

# ON THE EXTERIOR STABILITY OF NONLINEAR WAVE EQUATIONS

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**ABSTRACT.** We consider a general class of nonlinear wave equations, which admit trivial solutions and not necessarily verify any form of null conditions. For data prescribed on  $\mathbb{R}^3 \setminus B_R$  with small weighted energy, without some form of null conditions on the nonlinearity, the exterior stability is not expected to hold in the full domain of dependence, due to the known results of formation of shocks with data on annuli. The classical method can only give the well-posedness upto a finite time.

In this paper, we prove that, there exists a constant  $R_0 \geq 2$ , if the weighted energy of the data is sufficiently small on  $\mathbb{R}^3 \setminus B_R$  with the fixed number  $R \geq R_0$ , then the solution exists and is unique in the entire exterior of a Schwarzschild cone initiating from  $\{|x| = R\}$  (including the boundary) with a small negative mass  $-M_0$ . Such  $M_0$  is determined according to the size of the initial data. The result is achieved by obtaining in the exterior region a series of sharp decay properties for the solution, which are stronger than the general known behavior of free wave. For semi-linear equations, the stability region can be any close to  $\{|x| - t > R\}$  if the weighted energy of the data is sufficiently small on  $\{|x| \geq R\}$ . As a quick application, for Einstein (massive and massless) scalar fields, we show the solution converges to a small static solution, stable in the entire exterior of a Schwarzschild cone with positive mass, and hence patchable to the interior results.

## 1. Introduction

In this paper, we consider nonlinear wave equations in  $\mathbb{R}^{3+1}$  of the following form

$$\begin{cases} \mathbf{g}^{\alpha\beta}(\phi, \partial\phi) \partial_\alpha \partial_\beta \phi = \mathcal{N}^{\alpha\beta}(\phi) \partial_\alpha \phi \partial_\beta \phi + q(x) \phi \\ \phi|_{t=0} = \phi_0, \quad \partial_t \phi|_{t=0} = \phi_1 \end{cases} \quad (1.1)$$

with a smooth function  $0 \leq q(x) \leq 1$ , where  $\mathbf{g}(\phi, \partial\phi)$  is a Lorentzian metric with  $\mathbf{g}(y, \mathbf{P})$  being smooth on variables  $y \in \mathbb{R}$  and  $\mathbf{P} \in \mathbb{R}^4$ , and  $\mathbf{g}^{\alpha\beta}(0, \mathbf{0}) = \mathbf{m}^{\alpha\beta}$ . Here  $\mathbf{m}$  denotes the Minkowski metric.<sup>1</sup> The functions  $\mathcal{N}^{\alpha\beta}(y)$  are smooth for  $y \in \mathbb{R}$ .<sup>2</sup>

The most important case for us is  $q \equiv 0$ . For convenience, we assume the derivatives of  $q$  satisfy<sup>3</sup>

$$|r^i \partial^{(i)} q| \lesssim r^{-2-\eta} \text{ with } i = 1, \dots, n, \quad r = |x| \quad (1.2)$$

where  $\eta > 0$  is any fixed constant.<sup>4</sup>

Throughout this paper we set  $H^{\alpha\beta} = \mathbf{g}^{\alpha\beta} - \mathbf{m}^{\alpha\beta}$  and define  $\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta = \square_{\mathbf{g}}$ . In case  $\mathbf{g} \equiv \mathbf{m}$ , the quasilinear wave equation (1.1) becomes a semilinear equation.

**1.1. Main problem.** For the general class of wave equations (1.1), without assuming that the quadratic nonlinear term on the righthand side verifies any form of special structure, we construct the global-in-time classical solution for the generic Cauchy data with small weighted energy on  $\{|x| \geq R\}$ . Throughout this paper, we assume the initial data are not compactly supported in  $\mathbb{R}^3$  and consider the data with bounded weighted energy, which is defined as follows.

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<sup>1</sup>We fix the convention that, in the Einstein summation convention, a Greek letter is used for index taking values 0, 1, 2, 3.  $x^0 = t$  and  $\partial_0 = \partial_t$ .

<sup>2</sup>Our proof still works if  $\mathcal{N}^{\alpha\beta}$  also smoothly depend on  $\partial\phi$  and  $q$  also depends on  $t$  with nearly no modification. We keep them in the simple form for ease of exposition.

<sup>3</sup>For a differential operator  $P$ ,  $P^{(n)}$  means applying  $P$  to the  $n$ -th order,  $P^{(\leq n)} = \sum_{0 \leq i \leq n} P^{(i)}$ , and  $P^{(0)} = id$ .

<sup>4</sup> $A \lesssim B$  means  $A \leq cB$  with the constant  $c \geq 1$ .  $A \approx B$  means  $A \lesssim B$  and  $B \lesssim A$ .

Let  $1 < \gamma_0 < 2$  be a fixed constant and  $q_0 = \sup_{\{|x| \geq R\}} q(x)$ . We denote

$$\begin{aligned} \mathcal{E}_{k, \gamma_0, R, q_0} &= \int_{\Sigma_0 \cap \{r \geq R\}} (1+r)^{\gamma_0-2} |\phi_0|^2 dx \\ &+ \sum_{l \leq k} \int_{\Sigma_0 \cap \{r \geq R\}} (1+r)^{\gamma_0+2l} (|\partial \partial^l \phi_0|^2 + |\partial^l \phi_1|^2 + q_0 |\partial^l \phi_0|^2) dx, \end{aligned} \quad (1.3)$$

where  $\Sigma_0 = \{t = 0\}$ . The  $q_0$  in the subindex may be dropped when we only consider one single equation instead of an equation system. We may use  $\mathcal{E}_{k, \gamma_0}$  as a short-hand notation whenever there occurs no confusion. Here  $R$  is a fixed constant with  $R \geq 2$ .

For initial data with compact support, in either the semilinear or the quasilinear case, there holds only a semi-global existence result, with the time-span of the solution depending on the size of the small data (See [10, 12]). The finite life-span of the solution therein is actually sharp. In [8] and [9], examples of equations of the type (1.1) are constructed which does not admit global solution for data of any size. The potential flow of compressible Euler equations is also a typical example of (1.1) which does not verify null conditions. In the work of Christodoulou [3] on the relativistic Euler equation and the work of Speck [29] for the geometric wave equation, when a set of small Cauchy data is prescribed in an annulus, the singularity of the characteristic surfaces forms within finite time. The semi-global well-posedness of the solution holds until the formation of shock. Therefore for the generic data prescribed on  $\{|x| \geq R\}$ , one should not expect a global-in-time result in the full domain of dependence, which is exterior to the outgoing characteristic cone initiated from  $\{|x| = R\}$  upto the boundary.

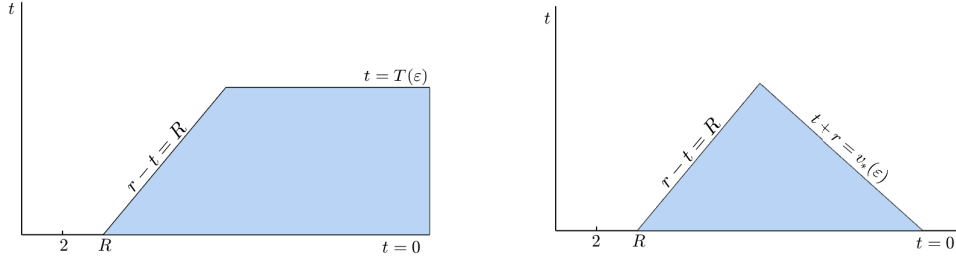


FIGURE 1. Illustration of the classical semi-global results

When the data are non-compactly supported, the known energy method can only give the local-in-time result of well-posedness, even if the data are small on  $\{|x| \geq R\}$ . See Figure 1 for the regions of well-posedness in the semilinear case by using the time foliation or the double null foliations. The  $T(\epsilon)$  and  $v_*(\epsilon)$  are both some finite numbers depending on the bound  $\mathcal{E}_{N, \gamma_0, R, 0}(\phi[0]) \leq \epsilon$ . We now state the first version of our global-in-time stability result.

**Theorem 1.1** (A rough statement of main result). *Let  $1 < \gamma_0 < 2$  be fixed. Consider (1.1). There exist a universal constant<sup>5</sup>  $C \geq 1$ , a small constant  $\delta_1 > 0$  and a constant  $R(\gamma_0, C) \geq 2$  such that, if  $\phi[0] = (\phi_0, \phi_1)$  verifies  $\mathcal{E}_{3, \gamma_0, R} \leq \delta_1$  with  $R \geq R(\gamma_0, C)$ , then with  $M_0 = C\delta_1^{\frac{1}{2}}$ , exterior to a Schwarzschild cone of mass  $-M_0$ <sup>6</sup> initiated from  $\{|x| = R\}$  (including the cone itself), there exists a unique global-in-time solution which converges to the trivial solution as  $r \rightarrow \infty$  for any  $t > 0$ . The solution not only has the standard asymptotic behavior of the free wave, but also has improved global decay properties.*

<sup>5</sup>Throughout the paper, a universal constant means a constant that depends only on the constant  $\eta$  and the bound in (1.2), the bounds of  $|D^{\leq 3} \mathbf{g}|$  and  $|D^{\leq 3} \mathcal{N}|$  on small compact domains.

<sup>6</sup>See the definition of the metric in (2.1).

Recall that for the generic data with finite weighted energy, there holds only the semi-global well-posedness result. If the data are compactly supported within  $\{|x| \leq R\}$ , the blow-up of the solution can only occur within the outgoing light cone initiated from a sphere  $\{|x| = R\}$ , due to the standard argument of the finite speed of propagation. If the data are not compactly supported, by applying our result to the generic data with bounded weighted energy, we conclude that the blow-up can only occur in the region interior to a Schwarzschild cone starting from  $\{|x| = R\}$  for all  $t > 0$  provided that  $R$  is sufficiently large.

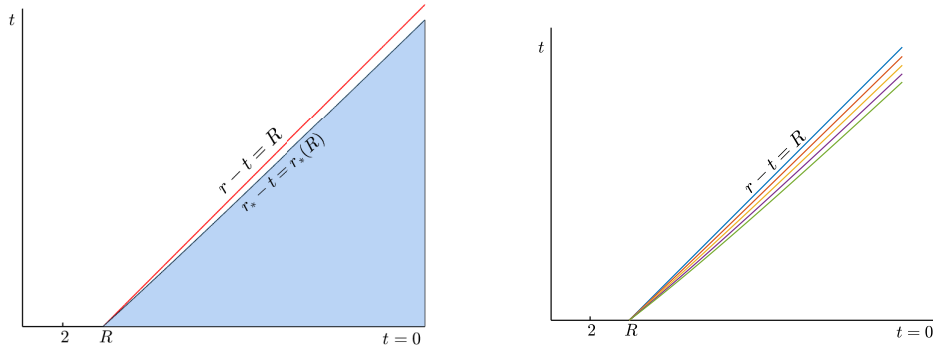


FIGURE 2. Illustration of the stability region of the main result for the semilinear equations.

We have several remarks on the above rough statement of our result.

- (1) In this result, the mass of the Schwarzschild metric is  $-M_0$  with  $|M_0| \ll \frac{1}{10}$  chosen according to the size of the initial data. The definition of the metric can be found in (2.1). For the general equation (1.1), we choose  $M_0 > 0$ . Correspondingly, the boundary of the exterior region is slightly spacelike<sup>7</sup>. For the semilinear case and Einstein scalar fields, we can have better results than the above statement. For the semilinear case, the stability region can be any close to  $\{r > t + R\}$ ; see Figure 2. For Einstein equations, we can choose  $M_0 \leq 0$ . The corresponding Schwarzschild cone is timelike or null. This makes the exterior result patchable with an interior result based on the Minkowskian hyperboloidal foliation if we further assume the smallness of the initial data in  $B_R$ .<sup>8</sup> We refer the readers to the main theorems, Theorem 2.1, Theorem 2.5 and Theorem 2.6, for detailed statements.
- (2) If the data are compactly supported, the choice of  $M_0 = C\delta_1^{\frac{1}{2}}$  leads to a semi-global result. The life-span coincides with the standard almost global result [10].

**1.2. Review of history and inspiration.** In general, one can construct global-in-time solution of (1.1) in  $\mathbb{R}^{3+1}$  for generic small data only when the quadratic nonlinearity  $\mathcal{N}^{\alpha\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi$  verifies certain null condition. The null conditions, which are important algebraic cancellation structures, can be found in various important geometric or physical field equations, such as wave maps, Maxwell-Klein-Gordon equations, Yang-Mills equations and Einstein equations. One can refer to [14, 2] for the results for quasilinear wave equations verifying null conditions. For the semilinear case, one can refer to [22, 31, 33] for the global asymptotic behavior of the massless Maxwell-Klein-Gordon equations, to [19, 26] for the massive case, to [5, 6] for the result of Yang-Mills equations; and to [15, 16, 20, 25] for the global well-posedness results of low or optimal regularity for Maxwell-Klein-Gordon equations and Yang-Mills equations. In the case that  $\mathcal{N}^{\alpha\beta} \equiv 0$ ,  $q \equiv 0$ , Lindblad ([23]) proved the stability result for (1.1) if the metric  $\mathbf{g}$  does not depend on  $\partial\phi$ , where the loss of sharp decay occurs for  $\phi$ . (See also [1].) The case for the general equation (1.1) with  $q > 0$  being a fixed constant is studied in [13] for small data with compact support without requiring null conditions.

<sup>7</sup>Throughout the paper, spacelike, null or timelike are in terms of the Minkowski metric.

<sup>8</sup>See a semilinear result [7] for an example of such direct patching.

The Einstein equation system is a quasilinear system that verifies null conditions in an intrinsic geometric framework, relative to the maximal foliation gauge. Under this gauge, the small data global-existence result was proved by Klainerman and Christodoulou in the monumental work [4]. Meanwhile under the wave coordinate gauge, the reduced Einstein equation system takes the form of (1.1), which further verifies the so-called weak null condition (see [21] and [24]). The global results for small data are proved by Lindblad and Rodnianski.

Now compare the simplest example of equations with the weak null condition,

$$\square_{\mathbf{m}}\phi_1 = -(\partial_t\phi_2)^2, \quad \square_{\mathbf{m}}\phi_2 = 0 \quad (1.4)$$

with the example

$$\square_{\mathbf{m}}\phi = -(\partial_t\phi)^2 \quad (1.5)$$

constructed by John in [8] which does not have a global solution for compactly supported data of any size. With  $L = \partial_t + \partial_r$ ,  $\underline{L} = \partial_t - \partial_r$ , we can decompose  $2\partial_t = L + \underline{L}$ . Thus the term of  $(\underline{L}f)^2$  appears in the quadratic term of both (1.4) and (1.5). The difference lies in that such bad term in (1.4) can be controlled by the better part of the system, since  $\phi_2$  is actually a free wave solution. Therefore the solution  $(\phi_1, \phi_2)$  exists globally. Nevertheless due to the appearance of such bad term, there merely holds the weaker decay property

$$(r+t+1)|\phi_1| \lesssim \ln(t+2). \quad (1.6)$$

Typically, the weak null system consists of good equations which verify null conditions, and bad equations which formally have the terms of  $\underline{L}f \cdot \underline{L}f$ . It is important that the function  $f$  verifies the good equations. In contrast, (1.5) is clearly an example that  $\underline{L}f \cdot \underline{L}f$  appears in the equation of  $f$  itself, which does not satisfy the weak null condition.

For ease of discussion, let us consider the data which have compact support for example. In the domain of influence, by running a standard energy argument for (1.5), we have

$$\|\partial\partial^{(n)}\phi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \lesssim \|\partial\partial^{(n)}\phi(0, \cdot)\|_{L^2(\mathbb{R}^3)} + \int_0^t \|\partial^{(n)}((\partial_t\phi)^2)(t', \cdot)\|_{L^2(\mathbb{R}^3)} dt'. \quad (1.7)$$

For simplicity we only consider one of the terms in  $\partial^{(n)}((\partial_t\phi)^2) = \sum_{a+b=n} \partial^{(a)}\partial_t\phi\partial^{(b)}\partial_t\phi$ , which is

$$\partial_t\phi \cdot \partial^{(n)}\partial_t\phi. \quad (1.8)$$

Note the standard decay of the free wave for  $\partial_t\phi$  with small data is

$$(|t-r|+1)^{\frac{1}{2}}(t+r+1)|\partial\phi| \lesssim \epsilon^{\frac{1}{2}}.$$

By a direct substitution,  $\|\partial\partial^{(n)}\phi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \lesssim t^{C\epsilon^{\frac{1}{2}}} \|\partial\partial^{(n)}\phi(0, \cdot)\|_{L^2(\mathbb{R}^3)}$ . However, with such energy growth, one can not recover the linear behavior to  $\partial_t\phi$  without loss of decay in  $t$ -variable. With a weaker decay for  $\partial_t\phi$ , we can not achieve the boundedness of energy even allowing growth. The only way to obtain the boundedness of energy is by setting  $t \leq T(\epsilon) < \infty$ , which implies the semi-global result.

Consider the quadratic nonlinearity verifies null conditions such as the null forms

$$\mathbf{Q}_0(\phi, \psi) = \partial^\mu\phi\partial_\mu\psi; \quad \mathbf{Q}_{\alpha\beta}(\phi, \psi) = \partial_\alpha\phi\partial_\beta\psi - \partial_\alpha\psi\partial_\beta\phi.$$

Due to

$$|\mathbf{Q}(\phi, \psi)| \lesssim |\bar{\partial}\phi| \cdot |\partial\psi| + |\bar{\partial}\psi| |\partial\phi|$$

where  $\bar{\partial} = (L, \nabla)$  and  $\nabla_i = \partial_i - \frac{x^i}{r}\partial_r$ ,  $i = 1, 2, 3$ , since the above structure is almost preserved if differentiated by the invariant vector fields of the Minkowski space, and since the decay of  $\bar{\partial}\phi$  can be improved to  $(1+r+t)^{-2}$  by using the commuting vector fields approach, we can obtain much more decay in the error integral in (1.7) compared with the case for the equation (1.5). This implies the boundedness of energy easily.

Based on the above examples, clearly the  $\underline{L}\phi_1 \cdot \underline{L}\phi_2$  type term in the quadratic nonlinearities significantly changes the asymptotic behavior of the solution. It either does not allow the local-in-time solution to be extended for all  $t > 0$ , or causes the global solution to lose the sharp decay, which is the case for the simplest system (1.4). Therefore without the structure of null

conditions, one should not expect the solution has the standard global linear behavior of a wave equation without loss.

Prescribing data on  $\{r \geq R\}$  does not improve decay property in terms of the parameter  $r$ . In the standard exterior stability results, [18, 17] for Einstein equations and [19] for the massive Maxwell-Klein-Gordon equation with arbitrary charge, the null conditions of the equations have played a crucial role. If both the quasilinear and semilinear nonlinearity verify the null condition, the solution of (1.1) is expected to be well-posed in the entire domain of dependence of  $\{r \geq R\}$  up to the characteristic boundary if the data are sufficiently small. Without any assumption on the nonlinear structures in (1.1), the solution may not exist in the entire  $\{r \geq R+t\}$  in the semilinear case, nor in the whole exterior of the global characteristic surfaces (upto the boundary) in the quasilinear case. For the quasilinear equations, the characteristic surface can be singular in finite time. Nevertheless, it is still possible that the solution remains regular in the majority of the domain of dependence, meanwhile concentrates in the remaining part. This inspires us to extend the solution in subregions of the domain of dependence. Our main result shows such subregion can be global in time, and in the semilinear case any close to  $\{r \geq t+R\}$ .

Geometrically, the lightcone of Schwarzschild spacetime intersects with any lightcone of Minkowski space within finite time. So in the semilinear case, we choose a Schwarzschild cone initiated from  $\{|x| = R\}$  can be the boundary surface. Such cone has to be spacelike since we need to obtain the positive energy flux on the boundary. Note that the region bounded by the Schwarzschild cones with the small negative mass  $-M_0$  can exhaust  $\{r > t+R\}$  by letting  $M_0 \rightarrow 0_+$ . Such region can be any close to  $\{r > t+R\}$ , and identical to  $\{r \geq t+R\}$  if and only if  $M_0 = 0$ . (See the second picture in Figure 2.)

After separating the region where the solution may concentrate away from the domain of dependence along a Schwarzschild cone, we then achieve a series of crucial improvements over the standard linear behavior of wave equations in the remaining region to control the nonlinearities.

**1.3. Strategy of the proofs.** The framework of our approach can be clearly seen in the proof for the semilinear case. The main idea is by constructing foliations of Schwarzschild light cones to obtain an improved set of decay properties for the solutions compared with the known free wave behavior. In this subsection, we will discuss the following aspects of our approach.

- (1) The improvements compared with the known standard linear behaviors and the control of the nonlinearity on the right of (1.1).
- (2) The influence of the variable potential  $q$  to our approach.
- (3) The difficulties in the quasilinear case caused by the nontrivial influence of the metric  $g(\phi, \partial\phi)$  and in the application to Einstein scalar fields.

We first explain how we treat the quadratic nonlinearity to achieve the boundedness of energy. For the free wave equation and data  $\phi[0]$  with  $\mathcal{E}_{2,\gamma_0,R} \leq \epsilon$ , we can apply commuting vector fields to derive the standard decay property in  $\{r \geq t+R\}$ ,

$$r|\partial\phi| \lesssim \epsilon^{\frac{1}{2}}(r-t+1)^{-\frac{1}{2}\gamma_0-\frac{1}{2}}. \quad (1.9)$$

Under the null frame  $L = \partial_t + \partial_r$ ,  $\underline{L} = \partial_t - \partial_r$ , and with  $\nabla_i = \partial_i - \frac{x^i}{r}\partial_r$ , the Cartesian component of covariant derivative on sphere  $S_{t,r}$ , the decay for  $L\phi$  and  $\nabla\phi$  can be improved. Nevertheless the decay rate in terms of  $r$  is unimprovable for  $\underline{L}\phi$  in the region  $\{r \geq t+R\}$ . By the energy argument for (1.5), we would only obtain a finite-in-time result.

We now improve the asymptotic behavior of  $\underline{L}\phi$  exterior to a spacelike Schwarzschild light cone. Let  $u_0(M_0) = -r_*(M_0, R)$  where  $r_*(M_0, r)$  is defined in (2.2). We may set  $u_0 = u_0(M_0)$  for convenience. Let  $u = t - r_*(M_0, r)$  and  $\underline{u} = t + r_*(M_0, r)$ . In the region with  $\{u \leq u_0\}$ , we adopt foliations by level sets of  $u$  and  $\underline{u}$ , denoted by  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$  respectively. The standard energy for  $\phi$  on  $\mathcal{H}_u$  takes the form of

$$\int_{\mathcal{H}_u} \left( \frac{M_0}{r} |\underline{L}\phi|^2 + |L\phi|^2 + |\nabla\phi|^2 \right) d\mu_{\mathcal{H}} \lesssim |u|^{-\gamma_0} \mathcal{E}_{0,\gamma_0,R}, \quad \gamma_0 > 1, \quad (1.10)$$

where  $d\mu_{\mathcal{H}}$  is the area element of  $\mathcal{H}_u$ , comparable to  $r^2 d\underline{u}' d\mu_{\mathbb{S}^2}$ , and  $\mathcal{E}_{0,\gamma_0,R}$  is defined in (1.3) for the initial data.

(1.10) gives us the crucial bound on  $\underline{L}\phi$ . To see how such improvement can help us to bound  $E[\partial^{(n)}\phi](\Sigma)$ , i.e. the standard energy on the hypersurfaces  $\Sigma$ , we next control the error integral in the standard energy estimate for (1.5). Let  $-\underline{u}_1 \leq u_1 \leq u_0$  and  $\underline{u}_1$  be arbitrarily large. The energy argument gives

$$\begin{aligned} & E[\partial^{(n)}\phi](\mathcal{H}_{\underline{u}_1}^{u_1}) + E[\partial^{(n)}\phi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) \\ & \lesssim \|\partial\partial^{(n)}\phi(0, \cdot)\|_{L^2(\Sigma_0 \cap \{u \leq u_1\})}^2 + \int_{\{-\underline{u}_1 \leq -\underline{u} < u \leq u_1\}} |\partial^{(n)}(\partial_t \phi \partial_t \phi) \partial \partial^{(n)}\phi|. \end{aligned} \quad (1.11)$$

Here the truncated hypersurfaces  $\mathcal{H}_{\underline{u}_1}^{u_1}$  and  $\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}$  are subsets of  $\mathcal{H}_{u_1}$  and  $\underline{\mathcal{H}}_{\underline{u}_1}$  respectively, both of which are defined in (2.12) in Section 2.

We treat the term (1.8) in the error term in (1.11) as an example. If we can recover the linear behavior (1.9) with  $r-t$  replaced by  $|u|$  to  $\partial\phi$  under the assumption that the data verify  $\mathcal{E}_{2,\gamma_0,R} \leq \epsilon$ , we have

$$\int_{\{-\underline{u}_1 \leq -\underline{u} < u \leq u_1\}} |\partial^{(n)} \partial_t \phi \cdot \partial_t \phi \cdot \partial \partial^{(n)} \phi| \lesssim M_0^{-1} \epsilon^{\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{1}{2}\gamma_0 - \frac{1}{2}} E[\partial^{(n)}\phi](\mathcal{H}_u^{u_1}) du. \quad (1.12)$$

With  $\gamma_0 > 1$ , we can achieve the boundedness of energy by the Gronwall's inequality. For the nonlinear problem itself, we certainly can not directly use the linear behavior to control error. The analysis will be based on a bootstrap argument. With  $\epsilon \leq CM_0^2$  and  $C \geq 1$ , we can close the bootstrap argument if  $c_0 R^{1-\gamma_0} < C^{-1}$ , where  $C$  is a fixed constant,  $c_0 \geq 1$  is a universal constant. This is achievable if  $R(\gamma_0)$ , the lower bound of  $R$ , satisfies the inequality.

Thus it is crucial for our nonlinear analysis to achieve the linear behavior of (1.9) with  $r-t$  replaced by  $|u|$ , without loss of the decay in  $r$ -variable. This requires us to perform our analysis in a no-loss regime. The analysis of the full nonlinearities is more involved than the sample term in (1.12). We explain our basic principle below.

If we write the spacetime error integral, such as the last term of (1.11) symbolically as

$$\int |B_1| |B_2| |B_3| dx dt. \quad (1.13)$$

If certain types of null conditions are satisfied, we may show one of the factors  $|B_i|$  has a stronger decay in  $r$  than (1.9); or show one factor achieves the standard global linear behavior, such as the pointwise decay  $(t+r+1)^{-1}$  which leads to the energy control with growth of  $t^C \epsilon^{\frac{1}{2}}$ . Without any of the extra structure, we rely on the sets of decay estimates in Section 4 and the energy flux of the Schwarzschild cone foliation to form the hierarchy of the analysis. For these bad terms  $B_i$ , we manage to recover the standard linear behavior, with the bound comparable to the size of the data,  $\Delta_0^{\frac{1}{2}} \approx \epsilon^{\frac{1}{2}}$ . They offer bounds of a general set of norms  $\|\cdot\|_G$ . We also achieve a set of integrated estimates with the bound  $\Delta_0^{\frac{1}{2}} M_0^{-\frac{1}{2}}$ , denoted by  $\|\cdot\|_S$ , which are stronger than the standard linear behavior. When applying Hölder's inequality, our basic principle is to bound the worst nonlinearity (1.13) by

$$\int |B_1| |B_2| |B_3| dx dt \lesssim \|B_1\|_G \|B_2\|_S \|B_3\|_S.$$

This allows us to improve the bootstrap argument and achieve the boundedness of the norms  $\|\cdot\|_G$  and  $\|\cdot\|_S$  for the bad terms.

To derive the set of improved estimates  $\|\cdot\|_S$  for bad terms, we note the linear behavior (1.10) shows that once the energy flux on  $\mathcal{H}_u$  can be bounded in terms of the initial data, since the weighted energy of data is bounded, the flux automatically decays nicely in  $|u|$ . We can combine this property with the weighted Sobolev inequalities in [19] to obtain more improvements, in

particular, on the integrated decay estimates. Below are some of the crucial improved estimates:

$$\|r^{\frac{1}{2}} \partial Z^{(b)} \phi\|_{L_{\underline{u}}^2 L_u^\infty L_\omega^4(u \leq u_1)}^2 \lesssim \epsilon M_0^{-1} |u_1|^{-\gamma_0 + 2\zeta(Z^{(b)})}, \quad b \leq n-1 \quad (1.14)$$

$$\|r^{\frac{1}{2}} \partial Z^{(l)} \phi\|_{L_{\underline{u}}^2 L^\infty(u \leq u_1)}^2 \lesssim \epsilon M_0^{-1} |u_1|^{-\gamma_0 + 2\zeta(Z^{(l)})}, \quad l \leq n-2 \quad (1.15)$$

$$\|r^{-\frac{1}{2}} Z^{(a)} \phi\|_{L^2(\mathcal{H}_{u_1})}^2 \lesssim \epsilon M_0^{-1} |u_1|^{-\gamma_0 + 2 + 2\zeta(Z^{(a)})}, \quad a \leq n, \quad (1.16)$$

where  $u_1 \leq u_0$  and  $\mathcal{E}_{n, \gamma_0, R}(\phi[0]) \leq \epsilon$ ,  $n = 2, 3$ . Here we denote the ordered product of vector fields as  $Z^k = Z_1 \cdots Z_k$ , with  $Z^{(k)}$  the corresponding differential operator of  $k$ -th order, where  $Z_l \in \{\Omega_{ij}, \partial\}$ ,  $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ ,  $1 \leq i < j \leq 3$ , and  $Z^{(0)} = id$ . The signature function is defined by

$$\zeta(Z^k) = \sum_{i=1}^k \zeta(Z_i), \quad \zeta(\Omega_{ij}) = 0, \quad \zeta(\partial) = -1.$$

See Proposition 4.1, Proposition 4.4 and Lemma 3.4 for the proofs of (1.14)-(1.16) and more improved estimates. (1.14) is crucial for treating other terms in (1.11) so as to achieve the boundedness of energy without any loss. The other two estimates are important for treating the quasilinear problem without loss. (1.15) is used to prove the weighted energy estimate. (1.16) is used to give an improved Hardy's inequality (4.5), which is crucial to treat  $[\Box_{\mathbf{g}}, Z^{(n)}]\phi$  in the general quasilinear case.

Next we comment on the influence of the potential  $q$  to our approach.

- (1) The scaling vectorfield  $S = t\partial_t + r\partial_r$  can not be used as a commuting vectorfield. This is similar to the massive case i.e.  $q > 0$  is a fixed constant.
- (2) The asymptotic behavior of the solution is similar to the massless case i.e.  $q = 0$ . Such set of decay is weaker than the standard decay of the massive case.

The problem with variable potential takes the essential difficulties from both the massive and massless wave equations. To treat the potential term, we take the spirit of the multiplier approach developed in [19], and yet have to make further improvements since the asymptotic behavior of the solution is much weaker. We adopt merely  $\{\Omega_{ij}, \partial\}$  to obtain good decay properties for  $\partial\phi$ ,

$$|u|^{\frac{1}{2}} r |\underline{L}\phi| + r^{\frac{3}{2}} |\nabla\phi| + r^{\frac{3}{2}} |L\phi| \lesssim \epsilon^{\frac{1}{2}} |u|^{-\frac{\gamma_0}{2}} \quad (1.17)$$

with decay of higher order derivatives included in Section 2, Section 4 and Section 6. Better decay is achievable by applying higher order derivatives. If  $q \equiv q_0 > 0$  is a constant, we achieve

$$q_0 r^{\frac{5}{2}} |\phi|^2 \lesssim \epsilon |u|^{-\gamma_0 + \frac{1}{2}}.$$

For the quasilinear problems, besides the difficulty caused by the semilinear quadratic error terms, we overcome the following difficulties caused by the metric  $\mathbf{g}(\phi, \partial\phi)$ .

- (1) Due to the influence of the metric, we have to construct the foliations of the spacetime carefully, and construct energy momentum tensor properly for obtaining positive energy without losing derivatives.
- (2) We use a  $r$ -weighted multiplier approach to recover the linear behavior for the nonlinear solution. It requires the metric verifies stronger decay property on  $H(\phi, \partial\phi)$  than the decay for the free wave.

Same as the semilinear cases, for the quasilinear cases the main task is to bound the standard energy and  $r$ -weighted energy in terms of initial data. It is well-known that the energy flux along the characteristic surfaces is positive. However since for the quasilinear equations, the characteristic surfaces can be singular in finite time. We can not obtain the energy flux along the characteristics with  $t \rightarrow \infty$ . Thus the first step is to determine the family of surfaces  $\mathcal{H}_u$  along which the energy flux is positive. We then prove the stability result exterior to the boundary  $\mathcal{H}_{u_0}$  with  $u_0 = \max u$ .

By constructing a proper modified energy momentum tensor, we manage to obtain that the energy density along  $\mathcal{H}_u$  takes the form of

$$\left(\frac{M_0}{r} - H\bar{L}\bar{L}\right)(\bar{L}\phi)^2 + H\partial\phi \cdot \bar{\partial}\phi + |\bar{\partial}\phi|^2. \quad (1.18)$$

(See the calculation in Lemma 6.3.) For Einstein equations, with  $M_0 = 0$ , the positivity of energy flux can be achieved if the data<sup>9</sup> are sufficiently small, since due to the positive mass theorem,

$$\lim_{|x| \rightarrow \infty} rH\bar{L}\bar{L}(x, 0) = -\frac{1}{2}m_0, \quad m_0 > 0.$$

In Section 7, we will take advantage of this fact to prove the improved result, Theorem 2.6.

In general, (1.18) is not coercively positive along the Minkowski cone, i.e.  $M_0 = 0$ . To achieve the positive energy flux, in Section 6, we choose  $M_0$  according to the size of the data, so that there exists a small constant  $M$  such that

$$r\left(\frac{M_0}{r} - H\bar{L}\bar{L}\right) > M > 0 \quad \text{and} \quad |M_0| \lesssim M. \quad (1.19)$$

Other error terms in (1.18) can then be absorbed. This treatment actually needs the smallness of  $|rH\bar{L}\bar{L}|$ . Although one can see from (1.6) that even for the system (1.4), which has better structure than the general case (1.1), the solution  $\phi$  does not have the sharp decay. But we manage to achieve it in the region  $\{u \leq u_0\}$ . There is a similar issue with the energy density on  $\mathcal{H}_{\underline{u}}$ , which can be solved in the same way.

Therefore with a suitable choice of the mass  $-M_0$  for the boundary cone, we can gain the control of  $Mr^{-1}|\bar{L}\phi|^2$  in the energy flux along the level set of  $u(M_0)$ . Hence (1.10) holds with  $M_0$  replaced by the small constant  $M > 0$ . The choice of  $M_0$  depends on the bound for the data, so does the size of  $M$ . This allows us to follow the treatment for (1.5) to control the quadratic nonlinearity.

In the semilinear case, in terms of the standard energy momentum tensor<sup>10</sup>  $Q_{\alpha\beta} = \partial_\alpha\varphi \cdot \partial_\beta\varphi - \frac{1}{2}\mathbf{m}_{\alpha\beta}\partial^\mu\varphi\partial_\mu\varphi$ , the standard energy is defined by  $\int_\Sigma Q_{\alpha\beta}X^\beta\mathbf{n}^\alpha d\mu_\Sigma$ , where  $X = \partial_t$  and  $\mathbf{n}^\alpha$  denotes the surface normal of  $\Sigma$ . The  $r$ -weighted energy is defined by using  $X = rL$  with suitable modifications in the energy current. The energy estimates are based on the following calculation

$$\partial^\alpha(Q_{\alpha\beta}X^\beta) = \square_{\mathbf{m}}\varphi X^\alpha + Q_{\alpha\beta}\partial^\alpha X^\beta.$$

For the quasilinear operator, we have to make a modification, otherwise the righthand side contains  $\partial^2\varphi$ . For this purpose, we construct the energy momentum tensor

$$\tilde{\mathcal{Q}}_{\alpha\beta}[\varphi] = \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}\mathbf{m}_{\alpha\beta}\mathbf{g}^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi + H_\alpha^\gamma\partial_\gamma\varphi\partial_\beta\varphi$$

which gives nice structures in the energy density under the Minkowski background.

In the quasilinear case, the form of the energy momentum tensor, the choice of multiplier and the modification to the energy current are all very sensitive for proving the  $r$ -weighted energy estimates. Since  $\bar{L}\varphi$  term does not appear in the  $r$ -weighted energy, we need to treat the error terms of  $f(H, \partial H)(\bar{L}\varphi)^2$  carefully. Typically, bounding  $r$ -weighted energy requires more decay on  $H$  than a free wave verifies. See [31, Section 1 (3)], where an additional  $r^{-\varepsilon}$  decay is assumed, with  $\varepsilon > 0$ , even for the equations with null condition therein. Our improved integrated decay estimates are weaker than this assumption. We carefully choose the multiplier  $X = r(L - H\bar{L}\bar{L})$ , which is inspired by considering the asymptotic equation (see [23, 24, 31]). In Lemma 6.5, it turns out the construction of energy current in (6.18) leads to a good structure in the error terms.

At last we consider the Einstein scalar fields. In comparison with Theorem 1.1 (or Theorem 2.5),  $H$  converges to a small static solution instead of 0. The static part slows down the decay properties of  $H$ . Fortunately for Theorem 2.5, the derivation of the inequalities of energy and

<sup>9</sup>See Theorem 2.6 and Section 7 for the set-up and the meaning of the data.

<sup>10</sup>For ease of discussion, we assume  $g \equiv 0$  in the sequel.



weighted energy relies more on the decay of  $\partial H$ , which is barely influenced. The framework of Section 6 still works through. However, borderline terms appear in the commutator  $[\Box_{\mathbf{g}}, Z^{(n)}]$ , since  $H$  has less decay in  $|u|$ . They are proved to be harmless, when we show the boundedness of energies for  $Z^{(n)}(\mathbf{h}^1, \phi)$ <sup>11</sup> with an induction on the signature  $\zeta(Z^n)$  from  $-n$ .

As further extensions, one may apply the approach to establish global result for the quasi-linear wave systems with certain weak null conditions, if the small weighted data are prescribed throughout the initial slice. The result of Theorem 2.5 can also be generalized to Euler equations with nontrivial vorticity.

**1.4. Structure of the paper.** In Section 2, we give the details of the geometric set-up and introduce the main theorems, which are Theorems 2.1, Theorem 2.5 and Theorem 2.6. In Section 3, we introduce the weighted Sobolev inequalities and derive some consequences of bounded standard and  $r$ -weighted energies, including some sharp  $L^p$  type estimates in Lemma 3.4. In Section 4, under the assumption of bounded energies up to  $n$ th-order, with  $n = 2$  or  $3$ , we derive the full set of decay properties in Proposition 4.1 and Proposition 4.4. In Section 5, we consider the semilinear case of (1.1) and prove Theorem 2.1 and 2.2. This section gives the main framework of our approach, which is divided into three steps. We first derive the energy inequalities. Under the bootstrap assumption of the smallness of energies up to  $n = 2$ , we then employ the decay results in Section 4 to analyse the error  $(\Box_{\mathbf{m}} - q)Z^{(n)}\phi$ . The final step is to achieve the boundedness theorem for the energies by substituting the error estimates into the energy inequalities. In Section 6, we prove Theorem 2.5. Due to the influence of metric, we need to make (1.19) hold in  $\{u \leq u_0\}$  which makes the bootstrap argument more involved. We also need to obtain higher order energy control for treating commutators  $[\Box_{\mathbf{g}}, Z^{(n)}]$ . We follow the steps in the semilinear case with more delicate analysis in each part. In Section 7, we prove Theorem 2.6. We need to show that the metric difference, which is one part of the solution, is convergent to a small static solution, while the scalar field converges to 0 as  $r \rightarrow \infty$ . By a simple reduction, we still solve the problem with data convergent to  $(0, 0)$  at the spatial infinity. The static part in  $H$  has slower decay property, which could change the behavior of the wave operator. In Proposition 7.4, we show that the inequalities for energy and weighted energy still hold under such background metric. We then analyze commutators in Lemma 7.5, which contain borderline terms. For the error terms not included in (1.1), we treat them in Lemma 7.7. At last we combine the error estimates in Section 6 to complete the proof.

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## 2. Set-up and main results

We first construct the foliations that will play a very crucial role in improving the asymptotic behavior in this paper. We also need the construction to determine the stability region in the main results.

Let  $|M_0| \ll 1$  be a constant. We consider the metric

$$\mathbf{g} = -\frac{r + M_0}{r - M_0} dt^2 + \frac{r - M_0}{r + M_0} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (2.1)$$

which has the same lightcones initiated from  $\{|x| = r, t = 0\}$  as the Schwarzschild metric with mass  $-M_0$

$$\mathbf{g}_s = -\frac{r + M_0}{r - M_0} dt^2 + \frac{r - M_0}{r + M_0} dr^2 + (r - M_0)^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Let

$$r_* = r_*(M_0, r) := r - 2M_0 \ln \frac{r + M_0}{1 + M_0}. \quad (2.2)$$

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<sup>11</sup>See Theorem 2.6 for the definitions of  $\mathbf{h}^1, \phi$ .

It is easy to see that

$$u = t - r_* \quad \text{and} \quad \underline{u} = t + r_*$$

form a pair of optical functions of  $\mathbf{g}$ . They can be used to set up the foliation of the Schwarzschild lightcones in  $(\mathbb{R}^{3+1}, \mathbf{m})$ . Clearly,  $u(0, 1) = -1$  and  $-u(0, r) \approx r$  when  $r \geq 2$ .<sup>12</sup> We set

$$h = M_0/r, \quad L' = L - h\underline{L}, \quad \underline{L}' = \underline{L} - hL, \quad \forall r \geq 1,$$

where  $L = \partial_t + \partial_r$  and  $\underline{L} = \partial_t - \partial_r$ . Let  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$  denote the respective level sets of  $u$  and  $\underline{u}$ . Direct calculation shows that the generators of the outgoing and incoming null geodesics are given by

$$-\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta = (1+h)^{-1} L', \quad -\mathbf{g}^{\alpha\beta} \partial_\alpha \underline{u} \partial_\beta = (1+h)^{-1} \underline{L}' \quad (2.3)$$

which are tangent to  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$  respectively. We denote by  $\mathcal{N}$ ,  $\underline{\mathcal{N}}$  the surface normals of  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$  in terms of the Minkowski metric, which are normalized in terms of  $\langle \mathcal{N}, \partial_t \rangle = -1$  and  $\langle \underline{\mathcal{N}}, \partial_t \rangle = -1$ . In view of (2.3), it is easy to compute that

$$\mathcal{N} = (1+h)^{-1}(L + h\underline{L}), \quad \underline{\mathcal{N}} = (1+h)^{-1}(\underline{L} + hL). \quad (2.4)$$

Let

$$u_0(M_0) = u_{M_0}(0, R) = -r_*(M_0, R) \quad (2.5)$$

with a fixed constant  $R \geq 2$ . In case there occurs no confusion, we may use  $u_0$  as a shorthand notation. We now consider the region in  $(\mathbb{R}^{3+1}, \mathbf{m})$  where  $u \leq u_0$ . By setting  $1 + \mathbf{b}^{-1} = \underline{L}(u)$ , we can easily calculate the lapse function

$$\mathbf{b}^{-1} = \frac{1-h}{1+h}. \quad (2.6)$$

Instead of using  $t$  to parameterize  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$ , we will use  $\underline{u}$  and  $u$ . It is straightforward to compute

$$Lu = \underline{Lu} = \frac{2M_0}{r+M_0} = 1 - \mathbf{b}^{-1} \quad L\underline{u} = \underline{Lu} = 2 - \frac{2M_0}{r+M_0} = 1 + \mathbf{b}^{-1}.$$

Thus we can obtain for  $u \leq u_0$  and  $-\underline{u} \leq u_0$  that

$$\frac{d}{d\underline{u}} = \frac{r}{2(r-M_0)} L' \text{ on } \mathcal{H}_u, \quad \frac{d}{du} = \frac{r}{2(r-M_0)} \underline{L}' \text{ on } \underline{\mathcal{H}}_{\underline{u}}. \quad (2.7)$$

This implies that

$$\partial_{\underline{u}} r = \frac{1}{2} \mathbf{b} \text{ on } \mathcal{H}_u, \quad \partial_u r = -\frac{1}{2} \mathbf{b} \text{ on } \underline{\mathcal{H}}_{\underline{u}}. \quad (2.8)$$

By using the level sets of  $u$  and  $\underline{u}$  to foliate the spacetime, the standard area element is

$$dxdt = \frac{r^2}{2r_*'(r)} du d\underline{u} d\omega = \frac{1}{2} \mathbf{b} r^2 du d\underline{u} d\omega,$$

where  $d\omega$  denotes the standard surface measure on the unit sphere  $\mathbb{S}^2$ . Thus in view of (2.6), the area elements of  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$  are

$$d\mu_{\mathcal{H}} = \frac{1}{2} \mathbf{b} r^2 du d\underline{u} d\omega, \quad d\mu_{\underline{\mathcal{H}}} = \frac{1}{2} \mathbf{b} r^2 du d\underline{u} d\omega. \quad (2.9)$$

Note that  $\mathbf{b}$  is an increasing function of  $h$  with  $\mathbf{b} = 1 + O(h)$ . Thus the area elements in (2.9) are comparable to  $\frac{1}{2} r^2 du d\underline{u} d\omega$  and  $\frac{1}{2} r^2 du d\underline{u} d\omega$ . Note also that  $\partial_r u = -\mathbf{b}^{-1} = -\partial_r \underline{u}$  on  $\Sigma_t$ , the area element on  $\Sigma_t$  is  $\mathbf{b} r^2 du d\underline{u} d\omega$ . Thus on  $\Sigma_t$ ,  $r^2 du d\underline{u} d\omega \approx dx \approx r^2 du d\underline{u} d\omega$ .

Let  $S_{u, \underline{u}} = \mathcal{H}_u \cap \underline{\mathcal{H}}_{\underline{u}}$ , where  $\underline{u} \geq -u$ . For smooth functions  $f$  we set<sup>13</sup>  $\int_{S_{u, \underline{u}}} f := \int_{S_{u, \underline{u}}} r^2 f d\omega$ . From the definition of  $u$  we note that  $1 - \frac{t-\underline{u}}{r} = M_0 O(\frac{\ln r}{r})$ . Thus we can derive that

$$r(S_{u, -u}) = -u(1 + o(M_0)), \quad r(S_{-\underline{u}, \underline{u}}) = \underline{u}(1 + o(M_0)), \quad (2.10)$$

where the second identity is an application of the first one, based on the fact that  $\mathcal{H}_{-\underline{u}}$  is initiated from  $S_{-\underline{u}, \underline{u}}$ .

<sup>12</sup>We assume  $r \geq 2$  throughout the paper if not stated otherwise.

