iii) *For the ω and M from Proposition 6.2.2,*

$$\omega \leq f'(x) \leq M, \tag{6.2.10}$$

and

$$\frac{|f'(x) - \hat{f}_0'(x)|}{x} + \frac{|f(x) - \hat{f}_0(x)|}{x^2} \leq C. \tag{6.2.11}$$

Moreover,

$$f'(x) = h(x, \psi) \qquad \text{for } x \in (0, \epsilon) \tag{6.2.12}$$

where $h \in C^\infty(\mathbb{R} \times \mathbb{R})$, and $\partial_x^{(k-1)} h(x, 0) = \hat{f}_0^{(k)}(x)$ for all $k \geq 1$.

(iv) *The coefficients $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)$ in (6.2.9) satisfy*

$$\hat{\beta}_1 \le -\frac{1}{C}, \qquad \hat{\beta}_2 \le -\frac{1}{C}, \tag{6.2.13}$$

and

$$\hat{\beta} \cdot \nu \ge \frac{1}{C}, \tag{6.2.14}$$

where $\boldsymbol{\nu}$ is the inner unit normal to Ω, expressed in (x, y) coordinates.

Proof. All the estimates above directly follow from (6.2.5) and the expressions and estimates in §3.5. In particular, (3.5.1) and (3.5.7) hold with M depending on the data and θ_w^*, due to (4.5.2). □

6.3 $C^{1,\alpha}_{(par)}$ estimates in the scaled Hölder norms near Γ_{sonic}

In this Section, we will make a regularity estimate of $\psi = c_2^2 - c^2$ and the shock curve $\{y = f(x)\}$ near Γ_{sonic}. For this, we will use the weighted norms introduced in the Appendix A.1.2, using the following simplified notations:

$$\|u\|^{(par)}_{m,\alpha,\mathcal{D}} := \|u\|^{1,(par)}_{m,\alpha,\mathcal{D}}, \qquad C^{m,\alpha}_{(par)}(\mathcal{D}) := C^{m,\alpha}_{1,(par)}(\mathcal{D}). \tag{6.3.1}$$

Furthermore, for a function $v \in C^2((0,\epsilon))$, the norm $\|v\|^{(par)}_{2,\alpha,(0,\epsilon)} \equiv \|v\|^{2,(par)}_{2,\alpha,(0,\epsilon)}$ is defined with only the x variable as follows;

$$\begin{aligned} \|v\|_{2,0,(0,\epsilon)} &= \sum_{i=1}^{2} \sup_{x\in(0,\epsilon)} x^{2-i}|v^{(i)}(x)|, \\ [v]_{2,\alpha,(0,\epsilon)} &= \sup_{x,y\in(0,\epsilon),x\neq y} \left(\min\{x^\alpha, y^\alpha\} \frac{|v''(x) - v''(y)|}{|x-y|^\alpha} \right), \end{aligned} \tag{6.3.2}$$

We also recall that the function $\hat{f}_0(x)$ from (6.2.1), where $\mathcal{S}_1 = \{y = \hat{f}_0(x)\}$ near Γ_{sonic}. Also, for any $\theta_w \in (\theta_{sonic}, \frac{\pi}{2}]$,

$$\Gamma^{(\theta_w)}_{sonic} = \{(0, y) \mid y \in (\theta_w, \theta_1)\}, \tag{6.3.3}$$

with $\theta_1 - \theta_w > 0$, and also $\theta_1 - \theta_w \to 0$ as $\theta_w \downarrow \theta_{sonic}$.

Proposition 6.3.1. *There exists $\sigma, C \in (0, \frac{\epsilon_0}{2})$ and $\alpha \in (0,1)$, depending on the initial data and θ_w^* such that the following holds: Let $l_{so} > 0$. Define*

$$\epsilon = \min\{\sigma, l_{so}^2\}. \tag{6.3.4}$$

If $\theta_w \in [\theta_{sonic}, \frac{\pi}{2})$ satisfies

$$\theta_1 - \theta_w \ge l_{so}, \tag{6.3.5}$$

and (ρ, r) is an admissible solution with the wedge angle θ_w, then $\psi = c_2^2 - c^2$ satisfies

$$\|\psi\|^{(par)}_{1,\alpha,\Omega_\epsilon} \le C. \tag{6.3.6}$$

Proof. We follow a similar argument to [26][Proposition 11.4.3], except the setting is that we now have L^∞ estimates and attain $C^{1,\alpha}$ estimates, rather than $C^{0,1}$ estimates and attaining $C^{2,\alpha}$ estimates.

Step 1 - Defining small rectangles, contained in $\Omega_{2\epsilon}$ We choose $\sigma \in (0, \frac{\epsilon_0}{2})$ small enough that results from §6.2 quoted below hold in $\Omega_{2\sigma}$; so that σ depends only on the data and θ_w^*. We possibly reduce it further below, depending only on (ρ_0, ρ_1, γ) again. For $z_0 := (x_0, y_0) \in \Omega_\epsilon$ and $r \in (0,1)$, we define

$$\begin{aligned} \tilde{R}_{z_0,r} &:= \Big((s,t) \mid |s-x_0| < \frac{r}{4}x_0,\, |t-y| < \frac{r}{4}\sqrt{x_0}\Big), \\ R_{z_0,r} &:= \tilde{R}_{z_0,r} \cap \Omega_{2\epsilon}. \end{aligned} \tag{6.3.7}$$

First we observe that

$$R_{z_0,r} \subset \Omega \cap \{(s,t) \mid \frac{3}{4}x_0 \le s \le \frac{5}{4}x_0\} \subset \Omega_{2\epsilon}, \tag{6.3.8}$$

so in particular,

$$R_{z_0,r} \cap \Gamma_{sonic} = \emptyset, \qquad \text{for all } z_0 \in \Omega_\epsilon. \tag{6.3.9}$$

Next, we observe that if σ is small, depending only on the data, then for any $z \in \Omega_\epsilon$, at least one of the following three cases holds:

(i) $z \in R_{\hat{z},1/20}$ for $R_{\hat{z},1/20} = \tilde{R}_{\hat{z},1/10}$,
(ii) $z \in R_{z_w,1/4}$ for $z_w = (x,0) \in \Gamma_{wedge} \cap \{0 < x < \epsilon\}$,
(iii) $z \in R_{z_s,1/4}$ for $z_s = (x, \hat{f}(x)) \in \Gamma_{shock} \cap \{0 < x < \epsilon\}$,

Thus, we only need to make estimates of ψ on the following rectangles:

(i) $R_{z_0,1/10}$ for $z_0 \in \Omega_\epsilon$ with $\tilde{R}_{z_0,1/10} \subset \Omega_{2\epsilon}$,
(ii) $R_{z_0,1}$ for $z_0 \in \Gamma_{wedge} \cap \{0 < x < \epsilon\}$,
(iii) $R_{z_0,1}$ for $z_0 \in \Gamma_{shock} \cap \{0 < x < \epsilon\}$.

Next, by (6.2.10), we see

$$\hat{f}(x) \ge \hat{f}(0) = y_{P_1} \ge l_{so} + \theta_w \qquad \text{for all } x \in (0, \epsilon),$$

so then due to the assumption $\epsilon \le l_{so}^2$, we see

$$\begin{aligned} R_{z_w,1} \cap \Gamma_{shock} &= \emptyset \qquad \text{for all } z_w \in \Gamma_{wedge} \cap \{0 < x < \epsilon\}, \\ R_{z_s,1} \cap \Gamma_{wedge} &= \emptyset \qquad \text{for all } z_w \in \Gamma_{shock} \cap \{0 < x < \epsilon\}. \end{aligned} \tag{6.3.10}$$

We now highlight the only place in the proof where the assumption $\epsilon \le l_{so}^2$ is used is (6.3.10). However this is a crucial restriction, which is why we cannot obtain uniform estimates in Ω_ϵ up to the sonic angle using the method in this proof. We now denote

$$Q_r := (-r, r)^2,$$

and for fixed $z_0 = (x_0, y_0) \in \Omega_\epsilon$, introduce the variables (S,T) by the invertible transformation

$$(x,y) = \Big(x_0 + \frac{x_0}{4}S, y_0 + \frac{\sqrt{x_0}}{4}T\Big), \tag{6.3.11}$$

so then for a subset $Q_r^{(z_0)} \subset Q_r$,

$$R_{z_0,r} = \Big\{(x,y) \mid (S,T) \in Q_r^{(z_0)}\Big\}. \tag{6.3.12}$$

Step 2 - Case (i) and a Quasilinear PDE for ψ in (S,T) coordinates We first consider Case **(i)** from Step 1. In this case,

$$Q_r^{(z_0)} = Q_r \qquad \text{for all } r \in (0, \frac{1}{10}]. \tag{6.3.13}$$

We now rescale ψ on $R_{z_0,1/10}$ by defining

$$\psi^{(z_0)}(S,T) := \frac{1}{x_0}\psi(x_0 + \frac{x_0}{4}S, y_0 + \frac{\sqrt{x_0}}{4}T) \qquad \text{for } (S,T) \in Q_{1/10}. \tag{6.3.14}$$

Then, the estimates (6.2.5) imply that in $Q_{1/10}$,

$$|\psi^{(z_0)}| \le L. \tag{6.3.15}$$

Changing to (S,T) coordinates in (3.5.4), we find $\psi^{(z_0)}$ satisfies

$$\begin{aligned} &A_{11}^{(z_0)}(\psi^{(z_0)}, S)\psi_{SS}^{(z_0)} + A_{22}^{(z_0)}(\psi^{(z_0)}, S)\psi_{TT}^{(z_0)} \\ &\quad + A_1^{(z_0)}(\psi_S^{(z_0)}, \psi^{(z_0)}, S)\psi_S^{(z_0)} + B_{11}^{(z_0)}(\psi_S^{(z_0)})^2 + B_{22}^{(z_0}(\psi_T^{(z_0)})^2 = 0, \end{aligned} \tag{6.3.16}$$

for $(\hat{A}_{ij}, \hat{A}_i, \hat{B}_{ij})$ from (6.1.1), where

$$\begin{aligned} A_{11}^{(z_0)}(z,S) &= x_0^{-1}\hat{A}_{11}(x_0 z, x_0(1+\frac{S}{4})), \\ A_{22}^{(z_0)}(z,S) &= \hat{A}_{22}(x_0 z, x_0(1+\frac{S}{4})), \\ A_1^{(z_0)}(z,S) &:= \frac{1}{4}\hat{A}_1(x_0 z, x_0(1+\frac{S}{4})), \\ B_{11}^{(z_0)}(z,S) &:= \hat{B}_{11}(x_0 z, x_0(1+\frac{S}{4})), \\ B_{22}^{(z_0)}(z,S) &:= x_0\hat{B}_{22}(x_0 z, x_0(1+\frac{S}{4})), \end{aligned} \tag{6.3.17}$$

from (6.2.5) and (6.1.5), we have

$$\frac{\lambda}{4}|\boldsymbol{\kappa}|^2 \le A_{11}^{(z_0)}(\psi^{(z_0)}, S)\kappa_{11} + A_{22}^{(z_0)}(\psi^{(z_0)}, S)\kappa_{22} \le \frac{4}{\lambda}|\boldsymbol{\kappa}|^2, \tag{6.3.18}$$

for $\boldsymbol{\kappa} \in \mathbb{R}^2$, $(S,T) \in Q_{1/10}^{(z_0)}$, where λ depends only on the data and θ_w^*. We also find, by first obtaining estimates for the original equation using the explicit expressions and(3.5.6), and then using the expressions in (6.3.17), that

$$\begin{aligned} \|(A_{ij}^{(z_0)}, A_i^{(z_0)}, B_{ij}^{(z_0)}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}\times Q_{1/10}^{(z_0)})} &\le C, \\ \|D_{(z,S,T)}(A_{ij}^{(z_0)}, A_i^{(z_0)}), B_{ij}^{(z_0)}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}\times Q_{1/10}^{(z_0)})} &\le C, \end{aligned} \tag{6.3.19}$$

with C only depending on the initial data. Thus the conditions of Proposition B.0.2 apply, and for each $\alpha \in (0,1)$,

$$\|\psi^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/20}})} \le C(\alpha). \tag{6.3.20}$$

We remark that we only require $C^{0,\alpha}$ regularity with respect to the (S,T) variables in the coefficients to attain $C^{1,\alpha}$ regularity, provided that α is small enough depending on the ellipticity of the rescaled equation. This will be used when applying the same argument to the iteration set equation.

Step 3 - Case (ii) and the slip boundary condition in (S,T) coordinates Let $z_0 = (x_0, 0) \in \Gamma_{wedge} \cap \{0 < x < \epsilon\}$. Then, by (6.3.9) and (6.3.10), and assuming ϵ is sufficiently small depending on the data,

$$Q_r^{(z_0)} = \{(S,T) \in Q_r \mid T > 0\} \qquad \text{for all } r \in (0,1]. \tag{6.3.21}$$

We now define $\psi^{(z_0)}(S,T)$ by (6.3.14) for $S_T \in Q_1^{(z_0)} \cap \{T > 0\}$. $\psi^{(z_0)}$ satisfies the same PDE (6.3.16) and estimates (6.3.15) in the new domain. Moreover, as $\partial_{\boldsymbol{\nu}}\psi = 0$ on Γ_{wedge},

$$\partial_T \psi^{(z_0)} = 0 \qquad \text{on } \{T = 0\} \cap Q_1.$$

So that by the same argument as Step 2, and reflecting in $\{T = 0\}$, we obtain

$$\|\psi^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/2}^{(z_0)}})} \leq C(\alpha). \tag{6.3.22}$$

Step 4 - Case (iii) and the Rankine-Hugoniot condition in (S,T) coordinates Using Lemma 6.2.4, in particular (6.2.7), we obtain that for each $r \in (0,1]$,

$$Q_r^{(z_0)} = \Big\{(S,T) \in Q_r \mid T < F^{(z_0)}(S)\Big\}, \tag{6.3.23}$$

where $F^{(z_0)}(S) = \frac{4}{\sqrt{x_0}}\Big(\hat{f}(x_0 + \frac{x_0}{4}S) - \hat{f}(x_0)\Big)$. Then by (6.2.11),

$$\|F^{(z_0)}\|_{C^{0,1}([-r,r])} \leq C\sqrt{x_0} \leq C\sqrt{\sigma}. \tag{6.3.24}$$

Now, by using (6.2.13), we can write

$$\partial_S \psi^{(z_0)} = x_0^{1/2} b^{(z_0)}(\psi^{(z_0)}, S, T)\partial_T \psi^{(z_0)} \qquad \text{on } \Gamma_{shock}^{(z_0)}, \tag{6.3.25}$$

where for $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)(x,\psi)$ from (3.5.14),

$$b^{(z_0)}(z,(S,T)) = \frac{\hat{\beta}_1}{\hat{\beta}_2}\Big(x_0 z, x_0(1 + \frac{S}{4})\Big). \tag{6.3.26}$$

Then, from (6.2.13), it follows that

$$\|b^{(z_0)}(z,S,T)\|_{C^1(\mathbb{R}\times(-1,1))} \leq C. \tag{6.3.27}$$

Thus for σ sufficiently small, we can apply Proposition B.0.7 to obtain

$$\|\psi^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/2}^{(z_0)}})} \leq C, \tag{6.3.28}$$

with $\alpha \in (0,1)$ and C depending only on the data and θ_w^*. We again remark that we only require $C^{0,\alpha}$ regularity with respect to the (S,T) variables in the coefficients to attain $C^{1,\alpha}$ regularity, provided that α is small enough depending on the ellipticity of the rescaled equation. This will be used when applying the same argument to the iteration set equation.

Step 5 - Concluding the result for ψ Combining (6.3.20), (6.3.22) and (6.3.28), we obtain

$$\|\psi^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/100}^{(z_0)}})} \leq C \qquad \text{for all } \overline{\Omega}_\epsilon \setminus \Gamma_{sonic}, \tag{6.3.29}$$

which implies (6.3.6).

□

We may now deduce the following global estimates by combining with with Proposition 5.2.4;

Corollary 6.3.2. *Let $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$, ϵ_0 as in Proposition 6.2.2. Then there exists $\alpha \in (0,1)$, depending on the data and θ_w^*, such that for each $\beta \in (0,1)$ and $\epsilon \in (0, \epsilon_0]$, there exists C depending on the initial data, ϵ, θ_w^* and β such that if (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, then*

$$\begin{aligned} &\psi \in C^{0,1}(\overline{\Omega} \setminus \{P_2\}), \\ &\|\psi\|^{(-\beta),\{P_2\}}_{1,\alpha,\Omega\setminus\Omega_{\epsilon/10}} + \|\psi\|^{(par)}_{1,\alpha,\Omega_\epsilon} \leq C, \\ &\|r\|^{(-1-\beta/2),\{\theta_1,\pi\}}_{2,\alpha,[\theta_1,\pi]} + \|f - \hat{f}_0^{(\theta_w)}\|^{(par)}_{2,\alpha,(0,\epsilon)} \leq C. \end{aligned} \tag{6.3.30}$$

Proof. The only thing to left to prove is the $C^{(par)}_{2,\alpha}$ estimate for $f - \hat{f}_0$. For this, we use (6.2.12) and the expressions (A.1.8) and (6.3.2). From (A.1.8), we find that for $M = \max\{\|\psi\|^{(par)}_{1,\alpha,\Omega_\epsilon}, Lip(f)\}$, and $\mathbf{x} = (x, f(x)), \tilde{\mathbf{x}} = (\tilde{x}, f(x)) \in \Gamma_{shock} \cap \partial\Omega_\epsilon$,

$$|\mathbf{x} - \tilde{\mathbf{x}}| \leq M|x - \tilde{x}|, \tag{6.3.31}$$

so that

$$\begin{aligned} &|\psi(\mathbf{x})| \leq Mx, \\ &|\partial_x\psi(\mathbf{x})| \leq M, \qquad |\partial_y\psi(\mathbf{x})| \leq Mx^{1/2}, \\ &|\partial_x\psi(\mathbf{x}) - \partial_x\psi(\tilde{\mathbf{x}})| \leq Mx^{-\alpha}|x - \tilde{x}|^\alpha, \\ &|\partial_y\psi(\mathbf{x}) - \partial_y\psi(\tilde{\mathbf{x}})| \leq Mx^{\frac{1}{2}-\alpha}|x - \tilde{x}|^\alpha. \end{aligned} \tag{6.3.32}$$

Then, using (6.2.12), we find that

$$\begin{aligned} f'(x) - f_0'(x) &= h(x,\psi) - h(x,0), \\ f''(x) = f_0''(x) &= \partial_x h(x,\psi) - \partial_x h(x,0) + \partial_z(\psi_x + f'(x)\psi_y), \end{aligned} \tag{6.3.33}$$

so that using (6.3.32), and the smooth dependence of h on (x,ψ) implies the result. □

Chapter 7

The Iteration Procedure and Existence of Solutions

We now carry out the iteration procedure, where we argue similarly to [1, 26] which showed existence for related problems, for the Potential Flow system. The first key difference here in the construction of the mapping between functions on Ω and function on the iteration region is that the iteration is now done in terms of two functions, the density ρ, and the boundary r. Another crucial change is that we must now deal with the degenerate obliqueness at the point P_2. As obliqueness is a property that depends both on the shape of the boundary and the values of the coefficients, this involves using a new technique, imposing a cutoff when deriving boundaries r from functions v on the unit interval.

7.1 Mapping to the iteration region

We now describe the mapping to the iteration region. Firstly when constructing our iteration region, we note that by convexity, Ω is always contained in the region bounded by $\mathcal{S}_1$, Γ_{shock}, Γ_{sym} and Γ_{wedge}; so in principle, we only need to map from this region. However, it is more convenient to expand this region slightly, so that Γ_{shock} is strictly within the interior of the region our mapping is defined on.

Fix a wedge angle $\theta_w \in (\theta_s, \frac{\pi}{2}]$. We replace the straight shock $\mathcal{S}_1 = \{\boldsymbol{\xi} \cdot \boldsymbol{\nu}_{S_1} = -\overline{c}_2\}$ from (3.4.4) by the line

$$\mathcal{S}_{1,\delta_*} := \{\boldsymbol{\xi} \cdot \boldsymbol{\nu}_{\mathcal{S}_1} = -\overline{c}_2 - \delta\}, \tag{7.1.1}$$

where $\delta_* > 0$ will be defined below. Let Q be the point defined in Definition 6.2.1. Due to continuous dependence on the wedge angle, there exists $\delta_* > 0$ such that

$$\mathcal{S}_{1,2\delta^*} \cap \Lambda \cap \mathcal{N}_{\epsilon_1}(\Gamma_{sonic}) \cap \{0 < x < \epsilon\} \subset \{\theta_w < \theta < \theta_Q - \delta\}, \tag{7.1.2}$$

where we have used the polar coordinates from Chapter 6. We now define $\hat{P}_1$ to be the closest point of intersection of $\mathcal{S}_{1,\delta_*}$ with the sonic circle for state (2), (which is in Λ due to (7.1.2)), and $\Gamma_{sonic}^{(\delta_*)}$ to be arc $P_4\hat{P}_1$.

We now note the region contained by S_{1,δ_*}, $\Gamma_{sonic}^{(\delta_*)}$, Γ_{wedge} and Γ_{sym} may be unbounded. To account for this, we restrict to $\{\xi > -2M\}$, where $M = \sup_{\theta_w \in (\theta_{sonic}, \frac{\pi}{2})} c_2(\theta_w)$, noting that for admissible solutions, $\Omega \subset B_{c_2}$.

Definition 7.1.1. Denote by $\mathcal{Q} \equiv \mathcal{Q}^{(\theta_w)}$ the subset of Λ bounded by $\mathcal{S}_{1,\delta^*}$, $\Gamma_{sonic}^{(\delta_*)}$, Γ_{wedge}, Γ_{sym}, and $\{\xi = -2M\}$.

7.1. Mapping to the iteration region

We now use (x, y) coordinates near $\Gamma_{sonic}^{(\delta_*)}$, and constant ϵ_1 from Proposition 6.2.4. For $\epsilon \in (0, \epsilon_1]$, denote

$$\mathcal{D}_{\delta_*,\epsilon} := \mathcal{Q} \cap \mathcal{N}_{\epsilon_1}(\Gamma_{sonic}^{(\delta_*)}) \cap \{0 < x < \epsilon\}. \tag{7.1.3}$$

Similarly to (6.2.1), for small enough δ_* there exists $f_{\mathcal{S}_{1,\delta_*}} \in C^\infty([-\epsilon_1, \epsilon_1])$ such that

$$\begin{aligned} \mathcal{D}_{\delta_*,\epsilon_1} &= \{(x, y) \mid x \in (0, \epsilon_1), \theta_w < y < f_{\mathcal{S}_{1,\delta_*}}(x)\}, \\ S_{1,\delta_*} \cap \mathcal{N}_{\epsilon_1}(\Gamma_{sonic}^{(\delta_*)} \cap \{0 < x < \epsilon_1\} &= \{(x, y) \mid x \in (0, \epsilon_1), y = f_{\mathcal{S}_{1,\delta_*}}(x)\}. \end{aligned} \tag{7.1.4}$$

We now use the construction from [26][Lemma 12.2.2] to construct our coordinate map. Though the form of the estimates we have on the shape Γ_{shock} are somewhat different, we find that Ω has suitably similar structure that we can use the same mapping $F_1 \equiv F_1^{(\theta_w)}$ to create our mapping to the iteration region.

Lemma 7.1.2. *There exists $\delta > 0$ and $C > 0$ depending only on the data, a one-to-one map*

$$F_1 \equiv F_1^{(\theta_w)} : \overline{\mathcal{Q}} \to [0, c_2(\theta_w)] \times [0, \infty) \qquad \text{for each } \theta_w \in [\theta_{sonic}, \frac{\pi}{2}],$$

and a smooth function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that, reducing ϵ_0 from Proposition 6.2.2 depending on the data, the following holds for $(F_1, \mathcal{Q})$:

(i) *$F_1(\mathcal{Q})$ is open, $F_1(\partial\mathcal{Q}) = \partial F_1(\mathcal{Q})$, and there exists a function $\eta \equiv \eta^{(\theta_w)} : [0, \hat{s}(\theta_w)] \to \mathbb{R}^+$ such that*

$$F_1(\mathcal{Q}) = \{(s, t) \mid s \in (0, c_2), t \in (0, \eta(s))\}, \tag{7.1.5}$$

and the rescaled functions $s \mapsto \eta^{(\theta_w)}\Big(\frac{s}{c_2}\Big)$ depend continuously on $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$ in the $C([0, 1])$ norm.

(ii) *$\|F_1\|_{C^3(\mathcal{Q})} + \|F_1^{-1}\|_{C^3(\overline{F_1(\mathcal{Q})})} \le C$, and $|\det(F_1^{-1})| \ge \frac{1}{C}$ in $F_1(\mathcal{Q})$, for all $\theta_w \in [\theta_{sonic}, \frac{\pi}{2}]$*

(iii) *F_1 depends smoothly on θ_w. More precisely, $\mathcal{Q}^\cup := \cup_{\theta_w \in (\theta_{sonic}, \frac{\pi}{2})} \mathcal{Q}^{(\theta_w)} \times \{\theta_w\}$ is open and bounded in $\mathbb{R}^2 \times \mathbb{R}$, and the map $(\boldsymbol{\xi}, \theta_w) \mapsto F_1^{(\theta_w)}(\boldsymbol{\xi})$ is in $C^\infty(\overline{\mathcal{Q}^\cup})$. Also, $F_1(\mathcal{Q}^\cup) := \cup_{\theta_w \in (\theta_{sonic}, \frac{\pi}{2})} F_1^{(\theta_w)}(\mathcal{Q}^{(\theta_w)}) \times \{\theta_w\}$ is open and bounded in $\mathbb{R}^2 \times \mathbb{R}$, and the map $((s, t), \theta_w) \mapsto (F_1^{(\theta_w)})^{-1}(s, t)$ is in $C^\infty(\overline{F_1(\mathcal{Q}^\cup)})$.*

(iv) *$F_1(P) = (x, y - \theta_w)$ for all $P \in \mathcal{D}_{\delta_*,\epsilon_0/10}$, where we have used the notation (7.1.3). In particular,*

$$\begin{aligned} F_1(P_1) = (0, \theta_1 - \theta_w), \quad F_1(P_4) = (0, 0), \\ F_1(\Gamma_{sonic}) = \{s = 0, t \in (0, \theta_1 - \theta_w)\}. \end{aligned} \tag{7.1.6}$$

(v) *$F_1(P_3) = (c_2(\theta_w), 0)$. Moreover, for any $P = (\xi, 0) \in \partial\mathcal{Q} \cap \{\eta = 0\}$ with $\xi \le 0$, $F_1(P) = (c_2, g^{(\theta_w)}(\xi))$, where $g \in C^3(\mathbb{R} \times \mathbb{R})$, $g^{(\theta_w)}(0) = 0$, and $g^{(\theta_w)}) \le -\delta$ on $\mathbb{R}$ for each θ_w.*

(vi) *$F_1(\Gamma_{wedge}) = \{(s, 0) \mid s \in (0, \hat{s}(\theta_w))\}$ and $\mathbf{e}_t \cdot \boldsymbol{\nu}_w \ge \delta$ on Γ_{wedge}.*

(vii) *$\partial_t (F_1)^{-1}(s, t) \cdot \mathbf{e}_\xi \le -\delta$ for any $(s, t) \in F_1(Q)$,*

(viii) *$\mathbf{e}_t \cdot \boldsymbol{\nu}_{sh} \le -\delta$ on Γ_{shock} for any admissible solution (r, ρ) with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, for δ depending on $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$ and the data.*

Proof. We verify that the same construction as in [26][Lemma 12.2.2] is appropriate for our case. As the Regular Shock Reflection-Diffraction structure of Ω is the same, except for the known estimates on Γ_{shock}, the only property that needs to be verified is **(viii)**. F_1 is constructed so that,

$$\begin{aligned} \mathbf{e}_t &= \mathbf{e}_y \text{ in } \mathcal{D}_{\delta_*,\epsilon_0/10}, \\ \mathbf{e}_t &= -\boldsymbol{\nu}_w \text{ in } (\mathcal{Q} \cap \{\eta > \frac{1}{2M}\}) \setminus \mathcal{D}_{\delta_*,\epsilon/5}, \\ \mathbf{e}_t &= -\mathbf{e}_\xi \text{ in } \mathcal{Q} \cap \{\eta < \frac{1}{M}\}, \end{aligned} \tag{7.1.7}$$

and smoothly interpolates between these values elsewhere, where we can choose M depending on the data and θ_w^*. Using (6.2.10) near Γ_{sonic}, (5.2.1) near $\{\eta = 0\}$, and (4.3.2), we obtain **(viii)** □

Lemma 7.1.3. *Let ϵ_0, ϵ_1 be as in Lemma 6.2.4 and $\overline{\alpha}$ be the α from Corollary 6.3.2. Fix $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$. Let (ρ, r) be an admissible solution for the wedge angle $\theta_w \in (\theta_w^*, \frac{\pi}{2}]$. Then for all $\beta \in (0, \frac{1}{2})$, and $\alpha \in (0, \overline{\alpha}]$, there exists C depending on θ_w^*, α, β and the initial data such that*

$$\begin{aligned} \tilde{\Omega} := F_1(\Omega) &= \{(s,t) \mid s \in (0, c_2), 0 < t < \mathfrak{g}_{sh}(s)\}, \\ F_1(\Gamma_{shock}) &= \{(s,t) \mid s \in (0, c_2), t = \mathfrak{g}_{sh}(s)\}, \end{aligned} \tag{7.1.8}$$

with

$$\|\mathfrak{g}_{sh}\|^{(-1-\beta),\{0,c_2\}}_{2,\alpha,[0,c_2]} \leq C, \tag{7.1.9}$$

Moreover, denoting

$$\mathcal{D}_{\epsilon_0} = \{\boldsymbol{\xi} \cdot \nu > -\overline{c}(\rho_2, \rho_1)\} \cap \Lambda \cap \mathcal{N}_{\epsilon_1}(\Gamma_{sonic}) \cap \{0 < x < \epsilon_0\}, \tag{7.1.10}$$

$$\begin{aligned} F_1(\mathcal{D}_{\epsilon_0}) &:= \{(s,t) \mid s \in (0, \epsilon_0), 0 < t < \mathfrak{g}_{\mathcal{S}_1}(s)\}, \\ \|\mathfrak{g}_{\mathcal{S}_1}\|_{C^3([0,\epsilon_0])} \leq C, C^{-1} &\leq \mathfrak{g}'_{\mathcal{S}_1} \leq C \qquad \text{on } [0, \epsilon_0), \end{aligned} \tag{7.1.11}$$

and

$$\|\mathfrak{g}_{sh} - \mathfrak{g}_{\mathcal{S}_1}\|^{(par)}_{2,\alpha,(0,\epsilon_0)} \leq C(\alpha). \tag{7.1.12}$$

Furthermore,

$$C^{-1} \leq \mathfrak{g}_{sh}(s) \leq \eta^{(\theta_w)}(s) - C^{-1}, \tag{7.1.13}$$

where $\eta^{(\theta_w)}(\cdot)$ is from Lemma 7.1.2 **(i)**.

Proof. Firstly, we observe that property **(viii)** in Lemma 7.1.2 implies that the region Ω can be written in the form (7.1.8). We also find that

$$\mathfrak{g}'_{sh}(s) = -\frac{|\partial_s F_1^{-1}|(\mathbf{e}_s \cdot \boldsymbol{\nu}_{sh})}{|\partial_t F_1^{-1}|(\mathbf{e}_t \cdot \boldsymbol{\nu}_{sh})}, \tag{7.1.14}$$

so that (7.1.9) follows from (6.3.30) and Lemma 7.1.2 **(viii)**. (7.1.11) and (7.1.12) follow from Proposition 6.2.2, Corollary 6.3.2, and Lemma 7.1.2 **(iv)**. The lower bound in (7.1.13) follows from the uniform separation of Γ_{shock} from Γ_{wedge}, and the upper bound follows from uniform separation of Γ_{shock} from $\mathcal{S}_{1,\delta_*}$ and $\{\xi = -2c_2\}$; in particular, as Γ_{shock} lays below $\mathcal{S}_1$, and within $\{\xi \geq -c_2\}$. □

7.1. Mapping to the iteration region

We now construct a function $\tilde{f}$ whose level sets are intended to approximate Γ_{shock}; the intention being to use it's level sets to map shock curves to and from the iteration region. Near Γ_{sonic}, Proposition 6.2.4 **(iii)** implies that $\mathcal{S}_1$ is a good approximation up to first order; but near P_2, (5.2.1) implies that Γ_{shock} is near vertical, so a vertical line is a better approximation. We also note that when $\theta_w = \frac{\pi}{2}$, $\mathcal{S}_1$ is vertical; so we choose the vertical approximation to be such that it agrees with $\mathcal{S}_1$ in the case where $\theta_w = \frac{\pi}{2}$, so that the zero level set is exactly $\mathcal{S}_1$ in that case.

Proposition 7.1.4. *For each $\theta_w \in (\theta_{sonic}, \frac{\pi}{2}]$, there exists a function $\tilde{f} \equiv \tilde{f}^{(\theta_w)}$ defined on $\mathcal{Q}$ such that for $\boldsymbol{\nu}_{\mathcal{S}_1}$ from* (3.4.4)*, and for C depending only on the data,*

$$\tilde{f} = \boldsymbol{\xi} \cdot \boldsymbol{\nu}_{\mathcal{S}_1} + \bar{c}_2 \text{ in } \overline{\mathcal{Q}} \text{ for } \theta_w = \frac{\pi}{2}, \tag{7.1.15}$$

$$\tilde{f} = \boldsymbol{\xi} \cdot \boldsymbol{\nu}_{\mathcal{S}_1} \text{ in } \mathcal{D}_{\delta_*, \epsilon_0/2}, \text{ for all } \theta_w \in [\theta_{sonic}, \frac{\pi}{2}], \tag{7.1.16}$$

$$\|\tilde{f}\|_{C^3(\mathcal{Q})} \leq C, \tag{7.1.17}$$

$$\partial_\eta \tilde{f}(\xi, 0) = 0 \text{ for } (\xi, 0) \in \mathcal{Q}, \tag{7.1.18}$$

$$\frac{\partial}{\partial t} \tilde{f} \circ F_1^{-1}(s,t) \leq -\delta \text{ for all } (s,t) \in F_1(\mathcal{Q}), \tag{7.1.19}$$

where $\tilde{f}$ depends smoothly on θ_w.

Proof. Define

$$f_{\mathcal{S}_1}(\boldsymbol{\xi}) = \boldsymbol{\xi} \cdot \boldsymbol{\nu}_{\mathcal{S}_1} + \bar{c}_2, \qquad f_0(\boldsymbol{\xi}) := \xi + \bar{c}_2, \tag{7.1.20}$$

and take

$$\tilde{f} = \zeta(\eta) f_{\mathcal{S}_1}(\boldsymbol{\xi}) + (1 - \zeta(\eta)) f_0(\boldsymbol{\xi}) \tag{7.1.21}$$

where $\zeta \in C^\infty(\mathbb{R})$ such that

$$\zeta(z) = \begin{cases} 0 & \text{for } z < \frac{1}{2M}, \\ 1 & \text{for } z > \frac{1}{M}, \end{cases} \tag{7.1.22}$$

where M is from the proof of Lemma 7.1.2. Then property (7.1.15) follows as $\boldsymbol{\nu}_{\mathcal{S}_1} = (1, 0)$ when $\theta_w = \frac{\pi}{2}$, so that $f_0 \equiv f_{\mathcal{S}_1}$. Properties (7.1.16)–(7.1.19) follow by (7.1.7), combined with (4.3.3) and Proposition 6.2.2. □

Now, given an admissible solution (ρ, r), define $\mathfrak{g}_{sh} : [0, c_2] \to \mathbb{R}$ using Lemma 7.1.3, and define a map

$$F_{2,\mathfrak{g}_{sh}}(\boldsymbol{\xi}) = \left(\frac{s}{c_2}, \frac{t}{\mathfrak{g}_{sh}(s)} \right) \qquad \text{for } (s,t) \in (0, c_2) \times (0, \infty), \tag{7.1.23}$$

so then, for $\tilde{\Omega}$ from (7.1.8),

$$F_{2,\mathfrak{g}_{sh}}(\tilde{\Omega}) = (0,1)^2 =: Q^{iter}, \tag{7.1.24}$$

and $F_{2,\mathfrak{g}_{sh}}$ on is invertible on $\tilde{\Omega}$, with $F_{2,\mathfrak{g}_{sh}} : Q^{iter} \to \tilde{\Omega}$ given by

$$F_{2,\mathfrak{g}_{sh}}^{-1}(s,t) = (c_2 s, t\mathfrak{g}_{sh}(c_2 s)). \tag{7.1.25}$$

Clearly, the regularity of $F_{2,\mathfrak{g}_{sh}}$ and $F_{2,\mathfrak{g}_{sh}}^{-1}$ is determined by the regularity of $\mathfrak{g}_{sh}$, whose regularity is determined by that of r (and hence $\mathfrak{g}_{sh}$. This allows us to use the superior

regularity for r (equivalently Γ_{shock}) over ρ (equivalently, ψ), to transfer our estimates to the corresponding u defined on the iteration region.

Now, for an admissible solution (ρ, r), define $u : Q^{iter} \to \mathbb{R}$, $v : (0,1) \to \mathbb{R}$ by

$$\begin{aligned} u &= \psi \circ F_1^{-1} \circ F_{2,\mathfrak{g}_{sh}}^{-1}(s,t), \\ v &= \tilde{f} \circ F_1^{-1} \circ F_{2,\mathfrak{g}_{sh}}^{-1}(s,1). \end{aligned} \tag{7.1.26}$$

for ψ determined from ρ and θ_w via (6.0.1), and $\tilde{f}$ from Proposition 7.1.4. For ϵ_0 from Lemma 7.1.3, we define

$$\epsilon_0' := \frac{\epsilon_0}{\max_{\theta_w \in [\theta_{sonic}, \frac{\pi}{2}]} c_2}, \tag{7.1.27}$$

which depends only on the initial data, and is well defined by continuous dependence of c_2 on the wedge angle.

Proposition 7.1.5. *Fix $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$ and define $\overline{\alpha} = \min\{\alpha, \frac{1}{2}\}$ for the α from Corollary 6.3.2, and let (ρ, r) be an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$. Then the functions u and v defined by* (7.1.26) *satisfy the property that for any $\alpha \in [0, \overline{\alpha}]$, $\beta \in (0, \frac{1}{2})$, there exists C depending on $\theta_w^*, \alpha, \beta$ and the initial data such that*

$$\begin{aligned} \|u\|^{(-\alpha),\{(1,0),(1,1)\}}_{1,\alpha,Q^{iter}\cap\{s>\epsilon_0'/10\}} + \|u\|^{(par)}_{1,\alpha,Q_{iter}\cap\{s<\epsilon_0'\}} &\leq C, \\ \|v\|^{(-1-\beta),\{0,1\}}_{2,\alpha,(0,1)} + \|v\|^{(par)}_{2,\alpha,(0,1)\cap\{s<\epsilon_0'\}} &\leq C, \end{aligned} \tag{7.1.28}$$

and

$$\begin{aligned} u \geq 0 \text{ on } Q^{iter}, v(1) \geq 0, \\ u = 0 \text{ on } \{s = 0\}, \\ v(0) = v'(0) = 0, \\ v'(1) = 0, \end{aligned} \tag{7.1.29}$$

with ϵ_0' from (7.1.27)*, and the parabolic norms are with respect to $s = 0$, where we use the 1-parabolic norm for u, and 2-parabolic norm for v.*

Proof. We first note that (7.1.28) follows from Corollary 6.3.2 combined with the C^3 regularity of F_1^{-1} from Lemma 7.1.2, and the properties of $\mathfrak{g}_{sh}$ from Lemma 7.1.3; in particular, the fact that $\mathfrak{g}_{sh}$ satisfies uniform $C^{1,\alpha}$ estimates and is bounded below with constant depending only on the data..

Now, for (7.1.29), we note from (7.1.26) and the construction of $\tilde{f}$,

$$\begin{aligned} v(1) = \tilde{f}(P_2) = \xi_{P_1} + \overline{c}(\rho_2, \rho_1) = -\overline{c}(\rho(P_2), \rho_1) + \overline{c}(\rho_2, \rho_1) \geq 0, \\ u(s,t) = (c_2^2 - c^2) \circ F_1^{-1} \circ F_{2,\mathfrak{g}_{sh}}^{-1}(s,t) \geq 0, \end{aligned} \tag{7.1.30}$$

as $\rho \leq \rho_2$ (and so $c_2^2 - c^2 \geq 0$) by (4.5.2). Next, we note that $u = 0$ on $\{s = 0\}$ follows because $\rho = \rho_2$ on Γ_{sonic}. Finally, $v(0) = v'(0) = 0$ and $v'(1) = 0$ follow by construction of $\tilde{f}$, and the fact that by (6.2.11), Γ_{shock} agrees with $\mathcal{S}_1$ up to first order at P_1, and by (5.2.1), Γ_{shock} is vertical at P_2. □

We now invert the mapping $(\rho, r) \mapsto (u, v)$, i.e. we will describe the process of obtaining (ρ, r) from suitable (u, v), to be able to determine our iteration equation from (u, v). For simplicity, we postpone imposition of the oblique cutoff, and then discuss in §7.1.1 how to add it to the construction.

