USING BLOCK DESIGNS IN CROSSING NUMBER BOUNDS

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Abstract. The *crossing number* $\text{CR}(G)$ of a graph $G = (V, E)$ is the smallest number of edge crossings over all drawings of G in the plane. For any $k \geq 1$, the *k-planar crossing number* of G, $\text{CR}_k(G)$, is defined as the minimum of $\text{CR}(G_1) + \text{CR}(G_2) + \ldots + \text{CR}(G_k)$ over all graphs $G_1, G_2, \ldots, G_k$ with $\cup_{i=1}^{k} G_i = G$. Pach et al. [*Computational Geometry: Theory and Applications* **68** 2–6, (2018)] showed that for every $k \geq 1$, we have $\text{CR}_k(G) \leq \left(\frac{2}{k^2} - \frac{1}{k^3}\right) \text{CR}(G)$ and that this bound does not remain true if we replace the constant $\frac{2}{k^2} - \frac{1}{k^3}$ by any number smaller than $\frac{1}{k^2}$. We improve the upper bound to $\frac{1}{k^2}(1 + o(1))$ as $k \to \infty$. For the class of bipartite graphs, we show that the best constant is exactly $\frac{1}{k^2}$ for every k. The results extend to the rectilinear variant of the k-planar crossing number.

1. Introduction

This note improves on results of Pach, Székely, Tóth, and Tóth [21]. We follow the introduction of that paper.

A *drawing* of a graph $G = (V, E)$ is a planar representation of G such that every vertex $v \in V$ corresponds to a point of the plane and every edge $uv \in E$ is represented by a simple continuous curve between the points corresponding to u and v, which does not pass through any point representing a vertex of G. We assume for simplicity that no two curves share infinitely many points, no two curves are tangent to each other, and no three curves pass through the same point. The *crossing number* $\text{CR}(G)$ of G is defined as the minimum number of edge crossings in a drawing of G. For surveys, see [24, 28], and the recent monograph [25]. Clearly, G is planar if and only if $\text{CR}(G) = 0$.

Selfridge (see [16]) noticed that by Euler's polyhedral formula, K_{11}, the complete graph on 11 vertices, cannot be written as the union of two planar graphs. Battle, Harary, and Kodama [5] and independently Tutte [32] proved that the same is true for K_9 and Beineke presented a self-complementary planar drawing which implies the biplanarity of K_8. This

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led Tutte [33] to introduce the *thickness* of a graph G, which is the minimum number of planar graphs that G can be decomposed into. The notion is relevant for VLSI chip design, where it corresponds to the number of layers required for realizing a network so that there is no crossing within a layer (see Mutzel, Odenthal, and Scharbrodt [18] for a survey). If the thickness of G is at most 2, G is called *biplanar*. Mansfield proved that it is an NP-complete problem to decide whether a graph is biplanar; see [6, 17].

Owens [19] defined the *biplanar crossing number* $\text{CR}_2(G)$ of G as the minimum sum of the crossing numbers of two graphs, G_0 and G_1, whose union is G. G is biplanar precisely when its biplanar crossing number is 0. Shahrokhi et al. [26] extended this notion as follows. For any positive integer $k \geq 1$, define the *k-planar crossing number* $\text{CR}_k(G)$ of G as the minimum of $\text{CR}(G_1) + \text{CR}(G_2) + \ldots + \text{CR}(G_k)$, where the minimum is taken over all graphs $G_1, G_2, \ldots, G_k$ whose union is G, that is, $\bigcup_{i=1}^{k} E(G_i) = E(G)$ and $V(G_i) = V(G)$ for all $i \in [k]$.

Spencer [27] showed that for sufficiently large c for all $p > c/n$ with high probability the biplanar crossing number of Erdős-Rényi random graphs is $\Theta(n^4p^2)$, and claimed a similar result for k-planar crossing numbers without proof. Asplund et al. [3] gave a proof and extended this result for random d-regular graphs, where d exceeds a certain threshold.

Czabarka, Sýkora, Székely, and Vrťo [12] proved that for every graph G, we have

$$\text{CR}_2(G) \leq \frac{3}{8}\text{CR}(G). \tag{1}$$

They also showed [11] that this inequality does not remain true if the constant $\frac{3}{8} = 0.375$ is replaced by anything less than $\frac{8}{119} \approx 0.067$.

