

# COCENTERS OF $p$ -ADIC GROUPS, III: ELLIPTIC AND RIGID COCENTERS

DAN CIUBOTARU AND XUHUA HE

ABSTRACT. In this paper, we show that the elliptic cocenter of the Hecke algebra of a connected reductive  $p$ -adic group is contained in the rigid cocenter. As applications, we prove the trace Paley-Wiener theorem and the abstract Selberg principle for mod- $l$  representations.

## INTRODUCTION

0.1. Let  $F$  be a nonarchimedean local field of residual characteristic  $p$ . Let  $\mathbb{G}$  be a connected reductive group over  $F$  and  $G = \mathbb{G}(F)$  be the group of  $F$ -points. The study of smooth admissible representations of  $G$  is a major topic in representation theory. In particular, the representation theory over complex numbers is a part of the local Langlands program, and it has been a central area of research in modern representation theory. The  $G$ -representations over an algebraically closed field  $R$  of characteristic  $l \neq p$  (in short, the mod- $l$  representations) is also a natural object to study, and it has attracted considerable interest recently due to the applications to number theory, e.g., congruences of the modular forms and the mod- $l$  Langlands program. Several important progresses have been achieved in this direction, e.g. Vignéras [24, 26] and Vignéras-Waldspurger [27]. However, less is known compared to the complex representations. One of the major difficulty is that the proofs of many key results for complex representations rely heavily on harmonic analysis methods, which are not always available for mod- $l$  representations.

The main purpose of this paper (as well as of the previous papers [13], [14]) is to develop a new approach towards the (complex and mod- $l$ ) representation theory of  $G$ . The approach is based on the relation between the cocenter  $\bar{H}_R = H_R/[H_R, H_R]$  of the Hecke algebra  $H_R = H(G)_R$  over the field  $R$  and the Grothendieck group  $\mathfrak{R}_R(G)$  of representations over  $R$  via the trace map

$$\mathrm{Tr}_R : \bar{H}_R \longrightarrow \mathfrak{R}_R(G)^*.$$

A detailed analysis on the cocenter side should lead to a deep understanding on the representation side. It is also worth mentioning that the study of the structure of the cocenter is influenced by, and relies on, certain recent developments in arithmetic geometry, in particular, the work the theory of  $\sigma$ -isocrystals and affine Deligne-Lusztig varieties.

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0.2. We explain the main results of this paper.

Let  $M$  be a standard Levi subgroup. On the representations side, there are two important functors: the parabolic induction functor  $i_M : \mathfrak{R}_R(M) \rightarrow \mathfrak{R}_R(G)$  and the Jacquet functor  $r_M : \mathfrak{R}_R(G) \rightarrow \mathfrak{R}_R(M)$ . On the cocenter side, there are also two important maps: the induction map  $\bar{i}_M : \bar{H}(M) \rightarrow \bar{H}(G)$  adjoint to the Jacquet functor and the restriction map  $\bar{r}_M : \bar{H}(G) \rightarrow \bar{H}(M)$  adjoint to the parabolic induction functor.

An important family of irreducible representations is formed by the elliptic representations, which can be viewed as the set of irreducible smooth representations that are not linear combinations of induced representations  $i_M(\sigma)$  for proper Levi subgroups (in the Grothendieck group). The analogous notion on the cocenter side is the elliptic cocenter  $\bar{H}_R^{\text{ell}}$ , the subspace of  $\bar{H}$  consisting of elements  $f$  such that  $\bar{r}_M(f) = 0 \in \bar{H}(M)$ .

Another important subspace of the cocenter  $\bar{H}_R$  is the rigid cocenter  $\bar{H}_R^{\text{rig}}$ . By definition,  $\bar{H}_R^{\text{rig}}$  consists of elements in the cocenter  $\bar{H}_R$  that are represented by functions with support in the subset of compact-modulo-center elements of  $G$ . As explained in [14, Theorem C], the rigid cocenters form the “building blocks” of the whole cocenter  $\bar{H}_R$ . We will also see later in Theorem C that there is a close relation between the rigid cocenter and the finitely generated projective representations of  $G$ .

The main result of this paper compares these two important subspaces of the cocenter. The study of the relation between the elliptic cocenter and the rigid cocenter (as well as the duality with relevant spaces of representations) is also motivated by the results for affine Hecke algebras obtained in [6].

**Theorem A** (Theorem 4.4). *The elliptic cocenter is contained in the rigid cocenter.*

As applications of Theorem A, we will also establish two important results for mod- $l$  representations: the trace Paley-Wiener theorem and the abstract Selberg principle.

**Theorem B** (Theorem 5.1). *Assume furthermore that the order of the relative Weyl group of  $G$  is invertible in  $R$ . Then the image of the trace map*

$$\text{Tr}_R : \bar{H}_R \longrightarrow \mathfrak{R}_R(G)^*$$

*consists of all the good linear forms on  $\mathfrak{R}_R(G)$ .*

Here “good forms” is used in the sense of Bernstein-Deligne-Kazhdan [1]. Roughly speaking, the set of irreducible representations of  $G$  has an algebraic variety structure and the good forms correspond to the algebraic functions with respect to this structure. We refer to §5.1 for the precise definition.

**Theorem C** (Theorem 6.1). *The rank map sends finitely generated projective representations of  $G$  over  $R$  into elements of the cocenter  $\bar{H}_R$ , which can be represented by functions with support in the subset of compact elements of  $G$ .*

The trace Paley-Wiener theorem was first established by Bernstein, Deligne and Kazhdan in [1] for complex representations. It was later generalized to a twisted version by Rogawski [23] and more recently, by Henniart and Lemaire [15] for complex  $\omega$ -representations arising from the theory of twisted endoscopy. A key

ingredient in these proofs is to establish a “finiteness result” for the discrete central characters via the study of tempered representations.

The abstract Selberg principle was first established by Blanc and Brylinski in [2] for complex representations using low-dimensional cyclic and Hochschild homology and the Fourier transform. A different proof for  $\text{char}(F) = 0$  was given by Dat in [9] using Clozel’s integration formula and the triviality of the action of unramified characters on the Grothendieck ring of projective modules of finite type.

It is worth pointing out that Theorem B and Theorem C hold beyond the so-called “banal cases” (i.e., the cases where the representation theory are known or expected to be similar to the representation theory over complex fields). For example, if  $l$  divides the order of  $\mathbb{G}(k)$ , where  $k$  is the residue field of  $F$ , then Density Theorem [16] does not hold in this case and there are “fewer” mod- $l$  irreducible representations than the irreducible complex representations.

We also expect that the twisted and even the endoscopic version of Theorem B and Theorem C hold and may be worked out along a similar strategy. However, this requires a much more elaborate work on the cocenter and we do not pursue it here.

Both results were not known for mod- $l$  representations before. In the rest of the introduction, we will sketch our strategy to prove these results.

0.3. Firstly, as we would like to develop a general machinery for arbitrary algebraically closed field of characteristic not equal to  $p$ , we work with the integral form  $H$  of  $H_R$ , the algebra of  $\mathbb{Z}[\frac{1}{p}]$ -valued functions, and its cocenter  $\bar{H} = H/[H, H]$ .

The starting point is the Newton decomposition introduced in [13]. Roughly speaking, we have decompositions

$$G = \bigsqcup_{\nu \in \aleph} G(\nu) \quad \text{and} \quad \bar{H} = \bigoplus_{\nu \in \aleph} \bar{H}(\nu),$$

where  $\aleph$  is the product of  $\pi_1(G)$  (the Kottwitz factor) and the set of dominant rational coweights of  $G$  (the Newton factor), and  $\bar{H}(\nu)$  is the  $\mathbb{Z}[\frac{1}{p}]$ -submodule of  $\bar{H}$  that can be represented by functions supported in  $G(\nu)$ . The subset of compact-modulo-center elements of  $G$  is the union of  $G(\nu)$  where the Newton factor of  $\nu$  is central. The subset of compact elements of  $G$  is the union of  $G(\nu)$  where the Newton factor of  $\nu$  is central and the Kottwitz factor of  $\nu$  is of finite order.

It is shown in [14] that the Newton decomposition on  $\bar{H}$  is compatible with the induction map. In section 2, we establish an explicit formula for the restriction map  $\bar{r}_M : \bar{H}(G) \rightarrow \bar{H}(M)$ , adjoint to the parabolic induction functor  $i_M : \mathfrak{R}_R(M) \rightarrow \mathfrak{R}_R(G)$ . In section 3, based on the explicit formula, we show that the restriction map  $\bar{r}_M$  is compatible with the Newton decomposition of  $\bar{H}(G)$  and  $\bar{H}(M)$ . This allows one to compute the image of  $\bar{r}_M$  component-wise. In section 4, we prove a Mackey-type formula at the level of the cocenter. Note that in small characteristics, the trace map  $\text{Tr}_R : \bar{H}_R \rightarrow \mathfrak{R}_R(G)^*$  is not injective and the desired formula for the cocenter does not follow from the Mackey formula for representations. Our proof of the Mackey-type formula for the cocenter is therefore rather involved.

After we establish all these ingredients, we are able to compute  $\bar{r}_M(f)$  for any  $f \in \bar{H}(G)$  in a fairly explicit way and to show that if  $\bar{r}_M(f) = 0$  for all proper

Levi subgroups  $M$ , then  $f \in \oplus \bar{H}(\nu)$ , where  $\nu$  runs over elements in  $\aleph$  with central Newton factor. Thus Theorem A is proved.

It is also worth mentioning that in the proof of Howe's conjecture in [13], one mainly uses only the fact that  $\bar{H}$  is spanned by  $\bar{H}(\nu)$ ; while in the proof of Theorem A here, we use the full strength of the Newton decomposition  $\bar{H} = \oplus \bar{H}(\nu)$ . The fact that this is a direct sum decomposition allows us to do the component-wise computations and it plays a crucial way in the argument here.

0.4. Now we discuss the strategy to prove the trace Paley-Wiener theorem.

We first follow the combinatorial argument of [1] and use the  $A$ -operator to reduce to the study of elliptic representations. This is the only part where we use the assumption that the order of the relative Weyl group is invertible in  $R$ . Then we establish the elliptic trace Paley-Wiener Theorem, based on a "finiteness result".

The finiteness result in [1] (as well as in [12], [15]) is that the set of discrete central characters in each given Bernstein component is a finite union of orbits under the action of the unramified characters. The finiteness result we establish here is different. We do not use the Bernstein components nor the irreducible representations, since for mod- $l$  representations, these objects are not well understood yet. We show instead that the image under the  $A$ -operators of the Grothendieck group of smooth admissible representations for a given depth is of finite rank. This result will be deduced from the analogous result for the cocenter: the image in the cocenter  $\bar{H}$  of the map  $\bar{A}$  on the space of functions on  $G$  of a given depth is finite dimensional.

Note that it suffices to consider the depth  $n - \epsilon$ , where  $n$  is a positive integer and  $\epsilon > 0$  is sufficiently small. In this case, one may just consider the functions in  $H(G, \mathcal{I}_n)$ , the compactly supported  $\mathcal{I}_n$ -biinvariant functions on  $G$ . Here  $\mathcal{I}_n$  is the  $n$ -th congruence subgroup of a fixed Iwahori subgroup  $\mathcal{I}$  of  $G$ . In this case, we have the Newton decomposition for depth  $n - \epsilon$  established in [13]:

$$\bar{H}(G, \mathcal{I}_n) = \bigoplus_{\nu \in \aleph} \bar{H}(G, \mathcal{I}_n; \nu).$$

The image under  $\bar{A}$  of the cocenter  $\bar{H}$  is exactly the elliptic cocenter that we discussed earlier. Now based on our main result Theorem A and the Newton decomposition for depth  $n - \epsilon$ , the image under  $\bar{A}$  of  $\bar{H}(G, \mathcal{I}_n)$  is contained in the rigid cocenter of  $\bar{H}(G, \mathcal{I}_n)$ , i.e. in  $\bigoplus \bar{H}(G, \mathcal{I}_n; \nu)$ , where  $\nu$  runs over elements in  $\aleph$  with central Newton factor. Now the desired "finiteness result" follows from the fact that each Newton component  $\bar{H}(G, \mathcal{I}_n; \nu)$  is finite dimensional ([13, Theorem 11]).

0.5. We now arrive at the abstract Selberg principle. For the purpose of the introduction, we assume that  $G$  is semisimple. Then every compact-modulo-center element is compact and we need to show that the image of the rank map is contained in the rigid cocenter  $\bar{H}_R^{\text{rig}}$ .

Let  $f \in \bar{H}_R$  be the image of the rank map of a finitely generated projective module of  $G$  over  $R$ . By the compatibility of the rank map with the Jacquet functor, and the inductive hypothesis, we have that  $\bar{r}_M(f) \in \bar{H}(M)_R^{\text{rig}}$ . Then we deduce that  $f \in \bar{H}_R^{\text{rig}}$  by

- the compatibility of the Newton decomposition on  $\bar{H}(G)$  and on  $\bar{H}(M)$ ;
- a stronger form of Theorem A, which says that for  $\nu \in \aleph$  whose Newton factor is non-central, if the  $\nu$ -component of  $\bar{r}_M(f)$  is trivial for all proper Levi  $M$ , then the  $\nu$ -component of  $f$  is also trivial.