Definition 7.1.6. For $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$ and $s^* \in (0, c_2(\theta_w))$, denote

$$\mathcal{Q}(s^*) := \mathcal{Q} \cap F_1^{-1}(\{s = s^*\}), \tag{7.1.31}$$

Let $v \in C^1([\theta_w^*, \frac{\pi}{2}])$ satisfy

$$\inf_{Q(s^*)} \tilde{f} < v(\frac{s^*}{c_2}) < \sup_{Q(s^*)} \tilde{f}, \tag{7.1.32}$$

for all $s^* \in (0, c_2]$. Let $\alpha, \beta \in (0,1)$, and consider the set $\mathfrak{S} \equiv \mathfrak{S}^{(\alpha,\beta)}$ defined by

$$\mathfrak{S} := \left\{ (u, v, \theta_w) \in C^{0,\alpha}(\overline{Q^{iter}}) \times C^{1,\beta}([0,1]) \times [\theta_w^*, \frac{\pi}{2}]) \,\middle|\, \begin{array}{r} u(0,\cdot) = v(0) = v'(0) = 0, \\ v'(1) = 0, v(1) \geq 0, v \text{ satisfies (7.1.32)}, \\ u \geq 0 \end{array} \right\} \tag{7.1.33}$$

We then define:

(i) A function $\mathfrak{h}_{sh} : (0,1) \to \mathbb{R}_+$ by for each $s^* \in (0,1)$, setting $\mathfrak{h}_{sh}(s^*) = t^*$, where $(c_2 s^*, t^*)$ is the unique point on $F_1(Q^{(\theta_w)}) \cap \{s = \hat{c}_2 s^*\}$ such that

$$\tilde{f}(F_1^{-1}(c_2 s^*, t^*)) = v(s^*). \tag{7.1.34}$$

such a point exists by (7.1.32) and is unique by (7.1.19).

(ii) A function $\mathfrak{g}_{sh} : (0, c_2) \to \mathbb{R}_+$ by setting $\mathfrak{g}_{sh}(s) = \mathfrak{h}_{sh}(\frac{s}{c_2})$,

(iii) A map $\mathfrak{F} : Q^{iter} \to \Lambda(\theta_w)$ defined by

$$\mathfrak{F} = F_1^{-1} \circ (F_{2,\mathfrak{g}_{sh}})^{-1} \tag{7.1.35}$$

(iv) Sets:

$$\begin{aligned} \Gamma_{shock} &:= F_1^{-1}(\{(s, \mathfrak{g}_{sh}(s)) \mid s \in (0, c_2)\}) \equiv \mathfrak{F}((0,1) \times \{1\}), \\ \Omega &:= F_1^{-1}(\{(s,t) \mid s \in (0,c_2), t \in (0, \mathfrak{g}_{sh}(s))\}) \equiv \mathfrak{F}(Q^{iter}) \end{aligned} \tag{7.1.36}$$

(v) Let $a, M > 0$. Consider $(u, v, \theta_w) \in \mathfrak{S}$ that satisfy

$$\begin{aligned} u &< c_2^2 - c_1^2, \\ a < \mathfrak{g}_{sh}^{(v,\theta_w)} &< \frac{1}{a}, \\ \|u\|_{C^{0,\alpha}(\overline{Q^{iter}})} + \|v\|_{C^{1,\beta}([0,1])} &\leq M, \\ \boldsymbol{\xi} \cdot \boldsymbol{\nu} &< -c_1 \qquad \text{on } \Gamma_{shock}, \end{aligned} \tag{7.1.37}$$

and then define

$$\mathfrak{S}_{a,M}^{(\alpha,\beta)} = \{(u, v, \theta_w) \in \mathfrak{S} \mid (7.1.37) \text{ hold } \}. \tag{7.1.38}$$

Then we first define ψ on Ω, depending on u by setting

$$c^2 = u(\mathfrak{F}^{-1}(\boldsymbol{\xi})), \tag{7.1.39}$$

and we can also define $\rho : \Omega \to \mathbb{R}$ and $r : [\theta_1, \pi] \to \mathbb{R}$, determined by u and v, such that

$$\begin{aligned} \rho &= (c_2^2 - \psi)^{\frac{1}{\gamma - 1}}, \\ \Gamma_{shock} &= \{(r, \theta) \mid r = r(\theta), \theta_1 < \theta < \pi\}, \\ \Omega &= \{r < r(\theta), r < c_2, \theta_w < \theta < \pi\}. \end{aligned} \tag{7.1.40}$$

where

$$\rho > \rho_1 \text{ in } \Omega. \tag{7.1.41}$$

We now state the following Lemma, relating properties of (u, v) to properties for ρ, r and ψ.

Lemma 7.1.7. *Let $\alpha, \beta \in [0, 1)$ and $a > 0$. Then for all $(u, v, \theta_w) \in \mathfrak{S} \equiv \mathfrak{S}^{(\alpha,\beta)}$, the following holds:*

(i) *$\mathfrak{h}_{sh} \in C^{1,\beta}([0, 1])$ and $\mathfrak{g}_{sh} \in C^{1,\beta}([0, c_2])$.*

(ii) *Γ_{shock} and Ω introduced in Definition 7.1.6 satisfy:*

- **a.** *$\Omega \cup \Gamma_{shock} \subset \mathcal{Q} \subset \Lambda$, and Γ_{shock} is continuous up to its endpoints $P_1(\theta_w)$ and $P_2 \in \{\eta = 0\}$.*
- **b.** *$P_1 = (F_1)^{-1}(0, \mathfrak{g}_{sh}(c_2))$ and $P_2 = (F_1)^{-1}(c_2, \mathfrak{g}_{sh}(c_2))$.*
- **c.** *Γ_{shock} is tangent to $\mathcal{S}_1$ at P_1, and is vertical at P_2, so that*

$$\begin{aligned} \mathfrak{g}_{sh}(0) = \hat{f}_0(0), \qquad \mathfrak{g}'_{sh}(0) = \hat{f}'_0(0), \\ \mathfrak{g}'_{sh}(c_2) = 0, \end{aligned} \tag{7.1.42}$$

where $\hat{f}_0$ is from (6.2.1). In particular, $\mathfrak{g}_{sh}$ is determined up to first order at P_1, by the wedge angle and initial data.

(iii) *$\partial\Omega$ consists of line segments and curves $\Gamma_{wedge}, \Gamma_{sonic}, \Gamma_{shock}$ and Γ_{sym} so that*

$$\begin{aligned} \Gamma_{shock} = \mathfrak{F}((0, 1) \times \{1\}), \qquad \Gamma_{sonic} = \mathfrak{F}(\{0\} \times (0, 1)), \\ \Gamma_{wedge} = \mathfrak{F}((0, 1) \times \{0\}, \qquad \Gamma_{sym} = \mathfrak{F}(\{1\} \times (0, 1)). \end{aligned} \tag{7.1.43}$$

and these segments do not intersect in their relative interiors.

(iv) *For ϵ_0 from Lemma 7.1.3, $\Omega \cap \mathcal{D}_{\delta_*,\epsilon}$ in the (x, y) coordinates has structure (6.2.7), with $f = \mathfrak{g}_{sh} + \theta_w$. Moreover, if for $\boldsymbol{\nu}$ being the inner unit normal to Γ_{shock} in Ω, $\boldsymbol{\xi} \cdot \boldsymbol{\nu} < -a$, Γ_{shock} is of the form $\{r = r(\theta) \mid \theta \in [\theta_1, \pi]\}$, with $r \in C^{1,\beta}([\theta_1, \pi])$, and $r'(\pi) = 0$.*

(v) *The corresponding $\psi = c_2^2 - c^2$ satisfies $\psi \in C^{0,\alpha}(\overline{\Omega})$, and*

$$\psi \geq 0 \text{ on } \overline{\Omega}, \text{ and } \psi = 0 \qquad \text{on } \Gamma_{sonic}, \tag{7.1.44}$$

7.1.1 Oblique cutoff for r near P_2

Here, we now add an oblique cutoff for r' to the process of obtaining (ρ, r) from (u, v), to ensure that our linearised boundary condition is degenerate oblique; more precisely, that it satisfies (5.1.1) with different constants. This cutoff is understood to be imposed immediately after defining r from v, but before defining ψ from u; ψ will be imposed on the cutoff region, not the original one. Let $\delta_3 = \delta/2$, for δ from (4.6.1), where we note that as $r^2 - \overline{c}^2 \geq 1$, $r'(\theta) \leq -\sqrt{\delta}(\pi - \theta)^{1/2}$ for admissible solutions, making explicit the relation

with (5.2.1). We set $\tilde{\delta}_3 = \sqrt{\delta_3}$. We also let $\sigma_1 > 0$, where $\sigma_1 > 0$ will be fixed later in the argument. Once we have constructed r as described in Definition 7.1.6, we define

$$\begin{aligned} \tilde{r} &= c_2 + \int_{\theta_1}^{\theta} \left(\zeta_2(\pi - s) r'(s) + (1 - \zeta_2(\pi - s))(\pi - s)^{1/2} \zeta_1\left(\frac{r'(s)}{(\pi - s)^{1/2}}\right) \right) ds, \\ \tilde{\Gamma}_{shock} &= \{ r = \tilde{r}(\theta) \mid \theta \in [\theta_1, \pi] \}. \end{aligned} \tag{7.1.45}$$

where ζ_1, ζ_2 satisfy

$$\zeta_1(t) = \begin{cases} t & \text{for } t > -\tilde{\delta}_3, \\ -\frac{1}{2}\delta & \text{for } t < -\frac{3}{4}\tilde{\delta}_3, \end{cases}, \qquad \zeta_2(t) = \begin{cases} 1 & \text{for } t > \sigma_1, \\ 0 & \text{for } t < \sigma_1/2. \end{cases} \tag{7.1.46}$$

and $0 \leq \zeta_1' \leq 2$, $\zeta(t) \leq t$, and $\zeta_2'(t) \geq 0$.

We need to check that $\tilde{r}$ arising from this cutoff still defines a suitable shock curve $\tilde{\Gamma}_{shock}$, such that we can obtain $\tilde{\mathfrak{g}}_{sh} : [0, \hat{s}(\theta_w)] \to \mathbb{R}$ that describes $\tilde{\Gamma}_{shock}$ in the iteration region $\mathcal{Q}$, so that we can define ψ from u via (7.1.39). Firstly, we note that because

$$|\zeta_1(t)| \leq C(|t| + \delta), \tag{7.1.47}$$

$\tilde{r}'(\pi) = 0$. Next, we need to check that r satisfies Lemma 7.1.2 **(viii)**. If σ sufficiently small, depending on δ, the region where the cutoff applies is contained in the region where $\mathbf{e}_t$ defined in Lemma 7.1.2 is equal to $-\mathbf{e}_\xi$. Possibly reducing σ further, as the cutoff increases the magnitude of r' to at most $\delta\sigma$. Thus the inner unit normal to $\tilde{\Gamma}_{shock}$ is sufficiently close to $\mathbf{e}_\xi$, such that the property $\mathbf{e}_t \cdot \tilde{\boldsymbol{\nu}}_{sh} < -\delta$, for δ depending only on the data still holds, and we can write $\tilde{\Gamma}_{shock}$ in the (s,t) coordinates as $\{(s,t) \mid t = \tilde{\mathfrak{g}}_{sh}(s), s \in (0, c_2)\}$, and we then define ψ from u using $\tilde{\mathfrak{g}}_{sh}$ instead of $\mathfrak{g}_{sh}$.

Next, we need to check the regularity of $\tilde{r}$ in terms of the regularity of r. We also need to check that $\tilde{r}$ depends continuously on r, so that necessary conditions for Γ_{shock} can be imposed as conditions for v. In these workings, we neglect the ζ_2 part of the cutoff, as owing to the fact $\zeta_2 \in C^\infty(\mathbb{R})$, it will only increase the regularity, and will not affect if $\tilde{r}$ depends continuously on r or not. We also note that due to the the smoothness of ζ_1, ζ_2, and the form of the mapping, the regularity of $\tilde{r}$ is the same as that of r away from $\theta = \pi$, so we need only consider regularity near $\theta = \pi$. We will show that the ζ_1 part of the cutoff preserves $C^{1,\beta}$ regularity for r, provided $\beta \in (0, \frac{1}{2}]$, but $C^{(-1-\beta),\{\pi\}}_{2,\alpha}$ regularity for r only leads to $C^{(-2\beta-\frac{1}{2}),\{\pi\}}_{2,\alpha}$ regularity, for similar reasons to the proof of Proposition 5.2.4.

To check the $C^{1,\beta}$ regularity, we define

$$g(z, s) := \zeta_1\left(\frac{z}{s}\right) s. \tag{7.1.48}$$

As $|\zeta_1(t)| \leq C(\delta + |t|)$, we find that

$$\begin{aligned} |\tilde{r}'(\theta)| &\leq C(\delta + \frac{|r'(\theta)|}{(\pi - \theta)^{1/2}})(\pi - \theta)^{1/2} \\ &\leq C(\delta(\pi - \theta_1)^{1/2} + \|r\|_{1,0,[\theta_1,\pi]}, \end{aligned} \tag{7.1.49}$$

so that

$$\|r\|_{1,0,[\theta_1,\pi]} \leq C \tag{7.1.50}$$

for C depending only on the data, θ_w^*, δ and $\|r\|_{1,0,[\theta_1,\pi]}$. For the difference quotient terms, we find that

$$\begin{aligned} |\partial_z g(z,s)| &= |\zeta_1'(\frac{z}{s})| \leq C, \\ |\partial_s g(z,s)| &= |\zeta_1(\frac{z}{s}) - \frac{z}{s}\zeta_1'(\frac{z}{s})| \leq C, \end{aligned} \tag{7.1.51}$$

for C depending only on the choice of ζ_1 (so on δ), so that

$$\begin{aligned} |\tilde{r}(\theta) - \tilde{r}(\tilde{\theta})| &= |g(r(\theta),\theta^{1/2}) - g(r(\tilde{\theta}),\tilde{\theta}^{1/2})| \\ &\leq C\Big(|r(\theta) - r(\tilde{\theta})| + |\theta^{1/2} - \tilde{\theta}^{1/2}|\Big) \\ &\leq C\Big(|r(\theta) - r(\tilde{\theta})| + |\theta - \tilde{\theta}|^{1/2}\Big), \end{aligned} \tag{7.1.52}$$

so that for $\beta \in (0, \frac{1}{2})$,

$$\|\tilde{r}\|_{1,\beta,[\theta_1,\pi]} \leq C, \tag{7.1.53}$$

with C depending only on δ, the data, and $\|r\|_{1,\beta,[\theta_1,\pi]}$. This also implies that $\tilde{r}$ depends continuously on r up to first order, hence on v up to first order.

We now look at the second derivatives. We find that

$$\begin{aligned} r''(\theta) &= \zeta_1'(\frac{r'(\theta)}{(\pi-\theta)^{1/2}})\Big(r''(\theta) + \frac{1}{2}\frac{r'(\theta)}{\pi-\theta}\Big) - \frac{1}{2}\zeta_1(\frac{r'(\theta)}{(\pi-\theta)^{1/2}})(\pi-\theta)^{-1/2} \\ &= \zeta_1'(\frac{r'(\theta)}{(\pi-\theta)^{1/2}})r''(\theta) + \frac{1}{2}\Big(\zeta_1'(\frac{r'(\theta)}{(\pi-\theta)^{1/2}})\frac{r'(\theta)}{(\pi-\theta)^{1/2}} - \zeta_1(\frac{r'(\theta)}{(\pi-\theta)^{1/2}})\Big)(\pi-\theta)^{-1/2} \end{aligned} \tag{7.1.54}$$

so that firstly, we find that, again as $r'(\pi) = 0$,

$$|r''(\theta)| \leq \|r\|_{1,0,[\theta_1,\pi]}^{(-1-\beta)}(\pi-\theta)^{\beta-\frac{1}{2}}\|r\|_{2,0,[\theta_1,\pi]}(\pi-\theta)^{\beta-1} + C(\pi-\theta)^{-1/2}, \tag{7.1.55}$$

so that

$$\|\tilde{r}\|_{2,0,[\theta_1,\pi]}^{(-2\beta-\frac{1}{2}),\{\pi\}} \leq C, \tag{7.1.56}$$

for C depending only on δ, the data, and $\|r\|_{2,0,[\theta_1,\pi]}^{(-1-\beta)}$, and $\|r\|_{1,\beta,[\theta_1,\pi]}$. To examine the difference quotient term, we define

$$h(z,w,s) := \zeta_1'(\frac{z}{s})\Big(w + \frac{1}{2}\frac{z}{s^2}\Big) - \zeta_1(\frac{z}{s})\frac{1}{2s}, \tag{7.1.57}$$

and find that

$$\begin{aligned} h_z(z,w,s) &= \zeta_1''(\frac{z}{s})\Big(\frac{w}{s} + \frac{1}{2}\frac{z}{s^3}\Big), \\ h_w(z,w,s) &= \zeta_1'(\frac{z}{s}), \\ h_s(z,w,s) &= \zeta_1''(\frac{z}{s})\Big(-\frac{wz}{s^2} - \frac{z^2}{s^4}\Big) - \frac{1}{2}\zeta_1'(\frac{z}{s})\frac{z}{s^3} + \zeta_1(\frac{z}{s})\frac{1}{s^2}, \end{aligned} \tag{7.1.58}$$

so that, noting that $|t^2\zeta_1''(t)|, |t\zeta_1''(t)| \leq C$, $|\zeta_1'(t)| \leq Ct$, and that $|\zeta_1(t) - t\zeta_1'(t)| \leq C$,

$$\begin{aligned} |h_z(z,w,s)| &\leq C\big(s^{-2} + |w|\big), \\ |h_w(z,w,s)| &\leq C|z|\,|w|^{-1}, \\ |h_s(z,w,s)| &\leq C(s^{-2} + s^{-1}|w|), \end{aligned} \tag{7.1.59}$$

so that for $\theta_1 \le \theta \le \tilde\theta \le \pi$,

$$\begin{aligned}|\tilde r''(\theta)-\tilde r''(\tilde\theta)| &\le C\Big(((\pi-\theta)^{-1}+\theta^{\beta-1}\|r\|^{(-1-\beta)}_{2,0,[\theta_1,\pi]})|r'(\theta)-r'(\tilde\theta)|\\ &\qquad +|r''(\theta)-r''(\tilde\theta)|+(\theta^{-1}+\theta^{\beta-\frac32})|\theta-\tilde\theta|^{1/2}\Big)\\ &\le C\Big(\|r\|_{1,\beta,[\theta_1,\pi]}(\theta^{-1}+\|r\|^{(-1-\beta)}_{2,0,[\theta_1,\pi]}\theta^{\beta-1})|\theta-\tilde\theta|^{\beta}\\ &\qquad +\|r\|_{2,\alpha,[\theta_1,\pi]}\theta^{\beta-\alpha-1}|\theta-\tilde\theta|^{\alpha}+(\theta^{-1}+\theta^{\beta-\frac32})|\theta-\tilde\theta|^{1/2}\Big)\end{aligned} \tag{7.1.60}$$

therefore, using the fact $0<\beta<\frac12$, and assuming $\beta\ge\alpha$,

$$\|\tilde r\|^{(-2\beta-\frac12),\{\pi\}}_{2,\alpha,[\theta_1,\pi]} \le C, \tag{7.1.61}$$

for C only depending on δ, the data, and $\|r\|^{(-1-\beta),\{\pi\}}_{2,0,[\theta_1,\pi]}$, and $\|r\|_{1,\beta,[\theta_1,\pi]}$. Because $2\beta+\frac12 < 1+\beta$, we conclude that there is a loss in regularity here; but it will turn out that the $C^{(-2\beta-\frac12)}_{2,\alpha}$ regularity for the original shock curve is still sufficient to get better-than $C^{(-1-\beta)}_{2,\alpha}$ regularity for the new shock curve obtained by the iteration problem.

7.2 Defining the iteration set

Let $Q^{iter}=(0,1)^2$. An iteration set in terms of (u,v), $\mathcal{K}\subset C^{0,\alpha}(\overline{Q^{iter}})\times C^{1,\beta}([0,1])\times[\theta^*_w,\frac{\pi}{2}]$ is defined below, for appropriate $\alpha,\beta\in(0,1)$, based on the estimates for admissible solutions. We will also define our iteration problem.

We use constants ϵ_0 and ϵ_1 from Lemma 6.2.4.

Definition 7.2.1. **(i)** Denote by $(u^{(norm)},v^{(norm)})$ the functions in the iteration region corresponding to the unique normal reflection solution. By Theorem 3.1.1 and (7.1.26), $(u^{norm},v^{norm})=(0,0)$.

(ii) For real $\sigma>0$ and $\alpha\in(0,1)$ we use the following norm for Q^{iter} with ϵ_0' from Proposition 7.1.5:

$$\|u\|^{*,\sigma}_{1,\alpha,Q^{iter}} := \|u\|^{(-\alpha),\{(1,0),(1,1)\}}_{1,\alpha/2,Q^{iter}\cap\{s>\epsilon_0'/10\}} + \|u\|^{(\sigma),(par)}_{1,\alpha/2,Q^{iter}\cap\{s<\epsilon_0'\}}, \tag{7.2.1}$$

and we denote by $C^{1,\alpha}_{*,\sigma}(Q^{iter})$ the space of u where (7.2.1) is finite. We use the $C^{(-1-\beta)}_{2,\alpha/2}((0,1))$ norm for v.

We now define the iteration set. Let $\alpha\in(0,1)$, and $\beta\in(0,1/2)$, and we always assume that

$$1+\alpha\le 2\beta-\frac12,\qquad \alpha/2\le\delta^*, \tag{7.2.2}$$

where $\delta^*=\frac18$. We set

$$\begin{aligned}\mathcal{B} &= C^{1,\alpha}_{*,\sigma}(Q^{iter})\times C^{(-1-\beta)}_{2,\alpha/2},\\ \mathcal{C} &:= \mathcal{B}\cap\{u(0,\cdot)=v(0)=v'(0)=v'(1)=0, v(1)\ge0, u\ge0\},\end{aligned} \tag{7.2.3}$$

and note that $\mathcal{B}$ is a Banach Space, and $\mathcal{C}$ is a closed, convex subset of $\mathcal{B}$. We will eventually show that for each $\theta_w\in[\theta^*_w,\frac{\pi}{2}]$, $\mathcal{K}(\theta_w)$ is relatively open in $\mathcal{C}$.

Definition 7.2.2. The iteration set $\mathcal{K} \subset \mathcal{C} \times [\theta_w^*, \frac{\pi}{2}]$ is the set of all (u, v, θ_w) such that;

(i) (u, v, θ_w) satisfy the estimates;

$$\begin{aligned} \|u\|^{*,\delta^*}_{1,\alpha,Q^{iter}} &< N_0, \\ \|v\|^{(-1-\beta),\{0,1\}}_{2,\alpha/2,(0,1)} &< N_0, \end{aligned} \tag{7.2.4}$$

Where $N_0 = 10M$ for the constant M from Proposition 7.1.5.

(ii) $(u, v, \theta_w) \in \mathfrak{S}$ as defined in (7.1.33), so that $\mathfrak{g}_{sh}, \Omega, \Gamma_{shock}$ and ψ are defined as in Definition 7.1.6. Also, denoting the inner unit normal to Γ_{shock}in $\boldsymbol{\xi}$ coordinates by $\boldsymbol{\nu}$,

$$\boldsymbol{\xi} \cdot \boldsymbol{\nu} < -c_1, \tag{7.2.5}$$

so that Γ_{shock} can be written in the form $\{r = r(\theta)\}$ for $r \in C^1 1, \beta([\theta_1, \pi]) \cap C^{(-2\beta-\frac{1}{2})}_{2,\alpha/2}([\theta_1, \pi])$, and has a cutoff as imposed in §7.1.1. Also, for $N_2 = 2C$ from (7.1.13),

$$\frac{1}{N_2} \leq \mathfrak{g}_{sh}(s) \leq \eta^{(\theta_w)}(s) - \frac{1}{N_2}. \tag{7.2.6}$$

(iii) ρ is well defined, and satisfies

$$\rho_1 + \tau_1 < \rho \leq \rho_2, \tag{7.2.7}$$

where $\tau_1 = \frac{\tau}{2}$, for τ as in Corollary 4.5.2.

(iv) ρ, c^2, $r(\theta)$ and $\psi = c_2^2 - c^2(\rho)$ determined by u and v satisfy;

$$|\psi| < (2c_2 - \delta_2)x \qquad \text{in } \overline{\Omega} \cap \left(\mathcal{D}_{\epsilon_0} \setminus \mathcal{D}_{\epsilon/10}\right), \tag{7.2.8}$$

$$|\psi| < (2c_2 - \delta_2)\epsilon \qquad \text{in } \mathcal{D}_{\epsilon_0}, \tag{7.2.9}$$

$$|r^2 - \overline{c}^2| < N_4\sigma, \quad |r'| < N_4\sigma^{1/2} \qquad \text{on } \Gamma_{shock} \cap \{\theta - \pi < \sigma\}, \tag{7.2.10}$$

$$2g(r, \rho) < r'(\theta) < \frac{1}{2}g(r, \rho) \qquad \text{on } \Gamma_{shock} \cap \{\pi - \theta > \sigma/10\}. \tag{7.2.11}$$

where $g(r, \rho)$ is defined in (2.2.11), $\delta_2 = \frac{\delta}{2}$ for δ from (6.2.5), and $N_4 = 2C$ for C from (5.2.1).

(v) Uniform ellipticity in $\overline{\Omega} \setminus \mathcal{D}_{\epsilon/10}$:

$$c^2 - r^2 > \lambda_1 \text{dist}(\boldsymbol{\xi}, \Gamma_{sonic}), \qquad \text{for all } \boldsymbol{\xi} \in \overline{\Omega} \setminus \mathcal{D}_{\epsilon/10}, \tag{7.2.12}$$

where $\lambda_1 = \frac{\lambda}{2}$, for λ from (4.6.1).

(vi) Uniform obliqueness away from P_2:

$$r^2 - \overline{c}^2 > \delta_3(\pi - \theta) \qquad \text{on } \Gamma_{shock} \cap \{\pi - \theta > \sigma/10\} \tag{7.2.13}$$

where $\delta_3 = \frac{\delta}{2}$, for δ is from (4.6.1), as set at the start of §7.1.1.

(vii) The boundary value problem, and updated boundary, determined by (u, v, θ_w), as defined below in Problem 7.2.7 has a solution $(\hat{\psi}, \hat{r})$ which satisfies;

$$\|\psi - \hat{\psi}\|^{*,\delta^*/2}_{1,\alpha/2,\Omega} < \delta_4, \tag{7.2.14}$$

and

$$\|r - \hat{r}\|_{1,\beta/2,[\theta_1,\pi]} < \delta_5. \tag{7.2.15}$$

Remark 7.2.3. By construction, the region Ω determined by $(u, v, \theta_w) \in \mathcal{K}$ satisfies

$$\Omega \subset \mathcal{Q} \subset B_{M_{dom}}(\mathbf{0}), \tag{7.2.16}$$

where M_{dom} only depends on the initial data, by the construction in §9.1.

Remark 7.2.4. Let $(u, v, \theta_w) \in \mathcal{K}$. Then, by Lemma 7.1.2, Proposition 7.1.4, Definition 7.1.6, (7.2.5), (7.1.53), (7.1.61), (7.2.6), and the assumption (7.2.2), we obtain

$$\begin{aligned} \|\psi\|^{(-\alpha),\{P_2,P_3\}}_{1,\alpha/2,\Omega\setminus\Omega_{\epsilon_0'/10}} + \|\psi\|^{(\sigma),(par)}_{1,\alpha/2,\Omega_{\epsilon_0'}} &< CN_0 := N_0^*, \\ \|r\|_{1,\beta,[\theta_1,\pi]} &< C^*, \\ \|r\|^{(-2\beta-\frac{1}{2}),\{\theta_1,\pi\}}_{2,\alpha/2,[\theta_1,\pi]} &< C^* \end{aligned} \tag{7.2.17}$$

where C only depends on the data, and N_2; thus on the data and θ_w^*. C^* depends on N_0, N_2 and the initial data.

7.2.1 Closure of the iteration set

We now look at properties of the closure of the iteration set; this will be necessary for defining the iteration map on $\overline{\mathcal{K}}$. We now introduce the following notation:

Definition 7.2.5. For the iteration set $\mathcal{K}$ defined in Definition 7.2.2,

(i) $\mathcal{K}^{ext}$ is the set of all $(u, v, \theta_w) \in \mathcal{C}$ which satisfy properties **(i)-(vi)** of Definition 7.2.2

(ii) $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}^{ext}}$ are the closures of $\mathcal{K}$ and $\mathcal{K}^{ext}$ in $\overline{\mathcal{C}}$, respectively.

(iii) For each $\mathfrak{C} \in \{\mathcal{K}, \mathcal{K}^{ext}\}, \overline{\mathcal{K}}, \overline{\mathcal{K}^{ext}}$, and for each $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$,

$$\mathfrak{C}(\theta_w) := \{(u, v) \mid (u, v, \theta_w) \in \mathfrak{C}\}. \tag{7.2.18}$$

We note that by definition,

$$\mathcal{K} \subset \mathcal{K}^{ext}, \qquad \overline{\mathcal{K}} \subset \overline{\mathcal{K}^{ext}}. \tag{7.2.19}$$

Lemma 7.2.6. *The sets introduced in Definition 7.2.5 satisfy the following properties;*

(i) $\overline{\mathcal{K}^{ext}} \subset \mathfrak{S}$*; any* $(u, v) \in \overline{\mathcal{K}^{ext}}$ *satisfies* (7.1.32).

(ii) *For any* $(u, v, \theta_w) \in \overline{\mathcal{K}^{ext}}$*, properties* **(i)-(vi)** *are satisfied with non-strict inequalities, and Remark 7.2.4 still holds.*

Proof. For **(i)**, we note that from (7.1.5), (7.1.19), and (7.2.6), for each $s^* \in (0, c_2)$, for $(u, v, \theta_w) \in \mathcal{K}^{ext}$,

$$\begin{aligned} \inf_{\mathcal{Q}_{bd}(s^*)} \tilde{f} = \tilde{f}(F_1^{-1}(s^*, \eta(s^*)) < \tilde{f}(F_1^{-1}(s^*, g_{sh}(s^*)) - \frac{1}{M} = v(s^*) - \frac{1}{M}, \\ \sup_{\mathcal{Q}_{bd}(s^*)} \tilde{f} = \tilde{f}(F_1^{-1}(s^*, 0) > \tilde{f}(F_1^{-1}(s^*, g_{sh}(s^*)) + \frac{1}{M} = v(s^*) + \frac{1}{M}, \end{aligned} \tag{7.2.20}$$

where M depends on N_2, hence on the data, and the same inequality will hold with a non-strict inequality when taking the closure. **(ii)** is a direct consequence of the continuous dependence of (ρ, r) on (u, v), which follows by the regularity of F_1^{-1} and the cutoff map. □

7.2.2 Boundary-value problem for the iteration

In order to complete Definition 7.2.2, we now set up the boundary value problem for a new density $\hat{\rho}$ in Ω, and a new shock curve, $\hat{\Gamma}_{shock} = \{\hat{r} = \hat{r}(\theta)\}$. We will in fact formulate the problem in terms of finding a new $\hat{\psi} = c_2^2 - c^2(\hat{\rho})$, noting that this is equivalent to finding a new ρ, except that no divergence form equation similar to (2.1.4) is available. However, we do need the divergence form here, as it was only used for obtaining $C^{0,\alpha}$ bounds for the oblique problem via Proposition B.0.4. Here, this will not be necessary, due to the coefficient regularity from the iteration set. Similar to the process in §3.5, we write (2.1.4) in non-divergence form, multiply by $(\gamma-1)\rho^{\gamma-2}$, and find we can write an equivalent equation of the form

$$\sum_{i,j=1}^{2} A_{ij}^{(o)}(\psi, \boldsymbol{\xi})\psi_{ij} + \sum_{i=1}^{2} A_i^{(o)}(\psi, \boldsymbol{\xi})\psi_i + \sum_{i,j=1}^{2} B_{ij}^{(o)}(\psi, \boldsymbol{\xi})\psi_i\psi_j = 0, \tag{7.2.21}$$

which when written in the (x, y) coordinates from §3.5, is equivalent to (3.5.4). We will use the (x, y) coordinates and (3.5.4) near Γ_{sonic}, and (7.2.21) away from Γ_{sonic}. We also note that (2.2.8) and (2.2.11) hold for ψ, simply replacing ρ with ψ by multiplying by $(\gamma-1)\rho^{\gamma-2}$.

For the elliptic operator, we cannot exclude the possibility that $A_{11}(\psi, x)$ is negative, based on the iteration set assumptions, as the bound (7.2.17) only implies $|\psi| \leq N_0 x^{\delta}$, so we must impose a cutoff. When defining the coefficients in the elliptic equation, we will define them separately on $\Omega \cap \mathcal{D}_{\epsilon_{eq}}$ and $\Omega \setminus \mathcal{D}_{\epsilon_{eq}/10}$, and then smoothly piece them together.

For the oblique boundary condition, as the cutoff (7.1.45) has been applied, we can simply linearise it. Due to the cutoff, combined with properties (7.2.10), (7.2.11) and (7.2.13), the property $r' = 0$ from Lemma 7.1.7 **(iv)**, and the cutoff in §7.1.1 imply obliqueness that degenerates at P_2, is small near P_2, and is strict away from P_2.

We assume that the ϵ in Definition 7.2.2 satisfies

$$\epsilon < \frac{\epsilon_{eq}}{2}. \tag{7.2.22}$$

Problem 7.2.7.

$$\mathcal{N}(\psi) := \sum_{i,j=1}^{2} A_{ij}\hat{\psi}_{\boldsymbol{\xi}_i\boldsymbol{\xi}_j} + \sum_{i,j=1}^{2} B_{ij}\hat{\psi}_{\boldsymbol{\xi}_i}\hat{\psi}_{\boldsymbol{\xi}_j} + \sum_{i=1}^{2} A_i\hat{\psi}_{\boldsymbol{\xi}_i} = 0 \qquad \text{in } \Omega, \tag{7.2.23}$$

$$\mathcal{M}\psi = \beta_1\hat{\psi}_r + \beta_2\hat{\psi}_\theta = 0 \qquad \text{on } \Gamma_{shock}, \tag{7.2.24}$$

$$\hat{\psi} = 0 \qquad \text{on } \Gamma_{sonic}, \tag{7.2.25}$$

$$\nabla\hat{\psi} \cdot \nu = 0 \qquad \text{on } \Gamma_{wedge} \cup \Gamma_{sym}, \tag{7.2.26}$$

$$\hat{\psi} = \overline{\psi} \qquad \text{at } P_2, \tag{7.2.27}$$

$$\hat{r}(\theta) = c_2 - \int_{\theta_1}^{\theta} \frac{r(s)}{\overline{c}(\hat{\rho}(r(s), s), \rho_1} \sqrt{r(s)^2 - \overline{c}^2(\hat{\rho}(r(s), s), \rho_1)}\, ds \qquad \text{for } \theta \in [\theta_1, \pi], \tag{7.2.28}$$

where $\overline{c}$ is as in (2.2.6), and $\overline{\psi}$ is defined by finding $\overline{\rho}$ such that $\overline{c}(\overline{\rho}, \rho_1) = r(P_2)$, and then setting $\overline{\psi} = c_2^2 - c^2(\overline{\rho})$.

7.2.3 The oblique boundary condition on Γ_{shock}

On Γ_{shock}, we linearise the boundary condition, freezing the coefficients at the ψ and r determined by u and v. We use the expressions (2.2.11) (multiplying by $(\gamma-1)\rho^{\gamma-2}$ to

obtain an equation for ψ) in the (r,θ) coordinates centred at the origin for the self-similar coordinates for $\boldsymbol{\xi}$:

$$\beta_1(\psi,r)\hat{\psi}_r+\beta_2(\psi,r)\hat{\psi}_\theta=0. \tag{7.2.29}$$

We note that when the boundary condition is linearised, we get the same expression as for the obliqueness as in (2.2.12), with all terms evaluated at ψ (equivalently ρ) from the iteration set;

$$\beta\cdot\boldsymbol{\nu}=-r'(\theta)r^2(c^2-\overline{c}^2), \tag{7.2.30}$$

so that (7.2.7), the cutoff (7.1.45) and (7.2.13) ensure uniform obliqueness away from P_2, the property (7.1.42) ensures the obliqueness degenerates at P_2, and Lemma 7.1.7 **(iv)**, and (7.2.10) ensure the boundary is almost vertical, and the obliqueness small near P_2. More precisely, due to the properties mentioned in the above, we find that for C depending only on the data,

$$C(\pi-\theta)^{1/2}\leq\beta\cdot\boldsymbol{\nu}\leq C\sigma^{1/2}. \tag{7.2.31}$$

Also, by Remark 7.2.4 and the assumption (7.2.2), for each $\tau>0$, there exists C only depending on N_0^* and hence the data, and $C(\tau)$ depending on N_0^* and τ such that

$$\begin{aligned}\|\beta_i\|_{0,\alpha,\Gamma_{shock}}&\leq C,\\ \|\beta_i\|^{(-\alpha),\{P_2\}}_{1,\alpha/2,\Gamma_{shock}\setminus\partial\Omega_s}&\leq C(\tau).\end{aligned} \tag{7.2.32}$$

We now remark on the boundary condition near P_2, with the intention that we will be able to use the arguments of Proposition 3.6.4 and Proposition 5.2.4, to be able to obtain a lower bound for $r^2-\overline{c}^2$, and estimates near P_2 for solution to the iteration problem. Using (7.2.10), (7.2.12) and the cutoff (7.1.45) we find that if we choose σ small enough depending only on the data, there exists C depending only on the data (and is uniform in σ once σ is chosen small enough) such that

$$-C\sigma^{1/2}\leq\beta_1\leq 0,\quad \beta_2\geq\frac{1}{C}\quad \text{on } \Gamma_{shock}\cap\{0\leq\pi-\theta<\sigma\}. \tag{7.2.33}$$

We next remark on the boundary condition near Γ_{sonic}, with the intention to use the argument of Proposition 6.3.1 to obtain $C^{(par)}_{1,\alpha}$ estimates near Γ_{sonic}. For small enough ϵ, by (7.2.11) and (7.2.13), $\Gamma_{shock}\cap\mathcal{N}_\epsilon(\Gamma_{sonic})$ can be written in the form $\{y=f(x)\}$, and the linearised boundary condition (7.2.24) is equivalent to (3.5.14) with coefficients evaluated at ψ and f;

$$\hat{\beta}_1(\psi,x)\psi_x+\hat{\beta}_2(\psi,x)\psi_y=0\qquad \text{on } \Gamma_{shock}\cap\mathcal{N}_\epsilon(\Gamma_{sonic}). \tag{7.2.34}$$

Then, by (6.2.13) and (7.2.8), for ϵ small enough depending on the data, we have the estimates of Proposition 6.2.4 **(iv)** for our linearised coefficients;

$$\hat{\beta}_1\leq-\frac{1}{C},\quad \hat{\beta}_2\leq-\frac{1}{C},\quad \hat{\beta}\cdot\boldsymbol{\nu}\geq\frac{1}{C}\qquad \text{on } \Gamma_{shock}\cap\mathcal{N}_\epsilon(\Gamma_{sonic}), \tag{7.2.35}$$

where $\boldsymbol{\nu}=(f'(x),-1)$ is the inner unit normal to Ω expressed in (x,y) coordinates.

7.2.4 The quasilinear equation in Ω, away from Γ_{sonic}

Away from Γ_{sonic}, in $\Omega\setminus\mathcal{D}_{\epsilon_{eq}/10}$, we use (7.2.21), with the coefficients frozen at ψ; i.e. for $(A^{(o)}_{ij},A^{(o)}_i,B^{(o)}_{ij})$ from (7.2.21), we set

$$(A^1_{ij},A^1_i,B^1_{ij})(\boldsymbol{\xi})=(A^{(o)}_{ij},A^{(o)}_i,B^{(o)}_{ij})(\psi(\boldsymbol{\xi}),\boldsymbol{\xi}), \tag{7.2.36}$$

and impose

$$\sum_{i,j=1}^{2} A^1_{ij}(\psi,\boldsymbol{\xi})\hat{\psi}_{\boldsymbol{\xi}_i\boldsymbol{\xi}_j} + \sum_{i,j=1}^{2} B^1_{ij}\hat{\psi}_{\boldsymbol{\xi}_i}\hat{\psi}_{\boldsymbol{\xi}_j} + \sum_{i=1}^{2} A^1_i\hat{\psi}_{\boldsymbol{\xi}_i} = 0 \text{ in } \Omega \setminus \mathcal{D}_{\epsilon_{eq}/10}. \tag{7.2.37}$$

The conditions (7.2.12) and (7.2.22) ensure that the equation is uniformly elliptic in $\Omega \setminus \mathcal{D}_{\epsilon_{eq}/10}$. Also, from Remark 7.2.4, and that the coefficients are smooth functions of $(\psi,\boldsymbol{\xi})$, the coefficients lie in $C^{(-\alpha),\{P_2,P_3\}}_{1,\alpha/2,\Omega\setminus\mathcal{D}_{\epsilon_{eq}/10}}$ with

$$\|(A^1_{ij},A^1_i,B^1_{ij})\|_{C^{(-\alpha),\{P_2,P_3\}}_{1,\alpha/2,\Omega\setminus\mathcal{D}_{\epsilon_{eq}/10}}} \leq C, \tag{7.2.38}$$

where C only depends on the data, ϵ_{eq} and $\hat{N}_0$ from Remark 7.2.4; thus on the data, ϵ_{eq} and θ^*_w.