Pach et al. [21] extended this investigation to the relationship between the k-planar crossing number and the (ordinary) crossing number of a graph. For every integer $k \geq 1$, they defined

$$\alpha_k = \sup \frac{\text{CR}_k(G)}{\text{CR}(G)},$$

where the supremum is taken over all *nonplanar* graphs G. The results mentioned from [12] yield $0.067 < \alpha_2 \leq \frac{3}{8} = 0.375$. Pach et al. [21] proved that for every positive integer k,

$$\frac{1}{k^2} \leq \alpha_k \leq \frac{2}{k^2} - \frac{1}{k^3}. \tag{2}$$

Note that for $k = 2$, (2) returns the value 3/8 given in (1), and the present paper does not improve this upper bound on α_2 either. In this paper, we show that the lower bound in (2) is asymptotically correct as $k \to \infty$.

Theorem 1. *$\alpha_k = \frac{1}{k^2}(1 + o(1))$ as $k \to \infty$.*

As Theorem 1 and its proof surrender control over the $o(1)$ term, it is of interest to determine the values of α_k for small k. To this end, we improve the upper bound $\frac{2}{k^2} - \frac{1}{k^3}$ for $3 \leq k \leq 10$, see Table 1, using the following theorem.

Theorem 2. *We have*

(i) $\alpha_4 \leq \frac{235}{2401}$;

k	α_k bound from (2)	α_k bound improved	lower bound
3	$\frac{5}{27} \lessapprox 0.1852$	$\frac{1}{6} \lessapprox 0.1667$ (v)	$\frac{1}{9} \gtrapprox 0.1111$
4	$\frac{7}{64} \lessapprox 0.1094$	$\frac{235}{2401} \lessapprox 0.0979$ (i)	$\frac{1}{16} = 0.0625$
5	$\frac{9}{125} = 0.072$	$\frac{1}{15} \lessapprox 0.0667$ (v)	$\frac{1}{25} = 0.04$
6	$\frac{11}{216} \lessapprox 0.0510$	$\frac{17}{432} \lessapprox 0.0394$ (iv)	$\frac{1}{36} \gtrapprox 0.0277$
7	$\frac{13}{343} \lessapprox 0.0380$	$\frac{1}{28} \lessapprox 0.0358$ (v)	$\frac{1}{49} \gtrapprox 0.2040$
8	$\frac{15}{512} \lessapprox 0.0293$	$\frac{85}{3375} \lessapprox 0.0252$ (ii)	$\frac{1}{64} = 0.015625$
9	$\frac{17}{729} \lessapprox 0.0234$	$\frac{13}{729} \lessapprox 0.0179$ (iv)	$\frac{1}{81} \gtrapprox 0.0123$
10	$\frac{19}{1000} = 0.019$	$\frac{325}{21952} \lessapprox 0.0149$ (iii)	$\frac{1}{100} = 0.01$

TABLE 1. Comparison of the best upper bounds for α_k from (2), due to Pach et al. [21], our upper bounds, and the lower bound $\frac{1}{k^2}$ for $3 \leq k \leq 10$. Roman numerals in the second column refer to cases of Theorem 2.

(ii) $\alpha_k \leq \frac{12k-11}{(2k-1)^3}$ *for* $k \equiv 2 \pmod 3$*;*
(iii) $\alpha_k \leq \frac{36k-35}{(3k-2)^3}$ *for* $k \equiv 2 \pmod 4$*;*
(iv) $\alpha_k \leq \frac{3k-1}{2k^3}$ *for* $k \equiv 0 \pmod 3$*;*
(v) $\alpha_k \leq \frac{2}{k(k+1)}$ *for odd* k*.*

Note that while for odd k the expression in (v) offers an improvement over (2) that is in diminishing proportion as $k \to \infty$, the gain is still meaningful for small values of k. In contrast, (ii), (iii), and (iv) also offer an asymptotic improvement over (2).

We also consider the restriction of the problem to bipartite graphs. To this end, define

$$\beta_k = \sup \frac{\text{CR}_k(G)}{\text{CR}(G)},$$

where the supremum is taken over all *nonplanar bipartite* graphs G. When restricted to bipartite graphs, we can show that the lower bound in (2) is exact.

Theorem 3. *For all* k*,* $\beta_k = \frac{1}{k^2}$*.*

The *rectilinear crossing number*, $\text{RCR}(G)$, of a graph G is the minimum number of crossings over all *straight-line* drawings of G, that is, where edges are represented by line segments. Obviously, we have $\text{CR}(G) \leq \text{RCR}(G)$ for every graph G. For every $t \geq 4$, Bienstock and Dean [7] constructed families of graphs whose crossing numbers are at most t and whose rectilinear crossing numbers are unbounded.

