Large Data Problems in
Fluid Dynamics and General
Relativity

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Contents

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Declaration

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at another university. Section 4 is joint work with Shengguo Zhu [8] and Section 5 is joint work with Xinliang An [4]. Both pieces of work have been submitted to ArXiv.

Abstract

In this thesis we contribute to the study of two large data problems within the realm of Hyperbolic Partial Differential Equations. The first result concerns the *Relativistic Euler equations*, which adequately describe fluid motion in the context of special relativity. In specially chosen Eulerian coordinates and on a fixed $(d+1)-$dimensional Minkowski background, they take the following form:

$$
\begin{cases}
\left(\dfrac{\rho + P|u|^2/c^4}{1 - |u|^2/c^2}\right)_t + \operatorname{div}\left(\dfrac{(\rho + P/c^2)u}{1 - |u|^2/c^2}\right) = 0, \\[2mm]
\left(\dfrac{(\rho + P/c^2)u}{1 - |u|^2/c^2}\right)_t + \operatorname{div}\left(\dfrac{(\rho + P/c^2)}{1 - |u|^2/c^2}\, u \otimes u\right) + \nabla P = 0.
\end{cases}
\tag{1}
$$

The constitutive relation $P = P(\rho)$ considered throughout most of this work is

$$
P(\rho) = k^2 \rho^\gamma,
\tag{2}
$$

for fixed constants $k > 0$ and $\gamma > 1$. The main result is the provision of a both necessary and sufficient condition on the initial data, uniformly away from vacuum, for the formation of singularities in finite time under the evolution of (1) in the case $d = 1$. This is joint work with Shengguo Zhu.

The second project developed in this thesis shifts focus from the area of conservation laws to that of mathematical General Relativity. We provide, in joint work with Xinliang An, a trapped surface formation criterion for the Einstein–Maxwell system

$$
R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu},
$$

where the right–hand side corresponds to the Maxwell energy–momentum tensor that models electromagnetic effects. In particular,

$$
T_{\mu\nu} = F_{\mu\alpha}F_\nu^{\ \alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},
$$

where F is an antisymmetric 2–form signifying the field strength of electromagnetism. This project builds on previous work of An [5] for the vacuum equations and generalizes a previous work of Yu [88] in terms of the size of the existence region.

Chapter 1

Overview of the results presented in this thesis

This thesis contains results on two problems within the area of Hyperbolic Partial Differential Equations. One concerns the Relativistic Euler equations and the other the Einstein–Maxwell system. What follows is a presentation of these results.

1.1 Relativistic Euler equations

Part of my doctorate has been towards establishing necessary and sufficient criteria for the blowup of C^1 solutions to the relativistic Euler equations (in the isentropic case) for 1 spatial dimension. An archetypal phenomenon in the study of hyperbolic systems of conservation laws is the development of singularities (in particular shocks) in finite time, no matter how smooth or small the initial data are. A series of works by Lax [46], John [40] et al. confirmed that for some important systems, when the initial data is a smooth small perturbation of a constant state, singularity formation in finite time is equivalent to the existence of compression in the initial data. In collaboration with Shengguo Zhu [8], we have established a necessary and sufficient criterion for the formation of singularities to the relativistic Euler equations even for large initial data uniformly away from vacuum. This criterion essentially comes down to examining whether or not there exists compression in the initial data, i.e. whether or not there exists an initial tendency of the characteristics that "drive" the solution to fall into each other. Compared to previous works, a novel idea for calculating a strong enough lower bound on the energy density was required.

1.1.1 The work with Zhu

Consider the isentropic, relativistic Euler equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(\dfrac{\rho + Pu^2/c^4}{1 - u^2/c^2}\Big) + \mathrm{div}\Big(\dfrac{(\rho + P/c^2)u}{1 - u^2/c^2}\Big) = 0, \\ \dfrac{\partial}{\partial t}\Big(\dfrac{(\rho + P/c^2)u}{1 - u^2/c^2}\Big) + \mathrm{div}\Big(\dfrac{(\rho + P/c^2)}{1 - u^2/c^2} u \otimes u\Big) + \nabla P = 0. \end{cases} \tag{1.1}$$

Here and throughout, $\rho \geq 0$ denotes the mass–energy density, $u = (u^{(1)}, ..., u^{(d)})^\top \in \mathbb{R}^d$ denotes the relativistic velocity, $d \geq 1$ is the dimension of the space, $c > 0$ is a sufficiently large constant, the speed of light, P denotes the pressure of the fluid, $x = (x^{(1)}, ..., x^{(d)})^\top \in \mathbb{R}^d$ denotes the Eulerian spatial coordinate and $t \in \mathbb{R}^+$ denotes the time coordinate. The constitutive relation of $P = P(\rho)$ we shall be working with is

$$P(\rho) = k^2 \rho^\gamma, \tag{1.2}$$

for both fixed constants $k > 0$ and $\gamma > 1$.

Let $d = 1$ in (1.1) . We first define the classical solutions as follows:

Definition 1. *Let $T > 0$ be some time. The pair $(\rho(t,x), u(t,x))$ is called a C^1 solution to the relativistic Euler equations (1.1) on $(0,T) \times \mathbb{R}$ if*

$$\rho > 0, \quad \rho \in C^1([0,T) \times \mathbb{R}), \ u \in C^1([0,T) \times \mathbb{R}),$$

and the equations (1.1)-(1.2) are satisfied in the pointwise sense on $(0,T) \times \mathbb{R}$. It is called a C^1 solution to the Cauchy problem (1.1) with given initial data (ρ_0, u_0) if it is a C^1 solution to the equations (1.1) on $(0,T) \times \mathbb{R}$ and admits the initial data .

It is well-known[1] that there exists a local-in-time C^1 solution (ρ, u) in $[0,T] \times \mathbb{R}$ for some $T > 0$, when $\inf_{x \in \mathbb{R}} \rho_0 > 0$ and

$$(\rho_0, \ u_0) \in C^1(\mathbb{R}). \tag{1.3}$$

The two eigenvalues λ_1 and λ_2 of equations (1.1) are given by

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \frac{u\sqrt{P'}}{c^2}} \qquad \text{and} \qquad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \frac{u\sqrt{P'}}{c^2}}. \tag{1.4}$$

We denote the directional derivatives as

$$\prime = \partial_t + \lambda_1 \partial_x, \quad \backslash = \partial_t + \lambda_2 \partial_x,$$

---

[1] See [82] and the references cited therein.

along the two characteristic directions

$$\frac{dx^1}{dt} = \lambda_1 \quad \text{and} \quad \frac{dx^2}{dt} = \lambda_2,$$

respectively and introduce the corresponding Riemann variables

$$w = \frac{c}{2}\ln\left(\frac{c+u}{c-u}\right) + \int_0^\rho \frac{\sqrt{P'(s)}}{s + \frac{P(s)}{c^2}}\,ds = r(u) + s(\rho), \tag{1.5}$$

$$z = \frac{c}{2}\ln\left(\frac{c+u}{c-u}\right) - \int_0^\rho \frac{\sqrt{P'(s)}}{s + \frac{P(s)}{c^2}}\,ds = r(u) - s(\rho). \tag{1.6}$$

Then, it is easy to see that w and z satisfy

$$w^\backslash = 0 \quad \text{and} \quad z' = 0. \tag{1.7}$$

Let h_1 and h_2 be functions satisfying

$$h_{1w} = \frac{\lambda_{1w}}{\lambda_1 - \lambda_2}, \quad h_{2z} = \frac{\lambda_{2z}}{\lambda_2 - \lambda_1}. \tag{1.8}$$

Define $\alpha = z_x, \beta = w_x$ and introduce

$$\tilde{\alpha} = e^{h_1}\alpha, \quad \tilde{\beta} = e^{h_2}\beta. \tag{1.9}$$

We continue with a simplification for $s(\rho)$, as can be found for example in [16],

$$s(\rho) = \frac{2c\sqrt{\gamma}}{\gamma - 1}\arctan\left(\frac{k\rho^{(\gamma-1)/2}}{c}\right). \tag{1.10}$$

Another important calculation is the following expression for $\sqrt{P'(\rho)}$ in terms of the Riemann invariants:

$$\sqrt{P'(\rho)} = k\sqrt{\gamma}\rho^{(\gamma-1)/2} = c\sqrt{\gamma}\tan\left[\frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}\right]. \tag{1.11}$$

1.1.1.1 Statement of main results

Our main contribution in this work takes the following form:

Assumption 1. *Assume that* $\rho_0(x), u_0(x)$ *are* C^1 *functions and*

- *there exists a uniform positive constant* M_0 *such that*

$$\|(\rho_0, u_0, z_0, w_0)(x)\|_{C^1} \le M_0;$$

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- *there exists a uniform positive constant ϵ such that $\inf_x(w_0 - z_0)(x) \geq \epsilon$;*

- *there holds*

$$w_{max} - z_{min} < \frac{4c\sqrt{\gamma}}{\gamma - 1} arctan\Big(\frac{1}{\sqrt{\gamma}}\Big),$$

where

$$w_{max} = \sup\Big\{w_0(x) \mid x \in \mathbb{R}\Big\}, \ z_{min} = \inf\Big\{z_0(x) \mid x \in \mathbb{R}\Big\}.$$

Theorem 1.1.1. *For $\gamma > 1$ and $d = 1$, if $(\rho_0(x), u_0(x))$ satisfy the conditions in Assumption 1, then the Cauchy problem (1.1) with initial data (w_0, z_0) has a unique global-in-time classical solution if and only if*

$$z_x(x, 0) \geq 0 \quad and \quad w_x(x, 0) \geq 0, \ for \ all \ \ x \in \mathbb{R}. \tag{1.12}$$

1.2 The work with An

1.2.1 Statement of main results

Our focus of study is the Einstein–Maxwell system

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu},$$

where

$$T_{\mu\nu} = F_{\mu\alpha}F_\nu^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},$$

for an anti–symmetric 2-form F. As is the norm for trapped surface formation problems, we work in a characteristic initial value problem setting. As we shall be imposing trivial data along the initial incoming hypersurface, the freedom in our initial data is given by the free specification of two parameters: the *shear* $\hat{\chi}$ and the 1–form α_F for the electromagnetic field along the outgoing hypersurface. In contrast to problems in spherical symmetry with non-trivial matter fields such as that of a scalar field, the main difficulty in establishing a formation of trapped surfaces statement is to prove the existence of the spacetime with initial data that are large in a controlled sense. This is the content of the first of two Theorems that we establish:

Theorem 1.2.1. *Consider the following characteristic initial value problem for the Einstein–Maxwell system*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu},$$

where

$$T_{\mu\nu} = F_{\mu\alpha}F_\nu^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},$$

for an anti–symmetric 2-form F. The initial incoming hypersurface $\underline{H}_0$ is required to coincide with a backwards light cone in Minkowski space. Along the outgoing hypersurface $u = u_\infty$, $0 \leq \underline{u} \leq 1$ we prescribe initial data $\hat{\chi}, \alpha_F$. Given $\mathcal{I}^{(0)}$, there exists a sufficiently large $a_0 = a_0(\mathcal{I}^{(0)})$ such that the following holds. For any $0 < a_0 < a$ and with smooth initial data $(\hat{\chi}, \alpha_F)$ satisfying

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{1}{2}} \|\nabla_4^k (|u_\infty|\nabla)^i (\hat{\chi}, \alpha_F)\|_{L^\infty(S_{u_\infty, \underline{u}})} \leq \mathcal{I}^{(0)}$ *along $u = u_\infty$,*

then the Einstein–Maxwell equations admit a unique smooth solution in the region

$$u_\infty \leq u \leq -a/4, \ \ 0 \leq \underline{u} \leq 1.$$

The second one steps directly on the first one and is a formation of trapped surfaces statement.

Theorem 1.2.2. *Given $\mathcal{I}^{(0)}$, there exists a sufficiently large $a_0 = a_0(\mathcal{I}^{(0)})$ such that the following holds. For any $0 < a_0 < a$, the unique smooth solution $(\mathcal{M}, g)$ of the Einstein–Maxwell equations from Theorem 1.2.1 with initial data satisfying*

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{1}{2}} \|\nabla_4^k (|u_\infty|\nabla)^i (\hat{\chi}, \alpha_F)\|_{L^\infty(S_{u_\infty, \underline{u}})} \leq \mathcal{I}^{(0)}$ *along $u = u_\infty$,*

- *Minkowskian initial data along $\underline{u} = 0$,*

- $\int_0^1 |u_\infty|^2 \left(|\hat{\chi}_0|^2 + |\alpha_{F0}|^2\right)(u_\infty, \underline{u}') \, d\underline{u}' \geq a$ *uniformly for every direction along $u = u_\infty$,*

has a trapped surface at $S_{-a/4,1}$.

This constitutes a trapped surface formation theorem for the Einstein–Maxwell system in a region close to past null infinity.

Chapter 2

Introduction to the theory of hyperbolic conservation laws

2.1 What is a conservation law

An admirable aspect of the mathematical modelling of physical processes is that many a physical system can be adequately described by pinpointing quantities that the system conserves upon change; and formalizing them in a mathematical way. As C. Dafermos points out in [29], perhaps the earliest examples of such kind of thinking can be traced back to the works of natural philosophers in the eighteenth century. It was the study, in particular, of the system of *gas dynamics* that started to form the skeleton for what would come to be the area of study of quasilinear hyperbolic systems in divergence form, namely hyperbolic conservation laws.

But how, in essence, should we think of a hyperbolic conservation law? Simply as a statement that stipulates that the time rate of change in the amount of an extensive quantity stored inside any subdomain of a body is balanced by the rate of flux of this quantity through the boundary of the subdomain (in the absence of any rate of production of the quantity inside the subdomain, otherwise we are dealing with a hyperbolic balance law). The point is that, in the absence of any such internal production, a quantity is conserved. The special feature that renders continuum physics amenable to analytical treatment is that, under quite natural assumptions, statements of gross balance, as above, reduce to field equations, meaning partial differential equations in divergence form.

2.2 The compressible Euler equations

The Euler equations of compressible fluid dynamics, a system of partial differential equations, have been a source of mathematical and physical interest since the formulation of the conservation of mass equation by Euler in the 1750s. The equations are the conservation of mass, momentum and energy for the motion of a compressible fluid and form an archetypal system for the theory of (hyperbolic) conservation laws. From here onwards in this section, we choose to follow the exposition in [15] closely. In d (classical) space dimensions, these equations take the form of $d + 2$ conservation laws:

$$
\begin{cases}
\partial_t \rho + \nabla \cdot \mathbf{m} = 0, \\
\partial_t \mathbf{m} + \nabla \cdot \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p = 0, \\
\partial_t E + \nabla \cdot \left(\frac{\mathbf{m}}{\rho} (E + p) \right) = 0.
\end{cases}
\tag{2.1}
$$

Here $(\mathbf{x}, t) \in \mathbb{R}^{d+1}_+$, where by $\mathbb{R}^{d+1}_+$ we mean $\mathbb{R}^d \times (0, \infty)$ and with initial data

$$
(\rho, \mathbf{m}, E)\big|_{t=0} = (\rho_0, \mathbf{m}_0, E_0).
\tag{2.2}
$$

System $(2.1) - (2.2)$ does not close by itself, but relies on the constitutive equations

$$
p = p(\rho, e), \quad E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\rho} + \rho e.
\tag{2.3}
$$

The physical meaning behind the variables is as follows. We have that $\tau = \frac{1}{\rho}$ is the deformation gradient (specific volume for fluids, strain for solids), $\mathbf{v} = (v_1, \ldots, v_d)^\top$ is the $d-$dimensional velocity, $\rho \mathbf{v} = \mathbf{m}$ is the momentum vector, p is the scalar pressure and E is the total energy. By e we denote the internal energy of the system, which is a given function of (τ, p) or (ρ, p) defined through thermodynamical relations. By $\mathbf{a} \otimes \mathbf{b}$ we mean the tensor product of the two vectors $\mathbf{a}$ and $\mathbf{b}$.

An alternative expression for constitutive relations

Another way to close the system above in terms of constitutive assumptions is to notice that there are another two variables of interest, the *temperature* θ and the *entropy* S. If (ρ, S) are chosen as the independent variables, the constitutive relations can be written in the following form

$$
(e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S)),
\tag{2.4}
$$

governed by the First Law of Thermodynamics

$$\theta \, dS = de + p \, d\tau = de - \frac{p}{\rho^2} \, d\rho. \tag{2.5}$$

For a so-called *polytropic* gas, there exist constants $R, c_v > 0$ with

$$p = R\rho\theta, \ \ e = c_v\theta, \ \ \gamma = 1 + \frac{R}{c_v} \tag{2.6}$$

and

$$p = p(\rho, S) = \kappa\rho^\gamma e^{S/c_v}, \ \ e = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}e^{S/c_v}. \tag{2.7}$$

The constant R can be taken to be the universal gas constant divided by the effective molecular weight of the particular gas, $c_v > 0$ is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent and $\kappa > 0$ can be any constant under scaling.

The equations in the isentropic case

A significant simplification to the system (2.1)-(2.2) occurs upon the imposition of the condition that the flow at question be *isentropic*. The Euler equations then take the following simpler form

$$\begin{cases} \partial_t\rho + \nabla \cdot \mathbf{m} = 0, \\ \partial_t\mathbf{m} + \nabla \cdot \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho}\right) + \nabla p = 0. \end{cases} \tag{2.8}$$

This time the pressure is regarded as a function of the density (and implicitly of a constant entropy S_0, so that $p = p(\rho, S_0)$). We can briefly motivate the derivation/form of (2.8) in the following simple way: It is well-known that, for smooth solutions to (2.1), the entropy $S(\rho, E)$ is conserved along the trajectories of fluid particles:

$$\partial_t(\rho S) + \nabla \cdot (\mathbf{m}S) = 0. \tag{2.9}$$

Moreover, the entropy conservation equation (2.9) is formally equivalent to the energy equation

$$\partial_t E + \nabla \cdot \left(\frac{\mathbf{m}}{\rho}(E + p)\right) = 0$$

for smooth solutions. This leads to the realization that, as long as one speaks for smooth solutions and one imposes the initial condition

$$S|_{t=0} = \text{constant},$$

the value of S in the slab of existence of a smooth solution simply propagates, thus decoupling the third equation from the first two in (2.1).

In the one-dimensional case, system (2.8) rewrites in Eulerian coordinates as

$$\begin{cases} \partial_t \rho + \partial_x (p\,v) = 0, \\ \partial_t (\rho\,v) + \partial_x \left(\rho\,v^2 + p \right) = 0, \end{cases} \tag{2.10}$$

while in Lagrangian coordinates it rewrites as

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0. \end{cases} \tag{2.11}$$

While the Eulerian and Lagrangian x-coordinates are different, we shall not distinguish them here.

2.2.1 Well-posedness of the Cauchy problem

In the previous section we discussed the process that translates the study of gas dynamics to a mathematical language through the Euler equations. Such a conversion would be devoid of meaning were it not to come with several benefits. Thankfully, it does and the most obvious one is the emergence of mathematically concrete and physically meaningful questions for one to ask about the system. More to the point, the introduction of the *Cauchy* problem (2.1)-(2.2) raises the question of its local or global *well-posedness*:

Definition 2. *A Cauchy problem is well-posed (in the sense of Hadamard) if*

- *There exists a solution.*

- *The solution is unique.*

- *The solution depends continuously on the initial data (under a suitable topology on the space of initial data).*

As stated, for example, in [68], the *existence* and *uniqueness* requirements above are reasonable, in that they attempt to guarantee the non-overdeterminacy and the non-underdeterminacy of the model. Together they encapture the *causality principle*, namely that given the present the future can be determined. The last point is more intricate and essentially asks whether the causality principle works in a stable way (even when approaching bifurcation). Solutions need some stability relative to the initial conditions giving rise to them to be safely regarded as observable and physical.

Moreover, without stability, there is no guarantee that a solution produced from rough data is in any sense close to the real solution of interest. When this is the case, a plethora of further headaches ensues for the modelling of real-world phenomena.

The subtle issue of topology: Smooth, BV, L^∞ solutions

Non-trivial, in general, is also the choice of the topology of initial data sets in which one seeks to prove well-posedness statements. Such a topology needs to be determined a priori and if it is the wrong one to start with, little or no progress can be made. The choice of function spaces is typically suggested by the structure of the problem and the physics behind it (energy, entropy and others). We can explain this better for the equations of gas dynamics.

The Cauchy problem for all the systems we have discussed above fits into the following general conservation form:

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \ \ \mathbf{u} \in \mathbb{R}^n, \ \ x \in \mathbb{R}^d, \tag{2.12}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x). \tag{2.13}$$

Here $\mathbf{f} = (\mathbf{f}_1, \ldots, \ldots, \mathbf{f}_n) : \mathbb{R}^n \to (\mathbb{R}^n)^d$ is a non-linear mapping with $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 1, \ldots, d$. The hyperbolicity of the system (2.12) means that, for every $\omega \in S^{d-1}$, the matrix $(\nabla \mathbf{f}(\mathbf{u}) \cdot \omega)_{n \times n}$ has n real eigenvalues $\lambda_i(\mathbf{u}, \omega)$ and is diagonalizable. The central difficulty of such systems stems from the fact that, even from smooth initial data, smooth solutions tend to break down and develop singularities in finite time. These singularities develop because of the physical phenomena of focusing and breaking of waves and the development of shock waves and vortices, among others.

The above already suggests that one cannot, in general, hope for a global well-posedness theory to systems (2.12)-(2.13) in the space of C^∞ or C^1 solutions. In C^∞, however, there exists a local well-posedness theory. A great deal of thought was spent by eminent mathematicians during the 20th century towards coming up with spaces of discontinuous functions in which local well-posedness can, to an extent, be salvaged. The two main frameworks that exist are:

- The framework of *bounded variation* (BV) solutions,

- The framework of *bounded* (L^∞) solutions.

Both solution spaces create frameworks that come with their own techniques and toolkits and both exhibit benefits and drawbacks relative to each other. Despite deep and far–reaching progress it can be said that, to date, there is no known space of functions that can lead to global well-posedness or even well-posedness in the large for systems (2.12)-(2.13) in any generality.

2.2.1.1 Local well-posedness for smooth solutions

We present here the standard argument, as found for example in [15], for the local well-posedness of smooth solutions to the three-dimensional Euler equations, expressed in Eulerian coordinates in terms of $(\rho, \mathbf{v}, S)$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \partial_t S + \mathbf{v} \cdot \nabla S = 0. \end{cases} \tag{2.14}$$

Let the equation of state be $p = p(\rho, S) = \rho^\gamma e^S$, $\gamma > 1$. It shall be more convenient at this stage to rewrite (2.14) in terms of $(p, \mathbf{v}, S)$. We obtain

$$\begin{cases} \partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \\ \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p = 0, \\ \partial_t S + \mathbf{v} \cdot \nabla S = 0, \end{cases} \tag{2.15}$$

with $\rho = \rho(p, S) = p^{\frac{1}{\gamma}} e^{-\frac{S}{\gamma}}$. We focus on the Cauchy problem for (2.15) with smooth initial data

$$(p, \mathbf{v}, S)\big|_{t=0} = (p_0, \mathbf{v}_0, S_0)(\mathbf{x}). \tag{2.16}$$

The following local existence theorem holds.

Theorem 2.2.1. *Assume $(p_0, \boldsymbol{v}_0, S_0) \in H^s \cap L^\infty(\mathbb{R}^3)$ with $s > 5/2$ and $p_0(\boldsymbol{x}) > 0$ (a physical assumption). Then there exists a finite time $T > 0$, depending on the H^s and L^∞ norms of the initial data, such that the Cauchy problem (2.15)-(2.16) has a unique bounded smooth solution $(p, \boldsymbol{v}, S) \in C^1(\mathbb{R}^3 \times [0, T])$ with $p(\boldsymbol{x}, t) > 0$ for all $(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, T]$ and $(p, \boldsymbol{v}, S) \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$.*

The idea behind the proof of Theorem 2.2.1 will be the use of a technique for more general systems, namely symmetric hyperbolic systems, a form into which the system (2.15)-(2.16) will be brought. Consider the Cauchy problem for the general hyperbolic system (2.12)-(2.13) with the values of $\mathbf{u}$ lying in the state space G, an open subset of $\mathbb{R}^n$. Assume now that (2.12) is endowed with the following structure of symmetric

hyperbolic systems: For all $\mathbf{u} \in G$, there is a positive definite matrix $A_0(\mathbf{u})$ that is smooth in $\mathbf{u}$ and satisfies

$$c_0^{-1}\,\mathbb{I}_n \leq A_0(\mathbf{u}) \leq c_0\,\mathbb{I}_n, \tag{2.17}$$

with a constant c_0 uniform for all $\mathbf{u}$ in any $G_1 \Subset G$, such that $A_i(\mathbf{u}) = A_0(\mathbf{u})\nabla \mathbf{f}_i(\mathbf{u})$ is symmetric, where $\nabla \mathbf{f}_i(\mathbf{u})$, $i = 1, \ldots, d$ are the $n \times n$ Jacobian matrices of the non-linearities and $\mathbb{I}_n$ is the $n \times n$ identity matrix. When the matrix $A_0(\mathbf{u})$ exists, we call it the *symmetrizing matrix* of system (2.12). Multiplying (2.12) by $A_0(\mathbf{u})$ and denoting $A(\mathbf{u}) = \big(A_0(\mathbf{u}), \ldots, A_d(\mathbf{u})\big)$ yields the system

$$A_0(\mathbf{u})\partial_t \mathbf{u} + A(\mathbf{u})\nabla \mathbf{u} = 0. \tag{2.18}$$

The equations in (2.15) are symmetrized in 3 dimensions by the following 5×5 matrix for polytropic gases:

$$A_0(p, S) = \begin{pmatrix} (\gamma p)^{-1} & 0 & 0 \\ 0 & \rho(p, S)\mathbb{I}_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.19}$$

Given the existence of (2.19), Theorem 2.2.1 follows as a corollary of the following Theorem on the local existence of smooth solutions, with the specific state space $G = \left\{ (p, \mathbf{v}, S)^\top \mid p > 0 \right\} \subset \mathbb{R}^5$ for the system (2.12):

Theorem 2.2.2. *Assume that $\mathbf{u}_0 : \mathbb{R}^d \to G$ is in $H^s \cap L^\infty$ with $s > (d+2)/2$. Then, for the Cauchy problem (2.12)-(2.13) there exists a finite time T depending on the H^s and L^∞ norms of the initial data, such that there exists a unique bounded classical solution $\boldsymbol{u} \in C^1(\mathbb{R}^d \times [0, T])$, with $\boldsymbol{u}(\boldsymbol{x}, t) \in G$ for $(\boldsymbol{x}, t) \in \mathbb{R}^d \times [0, T]$ and $\boldsymbol{u}(\boldsymbol{x}, t) \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$.*

The proof of Theorem 2.2.2 then rests on a classical iteration scheme, which we will omit here.

2.3 A cutoff between global well-posedness and singularity formation in the C^1 framework

Dual to the question of local well-posedness is the question of the maximal value of the time T of existence. In that aspect, as has been explained above, the phenomenon of wave breaking and shock formation means that global well-posedness for smooth solutions may only be expected to hold for particularly well-behaved classes

of initial data. We begin this section by exhibiting such a class in the case of the one-dimensional isentropic Euler equations:

Consider the Cauchy problem for the system

$$\begin{cases} \partial_t \rho + \partial_x (p\,v) = 0, \\ \partial_t(\rho\,v) + \partial_x \left(\rho\,v^2 + p\right) = 0, \end{cases} \tag{2.20}$$

with an equation of state $p = \rho^\gamma / \gamma$ and initial data

$$(\rho, v)\big|_{t=0} = (\rho_0, v_0)(x). \tag{2.21}$$

The eigenvalues of (2.20) are

$$\lambda_1 = v - c, \lambda_2 = v + c, \tag{2.22}$$

where $c = \rho^\theta$ and $\theta = \frac{\gamma-1}{2}$ is the sound speed. We focus on the physical regime $1 < \gamma \leq 3$, whence $\theta \in (0, 1]$. The Riemann invariants associated to (2.20) are

$$w_1 = w_1(\rho, v) = v + \frac{\rho^\theta}{\theta}, \quad w_2 = w_2(\rho, v) = v - \frac{\rho^\theta}{\theta}. \tag{2.23}$$

Finally, we denote by

$$w_{10}(x) = w_1\left(\rho_0(x), v_0(x)\right), \quad w_{20}(x) = w_2\left(\rho_0(x), v_0(x)\right)$$

the initial values of the Riemann invariants. The following theorem can then be established:

Theorem 2.3.1. *Suppose that the initial data $(\rho_0, v_0)(x)$, with $\rho_0 > 0$ and $1 < \gamma \leq 3$ are smooth and moreover that $\|(\rho_0, v_0)\|_{C^1(\mathbb{R})} < \infty$. Assume moreover that*

$$w'_{10}(x) > 0 \quad and \quad w'_{20}(x) > 0, \quad \forall x \in \mathbb{R}.$$

Then the Cauchy problem (2.20)-(2.21) has a unique global smooth solution $(\rho, v)(x, t)$ for all $x \in \mathbb{R}, t > 0$.

Proof. As an initial remark, the proof rests on the method of characteristics. It is instructive to note right from the start why the positivity condition on w'_{10} and w'_{20} is special. It expresses the lack of *compression* in the initial data, meaning that it expresses the lack of initial tendency of characteristics to fall onto each other. The spirit of the proof is then understood as showing that this tendency persists in time.

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We begin by noting the following local existence result (found for example in [49], [89]): Consider the Cauchy problem for the system

$$\begin{cases} s_t + a_2(r,s)s_x = 0, \\ r_t + a_1(r,s)r_x = 0, \end{cases} \tag{2.24}$$

$$(r,s)(0,x) = (r_0(x), s_0(x)), \tag{2.25}$$

where the initial data are smooth and the functions $a_i(r,s) \in C^{1,1}(\mathbb{R} \times \mathbb{R})$ satisfy

$$a_{ir} \geq 0, \quad a_{is} \geq 0, \quad i = 1, 2. \tag{2.26}$$

Then there exists a constant t_1 such that on the domain

$$R(t_1) = \left\{ 0 \leq t \leq t_1, \ -\infty < x < \infty \right\}$$

there exists a unique smooth solution $r(t,x), s(t,x)$ provided that the initial data are smooth functions with bounded C^1 norm. Crucially, the time t_1 depends only on the C^1 norm of the initial data.

Let

$$T_* := \sup \left\{ T \in \mathbb{R}^+ \mid (2.20) \text{ with } (2.21) \text{ has a smooth solution in } \mathbb{R} \times [0,T] \right\}.$$

We shall show that $T_* = \infty$. Notice first of all that, by the local existence result described above and by the fact that for smooth solutions, the system $(2.20), (2.21)$ is equivalent to

$$\begin{cases} w_{1t} + \lambda_1(w_1, w_2)w_{1x} = 0, \\ w_{2t} + \lambda_2(w_1, w_2)w_{2x} = 0, \end{cases} \tag{2.27}$$

$$(w_1, w_2)(0,x) = (w_{10}(x), w_{20}(x)), \tag{2.28}$$

with the λ_i satisfying (2.26) (since $1 < \gamma \leq 3$), we obtain that $T_* > 0$. We now proceed by contradiction. Assume that $T_* < \infty$. We claim that for all $0 < T < T_*$, the C^1 norm of the solution (ρ, u) is uniformly bounded (independently of T) in $\mathbb{R} \times [0,T]$. To that end, fix $T' < T_*$ and look at the solution in $\mathcal{D}' := \mathbb{R} \times [0,T']$.

For a smooth solution $(\rho, v)(x,t)$, a direct calculation shows that the derivatives of Riemann invariants along characteristics vanish:

$$w_1' = w_2' = 0. \tag{2.29}$$

Here we have denoted $\prime = \partial_t + \lambda_1 \partial_x, \grave{} = \partial_t + \lambda_2 \partial_x$. Differentiate $w_1' = 0$ from (2.29) with respect to x to obtain

$$\partial_{tx}^2 w_1 + \lambda_2 \partial_{xx}^2 w_1 + \partial_{w_1} \lambda_2 (\partial_x w_1)^2 + \partial_{w_2} \lambda_2 \, \partial_x w_1 \, \partial_x w_2 = 0.$$

Since $0 = w_2' = w_2' - 2c \partial_x w_2$, by setting $r = \partial_x w_1$ and noticing

$$\lambda_2 = \lambda_2(w_1, w_2) = \frac{1+\theta}{2} w_1 + \frac{1-\theta}{2} w_2, \quad \partial_x w_2 = \frac{w_2'}{2c},$$

one has

$$r' + \frac{1+\theta}{2} r^2 + \frac{1-\theta}{4c} w_2' r = 0.$$

Set

$$s = \frac{\theta-1}{2} \ln\rho = \frac{\theta-1}{2\theta} \ln(w_1 - w_2).$$

Then $\partial_{w_2} s = \frac{1-\theta}{4c}$ and $s' = w_2' \partial_{w_2} s = \frac{1-\theta}{4c} w_2'$. Thus

$$r' + \frac{1+\theta}{2} r^2 + s' r = 0.$$

Set $g = e^s r = \rho^{\frac{\theta-1}{2}} \partial_x w_1$. Then

$$g' = -\frac{1+\theta}{2} \left(\frac{\theta}{2} |w_1 - w_2| \right)^{\frac{1-\theta}{2\theta}} g^2. \tag{2.30}$$

Similarly, for $h = \rho^{\frac{\theta-1}{2}} \partial_x w_2$, we have

$$h\grave{} = -\frac{1+\theta}{2} \left(\frac{\theta}{2} |w_1 - w_2| \right)^{\frac{1-\theta}{2\theta}} h^2. \tag{2.31}$$

Let $x = x(\beta, t)$ be the forward characteristic passing through any fixed point $(\beta, 0)$ at $t = 0$, defined by

$$\frac{\mathrm{d}x(\beta, t)}{\mathrm{d}t} = \lambda_2 \left(w_1(x(\beta, t), t), w_2(x(\beta, t), t) \right), \quad x(\beta, 0) = \beta.$$

According to (2.29), w_1 is constant along characteristics, thus $w_1(x(\beta, t), t) = w_1(\beta, 0) = w_{10}(\beta)$. Moreover $\sup |w_1(x, t)| = \sup |w_{10}(x)|$. Similarly, $\sup |w_2(x, t)| = \sup |w_{20}(x)|$, with w_2 being constant along characteristics corresponding to the eigenvalue λ_1. For any given point $(x(\beta, t), t)$ on the forward characteristic $x = x(\beta, t)$, there exists a unique $\alpha = \alpha(\beta, t) \geq \beta$ such that $w_2(x(\beta, t), t) = w_{20}(\alpha)$. Therefore, along the characteristic $x = x(\beta, t)$, one has from (2.30) that

$$\begin{cases} \frac{\mathrm{d}g(x(\beta,t),t)}{\mathrm{d}t} = -\frac{1+\theta}{2} \left(\frac{\theta}{2} |w_{10}(\beta) - w_{20}(\alpha(\beta,t))| \right)^{\frac{1-\theta}{2\theta}} g(x(\beta,t),t)^2, \\ g|_{t=0} = \rho_0(\beta)^{\frac{\theta-1}{2}} w_{10}'(\beta). \end{cases} \tag{2.32}$$

Then

$$g(x(\beta,t),t) = \frac{\rho_0(\beta)^{\frac{\theta-1}{2}} w'_{10}(\beta)}{1 + \int_0^t K(\beta,\tau)\,\mathrm{d}\tau},$$

where

$$K(\beta,t) = \frac{1+\theta}{2}\left(\frac{\theta}{2}|w_{10}(\beta) - w_{20}(\alpha(\beta,t))|\right)^{\frac{1-\theta}{2\theta}} \rho_0(\beta)^{\frac{\theta-1}{2}} w'_{10}(\beta).$$

From the fact that $w'_{10} > 0$ initially, we conclude that $K(\beta,t) > 0$ and thus $g(x(\beta,t),t)$ is bounded. Moreover,

$$\partial_x w_1(x(\beta,t),t) = \left(\frac{\theta}{2}|w_{10}(\beta) - w_{20}(\alpha(\beta,t))|\right)^{\frac{1-\theta}{2\theta}} g(x(\beta,t),t)$$

is also bounded. A similar argument shows that $\partial_x w_2$ is bounded. As a consequence, the C^1 norms of $\rho = (\theta(w_1 - w_2)/2)^{\frac{1}{\theta}}$ and $v = (w_1 + w_2)/2$ are bounded on $\mathbb{R} \times [0, T']$. This concludes the proof of the theorem. □

2.4 The Relativistic Euler equations

The classical compressible Euler equations are a valid model for flows with speeds which are quite small compared to the speed of light c. When one wishes to study fluids in astrophysical settings, where the speeds can be comparable to $3 \cdot 10^8$ meters per second, this model loses its validity. The physics, instead, is better captured by the equation

$$\operatorname{div} T = 0, \tag{2.33}$$

where $T^{ij} = (p + \rho c^2)u^i u^j + p\eta^{ij}$. Here u is the relativistic velocity, p the pressure (related to ρ via a constitutive assumption), ρ is the mass–energy density and η is the flat Minkowski metric, with a chosen convention for signature . In Eulerian coordinates and on a fixed $(d + 1)$-dimensional Minkowski background, (2.33) reads:

$$\begin{cases} \left(\dfrac{\rho + P|u|^2/c^4}{1 - |u|^2/c^2}\right)_t + \operatorname{div}\left(\dfrac{(\rho + P/c^2)u}{1 - |u|^2/c^2}\right) = 0, \\[4mm] \left(\dfrac{(\rho + P/c^2)u}{1 - |u|^2/c^2}\right)_t + \operatorname{div}\left(\dfrac{(\rho + P/c^2)}{1 - |u|^2/c^2}\, u \otimes u\right) + \nabla P = 0. \end{cases} \tag{2.34}$$

In view of the above, the main theorem in Chapter 4 of this thesis is the provision of a necessary and sufficient criterion for the blow–up of the gradient of C^1 solutions to the 1-dimensional Relativistic Euler equations. The criterion is phrased in terms of the compression of the initial data and its importance stems from its physicality,

meaning that compression carries a well-defined (and easy to understand) physical interpretation. One can also argue that it is in this sense that the study of C^1 solutions is useful (as opposed to L^∞ or BV solutions), in that one can study the onset of singularity formation in a clean way even for large data, with the main mechanism being Lax's mechanism as found in [46].

Chapter 3

Trapped surfaces in General Relativity

3.1 Local well-posedness of the theory

The theory of General Relativity is a successful theory of gravitation. It postulates that a *spacetime* $(\mathcal{M}, g)$ is a 4-dimensional manifold equipped with a Lorentzian metric which satisfies, for a given energy-momentum tensor T of matter, the *Einstein equations*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}. \tag{3.1}$$

Here $R_{\mu\nu}$ and R denote the Ricci curvature tensor and scalar curvature respectively. In the absence of matter, (3.1) reduces to the vacuum equations

$$R_{\mu\nu} = 0. \tag{3.2}$$

As with any physical theory, a problem of great importance was the proof of its local well-posedness. However, the inherently geometric nature of the theory would mean that this was no easy feat. Perhaps surprisingly, the most challenging aspect of the proof of well-posedness was to find the proper notion of a Cauchy problem for the Einstein equations. A few important questions are the following:

- What constitutes a proper surface on which to describe initial data?

- What is the proper notion of initial data to impose on such a surface?

- In what way are (3.1) and (3.2) hyperbolic?

A great exposition on the basics of Lorenzian geometry, Cauchy surfaces and the notion of global hyperbolicity can be found for example in [81], [7]. We include a discussion on the last question here, following the exposition in [60] .

The Einstein equations in harmonic (wave) coordinates

Here and throughout, for the sake of simplicity, we focus on the vacuum equations (3.2). Recall the definition of the *Levi-Civita* connection ∇ of a metric g given, in coordinates, by

$$\nabla_{X^\alpha \frac{\partial}{\partial x^\alpha}} \left(Y^\beta \frac{\partial}{\partial x^\beta} \right) = X^\alpha \frac{\partial Y^\beta}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} + \Gamma^\mu_{\alpha\beta} X^\alpha Y^\beta \frac{\partial}{\partial x^\mu}, \tag{3.3}$$

where the *Christoffel symbols* $\Gamma^\mu_{\alpha\beta}$ are given by

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \left(g^{-1} \right)^{\mu\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right). \tag{3.4}$$

The Riemann curvature tensor is defined by

$$R^\gamma_{\alpha\mu\nu} \frac{\partial}{\partial x^\gamma} = \nabla_{\frac{\partial}{\partial x^\mu}} \nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\alpha} - \nabla_{\frac{\partial}{\partial x^\nu}} \nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\alpha},$$

with

$$R_{\beta\alpha\mu\nu} = g_{\beta\gamma} R^\gamma_{\alpha\mu\nu}. \tag{3.5}$$

An explicit computation then yields

$$R_{\mu\nu\alpha}{}^\gamma = -\frac{\partial}{\partial x^\mu} \Gamma^\gamma_{\nu\alpha} + \frac{\partial}{\partial x^\nu} \Gamma^\gamma_{\mu\alpha} + \dots \tag{3.6}$$

$$R_{\mu\nu} = -\frac{\partial}{\partial x^\mu} \Gamma^\gamma_{\gamma\nu} + \frac{\partial}{\partial x^\gamma} \Gamma^\gamma_{\mu\nu} + \dots \tag{3.7}$$

We can show, in particular, that in a local coordinate system the Ricci tensor takes the form

$$\begin{aligned}
R(g)_{\mu\nu} = &-\frac{1}{2} \left(g^{-1} \right)^{\alpha\beta} \partial^2_{\alpha\beta} g_{\mu\nu} - \frac{1}{2} \left(g^{-1} \right)^{\alpha\beta} \partial^2_{\mu\nu} g_{\alpha\beta} \\
&+ \frac{1}{2} \left(g^{-1} \right)^{\alpha\beta} \partial^2_{\alpha\nu} g_{\beta\mu} + \frac{1}{2} \left(g^{-1} \right)^{\alpha\beta} \partial^2_{\beta\mu} g_{\alpha\nu} + F(g, \partial g).
\end{aligned} \tag{3.8}$$

The beautiful realisation behind the introduction of so–called *wave* (or *harmonic*) coordinates is that, if one could introduce coordinates in which the second, third and fourth terms in (3.8) all vanish, the Einstein vacuum equation $R(g) = 0$ reduce to the following system of quasilinear wave equations on the underlying manifold:

$$\left(g^{-1} \right)^{\alpha\beta} \partial^2_{\alpha\beta} g_{\mu\nu} = 2F(g, \partial g). \tag{3.9}$$

Indeed, if one chooses a special system of coordinates that satisfy the *wave condition*

$$\Box_g x^\alpha = \frac{1}{\sqrt{-g}} \partial_\mu \left(\left(g^{-1} \right)^{\mu\nu} \sqrt{-g} \, \partial_\nu x^\alpha \right) = 0, \tag{3.10}$$

an explicit computation verifies that the afore–mentioned terms may not vanish, but can be rewritten as a sum of terms involving derivatives of the metric of order at most one. Upon introducing

$$\lambda_\sigma = \left(g^{-1}\right)^{\mu\alpha} \partial_\mu g_{\alpha\sigma} - \frac{1}{2} \left(g^{-1}\right)^{\alpha\beta} \partial_\sigma g_{\alpha\beta}, \tag{3.11}$$

the condition in (3.10) implies that λ_σ vanishes identically. Defining the *reduced Ricci curvature*

$$\widetilde{Ric}(g)_{\mu\nu} = -\frac{1}{2} \left(g^{-1}\right)^{\alpha\beta} \partial^2_{\alpha\beta} g_{\mu\nu} + \frac{1}{2} \left(g^{-1}\right)^{\alpha\sigma} \left(g^{-1}\right)^{\beta\rho} \partial_\mu g_{\sigma\rho} \partial_\beta g_{\alpha\nu}$$
$$+\frac{1}{2} \left(g^{-1}\right)^{\alpha\sigma} \left(g^{-1}\right)^{\beta\rho} \partial_\nu g_{\sigma\rho} \partial_\alpha g_{\beta\mu} - \frac{1}{2} \left(g^{-1}\right)^{\alpha\sigma} \left(g^{-1}\right)^{\beta\rho} \partial_\mu g_{\sigma\rho} \partial_\nu g_{\alpha\beta} + F_{\mu\nu}(g, \partial g),$$
$$\tag{3.12}$$

where F is as in (3.8), then it holds that

$$\widetilde{Ric}(g)_{\mu\nu} = R(g)_{\mu\nu} - \frac{1}{2}\partial_\mu \lambda_\nu - \frac{1}{2}\partial_\nu \lambda_\mu. \tag{3.13}$$

This means that, if the wave coordinate condition holds, the Einstein equations become the reduced Einstein equations

$$\widetilde{Ric}(g)_{\mu\nu} = 0, \tag{3.14}$$

which is a system of wave equations. The strategy adopted by Choquet-Bruhat in [18] to obtain her celebrated local existence result was two-fold:

- Use the theory of nonlinear wave equations to deduce a local existence result to the reduced Einstein system.

- Show that if the wave coordinate condition holds initially, it is then propagated by the flow.

Pivotal to this last step is the observation that λ_σ itself satisfies a wave equation.

Proposition 3.1.1. *Assume we are given a Lorentzian metric g that satisfies the reduced Einstein vacuum equations $\widetilde{Ric}(g) = 0$. Then λ, defined as in (3.11), satisfies the following wave equation*

$$\frac{1}{2} \left(g^{-1}\right)^{\sigma\mu} \partial^2_{\sigma\mu} \lambda_\nu + c_\nu^{\alpha\beta} \partial_\alpha \lambda_\beta = 0,$$

where the $c_\nu^{\alpha\beta}$ are smooth functions of g and its derivatives.

Proof. The proof is based on the well-known identity

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0.$$

The vanishing of $\widetilde{Ric}(g)$ translates to

$$R(g)_{\mu\nu} - \frac{1}{2}\partial_\mu \lambda_\nu - \frac{1}{2}\partial_\nu \lambda_\mu = 0.$$

Taking the trace, there holds

$$R - \left(g^{-1}\right)^{\mu\nu} \partial_\mu \lambda_\nu = 0.$$

Consequently, there holds

$$
\begin{aligned}
0 =& \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \\
=& -\left(g^{-1}\right)^{\sigma\mu} \partial_\sigma \left(\frac{1}{2}\partial_\mu \lambda_\nu + \frac{1}{2}\partial_\nu \lambda_\mu - \frac{1}{2} g_{\mu\nu} \left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \lambda_\beta \right) \\
& -\left(g^{-1}\right)^{\sigma\mu} \Gamma^\delta_{\sigma\mu} \left(\frac{1}{2}\partial_\delta \lambda_\nu + \frac{1}{2}\partial_\nu \lambda_\delta - \frac{1}{2} g_{\delta\nu} \left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \lambda_\beta \right) \\
& -\left(g^{-1}\right)^{\sigma\mu} \Gamma^\delta_{\sigma\nu} \left(\frac{1}{2}\partial_\mu \lambda_\delta + \frac{1}{2}\partial_\delta \lambda_\mu - \frac{1}{2} g_{\mu\delta} \left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \lambda_\beta \right) \\
=& -\frac{1}{2} \left(g^{-1}\right)^{\sigma\mu} \partial^2_{\sigma\mu} \lambda_\nu - c^{\alpha\beta}_\nu \partial_\alpha \lambda_\beta,
\end{aligned}
$$

(3.15)

where the $c^{\alpha\beta}_\nu$ are smooth functions of g and its derivatives. $\square$

Before stating the local existence result, we take a small detour and discuss the initial data that will be used in the Cauchy problem formulation. In formulating an initial value problem for the Einstein equations, two main issues must be addressed separately. The first one is to identify the nature of the initial data that should be specified. The second is to understand the constraints that need to be imposed on this data so that we can develop a theory of existence of solutions. In this endeavour, a starting point is to notice that the theory of general relativity is *diffeomorphism invariant*. This means that if spacetime is represented by a triple $(\mathcal{M}, g, \psi)$, where $\mathcal{M}$ and g are as usual and ψ denotes a matter field and $\phi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, then the triple $(\mathcal{M}, \phi_* g, \phi_* \psi)$ represents[1] the same spacetime and should thus be indistinguishable from the first triple. This means that the initial data should be geometric in nature (see also [7] for an exposition).

---

[1]Here ϕ_* represents the pullback of ϕ.

Given that we want to solve the reduced Einstein vacuum equations, which are a system of nonlinear wave equations, we need, in principle, to prescribe the initial g and $\partial_t g$. It turns out that we only need to prescribe the metric intrinsic to the initial hypersurface and the second fundamental form. The second fundamental form is defined as

$$\hat{k}_{ij} = \frac{1}{2} \left(\mathcal{L}_n g \right)_{ij} = \frac{1}{2} \left(n^\ell \partial_\ell g_{ij} + g_{j\ell} \partial_i n^\ell + g_{i\ell} \partial_j n^\ell \right).$$

Here n is the normal to the initial hypersurface such that $g(n, n) = -1$. The remaining components for g and $\partial_t g$ can be prescribed as coordinate conditions. Indeed, choosing ∂_t such that $g(\partial_t, \partial_t) = -1$ and such that it is orthogonal to the initial hypersurface, the second fundamental form becomes

$$\hat{k}_{ij} = \frac{1}{2} \partial_t g_{ij}.$$

The remaining freedom of choice for the initial conditions (namely $\partial_t g_{00}$ and $\partial_t g_{0i}$) is fixed by the wave coordinate condition. Indeed, $\lambda_i = 0$ fixes $\partial_t g_{0i}$ and $\lambda_0 = 0$ fixes $\partial_t g_{00}$. A final but very important observation is that the expressions for $R_{00} - \frac{1}{2} g_{00} R$ and R_{0i} do not contain any terms of the form $\partial_t^2 g$. In particular, if they are to vanish, it is necessary and sufficient that they do so on the initial hypersurface. This gives rise to the so–called *constraint equations*:

$$\nabla_i \hat{k}^i_j - \nabla_j \hat{k}^i_i = 0, \tag{3.16}$$

$$\hat{R}(\hat{g}) + (\hat{k}^i_i)^2 - \hat{k}^i_j k^j{}_i = 0. \tag{3.17}$$

Combining the above, we arrive at the local existence result of Choquet-Bruhat:

Theorem 3.1.1. *Given initial data $(\hat{g}_{ij}, \hat{k}_{ij})$ on $\mathbb{R}^n$ satisfying the constraint equations and such that $\sum_{i,j} |\hat{g}_{ij} - \delta_{ij}| < \frac{1}{20}$, there exists an interval I and a metric g on $I \times \mathbb{R}^n$ satisfying the Einstein vacuum equations and such that the induced metric and second fundamental form on $\{0\} \times \mathbb{R}^n$ coincide with $\hat{g}_{ij}, \hat{k}_{ij}$ respectively.*

Proof. Our starting point is the fact that the reduced Einstein vacuum equations are a system of nonlinear wave equations and as such admit a local existence result. We infer that there exists an I and a metric g on $I \times \mathbb{R}^n$ solving (3.14). To show that g is a solution to the Einstein vacuum equations, we need to show $\lambda_\sigma = 0$. Crucially,

since λ_σ satisfies a wave equation, all we have to show is that $(\lambda_\sigma, \partial_t \lambda_\sigma)\big|_{t=0} = 0$. As we have discussed, the vanishing of $\widetilde{Ric}$ yields

$$R(g)_{\mu\nu} - \frac{1}{2}\partial_\mu \lambda_\nu - \frac{1}{2}\partial_\nu \lambda_\mu = 0.$$

Taking the trace, there holds

$$R = \left(g^{-1}\right)^{\mu\nu} \partial_\mu \lambda_\nu.$$

Given that the initial data $(\hat{g}_{ij}, \hat{k}_{ij})$ satisfy the vacuum equations $R_{00} = R_{0i} = 0$, we have

$$\partial_t \lambda_i\big|_{t=0} = \partial_i \lambda_0\big|_{t=0} = 0$$

and

$$\left(\partial_t \lambda_0 + \frac{1}{2}\left(g^{-1}\right)^{\mu\nu} \partial_\mu \lambda_\nu\right)\bigg|_{t=0} = 0.$$

Consequently,

$$\partial_t \lambda_0\big|_{t=0} = -\frac{1}{2}\left(\left(g^{-1}\right)^{\mu\nu} \partial_\mu \lambda_\nu\right)\bigg|_{t=0} = 0.$$

This concludes the proof. $\qquad\square$

Remark 1. *The condition $\sum_{i,j}|\hat{g}_{ij} - \delta_{ij}| < \frac{1}{20}$ can be removed by localizing in neighbourhoods where the metric $\hat{g}$ is close to some constant coefficient metric.*

Remark 2. *This result can be mended to allow for initial data posed on any Riemannian manifold and not just one with the topology of $\mathbb{R}^n$.*

Typically, local existence and uniqueness results come with a notion of *maximality* of time of existence. In general relativity, maximalising Choquet-Bruhat's local statement is non-trivial. As Dafermos points out in [31], this is mainly because there is no common ambient structure on which all solutions are defined so as for them to be readily compared. Such a maximalisation was obtained in [19]:

Theorem 3.1.2. *Let (Σ^3, g, K) be a smooth vacuum initial data set. Then there exists a unique, smooth vacuum Cauchy development[2] $(\mathcal{M}, g)$ with the property that if $(\widetilde{\mathcal{M}}, \tilde{g})$ is any other vacuum Cauchy development, then there exists an isometric embedding $i : (\widetilde{\mathcal{M}}, \tilde{g}) \to (\mathcal{M}, g)$ commuting with the embeddings of Σ. We call $(\mathcal{M}, g)$ the maximal globally hyperbolic development (MGHD) of the initial data set.*

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[2]A vacuum Cauchy development $(\mathcal{M}, g)$ of a vacuum initial data set $(\Sigma, \bar{g}, \bar{k})$ is a 4-dimensional Lorentzian manifold solving the Einstein vacuum equations and such that there exists an embedding $\iota : \Sigma \to \mathcal{M}$ with pullbacks $\iota^* g = \bar{g}$ and $\iota^* K = \bar{k}$, where K is the second fundamental form of $\iota(\Sigma)$.

The proof of Theorem 3.1.2 crucially relied on the use of Zorn's lemma. A recent work by Sbierski [71] overcomes this necessity and gives an explicit construction of the maximal globally hyperbolic development, using axioms weaker than the axiom of choice.

A useful alternative: The characteristic initial value problem

There are some disadvantages when it comes to using the Cauchy problem framework to study the dynamics of the Einstein equations. The biggest one is that the constraint equations themselves are a difficult set of nonlinear Elliptic PDEs. So difficult, in fact, that the study of solutions to those PDEs is an active mathematical area of research in itself. An alternative way to study dynamics is provided by the characteristic initial value problem. Here the data are posed on two cones $\underline{H}_0$ and

Figure 3.1: The characteristic initial value problem setting.

H_0 intersecting on a 2–sphere $S_{0,0}$ and the cones are prescribed to be null in the spacetime to be constructed. The advantage of this framework is two–fold. First, the constraint equations in this setting take the type of transport equations and are hence much easier to analyze. Secondly, the null structure of the spacetime is more efficiently encoded in this framework. The first local existence result pertaining to this framework was given by Rendall [80]. By imposing data for a general quasilinear wave equation along the two hypersurfaces and using a clever argument to reduce the problem to a Cauchy problem, Rendall obtains a local existence result in the neighbourhood of the initial sphere $S_{0,0}$. This result is extended in a work of Luk [61] to a full neighbourhood of the two cones, but only works for equations admitting a special null structure and not for as general a class of equations as [80] permits. Luk's result includes, however, the Einstein vacuum equations and is obtained under a double null foliation. As far as trapped surfaces are concerned, most of the problems concerning their formation have been posed as characteristic initial value problems. The results of Section 5 are also obtained in a characteristic initial value setting and a precise set up of the problem can be found in Section 5.2. A very nice introduction to this framework is also given in Aretakis' [6].

3.1.1 Trapped surfaces and important results about them

A historical review of the trapped surface existence and formation problems can be found at the beginning of Chapter 5 in this thesis. Here, we restrict ourselves to the presentation of important results in the field. This subsection borrows from work carried out with Lesourd [9].

The definition of a trapped surface

A trapped surface is a 2-dimensional geometric object. It can be defined as follows. Assume we are given a $4-$dimensional time–oriented Lorentzian manifold $(\mathcal{M}, g)$ and within it a closed spacelike $2-$surface S. Since S has co–dimension 2, the tangent space at a point p on S, $T_p S$ has a 2-dimensional orthogonal complement in $T_p\mathcal{M}$. Let $l, \underline{l}$ denote a basis of this complement and extend $l, \underline{l}$ as vector fields. We define the following two fundamental forms $\chi, \underline{\chi}$ associated with the surface S:

$$\chi(X, Y) = g(\nabla_X l, Y), \underline{\chi}(X, Y) = g(\nabla_X \underline{l}, Y),$$

where X and Y are vector fields tangent to S. We look at the expansions $tr\chi, tr\underline{\chi}$. If both are pointwise negative on S, the surface is called trapped. A trapped surface is, therefore, a surface for which the area decreases for arbitrary infinitesimal displacements along the null generators of both null geodesic congruences normal to S. The importance of trapped surfaces stems from Penrose's celebrated *incompleteness* theorem:

Theorem 3.1.3. *For an Einstein vacuum or Einstein–Maxwell spacetime $(\mathcal{M}, g)$ containing a non-compact Cauchy hypersurface, if $\mathcal{M}$ contains a compact trapped surface, then it is future causally geodesically incomplete.*

In other words, under mild physical assumptions, the existence of a trapped surface implies geodesic incompleteness and hints at the existence of black holes in the spacetime. Be it noted that Penrose's theorem as stated above also works for the Einstein equations coupled to several other matter fields.

Results concerning trapped surfaces

The first trapped surface existence result is due to Schoen-Yau [83]. The result was refined by Yau [86]. These results are purely at the level of the initial data: one formulates conditions on a Riemannian manifold with boundary which imply the existence

of a trapped surface within. From a dynamical point of view, Christodoulou was the first one to study a toy problem for the Einstein vacuum equations in the absence of symmetry, namely the Einstein–scalar field equations in spherical symmetry. He managed to obtain a complete picture of gravitational collapse. In particular, he gave conditions on the inital data, free of trapped surfaces, that lead to the formation of a trapped surface in later time [23]. He also exhibited data that lead to naked singularities [24], meaning singularities that are not contained within a black hole, but then proceeded to show that these singularities are unstable [25]. In the absence of symmetry and in a landmark contribution, Christodoulou [22] found a way of forming a trapped surface *dynamically* for the vacuum Einstein system.

Theorem 3.1.4 (Christodoulou 2009). *Consider the characteristic initial value problem for (3.2) such that $\underline{H}_0$ coincides with a backwards lightcone in Minkowski space for $0 \leq u \leq 1$. For every $B > 0$ and $u_* \leq 1$, there exists $\delta = \delta(B, u_*) > 0$ sufficiently small such that if the initial $\hat{\chi}_0$, prescribed on H_1 for $0 \leq u \leq \delta$, satisfies*

$$\sum_{i \leq 5, j \leq 3} \delta^{\frac{1}{2}+j} ||\nabla^i \nabla_4^j \hat{\chi}_0||_{L_{\underline{u}}^\infty L^2(S_{u,\underline{u}})} \leq B \tag{3.18}$$

then the solution to the Einstein vacuum equations remains regular in $u_* \leq u \leq 1$, $0 \leq \underline{u} \leq \delta$. Moreover, if the initial data satisfy the lower bound

$$\inf_{\omega \in S_{1,0}} \int_0^\delta |\hat{\chi}_0(\underline{u}', \omega)|^2 d\underline{u}' \geq M_* \geq 2(1 - u_*) \tag{3.19}$$

then after choosing δ sufficiently small (depending on B, u_* and M_*) if necessary, the sphere $S_{u_*,\delta}$ is a trapped surface.

Christodoulou's argument relies on identifying a certain hierarchy among quantities that is preserved under the non-linear evolution of the Einstein vacuum system. Identifying and maintaining this hierarchy makes existence possible whilst permitting certain quantities to grow large, in particular those needed for trapped surface formation.

Shortly thereafter, Klainerman–Rodnianski [43] found a simplified and more direct argument for the formation of trapped surfaces. Their argument reduces the number of derivatives needed from two of curvature to one.

Another major contribution was brought by Li–Yu [58], who found a way to re–express a version of Theorem 3.1.4 in the language of Cauchy initial data. Their idea was to

improve the estimates of [22] in order to use the local deformation result of Corvino-Schoen [27], [28] and glue an asymptotically flat slice, isometric to a Kerr spacetime (see for example [32] for a definition and detailed properties of this spacetime) outside a compact set, onto the dynamical slab in [22]. To achieve this extra control, they insisted that the initial shear specified in Theorem 3.1.4 satisfy

$$m_0 = \frac{1}{4} \int_0^\delta |\hat{\chi}_0(\underline{u}', \omega)|^2 d\underline{u}' \tag{3.20}$$

for some constant m_0, so that the total shear along $\underline{u} \in [0, \delta]$ is independent of ω. With this assumption, they eventually obtained the following.

Theorem 3.1.5 (Li–Yu 2015). *Let Σ be a 3-dimensional differential manifold diffeomorphic to $\mathbb{R}^3$ and seperated into four concentric regions*

$$\Sigma = \Sigma_M \cup \Sigma_C \cup \Sigma_S \cup \Sigma_K$$

with Σ_M diffeomorphic to the 3-ball, Σ_C and Σ_S diffeomorphic to the 3-annulus and Σ_K diffeomorphic to $\mathbb{R}^3 \backslash B^3$. Then for any $\epsilon > 0$ sufficiently small, there is a Riemannian metric g and a symmetric two tensor k on Σ satisfying (3.16)–(3.17) such that

1. *Σ_M is a constant time slice in Minkowski spacetime $(g, k) = (\delta_{ij}, 0)$,*

2. *Σ_K is isometric to a constant time slice all the way to spacelike infinity in a Kerr spacetime with mass m and angular momentum $\boldsymbol{a}$ satisfying $|m - m_0| + |\boldsymbol{a}| \lesssim \epsilon$,*

3. *Σ is free of trapped surfaces,*

4. *there are trapped surfaces in the development of Σ.*

At around the same time, Klainerman–Luk–Rodnianski [42] were able to find an anisotropic mechanism to form trapped surfaces.

Theorem 3.1.6 (Klainerman–Luk–Rodnianski 2015). *Take as starting point the characteristic initial value problem of Theorem 3.1.4. If (3.19) is replaced with*

$$\sup_{\omega \in S_{1,0}} \int_0^\delta |\hat{\chi}_0(\underline{u}', \omega)|^2 d\underline{u}' \geq M_* > 0 \tag{3.21}$$

then, after choosing δ smaller if necessary, a compact trapped surface can be guaranteed to form to the future of the initial data, within the domain in which the solution remains regular.

In replacing 'inf' with 'sup', they only require the initial shear to be large in the neighborhood of a *single* geodesic. Theorem 3.1.6 thus yields an *anisotropic* formation result.

In a different direction, An–Luk [2] proved a trapped surface formation under 'scale critical' data for the Einstein vacuum system. Their result made use of techniques developed by Luk–Rodnianski [63], [64], which those authors had developed to study interacting impulsive waves. Though, in themselves, the two works of Luk–Rodnianski [63], [64] were not primarily meant as trapped surface formation statements, but rather as statements of existence and uniqueness for the Einstein vacuum equations with rough data, the introduction of the renormalization scheme in them proved a pivotal idea in the development of General Relativity this past decade. In particular, this renormalization proved crucial in [2].

Theorem 3.1.7 (An-Luk 2017)**.** *Consider the following characteristic initial value problem the Einstein vacuum system. The initial incoming hypersurface $\underline{H}_0$ is required to coincide with a backwards lightcone in Minkowski space with $0 \leq u \leq 1$. On the initial outgoing hypersurface H_1, the initial $\hat{\chi}_0$ satisfies*

$$\sum_{i \leq 7} ||\nabla^i \hat{\chi}_0||_{L_{\underline{u}}^\infty L^2(S_{u,\underline{u}})} \leq a^{1/2} \tag{3.22}$$

for $0 \leq \underline{u} \leq \delta$. There exists a universal large constant b_0 such that if $b_0 \leq b \leq a$ and $\delta a^{1/2} b < 1$, then the unique solution to the EVEs remains regular in the region $\delta a^{1/2} b \leq u \leq 1$, $0 \leq \underline{u}, \leq \delta$. Moreover, if the initial data also verify the lower bound

$$\inf_{\omega \in S_{1,0}} \int_0^\delta |\hat{\chi}_0(\underline{u}', \omega)|^2 d\underline{u}' \geq 4a^{1/2} b \delta \tag{3.23}$$

then the sphere $S_{b\delta a^{1/2}, \delta}$ is trapped.