0.6. Finally, in Appendix A, we present a different proof (based on certain classical results of Clozel) of Theorem A for Hecke algebras over  $R$  under the assumptions that  $F$  has characteristic 0.

## 1. PRELIMINARIES

1.1. We fix a minimal parabolic subgroup  $P_0$  of  $G$  and its Levi subgroup  $M_0$ . A parabolic subgroup is called *standard* if it contains  $P_0$ . A Levi subgroup  $M$  is called *standard* if  $M \supset M_0$  and it is a Levi subgroup of a standard parabolic subgroup. For any standard Levi subgroup  $M$ , we denote by  $W_M$  its relative Weyl group. We choose the Haar measure  $\mu_G$  on  $G$  in such a way that the volume of the pro- $p$  Iwahori subgroup  $\mathcal{I}'$  is 1. Then for any open compact subgroup  $\mathcal{K}$  of  $G$ ,  $\mu_G(\mathcal{K}) \in \mathbb{Z}[\frac{1}{p}]$ . The Haar measures on the standard Levi subgroups are chosen in the same way.

Let  $H$  be the Hecke algebra of  $G$  over  $\mathbb{Z}[\frac{1}{p}]$ , i.e., the space of locally constant, compactly supported  $\mathbb{Z}[\frac{1}{p}]$ -valued functions on  $G$ , endowed with convolution with respect to Haar measure  $\mu$ .

Let  $\bar{H} = H/[H, H]$  be the cocenter of  $H$ . If  $f \in H$ , denote by  $\bar{f}$  the image of  $f$  in  $\bar{H}$ .

We fix an algebraically closed field  $R$  of characteristic not equal to  $p$ . Set

$$H_R = H \otimes_{\mathbb{Z}[\frac{1}{p}]} R \text{ and } \bar{H}_R = \bar{H} \otimes_{\mathbb{Z}[\frac{1}{p}]} R.$$

Let  $\mathfrak{R}_R(G)$  be the  $R$ -vector space with basis  $\text{Irr} H_R$ , the isomorphism classes of irreducible smooth admissible  $R$ -representations of  $G$ . For every standard parabolic subgroup  $P = MN$ , denote by  $\pi_P : P \rightarrow M$  the natural projection map. We also denote by  $i_{M,R} : \mathfrak{R}_R(M) \rightarrow \mathfrak{R}_R(G)$  the functor of normalized induction and by  $r_{M,R} : \mathfrak{R}_R(G) \rightarrow \mathfrak{R}_R(M)$  the normalized Jacquet functor, the functor of  $N$ -coinvariants. Whenever we speak of normalized induction/restriction, we assume tacitly that  $p$  has a square root in  $R$  and we fix such a square root (see also [24, page 97]).

1.2. Now we recall the categorical description of  $\bar{H}_R$ , see [12, §F] and [8, §5.1]. The cocenter of  $H$  can be viewed as the quotient of the free abelian group on the symbols  $(\pi, u)$ , where  $\pi$  is a finitely generated projective  $H$ -module over  $R$ , and  $u \in \text{End}_{H_R}(\pi)$  modulo the relations

- (C1)  $(\pi, u) = (\pi_1, u_1) + (\pi_2, u_2)$  for any short exact sequence  $0 \rightarrow \pi_1 \rightarrow \pi \rightarrow \pi_2 \rightarrow 0$  that commutes with the  $u$ -action.
- (C2)  $(\pi, u) + (\pi, u') = (\pi, u + u')$ .
- (C3)  $(\pi, fg) = (\pi', gf)$  if  $f : \pi' \rightarrow \pi$  and  $g : \pi \rightarrow \pi'$ .

As explained in [8, §1.7], the functors  $i_{M,R}$  and  $r_{M,R}$  send finitely generated projective modules to finitely generated projective modules. Therefore they induce

maps on the symbols  $(\pi, u)$ , which preserve the above relations. We denote the induced map on the cocenter  $\bar{H}_R$  by

$$\bar{i}_{M,R}^{\text{cat}} : \bar{H}(M)_R \longrightarrow \bar{H}_R, \quad \bar{r}_{M,R}^{\text{cat}} : \bar{H}_R \longrightarrow \bar{H}(M)_R.$$

By Frobenius reciprocity and Bernstein's second adjointness theorem, we have

$$(1.1) \quad \begin{aligned} \text{Tr}_R^G(h_G, i_{M,R}(\sigma)) &= \text{Tr}_R^M(\bar{r}_{M,R}^{\text{cat}}(h_G), \sigma), \\ \text{Tr}_R^M(h_M, r_{M,R}(\pi)) &= \text{Tr}_R^G(\bar{i}_{M,R}^{\text{cat}}(h_M), \pi), \end{aligned}$$

for  $h_G \in \bar{H}_R, h_M \in \bar{H}(M)_R, \pi \in \mathfrak{R}_R(G), \sigma \in \mathfrak{R}_R(M)$ .

1.3. For every smooth admissible  $H_R$ -representation  $(\pi, V_\pi)$  and  $f \in \bar{H}_R$ , the operator  $\pi(f) \in \text{End}(V_\pi)$  is of trace class, and we denote by  $\text{Tr}_R(f, \pi)$  its trace. We have the induced trace map

$$\text{Tr}_R : \bar{H}_R \longrightarrow \mathfrak{R}_R(G)^*.$$

In the case where  $R = \mathbb{C}$ , we have the trace Paley-Wiener theorem [1] and spectral density theorem [16] which identifies the cocenter  $\bar{H}_{\mathbb{C}}$  with the space of “good” linear forms on  $\mathfrak{R}_{\mathbb{C}}(G)$  (we refer to §5.1 for the precise definition of good forms). In this case we may regard  $\bar{i}_{M,\mathbb{C}}^{\text{cat}}$  (resp.  $\bar{r}_{M,\mathbb{C}}^{\text{cat}}$ ) as the dual functor of  $r_{M,\mathbb{C}}$  (resp.  $i_{M,\mathbb{C}}$ ). However, if  $\text{char}(R)$  is small, then the trace map is not injective, and thus the relations about the functors on  $\mathfrak{R}_R(*)$  do not automatically imply the similar relations on  $\bar{H}(*)_R$ . However, it is easy to see, using (1.1), that  $\ker \text{Tr}_R$  is stable under the induction and restriction maps. We define the *reduced cocenter* by

$$\bar{H}_R^{\text{red}} = \bar{H}_R / \ker \text{Tr}_R.$$

For any open compact subgroup  $\mathcal{K}$ , we define  $\bar{H}_R(G, \mathcal{K})^{\text{red}}$  to be the image of  $\bar{H}_R(G, \mathcal{K})$  in  $\bar{H}_R^{\text{red}}$ .

1.4. We recall the Newton decomposition introduced in [13].

Let  $A$  be a maximal  $F$ -split torus in  $M_0$  and  $Z$  be the centralizer of  $A$ . We may identify the relative finite Weyl group  $W_G$  with  $N_G A(F)/Z(F)$ . We fix a special vertex in the fundamental alcove of the apartment corresponding to  $A$ . The Iwahori-Weyl group can be realized as

$$\tilde{W} \cong X_*(Z)_{\text{Gal}(\bar{F}/F)} \rtimes W_G = \{t^\lambda w; \lambda \in X_*(Z)_{\text{Gal}(\bar{F}/F)}, w \in W_G\}.$$

We set  $V = X_*(Z)_{\text{Gal}(\bar{F}/F)} \otimes \mathbb{R}$  and  $\Omega = \tilde{W}/W_a$ , where  $W_a$  is the affine Weyl group associated to  $\tilde{W}$ . Set  $\aleph = \Omega \times V_+$ , where  $V_+$  is the set of dominant elements in  $V$  with respect to the positive roots given by  $P_0$ . For any  $\nu \in \aleph$ , we have the corresponding Newton stratum  $G(\nu)$ . The precise definition is technical, and we refer to [13, §2.2] for more details. For any  $\nu \in \aleph$ , let  $H(G; \nu)$  be the subspace of  $H$  consisting of functions with support in  $G(\nu)$  and  $\bar{H}(G; \nu)$  be the image of  $H(\nu)$  in  $\bar{H}$ . We may simply write  $H(\nu)$  for  $H(G; \nu)$  and  $\bar{H}(\nu)$  for  $\bar{H}(G; \nu)$ . The following result is proved in [13, Theorem 3 & Theorem 10].

**Theorem 1.1.** *We have the Newton decompositions*

$$G = \sqcup_{\nu \in \aleph} G(\nu), \quad H = \oplus_{\nu \in \aleph} H(\nu), \quad \bar{H} = \oplus_{\nu \in \aleph} \bar{H}(\nu).$$

In this paper, we are mainly interested in the  $V$ -factor of  $\aleph$ . For  $v \in V_+$ , we set  $\bar{H}(v) = \bigoplus_{\nu=(\tau,v) \text{ for some } \tau \in \Omega} \bar{H}(\nu)$ .

Let  $\bar{H}_R^{\text{red}}(v)$  be the image of  $\bar{H}_R(v)$  in  $\bar{H}^{\text{red}}$ . The following result follows from [14, Theorem 6.3].

**Theorem 1.2.** *We have the Newton decomposition*

$$\bar{H}_R^{\text{red}} = \bigoplus_{v \in V_+} \bar{H}_R^{\text{red}}(v).$$

**Remark 1.3.** As explained in [14, Remark 6.4],  $\sum_{\nu \in \aleph} \bar{H}_R^{\text{red}}(\nu)$  may not be a direct sum in general.

Following [13, §6], we define

$$(1.2) \quad \bar{H}^{\text{rig}} = \bigoplus_{v \in V_+; M_v = G} \bar{H}(v), \quad \bar{H}_R^{\text{rig,red}} = \bigoplus_{v \in V_+; M_v = G} \bar{H}_R^{\text{red}}(v),$$

where  $M_v$  is the centralizer of  $v$  in  $G$ . We call  $\bar{H}^{\text{rig}}$  the *rigid cocenter* and  $\bar{H}_R^{\text{rig,red}}$  the reduced rigid cocenter.

## 2. THE RESTRICTION MAP

2.1. We give the formula for  $\bar{r}_M$  which is the cocenter dual of van Dijk's formula [11] for parabolic induction of characters.

Let  $\mathcal{K}^{\text{sp}}$  be the maximal special parahoric subgroup of  $G$  corresponds to the special vertex of the fundamental alcove we fixed in the beginning. For any  $w \in W_G$ , we choose a representative  $\dot{w}$  in  $\mathcal{K}^{\text{sp}}$ .

Let  $P = MN$  be a standard parabolic subgroup. In particular, the Iwasawa decomposition  $G = \mathcal{K}^{\text{sp}}P$  holds. Fix a Haar measure  $\mu_N$  on  $N$ . Let  $\mu_P = \mu_M \mu_N$  be the Haar measure on  $P = MN$ . Let  $\delta_P$  be the modulus function of  $P$ .

Let  $f \in H$  be given. The function  $f$  gives rise to two functions:  $f^{(P)}$ , a function on  $M$ , and  $\tilde{f}$ , a function on  $G$ , defined as follows:

$$f^{(P)}(m) = \delta_P^{\frac{1}{2}}(m) \int_N f(mn) dn, \text{ for } m \in M, \text{ and}$$

$$\tilde{f}(g) = \int_{\mathcal{K}^{\text{sp}}} f(k^{-1}gk) dk, \quad \text{for } g \in G.$$

It is immediate that

$$(2.1) \quad (L_{p_0} f)^{(P)} = \delta_P^{\frac{1}{2}}(\pi_P(p_0)) L_{\pi_P(p_0)} f^{(P)},$$

where  $p_0 \in P$ ,  $\pi_P : P \rightarrow M$  is the projection map and  $L_-$  denotes the left regular action. In other words, the map  $f \mapsto f^{(P)}$  is a homomorphism of  $P$ -modules  $H \rightarrow H(M)$ , where the action on the  $H$  is the left regular action, and the action on  $H(M)$  is via  $\delta_P^{\frac{1}{2}} \pi_P$ .

