Kinks in a model for two-phase lipid bilayer membranes

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A thesis submitted for the degree of
Doctor of Philosophy in Mathematics

Trinity 2011

Abstract

In the spontaneous curvature model for two-phase lipid bilayer membranes the shape of vesicles is governed by a combination of an elastic bending energy and an interface energy that penalises the size of phase boundaries. Each lipid phase induces a preferred curvature to the membrane surface, and these curvatures as well as phase boundaries may lead to the development of kinks.

In a rotationally symmetric setting we introduce a family of energies for smooth surfaces and phase fields for the lipid components and study convergence to a sharp-interface limit, which depends on the choice of the bending parameters of the phase field model. We prove that, if kinks are excluded, our energies Γ-converge to the commonly used sharp-interface spontaneous curvature energy with the additional assumption of C^1-regularity across interfaces. For a choice of parameters such that kinks may appear, we obtain a limit that coincides with the Γ-limit on all reasonable membranes and extends the classical model by assigning a bending energy also to kinks.

We illustrate the theoretical result by some numerical examples.

Acknowledgements

On my path to this dissertation I have enjoyed the company and support of several individuals and institutions.

First and foremost, I am indebted to my supervisor Barbara Niethammer. When I was an undergraduate she introduced me to nonlinear analysis and partial differential equations, later to phase field models and bilayer membranes. Since then she has offered advice in many discussions, and her guidance, encouragement and patience have always been of invaluable help to me. I am obliged to my colleague and friend Michael Herrmann whose support has also been fundamental. Besides his mathematical suggestions and his constant encouragement, I have enjoyed our many non-scientific conversations. I also thank him and Barbara Niethammer for reading several drafts of the dissertation and helping to improve the exposition to a great extent.

Other colleagues have contributed to my research through discussions and by pointing out new ideas or connections to other results. I am grateful for their support and in particular like to mention Matthias Röger, Andreas Rätz, Jan Kristensen and Endre Süli.

My life in Oxford would have been less enjoyable without my fellow students at the Oxford Centre for Nonlinear PDE (OxPDE), especially without my office mates and friends Filip Rindler and Parth Soneji. I am grateful that I have had the opportunity to meet them and thank OxPDE, the Engineering and Physical Sciences Research Council, and the MULTIMAT research and training network for their financial support.

Finally, I thank my family and friends in Germany who have always supported me during this project that has now come to an end.

Contents

List of symbols

The following list contains notation that is used throughout the dissertation. The number at the end of each entry is the page where the notation is defined or introduced.

$\mathcal{A}_\gamma, \mathcal{A}_\gamma(J)$	Area of M_γ or $M_\gamma(J)$. 8
$\lvert B\rvert$	Norm of the second fundamental form, $\lvert B\rvert = \sqrt{\kappa_1^2 + \kappa_2^2}$. 9
$\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$	Set of curves and phase fields for the approximate energies. 15
$\mathcal{C} \times \mathcal{P}$	Set of limit curves and phase fields for the regular approximation. 17
$C^{0,1}$	Space of Lipschitz continuous functions.
$\mathcal{D} \times \mathcal{Q}$	Set of limit curves and phase fields for the singular approximation. 37
$\mathcal{E}$	Limit energy in the regular setting. 17
$\mathcal{E}_\varepsilon$	Approximate energy in the regular setting. 17
$\mathcal{F}$	Limit energy in the singular setting. 39
$\mathcal{F}_\varepsilon$	Approximate energy in the singular setting. 33
$\mathcal{H}$	Helfrich energy in the limit of the singular approximation. 39
$\mathcal{H}_\varepsilon$	Helfrich energy in the singular approximation. 34
H	Mean curvature of a surface, $H = \kappa_1 + \kappa_2$. 9
$\mathcal{H}^k$	k-dimensional Hausdorff measure. 8
$\mathcal{I}$	Interface energy in the limit of the singular approximation. 39
$\mathcal{I}_\varepsilon$	Interface energy in the singular approximation. 34
K	Gauss curvature of a surface, $K = \kappa_1 \kappa_2$. 9
κ_1, κ_2	Principal curvatures of a surface; for a surface of revolution κ_1 is the curvature of the generating curve. 9
$\mathcal{L}_\gamma, \mathcal{L}_\gamma(J)$	Length of the curve γ or the segment $\gamma(J)$. 7
$M_\gamma, M_\gamma(J)$	Surface of revolution generated by the curve γ or the segment $\gamma(J)$. 7
μ	Area measure of a surface. 8
q_γ	Velocity $q_\gamma = \lvert\gamma'\rvert$ of the constant speed curve γ. 8
S_γ	Set of tangent discontinuities in $\{y > 0\}$ of $\gamma = (x, y) \in \mathcal{D}$. 37
S_u	Jump set in $\{y > 0\}$ of $u \in \mathcal{P}$ or $u \in \mathcal{Q}$ associated to $\gamma = (x, y)$; $s \in S_u$ is an interface point of (γ, u). 17
$W^{k,p}, W^{k,p}_{\mathrm{loc}}$	Sobolev space of k-times weakly differentiable functions with derivatives in L^p, L^p_{loc}.
$\omega, M_\gamma(\omega)$	Component of $\{y > 0\}$ or M_γ generated by $\gamma = (x, y)$. 11

Chapter 1

Introduction

Lipid bilayers are an integral part of biological cells, enclosing for instance the nucleus or separating the interior of the cell from the outside environment. Their mechanical properties influence not only the structure of the cell but also its functions [Boa02].

The building block of a bilayer is a lipid molecule that consists of a hydrophilic head and two hydrophobic tails. In an aqueous environment these molecules aggregate into two layers, as schematically shown in Figure 1.1, thereby shielding the tail from the water. Closed vesicles are formed due to unfavourable contact between the tails and the water at open ends of the bilayer. There is no covalent bonding between lipid molecules, the bilayer structure is solely caused by the hydrophobic effect.

Bilayer vesicles appear in a rich variety of shapes and are able to undergo shape transformations, including bud formation and vesicle fission [SBL91, DKN$^+$93]; two examples are shown in Figure 1.2. Membranes that consist of two or more lipid phases display an even more complex morphology such as micro-domains resembling lipid rafts, see for instance [BHW03, BDWJ05] and Figure 1.3 for experimental measurements. Phase separation in

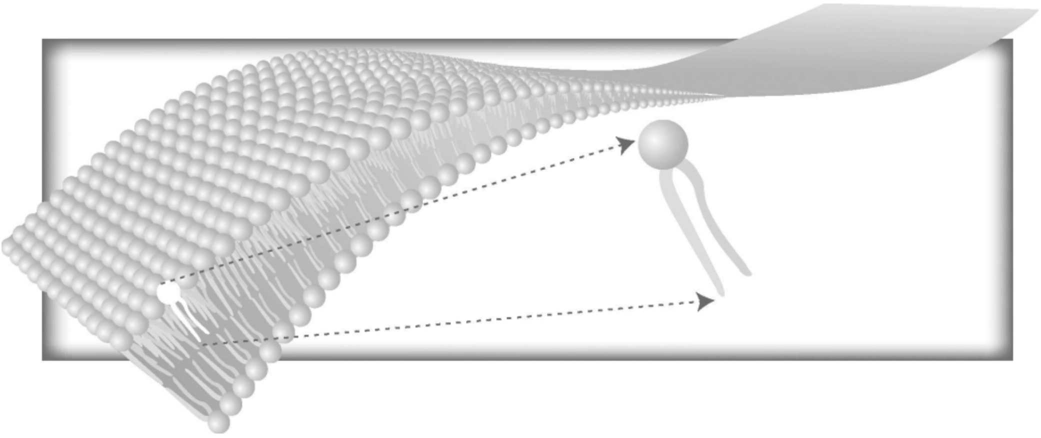

Figure 1.1: Lipid molecules in a bilayer structure. Reproduced from [PR09].
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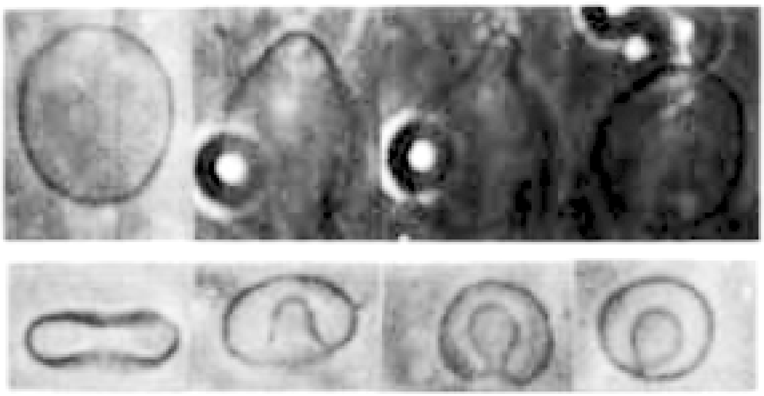

Figure 1.2: Budding transition (top) and discocyte-stomatocyte transition (bottom), both induced by raising temperature. Reproduced from [BKL$^+$90].
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such bilayers affects equilibrium shapes and shape transformations of the vesicles.

1.1 Spontaneous curvature model for lipid bilayers

In [Can70, Hel73, Eva74] Canham, Evans and Helfrich independently introduced energy-based methods to study the shapes of single-phase bilayers. The two layers of lipid molecules are represented by a single reference surface, and stable equilibrium shapes are described as closed two-dimensional surfaces minimising the bending energy

$$\int_M k\,(H - H_s)^2 + k_G K\, d\mu \tag{1.1}$$

among surfaces with prescribed area (and enclosed volume). Here H and K are the mean and Gauss curvature of the membrane surface M, and μ is its area measure; the bending rigidity k and the Gauss bending rigidity k_G are elastic material parameters; and H_s, the so-called spontaneous or preferred curvature, is supposed to reflect an asymmetry in the membrane introduced for instance by a different chemical environment on both sides of the bilayer or a different chemical composition of the two monolayers. The energy (1.1) is by now well-known by the name Helfrich energy.

In [JL93, SL95] this model was extended to vesicles made of two lipid phases $M = M^+ \cup M^- \cup \partial M^+$, $M^+ \cap M^- = \emptyset$ using an energy

$$\sum_{j=\pm} \int_{M^j} k^j (H - H_s^j)^2 + k_G^j K\, d\mu + \sigma \mathcal{H}^1(\partial M^+) \tag{1.2}$$

that consists of a bending term (1.1) for each component and an additional interface term $\sigma \mathcal{H}^1(\partial M^+)$ with line tension σ. The elastic parameters $k^\pm$ and $k_G^\pm$ and the spontaneous curvatures $H_s^\pm$ are constant within each lipid phase but in general different between the two phases. The area constraint translates into a constraint for each lipid phase.

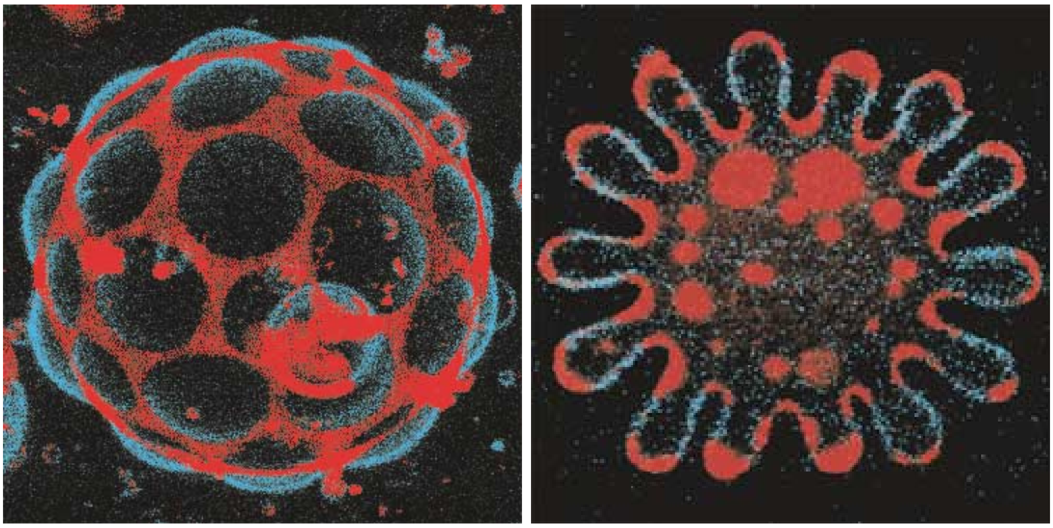

Figure 1.3: Hemispherical projection of image stacks (left) and equatorial section (right) of two-phase membranes from biological experiments. Reproduced from [BHW03].
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Studies of stationary shapes are usually based on solving the Euler-Lagrange equations of (1.1) or (1.2). Seifert et al. and Jülicher, Lipowsky investigate axially symmetric shapes for single-phase vesicles [SBL91] and for completely separated two-phase membranes [JL93, JL96]. In the latter case the authors briefly discuss the possibility of different smoothness conditions for the rotated curve at the interface, their analysis, however, is done for smooth membranes only.

In recent years, phase field approaches have been successfully applied in numerical simulations of membranes. For instance, Du, Liu, Wang [DLW04, WD08] and Lowengrub, Rätz, Voigt [LRV09] use diffuse interfaces for both the membrane surface and the lipid phases; Elliot and Stinner [ES10a, ES10b] consider a surface phase field model. Convergence to the sharp interface limit in these approaches is usually obtained by asymptotic analysis or under strong smoothness assumptions on the limit surface, which can not be justified from the energy.

Few analytical results for membrane models are known. For instance, Bellettini and Mugnai [BM10] propose a diffuse interface approximation for a single-phase membrane that slightly differs from the one used by the authors above, and they prove that under certain restrictions on the parameters its Γ-limit is given by the Helfrich energy on smooth limits. Interestingly, already in 1991 De Giorgi conjectured Γ-convergence of similar diffuse interface models to Helfrich type energies [DG91]. To our knowledge there is no analytical approach to multi-phase membranes that does not rely on strong regularity assumptions.

1.2 Diffuse interfaces without a priori smoothness

The aforementioned studies of two-phase membranes rely on the assumption that the membrane is smooth, at least C^1, across lipid interfaces. This assumption, however, is not justified from the energy (1.2). In this work we are interested in rotationally symmetric two-phase membranes, which are described by surfaces of revolution and rotationally symmetric lipid phases, without this a priori smoothness assumption. We study approximations by regular surfaces of revolution and phase fields for the lipid phases with the aim of obtaining two limits with different regularity properties at interfaces.

For a surface M_γ obtained by rotating a curve γ and an associated phase field $u\colon M_\gamma \to \mathbb{R}$ we consider an approximate energy of the form

$$\int_{M_\gamma} k(u,\varepsilon)\,(H - H_s(u))^2 + k_G(u,\varepsilon) K \, d\mu + \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u)\, d\mu, \tag{1.3}$$

where $H_s\colon \mathbb{R} \to \mathbb{R}$ is an extension of the preferred curvatures $H_s^\pm$ of each lipid phase and $W\colon \mathbb{R} \to [0,\infty)$ is a double-well potential such as $W(u) = (1-u^2)^2$.

Apart from the surface setting, the second integral in (1.3) is the usual diffuse interface energy derived from the Cahn-Hilliard theory of phase transitions [CH58]. Intuitively, if (u_ε) is a sequence with bounded interface energy as $\varepsilon \to 0$, the potential term forces the phase fields to ± 1, while the gradient term penalises transitions between these values. Both terms together fix the size of each transition layer of u_ε to be of order ε. In [MM77, Mod87] Modica and Mortola show that in the non-surface setting the Γ-limit as $\varepsilon \to 0$ of the diffuse interface energies is given by the size of the sharp interface times the tension $\sigma = 2\int_{-1}^{1} \sqrt{W(u)}\, du$.

The first integral in (1.3) accounts for the bending energy of the membrane. The parameter functions k and k_G have a twofold meaning: on the one hand they are extensions of the bending rigidities, on the other hand their precise dependence on u and ε determines the properties of the limit. It is reasonable to assume that $k^\pm > 0$ and that k_G^+ and k_G^- have the same sign. Then the extensions of these rigidities can be realised by continuous functions that are uniformly bounded away from zero, and the precise values of $k^\pm$ will not enter our arguments; we will thus assume that $k^+ = k^- = 1$. The relation between k and k_G is more subtle and discussed in Chapter 4.

We study the limit as $\varepsilon \to 0$ of (1.3) in two cases. When $k \equiv 1$ and for simplicity $k_G \equiv 0$, sequences $(M_{\gamma_\varepsilon}, u_\varepsilon)$ with uniformly bounded energy develop tangent singularities in the limit at the axis of revolution only. Limit surfaces hence consist of a finite number of regular topological spheres connected at points on the axis of revolution; in particular, the surfaces are smooth across interfaces. We refer to this case as the regular approximation.

In order that tangent discontinuities or kinks can appear in the limit, k and k_G have to be close to zero near phase transitions, that is when $u_\varepsilon \approx 0$. Motivated by the Ambrosio-Tortorelli approximation of free-discontinuity problems [AT90, Bra98], the choice $k \sim k_G \sim u^2$ seems reasonable. However, we will see in Chapter 4 that this choice is not sufficient to obtain compactness of uniformly energy-bounded sequences $(M_{\gamma_\varepsilon}, u_\varepsilon)$, because the first variations of the surfaces M_{γ_ε} are not necessarily uniformly bounded. We therefore introduce an additional ε-dependence and essentially consider $k \sim k_G \sim u^2 + \varepsilon$. Then we obtain a limit where kinks are possible and carry a certain amount of "bending energy" proportional to their size. We call this the singular approximation.

An interesting feature of the singular approximation is that tangent discontinuities are not restricted to interfaces, but may also appear inside a lipid phase. These *ghost interfaces* are unavoidable with a coupling of phase field and curvature like ours, and they might be useful for including curvature-induced budding in the model.

This thesis is organised as follows: We specify the rotationally symmetric setting in Chapter 2. Afterwards we derive the limit of the regular and singular approximation in Chapters 3 and 4. Finally, we illustrate the theoretical results with some numerical examples in Chapter 5.

Chapter 2

The rotationally symmetric setup

In this chapter we introduce our notation and recall some facts about surfaces of revolution. We also specify our setting for the approximation.

2.1 Surfaces of revolution

2.1.1 Basic definitions and notation

Let $I \subset \mathbb{R}$ be an open interval and $\gamma = (x, y)\colon I \to \mathbb{R}^2$ a Lipschitz parametrised curve in the upper half of the xy-plane, that is $y(t) \geq 0$ for all $t \in I$. By Lipschitz continuity $\gamma(\partial I)$ exists and we have $y(t) \geq 0$ for $t \in \partial I$. We denote by M_γ the surface in $\mathbb{R}^3$ obtained by rotating γ about the x-axis, thus M_γ is the image of $\overline{I} \times [0, 2\pi)$ under the Lipschitz continuous map

$$\Phi\colon (t, \theta) \mapsto (x(t), y(t)\cos\theta, y(t)\sin\theta);$$

γ is called generating curve of M_γ, see for instance [dC76] for a detailed discussion of surfaces.

Since γ is Lipschitz continuous, it is weakly and almost everywhere differentiable with bounded derivative γ' and the Fundamental Theorem of Calculus

$$\gamma(t_1) - \gamma(t_0) = \int_{t_0}^{t_1} \gamma'(t)\, dt$$

holds for all $t_0, t_1 \in \overline{I}$. The length of γ is well-defined and given by

$$\mathcal{L}_\gamma = \int_I |\gamma'(t)|\, dt.$$

If $I = (a, b)$, the function $l(s) = \int_a^{a+s} |\gamma'(t)|\, dt$, is continuous and increasing in I. It has at most countably many constancy intervals, because these are disjoint and each of them contains a rational number, and those intervals agree with the constancy intervals of γ. Removing the interior of the constancy intervals, pulling thus created holes together, and reparametrising linearly, we may assume that γ has no constancy intervals and that l is strictly increasing. After a further reparametrisation, we may assume that γ has constant speed $|\gamma'| \equiv q_\gamma = \mathcal{L}_\gamma/|I|$ almost everywhere in I [BGH98, Lemma 5.23].

By μ we denote the area measure of M_γ, that is $d\mu = |\partial_t\Phi \wedge \partial_\theta\Phi|\,dt\,d\theta = |\gamma'|y\,dt\,d\theta$. If γ is embedded, then also M_γ is, and μ is the two-dimensional Hausdorff measure $\mathcal{H}^2$ restricted to M_γ; in general, however, the multiplicity of μ may be larger than 1.

If J is a measurable subset of I, we denote by $M_\gamma(J)$ the part of M_γ obtained by rotating $\gamma(J)$, by $\mathcal{L}_\gamma(J)$ its length, and by

$$\mathcal{A}_\gamma(J) = \int_{M_\gamma(J)} d\mu = 2\pi \int_J |\gamma'|y\,dt$$

its area; we abbreviate $\mathcal{A}_\gamma = \mathcal{A}_\gamma(I)$, and if $J = (a,b)$ we write $\mathcal{A}_\gamma(a,b) = \mathcal{A}_\gamma((a,b))$ and similarly for $\mathcal{L}_\gamma$ and $\mathcal{M}_\gamma$.

The tangent space $\mathcal{T}_{(t_0,\theta_0)}M_\gamma$ exists for almost every $(t,\theta) \in I \times [0,2\pi)$ and is the plane spanned by the orthonormal vectors

$$\xi_1 = \frac{\partial_t\Phi}{|\partial_t\Phi|} = \frac{1}{|\gamma'|}\big(x', y'\cos\theta, y'\sin\theta\big) \qquad \text{and} \qquad \xi_2 = \frac{\partial_\theta\Phi}{|\partial_\theta\Phi|} = (0, -\sin\theta, \cos\theta)\,; \tag{2.1}$$

a unit normal is given by

$$\nu = \frac{\partial_t\Phi \wedge \partial_\theta\Phi}{|\partial_t\Phi \wedge \partial_\theta\Phi|} = \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{pmatrix} \cdot \frac{\gamma'^{\perp}}{|\gamma'|} = \frac{1}{|\gamma'|}\big(-y', x'\cos\theta, x'\sin\theta\big)\,, \tag{2.2}$$

where $\gamma'^{\perp} = (-y', x')$. We associate tangent space, normal and the quantities defined below to the parameter (t,θ) and not to the point $\Phi(t,\theta)$ on the surface, because the surface is not necessarily embedded. But if $t_0 \in I$ is a point such that γ is continuously differentiable in a neighbourhood of t_0 and $|\gamma'(t_0)| > 0$, then there is a small interval J around t_0 such that the restriction of γ to J is embedded. Thus, $M_\gamma(J)$ is embedded and $\mathcal{T}_{(t_0,\theta_0)}M_\gamma$ is the tangent space of $M_\gamma(J)$ at $\Phi(t_0,\theta_0)$.

We consider a function $f\colon M_\gamma \to \mathbb{R}^k$ to be a function $F\colon \overline{I} \times [0,2\pi) \to \mathbb{R}^k$ of the parameters. On embedded parts of M_γ this amounts to $f(\Phi(t,\theta)) = F(t,\theta)$. Given a tangent vector ξ at $(t_0,\theta_0) \in I \times (0,2\pi)$, the directional derivative of f in direction ξ is defined as

$$D_\xi f(t_0,\theta_0) = \frac{d}{ds}F(\eta(s))\bigg|_{s=0}\,,$$

where $\eta\colon (-\delta,\delta) \to I \times [0,2\pi)$ is any C^1-curve satisfying $\eta(0) = (t_0,\theta_0)$ and $\frac{d}{ds}\Phi(\eta(s))\big|_{s=0} = \xi$. The tangential gradient of $f\colon M_\gamma \to \mathbb{R}$ and the tangential divergence of a vector field $h\colon M_\gamma \to \mathbb{R}^3$ are

$$\nabla_{M_\gamma} f = (D_{\xi_1}f)\xi_1 + (D_{\xi_2}f)\xi_2$$

and

$$\operatorname{div}_{M_\gamma} h = D_{\xi_1} h \cdot \xi_1 + D_{\xi_2} h \cdot \xi_2,$$

where $\{\xi_1, \xi_2\}$ is any orthonormal basis of the tangent space [Sim83, Küh02]. For $\{\xi_1, \xi_2\}$ as in (2.1), we find $D_{\xi_1} f = |\gamma'|^{-1} \partial_t F$ and $D_{\xi_2} f = y^{-1} \partial_\theta F$, hence

$$\nabla_{M_\gamma} f = \frac{1}{|\gamma'|}(\partial_t F)\xi_1 + \frac{1}{y}(\partial_\theta F)\xi_2 = \frac{1}{|\gamma'|^2}(\partial_t F)\partial_t \Phi + \frac{1}{y^2}(\partial_\theta F)\partial_\theta \Phi.$$

In particular, if f is rotationally symmetric then

$$\nabla_{M_\gamma} f(t,\theta) = \frac{F'(t)}{|\gamma'(t)|}\xi_1(t,\theta) \qquad \text{and} \qquad |\nabla_{M_\gamma} f(t,\theta)| = \frac{|F'(t)|}{|\gamma'(t)|},$$

where $|\cdot|$ is the Euclidean norm of a vector in $\mathbb{R}^3$.

To consider curvatures we assume for the rest of this subsection that $\gamma \in W^{2,1}_{\mathrm{loc}}(I;\mathbb{R}^2)$ is twice weakly differentiable, thus twice differentiable almost everywhere, and that $y > 0$ in I. An orientation of M_γ is given by the unit normal ν from (2.2). Since ν is weakly differentiable, the shape operator

$$S\colon \mathcal{T}_{(t_0,\theta_0)}M \to \mathcal{T}_{(t_0,\theta_0)}M, \qquad S\zeta = D_\zeta \nu$$

and the second fundamental form

$$B\colon \mathcal{T}_{(t_0,\theta_0)}M \times \mathcal{T}_{(t_0,\theta_0)}M \to \mathbb{R}, \qquad (\zeta,\xi) \mapsto \xi \cdot D_\zeta \nu$$

are well-defined for almost every (t_0, θ_0). The matrix representation with respect to the basis $\{\xi_1, \xi_2\}$ in (2.1) of both is

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \qquad \text{with} \qquad \kappa_1 = \frac{-y''x' + y'x''}{|\gamma'|^3} = -\frac{\gamma'' \cdot \gamma'^{\perp}}{|\gamma'|^3} \quad \text{and} \quad \kappa_2 = \frac{x'}{y|\gamma'|}.$$

The eigenvalues κ_1, κ_2 of S are the principal curvatures of M_γ, and κ_1 is just the signed curvature of γ with respect to the normal $-\gamma'^{\perp}/|\gamma'|$. The mean curvature H and the Gauss curvature K of M_γ are

$$H = \operatorname{trace} S = \kappa_1 + \kappa_2 \qquad \text{and} \qquad K = \det S = \kappa_1 \kappa_2.$$

By $|S|^2 = \kappa_1^2 + \kappa_2^2$ we denote the squared Frobenius norm of S, and since $B(\zeta,\zeta) = \zeta \cdot S\zeta$, we also write $|B|^2 = |S|^2$. Obviously, $|B|^2 = H^2 - 2K$, and a simple calculation shows $H = \operatorname{div}_{M_\gamma} \nu$.

The signs of the principal curvatures and the mean curvature depend on the sign of the normal ν. Our choice above ensures that a unit ball has outer unit normal ν as in (2.2) and curvatures $\kappa_1 = \kappa_2 = +1$ when it is parametrised “from left to right” such that $x' \geq 0$, for instance by $\gamma(t) = (-\cos t, \sin t)$, $t \in [0, \pi]$. Note that the Frenet formulas for the plane curve γ then read

$$\left(\frac{\gamma'}{|\gamma'|}\right)' = -\kappa_1 \frac{\gamma'^{\perp}}{|\gamma'|} \qquad \text{and} \qquad \left(\frac{\gamma'^{\perp}}{|\gamma'|}\right)' = +\kappa_1 \frac{\gamma'}{|\gamma'|}.$$

Let $\varphi\colon I \to \mathbb{R}$ be an angle function for γ, that is, let $\varphi(t)$ be the angle between the positive x-axis and the tangent vector $\gamma'(t)$. Since $W^{2,1}_{\text{loc}}$ embeds into C^1_{loc}, the angle φ can be chosen continuously and is then uniquely determined up to multiples of 2π. In terms of φ, γ is characterised by fixing one point and

$$x' = |\gamma'| \cos\varphi, \qquad y' = |\gamma'| \sin\varphi.$$

The principal curvatures take the form

$$\kappa_1 = -\frac{\varphi'}{|\gamma'|}, \qquad \kappa_2 = \frac{\cos\varphi}{y},$$

and we have

$$K = -\frac{\varphi' \cos\varphi}{|\gamma'| y} = -\frac{(\sin\varphi)'}{|\gamma'| y} = -\frac{(y'/|\gamma'|)'}{|\gamma'| y}. \tag{2.3}$$

From (2.3) we see that for any $J = (a, b) \Subset I$ the integral

$$\int_{M_\gamma(J)} K d\mu = -2\pi \int_a^b (\sin\varphi)' \, dt = 2\pi \left(\sin\varphi(a) - \sin\varphi(b)\right). \tag{2.4}$$

depends only on the tangent angle at ∂J. If $\sin\varphi \in W^{1,1}(I)$, then (2.4) is by approximation also true for $J = I$, which is just the Gauss Bonnet Theorem for surfaces of revolution. In particular, if $y(\partial I) = \{0\}$ and M_γ is a C^1-surface, γ' is perpendicular to the axis of revolution at ∂I, thus $\varphi(a) = -\varphi(b) = \pi/2$ modulo 2π and $\int_{M_\gamma} K \, d\mu = 4\pi$.

Another consequence of (2.3) is that for γ parametrised with constant speed q_γ the integral

$$\int_{M_\gamma(J)} |K| \, d\mu = \frac{2\pi}{q_\gamma} \int_J |y''| \, dt \tag{2.5}$$

is the L^1-norm of y''. Moreover, in that case $|\gamma''|^2 = \varphi'^2 q_\gamma^2$ and therefore

$$\int_{M_\gamma(J)} \kappa_1^2 \, d\mu = \frac{2\pi}{q_\gamma} \int_J |\varphi'|^2 y \, dt = \frac{2\pi}{q_\gamma^3} \int_J |\gamma''|^2 y \, dt \tag{2.6}$$

is the weighted L^2-norm of φ' and γ''.

If M_γ is a closed surface, that is $y(\partial I) = \{0\}$, κ_2 seemingly degenerates at the axis of revolution. However, if M_γ is sufficiently smooth, say C^2, the principal curvatures are well-defined taking for instance another local parametrisation of M_γ; to compute κ_2 in the rotationally symmetric parametrisation, L'Hôpital's rule may be used and yields $\kappa_2 = \kappa_1$ at the axis of revolution [Küh02]. On the other hand, bounds on certain curvature expressions allow to derive some regularity properties of the generating curve γ.

2.1.2 Surfaces with L^2-bounded second fundamental form

In our setting we will encounter curves that satisfy only $y \geq 0$ in I, because the sharp inequality $y > 0$ is not conserved by the convergence of curves that our energy bounds yield. Also, we do not have a construction as in [Sch11, BD10, DDG08, DFGS08] where a priori bounds for rotated graphs are derived from bounds on the single-phase Helfrich energy inspired by analogy with elastic curves in the hyperbolic plane.

The set $\{y > 0\} = \{t \in I : y(t) > 0\}$ is open, hence its connected components are open intervals. Since the components are mutually disjoint and each component contains at least one rational number, $\{y > 0\}$ is the union of countably many disjoint intervals. In a slight abuse of language we refer to a component ω of $\{y > 0\}$ also as component of γ and call $M_\gamma(\omega)$ a component of M_γ. Thus, M_γ consists of at most countably many components which are connected at the axis of revolution.

In the following lemma and corollary we collect some regularity properties for γ and M_γ that can be derived from an L^2-bound on the second fundamental form. The most prominent result of such type is [Tor94, MŠ95], where the authors prove that a (generalised) surface with L^2-bounded second fundamental form has bi-Lipschitz coordinates. Here we already assume that γ is Lipschitz and twice weakly differentiable and focus on properties of γ near the axis of revolution.

Lemma 2.1. *Let $\gamma = (x, y) \colon I \to \mathbb{R}^2$, $y \geq 0$ be a Lipschitz curve that satisfies $\gamma \in W^{2,1}_{\text{loc}}(\{y > 0\}\,; \mathbb{R}^2)$, $|\gamma'| \equiv q_\gamma > 0$ in $\{y > 0\}$ and*

$$\int_{M_\gamma(\{y>0\})} |B|^2 \, d\mu < \infty.$$

Then $\gamma \in W^{2,2}_{\text{loc}}(\{y > 0\}\,; \mathbb{R}^2)$ and $y \in W^{2,1}(\{y > 0\}\,; \mathbb{R}^2)$. Moreover, for any connected component $\omega = (a, b)$ of $\{y > 0\}$ we have $\gamma \in C^1(\overline{\omega}; \mathbb{R}^2)$ with $\gamma'(a) = -\gamma'(b) = \pm(0, |\gamma'|)$, that is, γ is perpendicular to the axis of revolution. The number of components of $\{y > 0\}$ is finite.

Proof. Given $J \Subset \{y > 0\}$, y has a positive lower bound c_J in J, and $\gamma \in W^{2,2}(J; \mathbb{R}^2)$ follows from (2.6). Using $2|K| \leq |B|^2$ and (2.5) we obtain $y \in W^{2,1}(\{y > 0\})$. By embedding we have $x \in C^1_{\text{loc}}(\omega)$ and $y \in C^1(\overline{\omega})$ for any connected component $\omega = (a, b)$ of $\{y > 0\}$, that is, y' is continuous when approaching $\partial\omega$ from the interior of ω. Using this continuity and $x'^2 + y'^2 = q_\gamma^2$ in ω, we show that x' is continuous. Assume for contradiction that there are sequences $t_k \to a$, $s_k \to a$ in ω such that $\lim x'(t_k) \neq \lim x'(s_k)$; if such sequences cannot be found, $x'(t)$ converges as $t \searrow a$. Since $x'(t_k)^2$ and $x'(s_k)^2$ converge to $q_\gamma^2 - y'(a)^2$, we have $\lim x'(s_k) = -\lim x'(t_k) = m \neq 0$ and $x'(t_k) < -m/2$ and $x'(s_k) > m/2$ for sufficiently large k. Thus, there is $r_k \in (t_k, s_k)$ or (s_k, t_k) such that $x'(r_k) = 0$. Since $r_k \to a$, we find $y'^2(a) = q_\gamma^2$. Therefore, $x'^2(s_k) = q_\gamma^2 - y'^2(s_k) \to 0$ and the same holds for $x'(t_k)$ which contradicts the assumptions. The same argument holds at b.