Note that after choosing $a = B^2 \delta^{-1}$ and $b = b_0$ one basically recovers a version of Theorem 3.1.4 as a corollary.

Corollary 3.1.1 (An-Luk 2017)**.** *Replace (3.22) with*

$$\sum_{i \leq 7} \delta^{1/2} ||\hat{\chi}_0||_{L_{\underline{u}}^\infty L^2(S_{u,\underline{u}})} \leq B \tag{3.24}$$

for $0 \leq \underline{u} \leq \delta$. Then there exists a universal large constant b_0 such that the solution to the EVEs remains regular in $u_* \leq u \leq 1$, $0 \leq \underline{u} \leq \delta$ for $u_* = b_0 B \delta^{1/2}$. Moreover, if the initial data also verify the lower bound

$$\inf_{\omega \in S_{1,0}} \int_0^\delta |\hat{\chi}_0(\underline{u}', \omega)|^2 d\underline{u}' \geq 4b_0 B \delta^{1/2} \tag{3.25}$$

then the sphere $S_{u_*, \delta}$ is a trapped surface.

The significance of Theorem 3.1.7 lies in the fact that the size of the incoming radiation, given by

$$\inf_{\theta \in S_{1,0}} \int_0^\delta |\hat{\chi}|^2(\underline{u}', \theta) \, \mathrm{d}\underline{u}',$$

can be of the same order of magnitude as the length scale δ. In particular, there exist initial data satisfying the conditions of Theorem 3.1.7 for which the metric is only large in $H^{\frac{3}{2}}$ and small in H^s for all $s < \frac{3}{2}$. More precisely, one can construct initial data satisfying the conditions of Theorem 3.1.7 in which

$$\|\gamma\|_{H^s} \approx a^{\frac{1}{2}} \delta^{\frac{3}{2}-s}.$$

This is in contrast to Theorem 3.1.4, in which the data are large in H^s for all $s > 1$. The significance of the $H^{\frac{3}{2}}$ space is that it is a critical space in terms of scaling considerations for the Einstein vacuum equations. It is in this sense that the data are termed *mild*. Broadly speaking, therefore, scale-critical data can be thought of as the smallest initial data, in terms of size, known to produce a trapped surface in evolution.

In early 2019, An [5] extends [2] to a formation statement for the vacuum equations starting from a region close to past null infinity. Based on this work, we present in this thesis the work [4] which provides a similar statement for the Einstein–Maxwell system.

Remark 3. *Notice that, with the exception of [58], all the results above are obtained in the characteristic initial value problem setting. This is why we have chosen to include a short discussion on it in this introduction.*

Chapter 4

Formation of singularities for the relativistic Euler equations

4.1 Introduction

This section is devoted to the Cauchy problem of the relativistic Euler equations (henceforth denoted by RE) with large data. In fluid mechanics and astrophysics, the relativistic Euler equations are a generalization of the Euler equations that account for the effects of special relativity. On a fixed $(d+1)$-dimensional Minkowski background, they are given by

$$\begin{cases} \left(\dfrac{\rho + P|u|^2/c^4}{1 - |u|^2/c^2}\right)_t + \text{div}\left(\dfrac{(\rho + P/c^2)u}{1 - |u|^2/c^2}\right) = 0, \\[4mm] \left(\dfrac{(\rho + P/c^2)u}{1 - |u|^2/c^2}\right)_t + \text{div}\left(\dfrac{(\rho + P/c^2)}{1 - |u|^2/c^2}\, u \otimes u\right) + \nabla P = 0. \end{cases} \tag{4.1}$$

Here and throughout, $\rho \geq 0$ denotes the mass–energy density, $u = (u^{(1)}, ..., u^{(d)})^\top \in \mathbb{R}^d$ denotes the relativistic velocity, $d \geq 1$ the dimension of the space, $c > 0$ a large constant corresponding to the speed of light, P the pressure of the fluid, $x = (x^{(1)}, ..., x^{(d)})^\top \in \mathbb{R}^d$ the Eulerian spatial coordinate and finally $t \in \mathbb{R}_{\geq 0}$ denotes the time coordinate. The constitutive relation $P = P(\rho)$ considered throughout most of this chapter is

$$P(\rho) = k^2 \rho^\gamma, \tag{4.2}$$

for fixed constants $k > 0$ and $\gamma > 1$.

Abiding by the fact that the theory of special relativity, in a regime of low velocities, should reduce to the classical Newtonian theory, the system (4.1) formally reduces

to the classical d-dimensional isentropic compressible Euler equations (CE) when c approaches infinity:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = 0. \end{cases} \tag{4.3}$$

It is well-known that, for nonlinear hyperbolic conservation laws, a singularity can form in finite time from initial compression no matter how small or smooth the data are. Classical results including Liu [56], Li-Zhou-Kong [53] et al. confirm that when the initial data are small smooth perturbations of constant states, a blowup in the gradient of the solutions occurs in finite time *if and only if* the initial data contain any compression in a genuinely nonlinear characteristic field.

A natural follow-up question is whether this dichotomy persists, at least for archetypal systems of conservation laws such as the (1+1)-dimensional relativistic Euler equations (4.1) (resp.the classical compressible Euler equations (4.3)), when one passes to the framework of large data problems. It turns out, as we shall promptly explain in this work, that one of the key issues towards establishing a similar dichotomy for large data is finding an effective way to obtain sharp enough control on the lower bound of the mass–energy density (resp. the mass density). For the 1-dimensional classical compressible Euler equations (4.3), some important progress has been achieved for large data problems. When $\gamma \geq 3$ in the pressure law (4.2), the argument in Lax [46] for general 2×2 symmetric hyperbolic systems can be applied as is to the large data problem for the isentropic Euler equations (see Section 4.4 of the current section for CE)[1]. What renders this possible is that, when $\gamma \geq 3$, a lower bound control for the density is not needed. Therefore, the essence of the problem is to establish the finite time singularity formation for the compressible Euler equations in the most physically relevant case $1 < \gamma < 3$. For piecewise Lipschitz continuous solutions, in an interesting paper, Lin [55] argues that the density has a (sharp) $O(1+t)^{-1}$ lower bound and proceeds to infer the corresponding global well-posedness. However satisfying, Lin's result comes with an important caveat: It only applies to initial data that are purely rarefactive, i.e. devoid of any compression. For general C^1 solutions including compressions in the solution, further novelties were required. In a recent paper [11], Chen-Pan-Zhu find an $O(1+t)^{-\frac{4}{(3-\gamma)}}$ lower bound when $1 < \gamma < 3$ *and when the data is uniformly away from vacuum.* This result helps them to prove that gradient blowup of ρ and/or u happens in finite time *if and only if* the initial data are forward or backward compressive somewhere, thus establishing the same dichotomy

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[1]See also a generalization to full Euler equations by Chen-Young-Zhang [14].

observed for small initial data [53, 56]. Some further developments were achieved in [12] where, for general Lipschitz continuous solutions of (4.3) with $1 < \gamma < 3$, the authors improve the lower bound on the density from $O((1+t)^{-\frac{4}{(3-\gamma)}})$ in [11] to the optimal order of $O(1+t)^{-1}$. Finally, in Chen-Chen-Zhu [13] the authors provide a new method to extend the theory to more general initial density profiles including possible far field vacuum, such as when $\rho(0,x) \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Also, the authors provide the first global continuous non-isentropic solution including weak compression, using a new method involving solving an inverse Goursat problem. The solution they obtain is almost classical, except on two characteristics, along which the solution has a weak discontinuity (continuous but non-smooth). There is a large literature of works on sufficient conditions for formation of singularities in solutions to the compressible Euler equations and systems of hyperbolic conservation laws in multiple space dimensions. See a brief list in [20, 21, 29, 54, 61, 67, 78, 84].

In contrast, for the (1+1)-dimensional relativistic Euler equations, studies on the singularity formation and global well-posedness of solutions with large data are very few. Under the assumption that $P = \sigma^2 \rho$ for some positive constant $\sigma < c$, Smoller-Temple establish in their seminal work [85] the existence of a global BV weak solution to the Cauchy problem, by crucially noticing that the shock curves for (4.1) in one dimension satisfy very strong geometric properties. For a general γ-law $P(\rho)$, Chen [16] studies the corresponding Riemann problem. Later, Hsu-Lin-Markino [38] establish the existence of global L^∞ weak solutions with initial data containing the vacuum state. For smooth solutions, Ruan-Zhu [82] first prove the global well-posedness of C^1 solutions with large data to the Cauchy problem for the (1+1)-dimensional relativistic Euler equations if the initial data do not have compression (see Definition 4) and are away from vacuum. For the (d+1)-dimensional (d $\geq$ 1) relativistic fluids, Pan-Smoller [76] introduce two sufficient conditions for the formation of singularities of smooth solutions: the initial data are compactly supported, or the radial component of the initial generalized momentum is sufficiently large. However, for the sufficient and necessary condition for singularity formation in C^1 solutions with large data, according to our discussion in Section 4.5, due to lack of the lower bound estimate for the mass–energy density, the problem has remained hitherto unexplored. The current paper addresses this problem for the system (4.1) in the $(1 + 1)$-dimensional spacetime setting. This is a solid step in the study of relativistic Euler equations.

Our work is divided into 5 sections. In Section 4.2, we introduce some basic notations and equations. In Section 4.3, we state the main results. In Section 4.4, for the

smooth solutions with large data of the 1-dimensional (classical) isentropic Euler equations in Eulerian coordinates, we present our novel idea for obtaining a time-dependent mass density lower bound. The idea we present involves a careful study of the difference of the two Riemann invariants (and the study of certain weighted gradients of them). Naturally, why a new idea is needed in the first place is something that requires explanation. The reason is that in this problem, unlike the classical Euler equations, the introduction of Lagrangian coordinates cannot demystify the mathematical structure of the system under study in the same, efficient way it did in [11, 12, 13]. In other words, upon a thorough read of [11, 12, 13], one can see that the relation

$$(1/\rho)_t = \frac{r_x + s_x}{2},$$

where r and s are the Riemann invariants of the so-called p-system in the Lagrange coordinate setting, is the cornerstone of the argument for obtaining the desired lower bound estimate. In the Eulerian setting this relation, or others of similar simplicity, are unavailable. Indeed, here one has

$$(1/\rho)_t = -u(1/\rho)_x + (1/\rho)u_x$$

and therefore in order to get the mass density lower bound, i.e. $(1/\rho)'s$ upper bound, we should first have the upper bound of $-u(1/\rho)_x + (1/\rho)u_x$, which seems hard to obtain. In Section 4.5, for $1 < \gamma < 3$, combining an elaborate argument on a particular ODE inequality and introducing the crucial artificial quantity

$$\left(\frac{k\rho^{(\gamma-1)/2}/c}{\sqrt{1 + \frac{k^2\rho^{\gamma-1}}{c^2}}} \right)^{\frac{3-\gamma}{2\gamma-2}} \left(1 + \frac{k^2\rho^{\gamma-1}}{c^2} \right)^{\frac{\gamma+1}{4\gamma-4}} := \mathcal{Y}, \tag{4.4}$$

we apply our idea to get a lower bound estimate for the mass–energy density of the (1+1)-dimensional relativistic Euler equations. Ultimately, for C^1 solutions with large data and uniformly positive initial mass–energy density of the (1+1)-dimensional relativistic Euler equations, we give a necessary and sufficient condition for the formation of singularities in finite time.

4.2   Basic setup

Before introducing the main results of this work, we provide some equations and estimates for C^1 solutions of (4.1)-(4.2) or (4.3) with (4.2), together with initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{4.5}$$

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4.2.1 (1+1)-dimensional Relativistic Euler equations

We first define the C^1 solutions as follows:

Definition 3. *Let $T > 0$ be some time. The pair $(\rho(t, x), u(t, x))$ is called a C^1 solution to the relativistic Euler equations (4.1)-(4.2) on $(0, T) \times \mathbb{R}$ if*

$$\rho > 0, \quad \rho \in C^1([0, T) \times \mathbb{R}), \; u \in C^1([0, T) \times \mathbb{R}),$$

and the equations (4.1)-(4.2) are satisfied in the pointwise sense on $(0, T) \times \mathbb{R}$. It is called a C^1 solution to the Cauchy problem (4.1)-(4.2) with (4.5) if it is a C^1 solution to the equations (4.1)-(4.2) on $(0, T) \times \mathbb{R}$ and admits the initial data (4.5) continuously.

It is well-known[2] that there exists a local-in-time C^1 solution (ρ, u) in $[0, T] \times \mathbb{R}$ for some $T > 0$, when

$$\inf_{x \in \mathbb{R}} \rho_0 > 0, \quad (\rho_0, \; u_0) \in C^1(\mathbb{R}). \tag{4.6}$$

We proceed with a rudimentary analysis of the Cauchy problem (4.1)-(4.2) with (4.5). First, the two eigenvalues λ_1 and λ_2 of equations (4.1)-(4.2) can be given by

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \frac{u\sqrt{P'}}{c^2}} \quad \text{and} \quad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \frac{u\sqrt{P'}}{c^2}}. \tag{4.7}$$

We denote the directional derivatives as

$$\prime = \partial_t + \lambda_1 \partial_x, \quad \backprime = \partial_t + \lambda_2 \partial_x \tag{4.8}$$

along the two characteristic directions

$$\frac{dx^1}{dt} = \lambda_1 \quad \text{and} \quad \frac{dx^2}{dt} = \lambda_2, \tag{4.9}$$

respectively and introduce the corresponding Riemann variables

$$w = \frac{c}{2} \ln\left(\frac{c + u}{c - u}\right) + \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} \mathrm{d}\sigma, \tag{4.10}$$

$$z = \frac{c}{2} \ln\left(\frac{c + u}{c - u}\right) - \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} \mathrm{d}\sigma. \tag{4.11}$$

Then, it is easy to see that w and z satisfy

$$w^\backprime = 0 \quad \text{and} \quad z' = 0. \tag{4.12}$$

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[2]See [82] and the references cited therein.

Let h_1 and h_2 be functions satisfying

$$h_{1w} = \frac{\lambda_{1w}}{\lambda_1 - \lambda_2}, \quad h_{2z} = \frac{\lambda_{2z}}{\lambda_2 - \lambda_1}. \tag{4.13}$$

Define $\alpha = z_x, \beta = w_x$ and introduce

$$\xi = e^{h_1}\alpha, \quad \zeta = e^{h_2}\beta. \tag{4.14}$$

We continue with a simplification for $s(\rho)$, as can be found for example in [16],

$$s(\rho) = \frac{2c\sqrt{\gamma}}{\gamma - 1}\arctan\left(\frac{k\rho^{(\gamma-1)/2}}{c}\right). \tag{4.15}$$

Another important calculation is the following expression for $\sqrt{P'(\rho)}$ in terms of the Riemann invariants:

$$\sqrt{P'(\rho)} = k\sqrt{\gamma}\rho^{(\gamma-1)/2} = c\sqrt{\gamma}\tan\left(\frac{(w - z)(\gamma - 1)}{4c\sqrt{\gamma}}\right). \tag{4.16}$$

Finally, we define the compression and rarefaction characters.

Definition 4. *The local R/C character for a classical solution of* (4.1)-(4.2) *with* (4.5) *is*

Forward	R	iff	$w_x > 0$;	*Forward*	C	iff	$w_x < 0$;
Backward	R	iff	$z_x > 0$;	*Backward*	C	iff	$z_x < 0$.

Although this definition was not clearly provided in Lax [46], his result on some cases of 2×2 hyperbolic conservation laws can be explained as follows: a singularity forms in finite time if and only if there exists some backward or forward compression under Definition 4. According to the results obtained in this work, we see that this definition of compression and rarefaction can be used to give a necesary and sufficient criterion for singularity formation.

4.2.2   1-dimensional classical compressible Euler equations

Let $d = 1$ in (4.3) and (4.5). To make the corresponding statement precise, we first define the C^1 solutions as follows:

Definition 5. *Let $T > 0$ be some time. The pair $(\rho(t, x), u(t, x))$ is called a C^1 solution to the non-relativistic Euler equations* (4.3) *with* (4.2) *on* $(0, T) \times \mathbb{R}$ *if*

$$\rho > 0, \quad \rho \in C^1([0, T) \times \mathbb{R}), \ u \in C^1([0, T) \times \mathbb{R}),$$

and the equations (4.3) *with* (4.2) *are satisfied pointwise on* $(0, T) \times \mathbb{R}$. *It is called a* C^1 *solution to the Cauchy problem* (4.3) *with* (4.2) *and* (4.5) *if it is a* C^1 *solution to the equations* (4.3) *with* (4.2) *on* $(0, T) \times \mathbb{R}$ *and admits the initial data* (4.5) *continuously.*

As in the relativistic case, it is well-known that there exists a local-in-time C^1 solution (ρ, u) in $[0, T] \times \mathbb{R}$ for some $T > 0$, when (4.6) is satisfied. We proceed with a rudimentary analysis of the Cauchy problem (4.3) with (4.2) and (4.5). First, the two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of equations (4.3) with (4.2) can be given by

$$\tilde{\lambda}_1 = u - \sqrt{P'}, \quad \tilde{\lambda}_2 = u + \sqrt{P'}. \tag{4.17}$$

We denote the directional derivatives as

$$\partial_- = \partial_t + \tilde{\lambda}_1 \partial_x, \quad \partial_+ = \partial_t + \tilde{\lambda}_2 \partial_x \tag{4.18}$$

along the two characteristic directions

$$\frac{dy^1}{dt} = \tilde{\lambda}_1 \quad \text{and} \quad \frac{dy^2}{dt} = \tilde{\lambda}_1, \tag{4.19}$$

respectively and introduce the corresponding Riemann variables

$$\tilde{w} = u + \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma} \, ds, \quad \tilde{z} = u - \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma} \, d\sigma. \tag{4.20}$$

Then, it is easy to see that $\tilde{w}$ and $\tilde{z}$ satisfy

$$\partial_+ \tilde{w} = 0 \quad \text{and} \quad \partial_- \tilde{z} = 0. \tag{4.21}$$

Let $\tilde{h}_1, \tilde{h}_2$ be functions satisfying

$$\tilde{h}_{1\tilde{w}} = \frac{\tilde{\lambda}_{1\tilde{w}}}{\tilde{\lambda}_1 - \tilde{\lambda}_2}, \quad \tilde{h}_{2\tilde{z}} = \frac{\tilde{\lambda}_{2\tilde{z}}}{\tilde{\lambda}_2 - \tilde{\lambda}_1}. \tag{4.22}$$

Define $\tilde{\alpha} = \tilde{z}_x$, $\tilde{\beta} = \tilde{w}_x$ and introduce

$$\phi = \mathrm{e}^{\tilde{h}_1} \tilde{\alpha}, \quad \psi = \mathrm{e}^{\tilde{h}_2} \tilde{\beta}. \tag{4.23}$$

Finally, we define the compression and rarefaction characters.

Definition 6. *The local R/C character for a classical solution of* (4.3) *with* (4.2) *and* (4.5) *is*

$$\begin{array}{llll}
\textit{Forward} & R \;\; \textit{iff} \;\; \tilde{w}_x > 0; & \textit{Forward} & C \;\; \textit{iff} \;\; \tilde{w}_x < 0; \\
\textit{Backward} & R \;\; \textit{iff} \;\; \tilde{z}_x > 0; & \textit{Backward} & C \;\; \textit{iff} \;\; \tilde{z}_x < 0.
\end{array}$$

4.3 Statement of main results

From now on, we make the following assumption throughout the rest of this work:

Assumption 2. *Assume that*[3]

$$\inf_{x\in\mathbb{R}}(w_0 - z_0) > 0, \qquad \sqrt{P'\left(\mathcal{J}_{rel}^{-1}\left(\frac{w_{\max} - z_{\min}}{2}\right)\right)} < c,$$

$$(w_0,\ z_0) \in C^1(\mathbb{R}), \qquad \|(w_0, z_0)\|_{C^1} \leq M_0,$$

for some constant $M_0 > 0$; where $(w_0,\ z_0)(x) = (w(0,x),\ z(0,x))$ and

$$w_{max} = \sup\{w_0(x) \mid x \in \mathbb{R}\}, \ z_{min} = \inf\{z_0(x) \mid x \in \mathbb{R}\}.$$

We have also used

$$\mathcal{J}_{rel}(x) = \int_0^x \frac{\sqrt{P'(s)}}{s + \frac{P(s)}{c^2}}\,ds$$

and $\mathcal{J}_{rel}^{-1}$ denotes its inverse function.

Theorem 4.3.1. *For polytropic gases $\gamma > 1$ in (4.2), if $(w_0(x), z_0(x))$ satisfy the conditions in Assumption 2, then the Cauchy problem (4.1)-(4.2) with (4.5) has a unique global-in-time C^1 solution if and only if*

$$w_x(x,0) \geq 0 \quad and \quad z_x(x,0) \geq 0, \ for\ all\ \ x \in \mathbb{R}. \tag{4.24}$$

The proof of Theorem 4.3.1 can be found in Section 4.5. We make some necessary remarks on our conclusions at this point:

Remark 4. *The assumption $\sqrt{P'\left(\mathcal{J}_{rel}^{-1}\left(\frac{w_{\max} - z_{\min}}{2}\right)\right)} < c$ is imposed for one to be able to show that the local sound speed $\sqrt{P'(\rho)}$ is bounded away from the light speed c for C^1 solutions, which is natural in the sense of physics.*

Remark 5. *Our conclusion establishes a necessary and sufficient condition for the formation of singularities for the $(1+1)-$dimensional relativistic Euler equations, namely the existence of compression in the initial data, which gives a complete picture on the formation of singularities for the Cauchy problem of the system (4.1) in $(1+1)$-dimensional Minkowski spacetime.*

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[3]In the case where the gas is polytropic with $\gamma > 1$, the condition $\sqrt{P'\left(\mathcal{J}_{\text{rel}}^{-1}\left(\frac{w_{\max} - z_{\min}}{2}\right)\right)} < c$ rewrites as

$$w_{\max} - z_{\min} < \frac{4c\sqrt{\gamma}}{\gamma - 1}\arctan\left(\frac{1}{\sqrt{\gamma}}\right).$$

Remark 6. *A problem of physical significance is to determine the type of singularity that forms. It is generally expected that the discontinuities developed, among others, in the works of Chen-Pan-Zhu [11], Lax [46] and the current work are indeed discontinuity singularities, i.e. develop as shock waves. The proof of disproof of such a fact, in full generality, remains an open problem. The best partial results, to the author's knowledge, can be found in the paper of Kong [44]. There, for a general class of strictly hyperbolic 2×2 systems with two genuinely nonlinear characteristic fields, it is shown that if a singularity forms then it develops as a shock wave if either*

- *one of the two Riemann invariants, w or z, is initially a constant[4], or*

- *certain a priori conditions, essentially quantitative bounds on the size of the derivatives, hold at the blow-up point; conditions which are, however, difficult to verify dynamically.*

Thus, our Theorem above along with Kong's result readily implies that, if one of the two initial data variables w_0, z_0 is a constant, a shock forms if and only if there exists initial compression in the non-constant Riemann invariant variable. In any case, further study on the type of singularities obtained in this work and several others promises to be a meaningful and interesting direction for research. Another interesting problem, finally, would be to extend this result to initial data allowing for vacuum in the far field, following ideas in [13].

4.4 The mass density lower bound of 1-dimensional CE

Recall at this point the notation introduced in Subsection 4.2.2. We dedicate this section to the presentation of the new approach for obtaining the crucial lower bound estimate on the mass density for the classical compressible Euler equations (4.3). The main idea is that, instead of obtaining a transport equation for ρ and using it to obtain the estimate, we focus instead on the difference of the two Riemann invariants, in the classical case given by $\tilde{w} - \tilde{z}$. The function $\tilde{w} - \tilde{z}$ is an increasing function of ρ and therefore control on $\tilde{w} - \tilde{z}$ translates to control on ρ. To best exhibit our approach, we apply it in the first subsection to the classical Euler equations. In the final subsection, we lay down the main argument for singularity formation in finite

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[4]Thus the study of the system reduces to the study of a scalar conservation law, which in general has a complete theory. See also Lebaud [48] for the 1-D classical Euler equations.

time, which is essentially that of [46]. We explain then why a lower bound estimate is of such importance; and how we may use it to conclude our argument.

4.4.1 The mass density lower bound estimate in the Eulerian setting

We begin by noticing that the weighted gradients satisfy certain Riccati equations.

Lemma 4.4.1. *For the C^1 solution of the system (4.3), the following Riccati ODEs hold:*

$$\partial_- \phi = -\left(e^{-\tilde{h}_1} \tilde{\lambda}_{1\tilde{z}}\right)\phi^2, \quad \partial_+ \psi = -\left(e^{-\tilde{h}_2} \tilde{\lambda}_{2\tilde{w}}\right)\psi^2. \tag{4.25}$$

Moreover, let $y^i(t, y_0^i)$ $(i = 1, 2)$ be two characteristic curves (defined in Section 2.2) starting from $(0, y_0^i)$. One has

$$\frac{1}{\phi(t, y^1(t, y_0^1))} = \frac{1}{\phi(0, y_0^1)} + \int_0^t \left(e^{-\tilde{h}_1} \tilde{\lambda}_{1\tilde{z}}\right)(\sigma, y^1(\sigma, y_0^1)) \, d\sigma.$$

$$\frac{1}{\psi(t, y^2(t, y_0^2))} = \frac{1}{\psi(0, y_0^2)} + \int_0^t \left(e^{-\tilde{h}_2} \tilde{\lambda}_{2\tilde{w}}\right)(\sigma, y^2(\sigma, y_0^2)) \, d\sigma.$$

Proof. Differentiating the last equation of (4.21) with respect to x and recalling the definition of $\tilde{h}_1$ from (4.22), we arrive at

$$\partial_- \tilde{\alpha} + (\partial_- \tilde{h}_1)\tilde{\alpha} + \tilde{\lambda}_{1\tilde{z}} \tilde{\alpha}^2 = 0,$$

which, along with $\phi = e^{\tilde{h}_1}\tilde{\alpha}$, implies the desired ODE on ϕ. The formula of ϕ along the backward characteristic curve can be obtained by solving the Riccati ODE that we obtained. The proof for ψ is similar and we omit the details.

$\square$

Our strategy will be to work towards obtaining a time-dependent lower bound on $\tilde{w} - \tilde{z}$. To achieve this, we must first rewrite $\tilde{\lambda}_1, \tilde{\lambda}_2$ in terms of $\tilde{w}, \tilde{z}$:

$$\begin{cases} \tilde{\lambda}_1 = \dfrac{\tilde{w} + \tilde{z}}{2} - \dfrac{(\tilde{w} - \tilde{z})(\gamma - 1)}{4}, \\ \tilde{\lambda}_2 = \dfrac{\tilde{w} + \tilde{z}}{2} + \dfrac{(\tilde{w} - \tilde{z})(\gamma - 1)}{4}. \end{cases} \tag{4.26}$$

One can get that $\tilde{\lambda}_{1\tilde{z}} = \tilde{\lambda}_{2\tilde{w}} = \frac{\gamma+1}{4}$. Notice that in the classical case treated here, one can take

$$\tilde{h}_1 = \tilde{h}_2 = \tilde{h} = \frac{\gamma - 3}{2\gamma - 2} \ln(\tilde{w} - \tilde{z}), \tag{4.27}$$

which implies that

$$e^{-\tilde{h}_1}\tilde{\lambda}_{1\tilde{z}} = e^{-\tilde{h}_2}\tilde{\lambda}_{2\tilde{w}} = \frac{\gamma+1}{4}(\tilde{w}-\tilde{z})^{\frac{3-\gamma}{2\gamma-2}}. \tag{4.28}$$

Based on these observations and Lemma 4.4.1, standard ODE theory then leads us to the following result:

Proposition 4.4.1. *Denote*

$$\widetilde{Q}_1 = max\left\{0, \sup_x \phi(0,x)\right\}, \quad \widetilde{Q}_2 = max\left\{0, \sup_x \psi(0,x)\right\}.$$

For the C^1 solution of the system (4.3), there holds $\phi \leq \widetilde{Q}_1$, $\psi \leq \widetilde{Q}_2$.

Finally, we can get the desired lower bound estimates of the mass density.

Lemma 4.4.2. *let $y^i(t, y_0^i)$ $(i = 1, 2)$ be two characteristic curves starting from $(0, y_0^i)$. There holds*

$$\left(e^{-\tilde{h}}\tilde{\lambda}_{1\tilde{z}}\right)(t, y^1(t, y_0^1)) \geq \frac{1}{C_1 + C_2 t}, \quad \left(e^{-\tilde{h}}\tilde{\lambda}_{2\tilde{w}}\right)(t, y^2(t, y_0^2)) \geq \frac{1}{C_1 + C_2 t}, \tag{4.29}$$

for positive constants C_i $(i = 1, 2)$ independent of the time.

Proof. According to (4.21) and the definition of ψ, one can obtain that

$$\partial_-(\tilde{w}-\tilde{z}) = (\tilde{w}-\tilde{z})_t + \tilde{\lambda}_1(\tilde{w}-\tilde{z})_x = (\tilde{\lambda}_1-\tilde{\lambda}_2)\tilde{w}_x = -\frac{\tilde{\lambda}_2-\tilde{\lambda}_1}{e^{\tilde{h}_2}}\psi. \tag{4.30}$$

which, along with Proposition 4.4.1 and the relation (4.28), implies that

$$\partial_-(\tilde{w}-\tilde{z}) \geq -\frac{\tilde{\lambda}_2-\tilde{\lambda}_1}{e^{\tilde{h}_2}}\widetilde{Q}_2 = -\frac{(\gamma-1)\widetilde{Q}_2}{2}(\tilde{w}-\tilde{z})^{\frac{\gamma+1}{2\gamma-2}}. \tag{4.31}$$

Denoting $C' = \frac{(\gamma-1)\widetilde{Q}_2}{2}$ and integrating (4.31) along $y^1(t, y_0^1)$ over $[0, t]$, yields

$$(\tilde{w}-\tilde{z})(t, y^1(t, y_0^1)) \geq \left(\left((\tilde{w}-\tilde{z})(0, y_0^1)\right)^{\frac{2\gamma-2}{\gamma-3}} + \frac{3-\gamma}{2\gamma-2}C't\right)^{\frac{2\gamma-2}{\gamma-3}}. \tag{4.32}$$

In particular, taking (4.28) into account, one can obtain the time-dependent lower bound

$$\left(e^{-\tilde{h}}\tilde{\lambda}_{1\tilde{z}}\right)(t, y^1(t, y_0^1)) \geq \frac{1}{C_1 + C_2 t}, \tag{4.33}$$

for positive constants C_i $(i = 1, 2)$ independent of t. Similarly, we can obtain the second estimate. $\qquad\square$

Remark 7. *Taking into account that $w - z = C\rho^{\frac{\gamma-1}{2}}$, where C is a universal constant and Lemma 4.4.2, we recover precisely the result of [11]:*

$$\rho(x,t) \geq \left(\frac{1}{C_1 + C_2 t}\right)^{\frac{4}{3-\gamma}}.$$

4.4.2 Formation of singularity

It is important at this stage to highlight the main mechanism that shall be used throughout this work to obtain the formation of singularities. The argument within this subsection can be traced back to P.D. Lax in his 1964 paper on 2×2 systems. This is best described in the context of system (4.3). The Riccati type equations in Lemma 4.4.1 are precisely what gives us a clear passage to study the singularity formation and/or global existence of classical solutions for hyperbolic systems with two unknowns.

Without loss of generality, we assume that there exists a point $(0, y_0^1)$ on the initial data line $t = 0$ such that $\phi(0, y_0^1) < 0$. Then we see that a sufficient condition for the breakdown of the classical solution is

$$\int_0^\infty \left(e^{-\tilde{h}_1} \tilde{\lambda}_{1\tilde{z}} \right) (t, y^1(t, y_0^1)) \, \mathrm{d}t = \infty, \tag{4.34}$$

which, actually can be verified by the conclusions obtained in Lemma 4.4.2. The proof of our main theorem in the next section essentially comes down to establishing a statement of the form (4.34) for the system of $(1+1)-$dimensional relativistic Euler equations.

4.5 Formation of singularities for the RE equations in (1+1) dimensions

In this section we shall lay down the proof of Theorem 4.3.1. As we mentioned in Subsection 4.2.1, it is a well-known result that given initial data as in Theorem 4.3.1, there exists a $T \in (0, \infty)$ such that there exists a local-in-time C^1 solution to the Cauchy problem $(4.1) - (4.2)$ with (4.5). We proceed by obtaining estimates on the solution in the slab $[0, T] \times \mathbb{R}$.

4.5.1 Preliminaries

Before giving the detailed proof, we first give several fundamental lemmata for the RE equations. First, we show that the relativistic fluid velocity u is less than the light speed c.

Lemma 4.5.1. *For the C^1 solution of the Cauchy problem $(4.1)-(4.2)$ with (4.5), under Assumption 2, the absolute value $|u|$ of the velocity function is uniformly bounded away from the speed of light c.*

Proof. According to (4.21), one can obtain that

$$\left| \ln\left(\frac{c+u}{c-u}\right) \right| = \left| \frac{w+z}{c} \right| \leq \frac{2M_0}{c}. \tag{4.35}$$

That is to say,

$$e^{-\frac{2M_0}{c}} < \frac{c+u}{c-u} < e^{\frac{2M_0}{c}},$$

which implies that $|u|$ is uniformly bounded away from c. $\qquad\square$

Second, we confirm that the mass–energy density will keep the positivity property.

Lemma 4.5.2. *For the C^1 solution of the Cauchy problem $(4.1) - (4.2)$ with (4.5), under Assumption 2, $\rho > 0$.*

Proof. According to (4.16), one has

$$\rho = \left(\frac{c}{k}\tan\left(\frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}\right)\right)^{\frac{2}{\gamma-1}} := F(w-z).$$

Notice that $F(0) = 0$. Denote $\theta = w - z$. We can then rewrite

$$\lambda_1 - \lambda_2 = -\frac{2\sqrt{P'(\rho)}\,(1-u^2/c^2)}{1-u^2 P'(\rho)/c^4} = -\frac{2\sqrt{P'(F(\theta))}(1-u^2/c^2)}{1-u^2 P'(F(\theta))/c^4} = g(\theta, u).$$

Notice then that $g(0, u) = 0$ and

$$-\frac{2\sqrt{P'(F(\theta))}(1-u^2/c^2)}{1-u^2 P'(F(\theta))/c^4} = \frac{\partial g(\theta_1, u)}{\partial u}\theta,$$

where θ_1 is between 0 and θ. Thus,

$$(w-z)_t + \lambda_2(w-z)_x = \frac{\partial g(\theta_1, u)}{\partial u}(w-z)z_x.$$

Let $x^2 = x^2(t, x_0^2)$ denote the the forward characteristic curve starting from the point $(0, x_0^2)$. Integrating along this forward characteristic over $[0, t]$, one can get

$$(w-z)(t, x) = (w_0 - z_0)(x_0^2)\exp\left(\int_0^t \frac{\partial g(\theta_1, u)(\sigma, x^2(\sigma, x_0^2))}{\partial \theta} z_x(\sigma, x^2(\sigma, x_0^2))\,\mathrm{d}\sigma\right),$$

which, along with Assumption 2, implies the desired conclusion. Here we use that $w - z$ is a strictly increasing function of ρ. $\qquad\square$

Next, we show that the local sound speed $\sqrt{P'(\rho)}$ is also less than the light speed c.

Lemma 4.5.3. *For the C^1 solution of the Cauchy problem (4.1)-(4.2) with (4.5), there holds $\sqrt{P'(\rho)} < c$.*

Proof. According to (4.16), Assumption 2 and $w - z \le w_{max} - z_{min}$, one gets

$$\sqrt{P'(\rho)} \le c\sqrt{\gamma}\tan\left(\frac{(w_{max} - z_{min})(\gamma - 1)}{4c\sqrt{\gamma}}\right) < c, \tag{4.36}$$

which can be directly translated to an upper bound on the density ρ,

$$\rho < c^{\frac{2}{\gamma-1}} k^{-\frac{2}{\gamma-1}} \gamma^{-\frac{1}{\gamma-1}}. \tag{4.37}$$

$\square$

Lemmas 4.5.1–4.5.3 show that both $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2$, which implies that the system (4.1)-(4.2) is strictly hyperbolic. Now we note down the Riccati equations satisfied by ξ and ζ defined in Section 4.2.1.

Lemma 4.5.4. *For the C^1 solution of the Cauchy problem $(4.1) - (4.2)$ with (4.5), under Assumption 2, the following Riccati ODEs hold:*

$$\xi' = -\left(e^{-h_1}\lambda_{1z}\right)\xi^2, \quad \zeta^{\backprime} = -\left(e^{-h_2}\lambda_{2w}\right)\zeta^2. \tag{4.38}$$

Moreover, let $x^i(t, x_0^i)$ $(i = 1, 2)$ be two characteristic curves starting from $(0, x_0^i)$. One has

$$\frac{1}{\xi(t, x^1(t, x_0^1))} = \frac{1}{\xi(0, x_0^1)} + \int_0^t \left(e^{-h_1}\lambda_{1z}\right)(\sigma, x^1(\sigma, x_0^1))\, d\sigma.$$

$$\frac{1}{\zeta(t, x^2(t, x_0^2))} = \frac{1}{\zeta(0, x_0^2)} + \int_0^t \left(e^{-h_2}\lambda_{2w}\right)(\sigma, x^2(\sigma, x_0^2))\, d\sigma.$$

The proof is identical to that of Lemma 4.4.1. We omit the details. In what follows, it turns out that there is a clear distinction in the proof between the cases $\gamma \ge 3$ and the *physical* range $1 < \gamma < 3$. Because of that, we will lay down the proof for each of those two cases in separate subsections.

4.5.2 Proof for the case $\gamma \ge 3$ of Theorem 4.3.1

Instead of working with (4.1)-(4.2), here and throughout we will rephrase the problem entirely in the language of Riemann invariants w and z, i.e. system (4.12). To that end, we must first focus our attention on rewriting λ_1 and λ_2 as functions of w and z instead of ρ and u.

Lemma 4.5.5. *Define the following functions*

$$f(w, z) := \frac{w + z}{c} + \ln\left(\frac{1 - \sqrt{\gamma}\, tan\left(\frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}\right)}{1 + \sqrt{\gamma}\, tan\left(\frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}\right)}\right), \tag{4.39}$$

$$g(w, z) := \frac{w + z}{c} + \ln\left(\frac{1 + \sqrt{\gamma}\, tan\left(\frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}\right)}{1 - \sqrt{\gamma}\, tan\left(\frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}\right)}\right). \tag{4.40}$$

For the C^1 solution of the Cauchy problem $(4.1) - (4.2)$ with (4.5), under the Assumption 2, there holds

$$\lambda_1(w, z) = c\frac{e^f - 1}{e^f + 1}, \quad \lambda_2(w, z) = c\frac{e^g - 1}{e^g + 1}. \tag{4.41}$$

Proof. We notice that, the functions λ_1, λ_2, written in terms of ρ and u, are reminiscent of (in fact identical to) the relativistic addition formulae for u and $\sqrt{P'(\rho)}$. In particular,

$$\ln\left(\frac{c + \lambda_1}{c - \lambda_1}\right) = \ln\left(\frac{c + u}{c - u}\right) + \ln\left(\frac{c - \sqrt{P'}}{c + \sqrt{P'}}\right) = \tilde{f}(\rho, u) = f(w, z), \tag{4.42}$$

$$\ln\left(\frac{c + \lambda_2}{c - \lambda_2}\right) = \ln\left(\frac{c + u}{c - u}\right) + \ln\left(\frac{c + \sqrt{P'}}{c - \sqrt{P'}}\right) = \tilde{g}(\rho, u) = g(w, z), \tag{4.43}$$

which, along with (4.16) and solving for λ_1, λ_2, yields the desired relations. $\qquad\square$

Now we are ready to give the proof for the case $\gamma \geq 3$ of Theorem 4.3.1.

Proof. Assume now, without loss of generality[5], that there exists $x \in \mathbb{R}$ such that $z_0'(x) < 0$. According to Lemma 4.5.4, what we need to show is just the divergence of the integral

$$\int_0^\infty (e^{-h_1}\lambda_{1z})(\sigma, x^1(\sigma, x_0^1))\, d\sigma.$$

For this purpose, we divide the the rest of the proof into two steps.

Step 1: The detailed formula of $e^{-h_1}\lambda_{1z}$. Introduce, for convenience, the notation

$$Y = \frac{(w - z)(\gamma - 1)}{4\, c\sqrt{\gamma}}.$$

It follows from direct calculations that

$$\lambda_{1w} = \frac{e^{\frac{w+z}{c}}\left(2 - 2\gamma + (1 + \gamma)\cos(2Y)\right)\sec(Y)^2}{\left(1 + e^{\frac{w+z}{c}} - (e^{\frac{w+z}{c}} - 1)\sqrt{\gamma}\tan(Y)\right)^2}, \tag{4.44}$$

---

[5]If instead $w_0'(x) < 0$ the proof is precisely the same after relabelling the corresponding variables.

and

$$\lambda_1 - \lambda_2 = \frac{8\,c\,e^{\frac{w+z}{c}}\sqrt{\gamma}\tan(Y)}{-(1+e^{\frac{w+z}{c}})^2 + (e^{\frac{w+z}{c}}-1)^2\,\gamma\tan(Y)^2}. \tag{4.45}$$

Upon simplification, we have

$$\frac{\lambda_{1w}}{\lambda_1 - \lambda_2}$$
$$= \frac{\left(2-2\gamma+(1+\gamma)\cos(2Y)\right)\left((e^{\frac{w+z}{c}}-1)\sqrt{\gamma}+(1+e^{\frac{w+z}{c}})\cot(Y)\right)\operatorname{cosec}(Y)\sec(Y)}{8\,c\gamma\,(e^{\frac{w+z}{c}}-1) - 8\,c\sqrt{\gamma}\,(1+e^{\frac{w+z}{c}})\cot(Y)}.$$
$$\tag{4.46}$$

It should be pointed out that the complicated expression (4.46) can be explicitly integrated[6] with respect to w, which provides us with an explicit form for the function h_1 satisfying (4.13):

$$h = \frac{3\gamma-1}{2\gamma-2}\ln(\cos(Y)) + \frac{\gamma-3}{2\gamma-2}\ln(\sin(Y)) + \frac{w-z}{2c}$$
$$- \ln\left(\left(1+e^{\frac{w+z}{c}}\right)\cos(Y) - \left(-1+e^{\frac{w+z}{c}}\right)\sin(Y)\right). \tag{4.47}$$

We also notice that the term

$$\left(1+e^{\frac{w+z}{c}}\right)\cos(Y) - \left(-1+e^{\frac{w+z}{c}}\right)\sin(Y)$$

contained inside the ln-function is positive, as $\tan(Y) < \frac{1}{\sqrt{\gamma}} < 1$ because of (4.16) and Assumption 2. It follows from direct calculations that

$$\lambda_{1z} = \frac{e^{\frac{w+z}{c}}(1+\gamma)\cos(2Y)\sec(Y)^2}{\left(1+e^{\frac{w+z}{c}} - (e^{\frac{w+z}{c}}-1)\sqrt{\gamma}\tan(Y)\right)^2}. \tag{4.48}$$

It is, at this point, more useful to actually rewrite $e^{-h_1}\lambda_{1z}$ in terms of the original (ρ, u)-coordinates. Define for convenience

$$y = \frac{k\rho^{(\gamma-1)/2}}{c},$$

then the following formula holds:

$$e^{-h_1}\lambda_{1z} = \frac{c\,e^{-\frac{2\sqrt{\gamma}\arctan(y)}{\gamma-1}}(c+u)(\gamma+1)\left(\frac{y}{\sqrt{y^2+1}}\right)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}}(1-y^2)}{2c^2 - 2u\sqrt{P'(\rho)}} > 0. \tag{4.49}$$

---

[6]This was carried out on a computer with the help of Wolfram Mathematica.

Step 2: the uniform lower bound of $\mathrm{e}^{-h_1}\lambda_{1z}$. We analyze the above formulae of $\mathrm{e}^{-h_1}\lambda_{1z}$ term by term:

- the term $c\,\mathrm{e}^{-\frac{2\sqrt{\gamma}\arctan(y)}{\gamma-1}}$ can be bounded below by $c\,\mathrm{e}^{-\frac{\pi\sqrt{\gamma}}{\gamma-1}}$;

- the term $\frac{(c+u)(\gamma+1)}{2c^2-2u\sqrt{P'(\rho)}}$ can be uniformly bounded below by a positive constant C_0 depending only on the initial data and c, because of Lemmas 4.5.1 and 4.5.3;

- the term $\left(\frac{y}{\sqrt{y^2+1}}\right)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}}$ is bounded below by 1 for all $\gamma \geq 3$;

- the term $1-y^2$ can be bounded below by $1-\frac{1}{\gamma}$, as $\sqrt{P'(\rho)} < c$ implies

$$y < \gamma^{-\frac{1}{2}}, \tag{4.50}$$

which will play an important role in dealing with the physical case $1 < \gamma \leq 3$ later.

Then, according to the solution's formula shown in Lemma 4.5.4 and the uniform lower bound of $e^{-h_1}\lambda_{1z}$ obtained above, our theorem thus follows for the case $\gamma \geq 3$.

$\square$

Remark 8. *It is easy to see by the above discussion that the term which will ultimately dictate the divergence, or the (hopefully not to be encountered!) convergence of the integral of $e^{-h}\lambda_{1z}$ in time is*

$$\left(\frac{k\rho^{(\gamma-1)/2}/c}{\sqrt{1+\frac{k^2\rho^{\gamma-1}}{c^2}}}\right)^{\frac{3-\gamma}{2\gamma-2}}\left(1+\frac{k^2\rho^{\gamma-1}}{c^2}\right)^{\frac{\gamma+1}{4\gamma-4}} := \mathcal{Y}. \tag{4.51}$$

4.5.3 Proof for the case $1 < \gamma < 3$ of Theorem 4.3.1

Our starting point is Remark 8 from Subsection 4.5.2. We observe that the behavior of $\mathcal{Y}$ ultimately dictates whether or not the integral $\int_0^\infty \left(\mathrm{e}^{-h_1}\lambda_{1z}\right)(\sigma, x^1(\sigma, x_0^1))\,\mathrm{d}\sigma$ diverges. In particular, it becomes clear that in order to prove the Theorem 4.3.1 for $1 \leq \gamma < 3$, we are required to give a proper time-dependent lower bound on $\mathcal{Y}$, a function itself of the density ρ, strong enough such that $\int_0^\infty \mathcal{Y}(\sigma, x^1(\sigma))\,\mathrm{d}\sigma = \infty$. Before showing the detailed proof, we first give a clear outline of our strategy:

- (1)   We rewrite the equations (4.12) in the form of the difference $(w-z)$ of the Riemann invariants:

$$(w-z)' = (w-z)_t + \lambda_1(w-z)_x = (\lambda_1 - \lambda_2)w_x, \tag{4.52}$$

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which implies that,

$$(w - z)' = \frac{\lambda_1 - \lambda_2}{e^{h_2}} \zeta \geq -\frac{\lambda_2 - \lambda_1}{e^{h_2}} Q_2, \tag{4.53}$$

under the assumption that $\zeta = e^{h_2} w_x$ has a uniform upper bound Q_2 independent of the time. Such a kind of bound is established in Subsection 4.5.3.1;

- (2) In order to introduce a suitable ODEs inequality for $\mathcal{Y}$, we first obtain an ODEs inequality for y from (4.53), which requires us to rewrite the quantities $w - z$ and $\frac{\lambda_2 - \lambda_1}{e^{h_2}}$ explicitly in terms of ρ, u. To this end, there is a crucial observation that should be pointed out, that in $\frac{\lambda_2 - \lambda_1}{e^{h_2}}$, all the terms involving u have a uniform upper and lower bound. This can be seen in Subsection 4.5.3.2;

- (3) Based on the analysis on y and the relation between y and $\mathcal{Y}$, we successfully introduce a proper ODEs inequality for $\mathcal{Y}$, which could effectively control the behaviour of the density with respect to time. This can be seen in Subsection 4.5.3.3;

- (4) Finally, we show that the estimates obtained above are indeed sufficiently strong so that the desired quantity (4.51) has a divergent integral over time along the characteristic curve. Then we can obtain the desired conclusion stated in Theorem 4.3.1.

With that in mind, let us begin the proof.

4.5.3.1   Upper bound of the weighted gradients of the Riemann invariants

We now establish an upper bound for the weighted gradients of the Riemann invariants.

Lemma 4.5.6. *Define the non-negative constants*

$$Q_1 := max\left\{0, \sup_x \xi(x, 0)\right\}, \quad Q_2 := max\left\{0, \sup_x \zeta(x, 0)\right\}. \tag{4.54}$$

For the C^1 solution of the system (4.1)-(4.2), under Assumption 2, one has

$$\xi(x, t) \leq Q_1, \quad \zeta(x, t) \leq Q_2. \tag{4.55}$$

Proof. The key to the proof lies in showing that $e^{-h_1} \lambda_{1z}$ and $e^{-h_2} \lambda_{2w}$ are non-negative. For this purpose, according to (4.48) and the formula

$$\lambda_{2w} = \frac{e^{\frac{w+z}{c}} (1 + \gamma) \cos(2Y) \sec(Y)^2}{\left(1 + e^{\frac{w+z}{c}} + (e^{\frac{w+z}{c}} - 1) \sqrt{\gamma} \tan(Y)\right)^2},$$

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one gets that what we need is $\cos(2Y) > 0$, which, actually, can be obtained quickly from (4.50) and the following formulas

$$Y = \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}} = \arctan\left(\frac{k\rho^{(\gamma-1)/2}}{c}\right) \quad \text{and} \quad \cos(2\arctan(x)) = \frac{1-x^2}{1+x^2}.$$

Thus, the conclusion of this lemma follows from the Riccati ODEs established in Lemma 4.5.4.

□

It should be pointed out that, so far, (4.53) has been proved.

4.5.3.2   Establishing an ODE inequality for y

As mentioned before, now we need to rewrite the quantities $w-z$ and $\frac{\lambda_2-\lambda_1}{e^{h_2}}$ explicitly in terms of ρ, u. We note again that $\gamma > 1$ throughout. First, according to (4.16) and (4.39), one has

$$f = \ln\left(\frac{c+u}{c-u}\right) + \ln\left(\frac{1-\sqrt{\gamma}\frac{k\rho^{(\gamma-1)/2}}{c}}{1+\sqrt{\gamma}\frac{k\rho^{(\gamma-1)/2}}{c}}\right), \tag{4.56}$$

$$g = \ln\left(\frac{c+u}{c-u}\right) + \ln\left(\frac{1+\sqrt{\gamma}\frac{k\rho^{(\gamma-1)/2}}{c}}{1-\sqrt{\gamma}\frac{k\rho^{(\gamma-1)/2}}{c}}\right). \tag{4.57}$$

Secondly, from (4.2) and (4.7), one can get

$$\lambda_1 = \frac{c^2(u - k\sqrt{\gamma}\rho^{(\gamma-1)/2})}{c^2 - ku\sqrt{\gamma}\rho^{(\gamma-1)/2}}, \tag{4.58}$$

$$\lambda_2 = \frac{c^2(u + k\sqrt{\gamma}\rho^{(\gamma-1)/2})}{c^2 + ku\sqrt{\gamma}\rho^{(\gamma-1)/2}}, \tag{4.59}$$

whence the following simplified expression for $\lambda_2 - \lambda_1$ follows:

$$\lambda_2 - \lambda_1 = \frac{2c^2 k(c^2-u^2)\sqrt{\gamma}\rho^{(\gamma-1)/2}}{c^4 - k^2 u^2 \rho^{\gamma-1}}. \tag{4.60}$$

Moreover, the following explicit forms for h_1 and h_2 can be obtained:

$$\begin{aligned}
h_1 =& \frac{3\gamma-1}{2\gamma-2}\ln(\cos(Y)) + \frac{\gamma-3}{2\gamma-2}\ln(\sin(Y)) + \frac{w-z}{2c} \\
& - \ln\left(\left(1 + e^{\frac{w+z}{c}}\right)\cos(Y) - \left(-1 + e^{\frac{w+z}{c}}\right)\sqrt{\gamma}\sin(Y)\right), \\
h_2 =& \frac{3\gamma-1}{2\gamma-2}\ln(\cos(Y)) + \frac{\gamma-3}{2\gamma-2}\ln(\sin(Y)) + \frac{w-z}{2c} \\
& - \ln\left(\left(e^{\frac{2w}{c}} + e^{\frac{w-z}{c}}\right)\cos(Y) + \left(e^{\frac{2w}{c}} - e^{\frac{w-z}{c}}\right)\sqrt{\gamma}\sin(Y)\right).
\end{aligned}$$

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Here, as always, we denote $Y = \arctan\left(\frac{k\rho^{(\gamma-1)/2}}{c}\right) = \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}$.

Now we are ready to develop an ODE inequality for y from (4.53).

Lemma 4.5.7. *For the C^1 solution of the Cauchy problem $(4.1) - (4.2)$ with (4.5), under Assumption 2, there holds*

$$y' \geq -C_g\, y^{\frac{\gamma+1}{2\gamma-2}}\,(1+y^2)^{3/2}, \tag{4.61}$$

for some positive constant C_g independent of the time.

Proof. First of all, it follows from direct computations that the following simplified form for $\frac{\lambda_2-\lambda_1}{e^{h_2}}$ holds:

$$\frac{\lambda_2-\lambda_1}{e^{h_2}} = \frac{4ce^{\frac{2\sqrt{\gamma}\arctan(y)}{\gamma-1}}(c+u)\,c\sqrt{\gamma}\,y\left(\frac{y}{\sqrt{y^2+1}}\right)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}}}{c^2 - u\sqrt{P'(\rho)}}, \tag{4.62}$$

where $y = \frac{k\rho^{(\gamma-1)/2}}{c}$. Secondly, according to Lemmas 4.5.1-4.5.3, one can obtain that

$$C_g^{-1} \leq e^{\frac{2\sqrt{\gamma}\arctan(y)}{\gamma-1}} \leq C_g, \quad \text{and} \quad C_g^{-1} \leq \frac{c+u}{c^2 - u\sqrt{P'(\rho)}} \leq C_g, \tag{4.63}$$

for some positive constant C_g independent of the time, which, along with (4.53), implies

$$(w-z)' \geq -C_g\, y\left(\frac{y}{\sqrt{y^2+1}}\right)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}} = -C_g\, y^{\frac{\gamma+1}{2\gamma-2}}(1+y^2)^{\frac{1}{2}}. \tag{4.64}$$

Notice that

$$w - z = \frac{4c\sqrt{\gamma}}{\gamma-1}\arctan(y), \quad \text{and} \quad (w-z)' = C_g\frac{y'}{y^2+1},$$

so that we can rewrite (4.64) as (4.61). $\qquad\square$

4.5.3.3 Establishing an ODE inequality for $\mathcal{Y}$

We now recall that what determines the convergence/divergence of the integral of $e^{-h_1}\lambda_{1z}$ in time is the variable $\mathcal{Y}$ from (4.51), which can be rewritten as

$$\mathcal{Y} = \left(\frac{y}{\sqrt{y^2+1}}\right)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}}. \tag{4.65}$$

At this point, without loss of generality[7], that there exists $x \in \mathbb{R}$ such that $z_0'(x) < 0$. According to Lemma 4.5.4, what we need to show is just the divergence of the integral

$$\int_0^\infty (\mathrm{e}^{-h_1} \lambda_{1z})(\sigma, x^1(\sigma, x_0^1)) \, \mathrm{d}\sigma.$$

Then, according to fact on $\mathcal{Y}$ mentioned above, our task therefore reduces to showing that

$$\int_0^\infty \left(\left(\frac{y}{\sqrt{y^2+1}} \right)^{\frac{3-\gamma}{2\gamma-2}} (1+y^2)^{\frac{\gamma+1}{4\gamma-4}} \right)(t, x^1(t, x_0^1)) \mathrm{d}t$$

diverges, based on Lemma 4.5.7. However, the explicit solution of the differential equation

$$y' = -C_g y^{\frac{\gamma+1}{2\gamma-2}} (1+y^2)^{3/2}$$

is very hard to handle, as it involves hypergeometric functions. We instead adopt an indirect approach and look at a transport equation for the variable $\mathcal{Y}$ itself:

$$\mathcal{Y} = \left(\frac{y}{\sqrt{y^2+1}} \right)^{\frac{3-\gamma}{2\gamma-2}} (1+y^2)^{\frac{\gamma+1}{4\gamma-4}} = y^{\frac{3-\gamma}{2\gamma-2}} (1+y^2)^{\frac{1}{2}}. \tag{4.66}$$

It turns out that one can obtain the following Riccati–type inequality for $\mathcal{Y}$:

Lemma 4.5.8. *For the C^1 solution of the Cauchy problem $(4.1) - (4.2)$ with (4.5), under Assumption 2, there holds*

$$\mathcal{Y}' \geq -C_g \mathcal{Y}^2$$

for some universal constant C_g independent of the time.

Proof. First, it follows from direct calculation that

$$\mathcal{Y}' = \frac{y^{\frac{5-3\gamma}{2\gamma-2}} \left((\gamma+1)y^2 + (3-\gamma) \right) y'}{(2\gamma-2)\sqrt{y^2+1}}, \tag{4.67}$$

which, along with (4.61), implies that

$$\mathcal{Y}' \geq -C_g y^{\frac{6-2\gamma}{2\gamma-2}} (1+y^2) \left((\gamma+1)y^2 + (3-\gamma) \right), \tag{4.68}$$

Actually, the above ODE inequality can equivalently be rewritten as

$$\mathcal{Y}' \geq -C_g \mathcal{Y}^2 \left((\gamma+1)y^2 + (3-\gamma) \right). \tag{4.69}$$

---

[7]If instead $w_0'(x) < 0$ the proof is precisely the same after relabelling the corresponding variables.

Moreover, it follows from the fact $\sqrt{P'(\rho)} < c$ that

$$y = \frac{k\rho^{(\gamma-1)/2}}{c} < \gamma^{-\frac{1}{2}}.$$

Therefore

$$(\gamma + 1)y^2 + (3 - \gamma) \leq \frac{\gamma + 1}{\gamma} + (3 - \gamma),$$

which, along with (4.69), implies that

$$\mathcal{Y}' \geq -C_g \mathcal{Y}^2, \tag{4.70}$$

for some universal constant C_g independent of the time.

$\square$

4.5.3.4   Lower bound estimates on ρ and formation of singularity

Now, based on the conclusions obtained in Sections 4.5.3.1–4.5.3.3, we are ready to finish the proof of Theorem 4.3.1. From Lemma 4.5.8 one can obtain that

$$\mathcal{Y}(t) \geq \frac{1}{C_1 + C_2 t}, \tag{4.71}$$

for some universal constants C_i $(i = 1, 2)$ independent of the time. From (4.49), one can get

$$\mathrm{e}^{-h_1}\lambda_{1z} = \frac{c\,\mathrm{e}^{-\frac{2\sqrt{\gamma}\arctan(y)}{\gamma-1}}(c+u)(\gamma+1)(1-y^2)}{2c^2 - 2u\sqrt{P'(\rho)}}\mathcal{Y} = \mathcal{H}\mathcal{Y} > 0. \tag{4.72}$$

From the analysis shown in Step 2 of the proof for the case $\gamma \geq 3$ of Theorem 4.3.1 in Section 4.5.2, one obtains

$$C_g^{-1} \leq \mathcal{H} \leq C_g,$$

for some universal constant C_g independent of the time. Also, we know that $\mathcal{Y}(0)$ is positive. Then, finally, according to the solution formula shown in Lemma 4.5.4, the desired conclusion stated in Theorem 4.3.1 for $1 < \gamma < 3$ has been proved.

Remark 9. *Notice that* $\mathcal{Y} = C\rho^{\frac{3-\gamma}{4}}\sqrt{1 + \frac{k^2\rho^{\gamma-1}}{c^2}} < 2C\rho^{\frac{3-\gamma}{4}}$, *because of the upper bound on* ρ. *Together with (4.71), we obtain the same bound for* ρ *as in the classical case:*

$$\rho(x, t) = O(1 + t)^{-\frac{4}{3-\gamma}}.$$

It is interesting that precisely the same bound as in the classical case can be recovered.

Chapter 5

A trapped surface formation criterion for the Einstein–Maxwell equations

5.1 Introduction

5.1.1 Background

In this chapter we study the evolution of the Einstein–Maxwell system for a $(3+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$ and an electromagnetic 2–tensor $F_{\alpha\beta}$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{5.1}$$

where

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\lambda\tau}F^{\lambda\tau}. \tag{5.2}$$

Although the 2–parameter family of Reissner–Nördstrom electrovacuum spacetimes (see [62] for a reference) exhausts all possible spherically symmetric solutions to the Einstein–Maxwell system, in the absence of symmetry assumptions, the global dynamics of (5.1)-(5.2) are quite hard to study.

For the Einstein vacuum equations (where $T_{\mu\nu} = 0$), the study of (5.1)-(5.2) in the small data regime has been very successful. In a monumental work of Christodoulou and Klainerman [26], it is shown that the Minkowski spacetime is stable under small perturbations. Christodoulou and Klainerman showed that, for small perturbations of the trivial data, no singularity will form and all geodesics are complete. Later, Zipser [10] extended this result to the Einstein–Maxwell system.

One of the most fascinating aspects of the classical theory of general relativity is that it predicts the existence of a black hole. Historically, some notion of a black hole accompanies the theory of general relativity almost since its inception by Einstein in

1915. It was first encountered in an explicit solution to the Einstein vacuum equations and in particular the Schwarzschild solution

$$(\mathcal{M}, g)_{Schw} := {}^1 \left(\mathbb{R} \times (0, \infty) \times \mathbb{S}^2, -\left(1 - \tfrac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \tfrac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\mathbb{S}^2 \right),$$

communicated by Schwarzschild to Einstein in a letter about one month after the latter presented his field equations of general relativity at the Prussian academy of sciences. However, it was neither Schwarzschild nor Einstein who understood that what would come to be known as a black hole region featured prominently in the Schwarzschild solution. It was Lemaitre [51] who first observed, in 1932, that $(\mathcal{M}, g)_{Schw}$ contains a region $\mathcal{B}$ with the property that observers lying inside $\mathcal{B}$ cannot send signals to observers situated at an ideal conformal boundary *at infinity* $\mathcal{I}^+$ (this being defined in a rigorous and appropriate way). In the case of the Schwarzschild solution, the existence of a non-empty $\mathcal{B}$ is accompanied by another surprising, yet salient, feature:

Every observer in $\mathcal{B}$ lives for finite proper time[2] (future geodesic incompleteness).

When physicists and mathematicians first realized these two properties, they were hoping to be able to associate to them the characterization of an accident; a non-generic pathology, present only due to the high degree of symmetry imposed a priori on the solutions and that, in general solutions to the equations, such phenomena would not arise. Much to the surprise of the community, Penrose [77] in 1965 proved these hopes were ill–based through the following incompleteness theorem:

Theorem 5.1.1. *For a spacetime $(\mathcal{M}, g)$ containing a non-compact Cauchy hypersurface and $g_{\mu\nu}, F_{\mu\nu}$ satisfying (5.1)–(5.2), if $\mathcal{M}$ contains a compact trapped surface, then it is future causally geodesically incomplete.*

A trapped surface is a 2-dimensional geometric object. Assume we are given a $(3+1)$-dimensional, time-oriented Lorentzian manifold $(\mathcal{M}, g)$ and within it a closed, space-like $2-$surface S. Since S has co-dimension 2, the tangent space at a point p on S, T_pS has a 2-dimensional orthogonal complement in $T_p\mathcal{M}$. Let $l, \underline{l}$ denote a null

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[1]Here (r, θ, ϕ) represent the standard spherical coordinates and t the time coordinate. Moreover, $\mathrm{d}\mathbb{S}^2$ denotes the standard metric on the 2-sphere.

