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MULTIPRODUCT COURNOT OLIGOPOLY

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Abstract. We present a general Cournot model in which each firm may sell multiple quality-differentiated products. We use an upgrades approach, working not with the actual products, but instead with upgrades from one quality to the next. The properties of single-product Cournot models carry over to the supply of upgrades, but not necessarily to the supply of complete products. A firm's product line is determined by the properties of demand, its costs, and competitor characteristics. For symmetric firms, these determinants reduce to returns to quality and changes in demand elasticity as quality increases. For asymmetric firms whose (potentially endogenous) technological capabilities are defined by their maximum feasible qualities, gaps in product lines are determined precisely by the capabilities of lesser rivals. Strategic commitment to product lines prior to quantity competition is considered. Incentives to so commit are markedly different from those under price-setting models.

1. Introduction

Competition between multiproduct firms abounds. While understanding such competition is important, only limited progress has been made. We enhance understanding by presenting a general analysis of oligopolistic competition in quantities between firms offering multiple quality-differentiated products. We address three broad questions. First, when do the insights of single-product Cournot models carry over to a multiproduct world? Second, how is a firm's product line (that is, the qualities that it offers in positive supply) determined by the properties of demand, its costs, and competitor characteristics? Third, to what extent does the opportunity to strategically pre-commit to a product line influence our results?

The setting for these questions is a market populated by consumers with general preferences for quality, and an arbitrary number of firms. We follow earlier work (Johnson and Myatt, 2003) by taking an "upgrades approach." This approach re-casts competition to work not with actual products, but instead in terms of upgrades from one quality level to the next. For example, selling separate low and high quality products is equivalent to selling a low

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quality "baseline" product alongside an "upgrade" from low to high quality. A high quality product is then obtained by combining the baseline product with an upgrade. This helpfully allows us to think independently about the markets for the baseline product and upgrade. For instance, we show that the upgrade's price depends only upon its own supply, and not upon the supply of the baseline product. Equivalently, the theoretical baseline product and upgrade are neither substitutes nor complements. The only complication is that a monotonicity constraint must be obeyed: the baseline supply must weakly exceed the supply of upgrades (because a complete high quality product requires the baseline product plus an upgrade). The same technique works with more than two qualities, yielding a whole sequence of upgrades that are independent except for the fact that the supply of any lower upgrade must exceed that of a higher upgrade.

A benefit of the upgrades approach is that it helps answer our first question, which is, when do the insights of single-product Cournot models carry over to a multiproduct world? Such insights carry over when we think in terms of upgrades, but may fail when we think of the actual products themselves. For instance, when firms are symmetric an increase in the number of competitors will yield an expansion in the supply of each upgrade. The supply of a particular quality, however, might well decline. Similarly, in an asymmetric setting, firms with lower costs produce more of every upgrade, echoing results from single-product industries. Once again, however, this does not necessarily apply to the supply of a specific complete product. It is possible, for example, that lower cost firms produce zero units of some qualities while higher cost firms offer positive supplies.

We emphasize that upgrades correspond to measurable variables. In fact, as will be clear after presenting our formal model, predictions concerning an upgrade's supply correspond to predictions about the quantity of complete products supplied at or above a given quality.

Our second question is, how is a firm's product line determined by the properties of demand, its costs, and competitor characteristics? Precisely, we investigate how these factors influence firms' decisions either to sell complete product lines, or instead to offer lines that exhibit "gaps" so that some qualities are in zero supply. When firms have symmetric technological capabilities, the critical determinants are the returns to quality (that is, the change in the ratio of costs to willingness to pay as quality increases) and the changes in demand elasticity as quality increases.

Technological differences between firms also play an important role in determining equilibrium product lines. Such differences can take the form of either comparative or absolute differences in firms' costs for different qualities. One form of technology asymmetry involving both types of differences is variation in the maximum qualities that firms are capable of producing. We find that a gap in a firm's product line will contain the maximum feasible quality of a less capable firm. The intuition is simple. When considering an upgrade market just above the maximum that a rival can offer, more capable firms necessarily face one fewer rival in that market. This pushes more capable firms to expand their supply of upgrades to that quality level, which may lead to their monotonicity constraints binding (meaning that they supply as many upgrades to this higher quality level as they supplied to the lower quality level). This means these firms upgrade all consumers whom they serve to higher qualities, and hence their product lines exhibit a gap. Such asymmetries between firms naturally arise even when firms are symmetric *ex-ante*. We show this by generalizing our cost structure so that each firm incurs a fixed cost that is increasing in its maximum feasible quality. When firms simultaneously choose their maximum qualities and outputs (so there are no strategic effects), asymmetric technological capabilities arise endogenously, with the product ranges and gaps of each firm being determined by rivals' capabilities as just discussed.

Our third and final question is, to what extent does the opportunity to strategically pre-commit to a product line influence our results? We show that incentives for pre-commitment differ markedly from what prevails in price-setting models. Firms never pre-commit not to selling the highest quality good, since with strategic substitutes this makes them soft, which leads to an expansion of their rivals' output. Interestingly, a decision to not sell a lower quality product introduces conflicting effects. It toughens the committing firm's stance in the higher quality upgrade market but softens it in the lower quality upgrade market. The net effect on profits is indeterminate. However, we show that there is a bias towards not so committing, so that there are plausible circumstances in which firms compete head-to-head (that is, with complete product lines) even given the opportunity to avoid doing so. Thus, in such circumstances strategic effects can be safely ignored.

We close this section by relating our paper to the literature. Our "upgrades" technique builds upon earlier work (Johnson and Myatt, 2003) in which we considered a multiproduct incumbent's response to entry.[2] We investigated conditions under which the incumbent either lengthens (by using a lower-quality "fighting brand") or shortens (by "pruning" its lowest-quality goods) its product line.[3] Here we differ by providing general results on multiproduct oligopoly under wider specifications, rather than investigating different classes of incumbent response. We provide a thorough analysis under the assumption of decreasing marginal revenue, whereas the earlier paper offered only a partial analysis under this assumption (although there we considered product-line dynamics without it, but do not do so here).[4]

[2]Itoh's (1983) comparative-static analysis of a Mussa-Rosen (1978) model implicitly used an upgrade-type approach, and Fudenberg and Tirole (1998) studied the incentive to upgrade a previously sold product.

[3]The response depends on whether entry prompts an incumbent to expand or contract production. An expansion is accompanied by a fighting brand, and a contraction by product-line pruning.

[4]In particular, our earlier analysis of product line pruning presumed that the incumbent has a pronounced technological advantage over the entrant. Here we expand analysis of firms' product-line choices by dropping this assumption on technologies, and allow general differences in capability to emerge endogenously.

Other literature can be categorized according to whether a price-setting or quantity-setting approach is used, and by whether products are quality differentiated or horizontally differentiated. We focus on quality-driven models here.[5] Perhaps the most definitive work to date is that of Champsaur and Rochet (1989), where duopolists pre-commit to producing qualities within certain ranges prior to price competition. Equilibrium entails one firm selling a high-quality range and another selling a low-quality range, with no head-to-head competition. This contrasts with our study of strategic product line choice under quantity competition (that is, with strategic substitutes rather than strategic complements) in which, as discussed above, firms may well go head-to-head.[6]

Others have examined quantity competition. Gal-Or (1983), under a specification close to ours, provided sufficient conditions for the existence of a symmetric equilibrium in the special case of linear demand. De Fraja (1996) employed the consumer preference specification of Gabszewicz and Thisse (1979), and demonstrated that any equilibrium is symmetric when firms' technological capabilities are. He observed that there may be gaps in symmetric product lines determined by the shared cost structure of the firms.

Finally, our firms are engaged in second-order discriminatory behavior. Discrimination by a monopolist might involve quality (Mussa and Rosen, 1978), quantity (Spence, 1977, 1980), the date of delivery (Stokey, 1979) or even the damaging of goods (Deneckere and McAfee, 1996). Maskin and Riley (1984) provides a contract-theoretic approach, showing that a monopolist maximizes a virtual surplus term for each type that reflects the information rents paid to higher types. In contrast, our approach uses only familiar textbook concepts such as marginal revenue and marginal cost, and easily accommodates multiple firms.

In Section 2 we explain both our model and the upgrades approach to multiproduct oligopoly. Our three central questions are answered in Sections 3, 4, and 5, respectively. We offer some brief concluding remarks in Section 6.

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[5]The literature dealing with horizontal differentiation is expansive. Brander and Eaton (1984) and Klemperer (1992) studied duopolies in which firms commit to product lines before engaging in price competition. Caplin and Nalebuff (1991) and Anderson and de Palma (1992) led the way in generalizing such multiproduct competition. Eaton and Lipsey (1979) and Judd (1985) considered the issue of entry deterrence via brand proliferation. Horizontal differentiation raises the issue of optimal variety, an issue addressed by Dixit and Stiglitz (1977) for single-product firms. Anderson, de Palma, and Nesterov (1995) extended this analysis, while Anderson and de Palma (2002) considered firms capable of selling multiple products. Horizontal differentiation may also be combined with vertical differentiation. Price competition in this context was investigated by, for example, Gilbert and Matutes (1993), Stole (1995), Verboven (1999), and Ellison (2002).

[6]The price-setting models of Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982, 1983), while related, restrict each firm to a single product and hence are unable to offer insight into product lines.

2. Supply and Demand in a Multiproduct World

In this section we describe the basic structure of supply and demand in a market for quality-differentiated products, and define a game of Cournot quantity competition between multi-product firms. Our specification is a generalization of that in Johnson and Myatt (2003).

2.1. **Market Demand.** There are M distinct product qualities, where the quality of product i is denoted q_i. The products are indexed so that $0 < q_1 < q_2 < \cdots < q_M$.[7] A unit mass of consumers is indexed by the type parameter θ. The cumulative distribution function $G(\theta)$, which has support $[0, \bar{\theta}]$, is strictly increasing and continuously differentiable.

A consumer of type θ who consumes a product of quality q at price p enjoys utility $u(\theta, q) - p$, where $u(\theta, q)$ is strictly increasing in θ and q, twice continuously differentiable, and exhibits "increasing differences" so that $u(\theta, q) - u(\theta, q')$ is strictly increasing in θ whenever $q > q'$. Furthermore, we assume that $u(\theta, 0) = u(0, q) = 0$ for all θ and q. This last assumption equates a product of zero quality with no consumption, and ensures that the lowest type of consumer gains no value from a purchase. Given a set of prices $\{p_i\}_{i=1}^{M}$ for the M products, each consumer purchases a single unit of a product that maximizes her utility $u(\theta, q_i) - p_i$, unless doing so yields strictly negative utility, in which case she purchases nothing.

2.2. **Inverse Demand Functions.** We now derive the system of inverse demand functions in this marketplace. To this end, let the function $H(z)$ denote the type of consumer such that there is a mass z of consumers who value quality more highly. That is,

$$H(z) = \begin{cases} G^{-1}(1 - z) & \text{if } z < 1, \\ 0 & \text{if } z \geq 1. \end{cases} \tag{1}$$

Let $\{z_i\}_{i=1}^{M}$ be the set of industry supplies. Naturally, we must allow for the possibility that $z_i = 0$ for some i. However, it is conceptually easier to derive inverse demands assuming that all goods are in fact in positive supply. We do so here, and in Appendix A demonstrate that the same approach admits the possibility that $z_i = 0$ for some i.

When $\sum_{i=1}^{M} z_i < 1$ there is partial market coverage: not all consumers are able to purchase a good. Thus, given a set of supplies $\{z_i\}_{i=1}^{M}$, we require a set of positive prices $\{p_i\}_{i=1}^{M}$ such that exactly z_i consumers wish to purchase product i. If a lower quality product were priced no lower than a higher quality product, then it would attract no demand. There must, therefore, be a price premium for higher quality. Such higher quality products must be purchased by consumers with higher types: if a consumer θ is willing to pay a premium for

[7] Whereas this set of qualities is exogenous and distinct, we can allow M to be arbitrarily large, and the increments $q_i - q_{i-1}$ between successive qualities to be arbitrarily small. Hence we can allow suppliers to offer a large menu of different qualities drawn from an arbitrarily fine grid.

higher quality, then higher types will strictly wish to do so. Thus, the highest z_M consumers purchase product M, and the next z_{M-1} purchase product $M-1$, and so on.

Thus, the consumer with $\sum_{j=1}^{M} z_j$ others above her must be just indifferent between purchasing quality q_1 and not purchasing at all. Defining the cumulative variables $\{Z_i\}_{i=1}^{M}$ by $Z_i = \sum_{j=i}^{M} z_j$, so that the Z_i is the total quantity of all products supplied with quality q_i or above, we see that $p_1 = u(H(Z_1), q_1)$. Generally, the consumer of type $H(Z_i)$ with Z_i consumers above her must be just indifferent between products i and $i-1$, and so $p_i = p_{i-1} + [u(H(Z_i), q_i) - u(H(Z_i), q_{i-1})]$. We define $q_0 = p_0 = 0$ for convenience, and note that $p_i - p_{i-1}$ is the price differential between product i and $i-1$. This satisfies

$$p_i - p_{i-1} = P_i(Z_i) \quad \text{where} \quad P_i(Z) \equiv u(H(Z), q_i) - u(H(Z), q_{i-1}). \qquad (2)$$

$P_i(Z_i)$ is the increased utility received by a type $\theta = H(Z_i)$ from changing the product quality hypothetically received from q_{i-1} to q_i. Equivalently, it is the price of an "upgrade" from the lower quality to the higher quality.[8] We offer the following interpretation: the industry supplies Z_1 units of a "baseline" product of quality q_1, together with successive upgrades to this baseline product in order to achieve higher qualities. The strength of this "upgrades approach" is that the price of upgrade i depends only upon its own supply Z_i, and not on the supplies of any other upgrades. In contrast, the complete product i with quality q_i has a price p_i that depends upon the supplies of all n products.[9]

It turns out that (2) also describes the system of inverse demand when there is complete market coverage, so that $Z_1 \geq 1$.[10] Furthermore, defining prices according to (2) easily allows us to address the possibility that some products are in zero supply. We verify these claims in Appendix A, and confirm that (2) is correct for all circumstances.

2.3. **Market Supply.** The market is supplied by N firms. Firm r is able to manufacture a product of quality q at a constant marginal cost of $c_r(q)$, where $c_r(q)$ is continuously differentiable, strictly positive for $q > 0$, and satisfies $c_r(0) = 0$. We write $c_{ir} \equiv c_r(q_i)$ for the marginal cost of product i, and $C_{ir} \equiv c_{ir} - c_{(i-1)r}$ for the marginal cost of an upgrade from quality q_{i-1} to quality q_i.[11] Following our earlier notation, z_{ir} is firm r's supply of product i, and $Z_{ir} \equiv \sum_{j=i}^{M} z_{jr}$ is its supply of upgrade r, so that $z_i = \sum_{r=1}^{N} z_{ir}$ and $Z_i = \sum_{r=1}^{N} Z_{ir}$. Notice that $Z_{ir} - Z_{(i+1)r} = z_{ir} \geq 0$, and hence a firm's upgrades must form a monotonic

[8] $P_i(Z_i)$ is the inverse demand for the hypothetical upgrade from q_{i-1} to q_i. The assumptions made with respect to $u(\theta, q)$ and $G(\theta)$ ensure that $P_i(Z_i)$ is strictly decreasing and continuously differentiable in Z_i.