<sup>13</sup>We may hide the standard area elements for the integral on the corresponding hypersurfaces or spheres, and hide the area element  $dxdt$  if the integral is in a domain of the spacetime.

We also have the basic fact that  $r(u, \underline{u}, \omega)$  increases about  $\underline{u}$  for fixed  $(u, \omega)$  and decreases about  $u$  for fixed  $(\underline{u}, \omega)$ . Note also that  $r_*(r) \leq \underline{u} \leq 2r_*(r) - r_*(R)$ . These two facts together with (2.10) imply

$$-u \lesssim r_{\min}(\mathcal{H}_u), \quad \underline{u} \approx r \approx r_*(r). \quad (2.11)$$

We remark that (2.10), (2.11) and the fact  $\underline{u} \geq -u$  will be frequently used in our analysis, probably without being mentioned.

We use  $\mathcal{H}_u^{\underline{u}}$  and  $\underline{\mathcal{H}}_{\underline{u}}^u$  to denote the truncated level sets of  $u$  and  $\underline{u}$  respectively, i.e.

$$\begin{aligned} \mathcal{H}_u^{\underline{u}} &:= \{(t, x) : -u \leq \underline{u}' \leq \underline{u}\}, \quad \underline{\mathcal{H}}_{\underline{u}}^u := \{(t, x) : -\underline{u} \leq u' \leq u\}, \\ \mathcal{D}_u^{\underline{u}} &= \{(t, x) : -\underline{u} \leq -\underline{u}' \leq u' \leq u\}, \end{aligned} \quad (2.12)$$

where  $-\underline{u} \leq u \leq u_0$ . We denote

$$\Sigma_0^{u_1, \underline{u}_1} = \{-\underline{u}_1 \leq u \leq u_1, t = 0\} = \{-u_1 \leq \underline{u} \leq \underline{u}_1, t = 0\},$$

and may drop  $u_1$  when  $\underline{u}_1 = \infty$ .

We denote by  $E[f](\Sigma)$  and  $\mathcal{W}_1[f](\Sigma)$  the energy (flux) and weighted energy (flux) of the smooth function  $f$  on the hypersurface  $\Sigma$ . For the hypersurfaces of interest to us,

$$\begin{aligned} E[f](\Sigma_0^{u, \underline{u}}) &= \int_{\Sigma_0^{u, \underline{u}}} (|\partial f|^2 + qf^2) dx, \\ E[f](\mathcal{H}_u^{\underline{u}}) &= \int_{\mathcal{H}_u^{\underline{u}}} \frac{1}{2} r^2 \left( |Lf|^2 + \frac{M}{r} |\underline{L}f|^2 + |\nabla f|^2 + qf^2 \right) d\underline{u}' d\omega, \\ E[f](\underline{\mathcal{H}}_{\underline{u}}^u) &= \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \frac{1}{2} r^2 \left( |\underline{L}f|^2 + \frac{M}{r} |Lf|^2 + |\nabla f|^2 + qf^2 \right) du' d\omega, \\ \mathcal{W}_1[f](\mathcal{H}_u^{\underline{u}}) &= \int_{\mathcal{H}_u^{\underline{u}}} \frac{1}{2} \left( r(L(rf))^2 + r^3 \frac{M}{r} (|\nabla f|^2 + qf^2) \right) d\underline{u}' d\omega, \\ \mathcal{W}_1[f](\underline{\mathcal{H}}_{\underline{u}}^u) &= \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \frac{1}{2} r^3 \left( |\nabla f|^2 + \frac{M}{r} |Lf|^2 + qf^2 \right) du' d\omega, \\ \mathcal{W}_1[f](\mathcal{D}_u^{\underline{u}}) &= \int_{\mathcal{D}_u^{\underline{u}}} (r^{-2} |L(rf)|^2 + |\nabla f|^2) dx dt, \\ \mathcal{W}_1[f](\Sigma_0^{u, \underline{u}}) &= \int_{\Sigma_0^{u, \underline{u}}} (r^{-1} (L(rf))^2 + r(|\nabla f|^2 + qf^2)) dx, \end{aligned} \quad (2.13)$$

where  $M > 0$  is a fixed constant to be specified. Throughout the paper,  $M$  and  $h$  are chosen such that

$$r|h| \leq M. \quad (2.14)$$

Throughout the paper, we let  $Z^{(n)}$  be the  $n$ -th order differential operator  $Z_1 \cdots Z_n$ , with each  $Z_m \in \{\partial, \Omega_{ij}, 1 \leq i < j \leq 3\}$ . We are ready to state the main results of this paper.

**Theorem 2.1.** *Consider*

$$\square_{\mathbf{m}} \phi = \mathcal{N}^{\alpha\beta}(\phi) \partial_\alpha \phi \partial_\beta \phi + q(x) \phi \quad (2.15)$$

with  $0 \leq q \leq 1$  satisfying (1.2) for  $n = 2$ . Let  $1 < \gamma_0 < 2$  and  $C > 1$  be fixed constants. There exist a small constant  $0 < \delta_1 \ll \frac{1}{100}$  and a universal constant  $R(\gamma_0, C) \geq 2$  such that for any  $0 < M \leq \delta_1^{\frac{1}{2}}$ , if the initial data set  $\phi[0]$  on  $\{r \geq R\}$  with  $R \geq R(\gamma_0, C)$  verifies  $\mathcal{E}_{2, \gamma_0, R} \leq CM^2$ , there exists a unique solution in the entire region  $\{u(M) \leq u_0(M)\}$  for all  $t > 0$ . Here the function  $u(M) = t - r_*(M, r)$ , with  $r_*(M, r)$  defined in (2.2), and  $u_0(M)$  is defined in (2.5). There hold for any  $-\underline{u} \leq u \leq u_0(M)$  that

$$\begin{aligned} E[Z^{(n)} \phi](\mathcal{H}_u^{\underline{u}}) + E[Z^{(n)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) &\lesssim \mathcal{E}_{2, \gamma_0, R} |u|^{-\gamma_0 + 2\zeta(Z^n)}, \\ \mathcal{W}_1[Z^{(n)} \phi](\mathcal{H}_u^{\underline{u}}) + \mathcal{W}_1[Z^{(n)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)} \phi](\mathcal{D}_u^{\underline{u}}) &\lesssim \mathcal{E}_{2, \gamma_0, R} |u|^{1 - \gamma_0 + 2\zeta(Z^n)}, \end{aligned} \quad (2.16)$$

where  $n \leq 2$ ,  $u$  and  $\underline{u}$  are shorthand notations for  $u(M)$  and  $\underline{u}(M)$ ,  $Z \in \{\Omega_{ij}, \partial\}$ . There also hold the pointwise estimates

$$r^3|\nabla\phi|^2 + r^2|u||\underline{L}\phi|^2 + r^3|L\phi|^2 \lesssim \mathcal{E}_{2,\gamma_0,R}|u|^{-\gamma_0}.$$

**Theorem 2.2.** Consider (2.15) with  $0 \leq q \leq 1$  satisfying (1.2) for  $n = 2$ . Let  $R \geq 2$  and  $1 < \gamma_0 < 2$  be fixed. There exist small constants  $0 < \delta_1 \ll \frac{1}{100}$  and  $\delta_0 > 0$  such that for any  $0 < M \leq \delta_1^{\frac{1}{2}}$ , if the initial data set verifies

$$\mathcal{E}_{2,\gamma_0,R} \leq \delta_0 M^2, \quad (2.17)$$

there exists a unique solution in the entire region of  $\{u(M) \leq u_0(M)\}$  for all  $t > 0$ . There hold the same set of energy estimates as in (2.16) and the pointwise estimates for any  $-\underline{u} \leq u \leq u_0(M)$ .

*Remark 2.3.* Let  $S_M = \{u(M) \leq -r_*(R, M)\}$ , which is the exterior region to the Schwarzschild outgoing cone initiated from  $\{r = R\}$  including the boundary. Clearly  $S_{M_1} \subset S_{M_2}$  if  $M_1 > M_2$ . Note that as  $M \rightarrow 0$ ,  $u(M) \rightarrow t - r$  and  $S_M$  approaches the entire open exterior of  $\{r - t = R\}$ . Then Theorem 2.1 indicates that, for any  $0 < M \leq \delta_1^{\frac{1}{2}}$ , the stability result can always holds in  $S_M$  for the set of non-compactly supported data with the norm (1.3) bounded by  $M^2$ , provided that  $R \geq R(\gamma_0)$ . See Figure 2.

*Remark 2.4.* Theorem 2.2 is a consequence of the proof of Theorem 2.1 under a stronger assumption on data. One can further refine it by assuming part of the energy norm of data verifies (2.17).

**Theorem 2.5.** Consider (1.1) which verifies (1.2) for  $n = 3$ . There exist a universal constant  $C \geq 1$ , a small constant  $0 < \delta_1 < \frac{1}{100}$  and a constant  $R(\gamma_0, C) \geq 2$ , such that, if the initial data set  $\phi[0]$  satisfies that

$$\mathcal{E}_{3,\gamma_0,R} \leq \delta_1 \quad \text{with } R \geq R(\gamma_0, C), \quad (2.18)$$

there exists a unique solution in the entire region  $\{u(M) \leq u_0(M)\}$  with  $M = C\delta_1^{\frac{1}{2}}$ . The solution verifies the following energy estimates for  $-\underline{u} \leq u \leq u_0(M)$

$$\begin{aligned} E[Z^{(n)}\phi](\mathcal{H}_{\underline{u}}^u) + E[Z^{(n)}\phi](\underline{\mathcal{H}}_{\underline{u}}^u) &\lesssim \mathcal{E}_{3,\gamma_0,R}|u|^{-\gamma_0+2\zeta(Z^n)}, \\ \mathcal{W}_1[Z^{(n)}\phi](\mathcal{H}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}\phi](\underline{\mathcal{H}}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{D}_{\underline{u}}^u) &\lesssim \mathcal{E}_{3,\gamma_0,R}|u|^{1-\gamma_0+2\zeta(Z^n)}, \end{aligned}$$

where  $n \leq 3$ ,  $Z \in \{\Omega_{ij}, \partial\}$  and verifies the decay estimates

$$r^3|\nabla Z^{(l)}\phi|^2 + r^2|u||\underline{L}Z^{(l)}\phi|^2 + r^3|LZ^{(l)}\phi|^2 \lesssim \mathcal{E}_{3,\gamma_0}|u|^{-\gamma_0+2\zeta(Z^l)}, \quad l \leq 1.$$

We remark that the sharp local-wellposedness result [27, 30] implies (1.1) is local-in-time wellposed for data  $\phi[0] = (\phi_0, \phi_1)$  in the normed space  $H^{3+\epsilon}(\mathbb{R}^3) \times H^{2+\epsilon}(\mathbb{R}^3)$ ,  $\epsilon > 0$ . In terms of the order of derivative, our data are at the level of  $H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ . In a standard regime of commuting vector fields approach, it is optimal in terms of regularity.

As an application, we provide the following exterior stability results for Einstein scalar fields under the wave coordinates.

**Theorem 2.6.** Consider the Einstein scalar field system

$$\begin{cases} \mathbf{R}_{\alpha\beta}(\mathbf{g}) = \partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}q_0\mathbf{g}_{\alpha\beta}\phi^2, \\ \square_{\mathbf{g}}\phi = q_0\phi \end{cases} \quad (2.19)$$

where the constant  $q_0 \geq 0$ . Under the wave coordinate gauge, we set  $\mathbf{h}_{\mu\nu}^1 = \mathbf{g}_{\mu\nu} - \mathbf{m}_{\mu\nu} - \frac{m_0}{r}\delta_{\mu\nu}$ . For constants  $1 < \gamma_0 < 2$  and  $C_0 \geq 1$ , there exist a small constant  $\delta_1 > 0$  and a constant  $R(\gamma_0, C_0) \geq 2$ , such that, if the initial data set<sup>14</sup>  $(\mathbf{h}^1[0], \phi[0])$  verifies

$$\mathcal{E}_{3,\gamma_0,R,0}(\mathbf{h}^1[0]) + \mathcal{E}_{3,\gamma_0,R,q_0}(\phi[0]) \leq C_0 m_0^2, \quad 0 < m_0 < \delta_1,$$

<sup>14</sup>We assume they satisfy the constraint equations. See details of the data construction in [24, (2.3)-(2.5), Page 1410].

where  $R \geq R(\gamma_0, C_0)$ , then there exists a unique solution  $\Psi = (\mathbf{h}^1, \phi)$  for (2.19) in the entire region of  $\{u(-m) \leq u_0(-m)\}$ , where  $0 \leq m < \frac{1}{2}m_0$  is a fixed constant.<sup>15</sup> With  $u, \underline{u}$  the shorthand notations for  $u(-m)$  and  $\underline{u}(-m)$ , for  $-\underline{u} \leq u \leq u_0(-m)$  there hold

$$E[Z^{(n)}\Psi](\mathcal{H}_{\underline{u}}^u) + E[Z^{(n)}\Psi](\underline{\mathcal{H}}_{\underline{u}}^u) \lesssim m_0^2 |u|^{-\gamma_0+2\zeta(Z^n)},$$

$$\mathcal{W}_1[Z^{(n)}\Psi](\mathcal{H}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}\Psi](\underline{\mathcal{H}}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}\Psi](\mathcal{D}_{\underline{u}}^u) \lesssim m_0^2 |u|^{1-\gamma_0+2\zeta(Z^n)},$$

where  $n \leq 3$  and  $Z \in \{\Omega_{ij}, \partial\}$ . There also hold the decay estimates

$$r^3 |\nabla Z^{(l)}\Psi|^2 + r^2 |u| |\underline{L} Z^{(l)}\Psi|^2 + r^3 |L Z^{(l)}\Psi|^2 \lesssim m_0^2 |u|^{-\gamma_0+2\zeta(Z^l)}, \quad l \leq 1,$$

$$q_0 |r^{\frac{5}{4}} Z^{(l)}\phi|^2 \lesssim m_0^2 |u|^{-\gamma_0+2\zeta(Z^l)+\frac{1}{2}}, \quad l \leq 1.$$

The constants in the above inequalities are independent of  $q_0^{-1}$ .

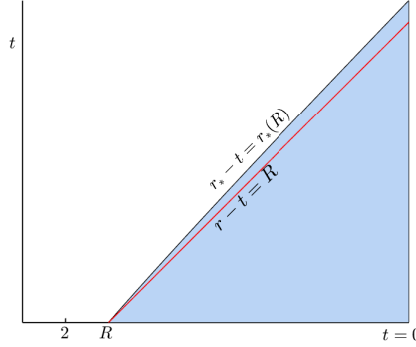


FIGURE 3. Illustration of the stability zone of Theorem 2.6

*Remark 2.7.* The energy estimates and pointwise decay in the above four theorems hold true if  $Z$  belongs to the generators of Poincaré group. If  $q \equiv 0$ , one can also extend the result to the set of vector fields  $\{\partial, x^\mu \partial_\mu, \Omega_{\mu\nu}\}$ .<sup>16</sup>

*Remark 2.8.* We do not rely on the weak null condition of the Einstein scalar field equation (2.19) to prove Theorem 2.6. If we further use the precise weak null structure of the reduced Einstein equation, the result can be proved up to  $R = 2$  under an extra smallness assumption of data.

### 3. Preliminary estimates

In this section, we adapt the Sobolev inequalities developed for the canonical null hypersurfaces in [19] to  $u$  and  $\underline{u}$  foliations. With the help of this set of Sobolev inequalities, we provide preliminary estimates in the region of  $\{u \leq u_0(M_0)\}$  for functions bounded in terms of the energy norms in (2.13). Some of the estimates, such as (3.32) and (3.30) in Lemma 3.4 are stronger than the known estimates for the free wave. They are crucial for the proof of boundedness of energies. We also provide estimates on the initial slice in this section.

For ease of exposition, we denote by  $\Omega$  any of the rotation vector fields in  $\{\Omega_{ij}, 1 \leq i < j \leq 3\}$  and by  $\Omega^{(k)}f$  any of the  $k$ -th order derivatives by the rotation vector fields.  $|\Omega f|^2 = \sum_{1 \leq i < j \leq 3} |\Omega_{ij} f|^2$ .  $|\Omega^{(k)} f|^2$  is the sum of all the combinations of  $k$ -th order derivatives by rotation vector fields. The same convention applies to  $|P\Omega^{(l)} f|$  if  $P$  is a linear differential operator.

With the help of (2.7) and  $|M_0| \ll 1$ , we may adapt from [19, Section 2.1] to obtain the following Sobolev inequalities.

<sup>15</sup>In our proof, we fix  $m = \frac{m_0}{20}$  for convenience.

<sup>16</sup> $\Omega_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$ ,  $0 \leq \mu < \nu \leq 3$  where  $x_\mu = \mathbf{m}_{\mu\nu} x^\nu$ .

**Lemma 3.1** (Sobolev inequalities). *For any smooth function  $f$  and constants verifying the relation  $2\gamma = \gamma'_0 + 2\gamma_2$ , we have, for all  $-\underline{u}_1 \leq -\underline{u} \leq u \leq u_0(M_0)$ ,*

$$\begin{aligned} \sup_{S_{u,\underline{u}}} |r^\gamma f|^4 &\lesssim \sum_{l \leq 1} \int_{S_{u,-u}} |r^\gamma \Omega^{(l)} f|^4 r^{-2} \\ &\quad + \sum_{k \leq 2} \int_{\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}} r^{2\gamma_2} |\Omega^{(k)} f|^2 r^{-2} \cdot \sum_{l \leq 1} \int_{\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}} r^{\gamma'_0} |L' \Omega^{(l)} (r^\gamma f)|^2 r^{-2}, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \int_{S_{u,\underline{u}}} |r^\gamma f|^4 r^{-2} &\lesssim \int_{S_{u,-u}} |r^\gamma f|^4 r^{-2} \\ &\quad + \int_{\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}} r^{\gamma'_0} |L' (r^\gamma f)|^2 r^{-2} \cdot \sum_{l \leq 1} \int_{\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}} r^{2\gamma_2} |\Omega^{(l)} f|^2 r^{-2}. \end{aligned} \quad (3.2)$$

The same estimates hold by using the incoming null hypersurface  $\underline{\mathcal{H}}_{\underline{u}}^u$ ; in this case  $L'$  is replaced by  $\underline{L}'$ , and the initial sphere is  $S_{-\underline{u},\underline{u}}$ .

Let (2.14) hold. We first give the following results in the initial slice.

**Proposition 3.2.** *Let  $1 < \gamma_0 < 2$  and the constant  $R \geq 2$  be fixed. With  $n \leq 3$ , there hold on  $\Sigma_0 \cap \{r \geq R\}$  the following estimates.*

(1) *With  $S_r = \{|x| = r\}$ , there hold*

$$\int_{S_r} r^{1+\gamma_0-2\zeta(Z^i)} |Z^{(i)} \phi|^2 d\omega \lesssim \mathcal{E}_{i,\gamma_0}, \quad i \leq n, \quad (3.3)$$

$$\int_{S_r} r^{2+2\gamma_0-4\zeta(Z^i)} |Z^{(i)} \phi|^4 d\omega \lesssim \mathcal{E}_{i,\gamma_0}^2, \quad i \leq n, \quad (3.4)$$

(2) *Let  $u_1, \underline{u}_1$  be a pair of fixed numbers verifying  $-\underline{u}_1 \leq u_1 \leq u_0$ . There hold*

$$\int_{-\underline{u}_1}^{\underline{u}_1} \left( \int_{S_{-\underline{u},\underline{u}}} r^{2-4\zeta(Z^i)} |Z^{(i)} \phi|^4 d\omega \right)^{\frac{1}{2}} d\underline{u} \lesssim |u_1|^{-\gamma_0+1} \mathcal{E}_{i,\gamma_0}, \quad i \leq n, \quad (3.5)$$

$$\int_{-\underline{u}_1}^{\underline{u}_1} \left( \int_{S_{-\underline{u},\underline{u}_1}} r^{2-4\zeta(Z^{i-1})} |\partial Z^{(i-1)} \phi|^4 d\omega \right)^{\frac{1}{2}} d\underline{u} \lesssim |u_1|^{-\gamma_0-1} \mathcal{E}_{i,\gamma_0}, \quad i \leq n. \quad (3.6)$$

The same estimates hold if the domain of integrals are changed to  $\int_{-\underline{u}_1}^{\underline{u}_1} \left( \int_{S_{u,-u}} \cdot d\omega \right)^{\frac{1}{2}} du$  for the same integrands. Moreover

$$\|r^{-\frac{1}{2}-\zeta(Z^i)} Z^{(i)} \phi\|_{L^2(\Sigma_0^{u_1,\underline{u}_1})}^2 \lesssim |u_1|^{-\gamma_0+1} \mathcal{E}_{i-1,\gamma_0}, \quad i \leq n+1, \quad (3.7)$$

$$\mathcal{W}_1[Z^{(i)} \phi](\Sigma_0^{u_1,\underline{u}_1}) \lesssim |u_1|^{-\gamma_0+1+2\zeta(Z^i)} \mathcal{E}_{i,\gamma_0}, \quad i \leq n. \quad (3.8)$$

(3) *For  $i \leq n$ , there holds on  $\Sigma_0 \cap \{r \geq R\}$  that*

$$r^2 |Z^{(i-1)} \phi(u, -u, \omega)|^2 \lesssim |u|^{-\gamma_0+1+2\zeta(Z^{i-1})} \mathcal{E}_{i,\gamma_0}. \quad (3.9)$$

*Proof.* Note that for  $i \leq n$ , due to  $\mathcal{E}_{n,\gamma_0,R} < \infty$ ,

$$\liminf_{r \rightarrow \infty} \int_{S_r} r^{\gamma_0-2\zeta(Z^i)} \left( |Z^{(i)} \phi|^2 + r^2 |\partial Z^{(i)} \phi|^2 \right) d\omega = 0. \quad (3.10)$$

By using the Sobolev embedding on  $\mathbb{S}^2$ , we have for any scalar function  $F$ ,

$$\left( \int_{S_r} r^4 |F|^4 d\omega \right)^{\frac{1}{2}} \lesssim \|\Omega F\|_{L^2(S_r)}^2 + \|F\|_{L^2(S_r)}^2.$$

Thus, with  $F = Z^{(i)} \phi$ ,  $i \leq n$ , by using (3.10) we have

$$\liminf_{r \rightarrow \infty} \int_{S_r} r^{2\gamma_0-4\zeta(Z^i)} |Z^{(i)} \phi|^4 d\omega = 0. \quad (3.11)$$

Now consider (3.3) with the help of (3.10). By integrating back from the spacelike infinity,

$$\begin{aligned} \int_{S_r} r^{1+\gamma_0-2\zeta(Z^i)} |Z^{(i)}\phi|^2 d\omega &\lesssim r^{1+\gamma_0-2\zeta(Z^i)} \int_r^\infty \int_{S_{r'}} |\partial_r Z^{(i)}\phi| |Z^{(i)}\phi| d\omega dr' \\ &\lesssim \|\partial Z^{(i)}\phi \cdot r^{1+\frac{\gamma_0}{2}-\zeta(Z^i)}\|_{L_r^2 L_\omega^2(r'\geq r)} \|Z^{(i)}\phi \cdot r^{\frac{\gamma_0}{2}-\zeta(Z^i)}\|_{L_r^2 L_\omega^2(r'\geq r)} \\ &\lesssim \mathcal{E}_{i,\gamma_0}, \end{aligned}$$

which gives (3.3).

By using (3.11) and the Hölder's inequality, we can derive that

$$\begin{aligned} \int_{S_r} |Z^{(i)}\phi|^4 d\omega &\lesssim \int_r^\infty \int_{S_{r'}} |\partial_r Z^{(i)}\phi| |Z^{(i)}\phi|^3 d\omega dr' \\ &\lesssim \|\partial Z^{(i)}\phi\|_{L_r^2 L_\omega^2\{r'\geq r\}} \| |Z^{(i)}\phi|^3 \|_{L_r^2 L_\omega^2\{r'\geq r\}}. \end{aligned} \quad (3.12)$$

Note that by using [4, (3.2.4a)]

$$\int_r^\infty \int_{S_{r'}} |F|^6 d\omega dr' \lesssim \int_r^\infty \left( \int_{S_{r'}} |F|^4 d\omega \right) \left( \int_{S_{r'}} (|F|^2 + |r\nabla F|^2) d\omega \right) dr'.$$

We then combine the above inequality with (3.12) to obtain

$$\begin{aligned} \sup_{r'\geq r} \left( \int_{S_{r'}} |Z^{(i)}\phi|^4 d\omega \right)^{\frac{1}{2}} &\lesssim \|\partial Z^{(i)}\phi\|_{L_r^2 L_\omega^2\{r'\geq r\}} \| |Z^{(i)}\phi| + |r\nabla Z^{(i)}\phi| \|_{L_r^2 L_\omega^2\{r'\geq r\}} \\ &\lesssim r^{-(1-2\zeta(Z^i)+\gamma_0)} \mathcal{E}_{i,\gamma_0}. \end{aligned}$$

This gives (3.4).

Note that  $\underline{u}_1 \geq \underline{u} \geq -u_1$ , and (2.10) implies  $r(S_{-\underline{u},\underline{u}}) \geq \frac{1}{2}\underline{u}$ . By integrating (3.4), we can obtain (3.5), which implies (3.6) immediately. (3.7) can be derived by using (3.3), the first identity in (2.10) and the definition (1.3). (3.8) is a consequence of (3.7).

Next we prove (3.9). Let  $r_1 \geq R$ . We adapt (3.1) to  $\Sigma_0 \cap \{r \geq R\}$  with  $\gamma'_0 = \gamma = 1$  and  $\gamma_2 = \frac{1}{2}$ . This implies for  $r \geq r_1$  that

$$\begin{aligned} \sup_{S_r} |rZ^{(i-1)}\phi|^4 &\lesssim \liminf_{r \rightarrow \infty} \int_{S_r} |r^\gamma \Omega^{(\leq 1)} Z^{(i-1)}\phi|^4 r^{-2} \\ &\quad + \int_{\Sigma_0 \cap \{r \geq r_1\}} r^{2\gamma_2} |\Omega^{(\leq 2)} Z^{(i-1)}\phi|^2 r^{-2} \cdot \int_{\Sigma_0 \cap \{r \geq r_1\}} r^{\gamma'_0} |\partial_r \Omega^{(\leq 1)}(r^\gamma Z^{(i-1)}\phi)|^2 r^{-2} \\ &\lesssim r_1^{-2\gamma_0+2+4\zeta(Z^{i-1})} \mathcal{E}_{i,\gamma_0}^2, \end{aligned}$$

where due to (3.4) and  $\gamma_0 > 1$ , the first term on the right vanished, and we also used the fact that  $|\Omega f| \lesssim r|\partial f|$  to treat the term of  $\Omega^{(\leq 2)} Z^{(i-1)}\phi$ . Thus, in view of (2.10), (3.9) is proved.  $\square$

The energy or weighted energy norms in (2.13) not only give control on  $\partial\varphi$ , they also control  $\varphi$  itself, which is given in the following result.

**Lemma 3.3.** *Let  $-\underline{u} \leq u \leq u_0(M_0)$  and  $\alpha > 0$  be fixed. There hold the following estimates*

$$\begin{aligned} \int_{\underline{\mathcal{H}}_{\underline{u}}} r\varphi^2 du' d\omega + \int_{\mathcal{D}_{\underline{u}}^u} \left\{ r^2(L\varphi)^2 + \alpha \left( \frac{|u|}{r} \right)^\alpha \varphi^2 \right\} du' d\omega \\ \lesssim \mathcal{W}_1[\varphi](\mathcal{D}_{\underline{u}}^u) + M \int_{-\underline{u}}^u |u'|^{-1} E[\varphi](\mathcal{H}_{u'}^u) du' + \int_{\Sigma_{\underline{u},\underline{u}}} r^{-1} \varphi^2 dx, \end{aligned} \quad (3.13)$$

$$\int_{\mathcal{H}_{\underline{u}}^u} r|L'(r\varphi)|^2 d\omega d\underline{u}' \lesssim \mathcal{W}_1[\varphi](\mathcal{H}_{\underline{u}}^u) + ME[\varphi](\mathcal{H}_{\underline{u}}^u) + M^2 \int_{\mathcal{H}_{\underline{u}}^u} r^{-1} |\varphi|^2 d\omega d\underline{u}', \quad (3.14)$$

$$\int_{S_{u,\underline{u}}} r\varphi^2 d\omega + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \varphi^2 du' d\omega \lesssim \int_{S_{-\underline{u},\underline{u}}} r\varphi^2 d\omega + E[\varphi](\mathcal{H}_{\underline{u}}^u), \quad (3.15)$$

$$\int_{\mathcal{H}_{\underline{u}}^u} \varphi^2 d\underline{u}' d\underline{w} \lesssim \int_{S_{-\underline{u}, \underline{u}}} r \varphi^2 d\underline{w} + E[\varphi](\mathcal{H}_{\underline{u}}^u) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u). \quad (3.16)$$

*Proof.* We first prove

$$\begin{aligned} \int_{\mathcal{H}_{\underline{u}}^u} r \varphi^2 d\underline{u}' d\underline{w} + \int_{\mathcal{D}_{\underline{u}}^u} \left\{ r^2 (L' \varphi)^2 + \alpha \left( \frac{|u|}{r} \right)^\alpha \varphi^2 \right\} d\underline{u}' d\underline{w} \\ \lesssim \int_{\mathcal{D}_{\underline{u}}^u} |L'(r\varphi)|^2 d\underline{u}' d\underline{w} + \int_{\Sigma_0^{u, \underline{u}}} r \varphi^2 d\underline{u}' d\underline{w}. \end{aligned} \quad (3.17)$$

Due to  $L'r = 1 + h$ , (2.7) and (2.8), by directly computing  $|L'(r\varphi)|^2$ , we obtain

$$\begin{aligned} \frac{(L'(r\varphi))^2}{2(1-h)(1+h)} &= \frac{r^2 (L'\varphi)^2}{2(1-h)(1+h)} + r \partial_{\underline{u}}(\varphi^2) + \frac{1+h}{2(1-h)} \varphi^2 \\ &= \frac{r^2 (L'\varphi)^2}{2(1-h)(1+h)} + \partial_{\underline{u}}(r\varphi^2). \end{aligned}$$

Integrating the above identity in  $\mathcal{D}_{\underline{u}}^u$  and using the smallness of  $|h|$  imply

$$\int_{\mathcal{H}_{\underline{u}}^u} r \varphi^2 d\underline{w} d\underline{u}' + \int_{\mathcal{D}_{\underline{u}}^u} r^2 (L'\varphi)^2 d\underline{u}' d\underline{w} \lesssim \int_{\Sigma_0^{u, \underline{u}}} r \varphi^2 d\underline{u}' d\underline{w} + \int_{\mathcal{D}_{\underline{u}}^u} |L'(r\varphi)|^2 d\underline{u}' d\underline{w}. \quad (3.18)$$

By using  $r \approx \underline{u}$  in (2.11), we have

$$\int_{\mathcal{D}_{\underline{u}}^u} r^{-\alpha} \varphi^2 d\underline{w} d\underline{u}' d\underline{w} \lesssim \alpha^{-1} |u|^{-\alpha} \sup_{-u \leq \underline{u}' \leq \underline{u}} \int_{\mathcal{H}_{\underline{u}'}^u} r \varphi^2 d\underline{w} d\underline{u}'.$$

Since (3.18) holds for any  $-u \leq \underline{u}' \leq \underline{u}$ , we can conclude (3.17).

We can prove (3.13) by using (3.17). Note that  $L'(r\varphi) = L(r\varphi) + h\underline{L}(r\varphi)$ . We have

$$\begin{aligned} \int_{\mathcal{D}_{\underline{u}}^u} |L'(r\varphi) - L(r\varphi)|^2 d\underline{u}' d\underline{w} &= \int_{\mathcal{D}_{\underline{u}}^u} |h\underline{L}(r\varphi)|^2 d\underline{u}' d\underline{w} \\ &\lesssim \int_{\mathcal{D}_{\underline{u}}^u} |rh\underline{L}\varphi - h\varphi|^2 d\underline{u}' d\underline{w}. \end{aligned}$$

Due to (2.11) and (2.14), we have

$$\int_{\mathcal{D}_{\underline{u}}^u} r^2 h^2 (\underline{L}\varphi)^2 d\underline{u}' d\underline{w} \lesssim M^2 \int_{\mathcal{D}_{\underline{u}}^u} r^{-1} (\underline{L}\varphi)^2 r d\underline{u}' d\underline{w} \lesssim M \int_{-\underline{u}}^u |u'|^{-1} E[\varphi](\mathcal{H}_{u'}^u) du'.$$

By using again (2.11) and (2.14), we have  $\int_{\mathcal{D}_{\underline{u}}^u} h^2 \varphi^2 d\underline{u}' d\underline{w} \lesssim \int_{\mathcal{D}_{\underline{u}}^u} M^2 \underline{u}'^{-3} r \varphi^2 d\underline{u}' d\underline{w}$ . This term can be absorbed by the first term on the left of (3.17) by using the Gronwall's inequality. Thus we can obtain (3.13) by using (3.17).

(3.14) follows by a direct calculation.

Next we prove (3.15). By using (2.8) we can derive on  $\mathcal{H}_{\underline{u}}^u$  that  $\partial_u(r\varphi^2) = r\partial_u(\varphi^2) - \frac{1}{2}\mathbf{b}\varphi^2$ . By integrating the above identity along  $\underline{\mathcal{H}}_{\underline{u}}^u$  with area element  $d\underline{u}' d\underline{w}$  and by using (2.7), we can obtain

$$\int_{S_{u, \underline{u}}} r \varphi^2 d\underline{w} + \frac{1}{2} \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \mathbf{b} \varphi^2 d\underline{w} d\underline{u}' = \int_{S_{-\underline{u}, \underline{u}}} r \varphi^2 d\underline{w} + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \frac{r^2}{(r - M_0)} \underline{L}' \varphi \cdot \varphi d\underline{w} d\underline{u}'.$$

By using Cauchy-Schwartz inequality,  $\mathbf{b} > \frac{3}{4}$ , and the smallness of  $|h|$ , we have

$$\int_{S_{u, \underline{u}}} r \varphi^2 d\underline{w} + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \varphi^2 d\underline{w} d\underline{u}' \lesssim \int_{S_{-\underline{u}, \underline{u}}} r \varphi^2 d\underline{w} + E[\varphi](\mathcal{H}_{\underline{u}}^u).$$

This proves (3.15). Next, we prove (3.16) in the same fashion. It is direct to derive along  $\mathcal{H}_u$  that  $\partial_{\underline{u}}(r\varphi^2) = r\partial_{\underline{u}}(\varphi^2) + \frac{1}{2}\mathbf{b}\varphi^2$ . Thus, by using (2.7), we may integrate along  $\mathcal{H}_{\underline{u}}^u$  with the area element  $d\underline{u}' d\underline{w}$  to derive that

$$\frac{1}{2} \int_{\mathcal{H}_{\underline{u}}^u} \mathbf{b} \varphi^2 d\underline{w} d\underline{u}' = \int_{S_{u, \underline{u}}} r \varphi^2 d\underline{w} - \int_{S_{u, -u}} r \varphi^2 d\underline{w} - \int_{\mathcal{H}_{\underline{u}}^u} \frac{r^2}{r - M_0} \underline{L}' \varphi \cdot \varphi d\underline{u}' d\underline{w}.$$



We then combine the estimate of (3.15) and Cauchy Schwartz inequality to derive

$$\int_{\mathcal{H}_{\underline{u}}^{\underline{u}}} \varphi^2 d\omega d\underline{u}' \lesssim \int_{S_{-\underline{u}, \underline{u}}} r \varphi^2 d\omega + E[\varphi](\mathcal{H}_{\underline{u}}^{\underline{u}}) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^{\underline{u}})$$

as desired in (3.16).  $\square$

**3.1. Simple facts of vector fields.** Before proceeding further, we give basic facts about the vector fields.

Let  $\nabla$  be the covariant derivative on  $S_{u, \underline{u}}$ . Its component under the Cartesian frame  $\partial_i, i = 1, 2, 3$  takes the form of  $\nabla_i = \partial_i - \omega^i \partial_r$ ,  $\omega^i = x^i/r$ . We set  $\bar{\partial} = (\nabla, L)$ ,  $\underline{\partial} = (\nabla, \underline{L})$ . For smooth functions  $f$ , we have

(1) By direct calculation, there hold

$$[\partial_\rho, \Omega_{\mu\nu}] = \mathbf{m}_{\rho\mu} \partial_\nu - \mathbf{m}_{\rho\nu} \partial_\mu, \quad 0 \leq \mu < \nu \leq 3, \quad (3.19)$$

$$[\Omega_{ij}, \nabla_l]f = -\delta_{li} \nabla_j f + \delta_{lj} \nabla_i f, \quad (3.20)$$

$$[L, \Omega_{ij}] = 0 = [\underline{L}, \Omega_{ij}] = [\partial_r, \Omega_{ij}], \quad [L, \underline{L}] = 0, \quad (3.21)$$

$$[L, \nabla]f = -r^{-1} \nabla f, \quad [\underline{L}, \nabla]f = r^{-1} \nabla f, \quad [\partial_r, \nabla]f = -r^{-1} \nabla f. \quad (3.22)$$

(2) For  $X = L, \underline{L}, L', \underline{L}'$ , due to  $X\omega^i = 0$ , there hold

$$|X(rLf)| + |X(r\underline{L}f)| \lesssim |X(r\partial f)|; \quad |XLf| + |X\underline{L}f| \lesssim |X\partial f|. \quad (3.23)$$

(3)

$$|\nabla \underline{L}f| + |\nabla Lf| + |\nabla \partial_r f| \lesssim |\nabla \partial f| + r^{-1} |\nabla f|, \quad (3.24)$$

$$|\partial f|^2 + |\Omega \partial_\mu f|^2 \approx |\partial f|^2 + |\partial_\mu \Omega f|^2, \quad \mu = 0, \dots, 3, \quad (3.25)$$

$$\Omega^{(\ell)} \mathcal{D}_* f = \mathcal{D}_* \Omega^{(\leq \ell)} f, \quad \text{if } \mathcal{D}_* \in \{\nabla, \underline{\partial}, \bar{\partial}\} \quad \ell \in \mathbb{N}, \quad (3.26)$$

where  $\Omega$  means one of  $\{\Omega_{ij}, 1 \leq i < j \leq 3\}$ .

Indeed, it is direct to check that

$$\nabla_l \omega^i = r^{-1} (\delta_l^i - \omega^l \omega^i); \quad \Omega_{lj} \omega^i = r^{-1} (x^l \delta_j^i - x^j \delta_l^i), \quad 1 \leq l < j \leq 3. \quad (3.27)$$

(3.24) follows by using the first identity. To see (3.25), in view of (3.19)  $|\Omega, \partial_i|f| \lesssim |\partial f|$ , by also using that  $\Omega \partial_0 = \partial_0 \Omega$ , (3.25) is proved.

By using (3.20), we can obtain

$$\begin{aligned} \Omega_{ij} \Omega_{mn} \nabla_l f &= \nabla_l \Omega_{ij} \Omega_{mn} f + [\Omega_{ij}, \nabla_l] \Omega_{mn} f + \Omega_{ij} [\Omega_{mn}, \nabla_l] f \\ &= \nabla_l \Omega_{ij} \Omega_{mn} f + \nabla \Omega_{mn} f + \nabla \Omega_{ij}^{(\leq 1)} f. \end{aligned}$$

Higher order calculation can be done by induction. This implies (3.26) with  $\mathcal{D}_* = \nabla$  for  $\ell \in \mathbb{N}$ . We then combine this calculation with (3.21) to conclude (3.26) if  $\mathcal{D}_*$  is one of the derivatives  $\nabla, \underline{\partial}, \bar{\partial}$ .

**3.2.  $L^4$  type estimates.** In order to give the product estimates for the nonlinear terms of (1.1), we will rely on  $L^4$ -type estimates. They can be derived by using the Sobolev inequality (3.2) and the energies in (2.13).

**Lemma 3.4.** *Let  $-\underline{u}_1 \leq u_1 \leq u_0(M_0)$  and  $\alpha > 0$  be fixed. For smooth functions  $F$  and  $\psi$ , there hold the following estimates*

$$\begin{aligned} \|r^{\frac{1}{2}} F\|_{L_u^2 L_\omega^4(\mathcal{H}_{\underline{u}_1}^{u_1})}^2 &\lesssim \mathcal{W}_1[F](\mathcal{H}_{\underline{u}_1}^{u_1}) + \mathcal{W}_1[F](\mathcal{D}_{u_1}^{u_1}) + \|r^{-\frac{1}{2}} F\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 \\ &\quad + M \int_{-\underline{u}_1}^{u_1} |u|^{-1} E[F](\mathcal{H}_u^{u_1}) du, \end{aligned} \quad (3.28)$$

$$\begin{aligned} \alpha \left\| \left( \frac{|u_1|}{r} \right)^\alpha F \right\|_{L_u^2 L_{\underline{u}}^2 L_\omega^4(\mathcal{D}_{u_1}^{u_1})}^2 &\lesssim \mathcal{W}_1[F](\mathcal{D}_{u_1}^{u_1}) + \|r^{-\frac{1}{2}} F\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 \\ &\quad + M \int_{-\underline{u}_1}^{u_1} |u|^{-1} E[F](\mathcal{H}_u^{u_1}) du, \end{aligned} \quad (3.29)$$

$$\|r^{\frac{1}{2}} \partial \psi\|_{L_u^2 L_{\underline{u}}^2 L_\omega^4(\mathcal{D}_{u_1}^{u_1})}^2 \lesssim M^{-1} \int_{-\underline{u}_1}^{u_1} E[\Omega^{(\leq 1)} \psi](\mathcal{H}_u^{u_1}) du, \quad (3.30)$$

$$\int_{S_{u_1, \underline{u}_1}} r^2 |F|^4 d\omega \lesssim \int_{S_{u_1, -u_1}} r^2 |F|^4 d\omega + \left( E[F](\mathcal{H}_{u_1}^{u_1}) + \int_{\mathcal{H}_{u_1}^{u_1}} |F|^2 d\omega d\underline{u} \right)^2, \quad (3.31)$$

$$\begin{aligned} \|r^{\frac{1}{2}} \partial \psi\|_{L_{\underline{u}}^2 L_u^\infty L_\omega^4(\mathcal{D}_{u_1}^{u_1})}^2 &\lesssim \int_{-u_1}^{u_1} \left( \int_{S_{-\underline{u}, \underline{u}}} r^2 |\partial \psi|^4 d\omega \right)^{\frac{1}{2}} d\underline{u} + M^{-1/2} \left( \int_{-\underline{u}_1}^{u_1} E[\Omega^{(\leq 1)} \psi](\mathcal{H}_u^{u_1}) du \right)^{\frac{1}{2}} \\ &\quad \cdot \left( M^{-1} \int_{-\underline{u}_1}^{u_1} E[\partial \psi](\mathcal{H}_u^{u_1}) du + |u_1|^{-2} \left( \sup_{-u_1 \leq \underline{u} < u_1} E[\psi](\underline{\mathcal{H}}_{\underline{u}}^{u_1}) \right. \right. \\ &\quad \left. \left. + \sup_{-\underline{u}_1 \leq u \leq u_1} E[\psi](\mathcal{H}_u^{u_1}) \right) \right)^{\frac{1}{2}}, \end{aligned} \quad (3.32)$$

$$\begin{aligned} \int_{S_{u_1, \underline{u}_1}} |r \partial \psi|^4 d\omega &\lesssim \int_{S_{-\underline{u}_1, \underline{u}_1}} |r \partial \psi|^4 d\omega + \left( E[\partial \psi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + |u_1|^{-2} E[\psi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) \right) \\ &\quad \cdot E[\Omega^{(\leq 1)} \psi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}), \end{aligned} \quad (3.33)$$

$$\begin{aligned} \int_{S_{u_1, -u_1}} |r L \psi|^4 d\omega &\lesssim \int_{S_{u_1, -u_1}} |r L \psi|^4 d\omega + \left( E[\partial \psi](\mathcal{H}_{u_1}^{u_1}) + |u_1|^{-2} E[\psi](\mathcal{H}_{u_1}^{u_1}) \right) \\ &\quad \cdot E[\Omega^{(\leq 1)} \psi](\mathcal{H}_{u_1}^{u_1}), \end{aligned} \quad (3.34)$$

$$\|r \partial \psi\|_{L_u^2 L_\omega^4(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})}^2 \lesssim E[\Omega^{(\leq 1)} \psi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}), \quad (3.35)$$

$$\begin{aligned} \|r L \psi\|_{L_u^2 L_\omega^4(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})}^2 &\lesssim \int_{-\underline{u}_1}^{u_1} \left( \int_{S_{u, -u}} |r L \psi|^4 d\omega \right)^{\frac{1}{2}} du + \left( \int_{-\underline{u}_1}^{u_1} E[\Omega^{(\leq 1)} \psi](\mathcal{H}_u^{u_1}) du \right)^{\frac{1}{2}} \\ &\quad \cdot \left( \int_{-\underline{u}_1}^{u_1} (E[\partial \psi](\mathcal{H}_u^{u_1}) + |u|^{-2} E[\psi](\mathcal{H}_u^{u_1})) du \right)^{\frac{1}{2}}. \end{aligned} \quad (3.36)$$

*Proof.* We first derive directly from the Sobolev inequality on unit sphere for  $-\underline{u} \leq u \leq u_0$ ,

$$\|r^{\frac{1}{2}} F\|_{L_\omega^4(S_{u, \underline{u}})} \lesssim \|r^{\frac{1}{2}} \nabla F\|_{L^2(S_{u, \underline{u}})} + \|r^{-1/2} F\|_{L^2(S_{u, \underline{u}})}. \quad (3.37)$$

Integrating the above inequality in  $u$  variable along  $\underline{\mathcal{H}}_{\underline{u}_1}$  implies

$$\|r^{\frac{1}{2}} F\|_{L_u^2 L_\omega^4(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})} \lesssim \mathcal{W}_1[F]^{\frac{1}{2}}(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + \|r^{-\frac{1}{2}} F\|_{L^2(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})}.$$

We then use (3.13) to obtain (3.28).

By using (3.37) and noting that  $|u_1|/r$  is a constant on  $S_{u, \underline{u}}$  we can obtain

$$\begin{aligned} \left\| \alpha \left( \frac{|u_1|}{r} \right)^\alpha F \right\|_{L_u^2 L_{\underline{u}}^2 L_\omega^4(\mathcal{D}_{u_1}^{u_1})} &\lesssim \left\| \alpha \left( \frac{|u_1|}{r} \right)^\alpha \nabla F \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} + \left\| r^{-1} \alpha \left( \frac{|u_1|}{r} \right)^\alpha F \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\ &\lesssim \|\nabla F\|_{L^2(\mathcal{D}_{u_1}^{u_1})} + \left\| \alpha \left( \frac{|u_1|}{r} \right)^\alpha r^{-1} F \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\ &\lesssim \mathcal{W}_1^{\frac{1}{2}}[F](\mathcal{D}_{u_1}^{u_1}) + \left\| \alpha \left( \frac{|u_1|}{r} \right)^\alpha r^{-1} F \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \end{aligned}$$

which together with (3.13) shows (3.29).