[2]Given a pseudo-Riemannian manifold with a local coordinates x^μ and equipped with a metric tensor $g_{\mu\nu}$, the proper time interval $\Delta\tau$ between two events along a timelike path P is given by the line integral $\Delta\tau = \frac{1}{c} \int_P \sqrt{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$.

basis[3] of this complement and extend $l, \underline{l}$ as vector fields. We define the following two fundamental forms $\chi, \underline{\chi}$ associated with the surface S:

$$\chi(X, Y) := g(\nabla_X l, Y), \quad \underline{\chi}(X, Y) := g(\nabla_X \underline{l}, Y)$$

where X and Y are vector fields tangent to S. We look at the expansions $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}$. If both are pointwise negative on S, then the surface is called trapped. A trapped surface is, therefore, a surface for which the area decreases for arbitrary infinitesimal displacements along the null generators of both null geodesic congruences normal to S. Penrose's theorem implies that the study of singularity formation for Einstein's equations can, in some generality, be reduced to the problem of trapped surface formation. This problem had, again, remained open for a long time.

5.1.1.1   The Einstein vacuum case

In a breakthrough work in 2008, Christodoulou solved this long–standing open problem (trapped surface formation for Einstein vacuum equations) with a 587–page monumental work [22]. He designed an open set of large initial data, which encode a special structure, the *short pulse* ansatz. In particular, this ansatz allows one to consider a hierarchy of large and small quantities, parametrized by a small parameter δ. For such initial data, these quantities behave differently, being of various sizes in term of δ. Moreover, their sizes form a hierarchy. But for each quantity, surprisingly, its size is almost preserved by the nonlinear evolution. Therefore, once this hierarchy is satisfied at the level of initial data, it persists for later time. With this philosophy, despite it being a large data problem, a long–time global existence theorem can be established. Moreover, these initial conditions indeed lead to trapped-surface formation in finite time.

Effort was consequently put towards simplifying Christodoulou's proof. In [43], an ingenious systematical approach was introduced by Klainerman–Rodnianski [43]. This approach was later extended by An in [1]. The Einstein vacuum equations are a nonlinear hyperbolic system, containing many unknowns. Christodoulou controlled all of them on a term-by-term basis. In [43], Klainerman and Rodnianski introduced a novel index s_1 which they termed *signature for short pulse*. With this index, Klainerman and Rodnianski systematically tracked the δ-weights used in the estimates and gave a shorter, simplified proof of the almost–preservation of the δ–hierarchy in a finite

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[3]That means $g(l, l) = 0, \ g(\underline{l}, \underline{l}) = 0, \ g(l, \underline{l}) = -2$.

region. In [22], besides δ–weights, Christodoulou also employed weights related to decay and proved his main theorem that *a trapped surface could form dynamically with initial data prescribed arbitrary dispersed at past null infinity*. In [1], An introduced a new index s_2 called *signature for decay rates*. With the help of this new index, An extended Klainerman and Rodnianski's result [43] from a finite region to an infinite region and re–proved Christodoulou's main theorem in [22] with around 120 pages.

Another important step of progress was made by An and Luk in [2]. By designing and employing a different hierarchy, in [2] they improved [22] and proved the first scale-critical result for the Einstein vacuum equations. With the same small parameter δ, with relatively larger initial data, Christodoulou formed a trapped surface of radius 1; while with much smaller initial data, An and Luk formed a trapped surface of radius δa, where a is a universal large constant like 1000.[4] In [2] the authors want to form, using as small data as possible (the smallness being defined in a precise sense), a tiny trapped surface with radius δa, hence they have to deal with the region very close to the center. In this region all the geometric quantities have growth rates. To bound these growth rates, they employ weighted estimates as well as several crucial geometric renormalizations.

Since [2] is scale critical, one can keep a as a universal constant and let $\delta \to 0$. Hence a series of trapped surfaces (with radius shrinking to 0) are obtained. In [3], An further explored this idea. Together with an elliptic approach to identify the boundary, An showed that a whole black hole region could emerge dynamically from just a "point" O in the spacetime. For an open set of initial data and appropriate control on all the derivatives of $\hat{\chi}_0$, this boundary (apparent horizon) is proved to be smooth except at O.

In early 2019, An [5] gave a different, 55–page proof of a trapped surface formation theorem that sharpens the previous results both of Christodoulou [22] and estimates in An–Luk [2]. The argument in [5] is based on a systematic extension of the scale-critical arguments in [2], connecting Christodoulou's short–pulse method and Klainerman–Rodnianski's signature counting argument to the peeling properties previously used in small–data results such as Klainerman–Nicolo. This in particular allows the author to avoid elliptic estimates and geometric renormalizations and gives new technical simplifications.

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[4]Letting $a = \delta^{-1}$, in a finite region they recover Christodoulou's main result of [22].

5.1.1.2 The Einstein–Maxwell case

For the Einstein–Maxwell system (5.1)-(5.2), important progress was made by Yu [87, 88]. In a finite region, Yu extended the result of Klainerman–Rodnianski [43] and obtained trapped surface formation for the Einstein–Maxwell system by using the signature for short-pulse. In the current work, combining the new ingredients in [5], we will extend Yu's results to obtain a trapped surface formation criterion for the Einstein–Maxwell system in a large spacetime region.
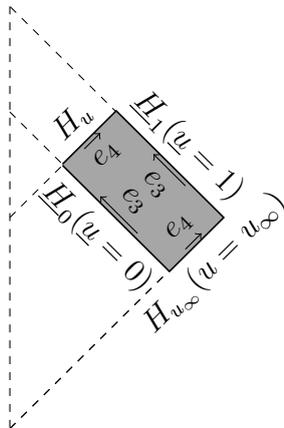
5.1.1.3 The Einstein–scalar field case

In a recent paper [57], Li and Liu studied Einstein-scalar field system:

$$\text{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu},$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\sigma\phi\partial_\sigma\phi.$$

With singular initial data prescribed along the incoming null hypersurface, an almost scale-critical trapped surface formation criterion was obtained. And renormalizations for scalar fields were used.

5.1.2 Main Results

We will introduce coordinates u and $\underline{u}$ in $(\mathcal{M}, g)$ through a *double null foliation*[5], where H_u and $\underline{H}_{\underline{u}}$ are incoming and outgoing characteristic cones, respectively. With coordinates $u, \underline{u}$, characteristic initial data will be prescribed along incoming null hypersurface $\underline{H}_0$, where $\underline{u} = 0$ and outgoing null hypersurface H_{u_∞}, where $u = u_\infty$.

Our main results can be summarized in two Theorems. The first one is a global existence result.

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[5] The detailed construction of double null foliation will be explained in Section 5.2.1.

Theorem 5.1.2. *Given $\mathcal{I}^{(0)}$, there exists a sufficiently large $a_0 = a_0(\mathcal{I}^{(0)})$ such that the following holds. For any $0 < a_0 < a$ and with initial data $(\hat{\chi}, \alpha_F)$ satisfying*

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{1}{2}} \|\nabla_4^k \left(|u_\infty| \nabla\right)^i (\hat{\chi}, \alpha_F)\|_{L^\infty(S_{u_\infty, \underline{u}})} \leq \mathcal{I}^{(0)}$ *along $u = u_\infty$,*

- *Minkowskian initial data along $\underline{u} = 0$,*

then the Einstein–Maxwell system admit a unique smooth solution in the region

$$u_\infty \leq u \leq -a/4, \ \ 0 \leq \underline{u} \leq 1.$$

The second one steps directly on the first one and is a formation of trapped surfaces statement.

Theorem 5.1.3. *Given $\mathcal{I}^{(0)}$, there exists a sufficiently large $a_0 = a_0(\mathcal{I}^{(0)})$ such that the following holds. For any $0 < a_0 < a$, the unique smooth solution $(\mathcal{M}, g)$ of the Einstein–Maxwell system from Theorem 5.1.2 with initial data satisfying*

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{1}{2}} \|\nabla_4^k \left(|u_\infty| \nabla\right)^i (\hat{\chi}, \alpha_F)\|_{L^\infty(S_{u_\infty, \underline{u}})} \leq \mathcal{I}^{(0)}$ *along $u = u_\infty$,*

- *Minkowskian initial data along $\underline{u} = 0$,*

- $\int_0^1 |u_\infty|^2 \left(|\hat{\chi}_0|^2 + |\alpha_{F0}|^2\right)(u_\infty, \underline{u}') \, d\underline{u}' \geq a$ *uniformly for every direction along $u = u_\infty$,*

has a trapped surface at $S_{-a/4, 1}$.

5.1.3 New Ingredients

Compared to the corresponding problem for the Einstein vacuum equations, the Einstein–Maxwell system gives rise to additional technical difficulties not encountered in the vacuum case. Here we list some important ones and outline the solutions.

1. We extend the systematical way of assigning signatures s_2 for the geometric quantities to the Maxwell field. To achieve this, we resort to the Bianchi equations, keeping the same s_2 values for the Ricci coefficients as those in the vacuum case. Crucially, the Maxwell equations expressed in a null frame have, in a sense, the same structure as the vacuum Bianchi equations. This means that such an assignment of signatures as in [1], [5] can be carried through in a cohesive and coherent way to the Einstein–Maxwell system.

2. The Ricci tensor is non-trivial for the Einstein–Maxwell system. As is typical in similar cases, the energy estimates will be carried out using the Weyl tensor components instead of the Riemann tensor components. We thus work with the Weyl components $\alpha_W, \underline{\alpha}_W, \beta_W, \underline{\beta}_W, \rho_W, \sigma_W$ and re-express all equations with respect to them. For simplicity, we still use $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, \sigma$ to mean $\alpha_W, \underline{\alpha}_W, \beta_W, \underline{\beta}_W, \rho_W, \sigma_W$.

3. We introduce and employ crucial renormalizations for β_a and $\underline{\beta}_a$. The reason for this stems from the Bianchi equations. For example, in the Bianchi equation involving $\nabla_4 \beta$, the right–hand side contains the term $D_4 R_{4A}$. This, in turn, contains the term $\nabla_4 \alpha_F$. When one attempts to estimate $\nabla_4 \alpha_F$, there is no available equation for it in the null Maxwell equations, thus making it difficult to estimate by itself. It is for this reason that we introduce the quantities

$$\tilde{\beta}_A := \beta_A - \frac{1}{2} R_{4A}, \quad \underline{\tilde{\beta}}_A := \beta_A - \frac{1}{2} R_{3A}.$$

We then rewrite the entire Bianchi equations in terms of those renormalized quantities and use them to do energy estimates. In this way, all the terms can be estimated directly through the null Maxwell equations and null Bianchi equations. In particular, terms like $\nabla_4 \alpha_F$ and $\nabla_3 \underline{\alpha}_F$ no longer appear this time.

4. We introduce and use elliptic estimates in the scale-invariant framework. This is achieved in Section 6. Its purpose is to allow us to close the energy estimates for up to 11 derivatives of the Maxwell field components and up to 10 derivatives of the Weyl curvature components. In the process, a control on 11 derivatives of the Ricci coefficients is required. We find a non-trivial way (via energy estimates) to incorporate the elliptic estimates into the systematical approach via the signature for decay rates s_2.

5.2 Setting, equations and notations

5.2.1 Double Null Foliation

We construct a double null foliation in a neighbourhood of $S_{u_\infty,0}$ as follows:

Starting from a point p on 2-sphere $S_{u_\infty,0}$, in 2-dimensional $T_p^\perp S_{u_\infty,0}$, we could find two future–directed vectors $L'_p, \underline{L}'_p$ such that

$$g(L'_p, L'_p) = 0, \ g(\underline{L}'_p, \underline{L}'_p) = 0, \ g(L'_p, \underline{L}'_p) = -2.$$

Note that $\{L'_p, \underline{L}'_p\}$ are uniquely determined up to a scaling factor $\lambda > 0$: $\{L'_p, \underline{L}'_p\} \to \{\lambda L'_p, \lambda^{-1}\underline{L}'_p\}$. Emanating from p and initially tangent to L'_p, a unique geodesic l_p is sent out. We extend L' along l_p such that $D_{L'}L' = 0$. We then have that l_p is null. This is because $g(L'_p, L'_p) = 0$ and

$$L'(g(L', L')) = 2g(D_{L'}L', L') = 0.$$

We hence have $g(L', L') = 0$ along l_p. Gathering all the $\{l_p\}$ together, we then have an outgoing null hypersurface called H_{u_∞}. Similarly, we obtain the incoming null hypersurface $\underline{H}_0$ emanating from $S_{u_\infty,0}$.

Note that, by above construction, for each point q on H_{u_∞} or $\underline{H}_0$, in $T_q H_{u_\infty}$ or $T_q \underline{H}_0$, there is a preferred null vector L'_q or $\underline{L}'_q$ associated with q.

We proceed to define the function Ω to be 1 on $S_{u_\infty,0}$ and extend Ω as a continuous function along H_{u_∞} and $\underline{H}_0$. [6] We consider vector fields

$$L = \Omega^2 L' \ \text{along} \ H_{u_\infty}, \ \text{and} \ \underline{L} = \Omega^2 \underline{L}' \ \text{along} \ \underline{H}_0$$

and define functions

$$\underline{u} \ \text{on} \ H_{u_\infty} \ \text{satisfying} \ L\underline{u} = 1 \ \text{and} \ \underline{u} = 0 \ \text{on} \ S_{u_\infty,0},$$

$$u \ \text{on} \ \underline{H}_0 \ \text{satisfying} \ \underline{L}u = 1 \ \text{and} \ u = u_\infty \ \text{on} \ S_{u_\infty,0}.$$

Let $S_{u_\infty,\underline{u}'}$ be the embedded 2-surface on H_{u_∞}, such that $\underline{u} = \underline{u}'$. Similarly, define $S_{u',0}$ to be the embedded 2-surface on $\underline{H}_0$, such that $u = u'$. At each point q on 2-surface $S_{u_\infty,\underline{u}'}$, we already have the preferred outgoing null vector L'_q tangent to H_{u_∞}. Hence, at q, we can also fix a unique incoming null vector $\underline{L}'_q$ via requiring

$$g(\underline{L}'_q, \underline{L}'_q) = 0 \quad \text{and} \quad g(\underline{L}'_q, L'_q) = -2\Omega^{-2}|_q.$$

---

[6]For a general double null foliation, we have the gauge freedom of choosing how to extend Ω along H_{u_∞} and $\underline{H}_0$. In this work, we extend $\Omega \equiv 1$ on both H_{u_∞} and $\underline{H}_0$.

There exists a unique geodesic $\underline{l}_q$ emitting from q with direction $\underline{L}'$. We then extend $\underline{L}'$ along $\underline{l}_q$ by imposing $D_{\underline{L}'}\underline{L}' = 0$. Gathering all the $\{\underline{l}_q\}$ for $q \in S_{u_\infty,\underline{u}'}$, we construct the incoming null hypersurface $\underline{H}_{\underline{u}'}$ emanating from $S_{u_\infty,\underline{u}'}$. Similarly, from $S_{u',0}$ we also construct the outgoing null hypersurface $H_{u'}$. We further define the 2-spheres $S_{u',\underline{u}'} := H_{u'} \cap \underline{H}_{\underline{u}'}$.

At each point p of $S_{u',\underline{u}'}$, we define the positive-valued function Ω via

$$g(L'_p, \underline{L}'_p) =: -2\Omega^{-2}|_p. \tag{5.3}$$

Note that L'_p is well-defined on $H_{u'}$, along an outgoing null geodesic l passing through p; $\underline{L}'_p$ is also well-defined on $\underline{H}_{\underline{u}'}$, along an incoming null geodesic $\underline{l}$ crossing p.

These 3-dimensional incoming null hypersurfaces $\{\underline{H}_{\underline{u}'}\}_{0 \le \underline{u}' \le 1}$, along with the outgoing null hypersurfaces $\{H_{u'}\}_{u_\infty \le u' \le -a/4}$ and their pairwise intersections $S_{u',\underline{u}'} = H_{u'} \cap \underline{H}_{\underline{u}'}$ give us the so-called *double null foliation*.

On $S_{u,\underline{u}}$, by (5.3) we have $g(L', \underline{L}') = -2\Omega^{-2}$. Thus, $g(\Omega L', \Omega \underline{L}') = -2$. Throughout this chapter we will work with the normalized null pair (e_3, e_4), namely

$$e_3 := \Omega \underline{L}', \quad e_4 := \Omega L', \text{ and } \quad g(e_3, e_4) = -2.$$

Moreover, for the imposition of our characteristic initial data we choose the following gauge:

$$\Omega \equiv 1 \quad \text{on } H_{u_\infty} \text{ and } \underline{H}_0.$$

Remark 10. *The functions u and $\underline{u}$ defined above also satisfy the eikonal equations*

$$g^{\mu\nu}\partial_\mu u \partial_\nu u = 0, \quad g^{\mu\nu}\partial_\mu \underline{u} \partial_\nu \underline{u} = 0.$$

And it is straight forward to check

$$L'^\mu = -2g^{\mu\nu}\partial_\nu u, \quad \underline{L}'^\mu = -2g^{\mu\nu}\partial_\nu \underline{u}, \quad L\underline{u} = 1, \quad \underline{L}u = 1.$$

Here $\underline{L} := \Omega^2 \underline{L}', \quad L := \Omega^2 L'$ are also called equivariant vector fields.

5.2.2   The Coordinate System

We shall use a coordinate system $(u, \underline{u}, \theta^1, \theta^2)$. Here u and $\underline{u}$ are defined as above. To get (θ^1, θ^2) for each point on $S_{u,\underline{u}}$, we follow the approach in Chapter 1 of [22]. We first define a coordinate system (θ^1, θ^2) on $S_{u_\infty,0}$. Since $S_{u_\infty,0}$ is the standard 2-sphere in Minkowski spacetime, here we use the coordinates of stereographic projection. Then we extend this coordinate system to $\underline{H}_0$ by requiring

$$\not{\mathcal{L}}_{\underline{L}}\theta^A = 0 \text{ on } \underline{H}_0.^7$$

Here $\not{\mathcal{L}}_L$ is the restriction of the Lie derivative to $TS_{u,\underline{u}}$. In other words, given a point p on $S_{u_\infty,0}$, assuming l_p is the incoming null geodesic on $\underline{H}_0$ emanating from p, then all the points along l_p are assigned the same angular coordinate (θ^1, θ^2). We further extend this coordinate system from $\underline{H}_0$ to the whole spacetime under the requirement

$$\not{\mathcal{L}}_L\theta^A = 0,$$

i.e. that all the points along the same outgoing null geodesics (along L) on H_u have the same angular coordinate. We have thus established a coordinate system in a neighborhood of $S_{u_\infty,0}$. With this coordinate system, we can rewrite e_3 and e_4 as

$$e_3 = \Omega^{-1}\left(\frac{\partial}{\partial u} + d^A \frac{\partial}{\partial \theta^A}\right), \, e_4 = \Omega^{-1}\frac{\partial}{\partial \underline{u}}.$$

The Lorentzian metric g takes the form

$$g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - d^A du) \otimes (d\theta^B - d^B du). \tag{5.4}$$

We require d^A to satisfy $d^A = 0$ on $\underline{H}_0$.

5.2.3   The equations

In this work, we study the Einstein–Maxwell equations for a 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with signature $\{-, +, +, +\}$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{5.5}$$

where

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\lambda\tau}F^{\lambda\tau}.$$

---

^{7}On $\underline{H}_0$, we have $\Omega = 1$ and $\not{\mathcal{L}}_{\underline{L}}\theta^A = \frac{\partial}{\partial u}\theta^A$.

Here $F_{\alpha\beta}$ is an anti-symmetric $2-$tensor representing the electromagnetic field. We introduce null tetrads $\{e_a, e_b, e_3, e_4\}$ where $a, b = 1, 2$ and require

$$g(e_a, e_b) = \delta_{ab}, \quad g(e_3, e_4) = -2, \quad g(e_a, e_3) = 0, \quad g(e_a, e_4) = 0.$$

For the Weyl curvature $W_{\mu\nu\lambda\tau}$, we define the null Weyl curvature components:

$$
\begin{aligned}
\alpha_{ab} &= W(e_a, e_4, e_b, e_4), & \underline{\alpha}_{ab} &= W(e_a, e_3, e_b, e_3), \\
\beta_a &= \frac{1}{2}W(e_a, e_4, e_3, e_4), & \underline{\beta}_a &= \frac{1}{2}W(e_a e_3, e_3, e_4), \\
\rho &= \frac{1}{4}W(e_4, e_3, e_4, e_3), & \sigma &= \frac{1}{4} {}^*W(e_4, e_3, e_4, e_3).
\end{aligned}
\tag{5.6}
$$

Here ${}^*W$ is the Hodge dual of W.

For the Riemann curvature tensor $R_{\mu\nu\lambda\tau}$, we define the null Riemann curvature components:

$$
\begin{aligned}
(\alpha_{\mathcal{R}})_{ab} &= R(e_a, e_4, e_b, e_4), & (\underline{\alpha}_{\mathcal{R}})_{ab} &= R(e_a, e_3, e_b, e_3), \\
(\beta_{\mathcal{R}})_a &= \frac{1}{2}R(e_a, e_4, e_3, e_4), & (\underline{\beta}_{\mathcal{R}})_a &= \frac{1}{2}R(e_a, e_3, e_3, e_4), \\
\rho_{\mathcal{R}} &= \frac{1}{4}R(e_4, e_3, e_4, e_3), & \sigma_{\mathcal{R}} &= \frac{1}{4} {}^*R(e_4, e_3, e_4, e_3).
\end{aligned}
\tag{5.7}
$$

Here ${}^*R$ is the Hodge dual of R.

Denote $D_A := D_{e_A}$. We define the Ricci coefficients:

$$
\begin{aligned}
\chi_{AB} &= g(D_A e_4, e_B), & \underline{\chi}_{AB} &= g(D_A e_3, e_B), \\
\eta_A &= -\frac{1}{2}g(D_3 e_A, e_4), & \underline{\eta}_A &= -\frac{1}{2}g(D_4 e_A, e_3), \\
\omega &= -\frac{1}{4}g(D_4 e_3, e_4), & \underline{\omega} &= -\frac{1}{4}g(D_3 e_4, e_3), \\
\zeta_A &= \frac{1}{2}g(D_A e_4, e_3).
\end{aligned}
\tag{5.8}
$$

We decompose χ and $\underline{\chi}$ into its trace and traceless parts. Denote by $\hat{\chi}_{AB}$ and $\underline{\hat{\chi}}_{AB}$ the traceless part of χ_{AB} and $\underline{\chi}_{AB}$ respectively.

We further define

$$(\alpha_F)_a = F_{a4}, \quad (\underline{\alpha}_F)_a = F_{a3}, \quad \rho_F = \frac{1}{2}F_{34}, \quad \sigma_F = F_{12}.$$

Note that (5.5) implies

$$R_{\mu\nu} = T_{\mu\nu} \quad \text{and} \quad R = 0.$$

Expressed in this double null frame, we have

$$\nabla_4 \operatorname{tr}\chi + \frac{1}{2} (\operatorname{tr}\chi)^2 = -|\hat{\chi}|^2 - 2\omega \operatorname{tr}\chi - \alpha_F^2, \tag{5.9}$$

$$\nabla_4 \hat{\chi} + \operatorname{tr}\chi \, \hat{\chi} = -2\omega \, \hat{\chi} - \alpha, \tag{5.10}$$

$$\nabla_3 \operatorname{tr}\underline{\chi} + \frac{1}{2} (\operatorname{tr}\underline{\chi})^2 = -|\hat{\underline{\chi}}|^2 - 2\underline{\omega} \operatorname{tr}\underline{\chi} - \underline{\alpha}_F^2, \tag{5.11}$$

$$\nabla_3 \hat{\underline{\chi}} + \operatorname{tr}\underline{\chi} \, \hat{\underline{\chi}} = -2\underline{\omega} \, \hat{\underline{\chi}} - \underline{\alpha}, \tag{5.12}$$

$$\nabla_4 \operatorname{tr}\underline{\chi} + \frac{1}{2} \operatorname{tr}\chi \operatorname{tr}\underline{\chi} = 2\omega \operatorname{tr}\underline{\chi} + 2\rho - \hat{\chi} \cdot \hat{\underline{\chi}} + 2\operatorname{div}\underline{\eta} + 2|\underline{\eta}|^2, \tag{5.13}$$

$$(\nabla_4 \hat{\underline{\chi}})_{ab} + \frac{1}{2} \operatorname{tr}\chi \, \hat{\underline{\chi}}_{ab} = (\nabla \widehat{\otimes} \underline{\eta})_{ab} + 2\omega \, \hat{\underline{\chi}}_{ab} - \frac{1}{2} \operatorname{tr}\underline{\chi} \, \hat{\chi}_{ab} + (\underline{\eta} \widehat{\otimes} \underline{\eta})_{ab} - \frac{1}{2} (\alpha_F \widehat{\otimes} \underline{\alpha}_F)_{ab}, \tag{5.14}$$

$$\nabla_3 \operatorname{tr}\chi + \frac{1}{2} \operatorname{tr}\chi \operatorname{tr}\underline{\chi} = 2\underline{\omega} \operatorname{tr}\chi + 2\rho - \hat{\chi} \cdot \hat{\underline{\chi}} + 2\operatorname{div}\eta + 2|\eta|^2, \tag{5.15}$$

$$(\nabla_3 \hat{\chi})_{ab} + \frac{1}{2} \operatorname{tr}\underline{\chi} \, \hat{\chi}_{ab} = (\nabla \widehat{\otimes} \eta)_{ab} + 2\underline{\omega} \, \hat{\chi}_{ab} - \frac{1}{2} \operatorname{tr}\chi \, \hat{\underline{\chi}}_{ab} + (\eta \widehat{\otimes} \eta)_{ab} - \frac{1}{2} (\alpha_F \widehat{\otimes} \underline{\alpha}_F)_{ab}. \tag{5.16}$$

Note that

$$\beta_a - \frac{1}{2} R_{a4} = (\beta_{\mathcal{R}})_a,$$

$$\underline{\beta}_a + \frac{1}{2} R_{a3} = (\underline{\beta}_{\mathcal{R}})_a,$$

$$\rho - \frac{1}{2} R_{43} = \rho_{\mathcal{R}}.$$

Moreover,

$$R_{11} = \frac{1}{2}\sigma_F^2 + \frac{1}{2}\rho_F^2 - (\alpha_F)_1(\underline{\alpha}_F)_1 + \frac{1}{2}\alpha_F \cdot \underline{\alpha}_F,$$

$$R_{22} = \frac{1}{2}\sigma_F^2 + \frac{1}{2}\rho_F^2 - (\alpha_F)_2(\underline{\alpha}_F)_2 + \frac{1}{2}\alpha_F \cdot \underline{\alpha}_F,$$

$$R_{4a} = R_{a4} = \rho_F(\alpha_F)_a - \sigma_F \epsilon_{ab}(\alpha_F)_b,$$

$$R_{3a} = R_{a3} = -\rho_F(\underline{\alpha}_F)_a - \sigma_F \epsilon_{ab}(\underline{\alpha}_F)_b,$$

$$R_{43} = \rho_F^2 + \sigma_F^2, \quad R_{44} = \alpha_F \cdot \alpha_F, \quad R_{33} = \underline{\alpha}_F \cdot \underline{\alpha}_F.$$

The other components satisfy the following transport equations:

$$\nabla_4 \eta_a = -\chi_{ab} \cdot (\eta - \underline{\eta})_b - \beta_a - \frac{1}{2} R_{a4}, \tag{5.17}$$

$$\nabla_3 \underline{\eta}_a = -\underline{\chi}_{ab} \cdot (\underline{\eta} - \eta)_b + \underline{\beta}_a - \frac{1}{2} R_{a3}, \tag{5.18}$$

$$\nabla_4 \underline{\omega} = 2\omega\underline{\omega} - \eta \cdot \underline{\eta} + \frac{1}{2}|\eta|^2 + \frac{1}{2}\rho + \frac{1}{4} R_{34}, \tag{5.19}$$

$$\nabla_3 \omega = 2\omega\underline{\omega} - \eta \cdot \underline{\eta} + \frac{1}{2}|\underline{\eta}|^2 + \frac{1}{2}\rho + \frac{1}{4} R_{34}, \tag{5.20}$$

as well as the constraint equations

$$\operatorname{div}\hat{\chi} = \frac{1}{2}\nabla\operatorname{tr}\chi - \frac{1}{2}(\eta - \underline{\eta}) \cdot (\hat{\chi} - \frac{1}{2}\operatorname{tr}\chi) - \beta_{\mathcal{R}}, \tag{5.21}$$

$$\operatorname{div}\underline{\hat{\chi}} = \frac{1}{2}\nabla\operatorname{tr}\underline{\chi} - \frac{1}{2}(\eta - \underline{\eta}) \cdot (\underline{\hat{\chi}} - \frac{1}{2}\operatorname{tr}\underline{\chi}) - \underline{\beta}_{\mathcal{R}}, \tag{5.22}$$

$$\operatorname{curl}\eta = -\operatorname{curl}\underline{\eta} = \sigma + \frac{1}{2}\underline{\hat{\chi}} \wedge \hat{\chi}, \tag{5.23}$$

$$K = -\rho_{\mathcal{R}} - \frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi} + \frac{1}{2}\hat{\chi} \cdot \underline{\hat{\chi}}. \tag{5.24}$$

Here K is the Gauss curvature of the spheres $S_{u,\underline{u}}$. The null curvature components satisfy the null Bianchi equations

$$\nabla_4\beta + 2\mathrm{tr}\chi\beta = \mathrm{div}\,\alpha - 2\omega\beta + \eta\cdot\alpha - \frac{1}{2}(D_A R_{44} - D_4 R_{4A}), \tag{5.25}$$

$$\nabla_3\beta + \mathrm{tr}\underline{\chi}\beta = \nabla\rho + {}^*\nabla\sigma + 2\hat{\chi}\cdot\beta + 2\underline{\omega}\beta + 3(\eta\rho + {}^*\eta\sigma) + \frac{1}{2}(D_4 R_{A3} - D_3 R_{A4}), \tag{5.26}$$

$$\nabla_4\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\nabla\rho + {}^*\nabla\sigma + 2\hat{\underline{\chi}}\cdot\beta + 2\omega\underline{\beta} - 3(\underline{\eta}\rho - {}^*\underline{\eta}\sigma) - \frac{1}{2}(D_4 R_{A3} - D_3 R_{A4}), \tag{5.27}$$

$$\nabla_3\underline{\beta} + 2\mathrm{tr}\underline{\chi}\,\underline{\beta} = -\mathrm{div}\,\underline{\alpha} - 2\underline{\omega}\,\underline{\beta} + \underline{\eta}\,\underline{\alpha} + \frac{1}{2}(D_A R_{33} - D_3 R_{3A}), \tag{5.28}$$

$$\nabla_4\underline{\alpha} + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha} = -\nabla\widehat{\otimes}\underline{\beta} + 4\omega\underline{\alpha} - 3(\hat{\underline{\chi}}\rho - {}^*\hat{\underline{\chi}}\sigma) + (\zeta - 4\underline{\eta})\widehat{\otimes}\underline{\beta} + \frac{1}{4}(D_4 R_{33} - D_3 R_{34})g_{ab}, \tag{5.29}$$

$$\nabla_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha = \nabla\widehat{\otimes}\beta + 4\underline{\omega}\alpha - 3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta + \frac{1}{4}(D_3 R_{44} - D_4 R_{43})g_{ab}, \tag{5.30}$$

$$\nabla_4\rho + \frac{3}{2}\mathrm{tr}\chi\rho = \mathrm{div}\,\beta - \frac{1}{2}\hat{\underline{\chi}}\cdot\alpha + \zeta\cdot\beta + 2\underline{\eta}\cdot\beta - \frac{1}{4}(D_3 R_{44} - D_4 R_{43}), \tag{5.31}$$

$$\nabla_3\rho + \frac{3}{2}\mathrm{tr}\underline{\chi}\rho = -\mathrm{div}\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + \zeta\cdot\underline{\beta} - 2\eta\cdot\underline{\beta} + \frac{1}{4}(D_3 R_{34} - D_4 R_{33}), \tag{5.32}$$

$$\nabla_4\sigma + \frac{3}{2}\mathrm{tr}\chi\sigma = -\mathrm{div}\,{}^*\beta + \frac{1}{2}\hat{\underline{\chi}}\cdot{}^*\alpha - \zeta\cdot{}^*\beta - 2\underline{\eta}\cdot{}^*\beta - \frac{1}{4}(D_\mu R_{4\nu} - D_\nu R_{4\mu})\epsilon^{\mu\nu}{}_{34}, \tag{5.33}$$

$$\nabla_3\sigma + \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma = -\mathrm{div}\,{}^*\underline{\beta} + \frac{1}{2}\hat{\chi}\cdot{}^*\underline{\alpha} - \zeta\cdot{}^*\underline{\beta} - 2\eta\cdot{}^*\underline{\beta} + \frac{1}{4}(D_\mu R_{3\nu} - D_\nu R_{3\mu})\epsilon^{\mu\nu}{}_{34}. \tag{5.34}$$

Here, the Schouten tensor $S_{\mu\nu}$ is equal to the Ricci tensor $R_{\mu\nu}$ because of the special form of the electromagnetic field tensor, namely the fact that the Ricci scalar R vanishes. Finally, the Maxwell equations are equivalent to the null Maxwell equations

$$\nabla_4\underline{\alpha}_F + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}_F = -\nabla\rho_F - {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F - 2\,{}^*\eta\cdot\rho_F + 2\omega\,\underline{\alpha}_F - \hat{\underline{\chi}}\cdot\alpha_F, \tag{5.35}$$

$$\nabla_3\alpha_F + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha_F = \nabla\rho_F + {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F + 2\underline{\eta}\cdot\rho_F + 2\underline{\omega}\alpha_F - \hat{\chi}\cdot\underline{\alpha}_F, \tag{5.36}$$

$$\nabla_4\rho_F = \mathrm{div}\,\alpha_F - \mathrm{tr}\chi\rho_F - (\eta - \underline{\eta})\cdot\alpha_F, \tag{5.37}$$

$$\nabla_4\sigma_F = -\mathrm{curl}\,\alpha_F - \mathrm{tr}\chi\sigma_F + (\eta - \underline{\eta})\cdot{}^*\alpha_F, \tag{5.38}$$

$$\nabla_3\rho_F + \mathrm{tr}\underline{\chi}\rho_F = \mathrm{div}\,\underline{\alpha}_F + (\eta - \underline{\eta})\cdot\underline{\alpha}_F, \tag{5.39}$$

$$\nabla_3\sigma_F + \mathrm{tr}\underline{\chi}\sigma_F = -\mathrm{curl}\,\underline{\alpha}_F + (\eta - \underline{\eta})\cdot{}^*\underline{\alpha}_F. \tag{5.40}$$

5.2.4 Integration

Let U be a coordinate patch on $S_{u,\underline{u}}$ and let p_U be the corresponding partition of unity. For a function ϕ, we define its integral on $S_{u,\underline{u}}$ and along $H_u, \underline{H}_{\underline{u}}$ by

$$\int_{S_{u,\underline{u}}} \phi := \sum_U \int_{-\infty}^\infty \int_{-\infty}^\infty \phi \cdot p_U \cdot \sqrt{\det\gamma}\, d\theta^1\, d\theta^2,$$

$$\int_{H_u^{(0,\underline{u})}} \phi := \sum_U \int_0^{\underline{u}} \int_{-\infty}^\infty \int_{-\infty}^\infty \phi \cdot 2\, p_U \cdot \Omega \cdot \sqrt{\det\gamma}\, d\theta^1\, d\theta^2\, d\underline{u}',$$

$$\int_{\underline{H}_{\underline{u}}^{(u_\infty,u)}} \phi := \sum_U \int_{u_\infty}^u \int_{-\infty}^\infty \int_{-\infty}^\infty \phi \cdot 2\, p_U \cdot \Omega \cdot \sqrt{\det\gamma}\, d\theta^1\, d\theta^2\, du'.$$

Let $\mathcal{D}_{u,\underline{u}}$ be the region $u_\infty \le u' \le u$, $0 \le \underline{u}' \le \underline{u}$. We define the integral of ϕ in the region $\mathcal{D}_{u,\underline{u}}$ by

$$\int_{\mathcal{D}_{u,\underline{u}}} \phi := \sum_U \int_{u_\infty}^u \int_0^{\underline{u}} \int_{-\infty}^\infty \int_{-\infty}^\infty \phi \cdot p_U \cdot \Omega^2 \cdot \sqrt{-\det g}\, d\theta^1\, d\theta^2\, d\underline{u}'\, du'.$$

We proceed to define, for $1 \le p < \infty$, the L^p-norms for an arbitrary tensor field ϕ:

$$\|\phi\|_{L^p(S_{u,\underline{u}})}^p := \int_{S_{u,\underline{u}}} \langle \phi, \phi \rangle_\gamma^{p/2},$$

$$\|\phi\|_{L^p(H_u)}^p := \int_{H_u} \langle \phi, \phi \rangle_\gamma^{p/2},$$

$$\|\phi\|_{L^p(\underline{H}_{\underline{u}})}^p := \int_{\underline{H}_{\underline{u}}} \langle \phi, \phi \rangle_\gamma^{p/2}.$$

When $p = \infty$, we define the L^∞ norm by

$$\|\phi\|_{L^\infty(S_{u,\underline{u}})} := \sup_{\theta \in S_{u,\underline{u}}} \langle \phi, \phi \rangle_\gamma^{\frac{1}{2}}(\theta).$$

5.2.5 Definition of signatures

We give the following table for signatures throughout this work:

	α	β	ρ	σ	$\underline{\beta}$	$\underline{\alpha}$	χ	ω	ζ	η	$\underline{\eta}$	$\mathrm{tr}\chi$	$\hat{\chi}$	$\underline{\omega}$	α_F	ρ_F	σ_F	$\underline{\alpha}_F$
s_2	0	0.5	1	1	1.5	2	0	0	0.5	0.5	0.5	1	1	1	0	0.5	0.5	1

This comes from wanting to have $s_2(\alpha_F) = s_2(\hat{\chi})$, $s_2(\underline{\alpha}_F) = s_2(\underline{\hat{\chi}})$ and $s_2(\rho_F, \sigma_F)$ such that the null Maxwell equations conserve signature.

5.2.6 Scale-invariant norms

For any horizontal tensor-field ϕ with signature $s_2(\phi)$, we define the following scale-invariant norms on $S_{u,\underline{u}}$:

$$\|\phi\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} := a^{-s_2(\phi)}|u|^{2s_2(\phi)+1}\|\phi\|_{L^\infty(S_{u,\underline{u}})},$$

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} := a^{-s_2(\phi)}|u|^{2s_2(\phi)}\|\phi\|_{L^2(S_{u,\underline{u}})},$$

$$\|\phi\|_{\mathcal{L}^1_{(sc)}(S_{u,\underline{u}})} := a^{-s_2(\phi)}|u|^{2s_2(\phi)-1}\|\phi\|_{L^1(S_{u,\underline{u}})}.$$

Along $H_u^{(0,\underline{u})}$ and $\underline{H}_{\underline{u}}^{(u_\infty,u)}$ we also define scale-invariant norms along null hypersurfaces

$$\|\phi\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} := \int_0^{\underline{u}} \|\phi\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\, \mathrm{d}\underline{u}',$$

$$\|\phi\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} := \int_{u_\infty}^u \frac{a}{|u'|^2}\|\phi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\, \mathrm{d}u'.$$

5.2.7 Conservation of signatures

Notice that under the table of signatures in Section 5.2.5 and the fact that the induced metric on a $2-$sphere $\gamma_{\alpha\beta}$ has $s_2(\gamma_{\alpha\beta}) = 0$, the following remarkable property follows for tensor fields ϕ_1 and ϕ_2:

$$s_2(\phi_1 \cdot \phi_2) = s_2(\phi_1) + s_2(\phi_2).$$

This ensures *signature conservation* for all null structure, Bianchi, constraint and null Maxwell equations. When working with scale-invariant norms, this key property enables us to treat all the terms on the right hand side as one term. For example, look at the null Maxwell equation for $\nabla_3 \alpha_F$:

$$\nabla_3 \alpha_F + \frac{1}{2}\mathrm{tr}\underline{\chi}\,\alpha_F = -\nabla\rho_F + {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F + 2\underline{\eta}\cdot\rho_F + 2\underline{\omega}\alpha_F - \hat{\underline{\chi}}\cdot\underline{\alpha}_F.$$

There holds

- $s_2(\nabla_3 \alpha_F) = s_2\alpha_F + 1 = 1,$

- $s_2(\mathrm{tr}\underline{\chi}\,\alpha_F) = s_2(\mathrm{tr}\underline{\chi}) + s_2(\alpha_F) = 1,$

- $s_2(\nabla\rho_F, {}^*\nabla\sigma_F) = \frac{1}{2} + s_2(\rho_F, \sigma_F) = \frac{1}{2} + \frac{1}{2} = 1,$

- $s_2(\underline{\eta}\cdot\rho_F, {}^*\underline{\eta}\cdot\sigma_F) = \frac{1}{2} + \frac{1}{2} = 1,$

- $s_2(\underline{\omega}\,\alpha_F) = 1 + 0 = 1$,

- $s_2(\hat{\chi} \cdot \underline{\alpha}_F) = 0 + 1 = 1$.

Thus, throughout the equation, there is a balance of signature.

5.2.8 Hölder's inequality in scale-invariant norms

Any two tensor fields satisfy the following scale-invariant Hölder inequalities:

$$\|\phi_1 \cdot \phi_2\|_{\mathcal{L}^1_{(sc)}(S_{u,\underline{u}})} \lesssim \frac{1}{|u|} \|\phi_1\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \|\phi_2\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}, \tag{5.41}$$

$$\|\phi_1 \cdot \phi_2\|_{\mathcal{L}^1_{(sc)}(S_{u,\underline{u}})} \leq \frac{1}{|u|} \|\phi_1\|_{\mathcal{L}^1_{(sc)}(S_{u,\underline{u}})} \|\phi_2\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}, \tag{5.42}$$

$$\|\phi_1 \cdot \phi_2\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{1}{|u|} \|\phi_1\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \|\phi_2\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}, \tag{5.43}$$

Also, the following inequality holds

$$\|\phi_1 \cdot \phi_2\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \leq \frac{1}{|u|} \|\phi_1\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \|\phi_2\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}. \tag{5.44}$$

Crucially, in the region of study $\frac{1}{|u|} \ll 1$. This means, when measuring the size of a product of terms in scale-invariant norms, this size is very small compared to the scale-invariant norms of the individual terms. Essentially, it is this crucial fact that allows us to close all bootstrap arguments throughout this chapter.

5.2.9 Norms

Let $\psi \in \left\{\omega, \mathrm{tr}\chi, \eta, \underline{\eta}, \underline{\omega}\right\}$, $\Psi \in \left\{\tilde{\beta}, \rho, \sigma, \underline{\tilde{\beta}}, \underline{\alpha}\right\}$ and $\Psi' \in \left\{\rho, \sigma, \underline{\tilde{\beta}}, \underline{\alpha}\right\}$. Here $\tilde{\beta} := \beta - \frac{1}{2}R_{4(\cdot)}$ and $\underline{\tilde{\beta}} := \underline{\beta} + \frac{1}{2}R_{3(\cdot)}$. Denote $\widetilde{\mathrm{tr}\underline{\chi}} = \mathrm{tr}\underline{\chi} + \frac{2}{|u|}$. Also, let $\Upsilon \in \left\{\rho_F, \sigma_F, \underline{\alpha}_F\right\}$. For $0 \leq i \leq 6$, we define

$$\mathcal{O}_{i,\infty}(u,\underline{u}) := \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\hat{\chi}\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i\psi\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} + \frac{a^{\frac{1}{2}}}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\underline{\hat{\chi}}\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}$$
$$+ \frac{a}{|u|^2}\|(a^{\frac{1}{2}}\nabla)^i\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} + \frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}, \tag{5.45}$$

$$\mathcal{R}_{i,\infty}(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})},$$

$$\mathcal{F}_{i,\infty}(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha_F\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i\Upsilon\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}.$$

For $0 \leq i \leq 9$ and $0 \leq j \leq 10$, we define

$$\mathcal{O}_{j,2}(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^j\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \frac{a^{\frac{1}{2}}}{|u|}\|(a^{\frac{1}{2}}\nabla)^j\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$
$$+ \frac{a}{|u|^2}\|(a^{\frac{1}{2}}\nabla)^j\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^j\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})},$$

$$\mathcal{R}_{i,2}(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})},$$
$$\mathcal{F}_{j,2}(u,\underline{u}) = \|(a^{\frac{1}{2}})^{j-1}\nabla^j\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j(\rho_F,\sigma_F,\underline{\alpha}_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

For $0 \leq i \leq 10$ and $0 \leq j \leq 11$ we define

$$\mathcal{R}_i(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})},$$
$$\underline{\mathcal{R}}_i(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} + \|(a^{\frac{1}{2}}\nabla)^i\Psi'\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})},$$
$$\mathcal{F}_j(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}})^{j-1}\nabla^j\alpha_F\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{j-1}\nabla^j(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})},$$
$$\underline{\mathcal{F}}_j(u,\underline{u}) = \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}})^{j-1}\nabla^j(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} + \|(a^{\frac{1}{2}})^{j-1}\nabla^j\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})}.$$

We now set $\mathcal{O}_{i,\infty}, \mathcal{O}_{i,2}, \mathcal{R}_{i,\infty}, \mathcal{R}_{i,2}, \mathcal{F}_{i,\infty}, \mathcal{F}_{i,2}$ to be the supremum over $u, \underline{u}$ in the space-time region of the norms $\mathcal{O}_{i,\infty}(u,\underline{u}), \mathcal{O}_{i,2}(u,\underline{u}), \mathcal{R}_{i,\infty}(u,\underline{u}), \mathcal{R}_{i,2}(u,\underline{u}), \mathcal{F}_{i,\infty}(u,\underline{u})$ and $\mathcal{F}_{i,2}(u,\underline{u})$ respectively. Finally, define $\mathcal{O}, \mathcal{R}$ and $\mathcal{F}$:

$$\mathcal{O} = \sum_{i \leq 6}\left(\mathcal{O}_{i,\infty} + \mathcal{R}_{i,\infty} + \mathcal{F}_{i,\infty}\right) + \sum_{0 \leq i \leq 9}\mathcal{R}_{i,2} + \sum_{0 \leq j \leq 10}\left(\mathcal{O}_{j,2} + \mathcal{F}_{j,2}\right),$$
$$\mathcal{R} = \sum_{0 \leq i \leq 10}\left(\mathcal{R}_i + \underline{\mathcal{R}}_i\right), \quad \mathcal{F} = \sum_{0 \leq j \leq 11}\left(\mathcal{F}_j + \underline{\mathcal{F}}_j\right).$$

5.2.10 An explicit form of the Bianchi equations

Recall the Bianchi equations (5.25)-(5.34). In this section, we shall use the schematic notation for various components as introduced in the preceding subsection. We will work on a term-by-term basis to give the explicit forms of the right hand sides (RHS) of these equations. We will use the property

$$(D_\alpha R)_{\beta\gamma} = D_\alpha(R_{\beta\gamma}) - R(D_\alpha e_\beta, e_\gamma) - R(e_\beta, D_\alpha e_\gamma),$$

as well as the following identities:

$$D_A e_B = \nabla_A e_B + \frac{1}{2}\chi_{AB}e_3 + \frac{1}{2}\underline{\chi}_{AB}e_4, \tag{5.46}$$

$$D_3 e_A = \nabla_3 e_A + \eta_A e_3, \quad D_4 e_A = \nabla_4 e_A + \underline{\eta}_A e_4, \tag{5.47}$$

$$D_A e_3 = \underline{\chi}_A{}^{\sharp B}e_B + \zeta_A e_3, \quad D_A e_4 = \chi_A{}^{\sharp B}e_B - \zeta_A e_4, \tag{5.48}$$

$$D_3 e_4 = 2\eta^{\sharp A}e_A + 2\underline{\omega}\,e_4, \quad D_4 e_3 = 2\underline{\eta}^{\sharp A}e_A + 2\omega\,e_3, \tag{5.49}$$

$$D_3 e_3 = -2\underline{\omega}e_3, \quad D_4 e_4 = -2\omega e_4. \tag{5.50}$$

We therefore compute

$$
\begin{aligned}
(D_A R)_{44} &= 2\alpha_F \cdot \nabla\alpha_F - 2R(D_A e_4, e_4)\\
&= 2\alpha_F \cdot \nabla\alpha_F - (\psi, \hat{\chi}) \cdot R(e_A, e_4) + \psi \cdot \alpha_F \cdot \alpha_F\\
&= 2\alpha_F \cdot \nabla\alpha_F + (\psi, \hat{\chi}) \cdot \Upsilon \cdot \alpha_F + \psi \cdot \alpha_F^2,
\end{aligned}
$$

$$
\begin{aligned}
(D_4 R)_{4A} &= D_4(R_{4A}) - R(D_4 e_4, e_A) - R(e_4, D_4 e_A)\\
&= D_4(R_{4(\cdot)})(e_A) + (\psi \cdot \Upsilon \cdot \alpha_F)\\
&= D_4(R_{4(\cdot)})(e_A) + \psi \cdot \Upsilon \cdot \alpha_F,
\end{aligned}
$$

$$
\begin{aligned}
(D_A R)_{43} &= D_A(R_{43}) - R(D_A e_4, e_3) - R(D_A e_3, e_4) = \Upsilon \cdot \nabla\Upsilon\\
&\qquad\qquad + (\psi, \hat{\underline{\chi}}, \hat{\chi}, \operatorname{tr}\underline{\chi}) \cdot (\Upsilon, \alpha_F) \cdot \Upsilon,
\end{aligned}
$$

$$
\begin{aligned}
(D_4 R)_{A3} &= D_4(R_{A3}) - R(D_4 e_A, e_3) - R(e_A, D_4 e_3)\\
&= \Upsilon \cdot \nabla_4 \Upsilon + (\psi \cdot \Upsilon \cdot (\Upsilon, \alpha_F))\\
&= \Upsilon \cdot \nabla(\Upsilon, \alpha_F) + (\psi, \hat{\underline{\chi}}) \cdot \Upsilon \cdot (\Upsilon, \alpha_F),
\end{aligned}
$$

since

$$\nabla_4 \Upsilon = \nabla(\Upsilon, \alpha_F) + (\psi, \hat{\underline{\chi}}) \cdot (\Upsilon, \alpha_F).$$

Continuing, we have

$$
\begin{aligned}
(D_3 R)_{A4} &= D_3(R_{A4}) - R(D_3 e_A, e_4) - R(e_A, D_3 e_4)\\
&= \alpha_F \cdot \nabla_3 \Upsilon + \Upsilon \cdot \nabla_3 \alpha_F + \psi \cdot \Upsilon \cdot (\Upsilon, \alpha_F)\\
&= (\alpha_F, \Upsilon) \cdot \nabla\Upsilon + (\psi, \operatorname{tr}\underline{\chi}) \cdot \alpha_F \cdot \Upsilon + (\psi, \hat{\underline{\chi}}, \hat{\chi}) \cdot \Upsilon \cdot \Upsilon,
\end{aligned}
$$

since we have

$$\nabla_3 \Upsilon = \nabla \Upsilon + (\mathrm{tr}\underline{\chi}, \psi) \cdot \Upsilon,$$

$$\nabla_3 \alpha_F = \nabla \Upsilon + (\psi, \mathrm{tr}\underline{\chi})\alpha_F + (\psi, \hat{\chi}) \cdot \Upsilon.$$

Continuing, we have

$$(D_A R)_{33} = D_A(R_{33}) - 2R(D_A e_3, e_3) = \Upsilon \cdot \nabla \Upsilon + (\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}) \cdot \Upsilon \cdot \Upsilon, \qquad (5.51)$$

$$(D_3 R)_{3A} = D_3(R_{3A}) - R(D_3 e_3, e_A) - R(e_3, D_3 e_A) = \nabla_3(R_{3A}) + \psi \cdot \Upsilon \cdot \Upsilon, \qquad (5.52)$$

$$\begin{aligned}(D_4 R)_{33} &= 2\underline{\alpha}_F \cdot \nabla_4 \underline{\alpha}_F - 2R(D_4 e_3, e_3) = 2\underline{\alpha}_F \cdot \nabla_4 \underline{\alpha}_F + \psi \cdot \Upsilon \cdot \Upsilon \\ &= \Upsilon \cdot \left(\nabla \Upsilon + (\psi, \hat{\underline{\chi}}) \cdot (\Upsilon, \alpha_F)\right) + \psi \cdot \Upsilon \cdot \Upsilon \\ &= \Upsilon \cdot \nabla \Upsilon + (\psi, \hat{\underline{\chi}}) \cdot \Upsilon \cdot (\Upsilon, \alpha_F),\end{aligned}$$

$$\begin{aligned}(D_3 R)_{34} &= D_3(R_{34}) - R(D_3 e_3, e_4) - R(e_3, D_3 e_4) \\ &= \Upsilon \cdot \nabla_3 \Upsilon + \psi \cdot \Upsilon \cdot \Upsilon \qquad (5.53) \\ &= \Upsilon \cdot \nabla \Upsilon + (\mathrm{tr}\underline{\chi}, \psi) \cdot \Upsilon \cdot \Upsilon,\end{aligned}$$

$$\begin{aligned}(D_3 R)_{44} &= 2\alpha_F \cdot \nabla_3 \alpha_F - 2R(D_3 e_4, e_4) = 2\alpha_F \cdot \nabla_3 \alpha_F + \psi \cdot \alpha_F \cdot (\alpha_F, \Upsilon) \\ &= 2\alpha_F \cdot \nabla \Upsilon + (\psi, \mathrm{tr}\underline{\chi}, \hat{\underline{\chi}})\alpha_F^2 + (\psi, \hat{\chi}) \cdot \alpha_F \cdot \Upsilon + \psi \cdot \alpha_F \cdot (\alpha_F, \Upsilon),\end{aligned}$$

$$\begin{aligned}(D_4 R)_{43} &= D_4(R_{43}) - R(D_4 e_4, e_3) - R(e_4, D_4 e_3) = \Upsilon \cdot \nabla_4 \Upsilon + \psi \cdot \Upsilon \cdot \Upsilon + \psi \cdot \Upsilon \cdot \alpha_F \\ &= \Upsilon \cdot \nabla(\Upsilon, \alpha_F) + \Upsilon \cdot (\psi, \hat{\underline{\chi}}) \cdot (\Upsilon, \alpha_F).\end{aligned}$$

The expressions for $(D_A R)_{4B}$ and $(D_A R)_{3B}$ are as follows:

$$\begin{aligned}(D_A R)_{4B} &= D_A(R_{4B}) - R(D_A e_4, e_B) - R(D_A e_B, e_4) = (\alpha_F \cdot \nabla \Upsilon + \Upsilon \cdot \nabla \alpha_F) \\ &\quad + \left((\psi, \hat{\chi}) \cdot \Upsilon \cdot \Upsilon + \psi \cdot \alpha_F \cdot \Upsilon\right) + \left((\psi, \hat{\chi}) \cdot \Upsilon \cdot \Upsilon + (\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}) \cdot \alpha_F \cdot \alpha_F\right) \\ &= \alpha_F \cdot \nabla \Upsilon + \Upsilon \cdot \nabla \alpha_F + (\psi, \hat{\underline{\chi}}, \hat{\chi}, \mathrm{tr}\underline{\chi}) \cdot (\alpha_F, \Upsilon) \cdot (\alpha_F, \Upsilon),\end{aligned}$$

$$(D_A R)_{3B} = D_A(R_{3B}) - R(D_A e_3, e_B) - R(D_A e_B, e_3) = \Upsilon \cdot \nabla \Upsilon + (\psi, \hat{\chi}, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}) \cdot \Upsilon \cdot \Upsilon.$$

$$(5.54)$$

5.2.10.1 An important renormalization

A novel ingredient in our analysis is the introduction of renormalized quantities for β and $\underline{\beta}$. The motivation behind the introduction of these two quantities stems from the Bianchi equations. Take for example the identity for β:

$$\nabla_4\beta + 2tr\chi\beta = \text{div}\alpha - 2\omega\beta + \eta\cdot\alpha - \frac{1}{2}(D_A R_{44} - D_4 R_{4A}).$$

Strictly speaking, this is an equality of 1-forms. In particular, if we evaluate on a vector e_A, we get

$$(\nabla_4\beta)(e_A) + 2tr\chi\beta_A = (\text{div}\alpha)(e_A) - 2\omega\beta_A + (\eta\cdot\alpha)_A$$
$$- \frac{1}{2}(D_A R)_{44} + \frac{1}{2}(D_4 R)_{4A} \Rightarrow$$
$$(\nabla_4\beta)(e_A) - \frac{1}{2}\nabla_4(R_{4(\cdot)})(e_A)$$
$$= \nabla\alpha + \psi\cdot\Psi + \alpha_F\cdot\nabla\alpha_F + (\psi,\hat{\chi})\cdot\Upsilon\cdot\alpha_F + \psi\cdot\alpha_F^2 + \psi\cdot\alpha \Rightarrow$$
$$\nabla_4(\beta - \frac{1}{2}R_{4(\cdot)}) = \nabla\alpha + \psi\cdot(\Psi,\alpha) + \alpha_F\cdot\nabla\alpha_F + (\psi,\hat{\chi})\cdot\Upsilon\cdot\alpha_F + \psi\cdot\alpha_F^2.$$

This motivates us to define the renormalized curvature component

$$\tilde{\beta} := \beta - \frac{1}{2}R_{4(\cdot)}. \tag{5.55}$$

Using the Bianchi equation for $\tilde{\beta}$, we succeed in getting rid of the term $(\rho_F, \sigma_F)\nabla_4\alpha_F$ coming from $D_4 R_{4A}$, which cannot be bounded directly from the Maxwell equations. Similarly, we need to define

$$\underline{\tilde{\beta}} := \underline{\beta} + \frac{1}{2}R_{3(\cdot)}. \tag{5.56}$$

The gain from (5.55) and (5.56) is that the Bianchi equations are now expressed in a way that the right-hand sides of the equations are controllable in terms of the Ricci coefficients and the curvature and Maxwell components.

5.2.10.2 The Bianchi equations in schematic form for the renormalized components

In this section we give, in schematic form, the Bianchi equations expressed in terms of $\left\{\alpha, \underline{\alpha}, \tilde{\beta}, \underline{\tilde{\beta}}, \rho, \sigma\right\}$. We explain our way of obtaining these. Take for example the transport equation for β:

$$\nabla_4\beta + 2\mathrm{tr}\chi\beta = \mathrm{div}\alpha - 2\omega\beta + \eta\cdot\alpha - \frac{1}{2}(D_A R_{44} - D_4 R_{4A}).$$

Then

$$\nabla_4\tilde{\beta} + 2\mathrm{tr}\chi\tilde{\beta} = (\nabla_4\beta + 2\mathrm{tr}\chi\beta) - \frac{1}{2}\nabla_4(R_{4(\cdot)})(e_A) - \mathrm{tr}\chi R_{4A}.$$

Working similarly, we obtain

$$\nabla_4\tilde{\beta} + 2\mathrm{tr}\chi\tilde{\beta} = \nabla\alpha + \psi\cdot(\alpha,\Psi) + \alpha_F\cdot\nabla\alpha_F + \psi\cdot(\Upsilon,\alpha_F)\cdot\alpha_F + (\psi,\hat{\chi})\cdot\Upsilon\cdot\alpha_F, \tag{5.57}$$

$$\nabla_3\tilde{\beta} + \mathrm{tr}\underline{\chi}\tilde{\beta} = \nabla\Psi + (\psi,\hat{\chi})\Psi + \Upsilon\nabla(\Upsilon,\alpha_F) + (\alpha_F,\Upsilon)\nabla\Upsilon + (\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot\Upsilon, \tag{5.58}$$

$$\nabla_4\underline{\tilde{\beta}} + \mathrm{tr}\chi\underline{\tilde{\beta}} = \nabla\Psi + (\psi,\hat{\underline{\chi}})\Psi + (\alpha_F,\Upsilon)\nabla\Upsilon + (\psi,\hat{\chi},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot\Upsilon, \tag{5.59}$$

$$\nabla_3\underline{\tilde{\beta}} + 2\mathrm{tr}\underline{\chi}\underline{\tilde{\beta}} = \nabla\Psi + \psi\cdot\Psi + \Upsilon\nabla\Upsilon + (\psi,\mathrm{tr}\underline{\chi},\hat{\underline{\chi}})\cdot\Upsilon\cdot\Upsilon, \tag{5.60}$$

$$\nabla_4\underline{\alpha} + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha} = \nabla\Psi + (\psi\hat{\underline{\chi}})\cdot\Psi + \Upsilon\cdot\nabla\Upsilon + (\psi,\mathrm{tr}\underline{\chi})\cdot\Upsilon\cdot\Upsilon + (\psi,\hat{\underline{\chi}})\cdot\Upsilon\cdot(\Upsilon,\alpha_F), \tag{5.61}$$

$$\begin{aligned}\nabla_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha &= \nabla\Psi + \alpha_F\cdot\nabla\Upsilon + \Upsilon\cdot(\nabla\alpha_F,\nabla\Upsilon) \\ &\quad + (\psi,\hat{\chi})\cdot\Psi + \psi\cdot\alpha + (\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot(\alpha_F,\Upsilon),\end{aligned} \tag{5.62}$$

$$\begin{aligned}\nabla_4\rho + \frac{3}{2}\mathrm{tr}\chi\rho &= \nabla\Psi + (\psi,\hat{\underline{\chi}})\cdot(\alpha,\Psi) + \alpha_F\cdot\nabla\Upsilon + \Upsilon\cdot(\nabla\alpha_F,\nabla\Upsilon) \\ &\quad + (\psi,\hat{\chi},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot(\alpha_F,\Upsilon),\end{aligned} \tag{5.63}$$

$$\nabla_3\rho + \frac{3}{2}\mathrm{tr}\underline{\chi}\rho = \nabla\Psi + \Upsilon\cdot\nabla\Upsilon + (\psi,\hat{\chi})\cdot\Psi + (\psi,\mathrm{tr}\underline{\chi})\cdot\Upsilon\cdot\Upsilon + (\psi,\hat{\underline{\chi}})\cdot(\Upsilon,\alpha_F)\cdot\Upsilon, \tag{5.64}$$

$$\begin{aligned}\nabla_4\sigma + \frac{3}{2}\mathrm{tr}\chi\sigma &= \nabla\Psi + (\psi,\hat{\underline{\chi}})\cdot(\alpha,\Psi) + \alpha_F\cdot\nabla\Upsilon \\ &\quad + \Upsilon\cdot\nabla\alpha_F + (\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi})\cdot(\alpha_F,\Upsilon)\cdot(\alpha_F,\Upsilon),\end{aligned} \tag{5.65}$$

$$\nabla_3\sigma + \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma = \nabla\Psi + \Upsilon\cdot\nabla\Upsilon + (\psi,\hat{\chi})\cdot\Psi + (\psi,\mathrm{tr}\underline{\chi})\cdot\Upsilon\cdot\Upsilon + (\psi,\hat{\underline{\chi}})\cdot(\Upsilon,\alpha_F)\cdot\Upsilon. \tag{5.66}$$

5.3 The preliminary estimates

5.3.1 Setting up the bootstrap argument

We shall employ a bootstrap argument to derive uniform upper bounds on $\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}}, \mathcal{F}, \underline{\mathcal{F}}$ for the nonlinear Einstein–Maxwell equations. Along H_{u_∞} and $\underline{H}_0$, by analysing the characteristic initial data, we can obtain the bounds

$$\mathcal{O}^{(0)} + \mathcal{R}^{(0)} + \underline{\mathcal{R}}^{(0)} + \mathcal{F}^{(0)} + \underline{\mathcal{F}}^{(0)} \lesssim \mathcal{I}^{(0)}. \tag{5.67}$$

Our goal is to show that in $\mathcal{D} = \left\{ (u, \underline{u}) \times \mathbb{S}^2 \mid u_\infty \leq u \leq -a/4, 0 \leq \underline{u} \leq 1 \right\}$ there holds

$$\mathcal{O}(u, \underline{u}) + \mathcal{R}(u, \underline{u}) + \underline{\mathcal{R}}(u, \underline{u}) + \mathcal{F}(u, \underline{u}) + \underline{\mathcal{F}}(u, \underline{u}) \lesssim \left(\mathcal{I}^{(0)} \right)^4 + \left(\mathcal{I}^{(0)} \right)^2 + \mathcal{I}^{(0)} + 1. \tag{5.68}$$

Once these uniform bounds are obtained, by a standard local existence result, the solutions can always be extended a bit towards the future direction of u. Hence, the uniform estimate (5.68) for $u_\infty \leq u \leq -a/4$ would imply that a solution to the Einstein–Maxwell equations exists in the slab $\mathcal{D}$. To derive the uniform bound (5.68), we make the bootstrap assumptions

$$\mathcal{O}(u, \underline{u}) \leq O, \ \ \mathcal{R}(u, \underline{u}) + \underline{\mathcal{R}}(u, \underline{u}) \leq R, \ \ \mathcal{F}(u, \underline{u}) + \underline{\mathcal{F}}(u, \underline{u}) \leq F, \tag{5.69}$$

for large numbers O, R and F such that

$$\left(\mathcal{I}^{(0)} \right)^4 + \left(\mathcal{I}^{(0)} \right)^2 + \mathcal{I}^{(0)} + 1 \ll \min \left\{ O, R, F \right\},$$

but also such that

$$(O + R + F)^{20} \leq a^{\frac{1}{16}}.$$

Define[8], $\mathcal{B} = \left\{ u \mid u_\infty \leq u \leq -a/4 \text{ and (5.69) holds for every } 0 \leq u' \leq u, 0 \leq \underline{u} \leq 1 \right\}$. We are hoping to prove that $\mathcal{B}$ is in fact equal as a set to the entire interval $[u_\infty, -a/4]$. To do this, we take advantage of the topology of the unit interval. In particular, since it is connected, it suffices to show that the set $\mathcal{B}$ is both closed and open.

