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Convergence of Almost Harmonic Maps to Geodesic Bubble Trees

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Abstract

We prove a sharp criterion on the decay of the tension of almost harmonic maps from degenerating surfaces that ensures that such maps subconverge to a limiting object that is made up entirely of harmonic maps.

Keywords Harmonic maps · Bubble trees · Degenerating domains

Mathematics Subject Classification 53C43 · 58E20

1 Introduction

A map u from a closed Riemannian surface (M, g) to a Riemannian manifold (N, g_N) is called harmonic if it is a critical point of the Dirichlet energy

$$E(u, g) = \frac{1}{2} \int_M |du|_g^2 dv_g.$$

Harmonic maps are characterised by $\tau_g(u) = 0$, where, viewing (N, g_N) as isometrically embedded in some $\mathbb{R}^N$ and writing A for the second fundamental form, the tension field $\tau_g(u)$ is given by

$$\tau_g(u) = \Delta_g u + A_g(u)(du, du) = \Delta_g u + g^{ij} A(u)(u_{x_i}, u_{x_j}).$$

For any *fixed* domain surface (M, g) the compactness results from [3, 6, 7, 11] ensure that for maps $u_i : (M, g) \to N$ with bounded energy which are almost harmonic in the sense that

$$\|\tau_g(u_i)\|_{L^2(M,g)} \to 0 \text{ as } i \to \infty \tag{1.1}$$

a subsequence converges to a bubble tree that consists of a harmonic base map $u_\infty : M \to N$ and a finite number of harmonic spheres $\omega_i : S^2 \to N$ that bubble off and that

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in this convergence there is no loss of energy nor formation of non-trivial necks. Hence for almost harmonic maps from any fixed domain surface the limit against which the sequence converges is made up entirely of objects that are themselves critical points of the Dirichlet energy, i.e. harmonic maps.

It is now natural to ask whether the same can be expected if we instead consider a sequence of maps $u_i : M \to N$ which are almost harmonic with respect to a sequence of metrics g_i on M. Unsurprisingly the answer is positive if the metrics themselves converge to a limiting metric g_∞, after potentially pulling-back by suitable diffeomorphisms. In particular, if we use the uniformisation theorem and the conformal invariance of the Dirichlet energy to restrict our attention to domain metrics of constant curvature $\kappa_{g_i} = 1, 0, -1$ for surfaces of genus $0, 1, \geq 2$ (with unit area if $\gamma = 1$) then this holds true unless the injectivity radius of (M, g_i) tends to zero.

If $\text{inj}(M, g_i) \to 0$ then the situation is more involved as parts of the surface degenerate. In this setting the compactness properties of *harmonic maps* were investigated in [1, 5, 14], while the compactness properties of almost harmonic maps were considered in [4, 9]. We note that such almost harmonic maps from degenerating surfaces in particular arise in the asymptotic analysis of Teichmüller harmonic map flow, see [4, 9, 10].

To describe these results we first recall some well known properties about degenerating hyperbolic surfaces, see [13] for more details.

So let (M, g_i) be a sequence of surfaces with Gauss-curvature $\kappa_{g_i} \equiv -1$ for which $\text{inj}(M, g_i) \to 0$. Then we can pass to a subsequence so that for some $k \in \{1, \ldots, 3(\gamma - 1)\}$ there are simple closed geodesics $\{\sigma_i^j\}_{j=1}^k$ of length $\ell_i^j \to 0$, a complete hyperbolic surface (Σ, h) with $2k$ punctures and diffeomorphisms $f_i : \Sigma \to M \setminus \bigcup_{j=1}^k \sigma_i^j$ so that

$$f_i^* g_i \to h \text{ smoothly locally on } \Sigma.$$

Furthermore we know that the degenerating parts of the surface (M, g_i) are contained in the union of so called collar neighbourhoods $\mathcal{C}(\sigma_i^j)$ around the collapsing geodesics which are isometric to hyperbolic cylinders $(\mathcal{C}(\ell_i^j), g_{\ell_i^j})$ characterised by

$$(\mathcal{C}(\ell), g_\ell) = \left([-X(\ell), X(\ell)] \times S^1, \rho_\ell^2(s)(ds^2 + d\theta^2)\right) \tag{1.2}$$

and

$$X(\ell) = \tfrac{2\pi}{\ell}\left(\tfrac{\pi}{2} - \arctan\left(\sinh\left(\tfrac{\ell}{2}\right)\right)\right) \text{ and } \rho_\ell(s) = \frac{\ell}{2\pi \cos(\frac{\ell}{2\pi}s)}. \tag{1.3}$$

If we now consider a sequence of maps $u_i : M \to N$ from such degenerating hyperbolic surfaces which satisfy $\|\tau_{g_i}(u_i)\|_{L^2(M, g_i)} \to 0$, then on compact subsets of Σ the almost harmonic maps $u_i \circ f_i : (\Sigma, f_i^* g_i) \to (N, g_N)$ exhibit the same type of convergence behaviour as obtained in [3, 6, 7, 11] in the non-degenerate case, that is subconvergence to a bubble tree of harmonic maps without loss of energy or formation of necks. Conversely, these results do not describe the behaviour of the maps u_i on the parts of the surface where $\text{inj} \to 0$.

As these degenerating regions are contained in the collar neighbourhoods described above we can equivalently consider the behaviour of almost harmonic maps u_i on hyperbolic cylinders $(\mathcal{C}(\ell_i), g_{\ell_i})$ with $\ell_i \to 0$. In [4] it is shown that such almost harmonic maps subconverge to a full bubble branch in the following sense:

Theorem 1.1 (Contents of Theorem 1.9 of [4]) *Let $(\mathcal{C}(\ell_i), g_{\ell_i})$, $\ell_i \to 0$, be a sequence of hyperbolic cylinders as in (1.2) and let $u_i : (\mathcal{C}(\ell_i), g_{\ell_i}) \to N$ be a sequence of smooth maps which have bounded energy and for which*

$$\|\tau_{g_i}(u_i)\|_{L^2(\mathcal{C}(\ell_i), g_{\ell_i})} \to 0.$$

Then, after passing to a subsequence, there exists a number $\bar{m} \in \mathbb{N}_0$ and sequences

$$-X(\ell_i) =: s_i^0 \ll s_i^1 \ll \ldots \ll s_i^{\bar{m}} := X(\ell_i)$$

so that for each $m \in \{1, \ldots, \bar{m}-1\}$ the shifted maps $u_i(\cdot + s_i^m, \cdot)$ converge to a non-trivial bubble branch locally on $\mathbb{R} \times S^1$ and so that u_i maps $[s_i^{m-1} + \Lambda, s_i^m - \Lambda] \times S^1$, Λ large, close to a curve in the sense that

$$\lim_{\Lambda\to\infty} \limsup_{i\to\infty} \sup_{s\in[s_i^{m-1}+\Lambda, s_i^m-\Lambda]} \operatorname{osc}_{\{s\}\times S^1} u_i = 0 \text{ for every } m \in \{1, \ldots, \bar{m}\}. \tag{1.4}$$

Here we recall from [4] that we say that maps from cylinders $[-Y_i, Z_i] \times S^1$ with $Y_i, Z_i \to \infty$ converge to a bubble branch if they converge to a harmonic limit ω_∞ weakly in $H^1_{loc}(\mathbb{R} \times S^1)$ and strongly in $H^2_{loc}(\mathbb{R} \times S^1 \setminus S)$ away from a finite (possibly empty) set of points $S \subset \mathbb{R} \times S^1$ where a finite number of bubbles ω_i form. We call such a bubble branch non-trivial if the limiting configuration consists of at least one non-trivial harmonic map, or equivalently we call a bubble branch trivial only if no bubbles form and if the map ω_∞ is constant.

Remark 1.2 We also recall that in this convergence of the shifted maps to a bubble branch there can be no loss of energy nor formation of necks on *compact* subsets of $\mathbb{R} \times S^1$. Namely for all suitably large (but fixed) numbers $\Lambda > 0$ we know that

$$E(u_i, [s_i^m - \Lambda, s_i^m + \Lambda] \times S^1) \to E(\omega_\infty, [-\Lambda, \Lambda] \times S^1) + \sum E(\omega_i) \tag{1.5}$$

as well as that the maps $u_i(\cdot - s_i^m)$ are close in $L^\infty([-\Lambda, \Lambda] \times S^1)$ to maps that are built out of ω_∞ and suitable rescalings of the bubbles, see [4] for details.

Conversely, energy can be lost on the longer and longer cylinders $[s_i^{m-1} + \Lambda, s_i^m - \Lambda] \times S^1$ in the domains that connect the different bubble regions and a description of this loss of energy in terms of the Hopf-differential was provided in [14] (for harmonic maps) and in [4] (for almost harmonic maps).

We note that while (1.4) ensures that the degenerating regions are always mapped close to the curves

$$\hat{u}_i(s) := \pi_N\left(\fint_{\{s\}\times S^1} u_i d\theta\right), \quad s \in [s_i^{m-1} + \Lambda, s_i^m - \Lambda], \quad m = 1, \ldots, \bar{m}, \tag{1.6}$$

π_N the nearest point projection from a neighbourhood of N to N, in the sense that

$$\lim_{\Lambda\to\infty}\limsup_{i\to\infty}\|u_i-\hat{u}_i\|_{L^\infty([s_i^{m-1}+\Lambda,s_i^m-\Lambda]\times S^1)}=0,\quad m=1,\ldots,\bar{m},\tag{1.7}$$

we cannot expect these curves to collapse to a point in the limit $i\to\infty$ and $\Lambda\to\infty$. In particular we cannot expect that the asymptotic values

$$p_\infty^{m-1,+}:=\lim_{\Lambda\to\infty}\lim_{i\to\infty}u_i(s_i^{m-1}+\Lambda,\cdot)\text{ and }p_\infty^{m,-}:=\lim_{\Lambda\to\infty}\lim_{i\to\infty}u_i(s_i^m-\Lambda,\cdot)\tag{1.8}$$

of neighbouring bubble branches agree.