Similar to $\text{CR}_k(G)$, we define the *rectilinear k-planar crossing number* of a graph G, denoted $\text{RCR}_k(G)$, as the minimum of $\text{RCR}(G_1) + \text{RCR}(G_2) + \ldots + \text{RCR}(G_k)$, where the minimum is taken over all graphs $G_1, G_2, \ldots, G_k$ whose union is G. It is likewise clear that $\text{CR}_k(G) \leq \text{RCR}_k(G)$ for every positive integer k. The analogue of α_k is

$$\overline{\alpha}_k = \sup \frac{\text{RCR}_k(G)}{\text{RCR}(G)},$$

where the supremum is taken over all *nonplanar* graphs G, and the analogue of β_k is

$$\overline{\beta}_k = \sup \frac{\text{RCR}_k(G)}{\text{RCR}(G)},$$

where the supremum is taken over all *bipartite nonplanar* graphs G (as planar is the same as rectilinear planar by [14]). We have

Theorem 4. *Theorems 1 and 2 remain true if we replace α_k with $\overline{\alpha}_k$ and β_k with $\overline{\beta}_k$, consequently the bounds for α_k in Table 1 apply for $\overline{\alpha}_k$ as well.*

2. Methodology

We generalize the procedure that was defined for two planes in [12] and extended to k planes in [21]. Given an integer $k > 1$, we create a k-planar drawing of G in the following way. We number the k planes with $1, 2, \ldots, k$, and describe a probabilistic procedure that assigns a plane to each edge, resulting in a graph G_i on the i-th plane. Let K_s^o denote the complete graph on the vertex set $\{1, 2, ..., s\}$, with a loop edge added at every vertex and let the graphs $H_1, H_2, ..., H_k$ partition the edge set of K_s^o. (In [12], where $k = 2$, the choice of s was 2, with $H_1 =$ a single edge and $H_2 =$ two loops; in [21] the choice was $s = k$, for odd k every H_i consisted of a single loop and a perfect matching on the remaining $k-1$ vertices, while for even k every H_i was either a perfect matching, or two loops and a perfect matching on the remaining $k-2$ vertices.) We call the connected components of $H_1, H_2, ..., H_k$ *types.* Note that two distinct components (no matter whether they are in the same H_i or not) are different types even if they are isomorphic. For an example, see Figure 1. We define the *type* of an edge of K_s^o (either loop or not) as the unique component of the subgraph H_i containing it.

Given a graph G, we distribute the edges of G into the k planes as follows. To each vertex v of G assign a value $\xi(v)$ randomly and uniformly chosen from $\{1, 2, ..., s\}$ where s is chosen carefully depending on values of k. If uv is an edge of G, assign the edge uv to the j-th plane if $\{\xi(u), \xi(v)\} \in E(H_j)$ (where $\{i\} = \{i, i\}$ is the loop on vertex i of K_s^o). As the edge sets of the graphs H_i partition $E(K_s^o)$, there is exactly one j assigned to each edge.

We will use an optimal drawing $\mathcal{D}$ of G realizing $\text{CR}(G)$ to create a k-planar drawing. It is well-known that in $\mathcal{D}$ every pair e, f of crossing edges has four distinct endvertices and the edges e, f have exactly one point in common [24, 25, 28]. Denote by G_j the subgraph of G containing the edges assigned to the j-th plane. Draw G_j in the j-th plane following the drawing $\mathcal{D}$, i.e. the drawing of each edge uv in G_j follows the curve representing the uv edge in $\mathcal{D}$.

Let $C_1, C_2, \ldots, C_m$ be the components of H_i. Then for $j = 1, 2, \ldots, m$, the subgraphs spanned by the vertices $\{v \in V(G_i) : \xi(v) \in V(C_j)\}$ are in different components of G_i. We modify our k-planar drawing to further reduce the number of crossing edge pairs by translating the drawings of subgraphs of G_i on the vertex sets $\{v \in V(G) : \xi(v) \in V(C)\}$ for the components of H_i far enough from each other so that if $e_1, e_2 \in E(G_i)$ and vertices of e_1 and e_2 are mapped to different components of H_i by ξ, then curves corresponding to e_1 and e_2 do not cross in the drawing of G_i.