[9] To see this, notice that $p_i = \sum_{j=1}^{i} P_j(Z_j) = \sum_{j=1}^{i} P_j\left(\sum_{k=j}^{M} z_k\right)$.

[10] Given that the lower bound of the support of $G(\theta)$ is zero, there will be only partial coverage in equilibrium. The consideration of complete coverage is to accommodate any out-of-equilibrium deviations. Allowing the support of $G(\theta)$ to be bounded away from zero would not affect our analysis in any interesting way.

[11] The only assumption needed to ensure that the upgrade approach works is that the marginal cost of a complete product of each given quality is constant with respect to quantity. This ensures that the marginal cost of supplying an upgrade does not depend on the supplies of other upgrades.

sequence: $1 \geq Z_{1r} \geq \cdots \geq Z_{Mr} \geq 0$.[12] Furthermore, whenever a monotonicity constraint binds, so that $Z_{ir} = Z_{(i+1)r}$, firm r is not supplying a product of quality q_i.

The profit enjoyed by firm r is simply the sum of profits associated with each of the M products. Equivalently, it is also the sum of profits associated with each upgrade:

$$\pi_r = \sum_{i=1}^{M} z_{ir}(p_i - c_{ir}) = \sum_{i=1}^{M}(Z_{ir} - Z_{(i+1)r})\left[\sum_{j\leq i}(P_j(Z_j) - C_{jr})\right] = \sum_{i=1}^{M} Z_{ir}(P_i(Z_i) - C_{ir}).$$

Hence we may construct our game in two ways. First, we may view each firm r as choosing a set of non-negative product supplies $\{z_{ir}\}_{i=1}^{M}$ to maximize its profits, fixing the quantities supplied by other firms. Alternatively, we may view each firm as choosing a set of upgrade supplies, subject to the aforementioned monotonicity constraints. The advantage of the upgrades approach is that the direct effect of Z_{ir} is felt only in the market for the ith theoretical upgrade, and hence standard single-product Cournot logic may be employed.

In summary, a multiproduct Cournot equilibrium (pure strategy Nash equilibrium in quantities) is a collection of upgrade supplies Z_{ir}^* for each i and r, where firm r solves

$$\max \sum_{i=1}^{M} \pi_{ir} \quad \text{where} \quad \pi_{ir} = Z_{ir}\left[P_i\left(Z_{ir} + \sum_{s\neq r} Z_{is}^*\right) - C_{ir}\right],$$

subject, of course, to the monotonicity constraints $1 \geq Z_{1r} \geq \cdots \geq Z_{Mr} \geq 0$ on the upgrade supplies. In Sections 3–4 we investigate the properties of such equilibria.

2.4. **Price Sensitivity, Demand Curvature, and Returns to Quality.** We introduce here two different indices to measure the shape of demand for different upgrades:

$$\eta_i(Z_i) \equiv -\frac{Z_i P_i'(Z_i)}{P_i(Z_i)} \quad \text{and} \quad \rho_i(Z_i) \equiv -\frac{Z_i P_i''(Z_i)}{P_i'(Z_i)}. \tag{3}$$

$\eta_i(Z_i)$ is the reciprocal of the (price) elasticity of demand for upgrade i. The second measure, $\rho_i(Z_i)$, is the elasticity of the slope of the inverse-demand curve for upgrade i. It is a classic measure of curvature; Robinson (1933) called it the "adjusted concavity" of inverse demand. For our analysis, we will wish to consider the relative price sensitivity and the relative inverse-demand curvature of neighboring upgrade markets. We adopt the following definition.

Definition. *For a quantity Z,* price sensitivity is (strictly) decreasing in quality *when $\eta_i(Z)$ is (strictly) increasing in i. Similarly,* the curvature of inverse demand is (strictly) increasing in quality *when $\rho_i(Z)$ is (strictly) increasing in i.*

When utility is multiplicative, so that $u(\theta, q) = \theta q$, the inverse demand for upgrade i satisfies $P_i(Z_i) = (q_i - q_{i-1})H(Z_i)$, and both price sensitivity and the curvature of inverse demand are

[12] While we must allow for the possibility that industry supply of an upgrade exceeds 1, we may safely restrict each firm's upgrade supplies to be less than 1, since producing more than that is strictly dominated.

constant with respect to quality. For more general utility, straightforward calculations show that price sensitivity is decreasing in quality if and only if the following inequality holds.[13]

$$\frac{\partial}{\partial \theta}\left[\frac{u(\theta, q_{i+1}) - u(\theta, q_i)}{u(\theta, q_i) - u(\theta, q_{i-1})}\right] \geq 0. \tag{4}$$

This is stronger than the usual sorting condition: a consumer's relative preference for higher-quality upgrades versus lower-quality upgrades is increasing in her type.[14]

The definitions of price sensitivity and demand curvature can be framed easily in terms of complete products. For example, suppose that price sensitivity is decreasing across the entire range of upgrade markets, and that the industry initially were supplying a total of Z units, all of quality q_i. If the industry instead were to supply Z units, all of quality q_{i+1}, demand would be less elastic at that point.

A supplier cares about the cost of each upgrade as well as the properties of demand. For this reason, we define a notion of returns to quality. For heightened quality, this relates the increase in a consumer's willingness to pay to any increase in cost. For this to be meaningful, the cost must actually go up and hence for the following definition (but not elsewhere) we restrict attention to production technologies for which c_{ir} is strictly increasing in i.[15]

Definition. *For a quantity Z, the production technology of firm r exhibits (strictly) decreasing returns to quality when $C_{ir}/P_i(Z)$ is positive and (strictly) increasing in i.*

When utility is multiplicative, so that $u(\theta, q) = \theta q$, returns to quality are decreasing when $(c_{ir} - c_{(i-1)r})/(q_i - q_{i-1})$ is increasing in i. This is true if $c(q)$ happens to be convex. Also, decreasing returns to quality implies in this case that, for each consumer type, the ratio of the cost to willingness-to-pay of the complete product is increasing in quality.

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[13]Price sensitivity is decreasing in quality if $\eta_i(Z)$ is increasing in i. $\eta_i(Z)$ satisfies

$$\eta_i(Z) = -\frac{\partial \log P_i(Z)}{\partial \log Z} = -ZH'(Z)\frac{\partial \log[u(H(Z), q_i) - u(H(Z), q_{i-1})]}{\partial \theta}.$$

Since $-ZH'(Z)$ is positive and constant with respect to i, $\eta_i(Z)$ is increasing in i if and only if

$$\frac{\partial \log[u(\theta, q_{i+1}) - u(\theta, q_i)]}{\partial \theta} \geq \frac{\partial \log[u(\theta, q_i) - u(\theta, q_{i-1})]}{\partial \theta} \quad \text{where} \quad \theta = H(Z),$$

which is equivalent to the inequality given in the text.

[14]Price sensitivity is strictly decreasing in quality and curvature is strictly increasing in quality if, respectively,

$$\frac{\partial^2}{\partial \theta \partial q}\left[\log\left(\frac{\partial u(\theta, q)}{\partial q}\right)\right] > 0 \quad \text{and} \quad \frac{\partial^2}{\partial \theta \partial q}\left[\log\left(\frac{\partial^2 u(\theta, q)}{\partial \theta \partial q}\right)\right] > 0. \tag{5}$$

These conditions also arise in models of matching and search. For instance, Shimer and Smith (2000) extended the Becker (1973) marriage-market model in which a match between x and y yields output $f(x, y)$. For assortative matching, they required both $\partial f/\partial x$ and $\partial^2 f/\partial x \partial y$ to be log-supermodular. These are exactly the conditions placed on $u(\theta, q)$ in (5).

[15]Requiring cost to increase with quality rules out the "product crimping" of Deneckere and McAfee (1996) via which lower qualities are produced by intentionally (and at some cost) damaging a higher quality product. Nevertheless, such damaged goods may be seen as a special case of increasing returns to quality.

3. Multiproduct Cournot Equilibrium

Here we take up the first of the questions posed at the beginning of the paper: when do the insights of single-product Cournot models carry over to a multiproduct world?

3.1. **Existence and Uniqueness.** Before studying an equilibrium, we first ensure that one exists, and describe some conditions under which it is unique. Recall that a standard textbook condition that is sufficient for existence in a single-product industry is the following.

Assumption. *We assume that* marginal revenue is decreasing in every upgrade market*, so that for all i and any fixed $Z_{ir} \geq 0$, marginal revenue $P_i(Z_i) + Z_{ir}P_i'(Z_i)$ is decreasing in Z_i.*

This implies that a firm's marginal revenue is decreasing in the output offered by other firms and hence is strictly decreasing in its own upgrade output. In terms of complete physical products, a unit increase in Z_i corresponds to a unit increase in z_i and a unit decrease in z_{i-1}, so that this assumption says that the marginal revenue of converting lower quality complete units into higher quality units is decreasing. This assumption also implies that the marginal revenue of increasing the supply of any product i is decreasing, everything else fixed.[16] This assumption can also be cast in terms of the underlying consumer preferences and type distribution, leading to a condition on the virtual surplus of consumers being well-behaved.[17]

If the different upgrade markets were truly independent, so that the upgrade constraints could be ignored, decreasing marginal revenue would be sufficient for uniqueness in each market just as it is sufficient in a single-product setting. Even incorporating the monotonicity constraints on upgrades, uniqueness can be established for a number of cases in a full multiproduct world. For the purposes of the following proposition, we say that firms r and s are of the same type if $c_{ir} = c_{is}$ for each i; that is, if they face the same cost structure.

Proposition 1. *An equilibrium exists. This equilibrium is unique if either (i) there are at most two different types of firms, or (ii) there at most two feasible quality levels.*

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[16]By the definition of upgrades as cumulative variables, a unit increase in z_i, fixing z_j for $j \neq i$, is equivalent to a unit increase in Z_j for all $j \leq i$. Since marginal revenue is decreasing in each upgrade market, it follows that the marginal revenue of increasing z_i is also decreasing.

[17]In terms of derivatives, decreasing marginal revenue in upgrade market i is equivalent to

$$\frac{\partial^2}{\partial\theta\partial q}\left[u(\theta_i, q_i) - \frac{1 - G(\theta_i)}{g(\theta_i)}\frac{\partial u(\theta_i, q_i)}{\partial\theta}\right] > 0, \text{ where } \theta_i = H(Z_i).$$

This is precisely the condition used by Champsaur and Rochet (1989), and plays a role in ensuring that pointwise maximization of virtual surplus generates a quality schedule that is non-decreasing in θ, so that solving a firm's maximization program pointwise while ignoring this constraint is legitimate.

An immediate corollary of this proposition is that there is a unique and symmetric equilibrium when firms are symmetric (that is, they share the same cost function).[18] In the next section, we conduct a number of comparative-static exercises under symmetric specifications, before moving on to the properties of an asymmetric industry.

3.2. **Symmetric Cournot Oligopoly.** When firms are symmetric, the unique equilibrium is easy to characterize. To see this, suppose for now that all products are offered in strictly positive supply, so that $Z_1^* > Z_2^* > \cdots > Z_M^* > 0$. Given that the equilibrium will be symmetric, so that each firm supplies a $1/N$ share of industry output in each upgrade market, the monotonicity constraints of firms do not bind. The equilibrium will be determined by a single-product Cournot first-order condition, applied to each individual upgrade market:

$$P_i(Z_i^*) + \frac{Z_i^* P_i'(Z_i^*)}{N} = C_i \quad \text{for} \quad i \in \{1, \ldots, M\}, \tag{6}$$

where the marginal cost $C_i = c_i - c_{i-1}$ of upgrade i is common to all firms. The familiar equalization of marginal revenue and marginal cost carries over to a multiproduct world when we work with upgrades. Moreover, it is clearly the case that, relative to the first best, too few of each upgrade are being produced. Consequently, we arrive at a result familiar from monopoly analysis of price discrimination, namely that the quality received by any consumer is distorted downwards from the first best.[19] We emphasize, however, that we now reach this conclusion in an oligopoly setting.

When (6) holds, a selection of comparative statics emerge naturally. Following the maintained assumption of decreasing marginal revenue, the upgrade supply Z_i^* is decreasing in its marginal cost C_i and increasing in the number of firms N. Before commenting further on these and other results, we consider what happens when some products are not supplied.

Suppose, for instance, that product i is not offered, so that $z_i^* = 0$. This means that the industry supplies of upgrades i and $i+1$ are exactly equal, $Z_i^* = Z_{i+1}^*$, so that a monotonicity constraint binds. The binding constraint means that (6) will not necessarily hold. However, since i and $i+1$ are identically supplied neighboring upgrades, we may amalgamate them. That is, we combine upgrade i and $i+1$ to obtain a new theoretical upgrade with a marginal cost of $\tilde{C}_i = C_i + C_{i+1} = c_{i+1} - c_{i-1}$, and an inverse demand function $\tilde{P}_i(Z_i) = P_i(Z_i) + P_{i+1}(Z_i)$ (recall that $Z_i = Z_{i+1}$ by assumption). The upgrades are then relabelled to yield a new system with $M-1$ upgrades. Given this amalgamation procedure, (6) holds when we replace C_i with $\tilde{C}_i$ and $P_i(Z_i)$ with $\tilde{P}_i(Z_i)$. Thus, even when some monotonicity constraints bind, we expect natural comparative statics to emerge.

[18]When firms are symmetric, if the unique equilibrium were asymmetric, then by relabelling we could produce a second equilibrium, contradicting uniqueness.

[19]To see this precisely, suppose a given consumer were consuming quality q_i in the oligopoly equilibrium. The first-best outcome would involve an increase in Z_i from Z_i^*. Therefore, this consumer must be consuming at least (and possibly strictly greater than) quality q_i in the first best.