To prove that γ is perpendicular to the axis of revolution at a, consider the second principal curvature. Since γ is Lipschitz, we have $y(t) \leq y(a) + q_\gamma(t-a) = q_\gamma(t-a)$ in ω, hence

$$\infty > \frac{q_\gamma^2}{2\pi} \int_{M(\omega)} \kappa_2^2 \, d\mu \geq \int_a^{a+\delta} \frac{x'^2}{t-a} \, dt \geq \left(\inf_{(a,a+\delta)} x'^2 \right) \int_a^{a+\delta} \frac{dt}{t-a}$$

uniformly in $\delta \in (0, b-a)$. The continuity of x' now implies $x'(a) = 0$, and similarly we get $x'(b) = 0$. As $|\gamma'| = q_\gamma$ and $y > 0$ in ω, we find $y'(a) = -y'(b) = q_\gamma$ or $y'(a) = -y'(b) = -q_\gamma$. By the Gauss-Bonnet formula (2.4) we have

$$\left| \int_{M(\omega)} K \, d\mu \right| = 4\pi,$$

so the number N_γ of components of $\{y > 0\}$ satisfies

$$N_\gamma \leq \frac{1}{4\pi} \sum_\omega \int_{M_\gamma(\omega)} |K| \, d\mu \leq \frac{1}{8\pi} \int_{M_\gamma(\{y>0\})} |B|^2 \, d\mu$$

and is thus finite. □

Corollary 2.2. *Let $\gamma = (x, y)$ be as in Lemma 2.1. Then M_γ has finitely many components which are connected at the axis of revolution. Each component is an immersed C^1-surface and a $W^{2,2}$-surface in $\{y > 0\}$, that is, away from the axis of revolution.*

Remark. The properties $y \in W^{2,1}(\{y > 0\})$ and $x \in C^1(\{y > 0\})$ but $x' \notin W^{2,1}(\{y > 0\})$ in Lemma 2.1 are sharp, as the following example shows. Let

$$\psi(t) = \frac{\sin \ln(1/t) + 1}{\ln(1/t)}$$

for $t \in (0, t_0)$ with t_0 sufficiently small that $\psi \in [0, 1]$ and consider

$$x'(t) = \cos(\pi/2 - \psi(t)) = \sin \psi(t), \qquad y'(t) = \sin(\pi/2 - \psi(t)) = \cos \psi(t)$$

with $x(0) = y(0) = 0$. As $t \to 0$, $\psi(t)$ converges to 0, so for small t we have $x'(t) \sim \psi(t)$, $y'(t) \sim 1$, and $y(t) \sim t$, where $a \lesssim b$ denotes $a \leq Cb$ with a constant $C > 0$ and $a \sim b$ means $a \lesssim b \lesssim a$. We obtain that

$$\int \kappa_2^2 \, d\mu \sim \int \frac{\psi^2}{t} \, dt \lesssim \int \frac{dt}{t(\ln(1/t))^2} < \infty.$$

Moreover, we have

$$\psi'(t) = -\frac{\cos \ln(1/t)}{t \ln(1/t)} + \frac{\sin \ln(1/t) + 1}{t(\ln(1/t))^2}$$

and

$$\int \kappa_1^2 \, d\mu \sim \int \psi'^2 t \, dt \lesssim \int \frac{dt}{t(\ln(1/t))^2} < \infty.$$

On the other hand, $x'' = \psi' \cos\psi \sim \psi'$ for small t and

$$\int |\psi'| \, dt \sim \int \frac{|\cos\ln(1/t)|}{t\ln(1/t)} \, dt = \infty,$$

thus $x \notin W^{2,1}((0,t_0))$. Furthermore, $y'' = -\psi' \sin\psi \sim -\psi'\psi$ and

$$\int |\psi'\psi|^p dt \sim \int \frac{dt}{t^p(\ln(1/t))^{2p}} = \infty$$

for any $p > 1$, thus $y'' \notin L^p((0,t_0))$. Note that M_γ is embedded due to $x' \geq 0$.

2.1.3 Diameter bounds

In order to establish compactness we look for uniform bounds on the generating curves that are derived from uniform bounds on the curvature integrals in the energy and on surface area. The following two well-known results relate the mean curvature of a surface M_γ to its external and internal diameter, thus to the L^∞-norm and the length of γ. Both Lemmas are valid for arbitrary two-dimensional surfaces that are smoothly immersed in $\mathbb{R}^n$, where for $n > 3$ H is the mean curvature vector [Sim83].

Lemma 2.3 ([Sim93, Lemma 1.1]). *If M is a closed (compact, no boundary) and connected surface that is smoothly immersed in $\mathbb{R}^n$, then*

$$\mu(M)\left(\int_M H^2 \, d\mu\right)^{-1} \leq d_{ext}(M)^2 \leq C\mu(M)\int_M H^2 \, d\mu,$$

where $\mu(M)$ is the area of M and d_{ext} the extrinsic diameter.

Lemma 2.4 ([Top08]). *If M is a closed, connected surface that is smoothly immersed in $\mathbb{R}^n$, then*

$$d_{int}(M) \leq C \int_M |H| \, d\mu,$$

where d_{int} is the intrinsic diameter.

The proofs of these lemmas hinge on the fact that inside an arbitrary extrinsic or intrinsic ball not both the area of M and the mean curvature integral can be small at the same time. The upper bounds on the diameters are then derived by exploiting this fact in some carefully chosen balls. For closed surfaces of revolution there is a straightforward proof that the mean curvature integral bounds the length of the generating curve.

Lemma 2.5. *Let $\gamma = (x, y) \in C^{0,1}(I;\mathbb{R}^2) \cap W^{2,1}_{\mathrm{loc}}(I;\mathbb{R}^2)$ be a curve such that $y(I) \subset (0,\infty)$ and $y(\partial I) = \{0\}$. Then*

$$\int_{M_\gamma} |H| d\mu \geq 2\pi \mathcal{L}_\gamma.$$

Proof. We assume that the mean curvature integral is finite because otherwise there is nothing to prove. Without loss of generality we also assume that $\gamma\colon (0, \mathcal{L}_\gamma) \to \mathbb{R}^2$ is parametrised by arc length. If $x' \geq 0$ in I, we can find an angle φ that is weakly differentiable in I and satisfies $\varphi \in [-\pi/2, \pi/2]$. Then

$$\begin{aligned}\int_{M_\gamma} H\, d\mu &= 2\pi \int_0^{\mathcal{L}_\gamma} \left(-\varphi' + \frac{\cos\varphi}{y}\right) y\, dt = 2\pi \int_0^{\mathcal{L}_\gamma} \varphi y' + \cos\varphi\, dt - 2\pi \varphi y\Big|_0^{\mathcal{L}_\gamma} \\ &= 2\pi \int_0^{\mathcal{L}_\gamma} \varphi \sin\varphi + \cos\varphi\, dt \\ &\geq 2\pi \mathcal{L}_\gamma,\end{aligned}$$

because $\varphi \sin\varphi + \cos\varphi \geq 1$. In general, when $x' \geq 0$ does not hold, consider the curve $\widetilde{\gamma} = (\widetilde{x}, \widetilde{y})$ defined by

$$\widetilde{y} = y \qquad \text{and} \qquad \widetilde{x}(t) = x(0) + \int_0^t |x'(s)|\, ds.$$

Obviously $|\widetilde{\gamma}'| = |\gamma'|$, and a simple calculation shows $\widetilde{H} = H \operatorname{sign} x'$ almost everywhere. Therefore

$$\int_{M_\gamma} |H|\, d\mu \geq \int_{M_{\widetilde{\gamma}}} \widetilde{H}\, d\widetilde{\mu} \geq 2\pi \mathcal{L}_{\widetilde{\gamma}} = 2\pi \mathcal{L}_\gamma. \qquad \square$$

Remark. The inequality in Lemma 2.5 is actually strict: equality in the above calculation means $\varphi \sin\varphi + \cos\varphi = 1$ in I, which holds for $\varphi \in [-\pi/2, \pi/2]$ only if $\varphi \equiv 0$, and is thus impossible for a nontrivial closed surface of revolution. Moreover, the inequality is sharp, as can be seen by a cylinder with spherical caps when the radius tends to zero.

2.2 The approximate energy

In this section we state the setting of the shared approximation for the singular and the regular case, that is everything but the parameter extensions k and k_G. The energy we consider is

$$\int_{M_\gamma} k(u,\varepsilon)\,(H - H_s(u))^2 + k_G(u,\varepsilon) K\, d\mu + \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u)\, d\mu \tag{2.7}$$

for surfaces of revolution M_γ with prescribed area $\mathcal{A}_\gamma = A_0 > 0$ and phase constraint $\int_{M_\gamma} u\, d\mu = m A_0$ with $m \in (-1, 1)$. If the areas of the two phases are given by A^+ and A^-, then $A_0 = A^+ + A^-$ and $m = (A^+ - A^-)/A_0$. The double-well potential $W\colon \mathbb{R} \to [0,\infty)$ is a continuous function that vanishes in ±1 only and for technical reasons C^2 around these two points. For notational simplicity we assume that W is symmetric with respect to the origin. The function $H_s \colon \mathbb{R} \to \mathbb{R}$ is a continuous and bounded extension of two prescribed preferred curvatures $H_s(\pm 1) = H_s^{\pm}$, for instance

$$H_s(u) = \begin{cases} H_s^- & \text{if } u \leq -1, \\ \frac{1}{2}(H_s^+ + H_s^-) + \frac{1}{2}(H_s^+ - H_s^-)u & \text{if } -1 \leq u \leq 1, \\ H_s^+ & \text{if } 1 \leq u. \end{cases}$$

We consider (2.7) for parametrisations $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ where

$$\begin{aligned} \mathcal{C}_\varepsilon := \Big\{ & \gamma = (x,y) \in C^{0,1}(I;\mathbb{R}^2) \cap W^{2,1}_{\text{loc}}(I;\mathbb{R}^2) : \\ & |\gamma'| = \text{const},\ y(\partial I) = \{0\}\,,\ y(I) \subset (0,\infty),\ x' \geq 0,\ \int_{M_\gamma} |B|^2 \, d\mu < \infty,\ \mathcal{A}_\gamma = A_0 \Big\} \end{aligned}$$

and

$$\mathcal{P}_\varepsilon := \Big\{ u \in W^{1,1}_{\text{loc}}(I) : \int_{M_\gamma} |\nabla_{M_\gamma} u|^2 \, d\mu < \infty,\ \|u\|_\infty \leq C_0,\ \int_M u \, d\mu = mA_0, \Big\}.$$

The first three conditions in the definition of $\mathcal{C}_\varepsilon$ ensure that γ is parametrised with constant speed and that M_γ is a closed surface. The requirement $x' \geq 0$ fixes the orientation and guarantees that M_γ is embedded. The L^2-bound on the second fundamental form of M_γ and the first two conditions on the phase fields ensure that the energy (2.7) is well-defined for $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ for the choices of k and k_G that we will make. If $\gamma \in \mathcal{C}_\varepsilon$, then by Lemma 2.1 and the subsequent corollary, M_γ is a C^1-surface and a $W^{2,2}$-surface away from the axis of revolution. The uniform bound $\|u\|_\infty \leq C_0$ with a large constant $C_0 \gg 1$ is actually slightly more than is needed for the energy to be well-defined, but we impose it for technical reasons. This does not seem to be a strong restriction, as one would expect phase fields with small energy to be roughly between $+1$ and -1 for small ε. The set $\mathcal{P}_\varepsilon$ depends on the chosen $\gamma \in \mathcal{C}_\varepsilon$ due to the phase integral constraint, but since we usually consider pairs or configurations (γ, u) we suppress this fact in the notation. Instead we highlight the affiliation to the approximate energy by the index ε in $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. In the following chapters we will write M_ε instead of M_{γ_ε} and so forth, when considering sequences (γ_ε) of curves. If necessary for clarification we will also add the curve or an ε as index to other quantities such as H_γ, μ_γ, or y_γ.

Obviously, (2.7) is invariant under reparametrisations that preserve the orientation and the regularity properties of γ. In particular, if γ satisfies all requirements of $\mathcal{C}_\varepsilon$ but only $|\gamma'| \neq 0$ instead of $|\gamma'| = \text{const}$, the corresponding constant speed parametrisation belongs to $\mathcal{C}_\varepsilon$ and has the same energy. Hence, considering only $|\gamma'| = \text{const}$ is no geometric restriction.

2.3 Γ-convergence

We study the limit of (2.7) in the sense of Γ-convergence introduced by De Giorgi and Franzoni in [DGF75], see also [DM93, Bra02] for systematic studies. If X is a normed linear space, for instance a function space, then a sequence of functionals $F_\varepsilon \colon X \to \mathbb{R}$ is said to Γ-converge to $F \colon X \to \mathbb{R}$, if

- for any sequence $(x_\varepsilon) \subset X$ that converges to some $x \in X$ there holds
$$\liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \geq F(x); \tag{2.8}$$
- for any $x \in X$ there is a sequence $(x_\varepsilon) \subset X$ that converges to x as $\varepsilon \to 0$ such that
$$\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \leq F(x). \tag{2.9}$$

Here and in the following we understand convergence with respect to the continuous parameter $\varepsilon \to 0$ as convergence for all sequences $\varepsilon_j \to 0$ as $j \to \infty$, but still write ε instead of ε_j. Inequality (2.8) is called the lower bound, (2.9) the upper bound inequality; the sequence (x_ε) in the latter is called recovery sequence for x and, due to (2.8), satisfies $F_\varepsilon(x_\varepsilon) \to F(x)$ as $\varepsilon \to 0$. It is sufficient to prove (2.9) for a subset $Y \subset X$ such that for any $x \in X$ there is a sequence $(x_\delta) \subset Y$ with $x_\delta \to x$ and $F(x_\delta) \to F(x)$ as $\delta \to 0$. Then a diagonal argument yields the existence of a recovery sequence for x.

Γ-convergence is most useful when combined with equi-coercivity. The functionals (F_ε) are called equi-coercive if any sequence $(x_\varepsilon) \subset X$ with uniformly bounded $F_\varepsilon(x_\varepsilon)$ admits a convergent subsequence. If (F_ε) is equi-coercive and Γ-converges to F, every almost minimising sequence (x_ε), that is, every sequence satisfying $F_\varepsilon(x_\varepsilon) = \inf_X F_\varepsilon + o(1)_{\varepsilon \to 0}$, has a subsequence that converges to a minimiser of F, whose existence is thus proved. Such an almost minimising sequence trivially exists if each F_ε is bounded from below.

The notion of Γ-convergence depends on the convergence in the underlying space X. Our singular and regular case are the limit of (2.7) with respect to two different notions of convergence. For the first in Chapter 3 we prove full Γ-convergence, for the second in Chapter 4 only a partial convergence result.

Chapter 3

The regular approximation

In this chapter we study the regular approximation where the energy is given by (2.7) with $k(u,\varepsilon) \equiv 1$ and $k_G(u,\varepsilon) \equiv 0$, that is

$$\mathcal{E}_\varepsilon(\gamma,u) = \int_{M_\gamma} (H - H_s(u))^2 \, d\mu + \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u) \, d\mu$$

for configurations $(\gamma,u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. The limit energy will be

$$\mathcal{E}(\gamma,u) = \int_{M_\gamma} (H - H_s(u))^2 \, d\mu + \sigma \mathcal{H}^1(M(S_u))$$

for curves with parametrisations γ in

$$\begin{aligned}\mathcal{C} := \Big\{ &\gamma = (x,y) \in C^{0,1}(I;\mathbb{R}^2) \cap W^{2,1}_{\text{loc}}(\{y>0\}\,;\mathbb{R}^2): \\ &|\gamma'| = \text{const},\ y(\partial I) = \{0\}\,,\ y \geq 0,\ \mathcal{H}^0(\{y=0\}) < \infty,\ x' \geq 0, \\ &\int_{M_\gamma(\{y>0\})} |B|^2 \, d\mu < \infty,\ \mathcal{A}_\gamma = A_0 \Big\}\end{aligned}$$

and associated phase fields u in

$$\mathcal{P} := \Big\{ u\colon I \to \{\pm 1\} \text{ piecewise constant}: \int_{M_\gamma} u \, d\mu = m A_0,\ \mathcal{H}^1(M_\gamma(S_u)) < \infty \Big\}.$$

Here $S_u \subset \{y>0\}$ denotes the countable jump set of u in $\{y>0\}$, and we call $s \in S_u$ and the corresponding circle $M_\gamma(\{s\})$ an interface of (γ,u). The constant σ is given by

$$\sigma = 2 \int_{-1}^{1} \sqrt{W(u)} \, du, \tag{3.1}$$

and

$$\mathcal{H}^1(M_\gamma(S_u)) = 2\pi \sum_{s \in S_u} y(s)$$

is the one-dimensional Hausdorff measure of the union of the countably many circles $M_\gamma(S_u)$.

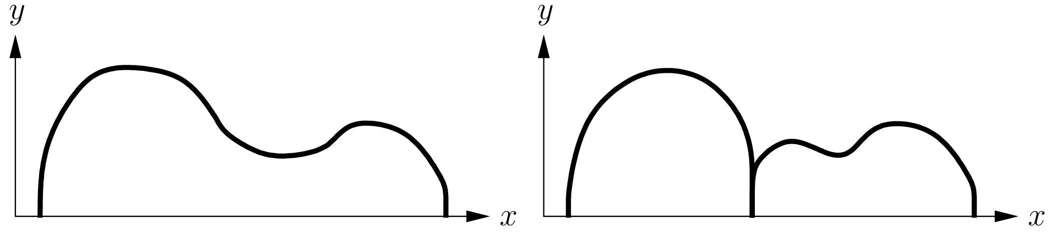

Figure 3.1: The curve $\gamma_\varepsilon \in \mathcal{C}_\varepsilon$ on the left can, as element of a sequence, lead to a limit $\gamma \in \mathcal{C}$ as on the right. The two components of M_γ touch each other along a vertical line segment.

The difference between $\mathcal{C}_\varepsilon$ and $\mathcal{C}$ is that $\gamma \in \mathcal{C}$ may touch the axis of revolution also in the interior of I, but this can happen only at finitely many points: For $\gamma \in \mathcal{C}$ we infer from Lemma 2.1 and the subsequent corollary that M_γ consists of finitely many components which are C^1-surfaces and $W^{2,2}$-surfaces away from the axis of revolution. Due to $x' \geq 0$ each component is embedded, but different components may touch. In fact, they can touch not only in $\{y = 0\}$, but also in $\{y > 0\}$ if and only if $x' \equiv 0$ in a neighbourhood of some $s \in \{y = 0\}$. Moreover, $\{y = 0\}$ is finite. See Figure 3.1 for an example.

The set $\mathcal{P}$ resembles the set of special functions of bounded variation SBV with values in $\{\pm 1\}$, weighted with the height y of the generating curve $\gamma \in \mathcal{C}$. Indeed, for $u \in \mathcal{P}$ and any $J \Subset \{y > 0\}$ we have $u \in SBV(J; \{\pm 1\})$, but as jumps of height 2 may accumulate near the axis of revolution, $u \notin SBV(I)$ in general. Points in $\{y = 0\}$ can be jump points of u or singular points where one or both one-sided limits are undefined. We emphasise that in our notation S_u only contains points in $\{y > 0\}$, because the restriction of u to $\{y = 0\}$ does not contribute to the limit energy $\mathcal{E}$.

We extend $\mathcal{E}_\varepsilon$ and $\mathcal{E}$ to $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$ by setting $\mathcal{E}_\varepsilon(\gamma, u) = \mathcal{E}(\gamma, u) = \infty$ whenever (γ, u) does not belong to $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ and $\mathcal{C} \times \mathcal{P}$, respectively. The result of this chapter is the following theorem.

Theorem 3.1. *The energies $\mathcal{E}_\varepsilon$ are equi-coercive, that is, any sequence $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ with uniformly bounded energy admits a subsequence that converges strongly in $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$ to some $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$. Furthermore, $\mathcal{E}_\varepsilon$ Γ-converges to $\mathcal{E}$ with respect to strong convergence in $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$ as $\varepsilon \to 0$.*

The proof of Theorem 3.1 is divided into the three steps equi-coercivity, lower bound, and upper bound, which are carried out in the following three sections.

3.1 Equi-coercivity

The weak coupling between phase fields and curves in $\mathcal{E}_\varepsilon$, which is present only in the interface energy, allows to consider both almost separately. Furthermore, $\mathcal{E}_\varepsilon$ bounds the

L^2-norm of the mean curvature of M_γ for $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ due to

$$\mathcal{E}_\varepsilon(\gamma, u) \geq \frac{1}{2} \int_{M_\gamma} H^2 \, d\mu - 2\|H_s\|_\infty^2 A_0.$$

Adding $-2\pi = -\frac{1}{2} \int_{M_\gamma} K \, d\mu$ and $2\|H_s\|_\infty^2 A_0$ to both sides, we find

$$\mathcal{E}_\varepsilon(\gamma, u) + \text{const} \geq \frac{1}{2} \int_{M_\gamma} |B|^2 \, d\mu \geq \int_{M_\gamma} |K| \, d\mu,$$

that is, $\mathcal{E}_\varepsilon(\gamma, u)$ also bounds the L^2-norm of the second fundamental form and the L^1-norm of the Gauss curvature of M_γ.

Lemma 3.2. *Let $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ be a sequence with uniformly bounded energy $\mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon)$. Then there are $\gamma = (x, y) \in \mathcal{C}$ and a subsequence, not relabelled, such that*

- $\gamma_\varepsilon \overset{\star}{\rightharpoonup} \gamma$ *in* $W^{1,\infty}(I; \mathbb{R}^2)$*;*
- $\gamma_\varepsilon \rightharpoonup \gamma$ *in* $W^{2,2}_{\text{loc}}(\{y > 0\}; \mathbb{R}^2)$*;*
- $\gamma_\varepsilon \to \gamma$ *in* $W^{1,p}(I; \mathbb{R}^2)$ *for any* $p \in [1, \infty)$*.*

Proof. Let $\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon)$ and $|\gamma_\varepsilon'| = q_\varepsilon$. With Lemma 2.5 and Hölder's inequality we find

$$2\pi q_\varepsilon |I| = 2\pi \mathcal{L}_\varepsilon \leq \int_{M_\varepsilon} |H_\varepsilon| \, d\mu_\varepsilon \leq \left(\mathcal{A}_\varepsilon \int_{M_\varepsilon} H_\varepsilon^2 \, d\mu_\varepsilon \right)^{1/2},$$

so the sequence (q_ε) is bounded from above. Since translations in x-direction do not change the energy, we may assume that all γ_ε have a common end point. Hence (γ_ε) is bounded in $W^{1,\infty}(I; \mathbb{R}^2)$ and we may extract a subsequence such that $q_\varepsilon \to q$ in $\mathbb{R}$ and $\gamma_\varepsilon \overset{\star}{\rightharpoonup} \gamma$ in $W^{1,\infty}(I; \mathbb{R}^2) = C^{0,1}(I; \mathbb{R}^2)$. By compact embedding, the convergence is uniform in $\overline{I}$, thus $y \geq 0$ and $y(\partial I) = \{0\}$. Since

$$A_0 = \mathcal{A}_\varepsilon = 2\pi q_\varepsilon \int_I y_\varepsilon \, dt \to 2\pi q \int_I y \, dt,$$

neither $q = 0$ nor $y \equiv 0$ in I. Without loss of generality we assume $q = 1$, thus $|\gamma'| \leq 1$ almost everywhere in I.

Taking into account only the just selected subsequence, let ε be sufficiently small so that $q_\varepsilon \leq 2$, and let $J \Subset \{y > 0\}$ and $c_J > 0$ such that $y \geq 2c_J$ in J. By uniform convergence of y_ε we then have $y_\varepsilon \geq c_J$, hence

$$\frac{1}{2\pi} \int_{M_\varepsilon} |B_\varepsilon|^2 \, d\mu_\varepsilon \geq \frac{1}{2\pi} \int_{M_\varepsilon(J)} \kappa_{1,\varepsilon}^2 \, d\mu_\varepsilon \geq \frac{c_J}{8} \int_J |\gamma_\varepsilon''|^2 \, dt \tag{3.2}$$

for all sufficiently small ε. Since the left hand side of (3.2) is uniformly bounded, a subsequence of (γ_ε'') converges weakly in $L^2(J; \mathbb{R}^2)$ to some γ_J''. The corresponding subsequence of (γ_ε) converges weakly in $W^{2,2}(J;\mathbb{R}^2)$, and from uniqueness of the weak limit we infer that γ''_J is the weak derivative of γ' in J and that the whole sequence converges. This proves $\gamma_\varepsilon \rightharpoonup \gamma$ in $W^{2,2}_{\text{loc}}(\{y>0\};\mathbb{R}^2)$. Consequently,

$$\int_{M(J)} |B|^2\,d\mu \le \liminf_{\varepsilon\to 0}\int_{M_\varepsilon(J)} |B_\varepsilon|^2\,d\mu_\varepsilon \le \liminf_{\varepsilon\to 0}\int_{M_\varepsilon} |B_\varepsilon|^2\,d\mu_\varepsilon$$

for any $J \Subset \{y>0\}$, and since the right hand side is finite and its value independent of J, we obtain

$$\int_{M(\{y>0\})} |B|^2\,d\mu < \infty$$

by exhausting $\{y>0\}$ with $J \Subset \{y>0\}$.

By compact embedding of $W^{2,2}$ into C^1, γ_ε converges strongly to γ in $C^1_{\text{loc}}(\{y>0\};\mathbb{R}^2)$, and we infer that $\gamma'_\varepsilon \to \gamma'$ pointwise in $\{y>0\}$. Therefore $x' \ge 0$ and $|\gamma'| = \lim|\gamma'_\varepsilon| = \lim q_\varepsilon = 1$ in $\{y>0\}$; the latter implies

$$\mathcal{A}_\gamma = 2\pi\int_I |\gamma'|y\,dt = 2\pi\int_{\{y>0\}} y\,dt = \lim_{\varepsilon\to 0} 2\pi\int_{\{y>0\}} q_\varepsilon y_\varepsilon\,dt = \lim_{\varepsilon\to 0}\mathcal{A}_\varepsilon = A_0.$$

To conclude $\gamma \in \mathcal{C}$, we have to show that $\{y=0\}$ is finite. This also yields strong convergence of (γ_ε) in $W^{1,p}(I;\mathbb{R}^2)$, because it implies $\gamma'_\varepsilon \to \gamma'$ almost everywhere in I and the Dominated Convergence Theorem applies. Assume for contradiction that J is a non-empty open subset of $\{y=0\}$. Then

$$\left(\int_J |x'_\varepsilon|\,dt\right)^2 = \left(\int_J \frac{|x'_\varepsilon|}{\sqrt{q_\varepsilon y_\varepsilon}}\sqrt{q_\varepsilon y_\varepsilon}\,dt\right)^2 \le \frac{\mathcal{A}_\varepsilon(J)}{4\pi^2}\int_{M_\varepsilon} \kappa_{2,\varepsilon}^2\,d\mu_\varepsilon$$

implies $x'_\varepsilon \to 0$ and $y'^2_\varepsilon = q_\varepsilon^2 - x'^2_\varepsilon \to 1$ in $L^1(J)$, which contradicts $y'=0$ almost everywhere in $\{y=0\}$. Consequently, $\{y=0\}$ does not contain interior points, and since by Lemma 2.1 the number of components of $\{y>0\}$ is finite, we conclude $\mathcal{H}^0(\{y=0\})<\infty$. □

Lemma 3.3. *Let $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ and $\gamma \in \mathcal{C}$ be as in Lemma 3.2. Then there exist a countable set $S \subset I$ that is finite in any $J \Subset \{y>0\}$ and $u \in \mathcal{P}$ with $S_u \subset S$ such that for a subsequence $u_\varepsilon \to u$ in measure, almost everywhere in I, and in $L^p(I)$ for $p\in[1,\infty)$. Moreover, in any $J \Subset \{y>0\}\setminus S$ there holds $|u_\varepsilon| \ge 1/2$ for all sufficiently small ε.*

Proof. According to Lemma 3.2 we restrict ourselves to a subsequence γ_ε that converges to γ; as before, $|\gamma'_\varepsilon| = q_\varepsilon$ and without loss of generality $|\gamma'|=1$. Uniform convergence implies that for $J \Subset \{y>0\}$ there is $c_J>0$ such that $y_\varepsilon \ge c_J$ in J for all sufficiently small ε. Therefore,

$$\frac{1}{2\pi}\int_{M_\varepsilon(J)} \varepsilon|\nabla_{M_\varepsilon}u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,d\mu_\varepsilon \ge c_J\int_J \frac{\varepsilon}{q_\varepsilon}|u'_\varepsilon|^2 + \frac{q_\varepsilon}{\varepsilon}W(u_\varepsilon)\,dt \tag{3.3}$$

and the well-known arguments of Modica and Mortola [Mod87, MM77] apply in J, see in particular [Bra02, Lemma 6.2 and Remark 6.3] for a proof in one dimension. The outcome is a finite set of points $S_J \subset J$ and a piecewise constant function $u\colon J \to \{\pm 1\}$ whose jump set is contained in S_J such that a subsequence of u_ε converges to u in measure and almost everywhere in $J \setminus S$. Since (u_ε) is uniformly bounded in $L^\infty(I)$, convergence in $L^p(J)$ for any $p < \infty$ follows. Moreover, in the one-dimensional setting we obtain that in any set compactly contained in $J \setminus S_J$ there holds $|u_\varepsilon| \geq 1/2$ for sufficiently small ε.

Exhausting $\{y > 0\}$ by a sequence of increasing sets such as $J_k = \{y > 1/k\}$ for $k \to \infty$ and taking a diagonal sequence, we find an at most countable set $S \subset \{y > 0\}$ and a function $u\colon \{y > 0\} \to \{\pm 1\}$ whose jump set is contained in S. A subsequence of (u_ε) converges to u in measure and almost everywhere in $\{y > 0\}$ and satisfies $|u_\varepsilon| \geq 1/2$ in $J \Subset \{y > 0\} \setminus S$ for all sufficiently small ε depending on J. The uniform L^∞-bound on u_ε together with $\mathcal{H}^0(\{y = 0\}) < \infty$ yields convergence in $L^p(I)$ for any $1 \leq p < \infty$. Taking convergence of y_ε and $|\gamma_\varepsilon'|$ into account, we obtain

$$mA_0 = \int_{M_\varepsilon} u_\varepsilon \, d\mu \to \int_{M_\gamma} u \, d\mu$$

as $\varepsilon \to 0$. The bound $\mathcal{H}^1(M(S_u)) < \infty$ follows from (3.3) and Young's inequality, the details are given in the lower bound section and are thus here omitted. □

Remark. In the classical one-dimensional setting without the area measure, a uniform L^∞-bound for the phase fields is in fact a result of the uniform energy bound, see [Bra02]. In our case, however, this bound depends in $J \Subset \{y > 0\}$ on the constant c_J, which is essentially the infimum of y on J, and might tend to infinity as $c_J \to 0$. Our assumption $\|u_\varepsilon\|_\infty \leq C_0$ is used to ensure L^p-convergence and can here be weakened to an L^1-bound.

3.2 Lower bound

Next we prove the lower bound inequality

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \geq \mathcal{E}(\gamma, u) \tag{3.4}$$

whenever $(\gamma_\varepsilon, u_\varepsilon)$ converges to (γ, u) in $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$. It suffices to examine the case when the left hand side of (3.4) is finite and to consider a subsequence such that the lower limit is attained. Then by definition $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, and our compactness argument yields $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ and the convergence properties listed in Lemmas 3.2 and 3.3. In particular, $\gamma_\varepsilon \rightharpoonup \gamma$ in $W^{2,2}_{\mathrm{loc}}(\{y>0\}\,;\mathbb{R}^2)$ and $u_\varepsilon \to u$ in $L^1(I)$ imply

$$\begin{aligned}\int_{M(J)} (H - H_s(u))^2\,d\mu &\le \liminf_{\varepsilon\to 0}\int_{M_\varepsilon(J)} (H_\varepsilon - H_s(u_\varepsilon))^2\,d\mu_\varepsilon\\ &\le \liminf_{\varepsilon\to 0}\int_{M_\varepsilon} (H_\varepsilon - H_s(u_\varepsilon))^2\,d\mu_\varepsilon\end{aligned}$$

for any $J \Subset \{y>0\}$, and since the right hand side is independent of J, we obtain the bulk lower bound inequality

$$\int_{M(\{y>0\})} (H - H_s(u))^2\,d\mu \le \liminf_{\varepsilon\to 0}\int_{M_\varepsilon} (H_\varepsilon - H_s(u_\varepsilon))^2\,d\mu_\varepsilon. \tag{3.5}$$

To analyse the interface energy let $s \in S_u$ and fix an interval $J \Subset \{y>0\}$ such that $\overline{J}\cap S_u = \{s\}$, which exists because $S_u \cap \{y > y(s)/2\}$ is finite. From the convergence of u_ε we deduce that there are points $a_\varepsilon, b_\varepsilon \in J$ with $a_\varepsilon < s < b_\varepsilon$ or $b_\varepsilon < s < a_\varepsilon$ such that $a_\varepsilon \to s$, $b_\varepsilon \to s$, $u_\varepsilon(a_\varepsilon) \to -1$ and $u_\varepsilon(b_\varepsilon) \to 1$ as $\varepsilon \to 0$. Assuming without loss of generality that $a_\varepsilon < b_\varepsilon$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{M_\varepsilon(a_\varepsilon,b_\varepsilon)} \varepsilon|\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,d\mu_\varepsilon &\ge \left(\inf_{(a_\varepsilon,b_\varepsilon)} y_\varepsilon\right)\int_{a_\varepsilon}^{b_\varepsilon} 2\sqrt{W(u_\varepsilon)}|u_\varepsilon'|\,dt\\ &\ge \left(\inf_{(a_\varepsilon,b_\varepsilon)} y_\varepsilon\right)\left|\int_{u_\varepsilon(a_\varepsilon)}^{u_\varepsilon(b_\varepsilon)} 2\sqrt{W(u)}\,du\right|\end{aligned}$$

thanks to Young's inequality and a change of variables. Taking the lower limit yields

$$\liminf_{\varepsilon\to 0}\int_{M_\varepsilon(a_\varepsilon,b_\varepsilon)} \varepsilon|\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,d\mu_\varepsilon \ge 2\pi y(s)\int_{-1}^{1} 2\sqrt{W(u)}\,du = 2\pi y(s)\sigma. \tag{3.6}$$

The above argument applies to each point of any finite subset S of S_u, and in addition we may extend the integral on the left hand side of (3.6) to the whole surface to obtain

$$\liminf_{\varepsilon\to 0}\int_{M_\varepsilon} \varepsilon|\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,d\mu_\varepsilon \ge \sigma\mathcal{H}^1(M(S)).$$

Since the left hand side is independent of S, the interface lower bound inequality

$$\liminf_{\varepsilon\to 0}\int_{M_\varepsilon} \varepsilon|\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,d\mu_\varepsilon \ge \sigma\mathcal{H}^1(M(S_u)) \tag{3.7}$$

follows from taking the supremum over all finite sets $S \subset S_u$. Combining (3.7) and (3.5) yields the lower bound inequality (3.4).

3.3 Upper bound

In the final section of the current chapter we construct a recovery sequence for (γ, u) with finite energy $\mathcal{E}(\gamma, u)$. To this end we first show that (γ, u) can be approximated by configurations with finitely many interfaces. We then construct a recovery sequence for such a configuration by changing the curve essentially only between components near the axis of revolution and the phase field only around interfaces.

Throughout this section we assume without loss of generality that γ is parametrised with constant speed $|\gamma'| \equiv 1$.