Similarly, by using the Sobolev inequality that

$$\|r^{\frac{1}{2}}\partial\psi\|_{L_u^2 L_{\underline{u}}^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \lesssim \|r^{\frac{1}{2}}\nabla\partial\psi\|_{L^2(\mathcal{D}_{u_1}^{u_1})} + \|r^{-\frac{1}{2}}\partial\psi\|_{L^2(\mathcal{D}_{u_1}^{u_1})},$$

$|\nabla f| \approx r^{-1}|\Omega f|$  and (3.25), we can obtain (3.30).

To prove (3.31), we may take  $\gamma'_0 = 1, \gamma = \frac{1}{2}, \gamma_2 = 0$  in (3.2) to derive

$$\begin{aligned} & \int_{S_{u_1, \underline{u}_1}} r^2 |F|^4 d\omega \\ & \lesssim \int_{S_{u_1, -u_1}} r^2 |F|^4 d\omega + \int_{\mathcal{H}_{u_1}^{u_1}} r |L'(r^{\frac{1}{2}}F)|^2 d\underline{u} d\omega \cdot \int_{\mathcal{H}_{u_1}^{u_1}} (|F|^2 + r^2 |\nabla F|^2) d\underline{u} d\omega. \end{aligned} \quad (3.38)$$

Note that  $r^{\frac{1}{2}}L'(r^{\frac{1}{2}}F) = rL'F + \frac{1}{2}(1+h)F$ . in view of the smallness of  $|h|$  in (2.14) we can obtain

$$\int_{\mathcal{H}_{u_1}^{u_1}} |r^{\frac{1}{2}}L'(r^{\frac{1}{2}}F)|^2 d\underline{u} d\omega \lesssim E[F](\mathcal{H}_{u_1}^{u_1}) + \int_{\mathcal{H}_{u_1}^{u_1}} |F|^2 d\omega d\underline{u}.$$

Substituting the above inequality into (3.38) implies (3.31).

Now we prove (3.32). Note that by taking  $\gamma'_0 = 0$  and  $\gamma_2 = \frac{1}{2} = \gamma$  in (3.2), we can derive for any smooth scalar function  $F$  and  $-u_1 \leq \underline{u} \leq u_1$  that

$$\begin{aligned} & \|r^{\frac{1}{2}}F\|_{L_u^\infty L_\omega^4(\mathcal{H}_{\underline{u}}^{u_1})}^2 \\ & \leq \left( \int_{S_{-\underline{u}, \underline{u}}} r^2 |F|^4 d\omega \right)^{\frac{1}{2}} + \left( \int_{\mathcal{H}_{\underline{u}}^{u_1}} |\underline{L}'(r^{\frac{1}{2}}F)|^2 d\underline{u} d\omega \right)^{\frac{1}{2}} \left( \int_{\mathcal{H}_{\underline{u}}^{u_1}} (|F|^2 + |\Omega F|^2) r d\underline{u} d\omega \right)^{\frac{1}{2}}. \end{aligned}$$

We then apply the above inequality to  $F = \partial\psi$ , followed with integrating in  $\underline{u}$  variable, to obtain

$$\begin{aligned} & \|r^{\frac{1}{2}}\partial\psi\|_{L_{\underline{u}}^2 L_u^\infty L_\omega^4(\mathcal{D}_{u_1}^{u_1})}^2 \lesssim \int_{-u_1}^{u_1} \left( \int_{S_{-\underline{u}, \underline{u}}} r^2 |\partial\psi|^4 d\omega \right)^{\frac{1}{2}} d\underline{u} \\ & + \left( \int_{-u_1}^{u_1} \int_{\mathcal{H}_{\underline{u}}^{u_1}} |\underline{L}'(r^{\frac{1}{2}}\partial\psi)|^2 d\underline{u} d\omega d\underline{u} \right)^{\frac{1}{2}} \left( \int_{-u_1}^{u_1} \int_{\mathcal{H}_{\underline{u}}^{u_1}} (|\partial\psi|^2 + |\Omega\partial\psi|^2) r d\underline{u} d\omega d\underline{u} \right)^{\frac{1}{2}}. \end{aligned} \quad (3.39)$$

Note also that  $|\underline{L}'(r^{\frac{1}{2}}\partial\psi)| \lesssim r^{\frac{1}{2}}|\underline{L}'\partial\psi| + r^{-\frac{1}{2}}(1+h)|\partial\psi|$ . By using the smallness of  $|h|$  we then have

$$\begin{aligned} & \int_{\mathcal{D}_{u_1}^{u_1}} |\underline{L}'(r^{\frac{1}{2}}\partial\psi)|^2 d\underline{u} d\omega d\underline{u} \\ & \lesssim M^{-1} \int_{-u_1}^{u_1} E[\partial\psi](\mathcal{H}_{\underline{u}}^{u_1}) d\underline{u} + |u_1|^{-2} \left( \sup_{-u_1 \leq \underline{u} \leq u_1} E[\psi](\mathcal{H}_{\underline{u}}^{u_1}) + \sup_{-u_1 \leq u \leq u_1} E[\psi](\mathcal{H}_u^{u_1}) \right), \end{aligned} \quad (3.40)$$

where the last line is the bound for  $\int_{\mathcal{D}_{u_1}^{u_1}} r^{-3} |\partial\psi|^2$ . It is achieved by using  $|\partial\psi| \lesssim |\underline{\partial}\psi| + |\bar{\partial}\psi|$ , (2.13) and (2.11), followed with integrating in  $u$  or  $\underline{u}$  variable. We then substitute (3.25) and (3.40) into (3.39), which implies (3.32).

Noting that  $|\underline{L}'(r\partial\psi)| \lesssim |r\underline{L}'\partial\psi| + |\partial\psi|$ , (3.33) can be proved by using (3.2) for  $\underline{\partial}\psi$  along  $\mathcal{H}_{\underline{u}_1}^{u_1}$  with  $\gamma'_0 = 0, \gamma_2 = \gamma = 1$ , with the help of (3.26).

Note that (3.23) and the smallness of  $|h|$  imply  $|L'(rL\psi)| \lesssim |L\psi| + r|L\partial\psi|$ . Applying (3.2) to  $L\psi$  along  $\mathcal{H}_{u_1}^{u_1}$  with the same combination of weight exponents, we can similarly obtain (3.34) with the help of (3.26).

The Sobolev embedding on  $\mathbb{S}^2$  gives

$$\|r\underline{\partial}\psi\|_{L_u^2 L_\omega^4(\mathcal{H}_{\underline{u}_1}^{u_1})} \lesssim \|r\underline{\Omega}\underline{\partial}\psi\|_{L_u^2 L_\omega^2(\mathcal{H}_{\underline{u}_1}^{u_1})} + \|r\underline{\partial}\psi\|_{L_u^2 L_\omega^2(\mathcal{H}_{\underline{u}_1}^{u_1})}.$$

Then with a direct substitution of (3.26), we can obtain (3.35).

(3.36) follows by a direct integration of (3.34) in  $-u_1 \leq u \leq u_1$ .  $\square$

#### 4. Decay estimates

In this section, we provide in Proposition 4.1 and Proposition 4.4 the decay properties for smooth functions  $\phi \in \mathbb{R}^{3+1}$  with bounded energies.

To be more precise, let  $n = 2$  or  $3$  be fixed. We suppose  $\phi$  verifies  $\mathcal{E}_{n,\gamma_0,R}(\phi[0]) < \infty$  for a fixed  $1 < \gamma_0 < 2$  and the following assumptions:

- Let  $\underline{u}_* > -u_0(M_0)$  be a fixed number and  $\Delta_0 > 0$  be a fixed small constant. There hold with  $0 \leq l \leq n$  that

$$\begin{aligned} E[Z^{(l)}\phi](\mathcal{H}_{\underline{u}}^u) + E[Z^{(l)}\phi](\underline{\mathcal{H}}_{\underline{u}}^u) &\leq 2\Delta_0|u|^{-\gamma_0+2\zeta(Z^l)}, \\ \mathcal{W}_1[Z^{(l)}\phi](\mathcal{H}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(l)}\phi](\underline{\mathcal{H}}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(l)}\phi](\mathcal{D}_{\underline{u}}^u) &\leq 2\Delta_0|u|^{-\gamma_0+1+2\zeta(Z^l)}, \end{aligned} \quad (\text{BA}_n)$$

for all  $-\underline{u}_* \leq -\underline{u} \leq u \leq u_0$ .

We first derive a set of estimates, including the pointwise decay estimates, integrated decay estimates and the improved Hardy's inequality.

**Proposition 4.1.** *Let  $n = 2$  or  $3$  be fixed and let the assumption  $(\text{BA}_n)$  hold.*

(1) *For any point  $(u, \underline{u}, \omega)$  with  $-\underline{u}_* \leq -\underline{u} \leq u \leq u_0$  and  $\omega \in \mathbb{S}^2$ , there hold for  $l \leq n - 2$  that*

$$r^3|\nabla Z^{(l)}\phi|^2 + r^2|u|\underline{L}Z^{(l)}\phi|^2 + r^3|LZ^{(l)}\phi|^2 \lesssim (\mathcal{E}_{l+2,\gamma_0} + \Delta_0)|u|^{-\gamma_0+2\zeta(Z^l)}, \quad (4.1)$$

$$|rZ^{(l)}\phi(u, v, \omega)|^2 \lesssim (\mathcal{E}_{l+2,\gamma_0} + \Delta_0)|u|^{-\gamma_0+1+2\zeta(Z^l)}. \quad (4.2)$$

(2) *Let  $(u_1, \underline{u}_1)$  be any pair of numbers verifying  $-\underline{u}_* \leq -\underline{u}_1 \leq u_1 \leq u_0$ . For  $a \leq n$  and  $l \leq n - 2$  there hold*

$$\|r^{\frac{1}{2}}\underline{L}Z^{(l)}\phi, r^{\frac{1}{2}}\partial Z^{(l)}\phi\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{\underline{u}_1}^u)}^2 \lesssim (\mathcal{E}_{l+2,\gamma_0} + \Delta_0 M^{-1})|u_1|^{-\gamma_0+2\zeta(Z^l)}, \quad (4.3)$$

$$\|r^{-\frac{1}{2}}Z^{(a)}\phi\|_{L^2(\mathcal{H}_{\underline{u}_1}^u)}^2 \lesssim |u_1|^{-\gamma_0+2+2\zeta(Z^a)}(\Delta_0 M^{-1} + \mathcal{E}_{a,\gamma_0}). \quad (4.4)$$

With  $p > -\frac{\gamma_0}{2} + \frac{3}{2}$ , there holds for  $a \leq n$  that

$$\| |u|^{-p} r^{-\frac{1}{2}} Z^{(a)} \phi \|_{L^2(\mathcal{D}_{\underline{u}_1}^u)} \lesssim |u_1|^{-\frac{\gamma_0}{2} + \frac{3}{2} - p + \zeta(Z^a)} \left( \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} + \mathcal{E}_{a,\gamma_0}^{\frac{1}{2}} \right). \quad (4.5)$$

*Remark 4.2.* The estimates with  $M^{-1}$  appeared in the bound are stronger than the standard estimates for the free wave in the region  $\{r \geq t + R\}$ . (4.3)-(4.5) are not used for the proofs for Theorem 2.1 and 2.2. They will be used in Section 6. In particular, (4.5) is an improved Hardy's inequality, which is proved by using the sharp improved estimate (4.4). The estimate (4.5) takes a weight of  $r^{\frac{1}{2}}$  up to the top order derivative, which is much stronger than the standard Hardy's type inequality. Such type of estimates will be crucial for the result for the quasilinear equations.

*Remark 4.3.* We emphasize that we only need  $(\text{BA}_2)$  and  $\mathcal{E}_{2,\gamma_0}(\phi[0])$  to be bounded in order to obtain the above results with  $n = 2$ . There involves neither the third order control from  $(\text{BA}_3)$  nor the bound of  $\mathcal{E}_{3,\gamma_0}$ .

*Proof.* We first consider the inequality for  $\underline{L}Z^{(l)}\phi$  in (4.1). With  $\gamma_2 = 1 = \gamma$ ,  $\gamma'_0 = 0$  in (3.1) and  $f = \underline{L}Z^{(l)}\phi$ , we have

$$\sup_{S_{u,\underline{u}}} |r^\gamma f|^4 \lesssim \int_{S_{-\underline{u},\underline{u}}} |r^\gamma \Omega^{(\leq 1)} f|^4 r^{-2} + \left( \int_{\underline{\mathcal{H}}_{\underline{u}}} r^{2\gamma_2} |\Omega^{(\leq 2)} f|^2 r^{-2} \right) \left( \int_{\underline{\mathcal{H}}_{\underline{u}}} r^{\gamma'_0} |\underline{L}' \Omega^{(\leq 1)}(r^\gamma f)|^2 r^{-2} \right).$$

Note that  $\underline{L}' \Omega^{(m)}(r^\alpha \underline{L} f) = \underline{L}'(r^\alpha \underline{L} \Omega^{(m)} f) = -\alpha(1+h)r^{\alpha-1} \underline{L} \Omega^{(m)} f + r^\alpha \underline{L}' \underline{L} \Omega^{(m)} f$  holds for any smooth function  $f$ . Due to the smallness of  $|h|$ , the definition of  $\underline{L}'$ ,  $[\underline{L}, \underline{L}'] = 0$  and (3.23), we have

$$|\underline{L}' \Omega^{(m)}(r^\alpha \underline{L} f)| \lesssim r^{\alpha-1} |\underline{L} \Omega^{(m)} f| + r^\alpha |\underline{L} \partial \Omega^{(m)} f|, \quad \alpha \geq 0. \quad (4.6)$$

With the help of (3.21) and (4.6) with  $\alpha = 1$ , we can derive that

$$\begin{aligned}
\sup_{S_{u,\underline{u}}} |r \underline{L} Z^{(l)} \phi|^4 &\lesssim \int_{S_{-\underline{u},\underline{u}}} |r \Omega^{(\leq 1)} \underline{L} Z^{(l)} \phi|^4 r^{-2} \\
&\quad + \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} |\underline{L} \Omega^{(\leq 2)} Z^{(l)} \phi|^2 \right) \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \left\{ |\underline{L} \Omega^{(\leq 1)} Z^{(l)} \phi|^2 + |r \underline{L} \partial \Omega^{(\leq 1)} Z^{(l)} \phi|^2 \right\} r^{-2} \right) \\
&\lesssim \int_{S_{-\underline{u},\underline{u}}} |r \Omega^{(\leq 1)} \underline{L} Z^{(l)} \phi|^4 r^{-2} \\
&\quad + E[\Omega^{(\leq 2)} Z^{(l)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) \left( |u|^{-2} E[\Omega^{(\leq 1)} Z^{(l)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) + E[\partial \Omega^{(\leq 1)} Z^{(l)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) \right) \\
&\lesssim (\mathcal{E}_{l+2,\gamma_0}^2 + \Delta_0^2) |u|^{-2-2\gamma_0+4\zeta(Z^l)},
\end{aligned}$$

where we employed (3.4) with  $Z^{(i)} = \partial \Omega_{ij}^{(\leq 1)} Z^{(l)}$  for  $l \leq n-2$ , (2.11) and (BA<sub>n</sub>).

Applying (3.1) along  $\underline{\mathcal{H}}_{\underline{u}}^u$  with  $\gamma_2 = \gamma = \frac{3}{2}$  and  $\gamma'_0 = 0$  to  $f = \nabla Z^{(l)} \phi$  implies that

$$\begin{aligned}
\sup_{S_{u,\underline{u}}} |r^{\frac{3}{2}} \nabla Z^{(l)} \phi|^4 &\lesssim \int_{S_{-\underline{u},\underline{u}}} |r^{\frac{3}{2}} \Omega^{(\leq 1)} \nabla Z^{(l)} \phi|^4 r^{-2} \\
&\quad + \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} r^3 |\Omega^{(\leq 2)} \nabla Z^{(l)} \phi|^2 r^{-2} \right) \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} |\underline{L}'(r^{\frac{3}{2}} \Omega^{(\leq 1)} \nabla Z^{(l)} \phi)|^2 r^{-2} \right). \quad (4.7)
\end{aligned}$$

By using (3.20), (3.22), (3.24),  $\underline{L}'(r) = -1 - h$  and the smallness of  $|h|$ , for  $m = 0, 1$  we have symbolically that

$$\begin{aligned}
|\underline{L}'(r^{\frac{3}{2}} \Omega^{(m)} \nabla f)| &\lesssim r^{\frac{1}{2}} |\nabla \Omega^{(\leq m)} f| + r^{\frac{3}{2}} |\underline{L}'(\nabla \Omega^{(m)} f + [\Omega^{(m)}, \nabla] f)| \\
&\lesssim r^{\frac{1}{2}} |\nabla \Omega^{(\leq m)} f| + r^{\frac{3}{2}} |\nabla \underline{L} \Omega^{(\leq m)} f| + r^{\frac{3}{2}} |h| |L \nabla \Omega^{(\leq m)} f| \\
&\lesssim r^{\frac{1}{2}} |\nabla \Omega^{(\leq m)} f| + r^{\frac{3}{2}} |\nabla \partial \Omega^{(\leq m)} f|.
\end{aligned}$$

By using the above calculation for  $f = Z^{(l)} \phi$  and (3.26), we deduce from (4.7), (3.4) and (2.11) that

$$\begin{aligned}
\sup_{S_{u,\underline{u}}} |r^{\frac{3}{2}} \nabla Z^{(l)} \phi|^4 &\lesssim \int_{S_{-\underline{u},\underline{u}}} |r^{\frac{3}{2}} \nabla \Omega^{(\leq 1)} Z^{(l)} \phi|^4 r^{-2} \\
&\quad + \mathcal{W}_1[\Omega^{(\leq 2)} Z^{(l)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) \left( \mathcal{W}_1[\partial \Omega^{(\leq 1)} Z^{(l)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) + |u|^{-1} E[\Omega^{(\leq 1)} Z^{(l)} \phi](\underline{\mathcal{H}}_{\underline{u}}^u) \right) \\
&\lesssim |u|^{-2\gamma_0+4\zeta(Z^l)} (\Delta_0^2 + \mathcal{E}_{l+2,\gamma_0}^2).
\end{aligned}$$

Thus the first two estimates in (4.1) are proved.

To prove the third one, we first need to prove

$$\sup_{S_{u,\underline{u}}} r |L(r Z^{(l)} \phi)|^2 \lesssim (\mathcal{E}_{l+2,\gamma_0} + \Delta_0) |u|^{-\gamma_0+2\zeta(Z^l)}, \quad l \leq n-2, \quad (4.8)$$

since so far we can not bound  $\|r^{\frac{1}{2}} L Z^{(l)} \phi\|_{L^2(\underline{\mathcal{H}}_{\underline{u}}^u)}$  without a loss in  $r$ . Instead of applying the Sobolev inequality (3.1) to  $L Z^{(l)} \phi$ , we apply it to  $f = r^{-1} L(r Z^{(l)} \phi)$  with  $\gamma_2 = \gamma = \frac{3}{2}$  and  $\gamma'_0 = 0$  to obtain

$$\begin{aligned}
\sup_{S_{u,\underline{u}}} |r^{\frac{3}{2}} \cdot r^{-1} L(r Z^{(l)} \phi)|^4 &\lesssim \int_{S_{u,-u}} |r^{\frac{1}{2}} \Omega^{(\leq 1)} L(r Z^{(l)} \phi)|^4 r^{-2} \\
&\quad + \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} r |\Omega^{(\leq 2)} (r^{-1} L(r Z^{(l)} \phi))|^2 \right) \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} |L' \Omega^{(\leq 1)} (r^{\frac{1}{2}} L(r Z^{(l)} \phi))|^2 r^{-2} \right). \quad (4.9)
\end{aligned}$$

Note that, due to (3.21) and the smallness of  $|h|$ ,

$$|L'(\Omega_{ij}(r^{\frac{1}{2}} L(r Z^{(l)} \phi)))| \lesssim r^{-\frac{1}{2}} |L(r \Omega_{ij} Z^{(l)} \phi)| + r^{\frac{1}{2}} |L'(L(r \Omega_{ij} Z^{(l)} \phi))|. \quad (4.10)$$

By using (3.23), we have

$$|L(\Omega_{ij}L(rZ^{(l)}\phi))| \lesssim |L\Omega_{ij}Z^{(l)}\phi| + |L(rL\Omega_{ij}Z^{(l)}\phi)| \lesssim |L(r\partial\Omega_{ij}Z^{(l)}\phi)| + |L\Omega_{ij}Z^{(l)}\phi|,$$

and, due to  $[L, \underline{L}] = 0$  and (3.23),

$$\begin{aligned} & |h||\underline{L}L(r\Omega_{ij}Z^{(l)}\phi)| \\ & \leq |h| \left( |L\Omega_{ij}Z^{(l)}\phi| + |L(r\underline{L}\Omega_{ij}Z^{(l)}\phi)| \right) \lesssim |h| \left( |L(r\partial\Omega_{ij}Z^{(l)}\phi)| + |L\Omega_{ij}Z^{(l)}\phi| \right). \end{aligned}$$

In view of the above two inequalities, the smallness of  $|h|$ , (4.10) and (2.11) we have

$$\begin{aligned} \int_{\mathcal{H}_u^u} |L'\Omega^{(\leq 1)}(r^{\frac{1}{2}}L(rZ^{(l)}\phi))|^2 r^{-2} & \lesssim \mathcal{W}_1[\partial\Omega Z^{(l)}\phi](\mathcal{H}_u^u) + E[\Omega Z^{(l)}\phi](\mathcal{H}_u^u)|u|^{-1} \\ & \lesssim \Delta_0|u|^{-\gamma_0-1+2\zeta(Z^l)}. \end{aligned}$$

Thus we can derive from (4.9) and (3.4) that

$$\begin{aligned} & \sup_{S_{u,\underline{u}}} |r^{\frac{3}{2}} \cdot r^{-1}L(rZ^{(l)}\phi)|^4 \\ & \lesssim \int_{S_{u,-u}} |r^{\frac{1}{2}}\Omega^{(\leq 1)}L(rZ^{(l)}\phi)|^4 r^{-2} + \mathcal{W}_1[\Omega^{(\leq 2)}Z^{(l)}\phi](\mathcal{H}_u^u) \cdot \Delta_0|u|^{-\gamma_0-1+2\zeta(Z^l)} \\ & \lesssim (\mathcal{E}_{l+2,\gamma_0}^2 + \Delta_0^2) |u|^{-2\gamma_0+4\zeta(Z^l)}, \end{aligned}$$

as desired in (4.8).

Next we prove (4.2). For any fixed point  $(u, \underline{u}, \omega)$ , we integrate the estimate of  $\underline{L}Z^{(l)}\phi$  in (4.1) along the ingoing integral curve of  $\partial_{u'}$  along  $\underline{\mathcal{H}}_{\underline{u}}^u$  from  $t = 0$ . For  $l \leq n - 2$ , we derive in view of (2.7) that

$$\begin{aligned} & |Z^{(l)}\phi(u, \underline{u}, \omega) - Z^{(l)}\phi(-\underline{u}, \underline{u}, \omega)| \lesssim \int_{-\underline{u}}^u \frac{1}{2}(1-h)^{-1} |\underline{L}'Z^{(l)}\phi|(u', \underline{u}, \omega) du' \\ & \lesssim \int_{-\underline{u}}^u \left\{ |\underline{L}Z^{(l)}\phi|(u', \underline{u}, \omega) + r^{-1}|h| \left( |L(rZ^{(l)}\phi)| + |Z^{(l)}\phi| \right) (u', \underline{u}, \omega) \right\} du'. \end{aligned}$$

The last term on the right can be treated by using the Gronwall's inequality,  $|h| \leq Mr^{-1}$ , (2.11) and  $-u \leq \underline{u}$ . The integration of the first and second term are bounded by  $(\Delta_0^{\frac{1}{2}} + \mathcal{E}_{l+2,\gamma_0}^{\frac{1}{2}})\underline{u}^{-1}|u|^{-\frac{\gamma_0+1}{2}+\zeta(Z^l)}$  by using  $\underline{u} \approx r$ , the second estimate of (4.1) and (4.8). Thus by using  $\underline{u} \approx r$  again, we can derive that

$$|rZ^{(l)}\phi(u, \underline{u}, \omega)| \lesssim \underline{u}|Z^{(l)}\phi(-\underline{u}, \underline{u}, \omega)| + \left( \Delta_0^{\frac{1}{2}} + \mathcal{E}_{l+2,\gamma_0}^{\frac{1}{2}} \right) |u|^{-\frac{\gamma_0+1}{2}+\zeta(Z^l)}.$$

With the help of (3.9) and  $r \approx \underline{u}$ , we can bound the first term on the right, which then implies that

$$|rZ^{(l)}\phi| \lesssim \left( \mathcal{E}_{l+2,\gamma_0}^{\frac{1}{2}} + \Delta_0^{\frac{1}{2}} \right) |u|^{-\frac{\gamma_0-1}{2}+\zeta(Z^l)}.$$

The proof of (4.2) is thus completed.

The last estimate in (4.1) can be obtained by using (4.2) and (4.8).

Next, we prove (4.3). It suffices to consider the estimate for  $\underline{L}Z^{(l)}\phi$ , other estimates follow from integrating the better estimates in (4.1). We first apply the Sobolev inequality (3.1) along  $\underline{\mathcal{H}}_{\underline{u}}^u$  to  $\underline{L}Z^{(l)}\phi$  with  $\gamma = \frac{1}{2} = \gamma_2, \gamma'_0 = 0$ , which yields

$$\begin{aligned} & \sup_{S_{u,\underline{u}}} r^2 |\underline{L}Z^{(l)}\phi|^4 \lesssim \int_{S_{-\underline{u},\underline{u}}} r^2 |\Omega^{(\leq 1)}\underline{L}Z^{(l)}\phi|^4 d\omega \\ & + \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} |\Omega^{(\leq 2)}(\underline{L}Z^{(l)}\phi)|^2 r du' d\omega \right) \left( \int_{\underline{\mathcal{H}}_{\underline{u}}^u} |\underline{L}'\Omega^{(\leq 1)}(r^{\frac{1}{2}}\underline{L}Z^{(l)}\phi)|^2 du' d\omega \right). \end{aligned}$$

We then derive

$$\begin{aligned} \int_{-u_1}^{u_1} \left( \sup_{-\underline{u} \leq u \leq u_1} \sup_{S_{u, \underline{u}}} r |\underline{L} Z^{(l)} \phi|^2 \right) d\underline{u} &\lesssim \int_{-u_1}^{u_1} \left( \int_{S_{-\underline{u}, \underline{u}}} r^2 |\Omega^{(\leq 1)} \underline{L} Z^{(l)} \phi|^4 d\omega \right)^{\frac{1}{2}} d\underline{u} \\ &+ \left( \int_{\mathcal{D}_{u_1}^{u_1}} |\Omega^{(\leq 2)} (\underline{L} Z^{(l)} \phi)|^2 r d\underline{u} d\underline{\omega} d\underline{u} \right)^{\frac{1}{2}} \left( \int_{\mathcal{D}_{u_1}^{u_1}} |\underline{L}' \Omega^{(\leq 1)} (r^{\frac{1}{2}} \underline{L} Z^{(l)} \phi)|^2 d\underline{u} d\underline{\omega} d\underline{u} \right)^{\frac{1}{2}}. \end{aligned} \quad (4.11)$$

We apply (3.6) to  $\underline{L} \Omega^{(\leq 1)} Z^{(l)} \phi$ , which then bounds the first term on the righthand side of the inequality by  $|u_1|^{-\gamma_0-1+2\zeta(Z^l)} \mathcal{E}_{l+2, \gamma_0}$ . For the second term, due to (3.21) we can derive that

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} |\Omega^{(\leq 2)} \underline{L} Z^{(l)} \phi|^2 r d\underline{u} d\underline{\omega} d\underline{u} &\lesssim M^{-1} \int_{-u_1}^{u_1} \int_{\mathcal{H}_u^{u_1}} M |\underline{L} \Omega^{(\leq 2)} Z^{(l)} \phi|^2 r d\underline{u} d\underline{\omega} d\underline{u} \\ &\lesssim M^{-1} \int_{-u_1}^{u_1} E[\Omega^{(\leq 2)} Z^{(l)} \phi](\mathcal{H}_u^{u_1}) du \\ &\lesssim M^{-1} \Delta_0 |u_1|^{-\gamma_0+1+2\zeta(Z^l)}. \end{aligned}$$

By using (3.23) and in view of (4.6) with  $\alpha = \frac{1}{2}$ ,  $f = Z^{(l)} \phi$ , we have

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} |\underline{L}' \Omega^{(\leq 1)} (r^{\frac{1}{2}} \underline{L} Z^{(l)} \phi)|^2 d\underline{u} d\underline{\omega} d\underline{u} &\lesssim \int_{\mathcal{D}_{u_1}^{u_1}} \left\{ r^{-1} |\underline{L} \Omega^{(\leq 1)} Z^{(l)} \phi|^2 + r |\underline{L} \partial \Omega^{(\leq 1)} Z^{(l)} \phi|^2 \right\} d\underline{u} d\underline{\omega} d\underline{u} \\ &\lesssim \int_{-u_1}^{u_1} M^{-1} E[\partial \Omega^{(\leq 1)} Z^{(l)} \phi](\mathcal{H}_u^{u_1}) du + |u_1|^{-2} \sup_{-u_1 \leq \underline{u} \leq u_1} E[\Omega^{(\leq 1)} Z^{(l)} \phi](\mathcal{H}_{\underline{u}}^{u_1}) \\ &\lesssim (|u_1|^{-1} + M^{-1}) \Delta_0 |u_1|^{-\gamma_0-1+2\zeta(Z^l)}. \end{aligned}$$

By substituting the above two estimates to (4.11), we therefore obtain (4.3).

Next, we prove (4.4). (4.5) follows as an immediate consequence by integrating in  $u$ -variable.

By applying  $\underline{L}'(rf^2) = (-1-h)f^2 + r\underline{L}'(f^2)$  to  $f = Z^{(a)} \phi$ , and in view of  $1+h > 0$  due to the smallness of  $h$ , we integrate the result in  $\mathcal{D}_{u_1}^{u_1}$  with the area element  $\frac{1}{2}(1-h)^{-1} d\underline{u} d\underline{\omega}$ , which yields,

$$\begin{aligned} \int_{\mathcal{H}_{u_1}^{u_1}} r |Z^{(a)} \phi|^2(u_1, \underline{u}, \omega) d\underline{u} d\underline{\omega} &\lesssim \int_{-u_1}^{u_1} \int_{\mathbb{S}^2} r |Z^{(a)} \phi|^2(-\underline{u}, \underline{u}, \omega) d\underline{u} d\underline{\omega} \\ &+ \int_{-u_1}^{u_1} \|r^{-\frac{1}{2}} \underline{L}' Z^{(a)} \phi\|_{L^2(\mathcal{H}_u^{u_1})} \|r^{-\frac{1}{2}} Z^{(a)} \phi\|_{L^2(\mathcal{H}_u^{u_1})} du, \end{aligned} \quad (4.12)$$

where we dropped the integral of  $(1+h)|Z^{(a)} \phi|^2$  due to the positivity. Note that

$$\|r^{-\frac{1}{2}} \underline{L}' Z^{(a)} \phi\|_{L^2(\mathcal{H}_u^{u_1})} \lesssim M^{-\frac{1}{2}} E[Z^{(a)} \phi]^{\frac{1}{2}}(\mathcal{H}_u^{u_1}) \lesssim |u|^{-\frac{\gamma_0}{2}+\zeta(Z^a)} M^{-\frac{1}{2}} \Delta_0^{\frac{1}{2}}, \quad a \leq n \quad (4.13)$$

and the first term on the right of (4.12) can be bounded by  $|u_1|^{-\gamma_0+1+2\zeta(Z^a)} \mathcal{E}_{a, \gamma_0}$  by using (3.7). By multiplying both sides by  $|u_1|^p$  with  $p \geq 1$ , followed with applying Gronwall's inequality,

$$\begin{aligned} |u_1|^{\frac{p}{2}} \left( \int_{\mathcal{H}_{u_1}^{u_1}} r |Z^{(a)} \phi|^2(u_1, \underline{u}, \omega) d\underline{u} d\underline{\omega} \right)^{\frac{1}{2}} \\ \lesssim |u_1|^{\frac{p-\gamma_0+1}{2}+\zeta(Z^a)} \mathcal{E}_{n, \gamma_0}^{\frac{1}{2}} + |u_1|^p \int_{-u_1}^{u_1} \|r^{-\frac{1}{2}} \underline{L}'(Z^{(a)} \phi)\|_{L^2(\mathcal{H}_u^{u_1})} |u|^{-\frac{p}{2}} du. \end{aligned}$$

We can obtain (4.4) for  $a \leq n$  by using (4.13).  $\square$

With the help of the assumption of  $(\text{BA}_n)$  with  $n = 2, 3$ , we can derive the following estimates of mixed  $L^p$  norms.

**Proposition 4.4.** *Let  $-\underline{u}_* \leq -\underline{u}_1 \leq u_1 \leq u_0$ ,  $b \leq n-1$ ,  $a \leq n$ ,  $\gamma > \frac{1}{2}$  and  $\alpha > 0$ . There hold*

$$\left\| r^{-\frac{1}{2}} \partial Z^{(a)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^a) + \frac{1}{2}} \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}}, \quad (4.14)$$

$$\left\| r^{-\gamma} \partial Z^{(a)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\gamma + \frac{1}{2} - \frac{1}{2}\gamma_0 + \zeta(Z^a)} \Delta_0^{\frac{1}{2}}, \quad (4.15)$$

$$\begin{aligned} \left\| r^{\frac{1}{2}} Z^{(a)} \phi \right\|_{L_u^\infty L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 + \alpha \left\| \left( \frac{|u_1|}{r} \right)^\alpha r^{-1} Z^{(a)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 \\ \lesssim |u_1|^{-\gamma_0 + 1 + 2\zeta(Z^a)} (\Delta_0 + \mathcal{E}_{a, \gamma_0}), \end{aligned} \quad (4.16)$$

$$\alpha \left\| \left( \frac{|u_1|}{r} \right)^\alpha Z^{(a)} \phi \right\|_{L_u^2 L_u^2 L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 \lesssim |u_1|^{-\gamma_0 + 1 + 2\zeta(Z^a)} (\Delta_0 + \mathcal{E}_{a, \gamma_0}), \quad (4.17)$$

$$\left\| r^{\frac{1}{2}} \partial Z^{(b)} \phi \right\|_{L^4(S_{u_1, \underline{u}_1})} \lesssim |u_1|^{-\frac{1}{2}\gamma_0 - \frac{1}{2} + \zeta(Z^b)} \left( \Delta_0^{\frac{1}{2}} + \mathcal{E}_{b+1, \gamma_0}^{\frac{1}{2}} \right), \quad (4.18)$$

$$\left\| r \partial Z^{(b)} \phi \right\|_{L_u^\infty L_u^2 L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^b)} \left( \Delta_0^{\frac{1}{2}} + \mathcal{E}_{b+1, \gamma_0}^{\frac{1}{2}} \right), \quad (4.19)$$

$$\left\| r \partial Z^{(b)} \phi \right\|_{L_u^2 L_u^\infty L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^b)} \left( \Delta_0^{\frac{1}{2}} + \mathcal{E}_{b+1, \gamma_0}^{\frac{1}{2}} \right), \quad (4.20)$$

$$\left\| r^{\frac{1}{2}} \partial Z^{(b)} \phi \right\|_{L_u^2 L_u^\infty L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{1}{2}\gamma_0 + \zeta(Z^b)} \left( \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} + \mathcal{E}_{b+1, \gamma_0}^{\frac{1}{2}} |u_1|^{-\frac{1}{2}} \right), \quad (4.21)$$

$$\left\| r^{\frac{1}{2}} \partial Z^{(b)} \phi \right\|_{L_u^2 L_u^2 L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{1}{2}\gamma_0 + \frac{1}{2} + \zeta(Z^b)} \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}}. \quad (4.22)$$

If the constant  $p > -\frac{\gamma_0}{2} + \frac{3}{2}$ , there hold

$$\left\| |u|^{-p} r^{\frac{1}{2}} Z^{(b)} \phi \right\|_{L_u^2 L_u^2 L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{\zeta(Z^b) - \frac{1}{2}\gamma_0 + \frac{3}{2} - p} \left( \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} + \mathcal{E}_{b+1, \gamma_0}^{\frac{1}{2}} \right), \quad (4.23)$$

$$\left\| r^{\frac{1}{2}} Z^{(a)} \phi \right\|_{L_u^2 L_\omega^4(\mathcal{H}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{\zeta(Z^a) - \frac{1}{2}\gamma_0 + \frac{1}{2}} \left( \Delta_0^{\frac{1}{2}} + \mathcal{E}_{a, \gamma_0}^{\frac{1}{2}} \right). \quad (4.24)$$

*Proof.* By using  $(BA_n)$  we can derive for  $a \leq n$  that

$$\left\| r^{-\frac{1}{2}} \partial Z^{(a)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim M^{-\frac{1}{2}} \left( \int_{-\underline{u}_1}^{u_1} E[Z^a \phi](\mathcal{H}_u^{\underline{u}_1}) du \right)^{\frac{1}{2}} \lesssim M^{-\frac{1}{2}} |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^a) + \frac{1}{2}} \Delta_0^{\frac{1}{2}}.$$

By using  $(BA_n)$ , (4.15) is a direct consequence of  $|\partial Z^{(a)} \phi| \lesssim |\underline{\partial} Z^{(a)} \phi| + |\bar{\partial} Z^{(a)} \phi|$ ; (4.16) is a consequence of (3.13) and (3.7).

Note that by using (3.29), (3.7) and  $(BA_n)$ , we can derive that

$$\begin{aligned} \alpha \left\| \left( \frac{|u_1|}{r} \right)^\alpha Z^{(a)} \phi \right\|_{L_u^2 L_u^2 L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 \\ \lesssim \mathcal{W}_1[Z^{(a)} \phi](\mathcal{D}_{u_1}^{\underline{u}_1}) + \left\| r^{-\frac{1}{2}} Z^{(a)} \phi \right\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 + M \int_{-\underline{u}_1}^{u_1} |u'|^{-1} E[Z^{(a)} \phi](\mathcal{H}_{u'}^{\underline{u}_1}) du' \\ \lesssim |u_1|^{-\gamma_0 + 1 + 2\zeta(Z^a)} (\Delta_0 + \mathcal{E}_{a, \gamma_0}) \end{aligned}$$

as desired in (4.17).

Next we consider (4.18). For the estimate of  $LZ^{(b)} \phi$ , we apply (3.34) to  $\psi = Z^{(b)} \phi$ , which yields for  $-\underline{u}_1 \leq -\underline{u} \leq u \leq u_1$  that

$$\begin{aligned} \int_{S_{u, \underline{u}}} \left| r L Z^{(b)} \phi \right|^4 d\omega \\ \lesssim \int_{S_{u, -u}} \left| r L Z^{(b)} \phi \right|^4 d\omega + \left( E[\partial Z^{(b)} \phi](\mathcal{H}_u^{\underline{u}}) + |u|^{-2} E[Z^{(b)} \phi](\mathcal{H}_u^{\underline{u}}) \right) E[\Omega^{(\leq 1)} Z^{(b)} \phi](\mathcal{H}_u^{\underline{u}}). \end{aligned}$$



The first term on the right can be bounded by applying (3.4) to  $\partial Z^{(b)}\phi$ , which is bounded by  $|u|^{-2-2\gamma_0+4\zeta(Z^b)}\mathcal{E}_{b+1,\gamma_0}^2$ . We then use  $(\text{BA}_n)$  to obtain

$$\int_{S_{u,\underline{u}}} \left| rLZ^{(b)}\phi \right|^4 d\omega \lesssim |u|^{-2-2\gamma_0+4\zeta(Z^b)} (\mathcal{E}_{b+1,\gamma_0}^2 + \Delta_0^2).$$

By repeating the same procedure for  $\partial\psi$  in view of (3.33), we can get the same estimate with  $L$  replaced by  $\partial$ . This implies (4.18).

Integrating (4.18) along  $\mathcal{H}_{\underline{u}}^{u_1}$  for any  $-u_1 \leq \underline{u} \leq u_1$  implies (4.19) directly.

We note also that (4.18) is independent of  $\underline{u}$ . For  $(u, \underline{u})$  satisfying  $-\underline{u}_1 \leq -\underline{u} \leq u \leq u_1$ , we can take supremum of (4.18) for  $-u \leq \underline{u} \leq u_1$ , followed with integrating  $u$  from  $-\underline{u}_1$  to  $u_1$ . Thus (4.20) is proved.

Next, we apply (3.32) to  $\psi = Z^{(b)}\phi$  for  $b \leq n-1$  and use  $(\text{BA}_n)$  to obtain

$$\left\| r^{\frac{1}{2}} \partial Z^{(b)}\phi \right\|_{L_{\underline{u}}^2 L_u^\infty L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 \lesssim \int_{-u_1}^{u_1} \left( \int_{S_{-\underline{u},\underline{u}}} r^2 \left| \partial Z^{(b)}\phi \right|^4 d\omega \right)^{\frac{1}{2}} d\underline{u} + M^{-1} \Delta_0 |u_1|^{2\zeta(Z^b)-\gamma_0}.$$

We then apply (3.5) to  $\partial Z^{(b)}\phi$ , which bounds the first term on the right by  $\mathcal{E}_{b+1,\gamma_0} |u_1|^{-\gamma_0-1+2\zeta(Z^b)}$ . The result (4.21) then follows by a direct substitution.

(4.22) follows by applying (3.30) to  $\psi = Z^{(b)}\phi$  and also using  $(\text{BA}_n)$ .

We can obtain (4.23) by applying the Sobolev embedding on the unit sphere

$$\left\| r^{\frac{1}{2}} |u|^{-p} Z^{(b)}\phi \right\|_{L_u^2 L_{\underline{u}}^2 L_\omega^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim \left\| r^{-\frac{1}{2}} |u|^{-p} \Omega^{(\leq 1)} Z^{(b)}\phi \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})},$$

followed by applying (4.5) to  $\Omega^{(\leq 1)} Z^{(b)}\phi$  with  $p > -\frac{\gamma_0}{2} + \frac{3}{2}$ .

We deduce by applying (3.28) to  $F = Z^{(a)}\phi$  followed with using (3.7) and  $(\text{BA}_n)$  that

$$\begin{aligned} \left\| r^{\frac{1}{2}} Z^{(a)}\phi \right\|_{L_u^2 L_\omega^4(\mathcal{H}_{u_1}^{\underline{u}_1})} &\lesssim \mathcal{W}_1^{\frac{1}{2}} [Z^{(a)}\phi](\mathcal{H}_{u_1}^{\underline{u}_1}) + \mathcal{W}_1^{\frac{1}{2}} [Z^{(a)}\phi](\mathcal{D}_{u_1}^{\underline{u}_1}) + \left\| r^{-\frac{1}{2}} Z^{(a)}\phi \right\|_{L^2(\Sigma_0^{u_1,\underline{u}_1})} \\ &\quad + M^{\frac{1}{2}} \left( \int_{-u_1}^{u_1} |u|^{-1} E[Z^{(a)}\phi](\mathcal{H}_u^{\underline{u}_1}) du \right)^{\frac{1}{2}} \\ &\lesssim |u_1|^{\zeta(Z^a)-\frac{1}{2}\gamma_0+\frac{1}{2}} \left( \Delta_0^{\frac{1}{2}} + \mathcal{E}_{a,\gamma_0}^{\frac{1}{2}} \right), \end{aligned}$$

as desired in (4.24).  $\square$

## 5. Semilinear wave equations

In this section we consider the equation (2.15) in  $\mathbb{R}^{3+1}$ , i.e.

$$\square_{\mathbf{m}}\phi = \mathcal{N}^{\alpha\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi + q(x)\phi$$

where  $0 \leq q(x) \leq 1$  satisfies (1.2) with  $n = 2$ . We will prove Theorem 2.1 and Theorem 2.2.

In Section 3 and 4, for functions with (1.3) bounded for  $k = 2$  or  $3$ , under the assumption of  $(\text{BA}_k)$ , we have obtained decay properties and a set of Sobolev inequalities. In this section, by a bootstrap argument, we will prove  $(\text{BA}_2)$  with  $\Delta_0$  comparable with  $\mathcal{E}_{2,\gamma_0,R}$ , provided that the latter is sufficiently small. The analysis in Sections 3 and 4 will play a crucial role for achieving the boundedness of the sets of energies.

We will give the fundamental standard and weighted energy inequalities for the  $u$  and  $\underline{u}$  foliations in Proposition 5.2 and Proposition 5.4. The goal is to justify the energy norms given in (2.13) can be achieved along the foliations  $\mathcal{H}_u$ ,  $\mathcal{H}_{\underline{u}}$  and  $\mathcal{D}_{\underline{u}}^u$  provided that  $(\square_{\mathbf{m}} - q)f$  can be bounded as desired.

Throughout this section,  $M > 0$  is a fixed small number, with the upper bound determined during the proofs of Theorem 2.1 and 2.2. Let

$$h = M/r, \quad u_0 = u_0(M, R),$$

where  $R \geq 2$ , whose lower bound will be finally determined at the end of the proof of Theorem 2.1.

For the wave equation  $\square_{\mathbf{m}}\varphi - q\varphi = F$ , we define the energy momentum tensor,

$$Q_{\alpha\beta}[\varphi] = \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}(\partial^\gamma\varphi\partial_\gamma\varphi + q\varphi^2)\mathbf{m}_{\alpha\beta}. \quad (5.1)$$

Recall  $\mathcal{N}, \underline{\mathcal{N}}$  from (2.4). It is straightforward to obtain the energy densities on  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$

$$\begin{aligned} Q[\varphi](\partial_t, \mathcal{N}) &= \frac{1}{2}(1+h)^{-1} \{ (L\varphi)^2 + h(\underline{L}\varphi)^2 + (1+h)(|\nabla\varphi|^2 + q\varphi^2) \}, \\ Q[\varphi](\partial_t, \underline{\mathcal{N}}) &= \frac{1}{2}(1+h)^{-1} \{ (\underline{L}\varphi)^2 + h(L\varphi)^2 + (1+h)(|\nabla\varphi|^2 + q\varphi^2) \}. \end{aligned} \quad (5.2)$$

Next, we give the fundamental energy inequality in  $\{u \leq u_0\}$ .