Proposition 2. *An increase in the number of firms results in an expansion in the supply of every upgrade, and hence a reduction in each of their prices. Industry profits fall.*

Proposition 2 reveals that a basic lesson of single-product Cournot models continues to hold in a multiproduct world so long as we think in terms of upgrades. Immediate corollaries flow from this proposition. First, since the price of a complete product with quality q_i is the sum of the component upgrade prices, its price p_i must fall with N. Second, recall that $Z_{ir} = \sum_{j=i}^{M} z_{jr}$, so that the supply Z_i^* of upgrade i is the number of units supplied at or above quality q_i. This means that an increase in N pushes the distribution of qualities sold upward. A special case of this is the fact that Z_1^* is increasing in N: more total units are provided by the industry when the number of competitors is higher.[20]

While the usual Cournot results concerning entry's effect on upgrade supplies hold in the space of upgrades, they may well fail when we think of the supplies of individual products. To see this, consider a two-quality world, and suppose that $Z_1^* > Z_2^*$ so that (6) holds in equilibrium. Increased competition increases both Z_1^* and Z_2^*; this implies that both total production and the production of the highest quality good rises. The supply of quality q_1, however, is $z_1^* = Z_1^* - Z_2^*$, and this can go either way. In particular, straightforward derivations lead to the following comparative static.[21]

$$\frac{\partial z_1^*}{\partial N} = \frac{1}{N}\left[\frac{Z_1^*}{(N+1)-\rho_1} - \frac{Z_2^*}{(N+1)-\rho_2}\right], \quad \text{where} \quad \rho_i = -\frac{Z_i^* P_i''(Z_i^*)}{P_i'(Z_i^*)}.$$

Here, ρ_i is the elasticity of the slope of inverse-demand from (3). By inspection, if $\rho_1 \geq \rho_2$, so the curvature of inverse demand is decreasing in quality, then the output of quality q_1 will rise following an increase in N. A special case of this is when demand for each upgrade is linear, so that $\rho_i = 0$ for each i.[22] If, on the other hand, $\rho_2 - \rho_1$ is sufficiently large, then the output of the second upgrade expands by more than the output of the baseline upgrade. A shift in supply from low to high quality overwhelms the overall expansion of the industry, meaning that the output of low quality units falls even though total output rises.[23]

We can tell a similar story when we consider industry-wide changes in marginal costs.

[20]We might also ask what would happen in a free-entry equilibrium in which any entering firm must bear a fixed cost $F > 0$. It is easy to show that the intuition of Mankiw and Whinston (1986) holds even in a multiproduct world. That is, when the number of firms is treated as a continuous variable, there is a tendency for over-entry. See Anderson and DePalma (1992, 2002) for results concerning the relation between socially optimal and equilibrium entry and product line decisions under price setting.

[21]Multiplying a firm's first-order condition for upgrade market i by N, applying the implicit function theorem, and simplifying by using the first-order condition, yields

$$\frac{\partial Z_i^*}{\partial N} = \frac{Z_i^* P_i'(Z_i^*)}{N\left[(N+1)P_i'(Z_i^*) + Z_i^* P_i''(Z_i^*)\right]} = \frac{Z_i^*}{N\left[(N+1) - \rho_i(Z_i^*)\right]}.$$

[22]In terms of primitives, setting $u(\theta, q) = \theta q$ and $\theta \sim U[0,1]$ yields $P_i(Z_i) = (q_i - q_{i-1})(1 - Z_i)$.

[23]More generally, the response of the shares of different qualities depends upon the relative curvature of the neighboring upgrade markets. The share of high quality products in the total supply of the industry is

Proposition 3. *An increase in the marginal cost of an upgrade results in a contraction in the supply of all upgrades, and a rise in their prices. If upgrade i is initially in strictly positive supply, then there is a strict reduction in its supply. Industry profits fall.*

If all products are offered (so that no monotonicity constraints bind) then the effect of a local increase in C_i is confined to Z_i^*.[24] If $i > 1$, then the total number of products brought to market will remain constant. However, the supply of product i, $z_i^* = Z_i^* - Z_{i+1}^*$, will fall, and the supply of product $i-1$, $z_{i-1}^* = Z_{i-1}^* - Z_i^*$, will rise. If the increase in C_i is sufficiently large, then the monotonicity constraint $Z_i \geq Z_{i+1}$ will bind, and product i will be eliminated from the product line offered by firms in the industry. Further increases in C_i may then cause the elimination of products $i+1$ and higher.

Similar logic reveals that an increase in the complete product cost c_i has intuitive effects on output.[25] To see this, note that such an increase corresponds to an increase in C_i and a corresponding decrease in C_{i+1}. Presuming that all products are in positive supply, then a small increase in c_i reduces Z_i and increases Z_{i+1}, which is equivalent to an increase in z_{i-1} and z_{i+1}, and a decrease in z_i that offsets these increases.

3.3. **Asymmetric Cournot Oligopoly.** Here we present a selection of results for asymmetric oligopoly. To begin, recall that a robust property of a single-product Cournot equilibrium is that a more efficient firm (that is, a firm with a lower marginal cost of production) produces more than a less efficient firm, and earns higher profits. To carry this result over, we must impose a more stringent notion of efficiency.

Definition. *Firm r has* lower costs *than firm s if $C_{ir} \leq C_{is}$ for all i.*

Notice that this definition implies that firm r faces a lower marginal cost in the production of any particular product: $c_{ir} \leq c_{is}$. It also requires firm r to face a lower marginal cost for increasing the quality of a product. Thus, it boils down to a sorting condition on the cost function of firms, similar to that on the preferences of consumers.

Proposition 4. *If firm r has lower costs than firm s, then $Z_{ir}^* \geq Z_{is}^*$ for all i, and firm r earns higher profits than s.*

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simply Z_2^*/Z_1^*. This increases with N if and only if

$$\frac{1}{Z_2^*}\frac{\partial Z_2^*}{\partial N} - \frac{1}{Z_1^*}\frac{\partial Z_1^*}{\partial N} > 0.$$

Based upon the derivations in the Footnote 21, this is true if and only if $\rho_2(Z_2^*) > \rho_1(Z_1^*)$, so that (evaluated at the equilibrium upgrade supply) the inverse-demand curve for the higher upgrade is more convex than that of the lower upgrade.

[24]Recall that an increase in C_i is equivalent to an identical increase in c_j for all $j \geq i$.

[25]The effect on profits from an increase in c_i is, however, ambiguous. See Section 5 and also the discussion following Proposition 5.

The first part of this result says that a firm with lower costs sells more of each upgrade. Equivalently, it supplies more complete units at and above any particular quality level. An implication is that a more efficient firm sells more products in total than a less efficient firm, and also sells more of the highest quality good M.

Importantly, this result does not, however, say that a firm with lower costs supplies more of any particular product $i < M$. Simply, the reason is that $z_{ir} = Z_{ir} - Z_{(i+1)r}$, and as both of these upgrades are higher for lower cost firms, in general their difference is indeterminate. In fact, one possibility is that a lower cost firm sells zero units of product i while a higher cost firm sells positive quantities. This may arise because, while a more efficient firm is pushed towards producing more of i due to its absolute cost advantage in its production, it also has an absolute advantage in the production of $i + 1$ which pushes it towards producing more of $i + 1$ and less of i. Therefore, Proposition 4 re-emphasizes that the logic from single product Cournot industries carries through when thinking in terms of upgrades, but not necessarily when thinking in terms of the actual products.

The second part of Proposition 4 says that lower cost firms earn higher equilibrium profits. This is a direct corollary of the first part of the proposition: since Z^*_{ir} is higher for firms with lower costs, total rivals' equilibrium output $\sum_{s \neq r} Z^*_{is}$ is lower for more efficient firms.

We now turn to comparative statics results. Such results require a unique equilibrium, and we focus on the case in which there are two possible qualities.[26] Suppose that the marginal cost of (say) upgrade 2 of firm r increases. Intuitively, the partial equilibrium effect is a reduction in Z_{2r}, to which other firms respond by raising their own output of upgrade 2. If this causes the upgrade constraints $Z_{1s} \geq Z_{2s}$ to bind for some firm s, then that firm's output in upgrade market 1 will also increase. In the resulting equilibrium, we expect that the firm whose costs have increased should face (weakly) higher rivals' output in each upgrade market, and hence be strictly worse off.

Proposition 5. *Suppose there are two qualities. If upgrade i is in positive supply for firm r, then an increase in r's marginal cost C_{ir} for upgrade i results in a strict contraction of the industry's supply of upgrade i (and hence a reduction in the industry supply of qualities i or higher), a strict contraction of r's output of upgrade i, and a strict reduction in the profits of r. Finally, rivals' equilibrium output $\sum_{s \neq r} Z^*_{js}$ increases for $j \in \{1, 2\}$.*

This proposition shows that a change in a firm's upgrade costs has intuitive effects. Note that thinking in terms of changes to the cost of a complete product does not ensure such

[26]It also can be shown that, for any given set of products, if there is an equilibrium in which each firm is selling exactly these products, then there is no other equilibrium in which each firm is selling exactly these same products. Restricting attention to such an equilibrium, comparative statics for local changes in upgrade costs mirror those from single-product markets. The reason is that each upgrade market can be considered (locally) to be a separate industry.

results. The reason is that it is not true that an increase in the cost of the complete low quality good is equivalent to an increase in the cost of upgrade 1.[27] Instead, it is equivalent to a simultaneous increase in the cost of upgrade 1 and a decrease of the same magnitude in the cost of upgrade 2. Intuitively, it becomes less costly to upgrade consumers to the high quality product.[28] Consequently, a change in the cost of a complete product (other than the highest quality product) will have conflicting effects. For example, a firm's profits could either increase or decrease, depending on how the outputs of other firms in the different upgrade markets equilibrate.[29]

This section answered the first of the questions posed at the beginning of the paper: when do the insights of single-product Cournot models carry over to a multiproduct world? We showed that some key insights carry over so long as we think in terms of upgrades as opposed to actual complete products. We re-emphasize that changes in the supply of upgrades correspond to measurable variables; a change in the supply of an upgrade corresponds to a change in the supply of complete products of that quality or above.

4. Product Lines

We turn to the second of our motivating questions. Referring to the set of qualities that a firm offers in positive supply as its product line, we ask, how is the structure of this product line determined by the properties of demand, its costs, and competitor characteristics?

In answering this question, we describe a firm's product line using the following terminology. Product lines are divided into sequences of positively supplied qualities, which we call *quality ranges*, and sequences of qualities that are not offered and yet are neighbored by qualities that are supplied, which we call *gaps*. Formally, there is a gap $\{j, \dots, k\}$ in the product line of firm r if $Z^*_{jr} = \cdots = Z^*_{(k+1)r} > Z^*_{(k+2)r}$ and either $j = 1$ or $j > 1$ and $Z^*_{(j-1)r} > Z^*_{jr}$. In an obvious way, quality ranges take up the space around gaps. Our formal results in this section will identify the number and location of quality ranges offered by each firm.

4.1. Returns to Quality and Price Sensitivity.

Returns to Quality and Price Sensitivity. We begin by focusing upon the first two determinants of a firm's product line, the properties of demand and its costs, and put aside (for now) the role of competitor characteristics by restricting attention to symmetric

[27]Generally, even with an arbitrary number of products, it is true however that an increase in cost of the complete highest quality good is equivalent to an increase in the cost of the highest upgrade.

[28]To see this algebraically, suppose that firm r's cost of producing the low quality product rises by κ, but the cost of the high quality product doesn't change. This firm's upgrade cost for product 1 rises by κ, whereas its upgrade cost for product 2 is $c_{2r} - (c_{1r} + \kappa) = (c_{2r} - c_{1r}) - \kappa$ (where c_{ir} denotes the initial product costs for $i \in \{1, 2\}$); the cost of upgrade 2 decreases by κ as the cost of product 1 increases by κ.

[29]In Section 5, we show how a similar logic prevails when we consider the incentives of firms to pre-commit strategically to a range of products before choosing exact quantities.

firms. Given symmetry, we know (from Proposition 1) that there is a unique and symmetric Cournot equilibrium; firms compete head-to-head with identical product lines.

Suppose product i is present in the product line offered by each firm. This means that $Z_i^* > Z_{i+1}^*$, so that the monotonicity constraint between upgrades i and $i+1$ does not bind for any firm r. This, in turn, means that marginal revenue in upgrade market i must weakly exceed marginal cost, and must strictly exceed it for industry output $Z < Z_i^*$. Similarly, the marginal cost in upgrade market $i+1$ must weakly exceed marginal revenue, and strictly exceed it for $Z > Z_{i+1}^*$. Combining these observations, when $C_i > 0$ and $C_{i+1} > 0$, gives

$$Z \in (Z_{i+1}^*, Z_i^*) \quad \Rightarrow \quad \frac{P_i(Z)}{C_i}\left[1 - \frac{\eta_i(Z)}{N}\right] > 1 > \frac{P_{i+1}(Z)}{C_{i+1}}\left[1 - \frac{\eta_{i+1}(Z)}{N}\right].$$

Hence the presence of quality q_i in the industry's product line is associated with (i) the effect of increased quality on the ratio of the marginal consumer's valuation for an upgrade to its cost, and (ii) the effect of an increase in quality on the elasticity of demand for upgrades. Recalling the definitions presented in Section 2.4, we have the following.

Proposition 6. *Suppose that product i is offered by a symmetric industry, with $C_i > 0$ and $C_{i+1} > 0$. Then, evaluated for any $Z \in (Z_i^*, Z_{i+1}^*)$, there must either be (i) decreasing returns to quality, or (ii) price sensitivity must be decreasing in quality. Conversely, if (i) and (ii) hold everywhere (with at least one holding strictly), and $P_M(0) > C_M$, then the industry offers a complete product line in which every quality is in positive supply.*

Proposition 6 reveals the simple economics of price discrimination by quantity-setting firms. The industry practices discrimination whenever the pressure to expand is lower for higher quality upgrades. This will be so when $P_i(Z)/C_i$ is lower for such upgrades (that is, when there are decreasing returns to quality) and when consumers are less sensitive to prices for higher quality upgrades. Only when at least one of these conditions fails will the product lines of symmetric firms exhibit gaps.[30]

4.2. **Differences in Technological Capability.** Here we begin our examination of how competitor characteristics, by which we mean technological differences between firms, influence product lines. We first show that, even when the hypotheses of Proposition 6 hold, so that each firm exhibits decreasing returns to quality and price sensitivity is decreasing in quality, firms may offer different product lines.

To investigate, we first consider the role of absolute cost differences by deriving a simple necessary condition for asymmetric firms to sell complete product lines. Supposing product lines are indeed complete for all firms, we sum over all firms the first-order conditions for some upgrade market i. After some algebra, we find that $P_i(Z_i^*) = (\sum_r C_{ir})/(N - \eta_i(Z_i^*))$.

[30]With decreasing returns to quality, the assumption $P_M(0) > C_M$ implies that no upgrade is in zero supply.

For this to be an equilibrium the price of upgrade i must exceed marginal cost C_{ir} for each firm r:

$$P_i(Z_i^*) \geq C_{ir} \quad \Leftrightarrow \quad \frac{C_{ir}}{\sum_s C_{is}} \leq \frac{1}{N - \eta_i(Z_i^*)}.$$

This says that firms cannot differ too much. If firm r is at a sufficient absolute disadvantage in upgrade market i (when $C_{ir}/\sum_s C_{is}$ is sufficiently large) then there cannot be an equilibrium in which all firms offer complete product lines.