3.3.1 Approximation by finite number of interfaces

Lemma 3.4. *Assume that $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ has countably many interfaces. Then there exists $(\gamma, u_\delta) \in \mathcal{C} \times \mathcal{P}$ for sufficiently small $\delta > 0$, each with a finite number of interfaces, such that $u_\delta \to u$ in $L^p(I)$ for any $p \in [1, \infty)$ and $\mathcal{E}(\gamma, u_\delta) \to \mathcal{E}(\gamma, u)$ as $\delta \to 0$.*

Proof. Let 3δ be smaller than the minimal length of γ restricted to a component of $\{y > 0\}$. We construct u_δ by omitting interfaces whose distance on γ to a component boundary is less than δ. More precisely, for a component $\omega = (a, b)$ of $\{y > 0\}$ let $a_\delta = a + \delta$ and $b_\delta = b - \delta$. By the choice of δ, $a_\delta < b_\delta$ and $\mathcal{L}_\gamma(a, a_\delta) = \mathcal{L}_\gamma(b_\delta, b) = \delta$. Define u_δ on ω to be equal to u in (a_δ, b_δ), and continuously extend it to $(a, a_\delta) \cup (b_\delta, b)$, that is

$$u_\delta = \begin{cases} u & \text{in } (a_\delta, b_\delta), \\ \lim\limits_{t \searrow a_\delta} u(t) & \text{in } (a, a_\delta], \\ \lim\limits_{t \nearrow b_\delta} u(t) & \text{in } [b_\delta, b) \end{cases}$$

in ω. Since the number of components N_γ is finite, this procedure can be applied to each component and yields a configuration (γ, u_δ) with finitely many interfaces. By construction $|u - u_\delta| \leq 2$ and $y \leq \delta$ in $(a, a_\delta) \cup (b_\delta, b)$, so we have

$$\int_{M_\gamma} |u - u_\delta|^p \, d\mu \leq 2^{p+1} N_\gamma \delta^2 \tag{3.8}$$

and $u_\delta \to u$ as $\delta \to 0$ in $L^p(I)$ for any $p \in [1, \infty)$. Furthermore,

$$|\mathcal{H}^1(M_\gamma(S_u)) - \mathcal{H}^1(M_\gamma(S_{u_\delta}))| \leq \mathcal{H}^1(M_\gamma(S_u \cap \{y \leq \delta\}))$$

and

$$\left| \int_{M_\gamma} (H - H_s(u))^2 - (H - H_s(u_\delta))^2 \, d\mu \right| \leq 4 \|H_s\|_\infty \int_{M_\gamma(\{y \leq \delta\})} \|H_s\|_\infty + |H| \, d\mu,$$

converge to 0, therefore $\mathcal{E}(\gamma, u_\delta) \to \mathcal{E}(\gamma, u)$. Finally, for sufficiently small δ there is an interface $s \in S_u \cap S_{u_\delta}$ that is independent of δ and whose distance to all other interface points is greater than δ. According to (3.8) the error in the phase constraint is at most of order δ^2, so it suffices to move s by an order of at most δ^2 to the left or right to recover the integral constraint $\int_{M_\gamma} u_\delta \, d\mu = mA_0$. This additional change yields $u_\delta \in \mathcal{P}$ and does obviously not disturb the convergence of the phase fields or the energy. $\square$

Thanks to Lemma 3.4 we can assume from now on that (γ, u) has only finitely many interfaces. Then the minimal distance between two interfaces and from an interface to the boundary of its component is strictly positive. Hence, for any interface $s \in S_u$ there is an interval $J \Subset \{y > 0\}$ that contains no other interface, and for any component boundary point $s \in \{y = 0\}$ there is an interval $J \subset I$ that contains no other component boundary point or interface. The recovery construction for interfaces and component boundaries is done separately for each of these points.

3.3.2 Curve approximation

Let $s \in \{y = 0\}$ be a point that connects two components of M_γ on the axis of revolution, and fix $J \subset I$ such that $\overline{J} \cap (\{y = 0\} \cup S_u) = \{s\}$. For simplicity of notation we assume $s = 0$ and $\gamma(0) = (0, 0)$.

We aim to replace γ in J by a sequence $(\gamma_\delta) \subset W^{2,2}(J; \mathbb{R}^2)$ that converges to γ and satisfies $\mathcal{E}(\gamma_\delta, u) \to \mathcal{E}(\gamma, u)$ as $\delta \to 0$. Since we are close to the axis of revolution, the dominating term in the energy is the squared mean curvature, in particular the second principal curvature. In order to control this term, the approximation is based on scaled catenoids which satisfy $\kappa_1 = -\kappa_2$, thus $H = 0$. Since both one-sided limit tangents of $\gamma = (x, y)$ are perpendicular to the axis of revolution at $s = 0$ and γ satisfies $x(t) \geq 0$ for $t \geq 0$, $x(t) \leq 0$ for $t \leq 0$, it suffices to study a construction for γ_δ in $J \cap \{t \geq 0\}$ that can be smoothly joined at $s = 0$ with its mirrored counterpart in $J \cap \{t \leq 0\}$.

Two situations can occur in $J \cap \{t \geq 0\}$: either γ is a vertical line segment near $s = 0$ or it is not. In the latter case, which is studied in Lemma 3.5, we glue the catenoid directly to the original surface; in the former case, considered in Lemma 3.6, an additional transition region is needed to join the catenoid to the vertical original surface, see Figure 3.2 for a sketch of the constructions. In both cases the construction is essentially local, but the original surface is slightly shifted in x-direction to gain some space for the catenoid and maintain $x' \geq 0$. These shifts obviously do not change energy, area or phase constraint, and they vanish as $\delta \to 0$.

We denote by $c_\delta = (i_\delta, j_\delta) \in C^\infty(\mathbb{R}; \mathbb{R}^2)$, where

$$i_\delta(t) = \delta \operatorname{arcsinh} \tfrac{t}{\delta}, \qquad j_\delta(t) = \sqrt{\delta^2 + t^2},$$

the δ-catenary, which is the generating curve of a δ-catenoid parametrised with unit speed. The δ-catenoid has principal curvatures

$$-\kappa_1(t) = \kappa_2(t) = \frac{\delta}{\delta^2 + t^2}$$

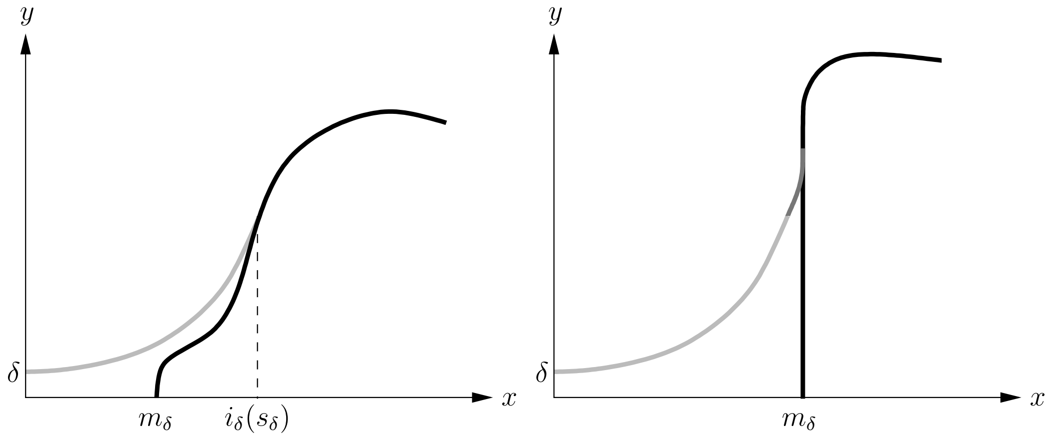

Figure 3.2: The construction for Lemma 3.5 (left) and Lemma 3.6 (right): original curve (black), δ-catenary (light grey), and circle segment (dark grey).

and thus satisfies

$$\int_{M_{c_\delta}(a,b)} |B_{c_\delta}|^2 \, d\mu = 4\pi \int_a^b \frac{\delta^2}{(\delta^2+t^2)^2} \sqrt{\delta^2+t^2} \, dt = 4\pi \left. \frac{t}{\sqrt{\delta^2+t^2}} \right|_a^b \leq 8\pi \tag{3.9}$$

for any $\delta > 0$ and all $a, b \in \mathbb{R}$, $a < b$.

Lemma 3.5. *Assume that γ is not a vertical line segment in $J = (0, t_0)$, that is, there exists a decreasing sequence $t_k \to 0$ as $k \to \infty$ such that $x'(t_k) > 0$. Then for all sufficiently small δ depending only on γ there is a curve $\gamma_\delta = (x_\delta, y_\delta) \in W^{2,2}(J; \mathbb{R}^2)$ such that*

- *γ_δ satisfies $y_\delta > 0$, $x'_\delta \geq 0$ and $|\gamma'_\delta| = 1$ in J;*
- *at the end points of $\gamma_\delta(J)$ we have*

$$\gamma_\delta(0) = (0, \delta), \quad \gamma'_\delta(0) = (1, 0), \quad \gamma_\delta(t_0) = (x(t_0) + o(1), y(t_0)), \quad \gamma'_\delta(t_0) = \gamma'(t_0);$$

- *$\gamma_\delta \to \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ for any $p \in [1, \infty)$ as $\delta \to 0$;*
- *$\mathcal{A}_\delta(J) = \mathcal{A}(J) + o(1)$ and $\int_{M_\delta(J)} u \, d\mu_\delta = \int_{M_\gamma(J)} u \, d\mu + o(1)$ as $\delta \to 0$;*
- *$\sup_{\delta > 0} \int_{M_\delta(J)} |B_\delta|^2 \, d\mu_\delta < \infty$; and*
- *$\int_{M_\delta(J)} (H_\delta - H_s(u))^2 \, d\mu_\delta \to \int_{M_\gamma(J)} (H - H_s(u))^2 \, d\mu$ as $\delta \to 0$.*

Proof. We replace γ in some interval $(0, s_\delta)$ with s_δ to be chosen by a δ-catenary, shifting the remaining part of γ slightly in x-direction to obtain a C^1-connection at s_δ, see Figure 3.2 on the left. Since γ' is continuous up to 0 and $y(t) > 0$ for $t > 0$, we have $\gamma'(t_k) \to (0, 1)$ as

$t_k \to 0$. By choosing t_1 sufficiently small we may assume $y'(t) > 0$ for all $0 \le t \le t_1$. We look for the connection point $s_\delta \in J$ and a shift $m_\delta \in \mathbb{R}$ such that

$$\frac{j'_\delta(s_\delta)}{i'_\delta(s_\delta)} = \frac{y'(s_\delta)}{x'(s_\delta)} \qquad \text{and} \qquad c_\delta(s_\delta) = (x(t_\delta) + m_\delta, y(t_\delta)). \tag{3.10}$$

The second equation in (3.10) ensures that the piecewise defined curve

$$\gamma_\delta(t) = (x_\delta(t), y_\delta(t)) = \begin{cases} c_\delta(t) & \text{if } 0 \le t \le s_\delta, \\ (x(t) + m_\delta, y(t)) & \text{if } s_\delta < t \end{cases}$$

is continuous; the first equation together with $|c'_\delta(s_\delta)| = |\gamma'(s_\delta)| = 1$ implies $c'_\delta(s_\delta) = \gamma'(s_\delta)$. Since $\gamma \in W^{2,2}(J \cap \{t > s_\delta\}; \mathbb{R}^2)$ for any $s_\delta > 0$ and $c_\delta \in C^\infty(\mathbb{R}; \mathbb{R}^2)$, $\gamma_\delta \in W^{2,2}(J; \mathbb{R}^2)$ follows.

To find s_δ we compute the derivatives of the catenary and rewrite (3.10) to obtain

$$\delta = s_\delta \frac{x'(s_\delta)}{y'(s_\delta)} \tag{3.11}$$

for $s_\delta \le t_1$. The function $g\colon [0, t_1] \to \mathbb{R}$, $t \mapsto tx'(t)/y'(t)$ is continuous and satisfies $g(0) = 0$ and $g_k := g(t_k) > 0$ for all $k \in \mathbb{N}$. After restriction to a subsequence, we may assume that (g_k) is decreasing. Given $\delta \in (0, g_1)$, let k_δ be the largest k, or equivalently g_{k_δ} the smallest g_k, such that $\delta \le g_{k_\delta}$. Then by the Intermediate Value Theorem there exists $s_\delta \in (0, t_{k_\delta})$ such that (3.11) holds. To satisfy the second condition in (3.10), we set $m_\delta = i_\delta(s_\delta) - x(s_\delta)$.

As δ tends to 0, also $s_\delta \le t_{k_\delta} \to 0$, $g_{k_\delta} = g(t_{k_\delta}) \to 0$, and $m_\delta \to 0$. Since both γ and c_δ are uniformly bounded in $W^{1,\infty}$, we obtain $\gamma_\delta \to \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ for any $p \in [1, \infty)$ and $\mathcal{A}_{\gamma_\delta}(J) \to \mathcal{A}_\gamma(J)$. The mean curvature H_{γ_δ} of γ_δ is given by $H_{\gamma_\delta}(t) = 0$ for $t < s_\delta$ and $H_{\gamma_\delta}(t) = H_\gamma(t)$ for $t > s_\delta$, and the spontaneous curvature $H_s(u)$ is constant in J. Hence,

$$\int_{M_{\gamma_\delta}(J)} (H_{\gamma_\delta} - H_s(u))^2 \, d\mu_\delta \to \int_{M_\gamma(J)} (H - H_s(u))^2 \, d\mu$$

as $\delta \to 0$. Moreover,

$$\int_{M_{\gamma_\delta}(J)} |B_{\gamma_\delta}|^2 \, d\mu_\delta \le \int_{M_{c_\delta}(\mathbb{R})} |B_{c_\delta}|^2 \, d\mu + \int_{M_\gamma} |B|^2 \, d\mu$$

is bounded because of (3.9) and $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$.

Finally, we estimate the error in the area constraint that is introduced by γ_δ. We find $\mathcal{A}_\gamma(0, s_\delta) = O(s_\delta^2)$ and

$$\begin{aligned} \frac{1}{2\pi} \mathcal{A}_{\gamma_\delta}(0, s_\delta) &= \int_0^{s_\delta} \sqrt{\delta^2 + t^2} \, dt = \frac{1}{2} \left(t\sqrt{\delta^2 + t^2} + \delta^2 \ln(t + \sqrt{\delta^2 + t^2}) \right) \Bigg|_0^{s_\delta} \\ &= \frac{1}{2} \left(s_\delta \sqrt{\delta^2 + s_\delta^2} + \delta^2 \ln\left(s_\delta + \sqrt{\delta^2 + s_\delta^2} \right) - \delta^2 \ln \delta \right), \\ &= \frac{1}{2} s_\delta^2 \left(\sqrt{h_\delta^2 + 1} + h_\delta^2 \ln \frac{1 + \sqrt{h_\delta^2 + 1}}{h_\delta} \right), \end{aligned}$$

where $h_\delta = \delta/s_\delta = x'(s_\delta)/y'(s_\delta) \to 0$ as $\delta \to 0$. Therefore,

$$\mathcal{A}_{\gamma_\delta}(0, s_\delta) \sim s_\delta^2 \left(1 + h_\delta^2 \ln \frac{1}{h_\delta}\right) \sim s_\delta^2 \,(1 + o(1))$$

and the error in the area constraint is of order $O(s_\delta^2)$. Since $u \in \{\pm 1\}$, the same is true for the phase integral constraint. □

Lemma 3.6. *Assume that γ is a vertical line segment in $J = (0, t_0)$, that is $\gamma(t) = (0, t)$ in J. Then there is γ_δ such that all conclusions of Lemma 3.5 except for $|\gamma_\delta'| = 1$ in J still hold. Instead, there exists $\widetilde{J} \Subset J$ that is independent of δ such that*

$$|\gamma_\delta'| = \begin{cases} 1 & \textit{if } t = J \setminus \widetilde{J}, \\ 1 + r_\delta & \textit{if } t \in \widetilde{J}, \end{cases}$$

with $r_\delta \to 0$ in $W^{1,2}(\widetilde{J}) \cap C^0(\widetilde{J})$ as $\delta \to 0$.

Proof. Now we replace γ by a δ-catenary in some interval $[0, s_\delta)$ and a segment of a circle of radius one in $[s_\delta, t_\delta)$, which connects the catenary and the shifted original curve, see Figure 3.2 on the right. Writing

$$k_\delta(t) = \sin(t - b_\delta) + \hat{k}_\delta \qquad \text{and} \qquad l_\delta(t) = -\cos(t - b_\delta) + \hat{l}_\delta$$

for the x- and y-coordinate of the circle and fixing δ and s_δ for the moment, we aim to determine t_δ, b_δ, $\hat{k}_\delta$, $\hat{l}_\delta$, and shifts m_δ, n_δ such that

$$\gamma_\delta(t) = \begin{cases} c_\delta(t) & \text{if } 0 \le t < s_\delta, \\ (k_\delta(t), l_\delta(t)) & \text{if } s_\delta \le t < t_\delta, \\ (m_\delta, t + n_\delta) & \text{if } t_\delta \le t \end{cases}$$

is continuously differentiable at s_δ and t_δ. The corresponding conditions are

$$\begin{aligned} &k_\delta'(s_\delta) = i_\delta'(s_\delta), && l_\delta'(s_\delta) = j_\delta'(s_\delta), && k_\delta(s_\delta) = i_\delta(s_\delta), && l_\delta(s_\delta) = j_\delta(s_\delta), \\ &k_\delta'(t_\delta) = 0, && l_\delta'(t_\delta) = 1, && k_\delta(t_\delta) = m_\delta, && l_\delta(t_\delta) = t + n_\delta, \end{aligned}$$

and a short calculation shows

$$\begin{aligned} b_\delta &= s_\delta - \arctan(s_\delta/\delta), & t_\delta &= \pi/2 + b_\delta, \\ \hat{k}_\delta &= i_\delta(s_\delta) - \sin(s_\delta - b_\delta), & \hat{l}_\delta &= j_\delta(s_\delta) + \cos(s_\delta - b_\delta), \\ m_\delta &= 1 + \hat{k}_\delta, & n_\delta &= \hat{l}_\delta - t_\delta. \end{aligned}$$

If we let $s_\delta = \delta^\beta$ for some $\beta \in (0, 3/4)$, we find

$$t_\delta \sim \delta^\beta + \delta^{1-\beta}, \qquad m_\delta \sim \delta \ln \delta^{\beta - 1}, \qquad n_\delta \sim \delta^\beta + \delta^{1-\beta},$$

that is, the catenary and circle vanish as $\delta \to 0$, and therefore $\gamma_\delta \to \gamma$ in $W^{1,p}(J;\mathbb{R}^2)$ for $p \in [1,\infty)$ and $\mathcal{A}_{\gamma_\delta}(J) \to \mathcal{A}_\gamma(J)$. The principal curvatures of the circle segment are

$$\kappa_1 = -1 \qquad \text{and} \qquad \kappa_2 = \frac{\cos(t-b_\delta)}{\hat{l}_\delta - \cos(t-b_\delta)},$$

thus its second fundamental form is estimated by

$$\begin{aligned}\frac{1}{2\pi}\int_{M_{\gamma_\delta}(s_\delta,t_\delta)} |B_\delta|^2\,d\mu_\delta &\le \left(1+\frac{\cos^2(s_\delta-b_\delta)}{j_\delta(s_\delta)}\right)\cdot(t_\delta-s_\delta)\\ &= \left(1+\frac{\delta^2}{(\delta^2+\delta^{2\beta})^{3/2}}\right)\cdot\left(\pi/2-\arctan\delta^{\beta-1}\right)\\ &\sim \left(1+\delta^{2-3\beta}\right)\cdot\delta^{1-\beta} = \delta^{1-\beta}+\delta^{3-4\beta},\end{aligned}$$

which vanishes as $\delta \to 0$ due to $0 < \beta < 3/4$. Therefore, convergence of the curvature integral and the uniform bound on the second fundamental form follow as in Lemma 3.5. Also as in Lemma 3.5, the area of the catenoid part is of order $s_\delta^2 = \delta^{2\beta}$, and a similar computation shows $\mathcal{A}_{\gamma_\delta}(s_\delta,t_\delta) \sim \delta + \delta^{2-2\beta}$ and $\mathcal{A}_\gamma(0,t_\delta) \sim \delta + \delta^{2\beta} + \delta^{2-2\beta}$. Hence we obtain

$$|\mathcal{A}_{\gamma_\delta}(0,t_\delta) - \mathcal{A}_\gamma(0,t_\delta)| \lesssim \delta + \delta^{2\beta} + \delta^{2-2\beta}.$$

To dispose of the shift n_δ in y-direction, fix $\widetilde{J} \Subset J$ with $t_\delta < \inf \widetilde{J}$ for all sufficiently small δ and a function $f \in C_c^\infty(\widetilde{J})$ with $\int_J f\,dt = 1$. The perturbed curve $\widetilde{\gamma}_\delta = (\widetilde{x}_\delta, \widetilde{y}_\delta) = (x_\delta, y_\delta - n_\delta F)$, where $F(t) = \int_0^t f(s)\,ds$, has the desired end point y-coordinate

$$\widetilde{y}_\delta(t_0) = y_\delta(t_0) - n_\delta = y(t_0).$$

Since $|n_\delta| \to 0$ as $\delta \to 0$, the perturbation vanishes in any function space to which γ_δ belongs. The second fundamental forms are still uniformly bounded in $L^2(J)$, because for all small δ the perturbation is supported in a vertical line segment of γ_δ, where both principal curvatures are equal to 0. Moreover, we have $\inf_{\widetilde{J}} \widetilde{y}_\delta > 0$ and

$$|\mathcal{A}_{\widetilde{\gamma}_\delta}(t_\delta,t_0) - \mathcal{A}_\gamma(t_\delta,t_0)| \sim |(t_0^2-(t_\delta+n_\delta)^2)-(t_0^2-t_\delta)^2| \sim n_\delta t_\delta + n_\delta^2 \sim \delta+\delta^{2\beta}+\delta^{2-2\beta},$$

so the error in the area constraint in J and in the phase integral constraint are of the same order as above. □

Next we use the local constructions to build global approximations of γ.

Corollary 3.7. *Let $(\gamma,u) \in \mathcal{C}\times\mathcal{P}$ have finitely many interfaces. Then there exists a sequence $(\gamma_\delta,u_\delta) \in \mathcal{C}\times\mathcal{P}$ such that $\gamma_\delta \to \gamma$ in $W^{1,p}(I;\mathbb{R}^2)$, $u_\delta \to u$ in $L^p(I)$ for any $p \in [1,\infty)$ and $\mathcal{E}(\gamma_\delta,u_\delta) \to \mathcal{E}(\gamma,u)$ as $\delta \to 0$. Moreover, there holds $\gamma_\delta \in \mathcal{C}_\varepsilon$ for any $\delta > 0$.*

Proof. Let $\{t \in I : y(t) = 0\} = \{s_1, \dots, s_n\}$ where $n = N_\gamma - 1$. We employ Lemma 3.5 or 3.6 and their mirrored versions successively for each s_k, taking the global shifts in x-direction into account. The result for sufficiently small δ is a sequence γ_δ that converges to γ in $W^{1,p}(I;\mathbb{R}^2)$ for any $p \in [1,\infty)$ and satisfies

$$\int_{M_{\gamma_\delta}} (H_\delta - H_s(u))^2 \, d\mu_\delta \to \int_{M_\gamma} (H - H_s(u))^2 \, d\mu$$

as $\delta \to 0$. New interfaces might have been introduced at $s_1, \dots, s_n$ if u has a jump at one or more of these points. Their energy, however, is bounded by $\sigma N_\gamma \delta$; hence $\mathcal{E}(\gamma_\delta, u) \to \mathcal{E}(\gamma, u)$ as $\delta \to 0$.

To obtain a configuration in $\mathcal{C} \times \mathcal{P}$ with $\gamma_\delta \in \mathcal{C}_\varepsilon$ we have to correct the area, the phase integral, and the velocity constraint $|\gamma_\delta'| = \text{const}$. For the first find $J \Subset \{y > 0\} \setminus S_u$ such that $\gamma_\delta = \gamma$ except for an x-shift in J for all sufficiently small δ, $x' > 0$ in $\overline{J}$ and $M_\gamma(J)$ is not part of a catenoid, that is, γ is not stationary for the area among graphs of revolution with the same boundary as γ in J. Such an interval exists, because otherwise γ restricted to any component of $\{y > 0\}$ would consist only of vertical lines and catenary segments which is impossible for a C^1-curve that starts and ends on the x-axis. Let $f \in C_c^\infty(J)$ and $\widetilde{\gamma}_{\delta,\alpha} = \gamma_\delta + \alpha f$. Since

$$\mathcal{A}_{\widetilde{\gamma}_{\delta,\alpha}} = \mathcal{A}_{\gamma_\delta} + \mathcal{A}_{\widetilde{\gamma}_{\delta,\alpha}}(J) - \mathcal{A}_\gamma(J),$$

the requirement $\mathcal{A}_\gamma = \mathcal{A}_{\widetilde{\gamma}_{\delta,\alpha}}$ is equivalent to

$$\mathcal{A}_{\widetilde{\gamma}_{\delta,\alpha}}(J) - \mathcal{A}_\gamma(J) = \mathcal{A}_\gamma - \mathcal{A}_{\gamma_\delta}. \tag{3.12}$$

The left hand side of (3.12) equals 0 for $\alpha = 0$ and depends continuously on α. It is strictly positive for one sign of α and strictly negative for the other, since γ is not stationary for the area in J. Note that we need it to be negative, since the area is increased when replacing the original curve with a catenoid. The right hand side of (3.12) vanishes as $\delta \to 0$, hence for all sufficiently small δ there is an α_δ such that (3.12) holds; moreover, $\alpha_\delta \to 0$ as $\delta \to 0$.

The phase integral constraint is recovered as in Lemma 3.4 by moving one or more interfaces by a distance $o(1)_{\delta\to 0}$ to the left or right in a way that does not interfere with the other approximation procedures applied so far. Hence, we obtain $(\gamma_\delta, u_\delta)$ that satisfies all conditions of $\mathcal{C}_\varepsilon$ and $\mathcal{P}$ except for the constant speed requirement. But since by construction $|\gamma_\delta'| = 1 + o(1)$ and the perturbation vanishes in $W^{1,2}$, the constant speed reparametrisations converge to the identity in $W^{2,2}(I)$ and the properties of $(\gamma_\delta, u_\delta)$ carry over to reparametrised curve and phase field. □

As a consequence of Corollary 3.7, it remains to construct a recovery sequence for (γ, u), where $\gamma \in \mathcal{C}_\varepsilon$ and $u \in \mathcal{P}$ has finitely many jumps.

3.3.3 Local interface recovery

The recovery of a phase field u with finitely many jumps follows the lines of the one-dimensional setting of the Modica-Mortola theory for phase transitions; the main difference is the inhomogeneity due to the area measure $d\mu = 2\pi y\, dt$. But since u is changed only in an interval of order $\sqrt{\varepsilon}$ around each interface, this difference is easy to cope with.

It is well known, see for instance [Alb00], that in the classical one-dimensional setting the ε-energy-minimal profile for a transition of u_ε from -1 to $+1$ is obtained by minimising

$$G_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon|u'|^2 + \frac{1}{\varepsilon}W(u)\, dt$$

among functions u that satisfy $u(0) = 0$ and $u(\pm\infty) = \pm 1$. Indeed, setting $u_\varepsilon(t) = u(t/\varepsilon)$ we observe

$$G_\varepsilon(u_\varepsilon) = G_1(u) \geq 2\int_{\mathbb{R}} \sqrt{W(u)}u'\, dt = 2\int_{\mathbb{R}} \sqrt{W(u)}\, du = \sigma.$$

Equality holds if and only if

$$u' = \sqrt{W(u)}, \tag{3.13}$$

which admits a local solution p with initial condition $p(0) = 0$ because $\sqrt{W(\cdot)}$ is continuous. Obviously, the constants $+1$ and -1 are a global super- and sub-solution of (3.13), hence p can be extended to the whole real line. Since $W(p) > 0$ for $p \in (-1, +1)$, $p(t)$ converges to ± 1 as $t \to \pm\infty$. Thus $p(t/\varepsilon)$ minimises G_ε. Due to the symmetry of W we can presume $-p(-t) = p(t)$ and need to know the profile only for $t \geq 0$.

Let $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$, $\gamma \in \mathcal{C}_\varepsilon$ have finitely many interfaces and consider $s \in S_u$ and $J \Subset \{y > 0\}$ such that $\overline{J} \cap S_u = \{s\}$. For simplicity of notation we assume $s = 0$. Using an appropriately scaled version of the optimal profile p and a linear interpolation, we aim to construct the recovery sequence by replacing $u = \operatorname{sign} t$ in J with

$$p_\varepsilon(t) = \begin{cases} p(t/\varepsilon) & \text{if } 0 \leq t < \sqrt{\varepsilon}, \\ p(1/\sqrt{\varepsilon}) + \frac{1}{\varepsilon}(t - \sqrt{\varepsilon}) & \text{if } \sqrt{\varepsilon} \leq t < \sqrt{\varepsilon} + \varepsilon(1 - p(1/\sqrt{\varepsilon})), \\ 1 & \text{if } \sqrt{\varepsilon} + \varepsilon(1 - p(1/\sqrt{\varepsilon})) \leq t \end{cases}$$

for $t \geq 0$ and $p_\varepsilon(t) = -p_\varepsilon(-t)$ for $t < 0$; if $u = -\operatorname{sign} t$ in J, we use $-p_\varepsilon$. But since γ is in general not symmetric around $s = 0$ we have to correct p_ε in order to conserve the phase integral constraint.

Lemma 3.8. *There is $u_\varepsilon \in W^{1,2}(J)$ with $\{u_\varepsilon \neq u\} \Subset J$ such that $\|u_\varepsilon\|_\infty \leq C_0$, $u_\varepsilon \to u$ in $L^1(J)$, $\int_{M_\gamma(J)} u_\varepsilon\, d\mu = \int_{M(J)} u\, d\mu$, and*

$$\limsup_{\varepsilon \to 0} \int_{M_\gamma(J)} \varepsilon|\nabla_{M_\gamma} u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\, d\mu \leq 2\pi\sigma y(s). \tag{3.14}$$

Proof. Convergence $p_\varepsilon \to u$ in $L^1(J)$ is obvious from the definition of p_ε, and the estimate (3.14) with p_ε instead of u_ε follows by taking the upper limit $\varepsilon \to 0$ in

$$\int_{M_\gamma(J)} \varepsilon|\nabla_{M_\gamma} p_\varepsilon|^2 + \frac{1}{\varepsilon}W(p_\varepsilon)\,d\mu \le 2\pi \left(\sup_{[-\varepsilon,\varepsilon]} y\right) \int_{-1\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} |p'(t)|^2 + W(p(t))\,dt$$
$$+ 2\pi\left(1 - p(1/\sqrt{\varepsilon})\right)\left(\sup_J y\right)\left(1 + \sup_{[-1,1]} W\right).$$

To recover the constraint let $f\colon J \to \mathbb{R}$ be smooth, have compact support in $J \cap \{t > 0\}$ and satisfy $\int_{M_\gamma(J)} f\,d\mu = 1$. Then the phase integral is conserved by $u_\varepsilon = p_\varepsilon + \alpha_\varepsilon f$ if

$$\alpha_\varepsilon = \int_{M_\gamma(J)} u - p_\varepsilon\,d\mu.$$

Since

$$\int_{M_\gamma(0,\sqrt{\varepsilon})} 1 - p_\varepsilon\,d\mu \le 2\pi\|y\|_\infty\sqrt{\varepsilon}\int_0^1 1 - p(t/\sqrt{\varepsilon})\,dt = o(\sqrt{\varepsilon}),$$

α_ε is also of order $o(\sqrt{\varepsilon})$. This is sufficient to still ensure convergence $u_\varepsilon \to u$ in $L^1(J)$ and the energy inequality

$$\limsup_{\varepsilon\to 0}\int_{M_\gamma(J)} \varepsilon|\nabla_{M_\gamma} u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,d\mu \le 2\pi\sigma y(s)$$

thanks to

$$\frac{1}{\varepsilon}W(\pm 1 + \alpha_\varepsilon f) = \frac{1}{\varepsilon}\Big(W(\pm 1) + \alpha_\varepsilon f W'(\pm 1) + O(\alpha_\varepsilon^2)\Big) = o(1).$$

By construction, we have $u_\varepsilon \in W^{1,2}(J)$ and $\|u_\varepsilon\|_\infty \le \|p_\varepsilon\|_\infty + |\alpha_\varepsilon|\|f\|_\infty \le C_0$ for all sufficiently small $\varepsilon > 0$. ☐

3.3.4 Summary

For convenience we summarise the construction of the recovery sequence.

Corollary 3.9. *For $(\gamma, u) \in \mathcal{C}\times\mathcal{P}$ there is a sequence $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ such that $(\gamma_\varepsilon, u_\varepsilon) \to (\gamma, u)$ in $W^{1,1}(I;\mathbb{R}^2)\times L^1(I)$ and $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \le \mathcal{E}(\gamma, u)$.*

Proof. By Lemma 3.4, Corollary 3.7, and a diagonal argument there is a sequence $(\gamma_\delta, u_\delta) \in \mathcal{C}\times\mathcal{P}$ that converges to (γ, u) in $W^{1,1}(I;\mathbb{R}^2)\times L^1(I)$ such that $\mathcal{E}(\gamma_\delta, u_\delta) \to \mathcal{E}(\gamma, u)$ as $\delta \to 0$. Each $(\gamma_\delta, u_\delta)$ satisfies $\gamma_\delta \in \mathcal{C}_\varepsilon$ and has finitely many interfaces; thus, applying Lemma 3.8 to each interface of any u_δ yields recovery sequences $(\gamma_\delta, u_{\delta,\varepsilon})$ for $(\gamma_\delta, u_\delta)$. Taking another diagonal sequence finishes the proof. ☐

Chapter 4

The singular approximation

In this chapter we study the singular approximation

$$\mathcal{F}_\varepsilon(\gamma,u)=\int_{M_\gamma} u^2\left(H-H_s(u)\right)^2-u^2K\,d\mu+\varepsilon\int_{M_\gamma}|B|^2\,d\mu+\int_{M_\gamma}\varepsilon|\nabla_{M_\gamma}u|^2+\frac{1}{\varepsilon}W(u)\,d\mu \tag{4.1}$$

for $(\gamma,u)\in\mathcal{C}_\varepsilon\times\mathcal{P}_\varepsilon$. Essentially, this amounts to choosing the bending parameter extensions $k(u,\varepsilon)=-k_G(u,\varepsilon)=u^2+\varepsilon$ in (2.7), since $\varepsilon\int H_s(u)^2\,d\mu$ vanishes as $\varepsilon\to 0$.

4.1 Approximate setting

As stated in the introduction, the factor u^2 in front of the curvature terms in (4.1) is based on the Ambrosio-Tortorelli approximation of free discontinuity problems [AT90, Bra98]. It allows curvatures to become large where the phase field is zero. The choice of the Gauss rigidity depends on the bending rigidity and is motivated as follows. In order that $\mathcal{F}_\varepsilon$ is bounded from below and to obtain the lower bound inequality, we require that an extension $k_G(u)$ of $k_G^\pm$ to continuous phase fields satisfies

$$\int_{M_\gamma} u^2\left(H-H_s(u)\right)^2+k_G(u)K\,d\mu\geq\int_{M_\gamma} g_1(u)|B|^2-g_2(u)\,d\mu, \tag{4.2}$$

where g_1 and g_2 are to be found such that $\int_{M_\gamma} g_2(u)\,d\mu$ is controlled by certain norms of u and g_1 is continuous, non-negative and satisfies $g_1(\pm 1)>0$. Expanding the quadratic term on the left hand side of (4.2) and applying Young's inequality with some $\delta>0$ to the mixed term $2HH_s$ yields

$$\begin{aligned}
&\int_{M_\gamma} u^2\left(H-H_s(u)\right)^2+k_G(u)K\,d\mu\\
&\quad=\int_{M_\gamma}-\frac{k_G(u)}{2}|B|^2+\left(u^2+\frac{k_G(u)}{2}\right)H^2+u^2H_s(u)^2-2u^2HH_s(u)\,d\mu\\
&\quad\geq\int_{M_\gamma}-\frac{k_G(u)}{2}|B|^2+\left(u^2(1-\delta)+\frac{k_G(u)}{2}\right)H^2-u^2\frac{1-\delta}{\delta}H_s(u)^2\,d\mu.
\end{aligned}$$

Hence, (4.2) holds with $g_1(u) = -k_G(u)/2$ and $g_2(u) = u^2 H_s(u)^2(1-\delta)/\delta$, provided that for all $u \in [-C_0, C_0]$ we have

$$k_G(u) \leq 0, \qquad k_G(\pm 1) = k_G^{\pm} < 0 \qquad \text{and} \qquad u^2(1-\delta) + \frac{k_G(u)}{2} \geq 0. \tag{4.3}$$

The first two inequalities require that the Gauss rigidities $k_G^{\pm}$ of the membrane and their continuous extension are non-positive. The last inequality can be restated as $k_G(u)/(2u^2) > -1$ uniformly in u, and $u = \pm 1$ implies $k_G^{\pm}/(2k^{\pm}) = k_G^{\pm}/2 > -1$. Letting $k_G(u) = u^2 \widetilde{k}_G(u)$ where $\widetilde{k}_G$ continuously extends $k_G^{\pm}$ such that $0 \geq \widetilde{k}_G(u) > -2$ uniformly in u, we find that (4.3) is satisfied for some $\delta > 0$. Then $\widetilde{k}_G$ can in all the following arguments be treated as the extension of the preferred curvatures, and for simplicity of notation we assume $\widetilde{k}_G(u) = k_G^{\pm} = -1$. Then (4.2) becomes

$$\begin{aligned} \int_{M_\gamma} u^2 \left(H - H_s(u)\right)^2 - u^2 K \, d\mu &\geq \int_{M_\gamma} \frac{1}{2} u^2 |B|^2 - u^2 H_s(u)^2 \, d\mu \\ &\geq \int_{M_\gamma} \frac{1}{2} u^2 |B|^2 \, d\mu - \|H_s\|_\infty^2 \|u\|_\infty^2 \mathcal{A}_\gamma. \end{aligned} \tag{4.4}$$

Including $\varepsilon \int |B|^2 \, d\mu$ we arrive at the energy (4.1). The necessity of this stabilising term will become clear in Lemma 4.1 and the subsequent remark.