7.2.5 The quasilinear equation & oblique boundary condition in Ω, near Γ_{sonic}

We now define the iteration equation in $\Omega \cap \mathcal{D}_\epsilon$, i.e. near Γ_{sonic}. We will work in the (x,y) coordinates given by (3.5.3) and use the form (6.1.1). We note that here, we need to use a cutoff based on the estimate (6.2.5); the estimate (7.2.17) only implies that $|\psi| \leq N_0 x^{\delta^*}$, so $\hat{A}_{11}(\psi,x)$ may be negative. We make a cutoff such that;

a. Any fixed point $\hat{\psi} = \psi$ satisfies the original equation,

b. The modified equation has degenerate ellipticity structure (6.1.5) for $(x,y) \in \Omega \cap \mathcal{D}_{eq}$,

c. $\hat{B}_{11}(x) \approx -1, A_1 \approx c_2$,

The cutoff is based on the estimate (6.2.5), and condition (7.2.8):. Let $\zeta_1 \in C^\infty(\mathbb{R})$ (which depends on θ_w via dependence on c_2, but its bounds will be uniform in $\theta_w \in [\theta^*_w, \frac{\pi}{2}]$) satisfy

$$\zeta_1(s) = \begin{cases} s & \text{if } |s| < 2c_2 - \frac{\delta_2}{5}, \\ 2c_2 - \frac{\delta_2}{10}\text{sign}(s) & \text{if } |s| > 2c_2 - \frac{\delta_2}{10}, \end{cases} \tag{7.2.39}$$

where

$$\begin{aligned} \zeta_1'(s) \geq 0 \text{ on } \mathbb{R}, \\ \zeta_1(-s) = -\zeta_1(s) \text{ on } \mathbb{R}. \end{aligned} \tag{7.2.40}$$

We now define the modified equation in the (x,y) coordinates as follows:

$$\begin{aligned} \left(2c_2x - x\zeta_1\left(\frac{\hat{\psi}}{x}\right) + \tilde{O}_1\right)\hat{\psi}_{xx} + \left(c_2 + \tilde{O}_2\right)\hat{\psi}_x - \left(1+\tilde{O}_3\right)\psi_x^2 \\ + \left(1+\tilde{O}_4\right)\hat{\psi}_{yy} - \left(\frac{1}{(\gamma-1)c_2^2} + \tilde{O}_5\right)\psi_y^2 = 0, \end{aligned} \tag{7.2.41}$$

where $\tilde{O}_i$, $i = 1\ldots5$ are defined as

$$\tilde{O}_i = O_i\left(x, x\zeta_1\left(\frac{\hat{\psi}}{x}\right)\right), \tag{7.2.42}$$

for O_i from (3.5.5).

Lemma 7.2.8. *Let $\epsilon_{eq} \in (0, \epsilon_0)$ and suppose (7.2.22) holds. If $(u, v, \theta_w) \in \mathfrak{S}$, and the corresponding ψ satisfies (7.2.8)-(7.2.9), with constants δ_2 and N_3 depending only on the data, then*

$$|\tilde{O}_1(x,y)| \leq Cx^2, \quad |\tilde{O}_i(x,y)| \leq Cx, \qquad \text{for } i = 2, \dots 5, \tag{7.2.43}$$

where C depends only on the data.

Proof. This follows directly from (3.5.5), and the fact ζ_1 is bounded with bound depending only on the initial data. □

Now, we write (7.2.41) in (x, y) coordinates in the form;

$$0 = \hat{\mathcal{N}}^2(\hat{\psi}) := \hat{A}^2_{11}\hat{\psi}_{xx} + \hat{A}_{22}\hat{\psi}_{yy} + \hat{A}_1\psi_x + \hat{B}_{11}\hat{\psi}_x^2 + \hat{B}_{22}\hat{\psi}_y^2, \tag{7.2.44}$$

Where the $\hat{A}^2_{ij}, \hat{A}^2_i, \hat{B}^2_{ij}$ are functions of (x, y), and an equivalent form in the $\boldsymbol{\xi}$ coordinates;

$$\mathcal{N}^2(\hat{\psi}) := \sum_{i,j=1}^{2} A^2_{ij}(\boldsymbol{\xi})\psi_{ij} + \sum_{i,j=1}^{2} B_{ij}(\boldsymbol{\xi})\psi_i\psi_j + \sum_{i=1}^{2} A_i(\boldsymbol{\xi})\psi_i = 0. \tag{7.2.45}$$

Now, by the construction of our coefficients, and Remark 7.2.4, we find that:

Lemma 7.2.9. *There exists $\lambda > 0, \epsilon_{eq} \in (0, \frac{\epsilon_0}{2})$, and $N_{eq} \geq 1$ depending only on the data such that*

(i) *For any $(x, y) \in \Omega \cap \mathcal{D}_{\epsilon_{eq}}$ and $\zeta \in \mathbb{R}^2$,*

$$\lambda|\zeta|^2 \leq \frac{\hat{A}^2_{11}(x,y)}{x}\zeta_{11}^2 + \hat{A}^2_{22}(x,y)\zeta_{22}^2 \leq \frac{1}{\lambda}|\zeta|^2. \tag{7.2.46}$$

(ii) $(\hat{A}^2_{ij}, \hat{A}^2_i, \hat{B}^2_{ij}) \in C^{1,\alpha/2}(\overline{\Omega \cap \mathcal{D}_{eq}} \setminus \Gamma_{sonic})$, *and*

$$\begin{aligned} \|(\hat{A}^2_{ij}, \hat{A}^2_i, \hat{B}^2_{ij})\|_{0,\alpha,\Omega\cap\mathcal{D}_{eq}} &\leq N_{eq}, \\ \|(\hat{A}^2_{ij}, \hat{A}^2_i, \hat{B}^2_{ij}))\|_{1,\alpha/2,(\Omega\cap\mathcal{D}_{eq})\setminus\mathcal{N}_s(\Gamma_{sonic})} &\leq N_{eq}s^{-4}. \end{aligned} \tag{7.2.47}$$

(iii) *For all $(x, y) \in \Omega \cap \mathcal{D}_{\epsilon_{eq}}$,*

$$\hat{A}_1 \leq c_2 + N_{eq}\epsilon_{eq}, \quad \hat{B}_{11} \leq -1 + N_{eq}\epsilon_{eq}, \quad \hat{B}_{22} \leq -\frac{1}{(\gamma-1)c_2^2} + N_{eq}\epsilon_{eq}, \tag{7.2.48}$$

(iv) *Suppose that $\psi = \hat{\psi}$, and ψ satisfies*

$$|\psi(x,y)| \leq (2c_2 - \delta_2/5)x \qquad \text{in } \Omega \cap \mathcal{D}_{\epsilon/10}, \tag{7.2.49}$$

then in $\Omega \cap \mathcal{D}_{\epsilon_{eq}}$, ψ satisfies (6.1.1) in $\Omega \cap \mathcal{D}_{\epsilon_{eq}}$.

Proof. (7.2.46) follows from the explicit expression (7.2.41), the estimate for $\tilde{O}_1$ in (7.2.43), and as $|\zeta_1(s)| \leq 2c_2 - \frac{\delta_2}{10}$. (7.2.48) follows from the explicit expression and the estimates for $\tilde{O}_2, \tilde{O}_3$ in (7.2.43). Assertion **(iv)** follows directly from the construction of ζ_1 in (7.2.39). We now prove (7.2.47). We note that from the explicit expressions in §3.5, the original coefficients $(\hat{A}_{ij}, \hat{A}_i, \hat{B}_{ij})(x, z)$ are smooth functions of (x, z), so have the same regularity as

ψ if evaluated at (x,ψ), but the cutoff means that we must make similar considerations to §7.1.1. We first find that

$$\begin{aligned} |x\,\zeta_1(\frac{z}{x})| &\leq Cx, \\ |\partial_x(x\,\zeta_1(\frac{z}{x}))| = |\zeta_1(\frac{z}{x}) - \frac{z}{x}\zeta_1'(\frac{z}{x})| &\leq C, \\ |\partial_z(x\,\zeta_1(\frac{z}{x}))| = |\zeta_1'(\frac{z}{x})| &\leq C, \end{aligned} \tag{7.2.50}$$

so that $(x,z)\mapsto x\,\zeta_1(\frac{z}{x})$ is Lipschitz, and thus the first estimate in (7.2.47) follows from (7.2.17) and the smooth dependence of the original coefficients on (x,z). Now, continuing the calculation (7.2.50), we find

$$|D^2_{(x,z)}(x\,\zeta_1(\frac{z}{x}))| \leq Cx^{-1}, \tag{7.2.51}$$

so that using (7.2.17), and similar considerations to the above, we attain (7.2.47). □

We are now ready to define the equation on Ω, by combining the equations above as follows. First, we define the cutoff function; for $\tau>0$, let $\zeta_2\equiv\zeta_2^{(\tau)}\in C^\infty(\mathbb{R})$ satisfy;

$$\zeta_2(s)=\begin{cases}0 & \text{if } s\leq\frac{\tau}{2},\\ 1 & \text{if } s\geq\tau,\end{cases}, \qquad \text{and } 0\leq\zeta_2'(s)\leq\frac{10}{\tau} \text{ on } \mathbb{R}. \tag{7.2.52}$$

We then define for $\theta_w\in[\theta_w^*,\frac{\pi}{2}]$, and $\tau\in(0,\frac{\epsilon_0}{2})$, with map $F_1\equiv F_1^{(\theta_w)}$ from Lemma 7.1.2,

$$\zeta_2^{(\tau,\theta_w)} := \zeta_2^{(\tau)}(F_1(\boldsymbol{\xi})) \qquad \text{for all } \boldsymbol{\xi}\in\mathcal{Q}, \tag{7.2.53}$$

so then, we note the following properties, for $\mathcal{Q}^\cup$ from Lemma 7.1.2,

$$\begin{aligned} &(\boldsymbol{\xi},\theta_w)\mapsto\zeta_2^{(\tau,\theta_w)}(\boldsymbol{\xi})\in C^3(\overline{\mathcal{Q}^\cup}),\\ &\|\zeta_2^{(\tau,\theta_w)}\|_{C^3(\mathcal{Q})}\leq C \text{ for all } \theta_w\in[\theta_{sonic},\frac{\pi}{2}],\\ \zeta_2^{(\tau,\theta_w)}=&\begin{cases}0 & \text{on } \Omega^{(v,\theta_w)}\cap\mathcal{D}_{\tau/2},\\ 1 & \text{on } \Omega^{(v,\theta_w)}\setminus\mathcal{D}_\tau\end{cases} \text{ for all } (u,v,\theta_w)\in\mathfrak{S}. \end{aligned} \tag{7.2.54}$$

We are now ready to define the coefficients in (7.2.23) by:

$$(A_{ij},A_i,B_{ij})(\boldsymbol{\xi}) = \zeta_2^{(\epsilon_{eq},\theta_w)}(\boldsymbol{\xi})(A^1_{ij},A^1_i,B^1_{ij})(\boldsymbol{\xi}) + (1-\zeta_2^{(\epsilon_{eq},\theta_w)}(\boldsymbol{\xi}))(A^2_{ij},A^2_i,B^2_{ij})(\boldsymbol{\xi}) \tag{7.2.55}$$

where the coefficients $(A^1_{ij},A^1_i,B^1_{ij})$ are from (7.2.37), and $(A^2_{ij},A^2_i,B^2_{ij})$ are from (7.2.41). We obtain the following properties:

Lemma 7.2.10. *Let the coefficients be defined as in* (7.2.55)*. Then the resulting equation satisfies:*

(i) *The exists $\lambda_0>0$, depending only on the data, such that for $\boldsymbol{\xi}\in\Omega$, and $\zeta\in\mathbb{R}^2$,*

$$\lambda_0\,\mathrm{dist}(\boldsymbol{\xi},\Gamma_{sonic})|\zeta|^2\leq\sum_{i,j=1}^2 A_{ij}(\boldsymbol{\xi})\zeta_i\zeta_j\leq\lambda_0^{-1}|\zeta|^2, \tag{7.2.56}$$

(ii) *For* $s \in (0, \frac{\epsilon_0}{2})$,

$$\begin{aligned} \|(A_{ij}, B_{ij}, A_i)\|_{0,\alpha,\Omega} &\leq N_{eq}, \\ \|(A_{ij}, B_{ij}, A_i)\|_{1,\alpha/2,\Omega\setminus\mathcal{D}_s} &\leq N_{eq} s^{-4}. \end{aligned} \tag{7.2.57}$$

(iii) *In the region* $\Omega \cap \mathcal{D}_{\epsilon_{eq}/2}$, (7.2.23) *in the* (x, y) *coordinates is* (7.2.41).

Proof. This follows directly from the construction of the coefficients, in particular using (7.2.17) again away from Γ_{sonic}. □

We now finish our formulation by stating the following criteria for removing the elliptic cutoff for a fixed point.

Lemma 7.2.11. *Let* ϵ_{eq} *be as in Lemma 7.2.9. If* $\epsilon \in (0, \frac{\epsilon_{eq}}{2}$ *in Definition 7.2.2, then the following holds. Let* $(u, v, \theta_w) \in \overline{\mathcal{K}}$ *satisfy* $\psi = \hat{\psi}$ *in* Ω, *for* $\hat{\psi}$ *being the solution of* (7.2.23). *Then, if in the* (x, y) *coordinates,*

$$|\psi| \leq (2c_2 - \mu_0/5)x \qquad \text{in } \Omega \cap \mathcal{D}_{\epsilon/10}, \tag{7.2.58}$$

then ψ *satisfies the the original nonlinear wave system* (7.2.37) *in* Ω.

Proof. This follows directly from the construction of the cutoff; firstly, we find that by (7.2.9), (7.2.58) holds in $\Omega \cap \mathcal{D}_{\epsilon_{eq}}$. Then, we see from (7.2.39), (7.2.54) and (7.2.55) that this is a sufficient condition for the cutoff to be removed. □

7.3 Solvability of the iteration problem and estimates of solutions

We now turn to solvability of the iteration problem. The main point here is that for our solutions, we attain a gain-in-regularity over the ingredients of the equation, corresponding to compactness of our iteration map.

We start by first obtaining existence, and very basic estimates.

Proposition 7.3.1. *Suppose that the parameters* σ_1, ϵ, *and* $\delta_{\mathcal{K}} > 0$ *are small depending only on the data and* θ_w^*. *Then for each* $(u, \theta_w) \in \overline{\mathcal{K}^{ext}}$ *which satisfies*

$$\|u^{\#} - u\|_{L^\infty(Q^{iter})} + \|v^{\#} - v\|_{C^1([0,1])} + |\theta_w^{\#} - \theta_w| \leq \delta_{\mathcal{K}}, \tag{7.3.1}$$

for some $(u^{\#}, v^{\#}, \theta_w^{\#}) \in \overline{\mathcal{K}}$, *then there exists a unique solution* $\hat{\psi} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus (\Gamma_{sonic} \cup \{P_2\}) \cap C(\overline{\Omega})$ *of Problem 7.2.7 determined by* (u, v, θ_w), *such that*

$$\begin{aligned} 0 \leq \hat{\psi}(x, y) \leq Cx \text{ in } \Omega \cap \mathcal{D}_{\epsilon_0}, \\ 0 \leq \hat{\psi}(x, y) \leq \overline{\psi} \text{ in } \overline{\Omega}, \end{aligned} \tag{7.3.2}$$

for C *depending only on the data, where* $\overline{\psi}$ *is from*(7.2.27).

Proof. This follows by Proposition B.0.8. □

We now turn to better estimates of $\hat{\psi}$, as well as establishing well-definedness of $\hat{r}$, aiming to gain regularity for (u, v), via showing that $(\hat{\psi}, \hat{r})$ have better regularity than (u, v). This will require very careful arguments, focusing on how the constant α in our *a priori* estimates arose; i.e. precisely how it arose from the data, or from improving previous estimates. We first focus on the estimates away from Γ_{sonic}, and the well-definedness of $\hat{r}$.

Proposition 7.3.2. *Let $\hat{\psi}$ be the solution to Problem 7.2.7 from Proposition 7.3.1. Then, $\hat{r}$ is well defined by* (7.2.28)*. Fix $s \in (0, \epsilon_0/2)$. Then there exists $\hat{\alpha} \in (0,1)$ depending on s, such that $\hat{\alpha}/2 \leq \delta_*$, where if we choose the parameters $\alpha = \hat{\alpha}/2$, $\beta > \hat{\alpha}$ in the iteration set, then the following holds (where we note the α, β in the following sentence are not the ones from the iteration set:*

For each $\alpha \in (0, \hat{\alpha})$ and $\beta \in (0, \frac{1}{2})$, there exists $C_{s,\alpha} \geq 1$ depending only on the data, θ_w^* and s, and $C_{s,\alpha,\beta}$ also depending on β such that $\hat{\psi}$ and $\hat{r}$ satisfy

$$\begin{aligned} \|\hat{\psi}\|_{1,\alpha/2,\Omega\setminus\mathcal{D}_s}^{(-\alpha),\Gamma_{sym}\cup\{P_2,P_3\}} &\leq C_{s,\alpha}, \\ \|\hat{r}\|_{2,\alpha/2,[s,\pi]}^{-(1+\beta)} &\leq C_{s,\alpha,\beta}. \end{aligned} \tag{7.3.3}$$

Proof. Firstly, by taking $\delta_{\mathcal{K}}$ and δ_4 sufficiently small, depending on σ_1, properties (7.2.13), (7.2.7) and (7.3.2) imply that $\hat{\rho}$ corresponding to $\hat{\psi}$ can be defined, and

$$\begin{aligned} r^2 - \overline{c}^2(\hat{\rho}) > \frac{\delta_3}{2}(\pi - \theta) \text{ on } \Gamma_{shock} \cap \{\pi - \theta > \sigma/10\}, \\ \hat{\rho} > \rho_1 + \frac{\tau_1}{2} \text{ in } \overline{\Omega}, \end{aligned} \tag{7.3.4}$$

thus using (7.2.31) and (7.2.33), we can use the argument of Proposition 3.6.4 to deduce that

$$r^2 - \overline{c}^2(\hat{\rho}) \geq \frac{\delta_3}{2}(\pi - \theta) \qquad \text{on } \Gamma_{shock}, \tag{7.3.5}$$

so that once we prove the bound for $\hat{\psi}$ in (10.3.4), the bound for $\hat{r}$ follows immediately by the argument in Proposition 5.2.4.

We now turn to the estimates of $\hat{\psi}$. Firstly, using [56], we can obtain $C^{0,\alpha}$ estimates near P_3, with α depending only on the ellipticity of the iteration equation near P_3 (thus the data, by (7.2.46)), and the wedge angle, but uniform in $\theta_w \in [\theta_{sonic}, \frac{\pi}{2}]$. We can then use a standard scaling argument to attain $C_{1,\alpha}^{(-\alpha),\{P_3\}}(B_R(P_3) \cap \Omega)$ regularity, for R depending on the data.

Next, we focus on P_2. Due to (7.2.33), we can obtain the algebraic growth bound as in Lemma 5.2.3, on $\Gamma_{shock} \cap B_R(P_2)$ for R depending only on the data. We then apply the argument of Proposition 5.2.4, to attain $C_{1,\alpha}^{(-\beta),\{P_2\}}(B_R(P_2) \cap \Omega))$ regularity, for some R depending on the data, R and θ_w^*. In particular, we note that in Proposition B.0.7, the exponent α only depends on the ellipticity, hence on the data and θ_w^* (7.2.56).

Now, we turn to estimates on Γ_{shock}, noting that we now only require estimates away from P_2, and on $\partial(\Omega \setminus \mathcal{D}_s)$. Due to (7.2.57) and (7.2.32), away from $P_2 \cup P_1$, the coefficients are $C^{0,1}$, and the boundary $C^{1,1}$, (with constants depending on s and the R from above) so the α in Proposition B.0.6 only depends on the data and s, via the ellipticity (7.2.56).

We now just need to obtain estimates in the interior of $\Omega \setminus \mathcal{D}_s$, and on the remaining parts of $\Gamma_{wedge} \cup \Gamma_{sym}$ by even reflection. By the first part of the proof of Proposition B.0.2, we can attain $C^{1,\alpha}$ estimates, with α depending only on the ellipticity.

We have now attained the required estimates on all of $\Omega \setminus \mathcal{D}_s$, and the proof is complete. □

We now turn to the estimate near Γ_{sonic}.

Proposition 7.3.3. *There exists $\hat{\epsilon}_{eq} \in (0, \epsilon_{eq})$, depending only on the data, such that possibly reducing the $\hat{\alpha}$ for $s = \hat{\epsilon}_{eq}/2$ in Proposition 7.3.1, depending on the data, if we make the same assumptions as Proposition 7.3.2, then*

$$\begin{aligned} \|\hat{\psi}\|^{(par)}_{1,\alpha,\Omega\cap\mathcal{D}_{\hat{\epsilon}_{eq}}} &\leq C_\alpha, \\ \|\hat{r}\|^{(-1-1)}_{2,\alpha,[\theta_1,\theta_1+\epsilon_{eq}]} &\leq C_\alpha. \end{aligned} \tag{7.3.6}$$

Proof. This follows by using the bound (7.3.2), and the argument for Proposition 6.3.1, with $\hat{\epsilon}_{eq}$ small enough. In particular, similarly to the argument used for regularity near P_2 in the above Proposition, $\hat{\alpha}$ is determined by λ_1 in (7.2.46) (hence on the data), as λ_1 determines the ellipticity of the rescaled equation used in the argument of Proposition 6.3.1. The bound for $\hat{r}$ follows as $C^{(par),1}_{1,\alpha}$ regularity implies $C^{0,1}$ regularity, and the estimate (7.3.4). □

We now can conclude the following global estimate, by combining the two Propositions, by taking $s < \frac{1}{2}\hat{\epsilon}_{eq}$;

Proposition 7.3.4. *Under the assumptions of Proposition 7.3.2 and Proposition 7.3.3, fixing any $\beta \in (0, \frac{1}{2})$ such that $\beta > \hat{\alpha}$ in the iteration set, (where we recall that $\alpha = \hat{\alpha}/2$ is assumed)*

$$\begin{aligned} \|\hat{\psi}\|^{(-\hat{\alpha}),\Gamma_{sym}}_{1,\hat{\alpha}/2,\Omega\setminus\Omega_{\epsilon_0'/10}} + \|\hat{\psi}\|^{(par)}_{1,\hat{\alpha}/2,\Omega_{\epsilon_0'}} &\leq C, \\ \|\hat{r}\|^{(-1-(\frac{1}{2}+\beta))}_{2,\hat{\alpha}/2,[\theta_1,\pi]} &\leq C. \end{aligned} \tag{7.3.7}$$

This leads to the following gain-in-regularity for u and v:

Corollary 7.3.5. *Suppose the assumptions of Proposition 7.3.4 hold. Let $\hat{u} : Q^{iter} \to \mathbb{R}$ and $v : (0,1) \to \mathbb{R}$ be determined as follows; define $\hat{u}$ by*

$$\hat{u} = \tilde{f} \circ F_1^{-1} \circ F^{-1}_{2,\mathfrak{g}_{sh}}(s,t), \tag{7.3.8}$$

where $\mathfrak{g}_{sh}$ was obtained from v. Also, we may take δ_3 small enough in the iteration set, depending on the data, so that $\hat{r}$ from (7.2.28) *satisfies*

$$\hat{\Gamma}_{shock} := \{r = \hat{r}(\theta) \mid \theta \in [\theta_1, \pi]\} \subset \mathcal{Q}, \tag{7.3.9}$$

where $\mathcal{Q}$ is from Lemma 7.1.2, and Lemma 7.1.2 **(viii)** *holds for $\hat{\Gamma}_{shock}$. Then define $\hat{v}$ by the same formula as* (7.1.26)*, using $\hat{\Gamma}_{shock}$. Then, $\hat{u}$ and $\hat{v}$ satisfy*

$$\|\hat{u}\|^{*,2}_{1,\hat{\alpha},Q^{iter}} \leq C, \qquad \|\hat{v}\|^{(-1-(\frac{1}{2}+\frac{\beta}{2}))}_{2,\hat{\alpha}/2,(0,1)} \leq C, \tag{7.3.10}$$

where C only depends on the data and θ_w^*.

Proof. This follows using Proposition 7.3.4 and the same argument as Proposition 7.1.5. In particular, the assumption $\beta > \hat{\alpha}$ ensures that the regularity of r is better than the regularity of $\hat{\psi}$ (via (7.2.17), thus the estimates transfer appropriately. □

7.3.1 Openness of the iteration set

In this subsection, we show that the iteration set $\mathcal{K}$ is relatively open in $\mathcal{C}$ from (7.2.3).

Lemma 7.3.6. *$\mathcal{K}$ is open in $\{v(1) \geq 0\} \cap \mathcal{C} \times [\theta_w^*, \frac{\pi}{2}]$.*

Proof. Properties **(i),(ii)**, the left hand side of **(iii)**, **(iv),(v),(vi)** are all open conditions as they involve strict inequalities, and due to the continuous dependence of ψ and r on (u, v) by the construction in §7.1. The right hand side of **(iii)** is a relatively open condition in $\mathcal{C}$ as it is equivalent to $u \geq 0$, by (7.1.26). Property **(viii)** follows by uniqueness of solutions to the iteration problem, using a contradiction argument; see [26][Proposition 12.8.2] for details of this argument. □

7.4 Iteration map

We now define the iteration map $\mathcal{I} \equiv \mathcal{I}^{(\theta_w)}$ on $\overline{\mathcal{K}}$, as well as prove the remaining technical conditions to apply the Degree Theory from §2.5. First, we will show continuity and compactness of the iteration map. We will then show that fixed points (u, v) of the iteration map correspond to admissible solutions. We next show that $\mathcal{I}^{(\frac{\pi}{2})}$ has non-zero degree, taking advantage of uniqueness of the normal reflection solution. We then show that fixed points cannot lie on the boundary of the iteration set. This will then complete the argument that fixed points exist, and correspond to admissible solutions. As this can be done for any $\theta_w^* \in [\theta_{sonic}, \frac{\pi}{2}]$, this shows existence of solutions up to the sonic angle. We define the iteration map $\mathcal{I}$ as follows:

Definition 7.4.1. The iteration map $\mathcal{I} : \overline{\mathcal{K}} \to \mathcal{C}$ is defined by

$$\mathcal{I}(u, v, \theta_w) = (\hat{u}, \hat{v}), \tag{7.4.1}$$

where $\hat{u}$ and $\hat{v}$ are defined in Corollary 7.3.5.

7.4.1 Continuity and compactness of the iteration map

In this Section, we establish Continuity and Compactness of the Iteration Map.

Lemma 7.4.2. *$\mathcal{I}$ is continuous, and i-compact.*

Proof. The continuity follows from the uniqueness of solutions to the iteration problem for a fixed (u, v, θ_w), and continuous dependence of the coefficients of the equation on (u, v, θ_w), by the same argument as [26][Lemma 13.2.2, Corollary 13.2.3]. The compactness follows from the estimate (7.3.10) and compact embedding results for Hölder spaces and Lemma A.1.3,, because $\delta^* < 1$, and $\hat{\alpha}/2 < \hat{\alpha}$. □

7.4.2 Fixed points are admissible solutions

In this subsection, we show that fixed points of the iteration map correspond to admissible solutions. We note that for a fixed point (u, v) of $\mathcal{I}^{(\theta_w)}$ for a wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, the corresponding (ρ, r) satisfy:

- The quasilinear equation (2.1.4) in $\Omega \setminus \Omega_{\epsilon_{eq}/2}$. In $\Omega_{\epsilon_{eq}/2}$, the equation in the (x, y) coordinates is (7.2.41)

- The Dirichlet conditions $\rho = \rho_2$ on Γ_{sonic}, and $r^2 = \overline{c}^2$ at P_2
- The nonlinear oblique boundary condition (2.2.10), with no cutoffs on the β_i coefficients, but the following cutoff on the relation for r' in terms of (ρ, r):

$$r'(\theta) = \zeta_2(\pi - \theta) g(r, \rho) + (1 - \zeta_2(\pi - \theta))(\pi - \theta)^{1/2} \zeta_1\left(\frac{g(r,\rho)}{(\pi-\theta)^{1/2}}\right), \tag{7.4.2}$$

  where g is from (2.2.10), and ζ_1 and ζ_2 are from (7.1.45).

Thus, to show that (ρ, r) is an admissible solution, we must remove these cutoffs. We first focus on the elliptic cutoff. By Lemma 7.2.11, we need to show that

$$|\psi| \leq (2c_2 - \frac{\delta_2}{5})x \text{ in } \{0 < x < \epsilon/10\}. \tag{7.4.3}$$

To show this condition, we will take advantage of (7.2.8), (7.2.22), (7.2.35) and (7.2.48).

Lemma 7.4.3. *Let (ρ, r) correspond to a fixed point (u, v) of $\mathcal{I}^{(\theta_w)}$ for some wedge angle θ_w. Then if ϵ in the iteration set is chosen sufficiently small depending on the data, ψ corresponding to ρ satisfies*

$$0 \leq \psi \leq (2 - \frac{\delta_2}{5})x \qquad \text{in } \overline{\Omega} \cap \{0 < x < \epsilon/10\}. \tag{7.4.4}$$

Proof. $\psi \geq 0$ follows from (7.3.2). For the other side, we essentially refine the constant M in that same estimate, by using the fact that due to ψ now being a fixed point, we can use (7.2.8) to control M. Let $A = 2 - \frac{\delta_2}{5}$, and set $v = Ax$. If we take ϵ_{eq} (and so ϵ) small enough so that (7.2.48) implies

$$A(C_1 + AB_{11}) \leq 0 \qquad \text{in } \Omega \cap \mathcal{D}_{eq}, \tag{7.4.5}$$

which is possible as $A > c_2$, then for $\mathcal{N}$ from (7.2.23),

$$\mathcal{N} v \leq 0 \qquad \text{in } \Omega. \tag{7.4.6}$$

We also find that

$$\nabla v \cdot \boldsymbol{\nu} = \partial_y v = 0 \qquad \text{on } \Gamma_{wedge}, \tag{7.4.7}$$

and by (7.2.35),

$$\mathcal{M} v = A\beta_1 \leq 0 \qquad \text{on } \Gamma_{shock}, \tag{7.4.8}$$

and by (7.2.8),

$$v \leq \psi \qquad \text{on } \{x = \epsilon/10\} \cap \Omega. \tag{7.4.9}$$

The result now follows by the comparison principle. □

We now focus on the oblique cutoff, taking advantage of similar considerations. From the construction of ζ_1 in §7.1.1, we need to show that

$$\frac{r}{\overline{c}}\sqrt{r^2 - \overline{c}^2} \geq \tilde{\delta}_3(\pi - \theta)^{1/2} \qquad \text{on } \Gamma_{shock} \cap \{0 < \pi - \theta < \sigma\}. \tag{7.4.10}$$

Lemma 7.4.4. *Let (ρ, r) correspond to a fixed point (u, v) of $\mathcal{I}^{(\theta_w)}$ for some wedge angle θ_w. Then ψ corresponding to ρ satisfies*

$$\frac{r}{\overline{c}}\sqrt{r^2 - \overline{c}^2} \geq \tilde{\delta}_3(\pi - \theta)^{1/2} \qquad \text{on } \Gamma_{shock} \cap \{0 < \pi - \theta < \sigma\}. \tag{7.4.11}$$

Proof. This follows by essentially the same argument as 7.4.3. Using Proposition 3.6.4, owing to (7.2.33), and (7.2.13), we deduce that

$$r^2 - \bar{c}^2 \geq \delta_3(\pi - \theta)^{1/2} > 0 \qquad \text{on } \Gamma_{shock} \cap \{0 < \pi - \theta < \sigma\}, \tag{7.4.12}$$

and then noting $\tilde{\delta}_3 = \delta_3^{1/2}$ to deduce (7.4.11). □

Thus we can conclude that;

Proposition 7.4.5. *Let (u, v, θ_w) be a fixed point of $\mathcal{I}$. Then the corresponding (ρ, r) is an admissible solution of Problem 2.4.1 in the sense of Definition 4.2.1.*

7.4.3 Unique fixed point & Fréchet derivative for $\theta_w = \frac{\pi}{2}$

We have shown in §7.4.2 that fixed points of the iteration map correspond to admissible solutions. In the case where $\theta_w = \frac{\pi}{2}$, we see that normal reflection is the unique admissible solution, which is a consequence of the convexity, as remarked at the end of Proposition 4.3.2. Therefore, the unique fixed point of $\mathcal{I}^{(\frac{\pi}{2})}$ is $(0, 0)$, i.e the (u, v) corresponding to normal reflection.

To establish that the fixed point degree of the iteration map for $\theta_w = \frac{\pi}{2}$ is non-zero, we must show that the unique fixed point $(u, v) = (0, 0)$ has non-zero index. Due to how the fixed point degree is defined in [87] for closed convex sets in Banach spaces, it is sufficient to show that the Frechét derivative of $\mathcal{I}$ satisfies the property that $I - D\mathcal{I}(0, 0, \frac{\pi}{2})$ is invertible.

As the mappings which take (u, v) to (ρ, r), and the mapping which takes $(\hat{\rho}, \hat{r})$ to $(\hat{u}, \hat{v})$ will be mutual inverses in a neighbourhood of the fixed point (as the cutoff map is equal to the identity in a neighbourhood of the fixed point), we need only consider the Fréchet derivative of the map $F : (\psi, r) \mapsto (\hat{\psi}, \hat{r})$, defined by Problem 7.2.7.

We write the Fréchet derivative of F evaluated at the normal reflection as $(A, B) = DF(\rho, r)(e, f)$, where A and e refer to the ρ direction, and (B, f) to the r direction (so that A and e are functions on Ω, and B and f are functions on $[\theta_1, \pi]$). Taking the directional derivative in direction e, f of Problem 7.2.7, we find that (A, B) satisfy the following:

$$\begin{aligned}
&\sum_{i,j=1}^{2} (A_{ij}\partial_{ij}A + \partial_z A_{ij}\partial_{ij}\psi e), \\
&\qquad + \sum_{i=1}^{2} (A_i \partial_i A + \partial_z A \partial_i \psi e) = 0 \\
&\qquad + \sum_{i,j=1}^{2} (B_{ij}\partial_i\psi\partial_j A + \partial_i A \partial_j \psi + \partial_z B_{ij}\partial_i\psi\partial_j\psi e) && \text{in } \Omega, \\
&A = 0 && \text{on } \Gamma_{sonic} \cup \{P_2\}, \\
&\nabla A \cdot \boldsymbol{\nu} = 0 && \text{on } \Gamma_{wedge} \cup \Gamma_{sym}, \\
&\beta_i \cdot \nabla A + (\partial_z \beta_i [\partial_r \rho B + A] + \partial_r B)\partial_i \rho = 0, && \text{on } \Gamma_{shock} \\
&B(\theta) = \int_{\theta_1}^{\theta} g_z(\hat{\rho}_r f(s) + A(r(s), s)) g_r B(s)\, ds && \text{for } \theta \in [\theta_1, \pi],
\end{aligned} \tag{7.4.13}$$

where (A_{ij}, A_i, B_{ij}) are from (7.2.21), and β_i are from (2.2.11). As ψ is constant, $\partial_j \psi = 0$ for all j. From (7.4.13), we see that A satisfies a strictly elliptic PDE in Ω, and either

oblique, or zero Dirichlet boundary conditions on the whole of $\partial\Omega$. Thus $A = 0$ on Ω. We now turn to B.

We find that, evaluated at the normal reflection solution, so that $\rho = \rho_2$ and $r = -\overline{c}_2 \sec\theta$,

$$g_r = \frac{\overline{c}^2 - 2r^2}{\overline{c}\sqrt{r^2 - \overline{c}^2}} = 2\tan\theta + \cot\theta, \tag{7.4.14}$$

so $B : [\theta_1, \pi] \to \mathbb{R}$ satisfies the following initial value problem;

$$\begin{aligned} B(\theta_1) &= 0, \\ B'(\theta) &= (2\tan\theta + \cot\theta)B(\theta), \end{aligned} \tag{7.4.15}$$

hence $B \equiv 0$ on $[\theta_1, \pi]$. Thus $\mathcal{I}^{(\frac{\pi}{2})}(0, 0, \frac{\pi}{2})(A, B) = (0, 0)$, the normal reflection solution is a regular value of the iteration map, and the fixed point degree is nonzero.

7.4.4 Fixed points do not lie on the boundary of the iteration set

The last remaining technical condition we require to apply the Leray-Schauder degree theory is that the fixed points of the iteration map do not lie on the relative boundary of $\mathcal{K}$ within $\mathcal{C}$. This follows directly because fixed points correspond to admissible solutions, and the choice of the constants in Definition 7.2.2 corresponding to open conditions were chosen to be strictly increased or decreased in comparison to those valid for admissible solutions. As the boundary is taken relative to $\mathcal{C}$, the conditions arising from being part of $\mathcal{C}$, such as the fact $\rho \leq \rho_2$ in Ω, do not imply that the fixed points lay on the boundary of $\mathcal{K}$ within $\mathcal{C}$.

7.4.5 Existence of admissible solutions up to the sonic angle

We have now proved that $\mathcal{I}^{(\frac{\pi}{2})}$ has non-zero index, and that all conditions required to apply the Leray-Schauder theory from §2.5 apply, and therefore $\mathcal{I}^{(\theta_w)}$ has a fixed point for any $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$. As $\theta_w^* \in (\theta_{sonic}, \frac{\pi}{2})$ is arbitrary, we have now proven existence admissible solutions for all supersonic wedge angles.

Part III

Part III - Existence up to the Detachment Angle

Chapter 8

Admissible solutions & Uniform Estimates up to the Detachment Angle

In this Chapter, we extend our definition of admissible solution and basic estimates in Chapter **??** to subsonic cases. Here, we fix $\theta_w^* \in (\theta_d, \frac{\pi}{2})$, and some of our estimates will depend on θ_w^*, and be valid when the wedge angle satisfies $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$; others will be uniform in $\theta_w \in (\theta_d, \frac{\pi}{2})$.

Remark 8.0.1. When defining the mapping to the iteration region, and constructing the iteration set, we will often need both the original estimates from Part II for supersonic cases, and estimates from Part III for subsonic cases. For convenience, we will always make clear in the following sections which estimate from Part II is being extended, and only cross-reference to the extended estimate in Part III.

8.1 Main theorem

In this Part, we will prove existence of solutions to Problem 2.4.2.

Theorem 8.1.1. *Let $\theta_w \in (\theta_d, \theta_{sonic}]$. Then there exists a region Ω of the form described by Figure 2.2, and functions $\rho \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \Gamma_{sonic}) \cap C(\overline{\Omega})$, $r \in C^2((\theta_w, \pi)) \cap C^1([\theta_w, \pi])$, that solve Problem 2.4.2.*

8.2 Admissible solutions & basic estimates for the subsonic case

8.2.1 Definition of admissible solutions

We start by extending Definition 4.2.1 to subsonic wedge angles.

Definition 8.2.1. Fix $\theta_w \in [\theta_d, \theta_{sonic}]$. A solution (ρ, r) to Problem 2.4.2 is called an admissible solution if it satisfies the following:

(i) The shock $\Gamma_{shock} = \{r = r(\theta) \mid \theta \in (\theta_1, \pi)\}$ satisfies

$$\Gamma_{shock} \subset (\Lambda \setminus B_{c_1}), \tag{8.2.1}$$

and Γ_{shock} is C^2 in it's relative interior. Moreover, Γ^{ext}_{shock}, the extension of Γ_{shock} by first reflecting in $\{\eta = 0\}$ and adding P_2 is C^1 in its relative interior, including P_2.

(ii) For some $\alpha \in (0,1)$, ρ and r satisfy

$$\begin{aligned} &\rho \in C^{0,\alpha}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega} \setminus \{P_1\}) \cap C^2(\Omega), \\ &r \in C^{1,\alpha}([\theta_w, \pi]) \cap C^2((\theta_w, \pi)) \end{aligned} \tag{8.2.2}$$

(iii) Equation (2.1.3) is strictly elliptic in $\overline{\Omega} \setminus \overline{\Gamma_{sonic}}$:

$$c^2 > |\boldsymbol{\xi}|^2 \qquad \text{in } \overline{\Omega} \setminus \overline{\Gamma_{sonic}}. \tag{8.2.3}$$

(iv) The boundary condition (2.2.11) is strictly oblique on the interior of Γ_{shock};

$$r^2 - \bar{c}^2 > 0 \qquad \text{on } \Gamma_{shock} \tag{8.2.4}$$

8.2.2 Monotonicity of ρ on Γ_{shock} and convexity of Ω

Here, we extend the monotonicity of ρ on Γ_{shock}, and the convexity of the elliptic region Ω, as described in Proposition 4.3.2 to the subsonic case with no changes, as the one-point Dirichlet condition can be treated in the same way as when the sonic arc has positive length.

Proposition 8.2.2. *Let (ρ, r) be an admissible solution of Problem 2.4.2 in the sense of Definition 8.2.1. Then ρ is monotone increasing from P_1 to P_2 on Γ_{shock}, and the elliptic region Ω is convex.*

We may then extend Corollary 4.3.3 in the following way, as it is a direct consequence of convexity combined with property (3.2.10) of the local reflection.