By assumption, at $u = u_\infty$, we have (5.67). By continuity of solutions (via local existence), there exists a small $\epsilon > 0$ such that it holds for $u_\infty \leq u \leq u_\infty + \epsilon$ we have

$$\mathcal{O}^{(0)} \lesssim \mathcal{I}^{(0)} \ll O, \ \ \mathcal{R}^{(0)} + \underline{\mathcal{R}}^{(0)} \lesssim \mathcal{I}^{(0)} \ll R, \ \ \mathcal{F}^{(0)} + \underline{\mathcal{F}}^{(0)} \lesssim \mathcal{I}^{(0)} \ll F,$$

$$\mathcal{O}(u, \underline{u}) \lesssim 2\mathcal{I}^{(0)} \ll O, \ \ \mathcal{R}(u, \underline{u}) + \underline{\mathcal{R}}(u, \underline{u}) \lesssim 2\mathcal{I}^{(0)} \ll R, \ \ \mathcal{F}(u, \underline{u}) + \underline{\mathcal{F}}(u, \underline{u}) \lesssim 2\mathcal{I}^{(0)} \ll F.$$

This implies in particular that $\mathcal{B}$ is not empty and in fact $[u_\infty, u_\infty + \epsilon] \subseteq \mathcal{B}$. At the same time, naturally there holds $\mathcal{B} \subseteq [u_\infty, -\frac{a}{4}]$. If we are able to prove that $\mathcal{B}$ as a set is both open and closed, we can conclude that in fact $\mathcal{B} \equiv [u_\infty, -\frac{a}{4}]$. Indeed, in the remainder of the paper we show the following estimates

---

[8]We assume the solution exists for each region of the form $0 \leq u' \leq u, 0 \leq \underline{u} \leq 1$, where $u \in \mathcal{B}$.

$$\mathcal{O}(u,\underline{u}) \lesssim \mathcal{I}^{(0)} + \mathcal{R}(u,\underline{u}) + \underline{\mathcal{R}}(u,\underline{u}) + \mathcal{F}(u,\underline{u}) + \underline{\mathcal{F}}(u,\underline{u}),$$

$$\mathcal{F}(u,\underline{u}) + \underline{\mathcal{F}}(u,\underline{u}) \lesssim \mathcal{R}^2(u,\underline{u}) + \underline{\mathcal{R}}^2(u,\underline{u}) + \left(\mathcal{I}^{(0)}\right)^2 + \mathcal{I}^{(0)} + 1,$$

$$\mathcal{R}(u,\underline{u}) + \underline{\mathcal{R}}(u,\underline{u}) \lesssim \left(\mathcal{I}^{(0)}\right)^2 + \mathcal{I}^{(0)} + 1.$$

Here by $A \lesssim B$ we mean that there exists a constant c with $A \leq cB$, where the constant is independent of $\mathcal{O}, \mathcal{R}, \mathcal{F}$ and $\mathcal{I}^{(0)}$. These are improvements of the upper bounds in bootstrap assumptions. By the continuity of solutions and local existence arguments, $\mathcal{B}$ can be extended a bit towards larger u. This implies that $\mathcal{B}$ is open. Together with closedness or $\mathcal{B}$, we conclude that $\mathcal{B} \equiv [u_\infty, -a/4]$ and in $\mathcal{B}$ the desired bounds hold.

5.3.2 Estimates for the metric components

Proposition 5.3.1. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), we have*

$$\|\Omega - 1\|_{L^\infty(S_{u,\underline{u}})} \lesssim \frac{O}{|u|}.$$

Proof. Consider the equation

$$\omega = -\frac{1}{2}\nabla_4(\log \Omega) = \frac{1}{2}\frac{\partial}{\partial \underline{u}}(\Omega^{-1}).$$

We integrate with respect to $d\underline{u}$. Since on $\underline{H}_0$ we have $\Omega^{-1} = 1$, we can obtain

$$\|\Omega^{-1} - 1\|_{L^\infty(S_{u,\underline{u}})} \lesssim \int_0^{\underline{u}} \|\omega\|_{L^\infty(S_{u,\underline{u}'})} d\underline{u}' \lesssim \frac{O}{|u|}. \tag{5.70}$$

Here we have used the bootstrap assumption. Finally, notice that

$$\|\Omega - 1\|_{L^\infty(S_{u,\underline{u}})} \leq \|\Omega\|_{L^\infty(S_{u,\underline{u}})} \|\Omega^{-1} - 1\|_{L^\infty(S_{u,\underline{u}})} \lesssim \frac{\frac{O}{|u|}}{1 + \frac{O}{|u|}} \lesssim \frac{O}{|u|}.$$

$\square$

We now control the induced metric γ on $S_{u,\underline{u}}$:

Proposition 5.3.2. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), we have for the metric γ on $S_{u,\underline{u}}$:*

$$c' \leq \det \gamma \leq C',$$

where the two constants depend only on the initial data. Moreover, in $\mathcal{D}$, there holds

$$|\gamma_{AB}| + |(\gamma^{-1})^{AB}| \leq C'.$$

Proof. We employ the first variation formula $\mathcal{L}_L \gamma = 2\Omega\chi$. In coordinates, this rewrites as

$$\frac{\partial}{\partial \underline{u}} \gamma_{AB} = 2\Omega\chi_{AB}. \tag{5.71}$$

This implies

$$\frac{\partial}{\partial \underline{u}} (\log(\det\gamma)) = 2\Omega\,\mathrm{tr}\chi.$$

Let $\gamma_0(u, \underline{u}, \theta^1, \theta^2) = \gamma(u, 0, \theta^1, \theta^2)$. Then with $2\Omega\,\mathrm{tr}\chi \lesssim \frac{O}{|u|}$, we have

$$\frac{\det\gamma}{\det\gamma_0} = e^{\int_0^{\underline{u}} 2\Omega\mathrm{tr}\chi\,\mathrm{d}\underline{u}'} \lesssim e^{\frac{O}{a}}.$$

Via Taylor expansion, this implies

$$|\det\gamma - \det\gamma_0| \lesssim \frac{O}{a}. \tag{5.72}$$

This gives uniform upper and lower bounds for $\det\gamma$. Let Λ be the larger eigenvalue of γ. We have

$$\Lambda \leq \sup\gamma_{AB},$$

$$\sum_{A,B=1,2} |\chi_{AB}| \leq \Lambda\|\chi\|_{L^\infty(S_{u,\underline{u}})},$$

$$|\gamma_{AB} - (\gamma_0)_{AB}| \leq \int_0^{\underline{u}} |\chi_{AB}|\mathrm{d}\underline{u}' \leq \Lambda \frac{a^{\frac{1}{2}}}{|u|} O \lesssim \frac{O}{a^{\frac{1}{2}}}.$$

$\square$

We will also need the following:

Proposition 5.3.3. *We continue to work under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69). Fix a point $(\underline{u}, \theta)$ on the initial hypersurface $\underline{H}_0$. Along the outgoing null geodesics emanating from (u, θ), define $\Lambda(\underline{u})$ and $\lambda(\underline{u})$ to be the larger and smaller eigenvalue of $\gamma^{-1}(u, 0, \theta)\gamma(u, \underline{u}, \theta)$. Then there holds*

$$|\Lambda(\underline{u}) - 1| + |\lambda(\underline{u}) - 1| \lesssim \frac{1}{a^{\frac{1}{2}}}.$$

Proof. Define $\nu(\underline{u}) := \sqrt{\frac{\Lambda(\underline{u})}{\lambda(\underline{u})}}$. Following the derivation of (5.93) in [22], by (5.71), we have

$$\nu(\underline{u}) \leq 1 + \int_0^{\underline{u}} |\Omega\hat{\chi}(\underline{u}')|_\gamma \nu(\underline{u}') \, d\underline{u}'.$$

Via Grönwall's inequality, this implies

$$|\nu(\underline{u})| \lesssim 1 \quad \text{and} \quad |\nu(\underline{u}) - 1| \leq \frac{a^{\frac{1}{2}} \cdot O}{|u|^2} \leq \frac{O}{a^{\frac{3}{2}}} \leq \frac{1}{a}. \tag{5.73}$$

The desired estimate follows from (5.72) and (5.73). $\qquad\square$

The above two propositions also imply

Proposition 5.3.4. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption* (5.69), *in the slab of existence $\mathcal{D}$ we have*

$$\sup_{\underline{u}} |Area(S_{u,\underline{u}}) - Area(S_{u,0})| \lesssim \frac{O^{\frac{1}{2}}}{a^{\frac{1}{2}}} |u|^2.$$

Proof. This follows from the definitions in Subsection 5.2.4 and the estimate (5.72). $\qquad\square$

5.3.3 Estimates for transport equations

In the sections to follow, we will employ the following propositions for the transport equations:

Proposition 5.3.5. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption* (5.69), *for an arbitrary $S-$tangent tensor ϕ of arbitrary rank, we have*

$$\|\phi\|_{L^2(S_{u,\underline{u}})} \lesssim \|\phi\|_{L^2(S_{u,\underline{u}'})} + \int_{\underline{u}'}^{\underline{u}} \|\nabla_4\phi\|_{L^2(S_{u,\underline{u}''})} \, d\underline{u}'', \tag{5.74}$$

$$\|\phi\|_{L^2(S_{u,\underline{u}})} \lesssim \|\phi\|_{L^2(S_{u',\underline{u}})} + \int_{u'}^{u} \|\nabla_3\phi\|_{L^2(S_{u'',\underline{u}})} \, du''. \tag{5.75}$$

Proof. Here we first prove (5.74). For a scalar function f, by variation of area formula, we have

$$\frac{d}{d\underline{u}} \int_{S_{u,\underline{u}}} f = \int_{S_{u,\underline{u}}} \left(\frac{df}{d\underline{u}} + \Omega\mathrm{tr}\chi f \right) = \int_{S_{u,\underline{u}}} \Omega\left(e_4(f) + \mathrm{tr}\chi f\right).$$

Taking $f = |\phi|_\gamma^2$, using Cauchy-Schwarz inequality on the sphere and L^∞ bounds for Ω and $\mathrm{tr}\chi$, we obtain

$$2\|\phi\|_{L^2(S_{u,\underline{u}})} \cdot \frac{d}{d\underline{u}} \|\phi\|_{L^2(S_{u,\underline{u}})} \lesssim \|\phi\|_{L^2(S_{u,\underline{u}})} \cdot \|\nabla_4\phi\|_{L^2(S_{u,\underline{u}})} + \frac{O}{|u|} \|\phi\|_{L^2(S_{u,\underline{u}})}^2.$$

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This implies
$$\frac{d}{d\underline{u}}\|\phi\|_{L^2(S_{u,\underline{u}})} \lesssim \|\nabla_4\phi\|_{L^2(S_{u,\underline{u}})} + \frac{O}{|u|}\|\phi\|_{L^2(S_{u,\underline{u}})}.$$

And (5.74) can be concluded by applying Grönwall's inequality for $\underline{u}$ variable.

Inequality (5.75) can be proved in a similar fashion. For a scalar function f, we arrive at
$$\underline{L}\int_{S_{u,\underline{u}}} f = \int_{S_{u,\underline{u}}} \left(\underline{L}f + \Omega\mathrm{tr}\underline{\chi}f\right) = \int_{S_{u,\underline{u}}} \Omega\left(e_3(f) + \mathrm{tr}\underline{\chi}f\right).$$

Taking $f = |\phi|_\gamma^2$, using Cauchy-Schwarz inequality on the sphere and the fact $\Omega > 0, \mathrm{tr}\underline{\chi} < 0$, we obtain
$$2\|\phi\|_{L^2(S_{u,\underline{u}})} \cdot \underline{L}\|\phi\|_{L^2(S_{u,\underline{u}})} \lesssim \|\phi\|_{L^2(S_{u,\underline{u}})} \cdot \|\nabla_3\phi\|_{L^2(S_{u,\underline{u}})}.$$

This implies $\underline{L}\|\phi\|_{L^2(S_{u,\underline{u}})} \lesssim \|\nabla_3\phi\|_{L^2(S_{u,\underline{u}})}$ and (5.75) follows. $\qquad\square$

We then rewrite the above inequalities in scale invariant norms:

Proposition 5.3.6. *There holds*
$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,0})} + \int_0^{\underline{u}} \|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}',$$

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a}{|u'|^2}\|\nabla_3\phi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'.$$

For equations along the incoming direction, sometimes the borderline terms necessitate more precise estimates. Typically, a borderline term contains $\mathrm{tr}\underline{\chi}$. It turns out that the coefficients in front of $\mathrm{tr}\underline{\chi}$ play an important role.

Proposition 5.3.7. *We continue to work under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69). Let v and Υ be $S_{u,\underline{u}}-$tangent tensor fields of rank k satisfying the transport equation*
$$\nabla_3 v_{A_1\ldots A_k} + \lambda_0 \mathrm{tr}\underline{\chi}\, v_{A_1\ldots A_k} = \Upsilon_{A_1\ldots A_k}.$$

If we define $\lambda_1 = 2\lambda_0 - 1$, we have
$$|u|^{\lambda_1}\|v\|_{L^2(S_{u,\underline{u}})} \lesssim |u_\infty|^{\lambda_1}\|v\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^{\lambda_1}\|\Upsilon\|_{L^2(S_{u',\underline{u}})} \, du'$$

where the implicit constant is allowed to depend on λ_0.

Proof. We use the variation of area formula for the equivariant vector $\underline{L} = \Omega e_3$ and a scalar function f:

$$\underline{L} \int_{S_{u,\underline{u}}} f = \int_{S_{u,\underline{u}}} \left(\underline{L} f + \Omega \mathrm{tr}\underline{\chi} f \right) = \int_{S_{u,\underline{u}}} \Omega \left(e_3(f) + \mathrm{tr}\underline{\chi} f \right).$$

With this identity, we obtain

$$
\begin{aligned}
&\underline{L}\Big(\int_{S_{u,\underline{u}}} |u|^{2\lambda_1}|\phi|^2 \Big) \\
&= \int_{S_{u,\underline{u}}} \Omega \Big(-2\lambda_1 |u|^{2\lambda_1-1}(e_3 u)|\phi|^2 + 2|u|^{2\lambda_1} <\phi, \nabla_3\phi>_\gamma + \mathrm{tr}\underline{\chi}|u|^{2\lambda_1}|\phi|^2 \Big) \\
&= \int_{S_{u,\underline{u}}} \Omega \Big(2|u|^{2\lambda_1} <\phi, \nabla_3\phi + \lambda_0 \mathrm{tr}\underline{\chi}\phi> \Big) \\
&\quad + \int_{S_{u,\underline{u}}} \Omega |u|^{2\lambda_1} \Big(-\frac{2\lambda_1(e_3 u)}{|u|} + (1-2\lambda_0)\mathrm{tr}\underline{\chi} \Big)|\phi|^2.
\end{aligned}
$$

(5.76)

Observe that we have

$$
\begin{aligned}
&-\frac{2\lambda_1(e_3 u)}{|u|} + (1-2\lambda_0)\mathrm{tr}\underline{\chi} \\
&= -\frac{2\lambda_1 \Omega^{-1}}{|u|} + (1-2\lambda_0)\mathrm{tr}\underline{\chi} \\
&= -\frac{2\lambda_1(\Omega^{-1}-1)}{|u|} + (1-2\lambda_0)(\mathrm{tr}\underline{\chi} + \frac{2}{|u|}) - \frac{2\lambda_1 + 2 - 4\lambda_0}{|u|} \\
&\lesssim \frac{O}{|u|^2}.
\end{aligned}
$$

(5.77)

For the last inequality, we employ (5.70), the bootstrap assumption and the fact that $\|\mathrm{tr}\underline{\chi} + \frac{2}{|u|}\|_{L^\infty(S_{u,\underline{u}})} \leq \frac{O}{|u|^2}$ and $\lambda_1 = 2(\lambda_0 - 1/2)$.

Using Cauchy-Schwarz for the first term and applying Grönwall's inequality for the second term, we obtain

$$
\begin{aligned}
&|u|^{\lambda_1}\|\phi\|_{L^2(S_{u,\underline{u}})} \\
&\lesssim e^{O\|u^{-2}\|_{L^1_u}} \Big(|u_\infty|^{\lambda_1}\|\phi\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^{\lambda_1}\|F\|_{L^2(S_{u',\underline{u}})}du' \Big) \\
&\lesssim |u_\infty|^{\lambda_1}\|\phi\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^{\lambda_1}\|F\|_{L^2(S_{u',\underline{u}})}du'.
\end{aligned}
$$

In the last step, we use $O\|u^{-2}\|_{L^1_u} \lesssim O/a \leq 1$. $\qquad\square$

5.3.4 Sobolev embedding

With the derived estimates for the metric γ, we can obtain a bound on the isoperimetric constant for a $2-$sphere S:

$$I(S) = \sup_{\substack{U \subset S \\ \partial U \in C^1}} \frac{\min\left\{\mathrm{Area}(U), \mathrm{Area}(U^c)\right\}}{\left(\mathrm{Perimeter}(\partial U)\right)^2}.$$

Proposition 5.3.8. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), the isoperimetric constant obeys the bound*

$$I(S_{u,\underline{u}}) \le \frac{1}{\pi}$$

for $u_\infty \le u \le -a/4$ and $0 \le \underline{u} \le 1$.

Proof. Fix u. Given $U_{\underline{u}} \subset S_{u,\underline{u}}$, denote by $U_0 \subset S_{u,0}$ the pullback image of $U_{\underline{u}}$ under the diffeomorphism generated by the equivariant vector field L. Using Propositions 5.3.2 and 5.3.3 we can obtain that

$$\frac{\mathrm{Perimeter}\left(\partial U_{\underline{u}}\right)}{\mathrm{Perimeter}\left(\partial U_0\right)} \ge \sqrt{\inf_{S_{u,0}} \lambda(\underline{u})},$$

$$\frac{\mathrm{Area}\, U_{\underline{u}}}{\mathrm{Area}\, U_0} \le \sup_{S_{u,0}} \frac{\det(\gamma_{\underline{u}})}{\det(\gamma_0)}, \quad \frac{\mathrm{Area}\, U_{\underline{u}}^c}{\mathrm{Area}\, U_0^c} \le \sup_{S_{u,0}} \frac{\det(\gamma_{\underline{u}})}{\det(\gamma_0)}.$$

Using the fact that $I(S_{u,0}) = \frac{1}{2\pi}$, as it is the standard sphere in Minkowski spacetime, the bounds in Propositions 5.3.2 and 5.3.3 yield the conclusion. $\square$

We shall be employing an $L^2 - L^\infty$ embedding statement in this work quite often. To derive it, in addition to 5.3.8, we require the two propositions below, whose proof is found in [22].

Proposition 5.3.9. *Suppose (S, γ) is a Riemannian $2-$manifold. There holds*

$$\left(Area(S)\right)^{-\frac{1}{p}} \|\phi\|_{L^p(S)} \le C_p \sqrt{\max\left\{I(S), 1\right\}} \left(\|\nabla\phi\|_{L^2(S)} + \left(Area(S)\right)^{-\frac{1}{2}} \|\phi\|_{L^2(S)}\right),$$
$$(5.78)$$

for any $2 < p < \infty$ and any tensor ϕ.

Proposition 5.3.10. *Suppose (S, γ) is a Riemannian $2-$manifold. There holds*

$$\|\phi\|_{L^\infty(S)} \le C_p \sqrt{\max\left\{I(S), 1\right\}} \left(Area(S)\right)^{\frac{1}{2}-\frac{1}{p}} \left(\|\nabla\phi\|_{L^p(S)} + \left(Area(S)\right)^{-\frac{1}{2}} \|\phi\|_{L^p(S)}\right),$$
$$(5.79)$$

for any $2 < p < \infty$ and any tensor ϕ.

Given Proposition 5.3.4, we know that $Area(S_{u,\underline{u}}) \approx |u|^2$. Substituting this into Propositions 5.3.9 and 5.3.10 and taking into account Proposition 5.3.8, we have the following $L^2 - L^\infty$ Sobolev embedding inequality:

Proposition 5.3.11. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), it holds*

$$\|\phi\|_{L^\infty(S_{u,\underline{u}})} \lesssim \sum_{0 \le i \le 2} \left\||u|^{i-1}\nabla^i\phi\right\|_{L^2(S_{u,\underline{u}})}.$$

In scale invariant norms:

$$\|\phi\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \lesssim \sum_{0 \le i \le 2} \|(a^{\frac{1}{2}}\nabla)^i\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

5.3.5 Commutation formulae

We list some useful commutation formulae that shall be used to give a schematic representation of repeated commutations. These are standard in the literature and can be found, for example, in [43], [5] and [22].

Proposition 5.3.12. *For a scalar function f, there holds*

$$[\nabla_4, \nabla]f = \frac{1}{2}(\eta + \underline{\eta})\nabla_4 f - \chi \cdot \nabla f,$$

$$[\nabla_3, \nabla]f = \frac{1}{2}(\eta + \underline{\eta})\nabla_3 f - \underline{\chi} \cdot \nabla f.$$

Proposition 5.3.13. *For an $S_{u,\underline{u}}-$tangent $1-$form U_b, there holds*

$$[\nabla_4, \nabla_a]U_b = -\chi_{ac}\nabla_c U_b + \epsilon_{ac}{}^*\beta_{\mathcal{R}b}U_c + \frac{1}{2}(\eta_a + \underline{\eta}_a)\nabla_4 U_b - \chi_{ac}\underline{\eta}_b U_c + \chi_{ab}\underline{\eta} \cdot U,$$

$$[\nabla_3, \nabla_a]U_b = -\underline{\chi}_{ac}\nabla_c U_b + \epsilon_{ac}{}^*\underline{\beta}_{\mathcal{R}b}U_c + \frac{1}{2}(\eta_a + \underline{\eta}_a)\nabla_3 U_b - \chi_{ac}\underline{\eta}_b U_c + \underline{\chi}_{ab}\eta \cdot U.$$

Proposition 5.3.14. *For an $S_{u,\underline{u}}-$tangent $2-$form V_{bc}, there holds*

$$[\nabla_4, \nabla_a]V_{bc} = \frac{1}{2}(\eta_a + \underline{\eta}_a)\nabla_4 V_{bc} - \underline{\eta}_b V_{dc}\chi_{ad} - \underline{\eta}_c V_{bd}\chi_{ad} - \epsilon_{bd}{}^*\beta_{\mathcal{R}a}V_{dc} - \epsilon_{cd}{}^*\beta_{\mathcal{R}c}V_{bd}$$
$$+ \chi_{ac}V_{bd}\underline{\eta}_d + \chi_{ab}V_{dc}\underline{\eta}_d - \chi_{ad}\nabla_d V_{bc},$$

$$[\nabla_3, \nabla_a]V_{bc} = \frac{1}{2}(\eta_a + \underline{\eta}_a)\nabla_3 V_{bc} - \eta_b V_{dc}\underline{\chi}_{ad} - \eta_c V_{bd}\underline{\chi}_{ad} - \epsilon_{bd}{}^*\underline{\beta}_{\mathcal{R}}a V_{dc} - \epsilon_{cd}{}^*\underline{\beta}_{\mathcal{R}}c V_{bd}$$
$$+ \underline{\chi}_{ac}V_{bd}\eta_d + \underline{\chi}_{ab}V_{dc}\eta_d - \underline{\chi}_{ad}\nabla_d V_{bc}.$$

Proposition 5.3.15. *Assume $\nabla_4\phi = F_0$. Let $\nabla_4\nabla^i\phi = F_i$. Then*

$$F_i = \sum_{i_1+i_2+i_3=i} \nabla^{i_1}(\eta + \underline{\eta})^{i_2}\nabla^{i_3}F_0 + \sum_{i_1+i_2+i_3+i_4=i-1} \nabla^{i_1}(\eta + \underline{\eta})^{i_2}\nabla^{i_3}\beta_{\mathcal{R}}\nabla^{i_4}\phi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}(\eta + \underline{\eta})^{i_2}\nabla^{i_3}\chi\nabla^{i_4}\phi.$$

Assume now that $\nabla_3\phi = G_0$. Let $\nabla_3\nabla^i\phi = G_i$. Then

$$G_i + \frac{i}{2}tr\underline{\chi}\nabla^i\phi = \sum_{i_1+i_2+i_3=i} \nabla^{i_1}(\eta + \underline{\eta})^{i_2}\nabla^{i_3}G_0$$
$$+ \sum_{i_1+i_2+i_3+i_4=i-1} \nabla^{i_1}(\eta + \underline{\eta})^{i_2}\nabla^{i_3}\underline{\beta}_{\mathcal{R}}\nabla^{i_4}\phi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i-1} \nabla^{i_1}(\eta + \underline{\eta})^{i_2}\nabla^{i_3}(\underline{\hat{\chi}}, \widetilde{tr\underline{\chi}})\nabla^{i_4}\phi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i-1} \nabla^{i_1}(\eta + \underline{\eta})^{i_2+1}\nabla^{i_3}tr\underline{\chi}\nabla^{i_4}\phi.$$

Finally, we can replace $\beta_{\mathcal{R}}, \underline{\beta}_{\mathcal{R}}$ by expressions involving Ricci coefficients, under the Codazzi equations:

$$\beta_{\mathcal{R}} = -\text{div}\hat{\chi} + \frac{1}{2}\nabla tr\chi - \frac{1}{2}(\eta - \underline{\eta}) \cdot (\hat{\chi} - \frac{1}{2}tr\chi),$$
$$\underline{\beta}_{\mathcal{R}} = \text{div}\underline{\hat{\chi}} - \frac{1}{2}\nabla tr\underline{\chi} - \frac{1}{2}(\eta - \underline{\eta}) \cdot (\underline{\hat{\chi}} - \frac{1}{2}tr\underline{\chi}).$$

That way, we arrive at the following:

Proposition 5.3.16. *Suppose $\nabla_4\phi = F_0$. Let $\nabla_4\nabla^i\phi = F_i$. Then*

$$F_i = \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}F_0 + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi})\nabla^{i_4}\phi.$$

Similarly, suppose $\nabla_3\phi = G_0$. Let $\nabla_3\nabla^i\phi = G_i$. Then

$$G_i + \frac{i}{2} tr\underline{\chi} \nabla^i \phi = \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} G_0 + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}, \widetilde{tr\underline{\chi}}) \nabla^{i_4} \phi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i-1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} tr\underline{\chi} \nabla^{i_4} \phi.$$

5.4 $\quad L^2(S_{u,\underline{u}})$-estimates for Ricci coefficients and Maxwell components

We start with some useful estimates. Let

$$\psi \in \left\{ \frac{\hat{\chi}}{a^{\frac{1}{2}}}, tr\chi, \omega, \eta, \underline{\eta}, \zeta, \underline{\omega}, \frac{a}{|u|} \widetilde{tr\underline{\chi}}, \frac{a^{\frac{1}{2}}}{|u|} \underline{\hat{\chi}}, \frac{a}{|u|^2} tr\underline{\chi} \right\}, \Psi \in \left\{ \frac{\alpha}{a^{\frac{1}{2}}}, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha} \right\},$$

$$\Upsilon \in \left\{ \frac{\alpha_F}{a^{\frac{1}{2}}}, \rho_F, \sigma_F, \underline{\alpha}_F \right\}.$$

Proposition 5.4.1. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), we have*

$$\sum_{i_1+i_2\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\psi^{i_2}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le |u|,$$

$$\sum_{i_1+i_2\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\psi^{i_2+1}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le O,$$

$$\sum_{i_1+i_2\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\psi^{i_2+2}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le \frac{O^2}{|u|},$$

$$\sum_{i_1+i_2\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\psi^{i_2+3}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le \frac{O^3}{|u|^2},$$

$$\sum_{i_1+i_2+i_3\le 9}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le O,$$

$$\sum_{i_1+i_2+i_3\le 9}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3+1}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le \frac{a^{\frac{1}{2}}}{|u|}O^2,$$

$$\sum_{i_1+i_2+i_3\le 9}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3+2}\nabla^{i_1}\psi^{i_2+2}\nabla^{i_3}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le \frac{a}{|u|^2}O^3,$$

$$\sum_{i_1+i_2\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\Upsilon^{i_2}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le |u|,$$

$$\sum_{i_1+i_2\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\Upsilon^{i_2+1}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le O,$$

$$\sum_{i_1+i_2\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\Upsilon^{i_2+2}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le \frac{O^2}{|u|},$$

$$\sum_{i_1+i_2+i_3\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le \frac{O^2}{|u|},$$

$$\sum_{i_1+i_2+i_3+i_4\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3+i_4}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\le \frac{O^2}{|u|},$$

$$\sum_{i_1+i_2+i_3\le 10}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2+2}\nabla^{i_3}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\lesssim \frac{O^3}{|u|^2}.$$

Proof. We focus on the last six statements. The ones before are similar and their proof can be found in Section 4 of [5].

- For the first one we distinguish two cases: If $i_2 = 0$, then the result holds trivially as $\|1\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = |u|$. For $i_2 \ge 1$, we can rewrite $\nabla^{i_1}\Upsilon^{i_2}$ as a product of i_2 terms

$$\nabla^{i_1}\Upsilon^{i_2} = \nabla^{j_1}\Upsilon\ldots\nabla^{j_{i_2}}\Upsilon, \quad \text{with } j_1 + \cdots + j_{i_2} = i_1.$$

Assume that j_{i_2} is the largest number. Then we rewrite

$$(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\Upsilon^{i_2} = (a^{\frac{1}{2}})^{i_2} \cdot (a^{\frac{1}{2}}\nabla)^{j_{i_2}}\Upsilon \cdot \prod_{k=1}^{i_2-1}(a^{\frac{1}{2}}\nabla)^{j_k}\Upsilon.$$

Now we bound $(a^{\frac{1}{2}}\nabla)^{j_{i_2}}\Upsilon$ in $\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})$ and the rest of the terms in $\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})$. We then have

$$\frac{1}{|u|}\sum_{i_1+i_2\leq 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\Upsilon^{i_2}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq\frac{1}{|u|}\sum_{i_1+i_2\leq 10}\frac{(a^{\frac{1}{2}})^{i_2}}{|u|^{i_2-1}}\|(a^{\frac{1}{2}}\nabla)^{j_{i_2}}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\prod_{k=1}^{i_2-1}\|(a^{\frac{1}{2}}\nabla)^{j_k}\Upsilon\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}$$

$$\leq\frac{(a^{\frac{1}{2}})^{i_2}O^{i_2}}{|u|^{i_2}}\leq 1.$$

- For the second one, if $i_2 = 0$, then the statement is true because of the definition of O. If $i_2 \geq 1$, then assume $i_1 = j_1 + \cdots + j_{i_2+1}$. Assume j_{i_2+1} is the largest. Then, as above,

$$\sum_{i_1+i_2\leq 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\Upsilon^{i_2+1}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq\sum_{i_1+i_2\leq 10}\frac{(a^{\frac{1}{2}})^{i_2}}{|u|^{i_2}}\|(a^{\frac{1}{2}}\nabla)^{j_{i_2+1}}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\prod_{k=1}^{i_2}\|(a^{\frac{1}{2}}\nabla)^{j_k}\Upsilon\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}$$

$$\leq\frac{(a^{\frac{1}{2}})^{i_2}O^{i_2+1}}{|u|^{i_2}}\leq O.$$

- We have

$$|u|\sum_{i_1+i_2\leq 10}\|(a^{\frac{1}{2}})^{i_1+i_2}\nabla^{i_1}\Upsilon^{i_2+2}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq|u|\cdot\frac{1}{|u|}\cdot\sum_{\substack{i_1+i_2\leq 10,\\i_3+i_4=i_1,i_3\leq i_4}}\|(a^{\frac{1}{2}}\nabla)^{i_3}\Upsilon\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}\|(a^{\frac{1}{2}})^{i_2+i_4}\nabla^{i_4}\Upsilon^{i_2+1}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})},$$

$$\leq O\cdot O = O^2.$$

- The proof of this is the same as the item above. If $i_3 \leq 6$ we bound the term $(a^{\frac{1}{2}}\nabla)^{i_3}\Upsilon$ in $\mathcal{L}^\infty_{(sc)}$, otherwise we bound it in $\mathcal{L}^2_{(sc)}$ and the rest of the terms in $\mathcal{L}^\infty_{(sc)}$.

- We have

$$
|u| \sum_{i_1+i_2+i_3+i_4 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2+i_3+i_4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} \Upsilon \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
\leq |u| \cdot \frac{1}{|u|} \cdot O \cdot \sum_{i_1+i_2+i_3 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2+i_3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

since one of i_3, i_4, without loss of generality say i_4, has to be at most 6 so that we can bound the term $\nabla^{i_4} \Upsilon$ in $\mathcal{L}^\infty_{(sc)}$. We have

$$
\sum_{i_1+i_2+i_3 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2+i_3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
\leq \frac{O}{|u|} \sum_{i_1+i_2 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2} \nabla^{i_1} \psi^{i_2} \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
+ \frac{O}{|u|} \sum_{i_1+i_2 \leq 3} \|(a^{\frac{1}{2}})^{i_1+i_2} \nabla^{i_1} \psi^{i_2} \|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}
$$

$$
\leq \frac{O}{|u|} \sum_{i_1+i_2 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2} \nabla^{i_1} \psi^{i_2} \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
+ \frac{O}{|u|} \sum_{i_1+i_2 \leq 5} \|(a^{\frac{1}{2}})^{i_1+i_2} \nabla^{i_1} \psi^{i_2} \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{O}{|u|} \cdot |u| = O.
$$

Here we have distinguished between the cases where i_3 is at most 6, in which case we bound it in $\mathcal{L}^\infty_{(sc)}$ and the case where $7 \leq i_3 \leq 10$, in which case we bound the term $\nabla^{i_3} \Upsilon$ in $\mathcal{L}^2_{(sc)}$ and use the Sobolev embedding theorem to bound $(a^{\frac{1}{2}})^{i_1+i_2} \nabla^{i_1} \psi^{i_2}$ in $\mathcal{L}^\infty_{(sc)}$. Putting everything together, we arrive at

$$
\sum_{i_1+i_2+i_3+i_4 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2+i_3+i_4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} \Upsilon \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{O^2}{|u|}.
$$

- There holds

$$
\sum_{i_1+i_2+i_3 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2+i_3} \nabla^{i_1} \psi^{i_2+2} \nabla^{i_3} \Upsilon \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
= \sum_{i_1+i_2+i_3+i_4+i_5 \leq 10} \|(a^{\frac{1}{2}})^{i_1+i_2+i_3+i_4+i_5} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi \nabla^{i_5} \Upsilon \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.
$$

(5.80)

Either i_3 or i_4 must be less than or equal to 6. Without loss of generality, assume $i_3 \leq 6$. We then bound $(a^{\frac{1}{2}} \nabla)^{i_3} \psi$ in $\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})$ and the remaining term

in $\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})$. For the latter term, we have already established an upper bound of $\frac{O^2}{|u|}$. Therefore,

$$\sum_{i_1+i_2+i_3+i_4+i_5\leq 10}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3+i_4+i_5}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\nabla^{i_5}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \frac{1}{|u|}\sum_{i_3\leq 6}\|(a^{\frac{1}{2}}\nabla)^{i_3}\psi\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})}\cdot\sum_{i_1+i_2+i_3\leq 10}\|(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \frac{1}{|u|}\cdot O\cdot\frac{O^2}{|u|}\leq\frac{O^3}{|u|^2}.$$

(5.81)

$\square$

5.4.1 $\quad L^2(S_{u,\underline{u}})$-estimates for the Ricci coefficients

Proposition 5.4.2. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions* (5.69), *we have*

$$\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\omega\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\lesssim\frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}(\underline{\mathcal{R}}[\rho]+1).$$

Proof. We use the following schematic null structure equation for ω:

$$\nabla_3\omega = \frac{1}{2}\rho+\psi\psi+\Upsilon\Upsilon.$$

Commuting i times with ∇ using Proposition 5.3.16 , we arrive at

$$\nabla_3\nabla^i\omega+\frac{i}{2}\mathrm{tr}\underline{\chi}\nabla^i\omega$$

$$=\sum_{i_1+i_2+i_3=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho+\psi\psi+\Upsilon\Upsilon)+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\omega$$

$$+\sum_{i_1+i_2+i_3+i_4=i-1}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\omega$$

$$=\nabla^i\rho+\sum_{i_1+i_2+i_3+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\rho+\sum_{i_1+i_2+i_3=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi\psi)$$

$$+\sum_{i_1+i_2+i_3=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\Upsilon\Upsilon)+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\omega$$

$$+\sum_{i_1+i_2+i_3+i_4=i-1}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\omega.$$

Now notice that for any j and $S-$tangent tensor fields ϕ_1,ϕ_2 we have the schematic equality $\nabla^j(\phi_1\cdot\phi_2)=\sum_{j_1+j_2=j}\nabla^{j_1}\phi_1\nabla^{j_2}\phi_2$. We can thus write

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$$\nabla_3 \nabla^i \omega + \frac{i}{2} \mathrm{tr}\underline{\chi} \nabla^i \omega$$
$$= \nabla^i \rho + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \rho + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} \Upsilon$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{i_4} \psi + \sum_{i_1+i_2+i_3+i_4=i-1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \mathrm{tr}\underline{\chi} \nabla^{i_4} \psi.$$

Rewrite the above as
$$\nabla_3 \nabla^i \omega + \frac{i}{2} \mathrm{tr}\underline{\chi} \nabla^i \omega = G.$$

Applying Proposition 5.3.7, there holds

$$|u|^{i-1} \|\nabla^i \omega\|_{L^2(S_{u,\underline{u}})} \le |u_\infty|^{i-1} \|\nabla^i \omega\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^{i-1} \|G\|_{L^2(S_{u',\underline{u}})} \mathrm{d}u'.$$

Multiplying both sides by $|u|$ and using $|u| \le |u'|, |u| \le |u_\infty|$ we get

$$|u|^i \|\nabla^i \omega\|_{L^2(S_{u,\underline{u}})} \le |u_\infty|^i \|\nabla^i \omega\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^i \|G\|_{L^2(S_{u',\underline{u}})} \mathrm{d}u'. \tag{5.82}$$

From the signature table we get

$$s_2(G) = s_2(\nabla_3 \nabla^i \omega) = 1 + s_2(\nabla^i \omega) = \frac{i}{2} + 1.$$

Using the definition of the scale-invariant norms $\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})$ we have

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = a^{-s_2(\phi)} |u|^{2s_2(\phi)} \|\phi\|_{L^2(S_{u,\underline{u}})}$$

and thus

$$\|\nabla^i \omega\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = a^{-\frac{i}{2}} |u|^i \|\nabla^i \omega\|_{L^2(S_{u,\underline{u}})}, \quad \|G\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = a^{-\frac{i}{2}-1} |u|^{i+2} \|G\|_{L^2(S_{u,\underline{u}})}.$$

Equivalently,

$$|u|^i \|\nabla^i \omega\|_{L^2(S_{u,\underline{u}})} = \|(a^{\frac{1}{2}} \nabla)^i \omega\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}, \quad |u|^i \|G\|_{L^2(S_{u,\underline{u}})} = \frac{a}{|u|^2} \|(a^{\frac{1}{2}})^i G\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

We can now write (5.82) in scale-invariant norms as

$$\|(a^{\frac{1}{2}}\nabla)^i\omega\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \le \|(a^{\frac{1}{2}}\nabla)^i\omega\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}}\nabla)^i\rho\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2}\|\sum_{i_1+i_2+i_3+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\rho\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2}\|\sum_{i_1+i_2+i_3+i_4=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a\,\mathrm{d}u'}{|u'|^2}\|\sum_{i_1+i_2+i_3+i_4=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2}\|\sum_{i_1+i_2+i_3+i_4+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'.$$

We look at each term separately. For the first one, since $\Omega\mid_{u=u_\infty}=1$, we note that $\omega = -\frac{1}{2}\nabla_4(\log\Omega)$, we have $\|(a^{\frac{1}{2}}\nabla)^i\omega\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} = 0$. For the second and third terms, we have

$$\int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}}\nabla)^i\rho\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u' + \int_{u_\infty}^u \frac{a}{|u'|^2}\|\sum_{i_1+i_2+i_3+1=i}(a^{\frac{1}{2}})^i\psi^{i_2+1}\nabla^{i_3}\rho\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\le \left(\int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}}\nabla)^i\rho\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'\right)^{\frac{1}{2}}\left(\int_{u_\infty}^u \frac{a}{|u'|^2}\mathrm{d}u'\right)^{\frac{1}{2}} + \int_{u_\infty}^u \frac{a}{|u'|^2}\cdot\frac{a^{\frac{1}{2}}}{|u|}\cdot O^2\mathrm{d}u'$$

$$= \|(a^{\frac{1}{2}}\nabla)^i\rho\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}})}\cdot\frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}} + \frac{a^{\frac{3}{2}}}{|u|^2}O^2 \le \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}\left(\mathcal{R}[\rho]+1\right).$$

For the next term, we have

$$\int_{u_\infty}^u \frac{a}{|u'|^2}\|\sum_{i_1+i_2+i_3+i_4=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\le \int_{u_\infty}^u \frac{a}{|u'|^2}\cdot\frac{O^2}{|u'|}\mathrm{d}u' = \frac{a}{|u|^2}O^2 \le \frac{a^{\frac{1}{2}}}{|u|}O^2 \le \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}.$$

For the last two terms, we have

$$\int_{u_\infty}^u \frac{a}{|u'|^2}\|\sum_{i_1+i_2+i_3+i_4=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\le \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|}\|\sum_{i_1+i_2+i_3+i_4=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{a^{\frac{1}{2}}}{|u'|}\psi,\frac{a^{\frac{1}{2}}}{|u'|}\hat{\underline{\chi}},\frac{a^{\frac{1}{2}}}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\le \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|}\cdot\frac{O^2}{|u'|}\mathrm{d}u' \le \frac{a^{\frac{1}{2}}}{|u|}O^2 \le \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}.$$

Moreover,

$$\int_{u_\infty}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+1=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} a^{\frac{1}{2}} \sum_{i_1+i_2+i_3+i_4+1=i} \|(a^{\frac{1}{2}})^{i-1} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi})\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} a^{\frac{1}{2}} \cdot \frac{O^3}{|u'|^2}\mathrm{d}u' \leq \frac{a^{\frac{1}{2}}}{|u|}O^3 \leq \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}.$$

Gathering all the estimates above and letting a be sufficiently large, we obtain

$$\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\omega\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}(\underline{\mathcal{R}}[\rho] + 1).$$

$\square$

Proposition 5.4.3. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), we have*

$$\sum_{i\leq 10}\frac{a^{\frac{1}{2}}}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1.$$

Proof. We look at the ∇_3-equation for $\underline{\hat{\chi}}$:

$$\nabla_3\underline{\hat{\chi}} + \mathrm{tr}\underline{\chi}\,\underline{\hat{\chi}} = -2\underline{\omega}\underline{\hat{\chi}} - \underline{\alpha}.$$

Commuting with i angular derivatives and using Proposition 5.3.6 we arrive at

$$\nabla_3\nabla^i\underline{\hat{\chi}} + \frac{i+2}{2}\mathrm{tr}\underline{\chi}\nabla^i\underline{\hat{\chi}}$$

$$=\nabla^i\underline{\alpha} + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\underline{\alpha} + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\underline{\hat{\chi}}$$

$$+ \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\underline{\hat{\chi}}.$$

Rewriting the above equation as

$$\nabla_3\nabla^i\underline{\hat{\chi}} + \frac{i+2}{2}\mathrm{tr}\underline{\chi}\nabla^i\underline{\hat{\chi}} = F,$$

an application of Proposition 5.3.7 gives us

$$|u|^{i+1}\|\nabla^i\underline{\hat{\chi}}\|_{L^2(S_{u,\underline{u}})} \leq |u_\infty|^{i+1}\|\nabla^i\underline{\hat{\chi}}\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^{u}|u'|^{i+1}\|F\|_{L^2(S_{u',\underline{u}})}\mathrm{d}u'. \tag{5.83}$$

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Rewriting (5.83) in scale-invariant norms, we arrive at

$$\frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{a}{|u_\infty|}\|(a^{\frac{1}{2}}\nabla)^i\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|(a^{\frac{1}{2}})^i F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'.$$

Multiplying this equation by $a^{-\frac{1}{2}}$ we get

$$\frac{a^{\frac{1}{2}}}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \frac{a^{\frac{1}{2}}}{|u_\infty|}\|(a^{\frac{1}{2}}\nabla)^i\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|(a^{\frac{1}{2}}\nabla)^i\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|\sum_{i_1+i_2+i_3+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|\sum_{i_1+i_2+i_3+i_4=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|\sum_{i_1+i_2+i_3+i_4+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'.$$

The initial data term is directly bounded by $\mathcal{I}^{(0)}(\underline{u}) \lesssim 1$. For the terms containing $\underline{\alpha}$, we have

$$\int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|(a^{\frac{1}{2}}\nabla)^i\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|\sum_{i_1+i_2+i_3+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \|(a^{\frac{1}{2}}\nabla)^i\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}})} \cdot \frac{a}{|u|^{\frac{3}{2}}} + \frac{a^2 \cdot O^2}{|u|^3} \leq 1.$$

The last two terms can be bounded as follows:

$$\int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|\sum_{i_1+i_2+i_3+i_4=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|\sum_{i_1+i_2+i_3+i_4+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u' \leq \frac{O^2+O^3}{a^{\frac{1}{2}}} \leq 1.$$

$\square$

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Proposition 5.4.4. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions* (5.69), *we have*

$$\sum_{i \leq 10} \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}}\nabla)^i \hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{R}[\alpha] + 1.$$

Proof. We look at the schematic equation

$$\nabla_4 \hat{\chi} = \psi \cdot \hat{\chi} + \alpha. \tag{5.84}$$

Commuting (5.84) with i angular derivatives we arrive at

$$\nabla_4 \nabla^i \hat{\chi} = \nabla^i \alpha + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\alpha + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\hat{\chi}. \tag{5.85}$$

We thus have, passing to scale-invariant norms,

$$\frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}}\nabla)^i \hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \frac{1}{a^{\frac{1}{2}}} \int_0^{\underline{u}} \|(a^{\frac{1}{2}}\nabla)^i \alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+1=i} \frac{1}{a^{\frac{1}{2}}} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \frac{1}{a^{\frac{1}{2}}} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$\leq \frac{1}{a^{\frac{1}{2}}} \left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}}\nabla)^i \alpha\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'\right)^{\frac{1}{2}} \left(\int_0^{\underline{u}} 1 \, d\underline{u}'\right)^{\frac{1}{2}}$$

$$+ \sum_{i_1+i_2+i_3+1=i} \int_0^{\underline{u}} \left\|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\left(\frac{\alpha}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$+ a^{\frac{1}{2}} \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \left\|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$\leq \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}}\nabla)^i \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \leq \mathcal{R}[\alpha] + \frac{4O^2}{a^{\frac{1}{2}}} \leq \mathcal{R}[\alpha] + 1.$$

The result follows. $\qquad\square$

Remark 11. *As will be shown in Proposition 5.5.1, there holds*

$$\sum_{i \leq 9} \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}}\nabla)^i \alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1.$$

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In particular, this means that when i is not of top order, the estimate above can be improved to

$$\sum_{i \leq 9} \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}}\nabla)^i \hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1. \tag{5.86}$$

Using Sobolev's embedding Theorem 5.3.11, there also holds

$$\sum_{i \leq 7} \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}}\nabla)^i \hat{\chi}\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \lesssim 1. \tag{5.87}$$

We proceed with estimates for $\underline{\omega}$.

Proposition 5.4.5. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), there holds*

$$\sum_{i \leq 10} \|(a^{\frac{1}{2}}\nabla)^i \underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{R}[\rho] + 1.$$

Proof. We have the schematic null structure equation

$$\nabla_4 \underline{\omega} = \rho + \psi \cdot \psi + \Upsilon \cdot \Upsilon$$

Commuting this equation with i angular derivatives, using Proposition 5.3.16, we obtain

$$\nabla_4 \nabla^i \underline{\omega} = \nabla^i \rho + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\rho + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4}\Upsilon$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\psi.$$

Multiplying by $(a^{\frac{1}{2}})^i$ and using Proposition 5.3.6 we get

$$\|(a^{\frac{1}{2}}\nabla)^i \underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \int_0^{\underline{u}} \|(a^{\frac{1}{2}}\nabla)^i \rho\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}' + \sum_{i_1+i_2+i_3+1=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\rho\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$\leq \left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}}\nabla)^i \rho\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}' \right)^{\frac{1}{2}} \left(\int_0^{\underline{u}} 1 \, d\underline{u}' \right)^{\frac{1}{2}}$$

$$+ \sum_{i_1+i_2+i_3+1=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\rho\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} a^{\frac{1}{2}} \left\|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'$$

$$\leq \|(a^{\frac{1}{2}}\nabla)^i \rho\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,\underline{u})})} + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \leq \mathcal{R}[\rho] + 1.$$

Here and throughout we have made use of Proposition 5.4.1. $\qquad\square$

Proposition 5.4.6. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), we have*

$$\sum_{i\leq 10} \|(a^{\frac{1}{2}}\nabla)^i \eta\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{R}[\tilde{\beta}] + 1.$$

Proof. We begin by recalling the structure equation (5.17) for η:

$$\nabla_4 \eta_a = -\chi_{ab} \cdot (\eta - \underline{\eta})_b - \beta_a - \frac{1}{2}R_{a4}.$$

Also recall that $\tilde{\beta} = \beta - \frac{1}{2}R_{4(\cdot)}$. We can therefore rewrite (5.17) in terms of $\tilde{\beta}$ as follows:

$$\nabla_4 \eta_a = -\chi_{ab} \cdot (\eta - \underline{\eta})_b - \tilde{\beta}_a - R_{a4}.$$

This leads us to the following schematic null structure equation:

$$\nabla_4 \eta = \tilde{\beta} + \psi \cdot (\psi,\hat{\chi}) + (\rho_F,\sigma_F) \cdot \alpha_F.$$

Commuting with i angular derivatives, using Proposition 5.3.16, we have

$$\nabla_4 \nabla^i \eta = \nabla^i \tilde{\beta} + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \tilde{\beta}$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}) \nabla^{i_4} \psi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\rho_F, \sigma_F) \nabla^{i_4} \alpha_F.$$

Working in scale-invariant norms, we get

$$\|(a^{\frac{1}{2}} \nabla)^i \eta\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$
$$\leq \int_0^{\underline{u}} \|(a^{\frac{1}{2}} \nabla)^i \tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}' + \sum_{i_1+i_2+i_3+1=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} a^{\frac{1}{2}} \left\| (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} a^{\frac{1}{2}} \left\| (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha_F}{a^{\frac{1}{2}}} \right) \nabla^{i_4} (\rho_F, \sigma_F) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$
$$\leq \mathcal{R}[\tilde{\beta}] + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \leq \mathcal{R}[\tilde{\beta}] + 1.$$

$\square$

Proposition 5.4.7. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), we have*

$$\sum_{i \leq 10} \|(a^{\frac{1}{2}} \nabla)^i tr\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{R}[\alpha] + \underline{\mathcal{E}}[\rho_F, \sigma_F] + 1.$$

Proof. We again start by considering the schematic equation

$$\nabla_4 tr\chi = \hat{\chi} \cdot \hat{\chi} + \alpha_F \cdot \alpha_F + \psi\psi.$$

By commuting with i angular derivatives, we arrive at

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$$\nabla_4 \nabla^i \mathrm{tr}\chi = \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\hat{\chi}\nabla^{i_4}\hat{\chi} + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\alpha_F \nabla^{i_4}\alpha_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\psi$$

$$= \sum_{i_1+i_2=i} \nabla^{i_1}\hat{\chi}\nabla^{i_2}\hat{\chi} + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\hat{\chi}\nabla^{i_4}\hat{\chi}$$

$$+ \sum_{i_1+i_2=i} \nabla^{i_1}\alpha_F\nabla^{i_2}\alpha_F + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\alpha_F\nabla^{i_4}\alpha_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\psi.$$

Taking this into account[9], we have

$$\left\|(a^{\frac{1}{2}}\nabla)^i \mathrm{tr}\chi\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \sum_{i_1+i_2=i}\int_0^{\underline{u}} a\left\|(a^{\frac{1}{2}})^i\nabla^{i_1}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\nabla^{i_2}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4+1=i}\int_0^{\underline{u}} a\left\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\nabla^{i_2}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i}\int_0^{\underline{u}} a^{\frac{1}{2}}\left\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\mathrm{d}\underline{u}'$$

$$\leq \frac{a}{|u|}O_2[\hat{\chi},\alpha_F]\cdot O_\infty[\hat{\chi},\alpha_F] + \frac{a}{|u|^2}O^3 + \frac{a^{\frac{1}{2}}}{|u|}O^2$$

$$\leq O_2[\hat{\chi},\alpha_F]\cdot O_\infty[\hat{\chi},\alpha_F] + 1 \lesssim \mathcal{R}[\alpha] + \underline{\mathcal{F}}[\rho_F,\sigma_F] + 1.$$

$$(5.88)$$

This is because the $O_2[\hat{\chi},\alpha_F]$–term can be bounded by $\mathcal{R}[\alpha]+\underline{\mathcal{F}}[\rho_F,\sigma_F]+1$ by virtue of Propositions 5.4.4 and 5.4.10, whereas $O_\infty[\hat{\chi},\alpha_F]$ can be bounded by 1 using the improved estimates in (5.87) and (5.97). $\qquad\square$

We move on to estimates for $\mathrm{tr}\underline{\chi}$.

Proposition 5.4.8. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), we have*

$$\sum_{i\leq 10}\frac{a}{|u|}\left\|(a^{\frac{1}{2}}\nabla)^i(\mathrm{tr}\underline{\chi}+\frac{2}{|u|})\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{R}[\rho] + \underline{\mathcal{R}}[\rho] + 1,$$

---

[9]In the following, even though we do not encounter cross terms of the form $\nabla^{i_1}\hat{\chi}\nabla^{i_2}\alpha_F$, we do not lose any control on the inequality by grouping the terms together and controlling schematically terms of the form $\nabla^{i_1}(\hat{\chi},\alpha_F)\nabla^{i_2}(\hat{\chi},\alpha_F)$.

$$\sum_{i \leq 10} \frac{a}{|u|^2} \|(a^{\frac{1}{2}} \nabla)^i tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1.$$

Proof. We have the schematic equation for $tr\underline{\chi}$

$$\nabla_3 tr\underline{\chi} + \frac{1}{2}(tr\underline{\chi})^2 = -|\hat{\underline{\chi}}|^2 - 2\underline{\omega} tr\underline{\chi} - |\underline{\alpha}_F|^2 := G_0.$$

Commuting this with i angular derivatives and temporarily writing $tr\underline{\chi} = \underline{\psi}$ symbolically, we get an equation of the form

$$
\begin{aligned}
\nabla_3 \nabla^i \underline{\psi} + \frac{i+1}{2} tr\underline{\chi} \nabla^i \underline{\psi} \\
= \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} G_0 + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{\underline{\chi}}, \widetilde{tr\underline{\chi}}) \nabla^{i_4} \underline{\psi} \\
+ \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} tr\underline{\chi} \nabla^{i_3} \underline{\psi} \\
:= G_i.
\end{aligned}
$$

We proceed to apply Proposition 5.3.7 with $\lambda_1 = i$. There holds

$$|u|^i \|\nabla^i tr\underline{\chi}\|_{L^2(S_{u,\underline{u}})} \lesssim |u_\infty|^i \|\nabla^i tr\underline{\chi}\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^{u} |u'|^i \|G_i\|_{L^2(S_{u',\underline{u}})} \, du' \qquad (5.89)$$

We convert (5.89) into the language of scale-invariant norms. There holds

$$
\begin{aligned}
\|(a^{\frac{1}{2}} \nabla)^i tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} &= a^{\frac{i}{2}} a^{-s_2(\nabla^i tr\underline{\chi})} |u|^{2s_2(\nabla^i tr\underline{\chi})} \|\nabla^i tr\underline{\chi}\|_{L^2(S_{u,\underline{u}})} \\
&= \frac{|u|^2}{a} \cdot |u|^i \|\nabla^i tr\underline{\chi}\|_{L^2(S_{u,\underline{u}})}
\end{aligned}
\qquad (5.90)
$$

and

$$\|(a^{\frac{1}{2}})^i G_i\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = \frac{|u|^4}{a^2} \cdot |u|^i \|G_i\|_{L^2(S_{u,\underline{u}})}.$$

Consequently, there holds

$$
\begin{aligned}
\frac{a}{|u|^2} \|(a^{\frac{1}{2}} \nabla)^i tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} &\lesssim \frac{a}{|u_\infty|^2} \|(a^{\frac{1}{2}} \nabla)^i tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} \\
&\quad + \int_{u_\infty}^{u} \frac{a^2}{|u'|^4} \|(a^{\frac{1}{2}})^i G_i\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'.
\end{aligned}
\qquad (5.91)
$$

- There holds

$$
\begin{aligned}
\int_{u_\infty}^{u} \frac{a^2}{|u'|^4} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \hat{\underline{\chi}} \nabla^{i_4} \hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du' \\
\lesssim \int_{u_\infty}^{u} \frac{a}{|u'|^2} \cdot \frac{O^2}{|u'|} \, du' \lesssim \frac{aO^2}{|u|^2}.
\end{aligned}
\qquad (5.92)
$$

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- Similarly, there holds

$$\int_{u_\infty}^u \frac{a^2}{|u'|^4} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \underline{\omega} \nabla^{i_4} \mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^u \frac{a}{|u'|^2} \cdot \frac{O^2}{|u'|} \, \mathrm{d}u' \lesssim \frac{aO^2}{|u|^2}. \tag{5.93}$$

- There holds

$$\int_{u_\infty}^u \frac{a^2}{|u'|^4} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \underline{\alpha}_F \nabla^{i_4} \underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^4} \cdot \frac{O^2}{|u'|} \, \mathrm{d}u' \lesssim \frac{a^2 O^2}{|u|^4}. \tag{5.94}$$

- There holds

$$\int_{u_\infty}^u \frac{a^2}{|u'|^4} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{i_4} \mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|} \cdot \frac{O^2}{|u'|} \, \mathrm{d}u' \lesssim \frac{a^{\frac{1}{2}} O^2}{|u|}. \tag{5.95}$$

- There holds

$$\int_{u_\infty}^u \frac{a^2}{|u'|^4} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \mathrm{tr}\underline{\chi} \nabla^{i_4} \mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' = (a^{\frac{1}{2}})^i \cdot$$
$$\int_{u_\infty}^u \Big\| \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \left(\frac{a}{|u'|^2} \mathrm{tr}\underline{\chi}\right) \nabla^{i_5} \left(\frac{a}{|u'|^2} \mathrm{tr}\underline{\chi}\right) \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^u \frac{a^{\frac{1}{2}} \cdot O^3}{|u'|^2} \, \mathrm{d}u' \lesssim \frac{a^{\frac{1}{2}} \cdot O^3}{|u|} \lesssim 1. \tag{5.96}$$

Here we have used the fourth statement of Proposition 5.4.1 for the case of three variables.

Consequently, there holds $\sum_{i \le 10} \frac{a}{|u|^2} \|(a^{\frac{1}{2}} \nabla)^i \mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1$. Using the Sobolev embedding theorem, in particular, there holds $\sum_{i \le 6} \frac{a}{|u|^2} \|(a^{\frac{1}{2}} \nabla)^i \mathrm{tr}\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \lesssim 1$. This will prove useful in the following lines for $\widetilde{\mathrm{tr}\underline{\chi}}$.

