Analysis of Nonlinear Conservation Laws of Mixed Type and Related Equations



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To my Mother and Father, with love and thanks for everything

Gyda diolch a chariad am bob dim i Mam a Dad

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Statement of Originality

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at any other university. The work in Chapter 4 is joint work with Gui-Qiang Chen, Feimin Huang, Song Liu, and Qin Wang that is contained in the preprint [17], to appear in J. Eur. Math. Soc. (JEMS), where my own contribution was significant.

Abstract

In this thesis we study the system of Euler equations for the potential flow of a compressible gas in two dimensions. The governing equations—the conservation of mass and the Bernoulli law—were among the first partial differential equations (PDE) to be written down, originating from the great eighteenth century minds of Euler and Bernoulli. However, despite their long history, not even the existence of global weak solutions is known.

We begin in Chapter 1 with a short survey of the classical and recent theory of conservation laws in one and two dimensions, before moving on to the mathematical preliminaries for two-dimensional (2-D) potential flow in Chapter 2. The 2-D Riemann problem with initial data given in the four quadrants of the plane is then classified into five (or six) genuinely distinct configurations depending on the adiabatic exponent.

In Chapter 3, we study the detailed properties of the shock polar curve for 2-D steady potential flow. In particular, we obtain a new property: the steady critical angles are strictly increasing functions of the upstream Mach number. In turn, we prove the uniqueness of the critical angles for the shock reflection-diffraction problem.

Finally, in Chapter 4, we construct global weak solutions to a 2-D Riemann problem for the interaction of four shockwaves. In this collaborative work, we reformulate the problem as a free boundary problem for a nonlinear PDE of mixed hyperbolic-elliptic type, and solve this problem via a sophisticated iteration procedure. Associated calculations and auxiliary results are given in the Appendices A and B.

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Chapter 1

Introduction

Conservation laws are systems of nonlinear partial differential equations in divergence form. Simply, these equations assert in mathematical language the physically apparent principle: the time rate of change in the amount of a quantity contained within a region must equal the rate of flux of this quantity through the boundary of that region. Often these systems are of hyperbolic type, whereby nonlinearities typically result in the breakdown of smooth solutions in finite time, and consequently the formation of jump discontinuities, known as shockwaves or shocks. Examples of shocks are found naturally in high-speed fluid flows, such as the airflow around supersonic or near-sonic aircraft, illustrated magnificently in Figure 1.1, where the resultant shocks may be heard audibly as sonic booms. These flows are typically modelled by the system of Euler equations for compressible gases, or variations thereof, including the equations of potential flow, which represent the physical laws of conservation of mass, momentum and energy, and serve as the prototype for more general hyperbolic systems of nonlinear conservation laws. Subsequently, the rigorous mathematical analysis of these equations will ultimately lead to greater understanding of gas dynamics, and the plethora of physical phenomena it pervades.

The general mathematical theory of conservation laws in one spatial dimension is well-developed, but surprisingly little progress has been made with the theory in higher dimensions, and many fundamental problems remain open. A major open problem is the resolution of the Riemann problem in two space dimensions—the most

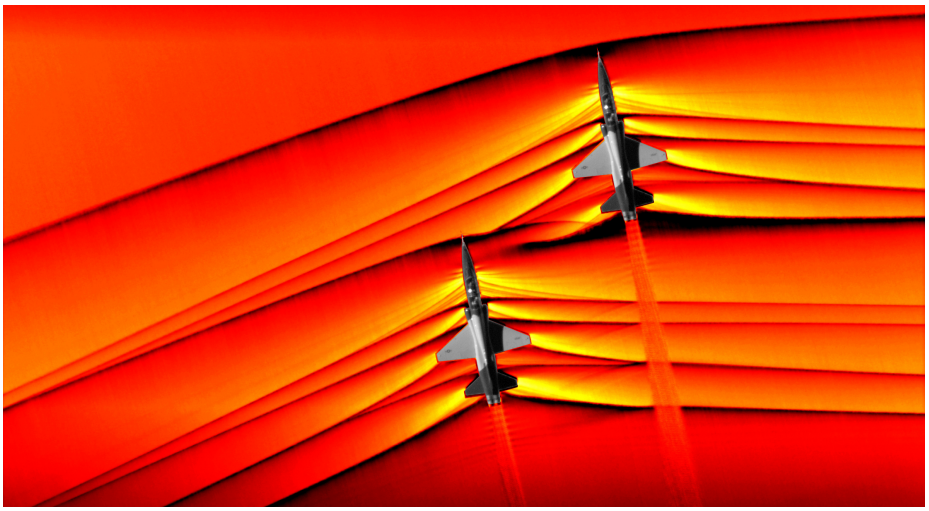


Figure 1.1: The complex interaction of shockwaves generated by two supersonic T-38 aircraft in flight, photographed by the NASA Quiet Supersonic Technology (QueSST) programme in developing quieter supersonic aircraft. *Credits: NASA Photo,* [46].

simple initial value problem with constant initial data given in sectors of the plane. Indeed, multidimensional Riemann problems are vastly more complicated than their one-dimensional counterparts; the multitude of complex wave interaction patterns displayed by numerical simulations, a sample of which are shown in Figure 1.2, greatly evidence the potential difficulties associated with their rigorous mathematical analysis. Nonetheless, the solutions to Riemann problems are fundamental to the general initial value problem in that they determine the local structure, and large-time asymptotics of solutions to hyperbolic systems of conservation laws. In this sense it is hoped that better understanding of wave interactions will facilitate development in both analytic and numerical techniques for the general initial value problem.

Recently, a handful of results have been obtained in simplified cases, including the shock reflection-diffraction problem [20], and the Prandtl–Meyer reflection problem [1], where important theoretical advances have been made in the theory of free boundary problems for mixed hyperbolic-elliptic type, and degenerate elliptic equations in domains with corners. Based on this work, the two-dimensional Riemann problem for the interaction of four shockwaves has been solved; see [17], and the summary of this work contained in Chapter 4.

1.1 Brief overview of mathematical gas dynamics

1.1.1 The Euler equations

The Euler equations are a successful model for gas dynamics, and serve as the prototypical example of a hyperbolic system of conservation laws. The equations may be obtained from the full system of Navier–Stokes equations under the simplifying assumptions of zero viscosity and zero thermal conductivity. In particular, they consist of the conservation laws of mass, momentum, and energy

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho \boldsymbol{u}\big) = 0\,, \\ \partial_t (\rho \boldsymbol{u}) + \operatorname{div}\big(\rho \boldsymbol{u} \otimes \boldsymbol{u} + pI\big) = 0\,, \\ \partial_t (\rho E) + \operatorname{div}\big((\rho E + p)\boldsymbol{u}\big) = 0\,, \end{cases} \tag{1.1.1}$$

where ρ is the density; $\boldsymbol{u}$ is the velocity field; p is the pressure; and E is the total energy per unit mass. The system (1.1.1) is closed by the constitutive relations

$$p = p(\rho, e)\,, \qquad E = \frac{1}{2}|u|^2 + e\,,$$

where e is the internal energy—a given function of (ρ, p) defined through the thermodynamic relations. In the case of a polytropic gas, we have

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}\,, \tag{1.1.2}$$

where $\gamma > 1$ is the adiabatic constant. There are two further thermodynamic variables of interest: the temperature T; and entropy S; which satisfy the fundamental thermodynamic relation

$$T\mathrm{d}S = \mathrm{d}e - \frac{p}{\rho^2}\mathrm{d}\rho\,.$$

For smooth solutions, it is simple to verify the transport equation for the entropy

$$(\partial_t + \boldsymbol{u} \cdot \nabla) S = 0\,, \tag{1.1.3}$$

or equivalently the companion conservation law for the so-called entropy-entropy flux pair $(-\rho S, -\rho S \boldsymbol{u})$

$$\partial_t(\rho S) + \operatorname{div}\big(\rho S \boldsymbol{u}\big) = 0\,,$$

from which the entropy is constant along pathlines, those being the integral curves of the flow $\frac{\mathrm{d}x}{\mathrm{d}t} = \boldsymbol{u}(x(t), t)$. It follows that any flow starting from isentropic (uniformly constant entropy) initial data, will remain isentropic so long as the flow remains smooth. This conclusion is no longer valid following the formation of discontinuities, where for instance the entropy production across a shock is of cubic order in the shock strength. Nevertheless, the isentropic approximation remains reasonable for flows having weak shocks, as is often the case in the transonic regime for aerodynamics.

An important consequence of the isentropic approximation is that the pressure and internal energy become functions of the density only. In the case of a polytropic gas we have

$$p(\rho) = \kappa \rho^\gamma\,, \qquad e(\rho) = \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1}\,, \tag{1.1.4}$$

for some constant $\kappa > 0$, which may be fixed through scaling. Subsequently, for smooth solutions, the conservation of energy in (1.1.1) becomes redundant, since it may be obtained directly from the conservation of mass and momentum after a short calculation employing the isentropic property. That is to say, the third equation in (1.1.1) is a companion conservation law to the system of isentropic Euler equations

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho \boldsymbol{u}\big) = 0\,, \\ \partial_t(\rho \boldsymbol{u}) + \operatorname{div}\big(\rho \boldsymbol{u} \otimes \boldsymbol{u} + p I\big) = 0\,, \end{cases} \tag{1.1.5}$$

under the barotropic equation of state $p = p(\rho)$.

Similar to the transport of entropy (1.1.3), for smooth solutions of the system of isentropic Euler equations (1.1.5), we have the transport equation for the vorticity $\boldsymbol{\omega} \coloneqq \operatorname{curl} \boldsymbol{u}$

$$(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} + \boldsymbol{\omega} \operatorname{div} \boldsymbol{u} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} \,,$$

and hence irrotational initial data generates irrotational flow, so long as the flow remains smooth. After the formation of singularities, the flow may cease to be irrotational, since the curvature of shock discontinuities is a generation mechanism for vorticity; see [43, 75].

For smooth irrotational flows, under the Hodge–Helmholtz decomposition, we may express the velocity as the gradient of a potential

$$\boldsymbol{u} = \nabla \Phi \,, \tag{1.1.6}$$

from which it is possible to derive an integral of motion: the Bernoulli law

$$\partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + h(\rho) = B \,, \tag{1.1.7}$$

where $h : \mathbf{R} \to \mathbf{R}$ is the enthalpy which satisfies $h'(\rho) = \frac{p'(\rho)}{\rho}$; and B is a constant, known as the Bernoulli constant. In the case of a barotropic polytropic gas

$$p(\rho) = \frac{1}{\gamma}\rho^\gamma \,, \qquad \text{and} \qquad h(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1} \,.$$

For smooth solutions, the conservation of momentum in (1.1.5) becomes redundant, since it may be obtained directly by combining the conservation of mass along with (1.1.6)–(1.1.7). Therefore, we feel free to drop the conservation of momentum

from (1.1.5), leaving behind the system of Euler equations for potential flow

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho \nabla \Phi\big) = 0\,, \\ \partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + h(\rho) = B\,. \end{cases} \tag{1.1.8}$$

1.1.2 A brief history of the Euler equations

The isentropic Euler equations (1.1.5) are the oldest known example of a system of hyperbolic conservation laws, first appearing in a 1757 paper of Euler [35]. The Bernoulli law (1.1.7) is even older, having origins in the 1738 book of Bernoulli [3], though it was not formally written down until the later 1754 work of Euler [36].