Corollary 8.2.3. *Let $\boldsymbol{\nu}_w$ denote the outer unit normal to Γ_{wedge}. Then, for any admissible solution to Problem 2.4.2 in the sense of Definition 8.2.1, denoting by $\boldsymbol{\nu}_{sh}$ the inner unit normal to Γ_{shock},*

$$\boldsymbol{\nu}_{sh} \cdot \boldsymbol{\nu}_w \leq -\delta, \tag{8.2.5}$$

where δ depends only on the initial data.

8.2.3 Estimates for the density ρ and the elliptic region Ω

We now extend the pointwise estimates of §4.4, noting that none of them relied on the positive length of the sonic arc, except the separation estimates. We first extend Proposition 4.4.2 and Proposition 4.4.3.

Proposition 8.2.4. *Let (ρ, r) be an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in (\theta_d, \frac{\pi}{2})$. Then there exists C depending only on the data such that*

$$\Omega \subset B_{c_2}, \tag{8.2.6}$$

$$\rho_1 < \rho \leq C, \tag{8.2.7}$$

$$\mathrm{Lip}(r) \leq C. \tag{8.2.8}$$

We now remark on the separation of the shock curve from the wedge. Clearly, the shock curve now meets the wedge at the reflection point; however, as Γ_{shock} is written in the form $\Gamma_{shock} = \{r = r(\theta)\}$, we can ensure uniform separation away from the reflection point. We will in fact find in §9.1, when doing our analysis near the reflection point that we have linear separation.

Proposition 8.2.5. *Let $r > 0$. Then there exists $C > 0$ depending only on (ρ_0, ρ_1, γ), and $C(r) > 0$ only depending on the data and r, such that for any admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1,*

$$\operatorname{dist}(P_1, \Gamma_{sym}) \geq \frac{1}{C}, \tag{8.2.9}$$

$$\operatorname{dist}(\Gamma_{shock} \setminus B_r(P_1), \Gamma_{wedge}) \geq \frac{1}{C(r)} \tag{8.2.10}$$

We now turn to the uniform lower bound for $\rho - \rho_1$. This again, follows by the same argument as for the supersonic case; the change from a Dirichlet condition on the sonic arc to a 1-point Dirichlet condition at the reflection point does not affect the argument.

Proposition 8.2.6. *Let $\theta_w^* \in (\theta_d, \frac{\pi}{2})$. There exists $\tau > 0$, depending only on the initial data, such that for any admissible solution (ρ, r) of Problem 2.4.2 in the sense of Definition 8.2.1 with wedge angle $\theta_w \in [\theta_w^*, \theta_{sonic}]$,*

$$\begin{aligned} \rho_1 + \tau < \rho \leq \rho_2 \text{ in } \overline{\Omega}, \\ r > c_1 + \tau \text{ on } \Gamma_{shock}. \end{aligned} \tag{8.2.11}$$

8.2.4 Uniform ellipticity & obliqueness estimates

We can now again use the global lower bound for $\rho - \rho_1$, and the results of §3.6 to deduce global ellipticity & obliqueness; more precisely, we extend Proposition 4.6.1 as follows:

Proposition 8.2.7. *Let $\theta_w^* \in (\theta_d, \frac{\pi}{2})$. There exists $\lambda > 0, \hat{\zeta} > 0, \sigma > 0$ depending only on the initial data and θ_w^*, such that for any admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in [\theta_w^*, \theta_{sonic}]$,*

$$\begin{aligned} c^2 - r^2 \geq \min\Big(c^2(P_1) - r^2(P_1) + \lambda|\boldsymbol{\xi} - P_1|, \hat{\zeta}\Big) \text{ in } \overline{\Omega}, \\ r^2 - \bar{c}^2 \geq \delta(\pi - \theta) \text{ on } \Gamma_{shock}. \end{aligned} \tag{8.2.12}$$

Proof. The only change here from Proposition 4.6.1 is the behaviour of the ellipticity and obliqueness at $\Gamma_{sonic} = \{P_1\}$ for subsonic cases. For the ellipticity estimate, the changed expression follows from the fact that on Γ_{sonic}, $c^2 - r^2 = c^2(P_1) - r^2(P_1) \geq 0$ for subsonic cases, rather than being equal to zero in supersonic cases; the constant $\hat{\zeta}$ arises from the condition (3.6.1). For the obliqueness estimate, we need to prove that $r^2 - \bar{c}^2$ is positive at the P_1 for subsonic solutions, with lower bound for it's value only on the data and θ_w^*. For this, we find that from (3.2.9), that at the reflection point,

$$r^2 - c^2 > \frac{2c_2^2\bar{c}_2^2}{c_2^2 + \bar{c}_2^2} - c_2^2 = \frac{c_2^2}{c_2^2 + \bar{c}_2^2}(c_2^2 - \bar{c}_2^2) > C, \tag{8.2.13}$$

where C depends only on the data, by (3.2.3) and continuous dependence on the parameters and the data. □

We remark that in the ellipticity estimate (8.2.12), if the wedge angle is subsonic but close to the sonic angle, $c^2(P_1) - r^2(P_1)$ will be small, and the first term will be the smaller one near the reflection point, but if the wedge angle is far away from the sonic angle, the second term is the smaller one. This continuous transition in elliptic structure will be crucial in our analysis.

8.2.5 Regularity away from the reflection point

We now conclude this chapter by remarking that as nothing has changed away from Γ_{sonic}, the results of Chapter 5 still apply with no changes, and we extend Propositions 5.2.1 and 5.2.4. We recall the definition of the region Ω_ϵ:

$$\Omega_\epsilon = \{\boldsymbol{\xi} \in \Omega \mid \text{dist}(\boldsymbol{\xi}, \Gamma_{sonic}) < \epsilon\}, \tag{8.2.14}$$

with the understanding that we denote $\Gamma_{sonic} = \{P_1\}$ in the subsonic case.

Proposition 8.2.8. *Let $\theta_w^* \in [\theta_d, \frac{\pi}{2})$. There exists $\alpha \in (0,1)$ and $\epsilon_0 > 0$, depending on the initial data and θ_w^*, such that for $0 < \epsilon < \epsilon_0$ and $\beta \in (0,1)$, there exists C depending on ϵ, β, θ_w^* and the data such that for any admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in [\theta_w^*, \theta_{sonic}]$,*

$$\begin{aligned} r^2 - \overline{c}^2 &\leq C(\pi - \theta) \text{ on } \Gamma_{shock}, \\ -C(\pi-\theta)^{1/2} \leq r'(\theta) &\leq -\frac{1}{C}(\pi-\theta)^{1/2} \text{ for } \theta \in [\theta_1, \pi], \\ \beta \cdot \boldsymbol{\nu} &\geq \frac{1}{C}(\pi-\theta)^{1/2} \text{ on } \Gamma_{shock}, \end{aligned} \tag{8.2.15}$$

and

$$\begin{aligned} \|\rho\|_{1,\alpha,\Omega\setminus\Omega_\epsilon}^{(-\beta),\{P_3\}} &\leq C, \\ \|r\|_{2,\alpha,[\theta_1+\epsilon,\pi]}^{(-1-\beta/2),\{\pi\}} &\leq C. \end{aligned} \tag{8.2.16}$$

Chapter 9

Regularity near Γ_{sonic} up to the Detachment Angle

In this Chapter, we will fix $\theta_w^* \in (\theta_d, \frac{\pi}{2}]$, and shall make *a priori* estimates up to Γ_{sonic}, uniform in $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$. We shall consider four separate cases;

- The *supersonic-away-from-sonic* case where the ellipticity degenerates at the sonic arc, and the sonic arc has a positive length bounded below by some constant, and the estimates will depend on this constant. This is precisely the case of Chapter 6.
- The *supersonic-near-sonic* case, where the sonic arc has a positive, but small length. In particular, we cannot use Proposition 6.3.1 to obtain estimates that are uniform as $\theta_w \downarrow \theta_{sonic}$, as the assumption (6.3.5) implies the estimates given are not uniform as the sonic arc shrinks to zero. We will derive algebraic growth estimates that are valid when the sonic arc is sufficiently small depending on the data, and use them alongside a different scaling to the supersonic-away-from-sonic case derive parabolic $C^{1,\alpha}$ estimates in this case.
- The *subsonic-near-sonic* case, where the sonic arc has zero length, but $c^2(P_1) - r^2(P_1)$ is still small. Using (8.2.12), the ellipticity is still small near the reflection point, and ρ still close to ρ_2, so that small elliptic structure and coefficient estimates, similar to the degenerate elliptic structure in the above two cases are still available, and it turns out that the method from the supersonic-near-sonic case can be used with little change.
- The *subsonic-away-from-sonic* case, where the sonic arc has zero length, and the ellipticity is strict. Also, only limited coefficient estimates are available near P_1, as $c^2 - r^2$ is no longer small, and we no longer know that $|\rho - \rho_2| \ll 1$. However, we can still use the fact $\rho \leq \rho_2$ and the fact we chose weak reflection so that(3.2.9) holds, to deduce that $\beta_2 \leq -\lambda$ near P_1. This allows us to obtain algebraic growth on Γ_{shock}, similar to the treatment of P_2, to allow us to obtain $C^{0,\beta}$ regularity up to the reflection point for any $\beta \in (0, 1)$.

Remark 9.0.1. In separating out our cases, we note that from how we define state (2) in Section 3.3, ρ is a strictly decreasing function of $\theta_w \in [\theta_d, \frac{\pi}{2}]$, and $r(P_1)$ is a strictly increasing function of θ_w, so that

$$\begin{aligned} e_{so}(\theta_w) &:= c_2^2 - r^2(P_1), \\ d_{so}(\theta_w) &:= x_{P_1} \end{aligned} \tag{9.0.1}$$

are both strictly increasing functions of θ_w. Also, from (3.3.6), $\theta_w - \theta_s$ is a strictly decreasing function of θ_w, and from (3.4.2), $\theta_1 - \theta_s$ is a strictly increasing function of θ_w, so that

$$b_{so}(\theta_w) := \theta_1(\theta_w) - \theta_w \tag{9.0.2}$$

is a strictly increasing function of θ_w. Thus we remark that controlling the ellipticity, the length of the sonic arc, and the wedge angle are completely equivalent. Thus we will characterise the four cases described above by the wedge angle; i.e. for some small $\delta_w^* > 0$, they will correspond to $\theta_w \in [\theta_{sonic} + \delta_w^*, \frac{\pi}{2}], (\theta_{sonic}, \theta_{sonic} + \delta_w^*), [\theta_{sonic} - \delta_w^*, \theta_{sonic}]$ and $(\theta_d, \theta_{sonic} - \delta_w^*]$ respectively.

9.1 Pointwise estimates for admissible solutions up to the detachment angle

In this Section, we extend the estimates made in Section 6.2, but with some differences. The key point will be that in the subsonic-near-sonic case, we have the smallness of ψ via (6.2.5) available, so all estimates of Proposition 6.2.4 still hold. In the subsonic-away-from-sonic case, this is no longer true, but we may use (3.2.9), and the fact $\rho \leq \rho_2$ from (8.2.11) to obtain useful estimates.

We first make a similar definition to Definition 6.2.1 for subsonic solutions, except now consider intersections of $\mathcal{S}_1$ with the circle of radius $r(P_1)$. Similarly to (3.4.2), these intersections are given by

$$\theta = \theta_s + \sin^{-1}\left(\frac{\overline{c}_2}{r(P_1)}\right), \tag{9.1.1}$$

where one solution, the smaller one, is $\theta = \theta_w$ by definition of P_1, and second one $\overline{\theta} \in (\theta_w, \pi)$ exists, as by (3.2.9), $r_{P_1} \geq \frac{\overline{c}_2^2}{\gamma}$, using $\overline{c}^2(\rho_2, \rho_1) \geq \overline{c}^2(\rho_2, 0) = \frac{c_2^2}{\gamma}$.

Definition 9.1.1. Let $\theta_w \in (\theta_d, \theta_{sonic}]$ be a subsonic or sonic wedge angle. $P_0 = P_1 = P_4$, and $\overline{\Gamma_{sonic}} = \{P_1\}$, so we define the following points:

- $\overline{P}_1$ as the second intersection of $\mathcal{S}_1$ with $\partial B_{r(P_1)}$, which exists by the discussion above,
- Q as the midpoint of the arc $P_1\overline{P}_1$,
- Q' as the midpoint of the line segment $P_1\overline{P}_1$.

We note that from how $P_1, \overline{P}_1$ and Q are defined here,

$$r_Q = r_{P_1}, \text{ and } \theta_{\overline{P_1}} = \frac{\pi}{2} + \theta_s \in (\theta_w, \pi),$$

We note that again, that $\mathcal{S}_1 \cap \{\theta \in [\theta_w, \theta_Q)\}$ can be written as $\{y = \hat{f}_0(x)\}$, for $\hat{f}_0(x)$ given by (6.2.1), and Proposition 6.2.2 can be extended as follows:

Proposition 9.1.2. *Let $\theta_w \in (\theta_d, \frac{\pi}{2}]$. Then there exists $\epsilon_0 > 0, \omega > 0$ and $M \geq 1$ such that*

$$2\omega < f_0'(x) \leq M \qquad \text{for } x \in (-\epsilon_0, \epsilon_0), \tag{9.1.2}$$

and

$$\|\hat{f}_0\|_{C^3([-\epsilon_0, \epsilon_0])} \leq M. \tag{9.1.3}$$

We will now establish estimates for ψ near Γ_{sonic}, that are uniform up to the detachment angle. We first obtain from Proposition 8.2.6 that

Lemma 9.1.3. *Let (ρ, r) be an admissible solution of Problem 2.4.1 in the sense of Definition 8.2.1, and $\psi = c_2^2 - c^2(\rho)$. Then there exists C depending only on the initial data such that;*

$$0 \leq \psi \leq C \qquad \text{in } \overline{\Omega} \tag{9.1.4}$$

We now extend Proposition 6.2.4. The key point here is that though we no longer yet have any estimate of the smallness of ψ, we can instead use (8.2.11) to deduce that we can still write Ω_ϵ in the (x, y) coordinates from §3.5. We can also use the fact we chose weak reflection, to use (3.2.9) to attain estimates of β_1, β_2.

Proposition 9.1.4. *Let $\theta_w^* \in (\theta_d, \frac{\pi}{2})$. There exists $C > 0$, and $\epsilon_1 > 0$, depending on θ_w^*, such that if (ρ, r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, and $\epsilon \in (0, \epsilon_1)$,*

(i) *Ω_ϵ can be written in the form*

$$\Omega_\epsilon = \{0 < x - x_{P_1} < \epsilon, \theta_w < y < f(x)\}, \tag{9.1.5}$$

where $f'(x)$ is given by the formula (3.5.11)*, and*

$$\partial\Omega_\epsilon \cap \Gamma_{shock} = \{(x, y) \mid 0 < x - x_{P_1} < \epsilon, y = f(x)\}. \tag{9.1.6}$$

(ii) *The boundary condition on $\Gamma_{shock} \cap \{0 < x - x_{P_1} < \epsilon\}$ can be written in the form*

$$\hat{\beta} \cdot \nabla\psi = \hat{\beta}_1 \psi_x + \hat{\beta}_2 \psi_y = 0, \tag{9.1.7}$$

with expressions as in (3.5.14)*.*

(iii) *For some $\omega > 0$, $M > 0$,*

$$\omega \leq f'(x) \leq M, \tag{9.1.8}$$

and

$$f'(x) = h(x, \psi) \qquad \text{for } x \in (0, \epsilon), \tag{9.1.9}$$

where $h \in C^\infty(\mathbb{R} \times \mathbb{R})$, and $\partial_x^{(k-1)} h(x, 0) = \hat{f}_0^{(k)}(x)$ for all $k \geq 1$,

(iv) *The coefficients $(\hat{\beta}_1, \hat{\beta}_2)$ in the oblique boundary condition satisfy*

$$\begin{aligned} \hat{\beta} \cdot \boldsymbol{\nu} &\geq \frac{1}{C}, \\ \hat{\beta}_2 &\leq -\frac{1}{C} \end{aligned} \tag{9.1.10}$$

where $\boldsymbol{\nu}$ is the inner unit normal to Ω, expressed in (x, y) coordinates.

Proof. For $(x, y) \in \Omega_\epsilon$, using the fact $\rho \leq \rho_2$ from Proposition 8.2.6, and the bound (3.2.9), we find that for ϵ_1 sufficiently small depending only on the data

$$\begin{aligned} r^2 - \bar{c}^2 &\geq r(P_1)^2 - \bar{c}_2^2 - C\epsilon_1 \\ &\geq \frac{\bar{c}_2^2(c_2^2 - \bar{c}_2^2)}{c_2^2 + \bar{c}_2^2} - C\epsilon_1 \\ &\geq \frac{1}{C}, \end{aligned} \tag{9.1.11}$$

which alongside (9.1.4) and the explicit expressions (3.5.11), (3.5.14) and (3.5.16), implies **(i),(ii),(iii),(iv)** and the first inequality in **(v)**. For the final inequality in **(v)**, we note that from the explicit expression in (3.5.14), and conditions (3.2.9) and (8.2.11),

$$\begin{aligned}\hat{\beta}_2 &= f'(x)\big(4c^2\bar{c}^2 - r^2(3c^2+\bar{c}^2\big)\\ &\leq \frac{f'(x)}{c_2^2+\bar{c}_2^2}(4c^2c_2^2\bar{c}^2 + 4c^2\bar{c}^2\bar{c}_2^2 - 3c^2c_2^2 - c_2^2\bar{c}^2\bar{c}_2^2)\\ &\leq \frac{f'(x)c_2^2\bar{c}_2^2}{c_2^2+\bar{c}_2^2}\big(\bar{c}^2-c^2\big)\\ &\leq -\frac{1}{C}.\end{aligned} \tag{9.1.12}$$

□

We now extend Lemma 6.2.3 from supersonic cases, to the subsonic-near-sonic case.

Lemma 9.1.5. *There exists $\epsilon_1 \in (0,\epsilon_0)$ $\delta > 0$ and $\delta_w^* > 0$, depending only on (ρ_0,ρ_1,γ), such that if (ρ,r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with $\theta_w \in [\theta_{sonic}-\delta_w^*,\frac{\pi}{2}]$, then the corresponding ψ satisfies*

$$0 \leq \psi \leq (2c_2-\delta)x \qquad \text{for } x \in (0,\epsilon_1). \tag{9.1.13}$$

Proof. This follows in the same way as Lemma 6.2.3. If δ_w^* is sufficiently small depending only on the data, then by Remark 9.0.1, $c_2^2 - r^2(P_1)$ is sufficiently small that the first term is the smaller one in (8.2.12) and

$$\lambda(x-x_{P_1}) \leq c^2 - (c_2-x)^2 = -\psi + 2c_2x - x^2, \tag{9.1.14}$$

so the result follows in the same way as Lemma 6.2.3. □

With this bound in hand, we may now fully extend Proposition 6.2.4 to subsonic-near-sonic wedge angles.

Lemma 9.1.6. *There exists $\delta_w^* > 0$ such that if $\rho, r)$ $\theta_w \in [\theta_{sonic}-\delta_w^*,\frac{\pi}{2}]$, then the remaining estimates of Proposition 6.2.4 hold; i.e. in addition to the estimates of Proposition 9.1.4 above, we also have;*

(ii)'

$$\frac{|f'(x)-\hat{f}_0'(x)|}{x} + \frac{|f(x)-\hat{f}_0(x)|}{x^2} \leq C, \tag{9.1.15}$$

(iv)'

$$\hat{\beta}_1 \leq -\frac{1}{C}. \tag{9.1.16}$$

We also have the following estimate, which extends (9.1.13) to all subsonic wedge angles, but with a weaker constant. This estimate is somewhat artificial, but is necessary when defining the iteration set to allow a continuous transition between cases.

Lemma 9.1.7. *There exists $\epsilon_0 > 0$ and C depending only on $(\rho_0,\rho_1,\gamma,\theta_w^*$ such that if (ρ,r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1,*

$$0 \leq \psi \leq Cx \qquad \text{in } \Omega_{\epsilon_0}. \tag{9.1.17}$$

Proof. From Lemma 9.1.5, there exists $\delta_w^* > 0$, depending only on the data, such that if the wedge angle θ_w satisfies $\theta_w \geq \theta_{sonic} - \delta_w^*$, then (9.1.17) holds. Then, from Remark 9.0.1, there exists $\hat{d} > 0$, such that if $\theta_w \in (\theta_{detach}, \theta_{sonic} - \delta_w^*]$, then $x_{P_1} \geq \hat{d}$, so that from (9.1.5), $x \geq \hat{d}$ in Ω_{ϵ_0}. Thus, by (9.1.4),

$$0 \leq \psi \leq \frac{C}{\hat{d}} x \qquad \text{in } \Omega_{\epsilon_0}, \text{ if } \theta_w \in (\theta_d, \theta_{sonic} - \delta_w^*], \tag{9.1.18}$$

so combining (9.1.13) and (9.1.18), we obtain (9.1.17). □

9.2 Estimates up to the sonic arc or reflection point

9.2.1 The supersonic-away-from-sonic case

We first examine at the *supersonic-away-from-sonic* case. This is simply the case from Part II, with estimates given by Corollary 6.3.2, by choosing $\theta_w^* = \theta_{sonic} + \delta_w^*$ for $\delta_w^* > 0$.

Corollary 9.2.1. *Let $\delta_w^* > 0$, ϵ_0 as in Proposition 9.1.2. Then there exists $\alpha \in (0,1)$, depending on the data and δ_w^*, such that for each $\beta \in (0,1)$ and $\epsilon \in (0, \epsilon_0]$, there exists C depending on the initial data, ϵ, δ_w^* and β such that if (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_{sonic} + \delta_w^*, \frac{\pi}{2}]$, then*

$$\begin{aligned} &\psi \in C^{0,1}(\overline{\Omega} \setminus \{P_2\}), \\ &\|\psi\|^{(-\alpha),\{P_2\}}_{1,\beta,\Omega\setminus\Omega_{\epsilon/10}} + \|\psi\|^{(par)}_{1,\alpha,\Omega_\epsilon} \leq C, \\ &\|r\|^{(-1-\beta/2),\{\theta_1,\pi\}}_{2,\alpha,[\theta_1,\pi]} + \|f - \hat{f}_0^{(\theta_w)}\|^{(par)}_{2,\alpha,(0,\epsilon)} \leq C. \end{aligned} \tag{9.2.1}$$

9.2.2 The supersonic-near-sonic case

As $\theta_w \downarrow \theta_{sonic}$, Proposition 6.3.1 no longer provides estimates in Ω_ϵ, uniform in ϵ, so a different technique must be used. The aim here is to find estimates for admissible solutions with wedge angles $\theta_w \in (\theta_{sonic}, \theta_{sonic} + \delta_w^*)$, for δ_w^* small. The outline of the procedure is as follows:

(i) Firstly, using Proposition 6.3.1, we may obtain estimates in $\Omega_{b_{so}^2}$ for $b_{so}(\theta_w)$ as defined in Remark 9.0.1. We note that $b_{so}(\theta_w) > 0$, but $b_{so}(\theta_w) \downarrow 0$ as $\theta_w \downarrow \theta_{sonic}$, so the estimates we obtain here are not uniform in θ_w up to the sonic angle.

(ii) We next, using Proposition 9.1.4, in particular (9.1.5) and (9.1.8), note that for k depending only on the initial data,

$$\begin{aligned} &\Omega_{\epsilon_0} \subset \{x \in (0, \epsilon_0), y \in (0, b_{so} + kx)\}, \\ &\{x \in (0, \epsilon_0), y \in (0, b_{so} + \frac{1}{k}x)\} \subset \Omega_{\epsilon_0}. \end{aligned} \tag{9.2.2}$$

where $k > 1$ can be chosen uniformly for all supersonic reflection solutions, depending only on the data. We will show that for all integers $m \geq 2$, and sufficiently small b_{so} depending only on the data and m, a solution $\psi(x,y)$ has the following growth in $\Omega_\epsilon \cap \{x > \frac{b_{so}^2}{10}\}$:

$$0 \leq \psi(x,y) \leq Cx^m \qquad \text{in } \Omega_\epsilon \cap \{x > \frac{b_{so}^2}{10}\}, \tag{9.2.3}$$

for some $\epsilon > 0$ and C, both depending only on the data and m. In the proof, we use the smallness of b_{so}, and the elliptic degeneracy near Γ_{sonic}. The main point is that b_{so} can be arbitrarily small, but once b_{so} is small enough depending on the data, C is independent of b_{so}.

(iii) We use the algebraic growth with $m = 3$ to obtain estimates of ψ in $\Omega_\epsilon \cap \{x > \frac{b_{so}^2}{10}\}$ by using a different scaling from that in the supersonic-away-from-sonic case. As the length of the sonic arc is now small, rectangles defined by (6.3.7) do not satisfy the property (6.3.10). Instead, we note that by (9.2.2), we can use a rectangle with side length proportional to x_0 to have a property similar to (6.3.10). Also, for the degenerate elliptic structure (8.2.12) to become a uniformly elliptic equation for the rescaled function, we require the ratio of the side lengths in the $x-$ and $y-$ directions to be proportional to $\sqrt{x_0}$. Therefore, for $(x_0, y_0) \in \Omega_\epsilon \cap \{x > \frac{b_{so}^2}{10}\}$, we consider the rectangle;

$$\hat{R}_{(x_0,y_0)} := \{|x - x_0| < x_0^{3/2}, |y - y_0| < \frac{x_0}{10k}\} \cap \Omega, \tag{9.2.4}$$

so that the ratio between the side lengths is $10k\sqrt{x_0}$; i.e. the same as (6.3.7), up to a multiplicative constant that is uniform in θ_w. These rectangles do satisfy the property that they *fit in* to Ω; i.e. rectangles centred on points on Γ_{wedge} do not intersect Γ_{shock}, and rectangles centred on point on Γ_{shock} do not intersect Γ_{wedge}. Then, rescaling to a portion of the square $(-1, 1)^2$,

$$\hat{Q}_{(x_0,y_0)} := \{(S, T) \in (-1, 1)^2 \mid (x_0 + x_0^{3/2} S, y_0 + \frac{x_0}{10k} T) \in \Omega\}, \tag{9.2.5}$$

we obtain a uniformly elliptic equation for

$$\psi^{(x_0,y_0)}(S, T) := \frac{1}{x_0^3} \psi(x_0 + x_0^{3/2} S, y_0 + \frac{x_0}{10k} T) \quad \text{for } (S, T) \in \hat{Q}_{(x_0,y_0)}, \tag{9.2.6}$$

and the algebraic growth of ψ when $m = 3$ implies $|\psi^{(x_0,y_0)}| \leq M$, for M depending only on the data. Then, a similar argument to Proposition 6.3.1 implies $C^{1,\alpha}$ estimates for the rescaled functions. Then, combining with the estimates from Proposition 6.3.1 mentioned in part **(i)**, we obtain the $C^{(par)}_{1,\alpha}$ estimates for ψ.

We now prove our algebraic growth bounds, working in the (x, y) coordinates from §3.5. We consider equation (3.5.4) as a quasilinear equation, obtained by freezing the coefficients at ψ;

$$\mathcal{Q}\psi := \sum_{i,j=1}^{2} a_{ij}(x, y)\partial_{ij}\psi + a_1(x, y)\psi_x + \sum_{i=1}^{2} b_{ii}(x, y)(\partial_i \psi)^2 = 0. \tag{9.2.7}$$

Using (3.5.5) and (6.2.5), we have the following structure for (9.2.7) in Ω_ϵ, with sufficiently small $\epsilon > 0$ depending only on the data;

$$\begin{aligned} &C^{-1}x \leq a_{11}(x, y) \leq Cx, \quad \frac{1}{C} \leq a_{22}(x, y) \leq C, \\ &|(a_1, b_{11}, c_{11})(x, y)| \leq C, \qquad\qquad b_{11}, b_{22} \leq 0. \end{aligned} \tag{9.2.8}$$

where C only depends on the data. Furthermore, using (3.5.13), the boundary condition on Γ_{shock} can be written as a linear condition;

$$\mathcal{N}\psi := \mathbf{b}(x, y) \cdot \nabla\psi = 0 \qquad \text{on } \Gamma_{shock} \cap \partial\Omega_\epsilon, \tag{9.2.9}$$

where from (9.1.10) and (9.1.16),

$$-C \leq b_i \leq -\frac{1}{C} \qquad \text{on } \Gamma_{shock} \cap \partial\Omega_\epsilon \text{ for } i = 1, 2, 3, \tag{9.2.10}$$

and the boundary condition (9.2.9) is strictly oblique in the sense that

$$\mathbf{b} \cdot \boldsymbol{\nu} > \frac{1}{C}. \tag{9.2.11}$$

Also, the slip boundary condition (2.2.8) is

$$\partial_y \psi = 0 \qquad \text{on } \Gamma_{wedge} \cap \partial\Omega_\epsilon. \tag{9.2.12}$$

Also, recall that from (9.1.4), ψ satisfies

$$\begin{aligned} \psi &= 0 \text{ on } \Gamma_{sonic}, \\ \psi &\leq C \text{ on } \Omega_\epsilon. \end{aligned} \tag{9.2.13}$$

Also, from the structure estimates in Proposition 9.1.4, in particular (9.1.8), there exists $k > 0$ depending on (ρ_0, ρ_1, γ) such that

$$\begin{aligned} \Omega_\epsilon \subset \{x \in (0, \epsilon), 0 < y - \theta_w < 0, b_{so} + kx\}, \\ \{x \in (0, \epsilon), 0 < y - \theta_w < b_{so} + \frac{1}{k}x \subset \Omega_\epsilon\}. \end{aligned} \tag{9.2.14}$$

We now show the desired algebraic growth bound by constructing a suitable supersolution to (9.2.7) with boundary conditions (9.2.9) and (9.2.12).

Lemma 9.2.2. *Let $m \geq 2$ be an integer. Then there exists $\delta_w^* > 0, \hat{\epsilon} > 0$ and $\hat{C} > 0$ depending only on m and the initial data, such that (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with $\theta_w \in (\theta_{sonic}, \theta_{sonic} + \delta_w^*)$, and $\epsilon \leq \hat{\epsilon}$, we have*

$$\psi(x, y) \leq \hat{C} \frac{\max_{\partial\Omega_\epsilon \cap \{x=\epsilon\}} \psi}{\epsilon^m} x^m \qquad \text{in } \Omega_\epsilon \cap \{x > \frac{b_{so}^2}{10}\}, \tag{9.2.15}$$

for $b_{so}(\theta_w)$ as defined in Remark 9.0.1.

Proof. We use a similar barrier function to [26][Lemma 16.4.1]. The difference is the presence of the quasilinear b_{11}, b_{22} terms (which are negative, so do not hinder us), and the a_1 term is now positive. We write out the full construction & proof here. Without loss of generality, applying a linear transformation, we may assume $\theta_w = 0$. We now show that for suitable $M, L > 0$, for any $A > 0$, the following function

$$v(x, y) = A\big((x + Mb_{so}^2)^m - L(x + Mb_{so}^2)^{m-1} y^2\big) \tag{9.2.16}$$

is a positive supersolution of equation (9.2.7) with boundary conditions (9.2.9) and (9.2.12). Then, choosing A suitably implies (9.2.15). In the following proof, the dependence is as follows; we set $M = NL$, and choose L, N large depending only on the data via (9.2.8) and (9.2.10). The smallness condition is explicitly:

$$b_{so} \leq \frac{1}{LMN}, \qquad \epsilon \leq \frac{1}{LN}, \tag{9.2.17}$$

which in particular implies $b_{so} \leq \frac{1}{N}$. We now use (9.2.14) to obtain that for $(x, y) \in \Omega_\epsilon$,

$$\begin{aligned} \frac{1}{A(x + Mb_{so}^2)^{m-1}} v(x, y) &= x + Mb_{so}^2 - Ly^2 \\ &\geq x + Mb_{so}^2 - L(b_{so} + kx)^2 \\ &\geq x + Mb_{so}^2 - 2Lb_{so}^2 - 2Lk^2x^2. \end{aligned} \tag{9.2.18}$$

Then, choosing $M = NL \geq 1$, and assuming ϵ satisfies (9.2.17), and choosing $N = 4\max\{1, k^2\}$, and using that $x \in (0, \epsilon)$ in Ω_ϵ, we have

$$v(x, y) \geq \frac{A}{2}(x + Mb_{so}^2)^m > 0. \tag{9.2.19}$$

Similarly, we find that

$$v_x(x, y) \geq A\frac{m}{2}(x + Mb_{so}^2)^{m-1} > 0, \tag{9.2.20}$$

and

$$v_{xx}(x, y) \geq A\frac{m(m-1)}{2}(x + Mb_{so}^2)^{m-2} \geq 0, \tag{9.2.21}$$

by further increasing N if necessary. We also find that

$$|v_{xx}(x, y)| \leq Am(m-1)(x + Mb_{so}^2)^{m-2}. \tag{9.2.22}$$

We now turn to estimating v_y and v_{yy}. Again, using (9.2.14) and (9.2.17) Ω_ϵ, so

$$\begin{aligned} |v_y(x, y)| &= A| - 2L(x + Mb_{so}^2)^{m-1}y| \\ &\leq 2AL(x + Mb_{so}^2)^{m-1}(b_{so} + kx) \\ &\leq \frac{2AC}{N}(x + Mb_{so}^2)^{m-1} \end{aligned} \tag{9.2.23}$$

and

$$v_{yy}(x, y) = -2AL(x + Mb_{so}^2)^{m-1} \tag{9.2.24}$$

Now, we combine these estimates to show that v is a supersolution; we first focus on (9.2.7). We find that

$$\begin{aligned} &|a_{11}v_{xx}| \leq ACx(x + Mb_{so}^2)^{m-2} \leq C(x + Mb_{so}^2)^{m-1}, \\ &a_{22}v_{yy} \leq -2LAC(x + Mb_{so}^2)^{m-1}, \\ &b_{ii}v_{ii} \leq 0, \\ &|a_1v_x| \leq C(x + Mb_{so}^2)^{m-1}, \end{aligned} \tag{9.2.25}$$

so that taking L large ensures $Qv \leq 0$. Next, using (9.2.10), (9.2.20) and (9.2.23), we can ensure $\mathcal{N}v \leq 0$ by taking N large. Also $v_y = 0$ on $\{y = 0\}$; so we have now shown that v is a supersolution of (9.2.7), (9.2.9) and (9.2.12). To complete the proof, we note that if we chose $A = 2\frac{\max_{\partial\Omega_\epsilon \cap \{x=\epsilon\}} \psi}{\epsilon^m}$, then from (9.2.19),

$$\begin{aligned} &v \geq 0 = \psi && \text{in } \Omega_\epsilon, \\ &v \geq \max_{\Omega_\epsilon \cap \{x=\epsilon\}} \psi \geq \psi && \text{on } \Omega_\epsilon \cap \{x = \epsilon\}, \end{aligned} \tag{9.2.26}$$

so that by the maximum principle,

$$\psi \leq v \qquad \text{on } \Omega_\epsilon, \tag{9.2.27}$$

and thus (9.2.15) follows, as when $x > b_{so}^2/10$, (9.2.16) implies

$$v(x,y) \leq A(1+\frac{M}{10})^m x^m. \tag{9.2.28}$$

□

Combining with (9.1.4) allows us to deduce the following:

Proposition 9.2.3. *For every integer $m \geq 2$, there exists small $\delta_w^*, \epsilon > 0$ and large C, depending only on the initial data and m, such that if (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in (\theta_{sonic}, \theta_{sonic} + \delta_w^*)$, then*

$$0 \leq \psi(x,y) \leq Cx^m \quad \text{in } \Omega_\epsilon \cap \{x > \frac{b_{so}^2(\theta_w)}{10}\}, \tag{9.2.29}$$

where b_{so} is defined in Remark 9.0.1.

Now, using the growth estimate in Proposition 9.2.3, we will prove *a priori* estimates of solutions in the weighted and scaled $C^{1,\alpha}$ spaces. Analogously to [26][Lemma 16.4.3], we extend the argument of Proposition 6.3.1. We will prove results for ψ that satisfy an equation, boundary condition and growth conditions on a domain of appropriate shape, with more general rescalings, without necessarily assuming ψ is a regular reflection solution. This result will also be used for the subsonic-near-sonic case, an for the iteration problem.

Lemma 9.2.4. *Let $\theta_w \in \mathbb{R}$, and $\epsilon^* > 10x_{P_1} \geq 0$, and let Ω_{ϵ^*} be of structure* (9.1.5) *and* (9.1.8)*, with $f(0) \geq \theta_w$; i.e. including both supersonic and subsonic cases. Note that $x \equiv x_{P_1}$ on $\overline{\Gamma_{sonic}}$. Denote*

$$d_{so}(x) := x - x_{P_1}, \tag{9.2.30}$$

so $d_{so} > 0$ in Ω_{ϵ^*}, and $d_{so} = 0$ on $\overline{\Gamma_{sonic}}$. Let $g, h \in C([x_{P_1}, x_{P_1} + \epsilon^*])$ satisfy that on $(x_{P_1}, x_{P_1} + \epsilon^*]$,

$$g(\cdot) \text{ and } h(\cdot) \text{ are monotone increasing}, \tag{9.2.31}$$

$$0 < h(x) \leq \frac{d_{so}(x)}{4\sqrt{x}}, \tag{9.2.32}$$

$$0 < g(x) \leq x^{1/2} h(x), \tag{9.2.33}$$

$$\frac{g(x)}{g(x - d_{so}(x)/2)} \leq M, \qquad \frac{h(x)}{h(x - d_{so}(x)/2)} \leq M, \tag{9.2.34}$$

Let $\epsilon \in (0, \frac{\epsilon^*}{2})$ and let $\psi \in C(\overline{\Omega}_{2\epsilon}) \cap C^1(\overline{\Omega}_{2\epsilon} \cap \{x_{P_1} < x < x_{P_1} + 2\epsilon\}) \cap C^2(\Omega_{2\epsilon})$ satisfy the equation;

$$\sum_{i,j=1}^{2} A_{ij}(\psi, x, y)\partial_{ij}^2 \psi + \sum_{i=1}^{2} A_i \partial_i \psi + \sum_{i,j=1}^{2} B_{ij}\psi_i \psi_j = 0, \tag{9.2.35}$$

with the degenerate elliptic structure

$$\lambda|\zeta|^2 \leq \frac{A_{ij}}{x^{2-\frac{i+j}{2}}}\zeta_i\zeta_j \leq \lambda^{-1}|\zeta|^2, \tag{9.2.36}$$

and the bounds

$$\begin{aligned}
&|A_{ij}(\psi(x,y),x)| \le Mx^{2-\frac{i+j}{2}},\\
&\|(A_i,B_{ij})\|_{L^\infty(\mathbb{R}\times\Omega_{2\epsilon})} + \|D_{z,x}(A_{ij},A_i,B_{ij})\|_{L^\infty(\mathbb{R}\times\Omega_{2\epsilon})} \le M,\\
&|x^{\frac{i+j-3}{2}}h(x)\partial_x A_{ij}(z,x,y)| + |x^{\frac{i+j}{2}-2}h(x)\partial_y A_{ij}(z,x,y)| \le M,\\
&|x^{\frac{i-1}{2}}h(x)^2\partial_x A_i(z,x,y)| + |x^{\frac{i}{2}-1}h^2(x)\partial_y A_i(z,x,y)| \le M,\\
&|x^{\frac{i+j-3}{2}}g(x)h(x)\partial_x B_{ij}(z,x,y)| + |x^{\frac{i+j}{2}-2}g(x)h(x)\partial_y B_{ij}(z,x,y)| \le M.
\end{aligned} \tag{9.2.37}$$

Also, let ψ satisfy the boundary conditions and estimates:

$$\begin{aligned}
&\psi_x = \beta(\psi(x,y),x,y)\psi_y \quad &&\text{on } \Gamma_{shock}\cap\partial\Omega_{2\epsilon},\\
&\psi_y = 0 &&\text{on } \Gamma_{wedge}\cap\partial\Omega_{2\epsilon},
\end{aligned} \tag{9.2.38}$$

where

$$\begin{aligned}
\|\beta(\cdot,\mathbf{x})\|_{C^{0,1}(\mathbb{R})} &\le M,\\
\|\beta(z,\cdot)\|_{C^{0,\alpha}(\Gamma_{shock}\cap\partial\Omega_{2\epsilon})} &\le M,\\
\|f\|_{C^{0,1}([x_{P_1},x_{P_1}+\epsilon])} &\le M.
\end{aligned} \tag{9.2.39}$$

Assume $x_{bot}\in[x_{P_1},x_{P_1}+\epsilon)$ and

$$|\psi(x,y)| \le Mg(x) \qquad \text{for all } (x,y)\in\Omega_{2\epsilon}\cap\{d_{so}(x) > \frac{d_{so}(x_{bot})}{2}\}. \tag{9.2.40}$$

For $z_0=(x_0,y_0)\in(\Omega_\epsilon\cap\{x>x_{bot}\}$ and $r\in(0,1]$, consider the rectangle:

$$\begin{aligned}
\tilde{R}_{z_0,r} &:= \{(x,y)\mid |x-x_0| < r\sqrt{x_0}h(x_0),\quad |y-y_0| < rh(x_0)\},\\
\hat{R}_{z_0,r} &:= \tilde{R}_{z_0,r}\cap\Omega_{2\epsilon}.
\end{aligned} \tag{9.2.41}$$

We assume that $f(\cdot)$, $h(\cdot)$ and ϵ are such that

$$\begin{aligned}
&\hat{R}_{z_w,1}\cap\Gamma_{shock} = \emptyset \qquad \text{for all } z_w=(x,0)\in\Gamma_{wedge}\cap\{x_{bot}<x<x_{P_1}+\epsilon\},\\
&\hat{R}_{z_s,1}\cap\Gamma_{wedge} = \emptyset \quad \text{for all } z_s=(x,f(x))\in\Gamma_{shock}\cap\{x_{bot}<x<x_{P_1}+\epsilon\}.
\end{aligned} \tag{9.2.42}$$

Furthermore, introducing by (S,T) the invertible change of variables

$$(x,y) = (x_0+\sqrt{x_0}h(x_0)S, y_0+h(x_0)T), \tag{9.2.43}$$

we find there exists $Q_r^{(z_0)}\subset Q_r := (-r,r)^2$ such that

$$\hat{R}_{z_0,r} := \Big\{(x_0+\sqrt{x_0}h(x_0)S, y_0+h(x_0)T)\mid (S,T)\in Q_\rho^{(z_0)}\Big\}. \tag{9.2.44}$$

Define

$$\psi^{(z_0)}(S,T) := \frac{1}{g(x_0)}\psi(x_0+\sqrt{x_0}h(x_0)S, y_0+h(x_0)T). \tag{9.2.45}$$

Then, there exists $\epsilon_1^*\in(0,\epsilon^*)$ and $\alpha\in(0,1)$ depending only the constant M, and the constants ω and M from the structure (9.1.8)*, such that if $\epsilon\le\frac{\epsilon_1^*}{2}$, $x_{P_1}\le\frac{\epsilon_1^*}{2}$,*

$$\|\psi^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/100}^{(z_0)}})} \le C(\alpha) \qquad \text{for } z_0\in\overline{\Omega}_\epsilon\cap\{x>x_{bot}\}. \tag{9.2.46}$$

Proof. We highlight the changes required to extend Proposition 6.3.1, which is the special case where $x_{P_1} = x_{bot} = 0$, $h(x) = \frac{\sqrt{x}}{4}$, $g(x) = \frac{x}{4}$.