The condition above is reminiscent of those that must hold in a single-product world for all firms to be active, and is a statement about absolute cost differences. But in a multiproduct world, it is not only absolute capability that is important but also comparative capability, as we now explain. Imagine that two qualities are supplied by two firms, and suppose that $C_{1r} = C_{1s}$ and $C_{2r} < C_{2s}$, so that firm r has lower costs than firm s, and hence has a weak absolute advantage in both upgrade markets. Moreover, since $C_{1r}/C_{1s} > C_{2r}/C_{2s}$, heuristically firm r has a comparative advantage in the high quality upgrade market; equivalently, it is at a comparative disadvantage in the low quality upgrade market. If C_{2r}/C_{2s} is small, Z_{2r}^* will tend to be large. This may lead to a binding constraint $Z_{1r}^* = Z_{2r}^*$, meaning that firm r's comparative advantage in the high quality upgrade causes it to eliminate the low quality from its product line.

To state this intuition formally, we consider a setting in which preferences are multiplicative $u(\theta, q) = \theta q$, so that the both the price sensitivity and inverse-demand curvature are constant across upgrades, and that $c_r(q)$ is convex. This ensures that, were firms symmetric, Proposition 6 would apply so that they would sell complete product lines.

Proposition 7. *Suppose that all firms offer complete product lines. So long as $q_i - q_{i-1}$ is sufficiently small, then Z_{ir}^* will be arbitrarily close to Z_{qr}^* for $q = q_i$ and all r where*

$$\frac{\partial u(H(Z_q^*), q)}{\partial q} + Z_{qr}^* H'(Z_q^*) \frac{\partial^2 u(H(Z_q^*), q)}{\partial \theta \partial q} = c_r'(q). \tag{7}$$

If preferences are multiplicative, $u(\theta, q) = \theta q$, and $c_r(q)$ is convex, then Z_{qr}^* is decreasing in q for each r only if

$$\frac{c_r''(q)}{\sum_s c_s''(q)} \geq \frac{1 - \lambda_{qr}\rho(Z_q^*)}{N + 1 - \rho(Z_q^*)} \quad \textit{where} \quad \rho(Z) = -\frac{ZH''(Z)}{H'(Z)} \quad \textit{and} \quad \lambda_{qr} = \frac{Z_{qr}^*}{Z_q^*}. \tag{8}$$

Thus, if all firms offer complete product lines and quality increments are small, then any change in the relative capability of firms cannot be too great as we move through qualities.

To interpret this first note that when the quality increments are small, $c_r'(q)$ gives firm r's upgrade costs, and so $c_r''(q)$ measures the rate of change of these costs. Next, note that (7) is simply the condition that marginal revenue equals marginal cost, in terms of the underlying preferences. Thus, (8) says there cannot be too much asymmetry in how firms' upgrade costs change if product lines are complete; this condition fails for r when its comparative cost advantage is improving quickly, for in that case $c_r''(q)$ is relatively small.

Our overall conclusion is that gaps in product lines are driven by either absolute cost disadvantages or comparative cost advantages. We now turn to a more structured setting to investigate further.

4.3. **Technological Limitations and Quality Ranges.** Here we continue our investigation of how technological differences influence product lines. We consider a specific kind of technological limitation, namely that a firm's technological capability corresponds to an upper bound on the quality that it is able to supply. Firm r may manufacture any product $i \leq m_r$ at a marginal cost of c_i (independent of r), but is forced to set $z_{ir} = 0$ and hence $Z_{ir} = 0$ for all $i > m_r$. This is equivalent to assuming that firm r's costs are sufficiently high in all markets $i > m_r$. Following our earlier definition, we see that firm r has lower costs than firm s when $m_r \geq m_s$. We discuss an interpretation of this assumption at the end of this section, where we also discuss the role fixed costs may play.

Without loss of generality, we order firms so that $s > r$ implies $m_s \geq m_r$. By assumption, if $m_r = i < m_s$, then r has an (extreme) absolute cost disadvantage in $i + 1$ and above, and hence does not produce products $i + 1$ and above. But our discussion in Section 4.2 also emphasized the role of comparative cost advantages, and we see that firm s also has a comparative cost advantage in market $i + 1$.[31] This suggests that s, the more-capable firm, might not sell product i. We show that this is correct, so that any gaps in firm's product lines are determined in a precise way by the technological limitations of their rivals, which correspond to changes in comparative cost advantages.

Before stating the formal result, we provide more precise intuition. Suppose that $m_r = i$ and $m_s > i$. Heuristically, firms r and s are symmetric competitors in upgrade markets i and below. In market $i + 1$, however, firm r is restricted to setting $Z_{(i+1)r} = 0$, so that firm s faces one less competitor. Even though decreasing returns to quality and decreasing price sensitivity might well push s towards producing less in upgrade market $i + 1$, the decline in the number of competitors pushes it towards producing more. If the effect from decreased competition is strong enough, this will lead to a binding constraint $Z^*_{is} = Z^*_{(i+1)s}$, so that firm s eliminates product i from its line.

To formalize this intuition, we require an additional bit of notation. Let $\bar{r}$ denote the number of firms with distinct technological profiles strictly inferior to that of firm r.[32]

[31]To be absolutely clear, the reason is that firms r and s have identical marginal upgrade costs for $j \leq m_r$, but r is unable to produce products $j > m_r$ which is equivalent to having very high upgrade costs there, implying a comparative advantage for s.

[32]Formally, $\bar{r}$ is the cardinality of the set $\{i : i = m_s < m_r\}$.

Proposition 8. *Suppose that returns to quality are decreasing and price sensitivity is decreasing in quality, and at least one is strictly decreasing. Then:*

(1) *A gap in a firm's product line includes the maximum quality of a less-able rival.*
(2) *Firm r offers a product line consisting of at most $\bar{r}+1$ quality ranges.*
(3) *Fix $\varepsilon > 0$. For $q_i - q_{i-1}$ and $q_{i+1} - q_i$ sufficiently small, either (i) firm r produces less than ε units of upgrade i or, (ii) any firm $s > r$ with strictly higher technological capability $m_s > m_r$ does not supply product i.*

Summarizing, this proposition says that a firm's product line consists of a sequence of quality ranges separated by gaps, where such gaps correspond to the technological limitations of less able firms (which in turn are linked to increases in more-able firms' comparative cost advantages). The additional hypotheses of claim (3) ensure that neighboring upgrade markets are very similar, so that the effect from decreased competition is dominant.

We close this section by discussing an interpretation of the assumption on technological limitations employed here, and by considering the role of fixed costs. One way to justify the assumption that firms are characterized by maximum feasible qualities is to assume that the production of higher quality products requires higher levels of scientific knowledge and technical expertise. If such knowledge and expertise can indeed be ranked, then firms capable of producing high quality goods necessarily will be capable of producing lower quality goods.

This leads to the question of why firms differ in their knowledge bases. While these differences may be exogenously given, another possibility is that they emerge endogenously. Suppose that firms simultaneously chooses both their maximum quality level and their outputs.[33] If firm s chooses a maximum quality of q_i, it pays a fixed cost $F_{is} \equiv F_s(q_i)$, which is strictly increasing and continuously differentiable in q_i.

For the sake of discussion, suppose for the moment there are two firms and two quality levels. With decreasing returns to quality and decreasing price sensitivity, if equilibrium is such that each firm s chooses $m_s = 2$, Proposition 6 ensures that $Z^*_{1s} > Z^*_{2s}$ for each firm s. However, if the duopoly profits in upgrade market 2 do not cover the incremental fixed costs $F_{2s} - F_{1s}$, then there is no (pure strategy) equilibrium in which both firms develop the capability to produce both products. Moreover, so long as the equilibrium monopoly profits available in upgrade market 2 exceed some firm's incremental fixed costs, and both firms can cover their fixed costs in upgrade market 1, then the overall equilibrium must exhibit

[33]We assume firms choose qualities and quantities simultaneously for two reasons. First, this abstracts away from strategic incentives regarding product line choice, which are considered in Section 5. Second, a simultaneous-move game is often an appropriate formulation, since quantities may be difficult (or costly) to adjust quickly and since firms may be able to invest in technical expertise secretly.

differences in the technological capabilities of firms.[34] This logic clearly carries over to cases in which there are N firms and M quality levels. A formal result follows.

Proposition 9. *Suppose firms simultaneously choose their maximal quality levels and outputs, where firm each firm r must bear a fixed cost F_{is} if its maximal quality is q_i. Then:*

(1) *If F_{Ms} is sufficiently small for each s, then all firms offer all products in equilibrium, and firms produce identical outputs.*
(2) *If $q_i - q_{i-1}$ and $q_{i+1} - q_i$ are sufficiently small, then, in a pure strategy equilibrium, if firm r chooses a maximum technological capability $m_r = i$ where $i < M$, then no other firm s chooses the same technological capability: $m_s \neq m_r$.*

This proposition indicates when we might expect either a symmetric industry or an asymmetric one to arise endogenously due to fixed costs. Heuristically, when fixed costs are not too significant, all firms will invest so as to be able to produce all M products and the equilibrium will be symmetric. (Our interpretation is that M represents an upper bound on what is currently achievable in an industry, and technically could be interpreted as large discrete jump upwards in the functions F_{is} at $i = M$.) On the other hand, when $q_i - q_{i-1}$ and $q_{i+1} - q_i$ are sufficiently small, then we expect an asymmetric equilibrium whenever it is not the case that all firms choose $m_r = M$.[35] The reason is that upgrade markets i and $i+1$ look very similar, so that if two firms s and r were choosing $m_r = m_s = i < M$, then the gains to firm s of instead expanding to market $i+1$, where there is less competition, would be worthwhile. A corollary is that if no more than one firm chooses $m_r = M$, then no two firms choose the same maximal qualities.[36] In this case, all firms will differ by their maximal feasible quality, and any gaps in the products line of a firm will be determined by the limitations of less-able rivals, as given in Proposition 8.

5. Strategic Product Line Choice

Here we take up our third main question by exploring the possibility of strategic product line commitment, and ask how this influences our earlier results. We suppose that all firms, prior to setting quantities, simultaneously decide whether to impose restrictions on their production capabilities; equivalently, firms decide which products they are capable of producing. Each firm's choice is observed by all other firms, and then quantity competition occurs (we

[34]Strictly speaking, it must be that the monopoly profits in upgrade market 2 exceed the incremental fixed costs, even respecting the fact that a firm entering that market must obey its monotonicity constraints.

[35]It will certainly be the case that not all firms choose to be able to produce the highest quality product if doing so is sufficiently expensive.

[36]If the equilibrium is in mixed strategies, then with positive probability the realized product lines are still asymmetric.

look at subgame perfect Nash equilibria). This two-stage formulation of multiproduct quality competition captures the possibility that there may be a long-run component to product line choice, causing strategic effects. Since we are mostly interested in understanding such effects, we ignore direct costs associated with the first stage.

We restrict formal attention to the two-product case, but discuss at the end of this section how relaxing this assumption influences the analysis. In the first stage, therefore, each firm chooses whether to restrict itself to selling just the high quality product or just the low quality product, or instead not to restrict itself.[37] For convenience, we also suppose that if a firm s has not restricted itself to sell only the low-quality product, then $Z^*_{2s} > 0$.[38]

5.1. **Committing to Low Quality.** Consider the possibility that some firm r would commit to producing only the low-quality product, given whatever product line choices other firms are making; this amounts to the restriction that $Z_{2r} = 0$ in the second stage. Intuitively, in partial equilibrium (that is, fixing the outputs of other firms) firm r is forced to lower Z_{2r}, and may also choose to lower Z_{1r} if $Z^*_{1r} = Z^*_{2r}$ originally. Firms $s \neq r$, meanwhile, will respond to the lowering of Z_{2r} by expanding their own outputs in upgrade market 2. This may lead them to also raise Z_{1s}, and Z_{1s} may also increase if Z_{1r} decreased. The net equilibrium effect is an increase in $\sum_{s \neq r} Z^*_{js}$ for $j \in \{1, 2\}$.

We conclude that there may well be strategic effects associated with committing to selling only the low-quality product. However, these effects work strictly against the firm that so commits: they lead to an increase in the sum of other firms' outputs in each upgrade market and also constrain the firm's ability to respond optimally.

Proposition 10. *No firm restricts itself to selling only the low-quality product.*

Another way of thinking about Proposition 10 is that committing to only selling the low quality product is equivalent to an increase in C_{2r} by some large amount κ. Proposition 5 indicates that this leads to a decrease in profits for firm r, and indeed it can be shown that the current proposition is a corollary of the earlier one.

5.2. **Committing to High Quality.** We turn now to the possibility that some firm r might commit to selling only the high quality product, which is equivalent to imposing the constraint $Z_{1r} = Z_{2r}$. This induces conflicting effects, and is optimal in some circumstances but not in others. To understand this, first note that the partial equilibrium effect is for r to lower Z_{1r} and raise Z_{2r} from their initial levels. Firms $s \neq r$ will tend to respond by raising Z_{1s} and lowering Z_{2s}.

[37]Our earlier uniqueness result for the case of two products is easily extended to this setting, ensuring a unique equilibrium in each subgame. Since firms have a finite number of choices in the first stage, we can also be certain that an equilibrium of the overall two-stage game exists (potentially in mixed strategies).
[38]This is to rule out uninteresting cases.

Proposition 11. *Suppose that firms $s \neq r$ are not restricting their product lines in stage 1, and that when firm r does not do so $Z^*_{1s} > Z^*_{2s}$ for all s. Then when firm r moves from not restricting itself to only selling product 2, the second-stage equilibrium changes as follows:*

(1) $\sum_{s \neq r} Z^*_{1s}$ *strictly increases and* $\sum_{s \neq r} Z^*_{2s}$ *strictly decreases,*
(2) Z^*_{1r} *strictly decreases and* Z^*_{2r} *strictly increases,*
(3) Z^*_1 *strictly decreases and* Z^*_2 *strictly increases.*

This proposition can be stated in terms of complete products. For instance, the first claim says that total rivals' output of the low quality good increases while their output of the high quality good decreases, and the second claim says that the opposite happens for firm r.

Committing to selling only the high quality good has conflicting effects on r. On the one hand, r is "tougher" in upgrade market 2, leading others to curb production there to the benefit of r. On the other hand, r is "softer" in upgrade market 1, causing others to increase production to the detriment of r.