Instead it is natural to ask whether we can expect the connecting curves $\hat{u}_i$ to 'look like' (possibly longer and longer) geodesics. If so this would mean that in the limit $i\to\infty$ we again only obtain objects that are harmonic maps themselves, namely a harmonic map from the limiting surface Σ, a finite number of harmonic spheres $\omega_i : S^2\to N$ and a number of (possibly infinite length) geodesics, i.e. harmonic maps from suitable intervals. In a situation like that we would then say that the sequence of maps $u_i : M\to N$ converges to a *geodesic bubble tree*.

For maps u_i which are harmonic, rather than just almost harmonic, the results of Chen-Tian [1] and Li-Wang [5] establish such a convergence to a geodesic bubble tree. Indeed they prove that after passing to a subsequence the connecting curves $\hat{u}_i^m$ either collapse to a point, converge (after reparametrising) to a finite length geodesic or that in the limit their image contains an infinite length geodesic. The results of [1] furthermore exclude the last possibility in the case of energy minimising maps.

Conversely, as explained in [4], we cannot expect to obtain convergence to a *geodesic* bubble tree if we consider the more general case of maps from degenerating hyperbolic surfaces that are almost harmonic in the sense that $\|\tau_{g_i}(u_i)\|_{L^2(M,g_i)}\to 0$. Indeed, as pointed out in Proposition 1.14 of [4] we can obtain any C^2 curve in (N,g_N) as limit of connecting curves for maps with tension $\|\tau_{g_i}(u_i)\|_{L^2(M,g_i)}\leq C\,\mathrm{inj}(M,g_i)^{\frac{1}{2}}\to 0$ simply by reparametrising such a curve along the collar.

The purpose of this paper is to close the gap left between the results of [1, 5] that ensure convergence to a *geodesic* bubble tree for *harmonic* maps and the examples of [4] that show that no such result can be true for almost harmonic maps whose tension decays no faster than $\mathrm{inj}(M,g_i)^{\frac{1}{2}}$.

Indeed, our main result shows that any rate of decay of the tension that is faster than $\mathrm{inj}(M,g_i)^{\frac{1}{2}}$ will force the connecting curves to look like geodesics and hence ensure subconvergence to a geodesic bubble tree. To prove this it suffices to prove the corresponding result for almost harmonic maps from hyperbolic cylinders with $\ell_i\to 0$.

Given such a sequence of maps $u_i : \mathcal{C}(\ell_i)\to N$ with $\|\tau_{g_{\ell_i}}(u_i)\|_{L^2(\mathcal{C}(\ell_i),g_{\ell_i})}=o(\ell_i^{\frac{1}{2}})$ we can first apply Theorem 1.1 to locate the points s_i^m at which the different bubble branches form. Instead of splitting the cylinder $[-X(\ell_i),X(\ell_i)]\times S^1$ into fixed size bubble regions $[s_i^m-\Lambda,s_i^m+\Lambda]\times S^1$ and the longer and longer cylinders between these sets, we instead want to consider *extended bubble regions* $B_i^m=[a_i^m,b_i^m]\times S^1$

around $\{s_i^m\}\times S^1$ with $b_i^m - a_i^m \to \infty$ and the *connecting cylinders* $[b_i^{m-1}, a_i^m]\times S^1$ between these regions, where a_i^m and b_i^m will be carefully chosen numbers with

$$s_i^{m-1} \ll b_i^{m-1} \le a_i^m \ll s_i^m \text{ for } m = 1, \dots, \bar{m} \text{ while } a_i^0 := -X(\ell_i) \text{ and } b_i^{\bar{m}} := X(\ell_i). \tag{1.9}$$

On the one hand, to ensure that there can be no loss of energy or formation of necks on the extended bubble regions we need to know that

$$\lim_{\Lambda\to\infty} \limsup_{i\to\infty} E(u_i, [s_i^{m-1}+\Lambda, b_i^{m-1}]\times S^1) + E(u_i, ([a_i^m, s_i^m - \Lambda]\times S^1) = 0 \tag{1.10}$$

and

$$\lim_{\Lambda\to\infty} \limsup_{i\to\infty} \operatorname*{osc}_{[s_i^{m-1}+\Lambda, b_i^{m-1}]\times S^1} u_i + \operatorname*{osc}_{[a_i^m, s_i^m-\Lambda]\times S^1} u_i = 0, \tag{1.11}$$

hold true, compare Remark 1.2.

On the other hand, we want a_i^m and b_i^m to be so that, after reparametrisation by arclength, the restriction of the maps u_i onto the connecting cylinders $[b_i^{m-1}, a_i^m]\times S^1$ is essentially described by a (trivial, finite length or infinite length) geodesic.

As we shall see, it is possible to satisfy both of these properties simultaneously if and only if we know that the tension decays strictly faster than $\ell_i^{\frac{1}{2}}$ and indeed we can prove

Theorem 1.3 *Let $u_i : (\mathcal{C}(\ell_i), g_{\ell_i}) \to (N, g_N)$ be a sequence of maps from hyperbolic cylinders with $\ell_i \to 0$ with bounded energy and*

$$\ell_i^{-\frac{1}{2}} \|\tau_{g_{\ell_i}}(u_i)\|_{L^2(\mathcal{C}(\ell_i), g_{\ell_i})} \to 0 \tag{1.12}$$

which converges to a full bubble branch as described in Theorem 1.1 above.

Then we can choose a_i^m, b_i^m as in (1.9) so that (1.10) and (1.11) hold, and hence so that there can be no loss of energy or formation of necks on the extended bubble regions $[a_i^m, b_i^m]\times S^1$, and so that the images $u_i([b_i^{m-1}, a_i^m]\times S^1)$ of the connecting cylinders subconverge to geodesics in the following sense.

Let $\hat{u}_i$ be the connecting curves defined by (1.6) which approximate u_i as described in (1.7) and let $v_i^m : [-c_i^m, c_i^m] \to N$ be the reparametrisation of $\hat{u}_i|_{[b_i^{m-1}, a_i^m]}$ by arclength. Then

$$\|\tau(v_i^m)\|_{L^p([-c_i^m, c_i^m])} \to 0 \textit{ for every } p \in [1, 2]$$

and hence, after passing to a subsequence, we have

(1) (Trivial neck case) If $c_i^m \to 0$ then the connecting curves $\hat{u}_i^m$ collapse to a point so no neck forms between the two bubble branches against which $u_i(\cdot - s_i^{m-1}, \cdot)$ and $u_i(\cdot - s_i^m, \cdot)$ converge.

(2) (Finite length case) If $c_i^m \to c_m \in (0, \infty)$ then the curves $v_i^m(\frac{c_m^i}{c_m}\cdot)$ converge strongly in $W^{2,2}([-c_m, c_m], N)$ to a geodesic which connects the points $p_\infty^{m-1,+}$

and $p_\infty^{m,-}$ in the images of the bubble branches against which $u_i(\cdot - s_i^{m-1})$ and $u_i(\cdot - s_i^m)$ converge.

(3) (Infinite length case) If $c_i^m \to \infty$ then the curves $v_i^m(\cdot + c_i^m)$ and $v_i^m(c_i^m - \cdot)$ converge in $W^{2,2}_{loc}([0,\infty))$ to infinite length geodesics which originate at the points $p_\infty^{m-1,+}$ and $p_\infty^{m,-}$.

Here $p_\infty^{m-1,+}$ and $p_\infty^{m,-}$ are given by (1.8).

The analogue result also holds true for maps from degenerating tori, albeit with a different sharp rate on the decay of the tension. Namely we show

Theorem 1.4 *Let (T^2, g_i) be a sequence of flat unit area tori whose injectivity radius converges to zero and let $u_i : T^2 \to N$ be a sequence of maps whose tension satisfies*

$$\|\tau_{g_i}(u_i)\|_{L^2(T^2,g_i)}\, \mathrm{inj}(T^2, g_i)^{-2} \to 0.$$

Then, after passing to a subsequence, the maps u_i converge to a geodesic bubble tree as described in Theorem 1.3.

This result is also sharp as we cannot expect the connecting curves to look like geodesics if the rate of decay of the tension is no faster than $\mathrm{inj}(T^2, g_i)^2$, compare Section 4.

One of the main difficulties in the proof of the above results is that if we reparametise a curve with small velocity by arclength then this leads to a sharp increase of the tension. Roughly speaking for a curve to look like a geodesic we need its tension to be small compared to the square of its velocity. A key step towards proving that the connecting curves $\hat{u}_i$ converge to geodesics is hence to obtain a *lower bound* on the velocity with which these curves are parametrised. To prove our results we then want to split each interval $[s_i^{m-1} + \Lambda, s_i^m - \Lambda]$ into a

- 'relevant' region on which the velocity of $\hat{u}_i$ is large enough that even after reparametrising by arclength the tension still tends to zero.
- its complement in $[s_i^{m-1} + \Lambda, s_i^m - \Lambda]$ for which we want to prove that the velocity of $\hat{u}_i$ is so small that the restriction of u_i to this set cannot make a relevant contribution to the limiting connecting curve nor lead to a loss of energy in the limit $i \to \infty$ and $\Lambda \to \infty$.

Importantly, we will be able to prove that the 'relevant' region of each such interval is connected. This is crucial as we cannot hope to control the tension of the reparametrised curves on sets where $|\hat{u}_i'|$ is too small. If the relevant region was not an interval then the curve v_i^m could change its direction in an uncontrolled way on the irrelevant set, so the best we could hope for would be to obtain a limiting curve that is piecewise geodesic but has corners, rather than a single geodesic arc between the images of neighbouring bubble branches.