Step 1 From (9.1.5) and (9.1.8), it follows that for any $(x_0, y_0) \in \Omega_\epsilon$ and $r \in (0, 1]$,

$$\begin{aligned}\hat{R}_{z_0,r} &\subset \Omega \cap \{(x,y) \mid x - \frac{d_{so}(x_0)}{4} < x < x_0 + \frac{d_{so}(x_0)}{4}\} \\ &\subset \Omega \cap \{(x,y) \mid \frac{3x_0}{4} < x < \frac{5x_0}{4}\} \subset \Omega_{2\epsilon}.\end{aligned} \tag{9.2.47}$$

In particular,

$$\overline{\hat{R}_{z,r}} \cap \Gamma_{sonic} = \emptyset \qquad \text{for all } z \in \Omega_\epsilon. \tag{9.2.48}$$

Next, as in the proof of Proposition 6.3.1, we consider the three cases of rectangles $R_{z_0,r}$;

(i) $\hat{R}_{z_0,1/10}$, for $z_0 \in \Omega_\epsilon \cap \{x > x_{bot}\}$ where $\hat{R}_{z_0,1/10} = \tilde{R}_{z_0,1/10}$,

(ii) $\hat{R}_{z_0,1/2}$, for $z_0 \in \Gamma_{wedge} \cap \{x_{bot} < x < x_{P_1} + \epsilon\}$,

(iii) $\hat{R}_{z_0,1}$ for $z_0 \in \Gamma_{shock} \cap \{x_{bot} < x < x_{P_1} + \epsilon\}$.

Then, using structure (9.1.8), we see this is equivalent to taking three cases for $Q_r^{(z_0)}$;

(i) $Q_{10}^{(z_0)} = Q_{1/10}$,

(ii) $Q_{1/2}^{(z_0)} = Q_{1/2} \cap \{T > 0\}$,

(iii)

$$Q_1^{(z_0)} = \Big\{(S,T) \in Q_1 \mid -1 < T < F^{(z_0}(S)\Big\}, \tag{9.2.49}$$

where

$$F^{(z_0)}(S) := \frac{\hat{f}(x_0 + \sqrt{x_0}h(x_0)S) - \hat{f}(x_0)}{h(x_0)} \tag{9.2.50}$$

Where we note that by using (9.2.39),

$$\|F^{(z_0)}\|_{C^{0,1}([-1,1])} \leq C\sqrt{x_0} \leq C\epsilon_1^* \tag{9.2.51}$$

Step 2 Now, similarly to Proposition 6.3.1, we obtain a uniformly elliptic equation for $\psi^{(z_0)}$ from (9.2.45), and also boundary conditions for the cases **(ii)** and **(iii)** Firstly, we compute that

$$\begin{aligned}\psi_i(x,y) &= \frac{g(x_0)}{h(x_0)} x_0^{\frac{i}{2}-1} \psi_i^{(z_0)}, \\ \psi_{ij}(x,y) &= \frac{g(x_0)}{h(x_0)^2} x_0^{\frac{i+j}{2}-2} \psi_{ij}^{(z_0)}.\end{aligned} \tag{9.2.52}$$

So that after multiplying (9.2.35) by $\frac{h(x_0)^2}{g(x_0)}$, $\psi^{(z_0)}$ satisfies

$$\begin{aligned}\sum_{i,j=1}^{2} A_{ij}^{(z_0)}(\psi^{(z_0)}, S, T)\partial_{ij}\psi^{(z_0)} + \sum_{i=1}^{2} A_i^{(z_0)}(\psi^{(z_0)}, S, T)\partial_i\psi^{(z_0)} \\ + \sum_{i,j=1}^{2} B_{ij}^{(z_0)}(\psi^{(z_0)}, S, T)\partial_i\psi^{(z_0)}\partial_j\psi^{(z_0)} = 0.\end{aligned} \tag{9.2.53}$$

where

$$\begin{aligned} A_{ij}^{(z_0)}(z,S,T) &= x_0^{\frac{i+j}{2}-2} A_{ij}(g(x_0)z,x,y), \\ A_i^{(z_0)}(z,S,T) &= h(x_0)x_0^{\frac{i}{2}-1} A_i(g(x_0)z,x,y), \\ B_{ij}^{(z_0)}(z,S,T) &= g(x_0)x_0^{\frac{i+j}{2}-2} B_{ij}(g(x_0)z,x,y). \end{aligned} \tag{9.2.54}$$

We now note that, by (9.2.31)–(9.2.33),

$$0 < h(x) \leq x^{1/2}, \qquad 0 < g(x) \leq x, \tag{9.2.55}$$

so we find that by (9.2.37)

$$|(A_{ij}^{(z_0)}, A_i^{(z_0)}, B_{ij}^{(z_0)})(\psi^{(z_0)}(S,T),S,T)| \leq M, \tag{9.2.56}$$

and also using (9.2.31)–(9.2.34), as well as (9.2.37), we find that

$$\|D_{(z,S,T)}(A_{ij}^{(z_0)}, A_i^{(z_0)}, B_{ij}^{(z_0)})\|_{L^\infty} \leq M, \tag{9.2.57}$$

We show the calculations for the $A_{11}^{(z_0)}$ term to demonstrate; the calculations for the other terms are similar. Firstly, we find that for $S,T \in [-1,1]^2$, (x,y) given by (9.2.43) satisfy

$$|x - x_0| = |\sqrt{x_0}h(x_0)S| \leq \frac{1}{4}d_{so}(x_0) \leq \frac{1}{4}x_0 \tag{9.2.58}$$

by (9.2.32).

Next, we find from (9.2.54) that

$$|\partial_z A_{ij}^{(z_0)}(z,S,T)| = x_0^{-1}g(x_0)|\partial_z A_{ij}(g(x_0)z,x,y)| \leq C, \tag{9.2.59}$$

using the fact that by (9.2.32) and (9.2.33), $g(x_0) \leq x_0$. Next, we find that from (9.2.37) and (9.2.34) combined with (9.2.58),

$$\begin{aligned} |\partial_S A_{11}^{(z_0)}(z,S,T)| &= |x_0^{-1/2}h(x_0)\partial_x A_{11}(g(x_0)z,x,y)| \\ &\leq M\frac{h(x_0)}{x_0^{1/2}}\frac{x^{1/2}}{h(x)} \leq C, \\ |\partial_T A_{11}^{(z_0)}(z,S,T)| &= |x_0^{-1}h(x_0)\partial_y A_{11}(g(x_0)z,x,y)| \\ &\leq M\frac{h(x_0)}{x_0}\frac{x}{h(x)} \leq C. \end{aligned} \tag{9.2.60}$$

So that by (9.2.36), for $\zeta \in \mathbb{R}^2$,

$$\lambda|\zeta|^2 \leq A_{ij}^{(z_0)}(\psi^{(z_0)}(S,T),S,T)\zeta_i\zeta_j \leq \lambda^{-1}|\zeta|^2. \tag{9.2.61}$$

We now focus on the oblique boundary condition. We find from (9.2.38),

$$\psi_S = x_0^{1/2}\beta(g(x_0)z, x_0 + \sqrt{x_0}h(x_0)S, y_0 + h(x_0)T), \tag{9.2.62}$$

so that the boundary condition scales into a near-tangential boundary condition in the same way as Proposition 6.3.1.

Step 3 From (9.2.40) and as g is increasing, ψ is bounded, we have

$$|\psi^{(z_0)}| \leq C, \tag{9.2.63}$$

and from here we have all necessary estimates in hand to follow the same proof as Proposition 6.3.1.

□

We can now prove the *a priori* estimates for admissible solutions in the supersonic-close-to-sonic case.

Proposition 9.2.5. *There exists $\delta_w^* > 0$, $\alpha \in (0,1)$ and $\epsilon \in (0, \frac{\epsilon_0}{2})$ depending only on the initial data such that the following holds; if (ρ, r) is a admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in (\theta_{sonic}, \theta_{sonic} + \delta_w^*)$, then the corresponding ψ satisfies*

$$\|\psi\|^{(par)}_{1,\alpha,\Omega_\epsilon} \leq C. \tag{9.2.64}$$

Proof. **Step 1.** Let (δ_w^*, ϵ) be as in Proposition 9.2.3 with $m = 3$. Moreover, reducing ϵ if necessary, we can assume $\epsilon \leq \sigma$, for σ from Proposition 6.3.1. By taking δ_w^* smaller,we can assume $b_{so}^2 < \epsilon$, and obtain the estimate

$$\|\psi\|^{(par)}_{1,\alpha,\Omega_{b_{so}^2}} \leq C. \tag{9.2.65}$$

Step 2. Now, we estimate ψ in $\Omega_\epsilon \cap \{\frac{b_{so}^2}{5} < x < \epsilon\}$. From our choice of δ_w^* and ϵ, we apply Proposition 9.2.3 with $m = 3$ and further reduce ϵ if needed to obtain the following growth estimate in $\Omega_{2\epsilon}$:

$$0 \leq \psi(x,y) \leq Cx^3 \qquad \text{in } \Omega_\epsilon \cap \{x > \frac{b_{so}^2(\theta_w)}{10}\}. \tag{9.2.66}$$

Moreover, due to (9.1.8), there exists $k > 1$ depending only on the data such that

$$\{0 < x < \epsilon, 0 < y - \theta_w < b_{so} + \frac{x}{k}\} \subset \Omega_{2\epsilon}. \tag{9.2.67}$$

We now define the rectangle for the present case. We consider

$$\tilde{R}_{z_0,r} := \left\{(x,y) \mid |x - x_0| < \frac{\rho}{10k} x_0^{\frac{3}{2}}, |y - y_0| < \frac{\rho}{10k} x_0\right\}, \tag{9.2.68}$$

and set

$$\hat{R}_{z_0,r} := \tilde{R}_{z_0,r} \cap \Omega_{2\epsilon}. \tag{9.2.69}$$

Then, using (6.2.10), we have

$$\begin{aligned} &\hat{R}_{z_w,1} \cap \Gamma_{shock} = \emptyset \qquad \text{for all } z_w = (x,0) \in \Gamma_{wedge} \cap \{x < \epsilon\}, \\ &\hat{R}_{z_s,1} \cap \Gamma_{wedge} = \emptyset \quad \text{for all } z_s = (x, f(x)) \in \Gamma_{shock} \cap \{x < \epsilon\}, \end{aligned} \tag{9.2.70}$$

i.e. the rectangles in (9.2.68) fit in the domain $\Omega_{2\epsilon}$. Now, using (9.2.70), we can apply Lemma 9.2.4 with the following choices:

$$x_{P_1} = 0, h(x) = \frac{x}{10k}, g(x) = \frac{x^3}{10k}, x_{bot} = \frac{b_{so}^2}{5}, \tag{9.2.71}$$

and then deduce that for rectangles $\hat{R}_{z_0,r}$ defined by (9.2.68), and the corresponding $Q_r^{(z_0)}$ and $\psi^{(z_0)}$ that there exists $\alpha \in (0,1)$ depending on the data such that:

$$\|\psi^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/100}^{(z_0)}})} \leq C. \tag{9.2.72}$$

Now if we set $\mathcal{D} := \Omega_\epsilon \cap \{x > \frac{b_{so}^2}{5}\}$, and repeat the proof of Lemma A.1.1, we obtain

$$\begin{aligned}&\sum_{0\leq k+l\leq 1} \sup_{z\in\mathcal{D}}\Big(x^{\frac{3k}{2}+l-3}|\partial_x^k\partial_y^l\psi(z)|\Big)\\ &+\sum_{k+l=1}\sup_{z,\tilde{z}\in\mathcal{D},z\neq\tilde{z}}\left(\min(x^{\frac{3}{2}(\alpha+k)+l-3},\tilde{x}^{\frac{3}{2}(\alpha+k)+l-3})\frac{|\partial_x^k\partial_y^l\psi(z)-\partial_x^k\partial_y^l\psi(\tilde{z})|}{\delta_\alpha^{(par)}(z,\tilde{z})}\right)\leq C.\end{aligned} \tag{9.2.73}$$

Now, comparing (9.2.73) to the right hand side of the definition of $\|\psi\|_{1,\alpha,\mathcal{D}}^{(1),(par)}$ in (A.1.8) we see that, the exponent of x is larger in the expression for $\|\psi\|_{1,\alpha,\mathcal{D}}^{(1),(par)}$ than the above. So assuming $\epsilon \leq 1$, we obtain

$$\|\psi\|_{1,\alpha,\Omega_\epsilon\cap\{x>b_{so}^2/5\}}^{(par)} \leq C, \tag{9.2.74}$$

where C is from (9.2.73). We now combine with (9.2.74) to obtain (9.2.64). □

Now, combining with Proposition 8.2.8, we find the following global regularity for the supersonic-near-sonic case:

Corollary 9.2.6. *Let δ_w^* be as in Proposition 9.2.5 and ϵ_0 as in Proposition 9.1.2. Then there exists $\alpha \in (0,1)$, depending on the data, such that for each $\beta \in (0,1)$ and $\epsilon \in (0,\epsilon_0]$, there exists C depending on the initial data, ϵ, and β such that if (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_{sonic} + \delta_w^*, \frac{\pi}{2}]$, then*

$$\begin{aligned}&\psi \in C^{0,1}(\overline{\Omega}\setminus\{P_2\}),\\ &\|\psi\|_{1,\alpha,\Omega\setminus\Omega_{\epsilon/10}}^{(-\beta),\{P_2\}} + \|\psi\|_{1,\alpha,\Omega_\epsilon}^{(par)} \leq C,\\ &\|r\|_{2,\alpha,[\theta_1,\pi]}^{(-1-\beta/2),\{\theta_1,\pi\}} + \|f-\hat{f}_0^{(\theta_w)}\|_{2,\alpha,(0,\epsilon)}^{(par)} \leq C.\end{aligned} \tag{9.2.75}$$

9.2.3 The subsonic-near-sonic case

We now consider the subsonic near sonic case; i.e. here we will obtain estimates for the case $\theta_w \in (\theta_{sonic} - \delta_w^*, \theta_{sonic})$, for small enough δ_w^*. In this case, all estimates in Proposition 8.2.7, Lemma 9.1.5, and Lemma 9.1.6 are available to us, so we have small elliptic structure, which takes the same form as the degenerate elliptic structure in supersonic cases. Firstly, we look at the main steps involved:

(i) From (9.1.8) in the subsonic case, the domain Ω_ϵ satisfies the following;

$$\begin{aligned}&\Omega_\epsilon \subset \{0 < x - x_{P_0} < \epsilon, 0 < y - \theta_w < kd_{so}(x)\},\\ &\{0 < x - x_{P_0} < \epsilon, 0 < y - \theta_w < \frac{1}{k}d_{so}(x)\} \subset \Omega_\epsilon,\end{aligned} \tag{9.2.76}$$

where $k > 1$ can be chosen uniformly for any subsonic reflection solution, and $P_1 = P_0$ in this case, where

$$d_{so}(x) := x - x_{P_1}, \tag{9.2.77}$$

as introduced in Remark 9.0.1. We then show, for each integer $m \geq 3$, and sufficiently small b_{so} (equivalently, sufficiently small δ_w^*), depending only on the data and m, $\psi(x,y)$ has growth $d_{so}(x)^m$ in Ω_ϵ:

$$0 \leq \psi(x,y) \leq C d_{so}(x)^m \qquad \text{in } \Omega_\epsilon, \tag{9.2.78}$$

where $\epsilon > 0$ and C depend only on the data and m. In this proof, we take advantage of the small ellipticity, and the shape of the region (9.1.5).

(ii) Using the algebraic growth, with $m = 4$, we obtain estimates of ψ in Ω_ϵ by using a different scaling from the supersonic cases. To have a property like (6.3.10), we now need to use a rectangle with side length approximately $\frac{1}{k} d_{so}(x_0)$ in the y direction. Again, for the small elliptic structure from , the ratio of the sides of the rectangle must be proportional to $\sqrt{x_0}$, to obtain a uniformly elliptic equation after rescaling to a square. Therefore, for $z_0 = (x_0, y_0) \in \overline{\Omega}_\epsilon \setminus \{P_0\}$, we consider the rectangles:

$$\hat{R}^{z_0} := \{|x - x_0| < \frac{\sqrt{x_0}}{10k} d_{so}(x_0), |y - y_0| < \frac{1}{10k} d_{so}(x_0)\} \cap \Omega, \tag{9.2.79}$$

and the corresponding square

$$\hat{Q}^{(z_0)} := \{(S,T) \in (-1,1)^2 \mid (x_0 + \frac{\sqrt{x_0}}{10k} d_{so}(x_0) S, y_0 + \frac{1}{10k} d_{so}(x) T) \in \Omega\}, \tag{9.2.80}$$

and then then a very similar argument to the supersonic-near-sonic case can be used.

We again consider ψ in the (x,y) coordinates from §3.5, using the expressions and estimates (9.2.7)–(9.2.14), where (9.2.14) is amended to:

$$\begin{aligned} &\Omega_\epsilon \subset \{0 < x - x_{P_1} < \epsilon, 0 < y - \theta_w < k d_{so}(x)\}, \\ &\{0 < x - x_{P_1} < \epsilon, 0 < y - \theta_w < \frac{1}{k} d_{so}(x)\} \subset \Omega_\epsilon. \end{aligned} \tag{9.2.81}$$

We now prove a similar algebraic growth bound to Lemma 9.2.2.

Lemma 9.2.7. *Let $m \geq 2$ be an integer. Then there exists $\delta_w^* > 0, \hat{\epsilon} > 0$ and $\hat{C} > 0$ depending only on m and the initial data, such that (ρ, r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with $\theta_w \in (\theta_{sonic} - \delta_w^*, \theta_{sonic}]$, and $\epsilon \leq \hat{\epsilon}$, we have*

$$\psi(x,y) \leq \hat{C} \frac{\max_{\partial\Omega_\epsilon \cap \{x = x_{P_1} + \epsilon\}} \psi}{\epsilon^m} (x - x_{P_1})^m \qquad \text{in } \Omega_\epsilon, \tag{9.2.82}$$

for $b_{so}(\theta_w)$ as defined in Remark 9.0.1.

Proof. We argue similarly to Lemma 9.2.2, this time similarly to [26][Lemma 16.5.1]. Firstly, we may replace y by $y - \theta_w$, so that (9.2.81) is now

$$\begin{aligned} &\{0 < x - x_{P_1} < \epsilon_0, 0 < y < \frac{1}{k}(x - x_{P_1})\} \subset \Omega_{\epsilon_0}, \\ &\Omega_{\epsilon_0} \subset \{0 < x - x_{P_1} < \epsilon_0, 0 < y < k(x - x_{P_1})\}. \end{aligned} \tag{9.2.83}$$

We shall show that if $L > 0$ is sufficiently large, and $\epsilon_0 > 0$ is sufficiently small, then whenever $x_{P_1} \in [0, \epsilon)$, then for any $A > 0$, the following function:

$$v(x,y) = A\left((x - x_{P_1})^m - \frac{1}{L}(x - x_{P_1})^{m-2} y^2\right) \tag{9.2.84}$$

is a supersolution of (9.2.7), with boundary conditions (9.2.9) and (9.2.12). We employ (9.2.83) to find that if $L \geq 2k^2$,

$$v(x,y) \geq (x-x_{P_1})^m - \frac{k^2}{L}(x-x_{P_1})^{m-2}y^2 \geq \frac{1}{2}(x-x_{P_1})^m \text{ for } (x,y) \in \Omega_\epsilon. \tag{9.2.85}$$

We also find similarly that if $L \geq \frac{2(m-2)}{m}k^2$,

$$\begin{aligned} v_x(x,y) &= (x-x_{P_1})^{m-3}\bigg(m(x-x_{P_1})^2 - \frac{m-2}{L}y^2\bigg), \\ &\geq (x-x_{P_1})^{m-3}\bigg(m(x-x_{P_1})^2 - \frac{m-2}{L}k^2(x-x_{P_1})^2\bigg) \\ &\geq \frac{m}{2}(x-x_{P_1})^{m-1} > 0. \end{aligned} \tag{9.2.86}$$

Similarly, by further increasing L;

$$\begin{aligned} v_{xx}(x,y) &= (x-x_{P_1})^{m-4}\bigg(m(m-1)(x-x_{P_1})^2 - \frac{(m-2)(m-3)}{L}y^2\bigg) \\ &\geq (x-x_{P_1})^{m-4}\bigg(m(m-1)(x-x_{P_1})^2 - k^2\frac{(m-2)(m-3)}{L}(x-x_{P_1})^2\bigg) \\ &\geq \frac{m(m-1)}{2}(x-x_{P_1})^{m-2} \geq 0, \end{aligned} \tag{9.2.87}$$

and so

$$|v_{xx}(x,y)| \leq m(m-1)(x-x_{P_1})^{m-2}. \tag{9.2.88}$$

We now turn to v_{yy} and v_y. Using (9.2.83), in particular the fact $\Omega_\epsilon \subset \{0 \leq y \leq k(x-x_{P_1})\}$,

$$v_{yy}(x,y) = -\frac{2}{L}(x-x_{P_1})^{m-2}, \tag{9.2.89}$$

$$|v_y(x,y)| = -\frac{2}{L}(x-x_{P_1})^{m-2}y \leq -\frac{C}{L}(x-x_{P_1})^{m-1}. \tag{9.2.90}$$

Now that we have obtained a similar supersolution the one in Lemma 9.2.2, we can conclude the result by a similar argument. □

We may now deduce the following;

Proposition 9.2.8. *For every integer $m \geq 2$, there exists small $\delta_w^*, \epsilon > 0$ and large C, depending only on the initial data and m, such that if (ρ, r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in (\theta_{sonic} - \delta_w^*, \theta_{sonic}]$, then*

$$\begin{aligned} x_{P_1}(\theta_w) &\leq \frac{\epsilon}{10}, \\ 0 \leq \psi(x,y) &\leq Cx^m \quad \text{in } \Omega_\epsilon. \end{aligned} \tag{9.2.91}$$

Proof. The first inequality follows by Remark 9.0.1; once ϵ is fixed, δ_w^* can be reduced further to establish this. The second inequality follows from (9.2.82), using (9.1.4) and the same argument as Lemma 5.2.3. □

We now deduce the following regularity near the reflection point:

Proposition 9.2.9. *There exists $\delta_w^* > 0$, $\epsilon \in (0, \frac{\epsilon_0}{2})$, $C > 0$ and α depending only on the data, such that if (ρ, r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in (\theta_{sonic} - \delta_w^*, \theta_{sonic}]$, then the corresponding ψ satisfies*

$$\begin{aligned} \|\psi\|_{C^{1,\alpha}(\overline{\Omega}_\epsilon)} &\leq C, \\ (\psi(P_0), \nabla\psi(P_0)) &= (0, \mathbf{0}), \end{aligned} \tag{9.2.92}$$

where the last assertion strengthens the Dirichlet boundary condition given by Problem 2.4.2 **(iv)**.

Proof. This follows similarly to Proposition 9.2.5, except with different scaling. We follow the proof of [26][Proposition 16.5.3].

Firstly, we take $\delta_w^* > 0$ to be the one from Proposition 9.2.8, and also take ϵ to be half the one from that Proposition. We then apply Proposition 9.2.8 with $m = 4$, to obtain the growth estimate

$$0 \leq \psi(x, y) \leq C(x - x_{P_1})^4 \qquad \text{in } \Omega_{2\epsilon}. \tag{9.2.93}$$

We now, as described earlier, for each $z_0 = (x_0, y_0) \in \overline{\Omega}_\epsilon \setminus \overline{\Gamma_{sonic}}$, we consider the rectangle

$$\tilde{R}_{z_0,r} := \{|x - x_0| < \frac{r\sqrt{x_0}}{10k} d_{so}(x_0), |y - y_0| < \frac{r}{10k} d_{so}(x_0)\}, \hat{R}_{z_0,r} \quad := \tilde{R}_{z_0,r} \cap \Omega_{2\epsilon}, \tag{9.2.94}$$

so that (9.2.42) holds for ϵ small, i.e. the rectangles in (9.2.94) fit into $\Omega_{2\epsilon}$. Now, we can apply Lemma 9.2.4 to obtain, that for the rescaled functions

$$\psi^{(z_0)}(S, T) := \frac{10k}{d_{so}(x_0)^4} \psi(x_0 + \frac{\sqrt{x_0}}{10k} d_{so}(x_0) S, y_0 + \frac{1}{10k} d_{so}(x_0) T) \text{ on } Q_r^{(z_0)}, \tag{9.2.95}$$

that the following estimate holds:

$$\|\psi^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/100}^{(z_0)}})} \leq C \qquad \text{for } z_0 \in \Omega_\epsilon. \tag{9.2.96}$$

for C, α depending on the data.

Now, similarly to Proposition 9.2.5, using the fact that when k, l are positive integers with $k + l = 1$, $\alpha \in (0, 1)$, and $x_{P_1} \geq 0$,

$$g_{k,l,\alpha} := (x - x_{P_1})^{k+l+\alpha-4} x^{\frac{k+\alpha}{2}} \tag{9.2.97}$$

is a decreasing function of $x \in (x_{P_1}, \infty)$, we have

$$\begin{aligned} &\sum_{0 \leq k+l \leq 1} \sup_{z \in \Omega_\epsilon} \Big((x - x_{P_1})^{k+l-4} x^{k/2} |\partial_x^k \partial_y^l \psi(z)| \Big) \\ &\quad + \sum_{k+l=1} \sup_{z, \tilde{z} \in \Omega_\epsilon, z \neq \tilde{z}} \left((\max(x, \tilde{x}) - x_{P_1})^{k+l-\alpha-4} (\max(x, \tilde{x}))^{\frac{k+\alpha}{2}} \frac{|\partial_x^k \partial_y^l \psi(z) - \partial_x^k \partial_y^l \psi(\tilde{z})|}{\delta_\alpha^{(par)}(z, \tilde{z})} \right) \leq C, \end{aligned} \tag{9.2.98}$$

for C depending only α and the C from (9.2.96), so only on the data and α, and $\delta_\alpha^{(par)}$ is defined in (A.1.7). We now note that the norm (9.2.98) is stronger than the $C^{1,\alpha}(\overline{\Omega}_\epsilon)$ norm.

Indeed, we note that as $x_{P_1} \in [0, \epsilon)$, as $k + l + \alpha - 4 < 0$ and $\frac{3}{2}(k + \alpha) + l - 4 < 0$ for all (k, l, α) with $0 \leq k, l \leq 1$ with $k + l = 1$, and $\alpha \in (0, 1)$, we find that for $x, \tilde{x} \in (x_{P_1}, x_{P_1+\epsilon}$,

$$\begin{aligned} (x - x_{P_1})^{k+l-4} x^{\frac{k}{2}} &\geq x^{\frac{3}{2}k+k-4} \geq \epsilon^{\frac{3}{2}k+l-4}, \\ (\max(x, \tilde{x}) - x_{P_1})^{k+l+\alpha-4} (\max(x, \tilde{x}))^{\frac{k+\alpha}{2}} &\geq (\max(x, \tilde{x}))^{\frac{3}{2}(k+\alpha)+l-4}, \end{aligned} \tag{9.2.99}$$

and also assuming without loss of generality that $\epsilon \leq 1$, (A.1.7) implies

$$\delta_{\alpha}^{(par)}(z, \tilde{z}) \leq |z - \tilde{z}|^{\alpha}. \tag{9.2.100}$$

Thus the expression on the right hand side is stronger than the $C^{1,\alpha}(\overline{\Omega}_\epsilon)$ norm, with constant depending only on (ϵ, α), i.e. depending on the data and α. We also note that from (9.2.98) that

$$|\nabla \psi(z)| \leq C(x - x_{P_1})^2, \tag{9.2.101}$$

so that $\nabla \psi(0) = \mathbf{0}$. □

Now, combining with Proposition 8.2.8, we find

Corollary 9.2.10. *Let δ_w^* be as in Proposition 9.2.9.Then there exists $\alpha \in (0, 1)$, depending on the data, such that for each $\beta \in (0, 1)$, there exists C depending on the initial data, and β such that if (ρ, r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in [\theta_{sonic} - \delta_w^*, \theta_{sonic}]$, then*

$$\begin{aligned} &\psi \in C^{1,\alpha}(\overline{\Omega} \setminus \{P_2\}), \\ &\|\psi\|_{1,\alpha,\Omega}^{(-\beta),\{P_2\}} \leq C, \\ &\|r\|_{2,\alpha,[\theta_1,\pi]}^{(-1-\beta/2),\{\pi\}} \leq C. \end{aligned} \tag{9.2.102}$$

and

$$\begin{aligned} \psi(P_1) = 0, \quad \nabla \psi(P_1) = 0, \\ f(x_{P_1}) = \hat{f}_0(x_{P_1}), \quad f'(x_{P_1}) = \hat{f}_0'(x_{P_1}), \quad f''(x_{P_1}) = \hat{f}_0''(x_{P_1}). \end{aligned} \tag{9.2.103}$$

9.2.4 The subsonic-away-from-sonic case

In this Section, we prove *a priori* estimates for the subsonic-away-from sonic case, where we now have strict ellipticity, and a one-point Dirichlet condition at P_1. We note that we do not have obliqueness at P_1 in the sense of [56, 57], or the Harnack inequality would imply that it is not possible to impose the one-point Dirichlet boundary condition. We also note that we do not have *a priori* $C^{0,1}$ estimates up to P_1, so the results of [58] also do not apply. Thus we must use (9.1.10), alongside the structure of the region Ω_ϵ, to apply the same technique as Proposition 5.2.4 and attain $C^{0,\beta}(P_1)$ estimates for each $\beta \in (0, 1)$, using the same argument as that used near P_2. We will fix $\theta_w \in (\theta_d, \frac{\pi}{2}]$, and attain estimates for $\theta_w \in [\theta_w^*, \theta_{sonic} - \delta_w^*]$, where $\delta_w^* > 0$, and the estimates depend on θ_w^*, δ, and the data.

We now outline the main steps:

(i) We first take advantage of (9.1.10) to obtain algebraic growth of ρ on Γ_{shock}. Using the same argument as the obliqueness principle in §3.6.2, similar to the treatment of P_2 in §5.2, we can show that for any $A > 0$ and integer $m \geq 2$,

$$A(y - \theta_w)^m - \psi \text{ does not achieve a minimum on } \Gamma_{shock} \cap \partial\Omega_\epsilon, \tag{9.2.104}$$

so that using (9.1.4), we find that for A depending only on the data,

$$0 \leq \psi \leq A(y-\theta_w)^m \qquad \text{on } \Gamma_{shock} \cap \partial\Omega_\epsilon. \tag{9.2.105}$$

(ii) Now, owing to the structure in (9.1.5), Ω_ϵ satisfies an exterior ball condition at $\mathbf{0}$, so we may use the barrier function from [43][Theorem 1.1.15] to deduce that for any $\beta \in (0,1)$, there exists C depending only on β, the data and θ_w^* such that

$$0 \leq \psi \leq C|\mathbf{x}-\mathbf{x}_{P_1}|^\beta \qquad \text{in } \Omega_\epsilon, \tag{9.2.106}$$

(iii) We can then, using a standard scaling argument, combined with the obliqueness in (3.5.16), deduce $C^{(\beta)}_{1,\alpha}$ regularity, for α depending only on the initial data. We note the difference here from the treatment at P_2 is the obliqueness is now strict, rather than small.

We first obtain the algebraic growth of ψ on Γ_{shock}.

Lemma 9.2.11. *Let $A > 0$, and $m \geq 1$ be an integer, and (ρ, r) be an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in (\theta_d, \frac{\pi}{2})$. Then $A(y-\theta_w)^m - \psi$ cannot achieve a minimum on the interior of $\Gamma_{shock} \cap \partial\Omega_\epsilon$.*

Proof. We observe that, for $\tilde{\beta}$ from (3.5.14), using (9.1.10),

$$\tilde{\beta} \cdot \nabla(A(y-\theta_w)^m - \psi) = 2Am\beta_2(y-\theta_w)^{m-1} < 0 \qquad \text{on } \Gamma_{shock} \cap \partial\Omega_\epsilon, \tag{9.2.107}$$

and then we can use the same argument as Lemma 5.2.2 to deduce the required result. □

Corollary 9.2.12. *Let (ρ, r) be an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in (\theta_d, \frac{\pi}{2})$. Then for any integer $m \geq 1$, there exists A depending only on the initial data and m such that*

$$0 \leq \psi \leq A(y-\theta_w)^m \qquad \text{on } \Gamma_{shock} \cap \partial\Omega_\epsilon. \tag{9.2.108}$$

Proof. Similarly to Lemma 5.2.2, using, (9.1.5), the relative boundary of $\Gamma_{shock} \cap \partial\Omega_\epsilon$ in $\partial\Omega_\epsilon$ is equal to

$$\{(0, f(0)), (\epsilon, f(\epsilon))\}. \tag{9.2.109}$$

as $f(0) = \theta_w$, we find that at $(0, f(0))$, $A(y-\theta_w)^2 - \psi = 0$. At $(\epsilon, f(\epsilon))$, $f(\epsilon) \geq \frac{1}{C}\epsilon$, so

$$Ay^2 - \psi \geq \frac{A}{C^m}\epsilon^m - M \geq 0, \tag{9.2.110}$$

by taking A sufficiently large, depending on ϵ and M, hence only on the data. □

Now, using the same argument used for P_2 in Proposition 5.2.4, except using Proposition B.0.6 instead of Proposition B.0.7, we find that

Proposition 9.2.13. *Then there exists $\alpha \in (0,1)$, depending only on the data, δ_w^*, θ_w^*, such that for any $\beta \in (0,1)$, there exists $C > 0$ depending on the data, δ_w^* and θ_w^* such that if (ρ, r) is an admissible solution of Problem 2.4.2, in the sense of Definition 8.2.1, with wedge angle $\theta_w \in [\theta_w^*, \theta_{sonic} - \delta_w^*)$,*

$$\|u\|^{(-\beta),\{P_1\}}_{1,\alpha,\Omega_\epsilon} \leq C. \tag{9.2.111}$$

Now, again combining with Proposition 8.2.8, we deduce

Corollary 9.2.14. *Let θ_w^* be as in Proposition 9.2.13, ϵ_0 as in Proposition 9.1.2. Then there exists $\alpha \in (0,1)$, depending on the data and θ_w^*, such that for each $\beta \in (0,1)$ there exists C depending on the initial data, θ_w^* and β such that if (ρ, r) is an admissible solution of Problem 2.4.1, in the sense of Definition 4.2.1, with wedge angle $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, then*

$$\begin{aligned} \|\psi\|_{1,\alpha,\Omega}^{(-\beta),\{P_1,P_2\}} &\leq C, \\ \|r\|_{2,\alpha,[\theta_1,\pi]}^{(-1-\beta/2),\{\theta_1,\pi\}} &\leq C. \end{aligned} \tag{9.2.112}$$

Chapter 10

Existence Theory up to the Detachment Angle

We now turn to the existence theory up to the detachment angle. The first key point here is that we must combine the estimates near Γ_{sonic} for ψ, into a unified estimate for u in the iteration set. For this, we note that in subsonic cases, due to Γ_{sonic} now being a single point, the map from the physical region Ω to the iteration region $Q^{iter} = (0,1)^2$ is no longer one-to-one on $\{s = 0\}$. It turns out the linear separation of Γ_{shock} and Γ_{wedge}, given by (9.1.8), allows us to find estimates for u in a specially constructed norm, given by (10.1.19). We are then able to relate this norm to the parabolic norm (A.1.8), which assists in finding an appropriate unified estimate.

The other key point will be to extend the definition of the iteration set in Definition 7.2.2, to subsonic cases. In particular, we need to ensure that (7.2.23) satisfies an elliptic structure like (8.2.12). We note that the cutoff used in supersonic cases is only suitable when the first term in the ellipticity of (8.2.12) is the smaller one. in particular, we essentially need (9.1.13) and the full estimates of Proposition 6.2.4 to remove the cutoff used for the supersonic case; this estimate is not available in subsonic-away-from-sonic cases, so the cutoff needs to be removed for subsonic-away-from-sonic cases.

10.1 Mapping to the iteration region

We start by following the construction in Chapter 7, extending to all wedge angles $\theta_w \in [\theta_d, \frac{\pi}{2}]$. The main changes are to the notation in the subsonic case, in which we employ the convention that $P_0 = P_1 = P_4$ and $\overline{\Gamma_{sonic}} = \{P_0\}$, and to the treatment of the region near $\overline{\Gamma_{sonic}}$. We define

$$\hat{s}(\theta_w) = |P_4|(\theta_w), \tag{10.1.1}$$

noting that for supersonic wedge angles, $\hat{s}(\theta_w) = c_2(\theta_w)$, and for subsonic/ (sonic) wedge angles, $\hat{s}(\theta_w) < (=)c_2(\theta_w)$. Using Definitions 6.2.1 and 9.1.1, we can find $\delta_* > 0$ so that (7.1.2) still holds. We can then define S_{1,δ_*} as in (7.1.1), point $\hat{P}_1$ as the closest point of intersection of $\mathcal{S}_{1,\delta_*}$ with $\partial_{\hat{s}(\theta_w)}$, the arc $\Gamma_{sonic}^{(\delta_*)}$ as the smaller arc $\hat{P}_1 P_4$ of $\partial B_{\hat{s}(\theta_w)}$, which is always an arc, and $\mathcal{Q}$ as in Definition 7.1.1.

Now, to define the region $\mathcal{D}_{\delta_*,\epsilon}$, for $\epsilon \in (0, \epsilon_0]$, we modify (7.1.3) in the following way.

We let ϵ_1 now be from Lemma 9.1.4, and denote for $\epsilon \in (0, \epsilon_1]$,

$$\begin{aligned}\mathcal{D}^{\theta_w}_{\delta_*,\epsilon} &:= \mathcal{Q} \cap \Lambda \cap \mathcal{N}_{\epsilon_1}(\Gamma^{(\delta_*)}_{sonic}) \cap \{x_{P_1} < x < x_{P_1} + \epsilon\} \\ &\equiv \mathcal{Q} \cap \mathcal{N}_{\epsilon_1}(\Gamma^{(\delta_*)}_{sonic}) \cap \{x_{P_1} < x < x_{P_1} + \epsilon\},\end{aligned} \tag{10.1.2}$$

so that similarly to (6.2.1) for small enough δ_*, there exists $f_{0,\delta_*} \in C^\infty([0,\epsilon_1])$ such that, similarly to (7.1.4),

$$\begin{aligned}&\mathcal{D}_{\delta_*,\epsilon_1} = \{(x,y) \mid x \in (x_{P_1}, x_{P_1}+\epsilon_1) \mid \theta_w < y < f_{0,\delta_*}(x)\}, \\ &S_{1,\delta_*} \cap \mathcal{N}_{\epsilon_1}(\Gamma^{(\delta_*)}_{sonic}) \cap \{x_{P_1} < x < x_{P_1} + \epsilon\} = \{(x,y) \mid x \in (x_{P_1}, x_{P_1}+\epsilon), y = f_{0,\delta_*}(x)\}.\end{aligned} \tag{10.1.3}$$

We now state the following extension of Lemma 7.1.2.