**Lemma 2.1.** *The assignment  $f \mapsto \tilde{f}^{(P)}$  defines a linear map  $H \rightarrow H(M)$ .*

*Proof.* It suffices to consider the case where  $f = \delta_X$ , where  $X$  is an open compact subset of  $G$ . Let  $\mathcal{K}' \subset \mathcal{K}^{\text{sp}}$  be an open compact subgroup such that  $X$  is stable under the left and right multiplication of  $\mathcal{K}$ . For any  $m$ ,

$$(2.2) \quad \tilde{f}^{(P)}(m) = \mu(\mathcal{K}') \sum_{g \in \mathcal{K}^{\text{sp}}/\mathcal{K}'} \sum_{m \in \pi_P(g^{-1}X \cap P)} \delta_P^{\frac{1}{2}}(m) \mu_N((g^{-1}X)m^{-1} \cap N);$$

recall that  $\pi_P : P \rightarrow M$  is the natural projection map. Note that  $\pi_P(g^{-1}X \cap P)$  is an open compact subset of  $M$ , and the functions  $\delta_P^{\frac{1}{2}}(m)$  and  $\mu_N((g^{-1}X)m^{-1} \cap N)$  are locally constant  $\mathbb{Z}[\frac{1}{p}]$ -valued functions on  $M$ . Thus  $\tilde{f}^{(P)} \in H(M)$ .  $\square$

**Lemma 2.2.** *For every  $f \in H$ ,  $\tilde{f} \equiv \mu(\mathcal{K}^{\text{sp}})f \pmod{[H, H]}$ .*

*Proof.* Let  $\mathcal{K} \subset \mathcal{K}^{\text{sp}}$  be a compact open subgroup such that  $f$  is  $\mathcal{K}$ -biinvariant. Then  $\tilde{f}(g) = \sum_{\bar{k} \in \mathcal{K}^{\text{sp}}/\mathcal{K}} f(\bar{k}^{-1}g\bar{k})\mu(K)$ . For  $x \in G$ , denote by  ${}^x f$  the function  ${}^x f(g) = f(x^{-1}gx)$ . Then

$$\tilde{f} = \mu(\mathcal{K}) \sum_{\bar{k} \in \mathcal{K}^{\text{sp}}/\mathcal{K}} \bar{k} f \text{ in } H.$$

By [13, Proposition 1] in the cocenter  $\bar{H}$ ,  $\bar{f} = \overline{{}^x f}$ . The claim follows.  $\square$

2.2. Now we define  $r_P(f) \in \bar{H}(M)$ . We set  $r_P(f)$  to be the unique element in  $\bar{H}(M) \otimes \mathbb{Q}$  such that

$$(2.3) \quad \mu_P(\mathcal{K}^{\text{sp}} \cap P) r_P(f) \text{ equals the image of } \tilde{f}^{(P)} \text{ in } \bar{H}(M).$$

Note that  $\mathbb{Z}[\frac{1}{p}]$  is an integral domain. Thus the element  $r_P(f)$ , as an element in  $\bar{H}(M) \otimes \mathbb{Q}$ , is uniquely determined by (2.3). In the case where  $\mu_P(\mathcal{K}^{\text{sp}} \cap P)$  is invertible in  $R$ , one may also regard  $r_P(f)$  as a function in  $H(M)_R$ . However, in general,  $r_P(f)$  is not represented by a function in  $H(M)$  in a natural way. But as we will see now,  $r_P$  is well defined at the level of the cocenter, i.e.,  $r_P(\bar{H}) \subseteq \bar{H}(M)$ .

**Lemma 2.3.** *The restriction map  $r_P$  preserves the  $\mathbb{Z}[\frac{1}{p}]$ -structure, i.e.,  $r_P(\bar{H}) \subseteq \bar{H}(M)$ .*

*Proof.* By Lemma 2.2,  $\tilde{f} = \mu(\mathcal{K}^{\text{sp}})f$  in  $\bar{H}$ , hence given (2.2), it is sufficient to show that  $\frac{\mu(\mathcal{K}^{\text{sp}})}{\mu_M(\mathcal{K}^{\text{sp}} \cap M)\mu_N(\mathcal{K}^{\text{sp}} \cap N)} \in \mathbb{Z}[\frac{1}{p}]$ . Clearly,  $\mu_N(\mathcal{K}^{\text{sp}} \cap N)^{-1} \in \mathbb{Z}[\frac{1}{p}]$ . Moreover, with our normalization of measures  $\frac{\mu(\mathcal{K}^{\text{sp}})}{\mu_M(\mathcal{K}^{\text{sp}} \cap M)} = \frac{\#\bar{\mathcal{K}}^{\text{sp}}}{\#\bar{M}}$ , where  $\bar{\mathcal{K}}^{\text{sp}}$  and  $\bar{M}$  are the reductive quotients of  $\mathcal{K}^{\text{sp}}$  and  $\mathcal{K}^{\text{sp}} \cap M$ , respectively. These are finite reductive groups and  $\bar{M}$  is a Levi subgroup of  $\bar{\mathcal{K}}^{\text{sp}}$ , hence the ratio is an integer.  $\square$

The normalization scalar  $\mu_P(\mathcal{K}^{\text{sp}} \cap P)$  in the definition is needed in order for  $r_P(f)$  to be in duality with parabolic induction. We explain this in an example. Notice first that if  $\mathcal{K}$  is any compact open subgroup of  $G$  and  $\pi'$  is a representation of  $G$ , then  $\text{Tr}(\mathbb{1}_{\mathcal{K}}, \pi') = \mu(\mathcal{K})(\pi')^{\mathcal{K}}$ . Then for any  $\pi \in \mathfrak{R}_R(M)$ , we have

$$\text{Tr}(\mathbb{1}_{\mathcal{K}^{\text{sp}}}, i_M^G(\pi)) = \mu(\mathcal{K}^{\text{sp}}) \dim(i_M^G(\pi)^{\mathcal{K}^{\text{sp}}}) = \mu(\mathcal{K}^{\text{sp}}) \dim((\pi')^{\mathcal{K}^{\text{sp}} \cap M}).$$

Note that this identity is particular to  $\mathcal{K}^{\text{sp}}$ . We also have that  $\tilde{\mathbb{1}}_{\mathcal{K}^{\text{sp}}} = \mu(\mathcal{K}^{\text{sp}})\mathbb{1}_{\mathcal{K}^{\text{sp}}}$  and then  $\tilde{\mathbb{1}}_{\mathcal{K}^{\text{sp}}}^{(P)} = \mu(\mathcal{K}^{\text{sp}})\mu_N(\mathcal{K}^{\text{sp}} \cap N)\mathbb{1}_{\mathcal{K}^{\text{sp}} \cap M}$ , where we used that  $mn \in \mathcal{K}^{\text{sp}}$  if and only if  $m \in \mathcal{K}^{\text{sp}} \cap M$  and  $n \in \mathcal{K}^{\text{sp}} \cap N$  and that  $\delta_P = 1$  on  $\mathcal{K}^{\text{sp}} \cap M$ . Therefore

$$\text{Tr}(\tilde{\mathbb{1}}_{\mathcal{K}^{\text{sp}}}^{(P)}, \pi) = \mu(\mathcal{K}^{\text{sp}})\mu_N(\mathcal{K}^{\text{sp}} \cap N)\mu_M(\mathcal{K}^{\text{sp}} \cap M) \dim((\pi')^{\mathcal{K}^{\text{sp}} \cap M})$$

and so

$$\text{Tr}(\mathbb{1}_{\mathcal{K}^{\text{sp}}}, i_M^G(\pi)) = \text{Tr}(r_P(\mathbb{1}_{\mathcal{K}^{\text{sp}}}), \pi).$$

**Theorem 2.4.** *The map  $H \rightarrow \bar{H}(M)$ ,  $f \mapsto r_P(f)$  induces a map  $\bar{r}_M : \bar{H} \rightarrow \bar{H}(M)$ .*



*Proof.* It is sufficient to show that  $f \in H$  and  ${}^x f$ ,  $x \in G$ , have the same image in  $\bar{H}(M)$ . It is clear that  $\widetilde{f} = \widetilde{k}f$ , for all  $k \in \mathcal{K}^{\text{sp}}$ , and therefore, by the Iwasawa decomposition, it is sufficient to consider the case  $x = p_0 \in P$ .

Note that for any  $p \in P$  and  $f \in H$ , we have that  $({}^p f)^{(P)} = \pi_P(p)(f^{(P)})$  and thus

$$({}^p f)^{(P)} \equiv f^{(P)} \pmod{[H(M), H(M)]}.$$

Now we have that (as functions of  $m \in M$ )

$$\begin{aligned} \delta_P^{-1/2} \widetilde{{}^{p_0} f}^{(P)} &= \int_N \int_{\mathcal{K}^{\text{sp}}} f(p_0^{-1} k^{-1} \cdot n k p_0) dk dn = \int_N \int_{\mathcal{K}^{\text{sp}}} ({}^{k p_0} f)(\cdot n) dk dn \\ &= \int_{\mathcal{K}^{\text{sp}}} ({}^{k p_0} f)^{(P)} dk = \int_{\mathcal{K}^{\text{sp}}} ({}^{p_0, k} f)^{(P)} dk \\ &\equiv \int_{\mathcal{K}^{\text{sp}}} ({}^{k_1} f)^{(P)} dk. \end{aligned}$$

In this calculation, the congruences are in  $\bar{H}(M)$ . Notice that we swapped the order of integration and used  $\mathcal{K}^{\text{sp}} P = P \mathcal{K}^{\text{sp}}$  in order to write  $k p_0 = p_{0, k} k_1$ . Here  $k_1$  (and  $p_{0, k}$ ) is not uniquely determined by  $k$  and  $p_0$ , but the right coset  $(\mathcal{K}^{\text{sp}} \cap P) k_1$  is.

The claim then follows since  $\mu$  is an invariant measure and the projection

$$\pi_2 : G \longrightarrow (\mathcal{K}^{\text{sp}} \cap P) \backslash \mathcal{K}^{\text{sp}}, \quad P k' \longmapsto (\mathcal{K}^{\text{sp}} \cap P) k'$$

gives a homeomorphism from  $(\mathcal{K}^{\text{sp}} \cap P) \backslash (\mathcal{K}^{\text{sp}} p_0)$  onto  $(\mathcal{K}^{\text{sp}} \cap P) \backslash \mathcal{K}^{\text{sp}}$  which preserves the Haar measure.  $\square$

2.3. We would like to compare the extension of  $\bar{r}_M$  over  $R$  with the categorical restriction functor  $\bar{r}_{M, R}^{\text{cat}}$ . To this end, we recall certain basic facts about the Jacquet functor.

For every open compact subgroup  $\mathcal{K}$  in a group  $L$  (which could be  $G$ ,  $P$ , or  $M$ ), such that  $\mu_L(\mathcal{K})$  is invertible in  $R$ , denote by  $e_{\mathcal{K}x\mathcal{K}} = \mu_L(\mathcal{K})^{-1} \mathbb{1}_{\mathcal{K}x\mathcal{K}}$  and  $e_{x\mathcal{K}} = \mu_L(\mathcal{K})^{-1} \mathbb{1}_{x\mathcal{K}}$ . These are elements of  $H(L)_R$ . When  $\mathcal{K}$  is a subgroup of  $G$  (and similarly for  $M$ ) we will implicitly choose  $\mathcal{K}$  sufficiently small (e.g., a subgroup of the pro- $p$  Iwahori subgroup) so that  $\mu_G(\mathcal{K})$  is invertible in  $R$ .

Now suppose  $\mathcal{K}$  is a compact open subgroup of  $P$  with a decomposition  $\mathcal{K} = (\mathcal{K} \cap M)(\mathcal{K} \cap N)$ . Then it is easy to check that, for  $p \in P$ ,  $\mathbb{1}_{p\mathcal{K}}^{(P)} = \mu_N(\mathcal{K} \cap N) \delta_P^{\frac{1}{2}} \mathbb{1}_{m(\mathcal{K} \cap M)}$  and therefore

$$(2.4) \quad e_{p\mathcal{K}}^{(P)} = \delta_P^{\frac{1}{2}} e_{\pi_P(p\mathcal{K})}.$$

The map  $\pi_P : P \rightarrow M$  makes  $H(M)_R$  into an  $H(P)_R$ -bimodule. More precisely, for every  $h \in H(P)$ , define  $\pi_P(h) \in H(M)$  by extending linearly the definition  $\pi_P(e_{p\mathcal{K}}) = e_{\pi_P(p\mathcal{K})}$ , see (2.4). Then the right action of  $H(P)$  on  $H(M)$  is given by

$$(2.5) \quad f_M \cdot h = f_M \star \pi_P(h), \quad \text{for all } h \in H(P)_R, f_M \in H(M)_R,$$

in the right hand side, the convolution being in  $H(M)_R$ .

Every smooth  $G$ -representation  $V$  gives rise to a smooth  $P$ -representation by restriction. Using the equivalence of categories, this implies that every  $H_R$ -module can be viewed as an  $H(P)_R$ -module. Precisely, if  $v \in V$  is fixed by an open compact subgroup  $\mathcal{K}$  of  $G$ , and  $\mathcal{K}'$  is a compact open subgroup of  $P$  such that  $\mathcal{K}' \subset \mathcal{K}$ , then

$$\pi(e_{p\mathcal{K}'} v) = \pi(e_{p\mathcal{K}}) v, \quad p \in P.$$

We will need the following result.