The idea of discontinuous motions, *i.e.* shockwaves, was first considered in 1848 by Stokes [81], who derived the Rankine–Hugoniot jump conditions in the context of isothermal gases, $\gamma = 1$. The pioneering work of Stokes apparently drew attention and criticism from the eminent Lord Kelvin and Lord Rayleigh, who argued that the discontinuous solutions posed by Stokes did not conserve mechanical energy, and thus were unphysical. Ultimately their objections would prove to be ill-founded, as we now understand that the mechanical energy is a convex entropy for the system (1.1.5), and is therefore dissipated by physically admissible shocks. Around the same time, in 1860, Riemann [71] formulated and solved what is now known as the Riemann problem for one dimensional isentropic flow.

Towards the end of the nineteenth century, the need for an additional energy equation became clear, thus leading to the full system of Euler equations (1.1.1). Later as the subject of thermodynamics advanced, the relation between the second law of thermodynamics and physically meaningful shocks was established, and other criteria to identify admissible shocks, such as the Lax E-condition and the concept of entropy-entropy flux pairs were developed.

During the first half of the twentieth century, faced with the austere reality of war, the field of gas dynamics became a highly prominent area of research, attracting attention from the World's leading scientists and mathematicians. Of particular importance during this era were the novel regimes of high speed, high temperature,

and high pressure flows, which moved beyond the physical validity of the polytropic gas models discussed above. The intense period of war-time research paved the way for incredible advances in aerodynamics, including the development of jet engines and supersonic swept-wing aircraft, and later, the development of rocket engines, and hypersonic vehicles. The state-of-the-art knowledge, as it was during the 1940s, is summarised in the legendary text by Courant and Friedrichs [26].

It was not until the mid-to-late twentieth century that the Euler equations, and more generally hyperbolic systems of conservation laws, were given their rigorous mathematical underpinning. The most famous contribution of the period, the 1957 seminal work of Lax [53], introduces many fundamental concepts, including strict hyperbolicity, genuine nonlinearity, simple waves, and the Lax E-condition. Based on these foundations, progress has accelerated and the subject has grown vast, and only now are we beginning to probe the most fundamental and challenging of questions.

1.2 Systems of conservation laws in one space dimension

Hyperbolic systems of conservation laws in one space dimension enjoy a rich and increasingly vast theory originating from around the mid-nineteenth century. At present, there are two main branches of the existence theory: the BV formulation introduced in the seminal works of Lax [53] and Glimm [41], based on wave interaction estimates; and the L^∞ formulation introduced originally by DiPerna [31], based on weak convergence methods and compensated compactness. The relation between these two approaches is an active area of research, primarily focusing on the uniqueness and stability of these solutions within certain classes. Important milestones in this direction include the L^1-stability theory of Bressan among others [8, 10, 63, 4], and the L^2-stability theory of Vasseur [48, 15].

We discuss briefly the theory in the BV setting, for it is the most relevant in the development of this thesis, and we do not pursue the latter approach here. The

material presented below is standard, and may be found easily within the excellent, encyclopaedic references [80, 27].

1.2.1 Preliminaries on conservation laws

We consider the general initial value problem for a system of n conservation laws in one space dimension given by

$$\begin{cases} \partial_t U + \partial_x F(U) = 0\,, \\ U(0,x) = U_0(x)\,, \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbf{R}\,, \tag{1.2.1}$$

where the flux $F : \mathbf{R}^n \to \mathbf{R}^n$ is smooth in some open set $N \subset \mathbf{R}^n$, called the state space; and the initial data $U_0 : \mathbf{R} \to \mathbf{R}^n$ is of bounded variation. Associated to (1.2.1) are two important structural concepts, introduced by Lax [53].

Definition 1.1 (Hyperbolic and genuinely nonlinear)**.** We say the system (1.2.1) is hyperbolic in N if, for every state $U \in N$, the Jacobian matrix $DF(U)$ has real eigenvalues $\lambda_1(U) \leq \cdots \leq \lambda_n(U)$, and n linearly independent eigenvectors $r_1(U)\,, \cdots, r_n(U)$. Furthermore, the system (1.2.1) is strictly hyperbolic in N if the eigenvalues are distinct. We say the k-characteristic family is genuinely nonlinear in N if, without loss of generality,

$$D\lambda_k \cdot r_k = 1\,,$$

whilst we say it is linearly degenerate if

$$D\lambda_k \cdot r_k = 0\,.$$

Remark 1.2. In one space dimension, the isentropic system of Euler equations (1.1.5) is strictly hyperbolic in the state space $\{p'(\rho) > 0\}$, and all characteristic families are genuinely nonlinear in $\{2p'(\rho) + \rho p''(\rho) > 0\}$. For the full system of Euler equations (1.1.1), the intermediate characteristic family is linearly degenerate.

Under reasonable assumptions, the local existence of C^1–solutions to (1.2.1) was established by Friedrichs [38], however, in general, the nonlinear structure of the characteristics leads to the development of discontinuities in finite time; see [54, 62]. Given that we expect the breakdown of smooth solutions in finite time, the solutions $U : [0, \infty) \times \mathbf{R} \to \mathbf{R}^n$ to (1.2.1) should be understood in the following weak sense.

Definition 1.3 (Weak solutions)**.** A bounded measurable function $U : [0, \infty) \times \mathbf{R} \to \mathbf{R}^n$ is called a weak solution of the initial value problem (1.2.1) with bounded and measurable initial data $U_0 : \mathbf{R} \to \mathbf{R}^n$ if, for any $\zeta \in C_c^1([0, \infty) \times \mathbf{R})$

$$\int_{[0,\infty)\times\mathbf{R}} \big(U\partial_t\zeta + F(U)\partial_x\zeta\big)\,\mathrm{d}t\mathrm{d}x + \int_{\mathbf{R}} U_0\zeta(0,\,\cdot\,)\,\mathrm{d}x = 0\,.$$

Clearly, any classical solution of (1.2.1) is a weak solution. On the other hand, suppose that U is a weak solution of (1.2.1), and that $U \in C^1([0, \infty) \times \mathbf{R}; \mathbf{R}^n)$. Then it is simple to verify that U is in fact a classical solution to (1.2.1) with $U(0, x) = U_0(x)$ almost everywhere.

It is well known that weak solutions of (1.2.1) are in general not unique; see *e.g.*, the counter example in [27, §4.4], and thus additional admissibility criteria or selection principles are required in an attempt to regain uniqueness. In 1-D, the concept of entropy solutions has been particularly successful. As previously alluded, a system of conservation laws (1.2.1) often admits a companion conservation law—a conserved quantity $\eta : N \to \mathbf{R}$ and flux $q : N \to \mathbf{R}$ formally satisfying

$$\partial_t\eta(U) + \partial_x q(U) = 0\,,$$

for any smooth solution U of (1.2.1). We refer to such a pair (η, q) as an entropy-entropy flux pair for (1.2.1), and a convex entropy-entropy flux pair when η is a convex function of U. Examples of entropy-entropy flux pairs include the (negative) physical entropy $(-\rho S, -\rho S\boldsymbol{u})$ for the full system of Euler equations (1.1.1), and the mechanical energy $(\rho E, (\rho E + p)\boldsymbol{u})$ for the system of isentropic Euler equations (1.1.5), which follow due to the Second Law of thermodynamics.

When one considers weak solutions, the companion conservation laws will no longer hold as equalities, rather, taking inspiration from the Second Law, we impose that they should hold in the sense of distributions as inequalities.

Definition 1.4 (Entropy solutions)**.** A weak solution $U : [0,\infty) \times \mathbf{R} \to \mathbf{R}^n$ of (1.2.1) is an entropy solution if, for any $\zeta \in C_c^1([0,\infty) \times \mathbf{R})$

$$\int_{[0,\infty)\times\mathbf{R}} \big(\eta(U)\partial_t\zeta + q(U)\partial_x\zeta\big)\,\mathrm{d}t\mathrm{d}x + \int_{\mathbf{R}} \eta(U_0)\zeta(0,\,\cdot\,)\,\mathrm{d}x \geq 0 \tag{1.2.2}$$

for every convex entropy-entropy flux pair (η, q).

Later, in §1.2.3–1.2.4, we discuss briefly the existence, uniqueness and stability of entropy solutions of (1.2.1), *i.e.*, the question of well-posedness.

1.2.2 Riemann problems in one space dimension

The Riemann problem is the simplest initial value problem having discontinuous initial data which are piecewise constant and invariant under scaling. The initial data takes the form

$$U(0,x) = \begin{cases} U_{\mathrm{L}}\,, & x < 0\,, \\ U_{\mathrm{R}}\,, & x > 0\,, \end{cases} \tag{1.2.3}$$

for constant states $U_{\mathrm{L}}, U_{\mathrm{R}} \in N$.

The problem is endowed with Riemann's name, who was the first to introduce and solve the problem in the context of the one dimensional isentropic Euler equations for gas dynamics [71]. This pioneering work is often criticised due to a supposed flaw within his treatment of shocks. In particular, since Riemann made use of the isentropic model, the resulting shocks he derived do not conserve mechanical energy. However, we echo the point of view found in [27], and refute the criticism on the grounds that the analysis provided by Riemann is entirely consistent and mathematically accurate within the framework of the isentropic Euler equations under his consideration. Indeed

the mechanical energy provides a convex entropy-entropy flux pair for the isentropic Euler equations and should therefore be dissipated by admissible (entropy) solutions.

The general theory for Riemann problems in one space dimension was established by Lax in his seminal work [53]. The paper contains many fundamental concepts such as hyperbolicity and genuine nonlinearity as introduced above, and further the k-wave curves and Lax E-condition, which we shall briefly introduce below.

1.2.2.1 Elementary waves

We begin with three important classes of solutions to (1.2.1), (1.2.3), called elementary waves. These elementary waves are the building blocks of the Riemann solutions, which are in turn used to construct solutions to (1.2.1) for general initial data.

The first elementary waves we consider are centred k-rarefaction waves, which are examples of more general k-simple waves—solutions which remain constant along any k-characteristic curve.

Definition 1.5 (Rarefaction waves)**.** Let $\mathcal{D} \subset (0,\infty) \times \mathbf{R}$. A solution $U \in C^1(\mathcal{D}; \mathbf{R}^n)$ to (1.2.1) is called a centred k-rarefaction wave if $U(t,x) \equiv V(\xi)$ for $\xi \coloneqq \frac{1}{t}x$, and some V satisfying

$$V'(\xi) = r_k(V(\xi)), \qquad \lambda_k(V(\xi)) = \xi. \tag{1.2.4}$$

We say the states U_{L}, U_{R} are connected by a centred k-rarefaction wave if $\lambda_k(U_{\mathrm{L}}) < \lambda_k(U_{\mathrm{R}})$ and there exists a centred k-rarefaction wave $V : [\lambda_k(U_{\mathrm{L}}), \lambda_k(U_{\mathrm{R}})] \to N$ such that

$$V(\lambda_k(U_{\mathrm{L}})) = U_{\mathrm{L}}, \qquad V(\lambda_k(U_{\mathrm{R}})) = U_{\mathrm{R}}.$$

Note that centred k-rarefaction waves exist only when the k-characteristic family is genuinely nonlinear. Indeed, if the k-characteristic family were linearly degenerate, then differentiating (1.2.4) with respect to ξ would yield the contradiction $0 = 1$.