Lemma 10.1.1. *Lemma 7.1.2 holds for any wedge angle $\theta_w \in [\theta_d, \frac{\pi}{2}]$, with $c_2(\theta_w)$ generalised to $\hat{s}(\theta_w)$ defined by* (10.1.1)*, constants $\delta, C > 0$ depending only on the data, with the following notational changes for subsonic/sonic wedge angles; $P_0 = P_1 = P_4$ for subsonic wedge angles, and assertion* **(iv)** *is replaced by the following (where the assertion applies to all wedge angles):*

(iv)' *$F_1(P) = (x_P - x_{P_1}, y_P - \theta_w)$ for all $P \in \mathcal{D}^{(\theta_w)}_{\delta_*,3\epsilon_0/4}$ under the notation of* (10.1.2)*. In particular,*

$$\begin{aligned}&F_1(P_1) = (0, y_{P_1} - y_{P_4}), \quad F_1(P_4) = \mathbf{0}, \\ &F_1(\overline{\Gamma_{sonic}}) = \{s = 0, 0 \le t \le y_{P_1} - y_{P_4}\}.\end{aligned} \tag{10.1.4}$$

Also, $F_1(\mathcal{Q}_{bd} \setminus \mathcal{D}^{(\theta_w)}_{\delta_*,3\epsilon_0/4}) = F_1(\mathcal{Q}_{bd}) \cap \{s \ge \frac{3\epsilon_0}{4}\}$.

Proof. This follows exactly as Lemma 7.1.2, with the use of the extended estimates for the shape of Γ_{shock} given by Corollary 4.3.3, (8.2.15) and Lemma 9.1.4. □

Next, we extend Lemma 7.1.3, with the change that owing to (9.1.8), the strict positivity of $\mathfrak{g}_{sh}$ is replaced by linear separation.

Lemma 10.1.2. *Let $\theta_w^* \in (\theta_d, \frac{\pi}{2})$. Then Lemma 7.1.3 holds for any admissible solutions with $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, with constants $C, M > 0$ depending on the data, given the following changes;*

(i) *Instead of* (7.1.9)*, there exists $\alpha \in (0,1)$ such that for any $\beta \in (0, \frac{1}{2})$, there exists $C(\beta)$ such that*

$$\|\mathfrak{g}_{sh}\|^{(-1-\beta),\{0,\hat{s}(\theta_w)\}}_{2,\alpha,[0,\hat{s}(\theta_w)]} \le C(\beta). \tag{10.1.5}$$

(ii) *Instead of* (7.1.13)*, for any $s \in (0, \hat{s}(\theta_w))$ and $\theta_w \in [\theta_w^*, \frac{\pi}{2})$,*

$$\min\left(\mathfrak{g}_{sh}(0) + \frac{s}{M}, \frac{1}{M}\right) \le \mathfrak{g}_{sh}(s) \le \min\left(\mathfrak{g}_{sh}(0) + Ms, \eta^{(\theta_w)}(s) - \frac{1}{M}\right), \tag{10.1.6}$$

where $\mathfrak{g}_{sh}(0) = \theta_{P_1} - \theta_w \ge 0$, and $M \ge 1$ only depends on the initial data.

Proof. (10.1.5) follows by using the regularity of the shock expressed in Corollaries 9.2.1, 9.2.6, 9.2.10 and 9.2.14, combined with the original argument in Lemma 7.1.3. (10.1.6) follows from (9.1.8) combined with (10.1.4). □

10.1. Mapping to the iteration region

We next turn to the function intended to approximate the shock curve. Due to the fact we have extended the estimates for the straight shock in Proposition 9.1.2, and the definition of the coordinate map from Lemma 7.1.2 to Lemma 10.1.1, we find that Proposition 7.1.4 holds without change.

Proposition 10.1.3. *For each $\theta_w \in (\theta_{detach}, \frac{\pi}{2}]$, there exists a function $\tilde{f} \equiv \tilde{f}^{(\theta_w)}$ defined on $\mathcal{Q}$ such that for $\boldsymbol{\nu}_{\mathcal{S}_1}$ from (3.4.4), and for C depending only on the data,*

$$\tilde{f} = \boldsymbol{\xi} \cdot \boldsymbol{\nu}_{\mathcal{S}_1} + \bar{c}_2 \text{ in } \overline{\mathcal{Q}} \text{ for } \theta_w = \frac{\pi}{2}, \tag{10.1.7}$$

$$\tilde{f} = \boldsymbol{\xi} \cdot \boldsymbol{\nu}_{\mathcal{S}_1} \text{ in } \mathcal{D}_{\delta_*, \epsilon_0/2} \text{ for all } \theta_w \in [\theta_{sonic}, \frac{\pi}{2}], \tag{10.1.8}$$

$$\|\tilde{f}\|_{C^3(\mathcal{Q})} \leq C, \tag{10.1.9}$$

$$\partial_\eta \tilde{f}(\xi, 0) = 0 \text{ for } (\xi, 0) \in \mathcal{Q}, \tag{10.1.10}$$

$$\frac{\partial}{\partial t} \tilde{f} \circ F_1^{-1}(s,t) \leq -\delta \text{ for all } (s,t) \in F_1(\mathcal{Q}), \tag{10.1.11}$$

where $\tilde{f}$ depends smoothly on θ_w.

Now, in the same way as §7.1, given an admissible solution (ρ, r), we define $\mathfrak{g}_{sh} : [0, \hat{s}(\theta_w)] \to \mathbb{R}$, $F_{2,\mathfrak{g}_{sh}}^{-1} : Q^{iter} \to \tilde{\Omega}$ by (7.1.23) and (7.1.25), as well as $u : Q^{iter} \to \mathbb{R}$ and $v : (0,1) \to \mathbb{R}$ by (7.1.26). The key difference in this case is that $\mathfrak{g}_{sh}(0) = 0$ for subsonic angles, so the transformation $F_{2,\mathfrak{g}_{sh}}$ given by (7.1.23) is now singular near $\{s = 0\}$; but the linear separation (10.1.6) means we can still obtain estimates for u, in a weighted and scaled norm.

We now first study the regularity in the case where $\theta_w \in (\theta_d, \theta_{sonic}]$, with the aid of (10.1.6). Using the property **(iv)'** in Lemma 10.1.1, we can use the (x, y) coordinates in $\Omega \cap \mathcal{D}_{\epsilon_0/10}$ for any $\theta_w \in (\theta_d, \frac{\pi}{2})$. For for $\boldsymbol{\xi} \in \Omega_{\epsilon_0/10}$, we set $(s,t) = F_{2,\mathfrak{g}_{sh}} \circ F_1(\boldsymbol{\xi})$; then, the (x, y) coordinates for $\boldsymbol{\xi}$ satisfy, for $h = \hat{s}(\theta_w)$,

$$(x - x_{P_1}, y - \theta_w) = (hs, \mathfrak{g}_{sh}(hs)t) \qquad \text{for all } s \in (0, \epsilon_0'/10), t \in (0,1), \tag{10.1.12}$$

where ϵ_0' is defined similarly to (7.1.27), with θ_d replacing θ_{sonic};

$$\epsilon_0' := \frac{\epsilon_0}{\max_{\theta_w \in [\theta_d, \frac{\pi}{2}]} \hat{s}(\theta_w)}, \tag{10.1.13}$$

where ϵ_0 is from Proposition 9.1.4, and we have used the continuity of $\theta_w \mapsto \hat{s}(\theta_w)$.

Now, using (10.1.4), we find that for ψ corresponding to (ρ, r), we have

$$\psi(x,y) = u\left(\frac{x - x_{P_1}}{h}, \frac{y - \theta_w}{\mathfrak{g}_{sh}(x - x_{P_1})}\right) \qquad \text{for all } (x,y) \in \Omega_{\epsilon_0/10}. \tag{10.1.14}$$

Now, the estimates of Chapter 9 imply that in in all subsonic and sonic cases $\psi \in C_{1,\alpha,\Omega_\epsilon}^{(-\beta),\{P_1\}}$ with $\psi(P_1) = 0$, for any $\beta \in (0,1)$, and uniform $\epsilon > 0$. Thus we first investigate equivalent estimates for u, particularly assisted by (10.1.6). After possibly increasing M, noting that $\mathfrak{g}_{sh}(0) = 0$, we can write (10.1.6) in the following form:

$$\frac{s}{M} \leq \mathfrak{g}_{sh}(s) \leq Ms \qquad \text{for } s \in (0, h). \tag{10.1.15}$$

We now look at the properties of functions u obtained by formula (10.1.14) with when $\psi \in C_{1,\alpha,\Omega_\epsilon}^{(-\beta),\{P_1\}}$; where we obtain similar results to [26][§17.2], but with one less derivative expected, and we must also obtain estimates for when $\alpha \neq \beta$.

Let $h > 0$, $x_{P_1} \in \mathbb{R}$, and $\theta_w \in \mathbb{R}$, and let $\mathfrak{g}_{sh}(\cdot)$ be a function on $[0,h]$ satisfying the estimate (10.1.15). Then, we obtain a map $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^2$, defined by formula (10.1.12). Then, the map $\mathbf{x}$ is one-to-one on $(0, \frac{\epsilon_0'}{10}) \times (0,1)$, but not one-to-one on $[0, \frac{\epsilon_0'}{10}) \times (0,1)$; more precisely, $\mathbf{x}$ is not one-to-one on $\{s = 0\} \times [0,1]$.

For $\epsilon \in (0,h)$, define

$$\Omega_\epsilon := \mathbf{x}(Q_{\epsilon_0'}), \qquad P_1 = P_1 = P_4 := \mathbf{x}(0,0), \tag{10.1.16}$$

where $\epsilon_0' = \frac{\epsilon_0}{h}$. Then Ω_ϵ is of the same form as in (6.2.7), where $f(x) = \theta_w + \mathfrak{g}_{sh}(x - x_{P_1})$. Now, for $k \in \{0,1\}$ and $\beta \in (0,1)$ satisfying $k + \alpha \geq \beta$, denote

$$\hat{C}^{(-\beta),\{P_1\}}_{k,\alpha,\Omega_\epsilon} := \Big\{ \psi \in C^{(-\beta),\{P_1\}}_{k,\alpha,\Omega_\epsilon} \mid \psi(P_1) = 0 \Big\}. \tag{10.1.17}$$

We now begin to describe the space of functions u on $\mathcal{Q}_{\epsilon_0'}$ obtained by formula (10.1.14), with $\psi \in \hat{C}^{(-\alpha),\{P_1\}}_{k,\alpha',\Omega_\epsilon}$. For this purpose, we define a norm as follows. For $\alpha \in (0,1)$ and $\mathbf{s} = (s,t)$ and $\tilde{\mathbf{s}} = (\tilde{s},\tilde{t})$, denote

$$\delta^{(subs)}_\alpha(\mathbf{s},\tilde{\mathbf{s}}) := \big(|s-\tilde{s}|^2 + (\max(s,\tilde{s}))^2 |t-\tilde{t}|^2\big)^{\alpha/2}, \tag{10.1.18}$$

and then for $\sigma > 0, \alpha \in (0,1)$ and $m \in \mathbb{N}$, define

$$\begin{aligned}
\|u\|^{(\sigma),(subs)}_{m,0,\mathcal{Q}_\epsilon} &:= \sum_{0 \leq k+l \leq m} \sup_{\mathbf{s} \in Q_\epsilon} \Big(s^{k-\sigma} |\partial_s^k \partial_t^l u(\mathbf{s})| \Big), \\
[u]^{(\sigma),(subs)}_{m,\alpha,\mathcal{Q}_\epsilon} &:= \sum_{k+l=m} \sup_{\mathbf{s},\tilde{\mathbf{s}} \in \mathcal{Q}_\epsilon} \left(\min(s^{\alpha+k-\sigma}, \tilde{s}^{\alpha+k-\sigma}) \frac{|\partial_s^k \partial_t^l u(\mathbf{s}) - \partial_s^k \partial_t^l u(\tilde{\mathbf{s}})|}{\delta^{(subs)}_\alpha(\mathbf{s},\tilde{\mathbf{s}})} \right), \\
\|u\|^{(\sigma),(subs)}_{m,\alpha,\mathcal{Q}_\epsilon} &:= \|u\|^{(\sigma),(subs)}_{m,0,\mathcal{Q}_\epsilon} + [u]^{(\sigma),(subs)}_{m,\alpha,\mathcal{Q}_\epsilon},
\end{aligned} \tag{10.1.19}$$

where $Q = Q^{iter}$, $Q_\epsilon = Q \cap \{s < \epsilon\}$, and $k,l \in \mathbb{N}$. We define

$$C^{m,\alpha}_{\sigma,(subs)}(Q_\epsilon) := \{ u \in C^m(Q_\epsilon) \mid \|u\|^{(\sigma),(subs)}_{m,\alpha,\mathcal{Q}_\epsilon} < \infty \}.$$

We next note, that in (10.1.19), we can use $\min(s,\tilde{s})$ instead of $\max(s,\tilde{s})$ and obtain an equivalent norm, with constant depending only on (m,α):

Lemma 10.1.4. *Let $s,\tilde{s},t,\tilde{t} \in (0,1)$, then*

$$|s-\tilde{s}|^2 + (\max(s,\tilde{s}))^2 |t-\tilde{t}|^2 \leq 5\big(|s-\tilde{s}|^2 + (\min(s,\tilde{s}))^2 |t-\tilde{t}|^2\big). \tag{10.1.20}$$

Proof. This follows as in [26][Lemma 17.2.4]; assume $s,\tilde{s},t,\tilde{t} \in (0,1)$, and $\tilde{s} \leq s$. Firstly, if $s \leq 2\tilde{s}$, then as $\tilde{s} \leq s$, $\max(s,\tilde{s}) \leq 2\min(s,\tilde{s})$, so (10.1.20) holds.

In the opposite case, as $0 < \tilde{s} < \frac{s}{2}$,

$$s - \tilde{s} > \frac{s}{2} = \frac{1}{2}\max(s,\tilde{s}) \geq \frac{1}{2}\max(s,\tilde{s})|t-\tilde{t}|, \tag{10.1.21}$$

so that

$$5|s-\tilde{s}|^2 \geq |s-\tilde{s}|^2 + \max(s,\tilde{s})^2 |t-\tilde{t}|^2, \tag{10.1.22}$$

and we obtain (10.1.20). □

10.1. Mapping to the iteration region

We now turn to the first main fact about the spaces $C^{m,\alpha}_{\sigma,(subs)}(Q_\epsilon)$.

Lemma 10.1.5. *Let $M, M_1 \geq 1$, $x_{P_1} \in \mathbb{R}$, $m \in \{0,1\}$ and $\alpha, \beta \in (0,1)$ such that $m+\alpha \geq \beta$, and $\theta_w \in \mathbb{R}$. Let $h \in (\frac{1}{M_1}, M_1)$ and $\epsilon \in (0, \frac{h}{2})$ Assume that a function $\mathfrak{g}_{sh}(s)$ on $(0,\epsilon)$ satisfies* (10.1.15) *on $(0,\epsilon)$, and*

$$\|\mathfrak{g}_{sh}\|^{(-1),\{0\}}_{m,\alpha,(0,\epsilon)} \leq M. \tag{10.1.23}$$

Define a function u on $Q_{\epsilon'}$ by

$$u(s,t) = \psi(x_{P_1} + hs, \theta_w + \mathfrak{g}_{sh}(hs)t), \tag{10.1.24}$$

so (10.1.14) *holds. Then $\psi \in \hat{C}^{(-\beta),\{P_1\}}_{m,\alpha,\Omega_\epsilon}$ if and only if $u \in C^{m,\alpha}_{\beta,(subs)}$, and*

$$\frac{1}{C}\|\psi\|^{(-\beta),\{P_1\}}_{m,\alpha,\Omega_\epsilon} \leq \|u\|^{(\beta),(subs)}_{m,\alpha,Q_{\epsilon'}} \leq C|\psi|^{(-\beta),\{P_1\}}_{m,\alpha,\Omega_\epsilon}, \tag{10.1.25}$$

where C depends only on (M, M_1), and we have used the notation from (10.1.19).

Proof. The proof has three steps.

Step 1 It is clear, from the definition (A.1.5) that for $m = 0, 1$ and $\alpha, \beta \in [0,1)$ such that $m + \alpha \geq \beta$, the space $\hat{C}^{(-\beta),\{P_1\}}_{m,\alpha,\Omega_\epsilon}$ and be categorised the set of $\psi \in C^{(-\beta),\{P_1\}}_{m,\alpha,\Omega_\epsilon}$ such that the norm $\|\psi\|^{'(-\beta)}_{m,\alpha,\Omega_\epsilon}$ as defined below is finite;

$$\begin{aligned}
&\|\psi\|^{'(-\beta)}_{m,0,\Omega_\epsilon} := \sum_{0\leq|\gamma|\leq m} \sup_{\mathbf{x}\in\Omega_\epsilon} \Big(|x - x_{P_1}|^{|\gamma|-\beta}|D^\gamma v(\mathbf{x})|\Big),\\
&[\psi]^{'(-\beta)}_{m,\alpha,\Omega_\epsilon} := \sum_{|\gamma|=m} \sup_{\mathbf{x},\tilde{\mathbf{x}}\in\Omega_\epsilon,\mathbf{x}\neq\tilde{\mathbf{x}}} \left(\min(|x - x_{P_1}|^{m+\alpha-\beta}, |\tilde{x} - x_{P_1}|^{m+\alpha-\beta})\frac{|D^\gamma v(\mathbf{x}) - D^\gamma v(\tilde{\mathbf{x}})|}{|\mathbf{x}-\tilde{\mathbf{x}}|^\beta}\right),\\
&\|\psi\|^{'(-\beta)}_{m,\alpha,\Omega_\epsilon} := \|v\|^{'(-\beta)}_{m,0,\Omega_\epsilon} + [v]^{'(-\beta)}_{m,\alpha,\Omega_\epsilon}.
\end{aligned} \tag{10.1.26}$$

Moreover, the norms $\|\cdot\|^{(-\alpha)}_{m,\alpha,\Omega_\epsilon}$ and $\|\cdot\|^{'(-\alpha)}_{m,\alpha,\Omega_\epsilon}$ are equivalent on $\hat{C}^{(-\alpha),\{P_1\}}_{m,\alpha,\Omega_\epsilon}$ with constant C depending only on M, M_1. Thus it suffices to prove that

$$\frac{1}{C}\|\psi\|^{'(-\beta)}_{m,\alpha,\Omega_\epsilon} \leq \|u\|^{(\beta),(subs)}_{m,\alpha,Q_{\epsilon'}} \leq C\|\psi\|^{'(-\beta)}_{m,\alpha,\Omega_\epsilon}, \tag{10.1.27}$$

where C only depends on M, M_1.

Step 2 We now note that the first derivatives of ψ are expressed as follows;

$$\psi_x = \frac{1}{h}u_s - t\frac{\mathfrak{g}'}{\mathfrak{g}}, \qquad \psi_y = \frac{1}{\mathfrak{g}}u_t, \tag{10.1.28}$$

where $\mathfrak{g}$ is used to denote $\mathfrak{g}_{sh}$. We now directly show:

$$\frac{1}{C}\|\psi\|^{(-\beta),\{P_1\}}_{m,0,\Omega_\epsilon} \leq \|u\|^{(\beta),(subs)}_{m,0,Q_{\epsilon'}} \leq C|\psi|^{(-\beta),\{P_1\}}_{m,0,\Omega_\epsilon}. \tag{10.1.29}$$

Indeed, we examine the case $m = 1$. Then, for $u \in C^{1,0}_{\beta,(subs)}(Q_{\epsilon'})$, we have, for $L = \|u\|^{(\beta),(subs)}_{1,0,Q_{\epsilon'}}$,

$$|u(s,t)| \leq Ls^\beta, \quad |u_s(s,t)| \leq Ls^{\beta-1}, \quad |u_t(s,t)| \leq Ls^\beta. \tag{10.1.30}$$

Thus, using the expressions (10.1.28), and the lower bound $\mathfrak{g}(s) \geq \frac{s}{M}$ from (10.1.6),

$$|\psi_y| = |\frac{1}{\mathfrak{g}} u_t| \leq CL(x - x_{P_1})^{\beta-1}, \tag{10.1.31}$$

and similarly, again using (10.1.6), in addition to other properties, we have

$$|\psi_x| = |\frac{1}{h} u_s - t\frac{\mathfrak{g}'}{\mathfrak{g}} u_t| \leq CL(x - x_{P_1})^{\beta-1}, \tag{10.1.32}$$

and from (10.1.30), $|\psi| \leq CL(x - x_{P_1})^{\beta}$, so we conclude that

$$\|\psi\|^{\prime,(-\beta)}_{1,0,\Omega_\epsilon} \leq C\|u\|^{(\beta),(subs)}_{1,0,Q_{\epsilon'}}. \tag{10.1.33}$$

The case $m = 0$ is similar. The other side of the estimate can be obtained similarly, using only the upper bound $|\mathfrak{g}(s)| \leq Ms$.

Step 3 We now just need to show the estimates for the difference quotient terms, i.e.;

$$[\psi]^{\prime,(-\beta)}_{m,\alpha,\Omega_\epsilon} \leq C\|u\|^{(\beta),(subs)}_{m,\alpha,\Omega_{\epsilon'}}, \qquad [u]^{(\beta),(subs)}_{m,\alpha,\Omega_{\epsilon'}} \leq C\|\psi\|^{\prime,(-\beta)}_{m,\alpha,\Omega_\epsilon}, \tag{10.1.34}$$

We will show the details for just one term, in the case $m = 1$, where $k = 0$ and $l = 1$; i.e. we will show that

$$\sup_{\mathbf{s},\tilde{\mathbf{s}} \in Q_{\epsilon'}, \mathbf{s} \neq \tilde{\mathbf{s}}} \left(\min(s^{\alpha-\beta}, \tilde{s}^{\alpha-\beta}) \frac{|u_t(\mathbf{s}) - u_t(\tilde{\mathbf{s}})|}{\delta^{(subs)}_\beta(\mathbf{s}, \tilde{\mathbf{s}})} \right) \leq C\|\psi\|^{\prime,(-\beta)}_{1,\alpha,Q_\epsilon}. \tag{10.1.35}$$

To prove this, we use the following fact; there exists C depending only on (M, M_1), such that for the transformation $\mathbf{s} \mapsto \mathbf{x}(\mathbf{s})$ from (10.1.12),

$$\frac{1}{C}\delta^{(subs)}_\beta(\mathbf{s}, \tilde{\mathbf{s}}) \leq |\mathbf{x}(\mathbf{s}) - \mathbf{x}(\tilde{\mathbf{s}})|^\beta \leq C\delta^{(subs)}_\beta(\mathbf{s}, \tilde{\mathbf{s}}). \tag{10.1.36}$$

to see this; for fixed $\mathbf{s} = (s, t)$ and $\tilde{\mathbf{s}} = (\tilde{s}, \tilde{t})$,

$$|\mathbf{x}(\mathbf{s}) - \mathbf{x}(\tilde{\mathbf{s}})|^2 = h^2(s - \tilde{s})^2 + (\mathfrak{g}(hs)t - \mathfrak{g}(h\tilde{s})\tilde{t})^2. \tag{10.1.37}$$

without loss of generality, we can assume $\tilde{s} = \min(s, \tilde{s})$. We can now use the upper bound $|\mathfrak{g}(s)| \leq Ms$ from (10.1.6) to obtain;

$$\begin{aligned} |\mathbf{x}(\mathbf{s}) - \mathbf{x}(\tilde{\mathbf{s}})|^2 &= h^2(s - \tilde{s})^2 + \big(t(\mathfrak{g}(hs) - \mathfrak{g}(h\tilde{s})) + \mathfrak{g}(h\tilde{s})(t - \tilde{t})\big)^2 \\ &\leq C\big((s - \tilde{s})^2 + \tilde{s}^2(t - \tilde{t})^2\big), \end{aligned} \tag{10.1.38}$$

so taking the power $\beta/2$, the upper bound is proved. For the lower bound, we must consider two cases. Firstly, if $|s - \tilde{s}| \geq \frac{s}{2M^2}|t - \tilde{t}|$, we find

$$|\mathbf{x}(\mathbf{s}) - \mathbf{x}(\tilde{\mathbf{s}})|^2 \geq \frac{1}{C}|s - \tilde{s}|^2, \tag{10.1.39}$$

and in the opposite case, when $|s - \tilde{s}| \leq \frac{s}{2M^2}|t - \tilde{t}|$, we find, by first writing

$$\mathfrak{g}(hs)t - \mathfrak{g}(h\tilde{s})\tilde{t} = \tilde{t}(\mathfrak{g}(hs) - \mathfrak{g}(h\tilde{s})) + \mathfrak{g}(hs)(t - \tilde{t}), \tag{10.1.40}$$

and using the lower bound $\mathfrak{g}(s) \geq \frac{s}{M}$, that

$$\begin{aligned} |\mathbf{x}(\mathbf{s}) - \mathbf{x}(\tilde{\mathbf{s}})| &\geq |\mathfrak{g}(hs)(t-\tilde{t})| - |\mathfrak{g}(hs) - \mathfrak{g}(h\tilde{s})| \\ &\geq \frac{hs}{M}|t-\tilde{t}| - Mh|s-\tilde{s}| \\ &\geq \frac{hs}{2M}|t-\tilde{t}|, \end{aligned} \tag{10.1.41}$$

and so formula (10.1.38) is proved.

Now, we resume the proof of (10.1.35). Without loss of generality, we may assume $\tilde{s} \leq s$, so $\tilde{x} - x_{P_1} \leq x - x_{P_1}$, and using Lemma 10.1.4, we find

$$\begin{aligned} s^{\alpha-\beta}\frac{u_t(\mathbf{s}) - u_t(\tilde{\mathbf{s}})}{\delta_\beta^{(subs)}(\mathbf{s},\tilde{\mathbf{s}})} &\leq C(x-x_{P_1})^{\alpha-\beta}\frac{|\mathfrak{g}(x-x_{P_1})\psi_y(\mathbf{x}) - \mathfrak{g}(x-x_{P_1})\psi_y(\mathbf{x})|}{|\mathbf{x}-\tilde{\mathbf{x}}|^\beta} \\ &\leq C(x-x_{P_1})^{\alpha-\beta}\left(|\psi_y(\mathbf{x})|\frac{|\mathfrak{g}(x-x_{P_1}) - \mathfrak{g}(\tilde{x}-x_{P_1})|}{|\mathbf{x}-\tilde{\mathbf{x}}|^\beta}\right. \\ &\qquad \left. + \mathfrak{g}(\tilde{x}-x_{P_1})\frac{|\psi_y(\mathbf{x}) - \psi_y(\tilde{\mathbf{x}})|}{|\mathbf{x}-\tilde{\mathbf{x}}|^\beta}\right) \\ &\leq C(x-x_{P_1})^{\alpha-\beta}\left(\|\psi\|^{(-\beta)}_{1,\alpha,\Omega_\epsilon}(x-x_{P_1})^{\beta-1}\|\mathfrak{g}_{sh}\|^{(-1)}_{0,\beta,(0,\epsilon)}(x-x_{P_1})^{1-\alpha}\right. \\ &\qquad \left. + (x-x_{P_1})[\psi]^{\prime,(\beta)}_{1,\alpha,\Omega_\epsilon}(x-x_{P_1})^{\beta-1-\alpha}\right) \\ &\leq C\|\psi\|^{\prime,(-\beta)}_{1,\alpha,\Omega_\epsilon}. \end{aligned} \tag{10.1.42}$$

□

Corollary 10.1.6. *Let $\epsilon' \in (0,1)$, and $m, \tilde{m} \in \{0,1\}$, $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in (0,1)$ such that $m + \alpha < \tilde{m} + \tilde{\alpha}$, and $\beta \leq \tilde{\beta}$, $m + \alpha \geq \beta, \tilde{m} + \tilde{\alpha} \geq \tilde{\beta}$. Then if there exists $M > 0$ such that if $\|u_k\|^{(\tilde{\beta}),(subs)}_{\tilde{m},\tilde{\alpha},Q_{\epsilon'}} \leq M$ for a sequence of functions $\{u_k\}_{k=1}^\infty$ on $Q_{\epsilon'}$, then there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ converging in norm $\|\cdot\|^{(\beta),(subs)}_{m,\alpha,Q_{\epsilon'}}$.*

Proof. This follows by Lemma 10.1.5, by taking $h = 1$, $x_{P_1} = \theta_w = 0$, $\epsilon = h\epsilon_0' = \epsilon_0'$ and $\mathfrak{g}_{sh}(s) = s$, and using the compactness of $C^{(-\tilde{\beta}),\{P_1\}}_{m,\tilde{\alpha},\Omega_\epsilon}$ in $C^{(-\beta),\{P_1\}}_{m,\alpha,\Omega_\epsilon}$. □

We now turn to estimates that also apply in supersonic cases, to aid in the construction of the norm for our iteration set; in particular, we relate the norm subsonic norm (10.1.19) to the parabolic norm (A.1.8). Specifically, we now let $\mathfrak{g}_{sh}$ be a function on $[0,h]$ satisfying linear growth conditions as in (10.1.6):

$$\min\left(\mathfrak{g}_{sh}(0) + \frac{s}{M}, \delta_{sh}\right) \leq \mathfrak{g}_{sh}(s) \leq \mathfrak{g}_{sh}(0) + Ms, \tag{10.1.43}$$

where $\delta_{sh} > 0$, and $0 \leq \mathfrak{g}_{sh}(0) \leq \delta_{sh}$. Now, with $x_{P_1} \in \mathbb{R}$ fixed, we again use map (10.1.12); for $\epsilon \in (0,h)$, denote

$$\begin{aligned} &\Omega_\epsilon := \mathbf{x}(Q_{\epsilon'}) \qquad \text{with } \epsilon' = \frac{\epsilon}{h} \text{ and } Q_{\epsilon'} := Q^{iter} \cap \{s < \epsilon'\}, \\ &P_1 := \mathbf{x}(0,1) = (x_{P_1}, \theta_w), \quad P_4 := \mathbf{x}(0,0) = (x_{P_1}, \theta_w + \mathfrak{g}_{sh}(0)), \\ &\Gamma_{sonic} := \partial\Omega_\epsilon \cap \{x = x_{P_1}\} = \mathbf{x}(\{0\} \times (0,1)). \end{aligned} \tag{10.1.44}$$

and again, similarly to (10.1.17), for $k \in \{0,1\}$ and $\alpha' \in (0,1)$ with $k+\alpha' \geq \beta$, we define

$$\hat{C}^{(-\beta),\{\Gamma_{sonic}\}}_{k,\alpha',\Omega_\epsilon} := \{\psi \in C^{(-\beta),\{\Gamma_{sonic}\}}_{k,\alpha',\Omega_\epsilon} \mid \psi = 0 \text{ on } \Gamma_{sonic}\}. \tag{10.1.45}$$

Lemma 10.1.7. *Let $M, M_1, M_2 \geq 1$, $x_{P_1} \in \mathbb{R}$ and $\theta_w \in \mathbb{R}$. Let $h \in (\frac{1}{M_1}, M_1)$, $\delta_{sh} \geq \frac{1}{M_2}$ and $\epsilon \in (0, \frac{h}{2})$. Let $m \in \{0,1\}$ and $\alpha, \beta \in [0,1)$ with $m+\alpha \geq \beta$. Assume that a function $\mathfrak{g}_{sh}(s)$ on on $(0,\epsilon)$ satisfies* (10.1.6) *and*

$$\|\mathfrak{g}_{sh}\|^{(-1),\{0\}}_{m,\alpha,(0,\epsilon)} \leq M. \tag{10.1.46}$$

Let u be a function on $Q_{\epsilon'}$, and define ψ on Ω_ϵ by (10.1.14). *Then if $u \in C^{m,\alpha}_{\beta,(subs)}(Q_{\epsilon'})$, $\psi \in \hat{C}^{(-\beta),\{\Gamma_{sonic}\}}_{m,\alpha,\Omega_\epsilon}$, and*

$$\|\psi\|^{(-\beta),\{\Gamma_{sonic}\}}_{m,\alpha,\Omega_\epsilon} \leq C\|u\|^{(\beta),(subs)}_{m,\alpha,Q_{\epsilon'}}, \tag{10.1.47}$$

where C only depends on (M, M_1, M_2), and we have used notation from (10.1.19).

Proof. The proof is similar to [26][Lemma 17.2.8], except with $\alpha \neq \beta$, and one less derivative, and is similar to the one of Lemma 10.1.5. We firstly note that $x = x_{P_1}$ on Γ_{sonic}, so that similarly to Lemma 10.1.5, we can characterise $\hat{C}^{(-\alpha),\{\Gamma_{sonic}\}}_{m,\alpha,\Omega_\epsilon}$ can be characterised as the set of all $v \in C^{(-\beta),\{\Gamma_{sonic}\}}_{m,\alpha,\Omega_\epsilon}$ where $\|\psi\|'^{,(-\beta)}_{m,\alpha,\Omega_\epsilon} < \infty$, where $\|\psi\|'^{,(-\beta)}_{m,\alpha,\Omega_\epsilon}$ is as in (10.1.26). Next, we note that the norms are equivalent, with C depending on (M, M_1). Therefore, we only need to show that

$$\|\psi\|'^{,(-\beta)}_{m,\alpha,\Omega_\epsilon} \leq C\|u\|^{(\beta),(subs)}_{m,\alpha,Q_{\epsilon/h}}, \tag{10.1.48}$$

with C depending on (M, M_1, M_2).

Firstly, we consider the case where $\epsilon \leq \frac{M}{M_2}$; here, for $s \in (0,\epsilon)$,

$$\delta_{sh} \geq \frac{1}{M_2} \geq \frac{s}{M}, \tag{10.1.49}$$

so that by (10.1.43),

$$\mathfrak{g}_{sh}(s) \geq \frac{s}{M}, \tag{10.1.50}$$

so we can follow the proof of (10.1.48) in Lemma 10.1.5, noting that it only required the lower bound for $\mathfrak{g}_{sh}$, and we obtain (10.1.48) with constants depending only on M, M_1.

If $\epsilon > \frac{M}{M_2}$, we first observe (10.1.48) is already proved with $\tilde{\epsilon} = \frac{M}{M_2}$, instead of ϵ. Next, from (10.1.46) and (10.1.43), the transform (10.1.12) is one-to-one from $Q_{\epsilon/h} \setminus Q_{\tilde{\epsilon}/2h}$ onto $\Omega_\epsilon \setminus \Omega_{\tilde{\epsilon}/2}$, and both $\mathbf{x}, \mathbf{x}^{-1}$ are $C^{m,\alpha}$ on their respective domains, with norms only depending on M, M_1, M_2. Moreover, the norms $\|\cdot\|'^{,(-\beta)}_{m,\alpha,\Omega_\epsilon \setminus \Omega_{\tilde{\epsilon}/2}}$ and $\|\cdot\|^{(\beta),(subs)}_{m,\alpha,Q_{\epsilon/h} \setminus Q_{\tilde{\epsilon}/2h}}$ are equivalent to the standard $C^{m,\alpha}$ norms on their domains, with constants depending only on $\epsilon, \tilde{\epsilon}$; thus on (M, M_1, M_2), because $\frac{M_1}{2} \geq \epsilon \geq \tilde{\epsilon} = \frac{M}{M_2}$, so we obtain

$$\|\psi\|'^{,(-\alpha)}_{m,\alpha,\Omega_\epsilon \setminus \Omega_{\tilde{\epsilon}/2}} \leq C\|u\|^{(\alpha),(subs)}_{m,\alpha,Q_{\epsilon/h} \setminus Q_{\epsilon/2h}}, \tag{10.1.51}$$

for C depending only on (M, M_1, M_2). Then, combining with (10.1.48) for $\epsilon = \tilde{\epsilon} = \frac{M}{M_2}$, we obtain the result. □

We now derive results relating the $\|\cdot\|^{(\sigma),(par)}_{m,\alpha,Q_{\epsilon'}}$ and $\|\cdot\|^{(\sigma),(subs)}_{m,\alpha,Q_{\epsilon'}}$ norms.

Lemma 10.1.8. *Let $\epsilon' \in (0,1)$, $\alpha \in (0,1)$, $\sigma \geq 0$, and m a non-negative integer. Then the subsonic norm is stronger than the parabolic norm* (A.1.8)*; i.e.*

$$\|u\|^{(\sigma),(par)}_{m,\alpha,Q_{\epsilon'}} \leq \|u\|^{(\sigma),(subs)}_{m,\alpha,Q_{\epsilon'}}. \tag{10.1.52}$$

Proof. This assertion follows directly from comparing the expressions (10.1.18) and (10.1.19) with (A.1.7) and (A.1.8), noting that the weights on the corresponding terms are always larger in the subsonic norm than those in the parabolic norm, noting that in the region considered, $s, \tilde{s} \in (0,1)$. In particular, $s^{k+\frac{l}{2}-\sigma} \leq s^{k-\sigma}$ and $s^{\alpha+k+\frac{l}{2}-\sigma} \leq s^{\alpha+k-\sigma}$, and $\delta^{(par)}_{\alpha}(\mathbf{s},\tilde{\mathbf{s}}) \geq \delta^{(subs)}_{\alpha}(\mathbf{s},\tilde{\mathbf{s}})$. □

In this next Lemma, we restrict to $\alpha \in (0, \frac{1}{3}]$.

Lemma 10.1.9. *Let $\epsilon' \in (0,1)$ and $\alpha \in [0,1/3)$. If $u \in C^{1,0}_{1,(par)}(Q_{\epsilon'})$, then $u \in C^{0,\alpha}_{\alpha,(subs)}$, and*

$$\|u\|^{(\alpha),(subs)}_{0,\alpha,Q_{\epsilon'}} \leq 2\|u\|^{(1),(par)}_{1,0,Q_{\epsilon'}}, \tag{10.1.53}$$

where the parabolic norm is with respect to $\{s=0\}$.

Proof. Firstly, we observe that as $s \in (0,1)$ whenever $(s,t) \in Q_{\epsilon'_0}$,

$$\|u\|^{(\alpha),(subs)}_{0,0,Q_{\epsilon'}} = \|s^{-\alpha}u\|_{\infty} \leq \|s^{-1}u\|_{\infty} = \|u\|^{(1),(par)}_{0,0,Q_{\epsilon'}}. \tag{10.1.54}$$

Thus, we now just need to estimate the difference quotient term, i.e.:

$$\sup_{\mathbf{s},\tilde{\mathbf{s}} \in Q_{\epsilon'}, \mathbf{s} \neq \tilde{\mathbf{s}}} \frac{|u(\mathbf{s}) - u(\tilde{\mathbf{s}})|}{\delta^{(subs)}_{\alpha}(\mathbf{s},\tilde{\mathbf{s}})}. \tag{10.1.55}$$

Now, from (10.1.18), we find that because $\alpha \in (0,1/3]$,

$$\begin{aligned} &\sqrt{|s-\tilde{s}|} \leq |s-\tilde{s}|^{\alpha} \leq \delta^{(subs)}_{\alpha}(\mathbf{s},\tilde{\mathbf{s}}), \\ &\sqrt{\max(s,\tilde{s})|t-\tilde{t}|} \leq \big(\max(s,\tilde{s})|t-\tilde{t}|\big)^{\alpha} \leq \delta^{(subs)}_{\alpha}(\mathbf{s},\tilde{\mathbf{s}}). \end{aligned} \tag{10.1.56}$$

Thus, by (A.1.7) and (A.1.8), we find that for $\mathbf{s},\tilde{\mathbf{s}} \in Q_{\epsilon'}$ with $s \leq \tilde{s}$,

$$\begin{aligned} |u(\mathbf{s}) - u(\tilde{\mathbf{s}})| &\leq |u(s,t) - u(\tilde{s},t)| + |u(\tilde{s},t) - u(\tilde{s},\tilde{t})| \\ &\leq \max_{s' \in [s,\tilde{s}]} |u_s(s',t)||s-\tilde{s}| + \max_{t' \in [t,\tilde{t}]} |u_t(s,t')||t-\tilde{t}| \\ &\leq \|u\|^{(1),(par)}_{1,0,Q_{\epsilon'}} \Big(|s-\tilde{s}| + \sqrt{\tilde{s}}|t-\tilde{t}| \Big) \\ &\leq \|u\|^{(1),(par)}_{1,0,Q_{\epsilon'}} \Big(\sqrt{s-\tilde{s}} + \sqrt{\tilde{s}}\sqrt{t-\tilde{t}} \Big) \\ &\leq 2\|u\|^{(1),(par)}_{1,0,Q_{\epsilon'}} \delta^{(subs)}_{\alpha}(\mathbf{s},\tilde{\mathbf{s}}), \end{aligned} \tag{10.1.57}$$

where we used (10.1.56). □

We can now extend the estimates for (u,v) in Proposition 7.1.5 to wedge angles $\theta^*_w \in (\theta_{detach}, \frac{\pi}{2})$. We use ϵ'_0 as in (10.1.13).