**Proposition 2.5** ([22, III.2.10]). *For every  $V \in \mathfrak{R}_R(G)$ , there is a natural isomorphism*

$$r_{M,R}(V) \cong \delta_P^{\frac{1}{2}} H(M)_R \otimes_{H(P)_R} V.$$

*The isomorphism is induced by the map  $v \mapsto e_{\mathcal{K}}^{(P)} \otimes v$ ,  $v \in V$ , where  $\mathcal{K}$  is a compact open subgroup of  $P$  such that  $\mathcal{K} \cdot v = v$ .*

Recall that, via (1.1), we defined a categorical restriction map  $\bar{r}_M^{\text{cat}}$ .

**Theorem 2.6.** *The restriction functor  $\bar{r}_M^{\text{cat}}$  equals the extension of  $\bar{r}_M$  over  $R$ .*

*Proof.* As it is well known, the Jacquet functor maps finitely generated smooth  $G$ -modules to finitely generated smooth  $M$ -modules. The algebra  $H$  itself is not finitely generated as a left  $H$ -module (as it is not unital), so instead we need to work with the finitely generated projective modules  $C_c(G/\mathcal{K})_R = H \otimes_{H(\mathcal{K})} \mathbb{1}_{\mathcal{K},R}$ , where  $\mathcal{K}$  are compact open subgroups of  $G$ . Of course,  $C_c(G/\mathcal{K})$  is the space of right  $\mathcal{K}$ -invariant compactly supported  $R$ -functions. Set

$$V_{\mathcal{K}} = \delta_P^{\frac{1}{2}} H(M) \otimes_{H(P)} C_c(G/\mathcal{K})_R.$$

If  $f \in H$  is a locally constant function, the image  $\bar{f} \in \bar{H}$  is represented by the pair  $\varinjlim_{\mathcal{K}} [C_c(G/\mathcal{K})_R, m_f]$ , where  $\mathcal{K}$  is a compact open subgroup of  $G$  such that  $f$  is  $\mathcal{K}$ -biinvariant, and  $m_f : C_c(G/\mathcal{K})_R \rightarrow C_c(G/\mathcal{K})_R$  is right convolution by  $f$ . We therefore need to compute the image of  $\varinjlim_{\mathcal{K}} \text{Tr}_{H(M)}(V_{\mathcal{K}}, m_f)$  in the categorical description of the restriction  $\bar{r}_M^{\text{cat}}$ . Computing the trace is compatible with this direct system and it stabilizes, i.e., it does not change for sufficiently small open compact subgroups  $\mathcal{K}$ .

Let  $\mathcal{K}_1$  be a compact open subgroup of  $G$ , and let  $f$  be a right  $\mathcal{K}_1$ -invariant  $R$ -function. Without loss of generality, we may assume that  $f = \mathbb{1}_{g\mathcal{K}_1}$  for a fixed  $g \in G$ . Let  $\mathcal{K}$  be a compact open subgroup of  $\mathcal{I}'$  such that  $\mathcal{K} \subset \mathcal{K}_1$  and  $\mathcal{K}$  is normal in  $\mathcal{K}^{\text{sp}}$ . Using the Cartan decomposition  $G = P\mathcal{K}^{\text{sp}}$ , we may write  $G = \sqcup_i P g_i \mathcal{K}$ , where  $\{g_i\}$  are a (finite) set of representatives of  $\mathcal{K}^{\text{sp}} \cap P \backslash \mathcal{K}^{\text{sp}} / \mathcal{K}$ . Notice that, by our assumption on  $\mathcal{K}$ ,  $g_i \mathcal{K} g_i^{-1} = \mathcal{K}$ , a fact that we will use repeatedly below. The space  $C_c(G/\mathcal{K})_R$  is a free left  $H(P)$ -module of finite rank:

$$C_c(G/\mathcal{K})_R = \bigoplus_i H(P) \cdot \mathbb{1}_{g_i \mathcal{K}}.$$

Here  $H(P)$  acts of  $C_c(G/\mathcal{K})$  as explained in the paragraph before Proposition 2.5. Concretely,

$$\mathbb{1}_{pg_i \mathcal{K}} = \frac{1}{\mu_P(P \cap \mathcal{K})} \mathbb{1}_{(P \cap \mathcal{K})p} \star \mathbb{1}_{g_i \mathcal{K}}, \quad p \in P.$$

To compute the categorical restriction, we therefore have:

$$\bar{r}_M^{\text{cat}}(f) = \text{Tr}_{H(M)}(V_{\mathcal{K}}, m_f) = \delta_P^{\frac{1}{2}} \pi_P \left( \text{Tr}_{H(P)}(C_c(G/\mathcal{K})_R, m_f) \right),$$

where, **the first equality is the well-known equivalent description of the categorical restriction map**, while for the second equality, we used (2.1), translated into the

equivalent setting of  $H(P)$ -modules. We have  $\mathbb{1}_{g_i\mathcal{K}} \star f = \mathbb{1}_{g_i\mathcal{K}} \star \mathbb{1}_{g\mathcal{K}_1} = \mathbb{1}_{g_i\mathcal{K}g\mathcal{K}_1g_i^{-1}} \star \mathbb{1}_{g_i\mathcal{K}}$ . The part of  $\mathbb{1}_{g_i\mathcal{K}} \star f$  that contributes to  $\mathrm{Tr}_{H(P)}$  comes from

$$\begin{aligned} g_i\mathcal{K}g\mathcal{K}_1g_i^{-1} \cap Pg_i\mathcal{K} &\cong (\text{why?})(g_i\mathcal{K}g\mathcal{K}_1g_i^{-1} \cap P) \cdot (g_i\mathcal{K}g_i^{-1}) = (g_i\mathcal{K}g\mathcal{K}_1g_i^{-1} \cap P) \cdot \mathcal{K} \\ &\cong (g_i\mathcal{K}g\mathcal{K}_1g_i^{-1} \cap P) \times_{\mathcal{K} \cap P} \mathcal{K}. \end{aligned}$$

Thus

$$\mathrm{Tr}_{H(P)}(C_c(G/\mathcal{K})_R, m_f) = \sum_i \frac{\mu_G(\mathcal{K})}{\mu_P(\mathcal{K} \cap P)} \mathbb{1}_{g_i\mathcal{K}g\mathcal{K}_1g_i^{-1} \cap P}.$$

For every  $i$ , in the cocenter, we have  $\mathbb{1}_{g_i\mathcal{K}g\mathcal{K}_1g_i^{-1} \cap P} \equiv \mathbb{1}_{g'\mathcal{K}g\mathcal{K}_1(g')^{-1} \cap P}$ , for all  $g' \in (\mathcal{K}^{\mathrm{sp}} \cap P)g_i\mathcal{K}$ . Moreover,  $\mu_G((\mathcal{K}^{\mathrm{sp}} \cap P)g_i\mathcal{K}) = \frac{\mu_G(\mathcal{K})}{\mu_P(\mathcal{K} \cap P)} \mu_P(\mathcal{K}^{\mathrm{sp}} \cap P)$ . This means that:

$$\mu_P(\mathcal{K}^{\mathrm{sp}} \cap P) \mathrm{Tr}_{H(P)}(C_c(G/\mathcal{K}), m_f) = \int_{\mathcal{K}^{\mathrm{sp}}} \mathbb{1}_{g'\mathcal{K}g\mathcal{K}_1(g')^{-1} \cap P} dg'.$$

On the other hand, by direct computation

$$\tilde{f}|_P = \int_{\mathcal{K}^{\mathrm{sp}}} \mathbb{1}_{g'\mathcal{K}g\mathcal{K}_1(g')^{-1} \cap P} dg'.$$

By taking  $\mathcal{K}$  sufficiently small, we may assume that  $\mathcal{K} \subset g\mathcal{K}_1g^{-1}$ , and therefore:

$$\mu_P(\mathcal{K}^{\mathrm{sp}} \cap P) \mathrm{Tr}_{H(P)}(C_c(G/\mathcal{K}), m_f) = \tilde{f}|_P.$$

$\mu_P(\mathcal{K}^{\mathrm{sp}} \cap P) \bar{r}_M^{\mathrm{cat}}(f) = \delta_P^{\frac{1}{2}} \pi_P(\tilde{f}|_P) = \tilde{f}^{(P)}$ . Hence, on the level of the cocenter,  $\bar{r}_M^{\mathrm{cat}} = \bar{r}_M$ .  $\square$

2.4. We need to investigate the relation between  $\bar{r}_M$  and twists by  $w \in W$ . The following lemma will allow us to transfer known results from complex numbers to our setting.

**Lemma 2.7.** *The natural map  $\bar{H} \rightarrow \bar{H}_{\mathbb{C}}$  is injective.*

*Proof.* By [25, §2a] (“changement de base”), the natural homomorphism  $\bar{H} \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{C} \rightarrow \bar{H}_{\mathbb{C}}$  is an isomorphism. On the other hand  $\mathbb{C}$  is flat over  $\mathbb{Z}[\frac{1}{p}]$ , hence the claim follows.  $\square$

Now we recall the Spectral Density Theorem.

**Theorem 2.8.** *Let  $f \in H_{\mathbb{C}}$ . If  $\mathrm{Tr}_{\mathbb{C}}(f, \pi) = 0$  for all irreducible smooth  $\mathbb{C}$ -representations  $\pi$ , then  $f \in [H_{\mathbb{C}}, H_{\mathbb{C}}]$ .*

The Spectral Density Theorem was first proved by Kazhdan [16, 17], in particular [17, Theorem B(2)]. See also the recent work by Henniart and Lemaire in [15] on the twisted density theorem. Note that for algebraically closed field  $R$  of positive characteristic, the analogous statement (by replacing  $\mathbb{C}$  by  $R$ ) may fail.

Now we prove the following statement on the relation between  $\bar{r}_M$  and twisted by  $w \in W$ .

**Proposition 2.9.** *Let  $M, M'$  be standard Levi subgroups and  $w \in W_G$  with  $M' = {}^w M$ . Then*

$$(2.6) \quad \bar{r}_{M'} = \dot{w} \circ \bar{r}_M : \bar{H} \longrightarrow \bar{H}.$$

*Proof.* By the adjunction property (1.1) and by the compatibility of restrictions, Theorem 2.6, we see that

$$(2.7) \quad \mathrm{Tr}_{\mathbb{C}}^G(h_G, i_{M, \mathbb{C}}(\sigma)) = \mathrm{Tr}_{\mathbb{C}}^M(\bar{r}_{M, \mathbb{C}}(h_G), \sigma), \quad h_G \in \bar{H}_{\mathbb{C}}.$$

On the other hand, it is well known that

$$(2.8) \quad i_{M, \mathbb{C}} \circ \dot{w}^{-1} = i_{M', \mathbb{C}},$$

see for example [1]. Therefore by Theorem 2.8,  $\bar{r}_{M', \mathbb{C}} = \dot{w} \circ \bar{r}_{M, \mathbb{C}}$ . The claim follows then from Lemma 2.7. Notice that one only needs to use spectral density theorem for the proper Levi subgroups and for  $R = \mathbb{C}$ .  $\square$

### 3. SOME COMPATIBILITY RESULTS

3.1. We first discuss the compatibility between the Newton decomposition and the induction maps on the cocenter.

We recall the explicit formula in [14]. Let  $M$  be a standard Levi subgroup and  $v \in V_+$  with  $M = M_v$ . Set  $P = P_v$ . By [14, Theorem A], the map

$$\mathbb{1}_{m\mathcal{K}_M} \longmapsto \delta_P(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M \mathcal{I}_n)} \mathbb{1}_{m\mathcal{K}_M \mathcal{I}_n} + [H, H] \text{ for } n \gg 0$$

gives a well-defined surjection

$$\bar{i}_v : \bar{H}(M; v) \longrightarrow \bar{H}(v).$$

Here  $\bar{H}(M; v) \subset \bar{H}(M)$  is the Newton component corresponding to  $v$ . By [14, Theorem B], the extension of  $\bar{i}_v$  over  $R$  is adjoint to the Jacquet functor  $r_{M, R} : \mathfrak{R}_R(G) \rightarrow \mathfrak{R}_R(M)$ . Therefore, the extension of  $\bar{i}_v$  over  $R$  equals  $\bar{i}_{M, R}^{\mathrm{cat}}$  on the reduced cocenter. In particular, we have

**Proposition 3.1.** *Let  $v \in V_+$  and  $M = M_v$ . Then  $\bar{i}_{M, R}^{\mathrm{cat}}$  maps  $\bar{H}_R^{\mathrm{red}}(M; v)$  onto  $\bar{H}_R^{\mathrm{red}}(v)$ .*

Note that the maps  $\bar{i}_*$  are defined only for the Newton strata  $\bar{H}(M; v)$  of  $\bar{H}(M)$  with  $M = M_v$ . It is a challenging problem to give an explicit formula for  $\bar{i}_M : \bar{H}(M) \rightarrow \bar{H}$ .

3.2. Next we show that the Newton decomposition is compatible with the restriction maps on the cocenter.

Let  $V_+^M$  be the set of  $M$ -dominant elements in  $V$  and  $\Omega_M = \tilde{W}(M)/W_a(M)$ , where  $\tilde{W}(M)$  is the Iwahori-Weyl group of  $M$  and  $W_a(M)$  is the associated affine Weyl group. Set  $\aleph_M = \Omega_M \times V_+^M$ . As explained in [14, §1.5], there is a natural map

$$\aleph_M \longrightarrow \aleph, \quad \nu \longmapsto \bar{\nu}.$$

On the  $V$ -factor, this map sends any  $M$ -dominant element in  $V$  to the unique  $G$ -dominant element in its  $W_G$ -orbit.