Our proofs are based on suitable estimates for almost harmonic maps from degenerating cylinders which we derive in Section 2. We then use these estimates in Section 3 to prove our main result Theorem 1.3 on almost harmonic maps from degenerating hyperbolic surfaces. We conclude this paper with a short Section 4 where we explain how these arguments can be modified, and indeed simplified, in the case where the domain is a torus.

2 Estimates for Maps from Cylinders on Low Energy Regions

The proof of our main result is based on a precise understanding of the behaviour of almost harmonic maps away from the regions where bubbles from. We recall from [4] that if a sequence of almost harmonic maps converges to a full bubble branch as recalled in Theorem 1.1 then for any $\varepsilon > 0$ there exist Λ_0 and i_0 so that

$$E(u_i, \mathscr{C}_1(s)) \leq \varepsilon \text{ for all } i \geq i_0 \text{ and } s \in [s_i^{m-1} + \Lambda_0, s_i^m - \Lambda_0],$$

where here and in the following we write for short $\mathscr{C}_\lambda(s) := [s-\lambda, s+\lambda] \times S^1$ and $\mathscr{C}_\lambda = \mathscr{C}_\lambda(0)$.

In this section we recall and derive estimates on almost harmonic maps on such low energy regions. To state and prove some of these estimates it is more convenient to work with respect to the flat metric $g_E = ds^2 + d\theta^2$. As the energy is conformally invariant and as the euclidean and the hyperbolic tension of maps are related by (2.17) we can then easily translate these results to make them applicable to almost harmonic maps from hyperbolic collars.

So let $X \geq 2$ be any fixed number and let $u \in H^2(\mathscr{C}_X, N)$. We set

$$\mathcal{A}(u) = \{s : |s| \geq X - 1 \text{ or } E(u, \mathscr{C}_1(s)) \geq \varepsilon_0\}, \tag{2.1}$$

$\varepsilon_0 = \varepsilon_0(N) > 0$ determined below, and recall that away from $\mathcal{A}(u) \times S^1$ we have the following well known bound on the angular energy

$$\vartheta(s) := \int_{\{s\}\times S^1} |u_\theta|^2 d\theta. \tag{2.2}$$

Lemma 2.1 (Standard angular energy estimates, see e.g Lemma 2.1 [4] or Lemma 2.13 [12]) *There exists $\varepsilon_0 = \varepsilon_0(N) > 0$ so that for any $u \in H^2(\mathscr{C}_X, N)$ and any $s \in [-X, X] \setminus \mathcal{A}(u)$*

$$\vartheta(s) \leq Ce^{-dist(s,\mathcal{A}(u))} + C \int_{\mathscr{C}_X} |\tau_{g_E}(u)|^2 e^{-|s-q|} dq d\theta. \tag{2.3}$$

In this section C denotes a constant that only depends on N and an upper bound on the energy of u.

We note that while the angular energy estimates in [4] are only stated for maps with small tension, the H^2 estimates stated in Lemma 2.2 below and the trace theorem ensure that the above estimate is trivially true if the right hand side of (2.3) is of order one.

We also need estimates on the second derivatives of u. Throughout the paper we use the extrinsic viewpoint of considering maps u to $N \hookrightarrow \mathbb{R}^N$ as maps into the surrounding Euclidean space to define higher order derivatives. Away from the high energy region we have the following standard H^2 estimates.

Lemma 2.2 (see e.g. [3, Lemma 2.1] or [12, Lemma 2.9]) *There exists $\varepsilon_0 = \varepsilon_0(N) > 0$ so that for any $u \in H^2(\mathscr{C}_X, N)$ and any $s_0 \in [-X, X] \setminus \mathcal{A}(u)$*

$$\int \phi^2 |\nabla^2 u|^2 ds d\theta \leq C \|\phi \tau_{g_E}(u)\|^2_{L^2(\mathscr{C}_1(s_0), g_E)} + CE(u, \mathscr{C}_1(s_0)) \tag{2.4}$$

and

$$\int \phi^2 (|u_s|^4 + |u_\theta|^4) ds d\theta \leq CE(u, \mathscr{C}_1(s_0)) \cdot \big[\|\phi \tau_{g_E}(u)\|^2_{L^2(\mathscr{C}_1(s_0), g_E)} + E(u, \mathscr{C}_1(s_0))\big], \tag{2.5}$$

for $\phi \in C_c^\infty(s_0 - 1, s_0 + 1)$ with $\phi \equiv 1$ on $[s_0 - \frac{1}{2}, s_0 + \frac{1}{2}]$ and $|\phi'| \leq 4$.

In the following we fix $\varepsilon_0 = \varepsilon_0(N) > 0$ so that both of these lemmas apply and denote by $\mathcal{A}(u)$ the resulting set defined by (2.1).

For integrals that involve angular derivatives of u we obtain the following stronger estimates that only involve the angular energy $E_\theta(u, \Omega) := \frac{1}{2} \int_\Omega |u_\theta|^2 ds d\theta$ instead of the full energy.

Lemma 2.3 *For any $u \in H^2(\mathscr{C}_X, N)$ and any $s_0 \in [-X, X] \setminus \mathcal{A}(u)$ we have*

$$\int \phi^2 (|u_{\theta\theta}|^2 + |u_{s\theta}|^2) ds d\theta \leq C \|\phi \tau_{g_E}(u)\|^2_{L^2(\mathscr{C}_1(s_0), g_E)} + CE_\theta(u, \mathscr{C}_1(s_0)), \tag{2.6}$$

ϕ as in Lemma 2.2, as well as

$$\int \phi^2 |u_\theta|^4 ds d\theta \leq E_\theta(u, \mathscr{C}_1(s_0)) \cdot \big[\|\phi \tau_{g_E}(u)\|^2_{L^2(\mathscr{C}_1(s_0), g_E)} + E_\theta(u, \mathscr{C}_1(s_0))\big]. \tag{2.7}$$

Proof of Lemma 2.3 Let $I_1 := \int \phi^2 (|u_{s\theta}|^2 + |u_{\theta\theta}|^2) ds d\theta$ and $I_2 := \int \phi^2 |u_\theta|^4 ds d\theta$. Viewing I_2 as the square of the L^2 norm of $\phi |u_\theta|^2$ and using that $W_0^{1,1}(\mathscr{C}_1)$ embeds continuously into $L^2(\mathscr{C}_1)$ we get

$$I_2 \leq C \|\nabla(\phi |u_\theta|^2)\|^2_{L^1} \leq CE_\theta(u, \mathscr{C}_1(s_0)) \cdot \big(I_1 + E_\theta(u, \mathscr{C}_1(s_0))\big) \tag{2.8}$$

so it suffices to prove the claimed bound (2.6) on I_1. As (2.6) already follows from Lemma 2.2 if $\|\phi \tau_{g_E}(u)\|_{L^2} \geq 1$ we can furthermore assume that $\|\phi \tau_{g_E}(u)\|_{L^2} \leq 1$.

Integration by parts, using also that $u_{ss} + u_{\theta\theta} = \tau_{g_E}(u) - A(u)(\nabla u, \nabla u)$, yields

$$\begin{aligned}I_1 &= -\int \partial_s(\phi^2)u_\theta u_{s\theta} ds d\theta + \int \phi^2 u_{\theta\theta}(u_{ss}+u_{\theta\theta})d\theta ds\\ &\leq C I_1^{\frac{1}{2}} E_\theta(u, \mathscr{C}_1(s_0))^{\frac{1}{2}} + I_1^{\frac{1}{2}}\|\phi\tau_{g_E}(u)\|_{L^2} - \int \phi^2 u_{\theta\theta} A(u)(\nabla u, \nabla u) ds d\theta\\ &\leq \tfrac{1}{4} I_1 + C[\|\phi\tau_{g_E}(u)\|_{L^2}^2 + E_\theta(u, \mathscr{C}_1(s_0))] + \int \phi^2 u_\theta \partial_\theta(A(u)(\nabla u, \nabla u)) ds d\theta.\end{aligned}$$

Using (2.8) as well as that $\int \phi^2|\nabla u|^4 \leq C$, compare (2.5), we can bound

$$\begin{aligned}\int \phi^2 u_\theta \partial_\theta(A(u)(\nabla u, \nabla u)) ds d\theta &\leq C I_1^{\frac{1}{2}}\Big(\int \phi^2|\nabla u|^4\Big)^{\frac{1}{4}} \cdot I_2^{\frac{1}{4}} + C\Big(\int \phi^2 |\nabla u|^4\Big)^{\frac{1}{2}} \cdot I_2^{\frac{1}{2}}\\ &\leq \tfrac{1}{4} I_1 + C E_\theta(u, \mathscr{C}_1(s_0))\end{aligned}$$

which, when inserted into the previous estimate, gives (2.6). □

Combined with Lemma 2.1 we hence obtain that for any s with $\mathrm{dist}(s, \mathcal{A}(u)) \geq 2$

$$\||u_{\theta\theta}| + |u_{s\theta}| + |u_\theta|^2\|_{L^2(\mathscr{C}_1(s))}^2 + \vartheta(s) \leq C R_u(s), \tag{2.9}$$

where here and in the following proofs we use the shorthand

$$R_u(s) := \int_{\mathscr{C}_X} |\tau_{g_E}(u)|^2 e^{-|s-q|} dq d\theta + e^{-\mathrm{dist}(s, \mathcal{A}(u))}. \tag{2.10}$$