**Lemma 5.1.** *Consider the equation  $\square_{\mathbf{m}}\varphi - q(x)\varphi = F$ . There holds for a smooth vectorfield  $X$  that*

$$\partial^\alpha(Q_{\alpha\beta}[\varphi]X^\beta) = Q_{\alpha\beta}^{(X)}\pi^{\alpha\beta} + (\square_{\mathbf{m}}\varphi - q\varphi)X\varphi - \frac{1}{2}Xq \cdot \varphi^2, \quad (5.3)$$

where  $^{(X)}\pi_{\alpha\beta} = \frac{1}{2}(\langle\partial_\alpha X, \partial_\beta\rangle + \langle\partial_\beta X, \partial_\alpha\rangle)$ .

There also holds the following energy estimate,

$$E[\varphi](\mathcal{H}_u^u) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u) \lesssim \left| \int_{\mathcal{D}_u^u} F \cdot \partial_t \varphi \right| + E[\varphi](\Sigma_0^{u, \underline{u}}) \quad (5.4)$$

for all  $-\underline{u} \leq u \leq u_0$ .

*Proof.* For  $Q_{\alpha\beta}[\varphi]$  in (5.1), it is straightforward to derive

$$\partial^\alpha Q_{\alpha\beta}[\varphi] = (\square_{\mathbf{m}}\varphi - q\varphi)\partial_\beta\varphi - \frac{1}{2}\partial_\beta q \cdot \varphi^2.$$

(5.3) follows as a consequence.

Applying (5.3) with  $X = \partial_t$  gives  $\partial^\alpha(Q_{\alpha\beta}\partial_t^\beta) = (\square_{\mathbf{m}}\varphi - q\varphi)\partial_t\varphi$ . We then integrate the identity in  $\mathcal{D}_u^u$  to obtain

$$\int_{\mathcal{H}_u^u} Q(\mathcal{N}, \partial_t) d\mu_{\mathcal{H}} + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} Q(\underline{\mathcal{N}}, \partial_t) d\mu_{\underline{\mathcal{H}}} + \int_{\mathcal{D}_u^u} F \cdot \partial_t \varphi dx dt = \int_{\Sigma_0^{u, \underline{u}}} Q(\partial_t, \partial_t) dx.$$

(5.4) is proved in view of (5.2) □

As a consequence, we derive the following energy estimate.

**Proposition 5.2.** *Let  $\mathcal{I} = \{(u, \underline{u}) : -\underline{u}_2 \leq -\underline{u} \leq u \leq u_2 \leq u_0\}$ , where  $u_2$  and  $\underline{u}_2$  are fixed. Then for  $(u, \underline{u}) \in \mathcal{I}$  there holds the energy inequality*

$$\begin{aligned} &|u|^{-2p+\gamma_0} \left( E[\varphi](\mathcal{H}_u^u) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u) \right) \\ &\lesssim \sup_{(u_1, \underline{u}_1) \in \mathcal{I}} \left\{ |u_1|^{-2p+\gamma_0} E[\varphi](\Sigma_0^{u_1, \underline{u}_1}) + |u_1|^{-2p+\gamma_0} \left( \|rF^\flat\|_{L_{\underline{u}}^1 L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{u_1})}^2 + \|rF^\flat\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 \right) \right. \\ &\quad \left. + |u_1|^{1-2p+\gamma_0} M^{-1} \|r^{\frac{1}{2}} F^\sharp\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 \right\}, \end{aligned}$$

where  $p \leq 0$  is any constant<sup>17</sup>,  $F = F^\flat + F^\sharp$ , and the two smooth functions  $F^\flat$  and  $F^\sharp$  are in the corresponding normed spaces.

*Proof.* We apply Lemma 5.1 with  $(u, \underline{u}) \in \mathcal{I}$ . This implies that

$$\begin{aligned} &|u|^{-2p+\gamma_0} \left( E[\varphi](\mathcal{H}_u^u) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u) \right) \\ &\lesssim \sup_{(u_1, \underline{u}_1) \in \mathcal{I}} \left\{ |u_1|^{-2p+\gamma_0} \left| \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} F \cdot \partial_t \varphi dx dt \right| + |u_1|^{-2p+\gamma_0} E[\varphi](\Sigma_0^{u_1, \underline{u}_1}) \right\}. \end{aligned} \quad (5.5)$$

<sup>17</sup>The inequality holds uniformly for any  $p \leq 0$ .

To control the first term on the right, we first consider the  $\mathcal{F}^\flat$  term in the decomposition  $F = F^\sharp + F^\flat$ . We have

$$\left| \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} F^\flat \cdot \partial_t \varphi dx dt \right| \lesssim \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} |F^\flat \cdot L\varphi| dx dt + \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} |F^\flat \cdot \underline{L}\varphi| dx dt = I + II.$$

We can estimate  $I$  and  $II$  in the following way:

$$\begin{aligned} I &\lesssim \left\| |u|^{p-\frac{1}{2}\gamma_0} r F^\flat \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \left\| |u|^{-p+\frac{1}{2}\gamma_0} r L\varphi \right\|_{L_u^\infty L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ &\lesssim |u_1|^{p-\frac{1}{2}\gamma_0} \|r F^\flat\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \sup_{-\underline{u}_1 \leq u \leq u_1} \left( |u|^{-p+\frac{1}{2}\gamma_0} E[\varphi]^{\frac{1}{2}}(\mathcal{H}_u^{\underline{u}_1}) \right), \\ II &\lesssim \left\| r F^\flat \right\|_{L_{\underline{u}}^1 L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \|r \underline{L}\varphi\|_{L_{\underline{u}}^\infty L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ &\lesssim |u_1|^{p-\frac{1}{2}\gamma_0} \|r F^\flat\|_{L_{\underline{u}}^1 L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \sup_{-u_1 \leq \underline{u} \leq \underline{u}_1} \left( |u_1|^{-p+\frac{1}{2}\gamma_0} E[\varphi]^{\frac{1}{2}}(\underline{\mathcal{H}}_{\underline{u}}^{\underline{u}_1}) \right). \end{aligned}$$

Similarly, we can derive that

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} |F^\sharp \cdot \partial_t \varphi| dx dt &\leq M^{-\frac{1}{2}} \left\| r^{\frac{1}{2}} F^\sharp \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \left( \int_{-u_1}^{u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) du \right)^{\frac{1}{2}} \\ &\lesssim M^{-\frac{1}{2}} \left\| r^{\frac{1}{2}} F^\sharp \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} |u_1|^{-\frac{\gamma_0}{2}+\frac{1}{2}+p} \sup_{-\underline{u}_1 \leq u \leq u_1} \left( |u|^{\frac{\gamma_0}{2}-p} E^{\frac{1}{2}}[\varphi](\mathcal{H}_u^{\underline{u}_1}) \right). \end{aligned}$$

By multiplying the weight  $|u_1|^{-2p+\gamma_0}$  to the three inequalities, followed by using the Cauchy-Schwartz inequality, we can derive that

$$\begin{aligned} &|u_1|^{-2p+\gamma_0} \left| \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} F \cdot \partial_t \varphi dx dt \right| \\ &\lesssim \epsilon_1^{-1} \left( |u_1|^{-2p+\gamma_0} \left\| r F^\flat \right\|_{L_{\underline{u}}^1 L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 + |u_1|^{-2p+\gamma_0} \left\| r F^\flat \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 \right. \\ &\quad \left. + \left\| r^{\frac{1}{2}} F^\sharp \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 M^{-1} |u_1|^{\gamma_0-2p+1} \right) \\ &\quad + \epsilon_1 \left( \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{-2p+\gamma_0} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) + \sup_{-u_1 \leq \underline{u} \leq \underline{u}_1} |u_1|^{-2p+\gamma_0} E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^{\underline{u}_1}) \right). \quad (5.6) \end{aligned}$$

By substituting the above inequality to (5.5), followed with taking supremum for  $(u, \underline{u}) \in \mathcal{I}$ , the last line of (5.6) can be absorbed. Thus Proposition 5.2 is proved.  $\square$

Next, we give the weighted energy estimate.

**Lemma 5.3.** *For any  $\underline{u} \leq u \leq u_0$ , there holds*

$$\begin{aligned} &\int_{\mathcal{H}_{\underline{u}}} [r(L(r\varphi))^2 + hr^3(|\nabla\varphi|^2 + q\varphi^2)] d\underline{u}' d\omega + \int_{\underline{\mathcal{H}}_{\underline{u}}} r^3 (h(L\varphi)^2 + |\nabla\varphi|^2 + q\varphi^2) du' d\omega \\ &\quad + \frac{1}{2} \int_{\mathcal{D}_{\underline{u}}^{\underline{u}}} ((L(r\varphi))^2 + r^2 |\nabla\varphi|^2) du' du' d\omega \\ &\lesssim \mathcal{W}_1[\varphi](\Sigma_0^{u, \underline{u}}) + \int_{\mathcal{D}_{\underline{u}}^{\underline{u}}} |F \cdot L(r\varphi)| dx dt + \int_{-\underline{u}}^u E[\varphi](\mathcal{H}_{u'}^{\underline{u}}) du' + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^{\underline{u}}) \\ &\quad + \|r^{-1/2}\varphi\|_{L^2(\Sigma_0^{u, \underline{u}})}^2 + \int_{S_{-\underline{u}, \underline{u}}} r\varphi^2 d\omega, \end{aligned}$$

where  $q(x) \geq 0$  verifies (1.2) with  $n = 2$ .

*Proof.* We define

$${}^{(X)}\mathcal{P}_\alpha = Q_{\alpha\beta} X^\beta + \frac{1}{2} \partial_\alpha(\varphi^2) + Y_\alpha,$$

where

$$X = rL, \quad Y = \frac{1}{2}r^{-1}\varphi^2L.$$

Similar to the calculation in [19], we have

$$\partial^\alpha(X)\mathcal{P}_\alpha = (\square_{\mathbf{m}}\varphi - q\varphi)(X\varphi + \varphi) + \frac{1}{2}(r^{-2}(L(r\varphi))^2 + |\nabla\varphi|^2) - \frac{1}{2}(Xq + q)\varphi^2. \quad (5.7)$$

Indeed, by using (5.3), we can derive that

$$\partial^\alpha(X)\mathcal{P}_\alpha = (\square_{\mathbf{m}}\varphi - q\varphi)X\varphi - \frac{1}{2}Xq \cdot \varphi^2 + Q_{\alpha\beta}^{(X)}\pi^{\alpha\beta} + \frac{1}{2}\partial^\alpha\partial_\alpha(\varphi^2) + \partial^\alpha Y_\alpha. \quad (5.8)$$

One can check that the nonvanishing components of  $^{(X)}\pi$  for  $X = rL$  are

$$^{(X)}\pi_{\underline{L}\underline{L}} = 2, \quad ^{(X)}\pi_{L\underline{L}} = -1, \quad ^{(X)}\pi_{e_A e_B} = \delta_{AB}, \quad A, B = 1, 2,$$

where  $\{e_A\}$ ,  $A = 1, 2$  forms the orthonormal basis on  $S_{u, \underline{u}}$ , and

$$\begin{aligned} Q_{LL} &= (L\varphi)^2, & Q_{\underline{L}\underline{L}} &= (\underline{L}\varphi)^2, & Q_{L\underline{L}} &= |\nabla\varphi|^2 + q\varphi^2 \\ Q_{AB} &= \nabla_A\varphi\nabla_B\varphi - \frac{1}{2}\mathbf{m}_{AB}(-L\varphi\underline{L}\varphi + |\nabla\varphi|^2 + q\varphi^2). \end{aligned}$$

By combining the calculations of  $Q_{\alpha\beta}$  and  $^{(X)}\pi^{\alpha\beta}$ , we derive

$$Q_{LL}^{(X)}\pi^{LL} = \frac{1}{2}(L\varphi)^2, \quad Q_{L\underline{L}}^{(X)}\pi^{L\underline{L}} = -\frac{1}{4}(|\nabla\varphi|^2 + q\varphi^2), \quad Q_{AB}^{(X)}\pi^{AB} = \underline{L}\varphi L\varphi - q\varphi^2.$$

Thus

$$Q_{\alpha\beta}^{(X)}\pi^{\alpha\beta} = \frac{1}{2}((L\varphi)^2 - |\nabla\varphi|^2) + \underline{L}\varphi L\varphi - \frac{3}{2}q\varphi^2.$$

It is straightforward to have

$$\frac{1}{2}\partial^\alpha\partial_\alpha(\varphi^2) = \partial^\alpha\varphi\partial_\alpha\varphi + \square_{\mathbf{m}}\varphi \cdot \varphi = -L\varphi\underline{L}\varphi + |\nabla\varphi|^2 + \square_{\mathbf{m}}\varphi \cdot \varphi.$$

Note that  $\partial^\alpha L_\alpha = 2/r$ . We have

$$\begin{aligned} \partial^\alpha Y_\alpha &= \frac{1}{2}\partial^\alpha(r^{-1}\varphi^2 L_\alpha) = \frac{1}{2}(\varphi^2 L(r^{-1}) + r^{-1}L(\varphi^2) + r^{-1}\varphi^2\partial^\alpha L_\alpha) \\ &= \frac{1}{2}(r^{-1}L(\varphi^2) + r^{-2}\varphi^2). \end{aligned}$$

Combining the above calculations with (5.8) implies (5.7).

On the other hand, by the divergence theorem we have

$$\int_{\mathcal{H}_{\underline{u}}^u} \mathcal{N}^{\alpha(X)}\mathcal{P}_\alpha d\mu_{\mathcal{H}} + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \underline{\mathcal{N}}^{\alpha(X)}\mathcal{P}_\alpha d\mu_{\underline{\mathcal{H}}} = \int_{\Sigma_0^{u, \underline{u}}} ^{(X)}\mathcal{P}_\alpha(\partial_t)^\alpha dx - \int_{\mathcal{D}_{\underline{u}}^u} \partial^\alpha(X)\mathcal{P}_\alpha dx dt \quad (5.9)$$

where the area elements  $d\mu_{\mathcal{H}}$  and  $d\mu_{\underline{\mathcal{H}}}$  are given in (2.9). Direct substitution shows that

$$\begin{aligned} r^2(1+h)\mathcal{N}^{\alpha(X)}\mathcal{P}_\alpha &= r \left( (L(r\varphi))^2 + r^2h(|\nabla\varphi|^2 + q\varphi^2) - \frac{1}{2}r^{-1}L'(r^2\varphi^2) \right), \\ r^2(1+h)\underline{\mathcal{N}}^{\alpha(X)}\mathcal{P}_\alpha &= r^3(h(L\varphi)^2 + |\nabla\varphi|^2 + q\varphi^2) + \frac{1}{2}\underline{L}'(r^2\varphi^2) + rh(rL(\varphi^2) + \varphi^2), \\ r^2\partial_t^\alpha(X)\mathcal{P}_\alpha &= \frac{1}{2}r^3(r^{-2}|L(r\varphi)|^2 + |\nabla\varphi|^2 + q\varphi^2) - \frac{1}{2}\partial_r(r^2\varphi^2). \end{aligned} \quad (5.10)$$

Note that for any  $-\underline{u}_1 \leq u_1 \leq u_0$ , there holds

$$\int_{\mathcal{H}_{u_1}^{\underline{u}_1}} \frac{d}{d\underline{u}}(r^2\varphi^2) d\omega d\underline{u} - \int_{\underline{\mathcal{H}}_{\underline{u}_1}^{\underline{u}_1}} \frac{d}{du}(r^2\varphi^2) d\omega du - \int_{\Sigma_0^{u_1, \underline{u}_1}} \partial_r(r^2\varphi^2) d\omega dr = 0. \quad (5.11)$$

By adding this identity to (5.9), in view of (5.10) and (2.7) we can obtain

$$\begin{aligned}
& \int_{\mathcal{H}_{\underline{u}}^u} (r(L(r\varphi))^2 + hr^3(|\nabla\varphi|^2 + q\varphi^2)) (1+h)^{-1} \mathbf{b} du' d\omega \\
& + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} r^3 (h|L\varphi|^2 + |\nabla\varphi|^2 + q\varphi^2) (1+h)^{-1} \mathbf{b} du' d\omega \\
& + \frac{1}{2} \int_{\mathcal{D}_{\underline{u}}^u} ((L(r\varphi))^2 + r^2|\nabla\varphi|^2) \mathbf{b} du' d\omega \\
& \lesssim \mathcal{W}_1[\varphi](\Sigma_0^{u,\underline{u}}) + \int_{\mathcal{D}_{\underline{u}}^u} (|Xq|\varphi^2 + q\varphi^2) + \int_{\mathcal{D}_{\underline{u}}^u} |F \cdot L(r\varphi)| \\
& + \int_{\mathcal{H}_{\underline{u}}^u} r|h| |\varphi^2 + rL(\varphi^2)| du' d\omega.
\end{aligned}$$

By using (3.15) and the Cauchy-Schwartz inequality, the last term can be treated as

$$\begin{aligned}
\int_{\underline{\mathcal{H}}_{\underline{u}}^u} |h| (r|L(\varphi^2)| + \varphi^2) du' d\omega & \lesssim \int_{S_{-\underline{u},\underline{u}}} r\varphi^2 d\omega + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u) + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} |h| |rL\varphi|^2 du' d\omega \\
& \lesssim \int_{S_{-\underline{u},\underline{u}}} r\varphi^2 d\omega + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u).
\end{aligned}$$

Note that, due to (1.2),  $|Xq| \lesssim (1+r)^{-2-\eta}$ . We then have by using (3.13) that

$$\int_{\mathcal{D}_{\underline{u}}^u} |Xq|\varphi^2 \lesssim \int_{-u}^{\underline{u}} (1+\underline{u}')^{-1-\eta} \mathcal{W}_1[\varphi](\mathcal{D}_{\underline{u}'}^{\underline{u}'} ) d\underline{u}' + \int_{\Sigma_0^{u,\underline{u}}} r^{-1}\varphi^2 + M \int_{-u}^u |u'|^{-1} E[\varphi](\mathcal{H}_{u'}^{\underline{u}}) du'. \quad (5.12)$$

The first term can be absorbed by using Gronwall's inequality. The last term can be treated due to  $|u| \geq 1$ . Lemma 5.3 is thus proved.  $\square$

We then can derive the following result.

**Proposition 5.4.** *Let  $p \leq 0$  be any fixed number. For  $-\underline{u}_1 \leq u_1 \leq u_0$  there holds the energy inequality*

$$\begin{aligned}
& \mathcal{W}_1[\varphi](\mathcal{H}_{\underline{u}_1}^{u_1}) + \mathcal{W}_1[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + \mathcal{W}_1[\varphi](\mathcal{D}_{\underline{u}_1}^{u_1}) \\
& \lesssim \mathcal{W}_1[\varphi](\Sigma_0^{u_1,\underline{u}_1}) + \|r^{\frac{3}{2}}F\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{\underline{u}_1}^{u_1})}^2 + |u_1|^{-\gamma_0+2p+1} \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{-2p+\gamma_0} E[\varphi](\mathcal{H}_u^{u_1}) \\
& + E[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + \|r^{-1/2}\varphi\|_{L^2(\Sigma_0^{u_1,\underline{u}_1})}^2 + \int_{S_{-\underline{u}_1,\underline{u}_1}} r\varphi^2 d\omega.
\end{aligned}$$

**5.1. Preliminaries.** The proof of Theorem 2.1 is based on a bootstrap argument, with the assumption of (BA<sub>2</sub>) and  $\Delta_0 = C_1 \mathcal{E}_{2,\gamma_0}$  with  $C_1 > 1$  to be determined. We recast the assumption as follows.

Let  $\underline{u}_* > -u_0$  be any fixed large number. For  $0 \leq n \leq 2$ , we suppose that

$$E[Z^{(n)}\phi](\mathcal{H}_{\underline{u}}^u) + E[Z^{(n)}\phi](\underline{\mathcal{H}}_{\underline{u}}^u) \leq 2\Delta_0 |u|^{-\gamma_0+2\zeta(Z^n)}, \quad (5.13)$$

$$\mathcal{W}_1[Z^{(n)}\phi](\mathcal{H}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}\phi](\underline{\mathcal{H}}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{D}_{\underline{u}}^u) \leq 2\Delta_0 |u|^{-\gamma_0+1+2\zeta(Z^n)} \quad (5.14)$$

hold for all  $-\underline{u}_* \leq -\underline{u} \leq u \leq u_0$ .

The local well-posedness result, in  $\{-u(\epsilon) \leq -\underline{u} < u \leq u_0\}$  with  $\underline{u}(\epsilon)$  finite, can follow by running a standard iteration argument (see [28]), or by using the standard local existence result up to the characteristic boundary, i.e.  $\{r \geq t + R\}$ . Thus the above assumptions hold for some  $\underline{u}_* > -u_0$ . Our task is to show that the estimates in the assumption hold for any  $\underline{u}_* > -u_0$ , with the bound improved to be  $< 2\Delta_0$ .<sup>18</sup>

As a direct consequence of the bootstrap assumption, we have

<sup>18</sup>The same argument is employed for setting up the bootstrap assumptions in Section 6 and Section 7, which will not be repeated in later sections.

**Lemma 5.5.** *For  $Z = \Omega$  or  $\partial$ , there holds*

$$|Z\phi| \lesssim |u|^{-\frac{1}{2}-\frac{\gamma_0}{2}} \Delta_0^{\frac{1}{2}} r^{\zeta(Z)}. \quad (5.15)$$

*Proof.* It follows from (4.1) with  $l = 0$  that

$$r|\nabla\phi| \lesssim r^{-\frac{1}{2}}|u|^{-\frac{\gamma_0}{2}} \Delta_0^{\frac{1}{2}}, \quad r|\partial\phi| \lesssim |u|^{-\frac{1}{2}-\frac{\gamma_0}{2}} \Delta_0^{\frac{1}{2}}$$

which gives (5.15).  $\square$

Let  $0 \leq m \leq n$ . For the ordered product of vector fields,  $Z^n = Z_1 \cdots Z_n$ , we denote by  $Z^m \subset Z^n$  if  $Z^m = Z_{k_1} \cdots Z_{k_m}$  with  $1 \leq k_1 < k_2 < \cdots < k_m \leq n$ . By  $Z^a \sqcup Z^b = Z^n$ , we denote a decomposition of  $Z^n$  into  $Z^a$  and  $Z^b$ . It means,  $Z^a, Z^b \subset Z^n$  with  $Z^a = Z_{k_1} \cdots Z_{k_a}$  and  $Z^b = Z_{m_1} \cdots Z_{m_b}$ , none of the subindices among  $(k_1, \dots, k_a)$  and  $(m_1, \dots, m_b)$  are equal, i.e.,  $k_l \neq m_j$  and  $a + b = n$ .  $Z^{a_1} \sqcup \cdots \sqcup Z^{a_m}$  can be understood inductively.

If  $Z_1 Z_2 \cdots Z_n$  is regarded as a differential operator, we denote it as  $Z^{(n)}$ . We set  $Z^{(n \setminus i)} = Z_1 \cdots Z_{i-1} Z_{i+1} \cdots Z_n$ , for  $i = 1, \dots, n$ .  $Z^{n \setminus i}$  represents the corresponding product of vector fields.

**Lemma 5.6.** *For each killing vector field  $Z$ ,  $[Z, \partial_\alpha] = C_{Z\alpha}{}^\gamma \partial_\gamma$ , where  $C_{Z\alpha}{}^\gamma = -\partial_\alpha Z^\gamma$  is a  $(1, 1)$  tensor.  $C_{Z\alpha}{}^\gamma = 0$  if  $Z = \partial$ . Due to (3.19), the components of  $C_Z$  are 1 or  $-1$  if  $Z = \Omega_{\mu\nu}$ . Thus, symbolically, we may ignore the tensorial feature of  $C_Z$ , and regard  $C_Z$  as constants. The tensor products  $C_{Z^m} = C_{Z_1} \cdots C_{Z_m}$  may be regarded as a set of product of constants, since  $C_{Z_i}$  is understood as a set of constants with  $|C_{Z_i}| = 1$  if  $Z_i = \Omega$ ; and  $C_{Z_i} = 0$  if  $Z_i = \partial$ .<sup>19</sup>*

(1) For  $n = 1, 2, 3$ , there holds the symbolic identity

$$[\partial, Z^{(n)}]f = \sum_{Z^a \sqcup Z^b = Z^n, a \geq 1} C_{Z^a} \partial Z^{(b)} f. \quad (5.16)$$

Thus, if  $\zeta(Z^n) = -n$ ,  $[\partial, Z^{(n)}]f = 0$ .

(2) For  $n = 1, 2, 3$  there holds

$$|Z^{(n)} \partial f| \lesssim \sum_{Z^a \sqcup Z^b = Z^n} r^{\zeta(Z^a)} |\partial Z^{(b)} f|. \quad (5.17)$$

*Remark 5.7.* In applications, most of the time we will replace  $r^{\zeta(Z^a)}$  by  $|u|^{\zeta(Z^a)}$  which is a weaker version of the result.

*Proof.* (5.16) follows by direct calculation. It follows directly from (1) and the definition of  $\zeta(\cdot)$  that  $|[\partial, Z^{(n)}]f| \lesssim \sum_{Z^a \sqcup Z^b = Z^n, a \geq 1} r^{\zeta(Z^a)} |\partial Z^{(b)} f|$ . Thus (5.17) follows as a consequence.  $\square$

**Lemma 5.8.** *Let  $\varphi$  be a smooth function and  $n = 1, 2, 3$ . Under the assumption that  $|\mathcal{N}^{(i)}(\varphi)| \leq C$  with  $i = 1, \dots, n$ , there holds*

$$|Z^{(n)}(\mathcal{N}(\varphi))| \lesssim |Z^{(n)}\varphi| + \sum_{i=1}^n |Z^{(n \setminus i)}\varphi \cdot Z_i \varphi| + \left| \prod_{i=1}^n Z_i \varphi \right| \quad (5.18)$$

and consequently

$$|Z^{(n)}(\mathcal{N}(\varphi))| \lesssim \sum_{Z^a \sqcup Z^b = Z^n, a \geq 1} |Z^{(a)}\varphi| \Delta_0^{\frac{1}{2}b} |u|^{\zeta(Z^b)}. \quad (5.19)$$

*Remark 5.9.* Under the bootstrap assumption  $(BA_2)$ , (4.2) holds, which imply  $|\phi| \lesssim r^{-1}(\mathcal{E}_{2, \gamma_0}^{\frac{1}{2}} + \Delta_0^{\frac{1}{2}}) \lesssim 1$ . Since  $\mathcal{N}(y)$  is smooth, we can obtain  $|\mathcal{N}^{(i)}(\phi)| \lesssim 1$ ,  $i \leq k$  for any fixed  $k \in \mathbb{N}$ . So the assumption holds for  $\varphi = \phi$ . We also remark that we only used  $|Z\varphi| \lesssim \Delta_0^{\frac{1}{2}} |u|^{\zeta(Z)}$  to prove the above result.

<sup>19</sup> $C_{id} = 0$

*Proof.* It is straightforward to derive

$$\begin{aligned} Z^{(1)}(\mathcal{N}(\varphi)) &= \mathcal{N}'(\varphi)Z^{(1)}\varphi, \\ Z^{(2)}(\mathcal{N}(\varphi)) &= \mathcal{N}'(\varphi)Z^{(2)}\varphi + \mathcal{N}''(\varphi)Z_2\varphi Z_1\varphi, \\ Z^{(3)}(\mathcal{N}(\varphi)) &= \mathcal{N}'(\varphi)Z^{(3)}\varphi + \mathcal{N}''(\varphi)\sum_{i=1}^3 Z^{(n\setminus i)}\varphi \cdot Z_i\varphi + \mathcal{N}'''(\varphi)\prod_{i=1}^3 Z_i\varphi. \end{aligned}$$

We then can derive (5.18) for  $n = 1, 2, 3$ . (5.19) follows by using (5.15).  $\square$

**5.2. Error estimates.** We will improve the bootstrap assumptions (5.13) and (5.14) by deriving energy estimates, with the help of Proposition 5.2 and Proposition 5.4. For deriving both types of estimates for  $Z^{(i)}\phi$ , the main task is to obtain the error estimates on  $(\square_{\mathbf{m}} - q)Z^{(i)}\phi$  with  $i \leq 2$ . We analyze in the following result these major error terms.

**Lemma 5.10.** *Let  $\mathcal{F} = \mathcal{N}(\phi)\partial\phi \cdot \partial\phi$ ,  $\mathcal{F}^{\{n\}} = Z^{(n)}\mathcal{F}$  and  $\mathcal{F}^{\{0\}} = \mathcal{F}$ . For  $n = 0, \dots, 3$ , there hold*

$$|Z^{(n)}(\partial\phi \cdot \partial\phi)| \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u|^{\zeta(Z^c)} |\partial Z^{(b)}\phi| |\partial Z^{(a)}\phi| \quad (5.20)$$

and

$$\left| \mathcal{F}^{\{n\}} \right| \lesssim \mathcal{F}^{\{n\}}_{\mathcal{Q}} + \mathcal{F}^{\{n\}}_{\mathcal{C}} \quad (5.21)$$

where the quadratic part and the cubic part are

$$\begin{aligned} \mathcal{F}^{\{n\}}_{\mathcal{Q}} &= \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u|^{\zeta(Z^c)} \left| \partial Z^{(b)}\phi \right| \left| \partial Z^{(a)}\phi \right|, \\ \mathcal{F}^{\{n\}}_{\mathcal{C}} &= \sum_{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n, 1 \leq a \leq n} \left| Z^{(a)}\phi \right| \left| \partial Z^{(b)}\phi \right| \left| \partial Z^{(c)}\phi \right| |u|^{\zeta(Z^d)}. \end{aligned}$$

*Remark 5.11.* The result with  $n = 3$  will be used in Section 6 and 7.

*Proof.* We first can obtain (5.20) by using (5.17) and

$$Z^{(n)}(\partial\phi \cdot \partial\phi) = \sum_{Z^a \sqcup Z^b = Z^n} Z^{(a)}\partial\phi \cdot Z^{(b)}\partial\phi.$$

Next we derive the estimates of  $\mathcal{F}^{\{n\}}$  in view of

$$\mathcal{F}^{\{n\}} = \left( \sum_{a \geq 1} + \sum_{a=0} \right) \sum_{Z^a \sqcup Z^b = Z^n} Z^{(a)}(\mathcal{N}(\phi)) Z^{(b)}(\partial\phi \cdot \partial\phi). \quad (5.22)$$

The  $a = 0$  term can be bounded by using (5.20) directly. For the terms with  $a \geq 1$  in (5.22), we can apply (5.19) with  $n = a$  and (5.20) with  $n = b$  to derive the cubic type of terms. We then combine the estimates for  $0 \leq a \leq n$  to obtain

$$\begin{aligned} \left| \mathcal{F}^{\{n\}} \right| &\lesssim \sum_{Z^{a_1} \sqcup Z^{b_1} \sqcup Z^{c_1} \sqcup Z^{d_1} = Z^n} \sum_{1 \leq a_1 \leq a \leq n} \Delta_0^{\frac{1}{2}(a-a_1)} \left| Z^{(a_1)}\phi \right| \left| \partial Z^{(b_1)}\phi \right| \left| \partial Z^{(c_1)}\phi \right| |u|^{\zeta(Z^{d_1})} \\ &+ \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u|^{\zeta(Z^c)} \left| \partial Z^{(b)}\phi \right| \left| \partial Z^{(a)}\phi \right|. \end{aligned}$$

The second line is the quadratic term  $\mathcal{F}^{\{n\}}_{\mathcal{Q}}$ . The first line on the righthand side is a sum of cubic terms of  $\phi$ . By the boundedness of  $\Delta_0^{a-a_1}$ , we can obtain the formula for  $\mathcal{F}^{\{n\}}_{\mathcal{C}}$  in (5.21).  $\square$

As an important remark, we can write according to Lemma 5.10 that

$$\left| \mathcal{F}^{\{1\}} \right| \lesssim \left( |\partial Z \phi| + |u|^{\zeta(Z)} |\partial \phi| \right) |\partial \phi|,$$

for which the cubic term is already controlled by using (5.15). Thus, symbolically,

$$\left| \mathcal{F}^{\{1\}} \right| \lesssim \mathcal{F}^{\{1\}}_{\mathcal{Q}}. \quad (5.23)$$

**Proposition 5.12.** *For  $n \leq 2$  and  $-\underline{u}_* \leq -\underline{u}_1 \leq u_1 \leq u_0$ , the following estimates hold:*

$$|u_1|^{\frac{1}{2}\gamma_0 + \frac{1}{2} - \zeta(Z^n)} \left\| r^{\frac{1}{2}} \mathcal{F}^{\{n\}}_{\mathcal{Q}} \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{1}{2}\gamma_0 + \frac{1}{2}} \Delta_0 M^{-\frac{1}{2}}, \quad (5.24)$$

$$|u_1|^{\frac{1}{2}\gamma_0 - \frac{1}{2} - \zeta(Z^n)} \left\| r^{\frac{3}{2}} \mathcal{F}^{\{n\}}_{\mathcal{Q}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{\gamma_0}{2}} \Delta_0 M^{-\frac{1}{2}}, \quad (5.25)$$

$$\left\| r^{\frac{1}{2}} \mathcal{F}^{\{2\}}_{\mathcal{C}} \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{3\gamma_0}{2} - \frac{1}{2} + \zeta(Z^2)} \Delta_0^{\frac{3}{2}} M^{-\frac{1}{2}}, \quad (5.26)$$

$$\left\| r^{\frac{3}{2}} \mathcal{F}^{\{2\}}_{\mathcal{C}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{3}{2}\gamma_0 - \frac{1}{2} + \zeta(Z^2)} \Delta_0^{\frac{3}{2}}. \quad (5.27)$$

*Proof.* We first decompose  $\mathcal{F}^{\{n\}}_{\mathcal{Q}}$  as below

$$\begin{aligned} \mathcal{F}^{\{n\}}_{\mathcal{Q},1} &= \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n, b=0} |u|^{\zeta(Z^c)} \left| \partial Z^{(b)} \phi \right| \left| \partial Z^{(a)} \phi \right|, \\ \mathcal{F}^{\{n\}}_{\mathcal{Q},2} &= \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n, b \geq 1} |u|^{\zeta(Z^c)} \left| \partial Z^{(b)} \phi \right| \left| \partial Z^{(a)} \phi \right|, \end{aligned} \quad (5.28)$$

where we assume  $b \leq a$  without loss of generality. We will frequently use (5.13) and (5.14) in the sequel.

Note that with  $a \leq n$ , we can apply (4.1) and (4.14) to derive that

$$\begin{aligned} \left\| r^{\frac{1}{2}} \mathcal{F}^{\{n\}}_{\mathcal{Q},1} \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} &\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n, b=0} |u_1|^{\zeta(Z^c)} \left\| r \partial Z^{(b)} \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \left\| r^{-\frac{1}{2}} \partial Z^{(a)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ &\lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + \zeta(Z^n)}. \end{aligned} \quad (5.29)$$

We note by (5.23) that  $\mathcal{F}^{\{n\}}_{\mathcal{Q},2}$  vanishes in the case  $n \leq 1$ . If  $n = 2$ ,  $a = b = 1 \leq n - 1$ . Thus, in view of (4.19) and (4.21), we deduce for  $\mathcal{F}^{\{2\}}_{\mathcal{Q},2}$  that

$$\begin{aligned} &\left\| r^{\frac{1}{2}} \mathcal{F}^{\{n\}}_{\mathcal{Q},2} \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ &\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n, b \geq 1} |u|^{\zeta(Z^c)} \left\| r \partial Z^{(b)} \phi \right\|_{L_{\underline{u}}^\infty L_u^2 L_{\omega}^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \left\| r^{\frac{1}{2}} \partial Z^{(a)} \phi \right\|_{L_{\underline{u}}^2 L_u^\infty L_{\omega}^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ &\lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + \zeta(Z^n)}. \end{aligned}$$

(5.24) follows by combining the estimates of  $\mathcal{F}^{\{n\}}_{\mathcal{Q},1}$  and  $\mathcal{F}^{\{n\}}_{\mathcal{Q},2}$ .



Next, we prove (5.25). Using (4.1) and (4.14) it follows that

$$\begin{aligned}
& \left\| r^{\frac{3}{2}} \mathcal{F}_{\mathcal{Q},1}^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \sum_{Z^a \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^c)} \left\| r^{\frac{1}{2}} \partial Z^{(a)} \phi |u|^{-\frac{1+\gamma_0}{2}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \left\| r \partial \phi \cdot |u|^{\frac{1+\gamma_0}{2}} \right\|_{L^\infty(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \Delta_0^{\frac{1}{2}} \sum_{Z^a \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^c)} \left\| r^{-\frac{1}{2}} \partial Z^{(a)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \left\| |u|^{-\frac{1}{2}\gamma_0 - \frac{1}{2}} \right\|_{L_u^2 L^\infty(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{\zeta(Z^n) - \gamma_0 + \frac{1}{2}}.
\end{aligned}$$

In the case that  $1 \leq b \leq a$ , again  $a \leq n-1$ . By using (4.20) and (4.22), we have

$$\begin{aligned}
& \left\| r^{\frac{3}{2}} \mathcal{F}_{\mathcal{Q},2}^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n, b \geq 1} |u_1|^{\zeta(Z^c)} \left\| r \partial Z^{(b)} \phi \right\|_{L_u^2 L_{\underline{u}}^\infty L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \left\| r^{\frac{1}{2}} \partial Z^{(a)} \phi \right\|_{L_u^2 L_{\underline{u}}^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{\zeta(Z^n) - \gamma_0 + \frac{1}{2}}.
\end{aligned}$$

By combining the above two estimates, we can obtain (5.25).

Next we consider the estimates of  $\mathcal{F}_C^{\{2\}}$ . By using (4.1), (4.21) and (4.24) we have

$$\begin{aligned}
& \left\| r^{\frac{1}{2}} \mathcal{F}_C^{\{2\}} \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^2 \\ a \geq 1, b \leq c}} \left\| Z^{(a)} \phi \right\|_{L_{\underline{u}}^\infty L_u^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \left\| r \partial Z^{(b)} \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{u_1})} \left\| r^{\frac{1}{2}} \partial Z^{(c)} \phi \right\|_{L_{\underline{u}}^2 L_u^\infty L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \Delta_0^{\frac{3}{2}} M^{-\frac{1}{2}} |u_1|^{-\frac{3\gamma_0}{2} - \frac{1}{2} + \zeta(Z^2)}.
\end{aligned}$$

Here we assumed  $b \leq c$  without loss of generality, which implies  $b = 0$ . By using (4.1), (4.20) and (4.17), we have

$$\begin{aligned}
& \left\| r^{\frac{3}{2}} \mathcal{F}_C^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^2 \\ a \geq 1, b \leq c}} |u_1|^{\zeta(Z^d)} \left\| r^{-\frac{1}{2}} Z^{(a)} \phi \right\|_{L_u^2 L_{\underline{u}}^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \left\| r \partial Z^{(b)} \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{u_1})} \left\| r \partial Z^{(c)} \phi \right\|_{L_u^2 L_{\underline{u}}^\infty L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \Delta_0^{\frac{3}{2}} |u_1|^{\zeta(Z^2) - \frac{3}{2}\gamma_0 - \frac{1}{2}}.
\end{aligned}$$

Thus (5.26) and (5.27) are both proved.  $\square$

**Lemma 5.13.** *Under the assumption (1.2), for  $n \leq 3$  and  $-\underline{u}_* \leq -\underline{u}_1 \leq u_1 \leq u_0$  there hold*

$$\begin{aligned}
& |u_1|^{-\zeta(Z^n) + \frac{\gamma_0}{2}} \left( \left\| r[Z^{(n)}, q] \phi \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} + \left\| r[Z^{(n)}, q] \phi \right\|_{L_{\underline{u}}^1 L_u^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \right) \\
& \lesssim \mathcal{E}_{n,\gamma_0}^{\frac{1}{2}} + \sup_{-u_1 \leq \underline{u} \leq u_1} \sum_{Z^m \subsetneq Z^n} E[Z^{(m)} \phi]^{\frac{1}{2}}(\mathcal{H}_{\underline{u}}^{u_1}) |u_1|^{\frac{\gamma_0}{2} - \zeta(Z^m)}, \\
& |u_1|^{-\zeta(Z^n) + \frac{1}{2}\gamma_0 - \frac{1}{2}} \left\| r^{\frac{3}{2}} [Z^{(n)}, q] \phi \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \mathcal{E}_{n,\gamma_0}^{\frac{1}{2}} + \sum_{Z^m \subsetneq Z^n} |u_1|^{-\zeta(Z^m) + \frac{\gamma_0 - 1}{2}} \left( \mathcal{W}_1[Z^{(m)} \phi](\mathcal{D}_{u_1}^{u_1}) + M \int_{-u_1}^{u_1} |u|^{-1} E[Z^{(m)} \phi](\mathcal{H}_u^{u_1}) du \right)^{\frac{1}{2}}.
\end{aligned}$$

*Proof.* It is direct to obtain

$$[Z^n, q]\phi = \sum_{i=1}^n Z^{(i)} q Z^{(n-i)} \phi,$$

where all  $Z^{n-i} \subsetneq Z^n$ . Note that the assumption (1.2) on  $q$  implies

$$\left\| r^{\frac{3}{2} + \frac{1}{2}\eta} Z^{(i)} q \right\|_{L_u^2 L^\infty(\mathcal{D}_{u_1}^{u_1})} + \left\| r^{\frac{3}{2} + \frac{1}{2}\eta} Z^{(i)} q \right\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{u_1}^{u_1})} \lesssim |u_1|^{\zeta(Z^i)}. \quad (5.30)$$

Thus

$$\begin{aligned} \left\| r[Z^n, q]\phi \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{u_1})} + \left\| r[Z^n, q]\phi \right\|_{L_{\underline{u}}^1 L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{u_1})} \\ \lesssim \sum_{1 \leq i \leq n} |u_1|^{\zeta(Z^i)} \left\| r^{-\frac{3}{2} - \frac{1}{2}\eta} Z^{(n-i)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})}. \end{aligned}$$

By using (3.15), we can derive

$$\begin{aligned} \left\| r^{-\frac{3}{2} - \frac{1}{2}\eta} Z^{(n-i)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} &\lesssim \sup_{-u_1 \leq \underline{u} \leq u_1} \left\| r^{-1} Z^{(n-i)} \phi \right\|_{L^2(\mathcal{H}_{\underline{u}}^{u_1})} \\ &\lesssim \sup_{-u_1 \leq \underline{u} \leq u_1} \left( \int_{S_{-\underline{u}, \underline{u}}} r(Z^{(n-i)} \phi)^2 d\omega + E[Z^{(n-i)} \phi](\mathcal{H}_{\underline{u}}^{u_1}) \right)^{1/2}. \end{aligned}$$

We can apply (3.3) to  $Z^{(n-i)} \phi$  to control the first term. The first inequality of Lemma 5.13 holds by combining the above two estimates.

To see the second inequality, we have by using (5.30) that

$$\left\| r^{\frac{3}{2}} [Z^n, q]\phi \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \sum_{1 \leq i \leq n} |u_1|^{\zeta(Z^i)} \left\| r^{-1 - \frac{1}{2}\eta} Z^{(n-i)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})}.$$

It then follows from (3.13) that

$$\begin{aligned} \left\| r^{-1 - \frac{\eta}{2}} Z^{(n-i)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})}^2 &\lesssim \mathcal{W}_1[Z^{(n-i)} \phi](\mathcal{D}_{u_1}^{u_1}) + M \int_{-\underline{u}_1}^{u_1} |u|^{-1} E[Z^{(n-i)} \phi](\mathcal{H}_u^{u_1}) du \\ &\quad + \int_{\Sigma_0^{u_1, \underline{u}_1}} r(Z^{(n-i)} \phi)^2 dud\omega. \end{aligned}$$

By combining the above two inequalities and applying (3.7) to  $Z^{(n-i)} \phi$ , we can obtain Lemma 5.13.  $\square$

**5.3. Boundedness of energies.** Next, we will use the fact

$$\square_{\mathbf{m}} Z^{(n)} \phi - q Z^{(n)} \phi = \mathcal{F}^{\{n\}} + [Z^{(n)}, q]\phi, \quad (5.31)$$

Proposition 5.12, Lemma 5.13, Proposition 5.2 and Proposition 5.4 to prove the boundedness of energies in (2.16).

**Proposition 5.14.** *Let  $n \leq 2$ . For  $-\underline{u}_* \leq -\underline{u} \leq u \leq u_1 \leq u_0$  there hold*

$$\begin{aligned} |u|^{\gamma_0 - 2\zeta(Z^n)} \left( E[Z^{(n)} \phi](\mathcal{H}_u^u) + E[Z^{(n)} \phi](\mathcal{H}_{\underline{u}}^u) \right) &\lesssim \mathcal{E}_{n, \gamma_0} + M^{-2} \Delta_0^2 |u_1|^{1 - \gamma_0}, \quad (5.32) \\ |u|^{-2\zeta(Z^n) - 1 + \gamma_0} \left( \mathcal{W}_1[Z^{(n)} \phi](\mathcal{H}_u^u) + \mathcal{W}_1[Z^{(n)} \phi](\mathcal{H}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)} \phi](\mathcal{D}_{\underline{u}}^u) \right) \\ &\lesssim \mathcal{E}_{n, \gamma_0} + M^{-2} \Delta_0^2 |u_1|^{1 - \gamma_0}. \end{aligned}$$

*Proof.* In view of (5.31), we will use Proposition 5.2 with  $\mathcal{F}^\# = \mathcal{F}_Q^{\{n\}} + \mathcal{F}_C^{\{n\}}$  and  $\mathcal{F}^b = [Z^{(n)}, q]\phi$ . By using (5.24), (5.26) and Lemma 5.13, we have

$$\begin{aligned} & |u|^{\gamma_0 - 2\zeta(Z^n)} \left( E[Z^{(n)}\phi](\mathcal{H}_u^u) + E[Z^{(n)}\phi](\mathcal{H}_u^u) \right) \\ & \lesssim \mathcal{E}_{n, \gamma_0} + M^{-2} \Delta_0^2 |u_1|^{1-\gamma_0} + \sup_{-\underline{u}_* \leq -\underline{u} \leq u_1} \sum_{i=1}^n E[Z^{(n-i)}\phi](\mathcal{H}_{\underline{u}}^{u_1}) |u_1|^{\gamma_0 - 2\zeta(Z^{n-i})}, \end{aligned}$$

where the last term vanishes when  $n = 0$ . This implies the first estimate in Proposition 5.14 by induction.