Proposition 10.1.10. *Let $\theta_w^* \in (\theta_{detach}, \frac{\pi}{2})$, and $\beta \in (0, 1/2)$. Then there exists $M, M(\beta) \geq 0$ and $\overline{\alpha} \in (0, \frac{1}{3}]$, depending only on the data and θ_w^*, such that for any admissible solution (ρ, r) for $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, functions u and v defined by (7.1.26) satisfy that for any $\alpha \in [0, \overline{\alpha}]$,*

$$\begin{aligned} \|u\|^{(-\alpha),\{(1,0),(1,1)\}}_{1,\alpha,Q^{iter}\setminus Q_{\epsilon_0'/10}} + \|u\|^{(\alpha),(par)}_{1,\alpha,Q_{\epsilon_0'}} + \|u\|^{(\alpha),(subs)}_{0,\alpha,Q_{\epsilon_0'}} &\leq M, \\ \|v\|^{(-1-\beta),\{0,1\}}_{2,\alpha,(\epsilon_0'/10,1)} &\leq M(\beta), \end{aligned} \tag{10.1.58}$$

where $Q_{\epsilon_0'} := Q^{iter} \cap \{s < \epsilon_0'\}$, and the parabolic norms are with respect to $\{s = 0\}$

Proof. The estimate for u follows by combining all the global estimates we attained in Chapter 9 with the relations and estimates found above. More precisely, in all four cases, away from the Γ_{sonic}, we use (10.1.5) combined with (10.1.6); the difference is when near Γ_{sonic}. For the supersonic cases, we use Lemma 10.1.8 and Lemma 10.1.9 with (10.1.6) to attain the desired bound from Corollaries 9.2.1 and 9.2.6. For the subsonic cases, we use Lemma 10.1.5 with (10.1.6), and Lemma 10.1.8 to attain the desired bound from Corollaries 9.2.10 and 9.2.14.

The second estimate for v follows from (10.1.5), combined with (10.1.9) and (10.1.11). □

We now turn to mapping from the iteration region; i.e. inverting the map $(\rho, r) \mapsto (u, v)$. We describe here the changes required to extend the construction from Definition 7.1.6 to all wedge angles. We fix $\theta_w^* \in (\theta_d, \frac{\pi}{2})$. Because $P_1 = P_4$ in the subsonic or sonic case, and $P_1 \neq P_4$ otherwise, the inequality (7.1.32) now takes the form:

$$\inf_{\partial Q_{bd}^{(\theta_w)} \cap F_1^{-1}(\{s=0\})} \tilde{f} < 0 \leq \sup_{\partial Q_{bd}^{(\theta_w)} \cap F_1^{-1}(\{s=0\})} \tilde{f}, \tag{10.1.59}$$

with equality on the right if and only if $\theta_w \in (\theta_d, \theta_{sonic}]$.

We now note that the definition of $\mathfrak{S}$ from Definition 7.1.6 applies without changes; more precisely, $\mathfrak{S}$ and $\mathfrak{S}(\theta_w)$ are defined by 7.1.33. Also, Lemma 7.1.7 holds without change.

We now conclude the construction of the mapping to the iteration region by remarking that the oblique cutoff can be imposed here exactly as in §7.1.1 with no changes, as the cutoff only applies locally to P_2, i.e. away from Γ_{sonic}, due to (8.2.9).

10.2 Iteration set

We now extend our iteration set, up to the detachment angle. We fix $\theta_w \in (\theta_d, \frac{\pi}{2}]$, and define

Definition 10.2.1. We define norms for use in our iteration set as follows.

(i) For $\alpha \in (0, 1)$, we define

$$\|u\|^{(**)}_{1,\alpha,Q^{iter}} := \|u\|^{(-\alpha),\{(1,0),(1,1)\}}_{1,\alpha/2,Q^{iter}\cap\{s>\epsilon_0'/10\}} + \|u\|^{(\alpha),(par)}_{1,\alpha/2,Q_{\epsilon_0'}} + \|u\|^{(\alpha),(subs)}_{0,\alpha,Q_{\epsilon_0'}}, \tag{10.2.1}$$

where we use norms defined in (10.1.19) and (A.1.8), with respect to $\{s = 0\}$.

(ii) For $\theta_w \in [\theta_{detach}, \frac{\pi}{2})$, and $\sigma \in (0, \epsilon_0]$, denote

$$\mathcal{D}_{\sigma}^{(\theta_w)} := \{\theta_w < y < \hat{f}_0(x)\} \cap \Lambda \cap \mathcal{N}_{\epsilon_1}(\Gamma_{sonic}^{(\delta_*)} \cap \{x_{P_1} < x < x_{P_1} + \sigma\}, \tag{10.2.2}$$

$$\mathcal{D}_{0,\sigma}^{(\theta_w)} := \{\theta_w < y < \hat{f}_0\} \cap \Lambda \cap \mathcal{N}_{\epsilon_1}(\Gamma_{sonic}^{(\delta_*)} \cap \{x_{P_1} < x < \sigma\}, \tag{10.2.3}$$

where $\hat{f}_0$ is defined in (6.2.1), δ_* and $\Gamma_{sonic}^{\delta_*}$ are defined at the start of §10.1, and ϵ_1 is from Proposition 9.1.4. We also often drop the θ_w dependence from the notation.

Remark 10.2.2. From (10.2.2) and (10.2.3), it follows that for $\sigma \in (0, \epsilon_0]$;

- $\mathcal{D}_{0,\sigma} \subset \mathcal{D}_{\sigma}$ for all wedge angles,
- $\mathcal{D}_{0,\sigma} = \mathcal{D}_{\sigma}$ for all supersonic and sonic wedge angles,
- $D_{0,\sigma} = \emptyset$ for subsonic wedge angles satisfying $x_{P_0} > \sigma$,
- From (9.1.5), it follows that for any admissible solution,

$$\Omega_{\sigma} = \Omega \cap \mathcal{D}_{\sigma} \qquad \text{for any } \sigma \in (0, \epsilon_0). \tag{10.2.4}$$

We next note the following compactness property of $C_{(**)}^{2,\alpha}(Q^{iter})$, which follows from Corollary 10.1.6, Lemma A.1.3, and the standard compactness property of $C^{1,\beta}(0,1)$:

Lemma 10.2.3. *Let $0 \leq \tilde{\alpha} < \alpha < 1$. Then $C_{(**)}^{1,\alpha}(Q^{iter})$ is compactly embedded into $C_{(**)}^{1,\tilde{\alpha}}(Q^{iter})$.*

We now turn to defining the iteration set. Let $\alpha, \beta \in (0,1)$, and we make the following assumption;

$$1 + \alpha \leq 2\beta + \frac{1}{2}. \tag{10.2.5}$$

Again, we consider $\mathcal{K}$ as a subset of $\mathcal{C}$ from (7.2.3), but replace $C_{*,\sigma}^{1,\alpha}$ by $C_{(**)}^{1,\alpha}(Q^{iter})$. We also fix δ_w^* to be small enough that Lemma 9.1.5, Lemma 9.1.6, Corollary 9.2.6 and Corollary 9.2.10 all hold.

Definition 10.2.4. The iteration set $\mathcal{K} \subset \mathcal{C} \times [\theta_w^*, \frac{\pi}{2}]$ is defined as the set of (u, v, θ_w) such that;

(i) (u, v) satisfy the estimates;

$$\begin{aligned} \|u\|_{1,\alpha,Q^{iter}}^{(**)} &\leq N_0, \\ \|v\|_{2,\alpha,(0,1)}^{(-(1+\beta)),\{0,1\}} &\leq N_0, \end{aligned} \tag{10.2.6}$$

where $N_0 = 10M$ for the constant M from (10.1.58).

(ii) $(u, v, \theta_w) \in \mathfrak{S}$ as defined by (7.1.33), so that $\mathfrak{g}_{sh}, \Omega, \Gamma_{shock}$ and ψ are defined as in Definition 7.1.6. Also, denoting the inner unit normal to Γ_{shock} in $\boldsymbol{\xi}$ coordinates by $\boldsymbol{\nu}$,

$$\boldsymbol{\xi} \cdot \boldsymbol{\nu} < -c_1, \tag{10.2.7}$$

so that Γ_{shock} can be written in the form $\{r = r(\theta)$ for some $r \in C^{1,\beta}([\theta_1, \frac{\pi}{2}])$. Also, for $N_2 = 2C$ from (10.1.6),

$$\min\left(\mathfrak{g}_{sh}(0) + \frac{s}{N_2}, \frac{1}{N_2}\right) < \mathfrak{g}_{sh}(s) < \min\left(\mathfrak{g}_{sh}(0) + N_2 s, \eta^{(\theta_w)}(s) - \frac{1}{N_2}\right), \tag{10.2.8}$$

for any $s \in (0, \hat{s}(\theta_w)$, and $\theta_w \in [\theta_w^*, \frac{\pi}{2})$, where we note $\mathfrak{g}_{sh}(0) = t_{P_1}(\theta_w) \geq 0$.

(iii) ρ is well defined, and satisfies

$$\rho_1 + \tau_1 < \rho \leq \rho_2, \tag{10.2.9}$$

where $\tau_1 = \frac{\tau}{2}$, for τ from (8.2.11).

(iv) $\rho, c^2, r(\theta)$ and $\psi = c_2^2 - c^2(\rho)$, determined by u and v satisfy;

$$|\psi| \leq \eta_2(\theta_w)x \qquad \text{in } \overline{\Omega} \cap \big(\mathcal{D}_{\epsilon_0} \setminus \mathcal{D}_{0,\epsilon/10}\big), \tag{10.2.10}$$

$$|\psi| \leq N_3\epsilon \qquad \text{in } \overline{\Omega} \cap \mathcal{D}_{\epsilon}, , \tag{10.2.11}$$

$$|r^2 - \overline{c}^2| < N_4\sigma, |r'| < N_4\sigma^{1/2} \qquad \text{on } \Gamma_{shock} \cap \{\theta - \pi < \sigma_1\}, \tag{10.2.12}$$

$$2g(r,\rho) < r'(\theta) < \frac{1}{2}g(r,\rho) \qquad \text{on } \Gamma_{shock} \cap \{\pi - \theta > \sigma_1/10\}, \tag{10.2.13}$$

where $\mathcal{D}_\sigma$ and $\mathcal{D}_{0,\sigma}$ are as in (10.2.2) and (10.2.3) and $\eta_2 \in C(\mathbb{R})$ is defined by

$$\eta_2(\theta_w) := \begin{cases} 2 - \mu_0 & \text{if } \theta_w \in [\theta_{sonic} - \delta_w^*/2, \frac{\pi}{2}], \\ N_3 & \text{if } \theta_w \in (\theta_d, \theta_{sonic} - \delta_w^*], \\ \text{linear} & \text{elsewhere,} \end{cases} \tag{10.2.14}$$

where $\mu_0 = \delta/2$ for δ from Lemma 9.1.5, and $N_3 = 2C$ for C from Lemma 9.1.7.

(v) Uniform ellipticity in $\overline{\Omega} \setminus \mathcal{D}_{0,\epsilon/10}$:

$$c^2 - r^2 > \min\big(c_2^2 - r^2(P_1)(\theta_w) - \lambda_1 \mathrm{dist}(\boldsymbol{\xi}, \overline{\Gamma_{sonic}}), \lambda_2\big), \quad \text{for all } \boldsymbol{\xi} \in \overline{\Omega} \setminus \mathcal{D}_{0,\epsilon/10}, \tag{10.2.15}$$

where $\lambda_1 = \lambda/2$, and $\lambda_2 = \hat{\zeta}/2$, for the constants from (8.2.12).

(vi) Uniform obliqueness away from P_2:

$$r^2 - \overline{c}^2 \geq \delta_3(\pi - \theta) \qquad \text{on } \Gamma_{shock} \cap \{\pi - \theta > \sigma_1/10\}, \tag{10.2.16}$$

where $\delta_3 = \frac{\delta}{2}$, for δ from (8.2.12)

(vii) The boundary value problem determined by (u, v, θ_w) in Problem 7.2.7 has a solution $(\hat{\psi}, \hat{r})$ which satisfies

$$\|\psi - \hat{\psi}\|^{(**)}_{1,\alpha/2,\Omega} < \delta_4, \tag{10.2.17}$$

and

$$\|r - \hat{r}\|^{(-1-\beta/2),\{\theta_1,\pi\}}_{2,\alpha/2,(\theta_1,\pi)} < \delta_5. \tag{10.2.18}$$

We first extend Remark 10.2.5, noting that the unified estimate for u arose from different estimates for each case.

Remark 10.2.5. Let $(u, v, \theta_w) \in \mathcal{K}$. Firstly, we note that using the estimate for v in (10.2.6), that for the determined $\mathfrak{g}_{sh}$ and r satisfy, for C depending non-linearly on N_0 and the data (due to the oblique cutoff),

$$\begin{aligned} \|\mathfrak{g}_{sh}\|^{(-(2\beta+\frac{1}{2})),\{0,\hat{s}(\theta_w)\}}_{2,\alpha,(0,\hat{s}(\theta_w))} + \|\mathfrak{g}_{sh}\|_{1,\beta,[0,\hat{s}(\theta_w)]} \leq C, \\ \|r\|^{(-(2\beta+\frac{1}{2})),\{\theta_1,\pi\}}_{2,\alpha,[\theta_1,\pi]} + \|r\|_{1,\beta,[\theta_1,\pi]} \leq C. \end{aligned} \tag{10.2.19}$$

Next, due to (10.2.8), the first and second terms in (10.2.1), the regularity of the ingredients in the explicit formula (7.1.26), and as $1+\alpha/2 \leq 1+\beta$, we find that for $\tau \in (0, \epsilon_0/2)$, there exists C_τ depending on the data (via N_0), θ_w^* and τ, such that for the corresponding ψ,

$$\|\psi\|_{1,\alpha/2,\Omega\setminus\mathcal{D}_{0,\tau}}^{(-\alpha),\{P_2,P_3\}} \leq C_\tau. \tag{10.2.20}$$

Finally, using the final term in (10.2.1) and the linear separation (10.2.8), as well as Lemma 10.1.7, we find that

$$\|\psi\|_{0,\alpha,\Omega} \leq C. \tag{10.2.21}$$

10.2.1 The equation and boundary conditions for the iteration

We now extend the definition of Problem 7.2.7 to subsonic wedge angles. The only change will be to the elliptic equation, but the estimates for the oblique boundary coefficients (and the reason why they hold) on Γ_{shock} will also change, similarly to Proposition 9.1.4 and Lemma 9.1.6. We will re-define (7.2.23) so that:

- The equation satisfies the same ellipticity structure as (8.2.12),
- For a fixed point $\psi = \hat{\psi}$, the equation (7.2.23) coincides with the original Nonlinear Wave System,
- The equation depends continuously on θ_w.

This will be done by using the same cutoff as the supersonic cases in some subsonic-near-sonic cases, but having no cutoff in subsonic-away-from-sonic cases.

The oblique boundary condition on Γ_{shock}

We first discuss the oblique boundary condition on Γ_{shock}. We note now, that analogously to Proposition 9.1.4 and Lemma 9.1.6, we no longer have the property that $|\rho - \rho_2| \ll 1$ near Γ_{sonic} in subsonic-away-from-sonic cases; this is highlighted in (10.2.10). However, the only properties that were required in Proposition 9.1.4 for the estimates near P_1 were $\rho \leq \rho_2$, which holds in the iteration set by (3.2.9) and (10.2.9), which is a property of weak regular reflection. Thus we find that the same argument as §7.2.3, that (7.2.30)–(7.2.33) and (7.2.34) hold without change, but (7.2.35) and the proceeding comment change as follows: For ϵ small enough only depending on the data,

$$\begin{aligned} \hat{\beta}_2 \leq -\frac{1}{C}, \hat{\beta}\cdot\hat{\boldsymbol{\nu}} \geq \frac{1}{C} \text{ on } \Gamma_{shock}\cap\mathcal{N}_\epsilon(\Gamma_{sonic}) \text{ for all wedge angles,} \\ \hat{\beta}_1 \leq -\frac{1}{C} \text{ on } \Gamma_{shock}\cap\mathcal{N}_\epsilon(\Gamma_{sonic}) \text{ if } \theta_w \in [\theta_{sonic}-\delta_w^*/2, \frac{\pi}{2}]. \end{aligned} \tag{10.2.22}$$

The quasilinear iteration equation

We now turn to extending the definition of the quasilinear coefficients. We first choose ϵ small in the iteration set, depending only on the data (via δ_w^*) to ensure that:

$$x_{P_1} > \epsilon/10 \text{ when } \theta_w < \theta_{sonic} - \delta_w^*/4, \tag{10.2.23}$$

so that either estimate (10.2.10) or (10.2.15) is always available to us; i.e. we always either have the degenerate or small elliptic structure, or strict ellipticity.

We will define our modified coefficients as follows:

- If $\theta_{sonic} > \theta_w > \theta_{sonic} - \delta_w^*/4$, then we have either small or degenerate elliptic structure, the equation in this section agrees completely with the one defined in (7.2.55) for the supersonic case, and the cutoff can be removed as in Lemma 7.4.3, again using (10.2.10).
- If $\theta_{sonic} - \delta_w^*/2 < \theta_w < \theta_{sonic} - \delta_w^*/4$, then the cutoff is automatically removed due to (10.2.10),(10.2.14), and (10.2.23), and we still have the small elliptic structure via (10.2.10).
- If $\theta_w < \delta_w^*/2$, then the equation does not contain a cutoff, i.e. it is (7.2.21) completely linearised with coefficients evaluated at ψ, and is strictly elliptic.

First, we let $\zeta_3 \in C^\infty(\mathbb{R})$ satisfy

$$\zeta_3(s) = \begin{cases} 0 & \text{if } s \geq \theta_{sonic} - \delta_w^*/4, \\ 1 & \text{if } s \leq \theta_{sonic} - \delta_w^*/2, \end{cases} \tag{10.2.24}$$

and $\theta'(s) \geq 0$ on $\mathbb{R}$.

We now define, denoting the coefficients from (7.2.55) used in the supersonic case by $(A_{ij}^{(sup)}, A_i^{(sup)}, B_{ij}^{(sup)})$, and using $(A_{ij}^1, A_i^1, B_{ij}^1)$ from (7.2.37) which are the original coefficients linearised with no cutoff:

$$(A_{ij}, A_i, B_{ij})(\boldsymbol{\xi}) = \zeta_3(\theta_w)(A_{ij}^1, A_i^1, B_{ij}^1)(\boldsymbol{\xi}) + (1-\zeta_3(\theta_w))(A_{ij}^{(sup)}, A_i^{(sup)}, B_{ij}^{(sup)})(\boldsymbol{\xi}), \tag{10.2.25}$$

and we denote by $(\hat{A}_{ij}, \hat{A}_i, \hat{B}_{ij}$ the coefficients of (7.2.23) written in the polar coordinates (x, y).

We now state the properties of the adjusted coefficients in the following Lemmas.

We first extend the estimates of Lemma 7.2.9 and Lemma 7.2.10 to subsonic-near-sonic cases, i.e. when $\theta_w > \theta_{sonic} - \delta_w^*$.

Lemma 10.2.6. *There exists $\lambda, \lambda_0 > 0$, $\epsilon_{eq} \in (0, \frac{\epsilon_0}{2})$, and $N_{eq} \geq 1$ depending only on the data, such that whenever $\theta_w \in [\theta_{sonic} - \delta_w^*, \frac{\pi}{2}]$,*

(i) *For any $(x, y) \in \Omega \cap \mathcal{D}_{\epsilon_{eq}}$, $\boldsymbol{\xi} \in \Omega$ and $\zeta \in \mathbb{R}^2$,*

$$\lambda|\zeta|^2 \leq \frac{\hat{A}_{11}(x,y)}{x}\zeta_{11}^2 + \hat{A}_{22}(x,y)\zeta_{22}^2 \leq \frac{1}{\lambda}|\zeta|^2, \tag{10.2.26}$$

and

$$\lambda_0 \text{dist}(\boldsymbol{\xi}, \Gamma_{sonic})|\zeta|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\boldsymbol{\xi})\zeta_i\zeta_j \leq \lambda_0^{-1}|\zeta|^2, \tag{10.2.27}$$

(ii) *For $s \in (0, \frac{\epsilon_0}{2})$,*

$$\begin{aligned} \|(A_{ij}, B_{ij}, A_i)\|_{0,\alpha,\Omega} &\leq N_{eq}, \\ \|(A_{ij}, B_{ij}, A_i)\|_{1,\alpha/2,\Omega\setminus\mathcal{D}_{0,s}} &\leq N_s. \end{aligned} \tag{10.2.28}$$

(iii) *For $(x, y) \in \Omega \cap \mathcal{D}_{\epsilon_{eq}}$,*

$$\hat{A}_1 \leq c_2 + N_{eq}\epsilon_{eq}, \quad \hat{B}_{11} \leq -1 + N_{eq}\epsilon_{eq}, \quad \hat{B}_{22} \leq -\frac{1}{(\gamma-1)c_2^2} + N_{eq}\epsilon_{eq}, \tag{10.2.29}$$

Proof. These estimates follow as in Lemma 7.2.9 and Lemma 7.2.10; the only change is that we now instead use (10.2.20) and (10.2.21) to obtain (10.2.28), and the second estimate no longer has an easy explicit dependence on s due to the form of (10.2.20), opposed to the parabolic norm in (7.2.17). □

We now state estimates valid for subsonic-away-from-sonic cases.

Lemma 10.2.7. *Let $\delta > 0$. Then there exists $C > 0$, $\lambda(\delta) > 0$, depending on the data, θ_w^* and δ such that whenever $\theta_w \in [\theta_w^*, \theta_{sonic} - \delta]$,*

(i) *For any $\boldsymbol{\xi} \in \Omega$, $\zeta \in \mathbb{R}$,*

$$\lambda(\delta)|\zeta|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\boldsymbol{\xi})\zeta_i\zeta_j \leq \lambda(\delta)^{-1}|\zeta|^2, \tag{10.2.30}$$

(ii) *For $s \in (0, \frac{\epsilon_0}{2})$, there exists N_{eq} depending on the data, θ_w^* and δ, and N_s also depending on s, such that*

$$\begin{aligned} \|(A_{ij}, B_{ij}, A_i)\|_{0,\alpha,\Omega} &\leq N_{eq}, \\ \|(A_{ij}, B_{ij}, A_i)\|_{1,\alpha/2,\Omega\setminus\mathcal{D}_{0,s}} &\leq N_s. \end{aligned} \tag{10.2.31}$$

Proof. (10.2.30) follows by (10.2.15) and the form of the coefficients. (10.2.31) follows from (10.2.20) and the smooth dependence of the coefficients on (x, z). □

We now write an analogous condition to Lemma 7.2.11 for removing the elliptic cutoff:

Lemma 10.2.8. *Let ϵ_{eq} be as in Lemma 10.2.6. If $\epsilon \in (0, \frac{\epsilon_{eq}}{2})$ in Definition 10.2.4, then the following holds. Let $(u, v, \theta_w) \in \overline{\mathcal{K}}$ satisfies $\psi = \hat{\psi}$, for ψ solving* (7.2.23)*. Then, if either:*

(i) *The wedge angle θ_w satisfies $\theta_w \in [\theta_w^*, \theta_{sonic} - \delta_w^*/4]$, or*

(ii) *In the (x, y) coordinates,*

$$|\psi| \leq (2c_2 - \mu_0/5)x \qquad \text{in } \Omega \cap \mathcal{D}_{0,\epsilon/10}, \tag{10.2.32}$$

Then, ψ satisfies the original nonlinear wave system (7.2.37) *in Ω.*

Proof. Firstly, we note that there is no cutoff when $\theta_w \in [\theta_w^*, \theta_{sonic} - \delta_w^*/2]$ by (10.2.25), and that when $\theta_w \in [\theta_{sonic} - \delta_w^*/4, \frac{\pi}{2}]$, the equation (10.2.25), and the relevant assumption (10.2.10) match the original equation and (7.2.8), so the conclusion follows as in Lemma 7.2.11. Thus we only need deal with the case $\theta_w \in [\theta_{sonic} - \delta_w^*/2, \theta_{sonic} - \delta_w^*/4]$. Here, we find that as (10.2.14) implies (10.2.32) in $\Omega \cap \mathcal{D}_{\epsilon_{eq}}$, as $\mathcal{D}_{0,\epsilon_0/10} = \emptyset$ by (10.2.23); thus the cutoff is removed by (10.2.10). □

10.3 Solutions of the iteration problem - Existence and estimates

We now turn to existence of solutions to the iteration problem. We will first show existence of solutions to the iteration problem, using Proposition B.0.8 and B.0.9. We then show estimates for the solution, $\hat{\psi}$ and $\hat{r}$, essentially using the same methods as used for admissible

solutions, but now with a linear equation, where we must show the solution is more regular than it's ingredients to attain the gain-in-regularity. A subtle point here is that the δ_w^* used to distinguish the near-sonic cases from the away-from-sonic cases relied on the data via bounds on the coefficients; in particular, through (9.2.8) and (9.2.10). The iteration equation satisfies the same bounds, but with different constants, still depending on the data. Thus, in this case, the near-sonic cases will be separated out from the away-from-sonic cases using some constant $0 < \delta_{it}^* < \delta_w^*$.

First, we apply our basic existence theorems Proposition B.0.8 and Proposition B.0.9, to deduce existence, and very basic estimates, to the iteration problem.

Proposition 10.3.1. *Suppose the parameters $\sigma_1, \epsilon,$ and $\delta_{\mathcal{K}} > 0$ are small depending only on the data and θ_w^*. Then, for each $(u, v, \theta_w) \in \overline{\mathcal{K}^{ext}}$ which satisfies*

$$\|u^{\#} - u\|_{L^\infty(Q^{iter})} + \|v^{\#} - v\|_{C^1([0,1])} + |\theta_w^{\#} - \theta_w| \leq \delta_{\mathcal{K}}, \tag{10.3.1}$$

for some $(u^{\#}, v^{\#}, \theta_w^{\#}) \in \overline{\mathcal{K}}$, then there exists a unique $\hat{\psi} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus (\overline{\Gamma_{sonic}} \cup \{P_2\}) \cap C(\overline{\Omega})$ of Problem 7.2.7 (updated for the subsonic cases), determined by (u, v, θ_w) such that

$$\begin{aligned} &0 \leq \hat{\psi} \leq \overline{\psi} \qquad \text{in } \Omega \cap \mathcal{D}_{\epsilon_0}, \\ &\|\hat{\psi}\|_{0,\Omega} \leq C, \end{aligned} \tag{10.3.2}$$

for C depending only on the data, and for $\overline{\psi}$ from (7.2.27). Moreover, when the wedge angle θ_w satisfies $\theta_w \in [\theta_{sonic} - \frac{3}{4}\delta_w^*, \frac{\pi}{2}]$, then

$$0 \leq \psi \leq Cx \qquad \text{in } \Omega_{\epsilon_0}. \tag{10.3.3}$$

We first remark that as nothing has changed away from the sonic arc, Proposition 7.3.2 extends immediately:

Proposition 10.3.2. *Let $\hat{\psi}$ be the solution to Problem 7.2.7 from Proposition 10.3.1. Then, $\hat{r}$ is well defined by (7.2.28). Fix $s \in (0, \epsilon_0/2)$. Then there exists $\hat{\alpha} \in (0, 1)$ depending on s, such that $\hat{\alpha}/2 \leq \delta_*$, where if we choose the parameters $\alpha = \hat{\alpha}/2$, $\beta > \hat{\alpha}$ in the iteration set, then the following holds (where we note the α, β in the following sentence are not the ones from the iteration set:*

For each $\alpha \in (0, \hat{\alpha})$ and $\beta \in (0, \frac{1}{2})$, there exists $C_{s,\alpha} \geq 1$ depending only on the data, θ_w^* and s, and $C_{s,\alpha,\beta}$ also depending on β such that $\hat{\psi}$ and $\hat{r}$ satisfy

$$\begin{aligned} \|\hat{\psi}\|_{1,\alpha/2,\Omega\setminus\mathcal{D}_s}^{(-\alpha),\Gamma_{sym}\cup\{P_2,P_3\}} &\leq C_{s,\alpha}, \\ \|\hat{r}\|_{2,\alpha/2,[s,\pi]}^{-(1+\beta)} &\leq C_{s,\alpha,\beta}. \end{aligned} \tag{10.3.4}$$

We now turn to the estimates away from Γ_{sonic} for (ψ, r) in each of the different cases, with the view to gain a unified estimate for (u, v). We first consider the case of supersonic-away-from-sonic wedge angles,

Proposition 10.3.3. *Let $\delta > 0$. There exists $\hat{\epsilon}_{eq} \in (0, \epsilon_{eq})$, depending only on the data and δ, such that possibly reducing the $\hat{\alpha}$ for $s = \hat{\epsilon}_{eq}/2$ in Proposition 10.3.1, depending on the data, if we make the same assumptions as Proposition 10.3.2, and the wedge angle satisfies $\theta_w \in [\theta_{sonic} + \delta, \frac{\pi}{2}]$, then*

$$\begin{aligned} \|\hat{\psi}\|_{1,\alpha,\Omega\cap\mathcal{D}_{\hat{\epsilon}_{eq}}}^{(par)} &\leq C_{\alpha,\delta}, \\ \|\hat{r}\|_{2,\alpha,[\theta_1,\theta_1+\epsilon_{eq}]}^{(-1-1)} &\leq C_{\alpha,\delta}. \end{aligned} \tag{10.3.5}$$

Proof. This is a restatement of Proposition 7.3.3, noting that the equation did not change from the one in the supersonic case, for any supersonic wedge angles. □

We now turn to supersonic-near-sonic wedge angles.

Proposition 10.3.4. *There exists $\hat{\epsilon}_{eq} \in (0, \epsilon_{eq})$, $\delta_{it} > 0$, depending only on the data, such that possibly reducing the $\hat{\alpha}$ for $s = \hat{\epsilon}_{eq}/2$ in Proposition 10.3.1, depending on the data, if we make the same assumptions as Proposition 10.3.2, and the wedge angle satisfies $\theta_w \in [\theta_{sonic}, \theta_{sonic} + \delta_{it}]$, then*

$$\begin{aligned} \|\hat{\psi}\|^{(par)}_{1,\alpha,\Omega\cap\mathcal{D}_{\hat{\epsilon}_{eq}}} &\leq C_\alpha, \\ \|\hat{r}\|^{(-1-1)}_{2,\alpha,[\theta_1,\theta_1+\epsilon_{eq}]} &\leq C_\alpha. \end{aligned} \tag{10.3.6}$$

Proof. This follows in a similar way to Proposition 10.3.3, applying the techniques used for admissible solutions, to the iteration problem. The key part in being able to attain the algebraic growth used for Proposition 9.2.5 was that the length of the sonic arc was small enough, dependent on the L^∞ bounds of the coefficients. We note we can attain (9.2.8), (9.2.10) and (9.2.11) from (7.2.31), (10.2.22), (10.2.26) and (10.2.29), but with different constants (still depending only on the data). Because the constants are different, a different smallness of b_{so} is needed. As b_{so} only depends on the wedge angle, we may take δ_{it} depending only the data. The other key point is to note that the α in Proposition B.0.7 only depends on the ellipticity of the rescaled equation (thus on λ in (10.2.26)), the and L^∞ bounds of the data; thus only on the data. □

We now turn to subsonic-near-sonic case, which follows by essentially the same adaptations as the above.

Proposition 10.3.5. *There exists $\hat{\epsilon}_{eq} \in (0, \epsilon_{eq})$, $\delta_{it} > 0$, depending only on the data, such that possibly reducing possibly reducing the $\hat{\alpha}$ for $s = \hat{\epsilon}_{eq}/2$ in Proposition 10.3.1, depending on the data, if we make the same assumptions as Proposition 10.3.2, and the wedge angle satisfies $\theta_w \in [\theta_{sonic} - \delta_{it}, \theta_{sonic}]$, then*

$$\begin{aligned} \|\hat{\psi}\|_{1,\alpha,\Omega\cap\mathcal{D}_{\hat{\epsilon}_{eq}}} &\leq C_\alpha, \\ \|\hat{r}\|^{(-1-1),\{\theta_1\}}_{2,\alpha,[\theta_1,\theta_1+\epsilon_{eq}]} &\leq C_\alpha. \end{aligned} \tag{10.3.7}$$

Proof. This follows by identically the same argument as Proposition 10.3.4, except this time Proposition 9.2.9 is adapted instead, noting the similarities between Proposition 9.2.5 and Proposition 9.2.9. □

Finally, we consider the subsonic-away-from-sonic case.

Proposition 10.3.6. *Let $\delta > 0$. Then there exists $\hat{\epsilon}_{eq}$, depending only on the data and δ, such that possibly reducing possibly reducing the $\hat{\alpha}$ for $s = \hat{\epsilon}_{eq}/2$ in Proposition 10.3.1, if we make the same assumptions as Proposition 10.3.2, and the wedge angle satisfies $\theta_w \in [\theta_w^*, \theta_{sonic} - \delta]$, then*

$$\begin{aligned} \|\hat{\psi}\|^{(\alpha),\overline{\Gamma_{sonic}}}_{1,\alpha,\Omega\cap\mathcal{D}_{\hat{\epsilon}_{eq}}} &\leq C_{\alpha,\delta}, \\ \|\hat{r}\|^{(-1-\alpha/2),\{\theta_1\}}_{2,\alpha,[\theta_1,\theta_1+\epsilon_{eq}]} &\leq C_\alpha. \end{aligned} \tag{10.3.8}$$

Proof. This follows by adapting Proposition 9.2.13. More precisely, the estimate (10.2.22) implies we can obtain the same algebraic growth. Then, following the argument of Proposition 9.2.13, using the estimates (10.2.22), (10.2.19), (10.2.28), we can attain (10.3.8), with $\hat{\alpha}$ depending only on the data, via the ellipticity (10.2.30), depending only on the data and δ. □

We now, using the same argument as Proposition 10.1.10 and the condition $\beta > \hat{\alpha}$ from Proposition 10.3.2 (which ensures r has better regularity than $\hat{\psi}$), attain a unified estimate for $(\hat{u}, \hat{v})$. This leads to an increase-in-regularity, extending Corollary 7.3.5.

Corollary 10.3.7. *Suppose the assumptions of Proposition 7.3.4 hold. Let $\hat{u} : Q^{iter} \to \mathbb{R}$ and $v : (0,1) \to \mathbb{R}$ be determined as follows; define $\hat{u}$ by*

$$\hat{u} = \tilde{f} \circ F_1^{-1} \circ F_{2,\mathfrak{g}_{sh}}^{-1}(s,t), \tag{10.3.9}$$

where $\mathfrak{g}_{sh}$ was obtained from v. Also, we may take δ_3 small enough in the iteration set, depending on the data, so that

$$\hat{\Gamma}_{shock} := \{r = \hat{r}(\theta) \mid \theta \in [\theta_1, \pi]\} \subset \mathcal{Q}, \tag{10.3.10}$$

where $\mathcal{Q}$ is from Lemma 7.1.2, and Lemma 7.1.2 **(viii)** *holds for $\hat{\Gamma}_{shock}$. Then define v by the same formula as* (7.1.26)*. Then, $\hat{u}$ and $\hat{v}$ satisfy*

$$\|\hat{u}\|_{1,\hat{\alpha},Q^{iter}}^{**,2} \leq C, \qquad \|\hat{v}\|_{2,\hat{\alpha}/2,(0,1)}^{(-1-(\frac{1}{2}+\frac{\beta}{2}))} \leq C, \tag{10.3.11}$$

where C only depends on the data and θ_w^*.

10.4 Existence of admissible solutions up to the detachment angle

Now that we have attained the gain-in-regularity, we conclude the proof of existence of admissible solutions up to the detachment angle in the same way as for supersonic wedge angles in Chapter 10. In particular, we note that we carefully set up the elliptic cutoff so that the same argument as Lemma 7.4.3 can be used to remove it when it is present. All other parts of the argument in Chapter 10 hold without change.

Chapter 11

Recovering the Momentum Components (m, n)

When formulating our problem in §2.1, we remarked that we can obtain (m, n) once we have attained a solution for ρ. We now describe how to attain the components, and discuss the regularity of (m, n) that arise from the regularity of ρ.

We find that in non-conservation form, (2.1.3) is equivalent to

$$\begin{aligned} m_\xi + n_\eta &= \boldsymbol{\xi} \cdot \nabla \rho, \\ r m_r &= \boldsymbol{\xi} \cdot \nabla m = p_\xi, \\ r n_r &= \boldsymbol{\xi} \cdot \nabla n = p_\eta, \end{aligned} \tag{11.0.1}$$

so that we can recover (m, n) within Ω by obtaining the values on $\Gamma_{sonic} \cup \Gamma_{shock}$, and then taking rays from the origin to determine (m, n) elsewhere.

We first define $R : [\theta_w, \pi] \to \mathbb{R}$ by

$$R(\theta) := \begin{cases} c_2 & \text{for } \theta \in [\theta_w, \theta_1), \\ r(\theta) & \text{for } \theta \in [\theta_1, \pi], \end{cases} \tag{11.0.2}$$

so that in the (r, θ) coordinates,

$$\Omega = \{(r, \theta) \mid 0 < r < R(\theta), \theta_w < \theta < \pi\}. \tag{11.0.3}$$

Next, we use (2.2.9) to determine the values of (m, n) on Γ_{shock}, defining:

$$(M, N)(\theta) := \begin{cases} (m_2, n_2) & \text{for } \theta_w \in [\theta_w, \theta_1), \\ (m_1 + \left(\frac{\cos\theta}{r} - \frac{r'(\theta)}{r}\sin\theta\right)[p], -\left(\frac{\sin\theta}{r} + \frac{r'(\theta)}{r}\cos\theta\right)[p]) & \text{for } \theta_w \in [\theta_1, \pi], \end{cases} \tag{11.0.4}$$

where all terms in the first part are evaluated at $(r(\theta), \theta)$. We now, by the second two conservation laws in (11.0.1), determine (m, n) as follows

$$(m, n)(r, \theta) := (M, N)(\theta) + \int_r^{R(\theta)} \frac{1}{s} \nabla \rho(s, \theta)\, ds. \tag{11.0.5}$$

We note that the second two conservation laws in (11.0.1) are valid due to the construction in the formula. Then, the first conservation law can be derived from satisfying the second two, combined with ρ satisfying (2.1.4) and (2.2.10), and (m, n) satisfying (11.0.4).

At first sight, it appears that (m, n) may be multi-valued at the origin; it is not clear that the formula (11.0.5) with $r = 0$ is independent of θ. Indeed, in the related problem of shock reflection-diffraction down a step in [13], only $C^{0,\alpha}$ regularity is obtained at the wedge vertex, so it is possible the momenta have a singularity. However, we can prove that in fact $(m, n) \in C^{(-\alpha),\{P_3\}}_{1,\alpha}$, and $(m, n) = (0, 0)$ at $\boldsymbol{\xi} = 0$.

We first note, that as the the slip boundary condition and $C^{1,\alpha}$ regularity of ρ up to P_3 implies that $\nabla\rho(\mathbf{0}) = \mathbf{0}$, and

$$|\nabla\rho| \leq Cr^{\alpha}, \tag{11.0.6}$$

where ∇ is in the $\boldsymbol{\xi}$ coordinates, and C depends on the $C^{1,\alpha}$ norm of ρ restricted to a neighbourhood of P_3. Next, using the slip boundary condition, on Γ_{sym} for the first inequality, and Γ_{wedge} for the second, we note that:

$$\begin{aligned} n(0,\pi) &= \int_0^{r(\pi)} \frac{1}{s}\rho_\eta(s,\pi)\,ds = 0, \\ (m,n)(0,\theta_w)\cdot\boldsymbol{\nu}_w &= \int_0^{R(\theta_w)} \frac{1}{s}\nabla\rho\cdot\boldsymbol{\nu}_w(s,\theta_w)\,ds = 0. \end{aligned} \tag{11.0.7}$$

Next, we find that from the second two conservation laws, and (11.0.6), that

$$|(m_r, n_r)| \leq Cr^{\alpha-1} \tag{11.0.8}$$

Next, returning to the original space-time system of conservation laws (1.2.1), we find

$$\partial_t(m_{x_2} - n_{x_1}) = (p_{x_1})_{x_2} - (p_{x_2})_{x_1} = 0, \tag{11.0.9}$$

so $m_{x_2} - n_{x_1}$ is independent of time; thus as the initial data satisfies $m_{x_2} - n_{x_1} = 0$ (in the sense of distribution, as the jump in $[m]$ is only in the x_1 direction), $m_{x_2} - n_{x_1} = 0$ for $t \geq 0$. Thus in the self-similar coordinates,

$$m_\eta - n_\xi = 0, \tag{11.0.10}$$

which in polar coordinates is equivalent to

$$r\sin\theta m_r - r\cos\theta n_r + \cos\theta m_\theta + \sin\theta n_\theta = 0, \tag{11.0.11}$$

so that from (11.0.8),

$$|\cos\theta m_\theta + \sin\theta n_\theta| \leq Cr^{\alpha}. \tag{11.0.12}$$

Also, the first conservation law is equivalent to

$$r\boldsymbol{\xi}\cdot\nabla\rho - r\cos\theta m_r - r\sin\theta n_r + sin\theta m_\theta - \cos\theta n_\theta = 0, \tag{11.0.13}$$

thus from (11.0.6) and (11.0.8),

$$|\sin\theta m_\theta - \cos\theta n_\theta| \leq Cr^{\alpha}. \tag{11.0.14}$$

Combining (11.0.12) and (11.0.14), we obtain

$$|(m_\theta, n_\theta)| \leq Cr^{\alpha} \tag{11.0.15}$$

Therefore, from (11.0.7), combined with (11.0.8) and (11.0.15) we deduce that for all $r > 0$,

$$|(m, n)|(r, \theta)| \leq C(\pi - \theta_w)r^{\alpha} = Cr^{\alpha}, \tag{11.0.16}$$

with C depending on the $C^{1,\alpha}$ norm of ρ near P_3. The above working also shows that (m,n) satisfy the slip boundary condition (1.2.6) on the whole wedge boundary ∂W.