We can use (2.9) to prove

Lemma 2.4 *For any $u \in H^2(\mathscr{C}_X, N)$ and any s_0 with dist$(s_0, \mathcal{A}(u)) \geq 2$ we have*

$$\operatorname*{osc}_{[s_0-1, s_0+1]} ⨍_{S^1} |\partial_s u(\cdot, \theta)| d\theta \leq C\|u_{\theta\theta}\|_{L^1(\mathscr{C}_1(s_0), g_E)} + C\|\tau_{g_E}(u)\|_{L^1(\mathscr{C}_1(s_0), g_E)} \leq C R_u(s_0)^{\frac{1}{2}} \tag{2.11}$$

as well as

$$\left|⨍_{S^1} u_s(s_0, \theta) d\theta\right| \geq ⨍_{S^1} |u_s(s_0, \theta)| d\theta - C R_u(s_0)^{\frac{1}{2}}. \tag{2.12}$$

We will apply these estimates later to analyse the curves $\hat{u}(s) := \pi_N(\bar{u}(s))$, where $\bar{u}(s) := ⨍_{\{s\}\times S^1} u d\theta$. Thanks to Lemma 2.1 $\hat{u}(s)$ is well defined for maps with suitably small tension and for s with suitably large distance from $\mathcal{A}(u)$. As $u_s = d\pi_N(u)(u_s)$ we can furthermore estimate

$$\begin{aligned}|\bar{u}' - \hat{u}'| = |\bar{u}' - d\pi_N(\bar{u})(\bar{u}')| &= \left|⨍_{S^1} [d\pi_N(u) - d\pi_N(\bar{u})](u_s) d\theta\right|\\ &\leq C \operatorname*{osc}_{S^1} u ⨍_{S^1} |u_s| \leq C\vartheta^{\frac{1}{2}} ⨍_{S^1} |u_s|.\end{aligned} \tag{2.13}$$

The lower bound (2.12) on $|\bar{u}'|$ from the above lemma hence yields the following lower bound on the velocity of the connecting curve $\hat{u}$

Corollary 2.5 *For any $u \in H^2(\mathscr{C}_X, N)$ and any s_0 with $dist(s_0, \mathcal{A}(u)) \geq 2$ for which $\hat{u}$ is well defined in a neighbourhood of s_0 we have*

$$|\hat{u}'(s_0)| \geq (1 - C\vartheta^{\frac{1}{2}}(s_0)) \fint_{S^1} |u_s(s_0,\theta)|d\theta - CR_u(s_0)^{\frac{1}{2}}. \tag{2.14}$$

Proof of Lemma 2.4 To establish the first claim of the lemma we can use that

$$\frac{1}{2}\partial_s |u_s|^2 = u_{ss} \cdot u_s = -u_{\theta\theta} \cdot u_s + \tau_{g_E}(u) \cdot u_s$$

and hence that $|\partial_s |u_s|| \leq |u_{\theta\theta}| + |\tau_{g_E}(u)|$ almost everywhere. This immediately implies the first estimate of (2.11) while the second estimate of (2.11) follows from (2.9) and the definition of R_u.

We now want to derive a similar estimate for the oscillation of $|\bar{u}'(s)|$ which we will later use to prove the second claim of the lemma. To this end we use that

$$\tfrac{d}{ds}|\bar{u}'(s)|^2 = 2\bar{u}'(s) \fint_{\{s\}\times S^1} u_{ss} d\theta = 2\bar{u}'(s) \fint_{\{s\}\times S^1} \tau_{g_E}(u) - A(u)(\nabla u, \nabla u) d\theta.$$

As $\partial_s u \perp A(u)(\nabla u, \nabla u)$ we can thus bound

$$\begin{aligned} |\tfrac{d}{ds}|\bar{u}'(s)|^2| &\leq 2|\bar{u}'(s)| \fint_{\{s\}\times S^1} |\tau_{g_E}(u)| d\theta + 2 \fint_{\{s\}\times S^1} |A(u)(\nabla u, \nabla u)| \cdot |u_s - \bar{u}'| d\theta \\ &\leq 2|\bar{u}'(s)| \fint_{\{s\}\times S^1} |\tau_{g_E}(u)| d\theta + C(\int_{\{s\}\times S^1} |\nabla u|^4)^{\frac{1}{2}} (\int_{\{s\}\times S^1} |u_{s\theta}|^2)^{\frac{1}{2}}. \end{aligned}$$

Integrating this estimate over $I := [s_0 - \frac{1}{2}, s_0 + \frac{1}{2}]$ and then using (2.9) and (2.5) gives

$$\begin{aligned} \operatorname*{osc}_I |\bar{u}'|^2 &\leq C \sup_I |\bar{u}'| \cdot \|\tau_{g_E}(u)\|_{L^2(\mathscr{C}_{\frac{1}{2}}(s_0))} + C\|\nabla u\|^2_{L^4(\mathscr{C}_{\frac{1}{2}}(s_0))} \|u_{\theta s}\|_{L^2(\mathscr{C}_{\frac{1}{2}}(s_0))} \\ &\leq C \sup_I |\bar{u}'| R_u(s_0)^{\frac{1}{2}} + CE(u, \mathscr{C}_1(s_0))^{\frac{1}{2}} R_u(s_0) + CE(u, \mathscr{C}_1(s_0)) R_u(s_0)^{\frac{1}{2}} \\ &\leq C \sup_I |\bar{u}'| R_u(s_0)^{\frac{1}{2}} + CR_u(s_0) + CE(u, \mathscr{C}_1(s_0))^{\frac{1}{2}} R_u(s_0)^{\frac{1}{2}}. \end{aligned} \tag{2.15}$$

As (2.9) furthermore allows us to bound

$$\begin{aligned} E(u, \mathscr{C}_1(s_0)) &\leq C \int_{s_0-1}^{s_0+1} \int_{S^1} |u_s(s,\theta) - \bar{u}'(s)|^2 d\theta + |\bar{u}'(s)|^2 + \vartheta(s) ds \\ &\leq C\|u_{s\theta}\|^2_{L^2(\mathscr{C}_1(s_0))} + C \sup_{[s_0-1, s_0+1]} |\bar{u}'|^2 + CR_u(s_0) \\ &\leq C \sup_{[s_0-1, s_0+1]} |\bar{u}'|^2 + CR_u(s_0), \end{aligned}$$

we hence know that $\operatorname{osc}_I |\bar{u}'|^2 \leq CR_u(s_0)^{\frac{1}{2}} \sup_I |\bar{u}'| + CR_u(s_0)$.

We now set $\alpha(s) := ⨍_{\{s\}\times S^1} |u_s| d\theta$ and use that $|\bar{u}'| \leq \alpha$ and that we have already established the bound (2.11) on the oscillation of α. This allows us to conclude that

$$\operatorname*{osc}_I |\bar{u}'|^2 \leq CR_u(s_0)^{\frac{1}{2}} \sup_I \alpha + CR_u(s_0) \leq C\alpha(s_0)R_u(s_0)^{\frac{1}{2}} + CR_u(s_0).$$

As $\int_I \alpha - |\bar{u}'| \leq \int_{I\times S^1} |u_s - ⨍_{S^1} u_s| \leq \|u_{s\theta}\|_{L^2(\mathscr{C}_{\frac{1}{2}}(s_0))} \leq CR_u(s_0)^{\frac{1}{2}}$ we furthermore have

$$\int_I \alpha^2 - |\bar{u}'|^2 \leq 2\sup_I \alpha \cdot \int_I \alpha - |\bar{u}'| \leq C(\alpha(s_0) + \operatorname*{osc}_I \alpha)\cdot R_u(s_0)^{\frac{1}{2}} \leq C\alpha(s_0)R_u(s_0)^{\frac{1}{2}} + CR_u(s_0).$$

Combined, these two estimates give

$$|\bar{u}'(s_0)|^2 \geq \int_I \alpha^2 - \int_I \alpha^2 - |\bar{u}'|^2 ds - \operatorname*{osc}_I |\bar{u}'|^2 \quad \geq \alpha(s_0)^2 - C[\alpha(s_0)R_u(s_0)^{\frac{1}{2}} + R_u(s_0)].$$

If $\alpha(s_0) \geq C_1 R_u(s_0)^{\frac{1}{2}}$ for a sufficiently large but fixed $C_1 \geq 1$, we hence deduce that

$$|\bar{u}'(s_0)|^2 \geq \alpha(s_0)(\alpha(s_0) - CR_u(s_0)^{\frac{1}{2}}) \geq |\bar{u}'(s_0)|(\alpha(s_0) - CR_u(s_0)^{\frac{1}{2}})$$

since $|\bar{u}'| \leq \alpha$ and since the term in the bracket is positive. We hence obtain the claimed bound (2.12) in this case which suffices to complete the proof of the lemma as (2.12) is trivially true (for $C \geq C_1$) if $\alpha(s_0) < C_1 R_u(s_0)^{\frac{1}{2}}$. □

To translate the estimates derived above to the hyperbolic setting we recall that the tension with respect to the hyperbolic metric $g = \rho^2 g_E$ is given by

$$\tau_g(u) = \rho^{-2}\tau_{g_E}(u) \tag{2.16}$$

which in particular implies that for any $I \subset [-X(\ell), X(\ell)]$

$$\|\tau_{g_E}(u)\|_{L^2(I\times S^1, g_E)} \leq \sup_I \rho \cdot \|\tau_g(u)\|_{L^2(I\times S^1, g)}. \tag{2.17}$$

We also recall that the conformal factor $\rho = \rho_\ell$ is bounded uniformly on collars $\mathcal{C}(\ell)$ for ℓ in a bounded range, say $\ell \in (0, \operatorname{arsinh}(1))$, and that for every $1 \leq \Lambda \leq X(\ell)$

$$C^{-1}\Lambda^{-1} \leq \rho(X(\ell) - \Lambda) \leq C\Lambda^{-1}. \tag{2.18}$$