Experimental measurements of the Gauss bending rigidity are difficult and scarce, but available data suggests that it is negative and that at least for some monolayers $-1 < k_G/(2k) < 0$ is satisfied [SK04, TKS98]. Thus our assumptions are not unrealistic, and relations between the bending rigidity and the Gauss bending rigidity have also been used in other mathematical approaches. For instance, in [BM10] the authors need similar conditions as above to establish convergence of diffuse approximations for the Helfrich energy of single-phase membranes.

In the following we write $\mathcal{F}_\varepsilon(\gamma, u) = \mathcal{H}_\varepsilon(\gamma, u) + \mathcal{I}_\varepsilon(\gamma, u)$ where

$$\mathcal{H}_\varepsilon(\gamma, u) = \int_{M_\gamma} u^2 \left(H - H_s(u)\right)^2 - u^2 K \, d\mu$$

is the Helfrich energy and

$$\mathcal{I}_\varepsilon(\gamma, u) = \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u) \, d\mu + \varepsilon \int_{M_\gamma} |B|^2 \, d\mu.$$

the interface energy. Moreover, for $J \subset I$ we will use the notation $\mathcal{F}_\varepsilon(\gamma, u, J)$, $\mathcal{H}_\varepsilon(\gamma, u, J)$, and $\mathcal{I}_\varepsilon(\gamma, u, J)$ to denote the corresponding integrals restricted to the set $M_\gamma(J)$. The inclusion of the curvature term $\varepsilon \int |B|^2 \, d\mu$ in the interface energy is due to its effect in the limit $\varepsilon \to 0$, where it contributes at (ghost) interfaces and the axis of revolution.

Inequality (4.4) provides a lower bound for $\mathcal{H}_\varepsilon$ on $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, hence $\mathcal{F}_\varepsilon$ is bounded from below on $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. Moreover, we have the individual bounds

$$|\mathcal{F}_\varepsilon(\gamma, u)| \leq C \left(\mathcal{F}_\varepsilon(\gamma, u) + \|H_s\|_\infty^2 C_0^2 A_0\right), \tag{4.5}$$

$$\mathcal{I}_\varepsilon(\gamma, u) \leq \mathcal{F}_\varepsilon(\gamma, u) + \|H_s\|_\infty^2 C_0^2 A_0, \tag{4.6}$$

$$\int_{M_\gamma} u^2 |B|^2 \, d\mu + \varepsilon \int_{M_\gamma} |B|^2 \, d\mu \leq C \left(\mathcal{F}_\varepsilon(\gamma, u) + \|H_s\|_\infty^2 C_0^2 A_0\right) \tag{4.7}$$

for all $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, where the constant $C > 0$ is independent of (γ, u) and all data of the problem. From (4.6) and (4.7) we derive a bound on the first variation of M_γ.

Lemma 4.1. *There is a constant $C = C(W, H_s, C_0, A_0)$ such that*

$$\frac{1}{\sqrt{2}} \int_{M_\gamma} |H| \, d\mu \leq \int_{M_\gamma} |B| \, d\mu \leq C(\mathcal{F}_\varepsilon(\gamma, u) + 1)$$

for all $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. Here $|B| = \sqrt{\kappa_1^2 + \kappa_2^2}$.

Proof. Splitting M_γ into two pieces where the phase field is small and large, respectively, and applying Hölder's inequality, we get

$$\begin{aligned} \int_{M_\gamma} |B| \, d\mu &\leq \int_{M_\gamma(\{|u| \leq 1/2\})} |B| \, d\mu + \int_{M_\gamma(\{|u| > 1/2\})} |B| \, d\mu \\ &\leq \left(\frac{1}{\varepsilon} \mathcal{A}_\gamma(\{|u| \leq 1/2\})\right)^{1/2} \left(\int_{M_\gamma} \varepsilon |B|^2 \, d\mu\right)^{1/2} + 2\sqrt{A_0} \left(\int_{M_\gamma} u^2 |B|^2 \, d\mu\right)^{1/2}. \end{aligned}$$

From the interface energy we obtain the estimate

$$\mathcal{I}_\varepsilon(\gamma, u) \geq \int_{M_\gamma(\{|u| \leq 1/2\})} \frac{1}{\varepsilon} W(u) \, d\mu \geq \left(\inf_{|u| \leq 1/2} W(u)\right) \frac{\mathcal{A}_\gamma(\{|u| \leq 1/2\})}{\varepsilon},$$

and since W has a positive minimum on $[-1/2, 1/2]$, we find

$$\int_{M_\gamma} |B| \, d\mu \leq C \left(\mathcal{I}_\varepsilon(\gamma, u)^{1/2} + 1\right) \left[\left(\int_{M_\gamma} \varepsilon |B|^2 \, d\mu\right)^{1/2} + \left(\int_{M_\gamma} u^2 |B|^2 \, d\mu\right)^{1/2}\right].$$

The conclusion now follows from (4.6), (4.7) and the elementary inequality $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)} \leq \sqrt{2}(\sqrt{a} + \sqrt{b})$ for $a, b \geq 0$. □

Remark. Combining Lemmas 4.1 and 2.5 we find that any sequence $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ with uniformly bounded energy has uniformly bounded length. For this conclusion we could have argued with $\int u^2 H^2 + \varepsilon H^2 \, d\mu$ directly, instead of using the second fundamental form in the proof of Lemma 4.1. The following example shows that an additional term like $\varepsilon \int H^2 \, d\mu$ or $\varepsilon \int |B|^2 \, d\mu$ is necessary to obtain the length bound. Let M_ε be a sequence of "dumbbells" that consist of two nearly balls of fixed size smoothly connected by a cylinder of length l_ε

and diameter h_ε, and let the phase field u_ε be 0 on the cylinder and $+1$ and -1 on the balls with exactly one transition at each connection. Then $\mathcal{H}_\varepsilon \sim 0$ on the cylinder and $\mathcal{H}_\varepsilon$ is bounded independently of ε on the balls. The energy of the two phase transitions and $u_\varepsilon \sim 0$ on the cylinder is

$$\int_{M_\varepsilon} \varepsilon|\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, d\mu_\varepsilon \sim h_\varepsilon + \frac{1}{\varepsilon} l_\varepsilon h_\varepsilon.$$

The smoothing of the cylinder ball connections can be done where $u_\varepsilon \sim 0$. Then its energy stems from $\mathcal{I}_\varepsilon$ only and tends to zero as $h_\varepsilon \to 0$; see the lower and upper bound sections for details.

If $l_\varepsilon \to \infty$ and $h_\varepsilon \to 0$ such that $l_\varepsilon h_\varepsilon = o(\varepsilon)$, the energy without $\varepsilon \int |B|^2\, d\mu$ is bounded, but the length of the generating curve is unbounded; $(\gamma_\varepsilon, u_\varepsilon)$ can easily be made admissible, since the area and phase constraint, which are disturbed by the vanishing cylinder with area $2\pi l_\varepsilon h_\varepsilon$, can be recovered by slightly perturbing the balls. On the other hand, M_ε satisfies

$$\varepsilon \int_{M_\varepsilon} |B_\varepsilon|^2\, d\mu_\varepsilon \sim \varepsilon \frac{l_\varepsilon}{h_\varepsilon},$$

and therefore $l_\varepsilon \to \infty$ is excluded for bounded total energy $\mathcal{F}_\varepsilon$.

Note that the scaling of ε in the stabilising term is critical. If the energy contained $\varepsilon^p \int |B|^2\, d\mu$ with $p > 1$, the above example would still work and there would be no length bound. If $p < 1$, there would be no tangent discontinuities in the limit, since an argument similar to the proof of Lemma 4.1 would yield an L^q-bound for some $q > 1$ on the second fundamental form and thus on κ_1, compare the equi-coercivity arguments in Section 4.4.

Remark. The bound $\|u\|_\infty \leq C_0$ used in the derivations of (4.6), (4.7) and Lemma 4.1 can for small ε be replaced by the assumption that W has quadratic growth, that is $W(u) \geq cu^2$ for all $|u| \geq u_0$ and some constants $c > 0$, $u_0 > 0$. Then

$$\int_M u^2\, d\mu \leq u_0^2 A_0 + \frac{1}{c} \int_M W(u)\, d\mu$$

leads to similar estimates for $\varepsilon \leq \varepsilon_0(W, H_s) = c/(2\|H_s\|_\infty^2)$.

4.2 Limit setting

The major technical difficulties in the limit of (4.1) as $\varepsilon \to 0$ stem from the appearance of kinks and from the axis of revolution. In particular, at the axis the compactness result for sequences $(\gamma_\varepsilon, u_\varepsilon)$ and the regularity properties of the limit are weaker than elsewhere. The

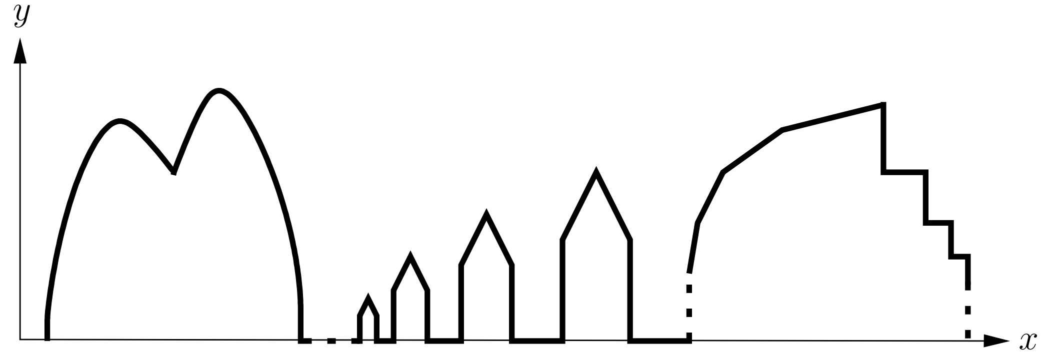

Figure 4.1: Example of a curve $\gamma \in \mathcal{D}$. The component on the left is regular except for one kink. In the centre there are countably many self-similar components decreasing from right to left. In the right component kinks accumulate at both ends; at the left end the limit tangent exist, at the right end it does not.

limit curves will have parametrisations in

$$\begin{aligned}\mathcal{D} := \Big\{ & \gamma = (x,y) \in C^{0,1}(I;\mathbb{R}^2) : \\ & |\gamma'| \equiv q_\gamma = \text{const in}\,\{y > 0\}\,,\ |\gamma'| = x' \leq q_\gamma \text{ in}\,\{y = 0\}\,, \\ & y(\partial I) = \{0\}\,,\ y \geq 0,\ x' \geq 0, \\ & \text{there is } S_\gamma \subset \{y > 0\} \text{ s.t. } \mathcal{H}^1(M_\gamma(S_\gamma)) < \infty \text{ and} \\ & \int_{M_\gamma(\{y>0\}\setminus S_\gamma)} |B|^2\, d\mu < \infty,\ \mathcal{A}_\gamma = A_0 \Big\}.\end{aligned}$$

A curve $\gamma \in \mathcal{D}$ is globally Lipschitz with Lipschitz constant q_γ and its restriction to $\{y > 0\}$ is a constant speed parametrisation. Due to $\mathcal{H}^1(M_\gamma(S_\gamma)) < \infty$ the set S_γ is countable and $S_\gamma \cap J$ is finite for any $J \Subset \{y > 0\}$. From the bound on the second fundamental form we infer $\gamma \in W^{2,2}(J \setminus S_\gamma;\mathbb{R}^2)$ for any $J \Subset \{y > 0\}$, and by embedding of $W^{2,2}(a,b)$ into $C^1([a,b])$, γ' is continuous from either side at any $s \in S_\gamma$. Thus S_γ contains the tangent discontinuities of γ in $\{y > 0\}$.

Since the bound on the second fundamental form is disturbed by the kinks and there is not enough control on the size of the kinks, see the definition of the limit energy below, there is no Gauss-Bonnet type argument to conclude that M_γ has finitely many components as in Chapter 3. Each component is piecewise embedded between kinks, but not necessarily globally. Moreover, if kinks accumulate at a component boundary, we cannot conclude that γ is perpendicular to the axis of revolution there, not even that the one-sided limit of the tangent vector exists. We obtain only the weak continuity result of the following lemma, see Figure 4.1 for an example of curve in $\mathcal{D}$.

Lemma 4.2. *Let* $\omega = (a,b)$ *be a component of* $\gamma = (x,y) \in \mathcal{D}$ *and assume that* $(s_j) \subset S_\gamma \cap \omega$ *is a decreasing sequence such that* $s_j \to a$ *as* $j \to \infty$ *and* $\gamma \in W^{2,2}(s_{j+1}, s_j)$ *for all* $j \in \mathbb{N}$. *Then the one-sided approximate limit of* x' *vanishes at* a, *that is*

$$\lim_{\rho \searrow 0} \frac{1}{\rho} \int_a^{a+\rho} x' \, dt = 0,$$

and $|y'|$ *has one-sided approximate limit* q_γ. *Moreover,* γ *is almost piecewise straight near* a *in the sense*

$$\lim_{j\to\infty} \operatorname*{osc}_{(s_{j+1},s_j)} \gamma' = \lim_{j\to\infty} \sup_{t,s\in(s_{j+1},s_j)} |\gamma'(t) - \gamma'(s)| = 0.$$

Proof. The approximate limit of x' is found similarly to the strong limit in the proof of Lemma 2.1: Lipschitz continuity implies $y(t) \le y(a) + q_\gamma(t-a) \le q_\gamma \rho$ in $(a, a+\rho)$, hence

$$\frac{1}{\rho} \int_a^{a+\rho} x'^2 \, dt \le q_\gamma^2 \int_a^{a+\rho} \frac{x'^2}{q_\gamma y} \, dt \le \frac{q_\gamma^2}{2\pi} \int_{M_\gamma((a,a+\rho)\setminus S_\gamma)} \kappa_2^2 \, d\mu.$$

The integral on the right hand side vanishes as $\rho \to 0$ due to the finite L^2-norm of the second fundamental form of $M_\gamma(\omega \setminus S_\gamma)$. The approximate limit of $|y'|$ follows from $y'^2 = q_\gamma^2 - x'^2$ almost everywhere in ω.

For the straightness we recall that $|B|^2 \ge |K|$ and $|K| \, d\mu = |y''| \, dt$ almost everywhere in ω, and thus we find

$$\sum_{j=1}^{\infty} \int_{s_{j+1}}^{s_j} |y''| \, dt \le \int_{M_\gamma(\omega\setminus S_\gamma)} |B|^2 \, d\mu < \infty.$$

Consequently,

$$\sup_{t,s\in(s_{j+1},s_j)} |y'(t) - y'(s)| \le \int_{s_{j+1}}^{s_j} |y''| \, dt \to 0$$

as $j \to \infty$, and likewise for x' due to

$$|x'(t) - x'(s)|^2 \le |x'^2(t) - x'^2(s)| = |y'^2(t) - y'^2(s)| \le 2q_\gamma |y'(t) - y'(s)|. \qquad \square$$

According to Lemma 4.2, γ consists roughly of straight line segments when approaching the component boundary, but the directions of these segments may vary as long as the approximate limit is perpendicular to the axis of revolution. Finally, note that for $\gamma \in \mathcal{D}$ there is no restriction on the size of $\{y = 0\}$ as it was the case for $\gamma \in \mathcal{C}$. We refer to Section 4.6 for examples of rather singular admissible limit curves.

To $\gamma \in \mathcal{D}$ we associate a phase field u in

$$\begin{aligned} \mathcal{Q} := \Big\{ & u \colon I \to [-C_0, C_0] : u \in \{\pm 1\} \text{ piecewise constant in } \{y > 0\}, \\ & \int_{M_\gamma} u \, d\mu = m A_0, \; \mathcal{H}^1(M_\gamma(S_u)) < \infty \Big\}. \end{aligned}$$

The only difference between $\mathcal{Q}$ and $\mathcal{P}$ is that $u \in \mathcal{Q}$ is not specified in the possibly large set where γ lies on the axis of revolution. For $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$ we consider the limit energy

$$\mathcal{F} = \mathcal{H} + \mathcal{I}$$

where

$$\mathcal{H}(\gamma, u) = \int_{M_\gamma(\{y>0\} \setminus S_\gamma)} (H - H_s(u))^2 - K \, d\mu$$

and

$$\mathcal{I}(\gamma, u) = 2\pi \sum_{s \in S_\gamma \cup S_u} \big(\sigma + \hat{\sigma}|[\gamma'](s)|\big)\, y(s) + 2\pi\hat{\sigma}\mathcal{L}_\gamma(\{y = 0\}). \tag{4.8}$$

Here $|[\gamma'](s)|$ denotes the modulus of the angle enclosed by the two one-sided tangent vectors at s modulo 2π, that is, the jump of the tangent vector because its length is fixed. The constant σ is given as before by (3.1), and $\hat{\sigma}$ is defined by

$$\hat{\sigma} = 2\sqrt{W(0)}.$$

In (4.8) an interface without kink is essentially penalised by its length, while kinks carry an additional "bending energy" proportional to their size. Kinks without phase field jumps, which we call ghost interfaces, are charged like corresponding kinks at interfaces. Segments of γ on the axis of revolution are penalised by their length. As for $\mathcal{F}_\varepsilon = \mathcal{H}_\varepsilon + \mathcal{I}_\varepsilon$, we find

$$\mathcal{H}(\gamma, u) \geq \frac{1}{2} \int_{M_\gamma(\{y>0\} \setminus S_\gamma)} |B|^2 \, d\mu - \|H_s\|_\infty^2 \mathcal{A}_\gamma$$

and bounds corresponding to (4.5)–(4.7).

4.3 Convergence theorem

We extend $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ to $C^0(I; \mathbb{R}^2) \times L^1(I)$ by setting $\mathcal{F}_\varepsilon(\gamma, u) = \mathcal{F}(\gamma, u) = \infty$ whenever (γ, u) does not belong to $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ or $\mathcal{D} \times \mathcal{Q}$, respectively. The result of this chapter is the following theorem.

Theorem 4.3. *The energies $\mathcal{F}_\varepsilon$ are equi-coercive: any sequence $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ with uniformly bounded energy $\mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon)$ admits a subsequence that converges in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ to some $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$. Furthermore, $\mathcal{F}_\varepsilon$ converges to $\mathcal{F}$ in the following sense:*

- *any sequence $(\gamma_\varepsilon, u_\varepsilon)$ that converges to (γ, u) in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ satisfies the lower bound inequality*

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \geq \mathcal{F}(\gamma, u);$$

- *for any* $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$ *such that* γ *is parametrised with constant speed in all of* I *there exists a sequence* $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ *that converges to* (γ, u) *in* $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ *and satisfies the upper bound inequality*
$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \le \mathcal{F}(\gamma, u).$$

Theorem 4.3 differs from Γ-convergence in two aspects. First, the underlying convergence of phase fields in $L^1(\{y > 0\})$ depends on the limit curve $\gamma = (x, y)$. This could be avoided by formulating the theorem with respect to the metrisable weak-$\star$ topology in $\{u \in L^\infty(I) : \|u\|_\infty \le C_0\}$. There is, however, not enough control on the phase fields in $\{y = 0\}$ by the approximate energy, and the actual value of the limit in $\{y = 0\}$ is not used. Therefore, we prefer to regard it as essentially undefined.

Second, only "good" limits (γ, u) with constant speed $|\gamma'|$ are recovered. Nevertheless it is still true that almost minimising sequences for $\mathcal{F}_\varepsilon$ can accumulate only in minimisers of $\mathcal{F}$. If $(\widetilde{\eta}, \widetilde{w}) \in \mathcal{D} \times \mathcal{Q}$ is arbitrary, one finds a good parametrisation of the same membrane by arguing as in Chapter 2: First, removing constancy intervals in $\{y = 0\}$, pulling holes together, and reparametrising linearly does not change the membrane, its area, energy $\mathcal{F}$ or phase integral. Then, if $\widetilde{\eta}$ has no constancy intervals and $I = (a, b)$, the function

$$\psi(t) = a + \frac{b - a}{\mathcal{L}_{\widetilde{\eta}}} \int_a^t |\widetilde{\eta}'(s)| \, ds$$

is strictly increasing, and the configuration $(\eta, w) = (\widetilde{\eta} \circ \psi^{-1}, \widetilde{w} \circ \psi^{-1})$ has constant speed $\mathcal{L}_{\widetilde{\eta}}/(b - a)$. Since ψ is affine in each component of $\widetilde{\eta}$, the pair (η, w) inherits its differentiability properties and bounds in $\{y_\eta > 0\}$ from $(\widetilde{\eta}, \widetilde{w})$ in $\{y_{\widetilde{\eta}} > 0\}$. Again, energy, area and phase integral are unchanged.

If $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ is an almost minimising sequence, then by equi-coercivity a subsequence converges in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ to some $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$. If $(\widetilde{\eta}, \widetilde{w}) \in \mathcal{D} \times \mathcal{Q}$ is arbitrary, (η, w) the corresponding constant speed parametrisation, and $(\eta_\varepsilon, w_\varepsilon)$ its recovery sequence, then

$$\mathcal{F}(\gamma, u) \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) = \liminf_{\varepsilon \to 0} (\inf \mathcal{F}_\varepsilon) \le \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\eta_\varepsilon, w_\varepsilon) \le \mathcal{F}(\eta, w) = \mathcal{F}(\widetilde{\eta}, \widetilde{w}).$$

Thus, (γ, u) has minimal energy $\mathcal{F}$.

From Theorem 4.3 we cannot deduce the full Γ-limit of $\mathcal{F}_\varepsilon$ on $\mathcal{D} \times \mathcal{Q}$. For instance, we do not know whether $\Gamma\text{-}\lim \mathcal{F}_\varepsilon$ is invariant under reparametrisations; in fact, the arguments for the lower bound and the example in Section 4.6.2 suggest that it is not. Hence, the above reparametrisation argument for $\mathcal{F}$ cannot be used for $\Gamma\text{-}\lim \mathcal{F}_\varepsilon$. The difficulty, however, appears at the axis of revolution only; we refer to Section 4.8 for a further discussion.

As in Chapter 3, we prove the theorem in three steps. We also consider some examples that lead to undesired effects in the limit and discuss some generalisations of Theorem 4.3.

4.4 Equi-coercivity

The equi-coercivity proof basically follows the lines of the corresponding arguments for the regular case. Since now, however, the coupling between curve and phase field is stronger than in $\mathcal{E}_\varepsilon$, both have to be considered simultaneously.

Lemma 4.4. *Let $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ be a sequence with uniformly bounded energy $\mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon)$. Then there exist $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$, $\gamma = (x, y)$, a countable set $S \subset \{y > 0\}$ with $S_\gamma \cup S_u \subset S$ and $S \cap J$ finite for any $J \Subset \{y > 0\}$, and a subsequence, not relabelled, such that*

- $\gamma_\varepsilon \overset{\star}{\rightharpoonup} \gamma$ *in* $W^{1,\infty}(I; \mathbb{R}^2)$;
- $u_\varepsilon \to u$ *in* $L^p(\{y > 0\})$ *for any* $p \in [1, \infty)$;
- $\gamma_\varepsilon \rightharpoonup \gamma$ *in* $W^{2,2}_{\text{loc}}(\{y > 0\} \setminus S; \mathbb{R}^2)$;
- *in any* $J \Subset \{y > 0\} \setminus S$ *there holds* $|u_\varepsilon| \geq 1/2$ *for all sufficiently small* ε.

Proof. Let $\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon)$ and $|\gamma'_\varepsilon| = q_\varepsilon$. By Lemma 4.1 and Lemma 2.5 the sequence (q_ε) is uniformly bounded from above. As in Lemma 3.2 we extract a subsequence $q_\varepsilon \to q$ in $\mathbb{R}$ and $\gamma_\varepsilon \overset{\star}{\rightharpoonup} \gamma$ in $W^{1,\infty}(I; \mathbb{R}^2)$ and strongly in $C^0(I; \mathbb{R}^2)$. The limit satisfies $y \geq 0$, $y(\partial I) = \{0\}$, $y \not\equiv 0$, $x' \geq 0$, $q > 0$, and $|\gamma'| \leq q$. Without loss of generality we assume $q = 1$.

We now apply the argument for the phase fields from the proof of Lemma 3.3 in $\{y > 0\}$ to find an at most countable set $S \subset \{y > 0\}$ with $S \cap J$ finite for any $J \Subset \{y > 0\}$, a function $u\colon \{y > 0\} \to \{\pm 1\}$ with $S_u \subset S$, and subsequence of (u_ε) that converges to u in measure and almost everywhere in $\{y > 0\}$ and that satisfies $|u_\varepsilon| \geq 1/2$ in $J \Subset \{y > 0\} \setminus S$ for all sufficiently small $\varepsilon > 0$ depending on J. From the uniform L^∞-bound on u_ε we infer convergence in $L^p(\{y > 0\})$ for any $p \in [1, \infty)$.

Along this subsequence we establish further compactness of the curves. Given $J \Subset \{y > 0\} \setminus S$, there holds $q_\varepsilon \leq 2$, $|u_\varepsilon| \geq 1/2$, and $y_\varepsilon \geq c_J$ for a constant $c_J > 0$ and all sufficiently small $\varepsilon > 0$. Therefore,

$$\int_{M_\varepsilon} u_\varepsilon^2 |B_\varepsilon|^2 \, d\mu_\varepsilon \geq \frac{1}{4} \int_{M_\varepsilon(J)} \kappa_{1,\varepsilon}^2 \, d\mu_\varepsilon \geq \frac{\pi c_J}{16} \int_J |\gamma''_\varepsilon|^2 \, dt$$

is uniformly bounded for all sufficiently small ε, and a subsequence of γ''_ε converges weakly to some γ''_J in $L^2(J; \mathbb{R}^2)$. From $\gamma_\varepsilon \overset{\star}{\rightharpoonup} \gamma$ in $W^{1,\infty}(I; \mathbb{R}^2)$ we obtain that γ''_J is the weak derivative of γ' in J and that the whole sequence γ_ε converges weakly in $W^{2,2}(J; \mathbb{R}^2)$. This shows $\gamma_\varepsilon \rightharpoonup \gamma$ in $W^{2,2}_{\text{loc}}(\{y > 0\} \setminus S; \mathbb{R}^2)$ and by embedding $\gamma_\varepsilon \to \gamma$ in $C^1_{\text{loc}}(\{y > 0\} \setminus S; \mathbb{R}^2)$ and $\gamma'_\varepsilon \to \gamma'$ pointwise in $\{y > 0\} \setminus S$. Hence, we find $S_\gamma \subset S$, $1 = \lim q_\varepsilon^2 = \lim(x_\varepsilon'^2 + y_\varepsilon'^2) = x'^2 + y'^2$ in $\{y > 0\} \setminus S$ and obtain

$$A_0 = \mathcal{A}_\varepsilon = 2\pi \int_{\{y>0\}} q_\varepsilon y_\varepsilon \, dt + 2\pi \int_{\{y=0\}} q_\varepsilon y_\varepsilon \, dt \to 2\pi \int_{\{y>0\}} y \, dt = \mathcal{A}_\gamma(\{y > 0\}) = \mathcal{A}_\gamma$$

as well as

$$mA_0 = \int_{M_\varepsilon} u_\varepsilon \, d\mu_\varepsilon \to \int_{M_\gamma} u \, d\mu.$$

To conclude $(\gamma, u) \in \mathcal{D}\times\mathcal{Q}$ we must show that $\mathcal{H}^1(M_\gamma(S_u \cup S_\gamma))$ and $\int_{M_\gamma(\{y>0\}\setminus S_\gamma)} |B|^2 \, d\mu$ are finite. From $\gamma_\varepsilon \rightharpoonup \gamma$ in $W^{2,2}(J;\mathbb{R}^2)$, $u_\varepsilon \to u \in \{\pm 1\}$ in $L^2(J)$, and $\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(J)} < \infty$ in any $J \Subset \{y > 0\} \setminus S$ we infer that

$$u_\varepsilon \kappa_{1,\varepsilon} \sqrt{q_\varepsilon y_\varepsilon} \rightharpoonup u\kappa_1 \sqrt{y} \qquad \text{and} \qquad u_\varepsilon \kappa_{2,\varepsilon} \sqrt{q_\varepsilon y_\varepsilon} \to u\kappa_2 \sqrt{y} \qquad \text{in } L^2(J),$$

hence

$$\int_{M_\gamma(J)} |B|^2 \, d\mu \le \liminf_{\varepsilon\to 0} \int_{M_\varepsilon(J)} u_\varepsilon^2 |B_\varepsilon|^2 \, d\mu_\varepsilon \le \liminf_{\varepsilon\to 0} \int_{M_\varepsilon} u_\varepsilon^2 |B_\varepsilon|^2 \, d\mu_\varepsilon.$$

Since the right hand side is bounded independently of J, we obtain

$$\int_{M_\gamma(\{y>0\}\setminus S)} |B|^2 \, d\mu \le \liminf_{\varepsilon\to 0} \int_{M_\varepsilon} u_\varepsilon^2 |B_\varepsilon|^2 \, d\mu_\varepsilon < \infty.$$

Similar to Chapter 3, the inequality $\mathcal{H}^1(M_\gamma(S_u \cup S_\gamma)) < \infty$ follows from Young's inequality applied to the interface energy and the fact that each kink or interface $s \in S_u \cup S_\gamma$ carries at least an energy of $2\pi\sigma y(s)$ in the limit $\varepsilon \to 0$. The details are given in the lower bound section and are thus here omitted. ☐

The following corollary states the convergence around possible kinks more precisely and will be used to establish the lower bound.

Corollary 4.5. *For any subsequence as in Lemma 4.4 there are angle functions $\varphi_\varepsilon \in L^\infty(I) \cap W^{1,2}_{\mathrm{loc}}(I)$ of γ_ε that converge weakly in $BV_{\mathrm{loc}}(\{y > 0\})$ to an angle function φ of γ in $\{y > 0\}$. Moreover, $\varphi \in W^{1,2}(J \setminus S)$ for any $J \Subset \{y > 0\}$.*

Proof. Without loss of generality let $I = (0, \mathcal{L}_\gamma)$ and $q_\gamma = 1$. Since $\gamma_\varepsilon \in W^{2,2}_{\mathrm{loc}}(I;\mathbb{R}^2)$ and $x'_\varepsilon \ge 0$ by definition of $\mathcal{C}_\varepsilon$, there are angle functions $\varphi_\varepsilon \in W^{1,2}_{\mathrm{loc}}(I) \cap L^\infty(I;[-\pi/2, \pi/2])$ of γ_ε. Recalling $\varphi'_\varepsilon = -\kappa_{1,\varepsilon} q_\varepsilon$, uniform convergence of y_ε and Lemma 4.1, we fix $J \Subset \{y > 0\}$ and obtain that

$$\int_J |\varphi'_\varepsilon| \, dt = \int_J |\kappa_{1,\varepsilon}| q_\varepsilon \, dt \le \frac{1}{2\pi c_J} \int_{M_\varepsilon} |B_\varepsilon| \, d\mu_\varepsilon$$

is uniformly bounded for sufficiently small $\varepsilon > 0$. Hence, φ_ε is uniformly bounded in $W^{1,1}(J)$ and there exists a subsequence that converges weakly in $BV(J)$ to some φ, that is, $\varphi_\varepsilon \to \varphi$ in $L^1(J)$ and $\kappa_{1,\varepsilon} \, dt$ restricted to J converges weakly to the measure $d\varphi'$. Consequently, $\gamma'_\varepsilon = q_\varepsilon(\cos\varphi_\varepsilon, \sin\varphi_\varepsilon) \to (\cos\varphi, \sin\varphi) = \gamma'$ in $L^p(J;\mathbb{R}^2)$. Since this argument can be applied to any subsequence, convergence of the whole sequence (φ_ε) in $BV(J)$ follows. Arguing as in Lemma 4.4 for γ''_ε, we obtain $\varphi_\varepsilon \rightharpoonup \varphi$ in $W^{1,2}(\widetilde{J})$ for any $\widetilde{J} \Subset \{y > 0\} \setminus S$ and

$$\int_{\widetilde{J}} |\varphi'|^2 \, dt \le \frac{4}{\pi c_{\widetilde{J}}} \liminf_{\varepsilon\to 0} \int_{M_\varepsilon} u_\varepsilon^2 \kappa_{1,\varepsilon}^2 \, d\mu_\varepsilon < \infty.$$

Exhausting $J \Subset \{y > 0\}$ by $\widetilde{J} \Subset J \setminus S$ yields $\varphi' \in L^2(J)$ and $\varphi \in W^{1,2}(J \setminus S)$. $\square$

Remark. In the proofs of Lemma 4.4 and Corollary 4.5 the L^∞-bound for u_ε is used to establish L^p-convergence in $\{y > 0\}$ and lower semi-continuity of the second fundamental form in $J \Subset \{y > 0\} \setminus S$. However, due to the one-dimensional setting u_ε is uniformly bounded in $L^\infty(J)$ with a bound depending on $J \Subset \{y > 0\} \setminus S$. Thus the global L^∞-bound can be weakened to an L^2-bound.

4.5 Lower bound

To prove the lower bound

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \geq \mathcal{F}(\gamma, u) \tag{4.9}$$

whenever $(\gamma_\varepsilon, u_\varepsilon)$ converges to (γ, u) in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$, it suffices to consider sequences such that the left hand side of (4.9) is finite and the limit inferior is attained. Then $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, and the equi-coercivity result yields $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$ and the convergence properties listed in Lemma 4.4 and Corollary 4.5. In the following we consider the bulk energy $\mathcal{H}(\gamma, u)$, kinks and interfaces, and the axis of revolution separately.

4.5.1 Bulk lower bound

Lemma 4.6. *There holds*

$$\liminf_{\varepsilon \to 0} \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, \{y > 0\}) \geq \mathcal{H}(\gamma, u).$$

Proof. Let $J \Subset \{y > 0\} \setminus S$. From $\gamma_\varepsilon \rightharpoonup \gamma$ in $W^{2,2}(J; \mathbb{R}^2)$, $u_\varepsilon \to u \in \{\pm 1\}$ in $L^2(J)$, and $\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(J)} < \infty$ we infer that

$$u_\varepsilon H_\varepsilon \sqrt{|\gamma_\varepsilon'| y_\varepsilon} \rightharpoonup u H \sqrt{|\gamma'| y} \quad \text{in } L^2(J) \qquad \text{and} \qquad u_\varepsilon^2 K_\varepsilon |\gamma_\varepsilon'| y_\varepsilon \rightharpoonup u^2 K |\gamma'| y \quad \text{in } L^1(J).$$

Moreover, we have

$$u_\varepsilon H_s(u_\varepsilon) \sqrt{|\gamma_\varepsilon'| y_\varepsilon} \to u H_s(u) \sqrt{|\gamma'| y} \qquad \text{in } L^2(\{y > 0\}). \tag{4.10}$$

Hence the inequality

$$\mathcal{H}(\gamma, u, J) + \int_{M_\gamma(J)} H_s(u)^2 \, d\mu \leq \liminf_{\varepsilon \to 0} \left(\mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) + \int_{M_\varepsilon(J)} u_\varepsilon^2 H_s(u_\varepsilon)^2 \, d\mu_\varepsilon \right) \tag{4.11}$$

holds. According to (4.4), the right hand side of (4.11) is non-negative, so we estimate it from above by extending the domain of integration to $M_\varepsilon(\{y > 0\})$. The right hand side is then independent of $J \Subset \{y > 0\} \setminus S$, and by exhausting $\{y > 0\} \setminus S$ we obtain

$$\begin{aligned}\mathcal{H}(\gamma, u) + \int_{M_\gamma(\{y>0\}\setminus S)} H_s(u)^2 \, d\mu \\ \le \liminf_{\varepsilon\to 0} \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, \{y > 0\}) + \limsup_{\varepsilon\to 0} \int_{M_\varepsilon(\{y>0\})} u_\varepsilon^2 H_s(u_\varepsilon)^2 \, d\mu_\varepsilon.\end{aligned}$$

The claim now follows from the convergence (4.10). □

4.5.2 Kinks and interfaces

Next we consider the interface energies $\mathcal{I}_\varepsilon$ and $\mathcal{I}$ in $\{y > 0\}$. Points in $S \setminus (S_u \cup S_\gamma)$ do not contribute to the limit energy $\mathcal{I}$, so it suffices to examine $s \in S_u \cup S_\gamma$. In the following let $J \Subset \{y > 0\}$ be an interval around s such that $\overline{J} \cap S = \{s\}$.