The remaining two elementary waves we consider are shocks and characteristic discontinuities, both of which have discontinuities. Suppose that U is a piecewise smooth weak solution of (1.2.1) in a bounded domain $\mathcal{D} \subset (0,\infty)\times\mathbf{R}$, having a jump discontinuity across a C^1-curve $\mathcal{S} \subset \mathcal{D}$. Applying the Gauss–Green formula to the definition of a weak solution results in the Rankine–Hugoniot condition

$$[U]n_t + [F(U)]n_x = 0\,, \tag{1.2.5}$$

where $\boldsymbol{n} = (n_t, n_x)$ is the unit normal to the curve $\mathcal{S}$, and $[w]$ denotes the change in a quantity w across $\mathcal{S}$. In the case that $\mathcal{S} = \{(t,x) \in \mathcal{D} : x = x(t)\}$, the condition (1.2.5) becomes

$$s[U] = [F(U)]\,, \qquad s = \frac{\mathrm{d}x}{\mathrm{d}t}\,. \tag{1.2.6}$$

Weak solutions of (1.2.1) are generally non-unique, and it is necessary to impose extra admissibility criteria to select physically relevant solutions. One choice of admissibility criterion is the Lax E-condition, originally introduced by Lax [53], which places restrictions on the permitted values of s in (1.2.6).

Definition 1.6 (Shockwaves)**.** We say the states U_{L} and U_{R} are connected by a k-shock if they satisfy (1.2.6) and the Lax E-condition

$$\lambda_k(U_{\mathrm{R}}) < s < \lambda_k(U_{\mathrm{L}})\,, \qquad \lambda_{k-1}(U_{\mathrm{L}}) < s < \lambda_{k+1}(U_{\mathrm{R}})\,.$$

Note that, when the system (1.2.1) is both strictly hyperbolic and genuinely nonlinear, the entropy inequality (1.2.2) is equivalent to the Lax E-condition, at least in the regime of weak shocks, though see [27, Theorem 8.5.3] for a precise statement of this fact. However, when characteristic families may vary between genuinely nonlinear and linearly degenerate, the Lax E-condition is no longer sufficient to select a unique solution to the Riemann problem, and instead the more stringent Liu E-condition must be imposed; see [27]. Another choice of entropy admissibility criteria is given by the maximal entropy rate admissibility condition; see [28, 27].

Like was the case for centred k-simple waves, k-shocks exist only when the k-characteristic family is genuinely nonlinear. When the k-characteristic family is linearly degenerate, we obtain the final type of elementary wave.

Definition 1.7 (Characteristic discontinuities)**.** We say the states U_{L} and U_{R} are connected by a k-characteristic discontinuity if they satisfy (1.2.6) and

$$\lambda_k(U_{\mathrm{R}}) = s = \lambda_k(U_{\mathrm{L}})\,.$$

1.2.2.2 Unique solvability of the Riemann problem

Fix any state $U_{\mathrm{L}} \in N$. The locus of states U_{R} that may be connected to U_{L} by an elementary k-wave forms a $C^{2,1}$-curve in the state space, called the forward k-wave curve. When the k-characteristic family is genuinely nonlinear near U_{L}, the forward k-wave curve consists of states which are connected to U_{L} by either a centred k-rarefaction wave, or a k-shock; whilst if the k-characteristic family is linearly degenerate near U_{L}, it consists of states which are connected to U_{L} by a k-characteristic discontinuity.

For hyperbolic systems, the forward k-wave curves allow us to construct a local coordinate system near any state U_{L}, which leads to the following existence and uniqueness theorem, originally due to Lax [53].

Theorem 1.8 (Lax [53])**.** *Suppose that system* (1.2.1) *is hyperbolic and that each characteristic field is either genuinely nonlinear or linearly degenerate in* N*. For* $U_{\mathrm{L}} \in N$*, there is a neighbourhood* $\widetilde{N} \subset N$ *of* U_{L} *such that, if* $U_{\mathrm{R}} \in \widetilde{N}$*, the Riemann problem* (1.2.1), (1.2.3) *has a unique global self-similar solution taking values in* $\widetilde{N}$*. This solution consists of at most* $n+1$ *constant states* $U_{\mathrm{L}} = U_0, U_1, \ldots, U_{n-1}, U_n = U_{\mathrm{R}}$*, where each state* U_k *lies on the forward* k*-wave curve of* U_{k-1}*.*

The restriction that $U_{\mathrm{R}} \in \widetilde{N}$ reflects the fact that, in general, the k-wave curves exist only locally around each state U_{L}. In the case of the full system of Euler equations (1.1.1) for a polytropic gas (1.1.2) in one space dimension, the k-wave curves are globally defined, and Theorem 1.8 can be improved to the following.

Theorem 1.9 ([26]). *Suppose* $U_{\mathrm{L}} = (u_{\mathrm{L}}, p_{\mathrm{L}}, \rho_{\mathrm{L}})$ *and* $U_{\mathrm{R}} = (u_{\mathrm{R}}, p_{\mathrm{R}}, \rho_{\mathrm{R}})$ *are any two states with* $p_{\mathrm{L}}\,,\ p_{\mathrm{R}}\,,\ \rho_{\mathrm{L}}\,,\ \rho_{\mathrm{R}} > 0\,.$ *There exists a unique solution to the Riemann problem* (1.1.1), (1.1.2), (1.2.3) *in the class of shocks, centred rarefaction waves and characteristic discontinuities if and only if*

$$u_{\mathrm{R}} - u_{\mathrm{L}} < \frac{2}{\gamma - 1}(c_{\mathrm{L}} + c_{\mathrm{R}})\,, \qquad c \coloneqq \sqrt{\frac{\gamma p}{\rho}}\,.$$

Otherwise a vacuum is present in the solution.

Allowing for the presence of vacuum states, the Riemann problem for the full system of Euler equations (1.1.1) is always solvable, and the exact qualitative form of the solution is known; see [80, Corollary 18.7].

1.2.3 The initial value problem for small BV initial data.

The unique solvability of the Riemann problem facilitates the construction of global solutions to the initial value problem (1.2.1) via a process known as the Glimm scheme, or the random choice method [41]. However, in general the global existence is conditional on the smallness of the initial data in the total variation sense.

Definition 1.10 (Functions of bounded variation). Let $\Omega \subset \mathbf{R}^n$ be open. A function $v \in L^1(\Omega)$ is of bounded variation on Ω if the distributional derivative Dv may be identified with a finite Radon measure $\mathcal{M}$, that is for any $\zeta \in C_c^\infty(\Omega; \mathbf{R}^n)$

$$\int_\Omega v \,\mathrm{div}\zeta \,\mathrm{d}x + \int_\Omega \zeta \,\mathrm{d}\mathcal{M} = 0\,.$$

The set of all functions of bounded variation on Ω is denoted by $\mathrm{BV}(\Omega)$. For $v \in \mathrm{BV}(\Omega)$, the total variation of v is given by

$$\mathrm{TV}(v) \coloneqq \sup\Big\{ \int_\Omega v \,\mathrm{div}\zeta \,\mathrm{d}x : \zeta \in C_c^\infty(\Omega; \mathbf{R}^n)\,, |\zeta| \le 1 \Big\}\,.$$

A vector-valued function $U \in L^1(\Omega; \mathbf{R}^n)$ is of bounded variation on Ω if each component is of bounded variation on Ω, and the total variation is given by the sum of the total

variation of each component.

Remark 1.11. The space of BV functions is a natural setting for solutions to conservation laws having shock discontinuities. Indeed, it is a general property of BV functions that their set of jump discontinuities are confined to rectifiable sets of codimension one; see [37] for a more precise discussion.

We may now state the following global existence theorem for solutions of the initial value problem (1.2.1), originally due to Glimm [41].

Theorem 1.12 (Glimm [41])**.** *Suppose the system* (1.2.1) *is strictly hyperbolic in a ball N centred at the origin. There exist positive constants δ_0 and δ_1 such that, whenever*

$$\|U_0\|_{L^\infty(\mathbf{R})} < \delta_0\,, \qquad \mathrm{TV}(U_0) < \delta_1\,,$$

there exists a solution $U : [0,\infty) \times \mathbf{R} \to N$ of (1.2.1)*, which is a function of locally bounded variation, and satisfies the entropy inequality* (1.2.2) *for any convex entropy-entropy flux pair (η, q). Furthermore, for any $t \in [0,\infty)$, $U(t,\cdot)$ is a function of bounded variation and*

$$\|U(t,\cdot)\|_{L^\infty(\mathbf{R})} \le c_0 \|U_0\|_{L^\infty(\mathbf{R})}\,, \qquad \mathrm{TV}(U(t,\cdot)) \le c_1 \mathrm{TV}(U_0)\,,$$
$$\int_{\mathbf{R}} |U(t_1,x) - U(t_2,x)|\,\mathrm{d}x \le c_2 |t_1 - t_2|\,\mathrm{TV}(U_0)\,,$$

where the positive constants $c_0\,, c_1$ and c_2 depend only on the flux F.

1.2.4 Uniqueness and stability of solutions to 1-D conservation laws

Previously, we introduced the notion of entropy solutions in an attempt to rectify the generic non-uniqueness of weak solutions to 1-D conservation laws. Under the same hypotheses as Theorem 1.12, it was shown by Bressan-Crasta-Piccoli [6] that the random choice method (Glimm scheme) produces a unique entropy solution that

has Lipschitz dependence on the initial data with respect to the L^1-norm. In the more general class of entropy solutions, without reference to any particular constructive procedure, the question of uniqueness has proved much harder, except in the scalar case, whereby the classical well-posedness theories of Oleĭnik [70] and Kružkov [51] are quite satisfactory. For 1-D systems, most results in this direction require further conditions on the regularity of the weak solutions, such as the *tame oscillation condition* [7] or the *bounded variation on space-like curves condition* [9]. Recently, promising results have been obtained under a weaker *strong normal trace condition* [15].

A related fundamental open question concerns the uniqueness and stability of solutions under passing to the inviscid limit from the Navier–Stokes system to the Euler equations. In this direction, several important results have been obtained within the past two years, including [48, 47, 15, 49, 14].

1.3 Systems of conservation laws in multiple space dimensions

Unlike the extensive theory that has been developed for systems of conservation laws in one space dimension, the theory in multiple space dimensions remains stark. Whilst there are similarities to the one-dimensional case—the local existence of classical solutions was established by Majda [65], and the generic breakdown of classical solutions in finite time by Majda and Sideris [65, 78, 79]—the reality is that multi-dimensional systems of conservation laws harbour untold difficulties. Undoubtedly, the existence of global weak solutions remains the ultimate open question in the field.