In view of (5.25) and (5.27) in Proposition 5.12 for  $\mathcal{F}_Q^{\{n\}}$  and  $\mathcal{F}_C^{\{n\}}$ , we can derive for  $-\underline{u}_1 \leq u_1 \leq u_0$  that

$$|u_1|^{-\zeta(Z^n) + \gamma_0 - \frac{1}{2}} \left\| r^{\frac{3}{2}} \mathcal{F}^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0 M^{-\frac{1}{2}}.$$

By using the above estimate, Proposition 5.4, (3.3), (3.7), (3.8), the second estimate in Lemma 5.13 and (5.32), we can derive for  $\underline{u}_1 \leq u_1 \leq u_0$  that

$$\begin{aligned} & |u_1|^{-2\zeta(Z^n) - 1 + \gamma_0} \left( \mathcal{W}_1[Z^{(n)}\phi](\mathcal{H}_{u_1}^{u_1}) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{H}_{u_1}^{u_1}) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{D}_{u_1}^{u_1}) \right) \\ & \lesssim \mathcal{E}_{n, \gamma_0} + \Delta_0^2 |u_1|^{-\gamma_0} (M^{-1} + M^{-2} |u_1|) + \sum_{i=1}^n |u_1|^{-2\zeta(Z^{n-i}) - 1 + \gamma_0} \mathcal{W}_1[Z^{(n-i)}\phi](\mathcal{D}_{u_1}^{u_1}). \end{aligned}$$

Note that when  $n = 0$  the last term on the right vanishes. The weighted energy estimate can then be derived by induction.  $\square$

**Improvement on the bootstrap assumption.** Let us denote the universal constant in  $\lesssim$  in the estimates of Proposition 5.14 as  $C_3 > 0$ . We need to show that

$$C_3 (\mathcal{E}_{2, \gamma_0} + M^{-2} \Delta_0^2 |u|^{1-\gamma_0}) < 2\Delta_0, \quad \forall u \leq u_0$$

Recall that  $\Delta_0 = C_1 \mathcal{E}_{2, \gamma_0}$  with  $C_1$  to be chosen, and  $\mathcal{E}_{2, \gamma_0} \leq CM^2$  with  $C \geq 1$  a fixed constant. Thus we need to choose  $C_1$  such that

$$C_3 (C_1^{-1} + M^{-2} \Delta_0 |u|^{1-\gamma_0}) < 2.$$

Let  $C_1 = 4C_3$ . It is reduced to  $M^{-2} \Delta_0 |u_0(R)|^{1-\gamma_0} < \frac{7}{4C_3}$ . Since  $\delta_1$  in Theorem 2.1 can be sufficiently small,  $M < \frac{1}{10}$  can be achieved, and thus  $|u_0(R)| > \frac{1}{2}R$  can be guaranteed. Thus we need

$$\left( \frac{R}{2} \right)^{1-\gamma_0} < \frac{7}{16C_3^2 C}. \quad (5.33)$$

Let  $R$  be fixed to satisfy the inequality of (5.33). Then the proof of Theorem 2.1 is completed. If  $R = 2$  but  $C$  is allowed to be chosen, with  $C < \delta_0$  such that  $\frac{16}{7} C_3^2 \delta_0 < (\frac{R}{2})^{\gamma_0 - 1} = 1$ , (5.33) can also be achieved. This proves Theorem 2.2.

## 6. Quasilinear equations

In this section, we consider the general quasilinear equations (1.1) in  $\mathbb{R}^{3+1}$  which verifies (1.2) for  $n = 3$ .  $\mathbf{g}(\phi, \partial\phi)$  and  $\mathcal{N}^{\alpha\beta}(\phi)$  are both smooth functions of their arguments,  $\mathbf{g}(0, \mathbf{0}) = \mathbf{m}$ . For convenience, we set  $\Phi = (\phi, \partial\phi)$ , which is a 5-vector valued function, and  $|\Phi| = |\phi| + \sum_{\mu=0}^3 |\partial^\mu \phi|$ . We will prove Theorem 2.5 in this section.

**6.1. Bootstrap assumptions.** Due to the influence of the metric  $\mathbf{g}^{\alpha\beta}$ , the bootstrap assumptions are more delicate than (5.13) and (5.14) in Section 5. We first need to fix the constant  $M_0$  since the region where the stability result holds is determined by  $u_0(M_0)$ . Let  $H^{\alpha\beta} := \mathbf{g}^{\alpha\beta} - \mathbf{m}^{\alpha\beta}$ . By using  $H^{\alpha\beta}(0, \mathbf{0}) = 0$  and the fact that  $H^{\alpha\beta}$  are smooth functions of the arguments, we can derive that

$$\sup_{0 \leq \alpha, \beta \leq n} |H^{\alpha\beta}(\phi, \partial\phi)| \lesssim |\phi| + |\partial\phi| \quad \text{on } \Sigma_0 \cap \{r \geq R\}.$$

From the above estimate and (3.9), it follows that

$$\sup_{\alpha, \beta} r |H^{\alpha\beta}| \leq C |u|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \mathcal{E}_{2, \gamma_0, R}^{\frac{1}{2}} \quad \text{on } \Sigma_0 \cap \{r \geq R\}. \quad (6.1)$$

We can choose  $M_0 = 3C\delta_1^{\frac{1}{2}}$  in the definition of (2.1) and  $h = \frac{M_0}{r}$ . By this choice and (3.9) on  $\Sigma_0 \cap \{r \geq R\}$  there holds

$$\sup_{\alpha, \beta} r(h - H^{\alpha\beta}) > C \left( 3\delta_1^{\frac{1}{2}} - |u|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \mathcal{E}_{2, \gamma_0, R}^{\frac{1}{2}} \right). \quad (6.2)$$

Since  $|u| \geq \frac{1}{2}R$ , with  $(\frac{R}{2})^{-\frac{\gamma_0}{2} + \frac{1}{2}} < \frac{1}{2}$ , on  $\Sigma_0 \cap \{r \geq R\}$  there holds

$$\sup_{\alpha, \beta} r(h - H^{\alpha\beta}) > \frac{5}{2} C \delta_1^{\frac{1}{2}} = \frac{5}{6} M, \quad M = M_0.$$

The bootstrap assumption for proving Theorem 2.5 consists of the control of the metric and the boundedness of energies.

Let  $\underline{u}_* > -u_0$  be a fixed number and let  $\mathcal{I} = \{(u, \underline{u}) : -\underline{u}_* \leq -\underline{u} \leq u \leq u_0\}$ . We assume that on  $\mathcal{I}$  there hold

$$r(h - H^{\underline{L}\underline{L}}) \geq \frac{M}{3}, \quad r(h - H^{LL}) \geq \frac{M}{3}, \quad (A1)$$

and for any  $(\underline{u}_1, u_1) \in \mathcal{I}$  and  $n \leq 3$ ,

$$E[Z^{(n)}\phi](\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}) + E[Z^{(n)}\phi](\mathcal{H}_{\underline{u}_1}^{u_1}) \leq 2\Delta_0 |u_1|^{-\gamma_0 + 2\zeta(Z^n)}, \quad (6.3)$$

$$\mathcal{W}_1[Z^{(n)}\phi](\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{H}_{\underline{u}_1}^{u_1}) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{D}_{\underline{u}_1}^{\underline{u}_1}) \leq 2\Delta_0 |u_1|^{-\gamma_0 + 1 + 2\zeta(Z^n)}, \quad (6.4)$$

where  $\Delta_0 \leq C_1 M^2$  with  $C_1 > 1$  to be chosen later and  $Z \in \{\Omega_{ij}, \partial\}$ .

As a consequence of the above assumptions, all the estimates in Section 4 hold. We can first summarize some of the decay estimates that will be frequently used in this section.

**Proposition 6.1** (Decay estimates). *There hold on  $\mathcal{I}$  the following decay estimates*<sup>20</sup>

$$r^2 \left| \partial^{(l)} Z^{(n)} \phi \right|^2 \lesssim \Delta_0 |u|^{-\gamma_0 + 1 - 2l + 2\zeta(Z^n)}, \quad l, n \leq 1 \quad (6.5)$$

$$r^2 \left| Z^{(n)} H \right|^2 \lesssim \Delta_0 |u|^{-\gamma_0 + 1 + 2\zeta(Z^n)}, \quad n \leq 1, \quad (6.6)$$

$$r^3 \left| \bar{\partial} H \right|^2 + r^2 |u| \left| \underline{L} H \right|^2 \lesssim \Delta_0 |u|^{-\gamma_0}, \quad (6.7)$$

$$\left\| r^{\frac{1}{2}} \partial H \right\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{\underline{u}_1}^{\underline{u}_1})}^2 \lesssim \Delta_0 M^{-1} |u_1|^{-\gamma_0}, \quad (6.8)$$

where  $(u_1, \underline{u}_1) \in \mathcal{I}$ .

*Proof.* If  $l = 1$ , (6.5) is a consequence of (4.1); if  $l = 0$ , it is the estimate (4.2). The result of (6.5) with  $n = 0$ ,  $H(0, \mathbf{0}) = 0$  and the fact that  $H$  is smooth imply that for all  $(u, \underline{u}) \in \mathcal{I}$  there holds

$$|H(\Phi)| \lesssim |\phi| + |\partial\phi|; \quad |(D^{(i)} H)(\Phi)| \lesssim 1 \quad \text{for } i \geq 1. \quad (6.9)$$

The  $n = 0$  case in (6.6) can then be derived by using (6.5) with  $(n, l) = (0, 1), (0, 0)$ . Due to (6.9),  $|Z^{(1)}(H(\Phi))| \lesssim |Z^{(1)}\phi| + |Z^{(1)}\partial\phi|$ , by also using (5.17) and (6.5) we have

$$r |Z^{(1)}\partial\phi| \lesssim |u|^{\zeta(Z^1) - \frac{1}{2} - \frac{\gamma_0}{2}} \Delta_0^{\frac{1}{2}}.$$

<sup>20</sup> $Z^{(n)} H$  is understood as  $Z^{(n)}(H(\Phi))$ .

The estimate for  $Z^{(1)}\phi$  follows from (6.5). Thus the case  $n = 1$  in (6.6) is treated. (6.7) follows by using (6.9) and (4.1); similarly, (6.8) can be proved by using (4.3).  $\square$

**6.2. Energy and weighted energy inequalities.** In this subsection, we derive the fundamental energy estimate and  $r$ -weighted energy estimate for (1.1).

We will always use the Minkowski metric to lift and lower the indices. For example  $H_\alpha^\beta := \mathbf{m}_{\alpha\alpha'} H^{\alpha'\beta}$ . We define the following  $(0, 2)$  tensor, which is not necessarily symmetric,

$$\tilde{\mathcal{Q}}_{\alpha\beta}[\varphi] = \mathcal{Q}_{\alpha\beta}[\varphi] + H_\alpha^\gamma \partial_\gamma \varphi \partial_\beta \varphi,$$

where

$$\mathcal{Q}_{\alpha\beta}[\varphi] = \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} \mathbf{m}_{\alpha\beta} (\mathbf{g}^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + q\varphi^2).$$

**Lemma 6.2.** *Let  $X$  be a smooth vector field. There holds*

$$\begin{aligned} \partial^\alpha (\tilde{\mathcal{Q}}_{\alpha\beta} X^\beta) &= X^\beta \left\{ (\square_{\mathbf{g}} \varphi - q\varphi + \partial^\alpha H_\alpha^\gamma \partial_\gamma \varphi) \cdot \partial_\beta \varphi - \frac{1}{2} \partial_\beta q \cdot \varphi^2 - \frac{1}{2} \partial_\beta H_\alpha^\gamma \partial_\gamma \varphi \cdot \partial^\alpha \varphi \right\} \\ &\quad + \mathcal{Q}_{\alpha\beta}^{(X)} \pi^{\alpha\beta} + H_\alpha^\gamma \partial_\gamma \varphi \partial_\beta \varphi \partial^\alpha X^\beta, \end{aligned} \quad (6.10)$$

where  $\square_{\mathbf{g}} := \mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta$ .

*Proof.* We have

$$\partial^\alpha (\tilde{\mathcal{Q}}_{\alpha\beta} X^\beta) = \partial^\alpha \tilde{\mathcal{Q}}_{\alpha\beta} X^\beta + \mathcal{Q}_{\alpha\beta}^{(X)} \pi^{\alpha\beta} + H_\alpha^\gamma \partial_\gamma \varphi \partial_\beta \varphi \partial^\alpha X^\beta.$$

By direct calculation we also have

$$\begin{aligned} \partial^\alpha \tilde{\mathcal{Q}}_{\alpha\beta} X^\beta &= \left( \square_{\mathbf{m}} \varphi \cdot \partial_\beta \varphi - \frac{1}{2} \partial_\beta (H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi) + \partial^\alpha (H_\alpha^\gamma \partial_\gamma \varphi \partial_\beta \varphi) \right) X^\beta \\ &= X^\beta \left\{ ((\square_{\mathbf{m}} - q)\varphi + H_\alpha^\gamma \partial^\alpha \partial_\gamma \varphi + \partial^\alpha H_\alpha^\gamma \partial_\gamma \varphi) \partial_\beta \varphi - \frac{1}{2} \partial_\beta H_\alpha^\gamma \partial_\gamma \varphi \partial^\alpha \varphi - \frac{1}{2} \partial_\beta q \varphi^2 \right\}. \end{aligned}$$

The desired identity thus follows.  $\square$

We now give the energy density on  $\mathcal{H}_u^u$ ,  $\underline{\mathcal{H}}_u^u$  and  $\Sigma_t$ .

**Lemma 6.3.** *There hold the following identities for energy densities:*

$$\begin{aligned} \tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\beta (L^\alpha + h \underline{L}^\alpha) &= \frac{1}{2} (\underline{L}\varphi)^2 (h + (h-1)H^{\underline{L}\underline{L}} - 2hH^{\underline{L}\underline{L}}) \\ &\quad + \frac{1}{2} ((\underline{L}\varphi)^2 + (1+h)(|\nabla\varphi|^2 + q\varphi^2)) + (H + hH)\partial\varphi\bar{\partial}\varphi, \end{aligned} \quad (6.11)$$

$$\begin{aligned} \tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\beta (\underline{L}^\alpha + h L^\alpha) &= \frac{1}{2} (\underline{L}\varphi)^2 (h + (h-1)H^{\underline{L}\underline{L}} - 2hH^{\underline{L}\underline{L}}) \\ &\quad + \frac{1}{2} ((\underline{L}\varphi)^2 + (1+h)(|\nabla\varphi|^2 + q\varphi^2)) + (H + hH)\underline{\partial}\varphi\partial\varphi, \end{aligned} \quad (6.12)$$

and

$$\tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\alpha \partial_t^\beta = \frac{1}{2} \{ -\mathbf{g}^{00} (\partial_t \varphi)^2 + \mathbf{g}^{ij} \partial_i \varphi \partial_j \varphi + q\varphi^2 \}, \quad (6.13)$$

where  $\bar{\partial} = (L, \nabla)$  and  $\underline{\partial} = (\underline{L}, \nabla)$ .

*Proof.* For the energy density on  $\mathcal{H}_u^u$ , we derive

$$\begin{aligned} &\tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\beta (L^\alpha + h \underline{L}^\alpha) \\ &= \frac{1}{2} (L\varphi + h \underline{L}\varphi) (L\varphi + \underline{L}\varphi) + \frac{1}{2} (h+1) (-\underline{L}\varphi L\varphi + |\nabla\varphi|^2 + q\varphi^2 + H^{\underline{L}\underline{L}} (\underline{L}\varphi)^2 + H \partial\varphi \cdot \bar{\partial}\varphi) \\ &\quad + (-2H^{\underline{L}\gamma} \partial_\gamma \varphi \cdot \partial_t \varphi + h H_{\underline{L}}^\gamma \partial_\gamma \varphi \partial_t \varphi) \\ &= \frac{1}{2} ((L\varphi)^2 + h(\underline{L}\varphi)^2) + \frac{1}{2} (h+1) (|\nabla\varphi|^2 + q\varphi^2 + H^{\underline{L}\underline{L}} (\underline{L}\varphi)^2 + H \partial\varphi \bar{\partial}\varphi) \\ &\quad - (H^{\underline{L}\underline{L}} (\underline{L}\varphi)^2 + H \partial\varphi \bar{\partial}\varphi) + h \left( \frac{1}{2} H_{\underline{L}}^{\underline{L}} (\partial_{\underline{L}} \varphi)^2 + H \bar{\partial}\varphi \partial\varphi \right). \end{aligned}$$

Thus, (6.11) is proved. The energy density (6.12) on  $\underline{\mathcal{H}}_{\underline{u}}$  can be derived by directly swapping  $L$  and  $\underline{L}$  in the above calculation.

The energy density on  $\Sigma_t$  can be derived by

$$\begin{aligned}\tilde{\mathcal{Q}}_{\alpha\beta}\partial_t^\alpha\partial_t^\beta &= (\partial_t\varphi)^2 + \frac{1}{2}q\varphi^2 + \frac{1}{2}\mathbf{g}^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi + H_0^\gamma\partial_\gamma\varphi\partial_t\varphi \\ &= \frac{1}{2}\{-\mathbf{g}^{00}(\partial_t\varphi)^2 + \mathbf{g}^{ij}\partial_i\varphi\partial_j\varphi + q\varphi^2\},\end{aligned}$$

where all other terms have been cancelled. This gives (6.13).  $\square$

With the help of Lemma 6.3, we give the fundamental energy estimates.

**Proposition 6.4** (Energy inequality). *Suppose that on  $\mathcal{I}$  there hold the assumptions (A1), (6.3), (6.4) and*

$$\tilde{C}M^{-1}\Delta_0^{\frac{1}{2}}\left(\frac{R}{2}\right)^{\frac{1}{2}-\frac{\gamma_0}{2}} \leq \frac{1}{6} \quad (\text{A2})$$

with the universal constant  $\tilde{C} \geq 1$  specified in the proof.

Let  $(u_2, \underline{u}_2) \in \mathcal{I}$ . For  $(u, \underline{u}) \in \mathcal{I}$  with  $u \leq u_2$ , there holds for any constant  $p \leq 0$  the energy inequality

$$\begin{aligned}|u|^{-2p+\gamma_0} &\left(E[\varphi](\mathcal{H}_u^u) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u)\right) \\ &\lesssim \sup_{(u_1, \underline{u}_1) \in \mathcal{I}, u_1 \leq u_2} \left\{|u_1|^{-2p+\gamma_0} \left(\|r\mathcal{F}^b\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2 + \|r\mathcal{F}^b\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2\right) \right. \\ &\quad \left. + |u_1|^{-2p+\gamma_0} E[\varphi](\Sigma_0^{u_1, \underline{u}_1}) + |u_1|^{1-2p+\gamma_0} M^{-1} \|r^{\frac{1}{2}} \mathcal{F}^\sharp\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})}^2\right\},\end{aligned}$$

where  $\square_{\mathbf{g}}\varphi - q\varphi = \mathcal{F}^b + \mathcal{F}^\sharp$ .

*Proof.* For convenience, we denote  $\mathcal{F} = \square_{\mathbf{g}}\varphi - q\varphi$ . We first show that, in  $\mathcal{D}_u^u$  with  $(u, \underline{u}) \in \mathcal{I}$  and  $u \leq u_2$ , there holds

$$\begin{aligned}E[\varphi](\mathcal{H}_u^u) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^u) \\ \lesssim E[\varphi](\Sigma_0^{u, \underline{u}}) + \int_{\mathcal{D}_u^u} \left|(\mathcal{F} + \partial^\alpha H_\alpha{}^\gamma \partial_\gamma \varphi) \partial_t \varphi - \frac{1}{2} \partial_t H_\alpha{}^\gamma \partial_\gamma \varphi \partial^\alpha \varphi\right|.\end{aligned} \quad (6.14)$$

If  $X = \partial_t$  in (6.10), the last two terms vanish due to  $\partial^\alpha \partial_t^\beta = 0$ . By the divergence theorem, we have

$$\int_{\mathcal{D}_u^u} \partial^\alpha (\tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\beta) = \int_{\Sigma_0^{u, \underline{u}}} \tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\beta \partial_t^\alpha - \left( \int_{\mathcal{H}_u^u} \tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\beta N^\alpha + \int_{\underline{\mathcal{H}}_{\underline{u}}^u} \tilde{\mathcal{Q}}_{\alpha\beta} \partial_t^\beta \underline{N}^\alpha \right).$$

Recall that the area element on  $\mathcal{D}_u^u$  is  $\frac{1}{2}\mathbf{b}dud\underline{u}d\omega$ . Recall also that  $d\mu_{\mathcal{H}} = \frac{1}{2}\mathbf{b}dud\underline{u}d\omega$  and  $d\mu_{\underline{\mathcal{H}}} = \frac{1}{2}\mathbf{b}d\underline{u}d\omega$  are the standard area elements on  $\mathcal{H}_u$  and  $\underline{\mathcal{H}}_{\underline{u}}$ . We may drop the standard area elements for the integral on the corresponding hypersurfaces or domains for convenience.

Combining the above identity with Lemma 6.2 and Lemma 6.3 we obtain

$$\begin{aligned}&\int_{\underline{\mathcal{H}}_{\underline{u}}^u} \frac{1}{2} \{(\underline{L}\varphi)^2 + (L\varphi)^2(h - H^{LL}) + (1+h)(|\nabla\varphi|^2 + q\varphi^2) + (H + hH)\partial\varphi\partial\varphi\} (1+h)^{-1} d\mu_{\underline{\mathcal{H}}} \\ &+ \int_{\mathcal{H}_u^u} \frac{1}{2} \{(L\varphi)^2 + (\underline{L}\varphi)^2(h - H^{LL}) + (1+h)(|\nabla\varphi|^2 + q\varphi^2) + (H + hH)\partial\varphi\bar{\partial}\varphi\} (1+h)^{-1} d\mu_{\mathcal{H}} \\ &= \frac{1}{2} \int_{\Sigma_0^{u, \underline{u}}} (-\mathbf{g}^{00}(\partial_t\varphi)^2 + \mathbf{g}^{ij}\partial_i\varphi\partial_j\varphi + q\varphi^2) dx \\ &\quad - \int_{\mathcal{D}_u^u} \left\{(\mathcal{F} + \partial^\alpha H_\alpha{}^\gamma \partial_\gamma \varphi) \partial_\beta \varphi - \frac{1}{2} \partial_\beta H_\alpha{}^\gamma \partial_\gamma \varphi \partial^\alpha \varphi\right\} \partial_t^\beta.\end{aligned}$$

Note that by (6.6) and the smallness of  $|h|$  we have

$$\begin{aligned} \int_{\mathcal{H}_{\underline{u}}^u} |(H + hH)\partial\varphi\partial\varphi| (1+h)^{-1} d\mu_{\mathcal{H}} &\lesssim \int_{\mathcal{H}_{\underline{u}}^u} |H| (|\partial\varphi\bar{\partial}\varphi| + |\partial\varphi|^2) d\mu_{\mathcal{H}} \\ &\lesssim \Delta_0^{\frac{1}{2}} \|\partial\varphi\|_{L^2(\mathcal{H}_{\underline{u}}^u)} \left( M^{-\frac{1}{2}} |u|^{-\frac{\gamma_0}{2}} \left\| M^{\frac{1}{2}} r^{-\frac{1}{2}} \bar{\partial}\varphi \right\|_{L^2(\mathcal{H}_{\underline{u}}^u)} + |u|^{-\frac{\gamma_0}{2} - \frac{1}{2}} \|\partial\varphi\|_{L^2(\mathcal{H}_{\underline{u}}^u)} \right) \\ &\lesssim \Delta_0^{\frac{1}{2}} \left( M^{-\frac{1}{2}} + |u|^{-\frac{1}{2}} \right) |u|^{-\frac{\gamma_0}{2}} E[\varphi](\mathcal{H}_{\underline{u}}^u). \end{aligned}$$

Similarly,

$$\int_{\mathcal{H}_{\underline{u}}^u} |(H + hH)\partial\varphi\bar{\partial}\varphi| (1+h)^{-1} d\mu_{\mathcal{H}} \lesssim \Delta_0^{\frac{1}{2}} \left( M^{-\frac{1}{2}} + |u|^{-\frac{1}{2}} \right) |u|^{-\frac{\gamma_0}{2}} E[\varphi](\mathcal{H}_{\underline{u}}^u).$$

Since  $\Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \lesssim M^{\frac{1}{2}}$ , the coefficient can be sufficiently small. Due to (A1), this pair of error terms will be absorbed by the leading positive terms on the lefthand side.

With the help of (6.1), we can derive that

$$\int_{\Sigma_0^{u,\underline{u}}} (-\mathbf{g}^{00}(\partial_t\varphi)^2 + \mathbf{g}^{ij}\partial_i\varphi\partial_j\varphi + q\varphi^2) dx \lesssim E[\varphi](\Sigma_0^{u,\underline{u}}).$$

Therefore

$$\begin{aligned} &\int_{\mathcal{H}_{\underline{u}}^u} \left( (L\varphi)^2 + |\nabla\varphi|^2 + q\varphi^2 + \frac{M}{r}(L\varphi)^2 \right) + \int_{\mathcal{H}_{\underline{u}}^u} \left( (L\varphi)^2 + \frac{M}{r}(L\varphi)^2 + |\nabla\varphi|^2 + q\varphi^2 \right) \\ &\leq \mathcal{C} \left( E[\varphi](\Sigma_0^{u,\underline{u}}) + \int_{\mathcal{D}_{\underline{u}}^u} \left| (\mathcal{F} + \partial^\alpha H_\alpha^\gamma \partial_\gamma\varphi) \partial_t\varphi - \frac{1}{2} \partial_t H_\alpha^\gamma \partial_\gamma\varphi \partial^\alpha\varphi \right| \right). \end{aligned} \quad (6.15)$$

Thus (6.14) is proved.

We remark that

$$\mathrm{Tr}[\varphi] = \partial^\alpha H_\alpha^\gamma \partial_\gamma\varphi \partial_t\varphi - \frac{1}{2} \partial_t H_\alpha^\gamma \partial_\gamma\varphi \partial^\alpha\varphi = \frac{1}{4} \underline{L} H \underline{L} (L\varphi)^2 + \overline{\mathrm{Tr}}(\partial H, \partial\varphi, \partial\varphi) \quad (6.16)$$

where the trilinear term  $\overline{\mathrm{Tr}}$  means that, in the product of the three terms, at least one of the  $\partial$  derivatives is  $\bar{\partial}$ .<sup>21</sup>

Symbolically,  $\mathrm{Tr}[\varphi] = \partial H \cdot \partial\varphi \cdot \partial\varphi$ . Let  $\alpha = -p + \frac{\gamma_0}{2}$  with the constant  $p \leq 0$ . For any  $(u_2, \underline{u}_2) \in \mathcal{I}$  we calculate via (6.7) and (6.8) to obtain

$$\begin{aligned} &|u_2|^{2\alpha} \int_{\mathcal{D}_{\underline{u}_2}^{u_2}} |\partial H \cdot \partial\varphi \cdot \partial\varphi| \\ &\lesssim |u_2|^{2\alpha} \left( \left\| r^{\frac{1}{2}} \partial H \right\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{\underline{u}_2}^{u_2})} \left\| r \partial\varphi \right\|_{L_{\underline{u}}^\infty L_u^2 L_\omega^2(\mathcal{D}_{\underline{u}_2}^{u_2})} + \left\| r^{\frac{1}{2}} \partial H \right\|_{L_u^2 L^\infty(\mathcal{D}_{\underline{u}_2}^{u_2})} \left\| r \bar{\partial}\varphi \right\|_{L_u^\infty L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{\underline{u}_2}^{u_2})} \right) \\ &\quad \times \left\| r^{-\frac{1}{2}} \partial\varphi \right\|_{L^2(\mathcal{D}_{\underline{u}_2}^{u_2})} \\ &\lesssim |u_2|^\alpha \left( \left\| r^{\frac{1}{2}} \partial H \right\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{\underline{u}_2}^{u_2})} + \left\| r^{\frac{1}{2}} \partial H \right\|_{L_u^2 L^\infty(\mathcal{D}_{\underline{u}_2}^{u_2})} \right) M^{-\frac{1}{2}} \left( \int_{-\underline{u}_2}^{u_2} E[\varphi](\mathcal{H}_{\underline{u}}^{u_2}) d\underline{u} \right)^{\frac{1}{2}} \\ &\quad \times \left( \sup_{-\underline{u}_2 \leq \underline{u} \leq u_2} |u_2|^\alpha E[\varphi]^{\frac{1}{2}}(\mathcal{H}_{\underline{u}}^{u_2}) + \sup_{-\underline{u}_2 \leq \underline{u} \leq u_2} |u|^\alpha E[\varphi]^{\frac{1}{2}}(\mathcal{H}_{\underline{u}}^{u_2}) \right) \\ &\lesssim M^{-1} \Delta_0^{\frac{1}{2}} |u_2|^{\frac{1}{2} - \frac{\gamma_0}{2}} \left( \sup_{-\underline{u}_2 \leq \underline{u} \leq u_2} |u|^{2\alpha} E[\varphi](\mathcal{H}_{\underline{u}}^{u_2}) + \sup_{-\underline{u}_2 \leq \underline{u} \leq u_2} E[\varphi](\mathcal{H}_{\underline{u}}^{u_2}) |u_2|^{2\alpha} \right). \end{aligned}$$

<sup>21</sup>In Section 7, we will take advantage of this structure when the wave coordinates condition is available.

To treat the integral of  $\mathcal{F} \cdot \partial_t \varphi$  in (6.14), we repeat the derivation in (5.6). Thus with  $\mathcal{F} = \mathcal{F}^b + \mathcal{F}^\sharp$ ,

$$\begin{aligned} & |u_2|^{-2p+\gamma_0} \int_{\mathcal{D}_{u_2}^{u_2}} (|\mathcal{F} \cdot \partial_t \varphi| + |\partial H \cdot \partial \varphi \cdot \partial \varphi|) \\ & \leq C(\epsilon_1) |u_2|^{-2p+\gamma_0} \left( \left\| r \mathcal{F}^b \right\|_{L_u^1 L_u^2 L_\omega^2(\mathcal{D}_{u_2}^{u_2})}^2 + \left\| r \mathcal{F}^\sharp \right\|_{L_u^1 L_u^2 L_\omega^2(\mathcal{D}_{u_2}^{u_2})}^2 + M^{-1} |u_2| \left\| r^{\frac{1}{2}} \mathcal{F}^\sharp \right\|_{L^2(\mathcal{D}_{u_2}^{u_2})}^2 \right) \\ & \quad + \left( \epsilon_1 + \tilde{C} M^{-1} \Delta_0^{\frac{1}{2}} |u_2|^{\frac{1}{2} - \frac{\gamma_0}{2}} \right) \\ & \quad \times \left( \sup_{-u_2 \leq u \leq u_2} |u|^{-2p+\gamma_0} E[\varphi](\mathcal{H}_u^{u_2}) + \sup_{-u_2 \leq u \leq u_2} |u_2|^{-2p+\gamma_0} E[\varphi](\mathcal{H}_{\underline{u}}^{u_2}) \right). \end{aligned}$$

We fix the constant in (A2) by  $\tilde{C} = C \cdot \tilde{C}$ . By using (A2) and choosing  $\epsilon_1$  sufficiently small, the last term can be absorbed. Substituting the above estimate into (6.15) completes the proof of Proposition 6.4.  $\square$

Next we establish the  $r$ -weighted energy inequality by first giving the following result.

**Lemma 6.5.** *Let  $-\underline{u}_* \leq -\underline{u}_1 \leq u_1 \leq u_0$ . Under the assumptions (6.3), (6.4) and (A1), for any constant  $p \leq 0$  there holds the estimate*

$$\begin{aligned} & \int_{\mathcal{D}_{u_1}^{u_1}} \left| (\Box_{\mathbf{g}} \varphi - q\varphi)(X\varphi + \varphi) + \frac{1}{2} (r^{-2}(L(r\varphi))^2 + |\nabla \varphi|^2) - \left( \partial^\alpha (X) \mathcal{P}_\alpha + \frac{1}{2} (Xq + q)\varphi^2 \right) \right| \\ & \lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \left\{ |u_1|^{1 - \frac{3\gamma_0}{2} + 2p} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0 - 2p} E[\varphi](\mathcal{H}_u^{u_1}) + |u_1|^{\gamma_0 - 2p} \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_{\underline{u}}^{u_1}) \right) \right. \\ & \quad \left. + \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} \mathcal{W}_1[\varphi](\mathcal{H}_u^{u_1}) du \right\}, \end{aligned} \quad (6.17)$$

where the energy current is defined by

$${}^{(X)}\mathcal{P}_\alpha = \tilde{\mathcal{Q}}_{\alpha\beta} X^\beta + \frac{1}{2} \mathbf{m}_{\alpha\alpha'} \mathbf{g}^{\alpha'\gamma} \partial_\gamma (\varphi^2) + Y_\alpha \quad (6.18)$$

with

$$X = r(L - H \underline{L} \underline{L}) \quad \text{and} \quad Y = \frac{1}{2} r^{-1} \varphi^2 L.$$

The proof of (6.17) relies on an important cancelation thanks to the construction of  $\mathcal{Q}_{\alpha\beta}$  and the choice of the multiplier  $X$ . The cancelation will be seen in the following proof. For convenience, we denote by  $\mathcal{P}$  the current  ${}^{(X)}\mathcal{P}$  from now on.

*Proof.* The proof is based on the following identity on  $\mathcal{D}_{u_1}^{u_1}$ :

$$\begin{aligned} \partial^\alpha \mathcal{P}_\alpha + \frac{1}{2} (Xq + q)\varphi^2 &= (\Box_{\mathbf{g}} \varphi - q\varphi)(X\varphi + \varphi) + \frac{1}{2} (r^{-2}(L(r\varphi))^2 + |\nabla \varphi|^2) \\ &\quad + I + II + III, \end{aligned} \quad (6.19)$$

where the error terms  $I$ ,  $II$  and  $III$  are

$$\begin{aligned} I &= \partial^\alpha H_\alpha{}^\gamma \partial_\gamma \varphi (X\varphi + \varphi) - \frac{1}{2} X H_\alpha{}^\gamma \partial_\gamma \varphi \cdot \partial^\alpha \varphi + \frac{1}{2} r L H \underline{L} \underline{L} (\underline{L}\varphi)^2, \\ II &= \frac{1}{2} \underline{L} (r H \underline{L} \underline{L}) |\nabla \varphi|^2 + H \partial \varphi \bar{\partial} \varphi, \\ III &= H_\alpha{}^\gamma \partial_\gamma \varphi \partial_\beta \varphi \partial^\alpha X^\beta + (H + r \partial H) (H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + q\varphi^2). \end{aligned} \quad (6.20)$$

In order to cancel the undesired term  $\int_{\{u \leq u_0\}} r |L H \underline{L} \underline{L} (\underline{L}\varphi)^2| dx dt$ , in the error integral, a further cancelation between the last two terms in  $I$  will be specified shortly.



To show (6.19), by using (6.10) we first have

$$\begin{aligned} \partial^\alpha \mathcal{P}_\alpha = X^\beta \left\{ (\Box_{\mathbf{g}} \varphi - q\varphi + \partial^\alpha H_\alpha^\gamma \partial_\gamma \varphi) \partial_\beta \varphi - \frac{1}{2} \partial_\beta q \varphi^2 - \frac{1}{2} \partial_\beta H_\alpha^\gamma \partial_\gamma \varphi \partial^\alpha \varphi \right\} \\ + H_\alpha^\gamma \partial_\gamma \varphi \partial_\beta \varphi \partial^\alpha X^\beta + \mathcal{E} \end{aligned} \quad (6.21)$$

where

$$\mathcal{E} = \mathcal{Q}_{\alpha\beta} \partial^\alpha X^\beta + \partial^\alpha \left( \frac{1}{2} \mathbf{m}_{\alpha\alpha'} \mathbf{g}^{\alpha'\gamma} \partial_\gamma (\varphi^2) + Y_\alpha \right).$$

It is straightforward to check that

$$\begin{aligned} {}^{(X)}\pi_{AB} &= \delta_{AB}(1 + H^{\underline{LL}}), & {}^{(X)}\pi_{L\underline{L}} &= -1 - H^{\underline{LL}} + r\underline{L}H^{\underline{LL}}, \\ {}^{(X)}\pi_{\underline{L}\underline{L}} &= 2, & {}^{(X)}\pi_{LL} &= 2(H^{\underline{LL}} + rLH^{\underline{LL}}). \end{aligned}$$

In view of the definition of  $\mathcal{Q}_{\alpha\beta}$  it is easy to check that

$$\begin{aligned} \mathcal{Q}_{LL} &= (L\varphi)^2, & \mathcal{Q}_{\underline{L}\underline{L}} &= (\underline{L}\varphi)^2, \\ \mathcal{Q}_{AB} &= \nabla_A \varphi \nabla_B \varphi - \frac{1}{2} \mathbf{m}_{AB} (-L\varphi \underline{L}\varphi + |\nabla \varphi|^2 + q\varphi^2 + H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi), \\ \mathcal{Q}_{\underline{L}\underline{L}} &= |\nabla \varphi|^2 + q\varphi^2 + H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi. \end{aligned}$$

By combining the lists of  $\mathcal{Q}_{\alpha\beta}$  and  ${}^{(X)}\pi^{\alpha\beta}$ , we derive that

$$\begin{aligned} \mathcal{Q}_{LL} {}^{(X)}\pi^{LL} &= \frac{1}{2} (L\varphi)^2, \\ \mathcal{Q}_{\underline{L}\underline{L}} {}^{(X)}\pi^{\underline{L}\underline{L}} &= \frac{1}{2} (\underline{L}\varphi)^2 (H^{\underline{LL}} + rLH^{\underline{LL}}), \\ \mathcal{Q}_{L\underline{L}} {}^{(X)}\pi^{L\underline{L}} &= \frac{1}{4} (-1 - H^{\underline{LL}} + r\underline{L}H^{\underline{LL}}) (|\nabla \varphi|^2 + q\varphi^2 + H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi), \\ \mathcal{Q}_{AB} {}^{(X)}\pi^{AB} &= (1 + H^{\underline{LL}}) (\underline{L}\varphi L\varphi - q\varphi^2 - H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi). \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{2} \partial^\alpha \left( \mathbf{m}_{\alpha\alpha'} \mathbf{g}^{\alpha'\gamma} \partial_\gamma (\varphi^2) \right) &= \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2) + \mathbf{g}^{\alpha'\gamma} \partial_{\alpha'} \varphi \partial_\gamma \varphi + \Box_{\mathbf{g}} \varphi \cdot \varphi \\ &= \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2) + \partial^\alpha \varphi \cdot \partial_\alpha \varphi + H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + \Box_{\mathbf{g}} \varphi \cdot \varphi. \end{aligned}$$

Since  $\partial^\alpha L_\alpha = 2/r$  and  $\partial^\alpha \underline{L}_\alpha = -2/r$ , we have

$$\begin{aligned} \partial^\alpha Y_\alpha &= \frac{1}{2} \partial^\alpha (r^{-1} \varphi^2 L_\alpha) = \frac{1}{2} \{ \varphi^2 L(r^{-1}) + r^{-1} L(\varphi^2) + r^{-1} \varphi^2 \partial^\alpha L_\alpha \} \\ &= \frac{1}{2} (r^{-1} L(\varphi^2) + r^{-2} \varphi^2). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{E} &= \Box_{\mathbf{g}} \varphi \cdot \varphi + \frac{1}{2} (r^{-2} (L(r\varphi))^2 + |\nabla \varphi|^2 + H^{\underline{LL}} (\underline{L}\varphi)^2) \\ &\quad + \left( -\frac{1}{2} - \frac{3}{2} H^{\underline{LL}} + \frac{1}{2} r \underline{L} H^{\underline{LL}} \right) H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + \frac{1}{2} (\underline{L}\varphi)^2 r L H^{\underline{LL}} + \frac{1}{2} \underline{L} (r H^{\underline{LL}}) |\nabla \varphi|^2 \\ &\quad + H^{\underline{LL}} \underline{L}\varphi L\varphi + q\varphi^2 \left( -\frac{3}{2} (1 + H^{\underline{LL}}) + \frac{1}{2} r \underline{L} H^{\underline{LL}} \right) + \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2). \end{aligned}$$

Note that

$$-H^{\underline{LL}} (\underline{L}\varphi)^2 + H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi = H \bar{\partial} \varphi \partial \varphi.$$

Hence we can conclude that

$$\begin{aligned} \mathcal{E} &= \Box_{\mathbf{g}} \varphi \cdot \varphi + \frac{1}{2} \partial_\alpha H^{\alpha\gamma} \partial_\gamma (\varphi^2) + \frac{1}{2} (r^{-2} (L(r\varphi))^2 + |\nabla \varphi|^2) + (H + r\partial H) (H^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + q\varphi^2) \\ &\quad + \frac{1}{2} r L H^{\underline{LL}} (\underline{L}\varphi)^2 + \frac{1}{2} \underline{L} (r H^{\underline{LL}}) |\nabla \varphi|^2 + H \partial \varphi \cdot \bar{\partial} \varphi - \frac{3}{2} q\varphi^2. \end{aligned}$$

Substituting the above formula into (6.21) yields (6.19).

Next we prove (6.17) by controlling the error terms in the identity. We claim

$$\begin{aligned} I &= rLH\partial\varphi \cdot \bar{\partial}\varphi + \partial H \cdot \partial\varphi \cdot L(r\varphi) + rH \cdot \partial H \cdot (\partial\varphi)^2, \\ II &= r\partial H |\bar{\partial}\varphi|^2 + H\partial\varphi \bar{\partial}\varphi. \end{aligned} \quad (6.22)$$

Indeed, the last term of  $I$  cancels completely the bad component of the second term, which can be seen as follows

$$\begin{aligned} & -\frac{1}{2}XH_\alpha^\gamma \partial_\gamma \varphi \partial^\alpha \varphi + \frac{1}{2}rLH\bar{L}\bar{L}(\underline{L}\varphi)^2 \\ &= \frac{1}{2}r(-LH^{\alpha\gamma} \partial_\gamma \varphi \partial_\alpha \varphi + LH\bar{L}\bar{L}(\underline{L}\varphi)^2) + \frac{1}{2}rH\bar{L}\bar{L}H_\alpha^\gamma \partial_\gamma \varphi \partial^\alpha \varphi \\ &= \frac{1}{2}rLH\partial\varphi \bar{\partial}\varphi + \frac{1}{2}rH\partial H(\partial\varphi)^2. \end{aligned}$$

By direct calculation, we can obtain the symbolic formula for  $I$ . The formula of  $II$  is a simple recast.

It remains to consider the error term  $III$  in (6.19). For the first term, note that

$$\partial^\alpha X^\beta = \partial^\alpha r(L - H\bar{L}\bar{L})^\beta + r\left(\partial^\alpha L^\beta - \partial^\alpha H\bar{L}\bar{L}^\beta - H\bar{L}\bar{L}\partial^\alpha \underline{L}^\beta\right)$$

and the nontrivial components of  $\partial^\alpha L^\beta$  and  $\partial^\alpha \underline{L}^\beta$  are  $\partial_A L_B = r^{-1}\delta_{AB}$  and  $\partial_A \underline{L}_B = -r^{-1}\delta_{AB}$ . By a direct substitution, we can obtain

$$|H_\alpha^\gamma \partial_\gamma \varphi \partial_\beta \varphi \partial^\alpha X^\beta| \lesssim |H\partial\varphi \cdot \bar{\partial}\varphi| + (r|\partial H| + |H|) \cdot |H|(\partial\varphi)^2$$

and then

$$|III| \lesssim |H\partial\varphi \cdot \bar{\partial}\varphi| + (r|\partial H| + |H|)(|H|(\partial\varphi)^2 + q\varphi^2). \quad (6.23)$$

Next we prove the following error estimate

$$\begin{aligned} & \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} (|I| + |II| + |III|) \\ & \lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{-\frac{3\gamma_0}{2}+1} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0} + \sup_{-u_1 \leq \underline{u} \leq \underline{u}_1} E[\varphi](\mathcal{H}_{\underline{u}}^{u_1}) |u_1|^{\gamma_0} \right) \\ & \quad + \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} \mathcal{W}_1[\varphi](\mathcal{H}_u^{\underline{u}_1}) du. \end{aligned} \quad (6.24)$$

Let us consider the error terms of  $I$  in (6.22). By using (6.6) and (6.8) we have

$$\begin{aligned} & \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} r|H||\partial H|(\partial\varphi)^2 \\ & \lesssim \|r\partial\varphi\|_{L_{\underline{u}}^\infty L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \left\| r^{-\frac{1}{2}} \partial\varphi \right\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \left\| r^{\frac{1}{2}} \partial H \right\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \|rH\|_{L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ & \lesssim M^{-1} \Delta_0 |u_1|^{-2\gamma_0+1} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E^{\frac{1}{2}}[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\frac{\gamma_0}{2}} \right) \left( \sup_{-u_1 \leq \underline{u} \leq \underline{u}_1} E^{\frac{1}{2}}[\varphi](\mathcal{H}_{\underline{u}}^{u_1}) |u_1|^{\frac{\gamma_0}{2}} \right). \end{aligned}$$

Similarly, by using (6.6) and (6.7) we can obtain

$$\begin{aligned} & \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} r(|LH| + |H||\partial H|) |\partial\varphi \cdot \bar{\partial}\varphi| \lesssim \int_{-\underline{u}_1}^{u_1} \|\bar{\partial}\varphi\|_{L^2(\mathcal{H}_u^{\underline{u}_1})} \left\| r^{-\frac{1}{2}} \partial\varphi \right\|_{L^2(\mathcal{H}_u^{\underline{u}_1})} |u|^{-\frac{\gamma_0}{2}} \Delta_0^{\frac{1}{2}} du \\ & \lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{-\frac{3\gamma_0}{2}+1} \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0}. \end{aligned} \quad (6.25)$$

By using (6.6), we can derive that

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} |L(r\varphi) \cdot \partial\varphi \cdot \partial H| &\lesssim \Delta_0^{\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} \left\| r^{-\frac{1}{2}} L(r\varphi) \right\|_{L^2(\mathcal{H}_u^{u_1})} \left\| r^{-\frac{1}{2}} \partial\varphi \right\|_{L^2(\mathcal{H}_u^{u_1})} du \\ &\lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} (\mathcal{W}_1[\varphi](\mathcal{H}_u^{u_1}) + E[\varphi](\mathcal{H}_u^{u_1})) du. \end{aligned} \quad (6.26)$$

Thus the error terms in  $I$  are all treated.

It again follows from (6.6) that

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} r |\bar{\partial}\varphi \cdot \bar{\partial}\varphi \cdot \partial H| &\lesssim \Delta_0^{\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} \|\bar{\partial}\varphi\|_{L^2(\mathcal{H}_u^{u_1})}^2 du \\ &\lesssim \Delta_0^{\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} E[\varphi](\mathcal{H}_u^{u_1}) du \\ &\lesssim \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{3\gamma_0}{2}+\frac{1}{2}} \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0} E[\varphi](\mathcal{H}_u^{u_1}), \\ \int_{\mathcal{D}_{u_1}^{u_1}} |H| |\partial\varphi| |\bar{\partial}\varphi| &\lesssim \Delta_0^{\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0}{2}} \|\bar{\partial}\varphi\|_{L^2(\mathcal{H}_u^{u_1})} \left\| r^{-\frac{1}{2}} \partial\varphi \right\|_{L^2(\mathcal{H}_u^{u_1})} du \\ &\lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{-\frac{3\gamma_0}{2}+1} \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0} E[\varphi](\mathcal{H}_u^{u_1}). \end{aligned}$$

Thus the terms of  $II$  are all treated.