For $\widetilde{\mathrm{tr}\underline{\chi}}$, we have the schematic null structure equation

$$\nabla_3 \widetilde{\mathrm{tr}\underline{\chi}} + \mathrm{tr}\chi \widetilde{\mathrm{tr}\underline{\chi}} = \frac{2}{|u|^2}(\Omega^{-1} - 1) + \widetilde{\mathrm{tr}\underline{\chi}} \widetilde{\mathrm{tr}\underline{\chi}} + \psi \mathrm{tr}\underline{\chi} - |\underline{\hat{\chi}}|^2 - |\underline{\alpha}_F|^2.$$

Commuting this equation with i angular derivatives, we have

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$$\nabla_3 \nabla^i \widetilde{\mathrm{tr}\underline{\chi}} + \frac{i+2}{2} \mathrm{tr}\underline{\chi}\widetilde{\mathrm{tr}\underline{\chi}}$$

$$= \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{2}{|u|^2}(\Omega^{-1}-1) + \widetilde{\mathrm{tr}\underline{\chi}}\widetilde{\mathrm{tr}\underline{\chi}} + \psi\mathrm{tr}\underline{\chi} - |\hat{\underline{\chi}}|^2 - |\underline{\alpha}_F|^2\right)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\widetilde{\mathrm{tr}\underline{\chi}} + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\widetilde{\mathrm{tr}\underline{\chi}}$$

$$:= \tilde{F}.$$

Rewriting in terms of scale-invariant norms,

$$\frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{a}{|u_\infty|}\|\|(a^{\frac{1}{2}}\nabla)^i\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})}$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|(a^{\frac{1}{2}})^i\tilde{F}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$= \frac{a}{|u_\infty|}\|(a^{\frac{1}{2}}\nabla)^i\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,$$

where

$$\frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{a}{|u_\infty|}\|\|(a^{\frac{1}{2}}\nabla)^i\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} \lesssim 1,$$

$$\mathcal{I}_1 = \int_{u_\infty}^u \frac{a}{|u'|}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a^{\frac{1}{2}}}{|u'|}\psi,\frac{a^{\frac{1}{2}}}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}},\frac{a^{\frac{1}{2}}}{|u'|}\hat{\underline{\chi}},\frac{a^{\frac{1}{2}}}{|u'|}\underline{\alpha}_F\right)$$

$$\times \nabla^{i_4}\left(\frac{a^{\frac{1}{2}}}{|u'|}\psi,\frac{a^{\frac{1}{2}}}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}},\frac{a^{\frac{1}{2}}}{|u'|}\hat{\underline{\chi}},\frac{a^{\frac{1}{2}}}{|u'|}\underline{\alpha}_F\right)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\lesssim O^2[\hat{\underline{\chi}}] + \frac{a^2}{|u|^3}O^2[\underline{\alpha}_F] + 1 \lesssim 1.$$

There holds

$$\mathcal{I}_2 = \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\omega}\nabla^{i_4}\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'$$

$$= \int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\omega}\nabla^{i_4+1}\left(\frac{a}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}}\right)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\omega}\cdot\left(\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi}\right)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'$$

$$\leq \int_{u_\infty}^u \frac{a}{|u'|}\cdot\frac{1}{|u'|}\cdot\left(O[\underline{\omega}]\cdot O_\infty[\mathrm{tr}\underline{\chi}]\right)du'$$

$$\lesssim O[\underline{\omega}]O_\infty[\mathrm{tr}\underline{\chi}] + 1 \lesssim \mathcal{R}[\rho] + 1,$$

by Proposition 5.4.5 and the fact that $\frac{a}{|u|^2}\|\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \lesssim 1$. There also holds

$$
\begin{aligned}
\mathcal{I}_3 &= \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{\Omega^{-1}-1}{|u'|^2})\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du' \\
&= \int_{u_\infty}^u |u'|^{i+1}\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{\Omega^{-1}-1}{|u'|^2})\|_{L^2(S_{u',\underline{u}})}du' \quad \text{(in standard norms)} \\
&= \int_{u_\infty}^u |u'|^{i+1}\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{\Omega^{-1}-1}{|u'|^2})\|_{L^2(S_{u',\underline{u}})}du' \\
&= \int_{u_\infty}^u |u'|^{i+1}\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}[\frac{1}{|u'|^2}\cdot\int_0^{\underline{u}} 2\omega(u',\underline{u}',\theta^1,\theta^2)d\underline{u}']\|_{L^2(S_{u',\underline{u}})}du' \\
&= \int_{u_\infty}^u |u'|^{i+1}\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}[\frac{1}{|u'|^2}\cdot\int_0^{\underline{u}} 2\nabla^{i_3}\omega(u',\underline{u}',\theta^1,\theta^2)d\underline{u}']\|_{L^2(S_{u',\underline{u}})}du' \\
&\leq |u'|^{i+1}\| \sum_{i_1+i_2+i_3=i} \frac{1}{|u'|^{i_1+i_2}}\cdot\frac{1}{|u'|^2}\cdot\frac{1}{|u'|^{i_3}}\cdot\frac{a^{\frac{1}{2}}}{|u'|^{\frac{1}{2}}}\cdot\Big(\mathcal{R}[\rho]+1\Big)du' \\
&\leq \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^{\frac{3}{2}}}\Big(\mathcal{R}[\rho]+1\Big)du' \lesssim \mathcal{R}[\rho]+1.
\end{aligned}
$$

Finally, there holds

$$
\begin{aligned}
\mathcal{I}_4 &= \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3=i-1} \nabla^{i_1}\psi^{i_2+1}\cdot\mathrm{tr}\underline{\chi}\cdot\nabla^{i_3}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{sc}(S_{u',\underline{u}})}du' \\
&= \int_{u_\infty}^u a^{\frac{1}{2}}\|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3=i-1} \nabla^{i_1}\psi^{i_2+1}\cdot\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi}\cdot\nabla^{i_3}(\frac{a}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}})\|_{\mathcal{L}^2_{sc}(S_{u',\underline{u}})}du' \\
&\leq \int_{u_\infty}^u a^{\frac{1}{2}}\cdot\frac{O^3}{|u'|^2}du' \leq 1 \quad \text{(by Proposition 5.82)}.
\end{aligned}
$$

In summary, we have obtained

$$
\sum_{i\leq 10} \frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^i(\mathrm{tr}\underline{\chi}+\frac{2}{|u|})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{R}[\rho]+\underline{\mathcal{R}}[\rho]+1.
$$

$\square$

Proposition 5.4.9. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), we have*

$$
\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \underline{\mathcal{R}}[\tilde{\underline{\beta}}]+\mathcal{R}[\tilde{\underline{\beta}}]+1.
$$

Proof. We use the following schematic null structure equation for $\underline{\eta}$:

$$\nabla_3 \underline{\eta} + \frac{1}{2} \text{tr}\underline{\chi}\,\underline{\eta} = \tilde{\underline{\beta}} + \text{tr}\underline{\chi}\eta + \hat{\underline{\chi}} \cdot \psi + \Upsilon \cdot \Upsilon.$$

Commuting with i angular derivatives, we have

$$\nabla_3 \nabla^i \underline{\eta} + \frac{i+1}{2} \text{tr}\underline{\chi}\nabla^i \underline{\eta}$$
$$= \nabla^i \tilde{\underline{\beta}} + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \tilde{\underline{\beta}} + \text{tr}\underline{\chi}\nabla^i \eta + \sum_{i_1+i_2+1=i} \nabla^{i_1+1} \text{tr}\underline{\chi}\nabla^{i_2}(\eta, \underline{\eta})$$
$$+ \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \psi \nabla^{i_4} \text{tr}\underline{\chi} + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4}(\hat{\underline{\chi}}, \widetilde{\text{tr}\underline{\chi}})$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} \Upsilon.$$

By passing to scale-invariant norms we have

$$\frac{1}{|u|} \|(a^{\frac{1}{2}}\nabla)^i \underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$
$$\leq \frac{1}{|u_\infty|} \|(a^{\frac{1}{2}}\nabla)^i \underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u_\infty,\underline{u})} + \int_{u_\infty}^u \frac{a}{|u'|^3} \|(a^{\frac{1}{2}}\nabla)^i \tilde{\underline{\beta}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$
$$+ \int_{u_\infty}^u \frac{a}{|u'|^3} \|\sum_{i_1+i_2+i_3+1=i} (a^{\frac{1}{2}}\nabla)^i \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \tilde{\underline{\beta}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$
$$+ \int_{u_\infty}^u \frac{a}{|u'|^3} \|\text{tr}\underline{\chi}(a^{\frac{1}{2}}\nabla)^i \eta\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$
$$+ \int_{u_\infty}^u \frac{a}{|u'|^3} \|\sum_{i_1+i_2+1=i} (a^{\frac{1}{2}}\nabla)^{i_1+1} \text{tr}\underline{\chi}(a^{\frac{1}{2}}\nabla)^{i_2} \eta\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$
$$+ \int_{u_\infty}^u \frac{a}{|u'|^3} \|\sum_{i_1+i_2+i_3+i_4+1=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \psi \nabla^{i_4} \text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$
$$+ \int_{u_\infty}^u \frac{a}{|u'|^3} \|\sum_{i_1+i_2+i_3+i_4=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4}(\hat{\underline{\chi}}, \widetilde{\text{tr}\underline{\chi}})\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$
$$+ \int_{u_\infty}^u \frac{a}{|u'|^3} \|\sum_{i_1+i_2+i_3+i_4=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} \Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$
$$\leq \frac{1}{|u_\infty|} + \frac{\mathcal{R}[\tilde{\underline{\beta}}] + \mathcal{R}[\tilde{\beta}] + 1}{|u|} + \int_{u_\infty}^u \frac{a}{|u'|^3} \cdot \frac{O^2}{|u'|} \, du' \lesssim \frac{\mathcal{R}[\tilde{\underline{\beta}} + \mathcal{R}[\tilde{\beta}] + 1}{|u|}.$$

$\square$

This concludes the L^2-estimates on Ricci coefficients.

5.4.2 $L^2(S_{u,\underline{u}})$-estimates for the Maxwell components

Proposition 5.4.10. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), we have*

$$\sum_{i\leq 10}\frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\leq \underline{\mathcal{F}}[\rho_F,\sigma_F]+1.$$

Proof. We have the schematic equation

$$\nabla_3\alpha_F+\frac{1}{2}\mathrm{tr}\underline{\chi}\alpha_F=\nabla(\rho_F,\sigma_F)+\psi\cdot(\rho_F,\sigma_F)+\hat{\underline{\chi}}\cdot\underline{\alpha}_F+\psi\cdot\alpha_F.$$

Commuting with i angular derivatives, we get

$$\begin{aligned}
&\nabla_3\nabla^i\alpha_F+\frac{i+1}{2}\mathrm{tr}\underline{\chi}\alpha_F\\
=&\nabla^{i+1}(\rho_F,\sigma_F)+\sum_{i_1+i_2+i_3+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}(\rho_F,\sigma_F)\\
&+\sum_{i_1+i_2+1=i}\nabla^{i_1+1}\mathrm{tr}\underline{\chi}\nabla^{i_2}\alpha_F+\sum_{i_1+i_2=i}\nabla^{i_1}\hat{\underline{\chi}}\nabla^{i_2}\alpha_F\\
&+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(\rho_F,\sigma_F)\\
&+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\hat{\underline{\chi}}\nabla^{i_4}\underline{\alpha}_F+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\alpha_F\\
&+\sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha_F+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\alpha_F.
\end{aligned}$$

Denote the right–hand side of the above as G. We then have

$$a^{-\frac{1}{2}}|u|^i\|\nabla^i\alpha_F\|_{L^2(S_{u,\underline{u}})}\leq a^{-\frac{1}{2}}|u_\infty|^i\|\nabla^i\alpha_F\|_{L^2(S_{u_\infty,\underline{u}})}+\int_{u_\infty}^u a^{-\frac{1}{2}}|u'|^i\|G\|_{L^2(S_{u',\underline{u}})}\mathrm{d}u'.$$

From the signature table one can read off

$$s_2(\alpha_F)=0\Rightarrow s_2(\nabla^i\alpha_F)=0+i\cdot\frac{1}{2}=\frac{i}{2}.$$

By conservation of signatures,

$$s_2(G)=s_2(\nabla_3\nabla^i\alpha_F)=\frac{i+2}{2}.$$

Taking into account now that

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}:=a^{-s_2(\phi)}|u|^{2s_2(\phi)}\|\phi\|_{L^2(S_{u,\underline{u}})},$$

we can conclude that

$$a^{-\frac{1}{2}}|u|^i\|\nabla^i\alpha_F\|_{L^2(S_{u,\underline{u}})} = a^{-\frac{1}{2}}\|(a^{\frac{1}{2}}\nabla)^i\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})},$$

$$a^{-\frac{1}{2}}|u|^i\|G\|_{L^2(S_{u,\underline{u}})} = \frac{a^{\frac{1}{2}}}{|u|^2}\|(a^{\frac{1}{2}})^iG\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Therefore,

$$a^{-\frac{1}{2}}|u|^i\|\nabla^i\alpha_F\|_{L^2(S_{u,\underline{u}})}$$

$$\leq a^{-\frac{1}{2}}|u|^i\|\nabla^i\alpha_F\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\nabla^{i+1}(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|\sum_{i_1+i_2+i_3+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|\sum_{i_1+i_2+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1+1}\mathrm{tr}\underline{\chi}\nabla^{i_2}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|\sum_{i_1+i_2=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\underline{\hat{\chi}}\nabla^{i_2}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\hat{\chi}\nabla^{i_4}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'.$$

For the first term, we have

$$a^{-\frac{1}{2}}|u|^i\|\nabla^i\alpha_F\|_{L^2(S_{u_\infty,\underline{u}})} \leq \mathcal{I}^{(0)}(\underline{u}) \lesssim 1.$$

For the two terms involving the highest number of derivatives, we have

$$\int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \|(a^{\frac{1}{2}})^i \nabla^{i+1}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$+ \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \|\sum_{i_1+i_2+i_3+1=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3+1} \Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$\leq \left(\int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \nabla^{i+1}(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u' \right)^{\frac{1}{2}} \left(\int_{u_\infty}^{u} \frac{1}{|u'|^2} \mathrm{d}u' \right)^{\frac{1}{2}}$$

$$+ \int_{u_\infty}^{u} \frac{1}{|u'|^2} \cdot \frac{a^{\frac{1}{2}}}{|u'|} \cdot O^2 \, \mathrm{d}u'$$

$$= \|(a^{\frac{1}{2}})^i \nabla^{i+1}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \cdot \frac{1}{|u|^{\frac{1}{2}}} + \frac{a^{\frac{1}{2}}}{|u|^2} \cdot O^2$$

$$= a^{-\frac{1}{2}} \|(a^{\frac{1}{2}})^i \nabla^{i+1}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \cdot \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}} + \frac{a^{\frac{1}{2}}}{|u|^2} \cdot O^2$$

$$\leq \underline{\mathcal{F}}[\rho_F, \sigma_F] \cdot \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}} + 1 \lesssim \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1.$$

For the next two terms, we have

$$\int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \Big\| \sum_{i_1+i_2+1=i} (a^{\frac{1}{2}})^i \nabla^{i_1+1} \mathrm{tr}\underline{\chi} \nabla^{i_2} \alpha_F \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$+ \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \Big\| \sum_{i_1+i_2=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \hat{\underline{\chi}} \nabla^{i_2} \alpha_F \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} \frac{\mathrm{d}u'}{|u'|} \Big\| \sum_{i_1+i_2+1=i} (a^{\frac{1}{2}})^i \nabla^{i_1+1} \left(\frac{a}{|u'|} \Big(\mathrm{tr}\underline{\chi} + \frac{2}{|u'|} \Big) \right) \nabla^{i_2} \left(\frac{\alpha_F}{a^{\frac{1}{2}}} \right) \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$+ \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|} \Big\| \sum_{i_1+i_2=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \Big(\frac{a^{\frac{1}{2}}}{|u'|} \hat{\underline{\chi}} \Big) \nabla^{i_2} \left(\frac{\alpha_F}{a^{\frac{1}{2}}} \right) \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} \frac{1}{|u'|} \cdot \frac{O^2}{|u'|} \, \mathrm{d}u' + \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|} \cdot \frac{O^2}{|u'|} \mathrm{d}u' \leq 1.$$

For the sixth, seventh and eighth terms, notice that

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$\leq \int_{u_\infty}^u \frac{a}{|u'|^2} \left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\Upsilon\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+\int_{u_\infty}^u \frac{a}{|u'|^2} \left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\leq \int_{u_\infty}^u \frac{a}{|u'|^2}\cdot\frac{O^2}{|u'|}\mathrm{d}u' \leq \frac{aO^2}{|u|^2} \leq 1.$$

For the last two terms, we can write

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$=\int_{u_\infty}^u \left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\left(\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi}\right)\nabla^{i_4}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+\int_{u_\infty}^u \frac{(a^{\frac{1}{2}})^{i+1}\mathrm{d}u'}{|u'|}\Big\|\sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a^{\frac{1}{2}}}{|u'|}\psi,\frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}},\frac{a^{\frac{1}{2}}}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}}\right)\nabla^{i_4}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$\leq \int_{u_\infty}^u \frac{O^3}{|u'|^2}\mathrm{d}u' + \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|}\cdot\frac{O^2}{|u'|}\mathrm{d}u' \leq \frac{O^3}{|u|} + \frac{a^{\frac{1}{2}}\cdot O^2}{|u|} \leq 1.$$

$\square$

Remark 12. *Notice here that, when $i \leq 9$, the term*

$$\|(a^{\frac{1}{2}}\nabla)^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

can itself be bounded by O without having to resort to an $\mathcal{L}^2_{(sc)}(\underline{H})$–norm. Then, the term $\underline{\mathcal{F}}[\rho_F,\sigma_F]$ goes away and hence we arrive at the improved bound

$$\sum_{i\leq 9}\|(a^{\frac{1}{2}}\nabla)^i\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \sum_{i\leq 7}\|(a^{\frac{1}{2}}\nabla)^i\alpha_F\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \lesssim 1. \tag{5.97}$$

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Proposition 5.4.11. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions 5.69, there holds*

$$\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{F}[\underline{\alpha}_F] + 1.$$

Proof. We have the following equations along the incoming direction:

$$\nabla_3\rho_F + \mathrm{tr}\underline{\chi}\,\rho_F = \mathrm{div}\,\underline{\alpha}_F + (\eta - \underline{\eta})\cdot\underline{\alpha}_F, \tag{5.98}$$

$$\nabla_3\sigma_F + \mathrm{tr}\underline{\chi}\,\sigma_F = -\mathrm{curl}\,\underline{\alpha}_F + (\eta - \underline{\eta})\cdot{}^*\underline{\alpha}_F. \tag{5.99}$$

Schematically, we can rewrite the above as

$$\nabla_3(\rho_F,\sigma_F) + \mathrm{tr}\underline{\chi}(\rho_F,\sigma_F) = \nabla\underline{\alpha}_F + \psi\cdot\Upsilon.$$

Commuting with i angular derivatives, we arrive at

$$\nabla_3\nabla^i(\rho_F,\sigma_F) + \frac{i+2}{2}\mathrm{tr}\underline{\chi}\nabla^i(\rho_F,\sigma_F)$$

$$= \nabla^{i+1}\underline{\alpha}_F + \sum_{i_1+i_2+i_3+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\underline{\alpha}_F + \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\Upsilon$$

$$+ \sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}(\rho_F,\sigma_F)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}(\rho_F,\sigma_F)$$

$$:= G.$$

Applying Proposition 5.3.7 with $\lambda_1 = i+1$, we have

$$|u|^{i+1}\|\nabla^i(\rho_F,\sigma_F)\|_{L^2(S_{u,\underline{u}})} \lesssim |u_\infty|^{i+1}\|\nabla^i(\rho_F,\sigma_F)\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^{i+1}\|G\|_{L^2(S_{u',\underline{u}})}\,\mathrm{d}u'.$$

We have

$$s_2\left(\nabla^i(\rho_F,\sigma_F)\right) = \frac{i+1}{2}, \quad s_2(G) = \frac{i+3}{2}.$$

Therefore,

$$\|(a^{\frac{1}{2}}\nabla)^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = (a^{\frac{1}{2}})^i\cdot a^{-\frac{i+1}{2}}|u|^{i+1}\|\nabla^i(\rho_F,\sigma_F)\|_{L^2(S_{u,\underline{u}})},$$

so that

$$|u|^{i+1}\|\nabla^i(\rho_F,\sigma_F)\|_{L^2(S_{u,\underline{u}})} = a^{\frac{1}{2}}\|(a^{\frac{1}{2}}\nabla)^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})},$$

as well as

$$\|(a^{\frac{1}{2}})^i G\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = (a^{\frac{1}{2}})^i a^{-\frac{i+3}{2}} |u|^{i+3} \|G\|_{L^2(S_{u,\underline{u}})},$$

whence we get

$$|u|^{i+1} \|(a^{\frac{1}{2}})^i G\|_{L^2(S_{u,\underline{u}})} = \frac{a^{\frac{3}{2}}}{|u|^2} \|(a^{\frac{1}{2}})^i G\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Passing to scale-invariant norms we have

$$\|(a^{\frac{1}{2}}\nabla)^i(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \|(a^{\frac{1}{2}}\nabla)^i(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i G\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$

$$\lesssim \|(a^{\frac{1}{2}}\nabla)^i(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \nabla^{i+1}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2} \Big\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\underline{\alpha}_F \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2} \Big\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\Upsilon \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2} \Big\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}(\rho_F, \sigma_F) \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2} \Big\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}(\rho_F, \sigma_F) \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$

$$:= J_1 + \cdots + J_6.$$

We treat $J_1, \ldots, J_6$ on a term-by-term basis.

- The initial data term J_1 is bounded by $\mathcal{I}^{(0)}(\underline{u}) \lesssim 1$.

- We have

$$J_2 \lesssim \frac{a}{|u|} \|(a^{\frac{1}{2}})^i \nabla^{i+1}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty, u)})} \lesssim \frac{a}{|u|} \mathcal{F}[\underline{\alpha}_F] \lesssim \mathcal{F}[\underline{\alpha}_F].$$

- For J_3 we have

$$\int_{u_\infty}^u \frac{a}{|u'|^2} \Big\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\underline{\alpha}_F \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'$$

$$\lesssim \int_{u_\infty}^u \frac{a}{|u'|^2} \Big\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\underline{\alpha}_F \Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' \qquad (5.100)$$

$$\lesssim \int_{u_\infty}^u \frac{a}{|u'|^2} \cdot \frac{O^2}{|u'|} \, \mathrm{d}u' \lesssim \frac{a \cdot O^2}{|u|^2} \lesssim 1.$$

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- For J_4, we have

$$\int_{u_\infty}^{u} \frac{a}{|u'|^2} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^{u} \frac{a}{|u'|^2} \cdot \frac{O^2}{|u'|} \mathrm{d}u' \lesssim \frac{a \cdot O^2}{|u|^2} \lesssim 1. \tag{5.101}$$

- For J_5, there holds

$$\int_{u_\infty}^{u} \frac{a}{|u'|^2} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}(\rho_F,\sigma_F) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^{u} a^{\frac{1}{2}} \cdot \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i-1} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\left(\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi} \right)\nabla^{i_4}(\rho_F,\sigma_F) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$
$$\mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}} \cdot O^3}{|u'|^2} \mathrm{d}u' \lesssim \frac{a^{\frac{1}{2}} \cdot O^3}{|u|} \lesssim 1. \tag{5.102}$$

- There holds

$$\int_{u_\infty}^{u} \frac{a}{|u'|^2} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}(\rho_F,\sigma_F) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a^{\frac{1}{2}}}{|u'|}(\psi,\underline{\hat{\chi}},\widetilde{\mathrm{tr}\underline{\chi}}) \right)\nabla^{i_4}(\rho_F,\sigma_F) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$
$$\mathrm{d}u'$$
$$\lesssim \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|} \cdot \frac{O^2}{|u'|} \lesssim \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \lesssim 1. \tag{5.103}$$

Combining these estimates together, we arrive at the desired result. $\qquad\square$

Proposition 5.4.12. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), we have*

$$\sum_{i\leq 10} \|(a^{\frac{1}{2}}\nabla)^i \underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{F}[\rho_F,\sigma_F] + \underline{\mathcal{F}}[\rho_F,\sigma_F] + 1.$$

Proof. For $\Upsilon \in \left\{ \rho_F, \sigma_F, \underline{\alpha}_F \right\}$ we have the following schematic equation

$$\nabla_4 \underline{\alpha}_F = \nabla(\rho_F,\sigma_F) + (\underline{\hat{\chi}},\psi) \cdot (\Upsilon,\alpha_F).$$

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Commuting this with i angular derivatives, we get

$$
\begin{aligned}
&\nabla_4\nabla^i\underline{\alpha}_F \\
&=\nabla^{i+1}(\rho_F,\sigma_F) + \sum_{i_1+i_2+i_3+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\rho_F,\sigma_F) \\
&\quad + \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\underline{\alpha}_F + \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}})\nabla^{i_4}(\Upsilon,\alpha_F).
\end{aligned}
$$

By multiplying with $(a^{\frac{1}{2}})^i$ on both sides and passing to scale-invariant norms, we have

$$
\begin{aligned}
&\|(a^{\frac{1}{2}}\nabla)^i\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \\
&\leq \int_0^{\underline{u}}\|(a^{\frac{1}{2}})^i\nabla^{i+1}(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \\
&\quad + \int_0^{\underline{u}}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \\
&\quad + \int_0^{\underline{u}}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \\
&\quad + \int_0^{\underline{u}}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}})\nabla^{i_4}(\Upsilon,\alpha_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \\
&\leq \mathcal{F}[\rho_F,\sigma_F] + \int_0^{\underline{u}}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \\
&\quad + \int_0^{\underline{u}}\|(a^{\frac{1}{2}})^{i+1}\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat{\chi}}{a^{\frac{1}{2}}})\nabla^{i_4}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \\
&\quad + \int_0^{\underline{u}}\frac{\mathrm{d}\underline{u}'}{a^{\frac{1}{2}}}|u|\|(a^{\frac{1}{2}})^{i+1}\sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\frac{a^{\frac{1}{2}}}{|u|}\psi,\frac{a^{\frac{1}{2}}}{|u|}\hat{\chi})\nabla^{i_4}(\frac{\Upsilon}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \\
&\quad + \int_0^{\underline{u}}a^{-\frac{1}{2}}|u|\|(a^{\frac{1}{2}})^{i+1}\sum_{i_1+i_2=i}\nabla^{i_1}(\frac{a^{\frac{1}{2}}}{|u|}\psi,\frac{a^{\frac{1}{2}}}{|u|}\hat{\chi})\nabla^{i_2}(\frac{\Upsilon}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \\
&\leq \mathcal{F}[\rho_F,\sigma_F] + O[\underline{\hat{\chi}}]O[\alpha_F] + 1 \lesssim \mathcal{F}[\rho_F,\sigma_F] + \underline{\mathcal{F}}[\rho_F,\sigma_F] + 1.
\end{aligned}
$$

In the last inequality we have used the refined estimates on $\hat{\chi}$ and α_F from Propositions 5.4.3 and 5.4.10 respectively. $\qquad\square$

5.5 $L^2(S_{u,\underline{u}})$-estimates for curvature

Proposition 5.5.1. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumption (5.69), we have*

$$\sum_{i\leq 9}\frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\lesssim 1.$$

Proof. Reading off equation (5.62), we have

$$\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\underline{\chi}\alpha=\nabla\tilde{\beta}+\alpha_F\cdot\nabla\Upsilon+\Upsilon\cdot(\nabla\alpha_F,\nabla\Upsilon)$$
$$+(\psi,\hat{\chi})\cdot\Psi+\psi\cdot\alpha+(\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot(\alpha_F,\Upsilon).$$

Commuting with i angular derivatives, we obtain

$$\nabla_3\nabla^i\alpha+\frac{i+1}{2}\mathrm{tr}\underline{\chi}\nabla^i\alpha$$
$$=\nabla^{i+1}\tilde{\beta}+\sum_{i_1+i_2+i_3+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\tilde{\beta}+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\alpha_F\nabla^{i_4+1}\Upsilon$$
$$+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\alpha_F,\Upsilon)+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\Psi$$
$$+\sum_{i_1+i_2+i_3+i_4+i_5=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}(\alpha_F,\Upsilon)$$
$$+\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\alpha+\sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha.$$

Denote the above as
$$\nabla_3\nabla^i\alpha+\frac{i+1}{2}\mathrm{tr}\underline{\chi}\nabla^i\alpha=G.$$

Using the definition of the $\mathcal{L}^2_{(sc)}(u,\underline{u})$-norms we have

$$\|\nabla^i\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}=a^{-\frac{i}{2}}|u|^i\|\nabla^i\alpha\|_{L^2(S_{u,\underline{u}})},\quad \|G\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}=a^{-\frac{i+2}{2}}|u|^{i+2}\|G\|_{L^2(S_{u,\underline{u}})},$$

which translates to

$$a^{-\frac{1}{2}}|u|^i\|\nabla^i\alpha\|_{L^2(S_{u,\underline{u}})}=a^{-\frac{1}{2}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})},$$

$$a^{-\frac{1}{2}}|u|^i\|G\|_{L^2(S_{u,\underline{u}})}=\frac{a^{\frac{1}{2}}}{|u|^2}\|(a^{\frac{1}{2}})^iG\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Hence we have

$$a^{-\frac{1}{2}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq a^{-\frac{1}{2}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{L^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{1}{|u'|^2}\|(a^{\frac{1}{2}}\nabla)^{i+1}\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|\sum_{i_1+i_2+i_3+1=i}(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\alpha_F,\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}(\alpha_F,\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$:=T_1 + T_2 + \cdots + T_9.$$

The first term can be bounded by the initial data, since

$$T_1 = a^{-\frac{1}{2}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{L^2_{(sc)}(S_{u_\infty,\underline{u}})} \leq \mathcal{I}^{(0)}(\underline{u}) \lesssim 1.$$

For the terms involving $\tilde{\beta}$, we have

$$T_2 + T_3 = \int_{u_\infty}^{u} \frac{1}{|u'|^2} \|(a^{\frac{1}{2}}\nabla)^{i+1}\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$+ \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \| \sum_{i_1+i_2+i_3+1=i} (a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$\leq \left(\int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}}\nabla)^{i+1}\tilde{\beta}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u' \right)^{\frac{1}{2}} \left(\int_{u_\infty}^{u} \frac{1}{a|u'|^2} \mathrm{d}u' \right)^{\frac{1}{2}}$$

$$+ \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \cdot \frac{O^2}{|u'|} \mathrm{d}u'$$

$$\leq a^{-\frac{1}{2}} \|(a^{\frac{1}{2}}\nabla)^{i+1}\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \cdot \frac{1}{|u|^{\frac{1}{2}}} + \frac{a^{\frac{1}{2}}\cdot O^2}{|u|^2}$$

$$\lesssim \mathcal{R}[\tilde{\beta}] \cdot \frac{1}{|u|^{\frac{1}{2}}} + \frac{a^{\frac{1}{2}}\cdot O^2}{|u|^2} \lesssim \frac{R}{|u|^{\frac{1}{2}}} + 1 \lesssim 1.$$

Here we have used the bootstrap assumptions (5.69).

Notice how the curvature term is actually bounded by 1. For the next two terms, we need to treat the cases where all the weight falls on i_4 separately. Look, first, at

$$T_4 = \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'.$$

If $i_4 = i$, then we can bound the term (α_F,Υ) below in $\mathcal{L}^\infty_{(sc)}$ to get

$$T_4 = \int_{u_\infty}^{u} \frac{1}{|u'|^2} \|(\alpha_F,\Upsilon)\cdot(a^{\frac{1}{2}}\nabla)^{i+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \cdot \frac{O}{|u'|} \cdot \|(a^{\frac{1}{2}}\nabla)^{i+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \cdot \frac{O}{|u'|} \cdot O \mathrm{d}u' \lesssim \frac{a^{\frac{1}{2}}\cdot O^2}{|u|^2} \lesssim 1.$$

If $i_4 < i \leq 9$ we distinguish two cases:

- There holds $i_4 + 1 \leq 6$. We then write

$$(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon$$

$$=(a^{\frac{1}{2}})^{i+1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{\alpha_F}{a^{\frac{1}{2}}},\frac{\Upsilon}{a^{\frac{1}{2}}})\nabla^{i_4+1}\Upsilon$$

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and bound $(a^{\frac{1}{2}}\nabla)^{i_4+1}\Upsilon$ in $\mathcal{L}^{\infty}_{(sc)}$. We have

$$\int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}\,\mathrm{d}u'}{|u'|^3}\cdot\|(a^{\frac{1}{2}}\nabla)^{i_4+1}\Upsilon\|_{\mathcal{L}^{\infty}_{(sc)}(S_{u',\underline{u}})}\|\sum_{i_1+i_2+i_3\leq 9}(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$\leq \frac{a^{\frac{1}{2}}\cdot O^2}{|u|} \leq \frac{O^2}{a^{\frac{1}{2}}} \leq 1.$$

- There holds $7 \leq i_4+1 \leq 9$. We then bound $(a^{\frac{1}{2}}\nabla)^{i_4+1}\Upsilon$ in $\mathcal{L}^2_{(sc)}$ and the rest of the terms in $\mathcal{L}^{\infty}_{(sc)}$. This gives

$$\int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}\,\mathrm{d}u'}{|u'|^3}\cdot\|(a^{\frac{1}{2}}\nabla)^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\|\sum_{i_1+i_2+i_3\leq 9}(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\|_{\mathcal{L}^{\infty}_{(sc)}(S_{u',\underline{u}})}$$

$$\leq \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2}\cdot\frac{1}{|u|'}\cdot O\cdot\frac{(a^{\frac{1}{2}})^{i_1+i_2+i_3}O^{i_1+i_2+i_3}}{|u'|^{i_1+i_2+i_3-1}}\,\mathrm{d}u' \leq \frac{a^{\frac{1}{2}}O}{|u'|} \leq 1.$$

We move on to T_5. If $i_4 = i$, we have

$$\int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \Upsilon\nabla^{i+1}(\alpha_F,\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^2}\cdot\frac{O}{|u'|}\cdot\frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^{i+1}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u' \leq \frac{a^{\frac{1}{2}}\cdot O^2}{|u|^2} \leq 1.$$

If $i_4 < i \leq 9$ we again distinguish two cases:

- There holds $i_4+1 \leq 6$. We then write

$$(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\alpha_F$$

$$= (a^{\frac{1}{2}})^{i+1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right) \quad (5.104)$$

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and bound $\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)$ in $\mathcal{L}^\infty_{(sc)}$. We have

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\cdot\frac{1}{|u'|}\left\|(a^{\frac{1}{2}}\nabla)^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^\infty_{(sc)}(S_{u',\underline{u}})}$$

$$\cdot\|\sum_{i_1+i_2+i_3\leq 9}(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\cdot\frac{1}{|u'|}\cdot O^2\mathrm{d}u' = \frac{a^{\frac{1}{2}}O^2}{|u|}\leq 1.$$

- There holds $7\leq i_4+1\leq 9$. We then bound $\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)$ in $\mathcal{L}^2_{(sc)}$ and the rest of the terms in $\mathcal{L}^\infty_{(sc)}$. We then have

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\left\|(a^{\frac{1}{2}})^{i+1}\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\frac{\alpha_F}{a^{\frac{1}{2}}})\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\cdot\frac{1}{|u'|}\left\|(a^{\frac{1}{2}}\nabla)^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$\cdot\|\sum_{i_1+i_2+i_3\leq 2}(a^{\frac{1}{2}})^{i_1+i_2+i_3}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\|_{\mathcal{L}^\infty_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\cdot\frac{O}{|u'|}\cdot\frac{(a^{\frac{1}{2}})^{i_1+i_2+i_3}O^{i_1+i_2+i_3}}{|u'|^{i_1+i_2+i_3-1}}\mathrm{d}u'\leq 1.$$

For T_6, we have

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat\chi)\nabla^{i_4}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$=\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\left\|(a^{\frac{1}{2}})^{i+1}\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat\chi}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\Psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\cdot\frac{a^{\frac{1}{2}}\cdot O^2}{|u'|}\,\mathrm{d}u' = \frac{a\cdot O^2}{|u|^2}\leq 1.$$

For the term T_7, which contains a triple anomaly, we have

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i \sum_i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat\chi},\mathrm{tr}\underline\chi,\hat\chi)\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}(\alpha_F,\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$=\int_{u_\infty}^u a^{\frac{1}{2}}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\Upsilon\nabla^{i_4}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq\int_{u_\infty}^u a^{\frac{1}{2}}\cdot\frac{O^3}{|u'|^2}\,\mathrm{d}u'\leq\frac{a^{\frac{1}{2}}\cdot O^3}{|u|}\leq 1.$$

For T_8, there holds

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\widetilde{\mathrm{tr}\chi})\nabla^{i_4}\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$=\int_{u_\infty}^u \frac{\mathrm{d}u'}{|u'|}\left\|(a^{\frac{1}{2}})^{i+1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a^{\frac{1}{2}}}{|u'|}\psi,\frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}},\frac{a^{\frac{1}{2}}}{|u'|}\widetilde{\mathrm{tr}\chi}\right)\nabla^{i_4}\left(\frac{\alpha}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$\leq\int_{u_\infty}^u \frac{1}{|u'|}\cdot\frac{a^{\frac{1}{2}}\cdot O^2}{|u'|}\,\mathrm{d}u' = \frac{a^{\frac{1}{2}}\cdot O^2}{|u|}\leq 1.$$

Finally, by the seventh inequality in Proposition 5.4.1, we can bound T_9 as follows:

$$\int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2}\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$=\int_{u_\infty}^u \frac{1}{a}\left\|(a^{\frac{1}{2}})^{i+2} \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\left(\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi}\right)\nabla^{i_4}\left(\frac{\alpha}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\leq\int_{u_\infty}^u \frac{1}{a}\cdot\frac{a^{\frac{3}{2}}\cdot O^3}{|u'|^2}\,\mathrm{d}u' \leq \frac{a^{\frac{1}{2}}\cdot O^3}{|u|}\leq 1.$$

Putting all the estimates together, the result follows.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We move on to estimates for the curvature components $\tilde{\beta},\rho,\sigma,\underline{\tilde{\beta}},\underline{\alpha}$.

Proposition 5.5.2. *Let $\Psi \in \left\{\tilde{\beta},\rho,\sigma,\underline{\tilde{\beta}},\underline{\alpha}\right\}$. Under the assumptions of Theorem 5.1.2 and (5.69), we have*

$$\sum_{i\leq 9}\|(a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\leq \mathcal{R}[\alpha]+\underline{\mathcal{F}}[\rho_F,\sigma_F]+1.$$

Proof. The terms Ψ satisfy the following schematic equations:

$$\nabla_4\Psi = \nabla(\Psi,\alpha)+(\psi,\underline{\hat{\chi}})\cdot(\Psi,\alpha)+(\alpha_F,\Upsilon)\cdot\nabla(\alpha_F,\Upsilon)+(\psi,\underline{\hat{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot(\alpha_F,\Upsilon).$$
$$(5.105)$$

Commuting (5.105) with i angular derivatives and using Proposition 5.3.16, we have

$$
\begin{aligned}
\nabla_4 \nabla^i \Psi = {} & \nabla^{i+1}(\Psi, \alpha) + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}(\Psi, \alpha) \\
& + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \underline{\hat{\chi}})\nabla^{i_4}(\Psi, \alpha) \\
& + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F, \Upsilon)\nabla^{i_4+1}(\alpha_F, \Upsilon) \\
& + \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \underline{\hat{\chi}}, \mathrm{tr}\underline{\chi}, \hat{\chi})\nabla^{i_4}(\alpha_F, \Upsilon)\nabla^{i_5}(\alpha_F, \Upsilon) \\
& + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi})\nabla^{i_4}\Psi.
\end{aligned}
$$

Applying Proposition 5.3.6 and multiplying both sides by $(a^{\frac{1}{2}})^i$ we get

$$\|(a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i\nabla^{i+1}(\Psi,\alpha)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+1=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}(\Psi,\alpha)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}})\nabla^{i_4}(\Psi,\alpha)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}(\alpha_F,\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_i \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\mathrm{tr}\underline{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}(\alpha_F,\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\leq \mathcal{R}[\alpha] + \frac{1}{a^{\frac{1}{2}}}\mathcal{R}[\Psi] + \sum_{i_1+i_2+i_3+1=i} \int_0^{\underline{u}} \left\|(a^{\frac{1}{2}})^{i+1}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\left(\frac{\Psi}{a^{\frac{1}{2}}},\frac{\alpha}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} |u|\left\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a^{\frac{1}{2}}}{|u|}\psi,\frac{a^{\frac{1}{2}}}{|u|}\underline{\hat{\chi}}\right)\nabla^{i_4}\left(\frac{\Psi}{a^{\frac{1}{2}}},\frac{\alpha}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \left\|(a^{\frac{1}{2}})^{i+2}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\alpha_F}{a^{\frac{1}{2}}},\frac{\Upsilon}{a^{\frac{1}{2}}}\right)\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}},\frac{\Upsilon}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_i |u|^2 \cdot$$

$$\int_0^{\underline{u}} \left\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a}{|u|^2}(\psi,\underline{\hat{\chi}},\mathrm{tr}\underline{\chi})\right)\nabla^{i_4}\left(\frac{\alpha_F}{a^{\frac{1}{2}}},\frac{\Upsilon}{a^{\frac{1}{2}}}\right)\nabla^{i_5}\left(\frac{\alpha_F}{a^{\frac{1}{2}}},\frac{\Upsilon}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \left\|(a^{\frac{1}{2}})^{i+1}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\Psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$:= \sum_{k=1}^7 J_k.$$

$$(5.106)$$

We focus on each J_k-term separately.

- We have $J_1 + J_2 \leq \mathcal{R}[\alpha] + 1$.

- We have
$$J_3 = \sum_{i_1+i_2+i_3+1=i} \int_0^{\underline{u}} \left\|(a^{\frac{1}{2}})^{i+1}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3+1}\left(\frac{\Psi}{a^{\frac{1}{2}}},\frac{\alpha}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' \leq \frac{a^{\frac{1}{2}}\cdot O^2}{|u|}.$$
$$(5.107)$$

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- We have

$$J_4 =$$

$$\sum_i \int_0^{\underline{u}} \frac{|u|}{a^{\frac{1}{2}}} \left\| (a^{\frac{1}{2}})^{i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{a^{\frac{1}{2}}}{|u|} \psi, \frac{a^{\frac{1}{2}}}{|u|} \hat{\underline{\chi}} \right) \nabla^{i_4} \left(\frac{\Psi}{a^{\frac{1}{2}}}, \frac{\alpha}{a^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$\leq \frac{|u|}{a^{\frac{1}{2}}} \cdot \frac{a^{\frac{1}{2}}}{|u|} (O[\hat{\underline{\chi}}] \cdot O[\alpha] + 1) \leq O[\alpha] + 1 \leq 1.$$

Note that here we have made use of Proposition 5.4.3 and Proposition 5.5.1 and used the improved (compared to the bootstrap assumptions) bounds on $\hat{\underline{\chi}}$ and α.

- We have

$$J_5 =$$

$$\sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \left\| (a^{\frac{1}{2}})^{i+2} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \nabla^{i_4+1} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$\lesssim \sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \| (a^{\frac{1}{2}})^{i+2} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_3} \Upsilon \nabla^{i_4+1} \Upsilon \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$\lesssim \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \lesssim 1.$$

Here we have used Proposition 5.4.1 and the fact that $i + 1 \leq 10$.

- We have

$$J_6 = \sum_{i_1+i_2+i_3+i_4+i_5=i} \frac{|u|^2}{a} \times \int_0^{\underline{u}} (a^{\frac{1}{2}})^{i+2}$$

$$\left\| \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{a}{|u|^2} \psi, \frac{a}{|u|^2} \hat{\underline{\chi}}, \frac{a}{|u|^2} \mathrm{tr}\underline{\chi} \right) \nabla^{i_4} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \nabla^{i_5} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}$$

$$\mathrm{d}\underline{u}'$$

$$\leq \frac{|u|^2}{a} \cdot a \cdot \left(\frac{O[\mathrm{tr}\underline{\chi}] \cdot O_{i_4}\left[\frac{\alpha_F}{a^{\frac{1}{2}}} \right] \cdot O_{i_5}\left[\frac{\alpha_F}{a^{\frac{1}{2}}} \right]}{|u|^2} \right) + 1.$$

The logic behind the bound above is as follows. If the term we wish to bound, schematically, is not in the form of a triple anomaly, then the estimates are not borderline and the term is bounded above by 1. The worst term is when we wish to bound

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$$\sum_{i_1+i_2+i_3+i_4+i_5=i} \int_0^{\underline{u}} \left\| (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \mathrm{tr}\underline{\chi} \nabla^{i_4} \alpha_F \nabla^{i_5} \alpha_F \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'.$$

This term can only be bounded by $O[\mathrm{tr}\underline{\chi}] \cdot O_{i_4} \left[\frac{\alpha_F}{a^{\frac{1}{2}}} \right] \cdot O_{i_5} \left[\frac{\alpha_F}{a^{\frac{1}{2}}} \right]$. We now use the improved bounds from Propositions 5.4.8 and 5.4.10 to bound $O[\mathrm{tr}\underline{\chi}] \leq 1$ and $O_{i_4} \left[\frac{\alpha_F}{a^{\frac{1}{2}}} \right] \cdot O_{i_5} \left[\frac{\alpha_F}{a^{\frac{1}{2}}} \right] \leq \left(\underline{\mathcal{F}}[\rho_F, \sigma_F] + 1 \right) \cdot 1$. This is because at least one of the indices i_4, i_5 will not be of top order, hence the estimate from Proposition 5.4.10 for that term will be better (this is the content of Remark 12). Combining these estimates, we arrive at

$$J_6 \leq \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1.$$

- The final term J_7 is handled as follows:

$$\sum_{i_1+i_2+i_3+i_4=i} \int_0^{\underline{u}} \left\| (a^{\frac{1}{2}})^{i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \nabla^{i_4} \Psi \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$\leq \int_0^{\underline{u}} \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \mathrm{d}\underline{u}' \leq \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \leq 1.$$

Combining all the estimates above, we arrive at the desired conclusion.

$\square$

5.6 Elliptic estimates for top-order derivatives of Ricci coefficients

5.6.1 General elliptic estimates for Hodge systems

We recall here the definition of divergence and curl for a symmetric, covariant tensor of arbitrary rank:

$$(\mathrm{div}\ \phi)_{A_1 \ldots A_r} = \nabla^B \phi_{BA_1 \ldots A_r},$$

$$(\mathrm{curl}\ \phi)_{A_1 \ldots A_r} = \not{\epsilon}^{BC} \nabla_B \phi_{CA_1 \ldots A_r}.$$

The trace of such a tensor is defined by

$$(\mathrm{tr}\phi)_{A_1 \ldots A_{r-1}} = (\gamma^{-1})^{BC} \phi_{BCA_1 \ldots A_{r-1}}.$$

The main elliptic estimate that will be used here is the following:

Proposition 5.6.1. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), let ϕ be a totally symmetric $(r+1)-$covariant tensor field on a metric $2-$sphere $(\mathbb{S}^2, \gamma)$, satisfying*

$$\text{div}\phi = f, \quad \text{curl}\phi = g, \quad \text{tr}\phi = h.$$

Then, for $1 \leq i \leq 11$, we have

$$\|(a^{\frac{1}{2}}\nabla)^i\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim a^{\frac{1}{2}} \sum_{j=0}^{i-1} \|(a^{\frac{1}{2}}\nabla)^j(f,g)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \sum_{j=0}^{i-1} \|(a^{\frac{1}{2}}\nabla)^j(\phi,h)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Proof. Recall the following identity from Chapter 7 in [22] that, for ϕ, f, g and h as above, there holds

$$\int_{S_{u,\underline{u}}} \left(|\nabla\phi|^2 + (r+1)K|\phi|^2\right) \mathrm{d}\mu_\gamma = \int_{S_{u,\underline{u}}} \left(|f|^2 + |g|^2 + rK|h|^2\right) \mathrm{d}\mu_\gamma. \qquad (5.108)$$

Here K denotes the Gauss curvature of the sphere. To prove the lemma for the case $i = 1$ first, we need to control K in L^∞. To that end, we will prove the following stronger lemma:

Lemma 5.6.1. *For $0 \leq k \leq 7$, there holds $\|(a^{\frac{1}{2}}\nabla)^k K\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \lesssim 1$.*

Proof. We begin by recalling that

$$K = -\rho_R - \frac{1}{4}\text{tr}\chi\,\text{tr}\underline{\chi} + \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} = -\rho - \frac{1}{4}\text{tr}\chi\,\text{tr}\underline{\chi} + \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} + \frac{1}{2}\left(|\rho_F|^2 + |\sigma_F|^2\right)$$

and $s_2(K) = 1$. By virtue of the scale-invariant version of the $L^2 - L^\infty$ Sobolev embedding inequality from Proposition 5.3.11, there holds

$$\sum_{0 \leq k \leq 7} \|(a^{\frac{1}{2}}\nabla)^k K\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \lesssim \sum_{0 \leq j \leq 9} \|(a^{\frac{1}{2}}\nabla)^j K\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}. \qquad (5.109)$$

We proceed to estimate, for a fixed $0 \leq i \leq 9$, the term $\|(a^{\frac{1}{2}}\nabla)^i K\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$. We have

$$\|(a^{\frac{1}{2}}\nabla)^i K\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \|(a^{\frac{1}{2}}\nabla)^i\rho\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \sum_{i_1+i_2=i} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\text{tr}\chi\nabla^{i_2}\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$+ \sum_{i_1+i_2=i} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\hat{\chi}\nabla^{i_2}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$+ \sum_{i_1+i_2=i} \|(a^{\frac{1}{2}})^i\nabla^{i_1}\Upsilon\nabla^{i_2}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

The first term above can be bounded by 1, by Proposition 5.5.2. For the second term, we have

$$
\sum_{i_1+i_2=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \mathrm{tr}\chi \nabla^{i_2} \mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
= \frac{|u|^2}{a} \sum_{i_1+i_2=i} \left\|(a^{\frac{1}{2}})^i \nabla^{i_1} \mathrm{tr}\chi \nabla^{i_2}\left(\frac{a}{|u|^2}\mathrm{tr}\underline{\chi}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \tag{5.110}
$$

$$
\lesssim \frac{|u|^2}{a} \cdot \frac{1}{|u|} \cdot \left(O_\infty[\mathrm{tr}\underline{\chi}] \cdot O_2[\mathrm{tr}\chi] + O_2[\mathrm{tr}\underline{\chi}] \cdot O_\infty[\mathrm{tr}\chi]\right).
$$

In the above inequality we have conditioned on the number of derivatives that fall on $\mathrm{tr}\chi$ and those that fall on $\mathrm{tr}\underline{\chi}$. Notice that, from Proposition 5.4.8, there holds $O_\infty[\mathrm{tr}\underline{\chi}] + O_2[\mathrm{tr}\underline{\chi}] \lesssim 1$. For $O_2[\mathrm{tr}\chi]$, from Proposition 5.4.7, we read off (5.88) that

$$
O_2[\mathrm{tr}\chi] \leq \frac{a}{|u|}O[\hat{\chi},\alpha_F] \cdot O[\hat{\chi},\alpha_F] + \frac{a}{|u|^2}O^3 + \frac{a^{\frac{1}{2}}}{|u|}O^2.
$$

Plugging this inequality in (5.110) and using $O[\hat{\chi},\alpha_F] \lesssim 1$ from Propositions 5.4.4 and 5.4.10 (remember crucially that we work with up to 9 derivatives at most, so the top order terms $\mathcal{R}[\alpha]$ and $\underline{\mathcal{F}}[\rho_F,\sigma_F]$ used to estimate $\hat{\chi}$ and α_F are redundant on the right–hand side) we arrive at

$$
\sum_{i_1+i_2=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \mathrm{tr}\chi \nabla^{i_2} \mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1.
$$

For the third term, there holds

$$
\sum_{i_1+i_2=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \hat{\chi} \nabla^{i_2} \underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} = |u| \sum_{i_1+i_2=i} \left\|(a^{\frac{1}{2}})^i \nabla^{i_1}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right) \nabla^{i_2}\left(\frac{a^{\frac{1}{2}}}{|u|}\underline{\hat{\chi}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
\lesssim O_\infty[\underline{\hat{\chi}}] \cdot O_2[\hat{\chi}] + O_2[\underline{\hat{\chi}}] \cdot O_\infty[\hat{\chi}] \lesssim 1 \cdot 1 = 1.
$$

Here we have used Proposition 5.4.3 and Proposition 5.4.4, where we achieve the better bound $O_2[\hat{\chi}] \lesssim 1$, given that we work up to 9 derivatives, not 10, which means that we can bound the curvature term in the $\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})$−norm instead of the $\mathcal{L}^2_{(sc)}(H^{0,\underline{u}}_u)$− norm. The bound on α is then invoked by Proposition 5.5.1. Finally, the fourth term is bounded by $O^2/|u| \lesssim 1$, using proposition 5.4.1. This concludes the proof of the lemma. $\qquad\square$

By applying the scale-invariant version of Hölder's inequality for $K|h|^2$ and using Lemma 5.6.1, we obtain the result for $i = 1$. For $i > 1$, the symmetrized angular

derivative of ϕ defined by

$$(\nabla\phi)^s_{BA_1\ldots A_{r+1}} = \frac{1}{r+2}\left(\nabla_B\phi_{A_1\ldots A_r} + \sum_{i=1}^{r+1}\nabla_{A_i}\phi_{A_1\ldots\langle A_i\rangle B\ldots A_{r+1}}\right)$$

satisfies the div-curl system

$$\begin{cases} \text{div }(\nabla\phi)^s = (\nabla f)^s - \frac{1}{r+2}(^*\nabla g)^s + (r+1)K\phi - \frac{2K}{r+1}(\gamma\otimes^s h), \\ \text{curl }(\nabla\phi)^s = \frac{r+1}{r+2}(\nabla g)^s + (r+1)K(^*\phi)^s, \\ \text{tr}(\nabla\phi)^s = \frac{2}{r+2}f + \frac{r}{r+2}(\nabla h)^s, \end{cases} \tag{5.111}$$

where

$$\gamma\otimes^s h := \gamma_{A_iA_j}\sum_{i\leq j=1,\ldots,r+1} h_{A_1\ldots\langle A_i\rangle\ldots\langle A_j\rangle\ldots A_{r+1}}$$

and

$$(^*\phi)^s_{A_1\ldots A_{r+1}} := \frac{1}{r+1}\sum_{i=1}^{r+1}\not\in_{A_i}{}^B\phi_{A_1\ldots\langle A_i\rangle B\ldots A_{r+1}},$$

where we recall that $\not\in$ is the volume form associated to the metric γ. Using (5.108) and iterating, we obtain that for $i\leq 11$ there holds

$$\|\nabla^i\phi\|^2_{L^2(S_{u,\underline{u}})}$$
$$\lesssim \|\nabla^{i-1}(f,g)\|^2_{L^2(S_{u,\underline{u}})} + \|K(|\nabla^{i-2}(f,g)|^2 + |\nabla^{i-1}(\phi,h)|^2)\|_{L^1(S_{u,\underline{u}})}$$
$$+ \left\|K\left(\sum_{i_1+2i_2+i_3=i-3}\nabla^{i_1}K^{i_2+1}\nabla^{i_3}(\phi,h)\right)^2\right\|_{L^1(S_{u,\underline{u}})}$$
$$+ \left\|K\left(\sum_{i_1+2i_2+i_3=i-4}\nabla^{i_1}K^{i_2+1}\nabla^{i_3}f\right)^2\right\|_{L^1(S_{u,\underline{u}})}$$
$$+ \sum_{i_1+2i_2+i_3=i-2}\|\nabla^{i_1}K^{i_2+1}\nabla^{i_3}(\phi,h)\|^2_{L^2(S_{u,\underline{u}})} + \sum_{i_1+2i_2+i_3=i-3}\|\nabla^{i_1}K\nabla^{i_2}(f,g)\|^2_{L^2(S_{u,\underline{u}})},$$
$$\tag{5.112}$$

where we have adopted the convention that $\sum_{i\leq -1}=0$. Whenever a K–term appears with at most 7 derivatives, we estimate it in L^∞ or equivalently in $\mathcal{L}^\infty_{(sc)}$. Whenever a K–term contains between 8 and 9 derivatives we shall estimate it in L^2 and the rest of the terms in L^∞, noting that we can estimate terms of the form $\|\nabla^i(f,g,\phi,h)\|_{L^\infty}$ with $i\leq 7$ by the corresponding norms in L^2 through the standard Sobolev embedding. By Lemma 5.6.1, after translating back to standard L^p norms, there holds

$$\sum_{i\leq 7}\||u|^i\nabla^i K\|_{L^\infty(S_{u,\underline{u}})} + \sum_{j\leq 9}\||u|^j\nabla^j K\|_{L^2(S_{u,\underline{u}})} \lesssim 1.$$

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Therefore, for $i \leq 11$, we have

$$\||u|^i \nabla^i \phi\|^2_{L^2(S_{u,\underline{u}})} \lesssim \sum_{j \leq i-1} \left(\||u|^{j+1} \nabla^j(f,g)\|^2_{L^2(S_{u,\underline{u}})} + \||u|^j \nabla^j(\phi,h)\|^2_{L^2(S_{u,\underline{u}})} \right).$$

Translating the above equation into scale-invariant norms and then multiplying it by $\frac{|u|^{2s_2(\phi)}}{a^{s_2(\phi)}}$, we arrive at

$$\|(a^{\frac{1}{2}}\nabla)^i \phi\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \sum_{j \leq i-1} \left(\|(a^{\frac{1}{2}})^{j+1} \nabla^j(f,g)\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j(\phi,h)\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right). \tag{5.113}$$

Taking square roots above yields Proposition 5.6.1. $\square$

Finally, for the special case where ϕ is a symmetric, traceless $2-$tensor, we need only know its divergence:

Proposition 5.6.2. *Suppose ϕ is a symmetric, traceless $2-$tensor satisfying*

$$\text{div } \phi = f.$$

Then, under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), for $1 \leq i \leq 11$, there holds

$$\|(a^{\frac{1}{2}}\nabla)^i \phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \sum_{j \leq i-1} \left(\|(a^{\frac{1}{2}})^{j+1} \nabla^j f\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j \phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right).$$

Proof. This is a direct application of Proposition 5.6.1, by noticing that

$$\text{curl } \phi = {}^* f.$$

This is a straightforward calculation, using that the $2-$tensor ϕ is symmetric and traceless. $\square$

5.6.2 Elliptic estimates for 11 derivatives of Ricci coefficients

We start this section with the following auxiliary bootstrap assumption. Introduce the top–order quantity

$$\begin{aligned}
\mathcal{O}_{11,2}(u,\underline{u}) =& \|(a^{\frac{1}{2}})^{10} \nabla^{11} \hat{\chi}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{10} \nabla^{11}(\text{tr}\chi,\omega,\eta)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \\
&+ \|a^5 \nabla^{11} \underline{\eta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} + \frac{a}{|u|} \|a^5 \nabla^{11} \underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \\
&+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u|^3} \|(a^{\frac{1}{2}})^{10} \nabla^{11} \underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du' + \|(a^{\frac{1}{2}})^{10} \nabla^{11} \underline{\omega}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \\
&+ \int_{u_\infty}^u \frac{a^2}{|u|^3} \|a^5 \nabla^{11} \text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'.
\end{aligned} \tag{5.114}$$

Throughout this section we shall work under the bootstrap assumption

$$\mathcal{O}_{11,2} \leq O_{11} \lesssim a^{\frac{1}{320}}. \tag{5.115}$$

The purpose of this section will be to obtain the improved bound

$$\mathcal{O}_{11,2} \lesssim 1 + \mathcal{R} + \underline{\mathcal{R}} + \mathcal{F} + \underline{\mathcal{F}}.$$

We begin with estimates for $\mathrm{tr}\chi$ and $\hat{\underline{\chi}}$.