If $s \in S_u \setminus S_\gamma$ is an interface without kink, we estimate the curvature term in $\mathcal{I}_\varepsilon$ from below by zero and use the same argument as in Chapter 3 for the other terms. Recall that there are points $a_\varepsilon, b_\varepsilon \in J$ such that $a_\varepsilon \to s$, $b_\varepsilon \to s$, $u_\varepsilon(a_\varepsilon) \to -1$, $u_\varepsilon(b_\varepsilon) \to +1$, and without loss of generality $a_\varepsilon < s < b_\varepsilon$. Hence we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_{M_\varepsilon(a_\varepsilon,b_\varepsilon)} \varepsilon|\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, d\mu_\varepsilon &\ge 2\pi \lim_{\varepsilon\to 0} \left(\left(\inf_{(a_\varepsilon,b_\varepsilon)} y_\varepsilon \right) \int_{u_\varepsilon(a_\varepsilon)}^{u_\varepsilon(b_\varepsilon)} 2\sqrt{W(u)} \, du \right) \\ &= 2\pi y(s)\sigma.\end{aligned} \tag{4.12}$$

If $s \in S_u \cap S_\gamma$ is a kink and a proper interface, let φ_ε be angle functions of γ_ε in J that by Corollary 4.5 converge weakly in $BV(J)$ to an angle function φ of γ. We then have

$$\int_J \varphi_\varepsilon' y_\varepsilon \, dt \to [\varphi](s) y(s) - \int_{J\setminus\{s\}} q_\gamma \kappa_1 y \, dt,$$

where $\kappa_1 = -\varphi'/q_\gamma \in L^2(J \setminus \{s\})$ is the curvature of γ in $J \setminus \{s\}$. Since $x_\varepsilon' \ge 0$, we may assume $\varphi_\varepsilon \in [-\pi/2, \pi/2]$, so that $[\varphi] = [\gamma'] \in [-\pi, \pi]$. The key step for the lower bound in J is to formalise the intuition that γ_ε approaches the kink where u_ε is close to zero.

Lemma 4.7. *For sufficiently small $\delta > 0$ let $J_{\varepsilon,\delta} = \{t \in I : |u_\varepsilon| \le \delta\}$. Then*

$$\liminf_{\varepsilon\to 0} \left| \int_{J\cap J_{\varepsilon,\delta}} \varphi_\varepsilon' y_\varepsilon \, dt \right| \ge y(s)|[\varphi](s)|.$$

Proof. We show that the complement of $J_{\varepsilon,\delta}$ in J contains only the absolutely continuous part of $d\varphi'$. Let $\beta > 0$ be arbitrary but fixed, and let $U_\beta = [s - \beta, s + \beta]$. As $J \setminus U_\beta$ is compactly contained in $\{y > 0\} \setminus S$, we have $|u_\varepsilon| \ge 2\delta$ in $J \setminus U_\beta$ for all sufficiently small ε, and therefore $J \cap J_{\varepsilon,\delta} \subset U_\beta$. Writing $w_\varepsilon = \varphi_\varepsilon' y_\varepsilon + q_\gamma \kappa_1 y$, we have

$$\left| \int_{J\setminus J_{\varepsilon,\delta}} w_\varepsilon \, dt \right| \le \left| \int_{J\setminus U_\beta} w_\varepsilon \, dt \right| + \int_{(J\setminus J_{\varepsilon,\delta})\cap U_\beta} |w_\varepsilon| \, dt$$

for all sufficiently small ε. The first term on the right hand side converges to 0 by weak convergence of w_ε in $J \setminus U_\beta$, and the second term is bounded by a constant times $\sqrt{\beta}$ due to Hölder's inequality and the uniform bound on the second fundamental forms of M_ε in $I \setminus J_{\varepsilon,\delta}$. As $\beta > 0$ is arbitrary, we obtain

$$\limsup_{\varepsilon\to 0} \left| \int_{J\setminus J_{\varepsilon,\delta}} w_\varepsilon \, dt \right| = 0,$$

and taking the lower limit in the inequality

$$\left| \int_{J\cap J_{\varepsilon,\delta}} w_\varepsilon \, dt \right| \geq \left| \int_J w_\varepsilon \, dt \right| - \left| \int_{J\setminus J_{\varepsilon,\delta}} w_\varepsilon \, dt \right|$$

yields the claim as $|J \cap J_{\varepsilon,\delta}| \to 0$ due to the uniform bound on the potential energy and $y\kappa_1 \in L^2(J \setminus \{s\})$. ☐

Using the above splitting of J into $J \cap J_{\varepsilon,\delta}$ where $|u_\varepsilon|$ is small and $J \setminus J_{\varepsilon,\delta}$ where the phase transition to ±1 takes place, we prove the lower bound inequality.

Lemma 4.8. *There holds*

$$\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) \geq 2\pi \left(\hat{\sigma}|[\gamma'](s)| + \sigma\right) y(s).$$

Proof. With the notation of the Lemma 4.7 we have

$$\frac{\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J)}{2\pi} \geq \int_{J\cap J_{\varepsilon,\delta}} \left(\frac{\varepsilon}{q_\varepsilon}|\varphi_\varepsilon'|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) \right) y_\varepsilon \, dt + \int_{J\setminus J_{\varepsilon,\delta}} \left(\frac{\varepsilon}{q_\varepsilon}|u_\varepsilon'|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) \right) y_\varepsilon \, dt.$$

Estimating the first term with Young's inequality we obtain

$$\begin{aligned} \int_{J\cap J_{\varepsilon,\delta}} \left(\frac{\varepsilon}{q_\varepsilon}|\varphi_\varepsilon'|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) \right) y_\varepsilon \, dt &\geq \int_{J\cap J_{\varepsilon,\delta}} 2\sqrt{W(u_\varepsilon)}|\varphi_\varepsilon'| y_\varepsilon \, dt \\ &\geq 2 \inf_{u\in[-\delta,\delta]} \sqrt{W(u)} \left| \int_{J\cap J_{\varepsilon,\delta}} \varphi_\varepsilon' y_\varepsilon \, dt \right|. \end{aligned}$$

With the second integral we deal as in (4.12); the only difference is that we now find an interval $(a_\varepsilon, b_\varepsilon) \subset J \setminus J_{\varepsilon,\delta}$ such that $u_\varepsilon(a_\varepsilon) \to \delta$, $u_\varepsilon(b_\varepsilon) \to 1$ on one side of s, and the same with $-\delta$ and -1 on the other side. Combining both estimates and taking the lower limit as $\varepsilon \to 0$ yields

$$\begin{aligned} \frac{1}{2\pi} \liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) &\geq 2y(s)|[\varphi](s)| \inf_{[-\delta,\delta]} \sqrt{W(u)} \\ &\quad + 2y(s) \int_\delta^1 \sqrt{W(u)} \, du + 2y(s) \int_{-1}^{-\delta} \sqrt{W(u)} \, du. \end{aligned}$$

Taking the supremum over $\delta > 0$ finishes the proof. ☐

Finally, if $s \in S_\gamma \setminus S_u$ is a ghost interface, then the phase field u is constant in $\overline{J}$, say $u \equiv 1$. Arguing as in Lemma 4.8, but with two transitions of u_ε from δ to 1, we find

$$\frac{1}{2\pi} \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) \geq \hat{\sigma}|[\gamma'](s)|y(s) + 4y(s) \int_0^1 \sqrt{W(u)}\, du,$$

and the right hand side is equal to $\hat{\sigma}|[\gamma'](s)|y(s) + \sigma y(s)$ due to the symmetry of W. The same argument holds when $u \equiv -1$ near s.

As in Chapter 3, the above reasoning extends to any finite subset S of $S_\gamma \cup S_u$, and we obtain the lower bound for kinks and interfaces

$$\liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, \{y > 0\}) \geq 2\pi \sum_{s \in S_\gamma \cup S_u} \big(\sigma + \hat{\sigma}|[\gamma'](s)|\big)\, y(s) = \mathcal{I}(\gamma, u, \{y > 0\}).$$

4.5.3 Axis of revolution

Let $R = \{y = 0\}$. To motivate the lower bound estimate below, we first consider the simple example of γ_ε being a straight horizontal line segment in R that converges to the axis of revolution. Then $\kappa_{1,\varepsilon} = 0$, while $\kappa_{2,\varepsilon} \sim 1/y_\varepsilon$ blows up and is likely to contribute to the limit energy. From the uniform bound on

$$\int_{M_\varepsilon(R)} u_\varepsilon^2 \kappa_{2,\varepsilon}^2 \, d\mu_\varepsilon \geq \frac{2\pi}{\sup_R y_\varepsilon} \int_R u_\varepsilon^2 x_\varepsilon'^2 \, dt$$

and $x_\varepsilon' = q_\varepsilon \to q = x'$ in R we conclude that u_ε converges to zero in R. Therefore, the potential term will contribute to the limit energy. On the other hand, there is no reason for u_ε to have a large gradient in R, and it is reasonable to assume that u_ε converges fast enough to 0 so that there is no contribution of $\mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R)$ in the limit. Then

$$\mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \sim \int_{M_\varepsilon(R)} \frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon \kappa_{2,\varepsilon}^2 \, d\mu_\varepsilon \sim \int_R \frac{y_\varepsilon}{\varepsilon} + \frac{\varepsilon}{y_\varepsilon} \, dt$$

is bounded as $\varepsilon \to 0$ if and only if $y_\varepsilon \sim \varepsilon$, and in this case $\mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \sim \mathcal{H}^1(R)$.

To extend the reasoning from the example to general $(\gamma_\varepsilon, u_\varepsilon)$, when in particular the behaviour of u_ε is not known, recall that $\mathcal{A}_\varepsilon(R) \to 0$ as $\varepsilon \to 0$ and $\|u_\varepsilon\|_\infty \leq C_0$. These properties imply

$$\int_{M_\varepsilon(R)} u_\varepsilon^2 H_s(u_\varepsilon)^2 \, d\mu_\varepsilon \to 0 \qquad \text{as} \qquad \varepsilon \to 0, \tag{4.13}$$

and with (4.4) we conclude

$$\liminf_{\varepsilon \to 0} \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) = \liminf_{\varepsilon \to 0} \left(\mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) + \int_{M_\varepsilon(R)} u_\varepsilon^2 H_s(u_\varepsilon)^2 \, d\mu_\varepsilon \right) \geq 0.$$

For the interface energy we consider again $J_{\varepsilon,\delta} = \{|u_\varepsilon| \le \delta\}$. Similar to the proof of Lemma 4.7, Hölder's inequality yields

$$\left(\delta \int_{M_\varepsilon(R\setminus J_{\varepsilon,\delta})} |B_\varepsilon|\, d\mu_\varepsilon\right)^2 \le \mathcal{A}_\varepsilon(R) \int_{M_\varepsilon} u_\varepsilon^2 |B_\varepsilon|^2\, d\mu_\varepsilon, \tag{4.14}$$

and the right hand side of (4.14) converges to zero as $\varepsilon \to 0$. Thus, using Young's inequality and $x'_\varepsilon \overset{\star}{\rightharpoonup} x'$ in $L^\infty(I)$, we estimate

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) &\ge \liminf_{\varepsilon\to 0} \int_{M_\varepsilon(R\cap J_{\varepsilon,\delta})} \varepsilon|B_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, d\mu_\varepsilon \\
&\ge 2\left(\inf_{u\in[-\delta,\delta]} \sqrt{W(u)}\right) \liminf_{\varepsilon\to 0} \int_{M_\varepsilon(R\cap J_{\varepsilon,\delta})} |B_\varepsilon|\, d\mu_\varepsilon \\
&= 2\left(\inf_{u\in[-\delta,\delta]} \sqrt{W(u)}\right) \liminf_{\varepsilon\to 0} \int_{M_\varepsilon(R)} |B_\varepsilon|\, d\mu_\varepsilon \\
&\ge 2\left(\inf_{u\in[-\delta,\delta]} \sqrt{W(u)}\right) \liminf_{\varepsilon\to 0} \int_{M_\varepsilon(R)} |\kappa_{2,\varepsilon}|\, d\mu_\varepsilon \\
&= 4\pi\left(\inf_{u\in[-\delta,\delta]} \sqrt{W(u)}\right) \int_R x'\, dt.
\end{aligned}$$

Taking the supremum over all $\delta > 0$ and combining with the estimate for $\mathcal{H}_\varepsilon$ yields

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \ge \hat\sigma \liminf_{\varepsilon\to 0} \int_{M_\varepsilon(R)} |B_\varepsilon|\, d\mu_\varepsilon \ge 2\pi\hat\sigma \int_R x'\, dt. \tag{4.15}$$

Due to $x' \ge 0$ and $y' = 0$ in R, the integral $\int_R x'\, dt$ measures the length of γ on the axis of revolution. Hence we may write

$$2\pi \int_R x'\, dt = 2\pi \int_R |\gamma'|\, dt = 2\pi \mathcal{L}_\gamma(R) = \mathcal{H}^1(M_\gamma(R)),$$

and this concludes the proof of the lower bound (4.9).

Since γ'_ε converges only weakly-$\star$ in $L^\infty(R;\mathbb{R}^2)$, the length $\mathcal{L}_\gamma(R)$ may be strictly less than $q\mathcal{H}^1(R) = \lim q_\varepsilon \mathcal{H}^1(R) = \lim \mathcal{L}_\varepsilon(R)$ where $|\gamma'_\varepsilon| \equiv q_\varepsilon \to q > 0$. As we will see in Section 4.6, it may even happen that segments of γ_ε with uniformly positive length collapse to a single point in the limit, that is, $x' = y' = 0$ in a set $J \subset R$ of positive measure $\mathcal{H}^1(J)$. Finally, for (4.13) and the subsequent reasoning to be true the uniform L^∞-bound on u_ε can be weakened, for instance to a uniform L^2-bound on $u_\varepsilon H_s(u_\varepsilon)$ in R.

4.6 Examples

Before proving the upper bound inequality we consider three examples of rather singular, non-membrane situations that may occur as limits of uniformly energy-bounded sequences. The first is the trivial note that a limit curve may consist of only vertical lines interrupted by kinks. The corresponding surface of revolution consists of circles and annuli that all have the same x-coordinate. Such a surface is unlikely to be a minimiser of $\mathcal{F}$ if $H_s(\pm 1) \neq 0$ but is easily approximated by a uniformly energy-bounded sequence.

4.6.1 A Cantor-type example

As second example we construct $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$, $\gamma = (x, y)$ and an approximation such that $R = \{y = 0\}$ has no interior points yet satisfies $\mathcal{L}_\gamma(R) = \mathcal{H}^1(R) = \mathcal{H}^1(\partial R) > 0$. The two ingredients for the example are a Cantor-type set and a curve segment that consists of an upward vertical line followed by a horizontal and a downward vertical. We only construct the interesting part of M_γ and assume that it is extended to an otherwise nice surface of revolution, for instance by two nearly ball-shaped segments. In particular, in the following all quantities $\mathcal{L}$, $\mathcal{A}$, $\mathcal{F}$ and so on refer only to the interesting segment of (γ, u) and not to the whole surface.

First, we briefly recall the construction of a generalised Cantor set [Mat95]. Let $0 < s < 1/2$, $l_k = (s/2)^k$ and $E_0 = [0, 1]$. Construct E_1 by deleting the open middle segment of length l_1 from E_0, so that E_1 consists of two disjoint closed intervals of length $(1 - l_1)/2 > l_2$. Inductively, if E_k consists of 2^k disjoint closed intervals of length $(1 - l_1 - 2l_2 - \cdots - 2^{k-1} l_k)/2^k > l_{k+1}$, delete the open middle segment of length l_{k+1} of each of these intervals to obtain E_{k+1}. Obviously, $E_{k+1} \subset E_k$ and

$$\mathcal{H}^1(\cap E_k) = \lim_{k\to\infty} \mathcal{H}^1(E_k) = 1 - \lim_{k\to\infty} \sum_{j=1}^{k} 2^{j-1} l_j = 1 - \frac{s}{2 - 2s} \in (1/2, 1).$$

We now construct a sequence (γ_k) of curves using the sequence (E_k), see Figure 4.2. With s from above let $0 < 4\sqrt{q} < s < r < 1/2$, and let γ_0 be the horizontal line segment of height $\varepsilon_0 = q^0$ above the E_0. Obtain γ_1 from γ_0 by changing the height of the horizontal from ε_0 to $\varepsilon_1 = q^1$ in E_1 and from ε_0 to $h_1 = r^1$ in $E_0 \setminus E_1$; then connect the horizontals by two vertical lines. Inductively, construct γ_{k+1} from γ_k by changing the height of the horizontals above E_{k+1} from $\varepsilon_k = q^k$ to ε_{k+1} and the height of the horizontals above $E_k \setminus E_{k+1}$ from ε_k to $h_{k+1} = r^{k+1}$; then introduce new verticals and adjust the length of existing ones to obtain a connected curve. Note that except for the latter adjustment γ_k is unchanged outside E_k.

We call the horizontals of heights $h_1, \ldots, h_k$ the top horizontals of γ_k and those of height ε_k the bottom horizontals. The sum of the lengths of all bottom horizontals β_k is obviously equal to $\mathcal{H}^1(E_k)$, and the sum of the lengths of all top horizontals is $\tau_k = 1 - \beta_k$. The length of γ_k is the length of all horizontals and all verticals, so

$$\mathcal{L}_{\gamma_k} = 1 + 2\sum_{j=1}^{k} 2^{j-1}(h_j - \varepsilon_k) = 1 + \sum_{j=1}^{k} (2r)^j - (2^{k+1} - 2) q^k \to \frac{1}{1 - 2r} \qquad \text{as } k \to \infty.$$

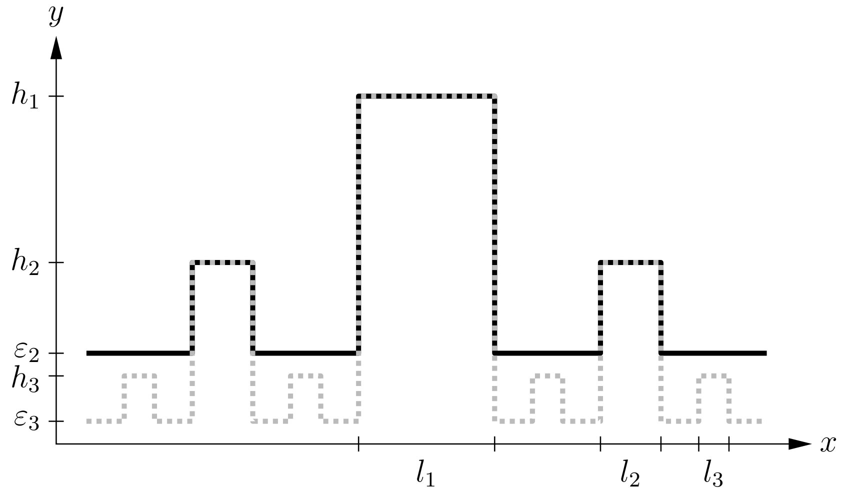

Figure 4.2: Sketch of the curves γ_2 (black) and γ_3 (grey).

Therefore, γ_k can be parametrised with constant speed converging to 1 on an interval of length $1/(1-2r)$. The area of M_{γ_k}, split in top, bottom, and vertical segments is computed as

$$\begin{aligned}\mathcal{A}_{\gamma_k} &= 2\pi\sum_{j=1}^{k} 2^{j-1} l_j h_j + 2\pi\beta_k\varepsilon_k + \pi\sum_{j=1}^{k} 2^j (h_j^2 - \varepsilon_k^2)\\ &= \pi\sum_{j=1}^{k}(sr)^j + o(1) + \pi\sum_{j=1}^{k}(2r^2)^j \to \frac{sr}{2-2sr} + \frac{2\pi r^2}{1-2r^2}.\end{aligned}$$

So for sufficiently large k we can change γ_k outside our construction slightly in order to keep the area constant.

To γ_k we associate a phase field u_k that is 0 on bottom horizontals and $+1$ on the top horizontals and the verticals. As the area, the phase integral can be kept constant by small changes outside this construction.

We will see in the next section that each kink of γ_k and each jump of u_k can be smoothed out in a neighbourhood of size of order $\sqrt{\varepsilon_k}$ and with cost $\mathcal{F}_{\varepsilon_k}$ proportional to its height above the axis of revolution. By our choice of $0 < 4\sqrt{q} < s < r$ this construction is possible and we obtain a sequence $(\gamma_k, u_k) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. The contribution of all interfaces and kinks to $\mathcal{F}_{\varepsilon_k}(\gamma_k, u_k)$ is of order

$$\sum_{j=1}^{k} 2^j (h_j + \varepsilon_k) \sim \sum_{j=1}^{k} (2r)^j + 2^{k+1} q^k \to \frac{2r}{1-2r}.$$

Note in particular that the contribution of the kinks at the bottom horizontals vanishes in the limit. Apart from the smoothing, the energy behaves like the energy for the verticals and horizontals. Since $\kappa_{1,\varepsilon_k} = \kappa_{2,\varepsilon_k} = 0$ on the verticals and $u_k = 0$ on the bottom horizontals, $\mathcal{H}_{\varepsilon_k}$ is non-zero only on the top horizontals, if we set $H_s \equiv 0$. Then we have

$$\mathcal{H}_{\varepsilon_k}(\gamma_k, u_k) \sim \sum_{j=1}^{k} 2^{j-1} \frac{l_j}{h_j} \sim \sum_{j=1}^{k} \left(\frac{s}{r}\right)^j \to \frac{s}{r-s}.$$

In the interface energy there is a contribution from $u_\varepsilon = 0$ at the bottom horizontals,

$$\int \frac{1}{\varepsilon_k} W(u_k)\, d\mu_{\varepsilon_k} \sim \frac{1}{\varepsilon_k}\beta_k \varepsilon_k \to 1 - \frac{s}{2-2s},$$

and another one from the second principal curvature on all horizontals

$$\varepsilon_k \int |B_{\varepsilon_k}|^2\, d\mu_{\varepsilon_k} \sim \varepsilon_k \left(\sum_{j=1}^{k} 2^{j-1} \frac{l_j}{h_j} + \frac{\beta_k}{\varepsilon_k} \right) \sim \beta_k.$$

In summary, $\mathcal{F}_{\varepsilon_k}(\gamma_k, u_k)$ is uniformly bounded, and setting $(\gamma_\varepsilon, u_\varepsilon) = (\gamma_k, u_k)$ for $\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$, we obtain a uniformly energy-bounded sequence defined for all sufficiently small $\varepsilon > 0$. By construction and the equi-coercivity result, the sequence $(\gamma_\varepsilon, u_\varepsilon)$ converges to a limit $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$. Also by construction, we have that $\{y = 0\} = \cap \{y_k = \varepsilon_k\}$,

$$\mathcal{L}_\gamma(\{y=0\}) = \mathcal{H}^1(\{y = 0\}) = \lim_{k\to\infty} \beta_k = 1 - \frac{s}{2-2s} \in (1/2, 1),$$

and that $\{y = 0\}$ has no interior points. Note finally that $\gamma_k' \sim (1, 0)$ in $\{y_k = \varepsilon_k\}$ converges strongly in L^p.

4.6.2 A constancy interval in the limit

Our third example consists of a sequence such that the corresponding limit parametrisation has a constancy interval. Again we consider only the interesting part of M_{γ_ε} and assume that it is extended to a nice surface of revolution. The example is based on a rectangular signal

$$\rho(t) = (x_\rho(t), y_\rho(t)) = \begin{cases} (0, t) & \text{if } 0 \le t < 1, \\ (t-1, 1) & \text{if } 1 \le t < \frac{3}{2}, \\ (\frac{1}{2}, \frac{5}{2} - t) & \text{if } \frac{3}{2} \le t < \frac{5}{2}, \\ (t-2, 0) & \text{if } \frac{5}{2} \le t < 3 \end{cases}$$

for $t \in [0, 3)$ and extended periodically. For a sequence $\varepsilon_k \to 0$ to be chosen below, let the approximate curve

$$\gamma_k(t) = (x_k(t), y_k(t)) = \begin{pmatrix} 0 \\ \varepsilon_k \end{pmatrix} + \begin{pmatrix} x_\rho(kt)/k^2 \\ y_\rho(kt)/k \end{pmatrix}$$

for $t \in [0, 3)$ consist of an appropriately scaled version of ρ, see Figure 4.3. By counting once again horizontals and verticals one easily checks that $\mathcal{L}_{\gamma_k} = 2 + 1/k$; hence γ_k can

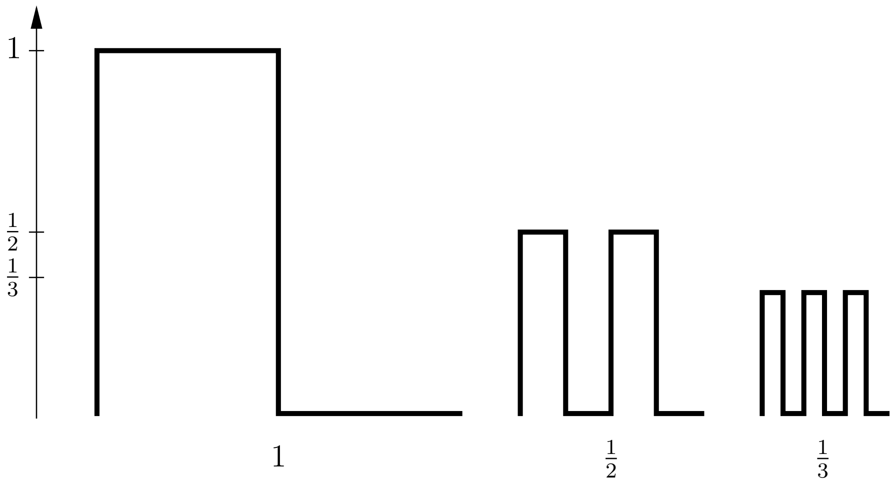

Figure 4.3: The shape of γ_1, γ_2, γ_3, neglecting the base height ε_k.

be parametrised with constant speed converging to 1 on the interval $[0, 2]$. Also, $\mathcal{A}_{\gamma_k} \leq 2\pi(\varepsilon_k + 1/k)\mathcal{L}_{\gamma_k} = o(1)$ as $k \to \infty$, so the area constraint can be conserved outside the construction.

To each γ_k we associate the phase field $u_k \equiv 0$. Then no phase transitions are necessary and each kink can be smoothed out in a neighbourhood of size of order ε_k with energy $\mathcal{F}_{\varepsilon_k}$ proportional to its height above the axis of revolution, see the next section for the details. Since the height of each rectangle in γ_k is $1/k$, the choice $\varepsilon_k = c/k$, where $0 < c \ll 1$ is a small constant, allows to do the smoothing and to obtain $(\gamma_k, u_k) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. The number of kinks of γ_k, both at the top and the bottom of the rectangle, is obviously proportional to k, so the total cost for smoothing out all kinks of γ_k is of order

$$k(\varepsilon_k + 1/k) \sim 1.$$

Apparently, $\mathcal{H}_{\varepsilon_k}(\gamma_k, u_k) = 0$ due to $u_k = 0$. The non-zero terms of the interface energy without the smoothing are of order

$$\int \varepsilon_k |B_{\varepsilon_k}|^2 \, d\mu_{\varepsilon_k} \lesssim \frac{1}{k},$$

because $y_k \geq \varepsilon_k$ and the total length of all horizontals is $1/k$, and

$$\int \frac{1}{\varepsilon_k} W(u_k) \, d\mu_{\varepsilon_k} \sim \mathcal{L}_{\gamma_k} \to 2.$$

In summary, $\mathcal{F}_{\varepsilon_k}(\gamma_k, u_k)$ is uniformly bounded, and $\gamma_k' \overset{\star}{\rightharpoonup} (0, 0)$ in L^∞ in a set of measure 2. Note that $\mathcal{F}_{\varepsilon_k}$ is of order $1 + \mathcal{L}_{\gamma_k}$ with non-zero contributions from the smoothing of kinks and the potential term as $\varepsilon_k \to 0$, while $\mathcal{F} = 0$. We will return to this degenerate behaviour in the last section of this chapter.

4.7 Upper bound inequality

This section is devoted to the upper bound inequality

$$\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(\gamma_\varepsilon,u_\varepsilon)\le\mathcal{F}(\gamma,u)$$

whenever $(\gamma,u)\in\mathcal{D}\times\mathcal{Q}$ and γ is parametrised with constant speed. Since the value of u in $\{y=0\}$ does not enter the energy or our arguments, we assume $u=0$ in $\{y=0\}$. Similar to the previous chapter we first approximate (γ,u) by a sequence $(\gamma_\delta,u_\delta)\in\mathcal{D}\times\mathcal{Q}$ of simple configurations that have a finite number of components and finitely many (ghost) interfaces and then construct a recovery sequence for such configurations.

Throughout this section we assume that $(\gamma,u)\in\mathcal{D}\times\mathcal{Q}$ has finite energy $\mathcal{F}(\gamma,u)$ and constant speed $|\gamma'|\equiv q_\gamma$ in I, without loss of generality $q_\gamma=1$.

4.7.1 Approximation by simple configurations

Lemma 4.9. *Assume that M_γ has infinitely many components. Then there is a sequence $(\gamma_\delta,u_\delta)\in\mathcal{D}\times\mathcal{Q}$ that converges to (γ,u) in $C^0(I;\mathbb{R}^2)\times L^1(\{y>0\})$ as $\delta\to 0$ such that $\mathcal{F}(\gamma_\delta,u_\delta)\to\mathcal{F}(\gamma,u)$ and each M_{γ_δ} has finitely many components.*

Proof. Given $\delta>0$ sufficiently small, we construct γ_δ by replacing components ω of $\{y>0\}$ with length $\mathcal{L}_\gamma(\omega)$ at most δ by horizontals on the x-axis of the same length. More precisely, if without loss of generality $I=(0,\mathcal{L}_\gamma)$, let

$$\gamma_\delta(0)=\gamma(0),\qquad \gamma_\delta'(t)=\begin{cases}(1,0) & \text{if } t\in\omega \text{ with } \mathcal{L}_\gamma(\omega)\le\delta,\\ \gamma'(t) & \text{otherwise}\end{cases}$$

and

$$u_\delta(t)=\begin{cases}0 & \text{if } t\in\omega \text{ with } \mathcal{L}_\gamma(\omega)\le\delta,\\ u(t) & \text{otherwise.}\end{cases}$$

Since length, area, and $|\mathcal{F}(\gamma,u)|$ are finite, the contributions of the small components vanish as $\delta\to 0$, that is

$$\sum_{\omega:\mathcal{L}_\gamma(w)\le\delta}|\mathcal{F}(\gamma,u,\omega)|+\mathcal{A}_\gamma(\omega)+\mathcal{L}_\gamma(\omega)+\mathcal{H}^1(\omega)\to 0\qquad\text{as}\qquad\delta\to 0. \tag{4.16}$$

Denoting by $U=\cup\{\omega:\mathcal{L}(\omega)\le\delta\}$ the union of all replaced components, we find

$$\begin{aligned}|x(t)-x_\delta(t)|&\le\int_0^t|x'(s)-x_\delta'(s)|\,ds\le\int_{(0,t)\cap U}1-x'(s)\,ds\\&\le\int_{(0,t)\cap U}1\,ds\le\sum_{\omega:\mathcal{L}_\gamma(\omega)\le\delta}\mathcal{L}_\gamma(\omega)\end{aligned}$$

and

$$\begin{aligned}|y(t)-y_\delta(t)| &= \begin{cases}0 & \text{if } t\in\omega \text{ with } \mathcal{L}_\gamma(\omega)>\delta \text{ or } y(t)=0,\\ y(t) & \text{otherwise}\end{cases}\\ &\le \delta,\end{aligned}$$

that is $\gamma_\delta \to \gamma$ in $C^0(I;\mathbb{R}^2)$ as $\delta\to 0$. Also,

$$\int_I |u-u_\delta|^p\,dt \le \int_U |u|^p\,dt = \int_U dt \le \sum_{\omega:\mathcal{L}_\gamma(w)\le\delta} \mathcal{H}^1(\omega),$$

hence $u_\delta \to u$ in $L^p(I)$ as $\delta\to 0$. The energy difference consists of the total energy of the removed components and the interface energy of the new horizontals on the axis of revolution. Thus we have

$$|\mathcal{F}(\gamma,u)-\mathcal{F}(\gamma_\delta,u_\delta)| \le \sum_{\omega:\mathcal{L}_\gamma(\omega)\le\delta} |\mathcal{F}(\gamma,u,\omega)| + 2\pi\hat{\sigma}\mathcal{L}_\gamma(\omega),$$

and both terms vanish as $\delta\to 0$ due to (4.16). The area of M_δ is smaller than the area of M_γ and the difference satisfies

$$\begin{aligned}\mathcal{A}_\gamma - \mathcal{A}_\delta &= \sum_{\omega:\mathcal{L}_\gamma(\omega)\le\delta} \mathcal{A}_\gamma(\omega) = 2\pi \sum_{\omega:\mathcal{L}_\gamma(\omega)\le\delta} \int_\omega |\gamma'|y\,dt\\ &\le 2\pi\delta \sum_{\omega:\mathcal{L}_\gamma(\omega)\le\delta} \mathcal{L}_\gamma(\omega) = o(\delta).\end{aligned}$$

The area constraint can for all sufficiently small δ be recovered as follows. If we find an interval $J \Subset \{y>0\}\backslash(S_\gamma \cup S_u)$ such that $x'>0$ in $\overline{J}$, we argue as in the proof of Corollary 3.7 and add a small perturbation supported in J to γ_δ that corrects the constraint and vanishes as $\delta\to 0$; a final reparametrisation ensures $\gamma_\delta\in\mathcal{D}$. Note that, in contrast to Corollary 3.7, we do not have to exclude that M_γ is a catenoid in J, because here the perturbation has to increase the area of M_δ. If there is no such interval J, then γ consists only of vertical line segments interrupted by kinks. In this case the area constraint is easily recovered by increasing the length of two adjacent line segments by an order less than δ, which together with the resulting energy vanishes as $\delta\to 0$.