For definiteness, throughout this section we focus on the full system of Euler equations (1.1.1), which are the prototypical example of a hyperbolic system of conservation laws. The physical significance of the Euler equations as a model for gas dynamics makes them an attractive subject of research, since one reasonably expects the existence of physically meaningful solutions.

1.3.1 The Riemann problem in four quadrants

We begin with a description of the Riemann problem in two space dimensions. However, as soon as we begin we face an obvious dilemma: unlike the Riemann problem in one dimension, in two dimensions there is no distinguished way to specify initial data that is piecewise constant and invariant under scaling. A natural approach and generalisation of the 1-D case, studied extensively through the 1990s in [83, 72, 73, 12, 13, 55], is to specify constant initial data in each of the four quadrants of the plane. The case of three sectors has been studied in [57], but is generally not as interesting due the severe restrictions upon the initial data in three sectors; *cf.* there does not exist a pure triple shock configuration in the plane [75]. Of course, more generally, one may consider dividing the plane arbitrarily into n sectors, or even specify initial data on the unit circle, although this is certainly an over-ambitious level of generality for our current analytic ability.

Returning to the 2-D Riemann problem in four quadrants, it is customary to place the following restriction on the initial data: *In the far-field region, adjacent states are connected by exactly one planar elementary wave, either: a forward/backward shock; a forward/backward rarefaction wave; or a positive/negative characteristic discontinuity (slip line).* Under this restriction, the 2-D Riemann problem in four quadrants has been classified into nineteen genuinely different cases [55]. However, we remark that the role of entropy waves (characteristic discontinuities of the full system of Euler equations for which only the entropy and density are discontinuous, whilst the pressure and velocity are continuous) was not made clear within these studies.

The analysis proceeds at first by exhaustively verifying the elementary wave relations for every possible combination of four elementary waves up to symmetries. It is found that certain combinations of waves are not permissible, since inconsistencies within the wave relations may be derived. For example, borrowing the somewhat standard notation from [72], the particular case $\overrightarrow{R}_{21}\overleftarrow{S}_{32}\overrightarrow{R}_{34}\overleftarrow{S}_{41}$, comprising two eastward-travelling rarefaction waves and two southward-travelling shocks, is found to be impossible.

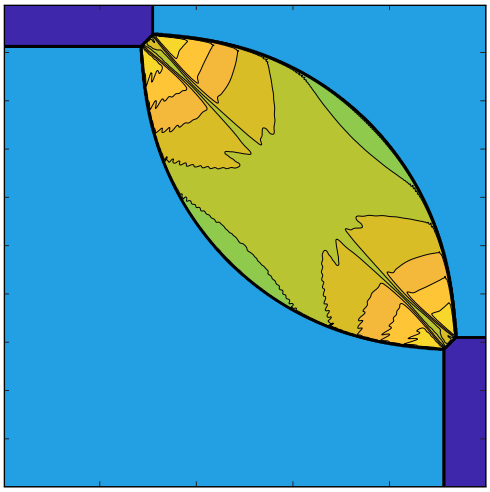
For each remaining configuration, the structure of the solution is analysed via the method of characteristics, and numerical solutions are presented for generic choices of initial data. Density contour plots for the numerical solutions corresponding to four different choices of initial data involving only shocks and slip lines are shown in Figure 1.2. In particular, Figure 1.2a shows the case of two pairs of shocks travelling away from a central interaction region, which is more or less equivalent to the 2-D Riemann problem under consideration in §4.

1.3.2 Analytic difficulties

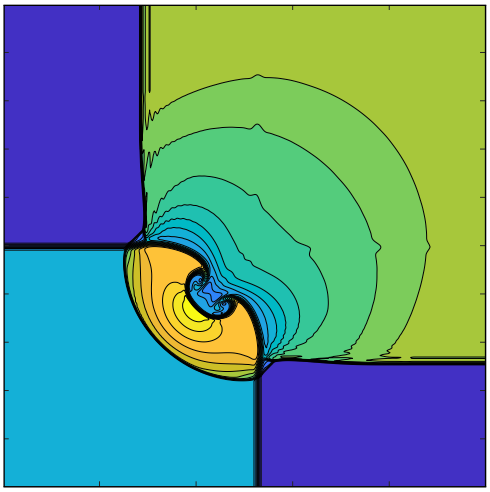
We reflect on some difficulties associated with the analysis of multi-dimensional conservation laws and Riemann problems, especially in the context of the full system of Euler equations (1.1.1). Since the Riemann problem is invariant under scaling, it is traditional to work in the self-similar formulation; see [20, §18.3] and §2. Within the self-similar formulation, in two dimensions the full system of Euler equations (1.1.1) becomes a system of two transport-type equations, corresponding to the two linearly degenerate characteristic families; and two nonlinear equations of mixed hyperbolic-elliptic type, corresponding to the two genuinely nonlinear characteristic families that exist within each hyperbolic region. At present, it is the transport-type equations that pose the greatest obstruction to analysis. In particular, due to the presence of rough coefficients and stagnation points, obtaining appropriate *a priori* estimates on the pressure and vorticity of solutions is not currently possible. There is strong evidence to suggest that both the pressure and vorticity may have singularities [75].

On the other hand, nonlinear equations of mixed hyperbolic-elliptic type appear more amenable to current analysis. Indeed, in the context of potential flow, where the governing equation is of mixed hyperbolic-elliptic type, the transonic free boundary formulation developed in [20] has overcome the essential difficulties faced in a number of shock reflection problems [20, 1]. We take the same approach for our own work on the interaction of four shockwaves; see §4 and [17].

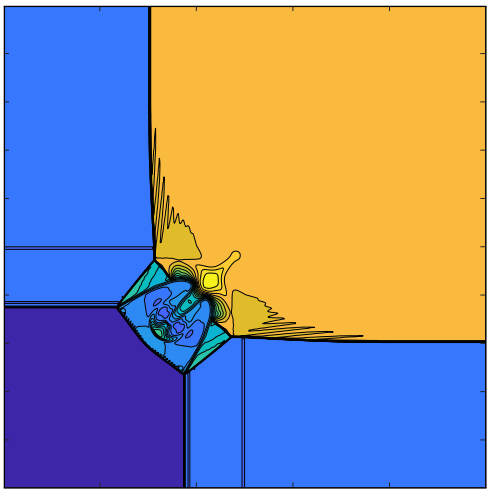
Another fundamental challenge is the determination of appropriate M-D admissibility/selection criteria. Indeed, in the past decade, the convex integration and



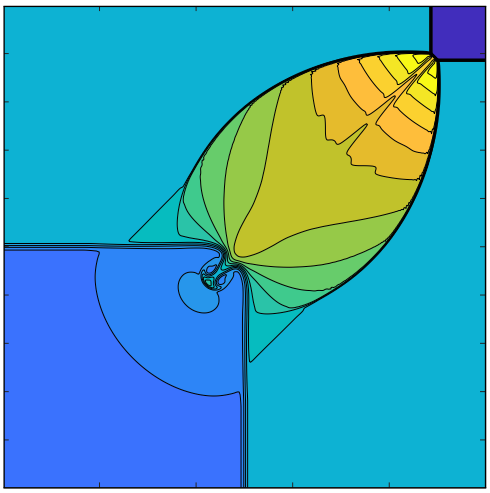
(a) $\overleftarrow{S}_{21}\,\overrightarrow{S}_{32}\,\overrightarrow{S}_{34}\,\overleftarrow{S}_{41}$. Two pairs of shocks moving apart, leaving behind the central interaction region. Configuration 4 in [55].



(b) $\overleftarrow{S}_{21}J^{+}_{32}J^{+}_{34}\overleftarrow{S}_{41}$. Two shocks (north/east) moving towards two slip lines (south/west). Configuration 11 in [55].



(c) $\overleftarrow{S}_{21}\,\overleftarrow{S}_{32}\,\overleftarrow{S}_{34}\,\overleftarrow{S}_{41}$. Four shocks moving westward/southward, with a complicated interaction pattern. Configuration 3 in [55].



(d) $\overrightarrow{S}_{21}J^{+}_{32}J^{+}_{34}\overrightarrow{S}_{41}$. Two shocks (north/east) moving away from two stationary slip lines (south/west). Configuration 12 in [55].

Figure 1.2: Density contour plots for the numerical solutions to four 2-D Riemann problem test cases involving the interaction of shocks and slip lines only. Plots generated via a monotonic upstream-centred scheme for conservation laws (MUSCL); see [56]. Blue corresponds to lower densities, and yellow higher densities.

Baire category theory methods of Gromov [42], and later Muller-Sverak [67], have bestowed a catastrophic non-uniqueness upon solutions of the incompressible Euler equations; see De Lellis-Székelyhidi [30, 29]. Then, in a series of works [25, 66, 50], the convex integration method for the incompressible Euler system was adapted to the compressible setting to construct infinitely many solutions to the M-D Riemann problem whenever the initial data contains a planar shock. In fact, these solutions, known as "wild" solutions, actually satisfy the entropy inequality (1.2.2), and even the entropy rate admissibility criterion proposed by Dafermos [28]. It is not yet known how to restore uniqueness, though perhaps vanishing viscosity limits will prove key, much like the 1-D case considered by Kružkov [51].

1.3.3 Recent progress towards analytical results

As discussed above, there are numerous difficulties to face when solving the M-D Riemann problem for the Euler equations. To make progress, it has been necessary to focus on specific configurations of initial data, and to introduce various simplifying assumptions, for example via flux-splitting. We will briefly discuss a number of recent results in this direction.

1.3.3.1 The equations of potential flow

Under the Hodge–Helmholtz decomposition, we may rewrite the flow velocity as

$$\boldsymbol{u} = \nabla\Phi + \boldsymbol{W}\,, \qquad \text{where } \operatorname{div}\boldsymbol{W} = 0\,,$$

in terms of the velocity potential Φ; and a vector field $\boldsymbol{W}$. For smooth solutions in two space dimensions, the vorticity $\boldsymbol{\omega} \coloneqq \nabla \times \boldsymbol{u} \equiv \nabla \times \boldsymbol{W}$ satisfies the linear transport equation

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}\operatorname{div}\boldsymbol{u} = 0\,,$$

and hence irrotational initial data generates irrotational flow, so long as the flow remains smooth. This observation demonstrates that the equations of potential flow, coupled with the incompressible-type Euler equations for $\boldsymbol{W}$, and the transport equation for S, are the core parts of the full system of Euler equations. Moreover, the equations of potential flow are actually *exact* in certain important regions of the flow; see [19]. An attractive quality of the potential flow formulation is that the only discontinuities it supports are shocks, in particular characteristic discontinuities like vortex sheets are not possible.