Heuristically, this also can be understood by supposing that firm r's upgrade cost in market 1 increases by some amount $\kappa > 0$ while its cost in upgrade market 2 decreases by a like amount. A suitably chosen value of κ leads to the same equilibrium outputs and profits for all firms as directly imposing the constraint $Z_{1r} = Z_{2r}$. To understand the total effect on profits from increasing κ, denote firm r's profits by $\pi_r(\kappa)$, and consider the marginal change in profits. Using the envelope theorem, we have precisely:

$$\frac{d\pi_r(\kappa)}{d\kappa} = (Z^*_{2r} - Z^*_{1r}) + Z^*_{1r} P'_1(Z^*_1) \sum_{s \neq r} \frac{\partial Z^*_{1s}}{\partial \kappa} + Z^*_{2r} P'_2(Z^*_2) \sum_{s \neq r} \frac{\partial Z^*_{2s}}{\partial \kappa}$$

The first term $(Z^*_{2r} - Z^*_{1r})$ is negative since the increase in upgrade market 1's costs is felt over more units than the decrease in market 2's costs.[39] The second term is the strategic effect on profits in upgrade market 1. For this upgrade, firm r's reaction curve is pulled down, causing its rivals to expand: the strategic price effect is negative. The third term is the strategic effect on profits in upgrade market 2, and is positive since r's reaction curve pushes outward, lowering the output of other firms so that the strategic price effect is positive.

The overall impact on profits from a marginal increase in κ is ambiguous, and depends on the magnitudes of the slopes of firms' reaction functions in the two upgrade markets, upon which we have put little structure. It follows that the total change in profits brought about by imposing the constraint $Z_{1r} = Z_{2r}$ (equivalently, by raising κ) is ambiguous. However, we do see that for a firm to gain by restricting itself to only selling the high quality good, the

[39] This term being negative is closely related to the fact that the firm's being constrained restricts its ability to respond to any fixed vector of other-firm outputs, and hence we could call this term the "sub-optimization effect." Increasing κ corresponds to lowering the value D on the constraint $Z_{1r} - Z_{2r} \leq D$ if one thinks in those terms, and hence an increase in D must make the firm worse off, neglecting the strategic effect.

positive strategic effect in market two must not only overwhelm the negative effect in market one, but also overcome the effect of being constrained. Note further that, because r produces more in upgrade market 1, for the strategic profit effect in upgrade market 2 to overwhelm that in upgrade market 1, a necessary condition is that the strategic price effect in market 2 strictly exceeds that in upgrade market 1: $\left|P_2'(Z_2^*)\sum_{s\neq r}\frac{\partial Z_{2s}^*}{\partial\kappa}\right| > \left|P_1'(Z_1^*)\sum_{s\neq r}\frac{\partial Z_{1s}^*}{\partial\kappa}\right|$. To understand what is needed for this condition to hold, consider a marketplace in which all firms are initially symmetric (that is, $C_{is} = C_i$ for each s and no firm has restricted its product line) and suppose firm r begins to increase κ away from zero. Calculations reveal that the condition described here holds if and only if $\rho_2(Z_2^*) < \rho_1(Z_1^*)$; that is, the curvature of inverse demand must be decreasing in quality.[40]

It is interesting to compare our results to those from the literature on commitment with price competition. Champsaur and Rochet (1989) find that price-setting duopolists have strong incentives to avoid competing head-to-head, with equilibrium involving each firm committing to an interval of qualities that does not intersect that offered by the other firm. Heuristically, since prices are strategic complements, firms achieve higher profits by offering non-overlapping lines since doing so leads each firm to raise its prices, thereby raising profits for each of the duopolists. Similarly, Anderson and de Palma (1992) show that firms hold back on their product lines to reduce the intensity of competition. We have shown that when firms choose quantities, which are strategic substitutes when marginal revenue is decreasing, the only clear-cut prediction is that all firms will offer the highest-quality product.

Because the possibility of head-to-head competition prevailing, even given the opportunity to strategically avoid it, is interesting and such a marked contrast to what arises in a price-setting world, we pursue the issue a little further. We describe the equilibrium outcome in the special case of multiplicative preferences and uniformly distributed consumer valuations. This yields linear upgrade demand curves $P_i(Z_i) = (q_i - q_{i-1})(1 - Z_i)$ for $i \in \{1, 2\}$.

Proposition 12. *Let demand be linear in each upgrade market. Suppose, if no firm restricts itself in stage one, that $Z_{1s}^* > Z_{2s}^*$ for each s. Then there exists a unique equilibrium to the two-stage game. In the equilibrium, no firm restricts its product line in the first period and each firm produces positive quantities of both complete products in the second period.*

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[40] Formally, adding κ to its costs of product i, the first-order condition of firm r is $P_i(Z_i^*)+Z_{ir}^*P_i'(Z_i^*) = C_i+\kappa$. Summing the first-order conditions of all firms we obtain $NP_i(Z_i^*) + Z_i^*P_i'(Z_i^*) = NC_i + \kappa$. Differentiating with respect to κ and evaluating at $\kappa = 0$ (so that $Z_{ir}^* = Z_i^*/N$) we obtain

$$\left[P_i'(Z_i^*) + \frac{Z_i^*P_i''(Z_i^*)}{N}\right]\frac{\partial Z_i^*}{\partial\kappa} + P_i'(Z_i^*)\frac{\partial Z_{ir}^*}{\partial\kappa} = 1 \quad\text{and}\quad [(N+1)P_i'(Z_i^*) + Z_i^*P_i''(Z_i^*)]\frac{\partial Z_i^*}{\partial\kappa} = 1.$$

Combining these expressions and re-arranging, we obtain

$$P_i'(Z_i^*)\sum_{s\neq r}\frac{\partial Z_{is}^*}{\partial\kappa} = P_i'(Z_i^*)\left[\frac{\partial Z_i^*}{\partial\kappa} - \frac{\partial Z_{ir}^*}{\partial\kappa}\right] = \frac{2N - \rho_i(Z_i^*)}{N\left[N+1-\rho_i(Z_i^*)\right]} - 1.$$

This ensures that the necessary condition stated in the text holds if and only if $\rho_2(Z_2^*) < \rho_1(Z_1^*)$.

With linear demand, the only equilibrium exhibits firms going head to head. The additional restriction in the hypothesis only serves to rule out multiple equilibria that are equivalent as far as second-period outcomes.[41]

We close by briefly explaining how our analysis changes when more than two upgrade markets are considered. With many upgrade markets, a commitment by r to not sell product j, amounting to the restriction $Z_{jr} = Z_{(j+1)r}$, may have strategic effects that spill over to upgrade markets other than j and $j+1$. To see why, note that firm r's partial equilibrium response is to lower Z_{jr} and raise $Z_{(j+1)r}$, which pushes other firms outputs up in j and down in $j+1$. This may cause the upgrade constraints of some of these other firms to begin binding in either market $j-1$ or $j+2$. This means it is possible that the strategic effects are felt for some number of upgrade markets below j and above $j+1$. This possibility, while certainly interesting, suggests the same conclusion, however: strategically committing to restrict ones product line induces positive effects in some markets and negative effects in others, so that the overall effect may go in either direction.

We conclude that allowing for strategic commitment introduces interesting results that stand in contrast to what happens in price-setting models. However, in some cases these effects can be safely ignored since firms would not choose to commit to restricted product lines.

6. Concluding Remarks

A general model of multiproduct Cournot competition was presented and analyzed using the upgrades approach. Using the upgrades approach approach allows intuition from single-product industries to be applied easily to multiproduct ones. Similarly, many properties of single-product industries carry over to multiproduct ones when we think in terms of upgrades.

We investigated the structure of firms' product lines, showing that when firms have identical technological capabilities, the critical determinants are the returns to quality and the changes in demand elasticity as quality increases. However, technological asymmetries such as differences in absolute or comparative cost advantages also play an important role in whether a firm offers a complete product line and, if not, where gaps in its line are.

We extended our basic model to allow for strategic product line choices. Such commitment possibilities have very different effects than in price-setting models, in that firms may choose to compete head-to-head even given the opportunity to doing so.

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[41]Suppose that the assumption that $Z^*_{1s} > Z^*_{2s}$ whenever no firm restricts itself in the first stage fails for some firm. Then, whether this firm restricts itself in the first stage or not has no effect on any firm's outputs or payoffs in the second period, but strictly speaking there would exist multiple equilibria categorized by this firm's first-stage decision. So, more generally, this proposition can be stated as follows: with linear demand, no firm restricts its product line unless doing so has no effect on the nature of the second-period subgame.

Appendix A. Construction of the Inverse Demand Functions

Here we show that the inverse demand functions derived in Section 2.2 are correct even when either market coverage is complete or when some products are in zero supply.

A.1. **Complete Market Coverage.** In the main text, our construction of the inverse demand functions considered the case of incomplete market coverage, so that $Z_1 \leq 1$. Our assumption that the lower bound of $G(\theta)$ is zero ensures that all prices will be positive in equilibrium, so that the market is not fully covered. We still must address the possibility that some prices are zero out of equilibrium, as we do here. That is, we ensure that (2) continues to hold when coverage is complete.

If $Z_1 = 1$, then p_1 must equal zero, given our assumptions on $G(\theta)$. If $Z_1 > 1$, then $p_1 = 0$ as well, because there is strictly excess supply. Similarly, $p_i - p_{i-1} = 0$ for any upgrade i satisfying $Z_i \geq 1$. But of course, if this holds then $H(Z_i) = 0$ by definition (see (1)) and hence (2) continues to hold. Notice that, if $Z_{j+1} \geq 1$ then there can be no demand for product j. The reason is that the total price of the upgrade to product $j+1$ is zero, so that all consumers will purchase a product at least of quality q_{j+1}.

A.2. **Products in Zero Supply.** We claimed that defining prices according to (2) admits the possibility that some products are in zero supply. To see how, suppose that the product supplies $\{z_i\}$ satisfy only that $z_i \geq 0$, and define the cumulative variables (that is, the upgrade supplies) $\{Z_i\}$ and prices $\{p_i\}$ exactly as in the body of the text. Note that a product i is in positive supply if $Z_i - Z_{i+1} > 0$, at least if we define $Z_{M+1} \equiv 0$, while a product is in zero supply of $Z_i - Z_{i+1} = 0$. If j is the first product in positive supply then a total of $Z_j = \sum_{i=j}^{M} z_i$ products are supplied to the market. If the prices defined above are correct in this case, it must be that consumer $\theta = H(Z_j)$ is just indifferent between buying good j and not. The price of good j can be obtained by adding up the incremental prices given by (2). Making use of the fact that for $k \leq j$ we have $Z_k = Z_j$,

$$p_j = \sum_{i=1}^{j} P_i(Z_i) = \sum_{i=1}^{j} P_i(Z_j) = u(H(Z_j), q_j).$$

Hence, a consumer with type $\theta = H(Z_j)$ is indifferent between buying good j and not buying at all. The only subtlety is that she is also indifferent between buying goods $k < j$ and not at the prices defined, but these are not in positive supply. Thus, we adopt the convention that when a consumer is indifferent between several products she purchases the one of highest quality. We have only dealt with the first good in positive supply, but a similar process can be applied recursively. The process just described can be extended to all possible supply configurations. Hence, given the convention that a consumer purchases the highest quality product when indifferent, the prices defined by (2) are correct for all circumstances.

Appendix B. Omitted Proofs

We prove Proposition 1 in steps. We begin by confirming that an equilibrium exists.

Proof of Proposition 1 (existence). Each firm's choice set is a compact and convex subset of a Euclidean space. A firm's profits are continuous in the choices of its opponents, and (given decreasing marginal revenue) concave in its own choice and hence quasi-concave in its own choice. It is well known that these results are sufficient for the existence of an equilibrium; see, for example, Fudenberg and Tirole (1991), page 34. □

To prove uniqueness for cases (i) and (ii), we will suppose that there are two distinct equilibria with equilibrium supplies Z^*_{ir} and Z^{**}_{ir}, and then argue by contradiction. Before proceeding, we present two lemmas. From the maintained assumption that marginal revenue is decreasing in every upgrade market, the following is immediate.

Lemma 1. *Suppose that $Z^{**}_i \geq Z^*_i$ and $Z^{**}_{ir} \geq Z^*_{ir}$. If one of these inequalities holds strictly then, neglecting monotonicity constraints, if firm r has a weak incentive to expand its supply of upgrade i in the $(**)$-equilibrium, then it will have a strict incentive to do so in the $(*)$-equilibrium. Similarly, if it has a weak incentive to contract its supply of upgrade i in the $(*)$-equilibrium, then it will have a strict incentive to do so in the $(**)$-equilibrium.*

The next lemma observes that when we compare two distinct equilibria, the industry output profiles cannot be completely ranked. When industry output of every upgrade is raised, a firm's marginal revenue in each upgrade market falls. Thus, for each individual firm to be reacting optimally, its output of each upgrade must fall. This leads to a contradiction.

Lemma 2. *There cannot be two distinct equilibria satisfying $Z^{**}_i \geq Z^*_i$ for every upgrade i.*

Proof. We argue by contradiction. If the equilibria differ and $Z^{**}_i \geq Z^*_i$ for every i, then there is some firm r that, for at least one upgrade, produces strictly more in the $(**)$-equilibrium than in the $(*)$-equilibrium. For such a firm, we can find a sequence of neighboring upgrade markets $\{j, \ldots, k\}$ for which $Z^{**}_{ir} > Z^*_{ir} \geq 0$ for all $i \in \{j, \ldots, k\}$. We can extend this sequence downward until either $j = 1$, or $Z^*_{(j-1)r} \geq Z^{**}_{(j-1)r}$. This ensures that there is no upward constraint on Z^*_{jr}; this is trivially true when $j = 1$, and for $j > 1$ we have $Z^*_{(j-1)r} \geq Z^{**}_{(j-1)r} \geq Z^{**}_{jr} > Z^*_{jr}$. Next, we extend the sequence upward until either $k = n$ or $k < n$ and $Z^*_{(k+1)r} \geq Z^{**}_{(k+1)r}$. This ensures that there is no downward constraint on Z^{**}_{kr}.

Under the $(*)$-equilibrium, firm r faces no binding upward constraint in market j. It could, therefore, simultaneously expand (that is, increase each upgrade output by an identical small amount) in all markets $i \in \{j, \ldots, k\}$. It does not, and so must have at least a weak incentive to contract simultaneously in all such markets. Since $Z^{**}_i \geq Z^*_i$ and $Z^{**}_{ir} > Z^*_{ir}$ for these i, under the $(**)$-equilibrium it has a strict incentive to contract simultaneously in all of these markets. It does not, and hence there must be a binding downward constraint in market k, so that $Z^{**}_{kr} = Z^{**}_{(k+1)r}$. This is a contradiction. □

Following Lemma 2, if there are two equilibria when there are only two qualities, then the industry outputs in these equilibria must satisfy either $Z^{**}_1 > Z^*_1$ and $Z^*_2 > Z^{**}_2$, or $Z^{**}_1 < Z^*_1$ and $Z^*_2 < Z^{**}_2$. We can label the firms to ensure that we deal with the former case. These observations leads to a proof of Proposition 1 for case (ii).