Finally, if there is at least one proper interface without kink in (γ,u) then this interface also belongs to (γ_δ,u_δ) for all sufficiently small δ. It can be moved by an order less than δ to recover the phase integral constraint. If (γ,u) contains no proper interface, introducing one or more new interfaces at a height of order less than $\sqrt{\delta}$ above the axis of revolution and flipping u_δ below these new interfaces recovers the constraint and vanishes as $\delta\to 0$. $\square$

Lemma 4.10. *Assume that M_γ has finitely many components. Then there is a sequence $(\gamma_\delta, u_\delta) \in \mathcal{D} \times \mathcal{Q}$ that converges to (γ, u) in $C^0(I;\mathbb{R}^2) \times L^1(\{y > 0\})$ as $\delta \to 0$ such that $\mathcal{F}(\gamma_\delta, u_\delta) \to \mathcal{F}(\gamma, u)$ and $\mathcal{H}^0(S_{\gamma_\delta} \cup S_{u_\delta}) < \infty$. Moreover, every component $\omega = (a, b)$ of M_δ meets the axis of revolution in a line perpendicular to it, that is $\gamma_\delta' = (0, 1)$ near a and $\gamma_\delta' = (0, -1)$ near b.*

Proof. Given $\delta > 0$ such that $\delta/2$ is less than the minimal length of a component of M_γ, we construct $(\gamma_\delta, u_\delta)$ by changing γ and u at points whose distance to the boundary of their component is less than δ. More precisely, let $\omega = (a, b)$ be a component of M_γ and let $b_\delta \in \omega$ be the point such that $\operatorname{dist}_\gamma(\gamma(b), \gamma(b_\delta)) = \delta$, that is $b_\delta = b - \delta$. In $[b_\delta, b]$ we change γ in the following way: starting from $\gamma(b_\delta)$ we move vertically down with speed $|\gamma'| = 1$ until we reach the axis of revolution at some $\hat{b}_\delta \in (b_\delta, b]$, where $\hat{b}_\delta = b$ corresponds to the case that $\gamma(\omega)$ is already a vertical line segment in $[b_\delta, b]$. We fill the remaining interval $(\hat{b}_\delta, b]$ by moving to the right with speed 1. Thus in $(b_\delta, b]$, γ_δ is given by

$$\gamma_\delta(t) = \begin{cases} (x(b_\delta), y(b_\delta) - t + b_\delta) & \text{if } t \in (b_\delta, \hat{b}_\delta), \\ (x(b_\delta) + t - \hat{b}_\delta, 0) & \text{if } t \in (\hat{b}_\delta, b), \end{cases}$$

where $\hat{b}_\delta = y(b_\delta) + b_\delta$. Note that the length of the new vertical as well the length of the new horizontal is less than $b - b_\delta = \delta$ and that $y_\delta \le y \le \delta$ in (b_δ, b). At the other component boundary a we do the same but with the horizontal to the left. Finally, we set the phase field u_δ to (for instance) $+1$ on the new verticals and 0 on the new horizontals.

Making this replacement for every component and moving remaining segments of γ slightly in x-direction to glue all parts together continuously we obtain $(\gamma_\delta, u_\delta)$ such that

$$|y(t) - y_\delta(t)| \le \delta, \qquad |x(t) - x_\delta(t)| \le 4N_\gamma \delta,$$

where N_γ denotes the number of components of γ. Hence, γ_δ converges to γ in $C^0(I;\mathbb{R}^2)$ as $\delta \to 0$. Obviously, $u_\delta \to u$ almost everywhere in I and thus in $L^p(I)$. Denoting by M_{orig} all the parts of M_γ that have been removed and by M_{hor} and M_{ver} the newly introduced horizontals and verticals, the Helfrich energy difference is bounded by

$$|\mathcal{H}(\gamma, u) - \mathcal{H}(\gamma_\delta, u_\delta)| \le \int_{M_{orig}} |H_\gamma - H_s(u)|^2 + |K_\gamma| \, d\mu + \int_{M_{ver}} H_s(u_\delta)^2 \, d\mu_\delta.$$

The second term is bounded by $\|H_s\|_\infty^2 \mu_\delta(M_{ver}) \to 0$ as $\delta \to 0$, and the first vanishes as $\delta \to 0$ due to $\mu(M_{orig}) \to 0$ and uniform continuity of the integral. The difference in the interface energy consists of original (ghost) interfaces that are omitted in $(\gamma_\delta, u_\delta)$, the two probably introduced kinks for each component, and the new pieces on the axis of revolution.

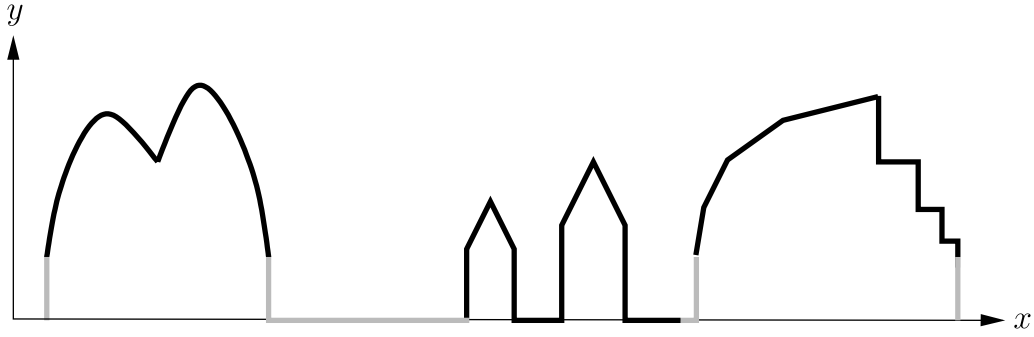

Figure 4.4: Approximation of the curve in Figure 4.1, original segments in black, new parts in grey. Small components are removed and the ends of the remaining components are replaced by line segments; between two adjacent components there is always a small horizontal segment on the axis of revolution. The number of kinks and interfaces is finite.

Therefore, we obtain

$$\frac{1}{2\pi}|\mathcal{I}(\gamma,u)-\mathcal{I}(\gamma_\delta,u_\delta)| \leq \sum_{\substack{s\in S_\gamma\cup S_u\\ y(s)\leq\delta}} (\sigma+\hat{\sigma}|[\gamma'](s)|)y(s) + 2N_\gamma(\sigma+\hat{\sigma}\pi)\delta + 2N_\gamma\hat{\sigma}\delta.$$

The first term converges to 0 as $\delta\to 0$ because the sum over all (ghost) interfaces is finite, thus the energy difference vanishes as $\delta\to 0$. Finally, the area difference is bounded by the area of the segments taken away plus the area of the newly introduced vertical segments. Hence,

$$|\mathcal{A}_\gamma-\mathcal{A}_\delta| \leq 2\pi\int_{\{y\leq\delta\}} |y(t)-y_\delta(t)|\,dt \leq 8\pi N_\gamma\delta^2,$$

and area and phase field constraint can be recovered as in the proof of Lemma 4.9. □

Combining Lemmas 4.9 and 4.10 and applying some minor cosmetics, namely removing horizontals at ∂I and adding a horizontal between adjacent components if necessary, yields the following approximation.

Corollary 4.11. *For* $(\gamma,u)\in\mathcal{D}\times\mathcal{Q}$ *with finite energy and* $|\gamma'|=\mathrm{const}$ *there is a sequence* $(\gamma_\delta,u_\delta)\in\mathcal{D}\times\mathcal{Q}$ *such that* $\mathcal{F}(\gamma_\delta,u_\delta)\to\mathcal{F}(\gamma,u)$ *as* $\delta\to 0$ *and each* (γ_δ,u_δ) *has finitely many components and finitely many (ghost) interfaces. Any component* ω *of* M_{γ_δ} *meets the axis of revolution in a straight line perpendicular to it; in particular,* γ_δ *is vertical near* ∂I*. Two adjacent components are always connected by a horizontal segment on the axis of revolution.*

From now on we assume that (γ,u) has the form of the approximations constructed above and summarised in Corollary 4.11. As an example see Figure 4.4 for the approximation of the curve in Figure 4.1.

4.7.2 Kinks and interfaces

Let $s \in S_\gamma$ be a kink, $S = S_\gamma \cup S_u$, and $J \Subset \{y > 0\}$ with $\overline{J} \cap S = \{s\}$. For simplicity of notation we formulate the following arguments for curves and phase fields given in an interval J around $s = 0$; recall that $|\gamma'| = 1$ in J.

First we smooth out kinks by a linear interpolation of the tangent angle of γ around $s = 0$. This local procedure disturbs area and phase integral constraint and disrupts the curve, so that we have to add some corrections and global shifts in x-direction similar to the curve approximation in Chapter 3.

Lemma 4.12. *Let $J = (-a, a)$. For all sufficiently small $\varepsilon > 0$ there is $\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon) \in W^{2,2}(J; \mathbb{R}^2)$ such that*

- *γ_ε satisfies $\inf_J y_\varepsilon > 0$ and $x'_\varepsilon \geq 0$;*
- *at the end points of $\gamma_\varepsilon(J)$ we have*
$$\gamma_\varepsilon(-a) = \gamma(-a), \quad \gamma'_\varepsilon(-a) = \gamma'(-a), \quad \gamma_\varepsilon(a) = (x(a) + o(1), y(a)), \quad \gamma'_\varepsilon(a) = \gamma'(a);$$
- *$\gamma_\varepsilon \to \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ for any $p \in [1, \infty)$ as $\varepsilon \to 0$;*
- *$\mathcal{A}_\varepsilon(J) = \mathcal{A}_\gamma(J) + O(\varepsilon)$ and $\displaystyle\int_{M_\varepsilon(J)} u \, d\mu_\varepsilon = \int_{M_\gamma(J)} u \, d\mu + O(\varepsilon)$ as $\varepsilon \to 0$; and*
- *with $J_\varepsilon = (-\delta_\varepsilon, \delta_\varepsilon)$, where*
$$\delta_\varepsilon = \frac{|[\gamma']|}{\hat{\sigma}} \varepsilon, \tag{4.17}$$
there holds
$$\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, 0, J_\varepsilon) = 2\pi\hat{\sigma}|[\gamma'](0)|y(0).$$

Moreover, $\gamma'_\varepsilon = \gamma' + r_\varepsilon$ in $J \setminus J_\varepsilon$ where $\operatorname{spt} r_\varepsilon \Subset J \setminus J_\varepsilon$ is independent of ε and $r_\varepsilon \to 0$ in $W^{1,2}(J; \mathbb{R}^2)$ as $\varepsilon \to 0$.

Proof. Let φ be an angle function for γ in J that is uniformly continuous on either side of $s = 0$ and satisfies $|\varphi| \leq \pi/2$. Denote by φ^+ and φ^- the one-sided limit of φ at $s = 0$ from the right and the left, respectively. Then the kink carries the "bending energy" $2\pi\hat{\sigma}y(s)|\varphi^+ - \varphi^-|$.

In the simple case that γ consists of two straight lines in J so that φ is constant on either side of $s = 0$, the linear interpolation $\varphi_\varepsilon \in W^{1,p}(J)$ of $\varphi^\pm$ in the interval $J_\varepsilon = (-\delta_\varepsilon, \delta_\varepsilon) \subset J$ is given by
$$\varphi_\varepsilon(t) = \begin{cases} \varphi^- & \text{if } t < -\delta_\varepsilon, \\ \dfrac{\varphi^+ - \varphi^-}{2\delta_\varepsilon} t + \dfrac{\varphi^+ + \varphi^-}{2} & \text{if } -\delta_\varepsilon \leq t < \delta_\varepsilon, \\ \varphi^+ & \text{if } \delta_\varepsilon \leq t. \end{cases}$$

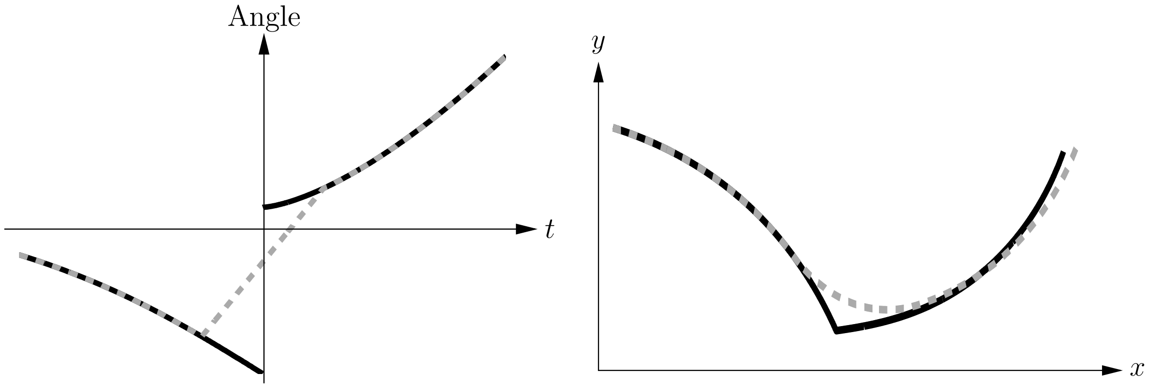

Figure 4.5: Linear interpolation of the angle (left) and the corresponding curve (right). Original angle and curve are black, interpolations grey.

The curve γ_ε, defined by φ_ε and the left endpoint of $\gamma_\varepsilon(J)$ equal to the corresponding endpoint of $\gamma(J)$, converges in $W^{1,p}(J;\mathbb{R}^2)$ to γ as $\varepsilon \to 0$, and using Young's inequality one easily verifies

$$\begin{aligned}\frac{1}{2\pi}\mathcal{I}_\varepsilon(\gamma_\varepsilon, 0, J_\varepsilon) &= \left(\frac{2\delta_\varepsilon}{\varepsilon}W(0) + \frac{\varepsilon}{2\delta_\varepsilon}(\varphi^+ - \varphi^-)^2\right)\frac{1}{2\delta_\varepsilon}\int_{-\delta_\varepsilon}^{\delta_\varepsilon} y_\varepsilon\,dt + \varepsilon\int_{-\delta_\varepsilon}^{\delta_\varepsilon}\kappa_{2,\varepsilon}^2 y_\varepsilon\,dt \\ &\geq \hat{\sigma}|\varphi^+ - \varphi^-|\frac{1}{2\delta_\varepsilon}\int_{-\delta_\varepsilon}^{\delta_\varepsilon} y_\varepsilon\,dt + \varepsilon\int_{-\delta_\varepsilon}^{\delta_\varepsilon}\kappa_{2,\varepsilon}^2 y_\varepsilon\,dt. \qquad (4.18)\end{aligned}$$

Equality holds if and only if δ_ε is as in (4.17). The first integral in (4.18) divided by $2\delta_\varepsilon$ converges to $y(0)$ as $\varepsilon \to 0$, and the second one vanishes because $\kappa_{2,\varepsilon} = x'_\varepsilon/y_\varepsilon$ is bounded in J. Thus $\mathcal{I}_\varepsilon(\gamma_\varepsilon, 0, J_\varepsilon) \to 2\pi\hat{\sigma}|[\gamma']|y(0)$ as desired.

The interpolation of a general angle function is

$$\varphi_\varepsilon(t) = \begin{cases}\varphi(t) & \text{if } |t| > \delta_\varepsilon, \\ \dfrac{(\varphi(\delta_\varepsilon) - \varphi(-\delta_\varepsilon))}{2\delta_\varepsilon}t + \dfrac{(\varphi(\delta_\varepsilon) + \varphi(-\delta_\varepsilon))}{2} & \text{if } |t| \leq \delta_\varepsilon,\end{cases}$$

see Figure 4.5, and similarly as above we get

$$\begin{aligned}\frac{1}{2\pi}\mathcal{I}_\varepsilon(\gamma_\varepsilon, 0, J_\varepsilon) &= \sqrt{W(0)}\left(|[\varphi]| + \frac{|\varphi(\delta_\varepsilon) - \varphi(-\delta_\varepsilon)|^2}{|[\varphi]|}\right)\frac{1}{2\delta_\varepsilon}\int_{-\delta_\varepsilon}^{\delta_\varepsilon} y_\varepsilon\,dt + \varepsilon\int_{-\delta_\varepsilon}^{\delta_\varepsilon}\kappa_{2,\varepsilon}^2 y_\varepsilon\,dt \\ &\to 2\sqrt{W(0)}|[\varphi]|y(s) = \hat{\sigma}|[\gamma']|y.\end{aligned}$$

By construction, $\varphi_\varepsilon \in [-\pi/2, \pi/2]$, that is $x'_\varepsilon \geq 0$, and $|\gamma'_\varepsilon| \equiv 1$ in J. Also, $\gamma_\varepsilon \to \gamma$ in $W^{1,p}(J;\mathbb{R}^2)$ because $\varphi_\varepsilon \to \varphi$ in $L^p(J)$ for any $p \in [1,\infty)$. Therefore, $y_\varepsilon \geq \inf_J y/2 > 0$ for all sufficiently small ε.

It remains to correct the y-coordinate of the right end point of $\gamma_\varepsilon(J)$ and to calculate the error in the area and phase integral constraint. We proceed similarly to the curve approximation in Lemma 3.6 and fix $\widetilde{J} \Subset J \setminus J_\varepsilon$ such that $\inf \widetilde{J} > \delta_\varepsilon$ for all small δ_ε and $f \in$

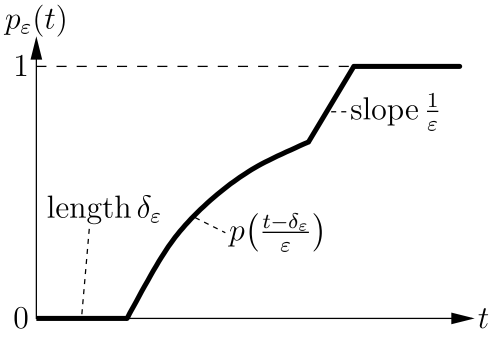

Figure 4.6: Construction of p_ε consisting of a "platform" for the curve recovery, the optimal profile, and the connection to 1.

$C_c^\infty(\widetilde{J})$ with $\int_{\widetilde{J}} f\,dt = 1$. The perturbed curve $\widetilde{\gamma}_\varepsilon = (x_\varepsilon, y_\varepsilon + \alpha_\varepsilon F)$, where $F(t) = \int_0^t f(s)\,ds$, has the desired end point y-coordinate for

$$\alpha_\varepsilon = y(a) - y_\varepsilon(a).$$

Since

$$|\gamma_\varepsilon(t) - \gamma(t)| \le \int_{-\delta_\varepsilon}^{\delta_\varepsilon} |\gamma_\varepsilon' - \gamma'|\,dt \le 4\delta_\varepsilon = O(\varepsilon),$$

also α_ε, $\|\widetilde{\gamma}_\varepsilon - \gamma\|_\infty$ and $\|\widetilde{\gamma}_\varepsilon' - \gamma'\|_{L^\infty(J\setminus J_\varepsilon)}$ are at most of order ε. The claims for area and phase constraint follow, and we have $r_\varepsilon = (0, \alpha_\varepsilon f)$. □

Next we construct a recovery sequence for the phase field in J which is in line with γ_ε of Lemma 4.12. The building block p_ε for the recovery connects $p_\varepsilon = 0$ and $p_\varepsilon = 1$ by a transition according to the optimal profile of the homogeneous one-dimensional phase transition problem and a linear function as in Chapter 3. But in addition, there is a "platform" $\{p_\varepsilon = 0\}$ to smooth out the kink, see Figure 4.6, and the profile is given by

$$p_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le \delta_\varepsilon, \\ p\left(\frac{t-\delta_\varepsilon}{\varepsilon}\right) & \text{if } \delta_\varepsilon < t \le \delta_\varepsilon + \sqrt{\varepsilon}, \\ p(1/\sqrt{\varepsilon}) + \frac{1}{\varepsilon}\left(t - \delta_\varepsilon - \sqrt{\varepsilon}\right) & \text{if } \delta_\varepsilon + \sqrt{\varepsilon} < t \le \delta_\varepsilon + \sqrt{\varepsilon} + \varepsilon\left(1 - p(1/\sqrt{\varepsilon})\right), \\ 1 & \text{if } \delta_\varepsilon + \sqrt{\varepsilon} + \varepsilon\left(1 - p(1/\sqrt{\varepsilon})\right) < t. \end{cases}$$

We now estimate the interface energy of γ_ε combined with a suitable phase field u_ε based on p_ε.

Lemma 4.13. *Let γ_ε be as in Lemma 4.12. Then there exists $u_\varepsilon \in W^{1,p}(J)$ such that $\|u_\varepsilon\|_{L^\infty(J)} \le C_0$, $u_\varepsilon = u$ on ∂J, $u_\varepsilon \to u$ in $L^p(J)$ for any $p \in [1,\infty)$, $\int_{M_\varepsilon(J)} u_\varepsilon\,d\mu_\varepsilon = \int_{M_\gamma(J)} u\,d\mu + o(\sqrt{\varepsilon})$, and*

$$\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) \le 2\pi(\sigma + \hat{\sigma}|[\gamma'](0)|)y(0),$$
$$\limsup_{\varepsilon\to 0} \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) \le \mathcal{H}(\gamma, u, J).$$

Proof. If $s = 0$ is a proper interface let

$$u_\varepsilon(t) = \begin{cases} p_\varepsilon(t) & \text{if } 0 \le t, \\ -p_\varepsilon(-t) & \text{if } 0 > t, \end{cases}$$

if $u(t) = \operatorname{sign} t$ in J, and the negative of it if $u(t) = -\operatorname{sign} t$; for a ghost interface take the combination of $p_\varepsilon(t)$ and $p_\varepsilon(-t)$ or its negative. Obviously, $u_\varepsilon \to u$ in $L^p(J)$, $|u_\varepsilon| \le C_0$ in J, and $u_\varepsilon = u$ on ∂J. For the energy estimates we assume $u(t) = \operatorname{sign} t$, the proof of the other cases works with the obvious changes.

Due to $u_\varepsilon \equiv 0$ in $J_\varepsilon = (-\delta_\varepsilon, \delta_\varepsilon)$, Lemma 4.12 provides

$$\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J_\varepsilon) \le 2\pi\hat{\sigma}|[\gamma'](0)|y(0).$$

Since $\gamma \in W^{2,2}(J \setminus \{0\}\,; \mathbb{R}^2)$ and $\gamma_\varepsilon = \gamma + o(1)$ in $W^{2,2}(J \setminus J_\varepsilon; \mathbb{R}^2)$, the curvature term in $\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J \setminus J_\varepsilon)$ vanishes as $\varepsilon \to 0$. The other terms are easily estimated by

$$\begin{aligned} \frac{1}{2\pi}\int_{M_\varepsilon(J\cap\{t>\delta_\varepsilon\})} \varepsilon|\nabla_{M_\varepsilon} p_\varepsilon|^2 + \frac{1}{\varepsilon}W(p_\varepsilon)\,d\mu_\varepsilon &\le \left(\sup_{[\delta_\varepsilon,\delta_\varepsilon+\sqrt{\varepsilon}]} y_\varepsilon\right)\int_0^{1/\sqrt{\varepsilon}} |p'(t)|^2 + W(p(t))\,dt \\ &\quad + \|y_\varepsilon\|_\infty \left(1 - p\left(1/\sqrt{\varepsilon}\right)\right)(1 + \sup_{[0,1]} W) \end{aligned}$$

on the positive side of $s = 0$, and similarly on the other side. Taking the limit superior as $\varepsilon \to 0$ proves the upper bound for the interface energy in J as in Lemma 3.8. The estimate for the Helfrich energy follows from $\mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) = \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J \setminus J_\varepsilon)$ and the Dominated Convergence Theorem using convergence of u_ε and $\gamma_\varepsilon\chi_{J\setminus J_\varepsilon}$.

Finally, one easily sees that

$$\left|\int_{M_\gamma(J)} u\,d\mu - \int_{M_\varepsilon(J)} u_\varepsilon\,d\mu_\varepsilon\right| \lesssim \delta_\varepsilon + \int_{\delta_\varepsilon}^{\delta_\varepsilon+\sqrt{\varepsilon}} (1 - p_\varepsilon)\,dt + \varepsilon\left(1 - p(1/\sqrt{\varepsilon})\right),$$

and since

$$\int_{\delta_\varepsilon}^{\delta_\varepsilon+\sqrt{\varepsilon}} (1 - p_\varepsilon)\,dt = \sqrt{\varepsilon}\int_0^1 1 - p(t/\sqrt{\varepsilon})\,dt = o(\sqrt{\varepsilon}),$$

the phase integral difference is also of order $\sqrt{\varepsilon}$. □

4.7.3 Axis of revolution

At the axis of revolution let $J_0 \subset \{y = 0\}$ be an interval that is enclosed by two intervals J_l, J_r such that $(J_l \cup J_0 \cup J_r) \cap S = \emptyset$,

$$\gamma'(t) = \begin{cases} (0,-1) & \text{in } J_l, \\ (1,0) & \text{in } J_0, \\ (0,1) & \text{in } J_r, \end{cases}$$

and $u = 0$ in J_0, $|u| = 1$ in $J_l \cup J_r$. The energy $\mathcal{F}(\gamma, u, J_0)$ is $2\pi\hat{\sigma}\mathcal{H}^1(J_0)$, and this is easily recovered by setting $u_\varepsilon = 0$ and $\gamma_\varepsilon = \gamma + (0, 2\varepsilon/\hat{\sigma})$ in J_0 because then

$$\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J_0) = 2\pi \int_{J_0} \left(\frac{1}{\varepsilon} W(0) + \varepsilon\kappa_{2,\varepsilon}^2 \right) y_\varepsilon \, dt = 2\pi \int_{J_0} \frac{2}{\hat{\sigma}} W(0) + \frac{\hat{\sigma}}{2} \, dt = 2\pi\hat{\sigma}\mathcal{H}^1(J_0).$$

In J_l and J_r we use the same construction as for kinks. If for simplicity of notation $J_r = (0, a)$, $\gamma(0) = (0, 0)$, and $u \equiv 1$ in J_r, we consider the approximate curve γ_ε given by $\gamma_\varepsilon(0) = (0, 2\varepsilon/\hat{\sigma})$ and the angle function

$$\varphi_\varepsilon(t) = \begin{cases} \dfrac{\pi}{2\alpha\varepsilon} t & \text{if } 0 < t < \alpha\varepsilon, \\ \dfrac{\pi}{2} & \text{if } t > \alpha\varepsilon \end{cases}$$

together with the phase field $u_\varepsilon = p_\varepsilon$ for $\delta_\varepsilon = \alpha\varepsilon$. The purpose of $\alpha = 2\pi/(\hat{\sigma}(\pi - 2))$ is to ensure the condition $y_\varepsilon(t) = t = y(t)$ for $t \geq \alpha\varepsilon$. Thanks to $y_\varepsilon \geq 2\varepsilon/\hat{\sigma}$ in J_r, we have

$$\frac{1}{2\pi} \int_{M_\varepsilon(J_r)} \varepsilon\kappa_{2,\varepsilon}^2 \, d\mu_\varepsilon = \varepsilon \int_0^{\alpha\varepsilon} \frac{x_\varepsilon'^2}{y_\varepsilon} \, dt \leq \frac{\hat{\sigma}}{2} \int_0^{\alpha\varepsilon} x_\varepsilon'^2 \, dt \leq \frac{\hat{\sigma}}{2} \alpha\varepsilon,$$

and since the computations for all other terms of $\mathcal{I}_\varepsilon$ from the kink and interface recovery still apply, we conclude $\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J_r) \to 0 = \mathcal{I}(\gamma, u, J_r)$. For the Helfrich energy we use that γ_ε is a vertical line where $u_\varepsilon \neq 0$ in J_r and obtain

$$\begin{aligned} \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J_r) = \mathcal{H}_\varepsilon((x_\varepsilon, t), u_\varepsilon, (\alpha\varepsilon, a)) &= \int_{M_\varepsilon(\alpha\varepsilon, a)} u_\varepsilon^2 H_s(u_\varepsilon)^2 \, d\mu_\varepsilon \\ &\to \int_{M(J_r)} H_s(u)^2 \, d\mu = \mathcal{H}(\gamma, u, J_r), \end{aligned}$$

where $x_\varepsilon = \int_0^{\alpha\varepsilon} \cos u_\varepsilon \, dt = 2\alpha\varepsilon/\pi$. The change in area when replacing γ by γ_ε is of order ε, and the difference in the phase integral is of order $o(\sqrt{\varepsilon})$ as in Lemma 4.13. Finally, applying the above reasoning to J_l finishes the construction at the axis of revolution.

4.7.4 Summary

Corollary 4.14. *Let* $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$ *satisfy* $|\gamma'| = \text{const}$ *in* I. *Then there exists a sequence* $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ *such that* $(\gamma_\varepsilon, u_\varepsilon) \to (\gamma, u)$ *in* $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ *and* $\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \leq \mathcal{F}(\gamma, u)$.

Proof. By Corollary 4.11 there is a sequence of simple configurations $(\gamma_\delta, u_\delta) \in \mathcal{D} \times \mathcal{Q}$ such that $(\gamma_\delta, u_\delta) \to (\gamma, u)$ in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ and $\mathcal{F}(\gamma_\delta, u_\delta) \to \mathcal{F}(\gamma, u)$ as $\delta \to 0$. Each $(\gamma_\delta, u_\delta)$ can be approximated by a sequence $(\gamma_{\delta_\varepsilon}, u_{\delta_\varepsilon})$ that results from combining the local constructions from Lemmas 4.12, 4.13 and the last subsection with the unchanged segments of $(\gamma_\delta, u_\delta)$, taking the x-shifts into account. These sequences satisfy $\mathcal{A}_{\delta_\varepsilon} = \mathcal{A}_\delta + O(\varepsilon) = A_0 + O(\varepsilon)$ and $\int_{M_{\delta_\varepsilon}} u_{\delta_\varepsilon} \, d\mu_{\delta_\varepsilon} = mA_0 + o(\sqrt{\varepsilon})$. The area constraint can be recovered as in the proof of Lemma 4.9, and the phase integral as in Lemma 3.8. A final reparametrisation to fix the constant speed requirement yields configurations in $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, and taking a diagonal sequence finishes the proof. □

4.8 Generalisations and open problems

Finally, we discuss possible extensions of Theorem 4.3 and problems related to them. First of all, the proof is easily adapted to non-symmetric potentials W. In this case one splits σ into two constants

$$\sigma^+ = \int_0^1 2\sqrt{W(u)}\,du \qquad \text{and} \qquad \sigma^- = \int_{-1}^0 2\sqrt{W(u)}\,du$$

and distinguishes proper interfaces and ghost interfaces in the different phases $u = \pm 1$ by the line tension $\sigma^+ + \sigma^-$, $2\sigma^+$, or $2\sigma^-$ instead of σ in the limit energy. One may also consider potentials such as $W(u) = (1-u)^2$ and drop the phase integral constraint for u_ε. Then there is only one lipid phase, and u_ε is merely an auxiliary variable that allows curvature induced kinks in the limit.

As already stated, bending rigidities other than $k^\pm = -k_G^\pm = 1$ can be considered without change of the proof as long as (4.3) holds. Also, $k(u) = u^2$ can be replaced by other continuous functions such as $k(u) = |u|$ or $k(u) = (\sqrt{u^2+\delta^2} - \delta)/(\sqrt{1+\delta^2} - \delta)$ with $\delta > 0$.

4.8.1 Open membranes

The arguments can be changed to handle open surfaces of revolution generated by curves $\gamma = (x, y)\colon I \to \mathbb{R} \times \mathbb{R}_{>0}$ with prescribed boundary conditions for γ and γ' at ∂I. The curve length is then controlled by energy, area and boundary conditions due to

$$2\pi \mathcal{L}_\gamma \leq \int_{M_\gamma} |H|\,d\mu + 2\pi \varphi y|_{\partial I}, \tag{4.19}$$

where φ is the tangent angle, see the proof of Lemma 2.5. Then the equi-coercivity argument still applies and the only additional issue in this situation is that kinks may appear at the boundary in the sense that the tangent vector of the limit curve differs from the prescribed one and contributes to the limit energy like a ghost interface. We refer to [Hel09] where we studied two-phase elastic curves as a one-dimensional analogue for membranes.

It is also possible to weaken the boundary conditions as long as (4.19) still ensures equi-coercivity. For instance, since $x' \geq 0$ we can always assume $\|\varphi\|_\infty \leq \pi/2$, hence requiring a uniform L^∞-bound on y at ∂I is sufficient. Such a bound can be derived from uniformly bounded energy $\mathcal{F}_\varepsilon + \mathcal{G}$ or $\mathcal{E}_\varepsilon + \mathcal{G}$, where

$$\mathcal{G}(\gamma) = \overline{\sigma} \int_{M_\gamma(\partial I)} d\mathcal{H}^1 = 2\pi\overline{\sigma} \sum_{s \in \partial I} y(s)$$

with a constant line tension $\overline{\sigma}$. Since $\mathcal{G}$ is continuous with respect to curve convergence in C^0, its presence does not influence the arguments of the current and previous chapter. The limit energies $\mathcal{E}+\mathcal{G}$ and $\mathcal{F}+\mathcal{G}$ model open lipid membranes, see [STTH98, TOY03, WD08] for experimental observations, modelling and numerical simulations of single-phase open membranes, respectively.

4.8.2 Relaxed and additional constraints

The constraint of prescribed area in the approximate setting can be relaxed to

$$0 < \inf_{\gamma\in\mathcal{C}_\varepsilon} \mathcal{A}_\gamma \leq \sup_{\gamma\in\mathcal{C}_\varepsilon} \mathcal{A}_\gamma < \infty,$$

and this chapter's arguments for equi-coercivity and lower bound still apply. The constraint can be incorporated as penalty term in the energy, for instance

$$\frac{1}{\varepsilon}\left(\mathcal{A}_\gamma - A_0\right)^2$$

or any other scale of ε, because we have recovered the constraint exactly. In the same way, the phase integral constraint can be replaced by a penalty term.

Other constraints that change continuously under the convergence proved in Lemma 4.4 can also be imposed, for instance on the enclosed volume $\mathcal{V}_\gamma(M) = \pi \int_{M_\gamma} x'y^2\,dt$.

4.8.3 The role of the Gauss curvature and the full Γ-limit of $\mathcal{F}_\varepsilon$

Another desirable extension is to drop the Gauss curvature in $\mathcal{F}_\varepsilon$ and to consider

$$\widetilde{\mathcal{F}}_\varepsilon(\gamma,u) = \int_{M_\gamma} u^2(H-H_s(u))^2\,d\mu + \mathcal{I}_\varepsilon(\gamma,u).$$

Since $\widetilde{\mathcal{F}}_\varepsilon$ still bounds the first variation of M_γ, see Lemma 4.1 and the remark thereafter, a uniformly energy-bounded sequence $(\gamma_\varepsilon,u_\varepsilon)\in\mathcal{C}_\varepsilon\times\mathcal{P}_\varepsilon$ admits a subsequence such that $\gamma_\varepsilon \overset{\star}{\rightharpoonup} \gamma = (x,y)$ in $W^{1,\infty}(I;\mathbb{R}^2)$. Using $\kappa_{2,\varepsilon} = x'_\varepsilon/q_\varepsilon y_\varepsilon \leq 2/\inf_J y$ and $|u_\varepsilon| \geq 1/2$ in $J \Subset \{y>0\}$ for sufficiently small $\varepsilon>0$, we obtain

$$\begin{aligned}\int_{M_\varepsilon(J)} u_\varepsilon^2(H-H_s(u_\varepsilon))^2\,d\mu_\varepsilon &\geq \frac{1}{8}\int_{M_\varepsilon(J)} H^2\,d\mu_\varepsilon - \frac{1}{2}\int_{M_\varepsilon(J)} H_s(u_\varepsilon)^2\,d\mu_\varepsilon \\ &\geq \frac{1}{16}\int_{M_\varepsilon(J)} \kappa_{1,\varepsilon}^2\,d\mu_\varepsilon - \frac{1}{4}\int_{M_\varepsilon(J)} \kappa_{2,\varepsilon}^2\,d\mu_\varepsilon - \frac{1}{2}\|H_s\|_\infty A_0,\end{aligned}$$

and thus an L^2-bound for γ''_ε in J as in the Equi-Coercivity Lemma 4.4. Then compactness and the lower bound in the bulk, at kinks and at interfaces follow as before; in $R=\{y=0\}$ the argument from Section 4.5.3 yields

$$\liminf_{\varepsilon\to 0} \widetilde{\mathcal{F}}_\varepsilon(\gamma_\varepsilon,u_\varepsilon,R) \geq \hat{\sigma} \liminf_{\varepsilon\to 0}\int_{M_\varepsilon(R)} |H|\,d\mu_\varepsilon \tag{4.20}$$

in place of (4.15). We would like to express the right hand side of (4.20) in terms of the limit curve γ, but we are not able to do this except for a special case. If $J \subset R$ is an interval, $|\gamma_\varepsilon'| \equiv q_\varepsilon$, and φ_ε an angle function for γ_ε we can integrate by parts as in Lemma 2.5 to find

$$\frac{1}{2\pi}\int_{M_\varepsilon(J)} H_\varepsilon \, d\mu_\varepsilon = q_\varepsilon \int_J \varphi_\varepsilon \sin\varphi_\varepsilon + \cos\varphi_\varepsilon \, dt - \varphi_\varepsilon y_\varepsilon|_{\partial J} \geq \int_J x_\varepsilon' \, dt - \varphi_\varepsilon y_\varepsilon|_{\partial J}, \tag{4.21}$$

where the last inequality holds due to $\varphi_\varepsilon \sin\varphi_\varepsilon \geq 0$. We conclude

$$\liminf_{\varepsilon\to 0} \int_{M_\varepsilon(\operatorname{int} R)} |H| \, d\mu_\varepsilon \geq 2\pi \int_{M(\operatorname{int} R)} x' \, dt = 2\pi \mathcal{L}_\gamma(\operatorname{int} R) \tag{4.22}$$

by exhausting $\operatorname{int} R$ with open intervals. In general, however, R is a closed set with $\partial R \neq \emptyset$, and we have already seen an example where $R = \partial R$ in Section 4.6; thus (4.22) is not sufficient. On the other hand, using simple measure theoretic arguments, one finds

$$\liminf_{\varepsilon\to 0} \int_{M_\varepsilon(R)} |H_\varepsilon| \, d\mu_\varepsilon = \liminf_{\varepsilon\to 0} \nu_\varepsilon(R) \geq \nu(R),$$

where $\nu_\varepsilon = |H_\varepsilon|\mu_\varepsilon \rightharpoonup \nu$ as finite Radon measures on R. But we are not able to connect ν to the limit curve γ, because we lack good bounds on the curves in R. In Section 4.5.3 this problem is circumvented by the estimate $|B_\varepsilon| \geq |\kappa_{2,\varepsilon}|$.