Proposition 5.6.3. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), there holds*

$$\|(a^{\frac{1}{2}})^{10}\nabla^{11}tr\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1 + \mathcal{F}[\alpha_F],$$

$$\|(a^{\frac{1}{2}})^{10}\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_u)} \lesssim 1 + \mathcal{R}[\tilde{\beta}] + \mathcal{R}[\alpha] + \mathcal{F}[\alpha_F].$$

Proof. Consider the following equation:

$$\nabla_4 \mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2 - |\alpha_F|^2 - 2\omega\,\mathrm{tr}\chi. \tag{5.116}$$

Commuting with angular derivatives i times, we arrive at

$$\begin{aligned}
\nabla_4\nabla^i\mathrm{tr}\chi &= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\hat{\chi}\nabla^{i_4}\hat{\chi} + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\alpha_F\nabla^{i_4}\alpha_F \\
&\quad + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\psi \\
&= \sum_{i_1+i_2=i} \nabla^{i_1}\hat{\chi}\nabla^{i_2}\hat{\chi} + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\hat{\chi}\nabla^{i_4}\hat{\chi} \\
&\quad + \sum_{i_1+i_2=i} \nabla^{i_1}\alpha_F\nabla^{i_2}\alpha_F + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\alpha_F\nabla^{i_4}\alpha_F \\
&\quad + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\psi.
\end{aligned}$$

Passing the above to scale-invariant norms and applying the triangle inequality, we

have

$$\|(a^{\frac{1}{2}})^{10}\nabla^{11}\text{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\leq \int_0^{\underline{u}} a\left\|(a^{\frac{1}{2}})^{10}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\nabla^{11}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+\sum_{\substack{i_1+i_2=11\\i_1,i_2\leq 10}}\int_0^{\underline{u}} a\left\|(a^{\frac{1}{2}})^{10}\nabla^{i_1}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\nabla^{i_2}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+\sum_{i_1+i_2+i_3+i_4=10}\int_0^{\underline{u}} a\left\|(a^{\frac{1}{2}})^{10}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}},\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+\sum_{i_1+i_2+i_3+i_4=11}\int_0^{\underline{u}} a^{\frac{1}{2}}\left\|(a^{\frac{1}{2}})^{10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}},\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\nabla^{i_4}\psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\leq \frac{a}{|u|}O_\infty[\hat{\chi}]\cdot\int_0^{\underline{u}}\left\|(a^{\frac{1}{2}})^{10}\nabla^{11}\left(\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+\frac{a}{|u|}O[\alpha_F]\cdot\mathcal{F}[\alpha_F]+\frac{a^{\frac{1}{2}}}{|u|}O_2[\hat{\chi},\alpha_F]\cdot O_\infty[\hat{\chi},\alpha_F]+\frac{a^{\frac{1}{2}}}{|u|^2}O^3+\frac{O^2}{|u|}+\frac{a^{\frac{1}{2}}}{|u|}O\cdot O_{11}$$

$$\leq \frac{a^{\frac{1}{2}}}{|u|}O_\infty[\hat{\chi}]\cdot\int_0^{\underline{u}}\|(a^{\frac{1}{2}})^{10}\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'+\frac{a}{|u|}O[\alpha_F]\cdot\mathcal{F}[\alpha_F]$$

$$+\frac{a^{\frac{1}{2}}}{|u|}(\mathcal{R}[\alpha]+1)(O[\alpha]+1)+1$$

$$\leq \frac{a^{\frac{1}{2}}}{|u|}O_\infty[\hat{\chi}]\cdot\int_0^{\underline{u}}\|(a^{\frac{1}{2}})^{10}\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'+\frac{a}{|u|}O_\infty[\alpha_F]\cdot\mathcal{F}[\alpha_F]+1.$$

$$(5.117)$$

For $\hat{\chi}$, we have

$$\text{div }\hat{\chi}=\frac{1}{2}\nabla\text{tr}\chi-\frac{1}{2}(\eta-\underline{\eta})\cdot(\hat{\chi}-\frac{1}{2}\text{tr}\chi\,\gamma)-\tilde{\beta}+\frac{1}{2}R_{a4}.$$

Schematically,

$$\text{div }\hat{\chi}-\frac{1}{2}\nabla\text{tr}\chi+\tilde{\beta}=\psi\cdot\psi+\alpha_F\cdot\Upsilon.$$

Applying Proposition 5.6.2, we arrive at

$$\|(a^{\frac{1}{2}})^{10}\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \sum_{i\leq 10}\frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\text{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}+\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}+\|a^5\nabla^{11}\text{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$+\sum_{i\leq 10}\sum_{i_1+i_2=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi\nabla^{i_2}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}+\sum_{i\leq 10}\sum_{i_1+i_2=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\alpha_F\nabla^{i_2}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$+\frac{1}{a^{\frac{1}{2}}}\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

$$(5.118)$$

By using the estimate on $\|(a^{\frac{1}{2}}\nabla)^i \mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$ from Proposition 5.4.7 and applying Grönwall's inequality, we get

$$
\begin{aligned}
&\|(a^{\frac{1}{2}})^{10}\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \\
&\lesssim 1 + \sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|a^5\nabla^{11}\mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \\
&\quad + \sum_{i\leq 10}\sum_{i_1+i_2=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi\nabla^{i_2}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \sum_{i\leq 10}\sum_{i_1+i_2=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\alpha_F\nabla^{i_2}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \\
&\quad + \frac{1}{a^{\frac{1}{2}}}\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.
\end{aligned}
\tag{5.119}
$$

Thus, using Proposition 5.4.4, we arrive at

$$
\|a^5\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \|a^5\nabla^{11}\mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + 1 + \mathcal{R}[\alpha] + \sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.
\tag{5.120}
$$

We now combine this with (5.117), which says that

$$
\|a^5\nabla^{11}\mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \frac{a^{\frac{1}{2}}}{|u|}(\mathcal{R}[\alpha]+1)\cdot\int_0^{\underline{u}}\|a^5\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}' + \mathcal{F}[\alpha_F] + 1.
\tag{5.121}
$$

Using Grönwall's inequality, we arrive at

$$
\|a^5\nabla^{11}\mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{F}[\alpha_F] + 1
\tag{5.122}
$$

and integrating (5.120) along the $\underline{u}$–direction, we also get

$$
\|a^5\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}[\alpha] + \mathcal{R}[\tilde{\beta}] + \mathcal{F}[\alpha_F] + 1.
\tag{5.123}
$$

$\square$

We now proceed with estimates for the highest number of derivatives in ω. Define the following Hodge operators acting on the leaves $S_{u,\underline{u}}$ of our double null foliation

- The operator $\mathcal{D}_1$ maps a $1-$form F to the pair of functions $(\mathrm{div}\,F, \mathrm{curl}\,F)$,

- The operator $\mathcal{D}_2$ maps an $S-$tangent, symmetric traceless tensor F into the $S-$tangent one-form $\mathrm{div}F$,

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- The operator $^*\mathcal{D}_1$ maps a pair of scalar functions (F_1, F_2) to the $S-$tangent $1-$form $-\nabla F_1 + {}^*\nabla F_2$,

- The operator $^*\mathcal{D}_2$ maps a $1-$form F to the $2-$covariant, symmetric, traceless tensor $-\frac{1}{2}\widehat{\mathcal{L}_F \gamma}$, where

$$\widehat{\mathcal{L}_F \gamma}_{ab} = \nabla_a F_b + \nabla_b F_a - (\mathrm{div}\, F)\gamma_{ab}.$$

Proposition 5.6.4. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions* (5.69)- (5.115), *there holds*

$$\|a^5 \nabla^{11}\omega\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}[\tilde{\beta}] + 1.$$

Proof. Introduce $\omega^\dagger$, defined as the solution to

$$\nabla_3 \omega^\dagger = \frac{1}{2}\sigma$$

with zero initial data on H_{u_∞}. Introduce the pair of scalars $\langle\omega\rangle = (-\omega, \omega^\dagger)$ and define κ by

$$\kappa := {}^*\mathcal{D}_1\langle\omega\rangle - \frac{1}{2}\tilde{\beta} = \nabla\omega + {}^*\nabla\omega^\dagger - \frac{1}{2}\tilde{\beta}.$$

We need to derive a transport equation for $^*\mathcal{D}_1\langle\omega\rangle$. To this end, recall the commutation formula

$$[\nabla_3, \nabla]f = -\frac{1}{2}\mathrm{tr}\underline{\chi}\nabla f - \underline{\hat{\chi}}\cdot\nabla f + \frac{1}{2}(\eta + \underline{\eta})\nabla_3 f,$$

$$[\nabla_3, {}^*\nabla]g = -\frac{1}{2}\mathrm{tr}\underline{\chi}\,{}^*\nabla g + \underline{\hat{\chi}}\cdot{}^*\nabla g + \frac{1}{2}({}^*\eta + {}^*\underline{\eta})\nabla_3 g.$$

Therefore,

$$[\nabla_3, {}^*\mathcal{D}_1](f, g) = -\frac{1}{2}\mathrm{tr}\underline{\chi}\,{}^*\mathcal{D}_1(f, g) + \underline{\hat{\chi}}\cdot(\nabla f + {}^*\nabla g) - \frac{1}{2}(\eta + \underline{\eta})\nabla_3 f + \frac{1}{2}({}^*\eta + {}^*\underline{\eta})\nabla_3 g.$$

Now recall that

$$\nabla_3\omega = \frac{1}{2}\rho + \psi\,\psi + \Upsilon\,\Upsilon := \frac{1}{2}\rho + \underline{F}.$$

This means that

$$\nabla_3\,{}^*\mathcal{D}_1\langle\omega\rangle = {}^*\mathcal{D}_1\left(-\frac{1}{2}\rho - \underline{F}, \frac{1}{2}\sigma\right) + [\nabla_3, {}^*\mathcal{D}_1]\langle\omega\rangle$$

$$= \frac{1}{2}\nabla\rho + \frac{1}{2}\,{}^*\nabla\sigma + \nabla\underline{F} - \frac{1}{2}\mathrm{tr}\underline{\chi}\,{}^*\mathcal{D}_1\langle\omega\rangle + \underline{\hat{\chi}}\cdot(-\nabla\omega + {}^*\nabla\omega^\dagger)$$

$$- \frac{1}{2}(\eta + \underline{\eta})\left(\frac{1}{2}\rho + \underline{F}\right) + \frac{1}{4}({}^*\eta + {}^*\underline{\eta})\sigma.$$

Schematically, we reduce this to the equation

$$\nabla_3 {}^*\mathcal{D}_1\langle\omega\rangle + \frac{1}{2}\mathrm{tr}\underline{\chi}\,{}^*\mathcal{D}_1\langle\omega\rangle - \frac{1}{2}(\nabla\rho + {}^*\nabla\sigma)$$
$$=\psi\nabla(\omega,\underline{\omega},\eta,\underline{\eta}) + \hat{\chi}\nabla(\omega,\omega^\dagger) + (\rho_F,\sigma_F)\nabla(\rho_F,\sigma_F) + \psi\cdot\Psi + \psi\cdot\psi\cdot\psi + \psi\cdot\Upsilon\cdot\Upsilon.$$
$$(5.124)$$

Recall also the ∇_3-direction schematic equation for $\tilde{\beta}$:

$$\nabla_3\tilde{\beta} + \mathrm{tr}\underline{\chi}\tilde{\beta} - \nabla\rho - {}^*\nabla\sigma = (\psi,\hat{\chi})\Psi + \alpha_F\nabla\underline{\alpha}_F + \underline{\alpha}_F\nabla\alpha_F + (\rho_F,\sigma_F)\nabla(\rho_F,\sigma_F)$$
$$+ (\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot\Upsilon.$$
$$(5.125)$$

From (5.124) and (5.125) we see that κ obeys the following schematic equation:

$$\nabla_3\kappa + \frac{1}{2}\mathrm{tr}\underline{\chi}\kappa = (\psi,\hat{\chi})\Psi + (\psi,\hat{\underline{\chi}})\nabla\psi + \Upsilon\nabla(\Upsilon,\alpha_F)$$
$$+ (\alpha_F,\Upsilon)\nabla\Upsilon + (\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\cdot(\alpha_F,\Upsilon)\cdot\Upsilon + \psi\cdot\psi\cdot\psi.$$
$$(5.126)$$

By commuting (5.126) with $i \leq 10$ angular derivatives, we arrive at

$$\nabla_3\nabla^i\kappa + \frac{i+1}{2}\mathrm{tr}\underline{\chi}\nabla^i\kappa = \sum_{i_1+i_2+i_3+i_4=i\leq 10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\Psi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i\leq 10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}})\nabla^{i_4+1}\psi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i\leq 10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\Upsilon,\alpha_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i\leq 10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon$$
$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i\leq 10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}\Upsilon$$
$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i\leq 10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\nabla^{i_5}\psi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i\leq 10}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\kappa$$
$$+ \sum_{i_1+i_2+i_3+i_4+1=i\leq 10}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\kappa := G.$$
$$(5.127)$$

Applying Proposition 5.3.7 with $\lambda_0 = \frac{i+1}{2}$, we get

$$|u|^i\,\|\nabla^i\kappa\|_{L^2(S_{u,\underline{u}})} \lesssim |u_\infty|^i\,\|\nabla^i\kappa\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^i\,\|G\|_{L^2(S_{u',\underline{u}})}\,\mathrm{d}u'.$$

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Now by definition, we have $s_2(\kappa) = s_2(\nabla\omega, \tilde\beta) = 0.5$. So $s_2(\nabla^i\kappa) = \frac{i+1}{2}$. This means that

$$
\begin{aligned}
\|(a^{\frac{1}{2}}\nabla)^i\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} &= a^{\frac{i-1}{2}}|u|^{i+1}\|\nabla^i\kappa\|_{L^2(S_{u,\underline{u}})} = (a^{\frac{i-1}{2}}|u|)\cdot|u|^i\|\nabla^i\kappa\|_{L^2(S_{u,\underline{u}})}\\
&\lesssim (a^{\frac{i-1}{2}}|u|)\cdot\left(|u_\infty|^i\|\nabla^i\kappa\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^i\|G\|_{L^2(S_{u',\underline{u}})}\,\mathrm{d}u'\right)\\
&\lesssim a^{\frac{i-1}{2}}|u_\infty|^{i+1}\|\nabla^i\kappa\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u a^{\frac{i-1}{2}}|u'|^{i+1}\|G\|_{L^2(S_{u',\underline{u}})}\,\mathrm{d}u'.
\end{aligned}
$$

In the last inequality we have used the facts that $|u|\le|u_\infty|$, $|u|\le|u'|$ for $|u'|$ in the range given above. From this we conclude that

$$
\begin{aligned}
&\|(a^{\frac{1}{2}}\nabla)^i\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\\
&\lesssim \|(a^{\frac{1}{2}}\nabla)^i\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^iG\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&\lesssim \|(a^{\frac{1}{2}}\nabla)^i\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat\chi)\nabla^{i_4}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&+\int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat\chi})\nabla^{i_4+1}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&+\int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\Upsilon,\alpha_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&+\int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\Upsilon,\alpha_F)\nabla^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&+\int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4+i_5=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat\chi},\mathrm{tr}\underline{\chi},\hat\chi)\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&+\int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4+i_5=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\nabla^{i_5}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&+\int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat\chi},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'\\
&+\int_{u_\infty}^u \frac{a}{|u'|^2}\sum_{i_1+i_2+i_3+i_4+1=i}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'.
\end{aligned}
$$

By raising the above to the second power and integrating in $\underline{u}$ we get

$$\int_0^{\underline{u}} \|(a^{\frac{1}{2}}\nabla)^i \kappa\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}^2 \, \mathrm{d}\underline{u}'$$

$$\lesssim \|(a^{\frac{1}{2}}\nabla)^i \kappa\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty})}^2 + \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i G\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$= \|(a^{\frac{1}{2}}\nabla)^i \kappa\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty})}^2 + \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i G\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}\underline{u}' \right) \mathrm{d}u'$$

$$\lesssim \|(a^{\frac{1}{2}}\nabla)^i \kappa\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty})}^2$$

$$+ \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat\chi)\nabla^{i_4}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat\chi})\nabla^{i_4+1}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\Upsilon,\alpha_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\Upsilon,\alpha_F)\nabla^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \frac{a\,\mathrm{d}\underline{u}'}{|u|} \int_{u_\infty}^u \frac{a\,\mathrm{d}u'}{|u'|^2} \sum_i \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat\chi},\mathrm{tr}\underline\chi,\hat\chi)\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2$$

$$+ \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+i_5=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\nabla^{i_5}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat\chi,\widetilde{\mathrm{tr}\underline\chi})\nabla^{i_4}\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+1=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline\chi\nabla^{i_4}\kappa\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$:= T_1 + \cdots + T_9.$$

We bound each term separately.

- There holds

$$T_1 = \|(a^{\frac{1}{2}}\nabla)^i \kappa\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty})}^2 \lesssim \|(a^{\frac{1}{2}}\nabla)^i \tilde\beta\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty})}^2 + \frac{1}{a}\sum_{i\leq 11}\|(a^{\frac{1}{2}}\nabla)^i \omega\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty})}^2$$

$$\lesssim \mathcal{R}[\tilde\beta]^2 + \left(\mathcal{I}^{(0)}\right)^2 + 1 \lesssim \mathcal{R}[\tilde\beta]^2 + 1.$$

- There holds

$$\int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{\chi}) \nabla^{i_4} \Psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$\lesssim \frac{a \cdot O^4}{|u'|^2} + \int_0^{\underline{u}} a \|\left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \cdot (a^{\frac{1}{2}} \nabla)^{10} \Psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$\lesssim \frac{a \cdot O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \mathcal{R}[\Psi]^2.$$

Therefore

$$T_2 \lesssim \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \cdot \left(\frac{a \cdot O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \mathcal{R}[\Psi]^2 \right) \, du' \lesssim \frac{a^3 \cdot O^4}{|u|^4} + \frac{a^3 \cdot O^2 \cdot R^2}{|u|^4} \lesssim 1. \tag{5.128}$$

- For the third term

$$\int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \underline{\hat{\chi}}) \nabla^{i_4+1} \psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

we bound separately the cases where $i_4 = 10$ and where not. In the former case, we need to distinguish the subcases where the ψ-term in $\nabla^{i_4+1}\psi$ belongs to those components that are bounded in the $\|\cdot\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$-norm (which in this case are ω, η and $\underline{\eta}$) and those bounded in the $\|\cdot\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})}$-norm (here this component is $\underline{\omega}$) in the bootstrap assumption (5.115).

  - When $i_4 < 10$ we can bound T_3 by 1, using Proposition 5.4.1.
  - When $i_4 = 10$ we have

  $$\int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \cdot \frac{|u'|^2}{a} \|(a^{\frac{1}{2}})^{10} \left(\frac{a^{\frac{1}{2}}}{|u'|} \psi, \frac{a^{\frac{1}{2}}}{|u'|} \underline{\hat{\chi}} \right) \nabla^{11} \psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

  $$= \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{O^2}{|u'|^2} \|(a^{\frac{1}{2}})^{10} \nabla^{11} \psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

  $$= \int_0^{\underline{u}} \frac{O^2}{|u|} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{10} \nabla^{11} \psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

  $$\lesssim \frac{O^2}{|u|} \cdot O_{11}^2 + \frac{a}{|u|} \cdot \frac{O^2}{|u|} \cdot O_{11}^2.$$

  In the last inequality we have distinguished the cases according to the exact form of ψ in (5.115), so that we are able to bound it by O_{11}.

- There holds

$$T_4 = \int_0^{\underline{u}} \frac{a\,\mathrm{d}\underline{u}'}{|u|} \int_{u_\infty}^{u} \frac{a\,\mathrm{d}u'}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\Upsilon,\alpha_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\lesssim \frac{a^2 \cdot O^4}{|u|^4} + \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a^2}{|u'|^2} \|(a^{\frac{1}{2}})^{10}\Upsilon\nabla^{11}\left(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}'$$

$$\lesssim \frac{a^2 \cdot O^4}{|u|^4} + \frac{a^3 \cdot O^2 \cdot \mathcal{F}^2[\alpha_F]}{|u|^4} + \frac{a^2 \cdot O^2 \cdot \mathcal{F}^2[\rho_F,\sigma_F]}{|u|^4}.$$

Here we have calculated explicitly all the possible pairs that appear in the schematic $\Upsilon\nabla(\Upsilon,\alpha_F)$ and those are $\rho_F\nabla\rho_F, \sigma_F\nabla\sigma_F$ and $\underline{\alpha}_F\nabla\alpha_F$.

- Similarly, there holds

$$T_5 = \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}'$$

$$\lesssim \frac{a^2 \cdot O^4}{|u|^4} + \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a^2}{|u'|^2} \|(a^{\frac{1}{2}})^{10}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}}\right)\nabla^{11}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}'$$

$$\lesssim \frac{a^2 \cdot O^4}{|u|^4} + \frac{a^3 \cdot O^2 \cdot \underline{\mathcal{F}}^2[\underline{\alpha}_F]}{|u|^4} + \frac{a^2 \cdot O^2 \cdot \mathcal{F}^2[\rho_F,\sigma_F]}{|u|^4}.$$

Here we have calculated explicitly all the possible pairs that appear in the schematic $(\alpha_F,\Upsilon)\nabla\Upsilon$ and those are $\alpha_F\nabla\underline{\alpha}_F, \rho_F\nabla\rho_F$ and $\sigma_F\nabla\sigma_F$.

- There holds

$$T_6$$
$$= \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \sum_{i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\lesssim \int_0^{\underline{u}} \frac{a}{|u|} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \cdot \frac{O^6}{a}\,\mathrm{d}u'\,\mathrm{d}\underline{u}' = \frac{a \cdot O^6}{|u|^2}.$$

Here we have estimated

$$\int_{u_\infty}^{u} \frac{a}{|u'|^2} \sum_{i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'$$

$$= \int_{u_\infty}^{u} |u'|^2 \cdot \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a(\psi,\underline{\hat{\chi}},\mathrm{tr}\underline{\chi},\hat{\chi})}{|u'|^2}\right)\nabla^{i_4}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}}\right)\nabla^{i_5}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\lesssim \int_{u_\infty}^{u} \frac{a}{|u'|^2} \cdot \frac{|u'|^4}{a^2} \cdot a \cdot \frac{O^6}{|u'|^4}\,\mathrm{d}u' \lesssim \frac{O^6}{u}.$$

- There holds

$$T_7 = \int_0^{\underline{u}} \frac{a \, d\underline{u}'}{|u|} \int_{u_\infty}^u \frac{a \, du'}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+i_5=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi \nabla^{i_5} \psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}^2$$

$$\lesssim \frac{a}{|u|} \cdot \frac{a \cdot O^6}{|u|^5}.$$

- The final two terms can be absorbed to the left by Grönwall's inequality, by virtue of schematically containing the term $\nabla^{i_4} \kappa$.

From the following div − curl system

$$\mathrm{div} \, \nabla\omega = \mathrm{div} \, \kappa + \frac{1}{2}\nabla\tilde{\beta},$$

$$\mathrm{curl} \, \nabla\omega = 0,$$

$$\mathrm{div} \, \nabla\omega^\dagger = \mathrm{curl} \, \kappa + \frac{1}{2}\mathrm{curl} \, \tilde{\beta},$$

$$\mathrm{curl} \, \nabla\omega^\dagger = 0$$

and Proposition 5.6.1 we have that

$$\|a^5 \nabla^{11}(\omega, \omega^\dagger)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \sum_{j=0}^{10} \|(a^{\frac{1}{2}}\nabla)^j \kappa\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i \tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \frac{1}{a^{\frac{1}{2}}} \sum_{j=0}^{10} \|(a^{\frac{1}{2}}\nabla)^j (\omega, \omega^\dagger)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Passing to $\|\cdot\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$-norms, we arrive at

$$\|(a^{\frac{1}{2}})^{10}\nabla^{11}(\omega, \omega^\dagger)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}[\tilde{\beta}] + 1. \tag{5.129}$$

$\square$

We move on to top order estimates for η.

Proposition 5.6.5. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69)-(5.115), there holds*

$$\|a^5\nabla^{11}\eta\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|a^5\nabla^{11}\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})} \lesssim \mathcal{R}[\alpha] + \mathcal{R}[\rho,\sigma] + \underline{\mathcal{R}}[\rho,\sigma] + \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1.$$

Proof. Introduce the quantity

$$\mu = -\text{div } \eta - \rho.$$

Our goal is to derive a ∇_4-transport equation for μ. Recall the commutation formula, for a $1-$form U

$$[\nabla_4, \text{div}] U = -\frac{1}{2}\text{tr}\chi \, \text{div } U - \hat{\chi} \cdot \nabla U$$
$$- \tilde{\beta} \cdot U + \frac{1}{2}(\eta + \underline{\eta}) \cdot \nabla_4 U - \underline{\eta} \cdot \hat{\underline{\chi}} \cdot U - \frac{1}{2}\text{tr}\chi \underline{\eta} \cdot U + \text{tr}\chi \underline{\eta} \cdot U.$$

In particular

$$\nabla_4 \text{div } \eta = \text{div } (\nabla_4 \eta) + [\nabla_4, \text{div}] \eta$$
$$= \text{div } (\hat{\chi}, \text{tr}\chi \, \gamma) \cdot (\eta - \underline{\eta}) + (\hat{\chi}, \text{tr}\chi \, \gamma) \cdot \text{div } (\eta - \underline{\eta})$$
$$- \text{div } \tilde{\beta} - \frac{1}{2}\text{tr}\chi \, \text{div } \eta - \hat{\chi} \cdot \nabla\eta - \tilde{\beta} \cdot \eta + \frac{1}{2}(\eta + \underline{\eta}) \cdot \nabla_4 \eta$$
$$- \underline{\eta} \cdot \hat{\underline{\chi}} \cdot \eta - \frac{1}{2}\text{tr}\chi \underline{\eta} \cdot \eta + \text{tr}\chi \underline{\eta} \cdot \eta.$$

Schematically, this rewrites as

$$\nabla_4(\text{div}\,\eta) + \text{div } \tilde{\beta} = (\psi, \hat{\chi}) \cdot \nabla(\eta, \underline{\eta}) + \psi \nabla(\psi, \hat{\chi}) + \psi \cdot \Psi + \psi \cdot (\psi, \hat{\chi}, \hat{\underline{\chi}}) \cdot \psi + \psi \cdot \alpha_F \cdot \Upsilon. \quad (5.130)$$

Moreover,

$$\nabla_4 \rho - \text{div } \tilde{\beta} = (\psi, \hat{\underline{\chi}}) \cdot (\alpha, \Psi) + \alpha_F \cdot \nabla(\rho_F, \sigma_F)$$
$$+ (\rho_F, \sigma_F) \cdot \nabla(\alpha_F, \rho_F, \sigma_F) + (\psi, \hat{\chi}, \text{tr}\underline{\chi}, \hat{\chi}) \cdot (\alpha_F, \Upsilon) \cdot (\alpha_F, \Upsilon).$$

Consequently, μ satisfies the following transport equation:

$$\nabla_4 \mu = (\psi, \hat{\chi}) \cdot \nabla(\eta, \underline{\eta}) + \psi \nabla(\psi, \hat{\chi}) + \alpha_F \cdot \nabla(\rho_F, \sigma_F)$$
$$+ (\rho_F, \sigma_F) \cdot \nabla(\alpha_F, \rho_F, \sigma_F) + (\psi, \hat{\underline{\chi}}) \cdot (\alpha, \Psi) \quad (5.131)$$
$$+ \psi \cdot (\psi, \hat{\chi}, \hat{\underline{\chi}}) \cdot \psi + (\psi, \hat{\chi}, \text{tr}\underline{\chi}, \hat{\chi}) \cdot (\alpha_F, \Upsilon) \cdot (\alpha_F, \Upsilon).$$

Commuting (5.131) with $i \leq 10$ angular derivatives we arrive at

$$\nabla_4 \nabla^i \mu$$
$$= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}) \nabla^{i_4+1} (\eta, \underline{\eta}) + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4+1} (\psi, \hat{\chi})$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \alpha_F \nabla^{i_4+1} (\rho_F, \sigma_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\rho_F, \sigma_F) \nabla^{i_4+1} (\alpha_F, \rho_F, \sigma_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \underline{\hat{\chi}}) \nabla^{i_4} (\alpha, \Psi)$$
$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\psi, \hat{\chi}, \underline{\hat{\chi}}) \nabla^{i_5} \psi$$
$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}, \mathrm{tr}\underline{\chi}, \hat{\chi}) \nabla^{i_4} (\alpha_F, \Upsilon) \nabla^{i_5} (\alpha_F, \Upsilon)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}) \nabla^{i_4} \mu.$$

We now pass to scale-invariant norms. Noticing that $(\nabla^i \mu)\big|_{\underline{H}_0} = 0$, we can apply Proposition 5.3.6 to obtain

$$\|(a^{\frac{1}{2}}\nabla)^i \mu\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \left\| (a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4+1}(\eta,\underline{\eta}) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4+1}(\psi,\hat{\chi})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\alpha_F\nabla^{i_4+1}(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho_F,\sigma_F)\nabla^{i_4+1}(\alpha_F,\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}})\nabla^{i_4}(\alpha,\Psi)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4+i_5=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(\psi,\hat{\chi},\underline{\hat{\chi}})\nabla^{i_4}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_i \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi},\mathrm{tr}\underline{\chi},\underline{\hat{\chi}})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}(\alpha_F,\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\mu\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}'$$

$$= I_1 + \cdots + I_8.$$

- We have

$$I_1 \le \int_0^{\underline{u}} \left\| (a^{\frac{1}{2}})^{i+1} \left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \nabla^{i+1}(\eta,\underline{\eta}) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}' + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|}$$

$$\lesssim \frac{a^{\frac{1}{2}} \cdot O}{|u|} \cdot \|a^5 \nabla^{11}(\eta,\underline{\eta})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \le \frac{O}{a^{\frac{1}{2}}} \cdot O_{11}[\eta,\underline{\eta}] + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \lesssim 1.$$

- We have

$$I_2 \lesssim \int_0^{\underline{u}} \left\| (a^{\frac{1}{2}})^{11}\psi\nabla^{11} \left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}' + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \lesssim \frac{a^{\frac{1}{2}} \cdot O \cdot O_{11}}{|u|} + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|}.$$
$$\tag{5.132}$$

- We have

$$I_3 \lesssim \int_0^{\underline{u}} \left\| \left(\frac{\alpha_F}{a^{\frac{1}{2}}} \right)(a^{\frac{1}{2}}\nabla)^{11}\Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}' + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \lesssim \frac{a^{\frac{1}{2}} \cdot O \cdot \mathcal{F}[\Upsilon]}{|u|} + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|}.$$
$$\tag{5.133}$$

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- Similarly, we have

$$I_4 \lesssim \int_0^{\underline{u}} \left\| \Upsilon (a^{\frac{1}{2}} \nabla)^{11} \left(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}' + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} \tag{5.134}$$

$$\lesssim \frac{a^{\frac{1}{2}} \cdot O \cdot \mathcal{F}[\alpha_F]}{|u|} + \frac{a^{\frac{1}{2}} \cdot O^2}{|u|}.$$

- There holds

$$I_5 \lesssim \int_0^{\underline{u}} \frac{|u|}{a^{\frac{1}{2}}} \cdot \left\| \left(\frac{a^{\frac{1}{2}}}{u} \psi, \frac{a^{\frac{1}{2}}}{u} \hat{\underline{\chi}} \right) \cdot (a^{\frac{1}{2}} \nabla)^{10}(\alpha, \Psi) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}' + 1$$

$$\lesssim \int_0^{\underline{u}} \frac{|u|}{a^{\frac{1}{2}}} \cdot \frac{O_\infty[\hat{\underline{\chi}}]}{|u|} \cdot \| (a^{\frac{1}{2}} \nabla)^{10}(\alpha, \Psi) \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \mathrm{d}\underline{u}' + 1 \lesssim \mathcal{R}[\alpha] + 1,$$

  since $O_\infty[\hat{\underline{\chi}}] \lesssim 1$ by Proposition 5.4.3.

- There holds
$$I_6 \lesssim \frac{O^3}{|u| \cdot a^{\frac{1}{2}}}. \tag{5.135}$$

- There holds
$$I_7 \lesssim \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1, \tag{5.136}$$

  just as in the term J_6 in the proof of Proposition 5.5.2.

- Finally, the term I_8, after expanding $\mu = -\mathrm{div}\, \eta - \rho$, can be controlled by $I_1 + I_5$.

Consequently,
$$\| (a^{\frac{1}{2}} \nabla)^i \mu \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \mathcal{R}[\alpha] + \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1. \tag{5.137}$$

Now observe the div–curl system (the second equation is given schematically):

$$\mathrm{div}\, \eta = -\mu - \rho,$$

$$\mathrm{curl}\, \eta = \sigma + \hat{\underline{\chi}} \wedge \hat{\chi} + \Upsilon \cdot \Upsilon.$$

So that, applying Proposition 5.6.1 , we have

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$$\|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \sum_{i \leq 10} \left(\|(a^{\frac{1}{2}}\nabla)^i \mu\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i (\rho, \sigma)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right.$$

$$\left. + |u| \cdot \left\|(a^{\frac{1}{2}}\nabla)^i \left(\frac{a^{\frac{1}{2}}}{u} \underline{\hat{\chi}} \cdot \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i (\Upsilon \cdot \Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right) \qquad (5.138)$$

$$+ \frac{1}{a^{\frac{1}{2}}} \sum_{i \leq 10} \|(a^{\frac{1}{2}}\nabla)^i \eta\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Integrating along the $\underline{u}$-direction and raising to the second power, we arrive at

$$\|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_u)} \lesssim \mathcal{R}[\alpha] + \mathcal{R}[\rho, \sigma] + \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1. \qquad (5.139)$$

In a similar way, using (5.138), we get

$$\|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}})} \lesssim \mathcal{R}[\alpha] + \underline{\mathcal{R}}[\rho, \sigma] + \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1.$$

$$\square$$

We move on to estimates for $\underline{\eta}$.

Proposition 5.6.6. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69)-(5.115), there holds*

$$\frac{a}{|u|} \|a^5 \nabla^{11} \underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_u)} \lesssim \mathcal{R}[\alpha] + \underline{\mathcal{R}}[\rho, \sigma] + \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1.$$

Proof. Introduce $\underline{\mu}$ defined by

$$\underline{\mu} = -\mathrm{div}\, \underline{\eta} - \rho.$$

We then have the Hodge system for $\underline{\eta}$

$$\mathrm{div}\, \underline{\eta} = -\underline{\mu} - \rho,$$

$$\mathrm{curl}\, \underline{\eta} = -\sigma - \frac{1}{2}\underline{\hat{\chi}} \wedge \hat{\chi}.$$

For a $1-$form U_b we have

$$[\nabla_3, \mathrm{div}]U = -\frac{1}{2}\mathrm{tr}\underline{\chi}\, \mathrm{div}U - \underline{\hat{\chi}} \cdot \nabla U - \tilde{\underline{\beta}} \cdot U + \frac{1}{2}(\eta + \underline{\eta})\nabla_3 U - \eta \cdot \underline{\hat{\chi}} \cdot U - \frac{1}{2}\mathrm{tr}\underline{\chi}\eta \cdot U + \mathrm{tr}\underline{\chi}\eta \cdot U.$$

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Consequently,

$$
\begin{aligned}
\nabla_3 \operatorname{div} \underline{\eta} =& \operatorname{div}\,(\nabla_3 \underline{\eta}) + [\nabla_3, \operatorname{div}]\underline{\eta} \\
=& \operatorname{div}\,\left(-\hat{\underline{\chi}} \cdot (\underline{\eta} - \eta) - \frac{1}{2}\operatorname{tr}\underline{\chi} \cdot (\underline{\eta} - \eta) + \tilde{\underline{\beta}} - \underline{\alpha}_F \cdot (\rho_F, \sigma_F)\right) + [\nabla_3, \operatorname{div}\,]\underline{\eta} \\
=& -(\operatorname{div}\,\hat{\underline{\chi}}) \cdot (\underline{\eta} - \eta) - \hat{\underline{\chi}} \cdot \operatorname{div}\,(\underline{\eta} - \eta) - \frac{1}{2}\operatorname{tr}\underline{\chi}\operatorname{div}\underline{\eta} + \frac{1}{2}\operatorname{tr}\underline{\chi}\operatorname{div}\eta \\
& - \frac{1}{2}(\underline{\eta} - \eta)\operatorname{div}\,(\widetilde{\operatorname{tr}\underline{\chi}}) + \operatorname{div}\,\tilde{\underline{\beta}} - \underline{\alpha}_F\operatorname{div}\,(\rho_F, \sigma_F) - (\rho_F, \sigma_F)\operatorname{div}\,\underline{\alpha}_F \\
& - \frac{1}{2}\operatorname{tr}\underline{\chi}\operatorname{div}\,\underline{\eta} - \hat{\underline{\chi}} \cdot \nabla\underline{\eta} - \tilde{\underline{\beta}} \cdot \eta + \frac{1}{2}(\eta + \underline{\eta}) \cdot \left(\psi \cdot (\hat{\underline{\chi}}, \operatorname{tr}\underline{\chi}) + \tilde{\underline{\beta}} + \Upsilon \cdot \Upsilon\right) \\
& - \eta \cdot \hat{\underline{\chi}} \cdot \underline{\eta} + \frac{1}{2}tr\underline{\chi} \cdot \eta \cdot \underline{\eta}.
\end{aligned}
$$

We thus have the semi–schematic identity

$$
\begin{aligned}
\nabla_3 \operatorname{div}\,\underline{\eta} + \operatorname{tr}\underline{\chi}\operatorname{div}\,\underline{\eta} - \operatorname{div}\,\tilde{\underline{\beta}} =& \psi \cdot \operatorname{div}\,\hat{\underline{\chi}} + \hat{\underline{\chi}}\nabla(\eta, \underline{\eta}) + \operatorname{tr}\underline{\chi}\nabla\eta \\
& + \psi\operatorname{div}\,(\widetilde{\operatorname{tr}\underline{\chi}}) + \underline{\alpha}_F\nabla(\rho_F, \sigma_F) \\
& + (\rho_F, \sigma_F)\nabla\underline{\alpha}_F + \psi \cdot \tilde{\underline{\beta}} \\
& + \psi \cdot (\hat{\chi}, \hat{\underline{\chi}}, \operatorname{tr}\underline{\chi}) \cdot \psi + \psi \cdot \Upsilon \cdot \Upsilon.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\nabla_3\rho + \operatorname{tr}\underline{\chi}\rho + \operatorname{div}\,\tilde{\underline{\beta}} =& -\frac{1}{2}\operatorname{tr}\underline{\chi} \cdot \rho + \hat{\chi} \cdot \underline{\alpha} + \psi \cdot \tilde{\underline{\beta}} + (\rho_F, \sigma_F)\nabla\underline{\alpha}_F \\
& + \underline{\alpha}_F\nabla(\rho_F, \sigma_F) + (\psi, \operatorname{tr}\underline{\chi}) \cdot \Upsilon \cdot \Upsilon + (\psi, \hat{\underline{\chi}}) \cdot (\Upsilon, \alpha_F) \cdot \Upsilon.
\end{aligned}
$$

Combining the above two equations, $\underline{\mu}$ satisfies the following transport equation:

$$
\begin{aligned}
& \nabla_3\underline{\mu} + \operatorname{tr}\underline{\chi}\,\underline{\mu} \\
=& \psi \cdot \nabla\hat{\underline{\chi}} + \hat{\underline{\chi}}\nabla(\eta, \underline{\eta}) + \operatorname{tr}\underline{\chi}\nabla\eta + \psi\nabla(\widetilde{\operatorname{tr}\underline{\chi}}) + \underline{\alpha}_F\nabla(\rho_F, \sigma_F) \\
& + (\rho_F, \sigma_F)\nabla\underline{\alpha}_F + \psi \cdot \tilde{\underline{\beta}} + \operatorname{tr}\underline{\chi} \cdot \rho + \psi \cdot (\hat{\chi}, \hat{\underline{\chi}}, \operatorname{tr}\underline{\chi}) \cdot \psi \\
& + (\psi, \operatorname{tr}\underline{\chi}) \cdot \Upsilon \cdot \Upsilon + (\psi, \hat{\underline{\chi}}) \cdot (\Upsilon, \alpha_F) \cdot \Upsilon.
\end{aligned} \tag{5.140}
$$

By commuting (5.140) with $i \leq 10$ angular derivatives, we arrive at

$$\nabla_3 \nabla^i \underline{\mu} + \frac{i+2}{2} \mathrm{tr}\underline{\chi}\,\underline{\mu}$$

$$= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4+1}\underline{\hat{\chi}} + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\hat{\chi}}\nabla^{i_4+1}(\eta,\underline{\eta})$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4+1}\eta + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4+1}\widetilde{\mathrm{tr}\underline{\chi}}$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\alpha_F\nabla^{i_4+1}(\rho_F,\sigma_F)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho_F,\sigma_F)\nabla^{i_4+1}\underline{\alpha}_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\underline{\tilde{\beta}} + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\rho$$

$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(\hat{\chi},\underline{\hat{\chi}},\mathrm{tr}\underline{\chi})\nabla^{i_5}\psi$$

$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\mathrm{tr}\underline{\chi})\nabla^{i_4}\Upsilon\nabla^{i_5}\Upsilon$$

$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}})\nabla^{i_4}(\Upsilon,\alpha_F)\nabla^{i_5}\Upsilon + \sum_{i_1+i_2+1=i} \nabla^{i_1+1}\mathrm{tr}\underline{\chi}\nabla^{i_2}\underline{\mu}$$

$$+ \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\underline{\mu} + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\underline{\mu}$$

$$:= G.$$

By Proposition 5.3.7 we can bound

$$|u|^{i+1}\|\nabla^i\underline{\mu}\|_{L^2(S_{u,\underline{u}})} \lesssim |u_\infty|^{i+1}\|\nabla^i\underline{\mu}\|_{L^2(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u |u'|^{i+1}\|G\|_{L^2(S_{u',\underline{u}})}\,\mathrm{d}u'.$$

We have $s_2(\nabla^i\underline{\mu}) = \frac{i+2}{2}$ and $s_2(G) = \frac{i+4}{2}$. By passing to scale-invariant norms, we

have

$$\frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \frac{a}{|u_\infty|}\|(a^{\frac{1}{2}}\nabla)^i\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} + \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|(a^{\frac{1}{2}})^i G\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\lesssim \frac{a}{|u_\infty|}\|(a^{\frac{1}{2}}\nabla)^i\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})}$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4+1}\hat{\underline{\chi}}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\hat{\underline{\chi}}\nabla^{i_4+1}(\eta,\underline{\eta})\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4+1}\eta\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4+1}\widetilde{\mathrm{tr}\underline{\chi}}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\alpha}_F\nabla^{i_4+1}(\rho_F,\sigma_F)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho_F,\sigma_F)\nabla^{i_4+1}\underline{\alpha}_F\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\underline{\tilde{\beta}}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\rho\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}(\hat{\chi},\hat{\underline{\chi}},\mathrm{tr}\underline{\chi})\nabla^{i_5}\psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\mathrm{tr}\underline{\chi})\nabla^{i_4}\Upsilon\nabla^{i_5}\Upsilon\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}(\Upsilon,\alpha_F)\nabla^{i_5}\Upsilon\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+1=i} \nabla^{i_1+1}\mathrm{tr}\underline{\chi}\nabla^{i_2}\underline{\mu}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\underline{\mu}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\underline{\mu}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$:=T_1 + \cdots + T_{15}.$$

We bound T_1 to T_{15} individually.

- Given that $\underline{\mu} = -\operatorname{div}\underline{\eta} - \rho$ and the fact that $\mathcal{I}^{(0)}$ bounds up to 14 derivatives for $\underline{\eta}$ and ρ, there holds

$$T_1 = \frac{a}{|u_\infty|}\|(a^{\frac{1}{2}}\nabla)^i\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}})} \lesssim \mathcal{I}^{(0)} \lesssim 1. \tag{5.141}$$

- There holds

$$T_2 = \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^i\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4+1}\hat{\underline{\chi}}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|a^5\cdot\psi\cdot\nabla^{11}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^2}\left\|(a^{\frac{1}{2}})^i\sum_{\substack{i_1+i_2+i_3+i_4=i,\\ i_4\leq 9}}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4+1}\left(\frac{a^{\frac{1}{2}}}{|u'|}\hat{\underline{\chi}}\right)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|a^5\cdot\psi\cdot\nabla^{11}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^2}\cdot\frac{1}{a^{\frac{1}{2}}}\sum_{i_1+i_2=i\leq 10}\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2+2}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u'$$

$$\lesssim 1 + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\cdot\frac{a^{\frac{1}{2}}\cdot O}{|u'|}\cdot\|a^5\nabla^{11}\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u' \lesssim 1 + \frac{O}{a^{\frac{1}{2}}}\cdot O_{11}[\hat{\underline{\chi}}]\,\mathrm{d}u' \lesssim 1.$$

Here we used the fact that $\frac{a^{\frac{1}{2}}\cdot O}{|u'|} \lesssim \frac{O}{a^{\frac{1}{2}}}$, given that $|u'| \geq a/4$, as well as Proposition 5.6.8.

- There holds

$$T_3 \lesssim \frac{a\cdot O^2}{|u|^2} + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^2}\cdot\frac{O}{|u'|}\|a^5\nabla^{11}(\eta,\underline{\eta})\,\mathrm{d}u'\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}$$

$$\lesssim \frac{a\cdot O}{|u|^{\frac{3}{2}}}\cdot\|a^5\nabla^{11}\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}\cdot O}{|u'|^3}\|a^5\nabla^{11}\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u' + 1$$

$$\lesssim \int_{u_\infty}^u \frac{a^{\frac{3}{2}}\cdot O}{|u'|^3}\|a^5\nabla^{11}\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\mathrm{d}u' + 1.$$

$$\tag{5.142}$$

Here we have used Proposition 5.6.5 which bounds $\|a^5\nabla^{11}\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})}$ by $R + F + 1$ using the bootstrap bounds (5.69).

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- There holds

$$
\begin{aligned}
T_4 &\lesssim \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} + \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \cdot \frac{|u'|^2}{a} \cdot \frac{O_\infty[\mathrm{tr}\underline{\chi}]}{|u'|} \|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \\
&\lesssim \int_{u_\infty}^{u} \frac{a \cdot O_\infty[\mathrm{tr}\underline{\chi}]}{|u'|^2} \|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' + 1 \\
&\lesssim \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' + 1 \\
&\lesssim \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}} \|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + 1.
\end{aligned}
\tag{5.143}
$$

Here we have used the fact that $O_\infty[\mathrm{tr}\underline{\chi}] \lesssim 1$, which was shown previously.

- There holds

$$
\begin{aligned}
T_5 &\lesssim \frac{a^{\frac{1}{2}} \cdot O^2}{|u|^2} + \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \cdot \frac{|u'|}{a} \cdot \frac{O}{|u'|} \|a^5 \nabla^{11} \widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' \\
&\lesssim \frac{a^{\frac{1}{2}} \cdot O^2}{|u|^2} + \frac{a^{\frac{1}{2}} \cdot O}{|u|^{\frac{3}{2}}} \|a^5 \nabla^{11} \widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \\
&\lesssim \frac{O}{|u|^{\frac{1}{2}} \cdot a^{\frac{1}{2}}} \cdot O_{11}[\widetilde{\mathrm{tr}\underline{\chi}}] + 1.
\end{aligned}
\tag{5.144}
$$

- There holds

$$
\begin{aligned}
T_6 &\lesssim \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \cdot \frac{1}{a^{\frac{1}{2}}} \cdot \frac{O^2}{|u'|} \, \mathrm{d}u' + \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \cdot \frac{O}{|u'|} \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' \\
&\lesssim \frac{a^{\frac{3}{2}} \cdot O^2}{|u|^3} + \left(\int_{u_\infty}^{u} \frac{a}{|u'|^2} \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' \right)^{\frac{1}{2}} \left(\int_{u_\infty}^{u} \frac{a^3 \cdot O^2}{|u'|^6} \, \mathrm{d}u' \right)^{\frac{1}{2}} \\
&\lesssim 1 + \frac{a^{\frac{3}{2}} \cdot O}{|u|^{\frac{5}{2}}} \cdot \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \lesssim 1 + \frac{a^2 \cdot O}{|u|^{\frac{5}{2}}} \cdot \underline{\mathcal{F}}[\rho_F, \sigma_F] \lesssim 1.
\end{aligned}
$$

- Similarly, there holds

$$
T_7 \lesssim \frac{a^{\frac{3}{2}} \cdot O^2}{|u|^3} + \frac{a^{\frac{3}{2}} \cdot O}{|u|^{\frac{5}{2}}} \cdot \|a^5 \nabla^{11} \underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \lesssim \frac{a^{\frac{3}{2}} \cdot O^2}{|u|^3} + \frac{a^{\frac{3}{2}} \cdot O}{|u|^{\frac{5}{2}}} \cdot \underline{\mathcal{F}}[\underline{\alpha}_F] \lesssim 1.
\tag{5.145}
$$

- There holds

$$T_8 \lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{O}{|u'|} \cdot \sum_{k \leq 10} \|(a^{\frac{1}{2}}\nabla)^k \underline{\tilde{\beta}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$

$$\lesssim \left(\sum_{k \leq 10} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}}\nabla)^k \underline{\tilde{\beta}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du' \right)^{\frac{1}{2}} \left(\int_{u_\infty}^u \frac{a^3 \cdot O^2}{|u'|^6} \, du' \right)^{\frac{1}{2}}$$

$$\lesssim \frac{a^{\frac{3}{2}} \cdot O}{|u|^{\frac{5}{2}}} \cdot \mathcal{R}[\underline{\tilde{\beta}}] \lesssim 1.$$

- There holds (this is the most marginal term)

$$T_9 \lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{O}{|u'|} \cdot \frac{|u'|^2}{a} \cdot \sum_{k \leq 10} \|(a^{\frac{1}{2}}\nabla)^k \rho\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$

$$\lesssim \left(\sum_{k \leq 10} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}}\nabla)^k \rho\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du' \right)^{\frac{1}{2}} \left(\int_{u_\infty}^u \frac{a \cdot O^2}{|u'|^2} \, du' \right)^{\frac{1}{2}}$$

$$\lesssim \frac{a^{\frac{1}{2}} \cdot O[\mathrm{tr}\underline{\chi}]}{|u|^{\frac{1}{2}}} \cdot \mathcal{R}[\rho] \lesssim \mathcal{R}[\rho].$$

Here we have used the fact that $O_\infty[\mathrm{tr}\underline{\chi}] \lesssim 1$.

- There holds

$$T_{11} + T_{12}$$

$$= \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \mathrm{tr}\underline{\chi})\nabla^{i_4}\Upsilon\nabla^{i_5}\Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi})\nabla^{i_4}(\Upsilon, \alpha_F)\nabla^{i_5}\Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$\lesssim \int_{u_\infty}^u \frac{a}{|u'|} \cdot \left\| (a^{\frac{1}{2}})^i \sum_i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a}{|u'|^2}(\psi, \mathrm{tr}\underline{\chi}) \right) \nabla^{i_4}\Upsilon\nabla^{i_5}\Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$+ \int_{u_\infty}^u \frac{a^3}{|u'|^3} \left\| (a^{\frac{1}{2}})^i \sum_i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \nabla^{i_4}\left(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}} \right) \nabla^{i_5}\Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$\lesssim \int_{u_\infty}^u \left(\frac{a}{|u'|} + \frac{a^3}{|u'|^3} \right) \cdot \frac{O^3}{|u'|^2} \, du' \lesssim \frac{a \cdot O^3}{|u|^2} \lesssim 1.$$

- The last three terms can be controlled by Grönwall's inequality. Indeed,

$$
\int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+1=i} \nabla^{i_1+1} \mathrm{tr}\underline{\chi} \nabla^{i_2} \underline{\mu} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
+ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \mathrm{tr}\chi \nabla^{i_4} \underline{\mu} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
+ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\chi}) \nabla^{i_4} \underline{\mu} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
\lesssim \int_{u_\infty}^{u} \frac{a}{|u'|^2} \left\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2=i} \nabla^{i_1} \left(\frac{a}{|u'|} \widetilde{\mathrm{tr}\underline{\chi}} \right) \nabla^{i_2} \underline{\mu} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
+ \int_{u_\infty}^{u} \frac{a}{|u'|} \cdot \left\| (a^{\frac{1}{2}})^i \sum_{i-1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \left(\frac{a}{|u'|^2} \mathrm{tr}\underline{\chi} \right) \nabla^{i_4} \underline{\mu} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
+ \int_{u_\infty}^{u} \frac{a^{\frac{3}{2}}}{|u'|^2} \cdot \left\| (a^{\frac{1}{2}})^i \sum_{i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{a^{\frac{1}{2}}}{|u'|} \psi, \frac{a^{\frac{1}{2}}}{|u'|} \hat{\underline{\chi}}, \frac{a^{\frac{1}{2}}}{|u'|} \widetilde{\mathrm{tr}\chi} \right) \nabla^{i_4} \underline{\mu} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
\lesssim \int_{u_\infty}^{u} \left(\frac{a \cdot O}{|u'|^3} + \frac{a \cdot O^2}{|u'|^3} + \frac{a^{\frac{3}{2}} \cdot O^2}{|u'|^3} \right) \cdot \left\| \sum_{i_1 \le i} (a^{\frac{1}{2}} \nabla)^{i_1} \underline{\mu} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'.
$$

When $i_1 = i$ the three terms in the parenthesis of the line above are integrable with respect to u and so Grönwall's inequality allows us to control the term. When $i_1 < i$, we can use the definition $\underline{\mu} = -\mathrm{div}\,\underline{\eta} - \rho$ and bound the terms by the already established estimates of the previous sections.

Consequently, there holds

$$
\frac{a}{|u|} \| (a^{\frac{1}{2}} \nabla)^i \underline{\mu} \|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}
$$

$$
\lesssim \int_{u_\infty}^{u} \frac{a^{\frac{3}{2}} \cdot O}{|u'|^3} \| a^5 \nabla^{11} \underline{\eta} \|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u' + \frac{a^{\frac{1}{2}}}{|u|^{\frac{1}{2}}} \| a^5 \nabla^{11} \eta \|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \underline{\mathcal{R}}[\rho] + 1. \tag{5.146}
$$

Now observe the $\mathrm{div} - \mathrm{curl}$ system (the second equation is given schematically):

$$
\mathrm{div}\,\underline{\eta} = -\underline{\mu} - \rho, \tag{5.147}
$$

$$
\mathrm{curl}\,\underline{\eta} = -\sigma - \hat{\underline{\chi}} \wedge \hat{\chi} + \Upsilon \cdot \Upsilon. \tag{5.148}
$$

Consequently,

$$\frac{a}{|u|}\|a^5\nabla^{11}\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \sum_{i\leq 10}\Big(\frac{a}{|u|}\|(a^{\frac{1}{2}}\nabla)^i\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i(\rho,\sigma)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$+ \|(a^{\frac{1}{2}}\nabla)^i(\underline{\hat{\chi}}\cdot\hat{\chi})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^i(\Upsilon\cdot\Upsilon)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\Big)$$

$$+ \frac{1}{a^{\frac{1}{2}}}\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

By raising the above to the second power, integrating along $\underline{u}$ and using (5.146) along with Grönwall's inequality we can get that

$$\frac{a}{|u|}\|a^5\nabla^{11}\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}[\alpha] + \underline{\mathcal{R}}[\rho,\sigma] + \mathcal{F}[\rho_F,\sigma_F] + 1. \tag{5.149}$$

$$\square$$

We now prove the highest order bounds for $\underline{\omega}$.

Proposition 5.6.7. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69)-(5.115), we have*

$$\|a^5\nabla^{11}\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})} \lesssim 1 + \underline{\mathcal{R}}[\tilde{\underline{\beta}}] + \mathcal{F}[\underline{\alpha}_F].$$

Proof. Define the auxiliary function $\underline{\omega}^\dagger$ by

$$\nabla_4\underline{\omega}^\dagger = \frac{1}{2}\sigma$$

and trivial initial data along $\underline{H}_0$. We then define $\underline{\kappa}$ by

$$\underline{\kappa} = -\nabla\underline{\omega} + {}^*\nabla\underline{\omega}^\dagger - \frac{1}{2}\tilde{\underline{\beta}}.$$

We need to obtain a transport equation for $\underline{\kappa}$. Notice that if we set $\langle\underline{\omega}\rangle = (\underline{\omega},\underline{\omega}^\dagger)$, then

$$\underline{\kappa} = {}^*\mathcal{D}_1\langle\underline{\omega}\rangle - \frac{1}{2}\tilde{\underline{\beta}}.$$

Recall the commutation formulae

$$[\nabla_4,\nabla]f = -\frac{1}{2}\mathrm{tr}\chi\nabla f - \hat{\chi}\cdot\nabla f + \frac{1}{2}(\eta+\underline{\eta})D_4 f,$$

$$[\nabla_4,{}^*\nabla]g = -\frac{1}{2}\mathrm{tr}\chi\,{}^*\nabla g + \hat{\chi}\cdot{}^*\nabla g + \frac{1}{2}({}^*\eta+{}^*\underline{\eta})D_4 g.$$

Thus, for a pair of scalars (f, g), there holds

$$[\nabla_4, {}^*\mathcal{D}_1](f, g) = -\frac{1}{2}\mathrm{tr}\chi\,{}^*\mathcal{D}_1(f, g) + \hat{\chi}\cdot(\nabla f + {}^*\nabla g) - \frac{1}{2}(\eta + \underline{\eta})\nabla_4 f + \frac{1}{2}({}^*\eta + {}^*\underline{\eta})\nabla_4 g. \tag{5.150}$$

Now recall that

$$\nabla_4\underline{\omega} = \frac{1}{2}\rho + 2\omega\underline{\omega} - \eta\cdot\underline{\eta} + \frac{1}{2}|\eta|^2 + \frac{1}{4}(\rho_F^2 + \sigma_F^2) := \frac{1}{2}\rho + F.$$

Therefore,

$$\nabla_4\,{}^*\mathcal{D}_1\langle\underline{\omega}\rangle + \frac{1}{2}\nabla\rho - \frac{1}{2}\,{}^*\nabla\sigma = \omega\nabla\underline{\omega} + \underline{\omega}\nabla\omega + \eta\nabla(\eta, \underline{\eta}) + \underline{\eta}\nabla\eta + (\mathrm{tr}\chi, \hat{\chi})\nabla(\omega, \underline{\omega}^\dagger)$$
$$+ (\rho_F, \sigma_F)\nabla(\rho_F, \sigma_F) + \psi\cdot(\rho, \sigma) + \psi\cdot\psi\cdot\psi + \psi\cdot\Upsilon\cdot\Upsilon. \tag{5.151}$$

Moreover,

$$\nabla_4\underline{\tilde{\beta}} + \mathrm{tr}\chi\underline{\tilde{\beta}} + \nabla\rho - {}^*\nabla\sigma = (\psi, \hat{\underline{\chi}})\cdot(\rho, \sigma, \underline{\tilde{\beta}}) + (\alpha_F, \Upsilon)\cdot\nabla\Upsilon + (\psi, \hat{\chi}, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\cdot(\alpha_F, \Upsilon)\cdot\Upsilon. \tag{5.152}$$

Consequently,

$$\nabla_4\underline{\kappa} = (\psi, \hat{\underline{\chi}})\cdot\Psi + (\alpha_F, \Upsilon)\cdot\nabla\Upsilon + (\psi, \hat{\chi})\cdot\nabla\psi + (\psi, \hat{\chi}, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\cdot(\alpha_F, \Upsilon)\cdot\Upsilon + \psi\cdot\psi\cdot\psi. \tag{5.153}$$

Commuting with $i \leq 10$ angular derivatives, we get

$$\begin{aligned}
\nabla_4\nabla^i\underline{\kappa} = &\sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\underline{\chi}})\nabla^{i_4}\Psi \\
&+ \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F, \Upsilon)\nabla^{i_4+1}\Upsilon \\
&+ \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi})\nabla^{i_4+1}\psi \\
&+ \sum_{i_1+i_2+i_3+i_4+i_5=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi}, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\nabla^{i_4}(\alpha_F, \Upsilon)\nabla^{i_5}\Upsilon \\
&+ \sum_{i_1+i_2+i_3+i_4+i_5=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\nabla^{i_5}\psi \\
&+ \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi})\nabla^{i_4}\underline{\kappa}.
\end{aligned}$$

Passing to scale-invariant norms, there holds

$$\|(a^{\frac{1}{2}}\nabla)^i\underline{\kappa}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}})\nabla^{i_4}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4+1}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4+i_5=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi},\underline{\hat{\chi}},\mathrm{tr}\underline{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}\Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4+i_5=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\psi\nabla^{i_4}\psi\nabla^{i_5}\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}\underline{\kappa}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$:= J_1 + \cdots + J_6.$$

We again estimate term by term.

- We have

$$\int_0^{\underline{u}} \sum_{i_1+i_2+i_3+i_4=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}})\nabla^{i_4}\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\lesssim \int_0^{\underline{u}} \frac{|u|}{a^{\frac{1}{2}}} \cdot \left\|\left(\frac{a^{\frac{1}{2}}}{|u|}\psi,\frac{a^{\frac{1}{2}}}{|u|}\underline{\hat{\chi}}\right)(a^{\frac{1}{2}}\nabla)^{10}\Psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \frac{|u|}{a^{\frac{1}{2}}} \cdot \sum_{\substack{i_1+i_2+i_3+i_4=i,\\ i_4<10}} \left\|(a^{\frac{1}{2}})^i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a^{\frac{1}{2}}}{|u|}\psi,\frac{a^{\frac{1}{2}}}{|u|}\underline{\hat{\chi}}\right)\nabla^{i_4}\Psi\right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \tag{5.154}$$

$$\lesssim \frac{O^2}{a^{\frac{1}{2}}} + \frac{O_\infty[\underline{\hat{\chi}}]}{a^{\frac{1}{2}}}\mathcal{R}[\Psi].$$

Consequently,

$$J_1 \lesssim \frac{O^2}{a^{\frac{1}{2}}} + \frac{O_\infty[\underline{\hat{\chi}}]\mathcal{R}[\Psi]}{a^{\frac{1}{2}}} \lesssim 1.$$

- We have

$$\|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \Upsilon) \nabla^{i_4+1} \Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}$$

$$= a^{\frac{1}{2}} \cdot \left\| (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \nabla^{i_4+1} \Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}$$

$$\lesssim \frac{O^2}{|u|} + \frac{a^{\frac{1}{2}}}{|u|} \cdot \left(O[\alpha_F] + \frac{O[\Upsilon]}{a^{\frac{1}{2}}} \right) \|a^5 \nabla^{11} \Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}$$

$$+ \frac{a^{\frac{1}{2}} \cdot O_\infty[\alpha_F]}{|u|} \|a^5 \nabla^{11} \underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}. \tag{5.155}$$

Consequently,

$$J_2 \lesssim \frac{a^{\frac{1}{2}} \cdot O^2}{|u|} + \mathcal{F}[\Upsilon] + \frac{a^{\frac{1}{2}} \cdot O_\infty[\alpha_F] \cdot \mathcal{F}[\Upsilon]}{|u|} + \frac{a^{\frac{1}{2}}}{|u|} \int_0^u \|a^5 \nabla^{11} \underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'.$$

- We have

$$\|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}) \nabla^{i_4+1} \psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}$$

$$= a^{\frac{1}{2}} \cdot \left\| (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \nabla^{i_4+1} \psi \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \tag{5.156}$$

Then, since all the ψ–terms apart from $\underline{\omega}, \underline{\omega}^\dagger$ can be bounded in the top order along the $\underline{u}$–direction, we have

$$J_3 \lesssim \frac{O^2}{|u|} + \frac{a^{\frac{1}{2}} \cdot O \cdot O_{11}}{|u|} + \frac{O \cdot O_{11}}{a^{\frac{1}{2}}} + \frac{a^{\frac{1}{2}}}{|u|} \int_0^u \|a^5 \nabla^{11} (\underline{\omega}, \underline{\omega}^\dagger)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}'.$$

- We have

$$\|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}, \underline{\hat{\chi}}, \mathrm{tr}\underline{\chi}) \nabla^{i_4} (\alpha_F, \Upsilon) \nabla^{i_5} \Upsilon\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}$$

$$= \frac{u^2}{a^{\frac{1}{2}}} \cdot \left\| (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{a}{u^2} \psi, \frac{a}{u^2} \hat{\chi}, \frac{a}{u^2} \underline{\hat{\chi}}, \frac{a}{u^2} \mathrm{tr}\underline{\chi} \right) \nabla^{i_4} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \nabla^{i_5} \Upsilon \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}$$

$$\lesssim \frac{u^2}{a^{\frac{1}{2}}} \cdot \frac{O^3}{u^2} \lesssim \frac{O^3}{a^{\frac{1}{2}}}. \tag{5.157}$$

Thus J_4 satisfies, upon integration, the same bound.

- There holds

$$\sum_{i_1+i_2+i_3+i_4+i_5=i} \|(a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi \nabla^{i_5} \psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \lesssim \frac{O^3}{|u|^2}.$$

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- The final term can be absorbed to the left by a Grönwall-type argument.

Overall,

$$\|(a^{\frac{1}{2}}\nabla)^i\underline{\kappa}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1 + \frac{a^{\frac{1}{2}}}{|u|}\int_0^{\underline{u}}\|a^5\nabla^{11}(\underline{\omega},\underline{\omega}^{\dagger})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,d\underline{u}'$$

$$+ \frac{a^{\frac{1}{2}}}{|u|}\int_0^{\underline{u}}\|a^5\nabla^{11}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}\,d\underline{u}'. \tag{5.158}$$

By the following div-curl system

$$\text{div }\nabla\underline{\omega} = -\text{div }\underline{\kappa} - \frac{1}{2}\text{div }\underline{\tilde{\beta}},$$

$$\text{curl }\nabla\underline{\omega} = 0,$$

$$\text{curl }\underline{\omega}^{\dagger} = \text{curl }\underline{\kappa} + \frac{1}{2}\text{curl }\underline{\tilde{\beta}},$$

$$\text{div }\nabla\underline{\omega}^{\dagger} = 0,$$

applying Proposition 5.6.1, we have

$$\|(a^{\frac{1}{2}})^{10}\nabla^{11}(\underline{\omega},\underline{\omega}^{\dagger})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \sum_{j=0}^{10}\left(\|(a^{\frac{1}{2}}\nabla)^j\underline{\kappa}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j\underline{\tilde{\beta}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\right)$$

$$+ \frac{1}{a^{\frac{1}{2}}}\sum_{j=0}^{10}\|(a^{\frac{1}{2}}\nabla)^j(\underline{\omega},\underline{\omega}^{\dagger})\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Using the bounds from (5.158), the right–hand–side term involving $\langle\underline{\omega}\rangle$ is absorbed to the left by Grönwall's inequality. Integrating in the $\underline{u}$–direction, the terms involving $\underline{\alpha}_F$ and $\underline{\tilde{\beta}}$ on the right–hand side can finally be controlled and we get

$$\|a^5\nabla^{11}(\underline{\omega},\underline{\omega}^{\dagger})\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}})} \lesssim 1 + \underline{\mathcal{R}}[\underline{\tilde{\beta}}] + \underline{\mathcal{F}}[\underline{\alpha}_F].$$

$$\square$$

Finally, we prove top order estimates for the remaining Ricci coefficients $\text{tr}\underline{\chi}, \underline{\hat{\chi}}$.

Proposition 5.6.8. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69)-(5.115), there holds*

$$\int_{u_\infty}^u \frac{a^2}{|u|^3}\|a^5\nabla^{11}tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,du' \lesssim \mathcal{R}[\rho] + \underline{\mathcal{R}}[\underline{\tilde{\beta}}] + \underline{\mathcal{F}}[\underline{\alpha}_F] + 1,$$

$$\int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|a^5\nabla^{11}\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,du' \lesssim 1.$$

Proof. We begin with the equation

$$\nabla_3 \widetilde{\text{tr}\underline{\chi}} + \text{tr}\underline{\chi}\widetilde{\text{tr}\underline{\chi}} = \frac{2}{|u|^2}(\Omega^{-1} - 1) + \widetilde{\text{tr}\chi}\widetilde{\text{tr}\underline{\chi}} + 2\underline{\omega}\text{tr}\underline{\chi} - |\underline{\hat{\chi}}|^2 - |\underline{\alpha}_F|^2.$$

Commuting this equation with i angular derivatives, we get

$$\nabla_3 \nabla^i \widetilde{\text{tr}\underline{\chi}} + \frac{i+2}{2}\text{tr}\underline{\chi}\widetilde{\text{tr}\underline{\chi}}$$

$$= \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{2}{|u|^2}(\Omega^{-1}-1) + \widetilde{\text{tr}\chi}\widetilde{\text{tr}\underline{\chi}} + 2\underline{\omega}\text{tr}\underline{\chi} - |\underline{\hat{\chi}}|^2 - |\underline{\alpha}_F|^2\right)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \underline{\hat{\chi}}, \widetilde{\text{tr}\underline{\chi}})\nabla^{i_4}\widetilde{\text{tr}\underline{\chi}}$$

$$+ \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\text{tr}\underline{\chi}\nabla^{i_4}\widetilde{\text{tr}\underline{\chi}} := \tilde{F}_i.$$

Rewriting in terms of scale-invariant norms,

$$\frac{a}{|u|}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\widetilde{\text{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{a}{|u_\infty|}\|\|(a^{\frac{1}{2}})^{i-1}\nabla^i\widetilde{\text{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,u})}$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3}\|(a^{\frac{1}{2}})^{i-1}\tilde{F}_i\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'$$

$$= \frac{a}{|u_\infty|}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\widetilde{\text{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,u})} + I_1 + I_2 + I_3,$$

where

$$\frac{a}{|u|}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\widetilde{\text{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \leq \frac{a}{|u_\infty|}\|\|(a^{\frac{1}{2}})^{i-1}\nabla^i\widetilde{\text{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,u})} \lesssim 1.$$

We proceed to estimate I_1 to I_3.