**Proposition 3.2.** *Let  $\nu \in \aleph$  and  $M$  be a standard Levi subgroup. Then*

$$\bar{r}_M(\bar{H}(\nu)) \subseteq \bigoplus_{\nu' \in \aleph_M; \bar{\nu}' = \nu} \bar{H}(M; \nu').$$

*Proof.* Let  $f = \mathbb{1}_X$ , where  $X \subset G(\nu)$ . By §2.1, the support of  $\tilde{f}^{(P)}$  equals  $\bigcup_{k \in \mathcal{K}^{\mathrm{sp}}} \pi_P(kXk^{-1} \cap P)$ . We have  $k^{-1}Xk \cap P \subset G(\nu) \cap P$ . Now the statement follows from §2.2 and the following Lemma.  $\square$

**Lemma 3.3.** *Let  $P = MN$  be a standard parabolic subgroup and  $\nu \in \mathfrak{N}$ . If  $p \in P \cap G(\nu)$ , then  $\pi_P(p) \in M(\nu')$  for some  $\nu' \in \mathfrak{N}_M$  with  $\bar{\nu}' = \nu$ .*

**Remark 3.4.** The original proof was more complicated. The following simplification was suggested by S. Nie.

*Proof.* Let  $p = mu$  with  $m \in M$  and  $u \in N$ . We assume that  $m \in M(\nu')$  for some  $\nu' \in \mathfrak{N}_M$ .

Let  $\lambda \in X_*(Z)$  such that  $\langle \lambda, \alpha \rangle = 0$  for any relative root  $\alpha$  in  $M$  and  $\langle \lambda, \alpha \rangle > 0$  for any relative root in  $N$ . By [14, Proposition 4.2],  $m \in G(\bar{\nu}')$ . By [13, §3.2 (a)], there exists  $n \in \mathbb{N}$  such that  $m\mathcal{I}_n \subset G(\bar{\nu}')$ . By our assumption on  $\lambda$ , there exists  $l \in \mathbb{N}$  such that  $t^{l\lambda}ut^{-l\lambda} \in \mathcal{I}_n \cap N$ . Hence

$$t^{l\lambda}pt^{-l\lambda} = mt^{l\lambda}ut^{-l\lambda} \in m\mathcal{I}_n \subset G(\bar{\nu}').$$

However,  $t^{l\lambda}pt^{-l\lambda}$  is conjugate to  $p$  and thus is contained in  $G(\nu)$ . By [13, Theorem 3], we must have  $\nu = \bar{\nu}'$ .  $\square$

#### 4. THE MAIN RESULT

We first prove the following weak form of the Mackey formula for the cocenter.

**Proposition 4.1.** *Let  $M$  be a standard Levi subgroup and  $v \in V_+$  with  $M = M_v$ . Then for any  $f \in \bar{H}(M; v)$ , we have*

$$\bar{r}_M \circ \bar{i}_v(f) \in f + \sum_{w \in {}^M W^M; w \neq 1} \bar{H}(M; w(v)),$$

where  ${}^M W^M$  is the subset of  $W_G$  consisting of elements of minimal length in their  $W_M \times W_M$ -cosets.

**Remark 4.2.** We expect that

$$\bar{r}_M \circ \bar{i}_v(f) = \sum_{w \in {}^M W^M} \bar{i}_{w(v)}^M \circ w \circ \bar{r}_{M \cap w^{-1}M}^M(f).$$

In the reduced cocenter  $\bar{H}_R^{\text{red}}$ , this can be obtained via the adjunction formula and the Mackey formula in  $\mathfrak{R}_R(G)$  [24, §I.5.5]. The equality in the reduced cocenter is enough for the application to the trace Paley-Wiener Theorem, but for the application to the abstract Selberg principle, we need an equality in the cocenter.

*Proof.* It suffices to consider the case where  $f = \mathbb{1}_{m\mathcal{K}_M}$ , where  $m \in M$  and  $\mathcal{K}_M$  is an open compact subgroup of  $M$  such that  $m\mathcal{K}_M \in M(v)$ . By definition,  $\mu_P(\mathcal{K}^{\text{sp}} \cap P)\bar{r}_M \circ \bar{i}_v(f)$  is represented by

$$\frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')} \int_{\mathcal{K}^{\text{sp}}} \mathbb{1}_{kXk^{-1} \cap P}^{(P)} dk,$$

where  $X = m\mathcal{K}_M\mathcal{K}'$  for some sufficiently small open compact subgroup  $\mathcal{K}' = \mathcal{K}'_N\mathcal{K}'_M\mathcal{K}'_{N^-}$  of  $G$ .

By [14, Proposition 2.3], any element in  $m\mathcal{K}_M\mathcal{K}'$  is conjugate by an element in  $\mathcal{K}^{\text{sp}}$  to an element in  $m\mathcal{K}_M$ .

Let  $w \in {}^M W^M$ . We first show that

$$(a) \quad \pi_P(km\mathcal{K}_Mk^{-1} \cap P) \subseteq M(w(v)) \quad \text{for any } k \in PwP.$$

Note that  $\pi_P(pX'p^{-1} \cap P) = \pi_P(p)\pi_P(X' \cap P)\pi_P(p)^{-1}$  for any  $p \in P$  and  $X' \subset G$ . It suffices to prove (a) for  $k \in \dot{w}P$ .

Let  $k = \dot{w}p$  for  $p \in P$ . Note that  $km\mathcal{K}_Mk^{-1} \cap P \subseteq \dot{w}P\dot{w}^{-1} \cap P$ . By [4, Theorem 2.8.7], we have

$$\dot{w}P\dot{w}^{-1} \cap P \cong M' \times (\dot{w}N\dot{w}^{-1} \cap M) \times (\dot{w}M\dot{w}^{-1} \cap N) \times (\dot{w}N\dot{w}^{-1} \cap N),$$

where  $M' = \dot{w}M\dot{w}^{-1} \cap M$  is a standard Levi subgroup of  $G$ . Then

$$\pi_P(km\mathcal{K}_Mk^{-1} \cap P) \subseteq \pi_P(\dot{w}\pi_P(pm\mathcal{K}_Mp^{-1})\dot{w}^{-1} \cap P)(\dot{w}N\dot{w}^{-1} \cap M).$$

Note that  $\pi_P(\dot{w}\pi_P(pm\mathcal{K}_Mp^{-1})\dot{w}^{-1} \cap P) \subseteq M'$ . We have

$$\begin{aligned} \pi_P(\dot{w}\pi_P(pm\mathcal{K}_Mp^{-1})\dot{w}^{-1} \cap P) &= \pi_{M'}(\dot{w}\pi_P(pm\mathcal{K}_Mp^{-1})\dot{w}^{-1} \cap P) \subseteq M'(w(v)) \\ &\subseteq M(w(v)). \end{aligned}$$

Here the first inequality follows from Proposition 3.2 and the second inequality follows from the fact that  $w(v)$  is  $M$ -dominant.

Hence we have  $\pi_P(km\mathcal{K}_Mk^{-1} \cap P) \subseteq M'(w(v))(\dot{w}N\dot{w}^{-1} \cap M)$ . As  $\langle w(v), \alpha \rangle > 0$  for any root  $\alpha$  in  $\dot{w}N\dot{w}^{-1} \cap M$ , by the proof of [14, Proposition 2.3 (1)], any element in  $M'(w(v))(\dot{w}N\dot{w}^{-1} \cap M)$  is conjugate in  $M$  to an element in  $M'(w(v))$ . Therefore  $\pi_P(km\mathcal{K}_Mk^{-1} \cap P) \subseteq M(w(v))$  and (a) is proved.

Now we compute the Newton component  $f'$  of  $\mu_P(\mathcal{K}^{\text{sp}} \cap P)\bar{r}_M \circ \bar{i}_v(f)$  with Newton point  $v$ . Note that  $\mathcal{K}^{\text{sp}} = \sqcup_{w \in {}^M W^M} \mathcal{K}^{\text{sp}} \cap P\dot{w}P$ . By (a),

$$f' = \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')} \int_Y \mathbb{1}_{kXk^{-1} \cap P}^{(P)} dk,$$

where  $Y$  is the subset of  $\mathcal{K}^{\text{sp}}$  consisting of elements of the form  $pk'$  with  $p \in \mathcal{K}^{\text{sp}} \cap P$  and  $k' \in \mathcal{K}^{\text{sp}}$  such that  $k'X(k')^{-1} \cap m\mathcal{K}_M \neq \emptyset$ . As  $\mathcal{K}'_N$  is sufficiently small, for any  $u \in \mathcal{K}^{\text{sp}} \cap N^-$ , we have that  $uXu^{-1} \subseteq (m\mathcal{K}_M\mathcal{K}'_N)N^-$  and if  $uXu^{-1} \cap m\mathcal{K}_M\mathcal{K}'_N \neq \emptyset$ , then  $uXu^{-1} \cap m\mathcal{K}_M\mathcal{K}'_N = m\mathcal{K}_M\mathcal{K}'_N$ . By the proof of [14, Proposition 2.3 (1)],  $Y = (\mathcal{K}^{\text{sp}} \cap P) \times Y'$ , where  $Y' = \{u \in \mathcal{K}^{\text{sp}} \cap N^-; m\mathcal{K}_M\mathcal{K}'_N \subseteq uXu^{-1}\}$  and  $\mu_{N^-}(Y') = \mu_{N^-}(\mathcal{K}'_{N^-})$ .

Note that for any  $p = m'u \in P$  with  $m_1 \in M$  and  $u \in N$ , we have that  $\pi_P(\mathbb{1}_{pm\mathcal{K}_M\mathcal{K}'_N}) = \mu_N(\mathcal{K}'_N)\mathbb{1}_{m'm\mathcal{K}_M(m')^{-1}}$ . Thus we have in  $\bar{H}(M)$ ,

$$\begin{aligned} f' &= \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')} \int_Y \mathbb{1}_{kXk^{-1} \cap P}^{(P)} dk \\ &= \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')} \mu_{N^-}(\mathcal{K}'_{N^-}) \int_{\mathcal{K}^{\text{sp}} \cap P} \mathbb{1}_{km\mathcal{K}_M\mathcal{K}'_Nk^{-1}}^{(P)} dk \\ &= \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')} \mu_{N^-}(\mathcal{K}'_{N^-}) \mu_N(\mathcal{K}^{\text{sp}} \cap N) \mu_N(\mathcal{K}'_N) \int_{\mathcal{K}^{\text{sp}} \cap M} \mathbb{1}_{m'm\mathcal{K}_M(m')^{-1}} dm' \\ &= \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')} \mu_{N^-}(\mathcal{K}'_{N^-}) \mu_N(\mathcal{K}^{\text{sp}} \cap N) \mu_N(\mathcal{K}'_N) \mu_M(\mathcal{K}^{\text{sp}} \cap M) f \\ &= \mu_P(\mathcal{K}^{\text{sp}} \cap P) f \in \bar{H}(M). \end{aligned}$$

Here the fourth equality follows from that  $\mathbb{1}_{m'm\mathcal{K}_M(m')^{-1}}$  and  $f = \mathbb{1}_{m\mathcal{K}_M}$  have the same image in  $\bar{H}(M)$  and the last equality follows from the fact that  $\mu_G(\mathcal{K}_M\mathcal{K}') = \mu_M(\mathcal{K}_M)\mu_N(\mathcal{K}'_N)\mu_{N^-}(\mathcal{K}'_{N^-})$  and  $\mu_P(\mathcal{K}^{\text{sp}} \cap P) = \mu_M(\mathcal{K}^{\text{sp}} \cap M)\mu_N(\mathcal{K}^{\text{sp}} \cap N)$ .  $\square$

Combining the weak form the Mackey formula with the Newton decomposition, we have the following result.

**Theorem 4.3.** *Let  $v \in V_+$  and  $M = M_v$ . Then the restriction maps  $\bar{r}_M : \bar{H}(G; v) \rightarrow \bar{H}(M)$  and  $\bar{r}_M : \bar{H}_R^{\text{red}}(G; v) \rightarrow \bar{H}_R^{\text{red}}(M)$  are injective.*

*Proof.* Let  $f \in \bar{H}(v)$ . By [14, Theorem A], we have  $f = \bar{i}_M(f')$  for some  $f' \in \bar{H}(M; v)$ . As  $W_M$  is the centralizer of  $\nu$  in  $W$ , for  $w \in {}^M W^M$ , we have  $w(v) = v$  if and only if  $w = 1$ .

If  $\bar{r}_M(f) = 0 \in \bar{H}(M)$ , then by the Newton decomposition on  $\bar{H}(M)$  (see [13, Theorem B]) and Proposition 4.1, we have  $f' = 0 \in \bar{H}(M)$ .