Assume that uv and wz are a pair of crossing edges in the optimal drawing $\mathcal{D}$ of G, and hence have 4 distinct endpoints. The probability that this edge pair is still crossing in the random k-planar drawing above, is exactly

$$q = \mathbb{P}[\text{type}(\xi(u), \xi(v)) = \text{type}(\xi(w), \xi(z))]. \tag{3}$$

The value of q does not depend on which crossing edge pair uv and wz was selected from $\mathcal{D}$, so the expected number of crossings in our random k-planar drawing is

$$q\,\text{CR}(G). \tag{4}$$

It follows that there exists a k-planar drawing of G which has at most $q\,\text{CR}(G)$ crossings. If this holds for a particular q for all graphs G, then we establish

$$\alpha_k \le q. \tag{5}$$

Note that this method can be further enhanced by replacing the base graph K_s^o by a graph that is missing some edges (but modifying ξ so no edge of G is matched to a missing edge) and allowing an edge of the base graph to appear in several of the H_i (and employing another probabilistic procedure to decide which plane we assign an edge uv to when $\xi(u)\xi(v)$ appears in several of the H_i). We will make use of these modifications in Sections 4.5 and 5, where we discuss them in more detail. For a warm-up, we start with $k = 4$.

3. Proof to Theorem 2(i): the case $k = 4$

Choose $s = 7$, and see Figure 1 for the partition of K_7^o into 12 types on 4 planes.

Take a crossing edge pair $\{ab, cd\}$. Given $V(G) = \{1, 2, ..., n\}$, without loss of generality we assume that $a < b, c < d$ and $a < c$. This determines the (a, b, c, d) quadruple uniquely for each crossing edge pair. We compute the probability that ab and cd still cross in the k-planar drawing we provided by counting the number of ways the edge pair can be labeled and remain crossing, and dividing it by 7^4, the total number of possible labelings.

(1) If $\{\xi(a), \xi(b)\} \cap \{\xi(c), \xi(d)\} = \emptyset$, then the edge pair does not remain crossing.
(2) If $\xi(a) = \xi(b) = \xi(c) = \xi(d)$, then the edge pair remains crossing, and 7 different labelings yield such a situation.

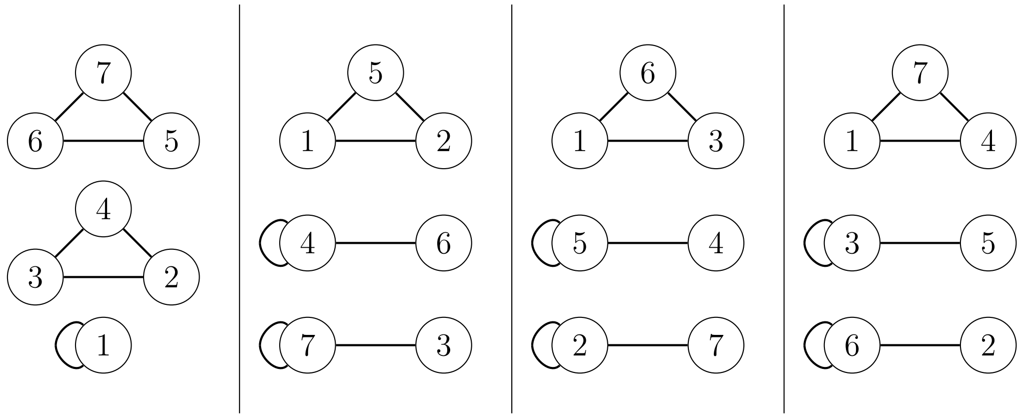

FIGURE 1. Partitioning K_7^o into 12 types in 4 planes.

(3) If for some $i \neq j$, $\{\{\xi(a), \xi(b)\}, \{\xi(c), \xi(d)\}\} = \{\{i\}, \{i, j\}\}$, then the edge pair remains crossing only if $i \neq 1$. There are $6 \cdot 4 = 24$ ways to label the vertices this way.
(4) If for some $i \neq j$, $\{\{\xi(a), \xi(b)\}, \{\xi(c), \xi(d)\}\} = \{\{i, j\}, \{i, j\}\}$, then the edge pair remains crossing, and there are $\binom{7}{2} \cdot 4 = 84$ ways to label the vertices this way.
(5) If $\{\{\xi(a), \xi(b)\}, \{\xi(c), \xi(d)\}\} = \{\{i, j\}, \{i, h\}\}$ for three different numbers i, j, h, then the edge pair remains crossing when i, j, h appear in some triangle in one of the planes in Figure 1, which can happen in $15 \cdot 8 = 120$ ways.