- We can rewrite $I_1 = I_{11} + I_{12} + I_{13} + I_{14} + I_{15}$ in the obvious way. We further decompose $I_{11} = I_{111} + I_{112}$. There holds

$$I_{111} = \int_{u_\infty}^u \frac{a^2}{|u'|^3}\left\|(a^{\frac{1}{2}})^{i-1}\sum_{i_1+i_2+i_3=i-1}\nabla^{i_1}\psi^{i_2}\nabla^{i_3+1}\frac{2}{|u|^2}(\Omega^{-1}-1)\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'$$

$$= \int_{u_\infty}^u \frac{a^2}{|u'|^5}\left\|(a^{\frac{1}{2}})^{i-1}\sum_{i_1+i_2=i-1}\nabla^{i_1}\psi^{i_2+1}\right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du' \lesssim \frac{a^2 \cdot O}{|u|^4}.$$

Also,

$$
\begin{aligned}
I_{112} &= \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \Big\| (a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Big(\frac{\Omega^{-1}-1}{|u'|^2}\Big)\Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du' \\
&= \int_{u_\infty}^{u} |u'|^{i+1} \Big\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Big(\frac{\Omega^{-1}-1}{|u'|^2}\Big)\Big\|_{L^2(S_{u',\underline{u}})} du' \\
&= \int_{u_\infty}^{u} |u'|^{i+1} \Big\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Big(\frac{\Omega^{-1}-1}{|u'|^2}\Big)\Big\|_{L^2(S_{u',\underline{u}})} du' \\
&= \int_{u_\infty}^{u} |u'|^{i+1} \Big\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Big[\frac{1}{|u'|^2}\cdot\int_0^{\underline{u}} 2\omega(u',\underline{u}',\theta^1,\theta^2)d\underline{u}'\Big]\Big\|_{L^2(S_{u',\underline{u}})} du' \\
&= \int_{u_\infty}^{u} |u'|^{i+1} \Big\| \sum_{i_1+i_2+i_3=i} \nabla^{i_1}\psi^{i_2}\Big[\frac{1}{|u'|^2}\cdot\int_0^{\underline{u}} 2\nabla^{i_3}\omega(u',\underline{u}',\theta^1,\theta^2)d\underline{u}'\Big]\Big\|_{L^2(S_{u',\underline{u}})} du' \\
&\leq \int_{u_\infty}^{u} |u'|^{i+1} \sum_{i_1+i_2+i_3=i} \frac{1}{|u'|^{i_1+i_2}}\cdot\frac{1}{|u'|^2}\cdot\frac{1}{|u'|^{i_3}}\cdot\frac{a^{\frac{1}{2}}}{|u'|^{\frac{1}{2}}}\cdot\Big(\underline{\mathcal{R}}[\rho]+1\Big)\,du' \\
&\leq \int_{u_\infty}^{u} \frac{a^{\frac{1}{2}}}{|u'|^{\frac{3}{2}}}\Big(\underline{\mathcal{R}}[\rho]+1\Big)\,du' \lesssim \underline{\mathcal{R}}[\rho]+1.
\end{aligned}
$$

Similarly, there holds

$$
\begin{aligned}
I_{12} &= \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \Big\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\widetilde{\mathrm{tr}\underline{\chi}}\nabla^{i_4}\widetilde{\mathrm{tr}\underline{\chi}}\Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,du' \\
&= \int_{u_\infty}^{u} \frac{du'}{|u'|} \Big\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Big(\frac{a}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}}\Big)\nabla^{i_4}\Big(\frac{a}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}}\Big)\Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \\
&\lesssim \int_{u_\infty}^{u} \frac{a^{\frac{3}{2}}}{|u'|^3}\cdot\frac{|u'|^2}{a^2}\cdot\frac{O^2}{|u'|}\,du' + \int_{u_\infty}^{u} \frac{a^2}{|u'|^3}\cdot\frac{|u'|}{a}\cdot\frac{O}{|u'|}\cdot\|a^5\nabla^{11}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,du'.
\end{aligned}
$$

Also,

$$
I_{13} = \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\omega}\nabla^{i_4}\mathrm{tr}\underline{\chi} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
= \int_{u_\infty}^u \frac{a}{|u'|} \cdot \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\omega}\nabla^{i_4} \left(\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
\lesssim \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3} \cdot \frac{|u'|^2}{a} \cdot \frac{O^2}{|u'|} \mathrm{d}u'
$$

$$
+ \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{|u'|^2}{a} \|a^5\nabla^{11}\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \cdot \frac{O_\infty[\mathrm{tr}\underline{\chi}]}{|u'|} \mathrm{d}u'
$$

$$
+ \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{|u'|}{a} \cdot \frac{O_\infty[\underline{\omega}]}{|u'|} \left\| a^5\nabla^{11} \left(\frac{a}{|u'|}\mathrm{tr}\underline{\chi} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
\lesssim \frac{a^{\frac{1}{2}}\cdot O^2}{|u|} + \int_{u_\infty}^u \frac{a}{|u'|^2}\|a^5\nabla^{11}\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
+ \text{(a term handled by Grönwall's inequality)}.
\tag{5.159}
$$

There also holds

$$
I_{14} = \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\hat{\chi}}\nabla^{i_4}\hat{\chi} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
= \int_{u_\infty}^u \frac{a}{|u'|} \cdot \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3} \left(\frac{a^{\frac{1}{2}}}{|u'|}\hat{\chi} \right) \nabla^{i_4} \left(\frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}
$$

$$
\lesssim \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3} \cdot \frac{|u'|^2}{a} \cdot \frac{O^2}{|u'|} \mathrm{d}u' + \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{|u'|}{a^{\frac{1}{2}}} \cdot \frac{O_\infty[\underline{\hat{\chi}}]}{|u'|} \|a^5\nabla^{11}\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
\lesssim \frac{a^{\frac{1}{2}}\cdot O^2}{|u|^2} + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3} \|a^5\nabla^{11}\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'.
\tag{5.160}
$$

Finally, for the term I_{15} we have

$$
\int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\alpha}_F\nabla^{i_4}\underline{\alpha}_F \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
$$

$$
\lesssim \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3} \cdot \frac{O^2}{|u'|} \mathrm{d}u' + \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{O}{|u'|} \cdot \|a^5\nabla^{11}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \mathrm{d}u'
\tag{5.161}
$$

$$
\lesssim 1 + \frac{a^{\frac{3}{2}}\cdot O}{|u|^{\frac{5}{2}}} \cdot \|a^5\nabla^{11}\underline{\alpha}_F\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \lesssim 1.
$$

- There holds

$$\int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{i_4} \widetilde{\mathrm{tr}\underline{\chi}} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$\lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2=i} \nabla^{i_1}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{i_2} \widetilde{\mathrm{tr}\underline{\chi}} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{i_4} \widetilde{\mathrm{tr}\underline{\chi}} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$\lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| a^5(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{11} \widetilde{\mathrm{tr}\underline{\chi}} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| a^5 \nabla^{11}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \widetilde{\mathrm{tr}\underline{\chi}} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du' + 1$$

$$\lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{|u'|}{a^{\frac{1}{2}}} \cdot \frac{|u'|}{a} \cdot \frac{O_\infty[\underline{\hat{\chi}}]}{|u'|} \cdot \left\| a^5 \nabla^{11} \left(\frac{a}{|u'|} \widetilde{\mathrm{tr}\underline{\chi}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$+ \int_{u_\infty}^u \frac{a^2}{|u'|^3} \cdot \frac{|u'|}{a} \cdot \frac{O_\infty[\widetilde{\mathrm{tr}\underline{\chi}}]}{|u'|} \cdot \| a^5 \nabla^{11}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du' + 1$$

$$\lesssim \int_{u_\infty}^u \frac{a^{\frac{1}{2}}}{|u'|^2} \cdot \left\| a^5 \nabla^{11} \left(\frac{a}{|u'|} \widetilde{\mathrm{tr}\underline{\chi}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^3} \cdot \| a^5 \nabla^{11}(\psi, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du' + 1.$$

Noting that $\psi \in \left\{ \eta, \underline{\eta} \right\}$ and recalling Propositions 5.6.5 and 5.6.6, using Grön-wall's inequality, we can bound this term by

$$\int_{u_\infty}^u \frac{a}{|u'|^3} \cdot \| a^5 \nabla^{11} \underline{\hat{\chi}} \|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du' + 1.$$

- For the last term there holds

$$\int_{u_\infty}^u \frac{a^2}{|u'|^3} \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \mathrm{tr}\underline{\chi} \nabla^{i_4} \widetilde{\mathrm{tr}\underline{\chi}} \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$= \int_{u_\infty}^u \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i-1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \left(\frac{a}{|u'|^2} \mathrm{tr}\underline{\chi} \right) \nabla^{i_4} \left(\frac{a}{|u'|} \widetilde{\mathrm{tr}\underline{\chi}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$\lesssim \int_{u_\infty}^u \left\| (a^{\frac{1}{2}})^{i-1} \sum_{i-1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \left(\frac{a}{|u'|^2} \mathrm{tr}\underline{\chi} \right) \nabla^{i_4} \left(\frac{a}{|u'|} \widetilde{\mathrm{tr}\underline{\chi}} \right) \right\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} du'$$

$$\lesssim \frac{O^3}{a^{\frac{1}{2}} \cdot |u|} \lesssim 1.$$

(5.162)

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We thus have, using Grönwall's inequality,

$$
\begin{aligned}
\frac{a}{|u|}\|a^5\nabla^{11}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} &\lesssim 1 + \mathcal{R}[\rho] + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|a^5\nabla^{11}\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u' \\
&\quad + \int_{u_\infty}^u \frac{a}{|u'|^2}\|a^5\nabla^{11}\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u' \\
&\lesssim 1 + \mathcal{R}[\rho] + \mathcal{R}[\underline{\tilde{\beta}}] + \mathcal{F}[\underline{\alpha}_F] \\
&\quad + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|a^5\nabla^{11}\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'.
\end{aligned}
\tag{5.163}
$$

For $\underline{\hat{\chi}}$, we have the constraint equation

$$
\mathrm{div}\,\underline{\hat{\chi}} = \frac{1}{2}\nabla\widetilde{\mathrm{tr}\underline{\chi}} - \frac{1}{2}(\underline{\eta}-\eta)(\underline{\hat{\chi}}-\frac{1}{2}\mathrm{tr}\underline{\chi}\gamma) + \underline{\tilde{\beta}}.
$$

Consequently,

$$
\begin{aligned}
\|a^5\nabla^{11}\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} &\lesssim \sum_{j\leq 10}\Big(\|(a^{\frac{1}{2}})^j\nabla^{j+1}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j\underline{\tilde{\beta}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \\
&\quad + \Big\|(a^{\frac{1}{2}})^j\sum_{j_1+j_2=j}\nabla^{j_1}(\eta,\underline{\eta})\nabla^{j_2}(\underline{\hat{\chi}},\mathrm{tr}\underline{\chi})\Big\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}\Big) \\
&\quad + \sum_{j\leq 10}\frac{1}{a^{\frac{1}{2}}}\cdot\|(a^{\frac{1}{2}}\nabla)^i\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.
\end{aligned}
$$

Integrating this along the incoming direction, we have

$$
\begin{aligned}
&\int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|a^5\nabla^{11}\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u' \\
&\lesssim \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\sum_{j\leq 10}\Big(\|(a^{\frac{1}{2}})^j\nabla^{j+1}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j\underline{\tilde{\beta}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \\
&\quad + \frac{|u'|^2}{a}\Big\|(a^{\frac{1}{2}})^j\sum_{j_1+j_2=j}\nabla^{j_1}(\eta,\underline{\eta})\nabla^{j_2}\Big(\frac{a}{|u'|^2}\underline{\hat{\chi}},\frac{a}{|u'|^2}\mathrm{tr}\underline{\chi}\Big)\Big\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\Big)\,\mathrm{d}u' \\
&\quad + \int_{u_\infty}^u \frac{a}{|u'|^3}\sum_{i\leq 10}\|(a^{\frac{1}{2}}\nabla)^i\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u' \\
&\lesssim \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|a^5\nabla^{11}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u' + \frac{a^{\frac{3}{2}}}{|u|^2}\cdot R[\underline{\tilde{\beta}}] \\
&\quad + \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\cdot\frac{|u'|^2}{a}\cdot\frac{O^2}{|u'|}\,\mathrm{d}u' + \int_{u_\infty}^u \frac{a}{|u'|^3}\cdot\frac{|u'|}{a^{\frac{1}{2}}}\cdot O\,\mathrm{d}u' \\
&\lesssim \int_{u_\infty}^u \frac{a^{\frac{3}{2}}}{|u'|^3}\|a^5\nabla^{11}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u' + 1.
\end{aligned}
\tag{5.164}
$$

Plugging this back to (5.163) and using Grönwall's inequality, we get

$$\frac{a}{|u|}\|a^5\nabla^{11}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim 1 + \underline{\mathcal{R}}[\rho] + \int_{u_\infty}^u \frac{a}{|u'|^2}\|a^5\nabla^{11}\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,du'$$

$$\lesssim 1 + \underline{\mathcal{R}}[\rho] + \frac{a^{\frac12}}{|u|^{\frac12}}\|a^5\nabla^{11}\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}})}$$

$$\lesssim \underline{\mathcal{R}}[\rho] + \underline{\mathcal{R}}[\tilde{\underline{\beta}}] + \underline{\mathcal{F}}[\underline{\alpha}_F] + 1.$$

Integrating in the u–direction we obtain

$$\int_{u_\infty}^u \frac{a^2}{|u'|^3}\|a^5\nabla^{11}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \lesssim \underline{\mathcal{R}}[\rho] + \underline{\mathcal{R}}[\tilde{\underline{\beta}}] + \underline{\mathcal{F}}[\underline{\alpha}_F] + 1.$$

Remark 13. *In a similar fashion, we can obtain the following estimates:*

$$\left(\int_{u_\infty}^u \frac{a^3}{|u'|^4}\|a^5\nabla^{11}\widetilde{tr\underline{\chi}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,du'\right)^{\frac12} + \left(\int_{u_\infty}^u \frac{a^2}{|u'|^4}\|a^5\nabla^{11}\hat{\underline{\chi}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,du'\right)^{\frac12}$$

$$\lesssim \underline{\mathcal{R}}[\rho] + \underline{\mathcal{R}}[\tilde{\underline{\beta}}] + \underline{\mathcal{F}}[\underline{\alpha}_F] + 1.$$

$$(5.165)$$

$\square$

5.7 Energy estimates

In this section with scale invariant norms we will derive energy estimates for curvature components and their angular derivatives. Our goal is to show that

$$\mathcal{R} + \underline{\mathcal{R}} + \mathcal{F} + \underline{\mathcal{F}} \lesssim \left(\mathcal{I}^{(0)}\right)^4 + \left(\mathcal{I}^{(0)}\right)^2 + \mathcal{I}^{(0)} + 1.$$

We begin with the integration by parts formula.

5.7.1 Integration by parts

The following holds. Define $\mathcal{D}_{u,\underline{u}} := (u_\infty, u) \times (0, \underline{u}) \times \mathbb{S}^2$. A direct computation yields the following:

Proposition 5.7.1. *Suppose ϕ_1, ϕ_2 are $r-$tensor fields. Then there holds*

$$\int_{\mathcal{D}_{u,\underline{u}}} \phi_1\nabla_4\phi_2 + \int_{\mathcal{D}_{u,\underline{u}}} \phi_2\nabla_4\phi_1 = \int_{\underline{H}^{(u_\infty,u)}_{\underline{u}}} \phi_1\phi_2 - \int_{\underline{H}^{(u_\infty,u)}_0} \phi_1\phi_2 + \int_{\mathcal{D}_{u,\underline{u}}} (2\omega - tr\chi)\phi_1\phi_2.$$

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Proposition 5.7.2. *Given an $r-$tensor field $^{(1)}\phi$ and an $(r-1)$-tensor field $^{(2)}\phi$, there holds*

$$\int_{\mathcal{D}_{u,\underline{u}}} {}^{(1)}\phi^{A_1...A_r} \nabla_{A_r} {}^{(2)}\phi^{A_1...A_{r-1}} + \int_{\mathcal{D}_{u,\underline{u}}} \nabla^{A_r} {}^{(1)}\phi_{A_1...A_r} {}^{(2)}\phi^{A_1...A_{r-1}}$$

$$= - \int_{\mathcal{D}_{u,\underline{u}}} (\eta + \underline{\eta}) \, {}^{(1)}\phi {}^{(2)}\phi.$$

Moreover, we shall require the following bound:

Proposition 5.7.3. *Suppose ϕ is an $r-$tensor field and let $\lambda_1 = 2\lambda_0 - 1$. Then there exists a function f, independent of ϕ and λ_0, such that there holds*

$$2\int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2\lambda_1}\phi(\nabla_3 + \lambda_0 tr\underline{\chi})\phi = \int_{H_u^{(0,\underline{u})}} |u|^{2\lambda_1}\phi^2 - \int_{H_{u_\infty}^{(0,\underline{u})}} |u_\infty|^{2\lambda_1}\phi^2 + \int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2\lambda_1} f\phi^2,$$

where f satisfies the bound

$$f \lesssim \frac{O}{|u|^2}.$$

Proof. There holds

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_{S_{u,\underline{u}}} |u|^{2\lambda_1}\Omega|\phi|^2\right) = \underline{L}\left(\int_{S_{u,\underline{u}}} |u|^{2\lambda_1}\Omega|\phi|^2\right)$$

$$= \int_{S_{u,\underline{u}}} \Omega^2 \left(2|u|^{2\lambda_1}\langle\phi, \nabla_3\phi + \lambda_0 tr\underline{\chi}\phi\rangle\right)$$

$$+ \int_{S_{u,\underline{u}}} \Omega^2 \left(|u|^{2\lambda_1}\left(-2\frac{\lambda_1(e_3(u))}{|u|} + (1 - 2\lambda_0)tr\underline{\chi} - 2\underline{\omega}\right)|\phi|^2\right).$$

Here we have used that $\underline{L} = \Omega e_3 = \frac{\partial}{\partial u} + b^A\frac{\partial}{\partial\theta^A}$. Immediate calculations imply that

$$\left|-2\frac{\lambda_1(e_3(u))}{|u|} + (1 - 2\lambda_0)tr\underline{\chi} - 2\underline{\omega}\right| \lesssim \frac{O}{|u|^2}.$$

The proposition then follows by integrating in the slab $\mathcal{D}_{u,\underline{u}}$ and using the fundamental theorem of calculus. $\square$

5.7.2 The Hodge structure as an aid for energy estimates

Observe that for

$$(\mathfrak{Y}_1, \mathfrak{Y}_2) \in \left\{(\alpha, \tilde{\beta}), (\tilde{\beta}, (\rho, \sigma)), ((\rho, \sigma), \underline{\tilde{\beta}}), (\underline{\tilde{\beta}}, \underline{\alpha})\right\} \cup \left\{(\alpha_F, (-\rho_F, \sigma_F)), ((\rho_F, -\sigma_F), \underline{\alpha}_F)\right\}$$

we can write the equations for $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ in the following form:

$$\nabla_3 \mathfrak{Y}_1 + \left(\frac{1}{2} + s_2(\mathfrak{Y}_1) \right) \mathrm{tr}\underline{\chi} \mathfrak{Y}_1 - \mathcal{D}\mathfrak{Y}_2 = P_0, \tag{5.166}$$

$$\nabla_4 \mathfrak{Y}_2 - {}^*\mathcal{D} \mathfrak{Y}_1 = Q_0. \tag{5.167}$$

Here by $\mathcal{D}$ we denote a differential operator on $S_{u,\underline{u}}$ and by ${}^*\mathcal{D}$ its L^2-adjoint. By commuting the above equations i times, we arrive at

$$\nabla_3 \nabla^i \mathfrak{Y}_1 + \left(\frac{i+1}{2} + s_2(\mathfrak{Y}_1) \right) \mathrm{tr}\underline{\chi} \nabla^i \mathfrak{Y}_1 - \mathcal{D}\nabla^i \mathfrak{Y}_2 = P_i, \tag{5.168}$$

$$\nabla_4 \nabla^i \mathfrak{Y}_2 - {}^*\mathcal{D} \nabla^i \mathfrak{Y}_1 = Q_i. \tag{5.169}$$

The purpose of this section is to prove the following:

Proposition 5.7.4. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69) and given a pair $(\mathfrak{Y}_1, \mathfrak{Y}_2)$ satisfying*

$$\nabla_3 \nabla^i \mathfrak{Y}_1 + \left(\frac{i+1}{2} + s_2(\mathfrak{Y}_1) \right) \mathrm{tr}\underline{\chi} \nabla^i \mathfrak{Y}_1 - \mathcal{D}\nabla^i \mathfrak{Y}_2 = P,$$

$$\nabla_4 \nabla^i \mathfrak{Y}_2 - {}^*\mathcal{D} \nabla^i \mathfrak{Y}_1 = Q,$$

the following inequality holds:

$$\int_{H_u^{(0,\underline{u})}} \|\nabla^i \mathfrak{Y}_1\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})}^2 \, d\underline{u}' + \int_{\underline{H}_{\underline{u}}^{(u_\infty,u)}} \frac{a}{|u'|^2} \|\nabla^i \mathfrak{Y}_2\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}^2 \, du'$$

$$\lesssim \int_{H_{u_\infty}^{(0,\underline{u})}} \|\nabla^i \mathfrak{Y}_1\|_{\mathcal{L}^2_{(sc)}(S_{u_\infty,\underline{u}'})}^2 \, d\underline{u}' + \int_{\underline{H}_0^{(u_\infty,u)}} \frac{a}{|u'|^2} \|\nabla^i \mathfrak{Y}_2\|_{\mathcal{L}^2_{(sc)}(S_{u',0})}^2 \, du'$$

$$+ \iint_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|} \|\nabla^i \mathfrak{Y}_1 \cdot P\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}' + \iint_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|} \|\nabla^i \mathfrak{Y}_2 \cdot Q\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'.$$

Proof. The Hodge structure will play a crucial role: For a pair $(\mathfrak{Y}_1, \mathfrak{Y}_2)$ or a pair $(\nabla^i \mathfrak{Y}_1, \nabla^i \mathfrak{Y}_2)$, the angular derivative operator $\mathcal{D}$ and its L^2 adjoint operator $\mathcal{D}^*$ form a Hodge system. Through Proposition 5.7.2, we have

$$\int_{S_{u,\underline{u}}} \mathfrak{Y}_1 \, \mathcal{D}\mathfrak{Y}_2 + \mathfrak{Y}_2 \, \mathcal{D}^*\mathfrak{Y}_1 = - \int_{S_{u,\underline{u}}} (\eta + \underline{\eta}) \mathfrak{Y}_1 \mathfrak{Y}_2,$$

$$\int_{S_{u,\underline{u}}} \nabla^i \mathfrak{Y}_1 \, \mathcal{D}\nabla^i \mathfrak{Y}_2 + \nabla^i \mathfrak{Y}_2 \, \mathcal{D}^*\nabla^i \mathfrak{Y}_1 = - \int_{S_{u,\underline{u}}} (\eta + \underline{\eta}) \nabla^i \mathfrak{Y}_1 \nabla^i \mathfrak{Y}_2. \tag{5.170}$$

We now move forward and apply Proposition 5.7.3 for $\nabla^i \mathfrak{Y}_1$. With

$$\lambda_0 = \frac{1+i}{2} + s_2(\mathfrak{Y}_1), \quad \lambda_1 := 2\lambda_0 - 1 = i + 2s_2(\mathfrak{Y}_1), \quad \text{we get}$$

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$$
2\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_1\left(\nabla_3+\left(\frac{1+i}{2}+s_2(\mathfrak{Y}_1)\right)\mathrm{tr}\underline{\chi}\right)\nabla^i\mathfrak{Y}_1
$$

$$
=\int_{H_u^{(0,\underline{u})}}|u|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2-\int_{H_{u_\infty}^{(0,\underline{u})}}|u_\infty|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2 \tag{5.171}
$$

$$
+\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}f|\nabla^i\mathfrak{Y}_1|^2,
$$

where $|f|\le O/|u'|^2$.

We also use Proposition 5.7.1, plugging in $\phi_1=\phi_2=|u|^{i+2s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_2$

$$
2\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_2\nabla_4\nabla^i\mathfrak{Y}_2
$$

$$
=\int_{\underline{H}_{\underline{u}}^{(u_\infty,u)}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2-\int_{\underline{H}_0^{(u_\infty,u)}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2 \tag{5.172}
$$

$$
+\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}(2\omega-\mathrm{tr}\chi)|\nabla^i\mathfrak{Y}_2|^2.
$$

Adding (5.171) and (5.172), we obtain

$$
2\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_1\left(\nabla_3+\left(\frac{1+i}{2}+s_2(\mathfrak{Y}_1)\right)\mathrm{tr}\underline{\chi}\right)\nabla^i\mathfrak{Y}_1
$$

$$
+2\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_2\nabla_4\nabla^i\mathfrak{Y}_2
$$

$$
=\int_{H_u^{(0,\underline{u})}}|u|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2-\int_{H_{u_\infty}^{(0,\underline{u})}}|u_\infty|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2
$$

$$
+\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}f|\nabla^i\mathfrak{Y}_1|^2
$$

$$
+\int_{\underline{H}_{\underline{u}}^{(u_\infty,u)}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2-\int_{\underline{H}_0^{(u_\infty,u)}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2
$$

$$
+\int_{\mathcal{D}_{u,\underline{u}}}|u'|^{2i+4s_2(\mathfrak{Y}_1)}(2\omega-\mathrm{tr}\chi)|\nabla^i\mathfrak{Y}_2|^2.
$$

We now take (5.168) and (5.169) into account. With the help of (5.170), we then

arrive at

$$\int_{H_u^{(0,\underline{u})}} |u|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2 + \int_{\underline{H}_{\underline{u}}^{(u_\infty,u)}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2$$

$$= \int_{H_{u_\infty}^{(0,\underline{u})}} |u_\infty|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2 + \int_{\underline{H}_0^{(u_\infty,u)}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2$$

$$+ 2\int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_1 \cdot P + 2\int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_2 \cdot Q$$

$$- 2\int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}(\eta + \underline{\eta})\nabla^i\mathfrak{Y}_1\nabla^i\mathfrak{Y}_2$$

$$+ \int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}f|\nabla^i\mathfrak{Y}_1|^2 + \int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}(2\omega - \mathrm{tr}\chi)|\nabla^i\mathfrak{Y}_2|^2.$$

Using $|(\eta + \underline{\eta})\nabla^i\mathfrak{Y}_1\nabla^i\mathfrak{Y}_2| \le |\eta + \underline{\eta}| \cdot (|\nabla^i\mathfrak{Y}_1|^2 + |\nabla^i\mathfrak{Y}_2|^2)$, and the fact

$$|\eta + \underline{\eta}| \le a^{\frac{1}{2}}O/|u'|^2, \quad |f| \le O/|u'|^2, \quad |2\omega - \mathrm{tr}\chi| \le O/|u'|,$$

by applying Grönwall's inequality twice (one for du, one for $d\underline{u}$), we obtain

$$\int_{H_u^{(0,\underline{u})}} |u|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2 + \int_{\underline{H}_{\underline{u}}^{(u_\infty,u)}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2$$

$$\lesssim \int_{H_{u_\infty}^{(0,\underline{u})}} |u_\infty|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2 + \int_{\underline{H}_0^{(u_\infty,u)}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2$$

$$+ 2\int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_1 \cdot P + 2\int_{\mathcal{D}_{u,\underline{u}}} |u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_2 \cdot Q.$$

Multiplying by $a^{-i-2s_2(\mathfrak{Y}_1)}$ on both sides, we get

$$\int_{H_u^{(0,\underline{u})}} a^{-i-2s_2(\mathfrak{Y}_1)}|u|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2 + \int_{\underline{H}_{\underline{u}}^{(u_\infty,u)}} a^{-i-2s_2(\mathfrak{Y}_1)}|u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2$$

$$\lesssim \int_{H_{u_\infty}^{(0,\underline{u})}} a^{-i-2s_2(\mathfrak{Y}_1)}|u_\infty|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_1|^2 + \int_{\underline{H}_0^{(u_\infty,u)}} a^{-i-2s_2(\mathfrak{Y}_1)}|u'|^{2i+4s_2(\mathfrak{Y}_1)}|\nabla^i\mathfrak{Y}_2|^2$$

$$+ 2\int_{\mathcal{D}_{u,\underline{u}}} a^{-i-2s_2(\mathfrak{Y}_1)}|u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_1 \cdot P + 2\int_{\mathcal{D}_{u,\underline{u}}} a^{-i-2s_2(\mathfrak{Y}_1)}|u'|^{2i+4s_2(\mathfrak{Y}_1)}\nabla^i\mathfrak{Y}_2 \cdot Q.$$

$$(5.173)$$

Taking into account the signature identities

$$s_2(\nabla^i\mathfrak{Y}_1) = \frac{i}{2} + s_2(\mathfrak{Y}_1), \quad s_2(\nabla^i\mathfrak{Y}_2) = \frac{i+1}{2} + s_2(\mathfrak{Y}_1),$$

$$s_2(P) = s_2(\nabla_3\nabla^i\mathfrak{Y}_1) = \frac{i+2}{2} + s_2(\mathfrak{Y}_1), \quad s_2(Q) = s_2(\mathcal{D}^*\nabla^i\mathfrak{Y}_1) = \frac{i+1}{2} + s_2(\mathfrak{Y}_1),$$

and definitions

$$\|\phi\|_{\mathcal{L}_{sc}^2(S_{u,\underline{u}})}^2 = a^{-s_2(\phi)}|u|^{2s_2(\phi)}\|\phi\|_{L^2(S_{u,\underline{u}})},$$

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$$\|\phi\|_{\mathcal{L}_{sc}^1(S_{u,\underline{u}})} = a^{-s_2(\phi)}|u|^{2s_2(\phi)-1}\|\phi\|_{L^1(S_{u,\underline{u}})},$$

we rewrite (5.173) as

$$\int_{H_u^{(0,\underline{u})}} \|\nabla^i \mathfrak{Y}_1\|_{\mathcal{L}_{sc}^2(S_{u,\underline{u}})}^2 + \int_{\underline{H}_u^{(u_\infty,u)}} \frac{a}{|u'|^2}\|\nabla^i \mathfrak{Y}_2\|_{\mathcal{L}_{sc}^2(S_{u',\underline{u}})}^2$$
$$\lesssim \int_{H_{u_\infty}^{(0,\underline{u})}} \|\nabla^i \mathfrak{Y}_1\|_{\mathcal{L}_{sc}^2(S_{u_\infty,\underline{u}})}^2 + \int_{\underline{H}_0^{(u_\infty,u)}} \frac{a}{|u_\infty|^2}\|\nabla^i \mathfrak{Y}_2\|_{\mathcal{L}_{sc}^2(S_{u_\infty,\underline{u}})}^2$$
$$+ 2\int_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|}\|\nabla^i \mathfrak{Y}_1 \cdot P\|_{\mathcal{L}_{sc}^1(S_{u',\underline{u}'})} + 2\int_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|}\|\nabla^i \mathfrak{Y}_2 \cdot Q\|_{\mathcal{L}_{sc}^1(S_{u',\underline{u}'})}.$$

Recalling the definitions

$$\|\phi\|_{\mathcal{L}_{sc}^2(H_u^{(0,\underline{u})})}^2 := \int_0^{\underline{u}} \|\phi\|_{\mathcal{L}_{sc}^2(S_{u,\underline{u}'})}^2 d\underline{u}',$$
$$\|\phi\|_{\mathcal{L}_{sc}^2(\underline{H}_{\underline{u}}^{(u_\infty,u)})}^2 := \int_{u_\infty}^{u} \frac{a}{|u'|^2}\|\phi\|_{\mathcal{L}_{sc}^2(S_{u',\underline{u}})}^2 du',$$

and substituting them in the above, we arrive at the desired result. $\qquad\square$

5.7.3   Energy estimates on the Maxwell components

Recall the null Maxwell equations

$$\nabla_4\underline{\alpha}_F + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}_F = -\nabla\rho_F - {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F - 2\,{}^*\eta\cdot\rho_F + 2\omega\,\underline{\alpha}_F - \hat{\underline{\chi}}\cdot\alpha_F, \quad (5.174)$$

$$\nabla_3\alpha_F + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha_F = \nabla\rho_F + {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F + 2\underline{\eta}\cdot\rho_F + 2\underline{\omega}\alpha_F - \hat{\chi}\cdot\underline{\alpha}_F, \quad (5.175)$$

$$\nabla_4\rho_F = \mathrm{div}\,\alpha_F - \mathrm{tr}\chi\rho_F - (\eta - \underline{\eta})\cdot\alpha_F, \quad (5.176)$$

$$\nabla_4\sigma_F = -\mathrm{curl}\,\alpha_F - \mathrm{tr}\chi\sigma_F + (\eta - \underline{\eta})\cdot{}^*\alpha_F, \quad (5.177)$$

$$\nabla_3\rho_F + \mathrm{tr}\underline{\chi}\rho_F = \mathrm{div}\underline{\alpha}_F + (\eta - \underline{\eta})\cdot\underline{\alpha}_F, \quad (5.178)$$

$$\nabla_3\sigma_F + \mathrm{tr}\underline{\chi}\sigma_F = -\mathrm{curl}\underline{\alpha}_F + (\eta - \underline{\eta})\cdot{}^*\underline{\alpha}_F. \quad (5.179)$$

Notice that for the pair $(\Upsilon_1, \Upsilon_2) = \left\{\alpha_F, (-\rho_F, \sigma_F)\right\}$ we have

$$\nabla_3\Upsilon_1 + \left(\frac{1}{2} + s_2(\Upsilon_1)\right)\mathrm{tr}\underline{\chi}\Upsilon_1 - {}^*\mathcal{D}_1\Upsilon_2 = \psi\cdot(\rho_F, \sigma_F) + \hat{\chi}\cdot\underline{\alpha}_F + \psi\cdot\alpha_F, \quad (5.180)$$

$$\nabla_4\Upsilon_2 + \mathcal{D}_1\Upsilon_1 = \mathrm{tr}\chi\cdot(\rho_F, \sigma_F) + (\eta, \underline{\eta})\cdot\alpha_F, \quad (5.181)$$

while for the pair $(\Upsilon_1, \Upsilon_2) = \left\{(\rho_F, -\sigma_F), \underline{\alpha}_F\right\}$ we have

$$\nabla_3 \Upsilon_1 + \left(\frac{1}{2} + s_2(\Upsilon_1) \right) \mathrm{tr}\underline{\chi} \Upsilon_1 - \mathcal{D}_1 \Upsilon_2 = (\eta, \underline{\eta}) \cdot \underline{\alpha}_F, \tag{5.182}$$

$$\nabla_4 \Upsilon_2 - {}^*\mathcal{D}_1 \Upsilon_1 = (\eta, \underline{\eta}) \cdot (\rho_F, \sigma_F) + \omega \cdot \underline{\alpha}_F + \hat{\underline{\chi}} \cdot \alpha_F. \tag{5.183}$$

We introduce the following proposition

Proposition 5.7.5. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69) and given a pair (Υ_1, Υ_2) satisfying*

$$\nabla_3 \nabla^i \Upsilon_1 + \left(\frac{i+1}{2} + s_2(\Upsilon_1) \right) \mathrm{tr}\underline{\chi} \nabla^i \Upsilon_1 - \mathcal{D} \nabla^i \Upsilon_2 = P,$$
$$\nabla_4 \nabla^i \Upsilon_2 - {}^*\mathcal{D} \nabla^i \Upsilon_1 = Q,$$

the following inequality holds:

$$\int_{H_u^{(0,\underline{u})}} \|\nabla^i \Upsilon_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}' + \int_{\underline{H}_{\underline{u}}^{(u\infty,u)}} \frac{a}{|u'|^2} \|\nabla^i \Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$

$$\lesssim \int_{H_{u\infty}^{(0,\underline{u})}} \|\nabla^i \Upsilon_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u\infty,\underline{u}'})} \, d\underline{u}' + \int_{\underline{H}_0^{(u\infty,u)}} \frac{a}{|u'|^2} \|\nabla^i \Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',0})} \, du'$$

$$+ \iint_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|} \|\nabla^i \Upsilon_1 \cdot P\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}' + \iint_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|} \|\nabla^i \Upsilon_2 \cdot Q\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'.$$

We begin with the pair $(\alpha_F, (-\rho_F, \sigma_F))$.

Proposition 5.7.6. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), for $i \le 11$, we have*

$$\frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}})^{i-1} \nabla^i \alpha_F\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})}$$

$$\le \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}})^{i-1} \nabla^i \alpha_F\|_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,\underline{u})})} + \frac{1}{a^{\frac{1}{3}}}.$$

Proof. We schematically have

$$\nabla_3 \alpha_F + \frac{1}{2} \mathrm{tr}\underline{\chi} \alpha_F - \mathcal{D}(-\rho_F, \sigma_F) = (\psi, \hat{\chi}) \cdot \Upsilon + \psi \cdot (\Upsilon, \alpha_F),$$

$$\nabla_4 (-\rho_F, \sigma_F) - {}^*\mathcal{D}\alpha_F = \psi \cdot (\Upsilon, \alpha_F).$$

Commuting with i angular derivatives we arrive at

$$\nabla_3 \nabla^i \alpha_F + \frac{i+1}{2} \mathrm{tr}\underline{\chi} \nabla^i \alpha_F - \mathcal{D}\nabla^i(\rho_F, \sigma_F)$$

$$= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta})\nabla^{i_4}(\rho_F,\sigma_F) + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\omega}\nabla^{i_4}\alpha_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\hat{\chi}\nabla^{i_4}\underline{\alpha}_F + \sum_{i-} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\alpha_F$$

$$:= P_1,$$

while for (ρ_F, σ_F) we similarly obtain

$$\nabla_4 \nabla^i(\rho_F,\sigma_F) - {}^*\mathcal{D}\nabla^i\alpha_F$$

$$= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta})\nabla^{i_4}\alpha_F + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\mathrm{tr}\chi\nabla^{i_4}(\rho_F,\sigma_F)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta},\hat{\chi})\nabla^{i_4}(\rho_F,\sigma_F)$$

$$:= Q_1.$$

We arrive at

$$\|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_u^{(u_\infty,u)})}$$

$$\leq \|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty}^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u_\infty,u)})} + N_1 + M_1,$$

where

$$N_1 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|}\|(a^{\frac{1}{2}})^{i-1}P_1 \cdot (a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}',$$

$$M_1 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|}\|(a^{\frac{1}{2}})^{i-1}Q_1 \cdot (a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}'.$$

Let us focus on the term N_1 first. Using the scale-invariant version of Hölder's inequality we get

$$N_1 \leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^{i-1}P_1\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}' \qquad (5.184)$$

$$\leq \int_{u_\infty}^u \frac{a}{|u'|^2}\left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1}P_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'\right)^{\frac{1}{2}}\,\mathrm{d}u' \cdot \sup_{u'}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})},$$

$$(5.185)$$

where

$$P_1 = \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta})\nabla^{i_4}(\rho_F,\sigma_F) + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\omega}\nabla^{i_4}\alpha_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\hat{\chi}\nabla^{i_4}\underline{\alpha}_F + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\mathrm{tr}\underline{\chi}\nabla^{i_4}\alpha_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\alpha_F$$

$$:= \sum_{j=1}^{5} P_{1j}.$$

Denote

$$H_1 = \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1}P_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'.$$

We further have

$$H_1 = \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1}P_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}' \lesssim \sum_{j=1}^{5} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1}P_{1j}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}' \qquad (5.186)$$

We treat each of those five terms separately.

- There holds

$$\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1}P_{11}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$= \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta})\nabla^{i_4}(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\lesssim \frac{O^4}{a\cdot|u'|^2} + \frac{O^2}{|u'|^2}\cdot\int_0^{\underline{u}}\|a^5\nabla^{11}(\eta,\underline{\eta})\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \frac{O^2}{|u'|^2}\cdot\int_0^{\underline{u}}\|a^5\nabla^{11}(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\lesssim \frac{O^4}{a\cdot|u'|^2} + \frac{O^2}{a^2}\cdot\int_0^{\underline{u}}(1+\mathcal{R}+\underline{\mathcal{R}})^2\,\mathrm{d}\underline{u}' + \frac{O^2}{|u'|^2}F[\rho_F,\sigma_F]^2.$$

Here we have used Propositions 5.6.5 and 5.6.6 as well as the bootstrap bounds (5.69).

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- There holds

$$\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1} P_{12}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$= \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \underline{\omega} \nabla^{i_4} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$\lesssim \frac{O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} \underline{\omega}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$+ \frac{a \cdot O^2}{|u'|^2} \cdot \frac{1}{a} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} a_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$\lesssim \frac{O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} \underline{\omega}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}' + \frac{a \cdot O^2}{|u'|^2} F[\alpha_F]^2.$$

Here we have made use of the bootstrap bounds (5.69).

- There holds

$$\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1} P_{13}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$= \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right) \nabla^{i_4} \underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$\lesssim \frac{O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} \left(\frac{\hat{\chi}}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$+ \frac{a \cdot O^2}{|u'|^2} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} \underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$\lesssim 1 + \frac{a \cdot O^2}{|u'|^2} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} \underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'.$$

Here we have made use of Proposition 5.6.3.

- The last two terms can be bounded above by

$$F^2[\alpha_F] \cdot O[\hat{\underline{\chi}}]^2 + 1,$$

using Grönwall's inequality and the elliptic estimates.

We arrive at the bound

$$H_1 \lesssim 1 + F^2[\alpha_F] \cdot O[\hat{\underline{\chi}}]^2 + \frac{a \cdot O^2}{|u'|^2} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} \underline{\omega}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u'$$
$$+ \frac{a \cdot O^2}{|u'|^2} \cdot \int_0^{\underline{u}} \|a^5 \nabla^{11} \underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u'. \tag{5.187}$$

The two integrals above cannot be estimated along the $\underline{u}$-direction, but only when you integrate them also along the $u-$direction and exchange the order of integration, as in (5.185). Integrating along the u-direction, these bounds translate to a bound on N_1

$$N_1 \lesssim \left(F[\alpha_F] \cdot O[\hat{\chi}] + 1 \right) \sup_{u'} \|(a^{\frac{1}{2}})^{i-1} \nabla^i \alpha_F\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}. \tag{5.188}$$

For the term M_1 we follow the same procedure. We have

$$
\begin{aligned}
M_1 &= \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^{i-1} Q_1 \cdot (a^{\frac{1}{2}})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \mathrm{d}u' \, \mathrm{d}\underline{u}' \\
&\leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_1\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \|(a^{\frac{1}{2}})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \mathrm{d}u' \mathrm{d}\underline{u}' \\
&\leq \int_0^{\underline{u}} \left(\int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \mathrm{d}u' \right)^{\frac{1}{2}} \|(a^{\frac{1}{2}})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}'}^{(u_\infty, u)})} \mathrm{d}\underline{u}' \\
&\leq \left(\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \mathrm{d}u' \, \mathrm{d}\underline{u}' \right)^{\frac{1}{2}} \cdot \\
&\quad \cdot \sup_{\underline{u}'} \|(a^{\frac{1}{2}})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}'}^{(u_\infty, u)})}.
\end{aligned}
$$

Here we recall that

$$
\begin{aligned}
Q_1 &= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\eta, \underline{\eta}) \nabla^{i_4} \alpha_F + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \mathrm{tr}\chi \nabla^{i_4} (\rho_F, \sigma_F) \\
&\quad + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\eta, \underline{\eta}, \hat{\chi}) \nabla^{i_4} (\rho_F, \sigma_F) \\
&:= Q_{11} + Q_{12} + Q_{13}.
\end{aligned}
$$

Let

$$J_1 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \mathrm{d}u' \, \mathrm{d}\underline{u}' := J_{11} + J_{12} + J_{13}.$$

We have

$$J_1 \lesssim \sum_{j=1}^3 \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_{1j}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \mathrm{d}u' \, \mathrm{d}\underline{u}'. \tag{5.189}$$

Then, separating the cases where $i_4 \neq i$ and $i_4 = i$ and treating each of the three terms separately, we get

- There holds

$$J_{11} = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a\,d\underline{u}'}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta})\nabla^{i_4}\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,d\underline{u}'$$

$$= \int_{u_\infty}^u \frac{a\,du'}{|u'|^2} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta})\nabla^{i_4}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,d\underline{u}'$$

$$\lesssim \int_{u_\infty}^u \frac{a}{|u'|^2}\left(\frac{O^4}{|u'|^2} + \frac{a\cdot O^2\cdot F[\alpha_F]^2}{|u'|^2} + \frac{O^2\cdot(1+R)^2}{a}\right)du' \lesssim 1.$$

We have made use of Propositions 5.6.5 and 5.6.6 here.

- There holds

$$J_{12} = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a\,du'}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\mathrm{tr}\chi\nabla^{i_4}(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,d\underline{u}'$$

$$= \int_{u_\infty}^u \frac{a\,du'}{|u'|^2} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\mathrm{tr}\chi\nabla^{i_4}(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,d\underline{u}'$$

$$\lesssim \int_{u_\infty}^u \frac{a}{|u'|^2}\left(\frac{O^4}{a\cdot|u'|^2} + \frac{O^2\cdot F[\rho_F,\sigma_F]^2}{|u'|^2} + \frac{O^2\cdot(1+R)^2}{|u'|^2}\right)du' \lesssim 1.$$

We have made use of Propositions 5.6.5 and 5.6.6 as well as the bootstrap assumptions (5.69).

- There holds

$$J_{13} = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta},\hat{\chi})\nabla^{i_4}(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$= \int_{u_\infty}^u \frac{a}{|u'|^2} \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\eta,\underline{\eta},\hat{\chi}}{a^{\frac{1}{2}}}\right)\nabla^{i_4}(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\lesssim \int_{u_\infty}^u \frac{a}{|u'|^2}\left(\frac{O^4}{|u'|^2} + \frac{a\cdot O^2\cdot F[\rho_F,\sigma_F]^2}{|u'|^2} + \frac{a\cdot O^2\cdot(1+R)^2}{|u'|^2}\right)du' \lesssim 1.$$

Here we have made use of Propositions 5.6.3, 5.6.5 and 5.6.6 as well as the bootstrap bounds 5.69.

Hence

$$M_1 \leq \sup_{\underline{u}'}\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}'})}. \tag{5.190}$$

Taking the bounds (5.188) and (5.190) into account, we get

$$a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})}$$

$$\leq a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,u)})} + a^{-1}(N_1 + M_1)$$

$$\leq a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,u)})}$$

$$+ a^{-\frac{1}{2}}(F \cdot O + 1) \cdot F + a^{-\frac{1}{2}}F$$

$$\leq a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + a^{-1}\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,u)})} + a^{-\frac{1}{4}}.$$

Thus

$$\mathcal{F}^2[\alpha_F] + \underline{\mathcal{F}}^2[\rho_F,\sigma_F] \leq \mathcal{F}_0^2[\alpha_F] + \underline{\mathcal{F}}_0^2[\rho_F,\sigma_F] + \frac{1}{a^{\frac{1}{4}}},$$

which translates to the desired energy bound

$$\Rightarrow \mathcal{F}[\alpha_F] + \underline{\mathcal{F}}[\rho_F,\sigma_F] \leq 2\mathcal{F}_0[\alpha_F] + 2\underline{\mathcal{F}}_0[\rho_F,\sigma_F] + \frac{1}{a^{\frac{1}{8}}}. \tag{5.191}$$

$\square$

Continuing the estimates for the Maxwell components, we shift attention to the pair $\left((\rho_F, -\sigma_F), \underline{\alpha}_F\right)$.

Proposition 5.7.7. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), we have*

$$\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, -\sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i\underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})}$$

$$\lesssim \mathcal{R}^4 + \underline{\mathcal{R}}^4 + \left(\mathcal{I}^{(0)}\right)^4 + \left(\mathcal{I}^{(0)}\right)^2 + 1.$$

Proof. We recall the following schematic equations for the pair

$$(\Upsilon_1, \Upsilon_2) = \left((\rho_F, -\sigma_F), \underline{\alpha}_F\right):$$

$$\nabla_3\Upsilon_1 + \left(\frac{1}{2} + s_2(\Upsilon_1)\right)\mathrm{tr}\underline{\chi}\Upsilon_1 - \mathcal{D}_1\Upsilon_2 = (\eta,\underline{\eta}) \cdot \underline{\alpha}_F, \tag{5.192}$$

$$\nabla_4\Upsilon_2 - {}^*\mathcal{D}_1\Upsilon_1 = (\eta,\underline{\eta}) \cdot (\rho_F,\sigma_F) + \omega \cdot \underline{\alpha}_F + \hat{\underline{\chi}} \cdot \alpha_F. \tag{5.193}$$

Commuting these equations i times with angular derivatives ∇ we arrive at the equation

$$\nabla_3 \nabla^i \Upsilon_1 + \left(\frac{i+1}{2} + s_2(\Upsilon_1) \right) \mathrm{tr}\underline{\chi} \nabla^i \Upsilon_1 - \mathcal{D}\nabla^i \Upsilon_2$$
$$= \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \underline{\eta}) \nabla^{i_4} \underline{\alpha}_F + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3}(\hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}) \nabla^{i_4}(\rho_F, \sigma_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \underline{\eta}, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{i_4}(\rho_F, \sigma_F)$$
$$:= P_2,$$

as well as the equation

$$\nabla_4 \nabla^i \Upsilon_2 - {}^*\mathcal{D}_1 \nabla^i \Upsilon_1$$
$$= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \underline{\eta}) \nabla^{i_4}(\rho_F, \sigma_F) + \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}\omega \nabla^{i_4} \underline{\alpha}_F +$$
$$+ \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}\hat{\underline{\chi}} \nabla^{i_4}\alpha_F + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \underline{\eta}, \hat{\chi}) \nabla^{i_4} \underline{\alpha}_F$$
$$:= Q_2.$$

Applying the proposition, we arrive at

$$\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i \underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})}$$
$$\leq \|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(H_{u_\infty}^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i \underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u_\infty,u)})} + N_2 + M_2,$$

where

$$N_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^{i-1}P_2 \cdot (a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, \sigma_F)\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}', \qquad (5.194)$$

$$M_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^{i-1}Q_2 \cdot (a^{\frac{1}{2}})^{i-1}\nabla^i \underline{\alpha}_F\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'. \qquad (5.195)$$

We focus on N_2 first. Using the same reasoning as for N_1, we have

$$N_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^{i-1}P_2 \cdot (a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, \sigma_F)\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$
$$\leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1}P_2\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \cdot \|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'.$$

Recall at this point the form that P_2 assumes:

$$P_2 = \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta})\nabla^{i_4}\underline{\alpha}_F$$
$$+ \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\nabla^{i_4}(\rho_F,\sigma_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta,\underline{\eta},\hat{\underline{\chi}},\widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}(\rho_F,\sigma_F) := P_{21} + P_{22} + P_{23}.$$

Consequently, we have the bound

$$N_2 \le \sum_{j=1}^{3}\int_0^{\underline{u}}\int_{u_\infty}^{u}\frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^{i-1}P_{2j}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\cdot\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}'$$
$$= N_{21} + N_{22} + N_{23}.$$

$$(5.196)$$

We estimate each term separately.

- There holds

$$N_{21} = \int_{u_\infty}^{u}\frac{a}{|u'|^2}\int_0^{\underline{u}}\|(a^{\frac{1}{2}})^{i-1}P_{21}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'\,\mathrm{d}u'$$

$$\lesssim \int_{u_\infty}^{u}\frac{a}{|u'|^2}\left(\frac{O^4}{a\cdot|u'|^2}+\frac{O^2}{|u'|^2}\|a^5\nabla^{11}(\eta,\underline{\eta})\|^2_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u'})}\right)^{\frac{1}{2}}$$
$$\cdot\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u'})}\,\mathrm{d}u'$$

$$+\int_0^{\underline{u}}\left(\int_{u_\infty}^{u}\frac{a}{|u'|^2}\cdot\frac{O^2}{|u'|^2}\|a^5\nabla^{11}\underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\right)^{\frac{1}{2}}$$
$$\cdot\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F,\sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\lesssim \int_{u_\infty}^{u}\frac{a}{|u'|^2}\left(\frac{O^4}{a\cdot|u'|^2}+\frac{O^2}{|u'|^2}\|a^5\nabla^{11}(\eta,\underline{\eta})\|^2_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u'})}\right)^{\frac{1}{2}}\cdot F\,\mathrm{d}u'$$

$$+\int_0^{\underline{u}}\left(\int_{u_\infty}^{u}\frac{a}{|u'|^2}\cdot\frac{O^2}{|u'|^2}\|a^5\nabla^{11}\underline{\alpha}_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\right)^{\frac{1}{2}}\cdot a^{\frac{1}{2}}\cdot F\,\mathrm{d}\underline{u}'$$

Here F is the bootstrap constant appearing in (5.69). Notice that what we have done in the above is to separate between three cases. The first is when neither $(\eta,\underline{\eta})$ nor $\underline{\alpha}_F$ have 11 derivatives, the second is when 11 derivatives fall on $(\eta,\underline{\eta})$

and finally the third case is for when 11 derivatives fall on $\underline{\alpha}_F$. The reason for this distinction is that we use Hölder's inequality in different directions, depending on what elliptic estimates and Maxwell norms we have. Making use of Propositions 5.6.5 and 5.6.6, we conclude that the last two terms are $\lesssim 1$, using the section on elliptic estimates. In particular,

$$N_{21} \lesssim 1.$$

- There holds

$$N_{22} = \int_{u_\infty}^u \frac{a \cdot F}{|u'|^2} \, \mathrm{d}u'$$

$$\left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\nabla^{i_4}(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}' \right)^{\frac{1}{2}}$$

$$= \int_{u_\infty}^u F \, \mathrm{d}u'$$

$$\left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i-1} \sum_{i-1} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}\left(\frac{a}{|u'|^2}(\hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}) \right) \nabla^{i_4}(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \right)^{\frac{1}{2}}$$

$$\lesssim \int_{u_\infty}^u \frac{1}{a^{\frac{1}{2}}} \cdot \frac{O^3}{|u'|^2} \, \mathrm{d}u' \cdot F \lesssim 1.$$

- There holds

$$N_{23} = \int_{u_\infty}^u \int_0^{\underline{u}} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1}P_{23}\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u'.$$

We first distinguish between the cases where $i < 11$ and $i = 11$. For the case $i < 11$, there holds

$$\|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\eta, \underline{\eta}, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \frac{1}{a^{\frac{1}{2}}} \cdot \frac{|u|}{a^{\frac{1}{2}}} \cdot \frac{O^2}{|u|}.$$

Moreover, since $i < 11$, we can bound

$$\|(a^{\frac{1}{2}})^{i-1}\nabla^i(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \frac{O}{a^{\frac{1}{2}}}.$$

Hence, when $i < 11$, it is easy to establish that $N_{23} \lesssim 1$. When $i = 11$, we distinguish between four cases. The first one is when neither $(\eta, \underline{\eta}, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}})$

nor (ρ_F, σ_F) have 11 derivatives. The second is when 11 derivatives fall on (ρ_F, σ_F). The third is when 11 derivatives fall on $(\eta, \underline{\eta})$ and the fourth is when 11 derivatives fall on $(\hat{\chi}, \widetilde{\mathrm{tr}\chi})$. We treat these cases below:

$$
\begin{aligned}
N_{23} &\lesssim \int_{u_\infty}^{u} \int_{0}^{\underline{u}} \frac{a}{|u'|^2} \cdot \frac{O^2}{a} \cdot \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}' \, \mathrm{d}u' \\
&+ \int_{u_\infty}^{u} \int_{0}^{\underline{u}} \frac{a}{|u'|^2} \cdot \frac{|u'|}{a^{\frac{1}{2}}} \cdot \frac{O}{|u'|} \cdot \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}' \, \mathrm{d}u' \\
&+ \int_{u_\infty}^{u} \int_{0}^{\underline{u}} \frac{a \cdot O}{|u'|^3} \cdot \|a^5 \nabla^{11}(\eta, \underline{\eta})\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}' \, \mathrm{d}u' \\
&+ \int_{u_\infty}^{u} \int_{0}^{\underline{u}} \frac{a \cdot O}{|u'|^3} \cdot \|a^5 \nabla^{11}(\hat{\chi}, \widetilde{\mathrm{tr}\chi})\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}' \, \mathrm{d}u' \\
&\lesssim 1 + \int_{u_\infty}^{u} \frac{a \cdot O}{|u'|^3} \|a^5 \nabla^{11}(\eta, \underline{\eta})\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} \cdot \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} \, \mathrm{d}u' \\
&+ \int_{0}^{\underline{u}} O \int_{u_\infty}^{u} \frac{a}{|u'|^3} \|a^5 \nabla^{11}(\hat{\chi}, \widetilde{\mathrm{tr}\chi})\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}' \\
&\lesssim 1 + \int_{u_\infty}^{u} \frac{a \cdot O}{|u'|^3} \cdot \frac{|u'|}{a} \cdot R \cdot F \, \mathrm{d}u' \\
&+ \sup_{\underline{u}'} \left(\int_{u_\infty}^{u} \frac{a^2 O_\infty^2[\rho_F, \sigma_F]}{|u'|^4} \|a^5 \nabla^{11}(\hat{\chi}, \widetilde{\mathrm{tr}\chi})\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2 \, \mathrm{d}u' \right)^{\frac{1}{2}} \cdot a^{-\frac{1}{2}} \\
&\quad \cdot \|a^5 \nabla^{11}(\rho_F, \sigma_F)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty, u)})} \\
&\lesssim 1 + \mathcal{I}^{(0)} \cdot (\mathcal{R} + \underline{\mathcal{R}} + 1) \lesssim (\mathcal{I}^{(0)})^2 + \mathcal{R}^2 + \underline{\mathcal{R}}^2 + 1.
\end{aligned}
$$

$$(5.197)$$

Here we have used the fact that $O_\infty[\rho_F, \sigma_F] \lesssim 1$ from Proposition 5.4.11, the energy estimates on $\underline{\mathcal{F}}[\rho_F, \sigma_F]$ to bound the term by the initial data, as well as Proposition 5.6.8 and in particular (5.165).

Combining these estimates, we arrive at

$$N_{23} \lesssim (\mathcal{I}^{(0)})^2 + \mathcal{R}^2 + \underline{\mathcal{R}}^2 + 1.$$

Putting everything together, there holds

$$N_2 \lesssim (\mathcal{I}^{(0)})^2 + \mathcal{R}^2 + \underline{\mathcal{R}}^2 + 1.$$

We move on to M_2. We have

$$\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^{i-1} Q_2 \cdot (a^{\frac{1}{2}})^{i-1} \nabla^i \Upsilon_2\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

$$\leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_2\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \|(a^{\frac{1}{2}})^{i-1} \nabla^i \Upsilon_2\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

$$\leq \int_0^{\underline{u}} \left(\int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \nabla^i \Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \right)^{\frac{1}{2}} \, d\underline{u}'$$

$$\leq \left(\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}' \right)^{\frac{1}{2}} \cdot \sup_{\underline{u}'} \|(a^{\frac{1}{2}})^{i-1} \nabla^i \Upsilon_2\|_{\mathcal{L}^2_{(sc)}(H^{(u_\infty,u)}_{\underline{u}'})}.$$

Denote
$$H_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'.$$

Recall at this point that

$$Q_2 = \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \underline{\eta}) \nabla^{i_4}(\rho_F, \sigma_F) + \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \omega \nabla^{i_4} \underline{\alpha}_F$$

$$+ \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \underline{\hat{\chi}} \nabla^{i_4} \alpha_F + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \underline{\eta}, \hat{\chi}) \nabla^{i_4} \underline{\alpha}_F$$

$$:= Q_{21} + Q_{22} + Q_{23} + Q_{24}.$$

Thus,

$$H_2 \leq \sum_{j=1}^4 \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_{2j}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'. \tag{5.198}$$

We estimate term by term.

- The first two terms

$$\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1}(Q_{21}, Q_{22})\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

can be bounded by 1 as before.

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- For the third term, there holds

$$\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_{23}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$= \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\underline{\hat\chi}\nabla^{i_4}\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$\lesssim \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \cdot \frac{1}{a} \cdot \frac{|u'|^2}{a} \cdot a \cdot \frac{O[\hat\chi]^2 O[\alpha_F]^2}{|u'|^2} \, \mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{a}{|u'|^2} \cdot \frac{|u'|^2}{a} \cdot \frac{O^2}{|u'|^2} \left(\int_0^{\underline{u}} \|a^5\nabla^{11}\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}' \right) \mathrm{d}u'$$

$$+ \int_0^{\underline{u}} \left(\int_{u_\infty}^u \frac{a}{|u'|^2} \cdot \frac{a \cdot O^2}{|u'|^2} \cdot \frac{|u'|^2}{a} \cdot \|a^5\nabla^{11}\left(\frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat\chi}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \right) \mathrm{d}\underline{u}'$$

$$\lesssim 1 + (\mathcal{R}^2 + \underline{\mathcal{R}}^2 + 1) \cdot (\mathcal{F}^2[\alpha_F] + \underline{\mathcal{F}}^2[\rho_F, \sigma_F] + 1)$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a^2 \cdot O_\infty[\alpha_F]^2}{|u'|^4} \cdot \|a^5\nabla^{11}\underline{\hat\chi}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

We focus on the term

$$\int_{u_\infty}^u \frac{a^2 \cdot O_\infty[\alpha_F]^2}{|u'|^4} \|a^5\nabla^{11}\underline{\hat\chi}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u' \lesssim \int_{u_\infty}^u \frac{a^2}{|u'|^4} \|a^5\nabla^{11}\underline{\hat\chi}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, \mathrm{d}u'.$$

Recall that, from the proof of Proposition 5.6.8, there holds

$$\|a^5\nabla^{11}\underline{\hat\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \sum_{j\leq 10} \left(\|(a^{\frac{1}{2}})^j \nabla^{j+1}\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j \underline{\tilde\beta}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right.$$

$$+ \left\| (a^{\frac{1}{2}})^j \sum_{j_1+j_2=j} \nabla^{j_1}(\eta,\underline{\eta})\nabla^{j_2}(\hat\chi, \mathrm{tr}\chi) \right\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right)$$

$$+ \sum_{j\leq 10} \frac{1}{a^{\frac{1}{2}}} \cdot \|(a^{\frac{1}{2}}\nabla)^i \underline{\hat\chi}\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

This implies that

$$\|a^5 \nabla^{11} \hat{\underline{\chi}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \sum_{j \leq 10} \left(\|(a^{\frac{1}{2}})^j \nabla^{j+1} \widetilde{\mathrm{tr}\underline{\chi}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \|(a^{\frac{1}{2}}\nabla)^j \tilde{\underline{\beta}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right.$$

$$+ \frac{|u|^4}{a^2}\left\|(a^{\frac{1}{2}})^j \sum_{j_1+j_2=j} \nabla^{j_1}(\eta,\underline{\eta})\nabla^{j_2}\left(\frac{a}{|u|^2}\hat{\underline{\chi}}, \frac{a}{|u|^2}\mathrm{tr}\underline{\chi}\right)\right\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \right)$$

$$+ \sum_{j \leq 10} \cdot \left\|(a^{\frac{1}{2}}\nabla)^i\left(\frac{\hat{\underline{\chi}}}{a^{\frac{1}{2}}}\right)\right\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}$$

$$\lesssim \frac{|u|^2}{a^2}(\mathcal{R}+\underline{\mathcal{R}}+1)^2 + \|(a^{\frac{1}{2}}\nabla)^j\tilde{\underline{\beta}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \frac{|u|^4}{a^2}\cdot\frac{O^4}{|u|^2} + \frac{1}{a}\cdot\frac{|u|^2}{a}$$

$$\lesssim \frac{|u|^2\cdot R^2}{a^2} + \|(a^{\frac{1}{2}}\nabla)^j\tilde{\underline{\beta}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} + \frac{|u|^4}{a^2}\cdot\frac{O^4}{|u|^2} + \frac{1}{a}\cdot\frac{|u|^2}{a}.$$

Multiplying the above by $\frac{a^2}{|u|^4}$ and taking the integral in the incoming direction, we have

$$\int_{u_\infty}^u \frac{a^2}{|u'|^4}\|a^5\nabla^{11}\hat{\underline{\chi}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$\lesssim \int_{u_\infty}^u \frac{R^2}{|u'|^2}\,\mathrm{d}u' + \int_{u_\infty}^u \frac{a^2}{|u'|^4}\|(a^{\frac{1}{2}}\nabla)^j\tilde{\underline{\beta}}\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}\,\mathrm{d}u'$$

$$+ \int_{u_\infty}^u \frac{O^4+1}{|u'|^2}\,\mathrm{d}u' \lesssim \frac{R^2+O^4+1}{|u|} + \frac{16}{a^2}\underline{\mathcal{R}}[\tilde{\underline{\beta}}]^2 \lesssim 1.$$

Here we have used Propositions 5.4.10 and 5.6.8 as well as the bootstrap bounds (5.69).

- The fourth term can be bounded by 1 using the same procedures as above.