Proof of Proposition 1 (uniqueness for case (ii)). Lemma 2 implies that two distinct equilibria must satisfy $Z^{**}_1 - Z^*_1 > 0 > Z^{**}_2 - Z^*_2$. Consider the subset of firms $R \subseteq \{1, \dots, N\}$ that satisfy $Z^{**}_{1r} > Z^*_{1r}$. Under the $(*)$-equilibrium, firm $r \in R$ faces no upward constraint in market $i = 1$, and hence must face a weak incentive to lower Z_{1r}. Since $Z^{**}_1 > Z^*_1$ and $Z^{**}_{1r} > Z^*_{1r} \geq 0$, firm r faces a strict incentive to contract under the $(**)$-equilibrium, and so its monotonicity constraint must bind: $Z^{**}_{1r} = Z^{**}_{2r}$. Recalling that $Z^*_{2r} \leq Z^*_{1r}$, we have

$$r \in R \quad \Rightarrow \quad Z^{**}_{2r} - Z^*_{2r} \geq Z^{**}_{1r} - Z^*_{1r} > 0. \tag{9}$$

Next, consider the subset of firms $S \subseteq \{1, \dots, N\}$ that satisfy $Z^{**}_{2s} < Z^*_{2s}$. Using an argument equivalent to that given above, we can conclude that $Z^{**}_{1s} = Z^{**}_{2s}$, and

$$s \in S \quad \Rightarrow \quad Z^*_{1s} - Z^{**}_{1s} \geq Z^*_{2s} - Z^{**}_{2s} > 0. \tag{10}$$

R contains all firms satisfying $Z^{**}_{1r} > Z^*_{1r}$, and no members of S. Since $Z^{**}_1 > Z^*_1$, this implies

$$\sum_{r \in R} (Z^{**}_{1r} - Z^*_{1r}) > \sum_{s \in S} (Z^*_{1s} - Z^{**}_{1s}) \geq \sum_{s \in S} (Z^*_{2s} - Z^{**}_{2s}), \tag{11}$$

where the weak inequality follows from (10). Similarly, since $Z^*_2 > Z^{**}_2$,

$$\sum_{s \in S} (Z^*_{2s} - Z^{**}_{2s}) > \sum_{r \in R} (Z^{**}_{2r} - Z^*_{2r}) \geq \sum_{r \in R} (Z^{**}_{1r} - Z^*_{1r}), \tag{12}$$

where this time the weak inequality follows from (9). Chaining the inequalities of (11) and (12) together, we reach the contradiction $\sum_{r \in R} (Z^{**}_{1r} - Z^*_{1r}) > \sum_{r \in R} (Z^{**}_{1r} - Z^*_{1r})$. □

Proof of Proposition 1 (uniqueness for case (i)). By assumption, there are at most two types of firms. We partition the firms into subsets R and S according to these types. Firms of the same type share the same cost structure, and hence offer the same outputs in equilibrium (this is implied by Proposition 4). For instance, if $\{r, r'\} \subseteq R$ then $Z^*_{ir} = Z^*_{ir'}$ and $Z^{**}_{ir} = Z^{**}_{ir'}$ for all i. Suppose that j is the first upgrade for which two equilibria differ. Without loss of generality, suppose that $Z^{**}_j \geq Z^*_j$. Given that the equilibria differ at j, one subset of firms must be producing strictly more in the $(**)$-equilibrium than in the $(*)$-equilibrium. Suppose it is subset R, so that $Z^{**}_{jr} > Z^*_{jr}$ for $r \in R$. Since j is the first upgrade where the equilibria differ, then either $j = 1$ or $j > 1$ and $Z^*_{(j-1)r} = Z^{**}_{(j-1)r} \geq Z^{**}_{jr} > Z^*_{jr}$, and hence firm $r \in R$ faces no upward constraint on Z^*_{jr}. Thus, it has a weak incentive to lower Z^*_{jr}. Since $Z^{**}_{jr} > Z^*_{jr} \geq 0$ and $Z^{**}_j \geq Z^*_j$ it has a strict incentive to lower Z^{**}_{jr} yet does not. If $j = n$, this is impossible and we have reached a contradiction. If $j < n$, however, then it must face a binding downward constraint, $Z^{**}_{jr} = Z^{**}_{(j+1)r}$, which in turn implies that $Z^{**}_{(j+1)r} - Z^*_{(j+1)r} \geq Z^{**}_{jr} - Z^*_{jr} > 0$. There are now two possibilities. The first is that $Z^{**}_{j+1} \geq Z^*_{j+1}$, and the second is that $Z^{**}_{j+1} < Z^*_{j+1}$.

We deal with the first possibility here. If $Z^{**}_{j+1} \geq Z^*_{j+1}$ then, since $Z^{**}_{(j+1)r} > Z^*_{(j+1)r}$ and firm r is free to expand in markets j and $j+1$ under the $(*)$-equilibrium, we can repeat our argument, and conclude that $Z^{**}_{(j+2)r} = Z^{**}_{(j+2)r}$ and $Z^{**}_{(j+2)r} - Z^*_{(j+2)r} \geq Z^{**}_{(j+1)r} - Z^*_{(j+1)r} > 0$. In fact, we iterate the same argument until either we reach an upgrade $k = n$, and therefore a contradiction, or we find an upgrade k where $Z^{**}_k - Z^*_k \geq 0 > Z^{**}_{k+1} - Z^*_{k+1}$ and $Z^{**}_{(k+1)r} - Z^*_{(k+1)r} \geq Z^{**}_{kr} - Z^*_{kr} > 0$ for all firms $r \in R$.

Finding such an upgrade k corresponds precisely to the second possibility. Since $Z^{**}_{k+1} < Z^*_{k+1}$ while $Z^{**}_{(k+1)r} > Z^*_{(k+1)r}$ for all $r \in R$, it must be the case that $Z^{**}_{(k+1)s} < Z^*_{(k+1)s}$ for all $s \in S$. Furthermore, given the other inequalities derived above,

$$\sum_{s \in S}(Z^*_{(k+1)s} - Z^{**}_{(k+1)s}) > \sum_{r \in R}(Z^{**}_{(k+1)r} - Z^*_{(k+1)r}) \geq \sum_{r \in R}(Z^{**}_{kr} - Z^*_{kr}) \geq \sum_{s \in S}(Z^*_{ks} - Z^*_{ks}),$$

which implies that $Z^{**}_{(k+1)s} < Z^{**}_{ks}$. We now see that under the $(*)$-equilibrium, any firm $s \in S$ is unconstrained upward. We can use identical logic to show that this means it has a strict incentive to contract $Z^*_{(k+1)s} > 0$. Once again, if $k+1 = n$ then, we have reached a contradiction. If not, then we can continue as before, only this time working with the subset of firms S. This procedure only stops once we reach upgrade n, and find that one subset of firms face binding downward monotonicity constraints on their strictly positive supply of that upgrade; this is where we always reach a contradiction. □

Proof of Proposition 2. We write Z^*_i and Z^{**}_i respectively for equilibrium industry output of upgrade i before and after an increase in the number of firms. Following either Proposition 1 or Proposition 4, any equilibrium will be symmetric. This means that a monotonicity constraint will bind for a firm if and only if it binds for output at the level of the industry. Hence, throughout the proof, we are able to work only with industry outputs and will not need to refer to individual firm outputs. For any upgrade i satisfying $Z^*_i = 0$, it is automatically true that $Z^{**}_i \geq Z^*_i$. We will show that if $Z^*_i > 0$ then $Z^{**}_i > Z^*_i$.

Suppose not. We may find a sequence of neighboring upgrades $\{j, \ldots, k\}$ for which $Z^{**}_i \leq Z^*_i$ and $Z^*_i > 0$ for all $i \in \{j, \ldots, k\}$. We extend this sequence downward (redefining j as required) until either $j = 1$ or $Z^{**}_{j-1} > Z^*_{j-1}$. If $j = 1$, then there is no upward constraint on Z^{**}_j. If $j > 1$, then we combine inequalities to obtain $Z^{**}_{j-1} > Z^*_{j-1} \geq Z^*_j \geq Z^{**}_j$. This ensures that $Z^{**}_{j-1} > Z^{**}_j$, and once again there is no upward constraint on Z^{**}_j. Next, we extend the sequence upward (here redefining k as required) until either $k = n$ or $Z^{**}_{k+1} > Z^{*}_{k+1}$. If $k = n$, there is downward constraint on Z^{*}_{k}. If $k < n$, then we may combine inequalities to obtain $Z^{*}_{k} \geq Z^{**}_{k} \geq Z^{**}_{k+1} > Z^{*}_{k+1}$. Again, since $Z^{*}_{k} > Z^{*}_{k+1}$, there is no downward constraint on Z^{*}_{k}.

Under the $(**)$-equilibrium each firm faces no upward constraint in market j, and so must face a weak incentive to simultaneously contract for all $i \in \{j, \ldots, k\}$. Under the $(*)$-equilibrium, industry output for each $i \in \{j, \ldots, k\}$ is weakly higher. Since there are strictly fewer firms, each individual firm's output for each one of these upgrades must be strictly higher. Hence, each firm faces a strict incentive to contract in all of these markets. No firm does, and so there must be a binding downward constraint in market k. This is a contradiction.

We have shown that the industry's output of each positively supplied upgrade strictly increases following entry, and hence the upgrade's price must strictly fall. The baseline upgrade (that is, $i = 1$) is always in strictly positive supply, and hence its price strictly falls. This implies, in turn, that the price of every complete product must strictly fall.

Finally, we turn to profits. It is straightforward to show that an individual firm's output declines following entry. For any upgrade satisfying $P_i(Z^{*}_{i}) \geq C_i$, an individual firm's profits must fall since it is selling less at a lower price. This argument can be extended to cover upgrades for which $P_i(Z^{*}_{i}) < C_i$. Hence incumbents' profits fall following entry. Furthermore, the industry's profits must also fall. The industry is producing more than a monopolist would, and hence further output expansions must reduce industry profits. □

Proof of Proposition 3. We write Z^{*}_{i} and Z^{**}_{i} respectively for equilibrium industry output of upgrade i before and after a weak increase in upgrade costs. If the proposition is false, then we may find a sequence $\{j, \ldots, k\}$ for which $Z^{**}_{i} > Z^{*}_{i}$ for all $i \in \{j, \ldots, k\}$. We extend the sequence downward (redefining j as required) until either $j = 1$ or $Z^{**}_{j-1} \leq Z^{*}_{j-1}$. An argument similar to that used in the proof of Proposition 2 ensures that there is no upward constraint on Z^{*}_{j}. We extend the sequence upward (here redefining k as required) until either $k = n$ or $Z^{**}_{k+1} \leq Z^{*}_{k+1}$. Again, an argument similar to that used in the proof of Proposition 2 ensures that there is no downward constraint on Z^{**}_{k}.

Under the $(*)$-equilibrium each firm faces no upward constraint in market j, and so must face a weak incentive to simultaneously contract for all $i \in \{j, \ldots, k\}$. Under the $(**)$-equilibrium and for all such $i \in \{j, \ldots, k\}$, industry output is strictly higher, firm outputs are strictly higher and costs are weakly higher (since we have not assumed that an upgrade cost for some i within this sequence $\{j, \ldots, k\}$ has increased). Each firm must face a strict incentive to contract for all $i \in \{j, \ldots, k\}$. No firm does so, and hence there must be a binding downward constraint in market k. This is a contradiction. Hence we conclude that $Z^{**}_{i} \leq Z^{*}_{i}$ for each $i \in \{j, \ldots, k\}$. The rise in the price of each upgrade follows directly.

We now show that if an upgrade is in positive supply and its cost strictly increase, then its supply must strictly fall. Suppose not. We know that the supply of upgrades cannot strictly rise. Thus, if the claim is false, we must be able to find a sequence $\{j, \ldots, k\}$ for which $Z_i^{**} = Z_i^* > 0$ for all $i \in \{j, \ldots, k\}$ and for which costs are strictly higher for some $i \in \{j, \ldots, k\}$. Once again, we expand the sequence to ensure that there is no upward constraint on Z_j^* and downward constraint on Z_k^{**}. Under the $(*)$-equilibrium each firm faces a weak incentive to simultaneously contract for all $i \in \{j, \ldots, k\}$. Due to the strictly higher cost for some upgrade in this sequence, there must be a strict incentive to contract under the $(**)$-equilibrium. Once again, this yields a contradiction.

The final claim is that industry profits fall following a change in costs. We omit a complete proof for space considerations, but describe the process. Amalgamate products as necessary, and apply the implicit function theorem to compute the change in output following an increase in costs. Then, the change in profits is easily computed, for example using the envelope theorem. It is easily shown that our definition of decreasing marginal revenue precludes a profit increase. The effect of discrete changes in costs may be calculated by integrating. If the set of products being offered changes as costs do, the integral must be computed over different regions but the result is the same. □

Proof of Proposition 4. Suppose that the result is false. Take the lowest j for which $Z_{jr}^* < Z_{js}^*$. Since this is the lowest such j, then either $j = 1$ or $j > 1$ and $Z_{(j-1)r}^* \geq Z_{(j-1)s}^*$, which implies that $Z_{(j-1)r}^* > Z_{jr}^*$. For both of these cases, firm r faces no upward monotonicity constraint in market j. Next, take the highest k for which $Z_{ks}^* = Z_{js}^*$. Since this is the highest k with this property, and since $Z_{js}^* > Z_{jr}^* \geq 0$, firm s faces no downward monotonicity constraint in market k. Hence firm s has a weak incentive to raise simultaneously the supply of all upgrades $i \in \{j, \ldots, k\}$. Since $Z_{is}^* > Z_{ir}^*$ for all $i \in \{j, \ldots, k\}$, and firm r has weakly lower costs, firm r must have a strict incentive to raise simultaneously all of these outputs. Since this is feasible by construction, we have a contradiction. □

Proof of Proposition 5. We consider an increase in C_{1r}. (The proof for an increase in C_{2r} follows from similar logic.) We write Z_{is}^* and Z_{is}^{**} for equilibrium upgrade outputs before and after this increase, and consider an exhaustive list of possibile cases.

Case (i): $Z_1^{**} \geq Z_1^*$ and $Z_2^{**} \geq Z_2^*$. Suppose that $Z_{1s}^{**} > Z_{1s}^* \geq 0$ for some firm s. It must have a weak incentive to lower Z_{1s}^* and hence a strict incentive to lower Z_{1s}^{**}, implying $Z_{1s}^{**} = Z_{2s}^{**} > Z_{2s}^* \geq 0$. Now, firm s must have a weak incentive to simultaneously raise Z_{1s}^{**} and Z_{2s}^{**} and thus a strict incentive to raise Z_{1s}^* and Z_{2s}^*. It does not, and we have a contradiction. We conclude that $Z_{1s}^{**} \leq Z_{1s}^*$ for all s. If $Z_{1s}^{**} < Z_{1s}^*$, then this would imply that $Z_1^{**} < Z_1^*$, which contradicts the original assumption that $Z_1^{**} \geq Z_1^*$. We conclude that $Z_{1s}^{**} = Z_{1s}^*$ for all s. Similar arguments ensure that $Z_{2s}^{**} \leq Z_{2s}^*$ for all s. Combining this fact with the assumption that $Z_2^{**} \geq Z_2^*$, we conclude that $Z_{2s}^{**} = Z_{2s}^*$ for all s. This is specifically true for firm r, and additionally we know that $Z_{1r}^* > 0$ by assumption. Firm r has a weak incentive to lower Z_{1r}^*. Since C_{1r} is strictly higher in the $(**)$-equilibrium, it must have a strict incentive to lower Z_{1r}^{**}. This is a contradiction if $Z_{1r}^{**} > Z_{2r}^{**}$. Suppose instead that $Z_{1r}^{**} = Z_{2r}^{**} > 0$, so that $Z_{1r}^* = Z_{2r}^* > 0$. Firm r must have a weak incentive to simultaneously lower Z_{1r}^* and Z_{2r}^*. Once again, in the $(**)$-equilibrium, where it produces the same output but has strictly higher costs, it must have a strict incentive to engage in simultaneous contraction, and this final contradiction rules out case (i) as a possibility.