As $|\log(\rho)'| = \rho|\sin(\frac{\ell}{2\pi}s)|$, and thus in particular $|\log(\rho)'| \leq \frac{1}{4}$ away from the ends of the collar, we can bound

$$\rho^2(s')e^{-\frac{1}{2}|s-s'|} \leq C\rho^2(s), \quad \text{for all } s, s' \in [-X(\ell), X(\ell)] \tag{2.19}$$

and hence get that the quantity R_u appearing in the above lemmas is bounded by

$$R_u(s) \leq C\rho^2(s) \int_{\mathcal{C}(\ell)} |\tau_{g_\ell}(u)(q,\theta)|^2 e^{-\frac{1}{2}|s-q|} dv_{g_\ell}(q,\theta) + e^{-\text{dist}(s,\mathcal{A}(u))}. \tag{2.20}$$

In particular, for every interval $I \subset [-X(\ell), X(\ell)] \setminus \mathcal{A}(u)$

$$\sup_I R_u + \int_I R_u \leq C \sup_I \rho^2 \cdot \|\tau_g(u)\|_{L^2}^2 + 2e^{-\text{dist}(I,\mathcal{A}(u))} \tag{2.21}$$

where here and in the following the norm of $\tau_g(u)$ is computed over $(\mathcal{C}(\ell), g)$ unless specified otherwise. We will furthermore use that

$$\begin{aligned}\int_I (R_u)^{\frac{1}{2}} &\leq (\int_I \rho^2 ds)^{\frac{1}{2}} \|\tau_g(u)\|_{L^2} + 4e^{-\frac{1}{2}\text{dist}(I,\mathcal{A}(u))} \\ &\leq C \sup_I \rho^{\frac{1}{2}} \cdot \|\tau_g(u)\|_{L^2} + 4e^{-\frac{1}{2}\text{dist}(I,\mathcal{A}(u))}.\end{aligned} \tag{2.22}$$

Here and in the following we still define the high energy set $\mathcal{A}(u)$ by the formula (2.1) that involves the energy of maps u on cylinders of unit length with respect to g_E in the corresponding collar coordinates. Similarly we continue to denote by $\text{dist}(\cdot, \mathcal{A}(u))$ the (euclidean) distance of numbers $s \in [-X(\ell), X(\ell)]$ respectively intervals $I \subset [-X(\ell), X(\ell)]$ from the high energy set $\mathcal{A}(u)$.

We first combine these estimate with the well known bounds on the angular energy that we recalled in Lemma 2.1 to show

Lemma 2.6 *Let $u : \mathcal{C}(\ell) \to N$ be a H^2-map from a hyperbolic cylinder $(\mathcal{C}(\ell), g)$ with $\ell \in (0, \text{arsinh}(1))$. Then for every $s \in [-X(\ell), X(\ell)]$ we can bound*

$$\int_{\{s\}\times S^1} |u_s|^2 - |u_\theta|^2 d\theta \leq C\ell + C\rho(s)\|\tau_g(u)\|_{L^2} \tag{2.23}$$

while for every interval $I \subset [-X(\ell), X(\ell)] \setminus \mathcal{A}(u)$

$$E(u, I\times S^1) \leq Ce^{-dist(I,\mathcal{A}(u))} + C\sup_I \rho^2 \cdot \|\tau_g(u)\|_{L^2}^2 + C|I|\cdot(\ell + \sup_I \rho \cdot \|\tau_g(u)\|_{L^2}) \tag{2.24}$$

and

$$\operatorname*{osc}_{I\times S^1} u \leq Ce^{-\frac{1}{2}dist(I,\mathcal{A}(u))} + C\sup_I \rho^{\frac{1}{2}} \|\tau_g(u)\|_{L^2} + C|I|\cdot(\ell + \sup_I \rho\|\tau_g(u)\|_{L^2})^{\frac{1}{2}}. \tag{2.25}$$

Proof of Lemma 2.6 Given u as in the lemma we let $\psi(s) = |u_s|^2 - |u_\theta|^2$ be the real part of the function that represents the Hopf-differential $\Phi(u) = (|u_s|^2 - |u_\theta|^2 - 2iu_s u_\theta)(ds + id\theta)^2$ in collar coordinates and set $I_\psi(s) := \int_{\{s\}\times S^1} \psi$. It is well known that the antiholomophic derivative of the Hopf-differential is bounded in terms of the tension and we exploit this to write

$$\partial_s I_\psi = \int_{S^1} \partial_s \psi = \int_{S^1} \partial_s \psi + \partial_\theta(2u_s u_\theta) = 2\int_{S^1}(u_{ss} + u_{\theta\theta})u_s = 2\int_{S^1} \tau_{g_E}(u)u_s.$$

Combined with (2.17) this implies that for any interval $J \subset [-X(\ell), X(\ell)]$

$$\operatorname*{osc}_J I_\psi \leq CE(u)^{\frac{1}{2}} \cdot \|\tau_{g_E}(u)\|_{L^2(J\times S^1, g_E)} \leq C \sup_J \rho \cdot \|\tau_g(u)\|_{L^2}. \tag{2.26}$$

We apply this estimate for $J = [s - \frac{1}{2}X(\ell), s]$ if $s \geq 0$ respectively for $J = [s, s + \frac{1}{2}X(\ell)]$ if $s < 0$ as this ensures that $\sup_J \rho \leq \sqrt{2}\rho(s)$. Indeed, if $|s| \geq \frac{X(\ell)}{2}$ then $\sup_J \rho = \rho(s)$ while otherwise $\sup_J \rho \leq \rho(\frac{X(\ell)}{2}) \leq \sqrt{2}\rho(0) \leq \sqrt{2}\rho(s)$. As $|J| = \frac{1}{2}X(\ell) \geq c\ell^{-1}$ for a universal $c > 0$, we can bound $\inf_J I_\psi \leq C\ell E(u) \leq C\ell$ and thus obtain from (2.26) that

$$I_\psi(s) \leq \inf_J I_\psi + \operatorname*{osc}_J I_\psi \leq C\ell + C\rho(s)\|\tau_g(u)\|_{L^2} \text{ for every } s \in [-X(\ell), X(\ell)], \tag{2.27}$$

as claimed in (2.23).

The second claim (2.24) of the lemma immediately follows by integrating (2.23) over the given interval $I \subset [-X(\ell), X(\ell)] \setminus \mathcal{A}(u)$ as Lemma 2.1 and (2.21) imply that

$$\sup_I \vartheta + \int_I \vartheta \leq C \sup_I \rho^2 \|\tau_g(u)\|^2_{L^2} + Ce^{-\operatorname{dist}(I, \mathcal{A}(u))}. \tag{2.28}$$

Similarly, from (2.23) we obtain

$$\begin{aligned}\operatorname*{osc}_{I\times S^1} u &\leq 2\sup_{s\in I} \operatorname*{osc}_{\{s\}\times S^1} u + \frac{1}{2\pi}\int_{I\times S^1} |u_s| ds d\theta \\ &\leq C\sup_I \vartheta^{\frac{1}{2}} + C\int_I \vartheta^{\frac{1}{2}} ds + C|I| \cdot \sup_I \Big(\int_{\{s\}\times S^1} |u_s|^2 - |u_\theta|^2\Big)^{\frac{1}{2}} \\ &\leq C\sup_I \vartheta^{\frac{1}{2}} + C\int_I \vartheta^{\frac{1}{2}} + C|I| \cdot (\ell + \sup_I \rho \cdot \|\tau_g(u)\|_{L^2})^{\frac{1}{2}}.\end{aligned}$$

As Lemma 2.1 and (2.22) imply that

$$\int_I \vartheta^{\frac{1}{2}} \leq C \sup_I \rho^{\frac{1}{2}} \cdot \|\tau_g(u)\|_{L^2} + Ce^{-\frac{1}{2}\operatorname{dist}(I, \mathcal{A}(u))} \tag{2.29}$$

we hence obtain the final claim (2.25) of the lemma. □

We shall later apply Lemma 2.6 to prove that there can be no loss of energy or formation of necks on regions of the collar with distance of order $O(|\log(\ell_i)|)$ from the high energy region $\mathcal{A}(u_i)$. Conversely, for points which have larger distance from the high energy set, and for which Lemma 2.1 hence gives a stronger bound on the angular energy, we will use the following lemma.

Lemma 2.7 *Let $u : \mathcal{C}(\ell) \to N$ be a map from a hyperbolic cylinder $(\mathcal{C}(\ell), g)$ with $\ell \in (0, \operatorname{arsinh}(1))$ and let $I \subset [-X(\ell), X(\ell)]$ be any interval with $dist(I, \mathcal{A}(u)) \geq 4|\log(\ell)|$. Then $\alpha(s) := \fint_{\{s\}\times S^1} |u_s| d\theta$ satisfies*

$$|\alpha(s)-\alpha(\tilde{s})| \leq C\ell^2+C\max(\rho(s),\rho(\tilde{s}))^{\frac{1}{2}}\|\tau_g(u)\|_{L^2(\mathcal{C}(\ell),g_\ell)} \text{ for all } s,\tilde{s}\in I \quad (2.30)$$

and, denoting by s_0 the element of $\bar{I}$ with $\rho(s_0) = \inf_I \rho$, we furthermore obtain that

$$\int_I \alpha \leq C\rho(s_0)^{-1}\alpha(s_0) + C\ell + C\rho(s_0)^{-\frac{1}{2}}\|\tau_g(u)\|_{L^2(\mathcal{C}(\ell),g_\ell)}. \quad (2.31)$$

Proof Since $e^{-\frac{1}{2}\operatorname{dist}(I,\mathcal{A}(u))} \leq C\ell^2$ we can integrate (2.11) from s to $\tilde{s}$ and use (2.22) to obtain the first claim of the lemma as the maximum of ρ on any closed interval is achieved at one of the endpoints.