Consequently,

$$H_2 \leq 1 + (\mathcal{R}^2 + \underline{\mathcal{R}}^2 + 1) \cdot (\mathcal{F}^2[\alpha_F] + \underline{\mathcal{E}}^2[\rho_F, \sigma_F] + 1)$$

and also

$$M_2 \leq H_2^{\frac{1}{2}}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})} \leq H_2 + \frac{1}{4}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})}.$$

We also know that $N_2 \leq 1$. Putting everything together, we have

$$\|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_1\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})}$$

$$\leq \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_1\|^2_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,u)})} + N_2 + M_2$$

$$\leq \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_1\|^2_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,u)})} + N_2 + H_2$$

$$+ \frac{1}{4}\|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})}.$$

The last term can be absorbed by the left-hand side. Thus,

$$\|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_1\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u\infty,u)})}$$

$$\lesssim \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_1\|^2_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + \|(a^{\frac{1}{2}})^{i-1}\nabla^i\Upsilon_2\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,u)})}$$

$$+ \left((\mathcal{I}^{(0)})^2 + \mathcal{R}^2 + \underline{\mathcal{R}}^2 + 1\right) + (1 + \mathcal{R}^2 + \underline{\mathcal{R}}^2)(\underline{\mathcal{F}}^2[\rho_F, \sigma_F] + \mathcal{F}^2[\alpha_F] + 1).$$

Thus, we arrive at the energy inequality

$$\mathcal{F}^2[\rho_F, \sigma_F] + \underline{\mathcal{F}}^2[\underline{\alpha}_F]$$

$$\leq \mathcal{F}_0^2[\rho_F, \sigma_F] + \underline{\mathcal{F}}_0^2[\underline{\alpha}_F] + (1 + \mathcal{R}^2 + \underline{\mathcal{R}}^2)(\mathcal{F}_0^2[\alpha_F] + \underline{\mathcal{F}}^2[\rho_F, \sigma_F] + a^{-\frac{1}{4}})$$

$$\leq (\mathcal{I}^{(0)})^2 + (1 + \mathcal{R}^2 + \underline{\mathcal{R}}^2)((\mathcal{I}^{(0)})^2 + \frac{1}{a^{\frac{1}{4}}}) \lesssim (\mathcal{I}^{(0)})^2 + (\mathcal{R}^4 + \underline{\mathcal{R}}^4 + 1) + ((\mathcal{I}^{(0)})^4 + \frac{1}{a^{\frac{1}{2}}})$$

$$\lesssim \mathcal{R}^4 + \underline{\mathcal{R}}^4 + (\mathcal{I}^{(0)})^4 + (\mathcal{I}^{(0)})^2 + 1.$$

$$(5.199)$$

Combining (5.191) and (5.199) we get the following

Theorem 5.7.1. *Under the assumptions of Theorem 5.1.2 and the bootstrap bounds* (5.69), *there holds*

$$\mathcal{F} + \underline{\mathcal{F}} \lesssim \mathcal{R}^2 + \underline{\mathcal{R}}^2 + (\mathcal{I}^{(0)})^2 + (\mathcal{I}^{(0)}) + 1.$$

We can now use Theorem 5.7.1 to control the top-order derivative terms containing Maxwell components that will naturally appear in the error terms as we obtain the estimates for curvature below. $\qquad\square$

5.7.4   Energy estimates for curvature

Again, for $(\Psi_1, \Psi_2) \in \left\{(\alpha, \tilde{\beta}), (\tilde{\beta}, (\rho, \sigma)), ((\rho, \sigma), \underline{\tilde{\beta}}), (\underline{\tilde{\beta}}, \underline{\alpha})\right\}$ we have the following:

Proposition 5.7.8. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), assuming we have a pair (Ψ_1, Ψ_2) satisfying*

$$\nabla_3 \nabla^i \Psi_1 + \left(\frac{1+i}{2} + s_2(\Psi_1) \right) tr\underline{\chi} \nabla^i \Psi_1 - \mathcal{D} \nabla^i \Psi_2 = P, \tag{5.200}$$

$$\nabla_4 \nabla^i \Psi_2 - {}^*\mathcal{D} \nabla^i \Psi_1 = Q, \tag{5.201}$$

with $(\mathcal{D}, {}^*\mathcal{D})$ forming a Hodge dual, it follows

$$\int_{H_u^{(0,\underline{u})}} \|\nabla^i \Psi_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}' + \int_{\underline{H}_u^{(u\infty,u)}} \frac{a}{|u'|^2} \|\nabla^i \Psi_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$

$$\lesssim \int_{H_{u\infty}^{(0,\underline{u})}} \|\nabla^i \Psi_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u\infty,\underline{u}'})} \, d\underline{u}' + \int_{\underline{H}_0^{(u\infty,u)}} \frac{a}{|u'|^2} \|\nabla^i \Psi_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',0})} \, du'$$

$$+ \iint_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|} \|\nabla^i \Psi_1 \cdot P\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}' + \iint_{\mathcal{D}_{u,\underline{u}}} \frac{a}{|u'|} \|\nabla^i \Psi_2 \cdot Q\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'.$$

With this in mind, we begin by considering the pair $(\alpha, \tilde{\beta})$:

Proposition 5.7.9. *Under the assumptions of Theorem 5.1.2 and the bootstrap assumptions (5.69), we have for $i \leq 10$ the following:*

$$\frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}} \nabla)^i \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}} \nabla)^i \tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_u^{(u\infty,u)})}$$
$$\leq \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}} \nabla)^i \alpha\|_{\mathcal{L}^2_{(sc)}(H_{u\infty}^{(0,\underline{u})})} + \frac{1}{a^{\frac{1}{2}}} \|(a^{\frac{1}{2}} \nabla)^i \tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u\infty,u)})} + \frac{1}{a^{\frac{1}{3}}}. \tag{5.202}$$

Proof. We have the schematic equations

$$\nabla_4 \tilde{\beta} - {}^*\mathcal{D}_2 \alpha = \psi(\tilde{\beta}, \alpha) + \alpha_F \cdot \nabla \alpha_F + (\psi, \hat{\chi}) \cdot (\Upsilon, \alpha_F) + \psi \cdot (\Upsilon, \alpha_F) \cdot \alpha_F, \tag{5.203}$$

$$\nabla_3 \alpha + \frac{1}{2} tr\underline{\chi} \alpha + \mathcal{D}_2 \tilde{\beta} = (\psi, \hat{\chi}) \cdot (\Psi, \tilde{\beta}, \alpha) + (\rho_F, \sigma_F) \cdot \nabla(\rho_F, \sigma_F, \alpha_F)$$
$$+ \alpha_F \cdot \nabla(\rho_F, \sigma_F) + (\psi, \underline{\hat{\chi}}, tr\underline{\chi}) \cdot (\Upsilon, \alpha_F) \cdot (\Upsilon, \alpha_F).$$

Commuting the above equations i times with ∇ we arrive at

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$$\nabla_3 \nabla^i \alpha + \frac{1+i}{2} \text{tr} \underline{\chi} \nabla^i \alpha - \mathcal{D} \nabla^i \tilde{\beta}$$

$$= \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} (\hat{\underline{\chi}}, \text{tr} \underline{\chi}) \nabla^{i_4} \alpha$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}, \hat{\underline{\chi}}, \widetilde{\text{tr} \underline{\chi}}) \nabla^{i_4} (\Psi, \tilde{\beta}, \alpha)$$

$$+ \sum_{i_1+i_2+i_3=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\Upsilon \cdot \nabla(\Upsilon, \alpha_F) + \alpha_F \cdot \nabla \Upsilon$$

$$+ (\psi, \hat{\underline{\chi}}, \text{tr} \underline{\chi}) \cdot (\Upsilon, \alpha_F) \cdot (\Upsilon, \alpha_F))$$

$$= \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} (\hat{\underline{\chi}}, \text{tr} \underline{\chi}) \nabla^{i_4} \alpha$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}, \hat{\underline{\chi}}, \widetilde{\text{tr} \underline{\chi}}) \nabla^{i_4} (\Psi, \tilde{\beta}, \alpha)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\rho_F, \sigma_F, \alpha_F) \nabla^{i_4+1} (\rho_F, \sigma_F)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\rho_F, \sigma_F) \nabla^{i_4+1} (\alpha_F)$$

$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\underline{\chi}}, \text{tr} \underline{\chi}) \nabla^{i_4} (\Upsilon, \alpha_F) \nabla^{i_5} (\Upsilon, \alpha_F)$$

$$:= F_1,$$

$$\nabla_4 \nabla^i \tilde{\beta} - {}^* \mathcal{D} \nabla^i \alpha$$

$$= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}) \nabla^{i_4} (\tilde{\beta}, \alpha)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \alpha_F \nabla^{i_4+1} \alpha_F$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}) \nabla^{i_4} (\Upsilon, \alpha_F)$$

$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \alpha_F \nabla^{i_5} \alpha_F$$

$$:= G_1.$$

This gives us

$$\| (a^{\frac{1}{2}} \nabla)^i \alpha \|^2_{\mathcal{L}^2_{(sc)} (H_u^{(0,\underline{u})})} + \| (a^{\frac{1}{2}} \nabla)^i \tilde{\beta} \|^2_{\mathcal{L}^2_{(sc)} (\underline{H}_u^{(u\infty,u)})}$$

$$\leq \| (a^{\frac{1}{2}} \nabla)^i \alpha \|^2_{\mathcal{L}^2_{(sc)} (H_{u\infty}^{(0,\underline{u})})} + \| (a^{\frac{1}{2}} \nabla)^i \tilde{\beta} \|^2_{\mathcal{L}^2_{(sc)} (\underline{H}_0^{(u\infty,u)})} + N_1 + M_1,$$

where

$$N_1 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^i F_1 \cdot (a^{\frac{1}{2}}\nabla)^i \alpha\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}', \qquad (5.204)$$

$$M_1 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^i G_1 \cdot (a^{\frac{1}{2}}\nabla)^i \tilde{\beta}\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'. \qquad (5.205)$$

By Hölder's inequality, as per the previous subsection, we can obtain

$$N_1 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^i F_1 \cdot (a^{\frac{1}{2}}\nabla)^i \alpha\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

$$\leq \int_{u_\infty}^u \frac{a}{|u'|^2} \left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i F_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}' \right)^{\frac{1}{2}} du' \cdot \sup_{u'} \|(a^{\frac{1}{2}}\nabla)^i \alpha\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})},$$

where we recall that

$$\begin{aligned}
F_1 = &\sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\hat{\underline{\chi}}, \text{tr}\underline{\chi})\nabla^{i_4}\alpha \\
&+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi}, \hat{\underline{\chi}}, \widetilde{\text{tr}\underline{\chi}})\nabla^{i_4}(\Psi, \tilde{\beta}, \alpha) \\
&+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho_F, \sigma_F, \alpha_F)\nabla^{i_4+1}(\rho_F, \sigma_F) \\
&+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho_F, \sigma_F)\nabla^{i_4+1}(\alpha_F) \\
&+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\underline{\chi}}, \text{tr}\underline{\chi})\nabla^{i_4}(\Upsilon, \alpha_F)\nabla^{i_5}(\Upsilon, \alpha_F).
\end{aligned}$$

Denote

$$H_1 = \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i F_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'.$$

We work on the term H_1.

$$H_1 = \int_0^u \|(a^{\frac{1}{2}})^i F_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$\leq \int_0^u a^{-1} \|\frac{a^{\frac{1}{2}}}{|u'|}\psi, \frac{a^{\frac{1}{2}}}{|u'|}\hat{\chi}, \frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}}, \frac{a^{\frac{1}{2}}}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}}\|^2_{\mathcal{L}^\infty_{(sc)}(S_{u',\underline{u}'})} \|(a^{\frac{1}{2}}\nabla)^i(\Psi, \beta, \alpha)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4+1=i} \int_0^u \frac{|u'|^4}{a^2} \|(a^{\frac{1}{2}})^{i+1}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\frac{a}{|u'|^2}\hat{\chi}, \frac{a}{|u'|^2}\mathrm{tr}\chi)\nabla^{i_4}(\frac{\alpha}{a^{\frac{1}{2}}})\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{\substack{i_1+i_2+i_3+i_4=i, \\ i_4 \leq i-1}} \int_0^u \frac{|u'|^2}{a} \cdot$$

$$\|(a^{\frac{1}{2}})^{i+1}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a^{\frac{1}{2}}}{|u'|}\psi, \frac{a^{\frac{1}{2}}}{|u'|}\hat{\chi}, \frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}}, \frac{a^{\frac{1}{2}}}{|u'|}\widetilde{\mathrm{tr}\underline{\chi}}\right)\nabla^{i_4}\left(\frac{\Psi}{a^{\frac{1}{2}}}, \frac{\beta}{a^{\frac{1}{2}}}, \frac{\alpha}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^u a\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}})\nabla^{i_4+1}(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^u a\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho_F, \sigma_F)\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \int_0^u |u'|^4 \cdot$$

$$\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{a}{|u'|^2}\psi, \frac{a}{|u'|^2}\underline{\hat{\chi}}, \frac{a}{|u'|^2}\mathrm{tr}\underline{\chi})\nabla^{i_4}(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}})\nabla^{i_5}(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}})\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

- The sum of the terms $J_1 + J_2 + J_3$ can be controlled just as in the vacuum case by

$$\mathcal{R}^2[\alpha] \cdot \left(O^2[\hat{\chi}] + O^2[\underline{\hat{\chi}}]\right) + O^6 + O^4.$$

- For the terms J_4 and J_5 we have

$$J_4 + J_5 = \sum_{i_1+i_2+i_3+i_4=i} \int_0^u a\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}})\nabla^{i_4+1}(\rho_F, \sigma_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \int_0^u a\|(a^{\frac{1}{2}})^i\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\rho_F, \sigma_F)\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$\lesssim \frac{O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \cdot \left(\mathcal{F}[\alpha_F]^2 + \mathcal{F}[\rho_F, \sigma_F]^2\right) \lesssim \frac{O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \cdot (R^4 + 1).$$

where we have used Theorem 6.1 in the last inequality.

- The final term J_6 can be bounded by O^6.

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Therefore, putting everything together, we have

$$H_1 \leq \mathcal{R}^2[\alpha] \cdot \left(O^2[\hat{\chi}] + O^2[\underline{\hat{\chi}}] \right) + \frac{a \cdot O^2}{|u'|^2} \cdot (R^4 + (\mathcal{I}^{(0)})^4 + (\mathcal{I}^{(0)})^2 + 1) + O^6 + O^4,$$

which translates to

$$N_1 \leq \left(\mathcal{R}[\alpha] \cdot \left(O[\hat{\chi}] + O[\underline{\hat{\chi}}] \right) + \frac{a^{\frac{1}{2}} \cdot O}{|u'|} \cdot (R^2 + (\mathcal{I}^{(0)})^2 + \mathcal{I}^{(0)} + 1) + O^3 + O^2 \right)$$
$$\cdot \sup_{u'} \|(a^{\frac{1}{2}}\nabla)^i \alpha\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u'})}.$$

(5.206)

We continue with the term M_1 in the same way. We have

$$M_1 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^i G_1 \cdot (a^{\frac{1}{2}}\nabla)^i \tilde{\beta}\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$
$$\leq \left(\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i G_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}' \right)^{\frac{1}{2}} \cdot \sup_{\underline{u}'} \|(a^{\frac{1}{2}}\nabla)^i \tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}'})}.$$

At this point we recall that

$$G_1 = \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}(\tilde{\beta},\alpha)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\alpha_F\nabla^{i_4+1}\alpha_F + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}(\Upsilon,\alpha_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi})\nabla^{i_4}(\Upsilon,\alpha_F)\nabla^{i_5}\alpha_F.$$

Define

$$K_1 := \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i G_1\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'.$$

We then have

$$K_1 \leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a^2}{|u'|^2} \|(a^{\frac{1}{2}})^i \left(\frac{(\psi,\hat{\chi})}{a^{\frac{1}{2}}}\right) \nabla^i \left(\frac{\tilde{\beta}}{a^{\frac{1}{2}}}, \frac{\alpha}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a \, \mathrm{d}u'}{|u'|^2} \| \sum_{\substack{i_1+i_2+i_3+i_4=i \\ i_4 \leq i-1}} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{(\psi,\hat{\chi})}{a^{\frac{1}{2}}}\right) \nabla^{i_4} \left(\frac{\tilde{\beta}}{a^{\frac{1}{2}}}, \frac{\alpha}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \alpha_F \cdot \nabla^{i+1} \alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \| \sum_{\substack{i_1+i_2+i_3+i_4=i \\ i_4 \leq i-1}} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \alpha_F \nabla^{i_4+1} \alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \| \sum_{i_1+i_2+i_3+i_4=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi,\hat{\chi}) \nabla^{i_4}(\Upsilon, \alpha_F)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \| \sum_i (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi,\hat{\chi}) \nabla^{i_4}(\Upsilon, \alpha_F) \nabla^{i_5} \alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

- The sum $I_1 + I_2$ can be bounded by $R^2 \cdot O^2 + O^4$.

- We have

$$I_3 + I_4 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right) \cdot \nabla^{i+1} \alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a \, \mathrm{d}u'}{|u'|^2} \| \sum_{\substack{i_1+i_2+i_3+i_4=i \\ i_4 \leq i-1}} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right) \nabla^{i_4+1} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'$$

$$\leq \int_{u_\infty}^u \frac{a}{|u'|^2} \cdot \frac{a \cdot O^2}{|u'|^2} \left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i \nabla^{i+1} \alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}\underline{u}'\right) \, \mathrm{d}u'$$

$$+ \int_{u_\infty}^u \int_0^{\underline{u}} \frac{a}{|u'|^2} \cdot \frac{a \cdot O^4}{|u'|^2} \, \mathrm{d}\underline{u}' \, \mathrm{d}u'$$

$$\lesssim \frac{a^3 \cdot O^2}{|u'|^3} \cdot \mathcal{F}[\alpha_F]^2 + 1 \lesssim O^2 \cdot (\mathcal{I}^{(0)})^2 + 1.$$

(5.207)

- We have

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$$I_5$$

$$= \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a \, \mathrm{d}u'}{|u'|^2} \| \sum_{i_1+i_2+i_3+i_4=i} (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{\chi}) \nabla^{i_4}(\Upsilon, \alpha_F) \|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \mathrm{d}\underline{u}'$$

$$= \int_0^{\underline{u}} \mathrm{d}\underline{u}' \int_{u_\infty}^u \frac{a^3 \, \mathrm{d}u'}{|u'|^2} \| \sum_i (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}}) \nabla^{i_4}(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}}) \|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\leq \frac{a^3 \cdot O^4}{|u|^3} \leq O^4.$$

$$(5.208)$$

- For the last term I_6 we have

$$I_6 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \| \sum_i (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{\chi}) \nabla^{i_4}(\Upsilon, \alpha_F) \nabla^{i_5} \alpha_F \|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$= \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a^4}{|u'|^2} \cdot$$

$$\| \sum_i (a^{\frac{1}{2}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\psi}{a^{\frac{1}{2}}}, \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \nabla^{i_4} \left(\frac{\Upsilon}{a^{\frac{1}{2}}}, \frac{\alpha_F}{a^{\frac{1}{2}}} \right) \nabla^{i_5} \left(\frac{\alpha_F}{a^{\frac{1}{2}}} \right) \|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a^4}{|u'|^2} \cdot \frac{O^6}{|u'|^4} \, \mathrm{d}u' \, \mathrm{d}\underline{u}' = \frac{a^4 \cdot O^6}{|u|^5} \leq \frac{O^6}{a}.$$

$$(5.209)$$

Putting everything together, we have

$$K_1 \leq R^2 \cdot O^2 + O^4 + O^6 + \frac{O^2}{|u|^3}(R^4 + (\mathcal{I}^{(0)})^4 + (\mathcal{I}^{(0)})^2 + 1), \qquad (5.210)$$

so that

$$M_1 \leq \left(R \cdot O + O^2 + O^3 + \frac{O}{|u|^{\frac{3}{2}}}(R^2 + (\mathcal{I}^{(0)})^2 + \mathcal{I}^{(0)} + 1) \right) \cdot \sup_{\underline{u}'} \| (a^{\frac{1}{2}} \nabla)^i \tilde{\beta} \|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty, u)}_{\underline{u}'})}.$$

$$(5.211)$$

Putting (5.206) and (5.211) together, we finally get

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$$\frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_u^{(u_\infty,u)})}$$

$$\leq \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty}^{(0,\underline{u})})} + \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u_\infty,u)})} + \frac{1}{a^{\frac{1}{2}}}N_1 + \frac{1}{a^{\frac{1}{2}}}M_1$$

$$\leq \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(H_{u_\infty}^{(0,\underline{u})})} + \frac{1}{a^{\frac{1}{2}}}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u_\infty,u)})}$$

$$+ \frac{1}{a^{\frac{1}{2}}}\left(\mathcal{R}[\alpha]\cdot\left(O[\hat{\chi}] + O[\underline{\hat{\chi}}]\right) + \frac{a^{\frac{1}{2}}\cdot O}{|u'|}\cdot(R^2 + (\mathcal{I}^{(0)})^2 + \mathcal{I}^{(0)} + 1) + O^3 + O^2\right)\cdot$$

$$\sup_{u'}\|(a^{\frac{1}{2}}\nabla)^i\alpha\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}$$

$$+ \frac{1}{a^{\frac{1}{2}}}\cdot\left(R\cdot O + O^2 + O^3 + \frac{O}{|u|^{\frac{3}{2}}}(R^2 + (\mathcal{I}^{(0)})^2 + \mathcal{I}^{(0)} + 1)\right)\cdot$$

$$\cdot\sup_{\underline{u}'}\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}'}^{(u_\infty,u)})}$$

from which (5.202) follows.

$\square$

5.7.4.1 Energy estimates on the remaining curvature components

We proceed to show the estimates for the pair $(\Psi_1, \Psi_2) = (\tilde{\beta}, (\rho, \sigma))$. The other two pairs are similar. Before we continue with the proof, we shall need the following helpful proposition:

Proposition 5.7.10 (Lemma 14.8 in [43]). *Let $f(x,y), g(x,y)$ be positive functions defined on the rectangle $U := \left\{(x,y) \mid 0 \leq x \leq x_0, 0 \leq y \leq y_0\right\}$. Suppose there exist nonnegative constants J, c_1, c_2 such that f and g verify the inequality*

$$f(x,y) + g(x,y) \leq J + c_1\int_0^x f(x',y)\,dx' + c_2\int_0^y g(x,y')\,dy'$$

for all $(x,y) \in U$. Then there holds

$$f(x,y) + g(x,y) \lesssim Je^{c_1 x + c_2 y}, \quad \forall(x,y) \in U.$$

We now proceed with the energy estimates.

Proposition 5.7.11. *There holds*

$$\|(a^{\frac{1}{2}}\nabla)^i(\tilde{\beta}, \rho, \sigma, \underline{\tilde{\beta}})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}}\nabla)^i(\rho, \sigma, \underline{\tilde{\beta}}, \underline{\alpha})\|_{\mathcal{L}^2_{(sc)}(\underline{H}_u^{(u_\infty,u)})} \lesssim (\mathcal{I}^{(0)})^2 + \mathcal{I}^{(0)} + 1.$$

Proof. We proceed to show the estimates for the pair $(\Psi_1, \Psi_2) = (\tilde{\beta}, (\rho, \sigma))$. The other two pairs are similar. We have the schematic equation

$$\nabla_3\tilde{\beta} + \mathrm{tr}\underline{\chi}\,\tilde{\beta} - \mathcal{D}(-\rho,\sigma) = (\psi, \hat{\chi})\Psi + \alpha_F\nabla(\underline{\alpha}_F, \rho_F, \sigma_F) + (\rho_F, \sigma_F)\nabla(\rho_F, \sigma_F, \alpha_F)$$
$$+ (\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}, \hat{\chi})\cdot(\alpha_F, \Upsilon)\cdot\Upsilon,$$

$$(5.212)$$

$$\nabla_4((-\rho,\sigma)) - {}^*\!\mathcal{D}\tilde{\beta} = (\psi, \hat{\chi})(\Psi, \alpha)$$
$$+ \alpha_F\nabla\Upsilon + \Upsilon\nabla(\Upsilon, \alpha_F) + (\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}, \hat{\chi})(\alpha_F, \Upsilon)(\alpha_F, \Upsilon).$$

$$(5.213)$$

Commuting with i angular derivatives we get

$$\nabla_3\nabla^i\tilde{\beta} + \frac{i+2}{2}\mathrm{tr}\underline{\chi}\nabla^i\tilde{\beta} - \mathcal{D}\nabla^i((-\rho,\sigma))$$
$$= \sum_{i_1+i_2+i_3+i_4+1=i}\nabla^{i_1}\psi^{i_2+1}\nabla^{i_3}(\hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\nabla^{i_4}\Psi + \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\chi}, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}})\nabla^{i_4}\Psi$$
$$+ \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F, \Upsilon)\nabla^{i_4+1}\Upsilon + \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\Upsilon, \alpha_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}, \hat{\chi})\nabla^{i_4}(\alpha_F, \Upsilon)\nabla^{i_5}\Upsilon$$
$$:= F_2,$$

while

$$\nabla_4\nabla^i((-\rho,\sigma)) - {}^*\!\mathcal{D}\nabla^i\tilde{\beta}$$
$$= \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\underline{\chi}}, \hat{\chi})\nabla^{i_4}(\Psi, \alpha)$$
$$+ \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F, \Upsilon)\nabla^{i_4+1}\Upsilon + \sum_{i_1+i_2+i_3+i_4=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}(\Upsilon, \alpha_F)$$
$$+ \sum_{i_1+i_2+i_3+i_4+i_5=i}\nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\nabla^{i_4}(\alpha_F, \Upsilon)\nabla^{i_5}(\alpha_F, \Upsilon)$$
$$:= G_2.$$

We have

$$\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}}\nabla)^i(\rho,\sigma)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_u^{(u_\infty,u)})}$$
$$\leq\|(a^{\frac{1}{2}}\nabla)^i\tilde{\beta}\|^2_{\mathcal{L}^2_{(sc)}(H_{u_\infty}^{(0,\underline{u})})} + \|(a^{\frac{1}{2}}\nabla)^i(\rho,\sigma)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(u_\infty,u)})} + N_2 + M_2,$$

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where

$$N_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^i F_2 \cdot (a^{\frac{1}{2}} \nabla)^i \tilde{\beta}\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}', \qquad (5.214)$$

$$M_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^i G_2 \cdot (a^{\frac{1}{2}} \nabla)^i (\rho, \sigma)\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'. \qquad (5.215)$$

We have

$$N_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|} \|(a^{\frac{1}{2}})^i F_2 \cdot (a^{\frac{1}{2}} \nabla)^i \tilde{\beta}\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'$$

$$\leq \int_{u_\infty}^u \frac{a}{|u'|^2} \left(\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i F_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}' \right)^{\frac{1}{2}} \, du' \cdot R,$$

where we recall

$$\begin{aligned} F_2 = &\sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} (\hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}) \nabla^{i_4} \Psi \\ &+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}) \nabla^{i_4} \Psi \\ &+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \Upsilon) \nabla^{i_4+1} \Upsilon + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4+1} (\Upsilon, \alpha_F) \\ &+ \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi}) \nabla^{i_4} (\alpha_F, \Upsilon) \nabla^{i_5} \Upsilon. \end{aligned}$$

Define $H_2 = \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^i F_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$. The first two terms are handled like in the vacuum case. For the last three terms, we have

- For the first of the last three terms, we control

$$\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i+1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \nabla^{i_4+1} \Upsilon \|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$\leq \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i+1} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \nabla^{i+1} \Upsilon \|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$+ \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i+1} \sum_{\substack{i_1+i_2+i_3+i_4=i \\ i_4 \leq i-1}} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}} \right) \nabla^{i_4+1} \Upsilon \|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, d\underline{u}'$$

$$\leq \frac{O^2}{|u'|^2} \int_0^{\underline{u}} \|(a^{\frac{1}{2}} \nabla)^{i+1} \Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} + \int_0^{\underline{u}} \frac{O^4}{|u'|^2} \, d\underline{u}' \leq \frac{a \cdot O^2}{|u'|^2} (\mathcal{F}[\Upsilon])^2 + \frac{O^4}{|u'|^2}. \qquad (5.216)$$

- For the middle term we have

$$\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i+1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\leq \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i+1}\Upsilon\nabla^{i+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i+1} \sum_{\substack{i_1+i_2+i_3+i_4=i \\ i_4 \leq i-1}} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\leq \frac{a \cdot O^2}{|u'|^2}(\mathcal{F}[\alpha_F])^2 + \frac{O^4}{|u'|^2}.$$

$$(5.217)$$

- For the last term we have

$$\int_0^{\underline{u}} \|(a^{\frac{1}{2}})^{i} \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\nabla^{i_4}(\alpha_F, \Upsilon)\nabla^{i_5}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\leq \int_0^{\underline{u}} \frac{|u'|^4}{a} \cdot \frac{O^6}{|u'|^4} = \frac{O^6}{a}.$$

This completes the bounds on N_2. For M_2, we have

$$M_2 = \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|}\|(a^{\frac{1}{2}})^i G_2 \cdot (a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^1_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\mathrm{d}\underline{u}'$$

$$\leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i G_2\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\|(a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}'$$

$$\leq \int_0^{\underline{u}} \left(\int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i G_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\right)^{\frac{1}{2}}\left(\int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}}\nabla)^i\Psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\right)^{\frac{1}{2}}$$

$$= \int_0^{\underline{u}} \left(\int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i G_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\right)^{\frac{1}{2}}\|(a^{\frac{1}{2}}\nabla)^i\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}'})}\,\mathrm{d}\underline{u}'$$

$$\leq \left(\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i G_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}'\right)^{\frac{1}{2}}\left(\int_0^{\underline{u}}\|(a^{\frac{1}{2}}\nabla)^i\Psi\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}'})}\,\mathrm{d}\underline{u}'\right)^{\frac{1}{2}}$$

$$\leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i G_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}\,\mathrm{d}u'\,\mathrm{d}\underline{u}' + \frac{1}{4}\int_0^{\underline{u}}\|(a^{\frac{1}{2}}\nabla)^i\Psi\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(u_\infty,u)}_{\underline{u}'})}\,\mathrm{d}\underline{u}'.$$

It is already clear that the last term above will eventually be handled by Grönwall's inequality. To this end, we shall make use of Proposition 5.7.10.

Before applying Grönwall's inequality, we first define

$$K_2 = \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i G_2\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}'.$$

We then have

$$
\begin{aligned}
K_2 \leq & \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi},\underline{\hat{\chi}})\nabla^{i_4}(\Psi,\alpha)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}' \\
& + \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F,\Upsilon)\nabla^{i_4+1}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}' \\
& + \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, du' \, d\underline{u}' \\
& + \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\underline{\hat{\chi}},\text{tr}\underline{\chi})\nabla^{i_4}(\alpha_F,\Upsilon)\nabla^{i_5}(\alpha_F,\Upsilon)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \\
:= & K_{21} + K_{22} + K_{23} + K_{24}.
\end{aligned}
$$

- The term K_{21} can be controlled as below

$$
\begin{aligned}
& K_{21} \\
\leq & \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i(\psi,\hat{\chi},\underline{\hat{\chi}})\nabla^{i}(\Psi,\alpha)\|^2_{\mathcal{L}^2_{sc}(S_{u',\underline{u}})} du' d\underline{u}' \\
& + \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{\substack{i_1+i_2+i_3+i_4=i \\ i_4 \leq i-1}} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi,\hat{\chi},\underline{\hat{\chi}})\nabla^{i_4}(\Psi,\alpha)\|^2_{\mathcal{L}^2_{sc}(S_{u',\underline{u}})} \, du' \, d\underline{u}' \\
\leq & \int_0^{\underline{u}} \int_{u_\infty}^{u} \frac{a}{|u'|^4} \|\psi,\hat{\chi},\underline{\hat{\chi}}\|^2_{\mathcal{L}^\infty_{sc}(S_{u',\underline{u}})} \|(a^{\frac{1}{2}})^i\nabla^{i}(\Psi,\alpha)\|^2_{\mathcal{L}^2_{sc}(S_{u',\underline{u}})} \, du' \, d\underline{u}' \\
& + \int_0^{\underline{u}} \int_{u_\infty}^{u} \|(a^{\frac{1}{2}})^{i+1} \sum_{\substack{i_1+i_2+i_3+i_4=i \\ i_4 \leq i-1}} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\frac{a^{\frac{1}{2}}}{|u'|}(\psi,\hat{\chi},\underline{\hat{\chi}}))\nabla^{i_4}(\frac{(\Psi,\alpha)}{a^{\frac{1}{2}}})\|^2_{\mathcal{L}^2_{sc}(S_{u',\underline{u}})} \\
& \quad du' \, d\underline{u}'
\end{aligned}
$$

$$\leq \int_0^{\underline{u}} \int_{u_\infty}^u a^{-1} \|\frac{a^{\frac{1}{2}}}{|u'|}\psi, \frac{a^{\frac{1}{2}}}{|u'|}\hat{\chi}, \frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}}\|^2_{\mathcal{L}^\infty_{(sc)}(S_{u',\underline{u}})} \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i\nabla^i\Psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'd\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \|\frac{a^{\frac{1}{2}}}{|u'|}\psi, \frac{a^{\frac{1}{2}}}{|u'|}\hat{\chi}, \frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}}\|^2_{\mathcal{L}^\infty_{(sc)}(S_{u',\underline{u}})} \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i\nabla^i(a^{-\frac{1}{2}}\alpha)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'd\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \left(\frac{a}{|u'|^2}O^2[\underline{\hat{\chi}}]O^2[\alpha] + \frac{a^{-\frac{1}{2}}\cdot a\cdot O^4}{|u'|^2}\right)du'd\underline{u}'$$

$$\leq \int_0^{\underline{u}} a^{-1}\sup_{u'}\left(\|\frac{a^{\frac{1}{2}}}{|u'|}\psi, \frac{a^{\frac{1}{2}}}{|u'|}\hat{\chi}, \frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}}\|^2_{\mathcal{L}^\infty_{(sc)}(S_{u',\underline{u}})}\right)\cdot$$

$$\cdot \left(\int_{u_\infty}^u \frac{a}{|u'|^2}\|(a^{\frac{1}{2}})^i\nabla^i\Psi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'\right)d\underline{u}'$$

$$+ \int_{u_\infty}^u \sup_{u'}\left(\|\frac{a^{\frac{1}{2}}}{|u'|}\psi, \frac{a^{\frac{1}{2}}}{|u'|}\hat{\chi}, \frac{a^{\frac{1}{2}}}{|u'|}\underline{\hat{\chi}}\|^2_{\mathcal{L}^\infty_{(sc)}(S_{u',\underline{u}})}\right)\cdot \frac{a}{|u'|^2}\cdot$$

$$\cdot \left(\int_0^{\underline{u}}\|(a^{\frac{1}{2}})^i\nabla^i(a^{-\frac{1}{2}}\alpha)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}d\underline{u}'\right)du'$$

$$+ \frac{a}{|u|}\left(O^2[\underline{\hat{\chi}}] + O^2[\hat{\chi}] + 1\right)O^2[\alpha] + \frac{a^{\frac{1}{2}}}{|u|}\cdot O^4$$

$$\leq a^{-1}\cdot O^2\cdot R^2 + \left(O^2[\underline{\hat{\chi}}] + O^2[\hat{\chi}] + 1\right)\cdot\left(\underline{\mathcal{R}}^2[\beta] + \frac{1}{a}\right)$$

$$+ \frac{a}{|u|}\left(O^2[\underline{\hat{\chi}}] + O^2[\hat{\chi}] + 1\right)\cdot\left(\mathcal{R}^2[\alpha] + O^2[\alpha]\right) + \frac{a^{\frac{1}{2}}}{|u|}\cdot O^4$$

$$\lesssim \frac{1}{a^{\frac{1}{3}}} + (1 + \mathcal{R}^2[\alpha])\cdot(\underline{\mathcal{R}}^2[\beta] + \mathcal{R}^2[\alpha] + 1) \leq \left(\mathcal{I}^{(0)}\right)^4 + 1.$$

- The sum $K_{22} + K_{23}$ can be controlled by

$$K_{22} + K_{23}$$

$$= \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\alpha_F, \Upsilon)\nabla^{i_4+1}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$\lesssim \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a^2}{|u'|^2} \|a^5 \left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}}\right) \nabla^{11}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i+1} \sum_{\substack{i_1+i_2+i_3+i_4=i, \\ i_4<10}} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{\alpha_F}{a^{\frac{1}{2}}}, \frac{\Upsilon}{a^{\frac{1}{2}}}\right) \nabla^{i_4+1}\Upsilon\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a^2}{|u'|^2} \|a^5 \Upsilon \nabla^{11}\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$+ \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i+1} \sum_{\substack{i_1+i_2+i_3+i_4=i, \\ i_4<10}} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\Upsilon\nabla^{i_4+1}\alpha_F\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})} \, \mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$\leq \int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \cdot \frac{a(O^4 + O^2 \cdot F^2)}{|u'|^2} \, \mathrm{d}u' \, \mathrm{d}\underline{u}' \lesssim 1.$$

$$(5.218)$$

- The final term K_{24} can be controlled in the same way

$$\int_0^{\underline{u}} \int_{u_\infty}^u \frac{a}{|u'|^2} \cdot$$

$$\|(a^{\frac{1}{2}})^i \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1}\psi^{i_2}\nabla^{i_3}(\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})\nabla^{i_4}(\alpha_F, \Upsilon)\nabla^{i_5}(\alpha_F, \Upsilon)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$= \int_0^{\underline{u}} \int_{u_\infty}^u a \cdot |u'|^2 \cdot$$

$$\|(a^{\frac{1}{2}})^i \sum_i \nabla^{i_1}\psi^{i_2}\nabla^{i_3}\left(\frac{a(\psi, \hat{\underline{\chi}}, \mathrm{tr}\underline{\chi})}{|u'|^2}\right) \nabla^{i_4}\left(\frac{(\alpha_F, \Upsilon)}{a^{\frac{1}{2}}}\right) \nabla^{i_5}\left(\frac{(\alpha_F, \Upsilon)}{a^{\frac{1}{2}}}\right)\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}$$

$$\mathrm{d}u' \, \mathrm{d}\underline{u}'.$$

$$(5.219)$$

Clearly, in the above, the worst bound that can be obtained is when there is a triple anomaly. This only holds when the Ricci coefficient is $\mathrm{tr}\underline{\chi}$ and both Maxwell components are α_F. Also, importantly, the number i of angular derivatives is at most 10 so that no elliptic estimates are required. In this case,

$$\int_0^{\underline{u}} \int_{u_\infty}^u a \cdot |u'|^2 \cdot \|(a^{\frac{1}{2}})^i \sum_i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{a \operatorname{tr}\chi}{|u'|^2} \right) \nabla^{i_4} \left(\frac{\alpha_F}{a^{\frac{1}{2}}} \right) \nabla^{i_5} \left(\frac{\alpha_F}{a^{\frac{1}{2}}} \right) \|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}'})}^2$$

$$\mathrm{d}u' \, \mathrm{d}\underline{u}'$$

$$\lesssim \sup_{\underline{u}} \int_{u_\infty}^u a \cdot |u'|^2 \cdot \frac{O^2[\operatorname{tr}\chi] O^4[\alpha_F]}{|u'|^4} \, \mathrm{d}u' \lesssim O^2[\operatorname{tr}\chi] O^4[\alpha_F].$$

$$(5.220)$$

Now $O[\operatorname{tr}\chi] \lesssim 1$ by Proposition 5.4.8 and $O[\alpha_F] \lesssim \underline{\mathcal{F}}[\rho_F, \sigma_F] + 1 \lesssim 2\underline{\mathcal{F}}_0[\rho_F, \sigma_F] + \frac{1}{a^{\frac{1}{8}}}$, where in the first inequality we use the scale-invariant L^2–estimates on the Maxwell components and in the second inequality the energy estimates. In particular, the inequality can be traced back to the initial data and this is a key point.
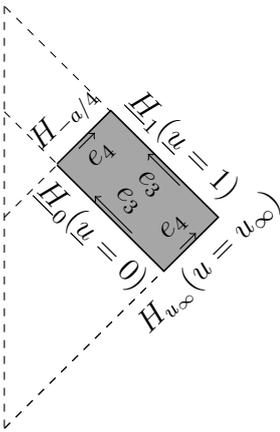
Finally, we use Grönwall's inequality for M_2 and conclude. In a similar way, we obtain

$$\|(a^{\frac{1}{2}}\nabla)^i (\rho, \sigma, \underline{\tilde{\beta}})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(a^{\frac{1}{2}}\nabla)^i (\underline{\tilde{\beta}}, \underline{\alpha})\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} \lesssim (\mathcal{I}^{(0)})^2 + \mathcal{I}^{(0)} + 1.$$

$\square$

5.8 The formation of trapped surfaces

In this section, we prove

Theorem 5.1.3 Given $\mathcal{I}^{(0)}$, there exists a sufficiently large $a_0 = a_0(\mathcal{I}^{(0)})$ such that the following holds. For any $0 < a_0 < a$, the unique smooth solution $(\mathcal{M}, g)$ of the Einstein–Maxwell equations from Theorem 5.1.2 with initial data satisfying

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{1}{2}} \|\nabla_4^k (|u_\infty|\nabla)^i (\hat{\chi}, \alpha_F)\|_{L^\infty(S_{u_\infty,\underline{u}})} \leq \mathcal{I}^{(0)}$ along $u = u_\infty$,

- Minkowskian initial data along $\underline{u} = 0$,

- $\int_0^1 |u_\infty|^2 \left(|\hat{\chi}_0|^2 + |\alpha_{F0}|^2 \right) (u_\infty, \underline{u}') \, \mathrm{d}\underline{u}' \geq a$ uniformly for every direction along $u = u_\infty$

has a trapped surface at $S_{-a/4,1}$.

Proof. We first derive pointwise estimates for $|\hat{\chi}|_\gamma^2$. Fix $(\theta^1, \theta^2) \in S^2$. We consider the following null structure equation

$$\nabla_3 \hat{\chi} + \frac{1}{2} \operatorname{tr}\underline{\chi} \hat{\chi} - 2\underline{\omega} \hat{\chi} = \nabla \widehat{\otimes} \eta - \frac{1}{2} \operatorname{tr}\chi \underline{\hat{\chi}} + \eta \widehat{\otimes} \eta.$$

We contract this 2-tensor with another 2-tensor $\hat{\chi}$ and get

$$\frac{1}{2}\nabla_3|\hat{\chi}|^2_\gamma + \frac{1}{2}\mathrm{tr}\underline{\chi}|\hat{\chi}|^2_\gamma - 2\underline{\omega}|\hat{\chi}|^2_\gamma = \hat{\chi}(\nabla\widehat{\otimes}\eta - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta). \tag{5.221}$$

Employing the fact $\underline{\omega} = -\frac{1}{2}\nabla_3(\log\Omega) = -\frac{1}{2}\Omega^{-1}\nabla_3\Omega$, we rewrite (5.221) as

$$\nabla_3(\Omega^2|\hat{\chi}|^2_\gamma) + \Omega^2\mathrm{tr}\underline{\chi}|\hat{\chi}|^2_\gamma = 2\Omega^2\hat{\chi}(\nabla\widehat{\otimes}\eta - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta).$$

Using $\nabla_3 = \frac{1}{\Omega}(\frac{\partial}{\partial u} + b^A\frac{\partial}{\partial\theta^A})$, we rewrite the above equation as

$$\frac{\partial}{\partial u}(\Omega^2|\hat{\chi}|^2_\gamma) + \Omega\mathrm{tr}\underline{\chi}\cdot\Omega^2|\hat{\chi}|^2_\gamma = 2\Omega^3\hat{\chi}(\nabla\widehat{\otimes}\eta - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta) - b^A\frac{\partial}{\partial\theta^A}(\Omega^2|\hat{\chi}|^2_\gamma).$$

Substitute $\Omega\mathrm{tr}\underline{\chi}$ with

$$\Omega\mathrm{tr}\underline{\chi} = \Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|}) - \Omega\frac{2}{|u|} = \Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|}) - (\Omega - 1)\frac{2}{|u|} - \frac{2}{|u|}$$

we have

$$\frac{\partial}{\partial u}(\Omega^2|\hat{\chi}|^2_\gamma) - \frac{2}{|u|}\Omega^2|\hat{\chi}|^2_\gamma = 2\Omega^3\hat{\chi}(\nabla\widehat{\otimes}\eta - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta) - b^A\frac{\partial}{\partial\theta^A}(\Omega^2|\hat{\chi}|^2_\gamma)$$
$$- \Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|})(\Omega^2|\hat{\chi}|^2_\gamma) + (\Omega - 1)\cdot\frac{2}{|u|}\cdot(\Omega^2|\hat{\chi}|^2_\gamma).$$

This gives

$$\frac{\partial}{\partial u}\left(u^2\Omega^2|\hat{\chi}|^2_\gamma\right) = 2\cdot|u|^2\cdot\Omega^3\hat{\chi}(\nabla\widehat{\otimes}\eta - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta) - |u|^2\cdot b^A\frac{\partial}{\partial\theta^A}(\Omega^2|\hat{\chi}|^2_\gamma)$$
$$- |u|^2\cdot\Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|})(\Omega^2|\hat{\chi}|^2_\gamma) + |u|^2\cdot(\Omega - 1)\cdot\frac{2}{|u|}\cdot(\Omega^2|\hat{\chi}|^2_\gamma). \tag{5.222}$$

For b^A, we have equation

$$\frac{\partial b^A}{\partial\underline{u}} = -4\Omega^2\zeta^A,$$

which is from

$$[L, \underline{L}] = \frac{\partial b^A}{\partial\underline{u}}\frac{\partial}{\partial\theta^A}.$$

Applying the identity $\zeta_A = \frac{1}{2}\eta_A - \frac{1}{2}\underline{\eta}_A$, Proposition 5.3.1 and the derived estimates on $\eta, \underline{\eta}$, we conclude that there holds in $\mathcal{D}_{u,\underline{u}}$

$$\|b^A\|_{L^\infty(S_{u,\underline{u}})} \le \frac{a^{\frac{1}{2}}}{|u|^2}.$$

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For the right hand side of (5.222), we have

$$\|2 \cdot |u|^2 \cdot \Omega^3 \hat{\chi} (\nabla \widehat{\otimes} \eta - \frac{1}{2} \mathrm{tr} \chi \underline{\hat{\chi}} + \eta \widehat{\otimes} \eta)\|_{L^\infty(S_{u,\underline{u}})} \leq |u|^2 \cdot \frac{a^{\frac{1}{2}}}{|u|} \cdot (\frac{a^{\frac{1}{2}}}{|u|^3} + \frac{a}{|u|^4}) \leq \frac{a}{|u|^2},$$

$$\||u|^2 \cdot b^A \frac{\partial}{\partial \theta^A}(\Omega^2 |\hat{\chi}|_\gamma^2)\|_{L^\infty(S_{u,\underline{u}})} \leq |u|^2 \cdot \frac{a^{\frac{1}{2}}}{|u|^2} \cdot \frac{a}{|u|^2} \leq \frac{a^{\frac{3}{2}}}{|u|^2},$$

$$\| - |u|^2 \cdot \Omega (\mathrm{tr} \underline{\chi} + \frac{2}{|u|})(\Omega^2 |\hat{\chi}|_\gamma^2)\|_{L^\infty(S_{u,\underline{u}})} \leq |u|^2 \cdot \frac{1}{|u|^2} \cdot \frac{a}{|u|^2} \leq \frac{a}{|u|^2},$$

$$\||u|^2 \cdot (\Omega - 1) \cdot \frac{2}{|u|} \cdot (\Omega^2 |\hat{\chi}|_\gamma^2)\|_{L^\infty(S_{u,\underline{u}})} \leq |u|^2 \cdot \frac{1}{|u|} \cdot \frac{2}{|u|} \cdot \frac{a}{|u|^2} \leq \frac{a}{|u|^2}.$$

In summary, we have

$$\frac{\partial}{\partial u}\left(u^2 \Omega^2 |\hat{\chi}|_\gamma^2\right) = M, \text{ and } |M| \lesssim \frac{a^{\frac{3}{2}}}{|u|^2} \ll \frac{a^{\frac{7}{4}}}{|u|^2},$$

which implies

$$-\frac{a^{\frac{7}{4}}}{|u|} + \frac{a^{\frac{7}{4}}}{|u_\infty|} \leq |u|^2 \Omega^2 |\hat{\chi}|_\gamma^2(u, \underline{u}, \theta^1, \theta^2) - |u_\infty|^2 \Omega^2 |\hat{\chi}|_\gamma^2(u_\infty, \underline{u}, \theta^1, \theta^2).$$

Recall $\Omega(u_\infty, \underline{u}, \theta^1, \theta^2) = 1$. We hence have

$$|u|^2 \Omega^2 |\hat{\chi}|_\gamma^2(u, \underline{u}, \theta^1, \theta^2) \geq |u_\infty|^2 |\hat{\chi}|_\gamma^2(u_\infty, \underline{u}, \theta^1, \theta^2) - \frac{a^{\frac{7}{4}}}{|u|}.$$

Integrating with respect to $\underline{u}$, we further have for $u_\infty \leq u \leq -a/4$

$$\int_0^1 |u|^2 \Omega^2 |\hat{\chi}|_\gamma^2(u, \underline{u}', \theta^1, \theta^2) d\underline{u}' \geq \int_0^1 |u_\infty|^2 |\hat{\chi}|_\gamma^2(u_\infty, \underline{u}', \theta^1, \theta^2) d\underline{u}' - \frac{a^{\frac{7}{4}}}{|u|}. \qquad (5.223)$$

In the same fashion, we derive pointwise estimates for $|\alpha_F|_\gamma^2$. Consider the null Maxwell equation

$$\nabla_3 \alpha_F + \frac{1}{2} \mathrm{tr} \underline{\chi} \alpha_F - 2\underline{\omega} \alpha_F = -\nabla \rho_F + {}^*\nabla \sigma_F - 2 \, {}^*\underline{\eta} \cdot \sigma_F + 2\underline{\eta} \cdot \rho_F - \hat{\chi} \cdot \underline{\alpha}_F.$$

We contract this 1-form with another 1-form α_F and get

$$\frac{1}{2}\nabla_3 |\alpha_F|_\gamma^2 + \frac{1}{2}\mathrm{tr}\underline{\chi}|\alpha_F|_\gamma^2 - 2\underline{\omega}|\alpha_F|_\gamma^2 = \alpha_F(-\nabla \rho_F + {}^*\nabla \sigma_F - 2 \, {}^*\underline{\eta} \cdot \sigma_F + 2\underline{\eta} \cdot \rho_F - \hat{\chi} \cdot \underline{\alpha}_F). \tag{5.224}$$

Employing the fact $\underline{\omega} = -\frac{1}{2}\nabla_3(\log \Omega) = -\frac{1}{2}\Omega^{-1}\nabla_3\Omega$, we rewrite (5.224) as

$$\nabla_3(\Omega^2 |\alpha_F|_\gamma^2) + \Omega^2 \mathrm{tr}\underline{\chi}|\alpha_F|_\gamma^2 = 2\Omega^2 \alpha_F(-\nabla \rho_F + {}^*\nabla \sigma_F - 2 \, {}^*\underline{\eta} \cdot \sigma_F + 2\underline{\eta} \cdot \rho_F - \hat{\chi} \cdot \underline{\alpha}_F).$$

Using $\nabla_3 = \frac{1}{\Omega}(\frac{\partial}{\partial u} + b^A \frac{\partial}{\partial \theta^A})$, we rewrite the above equation as

$$\frac{\partial}{\partial u}(\Omega^2|\alpha_F|_\gamma^2) + \Omega\mathrm{tr}\underline{\chi} \cdot \Omega^2|\alpha_F|_\gamma^2 = 2\Omega^3\alpha_F(-\nabla\rho_F + {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F + 2\underline{\eta}\cdot\rho_F - \hat{\chi}\cdot\underline{\alpha}_F)$$
$$- b^A\frac{\partial}{\partial\theta^A}(\Omega^2|\alpha_F|_\gamma^2).$$

Substituting $\Omega\mathrm{tr}\underline{\chi}$ with

$$\Omega\mathrm{tr}\underline{\chi} = \Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|}) - \Omega\frac{2}{|u|} = \Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|}) - (\Omega - 1)\frac{2}{|u|} - \frac{2}{|u|}$$

we have

$$\frac{\partial}{\partial u}(\Omega^2|\alpha_F|_\gamma^2) - \frac{2}{|u|}\Omega^2|\alpha_F|_\gamma^2 = 2\Omega^3\alpha_F(-\nabla\rho_F + {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F + 2\underline{\eta}\cdot\rho_F - \hat{\chi}\cdot\underline{\alpha}_F)$$
$$- b^A\frac{\partial}{\partial\theta^A}(\Omega^2|\alpha_F|_\gamma^2)$$
$$- \Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|})(\Omega^2|\alpha_F|_\gamma^2) + (\Omega - 1)\cdot\frac{2}{|u|}\cdot(\Omega^2|\alpha_F|_\gamma^2).$$

This gives

$$\frac{\partial}{\partial u}\left(u^2\Omega^2|\alpha_F|_\gamma^2\right) = 2\cdot|u|^2\cdot\Omega^3\alpha_F(-\nabla\rho_F + {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F + 2\underline{\eta}\cdot\rho_F - \hat{\chi}\cdot\underline{\alpha}_F)$$
$$- |u|^2\cdot b^A\frac{\partial}{\partial\theta^A}(\Omega^2|\alpha_F|_\gamma^2)$$
$$- |u|^2\cdot\Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|})(\Omega^2|\alpha_F|_\gamma^2) + |u|^2\cdot(\Omega - 1)\cdot\frac{2}{|u|}\cdot(\Omega^2|\alpha_F|_\gamma^2).$$
(5.225)

For b^A, we have the equation
$$\frac{\partial b^A}{\partial\underline{u}} = -4\Omega^2\zeta^A,$$

which is from
$$[L, \underline{L}] = \frac{\partial b^A}{\partial\underline{u}}\frac{\partial}{\partial\theta^A}.$$

Applying the identity $\zeta_A = \frac{1}{2}\eta_A - \frac{1}{2}\underline{\eta}_A$, Propositions 5.3.1, derived estimates of $\eta, \underline{\eta}$, it holds in $\mathcal{D}_{u,\underline{u}}$
$$\|b^A\|_{L^\infty(S_{u,\underline{u}})} \leq \frac{a^{\frac{1}{2}}}{|u|^2}.$$

For the right hand side of (5.222), we have

$$\|2\cdot|u|^2\cdot\Omega^3\alpha_F(-\nabla\rho_F + {}^*\nabla\sigma_F - 2\,{}^*\underline{\eta}\cdot\sigma_F + 2\underline{\eta}\cdot\rho_F - \hat{\chi}\cdot\underline{\alpha}_F)\|_{L^\infty(S_{u,\underline{u}})}$$
$$\leq |u|^2\cdot\frac{a^{\frac{1}{2}}}{|u|}\cdot(\frac{a^{\frac{1}{2}}}{|u|^3} + \frac{a}{|u|^4} + \frac{a^{\frac{3}{2}}}{|u|^4}) \leq \frac{a}{|u|^2} + \frac{a^2}{|u|^3},$$
(5.226)

$$\||u|^2 \cdot b^A \frac{\partial}{\partial \theta^A}(\Omega^2 |\alpha_F|_\gamma^2)\|_{L^\infty(S_{u,\underline{u}})} \leq |u|^2 \cdot \frac{a^{\frac{1}{2}}}{|u|^2} \cdot \frac{a}{|u|^2} \leq \frac{a^{\frac{3}{2}}}{|u|^2},$$

$$\| - |u|^2 \cdot \Omega(\mathrm{tr}\underline{\chi} + \frac{2}{|u|})(\Omega^2 |\alpha_F|_\gamma^2)\|_{L^\infty(S_{u,\underline{u}})} \leq |u|^2 \cdot \frac{1}{|u|^2} \cdot \frac{a}{|u|^2} \leq \frac{a}{|u|^2},$$

$$\||u|^2 \cdot (\Omega - 1) \cdot \frac{2}{|u|} \cdot (\Omega^2 |\alpha_F|_\gamma^2)\|_{L^\infty(S_{u,\underline{u}})} \leq |u|^2 \cdot \frac{1}{|u|} \cdot \frac{2}{|u|} \cdot \frac{a}{|u|^2} \leq \frac{a}{|u|^2}.$$

In summary, we have

$$\frac{\partial}{\partial u}\left(u^2 \Omega^2 |\alpha_F|_\gamma^2\right) = M, \text{ and } |M| \lesssim \frac{a^{\frac{3}{2}}}{|u|^2} \ll \frac{a^{\frac{7}{4}}}{|u|^2},$$

which implies

$$-\frac{a^{\frac{7}{4}}}{|u|} + \frac{a^{\frac{7}{4}}}{|u_\infty|} \leq |u|^2 \Omega^2 |\alpha_F|_\gamma^2(u, \underline{u}, \theta^1, \theta^2) - |u_\infty|^2 \Omega^2 |\alpha_F|_\gamma^2(u_\infty, \underline{u}, \theta^1, \theta^2).$$

Recall $\Omega(u_\infty, \underline{u}, \theta^1, \theta^2) = 1$. We hence have

$$|u|^2 \Omega^2 |\alpha_F|_\gamma^2(u, \underline{u}, \theta^1, \theta^2) \geq |u_\infty|^2 |\alpha_F|_\gamma^2(u_\infty, \underline{u}, \theta^1, \theta^2) - \frac{a^{\frac{7}{4}}}{|u|}.$$

Integrating with respect to $\underline{u}$, we further have for $u_\infty \leq u \leq -a/4$

$$\int_0^1 |u|^2 \Omega^2 |\alpha_F|_\gamma^2(u, \underline{u}', \theta^1, \theta^2) d\underline{u}' \geq \int_0^1 |u_\infty|^2 |\alpha_F|_\gamma^2(u_\infty, \underline{u}', \theta^1, \theta^2) d\underline{u}' - \frac{a^{\frac{7}{4}}}{|u|}.$$

Together with (5.223)

$$\int_0^1 |u|^2 \Omega^2 |\hat{\chi}|_\gamma^2(u, \underline{u}', \theta^1, \theta^2) d\underline{u}' \geq \int_0^1 |u_\infty|^2 |\hat{\chi}|_\gamma^2(u_\infty, \underline{u}', \theta^1, \theta^2) d\underline{u}' - \frac{a^{\frac{7}{4}}}{|u|}.$$

We conclude that

$$\int_0^1 |u|^2 \Omega^2 (|\hat{\chi}|_\gamma^2 + |\alpha_F|_\gamma^2)(u, \underline{u}', \theta^1, \theta^2) d\underline{u}'$$

$$\geq \int_0^1 |u_\infty|^2 (|\hat{\chi}_F|_\gamma^2 + |\alpha_F|_\gamma^2)(u_\infty, \underline{u}', \theta^1, \theta^2) d\underline{u}' - \frac{2a^{\frac{7}{4}}}{|u|}$$

$$\geq a - \frac{2a^{\frac{7}{4}}}{|u|} \geq a - \frac{8a^{\frac{7}{4}}}{a} \geq \frac{7a}{8}.$$

Pick $u = -a/4$. With the fact $\|\Omega - 1\|_{L^\infty(S_{u,\underline{u}})} \lesssim 1/a$, for sufficiently large a, we hence have

$$\left(-\frac{a}{4}\right)^2 \int_0^1 (|\hat{\chi}|_\gamma^2 + |\alpha_F|_\gamma^2)\left(-\frac{a}{4}, \underline{u}', \theta^1, \theta^2\right) d\underline{u}'$$

$$\geq \frac{6}{7} \cdot \int_0^1 \left(-\frac{a}{4}\right)^2 \Omega^2 (|\hat{\chi}|_\gamma^2 + |\alpha_F|_\gamma^2)\left(-\frac{a}{4}, \underline{u}', \theta^1, \theta^2\right) d\underline{u}'$$

$$\geq \frac{6}{7} \cdot \frac{7a}{8} = \frac{3a}{4}.$$

This implies

$$\int_0^1 (|\hat{\chi}|_\gamma^2 + |\alpha_F|_\gamma^2)(-\frac{a}{4}, \underline{u}', \theta^1, \theta^2) d\underline{u}' \geq \frac{3a}{4} \cdot \frac{16}{a^2} = \frac{12}{a} \tag{5.227}$$

We now consider the outgoing null structure equation for $\text{tr}\chi$,

$$\nabla_4 \text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 = -|\hat{\chi}|_\gamma^2 - 2\omega \text{tr}\chi - |\alpha_F|_\gamma^2.$$

Using $\omega = -\frac{1}{2}\nabla_4(\log\Omega)$, we have

$$\nabla_4 \text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 = -|\hat{\chi}|^2 - 2\omega \text{tr}\chi - |\alpha_F|_\gamma^2$$

$$= -|\hat{\chi}|_\gamma^2 + \nabla_4(\log\Omega)\text{tr}\chi - |\alpha_F|_\gamma^2 = -|\hat{\chi}|_\gamma^2 + \frac{1}{\Omega}\nabla_4\Omega \cdot \text{tr}\chi - |\alpha_F|_\gamma^2.$$

Hence,

$$\nabla_4(\Omega^{-1}\text{tr}\chi) = -\Omega^{-2}\nabla_4\Omega \cdot \text{tr}\chi + \Omega^{-1}\nabla_4\text{tr}\chi$$

$$= \Omega^{-1}(\nabla_4\text{tr}\chi - \Omega^{-1} \cdot \nabla_4\Omega \cdot \text{tr}\chi) = \Omega^{-1}\left(-\frac{1}{2}(\text{tr}\chi)^2 - |\hat{\chi}|_\gamma^2 - |\alpha_F|_\gamma^2\right).$$

With the fact $e_4 = \Omega^{-1}\frac{\partial}{\partial\underline{u}}$, we have

$$\frac{\partial}{\partial\underline{u}}(\Omega^{-1}\text{tr}\chi) = -\frac{1}{2}(\text{tr}\chi)^2 - |\hat{\chi}|_\gamma^2 - |\alpha_F|_\gamma^2. \tag{5.228}$$

For every $(\theta^1, \theta^2) \in \mathbb{S}^2$, along $\underline{H}_0$ we have

$$(\Omega^{-1}\text{tr}\chi)(-\frac{a}{4}, 0, \theta^1, \theta^2) = 1^{-1} \cdot \frac{2}{a/4} = \frac{8}{a}.$$

We then integrate (5.228). Using (5.227) we obtain

$$(\Omega^{-1}\text{tr}\chi)(-\frac{a}{4}, 1, \theta^1, \theta^2)$$

$$\leq (\Omega^{-1}\text{tr}\chi)(-\frac{a}{4}, 0, \theta^1, \theta^2) - \int_0^1 (|\hat{\chi}|_\gamma^2 + |\alpha_F|_\gamma^2)(-\frac{a}{4}, \underline{u}', \theta^1, \theta^2) d\underline{u}'$$

$$\leq \frac{8}{a} - \frac{12}{a} < 0.$$

Recall, finally, that in $\mathcal{D}_{u,\underline{u}}$ the following estimate holds

$$\|\text{tr}\underline{\chi} + \frac{2}{|u|}\|_{L^\infty(S_{u,\underline{u}})} \leq \frac{1}{|u|^2}.$$

In particular, this implies

$$\text{tr}\underline{\chi}(-\frac{a}{4}, 1, \theta^1, \theta^2) < 0 \text{ for every}(\theta^1, \theta^2) \in \mathbb{S}^2.$$

Therefore, we conclude that $S_{-\frac{a}{4},1}$ is a trapped surface.

Remark 14. *In the work presented above the concept of scale-invariant norms was crucial and was used to obtain a trapped surface formation theorem from a region as close to past null infinity as one wishes (in principle, one could take $u_\infty \to -\infty$ and obtain an asymptotic theorem from past null infinity, but we do not include the details here). An important extension of the above would be to obtain results similar to the above, using the introduction of similar scale-invariant norms, for different matter models. An important one would be the massive scalar field (Klein-Gordon) tensor. The main difficulty would be that the Klein-Gordon equation, expressed in double null coordinates, does not possess a Hodge pair structure like the Maxwell equations. The energy estimates would thus have to be different in techniques and the resulting paper would be of real technical interest.*

Remark 15. *As the external examiner also suggested, a real challenge in this area would be to obtain trapped surface formation results, in the absence of symmetry, for the Einstein-Euler system. In the Einstein–Euler system, the geometric intricacies of the theory of General Relativity couple to the volatile nature of the equations governing the motion of perfect fluids. This coupling creates novel difficulties and at the same time gives rise to new questions. It would be interesting to find initial data that guarantee the regularity of the spacetime up to the point of trapped surface formation, so that trapped surfaces form before shocks have a chance to.*

Remark 16. *Given the 2020 Nobel prize win of Sir Roger Penrose, which he earned mostly for establishing his Incompleteness Theorem (see Theorem 3.1.3 in the Thesis), the author expects interest in this research area to only increase in subsequent years.*

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