The von Neumann reflection-diffraction problem. In the monograph [20], a fundamental shock reflection-diffraction problem in potential flow is considered as an example of a 2-D Riemann problem. The Riemann initial data consists of two constant states separated by a vertical shock

$$(\rho, \Phi)(0, \boldsymbol{x}) = \begin{cases} (\rho_0, 0)\,, & \text{for } \boldsymbol{x} \in (\mathbf{R}^2 \cap \{x > 0\}) \setminus W\,, \\ (\rho_1, (u_1, 0) \cdot \boldsymbol{x})\,, & \text{for } \boldsymbol{x} \in \mathbf{R}^2 \cap \{x < 0\}\,, \end{cases}$$

where $W \coloneqq \{\boldsymbol{x} \in \mathbf{R}^2 : -\theta_{\mathrm{w}} < \theta < \theta_{\mathrm{w}}\}$ represents a solid wedge with wedge half-angle $\theta_{\mathrm{w}} \in (0, \frac{\pi}{2})$. Additionally the slip boundary condition $\boldsymbol{n} \cdot \nabla\Phi = 0$ on ∂W is imposed, which represents that no flow can pass through the solid wedge boundary. This initial boundary value problem is known as the von Neumann reflection-diffraction problem, which represents the physical phenomenon of a planar shock impinging upon a cone head-on.

By reformulating the above initial boundary value problem as a free boundary problem, the existence of a global solution having particular structure is obtained when $\theta_{\mathrm{w}} \in (\theta^{\mathrm{d}}, \frac{\pi}{2})$, where θ^{d} depends only on the initial data, and is known as the detachment angle. In particular, a long-standing open conjecture due to von Neumann is resolved: that $\theta_{\mathrm{w}} \in (\theta^{\mathrm{d}}, \frac{\pi}{2})$ is both necessary and sufficient for the existence of a regular reflection-diffraction configuration. The simplified structure of the solution is shown in Figure 1.3 (left).

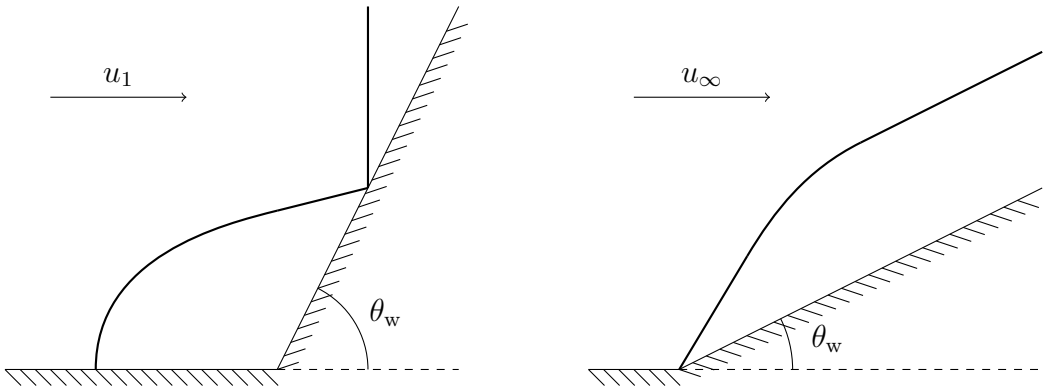


Figure 1.3: The von Neumann shock reflection-diffraction configuration (left), and the Prandtl–Meyer shock reflection configuration (right).

The Prandtl–Meyer reflection problem. Another fundamental shock reflection problem is considered in the monograph [1]. The Riemann initial data consists of a single constant state

$$(\rho, \Phi)(0, \boldsymbol{x}) = (1, (u_\infty, 0) \cdot \boldsymbol{x}) \,, \qquad \text{for } \boldsymbol{x} \in \mathbf{R}^2 \setminus W \,, \tag{1.3.1}$$

for a solid wedge W as above, along with the slip boundary condition on ∂W. It is assumed that $u_\infty > 1$, so that the incoming flow is supersonic. Since the initial data does not satisfy the slip boundary condition, it is expected that a boundary layer will be formed starting at $t = 0$, and correspondingly a reflected shock will appear. This initial boundary value problem is known as the Prandtl–Meyer shock reflection problem, which represents the physical phenomenon of a steady supersonic flow impinging upon a cone.

By reformulating the above initial boundary value problem as a free boundary problem, the existence of a global solution having particular structure is obtained when $\theta_{\rm w} \in [0, \theta^{\rm d})$, where once again $\theta^{\rm d}$ depends only on the initial data and is known as the detachment angle. In particular, the solution is carefully constructed to converge as $t \to \infty$ towards a steady Prandtl–Meyer weak reflection: that is for any $R > 0$

$$\lim_{t\to\infty} \left(\|\nabla\Phi(t,\cdot) - \nabla\bar{\Phi}^{\rm wk}\|_{L^1(B_R(0)\setminus W)} + \|\rho(t,\cdot) - \bar{\rho}^{\rm wk}\|_{L^1(B_R(0)\setminus W)} \right) = 0 \,, \tag{1.3.2}$$

where $(\bar{\rho}^{\rm wk}, \bar{\Phi}^{\rm wk})$ is the steady Prandtl–Meyer weak reflection corresponding to $\theta_{\rm w}$,

see [1, Definition 2.3 and Lemma 2.4]. The simplified structure of the solution is shown in Figure 1.3 (right).

In the same monograph, it is shown that no such solution exists when the steady Prandtl–Meyer weak reflection is replaced by the strong reflection. In this sense, a long-standing open conjecture of Prandtl is resolved: that only the weak shock reflection is physically admissible.

Loss of regularity. Despite the successes of the potential flow model in solving the above shock reflection problems, in certain innocuous physical situations it has been shown that the potential flow equations do not admit solutions due to an inherent loss of regularity. Though we will not discuss this issue here, it should always be kept in mind that the existence of solutions is not guaranteed. For reference, the best examples of this phenomenon can be found with the Lighthill shock diffraction problem [21], and the shock reflection-diffraction by a non-symmetric wedge [44].

1.3.3.2 The pressure gradient system

We may write the full system of Euler equations (1.1.1) in the form

$$\partial_t U + \operatorname{div}\big(F(U)\big) = 0\,,$$

for $U \coloneqq (\rho, \rho\boldsymbol{u}, \rho E)$, and $F(U) \coloneqq (\rho\boldsymbol{u}, \rho\boldsymbol{u}\otimes\boldsymbol{u} + pI, (\rho E + p)\boldsymbol{u})$. It is natural to split the flux F into two parts $F = F_1 + F_2$ with

$$F_1(U) \coloneqq (\rho\boldsymbol{u}, \rho\boldsymbol{u}\otimes\boldsymbol{u}, \rho E\boldsymbol{u})\,, \qquad F_2(U) \coloneqq (0, pI, p\boldsymbol{u})\,,$$

which represents the two physical mechanisms within the fluid motion: inertia and pressure difference. The Euler equations (1.1.1) may then be split into two sub-systems of conservation laws

$$\partial_t U + \operatorname{div}\big(F_1(U)\big) = 0\,, \qquad \partial_t U + \operatorname{div}\big(F_2(U)\big) = 0\,,$$

which are called the pressureless Euler system, and the pressure gradient system respectively.

In [24], the 2-D Riemann problem is considered for the pressure gradient system, for initial data consisting of a pair of adjacent shocks, and a pair of adjacent slip lines, corresponding to Figure 1.2b. Here the existence of a global solution is determined by reformulating the Riemann problem as a free boundary problem involving transonic shocks, similarly to [20, 1].

1.3.3.3 The system of isentropic Euler equations

In the context of the system of isentropic Euler equations (1.1.5), we present two results of note.

The expansion problem. One approach is to restrict attention to particularly simple classes of initial data, for which global solutions may be constructed more easily. A good example is the expansion problem of a wedge of gas into a vacuum, where the Riemann initial data is specified as

$$(\rho, u, v)(0, \boldsymbol{x}) = \begin{cases} (\rho_0, 0, 0)\,, & \text{for } \theta \in (-\theta_{\mathrm{w}}, \theta_{\mathrm{w}})\,, \\ \text{vacuum}, & \text{otherwise}, \end{cases}$$

where $\rho_0 > 0$ is a constant; $\theta \in (-\pi, \pi]$ is the plane polar angle of the point $\boldsymbol{x} \in \mathbf{R}^2$; and θ_{w} is the wedge half-angle.

The existence of solutions to this initial value problem has been fully determined in the context of the isentropic Euler equations (1.1.5) having a polytropic pressure law (1.1.4) for all $\gamma \geq 1$, and $\theta_{\mathrm{w}} \in (0, \frac{\pi}{2})$, see [58, 45, 84, 39]. These papers make use of special characteristic decompositions for the isentropic Euler equations, which are used to construct invariant regions and obtain *a priori* estimates on solutions.

In a similar setting, the Riemann problem for the interaction of four rarefaction waves was solved by characteristic analysis under the simplifying assumption of bi-symmetric initial data; see [59] for details.

The Chaplygin gas. The polytropic pressure law is ubiquitous throughout gas dynamics for its combination of apparent simplicity and physical relevance to the ideal gas, though other pressure laws can be of interest. A more exotic pressure law, called the Chaplygin gas, is given by $p(\rho) = a^2\big(\frac{1}{\rho_0} - \frac{1}{\rho}\big)$, for some constants $a > 0$, and $\rho_0 > 0$. The Chaplygin gas is rumoured to be a model for "dark fluids" in cosmology, though mathematically, it provides a good approximation of a real gas in a neighbourhood where $\frac{\partial(\rho c)}{\partial \rho}$ changes sign.

In [74], a Riemann problem for the interaction of four shockwaves was considered in the context of the Chaplygin gas, much like the problem considered in §4 in the context of potential flow. It is shown that the flow remains isentropic for all time, and that the position of the shocks are *a priori* known, since they are the characteristic waves corresponding to a linearly degenerate family. For these reasons, the problem reduces to solving a non-uniformly elliptic equation in an *a priori* fixed domain; see [74].

1.4 Structure of thesis and main contributions

In the above, we have presented a brief introduction to the subject of mathematical gas dynamics, including an overview of the theory of systems of conservation laws in one and multi-dimensions, and the associated challenges that we may face. For further details on any of these topics, we advise the reader at first to consult [27], and the endless supply of references contained therein. We turn our attention to the remainder of this thesis.

In Chapter 2, we present the mathematical preliminaries for two-dimensional potential flow, including the definitions of weak solutions and shockwaves. The self-similar formulation, which forms the basis for much of our subsequent analysis, is also introduced. The main contribution of this chapter is contained in §2.3, where, in analogy to [12, 72], the 2-D Riemann problem in four quadrants is classified into five (or six) genuinely different configurations.

In Chapter 3, the detailed properties of the shock polar curve for two-dimensional steady potential flow are considered. We prove that the steady critical angles—the

detachment and sonic angles—are strictly increasing functions of the upstream Mach number. Subsequently, we are able to prove Theorem 3.6, which provides an analogue of [11, 76] in the context of potential flow for the sharp transition criterion between the regular reflection and Mach reflection of a planar shock.

In Chapter 4, we consider the 2-D Riemann problem for the interaction of four shockwaves in potential flow, based on the preprint [17]. We obtain Theorem 4.9, the most significant result of this thesis, for the existence of admissible solutions to the four-shock interaction problem. The proof of Theorem 4.9 is extensive, involving many detailed *a priori* estimates and a sophisticated iteration procedure.