To obtain the second claim we then use that (2.30) gives

$$\alpha(s) \leq \alpha(s_0) + C\ell^2 + C(2^j\rho(s_0))^{\frac{1}{2}}\|\tau_g(u)\|_{L^2} \text{ for } s\in I \text{ with } 2^{j-1}\rho(s_0) \leq \rho(s) \leq 2^j\rho(s_0). \quad (2.32)$$

At the same time we can use that

$$\mathcal{L}^1(\{s\in[-X(\ell),X(\ell)] : \frac{1}{2}\eta \leq \rho(s) \leq \eta\}) \leq C\eta^{-1} \text{ for any } \eta > 0. \quad (2.33)$$

Indeed, for $\frac{1}{2}\eta \leq \sqrt{2}\rho(0) = \frac{\ell}{\sqrt{2}\pi}$ this follows as $X(\ell) \leq C\ell^{-1}$ while for larger values of η this set consists of two intervals on which $\cos(\frac{\ell}{2\pi}s) \leq \frac{1}{\sqrt{2}}$ and thus $|\log(\rho)'| = \rho|\sin(\frac{\ell}{2\pi}s)| \geq \frac{1}{\sqrt{2}}\rho \geq (2\sqrt{2})^{-1}\eta$. Hence these intervals have length no more than $2\sqrt{2}\log 2\eta^{-1}$.

From (2.32) and (2.33), which also implies that $|I| \leq C\rho(s_0)^{-1} \leq C\ell^{-1}$, we thus obtain the claimed estimate of

$$\begin{aligned}\int_I \alpha(s)ds &\leq (\alpha(s_0)+C\ell^2)|I| + C\|\tau_g(u)\|_{L^2}\sum_{j\geq 0}(2^j\rho(s_0))^{\frac{1}{2}}\cdot(2^j\rho(s_0))^{-1}\\ &\leq C\rho(s_0)^{-1}\alpha(s_0) + C\ell + C\rho(s_0)^{-\frac{1}{2}}\|\tau_g(u)\|_{L^2(\mathcal{C}(\ell))}.\end{aligned}$$

3 Proof of Theorem 1.3

We now turn to the proof of our main results in the hyperbolic setting. So let u_i be a sequence of almost harmonic maps from hyperbolic cylinders $(\mathcal{C}(\ell_i), g_i = g_{\ell_i})$, $\ell_i \to 0$, with

$$\varepsilon_i := \|\tau_{g_i}(u_i)\|_{L^2(\mathcal{C}(\ell_i),g_i)}\ell_i^{-\frac{1}{2}} \to 0 \quad (3.1)$$

which converges to a full bubble branch as recalled in the introduction, see [4] for more detail. The choice of the numbers $\{s_i^m\}_{m=0}^{\bar{m}}$ in [4] ensures that there exists a number Λ_0 (allowed to depend on the specific sequence u_i) for which $\mathcal{A}(u_i) \subset \bigcup_m [s_i^m - \Lambda_0, s_i^m + \Lambda_0]$ for all i. As the bubble branches against which the shifted maps $u_i(\cdot - s_i^m)$, $1 \leq m \leq \bar{m} - 1$ converge are non-trivial and as we always have $\pm X(\ell_i) \in \mathcal{A}(u_i)$ we furthermore know that $\mathcal{A}(u_i) \cap [s_i^m - \Lambda_0, s_i^m + \Lambda_0] \neq \emptyset$ for every $m = 0, \dots, \bar{m}$ and sufficiently large i.

In the following we can thus use that

$$\operatorname{dist}(s, \mathcal{A}(u_i)) - \Lambda_0 \leq \min_m |s - s_i^m| \leq \operatorname{dist}(s, \mathcal{A}(u_i)) + \Lambda_0 \tag{3.2}$$

for every $s \in [-X(\ell_i), X(\ell_i)]$, and hence in particular that factors like $e^{-\operatorname{dist}(s,\mathcal{A}(u))}$ are bounded by $Ce^{-\min_m |s-s_i^m|}$. Here and in the following C denotes a constant that is allowed to depend on the specific sequence of maps u_i, but is of course independent of i.

We first need to decide which parts of $[s_i^{m-1}+\Lambda_0, s_i^m - \Lambda_0]\times S^1$ we want to include in the extended bubble regions around s_i^{m-1} respectively s_i^m and which (central) part of this cylinder we want to view instead as the connecting cylinder $[b_i^{m-1}, a_i^m] \times S^1$ between the extended bubble regions.

To begin with we look at the auxiliary sets

$$I_i^m := \{s \in [s_i^{m-1}, s_i^m] : \operatorname{dist}(s, \mathcal{A}(u_i)) \geq 4|\log(\ell_i)| + 1\}, \tag{3.3}$$

where here and in the following we can fix $m \in \{1, \dots, \bar{m}\}$ and only need to consider sufficiently large indices i. We can hence in particular assume that $4|\log(\ell_i)| + 1 \geq \Lambda_0 + 2$ which ensures that these sets are closed (possibly empty) intervals.

If I_i^m is not empty we furthermore let t_i^m be the element of I_i^m for which the conformal factor is minimal, i.e. set $t_i^m = 0$ if $0 \in I_i^m$ while $t_i^m = \pm \min_{I_i^m} |s|$ if $I_i^m \subset \mathbb{R}^\pm$.

If, after passing to a subsequence, either $I_i^m = \emptyset$ for all i or $I_i^m \neq \emptyset$ but

$$\rho_i(t_i^m)^{-1}\alpha_i(t_i^m) \to 0 \text{ as } i \to \infty \tag{3.4}$$

for $\alpha_i(s) := ⨍_{\{s\}\times S^1} |\partial_s u_i|$ then we set $b_i^{m-1} = a_i^m = \frac{1}{2}(s_i^{m-1} + s_i^m)$. In this case we hence end up with a trivial set as connecting cylinder and will prove that no neck forms between the bubble branches against which $u_i(\cdot - s_i^{m-1})$ respectively $u_i(\cdot - s_i^m)$ converge.

Conversely if, after passing to a subsequence,

$$\rho_i(t_i^m)^{-1}\alpha_i(t_i^m) \geq c_0 \text{ for all } i \text{ and some } c_0 > 0, \tag{3.5}$$

then we choose b_i^{m-1} and a_i^m as the minimal and maximal element of I_i^m for which

$$\rho_i(s)^{-1}\alpha_i(s) \geq \delta_i := \max(\varepsilon_i, \ell_i)^{\frac{1}{2}}. \tag{3.6}$$

As $\delta_i \leq c_0$ (for sufficiently large i) these numbers are well defined and satisfy $b_i^{m-1} \leq t_i^m \leq a_i^m$.

We note that in both cases (1.9) is satisfied since $s_i^m - s_i^{m-1} \to \infty$ and since (3.2) ensures that $\text{dist}(I_i^m, \{s_i^{m-1}, s_i^m\}) \geq 4|\log(\ell_i)| - \Lambda_0 \to \infty$ if $I_i^m \neq \emptyset$.

As a first step towards the proof of our main theorem we now want to establish that this choice of a_i^m and b_i^m ensures that (1.10) and (1.11) hold true, and hence that there is no loss of energy nor formations of necks on the extended bubble regions $B_i^m = [a_i^m, b_i^m] \times S^1$.

By symmetry it suffices to analyse the behaviour of u_i for $s \in [a_i^m, s_i^m - \Lambda]$. To this end we first consider the set $[a_i^m, s_i^m - \Lambda] \setminus I_i^m$ which is an interval of length no more than $O(|\log \ell_i|)$ whose distance from $\mathcal{A}(u)$ is at least $\Lambda - \Lambda_0$. Using additionally that the conformal factors ρ_i are uniformly bounded we hence obtain from Lemma 2.6 and our main assumption (3.1) that

$$E(u_i, ([a_i^m, s_i^m - \Lambda] \setminus I_i^m) \times S^1) \leq Ce^{-\Lambda} + C\ell_i\varepsilon_i^2 + C|\log \ell_i|(\ell_i + \varepsilon_i \ell_i^{\frac{1}{2}})$$

and

$$\operatorname*{osc}_{([a_i^m, s_i^m - \Lambda] \setminus I_i^m) \times S^1} u_i \leq Ce^{-\Lambda/2} + C\ell_i^{\frac{1}{2}}\varepsilon_i + C|\log \ell_i|(\ell_i^{\frac{1}{2}} + \varepsilon_i^{\frac{1}{2}} \ell_i^{\frac{1}{4}})$$

both tend to zero as $i \to \infty$ and $\Lambda \to \infty$.

It thus remains to bound the energy and oscillation of u_i on $(I_i^m \cap \{s \geq a_i^m\}) \times S^1$ and to this end we want to prove that

$$\int_{I_i^m \cap \{s \geq a_i^m\}} \alpha_i(s) ds \to 0 \text{ as } i \to \infty. \tag{3.7}$$

This is trivially true if $I_i^m = \emptyset$ and is also true if (3.4) holds as in this situation Lemma 2.7 implies that

$$\int_{I_i^m} \alpha_i(s) ds \leq C\rho_i(t_i^m)^{-1}\alpha_i(t_i^m) + C\ell_i + C\rho_i(t_i^m)^{-\frac{1}{2}}\ell_i^{\frac{1}{2}}\varepsilon_i \to 0 \text{ as } i \to \infty.$$

In the remaining case where (3.5) holds we can instead use that a_i^m is chosen so that $\rho_i^{-1}(s)\alpha_i(s) \leq \delta_i \to 0$ for all $s \in I_i^m$ with $s \geq a_i^m$. We can hence apply the analogue argument (with t_i^m replaced by the element $\tilde{t}_i^m$ of $I_i^m \cap \{s \geq a_i^m\}$ which minimises ρ) to conclude that also in this case (3.7) holds.