Summing over all possible outcomes yields the probability that a crossing edge pair remains crossing in this k-planar drawing is

$$\frac{120 + 84 + 24 + 7}{7^4} = \frac{235}{2401} \lessapprox 0.0979.$$

4. RESOLVABLE BIBDs AND PROOFS OF THEOREM 1 AND 2

A *resolvable* BIBD, denoted as RBIBD(s, ℓ, λ), is a collection $P_1, \ldots, P_m$ of partitions of an underlying s-element set into ℓ-element subsets such that every 2-element subset of the s-element set is contained by exactly λ of the ms/ℓ ℓ-element sets listed in the partitions. We restrict ourselves to $\lambda = 1$, that is, each 2-element subset of the s-element set is contained in precisely one of the ℓ-element sets listed in the partitions.

Note that the existence of such a design implies that $|P_i| = \frac{s}{\ell}$ and $m\frac{s}{\ell}\binom{\ell}{2} = \binom{s}{2}$, i.e. $m = \frac{s-1}{\ell-1}$, which gives the well known necessary condition that $s \equiv \ell \pmod{\ell(\ell-1)}$ for the existence of such a resolvable BIBD. For the $\ell = 2$ case, which is the factorization of complete graphs into matchings, this condition is also sufficient. For the $\ell = 3$ case (known as a Kirkman triple system) it is also a sufficient condition [29], and for $\ell = 4$ the corresponding $s \equiv 4 \pmod{12}$ it is also a sufficient condition [15]. For every ℓ, the congruence is also a sufficient condition for all $s > s_0(\ell)$ [30]. Further, for every even $\ell \geq 4$, the congruence implies existence for $s > \exp\{\exp\{\ell^{18\ell^2}\}\}$ [9].

Assuming that a RBIBD$(s,\ell,1)$ exists, let $k=m+1$, and for $i=1,2,...,m$, let H_i be a disjoint union of K_ℓ's, whose vertex sets are the ℓ-element sets in the partition classes of the partition P_i. For $i=m+1$, we put the s loops into H_{m+1}. Following the drawing argument in Section 2, we evaluate the value of q.

Consider a crossing edge pair $\{ab,cd\}$ in G as we did in Section 3. The following ξ-assignments will leave the edge pair crossing:

(1) $\xi(a)=\xi(b)=\xi(c)=\xi(d)$: s different labelings of these 4 vertices yield such a situation.
(2) For some $i\neq j$, $\{\{\xi(a),\xi(b)\},\{\xi(c),\xi(d)\}\}=\{\{i,j\},\{i,j\}\}$: there are $\frac{s-1}{\ell-1}\cdot\frac{s}{\ell}\cdot\binom{\ell}{2}\cdot 4$ ways to label the vertices this way.
(3) $\{\{\xi(a),\xi(b)\},\{\xi(c),\xi(d)\}\}=\{\{i,j\},\{i,h\}\}$ for three different numbers i,j,h that appear together in some ℓ-set of some partition: there are $\frac{s-1}{\ell-1}\cdot\frac{s}{\ell}\cdot\ell\binom{\ell-1}{2}\cdot 8$ ways to label the vertices this way.
(4) $\{\{\xi(a),\xi(b)\},\{\xi(c),\xi(d)\}\}=\{\{i,j\},\{h,g\}\}$ for four different numbers i,j,h,g that appear together in some ℓ-set of some partition: there are $\frac{s-1}{\ell-1}\cdot\frac{s}{\ell}\cdot 3\binom{\ell}{4}\cdot 8$ ways to label the vertices this way.

Summing over all possibilities and dividing by s^4, the total number of labelings of the four vertices, we obtain

$$q=\frac{1+(s-1)(\ell^2-\ell)}{s^3}. \tag{6}$$

Now we are ready to show the main theorems.