Next we estimate  $III$  in view of (6.23). The first term on the right of (6.23) has been estimated in  $II$ . Note that by using (6.6),

$$r|\partial H| + \frac{r}{|u|}|H| \lesssim \Delta_0^{\frac{1}{2}} |u|^{-\frac{\gamma_0+1}{2}}, \quad r(r|\partial H||H| + |H|^2) \lesssim \Delta_0 |u|^{-\gamma_0}$$

which, in view of  $\gamma_0 > 1$ , imply

$$\begin{aligned} &\int_{\mathcal{D}_{u_1}^{u_1}} (r|\partial H| + |H|) (|H|(\partial\varphi)^2 + q\varphi^2) \\ &\lesssim \Delta_0 \int_{-\underline{u}_1}^{u_1} |u|^{-\gamma_0} \left\| r^{-\frac{1}{2}} \partial\varphi \right\|_{L^2(\mathcal{H}_u^{u_1})}^2 du + \Delta_0^{\frac{1}{2}} \int_{\mathcal{D}_{u_1}^{u_1}} |u|^{-\frac{\gamma_0+1}{2}} q\varphi^2 \\ &\lesssim \left( \Delta_0 M^{-1} + \Delta_0^{\frac{1}{2}} \right) |u_1|^{\frac{1}{2}-\frac{3}{2}\gamma_0} \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u|^{\gamma_0}. \end{aligned}$$

By combining the estimates for  $I$ ,  $II$  and  $III$ , (6.24) is proved since  $\Delta_0 M^{-1} \leq (\Delta_0 M^{-1})^{\frac{1}{2}}$ . Thus we can conclude the inequality in (6.17).  $\square$

Next, we give the  $r$ -weighted energy estimate.

**Proposition 6.6.** *Under the assumptions (6.3), (6.4) and (A1), with  $-\underline{u}_* \leq -\underline{u}_1 \leq u_1 \leq u_0$ , there holds for  $\square_{\mathbf{g}}\varphi - q\varphi = \mathcal{F}$  the following weighted energy estimate*

$$\begin{aligned} &\sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0-1-2p} \left( \mathcal{W}_1[\varphi](\mathcal{D}_u^{u_1}) + \mathcal{W}_1[\varphi](\mathcal{H}_u^{u_1}) + \mathcal{W}_1[\varphi](\mathcal{H}_{\underline{u}_1}^u) \right) \\ &\lesssim \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0-1-2p} \left\| r^{\frac{3}{2}} \mathcal{F} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})}^2 + \sup_{-\underline{u}_1 \leq -\underline{u} \leq u \leq u_1} \left\{ |u|^{\gamma_0-2p} (E[\varphi](\mathcal{H}_u^u) + E[\varphi](\mathcal{H}_{\underline{u}}^u)) \right\} \\ &\quad + \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0-1-2p} \left( \left\| r^{\frac{1}{2}} |\partial\varphi| + r^{-\frac{1}{2}} |\varphi| + q_0^{\frac{1}{2}} r^{\frac{1}{2}} |\varphi| \right\|_{L^2(\Sigma_0^{u, \underline{u}_1})}^2 + \int_{S_{u, -u}} r \varphi^2 d\omega \right), \end{aligned}$$

where  $p \leq 0$  is an any fixed constant.

*Proof.* With the help of Lemma 6.5, we will apply the divergence theorem to  $\partial^\alpha \mathcal{P}_\alpha$  in  $\mathcal{D}_{u_1}^u$ . We first confirm that the boundary terms give the desired weighted energy.

Let us first compute  $(1+h)\mathcal{P}_\alpha \mathcal{N}^\alpha$ . Recall  $X = r(L - H^{LL}\underline{L})$  and  $\mathcal{N}$  from (2.4). We have

$$\begin{aligned}
& (1+h)\tilde{\mathcal{Q}}_{\alpha\beta}X^\beta\mathcal{N}^\alpha \\
&= (L+h\underline{L})\varphi X\varphi - \frac{1}{2}r(L+h\underline{L}, L-H^{LL}\underline{L})(-\underline{L}\varphi L\varphi + |\nabla\varphi|^2 + H^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi + q\varphi^2) \\
&\quad + rH_\alpha{}^\gamma\partial_\gamma\varphi\partial_\beta\varphi(L+h\underline{L})^\alpha(L-H^{LL}\underline{L})^\beta \\
&= r\left\{(L\varphi - H^{LL}\underline{L}\varphi)(L\varphi + h\underline{L}\varphi - 2(H^{L\gamma} + hH^{L\gamma})\partial_\gamma\varphi) \right. \\
&\quad \left. + (h - H^{LL})(-\underline{L}\varphi L\varphi + |\nabla\varphi|^2 + q\varphi^2 + H^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi)\right\} \\
&= r\left\{(L\varphi - H^{LL}\underline{L}\varphi)(L\varphi + (h - 2(H^{LL} + hH^{LL}))\underline{L}\varphi - 2(H + hH)\bar{\partial}\varphi) \right. \\
&\quad \left. + (h - H^{LL})(-\underline{L}\varphi L\varphi + H^{LL}(\underline{L}\varphi)^2) + (h - H^{LL})(|\nabla\varphi|^2 + q\varphi^2 + H\partial\varphi\bar{\partial}\varphi)\right\}.
\end{aligned}$$

We can cancel the first term in the last line by noting that

$$L\varphi + (h - 2(H^{LL} + hH^{LL}))\underline{L}\varphi = (h - H^{LL})\underline{L}\varphi + L\varphi - H^{LL}\underline{L}\varphi - 2hH^{LL}\underline{L}\varphi,$$

where the first term on the right gives the cancelation after substitution. Hence

$$\begin{aligned}
& (1+h)\tilde{\mathcal{Q}}_{\alpha\beta}X^\beta\mathcal{N}^\alpha \\
&= r\left\{(L\varphi - H^{LL}\underline{L}\varphi)^2 + (h - H^{LL})(|\nabla\varphi|^2 + q\varphi^2) + (h - H^{LL})H\partial\varphi\bar{\partial}\varphi \right. \\
&\quad \left. + (L\varphi - H^{LL}\underline{L}\varphi)(-2hH^{LL}\underline{L}\varphi + (H + hH)\bar{\partial}\varphi)\right\}.
\end{aligned}$$

We remark that only the first term on the righthand side is involved with the further cancelations with the following two identities

$$\begin{aligned}
\frac{1}{2}(1+h)\mathcal{N}^\alpha\mathbf{m}_{\alpha\alpha'}\mathbf{g}^{\alpha'\gamma}\partial_\gamma(\varphi^2) &= \frac{1}{2}\mathbf{m}_{\alpha\alpha'}(H^{\alpha'\gamma} + \mathbf{m}^{\alpha'\gamma})\partial_\gamma(\varphi^2)(L^\alpha + h\underline{L}^\alpha) \\
&= \varphi\left\{(H_L{}^\gamma + hH_{\underline{L}}{}^\gamma)\partial_\gamma\varphi + (L\varphi + h\underline{L}\varphi)\right\},
\end{aligned}$$

and

$$(1+h)\mathcal{N}^\alpha Y_\alpha = -r^{-1}\varphi^2 h.$$

In view of the definition of  $\mathcal{P}^\alpha$  in (6.18) and the above three identities, we can derive that

$$\begin{aligned}
& r^2(1+h)\mathcal{N}^\alpha\mathcal{P}_\alpha \\
&= r\left\{(L(r\varphi) - rH^{LL}\underline{L}\varphi)^2 + r^2(h - H^{LL})(|\nabla\varphi|^2 + q\varphi^2) - r^{-1}\frac{1}{2}L(r^2\varphi^2) \right. \\
&\quad \left. + r[-2hH^{LL}\varphi\underline{L}\varphi - 2(H^{L\gamma} + hH^{L\gamma})\varphi\bar{\partial}\varphi + r^{-1}h\varphi\underline{L}(r\varphi)] \right. \\
&\quad \left. + r^2[(L\varphi - H^{LL}\underline{L}\varphi)(-2hH^{LL}\underline{L}\varphi + (H + hH)\bar{\partial}\varphi) + (h - H^{LL})H\partial\varphi\bar{\partial}\varphi]\right\}.
\end{aligned} \tag{6.27}$$

Thus the calculation of the energy density on  $\mathcal{H}_u^u$  is completed.

Next we consider the weighted energy on  $\underline{\mathcal{H}}_u^u$ . By calculation we first have

$$\begin{aligned}
& (1+h)\mathcal{Q}_{\alpha\beta}X^\beta\underline{\mathcal{N}}^\alpha \\
&= r \left\{ (\underline{L}\varphi + hL\varphi) (L\varphi - H^{\underline{L}\underline{L}}\underline{L}\varphi) - \frac{1}{2} (L - H^{\underline{L}\underline{L}}\underline{L}, \underline{L} + hL) ((H^{\rho\sigma} + \mathbf{m}^{\rho\sigma})\partial_\rho\varphi\partial_\sigma\varphi + q\varphi^2) \right\} \\
&= r \left\{ \underline{L}\varphi L\varphi(1 - hH^{\underline{L}\underline{L}}) + h(L\varphi)^2 - H^{\underline{L}\underline{L}}(\underline{L}\varphi)^2 \right. \\
&\quad \left. - \frac{1}{2}(-2 + 2hH^{\underline{L}\underline{L}}) (H^{\underline{L}\underline{L}}(L\varphi)^2 + H^{\underline{L}\underline{L}}(\underline{L}\varphi)^2 + H\underline{\partial}\varphi\bar{\partial}\varphi - \underline{L}\varphi L\varphi + |\nabla\varphi|^2 + q\varphi^2) \right\} \\
&= r \left\{ (L\varphi)^2 (h + (1 - H^{\underline{L}\underline{L}}h)H^{\underline{L}\underline{L}}) - (\underline{L}\varphi)^2 hH^{\underline{L}\underline{L}} + (1 - H^{\underline{L}\underline{L}}h)(|\nabla\varphi|^2 + q\varphi^2) \right. \\
&\quad \left. + (1 - H^{\underline{L}\underline{L}}h)H\underline{\partial}\varphi\bar{\partial}\varphi \right\} \tag{6.28}
\end{aligned}$$

and

$$\begin{aligned}
& (1+h)H_\alpha{}^\gamma\partial_\gamma\varphi\partial_\beta\varphi\underline{\mathcal{N}}^\alpha X^\beta \\
&= r \left\{ \underline{L}\varphi L\varphi (-2H^{\underline{L}\underline{L}} - 2hH^{\underline{L}\underline{L}} + 2H^{\underline{L}\underline{L}}(H^{\underline{L}\underline{L}} + hH^{\underline{L}\underline{L}})) + 2(\underline{L}\varphi)^2 H^{\underline{L}\underline{L}}(H^{\underline{L}\underline{L}} + hH^{\underline{L}\underline{L}}) \right. \\
&\quad \left. - 2(L\varphi)^2(H^{\underline{L}\underline{L}} + hH^{\underline{L}\underline{L}}) + (H + hH)\underline{\partial}\varphi\bar{\partial}\varphi \right\}. \tag{6.29}
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{2}(1+h)\mathbf{m}_{\alpha\alpha'}\mathbf{g}^{\alpha'\gamma}\partial_\gamma(\varphi^2)\underline{\mathcal{N}}^\alpha &= \frac{1}{2}\mathbf{m}_{\alpha\alpha'}(H^{\alpha'\gamma} + \mathbf{m}^{\alpha'\gamma})\partial_\gamma(\varphi^2)(\underline{L}^\alpha + hL^\alpha) \\
&= \frac{1}{2} \left\{ (\underline{L} + hL)(\varphi^2) + (H_\underline{L}{}^\gamma + hH_L{}^\gamma) \partial_\gamma(\varphi^2) \right\}
\end{aligned}$$

and

$$(1+h)\underline{\mathcal{N}}^\alpha Y_\alpha = \frac{1}{2}r^{-1}\varphi^2(L, \underline{L} + hL) = -r^{-1}\varphi^2.$$

We then combine the above calculations for the two terms to obtain

$$\begin{aligned}
& r^2(1+h)\underline{\mathcal{N}}^\alpha \left( \frac{1}{2}\mathbf{m}_{\alpha\alpha'}\mathbf{g}^{\alpha'\gamma}\partial_\gamma(\varphi^2) + Y_\alpha \right) \\
&= \frac{1}{2} \left\{ (\underline{L} + hL)(r^2\varphi^2) + r^2 (H_\underline{L}{}^\gamma + hH_L{}^\gamma) \partial_\gamma(\varphi^2) \right\} - rh\varphi^2. \tag{6.30}
\end{aligned}$$

Thus, by combining (6.28)-(6.30) and using the definition (6.18), we can obtain

$$\begin{aligned}
& (1+h)r^2\underline{\mathcal{N}}^\alpha \mathcal{P}_\alpha \\
&= r^3 \left\{ (L\varphi)^2(h - H^{\underline{L}\underline{L}} + hH^2 + hH) + (\underline{L}\varphi)^2(hH^2 + HH) + (|\nabla\varphi|^2 + q\varphi^2)(1 - H^{\underline{L}\underline{L}}h) \right. \\
&\quad \left. + H\underline{\partial}\varphi\bar{\partial}\varphi \right\} + \frac{1}{2} \left\{ (\underline{L} + hL)(r^2\varphi^2) + r^2(H + hH)\partial(\varphi^2) \right\} - rh\varphi^2, \tag{6.31}
\end{aligned}$$

where the term  $H(1+h+hH)\underline{\partial}\varphi \cdot \bar{\partial}\varphi$  is simplified to be  $H\underline{\partial}\varphi\bar{\partial}\varphi$  due to  $|h| + |H| \leq 1$ .

In the sequel, we will constantly use the fact that  $|h| + |H| \leq 1$  to shorten the symbolic formula. Recall from (2.9) for the area elements  $d\mu_{\mathcal{H}}$  and  $d\mu_{\underline{\mathcal{H}}}$ . By (6.27) we have

$$\begin{aligned}
I &= \int_{\mathcal{H}_{u_1}^{u_1}} (1+h) \mathcal{N}^\alpha \mathcal{P}_\alpha (1+h)^{-1} d\mu_{\mathcal{H}} \\
&= \int_{\mathcal{H}_{u_1}^{u_1}} \left\{ r(L(r\varphi) - rH^{\underline{L}\underline{L}}\underline{L}\varphi)^2 + r^3(h - H^{\underline{L}\underline{L}})(|\nabla\varphi|^2 + q\varphi^2) \right. \\
&\quad \left. - \frac{1}{2}(\underline{L} - h\underline{L})(r^2\varphi^2) \right\} \frac{r}{2(r-M_0)} d\omega d\underline{u} + \text{Er}_1, \\
\text{Er}_1 &= \int_{\mathcal{H}_{u_1}^{u_1}} \left\{ r^2(L(r\varphi) - rH^{\underline{L}\underline{L}}\underline{L}\varphi) \cdot (H\bar{\partial}\varphi + hH\underline{L}\varphi) + r^3(h - H^{\underline{L}\underline{L}})H\partial\varphi\bar{\partial}\varphi \right. \\
&\quad \left. + r^2\varphi H(\bar{\partial}\varphi + h\underline{\partial}\varphi) \right\} \frac{r}{2(r-M_0)} d\omega d\underline{u}. \tag{6.32}
\end{aligned}$$

By using (6.31), we have on  $\underline{\mathcal{H}}_{u_1}^{u_1}$  that

$$\begin{aligned}
II &= \int_{\underline{\mathcal{H}}_{u_1}^{u_1}} (1+h) \underline{\mathcal{N}}^\alpha \mathcal{P}_\alpha (1+h)^{-1} d\mu_{\underline{\mathcal{H}}} \\
&= \int_{\underline{\mathcal{H}}_{u_1}^{u_1}} \left\{ r^3[(L\varphi)^2(h - H^{\underline{L}\underline{L}} + hH) + (q\varphi^2 + |\nabla\varphi|^2)(1 - Hh) + (\underline{L}\varphi)^2HH + H\underline{\partial}\varphi\bar{\partial}\varphi] \right. \\
&\quad \left. + \frac{1}{2}[(\underline{L} - h\underline{L})(r^2\varphi^2) + r^2H\partial(\varphi^2)] + hL(r^2\varphi^2) - rh\varphi^2 \right\} \frac{r}{2(r-M_0)} d\omega du \\
&= \int_{\underline{\mathcal{H}}_{u_1}^{u_1}} \left\{ r^3((L\varphi)^2(h - H^{\underline{L}\underline{L}}) + |\nabla\varphi|^2 + q\varphi^2) + \frac{1}{2}(\underline{L} - h\underline{L})(r^2\varphi^2) \right\} \frac{r}{2(r-M_0)} d\omega du + \text{Er}_2, \\
\text{Er}_2 &= \int_{\underline{\mathcal{H}}_{u_1}^{u_1}} \left\{ r^2(H + h)\partial(\varphi^2) + rh\varphi^2 + r^3H(h(L\varphi)^2 + (\underline{L}\varphi)^2 + \underline{\partial}\varphi \cdot \bar{\partial}\varphi) \right. \\
&\quad \left. + qh\varphi^2 \right\} \frac{r}{2(r-M_0)} d\omega du. \tag{6.33}
\end{aligned}$$

In  $I$  and  $II$ , the coefficients of leading terms are precise, while  $\text{Er}_1$  and  $\text{Er}_2$  are symbolic formulas for the error terms. By applying the divergence theorem over the region  $\mathcal{D}_{u_1}^{u_1}$  it follows that

$$I + II - \int_{\Sigma_0^{u_1, \underline{u}_1}} \mathcal{P}_\alpha \partial_t^\alpha dx = - \int_{\mathcal{D}_{u_1}^{u_1}} \partial^\alpha \mathcal{P}_\alpha.$$

For convenience, we set

$$\begin{aligned}
\tilde{I} &= I + \int_{\mathcal{H}_{u_1}^{u_1}} \frac{1}{2}(\underline{L} - h\underline{L})(r^2\varphi^2) \frac{r}{2(r-M_0)} d\omega d\underline{u}; \\
\widetilde{II} &= II - \int_{\underline{\mathcal{H}}_{u_1}^{u_1}} \frac{1}{2}(\underline{L} - h\underline{L})(r^2\varphi^2) \frac{r}{2(r-M_0)} d\omega du; \\
\widetilde{III} &= \int_{\Sigma_0^{u_1, \underline{u}_1}} \left( \mathcal{P}_\alpha \partial_t^\alpha r^2 + \frac{1}{2} \partial_r(r^2\varphi^2) \right) d\omega dr.
\end{aligned}$$

Then, in view of (5.11) and (2.7), we have

$$\tilde{I} + \widetilde{II} - \widetilde{III} = - \int_{\mathcal{D}_{u_1}^{u_1}} \partial^\alpha \mathcal{P}_\alpha.$$

Combining (6.17) with the above identity and in view of the definitions of  $\tilde{I}, \widetilde{II}$ , we can derive that

$$\begin{aligned}
& \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} \frac{1}{2} (r^{-2} |L(r\varphi)|^2 + |\nabla\varphi|^2) \\
& + \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} \left\{ r (L(r\varphi) - rH^{\underline{L}\underline{L}}\underline{L}\varphi)^2 + r^3 (h - H^{\underline{L}\underline{L}}) (|\nabla\varphi|^2 + q\varphi^2) \right\} \frac{r}{2(r - M_0)} d\omega d\underline{u} \\
& + \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r^3 ((L\varphi)^2 (h - H^{\underline{L}\underline{L}}) + |\nabla\varphi|^2 + q\varphi^2) \frac{r}{2(r - M_0)} d\omega du \\
& \leq |\widetilde{III}| + |\text{Er}_1| + |\text{Er}_2| + \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} \left( |\mathcal{F}(X\varphi + \varphi)| + \frac{q + |Xq|}{2} \varphi^2 \right) \\
& + C\Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \left\{ |u_1|^{1 - \frac{3\gamma_0}{2} + 2p} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0 - 2p} + |u_1|^{\gamma_0 - 2p} \sup_{-u_1 \leq \underline{u} \leq \underline{u}_1} E[\varphi](\mathcal{H}_{\underline{u}}^{\underline{u}_1}) \right) \right. \\
& \left. + \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} \mathcal{W}_1[\varphi](\mathcal{H}_u^{\underline{u}_1}) du \right\}. \tag{6.34}
\end{aligned}$$

Moreover, by using (6.6), it is straightforward to obtain

$$\int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r^3 |H|^2 (\underline{L}\varphi)^2 d\underline{u} d\omega \lesssim E[\varphi](\mathcal{H}_{u_1}^{\underline{u}_1}) |u_1|^{-\gamma_0+1} \Delta_0 M^{-1},$$

which leads to

$$\begin{aligned}
& \left| \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r (L(r\varphi) - rH^{\underline{L}\underline{L}}\underline{L}\varphi)^2 \frac{r}{2(r - M_0)} d\omega d\underline{u} - \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r |L(r\varphi)|^2 \frac{r}{2(r - M_0)} d\omega d\underline{u} \right| \\
& \lesssim E[\varphi](\mathcal{H}_{u_1}^{\underline{u}_1}) (u_1)_+^{1-\gamma_0} \Delta_0 M^{-1}. \tag{6.35}
\end{aligned}$$

By using (A1), we can also derive that

$$\begin{aligned}
& \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} \frac{1}{2} (r^{-2} |L(r\varphi)|^2 + |\nabla\varphi|^2) + \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} \left\{ r (L(r\varphi))^2 + \frac{1}{3} r^2 M (|\nabla\varphi|^2 + q\varphi^2) \right\} \frac{r}{2(r - M_0)} d\omega d\underline{u} \\
& + \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r^3 \left( \frac{M}{3r} (L\varphi)^2 + |\nabla\varphi|^2 + q\varphi^2 \right) \frac{r}{2(r - M_0)} d\omega du \\
& \leq |\widetilde{III}| + |\text{Er}_1| + |\text{Er}_2| + \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} \left( |\mathcal{F}(X\varphi + \varphi)| + \frac{q + |Xq|}{2} \varphi^2 \right) \\
& + C\Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \left\{ |u_1|^{1 - \frac{3\gamma_0}{2} + 2p} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0 - 2p} + |u_1|^{\gamma_0 - 2p} \sup_{-u_1 \leq \underline{u} \leq \underline{u}_1} E[\varphi](\mathcal{H}_{\underline{u}}^{\underline{u}_1}) \right) \right. \\
& \left. + \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} \mathcal{W}_1[\varphi](\mathcal{H}_u^{\underline{u}_1}) du \right\}. \tag{6.36}
\end{aligned}$$

It is straightforward to derive

$$r^2 \mathcal{P}_\alpha \partial_t^\alpha + \frac{1}{2} \partial_r (r^2 \phi^2) = \frac{1}{2} (r (L(r\varphi))^2 + r^3 (|\nabla\varphi|^2 + q\varphi^2)) + r^3 H ((\partial\varphi)^2 + q\varphi^2 + \partial\varphi \cdot \varphi).$$

Note that with the help of (6.1),

$$|\widetilde{III}| \lesssim \left\| r^{\frac{1}{2}} \partial\varphi \right\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 + \left\| r^{-\frac{1}{2}} \varphi \right\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 + q_0 \left\| r^{\frac{1}{2}} \varphi \right\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2. \tag{6.37}$$

It remains to estimate  $\text{Er}_1$  and  $\text{Er}_2$ . We first estimate  $\text{Er}_1$  in view of

$$\begin{aligned} \left| r^{\frac{3}{2}} (H\bar{\partial}\varphi + hH\underline{L}\varphi) \right| &\lesssim (|rh\underline{L}\varphi| + r|\bar{\partial}\varphi|) \left| r^{\frac{1}{2}} H \right|, \\ r^3 |(h-H)H\partial\varphi\bar{\partial}\varphi| &\lesssim |r\bar{\partial}\varphi| \left| r^{\frac{1}{2}} \partial\varphi \right| \left| r(h-H) \right| \left| r^{\frac{1}{2}} H \right|, \\ r^2 |\varphi H(\bar{\partial}\varphi + h\underline{\partial}\varphi)| &\lesssim (|r\bar{\partial}\varphi| + |rh\underline{\partial}\varphi|) |\varphi| r |H|. \end{aligned}$$

By Hölder's inequality, (6.6) and  $r|h| \leq M$ , we have

$$\begin{aligned} \left\| r^{\frac{3}{2}} (H\bar{\partial}\varphi + hH\underline{L}\varphi) \right\|_{L_\omega^2 L_{\underline{u}}^2(\mathcal{H}_{u_1}^{u_1})} &\lesssim |u_1|^{-\frac{\gamma_0}{2}} \Delta_0^{\frac{1}{2}} E[\varphi]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{u_1}), \\ \int_{\mathcal{H}_{u_1}^{u_1}} r^3 |(h-H)H\partial\varphi\bar{\partial}\varphi| d\underline{u}d\omega &\lesssim E[\varphi](\mathcal{H}_{u_1}^{u_1}) \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0}{2}} \left( M^{-\frac{1}{2}} \Delta_0^{\frac{1}{2}} + M^{\frac{1}{2}} \right), \\ \int_{\mathcal{H}_{u_1}^{u_1}} r^2 |\varphi H| (|\bar{\partial}\varphi| + |h\underline{\partial}\varphi|) d\underline{u}d\omega &\lesssim E[\varphi]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{u_1}) \left( E[\varphi](\mathcal{H}_{u_1}^{u_1}) + E[\varphi](\underline{\mathcal{H}}_{u_1}^{u_1}) \right. \\ &\quad \left. + \int_{S_{-u_1, u_1}} r\varphi^2 d\omega \right)^{\frac{1}{2}} \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0-1}{2}}, \end{aligned}$$

where we employed (3.16) to derive the last inequality. Thus, noting that  $\frac{r}{r-M_0} \lesssim 1$  and  $\Delta_0 M^{-1} < 1$ , we conclude that

$$\begin{aligned} |\text{Er}_1| &\lesssim |u_1|^{-\frac{\gamma_0-1}{2}} \Delta_0^{\frac{1}{2}} \left\{ \left( |u_1|^{-1} \int_{\mathcal{H}_{u_1}^{u_1}} r(L(r\varphi) - rH\underline{L}\varphi)^2 d\underline{u}d\omega \right)^{\frac{1}{2}} E[\varphi]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{u_1}) \right. \\ &\quad \left. + E[\varphi](\mathcal{H}_{u_1}^{u_1}) + E[\varphi](\underline{\mathcal{H}}_{u_1}^{u_1}) + \sup_{-u_1 \leq u \leq u_1} \int_{S_{u, -u}} r\varphi^2 d\omega \right\} \\ &\lesssim |u_1|^{-\frac{\gamma_0-1}{2}} \Delta_0^{\frac{1}{2}} \left\{ |u_1|^{-1} \int_{\mathcal{H}_{u_1}^{u_1}} r(L(r\varphi))^2 d\underline{u}d\omega + E[\varphi](\mathcal{H}_{u_1}^{u_1}) + E[\varphi](\underline{\mathcal{H}}_{u_1}^{u_1}) \right. \\ &\quad \left. + \sup_{-u_1 \leq u \leq u_1} \int_{S_{u, -u}} r\varphi^2 d\omega \right\}, \end{aligned} \tag{6.38}$$

where we employed (6.35) to derive the last inequality. We remark that the term of  $L(r\varphi)$  can be absorbed when  $\text{Er}_1$  is substituted back into (6.36).

Next we control the error term in  $\text{Er}_2$  in a similar fashion. By using (6.6) and  $r|h| \leq M$ , we derive

$$\int_{\underline{\mathcal{H}}_{u_1}^{u_1}} r^3 (L\varphi)^2 |hH| d\underline{u}d\omega \lesssim \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0-1}{2}} \int_{\underline{\mathcal{H}}_{u_1}^{u_1}} Mr(L\varphi)^2 d\underline{u}d\omega \lesssim \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0-1}{2}} E[\varphi](\underline{\mathcal{H}}_{u_1}^{u_1}),$$



$$\begin{aligned}
\int_{\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}} r^3 |H| (|\underline{L}\varphi|^2 + q\varphi^2) d\omega du &\lesssim \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0-1}{2}} E[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}), \\
\int_{\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}} r^3 |H \underline{\partial} \varphi \bar{\partial} \varphi| d\omega du &\lesssim \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0-1}{2}} E[\varphi]^{\frac{1}{2}}(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) \\
&\quad \times \left( M^{-\frac{1}{2}} \left( \int_{\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}} M r^2 (L\varphi)^2 d\omega du \right)^{\frac{1}{2}} + E[\varphi]^{\frac{1}{2}}(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) \right), \\
\int_{\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}} r^2 (|H| + |h|) |\partial(\varphi^2)| d\omega du &\lesssim \left( \Delta_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0-1}{2}} + M \right) \|r^{-1}\varphi\|_{L^2(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})} \\
&\quad \times \left( \|\underline{\partial}\varphi\|_{L^2(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})} + M^{-\frac{1}{2}} \|M^{\frac{1}{2}} L\varphi\|_{L^2(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})} \right), \\
\int_{\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}} r |h| \varphi^2 d\omega du &\leq M \|r^{-1}\varphi\|_{L^2(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})}^2.
\end{aligned}$$

Thus by combining the above estimates, and by using (3.15) to treat the term  $\|r^{-1}\varphi\|_{L^2(\underline{\mathcal{H}}_{\underline{u}_1}^{u_1})}$  we can derive that

$$\begin{aligned}
|\text{Er}_2| &\lesssim \left( \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{-\frac{\gamma_0-1}{2}} + M^{\frac{1}{2}} \right) \left( \sup_{-\underline{u}_1 \leq u \leq u_1} \|r^{-\frac{1}{2}}\varphi\|_{L^2(S_{u,-u})}^2 + E[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + \mathcal{W}_1[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) \right) \\
&\lesssim M^{\frac{1}{2}} \left( \mathcal{W}_1[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + \sup_{-\underline{u}_1 \leq u \leq u_1} \|r^{-\frac{1}{2}}\varphi\|_{L^2(S_{u,-u})}^2 \right),
\end{aligned}$$

where the first term on the righthand side of the last inequality will be absorbed due to the smallness of  $M$ .

We then substitute the estimates of  $\text{Er}_1$ ,  $\text{Er}_2$ , (6.36) and (6.37) to derive that

$$\begin{aligned}
&\mathcal{W}_1[\varphi](\mathcal{D}_{u_1}^{u_1}) + \mathcal{W}_1[\varphi](\mathcal{H}_{u_1}^{u_1}) + \mathcal{W}_1[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) \\
&\lesssim \left\| r^{\frac{1}{2}} \underline{\partial} \varphi + r^{-\frac{1}{2}} |\varphi| + q_0^{\frac{1}{2}} r^{\frac{1}{2}} |\varphi| \right\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 + \int_{\mathcal{D}_{u_1}^{u_1}} \left( |\mathcal{F}(X\varphi + \varphi)| + \frac{q + |Xq|}{2} \varphi^2 \right) \\
&\quad + M^{\frac{1}{2}} \left( E[\varphi](\mathcal{H}_{u_1}^{u_1}) + E[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) + \sup_{-\underline{u}_1 \leq u \leq u_1} \int_{S_{u,-u}} r \varphi^2 d\omega \right) \\
&\quad + \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \left\{ |u_1|^{1-\frac{3\gamma_0}{2}+2p} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u|^{\gamma_0-2p} \right. \right. \\
&\quad \left. \left. + |u_1|^{\gamma_0-2p} \sup_{-\underline{u}_1 \leq u \leq \underline{u}_1} E[\varphi](\underline{\mathcal{H}}_{\underline{u}}^{u_1}) \right) + \int_{-\underline{u}_1}^{u_1} |u|^{-\frac{\gamma_0+1}{2}} \mathcal{W}_1[\varphi](\mathcal{H}_u^{u_1}) du \right\}. \tag{6.39}
\end{aligned}$$

Note that  $\int_{\mathcal{D}_{u_1}^{u_1}} q\varphi^2 \lesssim \int_{-\underline{u}_1}^{u_1} E[\varphi](\mathcal{H}_u^{u_1}) du$ . The term  $\int_{\mathcal{D}_{u_1}^{u_1}} |Xq|\varphi^2 dx dt$  can be treated exactly as in (5.12). The term  $\|r^{-\frac{1}{2}}(X\varphi + \varphi)\|_{L^2(\mathcal{H}_{u_1}^{u_1})}$  can be treated by applying (6.35) on each  $\mathcal{H}_u^{u_1}$ . By using Gronwall's inequality (see [19, Lemma 3]), the last term on the righthand side of the inequality and the first term on the right of (5.12) can both be absorbed. We summarize the result after the above treatments as below

$$\begin{aligned}
&\mathcal{W}_1[\varphi](\mathcal{D}_{u_1}^{u_1}) + \mathcal{W}_1[\varphi](\mathcal{H}_{u_1}^{u_1}) + \mathcal{W}_1[\varphi](\underline{\mathcal{H}}_{\underline{u}_1}^{u_1}) \\
&\lesssim C(\epsilon_1) \left\| r^{\frac{3}{2}} \mathcal{F} \right\|_{L_u^1 L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{u_1})}^2 + |u_1|^{-\gamma_0+2p+1} \epsilon_1 \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0-2p-1} \mathcal{W}_1[\varphi](\mathcal{H}_u^{u_1}) \\
&\quad + |u_1|^{1-\gamma_0+2p} \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u|^{\gamma_0-2p} + \left\| r^{\frac{1}{2}} |\underline{\partial}\varphi| + r^{-\frac{1}{2}} |\varphi| \right\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 + q_0 \left\| r^{\frac{1}{2}} \varphi \right\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2
\end{aligned}$$

$$+ \left( M^{\frac{1}{2}} + \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \right) \left( |u_1|^{1-\frac{\gamma_0}{2}} \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) + \sup_{-u_1 \leq u \leq u_1} \int_{S_{u,-u}} r \varphi^2 d\omega \right),$$

where  $p \leq 0$  is an any fixed constant. We then multiply both sides by  $|u_1|^{\gamma_0-1-2p}$  followed with taking supremum on  $u_1$  in  $-u_1 \leq u_1 \leq u_2 \leq u_0$  for a fixed  $u_2$ . In view of the assumption that  $\Delta_0 \lesssim M^2$ , by choosing the constant  $\epsilon_1 > 0$  sufficiently small, Proposition 6.6 can be proved.  $\square$

**6.3. Error estimates.** The main estimates of this part are the error estimates for controlling  $(\square_{\mathbf{g}} - q)Z^{(n)}\phi$  in Proposition 6.9. Comparing this result with Proposition 5.12 for the semilinear equation (2.15), one difference lies in that Proposition 6.9 copes with the terms of the form  $[\square_{\mathbf{g}}, Z^{(n)}]\phi$ , with  $n \leq 3$ . Such error terms arise due to the nontrivial influence of the metric  $\mathbf{g}(\phi, \partial\phi)$  and vanish in the semilinear case. The treatment needs a sharp decay property for the term  $ZH$ , particularly for  $Z = \Omega_{ij}$ , which requires us to bound energies for  $Z^{(3)}H$ . Therefore in Proposition 6.9 we treat  $\mathcal{F}^{\{n\}}$  with  $n \leq 3$  for the solution of quasilinear equation (1.1), while in Proposition 5.12 we only need to bound the terms of  $Z^{(n)}(\square\phi - q\phi)$  with  $n \leq 2$ .

**Lemma 6.7.** *Let  $\Phi = (\phi, \partial\phi)$ . For  $n \leq 3$  there holds*

$$\left| Z^{(n)}(H(\Phi)) \right| \lesssim \sum_{Z^a \sqcup Z^b = Z^n} \left| Z^{(a)}\Phi \right| |u|^{\zeta(Z^b)}. \quad (6.40)$$

Note that due to (6.5) and (5.17), (5.15) holds for  $\partial\phi$  as well. Thus, with the help of (6.9), we can repeat the proof of (5.19) with  $\mathcal{N}(\varphi)$  replaced by  $H(\Phi)$  to obtain (6.40).

**Proposition 6.8.** *For  $1 \leq n \leq 3$  there holds*

$$\begin{aligned} \left| [\square_{\mathbf{g}}, Z^{(n)}]\phi \right| &\lesssim \sum_{Z^{n-1} \sqcup Z^1 = Z^n} \left( |u|^{\zeta(Z^1)} |H| + |Z^{(1)}H| \right) \left| \partial^2 Z^{(n-1)}\phi \right| \\ &+ \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ b \leq n-2}} |u|^{\zeta(Z^c)} \left| Z^{(a)}\Phi \right| \left| \partial^2 Z^{(b)}\phi \right|, \end{aligned} \quad (6.41)$$

where the last term on the righthand side vanishes if  $n = 1$ .

*Proof.* Since  $Z \in \{\Omega, \partial\}$  are killing vector fields,  $[\square_{\mathbf{g}}, Z] = [H^{\mu\nu} \partial_\mu \partial_\nu, Z]$ . We can derive

$$[\square_{\mathbf{g}}, Z]\psi = -ZH^{\alpha\beta} \partial_\alpha \partial_\beta \psi + 2H^{\alpha\beta} C_{Z\alpha}{}^\gamma \partial_\gamma \partial_\beta \psi, \quad (6.42)$$

since

$$H^{\alpha\beta} [Z, \partial_\alpha] \partial_\beta \psi = H^{\alpha\beta} \partial_\alpha [Z, \partial_\beta] \psi = H^{\alpha\beta} C_{Z\alpha}{}^\gamma \partial_\gamma \partial_\beta \psi,$$

where  $C_{Z\alpha}{}^\gamma$  has been defined in Lemma 5.6. For convenience we will drop the coefficient 2 in the calculation, and we will adopt the convention in Lemma 5.6.

Similarly, we have

$$\begin{aligned} \square_{\mathbf{g}} Z^{(2)}\psi &= Z^{(2)} \square_{\mathbf{g}} \psi + Z_2 [Z_1, \square_{\mathbf{g}}] \psi + [Z_2, \square_{\mathbf{g}}] Z_1 \psi \\ &= Z^{(2)} \square_{\mathbf{g}} \psi + Z_2 [Z_1, H^{\alpha\beta} \partial_\alpha \partial_\beta] \psi + [Z_2, H^{\alpha\beta} \partial_\alpha \partial_\beta] Z_1 \psi. \end{aligned}$$

By using (6.42), we also have the commutation identities

$$\begin{aligned} \square_{\mathbf{g}} Z^{(2)}\psi &= Z^{(2)} \square_{\mathbf{g}} \psi + \sum_{X \sqcup Y = Z^2} \left\{ Z^{(2)} H^{\mu\nu} \partial_\mu \partial_\nu \psi + X H^{\mu\nu} \partial_\mu \partial_\nu Y \psi + X H^{\mu\nu} (C_Y \partial^2 \psi)_{\mu\nu} \right. \\ &\quad \left. + H^{\mu\nu} C_{X\mu}{}^\gamma \partial_\gamma \partial_\nu Y \psi + H^{\mu\nu} (C_X C_Y \partial^2 \psi)_{\mu\nu} \right\}, \end{aligned} \quad (6.43)$$

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<sup>22</sup>See the definition of  $\mathcal{F}^{\{n\}}$  in Lemma 5.10.

and

$$\begin{aligned}
\Box_{\mathbf{g}} Z^{(3)} \psi &= Z^{(3)} \Box_{\mathbf{g}} \psi + Z^{(3)} H^{\mu\nu} \partial_{\mu\nu}^2 \psi + \sum_{a=1}^3 Z^{(3 \setminus a)} H^{\mu\nu} (\partial_\mu \partial_\nu Z_a \phi + (C_{Z_a} \cdot \partial^2 \psi)_{\mu\nu}) \\
&+ \sum_{a=1}^3 Z_a H^{\mu\nu} \left( \partial_\mu \partial_\nu Z^{(3 \setminus a)} \psi + \sum_{X \sqcup Y = Z^{(3 \setminus a)}} (C_X \cdot \partial^2 Y \psi)_{\mu\nu} + (C_{Z^{(3 \setminus a)}} \cdot \partial^2 \psi)_{\mu\nu} \right) \\
&+ H^{\mu\nu} \left\{ \sum_{a=1}^3 \left( C_{Z_a \mu}{}^\gamma \partial_\gamma \partial_\nu Z^{(3 \setminus a)} \psi + (C_{Z^{(3 \setminus a)}} \partial^2 Z_a \psi)_{\mu\nu} \right) + (C_{Z^3} \partial^2 \psi)_{\mu\nu} \right\}. \quad (6.44)
\end{aligned}$$

We then summarize the terms in (6.42)–(6.44) into

$$\begin{aligned}
\left| [\Box_{\mathbf{g}}, Z^{(n)}] \phi \right| &\lesssim \sum_{Z^1 \sqcup Z^{n-1} = Z^n} \left( |u|^{\zeta(Z^1)} |H| + |Z^{(1)} H| \right) \left| \partial^2 Z^{(n-1)} \phi \right| \\
&+ \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ b \leq n-2}} |u|^{\zeta(Z^c)} \left| Z^{(a)} H \right| \left| \partial^2 Z^{(b)} \phi \right|, \quad (6.45)
\end{aligned}$$

where the last term on the righthand side vanishes if  $n = 1$ . By using (6.40) to treat the term  $Z^{(a)} H$ , the last term on the righthand side of (6.45) can be bounded by

$$\sum_{\substack{Z^{a_1} \sqcup Z^b \sqcup Z^{c_1} = Z^n \\ b \leq n-2}} |u|^{\zeta(Z^{c_1})} \left| Z^{(a_1)} \Phi \right| \left| \partial^2 Z^{(b)} \phi \right|.$$

Now we can combine the above estimates to conclude the proof of Proposition 6.8.  $\square$

In view of Proposition 6.8 and Lemma 5.10, we will prove the following result.

**Proposition 6.9.** *For  $0 \leq n \leq 3$ , there hold for  $(u_1, \underline{u}_1) \in \mathcal{I}$  that*

$$|u_1|^{-\zeta(Z^n) + \frac{1}{2}\gamma_0 + \frac{1}{2}} \left\| r^{\frac{1}{2}} [\Box_{\mathbf{g}}, Z^{(n)}] \phi \right\|_{L^2(\mathcal{D}_{\underline{u}_1}^{u_1})} \lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\frac{1}{2}\gamma_0 + \frac{1}{2}}, \quad (6.46)$$

$$|u_1|^{-\zeta(Z^n) + \frac{1}{2}\gamma_0} \left\| r^{\frac{3}{2}} [\Box_{\mathbf{g}}, Z^{(n)}] \phi \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{\underline{u}_1}^{u_1})} \lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\frac{1}{2}\gamma_0 + \frac{1}{2}}, \quad (6.47)$$

$$|u_1|^{-\zeta(Z^n) + \frac{1}{2}\gamma_0 + \frac{1}{2}} \left\| r^{\frac{1}{2}} \mathcal{F}_{\mathcal{Q}}^{\{n\}} \right\|_{L^2(\mathcal{D}_{\underline{u}_1}^{u_1})} \lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\frac{1}{2}\gamma_0 + \frac{1}{2}}, \quad (6.48)$$

$$|u_1|^{-\zeta(Z^n) + \frac{1}{2}\gamma_0} \left\| r^{\frac{3}{2}} \mathcal{F}_{\mathcal{Q}}^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{\underline{u}_1}^{u_1})} \lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\frac{1}{2}\gamma_0 + \frac{1}{2}}, \quad (6.49)$$

$$|u_1|^{-\zeta(Z^n)} \left\| r^{\frac{1}{2}} \mathcal{F}_{\mathcal{C}}^{\{n\}} \right\|_{L^2(\mathcal{D}_{\underline{u}_1}^{u_1})} \lesssim \Delta_0^{\frac{3}{2}} M^{-\frac{1}{2}} |u_1|^{-\frac{3\gamma_0}{2} - \frac{1}{2}}, \quad (6.50)$$

$$|u_1|^{-\zeta(Z^n)} \left\| r^{\frac{3}{2}} \mathcal{F}_{\mathcal{C}}^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{\underline{u}_1}^{u_1})} \lesssim \Delta_0^{\frac{3}{2}} |u_1|^{-\frac{3\gamma_0}{2} - \frac{1}{2}}. \quad (6.51)$$

*Proof.* If  $n = 0$ , the commutator is identically 0. Thus the corresponding estimates are trivially true. Therefore for the commutator estimates (6.46) and (6.47), we only need to consider the cases  $1 \leq n \leq 3$ . We first prove (6.46). Denote by  $I_n$  and  $II_n$  the terms on the right of (6.41) respectively.

We apply (6.6) to  $Z^{(i)} H$  with  $i \leq 1$  and bound  $\partial^2 Z^{(n-1)} \phi$  by using (4.14). Thus

$$\begin{aligned}
\left\| r^{\frac{1}{2}} I_n \right\|_{L^2(\mathcal{D}_{\underline{u}_1}^{u_1})} &\lesssim \sum_{Z^1 \sqcup Z^{n-1} = Z^n} \left\| r \left( |Z^{(1)} H| + |u|^{\zeta(Z^1)} |H| \right) \right\|_{L^\infty(\mathcal{D}_{\underline{u}_1}^{u_1})} \left\| r^{-\frac{1}{2}} \partial^2 Z^{(n-1)} \phi \right\|_{L^2(\mathcal{D}_{\underline{u}_1}^{u_1})} \\
&\lesssim M^{-\frac{1}{2}} \Delta_0 |u_1|^{-\gamma_0 + \zeta(Z^n)}.
\end{aligned}$$

For the term  $II_n = \sum_{Z^{a_1} \sqcup Z^{b_1} \sqcup Z^b = Z^n, b \leq n-2} |u|^{\zeta(Z^{b_1})} |Z^{(a_1)} \Phi| |\partial^2 Z^{(b)} \phi|$ , we first derive that

$$\begin{aligned} & \left\| r^{\frac{1}{2}} II_n \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\ & \lesssim \sum_{\substack{Z^{a_1} \sqcup Z^{b_1} \sqcup Z^b = Z^n \\ 1 \leq b \leq n-2}} |u_1|^{\zeta(Z^{b_1})} \left\| |u|^{-\frac{1}{2}\gamma_0 - \frac{3}{2}} r^{\frac{1}{2}} Z^{(a_1)} \Phi \right\|_{L_{\underline{u}}^2 L_u^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \left\| r |u|^{\frac{1}{2}\gamma_0 + \frac{3}{2}} \partial^2 Z^{(b)} \phi \right\|_{L^\infty L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \\ & + \sum_{Z^{a_1} \sqcup Z^{b_1} = Z^n} |u_1|^{\zeta(Z^{b_1})} \left\| |u|^{-\frac{1}{2}\gamma_0 - \frac{3}{2}} r^{-\frac{1}{2}} Z^{(a_1)} \Phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \left\| r |u|^{\frac{1}{2}\gamma_0 + \frac{3}{2}} \partial^2 \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{u_1})}. \end{aligned}$$

With the help of the following estimates

$$\left\| |u|^{-\frac{\gamma_0}{2} - \frac{3}{2}} r^{\frac{1}{2}} Z^{(a_1)} \Phi \right\|_{L_{\underline{u}}^2 L_u^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + \zeta(Z^{a_1})}, \quad a_1 \leq n-1 \quad (6.52)$$

$$\left\| |u|^{\frac{\gamma_0}{2} + \frac{3}{2}} r \partial^2 Z^{(b)} \phi \right\|_{L^\infty L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0^{\frac{1}{2}} |u_1|^{\zeta(Z^{b_1})}, \quad b \leq n-2 \quad (6.53)$$

$$\left\| |u|^{-\frac{\gamma_0}{2} - \frac{3}{2}} r^{-\frac{1}{2}} Z^{(a_1)} \Phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{\zeta(Z^{a_1}) - \gamma_0}, \quad a_1 \leq n \quad (6.54)$$

$$\left\| |u|^{\frac{\gamma_0}{2} + \frac{3}{2}} r \partial^2 \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0^{\frac{1}{2}}, \quad (6.55)$$

we can directly obtain

$$\left\| r^{\frac{1}{2}} II_n \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \lesssim M^{-\frac{1}{2}} \Delta_0 |u_1|^{-\gamma_0 + \zeta(Z^n)}.$$

Now we derive the estimates (6.52)-(6.55). (6.55) follows in view of (6.5). By applying (4.18) we can obtain (6.53). By using (4.23) and (4.5), we can obtain (6.52) and (6.54) if  $\Phi$  is simply  $\phi$ . By using (5.17) and (4.22), we can obtain (6.52) for  $\Phi$ ; finally by using (5.17) and (4.14), we obtain (6.54) for  $\Phi$ .