Having established (3.7), and hence that the oscillation of $\fint u_i(\cdot, \theta) d\theta$ over $I_i^m \cap \{s \geq a_i^m\}$ tends to zero, we immediately deduce that

$$\operatorname*{osc}_{(I_i^m \cap \{s \geq a_i^m\}) \times S^1} u_i \to 0$$

as the standard angular energy estimates recalled in Lemma 2.1 ensure that the oscillation of u_i over circles $\{s\} \times S^1$, $s \in I_i^m$, is bounded by $C\vartheta_i(s)^{\frac{1}{2}} \leq C\ell_i + C\ell_i^{\frac{1}{2}}\varepsilon_i \to 0$.

Combining (3.7) with Lemmas 2.1 and 2.3 furthermore allows us to deduce that

$$E((I_i^m \cap \{s \geq a_i^m\}) \times S^1) \leq C \int_{I_i^m \cap \{s \geq a_i^m\}} \alpha_i^2 + \vartheta_i + \int_{S^1} |u_{s\theta}|^2 d\theta\, ds$$
$$\leq C \int_{I_i^m \cap \{s \geq a_i^m\}} \alpha_i ds + C\ell_i^2 + C\ell_i \varepsilon_i^2 \to 0,$$

since Lemma 2.2 and the trace theorem ensure that the α_i are uniformly bounded away from $\mathcal{A}(u_i)$.

This completes the analysis of the maps u_i on the extended bubble regions and it remains to study the behaviour of the curves $\hat{u}_i$, and their reparametrisations v_i^m, on the intervals (b_i^{m-1}, a_i^m).

We recall that these intervals are non-empty only if (3.5) holds and note that by symmetry we can assume without loss of generality that $|a_i^m| \geq |b_i^{m-1}|$ and hence that $\rho \leq \rho(a_i^m)$ on $[b_i^{m-1}, a_i^m]$.

As the choice of a_i^m ensures that $\alpha_i(a_i^m) \geq \rho_i(a_i^m)\delta_i$ and as $\ell_i \leq 2\pi\rho_i$ we can first apply the estimate (2.30) from Lemma 2.7 to see that for every $s \in [b_i^{m-1}, a_i^m]$

$$\alpha_i(s) \geq \alpha_i(a_i^m) - C\ell_i^2 - C\rho_i(a_i^m)^{\frac{1}{2}} \|\tau_{g_i}(u_i)\|_{L^2}$$
$$\geq \rho_i(a_i^m)\big[\delta_i - C\ell_i - C\varepsilon_i\big] = (1 - o(1))\rho_i(a_i^m)\delta_i \geq (1 - o(1))\rho_i(s)\delta_i \tag{3.8}$$

where we use in the penultimate step that $\delta_i = (\max(\varepsilon_i, \ell_i))^{\frac{1}{2}} \gg \max(\varepsilon_i, \ell_i)$.

As $R_{u_i}(s) + \vartheta_i(s) \leq CR_{u_i}(s) \leq C\rho_i^2(s)\ell_i\varepsilon_i^2 + C\ell_i^4$, compare (2.20) and Lemma 2.1, we can use Corollary 2.5 to see that

$$|\hat{u}_i'| \geq (1 - C\vartheta_i^{\frac{1}{2}})\alpha_i - C\ell_i^2 - C\rho_i\ell_i^{\frac{1}{2}}\varepsilon_i \geq (1 - o(1) - C\ell_i\delta_i^{-1} - C\ell_i^{\frac{1}{2}}\delta_i^{-1}\varepsilon_i)\alpha_i$$
$$\geq (1 - o(1))\alpha_i \tag{3.9}$$

on $[b_i^{m-1}, a_i^m]$. In particular, for all sufficiently large i, we have

$$|\hat{u}_i'(s)| \geq \tfrac{3}{4}\alpha_i(s) \geq \frac{1}{2}\delta_i\rho_i(s) \text{ for } s \in [b_i^{m-1}, a_i^m]. \tag{3.10}$$

Next we prove that the (euclidean) tension of the curve $\hat{u}_i$ is bounded by

$$|\tau_{g_E}(\hat{u}_i)| \leq C \int_{S^1} |\tau_{g_E}(u_i)| d\theta + C\vartheta_i^{\frac{1}{2}}\left[|\hat{u}_i'|^2 + |\hat{u}_i'| \int_{S^1} |\partial_{s\theta} u_i| d\theta + \vartheta_i\right]. \tag{3.11}$$

To see this we first use that $\hat{u}_i' = d\pi_N(\bar{u}_i)(\bar{u}_i')$ for $\bar{u}_i(s) = ⨍_{\{s\}\times S^1} u_i$ and that $-d^2\pi(p)|_{T_pN \times T_pN} = A(p)$, $p \in N$, to write

$$\tau_{g_E}(\hat{u}_i) = \hat{u}_i'' + A(\hat{u}_i)(\hat{u}_i', \hat{u}_i') = d\pi_N(\bar{u}_i)(\bar{u}_i'') + d^2\pi_N(\bar{u}_i)(\bar{u}_i', \bar{u}_i') - d^2\pi_N(\hat{u}_i)(\hat{u}_i', \hat{u}_i'). \tag{3.12}$$

As $|\bar{u}_i - \hat{u}_i| \leq \operatorname{osc}_{S^1} u_i \leq C\vartheta_i^{\frac{1}{2}}$ and as $|\bar{u}_i' - \hat{u}_i'| \leq C\vartheta_i^{\frac{1}{2}}\alpha_i \leq C\vartheta_i^{\frac{1}{2}}|\hat{u}_i'|$, compare (2.13) and (3.10), we can bound

$$|d^2\pi_N(\bar{u}_i)(\bar{u}_i', \bar{u}_i') - d^2\pi_N(\hat{u}_i)(\hat{u}_i', \hat{u}_i')| \leq C\vartheta_i^{\frac{1}{2}}|\hat{u}_i'|^2,$$

so it remains to estimate the first term in (3.12).

As $\bar{u}_i''(s) = \fint_{S^1} \partial_{ss} u_i = \fint_{S^1} \Delta_{g_E} u_i = \fint_{S^1} \tau_{g_E}(u_i) + A(u_i)(\nabla u_i, \nabla u_i)$ and as $d\pi_N(u_i)(A(u_i)(\nabla u_i, \nabla u_i)) = 0$ we have

$$\begin{aligned}|d\pi_N(\bar{u}_i)(\bar{u}_i'')| &\leq C\int_{S^1} |\tau_{g_E}(u_i)| + \|u_i - \bar{u}_i\|_{L^\infty(S^1)} \int_{S^1} |\nabla u_i|^2 \\ &\leq C\int_{S^1} |\tau_{g_E}(u_i)| + C\vartheta_i^{\frac{1}{2}} \int_{S^1} |\nabla u_i|^2.\end{aligned}$$

This yields the claimed estimate (3.11) as $\alpha_i = \fint |\partial_s u_i| \leq 2|\hat{u}_i'|$ for all sufficiently large i and as we can hence bound

$$\begin{aligned}\int_{S^1} |\nabla u|^2 &\leq \vartheta_i + C\alpha_i \|\partial_s u_i\|_{L^\infty(S^1)} \leq \vartheta_i + C\alpha_i(\alpha_i + \int_{S^1} |\partial_{s\theta} u_i|) \\ &\leq \vartheta_i + C|\hat{u}_i'|(|\hat{u}_i'| + \int_{S^1} |\partial_{s\theta} u_i|).\end{aligned}$$

We now reparametrise $\hat{u}_i|_{[b_i^{m-1}, a_i^m]}$ by arclength and estimate the tension of the resulting curve $v_i^m = \hat{u}_i \circ s_i^m$, which is well defined as (3.10) ensures that $\hat{u}_i' \neq 0$ on this interval. To lighten the notation we write for short $v_i = v_i^m$ and $s_i = s_i^m$.

So let $c_i^m = \frac{1}{2}\int_{b_i^{m-1}}^{a_i^m} |\hat{u}_i'| ds$ and let $s_i : [-c_i^m, c_i^m] \to [b_i^{m-1}, a_i^m]$ be the increasing bijection with $|\dot{s}_i|^2 \cdot |\hat{u}_i' \circ s_i|^2 = 1$. Differentiating this relation and using that $\hat{u}_i'' \cdot \hat{u}_i' = \tau_{g_E}(\hat{u}_i)\hat{u}_i'$ yields

$$|\ddot{s}_i| \leq |\dot{s}_i|^2 |\hat{u}_i' \circ s_i|^{-1} |\tau_{g_E}(\hat{u}_i) \circ s_i| = |\hat{u}_i' \circ s_i|^{-3} |\tau_{g_E}(\hat{u}_i) \circ s_i|.$$

As $\tau_{g_E}(v_i) = |\dot{s}_i|^2 \tau_{g_E}(\hat{u}_i) \circ s_i + \ddot{s}_i \hat{u}_i' \circ s_i$ we can thus use (3.11) to bound

$$\begin{aligned}&|\tau_{g_E}(v_i) \circ s_i^{-1}| \leq 2|\hat{u}_i'|^{-2} |\tau_{g_E}(\hat{u}_i)| \\ &\leq C|\hat{u}_i'|^{-2} \int_{S^1} |\tau_{g_E}(u_i)| d\theta + C\vartheta_i^{\frac{1}{2}} \Big(1 + |\hat{u}_i'|^{-2}\vartheta_i + |\hat{u}_i'|^{-1} \int_{S^1} |\partial_{s\theta} u_i| d\theta\Big).\end{aligned}$$

For any $p \in [1, 2]$ we can hence estimate

$$\begin{aligned}&\|\tau_{g_E}(v_i)\|^p_{L^p([-c_i^m, c_i^m])} = \int |\hat{u}_i'| |\tau_{g_E}(v_i) \circ s_i^{-1}|^p ds \\ &\leq C\int |\hat{u}_i'|^{-2p+1} |\tau_{g_E}(u_i)|^p + \vartheta_i^{\frac{p}{2}} |\hat{u}_i'| + |\hat{u}_i'|^{-2p+1} \vartheta_i^{\frac{3p}{2}} + |\hat{u}_i'|^{-p+1} |\partial_{s\theta} u_i|^p ds d\theta\end{aligned}$$

where we integrate over $s \in [b_i^{m-1}, a_i^m]$ and $\theta \in S^1$. Combined with the lower bound (3.10) on the velocity of $\hat{u}_i$ and the fact that $\rho_i \geq c\ell_i$ we hence get

$$\begin{aligned}\|\tau_{g_E}(v_i)\|^p_{L^p([-c_i^m,c_i^m])} &\leq C\delta_i^{-2p+1}\ell_i^{1-p}\int(\rho_i^{-1}|\tau_{g_E}(u_i)|)^p ds d\theta + CE(\hat{u}_i)^{\frac{1}{2}}(\int \vartheta_i^p ds)^{\frac{1}{2}}\\ &+ C\delta_i^{-2p+1}\int \rho_i^{-(2p-1)}\vartheta_i^{\frac{3p}{2}} ds + C\delta_i^{-(p-1)}\int \rho_i^{-(p-1)}|u_{s\theta}|^p ds d\theta\\ &=: C(T_i^{(1)} + T_i^{(2)} + T_i^{(3)} + T_i^{(4)})\end{aligned}$$

and we will show that each of these terms tends to zero as $i \to \infty$.