Case (ii): $Z_1^{**} \geq Z_1^*$ and $Z_2^{**} < Z_2^*$. We claim that $Z_{2s}^{**} - Z_{2s}^* \geq Z_{1s}^{**} - Z_{1s}^*$ for all s. Once this claim is established, we will show that it leads to a contradiction. Recalling that $Z_{1s}^* \geq Z_{2s}^*$ is always satisfied, we note that the claim is automatically true when $Z_{1s}^{**} = Z_{2s}^{**}$. If $Z_{1s}^{**} > Z_{1s}^*$, then the argument at the beginning of case (i) ensures that $Z_{1s}^{**} = Z_{2s}^{**}$. This means that the only situation in which the claim might fail is when $Z_{1s}^* \geq Z_{1s}^{**} > Z_{2s}^{**}$. Firm s has a weak incentive to lower Z_{2s}^{**}. Since $Z_2^* > Z_2^{**}$, if $Z_{2s}^* > Z_{2s}^{**} \geq 0$ then firm s would have a strict incentive to lower Z_{2s}^*. This would be a contradiction, and hence $Z_{2s}^{**} \geq Z_{2s}^*$ while $Z_{1s}^{**} \leq Z_{1s}^{**}$ and hence $Z_{2s}^{**} - Z_{2s}^* \geq Z_{1s}^{**} - Z_{1s}^*$. We have proven that $Z_{2s}^{**} - Z_{2s}^* \geq Z_{1s}^{**} - Z_{1s}^*$ for all s. But,

$$0 > Z_2^{**} - Z_2^* = \sum\nolimits_s (Z_{2s}^{**} - Z_{2s}^*) \geq \sum\nolimits_s (Z_{1s}^{**} - Z_{1s}^*) = Z_1^{**} - Z_1^* \geq 0.$$

The final inequality is the case (ii) assumption that $Z_1^{**} \geq Z_1^*$. The contradiction $0 > 0$ rules out case (ii) as a possibility.

We ruled out cases (i) and (ii), and must conclude that $Z_1^{**} < Z_1^*$. This proves the claim that the industry's output Z_1 is strictly decreasing in C_{1r} when $Z_{1r}^* > 0$. For cases (iii) and (iv) we now show that firm r's competitors weakly expand in both upgrade markets.

Case (iii): $Z_1^{**} < Z_1^*$ and $Z_2^{**} < Z_2^*$. We can employ the logic used for case (i) to show that $Z_{1s}^{**} \geq Z_{1s}^*$ and $Z_{2s}^{**} \geq Z_{2s}^*$ for any $s \neq r$, and hence $\sum_{s \neq r} (Z_{is}^{**} - Z_{is}^*) \geq 0$ for $i \in \{1, 2\}$.

Case (iv): $Z_1^{**} < Z_1^*$ and $Z_2^{**} \geq Z_2^*$ with $Z_{2r}^{**} \leq Z_{2r}^*$. We will show that $Z_{1s}^{**} - Z_{1s}^* \geq Z_{2s}^{**} - Z_{2s}^*$ for any firm $s \neq r$. If $Z_{1s}^* = Z_{2s}^*$ then, since $Z_{1s}^{**} \geq Z_{2s}^{**}$, this is automatically true. Suppose instead that $Z_{1s}^* > Z_{2s}^*$. Firm s has a weak incentive to raise Z_{1s}^*. If $Z_{1s}^{**} \leq Z_{1s}^*$ then, since $Z_1^{**} < Z_1^*$, it would have a strict incentive to raise Z_{1s}^{**}, a contradiction. Thus $Z_{1s}^{**} > Z_{1s}^*$. A similar argument ensures that $Z_{2s}^{**} \leq Z_{2s}^*$. This implies that $Z_{1s}^{**} - Z_{1s}^* \geq Z_{2s}^{**} - Z_{2s}^*$. Hence,

$$\sum_{s \neq r} (Z_{1s}^{**} - Z_{1s}^*) \geq \sum_{s \neq r} (Z_{2s}^{**} - Z_{2s}^*) \geq 0,$$

where the second inequality follows directly from $Z_2^{**} \geq Z_2^*$ with $Z_{2r}^{**} \leq Z_{2r}^*$.

For cases (iii) and (iv) we have shown that firm r's competitors weakly expand in both upgrade markets following an increase in C_{1r}. In the $(**)$-equilibrium firm r faces weakly more competition and strictly higher costs, and hence must be strictly worse off given that

$Z_{1r}^* > 0$. Furthermore, using similar logic to that employed before, it is straightforward to confirm that $Z_{1r}^{**} < Z_{1r}^*$ for both cases, and $Z_{2r}^{**} \leq Z_{2r}^*$ for case (iii). This completes the proof for (iii) and (iv); we need only eliminate case (v).

Case (v): $Z_1^{**} < Z_1^*$ and $Z_2^{**} \geq Z_2^*$ with $Z_{2r}^{**} > Z_{2r}^*$. As for case (iv), $Z_{1s}^{**} - Z_{1s}^* \geq Z_{2s}^{**} - Z_{2s}^*$ for any firm $s \neq r$. By assumption, $Z_2^{**} - Z_2^* \geq 0 > Z_1^{**} - Z_1^*$. Combining these observations, it must be the case that $Z_{1r}^{**} - Z_{1r}^* < Z_{2r}^{**} - Z_{2r}^*$. Since $Z_{1r}^{**} \geq Z_{2r}^{**}$, it must be the case that $Z_{1r}^* > Z_{2r}^*$. Firm r has a weak incentive to lower Z_{2r}^*, and since $Z_2^{**} \geq Z_2^*$ with $Z_{2r}^{**} > Z_{2r}^* \geq 0$ it must have a strict incentive to lower Z_{2r}^{**}. This is a contradiction. □

Proof of Proposition 6. This follows from the argument given in the text. □

Proof of Proposition 7. If all firms offer complete product lines then first-order conditions must hold in every upgrade market for every firm. In particular, for upgrade i and firm r we have $P_i(Z_i^*) + Z_{ir}^* P_i'(Z_i^*) = C_{ir}$. Equivalently, on dividing through by $q_i - q_{i-1}$,

$$\frac{c_r(q_i) - c_r(q_{i-1})}{q_i - q_{i-1}} = \frac{u(H(Z_i^*), q_i) - u(H(Z_i^*), q_{i-1})}{q_i - q_{i-1}} + \frac{Z_{ir}^* H'(Z_i^*)}{q_i - q_{i-1}} \left[\frac{\partial u(H(Z_i^*), q_i)}{\partial \theta} - \frac{\partial u(H(Z_i^*), q_{i-1})}{\partial \theta} \right].$$

Exploiting uniform continuity, so long as $q_i - q_{i-1}$ is sufficiently small we can ensure that Z_i^* is close to Z_q^* and Z_{ir}^* is close to Z_{qr}^* for each r, where $q = q_i$ and where Z_{qr}^* and Z_q^* satisfy

$$c_r'(q) = \frac{\partial u(H(Z_q^*), q)}{\partial q} + Z_{qr}^* H'(Z_q^*) \frac{\partial^2 u(H(Z_q^*), q)}{\partial \theta \partial q}.$$

When payoffs are multiplicative, this simplifies to

$$H(Z_q^*) + Z_{qr}^* H'(Z_q^*) = c_r'(q) \quad \Rightarrow \quad NH(Z_q^*) + Z_q^* H'(Z_q^*) = \sum\nolimits_s c_s'(q).$$

Differentiating with respect to q, we obtain

$$\frac{\partial Z_{qr}^*}{\partial q} = \frac{c_r''(q)}{H'(Z_q^*)} - \frac{\partial Z_q^*}{\partial q} \left[1 + \frac{Z_{qr}^* H''(Z_q^*))}{H'(Z_q^*)} \right] \quad \text{and} \quad \frac{\partial Z_q^*}{\partial q} = \frac{\sum_s c_s''(q)}{(N+1)H'(Z_q^*) + Z_q^* H''(Z_q^*)}.$$

Combining these two expressions, and recalling that $H'(Z_q^*) < 0$, we obtain

$$\frac{\partial Z_{qr}^*}{\partial q} \leq 0 \quad \Leftrightarrow \quad \frac{c_r''(q)}{\sum_s c_s''(q)} \geq \frac{H'(Z_q^*) + Z_{qr}^* H''(Z_q^*)}{(N+1)H'(Z_q^*) + Z_q^* H''(Z_q^*)} = \frac{1 - \lambda_{qr} \eta(Z_q^*)}{N + 1 - \eta(Z_q^*)},$$

where $\lambda_{qr} = Z_{qr}^* / Z_q^*$ and $\eta(Z) = -ZH''(Z)/H'(Z)$. □

Lemmas 3 and 4 are used in the proof of Proposition 8.

Lemma 3. *Suppose that $m_s \geq m_r \geq k$. If firm s produces product k then firm r will also product product k. Moreover, $Z_{js}^* = Z_{jr}^*$ for all $j \leq k$, and hence $z_{js}^* = z_{jr}^*$ for all $j < k$.*

Proof. From Proposition 4, $Z^*_{is} \geq Z^*_{ir}$ for all i. Consider a product $k \leq m_r$ that is sold by s but not by r, so that $Z^*_{ks} > Z^*_{(k+1)s}$ and $Z^*_{kr} = Z^*_{(k+1)r}$, implying that $Z^*_{ks} > Z^*_{kr}$. Observe that firm s faces no downward monotonicity constraint in market k, and hence is able to simultaneously lower its supplies for all $i \in \{1, \ldots, k\}$. It does not, and so must have at least a weak incentive to raise them. Firm r faces the same costs in these markets, and produces weakly less. It produces strictly less in market k. It must have, therefore, a strict incentive to simultaneously expand for all $i \in \{1, \ldots, k\}$. This is a contradiction.

To prove the second claim, consider the highest j satisfying $j \leq k$ and $Z^*_{js} > Z^*_{jr}$. We can mimic the argument used to prove the first claim: firm s has a weak incentive to expand for all $i \in \{1, \ldots, j\}$, and hence firm r must have a strict incentive, which yields a contradiction.

The third claim follows directly from the second. □

Lemma 4. *If firm r's product line has a gap from j to k that does not include the maximum feasible quality of another firm, then industry output for $i \in \{j, \ldots, k+1\}$ is the same.*

Proof. If firm r has such a gap then $Z^*_{jr} = \cdots = Z^*_{(k+1)r} > Z^*_{(k+2)r} \geq 0$. Notice that firm r produces product $k+1$ and hence $m_r \geq k+1$. We will show that any other firm s satisfies $Z^*_{js} = \cdots = Z^*_{(k+1)s}$. By assumption, $m_s \notin \{j, \ldots, k\}$. If $m_s < j$ then $Z^*_{is} = 0$ for all $i \in \{j, \ldots, k+1\}$, and hence $Z^*_{js} = \cdots = Z^*_{(k+1)s}$. If $k < m_s \leq m_r$ then, since firm r produces product $k+1$, so must firm s (from Lemma 3). This means (again from Lemma 3) that $Z^*_{ir} = Z^*_{is}$ for all $i \leq k+1$, and once again $Z^*_{js} = \cdots = Z^*_{(k+1)s}$. Finally, suppose that $m_s > m_r$. If firm s produced a product $i \in \{j, \ldots, k\}$, then firm r would also do so following Lemma 3. This would yield a contradiction. Hence it must be the case, once again, that $Z^*_{js} = \cdots = Z^*_{(k+1)s}$. We have considered all cases, and conclude that $Z^*_j = \cdots = Z^*_{k+1}$. □

Proof of Proposition 8. We begin with claim (1). If firm r has a gap in its product line from j to k then either $j = 1$ or $Z^*_{(j-1)r} > Z^*_{jr}$. For both of these cases, there is no upward constraint on Z^*_{jr}. From the definition of a gap, firm r must sell product $k+1$, so that either $k+1 = m_r$ and $Z^*_{(k+1)r} > 0$, or $k+1 < m_r$ and $Z^*_{(k+1)r} > Z^*_{(k+2)r}$. Since its downward monotonicity constraint in market $k+1$ does not bind, firm r must have a weak incentive to raise $Z_{(k+1)r}$. Hence, omitting the dependencies of $P_i(Z^*_i)$ and $\eta_i(Z^*_i)$ on Z^*_i,

$$1 \leq \frac{P_{k+1}(Z^*_{k+1})}{C_{k+1}} \left[1 - \frac{\eta_{k+1}(Z^*_{k+1}) Z^*_{(k+1)r}}{Z^*_{k+1}}\right] < \frac{P_j(Z^*_{k+1})}{C_j} \left[1 - \frac{\eta_j(Z^*_{k+1}) Z^*_{(k+1)r}}{Z^*_{k+1}}\right] = \frac{P_j(Z^*_j)}{C_j} \left[1 - \frac{\eta_j(Z^*_j) Z^*_{jr}}{Z^*_j}\right]. \quad (13)$$

The strict inequality in (13) stems from the assumptions that (i) returns to quality and price sensitivity are both weakly decreasing, and (ii) at least one of them is strictly decreasing.

The equality emerges from the fact that $Z^*_{jr} = Z^*_{(k+1)r}$ (which is true by assumption, since firm r has a gap from j to k) and $Z^*_j = Z^*_{k+1}$ (which is a consequence of Lemma 4). This means that firm r has a strict incentive to expand output in upgrade market j. Since it does not, it must face a binding upward constraint on Z^*_{jr}. But this is a contradiction. This establishes claim (1).

Claim (2) now follows as a simple corollary. If firm r's product line consisted of more than $\bar{r}+1$ quality ranges, then it would also have at least $\bar{r}$ gaps. A gap must contain the maximum quality of a less-able rival, and there are at most $\bar{r}$ such distinct qualities.