Actually, (4.21) provides the estimate

$$\liminf_{\varepsilon\to 0} \int_{M_\varepsilon(\operatorname{int} R)} |H_\varepsilon| \, d\mu_\varepsilon \geq 2\pi \liminf_{\varepsilon\to 0} \mathcal{L}_\varepsilon(\operatorname{int} R), \tag{4.23}$$

using $\varphi_\varepsilon \sin\varphi_\varepsilon + \cos\varphi_\varepsilon \geq 1$ for $\varphi_\varepsilon \in [-\pi/2, \pi/2]$ instead of $\varphi_\varepsilon \sin\varphi_\varepsilon \geq 0$, and the right hand side of (4.23) may be strictly larger than $\mathcal{L}(\operatorname{int} R)$. Recall also the second example in Section 4.6 where $\liminf \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) > 0$ in a set $J \subset R$ with $\mathcal{H}^1(J) > 0$ and $\mathcal{L}_\gamma(J) = |\gamma'| = 0$. Both (4.23) and the example suggest that the full Γ-limit of $\mathcal{F}_\varepsilon$ or $\widetilde{\mathcal{F}}_\varepsilon$ contains a term that explicitly depends on the parametrisation and is not invariant under reparametrisation. The example also suggests that all three terms of $\mathcal{I}_\varepsilon$, and not only those considered here and in Section 4.5.3, can contribute to the limit.

On the other hand, due to $\liminf \mathcal{H}_\varepsilon(\cdot, \cdot, R) \geq 0$ and $\mathcal{I}_\varepsilon \geq 0$, the lower limit of $\mathcal{F}_\varepsilon$ at the axis of revolution is non-negative. Since changing $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$ at the axis of revolution does not change area or phase integral constraint, removing segments of γ at the axis is admissible and only reduces the limit energy. Minimisers of $\mathcal{F}$ as well as $\Gamma\text{-}\lim \mathcal{F}_\varepsilon$ should thus have no energy at the axis of revolution at all; for such membranes the two energies agree.

Chapter 5

Numerical examples

In the final chapter we present some numerical examples to compare the regular and singular approximation. We study numerical local minima of $\mathcal{E}_\varepsilon$ and $\mathcal{F}_\varepsilon$ that are obtained by a gradient flow type evolution, which has to our knowledge first been considered by Elliott and Stinner. In [ES10a] they numerically investigate the flow with $H_s \equiv 0$ and identical bending parameters in both phases; in [ES10b] they show convergence of the diffuse interface equilibrium equations to sharp interface equilibrium equations by means of formal asymptotic expansions. In the latter considerations, spontaneous curvature and bending parameters may differ between the two phases, but all surfaces, including the limit, are assumed to be C^1 across interfaces.

In this chapter, we fix $\varepsilon > 0$ and let

$$\begin{aligned}\mathcal{G}(M,u) = \int_M k(u)\,(H - H_s(u))^2 + k_G(u) K \, d\mu \\ + \widetilde{\sigma} \int_M \varepsilon |\nabla_M u|^2 + \frac{1}{\varepsilon} W(u)\, d\mu + k_B \int_M \varepsilon |B|^2 \, d\mu.\end{aligned}$$

be a general approximate energy. Here $k_B \geq 0$ and $\widetilde{\sigma} > 0$ are constants that we include to control the relative size of the different energy terms. For our simulations we use $W(u) = (1-u^2)^2$, so that $\mathcal{G}$ penalises an interface with the line tension

$$\widetilde{\sigma}\sigma = \widetilde{\sigma} \int_{-1}^{1} 2\sqrt{W(u)}\, du = \frac{8}{3}\widetilde{\sigma},$$

and kinks additionally with their size times

$$2\sqrt{k_B \widetilde{\sigma} W(0)} = 2\sqrt{k_B \widetilde{\sigma}}.$$

For $\widetilde{\sigma} = 1$, $k \equiv 1$, $k_G \equiv 0$ and $k_B = 0$ we have $\mathcal{G} = \mathcal{E}_\varepsilon$; for $\widetilde{\sigma} = 1$, $k(u) = -k_G(u) = u^2$ and $k_B = 1$ we obtain $\mathcal{G} = \mathcal{F}_\varepsilon$.

The evolution equations are described in full detail in [ES10b], but we include their derivation here for completeness. We first consider general surfaces and in a second step restrict the flow to rotationally symmetric membranes. Then we describe the numerical scheme, which is based on [MS02], and conclude the chapter with some numerical experiments.

5.1 The flow

5.1.1 More geometric analysis

To study the variation of the energy, we consider evolving surfaces indexed by a continuous time parameter τ. It is necessary that functions and vector fields given on a closed oriented surface M can be extended to a neighbourhood of M, and this is true if M is embedded and sufficiently smooth. For simplicity of notation we write for instance $f\colon \mathbb{R}^3 \to \mathbb{R}$ bearing in mind that existence in a neighbourhood of M is sufficient. We assume that all surfaces and functions are sufficiently regular such that all calculations are justified.

Recall from Section 2.1 that the tangential gradient of $f : \mathbb{R}^3 \to \mathbb{R}$ on M is

$$\nabla_M f = (D_{\xi_1} f)\xi_1 + (D_{\xi_2} f)\xi_2,$$

where $\{\xi_1, \xi_2\}$ is any orthonormal basis of the tangent space of M. For a surface of revolution parametrised by $\Phi(t,\theta) = (x(t), y(t)\cos\theta, y(t)\sin\theta)$ we choose

$$\xi_1 = \frac{\partial_t \Phi}{|\partial_t \Phi|} = \frac{1}{|\gamma'|}\left(x', y'\cos\theta, y'\sin\theta\right) \qquad \text{and} \qquad \xi_2 = \frac{\partial_\theta \Phi}{|\partial_\theta \Phi|} = (0, -\sin\theta, \cos\theta)\,.$$

Interpreting the tangent space $\mathcal{T}_p M$ at $p \in M$ as a subspace of $\mathbb{R}^3$ and thus $\nabla_M f \in \mathbb{R}^3$, we may write $\nabla_M f = (D_1 f, D_2 f, D_3 f)$, where

$$D_i f = \nabla_M f \cdot e_i \qquad (i = 1, 2, 3)$$

and $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. The definition of ∇_M is extended to vector fields $h = (h_1, h_2, h_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ by setting $(\nabla_M h)_{ij} = D_j h_i$. Since $\nabla f = (D_{\xi_1} f)\xi_1 + (D_{\xi_2} f)\xi_2 + (D_\nu f)\nu$, where ν is the outer unit normal to M, we find $\nabla_M f = P\nabla f$ and $\nabla_M h = \nabla h P$, where ∇f is the usual gradient of f, $(\nabla h)_{ij} = \partial_j h_i$ and P is the projection of $\mathbb{R}^3$ onto the tangent space of M.

Recall also that the tangential divergence of $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ on M is

$$\operatorname{div}_M h = D_{\xi_1} h \cdot \xi_1 + D_{\xi_2} h \cdot \xi_2.$$

Writing $h = \sum_{i=1}^3 h_i e_i$ in coordinates with respect to the basis $\{e_1, e_2, e_3\}$, we find

$$\operatorname{div}_M h = \sum_{j=1}^{2}\sum_{i=1}^{3} D_{\xi_j}(h_i e_i) \cdot \xi_j = \sum_{i=1}^{3} e_i \cdot \sum_{j=1}^{2} (D_{\xi_j} h_i)\xi_j = \sum_{i=1}^{3} D_i h_i.$$

The Laplace-Beltrami operator for $f:\mathbb{R}^3\to\mathbb{R}$ on M is defined by

$$\Delta_M f = \operatorname{div}_M \nabla_M f = \sum_{i=1}^3 D_i D_i f.$$

Finally, recall from Chapter 2 that $H = \operatorname{trace} S = \operatorname{div}_M \nu = \kappa_1 + \kappa_2$ and $K = \det S = \kappa_1\kappa_2$, where $S\colon \mathcal{T}_pM \to \mathcal{T}_pM$, $\xi \mapsto D_\xi \nu$ is the shape operator of M. Extending the domain of S to $\mathbb{R}^3$ by setting $S\zeta = SP\zeta$ for $\zeta\in\mathbb{R}^3$, we find

$$S\zeta = SP\zeta = D_{P\zeta}\nu = \nabla\nu P\zeta = \nabla_M\nu\zeta,$$

which means that $\nabla_M\nu$ is the matrix of the shape operator with respect to $\{e_1,e_2,e_3\}$. Since for a surface of revolution $S=\operatorname{diag}(\kappa_1,\kappa_2,0)$ with respect to the basis $\{\xi_1,\xi_2,\nu\}$ and since the Frobenius norm is invariant under an orthogonal change of basis, we conclude

$$|B|^2 = \kappa_1^2+\kappa_2^2 = |S|^2 = |\nabla_M\nu|^2 = \sum_{i=1}^3 |\nabla_M\nu_i|^2.$$

It is easy to see that this reasoning is also true for surfaces that are not rotationally symmetric.

The following Divergence Theorem, integration by parts formula and Green's identity hold on M, for a proof see for instance [GT01].

Lemma 5.1. *Let $f,g\colon\mathbb{R}^3\to\mathbb{R}$ be sufficiently smooth. Then*

$$\begin{aligned}\int_M D_i f\,d\mu &= \int_M f(\operatorname{div}_M\nu)\nu_i\,d\mu = \int_M fH\nu_i\,d\mu,\\ \int_M D_i f g + f D_i g\,d\mu &= \int_M fgH\nu_i\,d\mu,\\ \int_M f\Delta_M g\,d\mu &= \int_M \Delta_M f g\,d\mu.\end{aligned}$$

Let M_τ, $\tau\in(-\tau_0,\tau_0)$, $\tau_0>0$ be a surface that evolves smoothly with velocity $v_\tau\colon\mathbb{R}^3\to\mathbb{R}^3$. That means, if $\Phi_\tau\colon\Omega\subset\mathbb{R}^2\to U\subset\mathbb{R}^3$ parametrises $M_\tau\cap U$ there holds

$$\frac{d}{d\tau}\Phi_\tau(\cdot) = v_\tau(\Phi_\tau(\cdot)).$$

For simplicity of notation we will omit the dependence on τ when it is clear from the context whether we consider an evolving surface or a surface at a fixed time.

We denote by $\partial_\tau^\bullet f$ the material derivative of a function f on the evolving surface M_τ,

$$\partial_\tau^\bullet f = \partial_\tau f + \nabla f\cdot v = \frac{d}{d\tau} f(\Phi_\tau(\cdot),\tau),$$

and we call

$$\partial_\tau^\circ f = \partial_\tau f + (v\cdot\nu)\nabla f\cdot\nu$$

the normal time derivative, which is the material derivative taken only the normal velocity of M_τ into account. Obviously, $\partial^\bullet_\tau f = \partial^\circ_\tau f$ if v_τ is normal to M_τ.

The following Leibniz formulas for the derivative of a time-dependent integral hold, a proof can be found in [DE07].

Lemma 5.2. *Let* $f\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. *Then*

$$\frac{d}{d\tau}\int_{M_\tau} f \, d\mu = \int_{M_\tau} \partial^\bullet_\tau f + f \operatorname{div}_{M_\tau} v \, d\mu$$

and

$$\frac{d}{d\tau}\int_{M_\tau} |\nabla_{M_\tau} f|^2 \, d\mu = \int_{M_\tau} 2\nabla_{M_\tau} f \cdot \nabla_{M_\tau}\partial^\bullet_\tau f + |\nabla_{M_\tau} f|^2 \operatorname{div}_{M_\tau} v - 2D(v)\nabla_{M_\tau} f \cdot \nabla_{M_\tau} f \, d\mu$$

where $(D(v))_{ij} = (D_i v_j + D_j v_i)/2$.

If $w\colon \mathbb{R} \to \mathbb{R}$ and $v = w\nu$ is normal to M, we have $\operatorname{div}_M v = \nabla_M w \cdot \nu + w \operatorname{div} \nu = Hw$, and the first equation in Lemma 5.2 reads

$$\frac{d}{d\tau}\int_{M_\tau} f \, d\mu = \int_{M_\tau} \partial^\circ_\tau f + fHw \, d\mu.$$

Likewise, the second equation becomes

$$\frac{d}{d\tau}\int_{M_\tau} |\nabla_{M_\tau} f|^2 \, d\mu = \int_{M_\tau} 2\nabla_{M_\tau} f \cdot \nabla_{M_\tau}\partial^\circ_\tau f + |\nabla_{M_\tau} f|^2 Hw - 2(\nabla_{M_\tau}\nu)\nabla_{M_\tau} f \cdot \nabla_{M_\tau} f w \, d\mu.$$

Finally, the following formulas for derivatives of the normal and the mean curvature will be used to compute the first variation of the Helfrich energy:

$$\partial^\circ_\tau \nu = -\nabla_M (v \cdot \nu), \tag{5.1}$$

$$\Delta_M \nu = -|\nabla_M \nu|^2 \nu + \nabla_M H, \tag{5.2}$$

$$\partial^\circ_\tau H = -\Delta_M (v \cdot \nu) - |\nabla_M \nu|^2 (v \cdot \nu). \tag{5.3}$$

A proof of these identities can be found in [Gur00].

5.1.2 First variation of the energy

Given a membrane (M, u), we consider variations of $\mathcal{G}$ in direction (w, ψ), where $v = w\nu\colon M \to \mathbb{R}^3$ and $w, \psi\colon M \to \mathbb{R}$ are smooth; (M_τ, u_τ) is defined by

$$M_\tau = \{p_\tau = p + \tau w(p)\nu(p) : p \in M\},$$
$$u_\tau(p_\tau) = u(p) + \tau\psi(p).$$

Since M is embedded and smooth, there exists $\tau_0 > 0$ such that for all $|\tau| < \tau_0$ there is for each $p_\tau \in M_\tau$ a unique $p \in M = M_0$ such that $p_\tau = p + \tau w(p)\nu(p)$ and $u_\tau(p_\tau)$ is well-defined. The velocity of M is normal to M, hence we have

$$\partial_\tau^\bullet u_\tau\big|_{\tau=0} = \partial_\tau^\circ u_\tau\big|_{\tau=0} = \frac{d}{d\tau} u_\tau(p_\tau)\bigg|_{\tau=0} = \psi.$$

The first variation of $\mathcal{G}$ in direction (w,ψ) is defined as

$$\langle D\mathcal{G}(M,u),(w,\psi)\rangle = \frac{d}{d\tau}\mathcal{G}(M_\tau,u_\tau)\bigg|_{\tau=0}.$$

We split the energy $\mathcal{G}$ into $\mathcal{G} = \mathcal{MC} + \mathcal{K} + \mathcal{MM} + \mathcal{B}$, where

$$\begin{aligned}
\mathcal{MC}(M,u) &= \int_M k(u)(H - H_s(u))^2 \, d\mu,\\
\mathcal{K}(M,u) &= \int_M k_G(u) K \, d\mu,\\
\mathcal{MM}(M,u) &= \widetilde{\sigma}\int_M \varepsilon|\nabla_M u|^2 + \frac{1}{\varepsilon}W(u)\, d\mu,\\
\mathcal{B}(M) &= \int_M k_B \varepsilon |B|^2 \, d\mu,
\end{aligned}$$

and compute the first variation of each of these functionals separately.

Lemma 5.3. *The first variation of the Modica-Mortola energy in direction* (w,ψ) *is*

$$\begin{aligned}
\langle D\mathcal{MM}(M,u),(w,\psi)\rangle = &\int_M \widetilde{\sigma}\left\{-2\varepsilon\Delta_M u + \frac{1}{\varepsilon}W'(u)\right\}\psi\, d\mu\\
&+ \int_M \widetilde{\sigma}\left\{\left(\varepsilon|\nabla_M u|^2 + \frac{1}{\varepsilon}W(u)\right)H - 2\varepsilon(\nabla_M\nu)\nabla_M u\cdot\nabla_M u\right\} w\, d\mu.
\end{aligned}$$

Proof. Using both Leibniz formulas from Lemma 5.2 and integrating by parts, one obtains

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{MM}(M_\tau,u_\tau)\bigg|_{\tau=0} &= \widetilde{\sigma}\int_M 2\varepsilon\nabla_M u\cdot\nabla_M\psi - 2\varepsilon(\nabla_M\nu)\nabla_M u\cdot\nabla_M u w\\
&\qquad + \frac{1}{\varepsilon}W'(u)\psi + \left(\varepsilon|\nabla_M u|^2 + \frac{1}{\varepsilon}W(u)\right)Hw\, d\mu\\
&= \int_M \widetilde{\sigma}\left\{-2\varepsilon\Delta_M u + \frac{1}{\varepsilon}W'(u)\right\}\psi\, d\mu\\
&\quad + \int_M \widetilde{\sigma}\left\{\left(\varepsilon|\nabla_M u|^2 + \frac{1}{\varepsilon}W(u)\right)H - 2\varepsilon(\nabla_M\nu)\nabla_M u\cdot\nabla_M u\right\} w\, d\mu. \square
\end{aligned}$$

Lemma 5.4. *The first variation of the mean curvature energy in direction* (w,ψ) *is*

$$\begin{aligned}
\langle D\mathcal{MC}(M,u),(w,\psi)\rangle = &\int_M \Big\{k'(u)(H - H_s(u))^2 - 2k(u)(H - H_s(u))H_s'(u)\Big\}\psi\, d\mu\\
&- \int_M \Big\{2\Delta_M\Big(k(u)(H - H_s(u))\Big) + 2k(u)(H - H_s(u))|B|^2\\
&\qquad - k(u)(H - H_s(u))^2 H\Big\} w\, d\mu.
\end{aligned}$$

Proof. Using the first Leibniz formula and $\partial^\circ_\tau H = -\Delta_M w - |B|^2 w$ from (5.3), one computes

$$\begin{aligned}\frac{d}{d\tau}\mathcal{MC}(M_\tau, u_\tau)\Big|_{\tau=0} &= \int_M k'(H-H_s)^2\psi - 2k(H-H_s)H_s'\psi\, d\mu \\ &\quad - \int_M 2k(H-H_s)\Delta_M w + 2k(H-H_s)|B|^2 w - k(H-H_s)^2 H w\, d\mu.\end{aligned}$$

Applying Green's identity to the $\Delta_M w$ term yields the claim. ☐

Corollary 5.5. *There holds*

$$\langle D\mathcal{B}(M), w\rangle = -k_B\varepsilon \int_M \left\{2\Delta_M H + H(|B|^2 - 2K)\right\} w\, d\mu$$

and

$$\sum_{j=1}^{3} (\nabla_M \nu)\nabla_M \nu_j \cdot \nabla_M \nu_j - |B|^2 H + KH = 0.$$

Proof. On the one hand, we have $|B|^2 = H^2 - 2K$, and since $\int_{M_\tau} K\, d\mu$ is constant, Lemma 5.4 with $H_s \equiv 0$ and $k \equiv 1$ yields

$$\frac{d}{d\tau}\mathcal{B}(M_\tau)\Big|_{\tau=0} = \frac{d}{d\tau}\left(\int_{M_\tau} k_B\varepsilon H^2\, d\mu\right)\Big|_{\tau=0} = -\int_M k_B\varepsilon \Big\{2\Delta_M H + H(|B|^2 - 2K)\Big\} w\, d\mu.$$

On the other hand, $|B|^2 = |\nabla_M\nu|^2 = \sum_{j=1}^3 |\nabla_M \nu_j|^2$, and with the second Leibniz formula we obtain

$$\frac{d}{d\tau}\mathcal{B}(M_\tau)\Big|_{\tau=0} = k_B\varepsilon\sum_j \int_M 2\nabla_M\nu_j\cdot\nabla_M\partial^\circ_\tau\nu_j + |\nabla_M\nu_j|^2 H w - 2(\nabla_M\nu)\nabla_M\nu_j\cdot\nabla_M\nu_j w\, d\mu.$$

Integrating by parts the first term and using (5.1) and (5.2) for $\partial^\circ_\tau \nu$ and $\Delta_M \nu$ yields

$$\begin{aligned}\frac{d}{d\tau}\mathcal{B}(M_\tau)\Big|_{\tau=0} &= k_B\varepsilon\sum_j \int_M 2\Delta_M\nu_j D_j w + |\nabla_M\nu_j|^2 H w - 2(\nabla_M\nu)\nabla_M\nu_j\cdot\nabla_M\nu_j w\, d\mu \\ &= k_B\varepsilon \int_M -2\Delta_M H w + |B|^2 H w - 2\sum_j (\nabla_M\nu)\nabla_M\nu_j\cdot\nabla_M\nu_j w\, d\mu.\end{aligned}$$

Equating both expressions for $\langle D\mathcal{B}(M), w\rangle$ and using that w is an arbitrary smooth function yields the second claim. ☐

Lemma 5.6. *The first variation of the Gauss curvature energy in direction (w,ψ) is*

$$\langle D\mathcal{K}(M,u), (w,\psi)\rangle = \int_M k_G'(u) K\psi\, d\mu + \int_M \mathrm{div}_M\left(k_g'(u)(\nabla_M\nu - H\mathrm{id})\nabla_M u\right) w\, d\mu.$$

Proof. We use $K = (H^2 - |B|^2)/2$ and Lemma 5.4 with $H_s \equiv 0$ to get

$$\begin{aligned}\frac{d}{d\tau}\left(\int_{M_\tau}\frac{k_G}{2}H^2\, d\mu\right)\Big|_{\tau=0} &= \int_M \frac{k_G'}{2}H^2\psi\, d\mu \\ &\quad - \int_M \left\{\Delta_M(k_G H) + \frac{k_G}{2}H\left(2|B|^2 - H^2\right)\right\} w\, d\mu.\end{aligned}$$

With $|B|^2 = |\nabla_M \nu|^2$ and the second Leibniz formula we compute as in Corollary 5.5

$$
\begin{aligned}
&\frac{d}{d\tau}\left(\int_{M_\tau} \frac{k_G}{2}|B|^2\, d\mu\right)\bigg|_{\tau=0} \\
&\qquad = \int_M \frac{k_G'}{2}|B|^2\psi\, d\mu + \int_M \frac{k_G}{2}|B|^2 Hw\, d\mu \\
&\qquad\quad + \sum_j \int_M k_G \nabla_M \nu_j \cdot \nabla_M \partial_\tau^\circ \nu_j - k_G(\nabla_M \nu)\nabla_M \nu_j \cdot \nabla_M \nu_j\, d\mu \\
&\qquad = \int_M \frac{k_G'}{2}|B|^2\psi\, d\mu + \int_M \frac{k_G}{2}|B|^2 Hw\, d\mu \\
&\qquad\quad + \sum_j \int_M \left(k_G' \nabla_M u \cdot \nabla_M \nu_j + k_G \Delta_M \nu_j\right) D_j w - k_G(\nabla_M \nu)\nabla_M \nu_j \cdot \nabla_M \nu_j w\, d\mu \\
&\qquad = \int_M \frac{k_G'}{2}|B|^2\psi\, d\mu + \int_M \frac{k_G}{2}|B|^2 Hw\, d\mu \\
&\qquad\quad - \int_M \operatorname{div}_M \left(k_G' \nabla_M \nu \nabla_M u + k_G \nabla_M H\right) w + k_G \sum_j (\nabla_M \nu)\nabla_M \nu_j \cdot \nabla_M \nu_j w\, d\mu.
\end{aligned}
$$

Combining both expressions and employing the second claim of Corollary 5.5 yields

$$
\begin{aligned}
\frac{d}{d\tau}\left(\int_M k_G K\, d\mu\right)\bigg|_{\tau=0} &= \int_M k_G' K\psi\, d\mu \\
&\quad + \int_M \operatorname{div}_M \Big(k_G' \nabla_M \nu \nabla_M u + k_G \nabla_M H - \nabla_M (k_G H)\Big) w\, d\mu \\
&\quad + \int_M k_G \Big(HK - H|B|^2 + \textstyle\sum_j k_G (\nabla_M \nu)\nabla_M \nu_j \cdot \nabla_M \nu_j\Big) w\, d\mu \\
&= \int_M k_G' K\psi\, d\mu + \int_M \operatorname{div}_M \left(k_G'(\nabla_M \nu - H\mathrm{id})\nabla_M u\right) w\, d\mu.
\end{aligned}
$$

□

Lemma 5.7. *The first variations of the constraint functionals*

$$\mathcal{A}(M) = \int_M d\mu \qquad \textit{and} \qquad \mathcal{PI}(M,u) = \int_M u\, d\mu$$

in direction (w,ψ) *are*

$$\langle D\mathcal{A}(M), w\rangle = \int_M Hw\, d\mu \qquad \textit{and} \qquad \langle D\mathcal{PI}(M,u),(w,\psi)\rangle = \int_M \psi + uHw\, d\mu.$$

Proof. The assertion is obvious from the first Leibniz formula. □

5.1.3 Flow equations

The flow we consider is formally a weighted L^2 gradient flow for the membrane energy $\mathcal{G}$ on the manifold $\mathcal{N}$ of smooth oriented closed surfaces and smooth phase fields with prescribed area and phase integral. At $(M,u) \in \mathcal{N}$ the tangent space to $\mathcal{N}$ consists of smooth normal variations of M and variations of u such that the constraints are maintained, that is, according to Lemma 5.7,

$$\mathcal{T}_{(M,u)}\mathcal{N} = \left\{(w,\psi)\colon M \to \mathbb{R}^2 : \int_M Hw\, d\mu = \int_M \psi + uHw\, d\mu = 0\right\}.$$

We endow $\mathcal{N}$ with the weighted L^2 metric tensor

$$g_{(M,u)}\left((w,\psi),(v,\varphi)\right)=\int_M wv+\varepsilon\psi\varphi\,d\mu.$$

Neglecting the coupling between phase field and surface in the Helfrich energy, the ε scaling in the metric yields an Allen-Cahn equation for the phase field on the evolving surface. As $\varepsilon\to 0$, this equation approximates a geodesic curvature flow for the interface line, in analogy to the phase field approximation of the mean curvature flow [ES10a, DDE05].

The gradient flow for $\mathcal{G}$ with respect to g reads as follows: given an initial configuration $(M_0,u_0)\in\mathcal{N}$, find $(M_\tau,u_\tau)\in\mathcal{N}$ for $\tau>0$ such that

$$g_\tau((v_\tau,\partial_\tau^\circ u_\tau),(w,\psi))=-\langle D\mathcal{G}(M_\tau,u_\tau),(w,\psi)\rangle\qquad\text{for all }(w,\psi)\in\mathcal{T}_{(M_\tau,u_\tau)}\mathcal{N},\tag{5.4}$$

where v_τ is the normal velocity of M_τ and $g_\tau=g_{(M_\tau,u_\tau)}$. As for any gradient flow, the energy decreases along solutions, because

$$\frac{d}{d\tau}\mathcal{G}(M_\tau,u_\tau)=\langle D\mathcal{G}(M_\tau,u_\tau),(v_\tau,\partial_\tau^\circ u_\tau)\rangle=-g_\tau((v_\tau,\partial_\tau^\circ u_\tau),(v_\tau,\partial_\tau^\circ u_\tau))\leq 0.\tag{5.5}$$

In the numerical simulations we incorporate the constraints on (w,ψ) using Lagrange multipliers. Instead of (5.4) we look for (M_τ,u_τ) and $\lambda_{\mathcal{A},\tau},\lambda_{\mathcal{PI},\tau}\in\mathbb{R}$ for $\tau>0$ such that

$$\begin{aligned}&g_\tau((v_\tau,\partial_\tau^\circ u_\tau),(w,\psi))\\&\qquad=-\langle D\mathcal{G}(M_\tau,u_\tau)+\lambda_{\mathcal{A},\tau}D\mathcal{A}(M_\tau)+\lambda_{\mathcal{PI},\tau}D\mathcal{PI}(M_\tau,u_\tau),(w,\psi)\rangle\end{aligned}\tag{5.6}$$

for all smooth $(w,\psi)\colon M_\tau\to\mathbb{R}^2$ and

$$\mathcal{A}(M_\tau)=A_0,\qquad\mathcal{PI}(M_\tau,u_\tau)=mA_0.\tag{5.7}$$

Thanks to (5.7), we have

$$0=\frac{d}{d\tau}\mathcal{A}(M_\tau)=\langle D\mathcal{A}(M_\tau),v_\tau\rangle\quad\text{ and }\quad 0=\frac{d}{d\tau}\mathcal{PI}(M_\tau)=\langle D\mathcal{PI}(M_\tau,u_\tau),(v_\tau,\partial_\tau^\circ u_\tau)\rangle,$$

hence the energy inequality (5.5) also holds for $\mathcal{G}+\lambda_{\mathcal{A}}\mathcal{A}+\lambda_{\mathcal{PI}}\mathcal{PI}$. Since w and ψ are arbitrary smooth functions, we obtain from (5.6) the evolution equations

$$\begin{aligned}
v ={}& \Big\{2\Delta_M\Big(k(u)(H-H_s(u))\Big)+2k(u)(H-H_s(u))|B|^2-k(u)(H-H_s(u))^2H\Big\}\\
&-\operatorname{div}_M\left(k_G'(u)(\nabla_M\nu-H\mathrm{id})\nabla_M u\right)\\
&-\left\{\left(\varepsilon|\nabla_M u|^2+\frac{1}{\varepsilon}W(u)\right)H-2\varepsilon(\nabla_M\nu)\nabla_M u\cdot\nabla_M u\right\}\\
&+k_B\varepsilon\left\{2\Delta_M H+H(|B|^2-2K)\right\}\\
&-\lambda_{\mathcal{A}}H-\lambda_{\mathcal{PI}}uH, \qquad\qquad (5.8)\\
\varepsilon\partial_\tau^\circ u ={}& -\left\{k'(u)(H-H_s(u))^2-2k(u)(H-H_s(u))H_s'(u)\right\}\\
&-k_G'(u)K\\
&-\left\{-2\varepsilon\Delta_M u+\frac{1}{\varepsilon}W'(u)\right\}\\
&-\lambda_{\mathcal{PI}}.
\end{aligned}$$

5.1.4 Restriction to rotationally symmetric membranes

The flow (5.6), (5.7) is easily restricted to rotationally symmetric membranes by considering only rotationally symmetric variations (w,ψ), and one obtains the same evolution equations as in (5.8), denoting by v now the normal velocity of the generating curve. To see that the right hand sides of (5.8) depend only on the generating curve, we compute the terms involving the shape operator $\nabla_M\nu$; for all other terms this is obvious.

If M is a surface of revolution parametrised by $\Phi\colon(t,\theta)\mapsto(x(t),y(t)\cos\theta,y(t)\sin\theta)$, where $\gamma(t)=(x(t),y(t))$ with $|\gamma'|\equiv 1$, then $\xi_1=\partial_t\Phi$ and $\xi_2=y^{-1}\partial_\theta\Phi$ form an orthonormal basis for the tangent space of M. The tangential gradient of a rotationally symmetric function $f\colon M\to\mathbb{R}$ and the tangential divergence of a tangential vector field $h\colon M\to\mathbb{R}^3$ are then given by

$$\nabla_M f=(\partial_t f)\xi_1 \qquad\text{and}\qquad \operatorname{div}_M h=\partial_t h\cdot\xi_1+\frac{1}{y}\partial_\theta h\cdot\xi_2.$$

If the restriction of h to the curve γ is tangential to γ, we may write $h=h_1\xi_1$, where $(h_1,0,0)$ are the coordinate functions of h with respect to the basis $\{\xi_1,\xi_2,\nu\}$, to find

$$\operatorname{div}_M h=\partial_t h_1+\frac{y'}{y}h_1=\frac{1}{y}\partial_t(yh_1),$$

provided that $y>0$. Since $\nabla_M f$ is such a vector field, we obtain

$$\Delta_M f=\partial_{tt}f+\frac{y'}{y}\partial_t f.$$

Using these formulas and the matrix representation $\operatorname{diag}(\kappa_1,\kappa_2)$ for $\nabla_M\nu$ restricted to the tangent space of M, we obtain

$$(\nabla_M\nu)\nabla_M u\cdot\nabla_M u=\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix}\begin{pmatrix}\partial_t u\\ 0\end{pmatrix}\cdot\begin{pmatrix}\partial_t u\\ 0\end{pmatrix}=\kappa_1(\partial_t u)^2$$

and

$$\operatorname{div}_M \Big(k_G'(u)(\nabla_M \nu - H\mathrm{id})\nabla_M u\Big) = -\frac{1}{y}\partial_t(y\kappa_2 k_G'(u)\partial_t u).$$

If $y(t_0) = 0$, smoothness and rotational symmetry imply that $\partial_t f(t_0) = 0$, $h_1(t_0) = 0$ and that the curve is perpendicular to the x-axis at t_0. We thus find $y'(t) = \pm 1 + O(t - t_0)$ and $y(t) = \pm(t - t_0) + O((t - t_0)^2)$ and conclude

$$\lim_{t\to t_0} \frac{y'}{y} h_1 = \lim_{t\to t_0} \big(\pm 1 + O(t - t_0)\big)\frac{h_1(t) - h_1(t_0)}{\pm(t - t_0) + O((t - t_0)^2)} = \partial_t h_1(t_0).$$

Consequently, we have

$$\operatorname{div}_M h(t_0) = 2\partial_t h_1(t_0), \qquad \Delta_M f(t_0) = 2\partial_{tt} f(t_0),$$

and

$$\operatorname{div}_M \Big(k_G'(u)(\nabla_M \nu - H\mathrm{id})\nabla_M u\Big)(t_0) = -2\partial_t(\kappa_2 k_G'(u)\partial_t u)(t_0) = -2\left(\kappa_2 k_G'(u)\partial_{tt} u\right)(t_0).$$

Our model includes the assumptions $y > 0$ and $x' \geq 0$ in I. The numerical scheme, however, does not need $x' \geq 0$ and maintains both constraints locally in time when starting with initial data such that $x' > 0$. We therefore omit both constraints in our simulations.

5.2 Numerical scheme

We employ a numerical scheme developed by Mayer and Simonett in [MS02] for rotationally symmetric solutions of curvature driven free boundary problems. The scheme consists of a semi-implicit discretisation in time followed by a discretisation in space. In [MS02] the authors give numerous examples for the Willmore flow and provide numerical evidence for convergence. We extend the scheme to include the phase field in a straightforward manner; the description here follows [MS02].