If  $\bar{r}_M(f) \in \ker \text{Tr}_R^M$ , then by the compatibility between the Newton decomposition and the trace map (see [14, Theorem 6.3]), we have  $f' \in \ker \text{Tr}_R^M$  and thus  $f \in \ker \text{Tr}_R^G$ .  $\square$

4.1. We define the elliptic part of the cocenter by

$$\begin{aligned} \bar{H}^{\text{ell}} &= \{f \in \bar{H}; \bar{r}_M(f) = 0 \text{ for any proper standard Levi } M\}, \\ \bar{H}_R^{\text{ell,red}} &= \{f \in \bar{H}^{\text{red}}; \bar{r}_{M,R}^{\text{red}}(f) = 0 \text{ for any proper standard Levi } M\}. \end{aligned}$$

Now we compare the elliptic part and rigid part of the cocenter.

**Theorem 4.4.** *We have*

$$\bar{H}^{\text{ell}} \subset \bar{H}^{\text{rig}}, \quad \bar{H}_R^{\text{ell,red}} \subset \bar{H}_R^{\text{rig,red}}.$$

*Proof.* We prove the first inclusion. The second one is proved in the same way.

Let  $f \in \bar{H}$ . We may write  $f$  as  $f = \sum_{v \in V_+} f_v$ , where  $f_v \in \bar{H}(v)$  is the corresponding Newton component of  $f$ . By Proposition 3.2,

$$\bar{r}_M(f_v) \in \oplus_{v' \in W_G(v); v' \text{ is } M\text{-dominant}} \bar{H}(M; v').$$

Note that  $W_G(v_1) \cap W_G(v_2) = \emptyset$  for distinct  $v_1, v_2 \in V_+$ . By the Newton decomposition on  $\bar{H}(M)$  (see [13, Theorem B]), we have that  $\bar{r}_M(f_v) = 0$  for all proper Levi subgroups  $M$ . By Theorem 4.3,  $f_v = 0$  for all  $v \in V_+$  with  $M_v \neq G$ . Therefore  $f \in \bar{H}^{\text{rig}}$ .  $\square$

## 5. APPLICATION: THE TRACE PALEY-WIENER THEOREM

5.1. Let  $M$  be a standard Levi subgroup. Denote by  $M^0$  the subgroup of  $M$  generated by all the compact subgroups of  $M$ . Then  $M^0$  is open in  $M$  and  $M/M^0$  is an abelian group of finite rank [24, I.1.4]. An *unramified character* of  $M$  over  $R$  is a group homomorphism  $G \rightarrow R^\times$  which is trivial on  $M^0$ . We denote by  $\Psi(M)_R$  the set of all the unramified characters of  $M$  over  $R$ . This is a group under multiplication. In fact, as  $M/M^0 \cong \mathbb{Z}^n$  for  $n \geq 0$ , we have  $\Psi(M)_R \cong (R^\times)^n$ . The group  $\Psi(M)_R$  acts by multiplication on  $\mathfrak{R}_R(M)$ .

We recall the definition of good forms introduced in [1]. Let  $f \in \mathfrak{R}_R(G)^*$ . We say that  $f$  is *good* if:

- There exists an open compact subgroup  $\mathcal{K}$  such that  $f(\pi) = 0$  if  $\pi$  has no nonzero  $\mathcal{K}$ -fixed points.
- For every standard Levi subgroup  $M$  and  $\sigma \in \mathfrak{R}_R(M)$ , the function

$$\Psi(M)_R \longrightarrow R, \quad \psi \longmapsto f(i_M(\sigma \otimes \psi))$$

is a regular function.

Let  $\mathfrak{R}_R(G)_{\text{good}}^*$  be the set of all good forms in  $\mathfrak{R}_R(G)^*$ .

It is easy to see that  $\text{Tr}_R(f)$  is a good form for any  $f \in \bar{H}$ .

In the rest of this section, we show that if  $\sharp W_G$  is invertible in  $R$ , then the converse is also true, i.e., any good form comes from the cocenter of  $H$ .

**Theorem 5.1.** *Assume that  $\sharp W_G$  is invertible in  $R$ . Then the trace map  $\text{Tr}_R : \bar{H}_R \rightarrow \mathfrak{R}_R(G)_{\text{good}}^*$  is surjective.*

5.2. We first recall the  $A$ -operator introduced in [1, §5.5].

For any standard Levi subgroup  $M$ , let  $d(M) = \dim Z(M)$ , where  $Z(M)$  is the center of  $M$ . Let  $N_M = \{w \in W_M \setminus W_G/W_M; wW_Mw^{-1} = W_M\}$ . For any  $l \in \mathbb{N}$ , we define  $A_{l,R} = \prod_{M; d(M)=l} (i_{M,R} \circ r_{M,R} - \sharp N_M)$ . We define

$$A_R = A_{d(G),R} A_{d(G)-1,R} \cdots A_{1,R} : \mathfrak{R}_R(G) \longrightarrow \mathfrak{R}_R(G).$$

We have the Mackey formula [24, §I.5.5]

$$r_{M,R} \circ i_{M',R} = \sum_{w \in {}^M W^{M'}} i_{M \cap {}^w M',R}^M \circ {}^w \circ r_{M' \cap {}^w M,R}^{M'} : \mathfrak{R}_R(M') \longrightarrow \mathfrak{R}_R(M).$$

Combining Proposition 2.9 with Theorem 2.6, we have

**Lemma 5.2.** *Let  $M, M'$  be standard Levi subgroups of  $G$  and  $w \in W$  with  ${}^w M = M'$ . Then*

$$i_{M,R} = i_{M',R} \circ {}^w : \mathfrak{R}_R(M) \longrightarrow \mathfrak{R}_R(G).$$

**Remark 5.3.** This result is first proved in [1, Lemma 5.4 (iii)] for complex representations. The proof in *loc.cit.* is based on the Langlands classification. Notice that here, this follows from the cocenter results. Therefore Lemma 5.2 is valid for arbitrary algebraically closed field of characteristic not equal to  $p$ .

By the argument in [1, §5.4 & 5.5], we have

**Proposition 5.4.** *We have  $A_R^2 = aA_R$  for some positive integer  $a$  whose prime factors divides  $\sharp W_G$  and*

$$\ker A_R = \sum_{M \subsetneq G} i_{M,R}(\mathfrak{R}_R(M)).$$

Note that [1] only consider the case where  $R = \mathbb{C}$ . The essential ingredients used in *loc.cit.* are the Mackey formula and Lemma 5.2. As both the ingredients are known now for any algebraic closed field  $R$  of characteristic not equal to  $p$ , Proposition 5.4 is valid in this general situation as well.

5.3. Let  $\bar{r}_{M,R}^{\text{red}} : \bar{H}_R^{\text{red}} \rightarrow \bar{H}_R^{\text{red}}(M)$  be the map induced from the map  $\bar{r}_M$ . Let  $\bar{i}_{M,R}^{\text{red}} : \bar{H}_R^{\text{red}}(M) \rightarrow \bar{H}_R^{\text{red}}$  be the map adjoint to the Jacquet functor  $r_{M,R}$ . Let  $\bar{A}_R^{\text{red}} : \bar{H}_R^{\text{red}} \rightarrow \bar{H}_R^{\text{red}}$  be the map adjoint to the  $A$ -operator on  $\mathfrak{R}_R(G)$ , i.e. for  $f \in \bar{H}_R^{\text{red}}$  and  $\pi \in \mathfrak{R}_R(G)$ , we have

$$\text{Tr}_R^G(f, A_R(\pi)) = \text{Tr}_R^G(\bar{A}_R^{\text{red}}(f), \pi).$$

We also have the following description of the elliptic cocenter.

**Proposition 5.5.** *Assume that  $\sharp W_G$  is invertible in  $R$ . Then  $\bar{H}_R^{\text{ell},\text{red}} = \text{Im } \bar{A}_R^{\text{red}}$ .*



*Proof.* We have  $A_R \circ i_{M,R} = 0$  on  $\mathfrak{R}_R(M)$ . Therefore  $\bar{r}_M^{\text{red}} \circ \bar{A}_R^{\text{red}} = 0$  on  $\bar{H}_R^{\text{red}}$ . Hence  $\text{Im } \bar{A}_R^{\text{red}} \subset \bar{H}^{\text{ell,red}}$ . On the other hand, let  $f \in \bar{H}^{\text{ell,red}}$ , then by definition  $\bar{A}_R^{\text{cat}}(f) = \pm af$ , where  $a$  is the positive integer in Proposition 5.4. Thus  $f \in \text{Im}(\bar{A}_R^{\text{red}})$ .  $\square$

5.4. For any open compact subgroup  $\mathcal{K}$  of  $G$ , let  $H(G, \mathcal{K})$  be the space of compactly supported  $\mathcal{K} \times \mathcal{K}$ -biinvariant  $\mathbb{Z}[\frac{1}{p}]$ -valued functions on  $G$  and  $H_R(G, \mathcal{K}) = H(G, \mathcal{K}) \otimes_{\mathbb{Z}[\frac{1}{p}]} R$ . Then we have  $H = \varinjlim_{\mathcal{K}} H(G, \mathcal{K})$  and  $H_R = \varinjlim_{\mathcal{K}} H_R(G, \mathcal{K})$ . The trace map  $\text{Tr}_R : H_R \rightarrow \mathfrak{R}(H_R)^*$  induces

$$\text{Tr}_R : \bar{H}_R(G, \mathcal{K}) \longrightarrow \mathfrak{R}(H_R(G, \mathcal{K}))^*.$$

Let  $\mathcal{K}$  be an open compact subgroup of  $G$ . For any  $\nu \in \mathfrak{N}$ , let  $H(G, \mathcal{K}; \nu) = H(G, \mathcal{K}) \cap H(\nu)$  and  $\bar{H}(G, \mathcal{K}; \nu)$  be its image in  $\bar{H}$ . Then  $H(G, \mathcal{K}) \supseteq \bigoplus_{\nu} H(G, \mathcal{K}; \nu)$ . However, by [13, Theorem 11] and [14, Theorem 6.3], we have that

**Theorem 5.6.** *Let  $\mathcal{I}_n$  be the  $n$ -th congruence subgroup of the Iwahori subgroup of  $G$ . Then*

$$\bar{H}(G, \mathcal{I}_n) = \bigoplus_{\nu \in \mathfrak{N}} \bar{H}(G, \mathcal{I}_n; \nu), \quad \bar{H}_R^{\text{red}}(G, \mathcal{I}_n) = \bigoplus_{v \in V_+} \bar{H}_R^{\text{red}}(G, \mathcal{I}_n; v).$$

5.5. We recall the relevant properties of the Moy-Prasad filtration. The original references are [20, 21] where the notion of depth of a representation is defined for irreducible smooth complex representations. This is extended in [24] to the case of mod- $l$  representations. Let  $\bar{F}^u$  be a maximal unramified extension of  $F$  and  $\Gamma = \bar{F}^u/F$  be the Galois group. Let  $\mathcal{B}(\mathbb{G}, \bar{F}^u)$  be the Bruhat-Tits building of  $\mathbb{G}/\bar{F}^u$  and let  $\mathcal{B}(\mathbb{G}, F) = \mathcal{B}(\mathbb{G}, \bar{F}^u)^{\Gamma}$  be the building of  $\mathbb{G}/F$  as in [21, §3.1]. For every  $x \in \mathcal{B}(\mathbb{G}, F)$ , let  $\mathbb{G}_x$  be the parahoric subgroup defined by  $x$ , a subgroup of finite index in  $\mathbb{G}_x^{\dagger} = \{g \in \mathbb{G} \mid g \cdot x = x\}$ . For every  $r \geq 0$ ,  $\mathbb{G}_{x,r}$  denotes the Moy-Prasad filtration subgroup of  $\mathbb{G}_x$ . These subgroups are defined using the affine root subgroups  $\{U_{\psi} \mid \psi(x) \geq r\}$ , where  $\psi$  are the affine roots of  $\mathbb{G}$  defined with respect to a maximal  $\bar{F}^u$ -split torus  $T$ . Each  $\mathbb{G}_{x,r}$  is normal in  $\mathbb{G}_x$ ,  $\mathbb{G}_{x,s} \subseteq \mathbb{G}_{x,r}$  if  $s \geq r$ , and  $[\mathbb{G}_{x,r}, \mathbb{G}_{x,s}] \subseteq \mathbb{G}_{x,r+s}$ . Define

$$\mathbb{G}_{x,r+} = \bigcup_{s > r} \mathbb{G}_{x,s}.$$

If  $r > 0$ , then  $\mathbb{G}_{x,r}/\mathbb{G}_{x,r+}$  is abelian. Since  $\mathbb{G}_{x,r}$  is  $\Gamma$ -stable, set  $G_{x,r} = G_x \cap \mathbb{G}_{x,r}$  and  $G_{x,r+} = G_x \cap \mathbb{G}_{x,r+}$ . Then  $G_{x,r}$  and  $G_{x,r+}$  are open normal subgroups of the parahoric subgroup  $G_x$ .