4.1. Proof of Theorem 1: $\alpha_k=\frac{1}{k^2}(1+o(1))$ as $k\to\infty$.

Proof. Note that $\frac{1}{k^2}\leq\alpha_k$ from (2), so we only have to provide an upper bound. We will show that for any $\ell\geq 2$ we have $\alpha_k\leq\frac{\ell+1}{\ell-1}\cdot\frac{1}{k^2}(1+o(1))$ as $k\to\infty$. Letting $\ell\to\infty$ proves the claim. To this end, fix an $\ell\geq 2$. For a given k, set $s=s_k=(k-1)(\ell-1)+1$ (so $k=\frac{s-1}{\ell-1}+1$). If an RBIBD$(s,\ell,1)$ exists, then (6) gives

$$\alpha_k\leq q<\frac{k}{(k-1)^3}\cdot\frac{\ell}{\ell-1}+\frac{1}{(k-1)^3(\ell-1)^3}<\frac{k}{(k-1)^3}\cdot\frac{\ell+1}{\ell-1}.$$

While this may not be true, we know that if s' is sufficiently large and $s'\equiv\ell \pmod{\ell(\ell-1)}$ then an RBIBD$(s',\ell,1)$ does exist. This means that for k sufficiently large, there exists an s' such that $s_k\geq s'>s_k-\ell(\ell-1)$ and an RBIBD$(s',\ell,1)$ exists. Set $k'=\frac{s'-1}{\ell-1}+1$, an integer (so $s'=s_{k'}$). It is easy to see that $k\geq k'>k-\ell$, and

$$\alpha_k\leq\alpha_{k'}\leq\frac{k'}{(k'-1)^3}\cdot\frac{\ell+1}{\ell-1}\leq\frac{k}{(k-\ell-1)^3}\cdot\frac{\ell+1}{\ell-1}=\frac{\ell+1}{\ell-1}\cdot\frac{1}{k^2}\cdot(1+o(1)),$$

verifying our claim. □

Now we turn to the proof of Theorem 2. Note that Theorem 2(i) is already shown in Section 3.

4.2. **Proof of Theorem 2(ii): $\alpha_k \leq \frac{12k-11}{(2k-1)^3}$ for $k \equiv 2 \pmod 3$.**

Proof. In Equation (6), choose $\ell = 3$, and assume that $k \equiv 2 \pmod 3$. Then $k \equiv 2$ or $5 \pmod 6$. Set $s = 2k-1$. Easy calculation show that $s \equiv 3 \pmod 6$, and therefore a Kirkman triplet system exists on s vertices. Equation (6) gives $q = \frac{12k-11}{(2k-1)^3}$, proving Theorem 2(ii). □

4.3. **Proof of Theorem 2(iii): $\alpha_k \leq \frac{36k-35}{(3k-2)^3}$ for $k \equiv 2 \pmod 4$.**

Proof. In Equation (6), choose $\ell = 4$, and assume that $k \equiv 2 \pmod 4$. Then $3k \equiv 6 \pmod{12}$. Set $s = 3k-2$, giving $s \equiv 4 \pmod{12}$, and therefore a resolvable BIBD exists with $\ell = 4$ on s vertices. Equation (6) yields $q = \frac{36k-35}{(3k-2)^3}$, proving Theorem 2(iii). □

4.4. **Proof of Theorem 2(iv): $\alpha_k \leq \frac{3k-1}{2k^3}$ for $k \equiv 0 \pmod 3$.**

Proof. When $k \equiv 0 \pmod 3$, set $s = 2k$, which implies that $s \equiv 0 \pmod 3$. There exists a *resolvable group divisible* 3*-design of type* 2^k by [4, 8, 31], namely the $\binom{s}{2}$ edges of a complete graph on s vertices can be partitioned into $\frac{s-2}{2} = k-1$ sets that contain $\frac{s}{3}$ disjoint triangles each, and a k^{th} class, which is a perfect matching. Let $P_1, \ldots, P_{k-1}, P_k$ be the partition classes where for, $i < k$, P_i consists of the aforementioned disjoint triangles and P_k is the perfect matching. Define $H_1, H_2, ..., H_{k-1}$ as sets of vertex disjoint K_3's. Further, H_k will consist of the $k = \frac{s}{2}$ matching edges in P_k, with a loop added at both ends of each matching edges.

We will compute the probability that an ab, cd crossing edge pair remains crossed in the k-planar drawing, as before.

(1) Vertices a, b, c, d can map to the vertices of the same matching edge in 2^4 ways, for $\frac{s}{2}$ edges in $8s$ ways.
(2) Edges ab and cd can map to the same edge of a K_3 in $(k-1)s \cdot 4$ ways.
(3) Edges ab and cd can map to the different edges of a K_3 in $(k-1)s \cdot 8$ ways.

Summing over all possibilities and dividing by s^4, the total number of labelings of the four vertices, we obtain that

$$q = \frac{8}{s^3} + \frac{4k-4}{s^3} + \frac{8k-8}{s^3} = \frac{3k-1}{2k^3}.$$

□

4.5. **Proof to Theorem 2(v): The case when k is odd.**

Proof. Note that for $k = 1$, $\frac{2}{k(k+1)} = 1$. So we may further assume that $k > 1$. We modify our original method by allowing some of the edges of our base graph K_s^o to appear in several H_is.