By combining Corollaries 9.2.1, 9.2.6, 9.2.10 and 9.2.14 with the above, deduce the following regularity for the full system in (ρ,m,n), for the original self-similar problem in terms of (ρ,m,n), Problem 2.2.1:

Theorem 11.0.1. *For all $\theta_w \in (\theta_d, \frac{\pi}{2}]$, there exists a global solution to Problem 1.2.1, with Self-Similar Regular Shock-Reflection-Diffraction structure as described in §2.3. Moreover, the regularity within the elliptic region Ω is as follows. There exists θ_{crit} depending on the data, such that:*

- *If $\theta_w > \theta_{sonic}$, then for α depending only on the data, and any $\beta \in (0,1)$,*

$$\begin{aligned} \rho &\in C^{0,1}(\overline{\Omega}\setminus\{P_2\}) \cap C^{0,\beta}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}\setminus(\overline{\Gamma_{sonic}}\cup\{P_2\}),\\ (m,n) &\in C^{0,1}(\overline{\Omega}\setminus\{P_2,P_3\}) \cap C^{0,\beta}(\overline{\Omega}\setminus\{P_3\}) \cap C^{0,\alpha}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}\setminus(\overline{\Gamma_{sonic}}\cup\{P_2,P_3\}). \end{aligned} \tag{11.0.17}$$

- *If $\theta_w^* \in (\theta_{crit},\theta_{sonic})$ is fixed, then there exists α depending on the data and θ_w^* such that if $\theta_w \in [\theta_w^*,\theta_{sonic}$, and $\beta \in (0,1)$,*

$$\begin{aligned} \rho &\in C^{0,\beta}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}\setminus\{P_2\}),\\ (m,n) &\in C^{0,\beta}(\overline{\Omega}\setminus\{P_3\}) \cap C^{0,\alpha}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}\setminus\{P_2,P_3\}), \end{aligned} \tag{11.0.18}$$

- *If $\theta_w^* \in (\theta_d,\theta_{sonic})$ is fixed, then there exists α depending on the data and θ_w^* such that if $\theta_w \in [\theta_w^*,\theta_{sonic})$, and $\beta \in (0,1)$,*

$$\begin{aligned} \rho &\in C^{0,\beta}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}\setminus\{P_1,P_2\}),\\ (m,n) &\in C^{0,\beta}(\overline{\Omega}\setminus\{P_3\}) \cap C^{0,\alpha}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}\setminus\{P_1,P_2,P_3\}), \end{aligned} \tag{11.0.19}$$

Appendices

Appendix A

Relevant Results for Quasilinear Elliptic Equations

A.1 Norms & scaled Hölder spaces

In this section, we introduce notation for norms that will be used throughout the thesis. We use the same notation & proofs as [26][Chapter 4].

A.1.1 Hölder norms

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. We introduce the Hölder norms in Ω. For $\alpha \in (0,1)$ and $m \in \mathbb{N}$, define

$$\begin{aligned} \|u\|_{m,0,\Omega} &:= \sum_{0\le|\beta|\le m} \sup_{\mathbf{x}\in\Omega} |D^\beta u(\mathbf{x})|, \\ [u]_{m,\alpha,\Omega} &:= \sum_{|\beta|=m} \sup_{\mathbf{x},\mathbf{y}\in\Omega,\mathbf{x}\neq\mathbf{y}} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}, \\ \|u\|_{m,\alpha,\Omega} &:= \|u\|_{m,0,\alpha} + [u]_{m,\alpha,\Omega}, \end{aligned} \tag{A.1.1}$$

We denote by $C_{m,\alpha,\Omega}$ or $C^{m,\alpha}(\overline{\Omega})$ the space of functions with finite norm $\|u\|_{m,\alpha,\Omega}$. Let $\Sigma \subset \partial\Omega$. We now define partially interior norms in $\Omega \cup \Sigma$. For $\mathbf{x}, \mathbf{y} \in \Omega$, define

$$d_{\mathbf{x}} = \operatorname{dist}(\mathbf{x}, \partial\Omega \setminus \Sigma), \qquad d_{\mathbf{x},\mathbf{y}} = \min\{d_{\mathbf{x}}, d_{\mathbf{y}}\}. \tag{A.1.2}$$

For $\alpha \in (0,1)$ and $m \in \{0,1,\ldots\}$, the partially interior norms in $\Omega \cup \Sigma$ are

$$\begin{aligned} \|u\|^*_{m,0,\Omega\cup\Sigma} &:= \sum_{0\le|\beta|\le m} \sup_{\mathbf{x}\in\Omega} \Big(d_{\mathbf{x}}^{|\beta|} |D^\beta u(\mathbf{x})| \Big), \\ [u]^*_{m,\alpha,\Omega\cup\Sigma} &:= \sum_{|\beta|=m} \sup_{\mathbf{x},\mathbf{y}\in\Omega,\mathbf{x}\neq\mathbf{y}} \left(d_{\mathbf{x},\mathbf{y}}^{m+\alpha} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} \right), \\ \|u\|^*_{m,\alpha,\Omega\cup\Sigma} &:= \|u\|^*_{m,0,\Omega\cup\Sigma} + [u]^*_{m,\alpha,\Omega\cup\Sigma}. \end{aligned} \tag{A.1.3}$$

We then denote by $C_{m,\alpha,\Omega\cup\Sigma}$ or $C^{m,\alpha}(\Omega\cup\Sigma)$ the space of functions with finite norm $\|\cdot\|^*_{m,\alpha,\Omega\cup\Sigma}$. We finally define notation for the Hölder norms in Ω weighted by distance to Σ. Set

$$\delta_{\mathbf{x}} := \operatorname{dist}(\mathbf{x}, \Sigma), \qquad \delta_{\mathbf{x},\mathbf{y}} := \min\{\delta_{\mathbf{x}}, \delta_{\mathbf{y}}\} \qquad \text{for } \mathbf{x}, \mathbf{y} \in \Omega. \tag{A.1.4}$$

Then, for $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $m \in \mathbb{N}$, define

$$\begin{aligned}
\|u\|^{(k),\Sigma}_{m,0,\Omega} &:= \sum_{0\le|\beta|\le m} \sup_{\mathbf{x}\in\Omega}\Big(\delta_{\mathbf{x}}^{\max\{|\beta|+k,0\}}|D^\beta u(\mathbf{x})|\Big),\\
[u]^{(k),\Sigma}_{m,\alpha,\Omega} &:= \sum_{|\beta|=m} \sup_{\mathbf{x},\mathbf{y}\in\Omega,\mathbf{x}\neq\mathbf{y}}\left(\delta_{\mathbf{x},\mathbf{y}}^{\max\{m+\alpha+k,0\}}\frac{|D^\beta u(\mathbf{x})-D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}\right),\\
\|u\|^{(k),\Sigma}_{m,\alpha,\Omega} &:= \|u\|^{(k),\Sigma}_{m,0,\Omega} + [u]^{(k),\Sigma}_{m,\alpha,\Omega}.
\end{aligned} \tag{A.1.5}$$

We denote by $C^{(k),\Sigma}_{m,\alpha,\Omega}$ the space of functions with finite norm $\|\cdot\|^{(k),\Sigma}_{m,\alpha,\Omega}$. We also define the following interior norm for the case $\sigma = \partial\Omega$,

$$\|u\|^{(k)}_{m,\alpha,\Omega} := \|u\|^{(k),\partial\Omega}_{m,\alpha,\Omega}, \qquad k \in \mathbb{R}. \tag{A.1.6}$$

A.1.2 Hölder spaces with parabolic scaling

We now follow [26][§4.6], defining norms convenient for analysing degenerate elliptic equations, in an open domain $\mathcal{D} \subset \{(x_1,x_2 \mid x_1 > 0\} \subset \mathbb{R}^2$, with ellipticity degenerating on $\{x_1 = 0\}$. For $\alpha \in (0,1)$, denote

$$\delta^{(par)}_\alpha(\mathbf{x},\tilde{\mathbf{x}}) := \big(|x_1-\tilde{x}_1|^2 + \max\{x_1,\tilde{x}_1\}|x_2-\tilde{x}_2|^2\big)^{\alpha/2}. \tag{A.1.7}$$

For any non-negative m, real $\sigma > 0$, $\alpha \in (0,1)$, we define the parabolic Hölder norms, weighted and scaled by the distance to $\{x_1 = 0\}$ by:

$$\begin{aligned}
\|u\|^{(\sigma),(par)}_{m,0,\mathcal{D}} &:= \sum_{0\le k+l\le m}\sup_{z\in\mathcal{D}}\left(x_1^{k+\frac{l}{2}-\sigma}|\partial^k_{x_1}\partial^l_{x_2}u(\mathbf{x})|\right),\\
[u]^{(\sigma),(par)}_{m,\alpha,\mathcal{D}} &:= \sum_{k+l=m}\sup_{\mathbf{x},\tilde{\mathbf{x}}\in\mathcal{D},\mathbf{x}\neq\tilde{\mathbf{x}}}\left(\min\{x_1^{\alpha+k+\frac{l}{2}-\sigma},\tilde{x}_1^{\alpha+k+\frac{l}{2}-\sigma}\}\,\frac{|\partial^k_{x_1}\partial^l_{x_2}u(\mathbf{x})-\partial^k_{x_1}\partial^l_{x_2}u(\tilde{\mathbf{x}})|}{\delta^{(par)}_\alpha(\mathbf{x},\tilde{\mathbf{x}}}\right),\\
\|u\|^{(\sigma),(par)}_{m,\alpha,\mathcal{D}} &:= \|u\|^{(\sigma),(par)}_{m,0,\mathcal{D}} + [u]^{(\sigma),(par)}_{m,\alpha,\mathcal{D}}.
\end{aligned} \tag{A.1.8}$$

We denote by $C^{m,\alpha}_{\sigma,(par)}(\mathcal{D})$ the completion of the set $\{u \in C^\infty(\mathcal{D}) \mid \|u\|^{(\sigma),(par)}_{m,\alpha,\mathcal{D}} < \infty\}$ in the norm $\|\cdot\|^{(\sigma),(par)}_{m,\alpha,\mathcal{D}}$. We now state a characterisation of $C^{m,\alpha}_{\sigma,(par)}$ in terms of $C^{m,\alpha}$ norms of rescaled functions in parabolic rectangles.

Let $\epsilon > 0$ be fixed, and define $\mathcal{D}_\epsilon = \mathcal{D} \cap \{0 < x < \epsilon\}$. For $\mathbf{x} = (x_1,x_2) \in \mathcal{D}_\epsilon$ and $r \in (0,1]$, define

$$R_{\mathbf{x},\rho} := \Big\{(s,t) \mid |s-x_1| < \frac{\rho}{4}x_1, \quad |t-x_2| < \frac{\rho}{4}\sqrt{x_1}\Big\} \cap \mathcal{D}_\epsilon, \tag{A.1.9}$$

so that for $\rho \in (0,1)$,

$$R_{\mathbf{x},\rho} \subset \mathcal{D}_\epsilon \cap \{(s,t) \mid \frac{3}{4}x_1 < s < \frac{5}{4}x_1\}. \tag{A.1.10}$$

We then, for $\rho \in (0,1]$, denote $Q_\rho := (-\rho,\rho)^2$. Then, we find that

$$Q^{(\mathbf{x})}_\rho := \left((S,T) \in Q_\rho \mid (x + \frac{x_1}{4}S, x_2 + \frac{\sqrt{x_1}}{4}T) \in \mathcal{D}_\epsilon\right), \tag{A.1.11}$$

and denote by $u^{(\mathbf{x})}(S,T)$, the following function:

$$u^{(\mathbf{x})}(S,T) := \frac{1}{x_1^\sigma} u(x_1 + \frac{x_1}{4}S, x_2 + \frac{\sqrt{x_1}}{4}T) \qquad \text{for } (S,T) \in Q_\rho^{(\mathbf{x})}, \tag{A.1.12}$$

and then the following criteria, from [26][Lemma 4.6.1] holds:

Lemma A.1.1. *For any $\epsilon \in (0,1)$, and $\rho \in (0,1]$,*

$$C^{-1} \sup_{\mathbf{x}\in\mathcal{D}_{3\epsilon/4}} \|u^{(\mathbf{x})}\|_{m,\alpha,Q_\rho^{(\mathbf{x})}} \le \|u\|_{m,\alpha,\mathcal{D}_\epsilon}^{(\sigma),(par)} \le C \sup_{\mathbf{x}\in\mathcal{D}_\epsilon} \|u^{(\mathbf{x})}\|_{m,\alpha,Q_\rho^{(\mathbf{x})}}, \tag{A.1.13}$$

where C only depends on (m,α,σ,ρ); in particular, it does not depend on $\mathcal{D}$ or ϵ.

We now quote three more useful properties of the parabolic norms, which will be used in our analysis; see [26][Lemma 4.6.2, Lemma 4.6.3, Lemma 4.6.4] for details.

This first Lemma allows us to combine locally-obtained $C_{m,\alpha}^{(par),(\sigma)}$ estimates together. We note that this result highlights that a function being in $C_{m,\alpha}^{(par),(\sigma)}$, but not in $C_{m,\alpha}$ is a local property near $\{s=0\}$, because from (A.1.8), we see that if $\mathcal{D} \subset \{s > \epsilon\}$, then the $C_{m,\alpha}^{(par)}$ norm is equivalent to the $C_{m,\alpha}$ norm, with constant depending on ϵ.

Lemma A.1.2. *Let $\mathcal{D}$ be an open bounded subset in $\mathbb{R}^2$. Let $m \ge 0$ be an integer, $\alpha \in (0,1)$, and $\sigma > 0$. Then, there exists C, such that for any $\epsilon > 0$,*

$$\|u\|_{m,\alpha,\mathcal{D}}^{(\sigma),(par)} \le C\Big(\|u\|_{m,\alpha,\mathcal{D}\cap\{x_1>\epsilon\}}^{(\sigma),(par)} + \|u\|_{m,\alpha,\mathcal{D}\cap\{x_1<2\epsilon\}}^{(\sigma),(par)}\Big) \tag{A.1.14}$$

for any $u \in C_{\sigma,(par)}^{m,\alpha}(\mathcal{D})$.

We next state a compactness property, that will be used in the proof of gain-in-regularity for our iteration map.

Lemma A.1.3. *Let $\mathcal{D}$ be an open bounded subset of $\mathbb{R}^2$, with Lipschitz boundary. Let $m_1, m_2 \ge 0$ be integers, $\alpha_1, \alpha_2 \in [0,1)$, and $m_1 + \alpha_1 > m_2 + \alpha_2$. Let $\sigma_1 > \sigma_2 > 0$. Then $C_{\sigma_1,(par)}^{m_1,\alpha_1}(\mathcal{D})$ is compactly embedded into $C_{\sigma_2,(par)}^{m_2,\alpha_2}(\mathcal{D})$.*

We also now note that though $\sigma_1 > \sigma_2$ is necessary to obtain compactness, it is still possible to obtain an continuous embedding (with explicit constant) when $\sigma_1 = \sigma_2$.

Lemma A.1.4. *Let $0 \le \alpha_1 < \alpha_2 < 1$, $\sigma > 0$ and let m be a non-negative integer. Then,*

$$\|u\|_{m,\alpha_1,\mathcal{D}}^{(\sigma),(par)} \le 9\|u\|_{m,\alpha_2,\mathcal{D}}^{(\sigma),(par)}. \tag{A.1.15}$$

Appendix B

Regularity Results for Elliptic Equations

In this Chapter, we shall collect together some relevant existence & regularity results, that will be applicable to both the Nonlinear Wave System itself, and approximate solutions used in the iteration set. Most of our results arise from using the interpolation lemma from [77] to adapt the results in [26][Chapter 4], where typically, $C^{0,1}$ regularity is assumed and $C^{2,\alpha}$ regularity attained, to here, where, L^∞ regularity is assumed and $C^{1,\alpha}$ regularity is attained. Throughout this section, we will consider elliptic equations on a bounded, open region $\Omega \subset \mathbb{R}^2$, either of the Divergence form

$$\partial_i(A_{ij}(u(\mathbf{x}),\mathbf{x})\partial_j u) + B_i(u(\mathbf{x}),\mathbf{x})\partial_i u + cu = 0, \tag{B.0.1}$$

or the Quasilinear form

$$A_{ij}(u(\mathbf{x}),\mathbf{x})\partial_{ij} u + B_{ij}(u(\mathbf{x}),\mathbf{x})\partial_i u \partial_j u + A_i(u(\mathbf{x}),\mathbf{x})\partial_i u = 0. \tag{B.0.2}$$

We also consider oblique boundary conditions of the form

$$\beta \cdot \nabla u = \beta_i \partial_i u = 0, \text{ where } \beta \cdot \boldsymbol{\nu} > 0. \tag{B.0.3}$$

We note that provided A_{ij} is differentiable, (B.0.1) is a special case of (B.0.2). We will also assume the following strict ellipticity:

$$\lambda|\zeta|^2 \leq A_{ij}(u(\mathbf{x}),\mathbf{x})\zeta_i\zeta_j \leq \lambda^{-1}|\zeta|^2. \tag{B.0.4}$$

We will also, in results that involve boundary regularity, we assume assume that for some $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi(0) = 0$, Ω is of the form

$$\Omega := \{x_2 > \Phi(x_1)\}, \qquad \Gamma = \{x_2 = \Phi(x_1)\}, \tag{B.0.5}$$

and we define, for $r > 0$,

$$\Omega_r = \Omega \cap B_r(\mathbf{0}), \qquad \Gamma_r = \Gamma \cap B_r(\mathbf{0}). \tag{B.0.6}$$

B.0.1 An interpolation lemma

We quote a relevant interpolation result to deal with nonlinear terms; in particular, the quadratic terms in (B.0.2). From [77][Lemma 1],

Proposition B.0.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $w \in C^1(\Omega; \mathbb{R}^n)$, and suppose that for any ball $B = B_R(y) \subset\subset \Omega$ we have*

$$[Dw]^*_{\alpha,B} \leq \phi(RM_1)(1 + M_1), \tag{B.0.7}$$

where $0 < \alpha < 1$, $M_1 = |Dw|_{0,B}$, and $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function. Then we have the estimate

$$[w]^*_{1,\Omega} \leq C(\alpha, \phi(|u|_0))(|u|_0 + \operatorname{diam}(\Omega)). \tag{B.0.8}$$

We now note that as consequence, in the case where $\Omega = B_1(\mathbf{0})$, (B.0.8) implies that

$$|Dw|_{0,B_{3/4}(\mathbf{0})} \leq C, \tag{B.0.9}$$

and then using (B.0.7), we may obtain

$$|Dw|_{0,\alpha,B_{1/2}(\mathbf{0})} \leq C. \tag{B.0.10}$$

B.0.2 Interior estimates

We will start by deriving interior estimates.

Proposition B.0.2. *Suppose that $u \in C^2(\Omega)$ satisfies* (B.0.2) *in a region Ω, and $B_2 \subset\subset \Omega$, and that the coefficients satisfy* (B.0.4)*, and*

$$\begin{aligned} \|(A_{ij}, B_{ij}, A_i)\|_{C^{1,0}(\mathbb{R}\times\Omega)} &\leq M, \\ \|u\|_{0,\Omega} &\leq C. \end{aligned} \tag{B.0.11}$$

Then for any $\alpha > 0$, there exists C depending only on λ, M, α such that

$$\|u\|_{2,\alpha,B_1} \leq C. \tag{B.0.12}$$

Proof. By [37][Theorem 12.4], and rescaling from $B_R(y)$ to $B_1(0)$ then scaling back, there exists $\beta \in (0, 1)$ depending only λ such that for any $B = B_R(y) \subset\subset B_2$,

$$\begin{aligned} [u]^*_{1,\beta,B} &\leq C(\lambda)(\|u\|_{0,B} + \|R^2 B_{ij}\partial_i u \partial_j u + R A_i \partial_i u\|_{0,B} \\ &\leq C(\lambda)(\|u\|_{0,B} + R^2\|Du\|_{0,B} + R\|Du\|_{0,B}), \end{aligned} \tag{B.0.13}$$

so by the remark after Proposition B.0.1, we find that

$$\|u\|_{1,\beta,B_{3/2}} \leq C(\lambda, M). \tag{B.0.14}$$

In particular, combining (B.0.11) and (B.0.14), we find that

$$\begin{aligned} \|A_{ij}\|_{0,1,B_{3/2}} &\leq C(\lambda, M), \\ \|B_{ij}\partial_i u \partial_j u\|_{0,\beta,B_{3/2}} &\leq C(\lambda, M), \\ \|C_i \partial_i u\|_{0,\beta,B_{3/2}} &\leq C(\lambda, M), \end{aligned} \tag{B.0.15}$$

so that by [37][Theorem 6.2],

$$\|u\|_{2,\beta,B_{5/4}} \leq C(\lambda, M). \tag{B.0.16}$$

We can then repeat the step (B.0.15), but improving all three estimates to $C^{0,1}$ estimates. This allows us to deduce (B.0.12). □

Remark B.0.3. All that was required in the step (B.0.13), and thus in the $C^{1,\alpha}$ estimate (B.0.14) was the fact the coefficients are bounded when evaluated at u. The fact we can gain $C^{1,\alpha}$ regularity, depending only on the ellipticity of the second-order terms, and the boundedness of the coefficients, will be useful during the gain-in-regularity argument for the iteration map.

B.0.3 Strictly oblique boundary problems

We now assume u satisfies (B.0.3), and the region Ω and the boundary function Γ are of the form in (B.0.5) and (B.0.6),

$$\beta(u(\mathbf{x}), \mathbf{x}) \cdot \boldsymbol{\nu}(u(\mathbf{x}), \mathbf{x}) \geq \lambda \qquad \text{on } \Gamma, \tag{B.0.17}$$

where $\boldsymbol{\nu}$ is the inner unit normal to Γ. We will first derive a $C^{0,\alpha}$ regularity result for oblique boundary conditions of conormal form, for divergence form equations, which only depends on the ellipticity and L^∞ bounds on the coefficients.

Proposition B.0.4. *Let $u \in W^{1,2}(\Omega)$ satisfies* (B.0.1) *on a region Ω satisfying a uniform exterior ball condition, where the coefficients satisfy* (B.0.4)*, and we also have the following bounds:*

$$\begin{aligned} \|(A_{ij}, B_i, c\|_{0,\mathbb{R}\times\Omega} &\leq M, \\ c &\leq 0 \end{aligned} \tag{B.0.18}$$

Suppose that u satisfies the conormal boundary condition

$$A_{ij}\partial_i u \nu_j = 0 \qquad \text{on } \Gamma \subset \partial\Omega \tag{B.0.19}$$

in the weak sense. Then there exists $\alpha \in (0,1)$ depending on λ, and C depending on M, such that if $B_R \subset \Omega \cup \Gamma$,

$$[u]^*_{0,\alpha,B_{R/2}\cap\overline{\Omega}} \leq C|u|_0 \tag{B.0.20}$$

Proof. This follows in the same way as the interior estimates in [37][Theorem 8.24], noting that due to the definition of weak solution for the conormal problem, and using the exterior ball condition, Γ can be treated in the same way that the interior of Ω is treated in that proof. □

Remark B.0.5. If u satisfies a general oblique boundary condition $\beta_i\partial_i u = 0$, and we can extend the unit normal $\boldsymbol{\nu} = (\nu_1, \nu_2)$ on Γ to a function on Ω, we may adjust A_{ij} by an asymmetric part $\alpha_{ij} - \alpha_{ji}$ as described in [84]. Define

$$\begin{aligned} \theta &= \frac{A_{ij}\nu_i\nu_j}{\beta_i\nu_i}, \\ \tau_i &= \theta\beta_i - \nu_j A_{ij}, \\ \alpha_{ij} &= \nu_j\tau_i, \end{aligned} \tag{B.0.21}$$

and then set

$$\tilde{A}_{ij} = A_{ij} + (\alpha_{ij} - \alpha_{ij}). \tag{B.0.22}$$

Then, $\tilde{A}_{ij}$ has the same ellipticity structure as A_{ij}, as $A_{ij}\zeta_i\zeta_j = \tilde{A}_{ij}\zeta_i\zeta_j$ for any $\zeta \in \mathbb{R}^2$. If we assume that $\beta, \boldsymbol{\nu}$ only depend on $(u(\mathbf{x}), \mathbf{x})$, the adjusted divergence form equation is

$$\partial_i(\tilde{A}_{ij}\partial_j u) + (C_i + \partial_{x_j}(\alpha_{ij} - \alpha_{ji}))\partial_i u = 0, \tag{B.0.23}$$

noting that the adjustment does not add quadratic non-divergence terms, as these would be

$$(\partial_z \alpha_{ij} - \partial_z \alpha_{ji}) \partial_i u \partial_j u = 0. \tag{B.0.24}$$

Then, provided the boundary condition is strictly oblique, the derivatives of the coefficients are bounded with respect to $\mathbf{x}$, and the boundary is $C^{1,0}$, we have converted the general oblique boundary problem to a conormal one, satisfying the hypotheses of Proposition B.0.4.

We now turn to $C^{1,\alpha}$ estimates, adapting [26][Theorem 4.2.10], using Proposition B.0.1.

Proposition B.0.6. *Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies* (B.0.2) *and* (B.0.3). *Suppose that the equation* (B.0.2) *satisfies* (B.0.4). *Then there exists $\alpha \in (0,1)$, depending only on λ, such that if $\beta \in (0, \alpha]$, and*

$$\begin{aligned} \|(A_{ij}, B_{ij}, C_i)\|_{0,\Omega} &\leq M, \\ \|\Phi\|_{1,\beta,\mathbb{R}} &\leq M, \\ \|\beta_i\|_{0,\beta,\Omega} &\leq M, \\ \|u\|_{0,\Omega} &\leq M, \\ \beta \cdot \boldsymbol{\nu} &\geq \frac{1}{M}, \end{aligned} \tag{B.0.25}$$

then, there exists C depending on λ, β and M such that

$$\|u\|_{1,\beta,B_1} \leq C \tag{B.0.26}$$

Proof. Following the proof of [26][Theorem 4.2.10], our result follows via the following changes, and remarks about where certain assumptions were used.

Step 1 Rather than using a full flattening of the boundary, we may instead apply local flattenings of the boundary, given by

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= x_2 - \Phi'(0) x_1 - \Phi(0). \end{aligned} \tag{B.0.27}$$

This will reduce our problem to the case where

$$\beta_2(u(\mathbf{0}), \mathbf{0}) \geq \lambda, \tag{B.0.28}$$

and $\Phi(0) = \Phi'(0) = 0$. Due to (B.0.25), this coordinate transform is bounded in C^2 with bounds depending only on M, and is invertible.

Step 2 We also note that similarly to the original proof, we can make a linear change of variables to reduce to the case where $\beta(u(\mathbf{0}), \mathbf{0}) = (0, 1)$, defined by

$$\mathbf{X} = \frac{1}{\beta_2(\mathbf{0})} (\beta_2(\mathbf{0}) x_1 - \beta_1(\mathbf{0}) x_2, x_2), \tag{B.0.29}$$

and again, this coordinate change will be bounded & invertible, with bounds depending on λ and M.

Step 3 In the same way as the original argument (using a covering argument), combining with Proposition B.0.2, it suffices to prove that there exists $\epsilon \in (0,1)$ and C depending only on λ, M such that any $x \in \Gamma \cap B_{3/2}$,

$$\|u\|_{1,\beta,B_{2\epsilon}(x)\cap\Omega} \leq C. \tag{B.0.30}$$

We now scale from $B_{2\epsilon}(x)$ to Ω_2 by defining

$$v(\mathbf{x}) = \frac{1}{\epsilon}(u(\epsilon\mathbf{x}) - u(\mathbf{0})) \tag{B.0.31}$$

and then v satisfies

$$\begin{aligned} &\tilde{A}_{ij}\partial_j v + \tilde{B}_{ij}\partial_i v\partial_j v + \tilde{C}_i\partial_i v = 0, \quad &&\text{in } \Omega_2,\\ &\partial_2 v = \tilde{\beta}\cdot\nabla v &&\text{on } \Sigma_2, \end{aligned} \tag{B.0.32}$$

where

$$\begin{aligned} \tilde{A}_{ij}(z,\mathbf{x}) &= A_{ij}(\epsilon z + u(\mathbf{0}), \epsilon\mathbf{x}),\\ \tilde{B}_{ij}(z,\mathbf{x}) &= \epsilon B_{ij}(\epsilon z + u(\mathbf{0}), \epsilon\mathbf{x}),\\ \tilde{C}_i(z,\mathbf{x}) &= \epsilon C_i(\epsilon z + u(\mathbf{0}), \epsilon\mathbf{x}),\\ \tilde{\beta}(z,\mathbf{x}) &= (-\beta_1(\epsilon z + u(\mathbf{0}), \epsilon\mathbf{x}, -\beta_2(z,\epsilon\mathbf{x}) + 1). \end{aligned} \tag{B.0.33}$$

The boundary Σ_2 is now given by

$$\tilde{\Phi}(\mathbf{x}) = \frac{1}{\epsilon}\Phi(\epsilon x), \tag{B.0.34}$$

so that $\mathrm{Lip}(\tilde{\Phi}) = \mathrm{Lip}(\Phi)$. Similarly to [26], $w_2 = \partial_2 v$ is a weak solution of

$$\begin{aligned} \partial_{11}w_2 + \partial_2\left(\frac{2\tilde{A}_{12}}{\tilde{A}_{11}}\partial_1 w_2 + \frac{\tilde{A}_{22}}{\tilde{A}_{11}}\partial_2 w_2\right) &= -\partial_2\left(\frac{\tilde{B}_{ij}\partial_i v\partial_j v + \tilde{C}_i\partial_i v}{\tilde{A}_1 1}\right), \qquad \text{in } \Omega_2,\\ g &= \tilde{\beta}_2\partial_1 v \qquad \text{on } \Sigma_2, \end{aligned} \tag{B.0.35}$$

we find that, for $\mathcal{D} \subset \Omega$, using the $C^{0,\beta}$ regularity of β,

$$[g]_{0,\beta,\mathcal{D}} \leq C(\lambda, M)(\epsilon^\beta\|Dv\|_{0,\beta,\mathcal{D}} + \|Dv\|_{0,0,\mathcal{D}} + \|Dv\|^2_{0,0,\mathcal{D}}) \tag{B.0.36}$$

Using this estimate for the $C^{0,\beta}$ norm of g, and rescaling from balls $B_R \equiv B_R(z)$ to $B_1(0)$. We can then find α depending on λ (via [37][Theorem 8.29]), such that for any $\beta \in (0,\alpha]$, and $B_R \subset\subset \Omega \cup \Gamma$,

$$[u]^*_{1,\beta,B_R} \leq C(\lambda,\beta)\Big(R\|u\|_{1,0,B_R} + R^2\|u\|^2_{1,0,B_R} + \epsilon^{\alpha-\beta}[u]^*_{1,\beta,B_R}\Big). \tag{B.0.37}$$

Thus taking ϵ sufficiently small, and using Proposition B.0.1, we obtain the desired result.

□

B.0.4 Near-tangential problems

We now adapt [26][Theorem 4.2.4] to quasilinear equations.

Proposition B.0.7. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy* (B.0.2) *and* (B.0.4). *Then there exists $\alpha \in (0,1)$, depending only on λ, such that if $\beta \in (0, \alpha]$, then the following holds. Firstly, assume that*

$$\|(A_{ij}, B_{ij}, A_i)\|_{0,\Omega} \leq M. \tag{B.0.38}$$

Next, let $\Phi \in C^1(\mathbb{R})$ satisfy

$$\|\Phi\|_{1,0,\mathbb{R}} \leq M, \qquad \Phi(0) = 0, \tag{B.0.39}$$

and suppose Ω is of the form (B.0.5). *Let $\beta(z,x)$ satisfy*

$$\begin{aligned} |\beta| &\leq \epsilon, \\ \|\beta(\cdot, x)\|_{1,0,\mathbb{R}\times\Omega} &\leq M, \\ \|\beta(z, \cdot)\|_{0,\beta,\mathbb{R}\times\Omega} &\leq M, \end{aligned} \tag{B.0.40}$$

and

$$u_{x_1} = \beta(u(\mathbf{x}), \mathbf{x}) u_{x_2} \qquad \text{on } \Gamma. \tag{B.0.41}$$

Then there exists ϵ_0, depending on M, λ and β such that if $\epsilon < \epsilon_0$ in (B.0.40), *then there exists C depending on*

$$\|u\|_{1,\alpha,B_1} \leq C. \tag{B.0.42}$$

Proof. Again, this follows by the same argument as [26][Theorem 4.2.4], with some small changes, and the use of the interpolation inequality in Proposition B.0.1. Following the proof, we find instead that for $g = \beta(u(\mathbf{x}), \mathbf{x})$, for any $\mathcal{D} \subset \Omega$,

$$[g]_{0,\beta,\mathcal{D}} \leq C(\lambda)\big(\epsilon[Du]_{0,\beta,\mathcal{D}} + \|Du\|_{0,\mathcal{D}} + \|Du\|^2_{0,\mathcal{D}}\big), \tag{B.0.43}$$

and we obtain that for any $B_R \subset\subset B_2$, by scaling to B_1 and then back to B_R,

$$[u]^*_{1,\alpha,B_R} \leq C(\lambda)\big(\epsilon[u]^*_{1,\alpha,B_R} + R\|Du\|_{0,B_R} + R^2\|Du\|^2_{0,B_R}\big), \tag{B.0.44}$$

and the result follows by using Proposition B.0.1. □

B.0.5 Existence for the iteration problem

Here, we deal with existence to the Iteration Problem. These proofs are similar to [26][Proposition 4.7.2], noting that our iteration problem is a linear problem, owing to the lack of $\nabla\psi$ dependence in the coefficients. The main difference here in the treatment of the one-point Dirichlet condition, is that in the one-point Dirichlet case, we do not have the property $\beta_1 \leq -\lambda$, so we need to use the algebraic growth argument used in the subsonic-away-from-sonic case. Also, we need to deal with the degenerate obliqueness at P_2, which can be treated with the algebraic growth argument.

We will write these results to be specific to our case, though they may adapt to more general conditions with similar properties.

We first look at the supersonic and supersonic-near-sonic cases. We note that this existence result always applies in the section of Part II, but we refer to the extended properties stated in Part III.

Proposition B.0.8. *Suppose that the wedge angle θ_w satisfies $\theta_w \in [\theta_{sonic} - \frac{3}{4}\delta_w^*, \frac{\pi}{2}]$. Then there exists a unique solution $\hat{\psi} \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \Gamma_{sonic}) \cap C^2(\Omega)$ to (7.2.23)–(7.2.27), which satisfies*

$$\begin{aligned} 0 \leq \hat{\psi} \leq \overline{\psi} \text{ in } \Omega, \\ 0 \leq \hat{\psi} \leq Cx \text{ in } \Omega, \\ \|\hat{\psi}\|_{0,\Omega} \leq C, \end{aligned} \tag{B.0.45}$$

for C depending only on the data.

Proof. We will consider approximate problems defined as follows. Let $\gamma_1, \gamma_2 > 0$, and define

$$\begin{aligned} \Omega_{\gamma_1} &:= \Omega \cap \{x > x_{P_1} + \gamma_1\}, \\ \Gamma_{sonic}^{(\gamma_1)} &:= \Omega \cap \{x = x_{P_1} + \gamma_1\}. \end{aligned} \tag{B.0.46}$$

We consider the following approximations of Problem 7.2.7:

$$\begin{aligned} \mathcal{N}(\psi) &= 0 \text{ in } \Omega_{\gamma_1}, \\ M\psi &= 0 \text{ on } \Gamma_{shock} \cap \{\theta_1 < \theta < \pi - \gamma_2\}, \\ \hat{\psi} &= 0 \text{ on } \Gamma_{sonic}^{\gamma_1}, \\ \nabla\hat{\psi} &= 0 \text{ on } \Gamma_{wedge}, \\ \hat{\psi} &= \overline{\psi} \text{ on } \Gamma_{shock} \cap \{\pi - \gamma_2 < \theta < \pi\}. \end{aligned} \tag{B.0.47}$$

By [55][Lemma 3], using the regularity properties of the coefficients, we find that there exists a solution $\hat{\psi}_{\gamma_1,\gamma_2} \in C^2(\Omega) \cap C^1(\Omega \cap \Gamma_{shock} \cap \Gamma_{wedge}) \cap C(\overline{\Omega})$ which for some $\alpha(\gamma_1, \gamma_2) \in (0,1)$ depending on γ_1 (via the ellipticity of (7.2.23), which depends on γ_1, the length of $\Gamma_{sonic}^{(\gamma_1)}$, and the data via (10.2.27)) and $C > 0$ depending on the data (via parameters of the iteration set), γ_1 via the ellipticity in (7.2.56) and the length of $\Gamma_{sonic}^{\gamma_1}$, and γ_2 via the obliqueness in (7.2.31) and the length of $\Gamma_{shock} \cap \{\pi - \gamma_2 < \theta < \pi\}$,

$$\|\hat{\psi}_{\gamma_1,\gamma_2}\|_{2,\alpha(\gamma_1,\gamma_2),\Omega_{\gamma_1}}^{(-\alpha(\gamma_1,\gamma_2)),\partial\Omega} \leq C, \tag{B.0.48}$$

and also by the maximum principle,

$$0 \leq \hat{\psi}_{\gamma_1,\gamma_2} \leq \overline{\psi} \qquad \text{in } \overline{\Omega}_{\gamma_1}. \tag{B.0.49}$$

To take $\gamma_1, \gamma_2 \downarrow 0$, we need to obtain estimates that are uniform in γ_1, γ_2. The reason that (B.0.48) depend on γ_1, γ_2 is the degenerate ellipticity at Γ_{sonic}, and degenerate obliqueness at P_2. Thus we need to attain estimates near $\Gamma_{sonic}^{\gamma_1}$, and near P_2 that do not depend on γ_1 or γ_2, but need only be valid for small γ_1, γ_2.

We first focus on the behaviour of the approximate solutions near the point P_2. Using (10.2.22), we find that if $\gamma_2 < \sigma$, we can attain similar algebraic growth as in Lemma 5.2.2 (with $m = 1$),

$$|\hat{\psi}_{\gamma_1,\gamma_2} - \overline{\psi}| \leq C(\pi - \gamma_2 - \theta) \qquad \text{on } \Gamma_{shock} \cap \{\pi - \gamma_2 < \theta < \pi - \sigma\}, \tag{B.0.50}$$

for C depending on the parameters of the iteration set. So as by (10.2.19), the boundary is locally $C^{1,1}$ at every point, thus satisfies an exterior ball condition at $P_2^{(\gamma_2)} := (r(\pi - \gamma_2), \pi - \gamma_2)$ (in the (r, θ) coordinates). Thus,we find via the argument of Proposition 5.2.4,

that for some $\alpha \in (0,1), C > 0$ and $R > 0$ only depending on the parameters of the iteration set (i.e. not depending on (γ_1, γ_2)), that

$$\|\hat{\psi}_{\gamma_1,\gamma_2}\|_{1,\alpha,B_R(P_2)}^{(-\alpha),\Gamma_{shock}\cap\{\pi-\gamma_2<\theta<\pi\}} \leq C. \tag{B.0.51}$$

Next, we focus on the behaviour near Γ_{sonic}. Due to (10.2.26), (7.2.48), (10.2.22) and (B.0.49), we find that for C sufficiently large, depending only on the data, $x \mapsto C(x-\gamma_1)$ is a supersolution of (B.0.47) in $\Omega_{\gamma_1} \cap \mathcal{D}_{\epsilon_{eq}}$, so that

$$0 \leq \hat{\psi}_{\gamma_1,\gamma_2} \leq C(x-\gamma_1) \leq Cx \qquad \text{in } \Omega_{\gamma_1} \cap \mathcal{D}_{\epsilon_{eq}}. \tag{B.0.52}$$

Thus, taking a diagonal subsequence, we find that for a subsequence $\hat{\psi}_i$, there exists a limiting $\hat{\psi} \in C^2(\Omega) \cap C^1(\Omega \cap \Gamma) \cap C(\overline{\Omega})$,

$$\begin{aligned} &\hat{\psi}_i \to \hat{\psi} \text{ in } C^{2,\alpha(K)}(K) \text{ for any } K \subset\subset \Omega, \\ &\hat{\psi} \to \hat{\psi} \text{ in } C^{1,\hat{\alpha}(K)} \text{ for any } K \subset\subset \overline{\Omega} \setminus \{P_2 \cup \Gamma_{sonic}\}, \\ &\hat{\psi}_i \to \hat{\psi} \text{ pointwise on } \overline{\Omega}, \end{aligned} \tag{B.0.53}$$

so that using the fact $\hat{\psi}_i(P_2) = \overline{\psi}$ for all i, and (B.0.52), we find $\hat{\psi}$ satisfies (7.2.23)-(7.2.27), and (B.0.45). □

We then prove a result for the strict elliptic case.

Proposition B.0.9. *Let $\theta_w \in [\theta_w^*, \theta_{sonic}]$. Then there exists a solution $\hat{\psi} \in C(\overline{\Omega}) \cap C^1(\Omega \cup \Gamma) \cap C^2(\Omega)$ of (7.2.23)-(7.2.27) which satisfies*

$$0 \leq \psi \leq \overline{\psi} \text{ in } \overline{\Omega}. \tag{B.0.54}$$

Proof. The only difference here is in the treatment of Γ_{sonic}, where we no longer have the second estimate in (10.2.22), but now have access to (10.2.30). We can now treat P_1 similarly to the treatment of P_2, using the argument of Proposition 9.2.13 to obtain uniform $C^{0,\alpha}$ regularity up to $\Gamma_{shock}^{\gamma_1}$, which implies the result.

□

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