Following [21, Theorem 3.5] and [24, II.5] for every representation  $(\pi, V) \in \text{Irr}_R(G)$ , define the *depth* of  $\pi$  to be the rational number  $r(\pi) \geq 0$  such that for some  $x \in \mathcal{B}(\mathbb{G}, F)$ , the space  $V^{G_{x,r(\pi)+}}$  of  $G_{x,r+}$ -fixed vectors is nonzero and  $r(\pi)$  is the smallest number with this property. The next result relates this notion to parabolic induction and restriction.

**Theorem 5.7** ([21, Theorems 4.5 and 5.2], [24, II.5.12]). *The functors of parabolic induction  $i_{M,R}$  and parabolic restriction  $r_{M,R}$  preserve the depth of representations.*

As a corollary, we relate this to the case of representations with  $\mathcal{I}_n$ -fixed vectors.

**Corollary 5.8.** *Let  $M$  be a Levi subgroup of a parabolic subgroup  $P$  and let  $\mathcal{I}_n$  be the  $n$ -th level Iwahori filtration subgroup of  $G$ .*

- (1) The functor  $i_{M,R}$  maps  $\mathfrak{R}_R(H(M, \mathcal{I}_n \cap M))$  to  $\mathfrak{R}_R(H(G, \mathcal{I}_n))$  and  $r_{M,R}$  maps  $\mathfrak{R}_R(H(G, \mathcal{I}_n))$  to  $\mathfrak{R}_R(H(M, \mathcal{I}_n \cap M))$ .
- (2) The reduced parabolic functor  $i_{M,R}^{\text{red}}$  maps  $\bar{H}_R^{\text{red}}(M, \mathcal{I}_n \cap M)$  to  $\bar{H}_R^{\text{red}}(G, \mathcal{I}_n)$  and  $r_{M,R}^{\text{red}}$  maps  $\bar{H}_R^{\text{red}}(G, \mathcal{I}_n)$  to  $\bar{H}_R^{\text{red}}(M, \mathcal{I}_n \cap M)$ .
- (3) In particular,  $\bar{A}_R^{\text{red}}$  maps  $\bar{H}_R^{\text{red}}(G, \mathcal{I}_n)$  to itself.

*Proof.* Claim (3) follows from (2) since  $\bar{A}_R^{\text{red}}$  is the linear combination of composition of maps of the form  $i_{M,R}^{\text{red}} \circ r_{M,R}^{\text{red}}$ . Claim (2) follows from (1) by the adjunction properties (1.1). It remains to justify Claim (1). Let  $x_0 \in \mathcal{B}(\mathbb{G}, F)$  be such that  $\mathcal{I} = G_{x_0,0+}$ . We have  $\mathcal{I}_n = G_{x_0,(n-\epsilon)+}$  for  $\epsilon > 0$  infinitesimally small. But then Claim (1) follows from Theorem 5.7, since for every  $x \in \mathcal{B}(\mathbb{G}, F)$ , there exists  $g \in G$  such that  $\mathcal{I}_n \subseteq {}^g G_{x,(n-\epsilon)+}$ . (In other words, up to associates,  $\mathcal{I}_n$  is the smallest open compact subgroup of depth  $(n - \epsilon)^+$ .)  $\square$

Now we prove the elliptic trace Paley-Wiener theorem.

**Theorem 5.9.** *Assume that  $\sharp W_G$  is invertible in  $R$ . Let  $n$  be a positive integer. Then the trace map*

$$\text{Tr}_R : \bar{H}_R(G, \mathcal{I}_n) \longrightarrow A_R(\mathfrak{R}(H_R(G, \mathcal{I}_n)))_{\text{good}}^*$$

*is surjective.*

*In particular, the trace map  $\text{Tr}_R : \bar{H}_R \rightarrow A_R(\mathfrak{R}(G))_{\text{good}}^*$  is surjective.*

*Is it true that  $A_R(\mathfrak{R}(G))_{\text{good}}^*$  in the statement is  $\mathfrak{R}_R(G)_{\text{good}}^* |_{A_R(\mathfrak{R}(G))}$ ? If so, then we may modify it accordingly*

*Proof.* Using the action of the unramified central characters of  $G$ , we may reduce to the case where  $G$  is semisimple. (A similar reduction in the case of affine Hecke algebras is explicitly discussed in [6, Section 8.1]). In this case, there are only finitely many  $\nu \in \mathfrak{N}$  with  $M_\nu = G$  and  $G(\nu) \neq \emptyset$ . By [13, Proof of Theorem 20],  $\bar{H}_R(G, \mathcal{I}_n; \nu)$  is finite dimensional for each  $\nu$ . Hence  $\bar{H}_R(G, \mathcal{I}_n)^{\text{rig}}$  is finite dimensional.

Let  $f \in A_R(\mathfrak{R}(H_R(G, \mathcal{I}_n)))_{\text{good}}^*$ . Suppose that  $f$  is not contained in the image of  $\text{Tr}_R$ . Then there exists

$$\pi \in A_R(\mathfrak{R}(H_R(G, \mathcal{I}_n))) \subset \mathfrak{R}(H_R(G, \mathcal{I}_n))$$

such that  $\text{Tr}_R(\bar{H}_R(G, \mathcal{I}_n)^{\text{rig,red}}, \pi) = 0$  and  $f(\pi) \neq 0$ .

Let  $f' \in \bar{H}_R^{\text{red}}(G, \mathcal{I}_n)$ . Then

$$\text{Tr}_R(f', \pi) = \text{Tr}_R(f', \frac{1}{a} A_R(\pi)) = \frac{1}{a} \text{Tr}_R(f', A_R(\pi)) = \frac{1}{a} \text{Tr}_R(\bar{A}_R^{\text{red}}(f'), \pi).$$

By §5.5 and the Newton decomposition on  $\bar{H}_R^{\text{red}}(G, \mathcal{I}_n)$  (see Theorem 5.6), we have

$$\bar{A}_R^{\text{red}}(f') \in \bar{H}_R^{\text{red}}(G, \mathcal{I}_n) \cap \bar{H}_R^{\text{rig,red}} = \bar{H}_R(G, \mathcal{I}_n)^{\text{rig,red}}.$$

Thus by the assumption,  $\text{Tr}_R(f', \pi) = 0$  for any  $f' \in \bar{H}_R(G, \mathcal{I}_n)$ . Therefore  $\pi = 0 \in \mathfrak{R}(H_R(G, \mathcal{I}_n))$  and  $f(\pi) = 0$ . This is a contradiction.

The “in particular” part follows from the fact that  $H_R = \varinjlim_{\mathcal{I}_n} H_R(G, \mathcal{I}_n)$ .  $\square$

5.6. Now we explain how to deduce Theorem 5.1 from Theorem 5.9. The argument is the same as in [1]. We sketch it here for the convenience of the reader.

We assume that the trace Paley-Wiener theorem holds for all proper Levi subgroups. Let  $f \in R(G)^*_{\text{good}}$ . Then  $f|_{A_R(\mathfrak{R}_R(G))} \in A_R(\mathfrak{R}_R(G))^*_{\text{good}}$ . By Theorem 5.9, there exists  $h \in \bar{H}_R$  such that  $f - \text{Tr}_R(h)$  vanishes on  $A_R(\mathfrak{R}_R(G))$ . Therefore  $A_R^*(f - \text{Tr}_R(h)) = 0$  as a linear form on  $\mathfrak{R}_R(G)$ . We have  $A_R^* \text{Tr}_R(h) = \text{Tr}_R(\bar{A}_R^{\text{red}}(h))$  as a linear form on  $\mathfrak{R}_R(G)$ . Hence

$$A_R^*(f) = af - \sum_{M \text{ proper}} c_M r_{M,R}^* i_{M,R}^*(f) \in \text{Tr}_R(\bar{H}_R).$$

By [1, Proposition 3.2],  $i_{M,R}^*(f)$  is a good form on  $\mathfrak{R}_R(M)$ . By the inductive hypothesis on  $M$ , we have

$$r_{M,R}^* i_{M,R}^*(f) \in r_{M,R}^* \mathfrak{R}_R(M)^*_{\text{good}} = r_{M,R}^* \text{Tr}_R(\bar{H}_R(M)) \subset \text{Tr}_R(\bar{H}_R).$$

Therefore  $af \in \text{Tr}_R(\bar{H}_R)$  and  $f \in \text{Tr}_R(\bar{H}_R)$ .

5.7. We can now prove the rigid trace Paley-Wiener theorem. Define the rigid quotient of the Grothendieck group of representations

$$\mathfrak{R}_R(G)_{\text{rig}} = \mathfrak{R}_R(G) / \mathfrak{R}_R(G)_{\text{diff}},$$

where  $\mathfrak{R}_R(G)_{\text{diff}} \subset \mathfrak{R}_R(G)$  is spanned by  $i_M(\sigma) - i_M(\sigma \otimes \psi)$ . Here  $M$  ranges over the set of standard Levi subgroups,  $\sigma \in \mathfrak{R}_R(M)$  and  $\psi$  is an unramified character of  $M$  over  $R$  which is trivial on  $Z(G)^0$ , the identity component of the center of  $G$ .

This definition is motivated by [6], where the analogous notion for affine Hecke algebras was studied. For affine Hecke algebras, it is proved in [6] that for generic parameters, the trace map gives a perfect pairing between the rigid cocenter of the affine Hecke algebras and the rigid quotient of Grothendieck group of representations. The advantage of considering the rigid cocenter and the rigid quotient instead of the whole cocenter and the Grothendieck group is that for affine Hecke algebras with semisimple root data, both the rigid cocenter and the rigid quotient are finite dimensional and the dimension can be computed explicitly. This allows us to have a good understanding of the relation between the cocenter and representations.

We expect a similar phenomenon for the Hecke algebra and representations of  $p$ -adic groups. In the rest of this section, we establish the trace Paley-Wiener for the rigid cocenter.

Let  $(\mathfrak{R}_R(G))_{\text{rig}}^*$  be the  $R$ -linear functions on  $\mathfrak{R}_R(G)_{\text{rig}}$  and  $(\mathfrak{R}_R(G))_{\text{rig,good}}^* = (\mathfrak{R}_R(G))_{\text{rig}}^* \cap (\mathfrak{R}_R(G))_{\text{good}}^*$ .

**Proposition 5.10.** *The trace map  $\text{Tr}_R : \bar{H}_R^{\text{rig,red}} \rightarrow (\mathfrak{R}_R(G))_{\text{rig,good}}^*$  is surjective.*

*Proof.* Note first that  $\mathfrak{R}_R(G)_{\text{diff}}$  vanished on  $\bar{H}_R^{\text{rig}}$ . Hence the trace map induces a map  $\bar{H}_R^{\text{rig}} \rightarrow (\mathfrak{R}_R(G))_{\text{rig,good}}^*$ .

Now let  $f \in (\mathfrak{R}_R(G))_{\text{rig,good}}^*$ . By Theorem 5.1,  $f = \text{Tr}_R(h)$  for some  $h \in \bar{H}_R^{\text{red}}$ . We write  $h$  as  $h = \sum_{v \in V_+} h_v$ , where  $h_v \in \bar{H}(v)^{\text{red}}$ .

By [14, Theorem 4.1 and Theorem 6.1], for any  $v \in V_+$ ,  $\bar{i}_v : \bar{H}_R(M_v; v)^{\text{red}} \rightarrow \bar{H}_R(G; v)^{\text{red}}$  is bijective. Thus  $h_v = \bar{i}_v(h'_v)$  for some  $h'_v \in \bar{H}_R(M_v; v)^{\text{red}}$ . If  $h_v \neq 0$ , then  $h'_v \neq 0$  as an element in  $\bar{H}_R(M_v)^{\text{red}}$ . Thus by the definition of reduced

cocenter, there exists  $\sigma \in \mathfrak{R}_R(M_v)$  such that  $\mathrm{Tr}_R(h'_v, \sigma) \neq 0$ . By Theorem 2.6 and the Mackey-type formula (Proposition 4.1), we have that

$$\mathrm{Tr}_R(h_v, i_{M_v}(\sigma \circ \chi)) = \mathrm{Tr}_R(\bar{r}_{M_v} \bar{i}_v(h'_v), \sigma \circ \chi)$$

is a regular function on  $\chi$ , with leading term  $\mathrm{Tr}_R(h'_v, \sigma) \langle \chi, v \rangle$ .

By the definition of  $\mathfrak{R}(G)_{\mathrm{diff}}$ , we have that  $h_v = 0$  for any  $v \in V_+$  such that  $M_v \neq G$ . Therefore  $h \in \bar{H}_R^{\mathrm{rig}, \mathrm{red}}$ .  $\square$

## 6. APPLICATION: THE ABSTRACT SELBERG PRINCIPLE

### 6.1. Define

$$\aleph_c = \{\nu = (\tau, v) \in \aleph; \tau \text{ is of finite order in } \Omega, M_v = G\}.$$

Let  $G^c \subset G$  be the subset of compact elements. Then we have that

$$G^c = \sqcup_{\nu \in \aleph_c} G(\nu).$$

Let  $H^c = \oplus_{\nu \in \aleph_c} H(\nu)$  be the subset of  $H$  consisting of functions supported in  $G^c$  and  $\bar{H}^c = \oplus_{\nu \in \aleph_c} \bar{H}(\nu)$  be the image of  $H^c$  in  $\bar{H}$ . Note that by definition, the rigid cocenter  $\bar{H}^{\mathrm{rig}}$  consists of elements in the cocenter represented by functions with support in the subset of compact-modulo-center elements of  $G$ . Thus  $\bar{H}^c \subseteq \bar{H}^{\mathrm{rig}}$  and the equality holds if  $G$  is semisimple.