Set $s = k+1$, and note that s is even. It is well known that K_s admits a factorization into k perfect matchings, $M_1, M_2, ..., M_k$. H_i will be obtained from M_i by adding loops to every vertex, so edges that appear in more than one (in fact all) of the H_i are the loops. We still assign the $\xi(v)$ values randomly and uniformly from $\{1, 2, \ldots, s\}$ for $v \in V(G)$, but when an edge of G maps to a loop edge of K_s^o, we randomly and uniformly select an $1 \le i \le k$ and assign the edge to the i^{th} plane.

In the resulting random k-planar drawing of G, the probability q with which a crossing edge pair $\{ab, cd\}$ of the optimal planar drawing of G will cross is still independent of the selection of $\{ab, cd\}$ and is an upper bound for α_k.

If $|\{\xi(a), \xi(b)\}| = |\{\xi(c), \xi(d)\}| = 2$, then the edge-pair remains crossing in the k-planar drawing if $\{\xi(a), \xi(b)\} = \{\xi(c), \xi(d)\}$, which can happen in $\frac{s}{2} \cdot k \cdot 4$ ways, the probability of this is $\frac{2k}{s^3}$.

If $\xi(a) = \xi(b) = \xi(c) = \xi(d)$, then the edge-pair remains crossing in the k-planar drawing if ab and cd are assigned to the same plane. The probability of this is $s \cdot \frac{1}{s^4} \cdot \frac{1}{k} = \frac{1}{ks^3}$.

If for some $i \neq j$ we have $i = \xi(a) = \xi(b)$ and $j = \xi(c) = \xi(d)$, then the edge-pair remains crossing in the k-planar drawing if ab and cd is assigned to the t-th plane where $\{i, j\} \in M_t$. There are $\frac{s}{2} \cdot k$ matching edges, ab and cd can be assigned to different endvertices in 2 ways, among s^4 maps for these 4 vertices, and the images of ab and cd are present in this plane with probability $\frac{1}{k^2}$. The probability of this case is $\frac{1}{ks^3}$ again.

Finally, if for some $i \neq j$ we have $\{\{\xi(a), \xi(b)\}, \{\xi(c), \xi(d)\}\} = \{\{i, j\}, \{i\}\}$ then the edge pair remains crossing if they both get assigned to the t-th plane where $\{i, j\} \in M_t$ There are sk choices for the matching edge $\{i, j\}$ with a distinguished endvertex i, 2 ways to choose the edge that maps on the endvertex i, 2 ways to map the other edge to the matching edge, and the probability that the edge mapped to the loop gets assigned the right plane is $\frac{1}{k}$. The probability that the edge pair remains crossing is $\frac{4}{s^3}$.

Summing over all possibilities yields $\frac{2k}{s^3} + \frac{4}{s^3} + 2 \cdot \frac{1}{ks^3} = \frac{2}{k(k+1)}$. □

5. Proof to Theorem 3

Fix a $k > 1$ and assume that G is a bipartite graph, with bipartition A, B, so that $V(G) = A \cup B$.

In this section we modify our procedure by changing the base graph K_s^o to a complete bipartite graph $K_{k,k}$ with partite sets $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$. The graphs $H_1, H_2, ..., H_k$ are perfect matchings that make a factorization of $K_{k,k}$. As before, call the components (i.e. edges) of $H_1, H_2, ..., H_k$ *types*.

For a vertex $v \in A$, let $\xi(v)$ be a randomly and uniformly distributed value from $\{a_1, ..., a_k\}$, and for a vertex $v \in B$, $\chi(v)$ be a randomly and uniformly distributed value from $\{b_1, ..., b_k\}$. If uv is an edge of G ($u \in A, v \in B$), then we assign uv edge to the j-th plane, if $\{\xi(u), \chi(v)\} \in E(H_j)$. As we factorized a complete bipartite graph, there is one and only one such j. As before, we draw the uv edges in every plane following the curve representing the uv edge in an optimal drawing $\mathcal{D}$ of G in the plane.