Finally, in Appendix A, we present a number of detailed calculations that are necessary within the proof of Theorem 4.9, whilst in Appendix B, for the convenience of the reader, we collect a number of known results that have been referred to throughout.

Chapter 2

The Equations of Potential Flow

In this chapter we present the mathematical preliminaries for the equations of potential flow in two dimensions, and give a novel classification of the 2-D Riemann problems in four quadrants for potential flow.

2.1 Time-dependent potential flow

The system of Euler equations for potential flow are a model for irrotational isentropic flow, which are formally derivable from the full system of Euler equations (1.1.1) under the assumptions of isentropic and irrotational initial data, as explained in §1.1. They consist of the conservation law of mass, and the Bernoulli law

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho \nabla \Phi\big) = 0\,, \\ \partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + h(\rho) = B\,, \end{cases} \tag{2.1.1}$$

where ρ is the density; Φ is the velocity potential satisfying $\nabla \Phi \equiv \boldsymbol{u}$; the Bernoulli constant B is determined by the far field flow; and $h(\rho)$ is the enthalpy given by

$$h(\rho) = \int_1^\rho \frac{p'(s)}{s}\,\mathrm{d}s\,, \tag{2.1.2}$$

for the pressure $p(\rho)$.

The system (2.1.1) is closed by specifying an equation of state for the pressure.

We consider exclusively the polytropic equation of state

$$p(\rho) = \kappa\rho^{\gamma}\,, \tag{2.1.3}$$

for constants $\gamma \geq 1$, and $\kappa > 0$. We may fix $\kappa = \frac{1}{\gamma}$ through scaling. The enthalpy (2.1.2) is then given by

$$h(\rho) = \begin{cases} \frac{\rho^{\gamma-1}-1}{\gamma-1}\,, & \text{for } \gamma > 1\,, \\ \log\rho\,, & \text{for } \gamma = 1\,. \end{cases} \tag{2.1.4}$$

We observe that $h : (0,\infty) \to \mathbf{R}$ is injective, and thus we may express

$$\rho = \rho(\partial_t\Phi, |\nabla\Phi|) = h^{-1}\big(B - (\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2)\big)\,. \tag{2.1.5}$$

The system (2.1.1) may then be rewritten as the following second order nonlinear conservation law

$$\partial_t\rho(\partial_t\Phi, |\nabla\Phi|) + \operatorname{div}\big(\rho(\partial_t\Phi, |\nabla\Phi|)\nabla\Phi\big) = 0\,, \tag{2.1.6}$$

with $\rho(\partial_t\Phi, |\nabla\Phi|)$ given by (2.1.5).

Definition 2.1 (Constant state)**.** A solution Φ of (2.1.6) is called a constant state if the flow velocity $\boldsymbol{u} \equiv \nabla\Phi$, and the corresponding density $\rho(\partial_t\Phi, |\nabla\Phi|) > 0$ given by (2.1.5), are constant.

Observe that Φ is a constant state if and only if

$$\Phi(t, \boldsymbol{x}) = kt + (u, v)\cdot\boldsymbol{x} + \text{const.}\,, \tag{2.1.7}$$

for some constants (k, u, v). Furthermore the arbitrary additive constant has no effect on the resulting flow, and we may simply discard it.

2.1.1 Weak solutions of the potential flow equation

We consider weak solutions to the potential flow equation (2.1.6).

Definition 2.2 (Weak solutions)**.** A function $\Phi \in W^{1,\infty}_{\mathrm{loc}}([0,\infty)\times\mathbf{R}^2)$ is called a weak solution of the potential flow equation (2.1.6) in a domain $\mathcal{D}\subset[0,\infty)\times\mathbf{R}^2$ if Φ satisfies the following properties:

(i) $B-(\partial_t\Phi+\frac{1}{2}|\nabla\Phi|^2)\geq h(0^+)$ a.e. in $\mathcal{D}$;

(ii) $\big(\rho(\partial_t\Phi,|\nabla\Phi|),\rho(\partial_t\Phi,|\nabla\Phi|)|\nabla\Phi|\big)\in(L^1_{\mathrm{loc}}(\mathcal{D}))^2$;

(iii) $\int_{\mathcal{D}}\big(\rho(\partial_t\Phi,|\nabla\Phi|)\partial_t\zeta+\rho(\partial_t\Phi,|\nabla\Phi|)\nabla\Phi\cdot\nabla\zeta\big)\,\mathrm{d}t\mathrm{d}\boldsymbol{x}=0$, for every $\zeta\in C^\infty_c(\mathcal{D})$.

The requirement $\Phi\in W^{1,\infty}_{\mathrm{loc}}(\mathcal{D})$ implies that $\Phi\in C^0(\mathcal{D})$, though the derivatives of Φ need not be continuous. In particular, for a piecewise smooth weak solution of (2.1.6) with $(\partial_t\Phi,\nabla\Phi)$ having a jump discontinuity across an oriented surface $\mathcal{S}$ with time-space normal $\boldsymbol{n}=(n_t,\boldsymbol{n}_{\boldsymbol{x}})$, the requirement of being a weak solution in the sense of Definition 2.2 yields the Rankine–Hugoniot jump condition on $\mathcal{S}$

$$[\rho(\partial_t\Phi,|\nabla\Phi|)]n_t+[\rho(\partial_t\Phi,|\nabla\Phi|)\,\nabla\Phi]\cdot\boldsymbol{n}_{\boldsymbol{x}}=0\,, \tag{2.1.8}$$

where $[w]$ denotes the change in a quantity w across the oriented surface $\mathcal{S}$.

Definition 2.3 (Physical entropy condition)**.** For Φ as above, a discontinuity of $(\partial_t\Phi,\nabla\Phi)$ satisfying the Rankine–Hugoniot conditions on $\mathcal{S}$ is called a shock if furthermore Φ satisfies the physical entropy condition: the density $\rho(\partial_t\Phi,|\nabla\Phi|)$ increases across $\mathcal{S}$ in the relative flow direction with respect to $\mathcal{S}$.

Note that the relative flow direction across $\mathcal{S}$ is given by the sign of $\nabla\Phi\cdot\boldsymbol{n}_{\boldsymbol{x}}+n_t$, which has the same sign evaluated on either side of $\mathcal{S}$ by the Rankine–Hugoniot condition (2.1.8).

2.1.2 Planar shocks

Let us consider the most simple 2-D flow containing a shock

$$\Phi = \begin{cases} \Phi^- , & \text{for } \boldsymbol{x}\cdot\boldsymbol{n}_{\boldsymbol{x}} < st\,, \\ \Phi^+ , & \text{for } \boldsymbol{x}\cdot\boldsymbol{n}_{\boldsymbol{x}} > st\,, \end{cases} \tag{2.1.9}$$

with

$$\Phi^{\pm}(t,\boldsymbol{x}) \coloneqq k^{\pm}t + (u^{\pm},v^{\pm})\cdot\boldsymbol{x}\,,$$

which represents two constant states connected by a planar shock, having unit normal $\boldsymbol{n}_{\boldsymbol{x}}$ oriented from $(-)$ towards $(+)$; and propagating with normal speed $s\in\mathbf{R}$ in the direction $\boldsymbol{n}_{\boldsymbol{x}}$. Writing $[w]\coloneqq w^+ - w^-$, the continuity of Φ across the shock implies

$$[k] + s[\boldsymbol{u}]\cdot\boldsymbol{n}_{\boldsymbol{x}} = 0\,, \qquad [\boldsymbol{u}]\cdot\boldsymbol{t}_{\boldsymbol{x}} = 0\,, \tag{2.1.10}$$

where $\boldsymbol{t}_{\boldsymbol{x}}\coloneqq\boldsymbol{n}_{\boldsymbol{x}}^{\perp}$, whilst the Rankine–Hugoniot condition (2.1.8) with $\boldsymbol{n}=(-s,\boldsymbol{n}_{\boldsymbol{x}})$ yields

$$s[\rho] = [\rho\boldsymbol{u}]\cdot\boldsymbol{n}_{\boldsymbol{x}}\,, \qquad \text{or equivalently} \qquad [\rho(\boldsymbol{u}\cdot\boldsymbol{n}_{\boldsymbol{x}} - s)] = 0\,, \tag{2.1.11}$$

for $\rho^{\pm}\coloneqq\rho(\partial_t\Phi^{\pm},|\nabla\Phi^{\pm}|)$. The Bernoulli law reads

$$[k + \frac{1}{2}|\boldsymbol{u}|^2 + h(\rho)] = 0\,. \tag{2.1.12}$$

Observe that $[\rho]=0$ if and only if $\Phi^+=\Phi^-$ and so there is no shock in this case.

Combining (2.1.10)–(2.1.12) and observing the physical entropy condition gives

$$[u_n] = -\sqrt{\frac{2[\rho]^2[h(\rho)]}{[\rho^2]}}\,, \quad [u_t]=0\,, \quad [k] = -\frac{[\rho u_n][u_n]}{[\rho]}\,, \quad s=\frac{[\rho u_n]}{[\rho]}\,. \tag{2.1.13}$$

where we have abbreviated $u_n\coloneqq\boldsymbol{u}\cdot\boldsymbol{n}_{\boldsymbol{x}}$ and $u_t\coloneqq\boldsymbol{u}\cdot\boldsymbol{t}_{\boldsymbol{x}}$. Note that $[u_n]<0$ is a

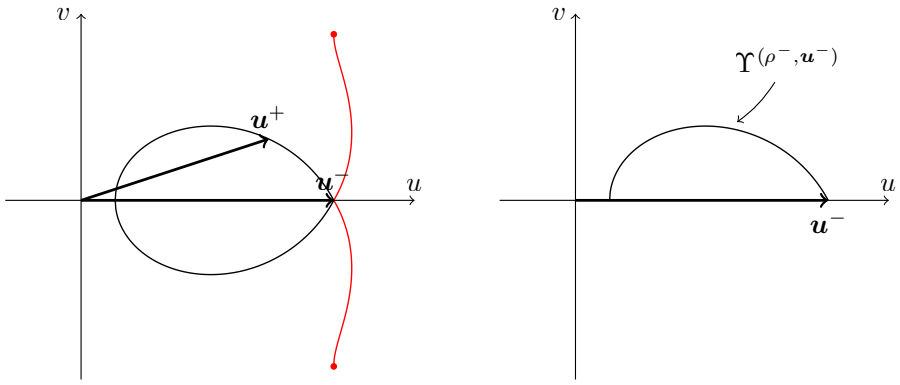


Figure 2.1: The full steady shock polar (left), and $\Upsilon^{(\rho^-,\boldsymbol{u}^-)}$ (right). Black: $\rho^+ > \rho^-$, Red: $\rho^+ < \rho^-$.

manifestation of the physical entropy condition. Indeed, by the Rankine–Hugoniot condition (2.1.11), the relative flow velocities with respect to $\mathcal{S}$, given by $u_n^\pm - s$, have the same sign. Suppose, $u_n^\pm - s$ is positive, then the physical entropy condition mandates that $\rho^+ > \rho^- > 0$, and thus the Rankine–Hugoniot condition implies $u_n^- - s > u_n^+ - s > 0$, and hence $[u_n] < 0$. The argument is similar when $u_n^\pm - s$ is negative.