Let  $\mathfrak{R}_R(G)$  be the  $R$ -vector space with a basis given by the isomorphism classes of indecomposable finitely generated projective  $G$ -modules over  $R$ . Let

$$\mathrm{Rk}_R : \mathfrak{R}_R(G) \longrightarrow \bar{H}_R$$

be the rank map, which sends a finitely generated projective module to the image in  $\bar{H}$  of the trace of the idempotent of  $M_n(H)$  defining it as a quotient of  $H^n$ . The main result of this section is the following abstract Selberg principle.

**Theorem 6.1.** *The image of the rank map  $\mathrm{Rk}_R$  is contained in  $\bar{H}_R^c$ .*

*Proof.* We argue by induction on the semisimple rank of  $G$ .

If the semisimple rank of  $G$  is 0, then  $G$  is compact modulo center. The statement is easy to prove in this case.

Now we assume that the semisimple rank of  $G$  is positive and that the statement holds for all the proper Levi subgroups of  $G$  (which have smaller semisimple rank). Let  $\Pi \in \mathfrak{R}_R(G)$ . By Frobenius reciprocity, we have that  $r_{M,R}(\Pi) \in \mathfrak{R}_R(M)$  for all Levi subgroup  $M$ . By the compatibility of restriction with rank map, we have  $\mathrm{Rk}_M r_{M,R}(\Pi) = \bar{r}_{M,R} \mathrm{Rk}_G(\Pi) \in \bar{H}_R(M)$ . By the inductive hypothesis,  $\bar{r}_{M,R} \mathrm{Rk}_G(\Pi) \in \bar{H}_R(M)^c$  for all proper Levi  $M$ .

We write  $\mathrm{Rk}_G(\Pi)$  as  $\mathrm{Rk}_G(\Pi) = \sum_{\nu \in G} f_\nu$ , where  $f_\nu$  is the corresponding Newton component of  $\mathrm{Rk}_G(\Pi)$ . Then by the Newton decomposition on  $\bar{H}_R(M)$  and Proposition 3.2, if  $\nu \notin \aleph_c$ , then  $\bar{r}_{M,R} f_\nu = 0$  for all proper Levi  $M$ . Now by Theorem 4.3, we have  $f_\nu = 0$  for  $\nu \notin \aleph_c$ . Therefore  $\mathrm{Rk}_G(\Pi) \in \bar{H}_R^c$ .  $\square$

**6.2.** We make some comments about the abstract Selberg principle. The classical statement [2] of the abstract Selberg principle is as follows.

Let  $f \in \bar{H}_{\mathbb{C}}$  be the image of the rank map of a finitely generated projective representation of  $G$  over  $\mathbb{C}$ . Then the orbital integral of  $f$ , relative to a non-compact element of  $G$  (in case  $\mathrm{char}(F) = 0$ ) and a non-compact semisimple element of  $G$  (in case  $\mathrm{char}(F) > 0$ ), vanishes.

A subtle issue here is that if  $\text{char}(F) > 0$ , then it is not known in general whether the orbital integral of an unipotent element converges, although most cases are settled by McNinch in [18].

The statement we have in this paper is somehow different. The statement says that the element  $f \in \bar{H}_R$  (for arbitrary algebraically closed field  $R$  of characteristic not equal to  $p$ ) can be represented by a function on  $G$  supported in compact elements. In characteristic 0, or more generally, if the characteristic  $l$  of  $R$  is not in a certain finite set of primes  $P$  [27, Théorème C.1], it is known that the following two conditions are equivalent:

- (1) An element  $f$  of the cocenter is represented by a function supported in compact elements;
- (2) The orbital integral of any regular semisimple non-compact element on  $f$  vanishes.

It is obvious that (1) implies (2). The equivalence of (1) and (2) follows from the geometric density theorem, which is known under the assumption that  $l \notin P$  [27, Théorème C.2]. We remark that the exact set  $P$  is not known in general, and that it is not sufficient to assume that  $l$  is banal in the sense of [24, II.3.9]. This is related to the question of characterization of the primes that divide the Assem number of orbital integrals, see [27, section B].

#### APPENDIX A. A DIFFERENT PROOF IN CHARACTERISTIC ZERO

In the appendix, we give a different proof that the elliptic reduced cocenter is contained in the rigid reduced cocenter, under the assumption that  $F$  is of characteristic 0. The proof is based on Clozel's integration formula [7, Proposition 1].

A.1. We recall several foundational results regarding characters of admissible representations motivated by [7, Proposition 1]. Let  $G_{\text{rs}}$  denote the set of regular semisimple elements of  $G$ . A (weak)  $R$ -analogue of the classical result of Harish-Chandra, see [27, E.3.4.4], says that for each admissible  $G$ -representation  $(\pi, V)$ , there exists a locally constant function

$$\text{Tr}_V : G_{\text{rs}} \longrightarrow R,$$

the character of  $(\pi, V)$ , such that, for every  $f \in \bar{H}$ ,

$$\text{Tr}_R(f, V) = \int_{G_{\text{rs}}} \text{Tr}_V(g) f(g) dg.$$

For every semisimple element  $s \in G$ , recall the parabolic (Deligne's parabolic)  $P_s = M_s N_s$  contracted by  $s$ , see [10, 5]. Let  $G_c$  be the "compact part" of  $G$  as defined in [7, §1], i.e. the set of semisimple elements  $s$  such that  $P_s = G$ . Then  $G_c$  is the set of semisimple elements in  $G^{\text{rig}}$ . We first have the generalization of Casselman's formula [5], see [24, II.3.7] and [19, Theorem 7.4]:

**Theorem A.1.** *For every admissible  $G$ -representation  $(\pi, V)$  and  $s \in G_{\text{rs}}$ ,*

$$\text{Tr}_V(s) = \text{Tr}_{r_{M_s}(\pi)}(s).$$

A.2. Let  $T \subset G$  be a Cartan subgroup and let  $W(G, T)$  is the Weyl group of  $G$  with respect to  $T$ . For every  $t \in T \cap G_{\text{rs}}$  and  $f \in \bar{H}$ , let

$$(A.1) \quad \mathcal{O}_t(f) = \int_{G/T} f(xtx^{-1})d(xT)$$

denote the orbital integral. Suppose  $T \subset M$ . Then for  $t \in T \cap G_{\text{rs}}$ , we have the known descent formula for orbital integrals:

$$(A.2) \quad \mathcal{O}_t(f) = \mathcal{O}_t^M(\bar{r}_M(f)),$$

where  $\mathcal{O}_t^M$  is the orbital integral taken in  $M$ .

We will use a variant of Weyl's integration formula as in [7, page 241]. Since  $T \cap G_{\text{rs}}$  is totally disconnected with a free action of  $W(G, T)$ , one may find an open and closed subset  $T_{\text{rs}}^0$  of  $T \cap G_{\text{rs}}$  such that  $T \cap G_{\text{rs}} = \bigsqcup_{w \in W(G, T)} wT_{\text{rs}}^0$ . Then the morphism

$$G/T \times T_{\text{rs}}^0 \longrightarrow G_{\text{rs}}, \quad (gT, t) \longmapsto \text{Ad}(g)t$$

is one-to-one. This is because  $\text{Ad}(g)t = t'$  implies  $\text{Ad}(g)T = T$  hence there exists  $w \in W(G, T)$  such that  $\text{Ad}(g) = w$  as homomorphisms of  $T$ . But since  $T_{\text{rs}}^0$  is a fundamental domain for the action of  $W(G, T)$ , it follows that  $w = 1$ . Then the form of Weyl's integration formula for a locally constant, compactly supported function  $F$  takes the form:

$$\int_G F(g)dg = \sum_{T \subset G/\sim} \int_{T_{\text{rs}}^0} |D_G(t)| \int_{G/T} F(xtx^{-1})d(xT)dt.$$

Here  $T$  runs over a set of representatives of Cartan subgroups of  $G$  (modulo  $G$ -conjugation) and  $D_G(t)$  is the coefficient of  $x^{\text{rank}(G)}$  in  $\det(x + 1 - \text{Ad}(t))$ . If  $f \in \bar{H}_R$ , specialize  $F = f \cdot \text{Tr}_V$ , which is compactly-supported locally constant (since  $\text{Tr}_V$  is locally constant on  $G_{\text{rs}}$ ), and arrive at the formula [7, page 241]:

$$(A.3) \quad \text{Tr}_R(f, V) = \sum_{T \subset G/\sim} \int_{T_{\text{rs}}^0} |D_G(t)| \text{Tr}_V(t) \mathcal{O}_t(f) dt.$$

Following [7, page 240], denote

$$\text{Tr}_{R,c}(f, V) = \int_{G_c \cap G_{\text{rs}}} f(g) \text{Tr}_V(g) dg.$$

Now we can show that the elements of  $\bar{A}_R^{\text{cat}}(\bar{H})$  are essentially supported on  $G^{\text{rig}}$ .

**Proposition A.2.** *For every  $f \in \bar{A}_R^{\text{cat}}(\bar{H})$  and  $(\pi, V)$  a smooth admissible  $G$ -representation,*

$$\text{Tr}_R(f, V) = \text{Tr}_{R,c}(f, V).$$

*Proof.* Following the proof of [7, Proposition 1], rewrite (A.3) according to the Deligne stratification of Cartan subgroups  $T = \cup S$ , where  $t, t' \in T$  are in the same stratum  $S$  if  $P_t = P_{t'}$ . In this case, write  $P_S = P_t$  for  $t \in S$ :

$$\text{Tr}_R(f, V) = \sum_{S \subset G/\sim} I(S), \quad I(S) = \int_{S \cap G_{\text{rs}}/\sim} |D_G(t)| \text{Tr}_V(t) \mathcal{O}_t(f) dt.$$

Since  $f \in \bar{A}_R^{\text{cat}}(\bar{H})$ , we have  $\bar{r}_M(f) = 0$  for all proper Levi subgroups  $M$  of  $G$ . The descent formula for orbital integrals implies therefore that if  $T \subset M$ , then  $\mathcal{O}_t(f) = 0$  for all  $t \in T \cap G_{\text{rs}}$ . This means that

$$\text{Tr}_R(f, V) = \sum_{S \subset G_c / \sim} I(S),$$

and for each such  $S$ ,  $P_S = G$ . The claim follows.  $\square$

A.3. Now we give an alternative proof that  $\bar{H}_R^{\text{ell}, \text{red}} \subset \bar{H}_R^{\text{rig}, \text{red}}$  under the assumption that  $F$  is of characteristic 0.

Let  $f \in \bar{H}^{\text{ell}, \text{red}}$ . Recall that  $\bar{H}_R^{\text{rig}, \text{red}} = \bigoplus_{v \in V_+; M_v = G} \bar{H}_R^{\text{red}}(v)$  and set  $\bar{H}_R^{\text{nrig}, \text{red}} = \bigoplus_{v \in V_+; M_v \neq G} \bar{H}_R^{\text{red}}(v)$ . Using Theorem 1.2, we write  $f$  as  $f = f_{\text{rig}} + f_{\text{nrig}}$ , where  $f_{\text{rig}} \in \bar{H}_R^{\text{rig}, \text{red}}$  and  $f_{\text{nrig}} \in \bar{H}_R^{\text{nrig}, \text{red}}$ . Since  $G_c \subset G^{\text{rig}}$ , we have

$$\begin{aligned} \text{Tr}_R(f, V) &= \text{Tr}_{R,c}(f, V) = \text{Tr}_{R,c}(f_{\text{rig}}, V) + \text{Tr}_{R,c}(f_{\text{nrig}}, V) \\ &= \text{Tr}_{R,c}(f_{\text{rig}}, V) = \text{Tr}_R(f_{\text{rig}}, V). \end{aligned}$$

Here the first equality follows from Proposition A.2, the second equality follows from the fact that the support of  $f_{\text{nrig}}$  does not intersect  $G_c$  and the third equality follows from the fact that the support of  $f_{\text{rig}}$  is contained in  $G^{\text{rig}}$ .

Therefore  $f - f_{\text{rig}} \in \ker \text{Tr}_R$ .

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(D. Ciubotaru) MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD, OX2 6GG, UNITED KINGDOM

*E-mail address:* dan.ciubotaru@maths.ox.ac.uk

(X. He) THE INSTITUTE OF MATHEMATICAL SCIENCES AND DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, N.T., HONG KONG.

*E-mail address:* xuhuahe@math.cuhk.edu.hk