Let $a_1 b_{\ell(1)}, a_2 b_{\ell(2)}, \dots, a_k b_{\ell(k)}$ be the edges of H_i. Then for $j = 1, 2, \dots, k$, the subgraphs spanned by the vertices $\{u \in A : \xi(u) = a_j\} \cup \{v \in B : \xi(v) = b_{\ell(j)}\}$ are in different components of G_i. We repeat this technique to reduce the number of crossing edge pairs in the k-planar drawing: we translate the subdrawings of G_i on the vertex sets $\{v \in V(G) : \xi(v) \in V(C)\}$ for the components of H_i so far from each other, such that for edges $C_1 \neq C_2$

of H_i, edges of G between vertices of C_1^{-1} and edges of G between vertices of C_2^{-1} should not cross.

Assume that uv and wz $(u, w \in A, v, z \in B)$ are a pair of crossing edges in the optimal drawing $\mathcal{D}$ of G, and have hence 4 distinct endpoints. The probability that this edge pair is still crossing in the random k-planar drawing above, is exactly

$$q = \mathbb{P}[\text{type}(\xi(u), \chi(v)) = \text{type}(\xi(w), \chi(z))]. \tag{7}$$

Note that the value of q does not depend on which crossing edge pair uv and wz was selected from $\mathcal{D}$. Hence the expected number of crossings in our random k-planar drawing is at most $q\text{CR}(G)$, and therefore some k-planar drawing of G has at most $q\text{CR}(G)$ crossings. If this holds with a certain q for all bipartite graphs G, then we have established

$$\beta_k \leq q. \tag{8}$$

If ab, cd are an edge pair that intersects in $\mathcal{D}$, then they remain intersecting in the k-planar drawing when they are exactly the same type (7). The probability of that happening is $q = \frac{k^2}{k^4} = \frac{1}{k^2}$, giving $\beta_k \leq \frac{1}{k^2}$ by (8).

Note that the lower bound $\alpha_k \geq 1/k^2$ in Pach et al. [21] depends on the existence of the *midrange crossing constant* $\kappa > 0$ from Pach, Spencer, and Tóth [20], but not on its value, which is not known. Let $\kappa(n, e)$ denote the minimum crossing number of a graph G with n vertices and at least e edges. That is,

$$\kappa(n, e) = \min_{\substack{|V(G)| = n \\ |E(G)| \geq e}} \text{CR}(G). \tag{9}$$

Then, according to [20], there exists a positive constant κ, such that the limit

$$\lim_{n \to \infty} \kappa(n, e) \frac{n^2}{e^3}$$

as $e/n \to \infty$ and $e = o(n^2)$, exists and is equal to κ. The existence of such a constant was conjectured by Erdős and Guy [13]. Czabarka, Reiswig, Székely and Wang [10] noted, that the existence of the midrange crossing constant for all graphs can be extended to the existence of the midrange crossing constant $\kappa_{\mathcal{C}}$ for certain graph classes $\mathcal{C}$, by requiring $G \in \mathcal{C}$ in (9), which may or not be equal to the midrange crossing constant κ for all graphs. In fact, the best known bounds for κ are $0.034 \leq \kappa \leq 0.09$; see [22, 1, 23], while Angelini, Bekos, Kaufmann, Pfister and Ueckerdt [2] implies that the midrange crossing constant for the class of bipartite graphs is at least $16/289 > 0.055$, making the conjecture that these two midrange crossing constants differ plausible.

The class of bipartite graphs is such a graph class that admits its midrange crossing constant, and therefore the proof of Pach et al. [21] to $\alpha_k \geq 1/k^2$ immediately extends to $\beta_k \geq 1/k^2$.

6. Theorem 4: rectilinear drawings

We repeat the arguments of Pach et al. [21] showing that our new upper bounds apply verbatim to the rectilinear k-planar crossing numbers.

The results in this paper on α_k are similarly applicable to $\overline{\alpha}_k$. Specifically, the upper bound starts from a fixed straight-line drawing of G with exactly $\textsc{rcr}(G)$ crossings. Our randomized procedure decomposes G into k graphs $G_1, \ldots, G_k$, each of which consists of vertex-disjoint subgraphs induced by the edge types. As the drawings of G_i follow a rectilinear drawing, and translations of drawings of components remain rectilinear and the argument still applies. The lower bound relies on the existence of a midrange crossing constant $\overline{\kappa} > 0$ for the *rectilinear* crossing number, which is established in [20] even though the constants κ and $\overline{\kappa}$ are not necessarily the same. Furthermore, our result in Theorem 1 on $\beta_k = 1/k^2$ also extends to $\overline{\beta}_k = 1/k^2$ and we leave the details to the reader.

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