5.2.1 Time discretisation

Denoting the right hand sides of (5.8) by $f(\kappa_1, \kappa_2, u, \lambda_\mathcal{A}, \lambda_{\mathcal{PI}})$ and $g(\kappa_1, \kappa_2, u, \lambda_{\mathcal{PI}})$, the flow reads

$$\begin{aligned} v &= f(\kappa_1, \kappa_2, u, \lambda_\mathcal{A}, \lambda_{\mathcal{PI}}), \\ \partial_\tau^\circ u &= g(\kappa_1, \kappa_2, u, \lambda_{\mathcal{PI}}), \\ \mathcal{A} &= A_0, \\ \mathcal{PI} &= mA_0, \end{aligned} \tag{5.9}$$

where $v = \partial_\tau \gamma \cdot \nu$ is the normal velocity of the generating curve $\gamma = \gamma_\tau(t)$. We omit the dependence on the curve parameter t in the notation and usually also the dependence on the time τ if the meaning is clear from the context.

A first-order finite difference approximation of (5.9) is given by

$$\begin{aligned}
\gamma_{\tau+h} &= \gamma_\tau + h v_\tau \nu_\tau, \\
v_\tau &= f(\kappa_{1,\tau+h}, \kappa_{2,\tau+h}, u_{\tau+h}, \lambda_{\mathcal{A},\tau}, \lambda_{\mathcal{PI},\tau}), \\
u_{\tau+h} &= u_\tau + h\partial_\tau^\circ u_\tau, \\
\partial_\tau^\circ u_\tau &= g(\kappa_{1,\tau+h}, \kappa_{2,\tau+h}, u_{\tau+h}, \lambda_{\mathcal{PI},\tau}).
\end{aligned}$$

It is implicit in κ_1, κ_2, u, and explicit in the Lagrange multipliers. The dependence of κ_1, κ_2 and u on the next time step $\tau + h$ is linearised by

$$\begin{aligned}
\kappa_{i,\tau+h} &\approx \kappa_{i,\tau} + hL_{i,\tau} v_\tau, \\
u_{\tau+h} &\approx u_\tau + h\partial_\tau^\circ u_\tau,
\end{aligned}$$

where the operators $L_{i,\tau}$, $i = 1, 2$ are formally defined by

$$L_{i,\tau} v_\tau = \frac{d}{dh}\kappa_{i,\tau+h}\Big|_{h=0}. \tag{5.10}$$

Taylor expansion of f in the first three arguments yields

$$\begin{aligned}
f(\kappa_{1,\tau+h}, \kappa_{2,\tau+h}, u_{\tau+h}, \lambda_{\mathcal{A},\tau}, \lambda_{\mathcal{PI},\tau}) \approx{} & f(\kappa_{1,\tau}, \kappa_{2,\tau}, u_\tau, \lambda_{\mathcal{A},\tau}, \lambda_{\mathcal{PI},\tau}) \\
& + \partial_{\kappa_1} f(\kappa_{1,\tau}, \kappa_{2,\tau}, u_\tau, \lambda_{\mathcal{A},\tau}, \lambda_{\mathcal{PI},\tau})\,(\kappa_{1,\tau+h} - \kappa_{1,\tau}) \\
& + \partial_{\kappa_2} f(\kappa_{1,\tau}, \kappa_{2,\tau}, u_\tau, \lambda_{\mathcal{A},\tau}, \lambda_{\mathcal{PI},\tau})\,(\kappa_{2,\tau+h} - \kappa_{2,\tau}) \\
& + \partial_u f(\kappa_{1,\tau}, \kappa_{2,\tau}, u_\tau, \lambda_{\mathcal{A},\tau}, \lambda_{\mathcal{PI},\tau})\,(u_{\tau+h} - u_\tau)\,,
\end{aligned}$$

and thus

$$f\Big|_{\tau+h} \approx \Big(f + h\partial_{\kappa_1} f L_1 v + h\partial_{\kappa_2} L_2 v + h\partial_u f \partial_\tau^\circ u\Big)\Big|_\tau = \Big(f + h\partial_\kappa f L v + h\partial_u f \partial_\tau^\circ u\Big)\Big|_\tau,$$

where $\partial_\kappa f = (\partial_{\kappa_1} f, \partial_{\kappa_2} f)$ and $L = (L_1, L_2)^T$. Using the same approximation for g, we obtain

$$\begin{pmatrix} \mathrm{id} - h\partial_\kappa f L & -h\partial_u f \\ -h\partial_\kappa g L & \mathrm{id} - h\partial_u g \end{pmatrix} \begin{pmatrix} v \\ \partial_\tau^\circ u \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{5.11}$$

Since f and g depend linearly on the Lagrange multipliers, (5.11) is a linear system in the variables $(v, \partial_\tau^\circ u, \lambda_\mathcal{A}, \lambda_{\mathcal{PI}})$. To complete the system, we linearise the constraints $\mathcal{A}_{\tau+h} = \mathcal{A}_\tau$ and $\mathcal{PI}_{\tau+h} = \mathcal{PI}_\tau$, that is, we complement (5.11) by

$$\int_M Hv\,d\mu = 0 \qquad \text{and} \qquad \int_M \partial_\tau^\circ u + uHv\,d\mu = 0. \tag{5.12}$$

When discretised, (5.11) and (5.12) yield a linear system for $(v, \partial_\tau^\circ u, \lambda_\mathcal{A}, \lambda_{\mathcal{PI}})$ at time τ.

5.2.2 Spatial discretisation

The generating curve $\gamma = (x, y)$ is subdivided into a finite number of arcs by points $\gamma_i = (x_i, y_i)$, $i = 0, \ldots, N$; since $y \geq 0$ and M_γ is a smooth closed surface, we have $y_0 = y_N = 0$ and $y_i > 0$ for $i = 1, \ldots, N-1$. It is not necessary to know any parametrisation of the curve; instead, the scheme uses the points γ_i. The phase field u is discretised by $u_i = u(\gamma_i)$.

Given a function w on γ, we approximate its integral over the surface segment obtained by rotating the arc (γ_i, γ_{i+1}) by

$$\int_{M_\gamma((\gamma_i,\gamma_{i+1}))} w\, d\mu \approx \frac{2}{3}\pi \left(w_{i+1}y_{i+1} + \frac{1}{2}w_{i+1}y_i + \frac{1}{2}w_i y_{i+1} + w_i y_i \right) |\gamma_{i+1} - \gamma_i|,$$

and the integral over M_γ by summation over all segments. In particular, if $w \equiv 1$, we obtain the exact area of the discrete surface as approximation to the area of the continuous surface.

The derivatives $\partial_t w$ and $\partial_{tt} w$ are approximated by the finite differences

$$\partial_t w(\gamma_i) \approx \frac{1}{2} \left(\frac{w(\gamma_{i+1}) - w(\gamma_i)}{|\gamma_{i+1} - \gamma_i|} + \frac{w(\gamma_i) - w(\gamma_{i-1})}{|\gamma_i - \gamma_{i-1}|} \right) \tag{5.13}$$

and

$$\partial_{tt} w(\gamma_i) \approx \frac{1}{D_i} \left(\frac{w(\gamma_{i+1}) - w(\gamma_i)}{|\gamma_{i+1} - \gamma_i|} - \frac{w(\gamma_i) - w(\gamma_{i-1})}{|\gamma_i - \gamma_{i-1}|} \right), \tag{5.14}$$

where $D_i = \left(|\gamma_{i+1} - \gamma_i| + |\gamma_i - \gamma_{i-1}|\right)/2$ is the average distance of γ_i to its two neighbours. For the points γ_0 and γ_N on the axis of revolution we may define

$$\gamma_{-1} = (x_1, -y_1) \quad \text{and} \quad \gamma_{N+1} = (x_{N-1}, -y_{N-1}) \tag{5.15}$$

as well as $w(\gamma_{-1}) = w(\gamma_1)$ and $w(\gamma_{N+1}) = w(\gamma_{N-1})$ by smoothness and rotational symmetry, so that the above formulas are valid for $i = 0, \ldots, N$.

From (5.13) with $w = y$ and (5.14) we obtain a discretisation of the Laplace-Beltrami operator; using $w = x$ in (5.13) yields the approximation

$$\kappa_2(\gamma_i) \approx \kappa_{2,i} := \frac{1}{2y_i} \left(\frac{x_{i+1} - x_i}{|\gamma_{i+1} - \gamma_i|} + \frac{x_i - x_{i-1}}{|\gamma_i - \gamma_{i-1}|} \right) \tag{5.16}$$

of the second principal curvature κ_2 for $y > 0$. If $y(t_0) = 0$, the same argument as for the tangential differential operators in the last section gives

$$\kappa_2(t_0) = \lim_{t\to t_0} \frac{x'(t)}{y(t)} = \lim_{t\to t_0} \frac{x'(t) - x'(t_0)}{\pm(t - t_0) + O((t - t_0)^2)} = \pm x''(t_0) = y'(t_0)x''(t_0) = \kappa_1(t_0),$$

and the approximation for $\kappa_1(t_0)$ below is also used for $\kappa_2(t_0)$.

To obtain an approximation for κ_1, recall that the variation of the length $\mathcal{L}_\gamma$ of a smooth plane curve γ under variations $\psi\nu$, where ν is the outer normal to γ, is given by

$$\langle D\mathcal{L}_\gamma, \psi \rangle = \int_\gamma \kappa_1 \psi\, ds. \tag{5.17}$$

Let $\eta_\delta\colon \mathbb{R}\to\mathbb{R}$ be the piecewise linear hat function with $\eta_\delta(0)=1$ and $\eta_\delta(t)=0$ for $|t|\geq\delta$, z_0 a point on γ and $\psi_\delta(z)=\eta_\delta(|z-z_0|)$. By (5.17) we then have

$$\langle D\mathcal{L}_\gamma,\psi_\delta\rangle=\int_{\gamma\cap B_\delta(z)}\left(1-\frac{1}{\delta}|z-z_0|\right)\kappa_1(z)\,ds(z)\approx\kappa_1(z_0)\int_{-\delta}^{\delta}\left(1-\frac{t}{\delta}\right)\,dt=\delta\kappa_1(z_0).$$

On the discretised curve let γ_i be a vertex and choose δ to be the average distance D_i from γ_i to its two neighbours. Consider the length of the curve as function of γ_i while all other vertices are fixed, that is, consider

$$\mathcal{L}_i(z)=C+|\gamma_{i+1}-z|+|z-\gamma_{i-1}|.$$

Replacing γ_i by $\gamma_i+h\xi$, where $\xi\in\mathbb{R}^2$ is a fixed vector, yields

$$\frac{d}{dh}\mathcal{L}_i(\gamma_i+h\xi)\bigg|_{h=0}=\left(\frac{\gamma_i-\gamma_{i-1}}{|\gamma_i-\gamma_{i-1}|}-\frac{\gamma_{i+1}-\gamma_i}{|\gamma_{i+1}-\gamma_i|}\right)\cdot\xi$$

and thus

$$\nabla\mathcal{L}_i:=\nabla\mathcal{L}_i(\gamma_i)=\frac{\gamma_i-\gamma_{i-1}}{|\gamma_i-\gamma_{i-1}|}-\frac{\gamma_{i+1}-\gamma_i}{|\gamma_{i+1}-\gamma_i|}.$$

One the one hand, $\mathcal{L}_i$ increases most for variations normal to γ at γ_i; on the other hand, $\nabla\mathcal{L}_i(z_i)$ points in the direction of greatest increase of $\mathcal{L}_i$. Thus we infer that

$$\nu_i=\pm\frac{\nabla\mathcal{L}_i}{|\nabla\mathcal{L}_i|}$$

is an approximation of the outer unit normal to γ at γ_i, provided that $\nabla\mathcal{L}_i\neq 0$. Here the positive sign applies, if $\nabla\mathcal{L}_i$ points towards the outside of γ, and the negative sign otherwise. Since the normal movement of one vertex of the discretised curve has by our choice of δ the same effect as the variation with ψ_δ, the above reasoning yields the approximation

$$\kappa_1(\gamma_i)\approx\kappa_{1,i}:=\frac{\nabla\mathcal{L}_i\cdot\nu_i}{D_i}\tag{5.18}$$

of the first principal curvature. If $\nabla\mathcal{L}_i=0$, the vectors $\gamma_{i+1}-\gamma_i$ and $\gamma_i-\gamma_{i-1}$ are parallel, and we have $\kappa_{1,i}=0$; the normal ν_i is obtained by normalising $\gamma_{i+1}-\gamma_i$ and rotating it by the angle $\pm\pi/2$. At the start and end point of γ on the axis of revolution the above formulas apply with γ_{-1} and γ_{N+1} from (5.15).

It remains to consider the linearisations of the principal curvatures. To this end, denote by v_i the normal velocity of the vertex γ_i, and by $\gamma_{i,\tau+h}=\gamma_i+hv_i\nu_i$ the vertices at the next time step. Since $\kappa_{1,i,\tau+h}=\nabla\mathcal{L}_{i,\tau+h}\cdot\nu_{i,\tau+h}/D_{i,\tau+h}$ from (5.18) and $\nabla\mathcal{L}_{i,\tau+h}$ is perpendicular to $\frac{d}{dh}\nu_{i,\tau+h}$, we find

$$\frac{d}{dh}\kappa_{1,i,\tau+h}\bigg|_{h=0}=\frac{1}{D_i}\frac{d}{dh}\nabla\mathcal{L}_{i,\tau+h}\bigg|_{h=0}\cdot\nu_i-\frac{\kappa_{1,i}}{D_i}\cdot\frac{d}{dh}D_{i,\tau+h}\bigg|_{h=0}.$$

Using the approximations

$$\nabla\mathcal{L}_{i,\tau+h} = \frac{\gamma_i + hv_i\nu_i - \gamma_{i-1} - hv_{i-1}\nu_{i-1}}{|\gamma_i + hv_i\nu_i - \gamma_{i-1} - hv_{i-1}\nu_{i-1}|} - \frac{\gamma_{i+1} + hv_{i+1}\nu_{i+1} - \gamma_i - hv_i\nu_i}{|\gamma_{i+1} + hv_{i+1}\nu_{i+1} - \gamma_i - hv_i\nu_i|}$$

and

$$D_{i,\tau+h} = \frac{1}{2}\Big(|\gamma_i + hv_i\nu_i - \gamma_{i-1} - hv_{i-1}\nu_{i-1}| + |\gamma_{i+1} + hv_{i+1}\nu_{i+1} - \gamma_i - hv_i\nu_i|\Big)$$

at $\tau + h$, one computes

$$\begin{aligned}
\frac{d}{dh}\nabla\mathcal{L}_{i,\tau+h}\bigg|_{h=0} \cdot \nu_i = &\left\{\frac{(\gamma_i - \gamma_{i-1})\cdot\nu_i\,(\gamma_i - \gamma_{i-1})\cdot\nu_{i-1}}{|\gamma_i - \gamma_{i-1}|^3} - \frac{\nu_i\cdot\nu_{i-1}}{|\gamma_i - \gamma_{i-1}|}\right\}v_{i-1} \\
&+ \left\{\frac{1}{|\gamma_i - \gamma_{i-1}|} + \frac{1}{|\gamma_{i+1} - \gamma_i|}\right. \\
&\qquad \left. - \frac{\big((\gamma_i - \gamma_{i-1})\cdot\nu_i\big)^2}{|\gamma_i - \gamma_{i-1}|^3} - \frac{\big((\gamma_{i+1} - \gamma_i)\cdot\nu_i\big)^2}{|\gamma_{i+1} - \gamma_i|^3}\right\}v_i \\
&+ \left\{\frac{(\gamma_{i+1} - \gamma_i)\cdot\nu_i\,(\gamma_{i+1} - \gamma_i)\cdot\nu_{i+1}}{|\gamma_{i+1} - \gamma_i|^3} - \frac{\nu_i\cdot\nu_{i+1}}{|\gamma_{i+1} - \gamma_i|}\right\}v_{i+1}
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
\frac{d}{dh}D_{i,\tau+h}\bigg|_{h=0} = &\left\{-\frac{(\gamma_i - \gamma_{i-1})\cdot\nu_{i-1}}{2|\gamma_i - \gamma_{i-1}|}\right\}v_{i-1} \\
&+ \left\{\frac{1}{2}\left(\frac{\gamma_i - \gamma_{i-1}}{|\gamma_i - \gamma_{i-1}|} - \frac{\gamma_{i+1} - \gamma_i}{|\gamma_{i+1} - \gamma_i|}\right)\cdot\nu_i\right\}v_i \\
&+ \left\{\frac{(\gamma_{i+1} - \gamma_i)\cdot\nu_{i+1}}{2|\gamma_{i+1} - \gamma_i|}\right\}v_{i+1}.
\end{aligned} \tag{5.20}$$

Combining (5.19) and (5.20) yields an expression for the operator L_1 from (5.10) of the form $(L_1 v)(\gamma_i) = \sum_{j=i-1}^{i+1} l^1_{i,j} v_j$. At the start point on the axis of revolution we use again $\gamma_{-1} = (x_1, -y_1)$ and $\kappa_{1,-1} = \kappa_{1,1}$; the same applies at the end point.

For the linearisation of the second principal curvature we use (5.16) if $y_i > 0$ and obtain

$$\begin{aligned}
\frac{d}{dh}\kappa_{2,i,\tau+h}\bigg|_{h=0} = &\frac{1}{2y_i}\left\{-\frac{\nu^x_{i-1}}{|\gamma_i - \gamma_{i-1}|} + \frac{(x_i - x_{i-1})\,(\gamma_i - \gamma_{i-1})\cdot\nu_{i-1}}{|\gamma_i - \gamma_{i-1}|^3}\right\}v_{i-1} \\
&+ \frac{1}{2y_i}\left\{-2\kappa_{2,i}\nu^y_i - \frac{\nu^x_i}{|\gamma_{i+1} - \gamma_i|} + \frac{\nu^x_i}{|\gamma_i - \gamma_{i-1}|}\right. \\
&\qquad \left. + \frac{(x_{i+1} - x_i)\,(\gamma_{i+1} - \gamma_i)\cdot\nu_i}{|\gamma_{i+1} - \gamma_i|^3} - \frac{(x_i - x_{i-1})\,(\gamma_i - \gamma_{i-1})\cdot\nu_i}{|\gamma_i - \gamma_{i-1}|^3}\right\}v_i \\
&+ \frac{1}{2y_i}\left\{\frac{\nu^x_{i+1}}{|\gamma_{i+1} - \gamma_i|} - \frac{(x_{i+1} - x_i)\,(\gamma_{i+1} - \gamma_i)\cdot\nu_{i+1}}{|\gamma_{i+1} - \gamma_i|^3}\right\}v_{i+1},
\end{aligned}$$

where (ν_i^x, ν_i^y) are the components of the approximate normal ν_i. As for $\kappa_{1,i}$ this yields an expression for L_2 of the form $(L_2 v)(\gamma_i) = \sum_{j=i-1}^{j=i+1} l_{i,j}^2 v_j$. At the start point on the axis of revolution we have $\kappa_{2,0} = \kappa_{1,0}$, so the linearisation of κ_2 is the one of κ_1 and $l_{0,j}^2 = l_{0,j}^1$; the same holds for the end point.

Remark. All spatial approximations are local in the sense that for an evaluation at a vertex γ_i only data from γ_i itself and the neighbouring vertices is needed. Since the evaluation of f, g and their derivatives is also local, the resulting linear system is sparse; the only dense rows and columns stem from the constraints and the Lagrange multipliers.

5.2.3 Implementation, mesh adaption and time stepping

We implemented the above scheme in Matlab. Discrete initial data was obtained by sampling continuous curves and phase fields at uniformly distributed points on the curve. Since regions of large phase field gradients and kinks are expected to be of order ε, the point distance was at least an order smaller than ε.

It has already been noted in [MS02], that the evolution moves some points closer to their neighbours and other points away from each other. We observed that this behaviour is more pronounced in regions, where the phase field is close to zero, than in other domains. We therefore readjust the position of a point, if the distance to its neighbours becomes too large or too small. The idea is to move such a point γ_i so that on the one hand its new location has the same distance to both neighbours γ_{i-1}, γ_{i+1} and on the other hand its discrete curvature $\kappa_{1,i}$ as in (5.18) is preserved. This means, we look for the new position on a circular arc through γ_{i-1} and γ_{i+1} of prescribed signed curvature $\kappa_{1,i}$. In a similar fashion we introduce new points with curvature and phase field equal to the mean value of the two new neighbours, and we also remove points that are too close to their neighbours. We refer to [MS02] for the details of the insertion and the computation of new locations.

Since interfaces and regions of high curvature may move, it is desirable to adapt the grid to be finer in such regions and coarser in less curved bulk domains. We implemented this idea by prescribing different thresholds for the point distance in these regions, which were identified by an upper bound on $|u_i|$ and a lower bound on $\kappa_{1,i}^2 + \kappa_{2,i}^2$, respectively. However, in our simulations the mesh adaption procedure often increased the energy. Additionally, the insertion of new points near highly curved interface regions sometimes produced locally large phase field gradients. In most cases we observed that the algorithm could recover and return close to the previous state; occasionally it failed, though. We therefore chose grid thresholds to keep the number of mesh adaptions small and we avoided mesh adaptions in consecutive time steps.

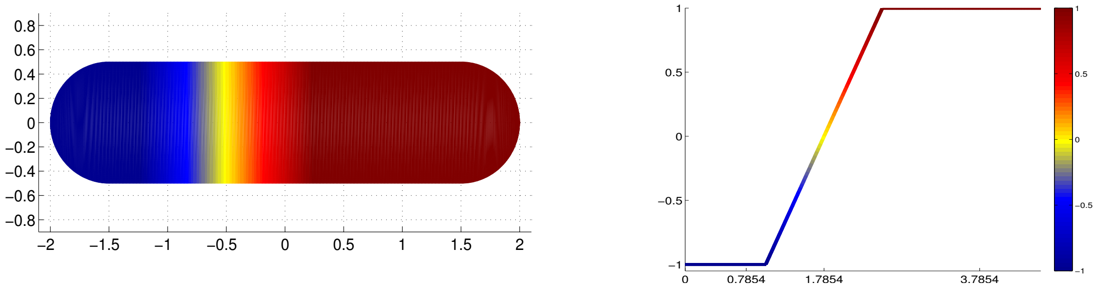

Figure 5.1: Initial data for the example in Section 5.3.1, cross section of the surface on the left, phase field over arclength on the right. The marks on the horizontal axis indicate the interface and the connection of the spherical caps to the cylinder.

The simulations presented below were carried out with a constant time step h. Stability considerations for explicit discretisation of the potential term in the Allen-Cahn equation yield a time step restriction of the form $h \lesssim \varepsilon^2$. In the simulations we chose $h \lesssim d^2$, where $d \lesssim \varepsilon$ denotes the initial minimum point distance; compare [ES10a]. We stopped the simulation, when the norm $g_{\mathcal{T}}(\cdot,\cdot)$ of the gradient was small.

5.3 Examples

5.3.1 A phase separated tube without kink

As an example we consider a cylinder of length 3 and radius 1/2 with spherical caps, see Figure 5.1. The cylinder is centred in the origin so that the x-coordinate ranges from -2 to 2, and the phase field is given by

$$u(x,y) = \begin{cases} -1 & \text{if } x \le -\frac{5}{4}, \\ \frac{4}{3}x + \frac{2}{3} & \text{if } -\frac{5}{4} < x < \frac{1}{4}, \\ 1 & \text{if } \frac{1}{4} \le x. \end{cases}$$

For the regular approximation we use $k \equiv 1$ and $k_G \equiv k_B \equiv 0$, for the singular approximation $k_G \equiv 0$, $k_B = 0.1$ and

$$k(u) = (\sqrt{u^2 + .01} - .1)/(\sqrt{1.01} - .1). \tag{5.21}$$

The function (5.21) is closer to $|u|$ than u^2 and is numerically more stable. In both cases we set $\widetilde{\sigma} = 1$, and let $H_s(u)$ in $[-1,1]$ be equal to the fifth-order polynomial interpolation of $H_s(1) = 2$ and $H_s(-1) = 1$ with zero first and second derivative at ± 1. When extended by the constants $H_s(\pm 1)$ to the whole real line, H_s belongs to $C^2(\mathbb{R})$.

Figure 5.2 shows the numerically stationary membranes for both approximations and $\varepsilon = 0.05$. The main observation is that the neck in the singular approximation has smaller

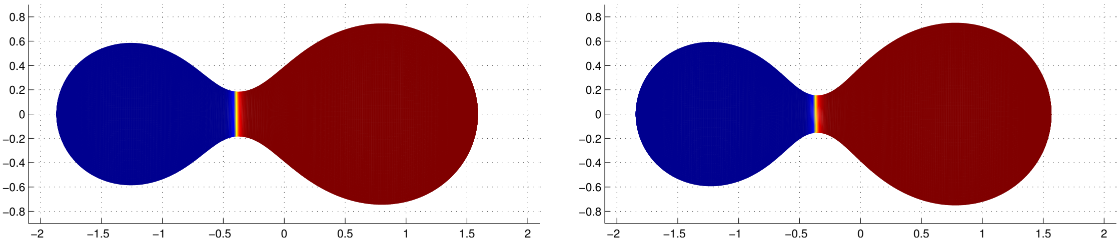

Figure 5.2: Numerically stationary shapes for the tube of Section 5.3.1 with $\varepsilon = 0.05$. The regular approximation (left) has a larger neck than the singular approximation (right), but there is no kink.

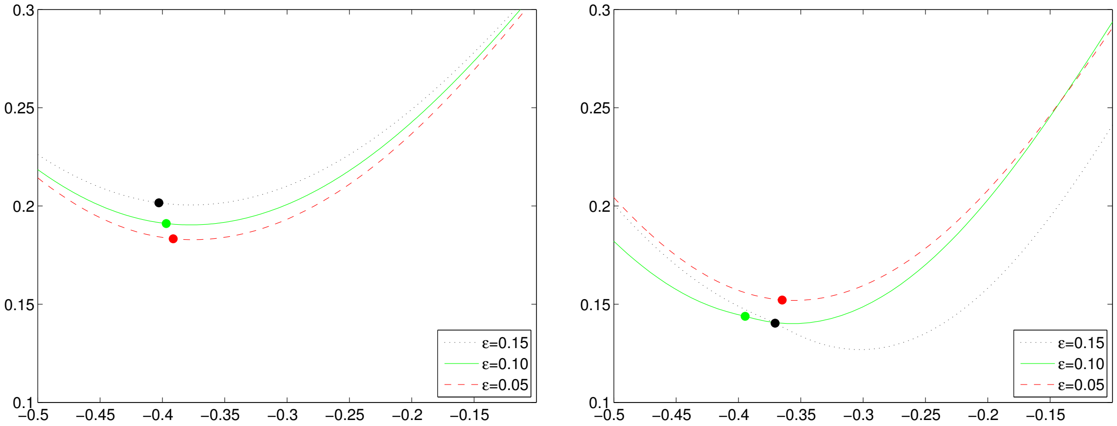

Figure 5.3: Numerically stationary generating curves for $\varepsilon = 0.15$, 0.1 and 0.05 in the neck regions for the regular (left) and singular approximation (right); the dots indicate the vertex where the phase field is closest to 0.

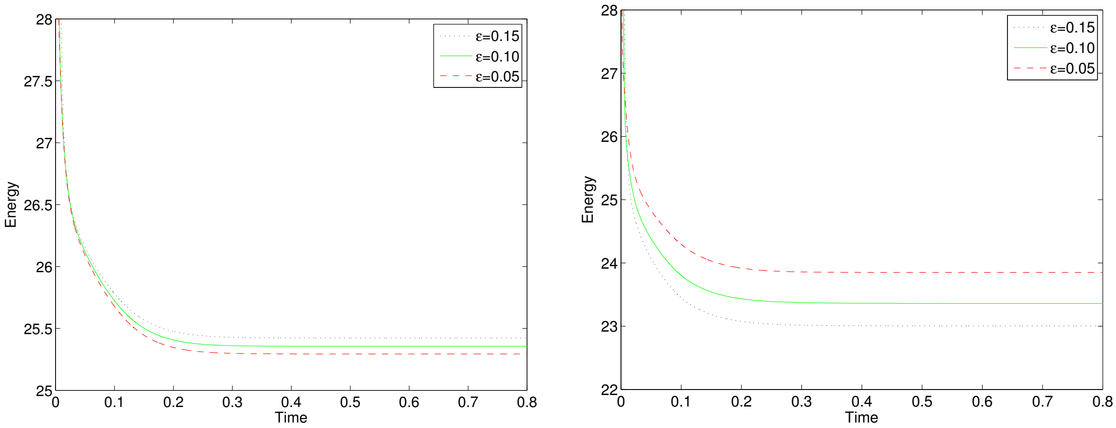

Figure 5.4: Evolution of energies for $\varepsilon = 0.15$, 0.1 and 0.05, regular approximation on the left, singular on the right. Note the different scaling of the energy axes.

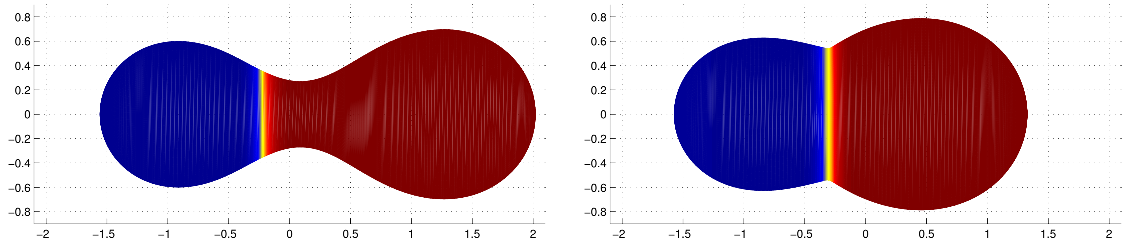

Figure 5.5: Numerically stationary membranes for the example of Section 5.3.2 with $\varepsilon = 0.1$, regular approximation on the left and singular approximation on the right.

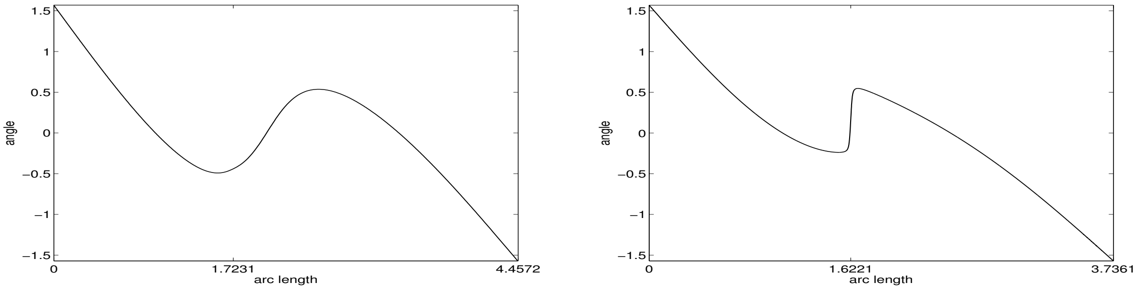

Figure 5.6: Angle of the generating curves of the membranes in Figure 5.5 over arc length, regular approximation on the left and singular approximation on the right.

diameter than in the regular approximation, there is, however, no kink. The neck regions are enlarged in Figure 5.3 for $\varepsilon = 0.15$, 0.1 and 0.05; Figure 5.4 shows the evolution of the membrane energies. It can be seen that the regular approximation seems to converge well as $\varepsilon \to 0$, including the interface position; convergence of the singular approximation is less visible. Furthermore, both approximations first quickly decrease the energy and evolve curve and phase field, later these quantities hardly change any more. Interestingly, the energy of the regular approximation decreases if ε decreases, the energy of the singular approximation increases. The change in area from time $\tau = 0$ to $\tau = 0.3$ in the above simulations is at most of order 10^{-3}, the change in the phase integral of order 10^{-4}. Hence, both constraints are preserved within the order of the linearisation.

5.3.2 A phase separated tube with kink

Next we provide an example in which both approximations behave differently and a kink in the singular approximation appears. We consider the same initial data and parameters as in the last example except for $\widetilde{\sigma} = 0.8$ and k_G. For the regular approximation we let $k_G(u)$ be the fifth-order polynomial interpolation of $k_G(-1) = -1$ and $k_G(+1) = -0.5$, and for the singular approximation we multiply this interpolation with a root function as in (5.21).

The computed stationary membranes for $\varepsilon = 0.1$ are shown in Figure 5.5, the angle functions of their generating curves in Figure 5.6. Obviously, while in the regular approx-

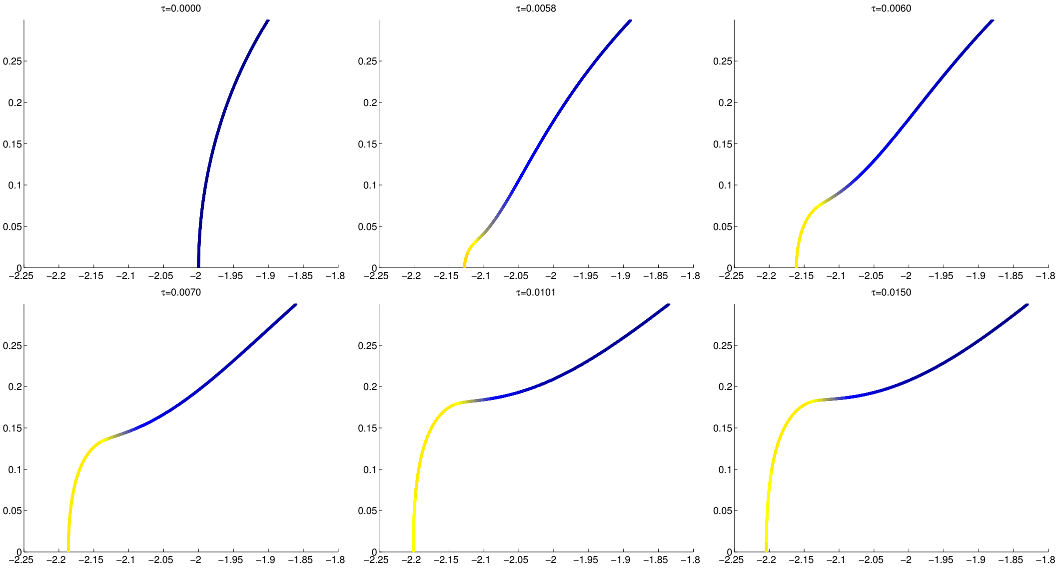

Figure 5.7: Left end of the generating curves for the example in Section 5.3.3 for $\varepsilon = 0.1$ and several time steps τ. The phase $u = -1$ is coloured blue, regions where $u = 0$ are yellow.

imation there is a rather ample neck region that lies completely in the red phase $u = 1$, the singular approximation makes a sharp turn at the interface: the angle almost jumps from about -0.24 to $+0.54$. Simulations with other values of ε led to the same result, thus we may conclude that there is a kink in the limit $\varepsilon \to 0$ and have omitted further figures.

5.3.3 Axis of revolution

The previous two examples show the potential of the singular approximation and its limit to model quite different shapes and behaviour of membranes – with or without kinks. The following example on the other hand emphasises once more the difficulties that arise at the axis of revolution. We used the same initial surface and parameters as in Section 5.3.1 but changed the initial phase field to

$$u(x,y) = \begin{cases} -1 & \text{if } x \leq -\pi/2 \text{ or } x \geq \pi/2, \\ \cos 6x & \text{if } -\pi/2 < x < \pi/2. \end{cases}$$

Hence, there are six interfaces on the surface.

Figure 5.7 shows the left end of the generating curve for several small time steps τ. The right end of the curve looks similar due to symmetry, and the middle segments, which contain the initial interfaces, are still far away from a stationary point and thus also omitted. It can be seen that the phase field u, which is initially equal to -1, quickly changes to 0 at the boundary. Afterwards the transition layer moves into the surface, thus creating a large region where $u = 0$ near the axis of revolution. Such regions cause a problem for the algorithm, which tends to move points out of it and towards or below the x-axis. Moreover, inside these regions the evolution of the surface is dominated by $\varepsilon k_B \int |B|^2 \, d\mu$ and not the membrane bending energy any more.

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