2.1.3 The steady shock polar

The case $s = 0$ in (2.1.9) corresponds to a steady state flow in which the shock remains at a fixed location in space. In this case, the shock relations (2.1.13) imply $[k] = 0$, and thus the potential Φ may be chosen to be independent of time after shifting $B \mapsto B - k$. Upon fixing $(\boldsymbol{u}^-, \rho^-)$, and setting the Bernoulli constant $B := \frac{1}{2}|\boldsymbol{u}^-|^2 + h(\rho^-)$, the collection of all velocities $\boldsymbol{u}^+$ for which the shock relations (2.1.13) may be satisfied, by some choice of unit normal $\boldsymbol{n}_{\boldsymbol{x}}$, forms a continuous curve called the (full) steady shock polar. Figure 2.1 (left) depicts the full steady shock polar when $\boldsymbol{u}^- = (u^-, 0)$ is aligned with the positive x-axis, and $u^- > c^- := (\rho^-)^{\frac{\gamma-1}{2}}$.

The full steady shock polar consists of two distinct parts: the part coloured black, where $|\boldsymbol{u}^-| > |\boldsymbol{u}^+|$ and $\rho^- < \rho^+$, meaning that $(-)$ is the upstream state, and $(+)$ is the downstream state; and the part coloured red, where $|\boldsymbol{u}^-| < |\boldsymbol{u}^+|$ and $\rho^- > \rho^+$, meaning that $(-)$ is the downstream state, and $(+)$ is the upstream state.

We denote by $\Upsilon^{(\rho^-,\boldsymbol{u}^-)}$ the arc of the full steady shock polar both lying in the upper half-plane $\{v \geq 0\}$ and coloured black, as depicted in Figure 2.1 (right). In §3 we will investigate thoroughly the fundamental properties of $\Upsilon^{(\rho^-,\boldsymbol{u}^-)}$.

Similar steady shock polar curves are obtained when considering the system of isentropic Euler equations (1.1.5), or the full system of Euler equations (1.1.1). In these cases, explicit algebraic formulae are known, and their properties have been well documented; see [26].

2.2 Self-similar potential flow

In §2.3, §3.3, and §4, our main focus will be on 2-D Riemann problems which are invariant under the scaling

$$(t,\boldsymbol{x}) \mapsto (\lambda t, \lambda \boldsymbol{x})\,, \quad (\rho,\Phi)(t,\boldsymbol{x}) \mapsto (\rho, \frac{1}{\lambda}\Phi)(\lambda t, \lambda \boldsymbol{x}) \qquad \text{for any } \lambda > 0\,. \tag{2.2.1}$$

Therefore, it is expedient to introduce a self-similar formulation for potential flow. We seek self-similar solutions having the form

$$\rho(t,\boldsymbol{x}) = \rho(\boldsymbol{\xi})\,, \qquad \Phi(t,\boldsymbol{x}) = t\,\phi(\boldsymbol{\xi})\,,$$

for the self-similar coordinates $\boldsymbol{\xi} = (\xi,\eta) \coloneqq \frac{1}{t}\boldsymbol{x}$. By direct computation, the functions (ρ,ϕ) satisfy

$$\begin{cases} -\boldsymbol{\xi}\cdot D\rho + \operatorname{div}\big(\rho\, D\phi\big) = 0\,, \\ -\boldsymbol{\xi}\cdot D\phi + \phi + \frac{1}{2}|D\phi|^2 + h(\rho) = B\,, \end{cases} \tag{2.2.2}$$

where D and div represent the gradient and divergence with respect to the self-similar coordinates. Note that the flow velocity is still given by the gradient of the potential function

$$\boldsymbol{u} \equiv \nabla\Phi \equiv D\phi\,. \tag{2.2.3}$$

It is convenient to introduce the pseudo-potential function $\varphi \coloneqq \phi - \frac{1}{2}|\boldsymbol{\xi}|^2$, for which the system (2.2.2) takes the more elegant form

$$\begin{cases} \operatorname{div}\big(\rho\, D\varphi\big) + 2\rho = 0\,, \\ \varphi + \frac{1}{2}|D\varphi|^2 + h(\rho) = B\,. \end{cases} \tag{2.2.4}$$

In particular, from (2.2.4) we see that the pseudo-potential φ satisfies the second order nonlinear pseudo-steady potential flow equation

$$\operatorname{div}\big(\rho(|D\varphi|,\varphi)\, D\varphi\big) + 2\rho(|D\varphi|,\varphi) = 0\,, \tag{2.2.5}$$

with

$$\rho(|D\varphi|,\varphi) \coloneqq h^{-1}(B - \varphi - \frac{1}{2}|D\varphi|^2)\,. \tag{2.2.6}$$

The term pseudo-steady reflects the fact that the principle part of this equation has the same form as the steady potential flow equations.

The pseudo-steady potential flow equation (2.2.5) can be written in non-divergence form

$$(c^2 - \varphi_\xi^2)\phi_{\xi\xi} - 2\varphi_\xi\varphi_\eta\phi_{\xi\eta} + (c^2 - \varphi_\eta^2)\phi_{\eta\eta} = 0\,, \tag{2.2.7}$$

where the sonic speed $c = c(|D\varphi|,\varphi) \coloneqq (\rho(|D\varphi|,\varphi))^{\frac{\gamma-1}{2}}$. It follows that equation (2.2.5) or (2.2.7) is of *mixed elliptic-hyperbolic type*: it is elliptic if and only if

$$|D\varphi| < c(|D\varphi|,\varphi) \iff |D\varphi| < c_\star(\varphi) \qquad \text{(pseudo-subsonic)},$$

and hyperbolic if and only if

$$|D\varphi| > c(|D\varphi|,\varphi) \iff |D\varphi| > c_\star(\varphi) \qquad \text{(pseudo-supersonic)},$$

where

$$c_\star(\varphi) \coloneqq \sqrt{\frac{2}{\gamma+1}\big(1+(\gamma-1)(B-\varphi)\big)}\,.$$

A survey of the recent progress and challenges in the area of nonlinear PDE of mixed type is given by [16].

Definition 2.4 (Uniform state)**.** A solution φ of (2.2.5) is called a uniform state if the corresponding density $\rho(|D\varphi|,\varphi) > 0$ given by (2.2.6) is constant.

Observe that $\rho(|D\varphi|,\varphi) > 0$ is constant if and only if the corresponding pseudo-potential φ has the following form

$$\varphi(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u,v)\cdot\boldsymbol{\xi} + k\,,$$

for some constants (u,v,k); see [18, §2.2]. The uniform states in pseudo-steady potential flow correspond precisely with the constant states in potential flow.

Analogously to Definition 2.2, we consider weak solutions to (2.2.5).

Definition 2.5 (Weak solutions)**.** A function $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is called a weak solution of the pseudo-steady potential flow equation (2.2.5) in a domain $\Omega \subset \mathbf{R}^2$ if φ satisfies the following properties:

(i) $B - (\varphi + \frac{1}{2}|D\varphi|^2) \geq h(0^+)$ a.e. in Ω;

(ii) $\big(\rho(|D\varphi|,\varphi), \rho(|D\varphi|,\varphi)|D\varphi|\big) \in (L^1_{\mathrm{loc}}(\Omega))^2$;

(iii) $\int_\Omega \big(\rho(|D\varphi|,\varphi)D\varphi\cdot D\zeta - 2\rho(|D\varphi|,\varphi)\zeta\big)\,\mathrm{d}\boldsymbol{\xi} = 0$, for every $\zeta \in C^\infty_c(\Omega)$.

The requirement $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ implies that $\varphi \in C^0(\Omega)$, though the derivatives of φ need not be continuous. In particular, for a piecewise smooth weak solution of (2.2.5) with $D\varphi$ having a jump discontinuity across a C^1-curve Γ with unit normal $\boldsymbol{\nu}$, the requirement of being a weak solution in the sense of Definition 2.5 yields the Rankine–Hugoniot jump condition on Γ

$$[\rho(|D\varphi|,\varphi)\,D\varphi]\cdot\boldsymbol{\nu} = 0\,, \tag{2.2.8}$$

where $[w]$ denotes the change in a quantity w across the curve Γ.

By Definition 2.5 and the Rankine–Hugoniot conditions (2.2.8), $D\varphi \cdot \boldsymbol{\nu}$ is either positive or negative across the discontinuity Γ. We say that $\boldsymbol{\nu}$ points from upstream to downstream if $D\varphi \cdot \boldsymbol{\nu} > 0$, and from downstream to upstream if $D\varphi \cdot \boldsymbol{\nu} < 0$.

Definition 2.6 (Physical entropy condition)**.** A discontinuity of $D\varphi$ satisfying the Rankine–Hugoniot conditions on Γ is called a *shock* if furthermore φ satisfies the *physical entropy condition*: the density $\rho = \rho(|D\varphi|, \varphi)$ increases across Γ from upstream to downstream, or equivalently, $|D\varphi \cdot \boldsymbol{\nu}|$ decreases across Γ from upstream to downstream.

The following lemma is fundamental in the study of 2-D Riemann problems for potential flow, in as much as it relates weak solutions of the pseudo-steady potential flow equation to weak solutions of the potential flow equation.

Lemma 2.7. *Suppose φ is a piecewise smooth weak solution of the pseudo-steady potential flow equation* (2.2.5) *for $\boldsymbol{\xi} \in \mathbf{R}^2$. Then in fact*

$$\Phi(t, \boldsymbol{x}) \coloneqq t\,(\frac{1}{2}|\boldsymbol{\xi}|^2 + \varphi(\boldsymbol{\xi}))\,, \qquad \textit{for } \boldsymbol{\xi} = \frac{1}{t}\boldsymbol{x}\,, \tag{2.2.9}$$

is a piecewise smooth weak solution of the potential flow equation (2.1.6) *for $(t, \boldsymbol{x}) \in (0, \infty) \times \mathbf{R}^2$.*

Proof. It can be checked that the properties in Definition 2.2(i)–(ii) follow directly from (2.2.9) and the analogous properties in Definition 2.5. For Definition 2.2(iii), we choose any test function $\zeta \in C_c^\infty((0, \infty) \times \mathbf{R}^2)$. Then

$$\begin{aligned} &\int_{(0,\infty)\times\mathbf{R}^2} \big(\rho(\partial_t\Phi, |\nabla\Phi|)\partial_t\zeta + \rho(\partial_t\Phi, |\nabla\Phi|)\nabla\Phi \cdot \nabla\zeta\big)\,\mathrm{d}t\mathrm{d}\boldsymbol{x} \\ &\quad = \int_{\mathbf{R}^2} \big(\rho(|D\varphi|, \varphi)D\varphi \cdot D\tilde{\zeta} - 2\rho(|D\varphi|, \varphi)\tilde{\zeta}\big)\,\mathrm{d}\boldsymbol{\xi}\,, \end{aligned} \tag{2.2.10}$$

where

$$\tilde{\zeta}(\boldsymbol{\xi}) \coloneqq \int_{(0,\infty)} t\zeta(t, t\boldsymbol{\xi})\,\mathrm{d}t\,.$$