First of all, the relation (2.17) between the euclidean and the hyperbolic tension and our main assumption (3.1) imply that

$$T_i^{(1)} \leq \delta_i^{-2p+1}\ell_i^{1-p}(2X(\ell_i))^{1-\frac{p}{2}}\|\tau_{g_i}(u_i)\|^p_{L^2(\mathcal{C}(\ell_i))} \leq C\delta_i^{-2p+1}\varepsilon_i^p \leq C\delta_i \to 0.$$

Since the endpoints of our interval $[b_i^{m-1}, a_i^m]$ have distance at least $4|\log \ell_i)| - \Lambda_0$ from the high energy set we can furthermore use Lemmas 2.1 and 2.3 as well as (2.20) to bound

$$\vartheta_i(s) + \int_{\mathscr{C}_1(s)} |\partial_{s\theta}u_i|^2 \leq C\rho_i^2\ell_i\varepsilon_i^2 + C\ell_i^2 e^{-\min(s-b_i^{m-1}, a_i^m - s)} \text{ for all } s \in [b_i^{m-1}, a_i^m]. \tag{3.13}$$

As $E(\hat{u}_i) \leq CE(u_i) \leq C$ this immediately implies that $T_i^{(2)} \to 0$ while also

$$T_i^{(3)} \leq C\delta_i^{-2p+1}\ell_i^{\frac{3p}{2}}\varepsilon_i^{3p}\int \rho_i^{p+1} + C\delta_i^{-2p+1}\ell_i^{p+1} \to 0.$$

Similarly (3.13) implies that

$$T_i^{(4)} \leq C\delta_i^{-(p-1)}\varepsilon_i^p\ell_i^{\frac{p}{2}}\int \rho_i + C\delta_i^{-(p-1)}\ell_i \leq C\ell_i^{\frac{p}{2}}|\log \ell_i| + C\delta_i^{-1}\ell_i \to 0$$

where we use in the penultimate step that $\int_{-X(\ell)}^{X(\ell)} \rho_\ell ds \leq C|\log \ell|$.

We hence conclude that $\|\tau(v_i)\|_{L^p([-c_i^m,c_i^m])} \to 0$ for every $p \in [1,2]$ as claimed in Theorem 1.3.

After passing to a subsequence we can now assume that $c_i^m \to c \in [0,\infty]$. If $c = 0$ then the curves simply collapse to a point so there is nothing to show.

If instead $c \in (0,\infty)$ then setting $\tilde{v}_i = v_i(\frac{c_i^m}{c}\cdot)$ we obtain curves in N that are parametrised over a fixed interval $I = [-c, c]$ with constant velocity $\lambda_i = \frac{c_i^m}{c} \to 1$ which satisfy

$$\tilde{v}_i'' = -A(\tilde{v}_i)(\tilde{v}_i', \tilde{v}_i') + g_i \tag{3.14}$$

for functions $g_i = \tau(\tilde{v}_i)$ with $\|g_i\|_{L^2(I)} = \lambda_i^{3/2}\|\tau(v_i)\|_{L^2([-c_i^m,c_i^m])} \to 0$. This implies in particular that the sequence $\tilde{v}_i$ is bounded in $W^{2,2}(I)$ and hence that it subconverges to a limit v_∞ weakly in $W^{2,2}(I)$ and strongly in $C^{1,\alpha}(I)$ for $\alpha < \frac{1}{2}$. Hence the right hand side of (3.14) converges strongly in L^2 to $-A(v_\infty)(\nabla v_\infty, \nabla v_\infty)$ which allows us to deduce that $\tilde{v}_i$ converges indeed strongly in $W^{2,2}$ to the limit v_∞ which satisfies $\tau_{g_E}(v_\infty) = v_\infty'' + A(v_\infty)(v_\infty, v_\infty) = 0$, i.e. is a geodesic.

Similarly, if $c_i^m \to \infty$ then we can apply the above argument locally on $[0, \infty)$ to conclude that the curves $v_i^m(\cdot - c_i^m)$ and $v_i^m(c_i^m - \cdot)$ subconverge strongly in $W^{2,2}_{loc}$ to infinite length geodesics.

4 Almost Harmonic Maps from Degenerating Tori

We recall that the moduli space of the torus is given by the quotients of $(\mathbb{C}, g_E)$ with respect to the lattices that are generated by 2π and $A + iB$ for $A \in (-\pi, \pi]$ and $B > 0$ with $|A + iB| \geq 2\pi$ (with strict inequality for $A < 0$).

We can thus view maps u_i from degenerating unit area tori (T^2, g_i) as maps from $\mathbb{R} \times S^1$ equipped with metric $g_i = \rho_i^2 g_E$ for $\rho_i = (2\pi B_i)^{-\frac{1}{2}}$ with periodicity $u_i(s, \theta) = u_i(s + B_i, \theta + A_i)$.

Given any closed curve $\alpha \in C^2(S^1, N)$ we can hence consider the maps $u_i(s, \theta) = \alpha(\frac{2\pi s}{B_i})$ whose tension decays according to

$$\|\tau_{g_i}(u_i)\|^2_{L^2(T^2, g_i)} \leq C \int_0^{B_i} \rho_i^{-2} B_i^{-4} = C B_i^{-2} = C\ell_i^4 \tag{4.1}$$

to see that we cannot expect almost harmonic maps to converge to *geodesic* bubble trees if the tension decays no faster than $\operatorname{inj}(M, g_i)^2$. Here ℓ_i denotes the length of the shortest closed geodesic in the domain (T^2, g_i) which is given by $\ell_i = 2\operatorname{inj}(M, g_i) = 2\pi\rho_i$ as $|A_i + iB_i| \geq 2\pi$.

The proof of our main result in the hyperbolic case now carries over to the new setting of maps from tori with only minor modifications and indeed simplifies since the conformal factors $\rho_i = \frac{\ell_i}{2\pi}$ are constant. To explain the necessary modifications we first note that $\|\tau_{g_i}(u_i)\|_{L^2(T^2, g_i)} = o(\ell_i^2)$ is equivalent to asking that $\|\tau_{g_E}(u_i)\|_{L^2([0,B_i]\times S^1, g_E)} = \rho_i \|\tau_{g_i}(u_i)\|_{L^2([0,B_i]\times S^1, g_i)} = o(\ell_i^3)$.

Defining the intervals I_i^m now so that $\operatorname{dist}(I_i^m, \{s_i^{m-1}, s_i^m\}) = 8|\log \ell_i|$ we then need to distinguish between the case where either $I_i^m = \emptyset$ or $\ell_i^{-2} \sup_{I_i^m} \alpha_i \to 0$ and the case where $\sup_{I_i^m} \ell_i^{-2} \alpha_i \geq c_0 > 0$. In the former situation we again set $b_i^{m-1} = a_i^m = \frac{1}{2}(s_i^{m-1} + s_i^m)$ so end up with a trivial connecting cylinder, while in the later case we choose a_i^m and b_i^{m-1} as the maximal and minimal elements of I_i^m with $\ell_i^{-2}\alpha_i(s) \geq \delta_i$ where we again set $\delta_i = \min(\varepsilon_i, \ell_i)^{\frac{1}{2}}$, now for $\varepsilon_i := \ell_i^{-3}\|\tau_{g_E}(u_i)\|_{L^2([0,B_i]\times S^1, g_E)} \to 0$. As $B_i \leq C\ell_i^{-2}$ we hence again get that $\int_{I_i^m \setminus [b_i^{m-1}, a_i]} \alpha_i \to 0$ and we can argue exactly as in the previous proof to obtain (1.10) and (1.11), i.e. to exclude the loss of energy or formation of necks on the extended bubble regions.

Integrating (2.11) over the interval $[b_i^{m-1}, a_i^m]$ whose length is bounded by $C\ell_i^{-2}$ and using that $\int_{I_i^m} R_{u_i}^{\frac{1}{2}} \leq C\ell_i^4 + C\varepsilon_i \ell_i^3 B_i^{\frac{1}{2}} \leq C(\ell_i^2 + \varepsilon_i)\ell_i^2$ then allows us to conclude that $\alpha_i \geq (1 - o(1))\delta_i \ell_i^2$ on this interval and hence that also $|\hat{u}_i'| \geq (1 - o(1))\delta_i \ell_i^2$ on the connecting cylinder. This allows us to carry out the rest of the proof exactly as in the hyperbolic case to complete the proof of Theorem 1.4.

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