By combining the estimates of  $I_n$  and  $II_n$ , we have therefore proved (6.46).

Next we prove (6.47). By using (6.6) and (6.3), we can derive that

$$\begin{aligned} & |u_1|^{-\zeta(Z^n) + \frac{1}{2}\gamma_0} \left\| r^{\frac{3}{2}} I_n \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\ & \lesssim |u_1|^{-\zeta(Z^{n-1}) + \frac{1}{2}\gamma_0} \Delta_0^{\frac{1}{2}} \left\| |u|^{-\frac{\gamma_0}{2} + \frac{1}{2}} r^{\frac{1}{2}} \partial^2 Z^{(n-1)} \phi \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\ & \lesssim |u_1|^{\frac{1}{2}\gamma_0} \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} \sup_{-u_1 \leq u \leq u_1} E^{\frac{1}{2}} [\partial^2 Z^{(n-1)} \phi](\mathcal{H}_u^{u_1}) |u|^{1 + \frac{\gamma_0}{2} - \zeta(Z^{n-1})} \int_{-u_1}^{u_1} |u|^{-\gamma_0 - \frac{1}{2}} du \\ & \lesssim |u_1|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \Delta_0 M^{-\frac{1}{2}}, \end{aligned}$$

which holds for any  $Z^{n-1} \subset Z^n$ .

To estimate  $II_n$ , by repeating the derivation of (6.52) and (6.54), we have

$$\begin{aligned} & \left\| |u|^{-\frac{\gamma_0}{2} - \frac{1}{2}} r^{\frac{1}{2}} Z^{(a_1)} \Phi \right\|_{L_{\underline{u}}^2 L_u^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + 1 + \zeta(Z^{a_1})}, \quad a_1 \leq n-1 \\ & \left\| |u|^{-\frac{\gamma_0}{2} - \frac{1}{2}} r^{-\frac{1}{2}} Z^{(a_1)} \Phi \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0^{\frac{1}{2}} M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + 1 + \zeta(Z^{a_1})}, \quad a_1 \leq n. \end{aligned}$$

Combining (6.53), (6.55) with the above two estimates, by a standard Hölder's inequality, we can deduce that

$$\left\| r^{\frac{3}{2}} II_n \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \left\| r^{\frac{1}{2}} |u| II_n \right\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \left\| |u|^{-1} \right\|_{L_u^2 L^\infty(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{\zeta(Z^n) - \gamma_0 + \frac{1}{2}}.$$

Thus (6.47) follows by combining the estimates for  $I_n$  and  $II_n$ .

We now consider  $\mathcal{F}_{\mathcal{Q}}^{\{n\}}$ . Recall from (5.28) in the proof of Proposition 5.12 that  $\mathcal{F}_{\mathcal{Q}}^{\{n\}} = \mathcal{F}_{\mathcal{Q},1}^{\{n\}} + \mathcal{F}_{\mathcal{Q},2}^{\{n\}}$ . We will control the terms  $\mathcal{F}_{\mathcal{Q},1}^{\{n\}}$  and  $\mathcal{F}_{\mathcal{Q},2}^{\{n\}}$  in a similar way.

Similar to the estimate of  $I_n$ , by using (6.5), (4.14) and  $b \leq n$ , we can obtain

$$\begin{aligned} \left\| r^{\frac{1}{2}} \mathcal{F}_{\mathcal{Q},1}^{\{n\}} \right\|_{L^2(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} &\lesssim \sum_{Z^b \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^c) - \frac{1}{2} - \frac{\gamma_0}{2}} \left\| |u|^{\frac{1}{2} + \frac{\gamma_0}{2}} r \partial \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \left\| r^{-\frac{1}{2}} \partial Z^{(b)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \\ &\lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + \zeta(Z^n)}. \end{aligned}$$

Note that  $1 \leq b \leq a \leq n-1$  in  $\mathcal{F}_{\mathcal{Q},2}^{\{n\}}$ . We can employ (4.22) for the term  $\partial Z^{(b)} \phi$  and apply (4.18) to  $\partial Z^{(a)} \phi$  to derive that

$$\begin{aligned} \left\| r^{\frac{1}{2}} \mathcal{F}_{\mathcal{Q},2}^{\{n\}} \right\|_{L^2(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} &\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^c)} \left\| r^{\frac{1}{2}} \partial Z^{(b)} \phi \right\|_{L_{\underline{u}}^2 L_u^2 L_\omega^4(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \left\| r \partial Z^{(a)} \phi \right\|_{L^\infty L_\omega^4(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \\ &\lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + \zeta(Z^n)}. \end{aligned}$$

By combining the above two estimates, we obtain (6.48).

Next we prove (6.49). We first consider  $\mathcal{F}_{\mathcal{Q},1}^{\{n\}}$ . By using (6.5) and (4.14), we have

$$\begin{aligned} \left\| r^{\frac{3}{2}} \mathcal{F}_{\mathcal{Q},1}^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} &\lesssim \sum_{Z^b \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^c)} \left\| r \partial \phi \right\|_{L_u^2 L^\infty(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \left\| r^{-\frac{1}{2}} \partial Z^{(b)} \phi \right\|_{L^2(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \\ &\lesssim \Delta_0 \sum_{Z^b \sqcup Z^c = Z^n} |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^c)} M^{-\frac{1}{2}} |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^b) + \frac{1}{2}} \\ &= \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + \frac{1}{2} + \zeta(Z^n)}. \end{aligned}$$

Due to  $1 \leq b \leq a \leq n-1$  in  $\mathcal{F}_{\mathcal{Q},2}^{\{n\}}$ , we can derive by using (4.20) and (4.22) that

$$\begin{aligned} \left\| r^{\frac{3}{2}} \mathcal{F}_{\mathcal{Q},2}^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} &\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^c)} \left\| r \partial Z^{(b)} \phi \right\|_{L_u^2 L_{\underline{u}}^\infty L_\omega^4(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \left\| r^{\frac{1}{2}} \partial Z^{(a)} \phi \right\|_{L_u^2 L_{\underline{u}}^2 L_\omega^4(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \\ &\lesssim \Delta_0 M^{-\frac{1}{2}} \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^b) + \zeta(Z^c)} |u_1|^{\zeta(Z^a) - \frac{\gamma_0}{2} + \frac{1}{2}} \\ &\lesssim \Delta_0 M^{-\frac{1}{2}} |u_1|^{-\gamma_0 + \frac{1}{2} + \zeta(Z^n)}. \end{aligned}$$

By combining the estimates for  $\mathcal{F}_{\mathcal{Q},1}^{\{n\}}$  and  $\mathcal{F}_{\mathcal{Q},2}^{\{n\}}$ , we obtain (6.49).

Finally we consider the term  $\mathcal{F}_{\mathcal{C}}^{\{n\}}$ . Recall the definition of  $\mathcal{F}_{\mathcal{C}}^{\{n\}}$  from Lemma 5.10, we first show (6.50). Note that at least one of  $b$  or  $c$  is  $\leq 1$ . Assume, without loss of generality,  $b \leq c$ .

Since  $a \geq 1$  in  $\mathcal{F}_{\mathcal{C}}^{\{n\}}$ ,  $b + c \leq n-1$ . Thus  $0 \leq b \leq c \leq n-1$  and  $b \leq 1$ . In view of (6.5), (4.21) and (4.24), we have

$$\begin{aligned} \left\| r^{\frac{1}{2}} \mathcal{F}_{\mathcal{C}}^{\{n\}} \right\|_{L^2(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} &\lesssim \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n \\ a \geq 1, b \leq c}} |u_1|^{\zeta(Z^d)} \left\| r \partial Z^{(b)} \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \\ &\quad \times \left\| r^{\frac{1}{2}} \partial Z^{(c)} \phi \right\|_{L_{\underline{u}}^2 L_u^\infty L_\omega^4(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \left\| Z^{(a)} \phi \right\|_{L_{\underline{u}}^\infty L_u^2 L_\omega^4(\mathcal{D}_{u_1}^{\frac{u_1}{1}})} \\ &\lesssim \Delta_0^{\frac{3}{2}} M^{-\frac{1}{2}} |u_1|^{-\frac{3}{2}\gamma_0 - \frac{1}{2} + \zeta(Z^n)}, \end{aligned}$$

which gives (6.50).

By using (6.5), (4.20) and (4.17), we have

$$\begin{aligned}
\left\| r^{\frac{3}{2}} \mathcal{F}_c^{\{n\}} \right\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} &\lesssim \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n \\ a \geq 1, b \leq c}} |u_1|^{\zeta(Z^d)} \left\| r \partial Z^{(b)} \phi \right\|_{L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
&\quad \times \left\| r \partial Z^{(c)} \phi \right\|_{L_u^2 L_{\underline{u}}^\infty L_{\omega}^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \left\| r^{-\frac{1}{2}} Z^{(a)} \phi \right\|_{L_u^2 L_{\underline{u}}^2 L_{\omega}^4(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
&\lesssim \Delta_0^{\frac{3}{2}} |u_1|^{-\frac{3\gamma_0}{2} - \frac{1}{2} + \zeta(Z^n)}.
\end{aligned}$$

Thus (6.51) is proved.  $\square$

**6.4. Boundedness theorem.** We now combine the energy and weighted energy inequalities in Section 6.2 and the error estimates in Section 6.3 to give the boundedness of the energy and the weighted energy. The proof follows similarly as for Proposition 5.14.

**Theorem 6.10** (Boundedness of energies). *For  $n \leq 3$ , under the assumptions (A1) and (A2), there hold for  $(u, \underline{u}) \in \mathcal{I}$  that*

$$\sup_{-\underline{u} \leq u' \leq u} |u'|^{\gamma_0 - 2\zeta(Z^n)} \left( E[Z^{(n)} \phi](\mathcal{H}_{u'}^{\underline{u}}) + E[Z^{(n)} \phi](\mathcal{H}_{\underline{u}}^{u'}) \right) \lesssim \mathcal{E}_{n, \gamma_0} + M^{-2} \Delta_0^2 |u|^{1-\gamma_0}, \quad (6.56)$$

$$\begin{aligned}
&\sup_{-\underline{u} \leq u' \leq u} |u'|^{-2\zeta(Z^n) - 1 + \gamma_0} \left( \mathcal{W}_1[Z^{(n)} \phi](\mathcal{H}_{u'}^{\underline{u}}) + \mathcal{W}_1[Z^{(n)} \phi](\mathcal{H}_{\underline{u}}^{u'}) + \mathcal{W}_1[Z^{(n)} \phi](\mathcal{D}_{u'}^{\underline{u}}) \right) \\
&\lesssim \mathcal{E}_{n, \gamma_0} + M^{-2} \Delta_0^2 |u|^{1-\gamma_0}.
\end{aligned} \quad (6.57)$$

*Remark 6.11.* We can find a universal constant  $C_3 \geq 1$  such that both of the inequalities are bounded by

$$C_3 \left( \mathcal{E}_{n, \gamma_0} + M^{-2} \Delta_0^2 |u|^{1-\gamma_0} \right).$$

*Proof.* We first consider (6.56). Similar to (5.31), we have  $\square_{\mathbf{g}} Z^{(n)} \phi - q Z^{(n)} \phi = \mathcal{F}$ , where

$$\mathcal{F}^\# = \left[ \square_{\mathbf{g}}, Z^{(n)} \right] \phi + \mathcal{F}_c^{\{n\}} + \mathcal{F}_q^{\{n\}}, \quad \mathcal{F}^\flat = \left[ Z^{(n)}, q \right] \phi.$$

We may apply the first inequality in Lemma 5.13 to treat  $\mathcal{F}^\flat$ , and apply (6.46), (6.48) and (6.50) to treat  $\mathcal{F}^\#$ . By using Proposition 6.4, we have due to  $\Delta_0 M^{-1} < 1$  that

$$\begin{aligned}
&\sup_{-\underline{u} \leq u' \leq u} |u'|^{\gamma_0 - 2\zeta(Z^n)} \left( E[Z^{(n)} \phi](\mathcal{H}_{u'}^{\underline{u}}) + E[Z^{(n)} \phi](\mathcal{H}_{\underline{u}}^{u'}) \right) \\
&\lesssim \mathcal{E}_{n, \gamma_0} + M^{-2} \Delta_0^2 \left( |u|^{1-\gamma_0} + M |u|^{-2\gamma_0} \right) + \sup_{-\underline{u} \leq u' \leq u} \sum_{i=1}^n E[Z^{(n-i)} \phi](\mathcal{H}_{\underline{u}}^{u'}) |u|^{\gamma_0 - 2\zeta(Z^{n-i})},
\end{aligned}$$

where the last term vanishes when  $n = 0$ . Thus under the assumptions (A1) and (A2), (6.56) holds true by induction.

Let  $(u_1, \underline{u}_1) \in \mathcal{I}$  be fixed. To see the weighted energy estimate for  $Z^{(n)} \phi$ , by using Proposition 6.6, (3.3), (3.7) and the fact that  $\|r^{\frac{1}{2}} \partial Z^{(a)} \phi\|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 \lesssim |u_1|^{-\gamma_0+1} \mathcal{E}_{a, \gamma_0}$  for  $a \leq n$ , we derive that

$$\begin{aligned}
&\sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0 - 1 - 2\zeta(Z^n)} \left( \mathcal{W}_1[Z^{(n)} \phi](\mathcal{D}_{u_1}^{\underline{u}_1}) + \mathcal{W}_1[Z^{(n)} \phi](\mathcal{H}_{u_1}^{\underline{u}_1}) + \mathcal{W}_1[Z^{(n)} \phi](\mathcal{H}_{\underline{u}_1}^u) \right) \\
&\lesssim |u_1|^{-\gamma_0} \Delta_0^2 (M^{-1} + 1) + \sum_{Z^m \subsetneq Z^n} |u_1|^{-2\zeta(Z^m) - 1 + \gamma_0} \mathcal{W}_1[Z^{(m)} \phi](\mathcal{D}_{u_1}^{\underline{u}_1}) \\
&\quad + \sup_{-\underline{u}_1 \leq -\underline{u} \leq u \leq u_1} |u|^{\gamma_0 - 2\zeta(Z^n)} \left( E[Z^{(n)} \phi](\mathcal{H}_{\underline{u}}^u) + E[Z^{(n)} \phi](\mathcal{H}_{\underline{u}}^u) \right) + \mathcal{E}_{n, \gamma_0},
\end{aligned}$$

where we also employed the second inequality in Lemma 5.13, (6.47), (6.49) and (6.51). We then substitute the result of (6.56) followed with an induction argument to derive that

$$\begin{aligned} \sup_{-\underline{u}_1 \leq u \leq \underline{u}_1} |u|^{\gamma_0-1+2\zeta(Z^n)} \left( \mathcal{W}_1[Z^n]\phi(\mathcal{D}_{\underline{u}_1}^{\underline{u}_1}) + \mathcal{W}_1[Z^n]\phi(\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}) + \mathcal{W}_1[Z^n]\phi(\mathcal{H}_{\underline{u}_1}^u) \right) \\ \lesssim |u_1|^{-\gamma_0} \Delta_0^2 (M^{-1} + 1) + M^{-2} \Delta_0^2 |u_1|^{1-\gamma_0} + \mathcal{E}_{n,\gamma_0}. \end{aligned}$$

Thus (6.57) is proved.  $\square$

**6.5. Proof of Theorem 2.5.** With  $\Delta_0 = \mathcal{C}_1 \mathcal{E}_{3,\gamma_0}$  and  $\mathcal{C}_1 = 4C_3C$ , in view of (6.56), (6.57) and  $M = 3C\delta_1^{\frac{1}{2}}$ , to improve the bootstrap assumptions (6.3) and (6.4), we need to have

$$C_3 (\mathcal{C}_1^{-1} \Delta_0 + (3C)^{-2} \delta_1^{-1} \Delta_0 \mathcal{C}_1 \mathcal{E}_{3,\gamma_0} |u|^{1-\gamma_0}) < 2\Delta_0,$$

where  $C, \mathcal{C}_1, C_3 > 1$ . Identically,

$$C_3 (\mathcal{C}_1^{-1} + (3C)^{-2} \mathcal{C}_1 |u|^{1-\gamma_0}) < 2$$

which requires

$$\frac{4}{9} C_3^2 C^{-1} |u|^{1-\gamma_0} \leq \frac{4}{9} C_3^2 C^{-1} |u_0(R)|^{1-\gamma_0} < \frac{7}{4}. \quad (6.58)$$

Next we determine  $R$  so that (A1) can be improved and (A2) can be satisfied.

Note that by using (6.6), for  $(u, \underline{u}) \in \mathcal{I}$  we have

$$r |H^{\alpha\beta}(u, \underline{u}, \omega)| \leq C_2 \Delta_0^{\frac{1}{2}} |u|^{-\frac{1}{2}\gamma_0 + \frac{1}{2}}. \quad (6.59)$$

Thus, in view of (6.2),

$$\sup_{\alpha, \beta} r(h - H^{\alpha\beta}) > 3C\delta_1^{\frac{1}{2}} - |u|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \mathcal{C}_1^{\frac{1}{2}} C_2 \mathcal{E}_{3,\gamma_0}^{\frac{1}{2}}.$$

With

$$|u|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \mathcal{C}_1^{\frac{1}{2}} C_2 \leq |u_0(R)|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \mathcal{C}_1^{\frac{1}{2}} C_2 < 2C, \quad (6.60)$$

the strict inequality in (A1) holds true.

By a direct substitution, to make (A2) hold, we need

$$\mathcal{C}_1^{\frac{1}{2}} \left( \frac{R}{2} \right)^{\frac{1}{2} - \frac{\gamma_0}{2}} < \frac{1}{2} C \tilde{C}^{-1}.$$

Thus we require  $u_0(R)$  to satisfy the second inequalities in (6.58), (6.60), and  $R$  to satisfy the above inequality. With the help of  $|u_0(R)| > \frac{1}{2}R$ , we can fix  $R(\gamma_0, C)$ , the lower bound of  $R$ , such that these inequalities hold.

## 7. Einstein scalar fields

In this section, we apply the approach in Section 6 to prove the nonlinear stability result for Einstein scalar fields, exterior to a Schwarzschild cone with small positive mass, which is stated in Theorem 2.6.

Under the wave coordinates <sup>23</sup>, let  $\mathbf{h}_{\mu\nu} = \mathbf{g}_{\mu\nu} - \mathbf{m}_{\mu\nu}$ . The Einstein equation with scalar fields takes the form

$$\begin{cases} \square_{\mathbf{g}} \mathbf{h}_{\mu\nu} = (A_{\mu\nu}^{\alpha\beta} + G_{\mu\nu}^{\alpha\beta}(\mathbf{h})) \partial_{\alpha} \mathbf{h} \partial_{\beta} \mathbf{h} + \partial_{\mu} \phi \cdot \partial_{\nu} \phi + q_0 \mathbf{g}_{\mu\nu} \phi^2 \\ \square_{\mathbf{g}} \phi - q_0 \phi = 0 \end{cases}, \quad (7.1)$$

where we assume the constant  $0 \leq q_0 \leq 1$  without loss of generality. For each fixed  $(\mu, \nu)$ ,  $A_{\mu\nu}$  is a matrix of constant components. For each fixed  $(\mu, \nu)$ ,  $G_{\mu\nu}^{\alpha\beta}(\mathbf{h})$  are smooth functions of  $\mathbf{h}$ . They represent the product  $\mathbf{h}_{\alpha\beta}$  or  $H^{\alpha\beta}$  with components of  $\mathbf{g}$ . We will symbolically represent

<sup>23</sup>The wave coordinates  $\{x^{\mu}\}_{\mu=0}^3$  are required to be the solution of  $\square_{\mathbf{g}} x^{\mu} = 0$  where  $\square_{\mathbf{g}}$  is the Laplace-Beltrami operator of the Einstein metric  $\mathbf{g}$ .

all such functions as  $G(\mathbf{h})$ . Such  $G(\mathbf{h})$  vanishes at  $(\mathbf{h}_{\alpha\beta}) \equiv (\mathbf{0})$ . Other constant coefficients on the righthand side of (7.1) have been simplified to be 1.<sup>24</sup>

Let  $m_0 > 0$  be a fixed small number. Denote  $\mathring{\mathbf{h}}_{\mu\nu} = \frac{m_0}{r} \delta_{\mu\nu}$ . We decompose

$$\mathbf{h}_{\mu\nu} = \mathbf{h}_{\mu\nu}^1 + \mathring{\mathbf{h}}_{\mu\nu}.$$

This reduces (7.1) to the equations for  $(\mathbf{h}^1, \phi)$ :

$$\begin{cases} \square_{\mathbf{g}} \mathbf{h}_{\mu\nu}^1 = \mathcal{N}(\mathbf{h})(\partial \mathring{\mathbf{h}} + \partial \mathbf{h}^1) \cdot \partial \mathbf{h}^1 + \partial \phi \cdot \partial \phi + q_0 \mathbf{g}_{\mu\nu} \phi^2 + \mathcal{S}_{\mu\nu}, \\ \square_{\mathbf{g}} \phi - q_0 \phi = 0, \end{cases} \quad (\text{ES})$$

where  $\mathcal{N}(\mathbf{h}) = 1 + G(\mathbf{h})$  symbolically and

$$S_{\mu\nu} = -\square_{\mathbf{g}} \mathring{\mathbf{h}}_{\mu\nu} + \mathcal{N}(\mathbf{h}) \partial \mathring{\mathbf{h}} \partial \mathring{\mathbf{h}}. \quad (7.2)$$

We remark that the structure of wave coordinates implies (see [24, Lemma 8.1])

$$|\underline{L}H^{\underline{L}L}| \lesssim |\bar{\partial}H| + |H \cdot \partial H| \quad (7.3)$$

which can provide some convenience to the proof of boundedness of energy. This will be shown shortly. In this section, we prove Theorem 2.6 by applying our approach in Section 6 to  $(\mathbf{h}^1, \phi)$  with potentials  $(0, q_0)$ .

Let  $1 < \gamma_0 < 2$  be fixed. We define for the initial data  $\mathbf{h}^1[0] = (\mathbf{h}^1(0), \partial_t \mathbf{h}^1(0))$  and  $\phi[0] = (\phi(0), \partial_t \phi(0))$  the weighted norm

$$\mathcal{E}_{3, \gamma_0, R}(\mathbf{h}^1, \phi) = \mathcal{E}_{3, \gamma_0, R, 0}(\mathbf{h}^1[0]) + \mathcal{E}_{3, \gamma_0, R, q_0}(\phi[0]).$$

The extra subindex  $C$  of  $\mathcal{E}_{3, \gamma_0, R, C}$  on the right denotes the constant potential function of each equation. In this section, we assume

$$\mathcal{E}_{3, \gamma_0, R}(\mathbf{h}^1, \phi) \leq C_0 m_0^2 \quad (7.4)$$

where  $C_0 \geq 1$  is a fixed constant,  $R \geq 2$  with the lower bound determined later.  $\mathcal{E}_{3, \gamma_0, R, 0}(\mathbf{h}^1[0]) < \infty$  implies that  $\liminf_{|x| \rightarrow \infty} \mathbf{g}_{\mu\nu}(0, x) = \mathring{\mathbf{g}}_{\mu\nu}$ , where  $\mathring{\mathbf{g}}_{\mu\nu} = \mathbf{m}_{\mu\nu} + \frac{m_0}{r} \delta_{\mu\nu}$ . It is direct to compute

$$\mathring{\mathbf{g}}^{\underline{L}L} = \mathring{\mathbf{g}}^{\underline{L}L} < \frac{-m_0}{2r}. \quad (7.5)$$

To prove Theorem 2.6, we fix

$$h = -\frac{m_0}{20r}, \text{ i.e. } M_0 = -\frac{m_0}{20}$$

and show that with  $M = m_0$  and the constant potentials  $(0, q_0)$ , all the norms in (2.13) for  $Z^{(i)}(\mathbf{h}^1, \phi)$ ,  $i \leq 3$  remain small in the region  $\{u \leq u_0(M_0)\}$  provided that  $m_0$  in (7.4) is sufficiently small and  $R \geq R_0(\gamma_0, C_0)$ . The constant  $R_0(\gamma_0, C_0)$  will be specified at the end of the proof.

### 7.1. Preliminaries.

**Lemma 7.1.** *If  $\mathcal{E}_{3, \gamma_0, R, 0}(\mathbf{h}^1[0]) \leq C_0 m_0^2$  with  $C_0 \geq 1$ , there exists a constant  $\tilde{C}_1 \geq 1$  depending on  $C_0$  and  $\gamma_0$  such that if  $\tilde{C}_1 R^{-\frac{\gamma_0}{2} + \frac{1}{2}} \leq 1$  then*

$$r(h - H^{\underline{L}L}) > \frac{m_0}{3}, \quad r(h - H^{LL}) > \frac{m_0}{3} \quad \text{on } \Sigma_0 \cap \{r \geq R\}. \quad (7.6)$$

*Proof.* It follows from (3.9) that

$$r|\partial^{(l)} \mathbf{h}^1(u, \underline{u}, \omega)| \lesssim |u|^{-\frac{\gamma_0}{2} + \frac{1}{2} - l} \mathcal{E}_{2, \gamma_0}^{\frac{1}{2}}, \quad l \leq 1 \quad (7.7)$$

Thus  $r|\mathbf{h}| \lesssim (\mathcal{E}_{2, \gamma_0}^{\frac{1}{2}} + m_0)$ .

<sup>24</sup>See [24] for the more detailed structure. We do not need the weak null structure to prove the result of Theorem 2.6.



For small  $\mathbf{h}$ ,  $H^{\mu\nu} = -\mathbf{h}^{\mu\nu} + \mathcal{O}^{\mu\nu}(\mathbf{h}^2)$  where  $\mathbf{h}^{\mu\nu} = \mathbf{m}^{\mu\mu'} \mathbf{m}^{\nu\nu'} \mathbf{h}_{\mu'\nu'}$  and  $\mathcal{O}^{\mu\nu}(\mathbf{h}^2)$  vanishes to second order at  $\mathbf{h} = 0$ . Thus

$$|H^{\mu\nu} - \overset{\circ}{H}{}^{\mu\nu}| \lesssim |\mathbf{h}^1| + |\mathbf{h}|^2 \quad (7.8)$$

where  $\overset{\circ}{H}{}^{\mu\nu} = \overset{\circ}{\mathbf{g}}{}^{\mu\nu} - \mathbf{m}^{\mu\nu}$ . By using the above two inequalities, (7.7) and (7.5), we can derive

$$r(h - H^{LL}) \geq \frac{9}{20}m_0 - C \left( |u|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \mathcal{E}_{2,\gamma_0}^{\frac{1}{2}} + r^{-1}(m_0 + \mathcal{E}_{2,\gamma_0}^{\frac{1}{2}})^2 \right),$$

where the universal constant  $C \geq 1$ . Due to  $|u(R)| > \frac{R}{2}$  and  $m_0 < 1$ , if we choose  $R$  such that

$$C \left( \left( \frac{R}{2} \right)^{\frac{1}{2} - \frac{\gamma_0}{2}} C_0^{\frac{1}{2}} + R^{-1}(1 + C_0^{\frac{1}{2}})^2 \right) < \frac{1}{10},$$

then (7.6) holds. The same treatment works the same for  $r(h - H^{LL})$ . The proof is therefore complete.  $\square$

The above result shows that the assumption (A1) holds true on  $\Sigma_0 \cap \{r \geq R\}$  with  $M = m_0$  if  $R^{\frac{1}{2} - \frac{\gamma_0}{2}} \leq \tilde{C}_1^{-1}$ . We will further specify the lower bound  $R_0(\gamma_0, C_0)$  during the proof of Theorem 2.6.

To prove Theorem 2.6, we make the following bootstrap assumptions:

Let  $\underline{u}_* > -u_0$  be any fixed number, where  $u_0 = -r_*(-\frac{m_0}{20}, R)$ . In  $\mathcal{I} = \{(u, \underline{u}), -\underline{u}_* \leq -\underline{u} \leq u \leq u_0\}$ , suppose that (A1) holds and that (BA<sub>3</sub>) holds for  $(\mathbf{h}^1, \phi)$  with  $\Delta_0 = C_1 m_0^2$ ,  $C_1 \geq 1$  to be chosen. (BA<sub>3</sub>) is restated as below for  $n \leq 3$  and  $Z \in \{\Omega, \partial\}$ ,

$$\begin{aligned} E[Z^{(n)}(\mathbf{h}^1, \phi)](\mathcal{H}_{\underline{u}}^u) + E[Z^{(n)}(\mathbf{h}^1, \phi)](\underline{\mathcal{H}}_{\underline{u}}^u) &\leq 2\Delta_0 |u|^{-\gamma_0 + 2\zeta(Z^n)} \\ \mathcal{W}_1[Z^{(n)}(\mathbf{h}^1, \phi)](\mathcal{H}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}(\mathbf{h}^1, \phi)](\underline{\mathcal{H}}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}(\mathbf{h}^1, \phi)](\mathcal{D}_{\underline{u}}^u) &\leq 2\Delta_0 |u|^{-\gamma_0 + 1 + 2\zeta(Z^n)} \end{aligned} \quad (7.9)$$

for all  $(u, \underline{u}) \in \mathcal{I}$ .

Thus the full set of decay estimates in Section 4 hold for  $(\mathbf{h}^1, \phi)$ . We will quote the results in the proofs whenever necessary.

Before starting to prove Theorem 2.6, we highlight the differences in analysis in comparison with Section 6.

- In the Einsteinian case, there holds  $H^{\mu\nu} = -\mathbf{h}^{\mu\nu} + \mathcal{O}(\mathbf{h}^2)$ . Thus  $H$  does not depend on  $\partial(\mathbf{h}^1, \phi)$ . In terms of regularity it seems better than in Section 6. Nevertheless, the small static part  $\overset{\circ}{\mathbf{h}}$  in  $\mathbf{h}$  slows down the decay of  $H$ . In the proofs of Theorem 2.1 and Theorem 2.5, we take advantage of the fact that  $R^{1-\gamma_0}$  can be small so as to improve the bootstrap assumption, while in this section, we lose such extra smallness for the critical terms. This requires us to separate carefully the evolutionary part of the metric  $\mathbf{h}^1$  from  $\overset{\circ}{\mathbf{h}}$  in analysis. For the borderline terms appeared in the error estimates, we bound them by energies which have been controlled in the induction. Although these energies are of the same order, the signature of the derivative  $Z^n$  is strictly smaller. See Lemma 7.5 and Section 7.3.
- Comparing the equations of (ES) with the standard equation (1.1), (ES) have quite a few extra error terms on the righthand side. The term  $q_0 \mathbf{g}_{\mu\nu} \phi^2$  in (ES) requires an improved decay estimate for the scalar field if  $q_0 > 0$ . The small static part  $\overset{\circ}{\mathbf{h}}$  of the metric also gives a set of error terms. Such set of error estimates are included in Lemma 7.7.
- The changed asymptotic behavior of the metric  $H$  influences the proofs of Proposition 6.4 and Proposition 6.6. We will go through the analysis in the proof of Proposition 7.4.

We first give an improve decay estimate of  $\phi$  compared with (4.2) in Proposition 4.1.

**Proposition 7.2.** *For  $(u, \underline{u}) \in \mathcal{I}$ , there holds*

$$q_0 |r^{\frac{5}{4}} Z^{(l)} \phi|^2 \lesssim (\mathcal{E}_{l+2,\gamma_0} + \Delta_0) |u|^{-\gamma_0 + 2\zeta(Z^{(l)}) + \frac{1}{2}}, \quad l \leq 1. \quad (7.10)$$

*Proof.* To begin with, noting that  $0 \leq q_0 \leq 1$  is now a fixed constant, (3.4) can be improved to

$$q_0^2 \int_{S_r} r^{6+2\gamma_0-4\zeta(Z^i)} |Z^i \phi|^4 d\omega \lesssim \mathcal{E}_{i+1, \gamma_0}^2, \quad i \leq 2. \quad (7.11)$$

Similar to [19, Section 3.2], by applying (3.1) to  $f = Z^l \phi$  with  $l \leq 1$ ,  $\gamma = \frac{5}{4}$ ,  $\gamma'_0 = \frac{1}{2}$  and  $\gamma_2 = 1$ , we have

$$\begin{aligned} & \sup_{S_{u, \underline{u}}} |r^{\frac{5}{4}} Z^l \phi|^4 \\ & \lesssim \int_{S_{u, -u}} r^3 |\Omega^{(\leq 1)} Z^l \phi|^4 + \left( \int_{\mathcal{H}_u^u} |\Omega^{(\leq 2)} Z^l \phi|^2 \right) \left( \int_{\mathcal{H}_u^u} r^{-\frac{3}{2}} |L'(\Omega^{(\leq 1)}(r^{\frac{5}{4}} Z^l \phi))|^2 \right). \end{aligned}$$

By using (3.14) and  $|L'(r^{\frac{5}{4}} f)| \lesssim r^{\frac{1}{4}}(|f| + |L'(rf)|)$ , we can derive that

$$\begin{aligned} \int_{\mathcal{H}_u^u} r^{-\frac{3}{2}} q_0 |L'(\Omega^{(\leq 1)}(r^{\frac{5}{4}} Z^l \phi))|^2 & \lesssim q_0 \int_{\mathcal{H}_u^u} r^{-1} \left( |\Omega^{(\leq 1)} Z^l \phi|^2 + |L'(r \Omega^{(\leq 1)} Z^l \phi)|^2 \right) \\ & \lesssim \mathcal{W}_1[\Omega^{(\leq 1)} Z^l \phi](\mathcal{H}_u^u) + |u|^{-1} E[\Omega^{(\leq 1)} Z^l \phi](\mathcal{H}_u^u) \\ & \lesssim |u|^{-\gamma_0+1+2\zeta(Z^l)} \Delta_0. \end{aligned}$$

Thus we conclude by using (7.11) that

$$\begin{aligned} q_0^2 \sup_{S_{u, \underline{u}}} |r^{\frac{5}{4}} Z^l \phi|^4 & \lesssim q_0^2 \int_{S_{u, -u}} r^3 |\Omega^{(\leq 1)} Z^l \phi|^4 + E[\Omega^{(\leq 2)} Z^l \phi](\mathcal{H}_u^u) |u|^{-\gamma_0+1+2\zeta(Z^l)} \Delta_0 \\ & \lesssim |u|^{-2\gamma_0+1+4\zeta(Z^l)} \Delta_0^2. \end{aligned}$$

□

If  $Z \in \{\partial, \Omega_{\mu\nu}, 0 \leq \mu < \nu \leq 3\}$ ,

$$|Z^{(l)} \overset{\circ}{\mathbf{h}}| \lesssim m_0 r^{-1+\zeta(Z^l)}, \quad l \geq 0. \quad (7.12)$$

If  $Z \in \{\Omega, \partial\}$ , we can use the fact

$$|Z^{(l)} \overset{\circ}{\mathbf{h}}| \lesssim m_0 r^{-1-l}, \quad l \geq 0 \quad (7.13)$$

to treat the error terms.

**Proposition 7.3** (Decay estimates).

$$|\mathbf{h}, H| \lesssim r^{-1} \left( m_0 + \Delta_0^{\frac{1}{2}} |u|^{-\frac{\gamma_0}{2} + \frac{1}{2}} \right), \quad (7.14)$$

$$|\mathcal{D}_* H| \lesssim |\mathcal{D}_* \mathbf{h}^1| + r^{-2} m_0, \quad \mathcal{D}_* \in \{\partial, \underline{\partial}, \bar{\partial}\}, \quad (7.15)$$

$$r|u| |\partial H| \lesssim \Delta_0^{\frac{1}{2}} + m_0, \quad (7.16)$$

$$|Z^{(n)} H, Z^{(n)} G(\mathbf{h})| \lesssim \sum_{Z^a \sqcup Z^b = Z^n} |u|^{\zeta(Z^b)} \left( |Z^{(a)} \mathbf{h}^1| + m_0 r^{-1+\zeta(Z^a)} \right), \quad n \leq 3, \quad (7.17)$$

$$|Z^{(n)} H| \lesssim \sum_{\substack{Z^a \sqcup Z^b = Z^n \\ a \geq 1}} |u|^{\zeta(Z^b)} \left( |Z^{(a)} \mathbf{h}^1| + m_0 r^{-1-a} \right), \quad 1 \leq n \leq 3. \quad (7.18)$$

*Proof.* The inequalities (7.14)-(7.16) can follow directly from the property of  $H^{\mu\nu}(\mathbf{h})$ ,  $\mathbf{h} = \overset{\circ}{\mathbf{h}} + \mathbf{h}^1$  and applying (4.1) and (4.2) to  $\mathbf{h}^1$ . (7.17) can be similarly proved as Lemma 5.8. (7.18) is an improved version, which relies on (7.13) instead of (7.12). □

**Proposition 7.4.** Consider

$$\square_{\mathbf{g}} \varphi = \mathcal{F} + q\varphi$$

where  $q \geq 0$  is a constant,  $\mathbf{h}^1$  in the Lorentzian metric  $\mathbf{g}_{\mu\nu} = \mathbf{h}_{\mu\nu}^1 + \overset{\circ}{\mathbf{h}}_{\mu\nu} + \mathbf{m}_{\mu\nu}$  satisfy the bootstrap assumptions (7.9) and (7.4). By assuming (A1), Proposition 6.6 holds true, and Proposition 6.4 holds if assuming (7.3) instead of (A2).

*Proof.* We first confirm that if the background metric is the Einstein metric  $\mathbf{g}(\mathbf{h})$ , the weighted energy estimate in Proposition 6.6 holds. The proof is carried out in two steps.

**Step 1.** For the terms  $I$ ,  $II$  and  $III$  defined in (6.20), we will show that for  $(u_1, \underline{u}_1) \in \mathcal{I}$  there holds

$$\begin{aligned} & \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} |I| + |II| + |III| \\ & \lesssim \left( \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} + m_0^{\frac{1}{2}} \right) \left\{ |u_1|^{-\frac{3\gamma_0}{2}+1} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0} + \sup_{-\underline{u}_1 \leq \underline{u} \leq u_1} E[\varphi](\mathcal{H}_{\underline{u}}^{\underline{u}_1}) |\underline{u}_1|^{\gamma_0} \right) \right. \\ & \quad \left. + \int_{-\underline{u}_1}^{u_1} \left( |u|^{-\frac{\gamma_0+1}{2}} + |u|^{\frac{\gamma_0}{2}-2} \right) \mathcal{W}_1[\varphi](\mathcal{H}_u^{\underline{u}_1}) du \right\}. \end{aligned} \quad (7.19)$$

This implies that (6.17) holds with the bound replaced by the righthand side of (7.19).

We first give the following straightforward calculations:

$$\int_{\mathcal{D}_{u_1}^{\underline{u}_1}} r^{-2} (\partial\varphi)^2 \lesssim |u_1|^{-1} \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}), \quad (7.20)$$

$$\int_{\mathcal{D}_{u_1}^{\underline{u}_1}} r^{-1} |\partial\varphi \cdot \bar{\partial}\varphi| \lesssim \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} r^{-1} (|\bar{\partial}\varphi| |\partial\varphi| + |\bar{\partial}\varphi|^2) \quad (7.21)$$

$$\begin{aligned} & \lesssim \|r^{-1} \partial\varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \|\bar{\partial}\varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} + \int_{-\underline{u}_1}^{u_1} |u|^{-1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) du \\ & \lesssim |u_1|^{-\gamma_0} \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u_1|^{\gamma_0} + \sup_{-\underline{u}_1 \leq \underline{u} \leq u_1} E[\varphi](\mathcal{H}_{\underline{u}}^{\underline{u}_1}) |\underline{u}_1|^{\gamma_0} \right), \end{aligned} \quad (7.22)$$

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} r^{-2} |L(r\varphi) \partial\varphi| & \lesssim \int_{-\underline{u}_1}^{u_1} \|r^{-\frac{1}{2}} L(r\varphi)\|_{L^2(\mathcal{H}_u^{\underline{u}_1})} \|r^{-\frac{1}{2}} \partial\varphi\|_{L^2(\mathcal{H}_u^{\underline{u}_1})} r^{-1} du \\ & \lesssim m_0^{-\frac{1}{2}} \int_{-\underline{u}_1}^{u_1} \left( \mathcal{W}_1[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\frac{\gamma_0}{2}-2} + |u|^{-\frac{3\gamma_0}{2}} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0} \right) du. \end{aligned} \quad (7.23)$$

We also note that, by using (7.14) and (7.15),

$$|H| |\partial H| \lesssim r^{-1} (m_0 + \Delta_0^{\frac{1}{2}}) (r^{-2} m_0 + |\partial \mathbf{h}^1|), \quad (7.24)$$

$$r(|LH| + |H| |\partial H|) \lesssim r |L \mathbf{h}^1| + (m_0 + \Delta_0^{\frac{1}{2}}) |\partial \mathbf{h}^1| + r^{-1} (m_0 + \Delta_0^{\frac{1}{2}}). \quad (7.25)$$

The parts of  $\partial \mathbf{h}^1$  will be treated similar to Section 6. We first control  $I$  and  $II$  in view of (6.22). By using (7.20), (7.24) and (4.3), we have

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} r |H| |\partial H| |\partial\varphi|^2 & \lesssim (m_0 + \Delta_0^{\frac{1}{2}}) \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} (|\partial \mathbf{h}^1| |\partial\varphi|^2 + m_0 r^{-2} |\partial\varphi|^2) \\ & \lesssim (m_0 + \Delta_0^{\frac{1}{2}}) \left( \|r^{\frac{1}{2}} \partial \mathbf{h}^1\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \|r^{-\frac{1}{2}} \partial\varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \|r \partial\varphi\|_{L_{\underline{u}}^\infty L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \right. \\ & \quad \left. + m_0 |u_1|^{-1} \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) \right) \\ & \lesssim \left( m_0 + \Delta_0^{\frac{1}{2}} \right) \left( \Delta_0^{\frac{1}{2}} m_0^{-1} + m_0 \right) |u_1|^{-\frac{3}{2}\gamma_0 + \frac{1}{2}} \\ & \quad \times \left( \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0} + \sup_{-\underline{u}_1 \leq \underline{u} \leq u_1} E[\varphi](\mathcal{H}_{\underline{u}}^{\underline{u}_1}) |\underline{u}_1|^{\gamma_0} \right). \end{aligned}$$

Next by using (7.25), (7.22), (4.1) and (7.14), we can derive that

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} (r(|LH| + |H||\partial H|) + |H|) |\partial\varphi \cdot \bar{\partial}\varphi| &\lesssim \int_{\mathcal{D}_{u_1}^{u_1}} \left( r|L\mathbf{h}^1| + r^{-1}(m_0 + \Delta_0^{\frac{1}{2}}) \right) |\partial\varphi \cdot \bar{\partial}\varphi| \\ &\lesssim \left( \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} + m_0 \right) |u_1|^{-\frac{3}{2}\gamma_0+1} \left( \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u|^{\gamma_0} + \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u_1|^{\gamma_0} \right), \end{aligned}$$

where we used the estimate (6.25) to treat the product term with  $L\mathbf{h}^1$ .

By using (7.15) and (7.16), we have

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} r|\partial H||\bar{\partial}\varphi|^2 &\lesssim \left( \Delta_0^{\frac{1}{2}} + m_0 \right) \int_{-u_1}^{u_1} |u|^{-1} \|\bar{\partial}\varphi\|_{L^2(\mathcal{H}_u^{u_1})}^2 du \\ &\lesssim \left( \Delta_0^{\frac{1}{2}} + m_0 \right) |u_1|^{-\gamma_0} \sup_{-u_1 \leq u \leq u_1} |u|^{\gamma_0} E[\varphi](\mathcal{H}_u^{u_1}) \end{aligned}$$

and

$$\int_{\mathcal{D}_{u_1}^{u_1}} |L(r\varphi)\partial\varphi \cdot \partial H| \lesssim \int_{\mathcal{D}_{u_1}^{u_1}} |L(r\varphi) \cdot \partial\varphi \cdot \partial\mathbf{h}^1| + \int_{\mathcal{D}_{u_1}^{u_1}} r^{-2} m_0 |L(r\varphi)| |\partial\varphi|.$$

Note that the first term on the right can be treated as (6.26) and the second term is treated by using (7.23). Thus

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} |L(r\varphi)\partial\varphi \cdot \partial H| &\lesssim \left( \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} + m_0^{\frac{1}{2}} \right) \left( \int_{-u_1}^{u_1} \left( |u|^{-\frac{\gamma_0+1}{2}} + |u|^{\frac{\gamma_0}{2}-2} \right) \mathcal{W}_1[\varphi](\mathcal{H}_u^{u_1}) du \right. \\ &\quad \left. + |u_1|^{-\frac{3}{2}\gamma_0+1} \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u|^{\gamma_0} \right). \end{aligned}$$

Hence we completed the estimates for  $I$  and  $II$  in view of the symbolic formula in (6.22).