Turning to claim (3), we begin by fixing $\varepsilon > 0$, and take a firm r satisfying $m_r = i$ and a second firm s satisfying $m_s > i$. We also construct a strictly decreasing sequence $\{\delta_t\}$ for each positive integer t satisfying $\delta_t > 0$ and $\lim_{t\to\infty} \delta_t = 0$. If the claim is false, then for each t we may find a pattern of qualities and equilibrium upgrade outputs, labelled by t, such that $Z^{*(t)}_{ir} \geq \varepsilon$, $Z^{*(t)}_{is} > Z^{*(t)}_{(i+1)s}$, $q^t_i - q^t_{i-1} < \delta_t$, and $q^t_{i+1} - q^t_i < \delta_t$.

We may restrict attention to upgrade outputs and qualities drawn from a compact set. Since there is a finite number of firms, and the upgrade choices for each firm lie in the same compact set at each point t in the sequence defined above, there is a subsequence of the original sequence such that the upgrade outputs and qualities converge. We restrict attention to this subsequence and write $q_j = \lim_{t\to\infty} q^t_j$, $Z^*_{is} = \lim_{t\to\infty} Z^{*(t)}_{is}$, and $Z^*_i = \lim_{t\to\infty} Z^{*(t)}_i$.

$Z^{*(t)}_{is} > Z^{*(t)}_{(i+1)s}$ for each t, so that firm s produces product i. Firm s faces no downward constraint on $Z^{*(t)}_{is}$, and hence must have a weak incentive to expand output of that upgrade:

$$\frac{c_s(q^t_i) - c_s(q^t_{i-1})}{q^t_i - q^t_{i-1}} \leq \frac{P_i(Z^{*(t)}_i) + Z^{*(t)}_{is} P'_i(Z^{*(t)}_i)}{q^t_i - q^t_{i-1}} = \frac{u(H(Z^{*(t)}_i), q^t_i) - u(H(Z^{*(t)}_i), q^t_{i-1})}{q^t_i - q^t_{i-1}} + \frac{Z^{*(t)}_{is} H'(Z^{*(t)}_i)}{q^t_i - q^t_{i-1}} \left[\frac{\partial u(H(Z^{*(t)}_i), q^t_i)}{\partial \theta} - \frac{\partial u(H(Z^{*(t)}_i), q^t_{i-1})}{\partial \theta} \right].$$

Taking $t \to \infty$, and exploiting continuity, we obtain

$$c'_s(q_i) \leq \frac{\partial u(H(Z^*_i), q_i)}{\partial q} + Z^*_{is} H'(Z^*_i) \frac{\partial^2 u(H(Z^*_i), q_i)}{\partial \theta \partial q}.$$

Turning to upgrade market $i+1$, since $Z^{*(t)}_{is} > Z^{*(t)}_{(i+1)s}$ for each t, firm s faces no upward constraint on $Z^{*(t)}_{(i+1)s}$, and hence must have a weak incentive to contract output of that upgrade. Taking $t \to \infty$ as before,

$$c'_s(q_i) \geq \frac{\partial u(H(Z^*_{i+1}), q_i)}{\partial q} + Z^*_{(i+1)s} H'(Z^*_{i+1}) \frac{\partial^2 u(H(Z^*_{i+1}), q_i)}{\partial \theta \partial q} \geq \frac{\partial u(H(Z^*_{i+1}), q_i)}{\partial q} + Z^*_{is} H'(Z^*_{i+1}) \frac{\partial^2 u(H(Z^*_{i+1}), q_i)}{\partial \theta \partial q}.$$

Combining inequalities, we obtain

$$\frac{\partial u(H(Z^*_{i+1}), q_i)}{\partial q} + Z^*_{is} H'(Z^*_{i+1}) \frac{\partial^2 u(H(Z^*_{i+1}), q_i)}{\partial \theta \partial q} \leq \frac{\partial u(H(Z^*_i), q_i)}{\partial q} + Z^*_{is} H'(Z^*_i) \frac{\partial^2 u(H(Z^*_i), q_i)}{\partial \theta \partial q}.$$

Since $Z^*_i - Z^*_{i+1} \geq Z^*_{ir} - Z^*_{(i+1)r} \geq \varepsilon > 0$, this contradicts the maintained assumption that marginal revenue is strictly decreasing, and we have reached a contradiction. □

Proof of Proposition 9. We follow the approach used for claim (3) of Proposition 8. We construct a sequence $\{\delta_t\}$ satisfying $\delta_t > 0$ and $\lim_{t\to\infty} \delta_t = 0$. If the result is false, then we may find qualities and equilibrium outputs for each t such that $m^{*(t)}_r = m^{*(t)}_s = i < M$ for a pair of firms r and s, $q^t_i - q^t_{i-1} < \delta_t$, and $q^t_{i+1} - q^t_i < \delta_t$. As in the proof of Proposition 8, we may restrict attention to a convergent subsequence. Having done so, we write $q_j = \lim_{t\to\infty} q^t_j$, $Z^*_{jr} = \lim_{t\to\infty} Z^{*(t)}_{jr}$, $Z^*_{js} = \lim_{t\to\infty} Z^{*(t)}_{js}$, and $Z^*_j = \lim_{t\to\infty} Z^{*(t)}_j$.

By construction, $Z^{*(t)}_{(i+1)r} = Z^{*(t)}_{(i+1)s} = 0$, and hence $Z^{*(t)}_i - Z^{*(t)}_{i+1} \geq Z^{*(t)}_{ir} + Z^{*(t)}_{is}$. This holds in the limit as $t \to \infty$, and hence $Z^*_i - Z^*_{i+1} \geq Z^*_{ir} + Z^*_{is}$, or equivalently

$$Z^*_i \geq Z^*_{i+1} + Z^*_{ir} + Z^*_{is}. \tag{14}$$

The sequence of steps that we perform just below will lead to a contradiction of (14).

For each t in the sequence, firms r and s choose the same capabilities: $m^t_r = m^t_s = i$. One possible deviation would be for firm r to switch to $m_r = i-1$ and hence $Z_{ir} = 0$, while retaining its supplies of all other upgrades $j \leq i-1$. Since such a deviation is not profitable,

$$Z^{*(t)}_{ir} \left[\frac{P_i(Z^{*(t)}_i)}{q^t_i - q^t_{i-1}} - \frac{C_{ir}}{q^t_i - q^t_{i-1}} \right] \geq \frac{F_r(q^t_i) - F_r(q^t_{i-1})}{q^t_i - q^t_{i-1}}.$$

The left-hand side of the inequality is the loss in profits from exiting upgrade market i, while the right-hand side is the associated fixed cost that would be saved. Taking $t \to \infty$,

$$Z^*_{ir} \left[\frac{\partial u(H(Z^*_i), q_i)}{\partial q} - c'_r(q_i) \right] \geq F'_r(q_i). \tag{15}$$

Given that $F'_r(\cdot) > 0$ by assumption, (15) implies that $Z^*_{ir} > 0$. We can make an identical argument for firm s, and establish that $Z^*_{is} > 0$.

A second possible deviation would be for firm r to switch to $m_r = i+1$ and set $Z_{(i+1)r} = Z^{*(t)}_{ir}$, while retaining its supplies of all other upgrades $j \leq i$. Since such a deviation is not profitable,

$$Z^{*(t)}_{ir} \left[\frac{P_{i+1}(Z^{*(t)}_{i+1} + Z^{*(t)}_{ir})}{q^t_{i+1} - q^t_i} - \frac{C_{(i+1)r}}{q^t_{i+1} - q^t_i} \right] \geq \frac{F_r(q^t_{i+1}) - F_r(q^t_i)}{q^t_{i+1} - q^t_i}.$$

Taking the limit as $t \to \infty$ once again, we obtain

$$Z^*_{ir} \left[\frac{\partial u(H(Z^*_{i+1} + Z^*_{ir}), q_i)}{\partial q} - c'_r(q_i) \right] \leq F'_r(q_i). \tag{16}$$

Combining (15) and (16), and recalling that $Z^*_{ir} > 0$, we obtain

$$\frac{u(H(Z^*_{i+1} + Z^*_{ir}), q_i)}{\partial q} \leq \frac{u(H(Z^*_i), q_i)}{\partial q} \quad \Rightarrow \quad Z^*_{i+1} + Z^*_{ir} \geq Z^*_i,$$

where the implication follows from the sorting condition imposed on $u(\theta, q)$ and the fact that $H(Z)$ is strictly decreasing. Following (15) we established that $Z^*_{is} > 0$, and hence $Z^*_{i+1} + Z^*_{ir} + Z^*_{is} > Z^*_i$. But this last strict inequality contradicts (14). □

Proof of Proposition 10. Suppose that firm r is considering whether to commit to omitting the high-quality product from its product line. For such a restriction to be relevant, it must be the case that $Z^*_{2r} > 0$ initially. A commitment not to produce is equivalent to imposing a sufficiently high cost c_{2r} of producing quality q_2. In turn, this is equivalent to imposing a sufficiently high cost C_{2r} of producing the upgrade $q_2 - q_1$. Proposition 5 demonstrates that such an increase will strictly harm the profits of firm r. The only subtlety is that in Proposition 5 no other firm s has restricted itself to sell only the high-quality or only the low-quality product. But such restrictions would correspond to setting either c_{1s} or c_{2s} to sufficiently high levels; these possibilities are accommodated by Proposition 5. □

Proof of Proposition 11. As discussed in the text, a commitment by firm r only to sell quality q_2 is equivalent to adding $\kappa > 0$ to c_{1r} where κ is sufficiently large. Equivalently, firm r operates with upgrade costs of $C_{1r} + \kappa$ and $C_{2r} - \kappa$. Beginning from $\kappa = 0$, all firms sell both products, and hence first order conditions are satisfied in both upgrade markets. A marginal increase in κ results in an expansion in Z^*_{1s} and a contraction in Z^*_{2s} for each $s \neq r$. This ensures that we may safely ignore the monotonicity constraints of firms $s \neq r$ as we increase κ. It is straightforward to confirm that Z^*_1 and Z^*_{1r} shrink while Z^*_2 and Z^*_{2r} expand as κ increases. We keep increasing κ until the constraint $Z^*_{1r} \geq Z^*_{2r}$ binds. □

Proof of Proposition 12. We will show that, no matter what the decisions of others, firm r will never wish to restrict itself in the first stage. As noted in the text, one way of restricting itself to producing only the high quality would be for firm r to increase C_{1r} by κ while reducing C_{2r} by κ, where $\kappa > 0$ is chosen to be sufficiently large. We will show the stronger result that any increase in κ harms firm r so long as $Z^*_{1r} > Z^*_{2r}$ initially. Consider any second-stage equilibrium. For each firm s, summing over $i \in \{1, 2\}$ gives:

$$\sum\nolimits_{i=1}^{2} [(q_i - q_{i-1})(1 - Z^*_i - Z^*_{is})] = C_{1s} + C_{2s} = c_{2s} \tag{17}$$

This is just the first-order condition for firm s if $Z^*_{1s} = Z^*_{2s}$, or the sum of its two first-order conditions if $Z^*_{1s} > Z^*_{2s}$. Summing (17) over s and re-arranging, we obtain

$$\sum\nolimits_{i=1}^{2} (q_i - q_{i-1}) Z^*_i = \frac{N q_2}{N+1} - \frac{\sum_s c_{2s}}{N+1} \tag{18}$$

Note that (18) does not depend on c_{1s} for any s, and in particular c_{1r} for firm r. It follows that whatever firm r does, $\sum_{i=1}^{2}(q_i - q_{i-1})Z_i^*$ is identical across each second-stage equilibrium.

Compare the second-stage equilibrium in which no firm has restricted itself in the first stage to that in which a single firm, say firm 1, has restricted itself. Using arguments similar to those in other proofs, the effect is a strict decrease in Z_1^* and a strict increase in Z_2^*. For each firm $s \neq 1$, Z_{1s}^* has increased and Z_{2s}^* has decreased, so that it is still the case that $Z_{1s}^* > Z_{2s}^*$. Now consider also restricting firm $s = 2$ to offer only the high quality product. An inspection of firm 1's first-order condition reveals that firm 1 does not change its equilibrium output since $\sum_{i=1}^{2}(q_i - q_{i-1})Z_i^*$ is the same. From this it can be argued again that Z_1^* has decreased, Z_2^* has increased, and firms $s > 2$ are still choosing $Z_{1s}^* > Z_{2s}^*$. Continuing iteratively, we conclude that whenever firm r has not restricted itself, it must be choosing $Z_{1r}^* > Z_{2r}^*$.

Now we allow firm r to restrict itself by changing its two upgrade costs as described in earlier. Consider a marginal increase in κ. Following the arguments used in earlier proofs, this leads to a decrease in Z_1^* and an increase in Z_2^*, where these are strict changes unless κ is large enough that firm r is setting $Z_{1r}^* = Z_{2r}^*$ in which case any further increase in κ has no effect. Now, as argued just above, any firm $s \neq r$ that has restricted itself does not change its equilibrium outputs. For any unrestricted firm $s \neq r$, we know that $Z_{1s}^* > Z_{2s}^*$ and so this firm's two first-order conditions hold. Manipulating them we see

$$\frac{\partial Z_{is}^*}{\partial \kappa} = -\frac{\partial Z_i^*}{\partial \kappa} \quad \text{for} \quad i \in \{1, 2\}.$$

We now compute the equilibrium change on firm r's profits from a small increase in κ (in a region where firm r chooses $Z_{1r}^* > Z_{2r}^*$). Denote by S the set of firms $s \neq r$ that are not restricted in this subgame, and by $|S|$ the number of such firms in S. Then

$$\begin{aligned}
\frac{d\pi_r(\kappa)}{d\kappa} =& (Z_{2r}^* - Z_{1r}^*) + Z_{1r}^* P_1'(Z_1^*) \sum_{s \neq r} \frac{\partial Z_{1s}^*}{\partial \kappa} + Z_{2r}^* P_2'(Z_2^*) \sum_{s \neq r} \frac{\partial Z_{2s}^*}{\partial \kappa} \\
=& (Z_{2r}^* - Z_{1r}^*) + |S| \left[Z_{1r}^*(q_1 - q_0) \frac{\partial Z_1^*}{\partial \kappa} + Z_{2r}^*(q_2 - q_1) \frac{\partial Z_2^*}{\partial \kappa} \right] \\
=& (Z_{2r}^* - Z_{1r}^*) + |S|(q_1 - q_0) \frac{\partial Z_1^*}{\partial \kappa} (Z_{1r}^* - Z_{2r}^*) < Z_{2r}^* - Z_{1r}^* < 0.
\end{aligned}$$

The third equality is implied by the fact that $\sum_{i=1}^{2}(q_i - q_{i-1})Z_i^*$ is constant as κ changes. This implies, in turn, that $(q_1 - q_0)(\partial Z_1^*/\partial \kappa) = -(q_2 - q_1)(\partial Z_2^*/\partial \kappa)$. The penultimate inequality follows from $Z_{1r}^* > Z_{2r}^*$ and $\partial Z_1^*/\partial \kappa < 0$. Hence an increase in κ strictly harms firm r. It follows that firm r is worse off in each possible subgame, when it restricts itself. □

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