Next we consider the term  $III$  defined in (6.20) with the bound given in (6.23). The first term on the right of (6.23) has been treated. We will treat the remaining terms in the sequel. We note that, by using Proposition 7.3,

$$(r|\partial H| + |H|)|H| \lesssim r^{-2}(m_0 + \Delta_0^{\frac{1}{2}})^2 + (m_0 + \Delta_0^{\frac{1}{2}})|\partial\mathbf{h}^1|. \quad (7.26)$$

Thus, by using (4.1) for  $\partial\mathbf{h}^1$ ,

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} (r|\partial H| + |H|)|H||\partial\varphi|^2 &\lesssim (m_0 + \Delta_0^{\frac{1}{2}}) \int_{\mathcal{D}_{u_1}^{u_1}} \left( r^{-1}(m_0 + \Delta_0^{\frac{1}{2}}) + \Delta_0^{\frac{1}{2}} |u|^{-\frac{\gamma_0+1}{2}} \right) r^{-1} |\partial\varphi|^2 \\ &\lesssim m_0^{-1} (m_0 + \Delta_0^{\frac{1}{2}})^2 |u_1|^{-\gamma_0} \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u|^{\gamma_0}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{\mathcal{D}_{u_1}^{u_1}} (r|\partial H| + |H|)q\varphi^2 &\lesssim (m_0 + \Delta_0^{\frac{1}{2}}) \int_{\mathcal{D}_{u_1}^{u_1}} |u|^{-1} q\varphi^2 \\ &\lesssim (m_0 + \Delta_0^{\frac{1}{2}}) |u_1|^{-\gamma_0} \sup_{-u_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{u_1}) |u|^{\gamma_0}. \end{aligned}$$

Thus the estimate for  $III$  is completed and (7.19) is proved.

**Step 2.** We give the estimates of  $\text{Er}_1$  defined in (6.32) and  $\text{Er}_2$  defined in (6.33).

Noting that (7.14) implies  $r|H| \lesssim m_0 + \Delta_0^{\frac{1}{2}}$ , we can follow the same treatment as in the proof of Proposition 6.6 to obtain

$$\begin{aligned} |\text{Er}_1| &\lesssim \left( \Delta_0^{\frac{1}{2}} + m_0 \right) \left( |u_1|^{-1} \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r(L(r\varphi) - H^{\underline{L}\underline{L}}\underline{L}\varphi)^2 d\underline{u}d\underline{\omega} \right. \\ &\quad \left. + E[\varphi](\mathcal{H}_{u_1}^{\underline{u}_1}) + E[\varphi](\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}) + \sup_{-\underline{u}_1 \leq u \leq u_1} \int_{S_{u,-u}} r\varphi^2 d\omega \right), \\ |\text{Er}_2| &\lesssim \left( \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} + m_0^{\frac{1}{2}} \right) \left( \mathcal{W}_1[\varphi](\mathcal{H}_{u_1}^{\underline{u}_1}) + E[\varphi](\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}) + \sup_{-\underline{u}_1 \leq u \leq u_1} \|r^{-\frac{1}{2}}\varphi\|_{L^2(S_{u,-u})}^2 \right). \end{aligned}$$

By substituting these two estimates into (6.34) and noting that

$$\begin{aligned} \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r^3 |H|^2 |\underline{L}\varphi|^2 d\underline{u}d\underline{\omega} &\lesssim \int_{\mathcal{H}_{u_1}^{\underline{u}_1}} r(\Delta_0^{\frac{1}{2}} + m_0)^2 |\underline{L}\varphi|^2 d\underline{u}d\underline{\omega} \\ &\lesssim (\Delta_0^{\frac{1}{2}} + m_0)^2 m_0^{-1} E[\varphi](\mathcal{H}_{u_1}^{\underline{u}_1}), \end{aligned}$$

also using (7.19), we have

$$\begin{aligned} &\mathcal{W}_1[\varphi](\mathcal{D}_{u_1}^{\underline{u}_1}) + \mathcal{W}_1[\varphi](\mathcal{H}_{u_1}^{\underline{u}_1}) + \mathcal{W}_1[\varphi](\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}) \\ &\lesssim \| |r^{\frac{1}{2}} \partial \varphi| + r^{-\frac{1}{2}} |\varphi| + q_0^{\frac{1}{2}} r^{\frac{1}{2}} |\varphi| \|_{L^2(\Sigma_0^{u_1, \underline{u}_1})}^2 + \int_{\mathcal{D}_{u_1}^{\underline{u}_1}} \left( |\mathcal{F}(X\varphi + \varphi)| + \frac{q}{2} \varphi^2 \right) \\ &\quad + \left( m_0^{\frac{1}{2}} + \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} \right) \left( E[\varphi](\mathcal{H}_{u_1}^{\underline{u}_1}) + E[\varphi](\mathcal{H}_{\underline{u}_1}^{\underline{u}_1}) + \sup_{-\underline{u}_1 \leq u \leq u_1} \int_{S_{u,-u}} r\varphi^2 d\omega \right) \\ &\quad + \left( \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} + m_0^{\frac{1}{2}} \right) \left\{ |u_1|^{1-\frac{3\gamma_0}{2}+2p} \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) |u|^{\gamma_0-2p} \right. \\ &\quad \left. + |u_1|^{1-\frac{\gamma_0}{2}} \sup_{-\underline{u}_1 \leq u \leq u_1} E[\varphi](\mathcal{H}_u^{\underline{u}_1}) + \int_{-\underline{u}_1}^{u_1} \left( |u|^{-\frac{\gamma_0+1}{2}} + |u|^{\frac{\gamma_0}{2}-2} \right) \mathcal{W}_1[\varphi](\mathcal{H}_u^{\underline{u}_1}) du \right\}, \end{aligned}$$

where  $p \leq 0$  is an any constant. By repeating the same treatment on (6.39), we can derive the same inequality in Proposition 6.6.

Next we show Proposition 6.4 holds without the assumption (A2). Indeed, note that due to the wave coordinate condition (7.3), we can derive in view of (4.1), (7.15), (7.16) and (6.16) that

$$|\text{Tr}[\varphi]| \lesssim \left( \Delta_0^{\frac{1}{2}} + m_0 \right) r^{-1} |u|^{-1} \left( \left( \frac{|u|}{r} \right)^{\frac{1}{2}} |\underline{\partial}\varphi|^2 + |\partial\varphi| |\bar{\partial}\varphi| \right). \quad (7.27)$$

Substituting the above inequality into (6.14) and then using the Gronwall inequality, we can obtain Proposition 6.4 without the assumption (A2).  $\square$

**7.2. Error estimates.** The commutator  $[\square_{\mathbf{g}}, Z^{(n)}]\varphi$  with  $\varphi \in \{\mathbf{h}^1, \phi\}$  contains borderline terms, which are treated in the following result.

**Lemma 7.5.** *For  $\varphi \in \{\mathbf{h}^1, \phi\}$  and  $n \leq 3$ , there hold for  $(u_1, \underline{u}_1) \in \mathcal{I}$  that*

$$\begin{aligned} &|u_1|^{-\zeta(Z^n) + \frac{\gamma_0}{2} + \frac{1}{2}} \|r^{\frac{1}{2}} [\square_{\mathbf{g}}, Z^{(n)}]\varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ &\lesssim \left( \Delta_0 m_0^{-\frac{1}{2}} + m_0 \Delta_0^{\frac{1}{2}} \right) |u_1|^{-\frac{\gamma_0}{2} + \frac{1}{2}} + m_0^{\frac{1}{2}} \sup_{\substack{-\underline{u}_1 \leq u \leq u_1 \\ Z^b \subset Z^{n-1} \subset Z^n}} E[\partial Z^{(b)} \varphi]^{\frac{1}{2}} (\mathcal{H}_u^{\underline{u}_1}) |u|^{-\zeta(Z^b) + \frac{\gamma_0}{2} + 1}, \quad (7.28) \end{aligned}$$

$$|u_1|^{-\zeta(Z^n) + \frac{\gamma_0}{2} - \frac{1}{2}} \|r^{\frac{3}{2}} [\square_{\mathbf{g}}, Z^{(n)}]\varphi\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim \left( \Delta_0 m_0^{-\frac{1}{2}} + m_0 \Delta_0^{\frac{1}{2}} \right) |u_1|^{-\frac{\gamma_0}{2}} + \Delta_0^{\frac{1}{2}} m_0^{\frac{1}{2}} |u_1|^{-\frac{1}{2}}, \quad (7.29)$$

where the last term in (7.28) vanishes if  $Z^n = \partial^n$ .

*Proof.* We first rewrite the terms in (6.42)-(6.44) into

$$|[\tilde{\mathbf{g}}, Z^{(n)}]\varphi| \lesssim |H| \sum_{\substack{Z^a \sqcup Z^b = Z^n \\ b \leq n-1}} |u|^{\zeta(Z^a)} |\partial^2 Z^{(b)}\varphi| + \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} |u|^{\zeta(Z^c)} |Z^{(a)}H| |\partial^2 Z^{(b)}\varphi|, \quad (7.30)$$

where the first term on the right vanishes completely if  $Z^n = \partial^n$ .

By using (7.18), we have

$$\sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} |Z^{(a)}H| |\partial^2 Z^{(b)}\varphi| \lesssim \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} |u|^{\zeta(Z^c)} \left( |Z^{(a)}\mathbf{h}^1| + r^{-1-a}m_0 \right) |\partial^2 Z^{(b)}\varphi|, \quad (7.31)$$

where  $b \leq n-1$ . Thus

$$\begin{aligned} & \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} |u|^{\zeta(Z^c)} \|r^{\frac{1}{2}} Z^{(a)}H \cdot \partial^2 Z^{(b)}\varphi\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\ & \lesssim \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} |u|^{\zeta(Z^c)} \left( \|r^{\frac{1}{2}} Z^{(a)}\mathbf{h}^1 \cdot \partial^2 Z^{(b)}\varphi\|_{L^2(\mathcal{D}_{u_1}^{u_1})} + \|r^{-\frac{1}{2}-a}m_0 \partial^2 Z^{(b)}\varphi\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \right). \end{aligned}$$

Since  $\mathbf{h}^1$  verifies (6.5) which is stronger than the estimate (6.6) of  $H$  for (1.1), moreover  $\varphi \in (\mathbf{h}^1, \phi)$  verifies all the estimates in Section 4, by repeating the same argument for proving (6.46) and (6.47) in Proposition 6.9, we have

$$\begin{aligned} & \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} \| |u|^{\zeta(Z^c)} r^{\frac{1}{2}} Z^{(a)}\mathbf{h}^1 \cdot \partial^2 Z^{(b)}\varphi \|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\ & + \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} |u_1|^{-\frac{1}{2}} \| |u|^{\zeta(Z^c)} r^{\frac{3}{2}} Z^{(a)}\mathbf{h}^1 \cdot \partial^2 Z^{(b)}\varphi \|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \lesssim m_0^{-\frac{1}{2}} \Delta_0 |u_1|^{-\gamma_0 + \zeta(Z^n)}. \end{aligned}$$

By the definition of the standard energies followed with direct integrations, we derive, with  $0 \leq \alpha < \frac{1}{2}$  and  $a \geq 1$ , that

$$\begin{aligned} & |u_1|^{-\alpha} \| |u|^{\frac{1}{2}+\alpha} r^{-\frac{1}{2}-a} \partial^2 Z^{(b)}\varphi \|_{L^2(\mathcal{D}_{u_1}^{u_1})} + \| r^{-a+\frac{1}{2}} \partial^2 Z^{(b)}\varphi \|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\ & \lesssim |u_1|^{-a-\frac{1}{2}\gamma_0-\frac{1}{2}+\zeta(Z^b)} \left\{ \sup_{-u_1 \leq u \leq u_1} |u|^{\frac{\gamma_0}{2}+1-\zeta(Z^b)} E[\partial Z^{(b)}\varphi]^{\frac{1}{2}}(\mathcal{H}_u^{u_1}) \right. \\ & \quad \left. + |u_1|^{\frac{\gamma_0}{2}+1-\zeta(Z^b)} \sup_{-u_1 \leq u \leq u_1} E[\partial Z^{(b)}\varphi]^{\frac{1}{2}}(\mathcal{H}_{\underline{u}}^{u_1}) \right\}. \quad (7.32) \end{aligned}$$

Noting that  $1 < \gamma_0 < 2$ , by combining the above two sets of estimates and using  $-a \leq \zeta(Z^a)$ , we obtain

$$\begin{aligned} & \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} \| |u|^{\zeta(Z^c)} r^{\frac{1}{2}} Z^{(a)}H \cdot \partial^2 Z^{(b)}\varphi \|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\ & + \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ a \geq 1}} |u_1|^{-\frac{1}{2}} \| |u|^{\zeta(Z^c)} r^{\frac{3}{2}} Z^{(a)}H \cdot \partial^2 Z^{(b)}\varphi \|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\ & \lesssim (m_0^{-\frac{1}{2}} \Delta_0 + m_0 \Delta_0^{\frac{1}{2}}) |u_1|^{-\gamma_0 + \zeta(Z^n)}. \quad (7.33) \end{aligned}$$

Next we consider the borderline terms in (7.30). By using (7.14), we can derive that

$$\begin{aligned}
& \sum_{\substack{Z^a \sqcup Z^b = Z^n \\ b \leq n-1}} \|r^{\frac{1}{2}} |u|^{\zeta(Z^a)} H \partial^2 Z^{(b)} \varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
& \lesssim \sum_{\substack{Z^a \sqcup Z^b = Z^n \\ b \leq n-1}} \|r^{\frac{1}{2}} (r^{-1} m_0 + r^{-1} |u|^{\frac{1-\gamma_0}{2}} \Delta_0^{\frac{1}{2}}) |u|^{\zeta(Z^a)} \partial^2 Z^{(b)} \varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
& \lesssim |u_1|^{\zeta(Z^n)} \left( m_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0}{2} - \frac{1}{2}} + \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} |u_1|^{-\gamma_0} \right) \sup_{\substack{-\underline{u}_1 \leq u \leq u_1 \\ Z^b \subset Z^{n-1}}} E[\partial Z^{(b)} \varphi]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{\underline{u}_1}) |u|^{-\zeta(Z^b) + \frac{\gamma_0}{2} + 1}.
\end{aligned}$$

We then substitute (7.9) into the inequality if the product contains  $\Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}}$ . Combining the results with (7.33) implies (7.28). Similarly,

$$\begin{aligned}
& \sum_{\substack{Z^a \sqcup Z^b = Z^n \\ b \leq n-1}} |u_1|^{-\frac{1}{2}} \|r^{\frac{3}{2}} |u|^{\zeta(Z^a)} H \partial^2 Z^{(b)} \varphi\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
& \lesssim \sum_{\substack{Z^a \sqcup Z^b = Z^n \\ b \leq n-1}} |u_1|^{-\frac{1}{2}} \|r^{\frac{3}{2}} (r^{-1} m_0 + r^{-1} |u|^{\frac{1-\gamma_0}{2}} \Delta_0^{\frac{1}{2}}) |u|^{\zeta(Z^a)} \partial^2 Z^{(b)} \varphi\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
& \lesssim |u_1|^{\zeta(Z^n)} \left( m_0^{\frac{1}{2}} |u_1|^{-\frac{\gamma_0}{2} - \frac{1}{2}} + \Delta_0^{\frac{1}{2}} m_0^{-\frac{1}{2}} |u_1|^{-\gamma_0} \right) \sup_{\substack{-\underline{u}_1 \leq u \leq u_1 \\ Z^b \subset Z^{n-1}}} E[\partial Z^{(b)} \varphi]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{\underline{u}_1}) |u|^{-\zeta(Z^b) + \frac{\gamma_0}{2} + 1}.
\end{aligned}$$

(7.29) then follows with the substitution of (7.9).  $\square$

*Remark 7.6.* If  $Z \in \{\partial, \Omega_{\mu\nu}, 0 \leq \mu < \nu \leq 3\}$ , the result in this lemma still holds. For the proof, we only need to modify (7.31) by separating the case that  $Z^n = \partial^n$  from the general case.

**Lemma 7.7.** For  $(u_1, \underline{u}_1) \in \mathcal{I}$  and  $n \leq 3$ , there hold

$$\|r Z^{(n)} \mathcal{S}\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} + \|r Z^{(n)} \mathcal{S}\|_{L_{\underline{u}}^1 L_u^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-\frac{3}{2} + \zeta(Z^n)} (\Delta_0 + m_0^2), \quad (7.34)$$

$$\|r^{\frac{3}{2}} Z^{(n)} \mathcal{S}\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{-1 + \zeta(Z^n)} (\Delta_0 + m_0^2), \quad (7.35)$$

$$\|r^{\frac{1}{2}} q_0 Z^{(n)} (\mathbf{g}_{\mu\nu} \phi^2)\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim \Delta_0 |u_1|^{-\gamma_0 + \zeta(Z^n)}, \quad (7.36)$$

$$\|r^{\frac{3}{2}} q_0 Z^{(n)} (\mathbf{g}_{\mu\nu} \phi^2)\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim \Delta_0 |u_1|^{-\gamma_0 + \frac{1}{2} + \zeta(Z^n)}. \quad (7.37)$$

For  $\varphi \in (\mathbf{h}^1, \phi)$ , there hold

$$\begin{aligned}
& \|r Z^{(n)} (\mathcal{N}(\mathbf{h}) \partial \overset{\circ}{\mathbf{h}} \partial \varphi)\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} + \|r Z^{(n)} (\mathcal{N}(h) \partial \overset{\circ}{\mathbf{h}} \partial \varphi)\|_{L_{\underline{u}}^1 L_u^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
& \lesssim \left( |u_1|^{-1} m_0 + |u_1|^{-1 - \frac{\gamma_0}{2}} \Delta_0^{\frac{1}{2}} \right) |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^n)} \Delta_0^{\frac{1}{2}}, \quad (7.38) \\
& \|r^{\frac{3}{2}} Z^{(n)} (\mathcal{N}(\mathbf{h}) \partial \overset{\circ}{\mathbf{h}} \partial \varphi)\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim \left( |u_1|^{-\frac{1}{2}} m_0 + |u_1|^{-\frac{\gamma_0}{2} - \frac{1}{2}} \Delta_0^{\frac{1}{2}} \right) |u_1|^{-\frac{\gamma_0}{2} + \zeta(Z^n)} \Delta_0^{\frac{1}{2}}. \quad (7.39)
\end{aligned}$$

*Proof.* We first analyse  $\mathcal{S}$  which is defined in (7.2). By using (7.12) and (7.17), we have

$$\begin{aligned}
& |Z^{(n)} (H^{\mu\nu} \partial_{\mu} \partial_{\nu} \overset{\circ}{\mathbf{h}})| \lesssim m_0 \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} \left( |Z^{(a)} \mathbf{h}^1| + |Z^{(a)} \overset{\circ}{\mathbf{h}}| \right) r^{-3 + \zeta(Z^b)} |u|^{\zeta(Z^c)} \\
& \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} m_0 \left( |Z^{(a)} \mathbf{h}^1| + m_0 r^{-1 + \zeta(Z^a)} \right) r^{-3 + \zeta(Z^b)} |u|^{\zeta(Z^c)},
\end{aligned}$$

$$\begin{aligned}
|Z^{(n)}(G(\mathbf{h})\partial \overset{\circ}{\mathbf{h}} \partial \overset{\circ}{\mathbf{h}})| &\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n} |Z^{(a)}\mathbf{h}| |Z^{(b)}\partial \overset{\circ}{\mathbf{h}}| |Z^{(c)}\partial \overset{\circ}{\mathbf{h}}| |u|^{\zeta(Z^d)} \\
&\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n} m_0^2 |Z^{(a)}\mathbf{h}| r^{-4+\zeta(Z^b Z^c)} |u|^{\zeta(Z^d)} \\
&\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n} m_0^2 \left( |Z^{(a)}\mathbf{h}^1| + m_0 r^{-1+\zeta(Z^a)} \right) r^{-4+\zeta(Z^b Z^c)} |u|^{\zeta(Z^d)},
\end{aligned}$$

and

$$|Z^{(n)}(\partial \overset{\circ}{\mathbf{h}} \partial \overset{\circ}{\mathbf{h}})| \lesssim m_0^2 r^{-4+\zeta(Z^n)}.$$

We combine the above calculations, in view of  $\mathcal{N}(\mathbf{h}) = 1 + G(\mathbf{h})$  we obtain

$$|Z^{(n)}\mathcal{S}| \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} m_0 \left( |Z^{(a)}\mathbf{h}^1| + m_0 r^{-1+\zeta(Z^a)} \right) r^{-3+\zeta(Z^b)} |u|^{\zeta(Z^c)}. \quad (7.40)$$

By using (3.15) and (3.3), we can obtain for  $(u, \underline{u}) \in \mathcal{I}$  that

$$\|r^{-\frac{1}{2}} Z^{(a)}(\mathbf{h}^1, \phi)\|_{L^2(S_{u, \underline{u}})} \lesssim |u|^{-\frac{1}{2}\gamma_0 + \zeta(Z^a)} \left( \mathcal{E}_{a, \gamma_0}^{\frac{1}{2}} + \Delta_0^{\frac{1}{2}} \right).$$

We then combine the above two inequalities and integrate in  $\mathcal{D}_{u_1}^{\underline{u}_1}$  to obtain

$$\begin{aligned}
&\|r Z^{(n)}\mathcal{S}\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} + \|r Z^{(n)}\mathcal{S}\|_{L_{\underline{u}}^1 L_u^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
&\lesssim m_0 \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} \sup_{-\underline{u} \leq u \leq u_1} \|r^{-\frac{1}{2}} Z^{(a)}\mathbf{h}^1\|_{L^2(S_{u, \underline{u}})} |u_1|^{-1+\zeta(Z^b Z^c)} + m_0^2 |u_1|^{-\frac{3}{2}+\zeta(Z^n)} \\
&\lesssim m_0 \left( m_0 + \Delta_0^{\frac{1}{2}} |u_1|^{\frac{1}{2}-\frac{\gamma_0}{2}} \right) |u_1|^{-\frac{3}{2}+\zeta(Z^n)}.
\end{aligned}$$

Similarly, we can obtain

$$\|r^{\frac{3}{2}} Z^{(n)}\mathcal{S}\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim m_0 \left( m_0 + \Delta_0^{\frac{1}{2}} |u_1|^{\frac{1}{2}-\frac{\gamma_0}{2}} \right) |u_1|^{-1+\zeta(Z^n)}.$$

Thus we have completed the proof of (7.34) and (7.35) for  $\mathcal{S}$  defined in (7.2).

Next, by using (7.12) we have

$$\begin{aligned}
&|Z^{(n)}(\mathbf{g}_{\mu\nu} \cdot \phi^2)| \\
&\lesssim \sum_{Z^a \sqcup Z^b = Z^n} |Z^{(a)}\phi| |Z^{(b)}\phi| + \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |Z^{(a)}\mathbf{h} \cdot Z^{(b)}\phi \cdot Z^{(c)}\phi| \\
&\lesssim \sum_{Z^a \sqcup Z^b = Z^n} |Z^{(a)}\phi| |Z^{(b)}\phi| + \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} \left( m_0 r^{-1+\zeta(Z^a)} + |Z^{(a)}\mathbf{h}^1| \right) |Z^{(b)}\phi| |Z^{(c)}\phi| \\
&\lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} \left( r^{\zeta(Z^a)} + |Z^{(a)}\mathbf{h}^1| \right) |Z^{(b)}\phi| |Z^{(c)}\phi|. \quad (7.41)
\end{aligned}$$

Assume without loss of generality that  $b \leq c$ . Thus  $b \leq 1$ . By using (7.10), we can derive that

$$\begin{aligned}
q_0 \|r^{\frac{1}{2}} Z^{(b)}\phi \cdot Z^{(c)}\phi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} &\lesssim \|r^{\frac{5}{4}} |u|^{-\frac{1}{4}+\frac{\gamma_0}{2}} q_0^{\frac{1}{2}} Z^{(b)}\phi\|_{L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \|q_0^{\frac{1}{2}} |u|^{\frac{1}{4}-\frac{\gamma_0}{2}} r^{-\frac{3}{4}} Z^{(c)}\phi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
&\lesssim \left( \int_{-\underline{u}_1}^{u_1} E[Z^{(c)}\phi](\mathcal{H}_u^{\underline{u}_1}) |u|^{-\gamma_0} du \right)^{\frac{1}{2}} \Delta_0^{\frac{1}{2}} |u_1|^{\zeta(Z^b)-\frac{1}{2}} \\
&\lesssim \Delta_0 |u_1|^{-\gamma_0+\zeta(Z^b Z^c)}
\end{aligned}$$

and

$$\begin{aligned}
&q_0 \|r^{\frac{3}{2}} Z^{(b)}\phi \cdot Z^{(c)}\phi\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\
&\lesssim \|r^{\frac{5}{4}} |u|^{-\frac{1}{4}+\frac{\gamma_0}{2}} q_0^{\frac{1}{2}} Z^{(b)}\phi\|_{L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \|q_0^{\frac{1}{2}} |u|^{\frac{1}{4}-\frac{\gamma_0}{2}} r^{\frac{1}{4}} Z^{(c)}\phi\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim \Delta_0 |u_1|^{-\gamma_0+\frac{1}{2}+\zeta(Z^b Z^c)}.
\end{aligned}$$

In view of  $|r^{\zeta(Z^a)}| \lesssim |u_1|^{\zeta(Z^a)}$ , we can obtain the estimates for the first term on the right of (7.41). Next we consider the other term. Since at least two of the indices  $a, b, c$  are  $\leq 1$ , and



for convenience, we denote by  $\Psi = (\mathbf{h}^1, \phi)$  and assume  $a \leq b \leq c$ . By using (3.16) and (3.3), for any  $(u, \underline{u}) \in \mathcal{I}$  there holds

$$\|r^{-1}Z^{(c)}\Psi\|_{L^2(\mathcal{H}_{\underline{u}}^u)} \lesssim |u|^{-\frac{\gamma_0}{2}} \left( \mathcal{E}_{c, \gamma_0}^{\frac{1}{2}} + \Delta_0^{\frac{1}{2}} \right).$$

We also note that in the symbolic notation  $Z^{(a)}\Psi$ ,  $Z^{(b)}\Psi$ , at least one of the functions  $\Psi$  is actually  $\phi$ . In this case, by using (4.2), (7.10) and the above inequality, we have

$$\begin{aligned} & \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} q_0 \|r^{\frac{1}{2}} Z^{(a)}\Psi \cdot Z^{(b)}\Psi \cdot Z^{(c)}\Psi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ & \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} q_0 \| |u|^{\gamma_0 - \frac{3}{4}} r^{2 + \frac{1}{4}} Z^{(a)}\Psi Z^{(b)}\Psi \|_{L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \| |u|^{\frac{3}{4} - \gamma_0} r^{-\frac{7}{4}} Z^{(c)}\Psi \|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \\ & \lesssim |u_1|^{\zeta(Z^n) + \frac{1}{2} - \frac{3}{2}\gamma_0} \Delta_0^{\frac{3}{2}} \end{aligned}$$

and

$$\sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} q_0 \|r^{\frac{3}{2}} Z^{(a)}\Psi \cdot Z^{(b)}\Psi \cdot Z^{(c)}\Psi\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{\zeta(Z^n) + 1 - \frac{3}{2}\gamma_0} \Delta_0^{\frac{3}{2}}.$$

Thus we completed the proof of (7.36) and (7.37).

Finally we prove (7.38) and (7.39). By using (5.17) and (7.12), we derive

$$|Z^{(n)}(\partial \overset{\circ}{\mathbf{h}} \cdot \partial \varphi)| \lesssim \sum_{Z^b \sqcup Z^c = Z^n} m_0 r^{-2 + \zeta(Z^b)} |\partial Z^{(c)}\varphi|.$$

For the cubic part  $Z^{(n)}(G(\mathbf{h})\partial \overset{\circ}{\mathbf{h}} \partial \varphi)$ , we first derive by using Lemma 5.6, (7.17) and (7.12) that

$$\begin{aligned} & |Z^{(n)}(G(\mathbf{h})\partial \overset{\circ}{\mathbf{h}} \partial \varphi)| \\ & \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n} |u|^{\zeta(Z^d)} \left( |Z^{(a)}\mathbf{h}^1| + |Z^{(a)}\overset{\circ}{h}| \right) |Z^{(b)}\partial \overset{\circ}{\mathbf{h}}| |\partial Z^{(c)}\varphi| \\ & \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n} m_0 |u|^{\zeta(Z^d)} r^{-2 + \zeta(Z^b)} \left( |Z^{(a)}\mathbf{h}^1| + m_0 r^{-1 + \zeta(Z^a)} \right) |\partial Z^{(c)}\varphi|. \end{aligned}$$

Thus

$$\begin{aligned} & |Z^{(n)}(\mathcal{N}(\mathbf{h})\partial \overset{\circ}{\mathbf{h}} \partial \varphi)| \\ & \lesssim \sum_{Z^a \sqcup Z^b \sqcup Z^c \sqcup Z^d = Z^n} m_0 |u|^{\zeta(Z^d)} r^{-2 + \zeta(Z^b)} \left( |Z^{(a)}\mathbf{h}^1| + r^{\zeta(Z^a)} \right) |\partial Z^{(c)}\varphi|. \end{aligned} \quad (7.42)$$

We first note by (4.15) that, with  $0 \leq \alpha < \frac{1}{2}$ ,

$$\|r^{-1 + \alpha} \partial Z^{(c)}\varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \lesssim |u_1|^{\alpha - \frac{1}{2} - \frac{\gamma_0}{2} + \zeta(Z^c)} \Delta_0^{\frac{1}{2}}, \quad c \leq 3. \quad (7.43)$$

For simplicity, we denote by  $\|\cdot\|_{\flat}$  either the norm  $\|\cdot\|_{L_u^1 L_{\underline{u}}^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}$  or  $\|\cdot\|_{L_{\underline{u}}^1 L_u^2 L_\omega^2(\mathcal{D}_{u_1}^{\underline{u}_1})}$ . Thus, by using the above inequality with  $\alpha = 0$ , we have

$$\begin{aligned} & \sum_{Z^b \sqcup Z^c \sqcup Z^d = Z^n} \|r |u|^{\zeta(Z^d)} m_0 r^{-2 + \zeta(Z^b)} \partial Z^{(c)}\varphi\|_{\flat} \\ & \lesssim \sum_{Z^{b'} \sqcup Z^c = Z^n} m_0 |u_1|^{\zeta(Z^{b'})} \|r^{-1} \partial Z^{(c)}\varphi\|_{L^2(\mathcal{D}_{u_1}^{\underline{u}_1})} \left( \|r^{-1}\|_{L_{\underline{u}}^2 L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} + \|r^{-1}\|_{L_u^2 L^\infty(\mathcal{D}_{u_1}^{\underline{u}_1})} \right) \\ & \lesssim |u_1|^{-1 - \frac{\gamma_0}{2} + \zeta(Z^n)} m_0 \Delta_0^{\frac{1}{2}}. \end{aligned} \quad (7.44)$$

With  $0 < \alpha < \frac{1}{2}$  in (7.43) and  $|u| \lesssim r$ , we have

$$\begin{aligned}
& \sum_{Z^b \sqcup Z^c \sqcup Z^d = Z^n} \|r^{\frac{3}{2}} |u|^{\zeta(Z^d)} m_0 r^{-2+\zeta(Z^b)} \partial Z^{(c)} \varphi\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim m_0 \sum_{Z^{b'} \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^{b'})} \| |u|^{\alpha} r^{-1} \partial Z^{(c)} \phi \|_{L^2(\mathcal{D}_{u_1}^{u_1})} \|r^{-\frac{1}{2}} |u|^{-\alpha}\|_{L_u^2 L^{\infty}(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim |u_1|^{-\frac{1}{2}-\frac{\gamma_0}{2}+\zeta(Z^n)} m_0 \Delta_0^{\frac{1}{2}}.
\end{aligned} \tag{7.45}$$

By the Sobolev embedding on the unit sphere and (7.43), we have for  $c \leq 2$  and  $0 \leq \alpha < \frac{1}{2}$  that

$$\begin{aligned}
\|r^{\alpha} \partial Z^{(c)} \varphi\|_{L_u^2 L_{\underline{u}}^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} & \lesssim \|r^{\alpha-1} \partial \Omega^{(\leq 1)} Z^{(c)} \varphi\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim |u_1|^{\alpha-\frac{1}{2}-\frac{\gamma_0}{2}+\zeta(Z^c)} \Delta_0^{\frac{1}{2}}.
\end{aligned} \tag{7.46}$$

We can also use (3.31), (3.4) and the bootstrap assumption to obtain for  $a \leq 3$  that

$$\|r^{\frac{1}{2}} Z^{(a)} \mathbf{h}^1\|_{L_{\omega}^4(S_{u, \underline{u}})} \lesssim |u|^{-\frac{\gamma_0}{2}+\zeta(Z^a)} \Delta_0^{\frac{1}{2}}, \quad (u, \underline{u}) \in \mathcal{I}. \tag{7.47}$$

With  $0 < \alpha < \frac{1}{2}$ , we then employ (7.46) and (7.47) to treat the first term on the right if  $c \leq n-1$

$$\begin{aligned}
& \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ c \leq n-1}} |u_1|^{\zeta(Z^b)} m_0 \|r^{-\frac{1}{2}+\alpha} |u|^{\frac{1}{2}} \partial Z^{(c)} \varphi \cdot Z^{(a)} \mathbf{h}^1\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim \sum_{\substack{Z^a \sqcup Z^b \sqcup Z^c = Z^n \\ c \leq n-1}} |u_1|^{\zeta(Z^b)} m_0 \|r^{\alpha} \partial Z^{(c)} \varphi\|_{L_u^2 L_{\underline{u}}^2 L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \|r^{\frac{1}{2}} |u|^{\frac{1}{2}} Z^{(a)} \mathbf{h}^1\|_{L^{\infty} L_{\omega}^4(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim |u_1|^{\alpha-\gamma_0+\zeta(Z^n)} m_0 \Delta_0.
\end{aligned}$$

When  $c = n$ , by using (7.43) and (4.2) for  $\mathbf{h}^1$ , we have

$$\begin{aligned}
m_0 \|r^{-\frac{1}{2}+\alpha} |u|^{\frac{1}{2}} \partial Z^{(n)} \varphi \cdot \mathbf{h}^1\|_{L^2(\mathcal{D}_{u_1}^{u_1})} & \lesssim m_0 \|r^{-1+\alpha} \partial Z^{(n)} \varphi\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \| |u|^{\frac{1}{2}} r^{\frac{1}{2}} \mathbf{h}^1\|_{L^{\infty}(\mathcal{D}_{u_1}^{u_1})} \\
& \lesssim |u_1|^{\alpha-\gamma_0+\zeta(Z^n)} m_0 \Delta_0.
\end{aligned}$$

Thus

$$\sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^b)} m_0 \|r^{-\frac{1}{2}+\alpha} |u|^{\frac{1}{2}} \partial Z^{(c)} \varphi \cdot Z^{(a)} \mathbf{h}^1\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \lesssim |u_1|^{\alpha-\gamma_0+\zeta(Z^n)} m_0 \Delta_0.$$

Also by using Hölder's inequality in  $\mathcal{D}_{u_1}^{u_1}$ ,

$$\sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^b)} m_0 \|r^{-1} \partial Z^{(c)} \varphi \cdot Z^{(a)} \mathbf{h}^1\|_b \lesssim |u_1|^{-1-\gamma_0+\zeta(Z^n)} m_0 \Delta_0$$

and

$$\sum_{Z^a \sqcup Z^b \sqcup Z^c = Z^n} |u_1|^{\zeta(Z^b)} m_0 \|r^{-\frac{1}{2}} \partial Z^{(c)} \varphi \cdot Z^{(a)} \mathbf{h}^1\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \lesssim |u_1|^{-\gamma_0-\frac{1}{2}+\zeta(Z^n)} m_0 \Delta_0.$$

In view of (7.42), we may combine these two estimates with (7.44) and (7.45) to obtain (7.38) and (7.39).  $\square$

**7.3. Boundedness of energy.** Note that, according to the notation in Lemma 5.10, for the equation (ES), we can set

$$\mathcal{F}[\mathbf{h}^1] = \mathcal{N}(\mathbf{h}) \partial \mathbf{h}^1 \partial \mathbf{h}^1 + \partial \phi \cdot \partial \phi; \quad \mathcal{F}[\phi] = 0.$$

When considering the energy of  $\mathbf{h}^1$ , we decompose  $\mathcal{F}_{\mathbf{h}^1}$  into two parts

$$\mathcal{F}_{\mathbf{h}^1}^{\sharp} = [\square_{\mathbf{g}}, Z^{(n)}] \mathbf{h}^1 + \mathcal{F}^{\{n\}}[\mathbf{h}^1] + q_0 Z^{(n)}(\mathbf{g}_{\mu\nu} \phi^2); \quad \mathcal{F}_{\mathbf{h}^1}^{\flat} = Z^{(n)} \mathcal{S} + Z^{(n)}(\mathcal{N}(\mathbf{h}) \partial \mathbf{h}^1 \partial \mathbf{h}^1)$$

and for the energy of the scalar field  $\phi$ , we have

$$\mathcal{F}_{\phi} = \mathcal{F}_{\phi}^{\sharp} = [\square_{\mathbf{g}}, Z^{(n)}] \phi.$$

Hence for  $\mathbf{h}^1$ , we can treat  $\mathcal{F}_{\mathbf{h}^1}^b$  by using (7.38) and (7.34). To treat  $\mathcal{F}_{\mathbf{h}^1}^\sharp$ , we can apply (6.48)-(6.51) to treat  $\mathcal{F}^{\{n\}}[\mathbf{h}^1]$  and treat the remaining terms by using (7.36), (7.37) and Lemma 7.5:

$$\begin{aligned}
& \|r\mathcal{F}_{\mathbf{h}^1}^b\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} + \|r\mathcal{F}_{\mathbf{h}^1}^b\|_{L_{\underline{u}}^1 L_u^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \lesssim |u_1|^{-\frac{3}{2}+\zeta(Z^n)}(\Delta_0 + m_0^2) \\
& \|r^{\frac{3}{2}}\mathcal{F}_{\mathbf{h}^1}\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \left(\Delta_0 m_0^{-\frac{1}{2}}|u_1|^{-\gamma_0+\frac{1}{2}} + \Delta_0|u_1|^{-1}\right)|u_1|^{\zeta(Z^n)} + |u_1|^{-\frac{1}{2}\gamma_0+\zeta(Z^n)}m_0^{\frac{1}{2}}\Delta_0^{\frac{1}{2}} \\
& \lesssim \left(\Delta_0 m_0^{-\frac{1}{2}} + \Delta_0^{\frac{1}{2}}m_0^{\frac{1}{2}}\right)|u_1|^{-\frac{1}{2}\gamma_0+\zeta(Z^n)} \\
& \|r^{\frac{1}{2}}\mathcal{F}_{\mathbf{h}^1}^\sharp\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0 m_0^{-\frac{1}{2}}|u_1|^{-\gamma_0+\zeta(Z^n)} \\
& + |u_1|^{-\frac{1}{2}\gamma_0-\frac{1}{2}+\zeta(Z^n)}m_0^{\frac{1}{2}} \sup_{\substack{-\underline{u}_1 \leq u \leq u_1 \\ Z^b \subset Z^{n-1} \subset Z^n}} E[\partial Z^{(b)}\mathbf{h}^1]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{u_1})|u|^{-\zeta(Z^b)+\frac{1}{2}\gamma_0+1}, \quad (7.48)
\end{aligned}$$

where the last term in the last inequality vanishes if  $\zeta(Z^n) = -n$ .

We use Lemma 7.5 to treat  $\mathcal{F}_\phi$  and obtain

$$\begin{aligned}
& \|r^{\frac{3}{2}}\mathcal{F}_\phi\|_{L_u^1 L_{\underline{u}}^2 L_{\omega}^2(\mathcal{D}_{u_1}^{u_1})} \lesssim |u_1|^{-\frac{1}{2}\gamma_0+\zeta(Z^n)} \left(m_0^{\frac{1}{2}}\Delta_0^{\frac{1}{2}} + \Delta_0 m_0^{-\frac{1}{2}}\right) \\
& \|r^{\frac{1}{2}}\mathcal{F}_\phi\|_{L^2(\mathcal{D}_{u_1}^{u_1})} \lesssim \Delta_0 m_0^{-\frac{1}{2}}|u_1|^{-\gamma_0+\zeta(Z^n)} \\
& + |u_1|^{-\frac{1}{2}\gamma_0-\frac{1}{2}+\zeta(Z^n)}m_0^{\frac{1}{2}} \sup_{\substack{-\underline{u}_1 \leq u \leq u_1 \\ Z^b \subset Z^{n-1} \subset Z^n}} E[\partial Z^{(b)}\phi]^{\frac{1}{2}}(\mathcal{H}_{u_1}^{u_1})|u|^{-\zeta(Z^b)+\frac{1}{2}\gamma_0+1}, \quad (7.49)
\end{aligned}$$

where the last terms in the above inequalities vanish if  $\zeta(Z^n) = -n$ .

By a substitution of (7.48) and (7.49) into the energy inequalities in Proposition 6.4 and Proposition 6.6, we can obtain the boundedness of energies for  $\Psi = (\mathbf{h}^1, \phi)$  on  $\mathcal{I} = \{(u, \underline{u}) : -\underline{u}_1 \leq -\underline{u} \leq u \leq u_1\}$ :

$$\begin{aligned}
& \sup_{(u, \underline{u}) \in \mathcal{I}} |u|^{-2\zeta(Z^n)+\gamma_0} \left(E[Z^{(n)}\Psi](\mathcal{H}_u^u) + E[Z^{(n)}\Psi](\mathcal{H}_{\underline{u}}^u)\right) \\
& \lesssim \mathcal{E}_{n, \gamma_0}(\mathbf{h}^1, \phi) + |u_1|^{\gamma_0-3}(\Delta_0^2 + m_0^4) + |u_1|^{1-\gamma_0}m_0^{-2}\Delta_0^2 \\
& + \sup_{\substack{-\underline{u}_1 \leq u \leq u_1 \\ Z^b \subset Z^{n-1} \subset Z^n}} E[\partial Z^{(b)}\Psi](\mathcal{H}_u^{u_1})|u|^{-2\zeta(Z^b)+\gamma_0+2}, \quad (7.50)
\end{aligned}$$

where the last term in the above inequality vanishes if  $Z^n = \partial^n$ . We then run the induction on the signature  $\zeta(Z^n)$ . When  $\zeta(Z^n) = -n + l$ ,  $1 \leq l \leq n$ , we substitute the estimates for  $Z^{(n)}\Psi$  with  $\zeta(Z^n) = -n + l - 1$ . This procedure allows us to conclude that (7.50) can be bounded by

$$C_3 \left(m_0^4|u_1|^{\gamma_0-3} + |u_1|^{1-\gamma_0}m_0^{-2}\Delta_0^2 + \mathcal{E}_{n, \gamma_0}(\mathbf{h}^1, \phi)\right). \quad (7.51)$$

We also have the bound for the  $r$ -weighted energy:

$$\begin{aligned}
& \sup_{-\underline{u}_1 \leq u \leq u_1} |u|^{\gamma_0-1-2\zeta(Z^n)} \left(\mathcal{W}_1[Z^{(n)}\Psi](\mathcal{D}_{u_1}^{u_1}) + \mathcal{W}_1[Z^{(n)}\Psi](\mathcal{H}_{u_1}^{u_1}) + \mathcal{W}_1[Z^{(n)}\Psi](\mathcal{H}_{\underline{u}_1}^u)\right) \\
& \lesssim |u_1|^{-1}(\Delta_0 m_0 + \Delta_0^2 m_0^{-1}) + \mathcal{E}_{n, \gamma_0}(\mathbf{h}^1, \phi) \\
& + \sup_{-\underline{u}_1 \leq -\underline{u} \leq u \leq u_1} |u|^{\gamma_0-2\zeta(Z^n)} \left(E[Z^{(n)}\Phi](\mathcal{H}_u^u) + E[Z^{(n)}\Phi](\mathcal{H}_{\underline{u}}^u)\right). \quad (7.52)
\end{aligned}$$

For the last line, we directly substitute the bound (7.51), thus we conclude that (7.52) is bounded by (7.51) with a larger constant  $C_3$ .

Since  $\mathcal{E}_{n, \gamma_0}(\mathbf{h}^1, \phi) \leq C_0 m_0^2$ , with  $\Delta_0 = C_1 m_0^2$  and  $C_0, C_1 \geq 1$  we need

$$C_3 \left(C_0 C_1^{-1} \Delta_0 + |u|^{\gamma_0-3} \Delta_0^2 C_1^{-2} + |u|^{1-\gamma_0} \Delta_0 C_1\right) < 2\Delta_0.$$

Let  $C_1 = 4C_0C_3$  and  $\Delta_0 < 1$ , due to  $|u|^{-\gamma_0+1} \leq (\frac{R}{2})^{1-\gamma_0}$ , as long as

$$2C_3C_1 \left(\frac{R}{2}\right)^{1-\gamma_0} < \frac{7}{4}, \quad (7.53)$$

we can obtain for  $n \leq 3$  and  $-\underline{u}_* \leq -\underline{u} \leq u \leq u_0$  that

$$\begin{aligned} E[Z^{(n)}(\mathbf{h}^1, \phi)](\mathcal{H}_{\underline{u}}^u) + E[Z^{(n)}(\mathbf{h}^1, \phi)](\underline{\mathcal{H}}_{\underline{u}}^u) &< 2\Delta_0|u|^{-\gamma_0+2\zeta(Z^n)}, \\ \mathcal{W}_1[Z^{(n)}(\mathbf{h}^1, \phi)](\mathcal{H}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}(\mathbf{h}^1, \phi)](\underline{\mathcal{H}}_{\underline{u}}^u) + \mathcal{W}_1[Z^{(n)}\phi](\mathcal{D}_{\underline{u}}^u) &< 2\Delta_0|u|^{-\gamma_0+1+2\zeta(Z^n)}. \end{aligned}$$

It remains to improve (A1) with  $\geq$  replaced by  $>$ . Due to (7.8), (4.2) and (7.14),

$$r|H^{\underline{LL}} - \overset{\circ}{H}^{\underline{LL}}| \leq C \left( \Delta_0^{\frac{1}{2}}|u|^{-\frac{\gamma_0}{2}+\frac{1}{2}} + r^{-1}(m_0 + \Delta_0^{\frac{1}{2}})^2 \right).$$

Similar to Lemma 7.1, with  $0 < m_0 < 1$ , we choose

$$C \left( C_1^{\frac{1}{2}} \left(\frac{R}{2}\right)^{\frac{1-\gamma_0}{2}} + R^{-1} \left(1 + C_1^{\frac{1}{2}}\right)^2 \right) < \frac{1}{10} \quad (7.54)$$

which implies that

$$r(h - H^{\underline{LL}}), \quad r(h - H^{LL}) > \frac{7m_0}{20}$$

which improves (A1). By choosing the lower bound  $R(\gamma_0, C_0)$  of  $R$  such that (7.54), (7.53) and  $\tilde{C}_1 R^{-\frac{\gamma_0}{2}+\frac{1}{2}} \leq 1$  as requested in Lemma 7.1 hold, the proof of Theorem 2.6 is completed.

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