Then Definition 2.2(iii) follows from (2.2.10), Definition 2.5(iii) and the dominated convergence theorem after approximating $\tilde{\zeta}$ by a sequence of compactly supported test functions. □

Remark 2.8. Weak solutions of the potential flow equation are invariant under reflections and Galilean transformations, and correspondingly solutions of the pseudo-steady potential flow equation are invariant under reflections, rotations and translations. Moreover, the pseudo-steady potential flow equation has the following scale invariance

$$\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}}{c_0}\,, \quad (\rho, \varphi, B) \mapsto \big(\frac{\rho}{\rho_0}, \frac{\varphi}{c_0^2}, \frac{1}{c_0^2}(B - h(\rho_0))\big)\,, \qquad \text{for any } c_0 \coloneqq \rho_0^{\frac{\gamma-1}{2}} > 0\,.$$

2.2.1 Pseudo-steady shocks

Corresponding to planar shocks in potential flow, pseudo-steady shocks in self-similar pseudo-steady potential flow have the form

$$\varphi = \begin{cases} \varphi^-\,, & \text{for } \boldsymbol{\xi}\cdot\boldsymbol{\nu} < s\,, \\ \varphi^+\,, & \text{for } \boldsymbol{\xi}\cdot\boldsymbol{\nu} > s\,. \end{cases}$$

with

$$\varphi^\pm(\boldsymbol{\xi}) \coloneqq -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u^\pm, v^\pm)\cdot\boldsymbol{\xi} + k^\pm\,.$$

Direct computation reveals that the Rankine–Hugoniot shock relations for pseudo-steady potential flow, when written in the above form, coincide with (2.1.13) for potential flow. This is because the principle parts of both equations have the same form. Equivalently, we can think of the shock relations for pseudo-steady potential flow in the following geometric way: the uniform states φ^+ and φ^- are connected by a pseudo-steady shock passing through some point $P_0 \in \mathbf{R}^2$ if and only if the $(+)$ state $(\rho^+, \boldsymbol{u}^+ \coloneqq D\varphi^+(P_0))$ lies on the full steady shock polar curve corresponding to the $(-)$ state $(\rho^-, \boldsymbol{u}^- \coloneqq D\varphi^-(P_0))$.

2.3 Classification of the 2-D Riemann problems for potential flow

We consider the classification of the 2-D Riemann problems for potential flow with constant initial data given in the four quadrants of the plane, under the assumption that there is exactly one plane wave connecting each state at infinity. This classification is motivated by, and analogous to, a series of work throughout the 1990s on the corresponding classification of the two-dimensional Riemann problems for the compressible Euler equations; see [83, 12, 13, 72, 77], and also [73, 55] for numerical simulations. In these works, it was shown that there are 19 genuinely different four-wave interactions for the full system of Euler equations, whilst for the isentropic system of Euler equations there are 20. The configurations are listed in Table 2.1.

Remark 2.9. We choose to use a pictorial style of notation similar to [12] to represent the 2-D Riemann initial data since the meaning is clearer at face value as compared to the more common notation in [72]. In our notation, the letter represents the type of wave, and the arrow shows the direction of propagation of the wave.

2.3.1 Mathematical formulation

The initial data under consideration comprise of four constant states $U_i = (\rho_i, u_i, v_i)$ given in the four quadrants of the plane

$$(\rho, \Phi)(0, \boldsymbol{x}) = (\rho_i, (u_i, v_i) \cdot \boldsymbol{x}), \qquad \text{for } \boldsymbol{x} \text{ in the } i^{\text{th}} \text{ quadrant}, \tag{2.3.1}$$

as shown in Figure 2.2. Furthermore, the initial data should satisfy that, in the far field region, exactly one elementary planar wave is generated between each adjacent state. Since potential flow does not support characteristic discontinuities, the elementary waves considered are either rarefaction waves or shocks.

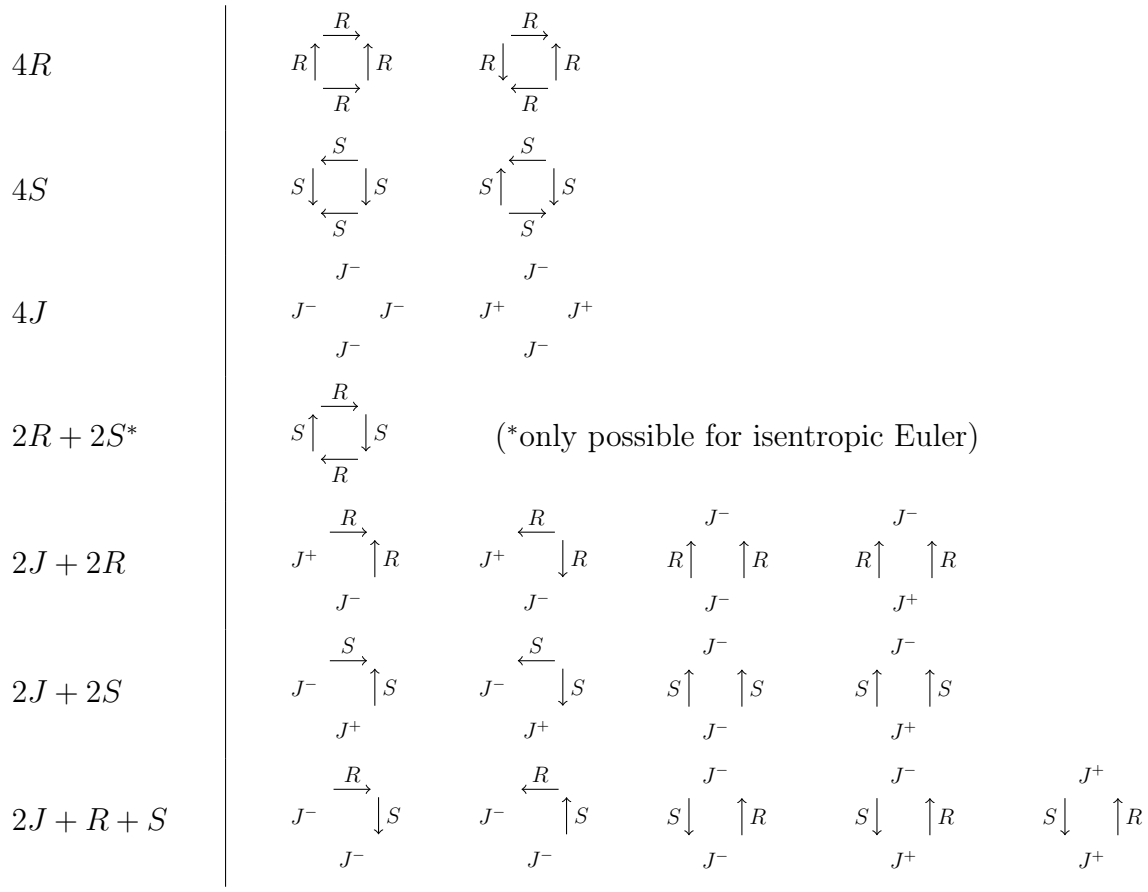


Table 2.1: The 19 genuinely different four-wave interactions for the full system of Euler equations; reproduced from [55]. The configuration $2R + 2S$ is only possible for the isentropic Euler equations, making 20 genuinely different four-wave interactions in this case; see [72]. R = rarefaction wave, S = shockwave, J = characteristic discontinuity.

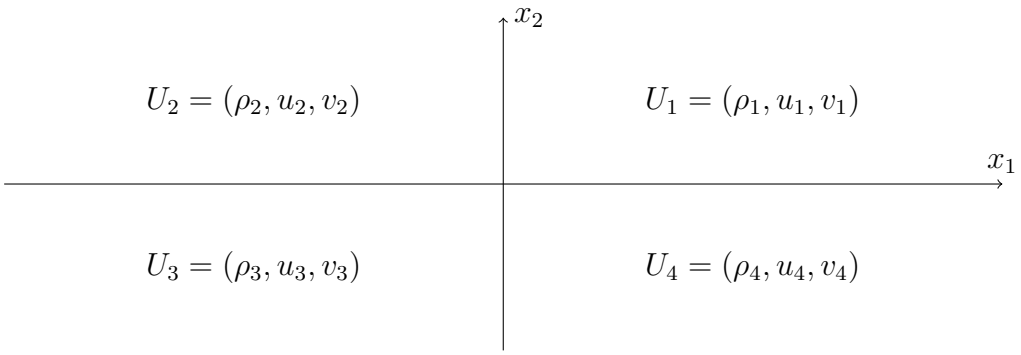


Figure 2.2: The 2-D Riemann initial data in the four quadrants of the plane.

In terms of the flow parameters (ρ, u, v), the elementary wave relations between the first and second quadrants may be summarised as follows:

$$\begin{aligned}
\xrightarrow{R} \;:\; &\begin{cases} \rho_1 > \rho_2 \\ u_1 - u_2 = +\varpi(\rho_1, \rho_2) > 0 \\ v_1 - v_2 = 0 \end{cases} \qquad \xleftarrow{R} \;:\; \begin{cases} \rho_1 < \rho_2 \\ u_1 - u_2 = -\varpi(\rho_1, \rho_2) > 0 \\ v_1 - v_2 = 0 \end{cases} \\
\xrightarrow{S} \;:\; &\begin{cases} \rho_1 < \rho_2 \\ u_1 - u_2 = -\ell(\rho_1, \rho_2) < 0 \\ v_1 - v_2 = 0 \end{cases} \qquad \xleftarrow{S} \;:\; \begin{cases} \rho_1 > \rho_2 \\ u_1 - u_2 = -\ell(\rho_1, \rho_2) < 0 \\ v_1 - v_2 = 0 \end{cases}
\end{aligned} \tag{2.3.2}$$

where

$$\varpi(\rho_1, \rho_2) \coloneqq \tfrac{2}{\gamma-1}\big(\rho_1^{\frac{\gamma-1}{2}} - \rho_2^{\frac{\gamma-1}{2}}\big)\,, \qquad \ell(\rho_1, \rho_2) \coloneqq \sqrt{\frac{2(\rho_1^{\gamma-1} - \rho_2^{\gamma-1})(\rho_1 - \rho_2)}{(\gamma-1)(\rho_1 + \rho_2)}}\,. \tag{2.3.3}$$

The wave relations between the other adjacent quadrants are similar up to rotation. It is clear that the function ℓ is symmetric, and direct calculation reveals the following strict monotonicity property: for any $\bar\rho > 0$,

(2.3.4)

$\rho \mapsto \ell(\rho, \bar\rho)$ is strictly decreasing on $(0, \bar\rho)$, and strictly increasing on $(\bar\rho, \infty)$.

The analysis for classification proceeds in the first instance analogously to [72], enumerating all possible configurations of rarefaction waves and shocks, and then analysing each configuration on a case by case basis. Up to symmetries, we find there are nine such configurations to consider. In each case, we may assume that $\rho_1 > \max\{\rho_2, \rho_4\}$. In the following, we use the abbreviation $\varpi_{ij} \coloneqq \varpi(\rho_i, \rho_j)$, and $\ell_{ij} \coloneqq \ell(\rho_i, \rho_j)$.

2.3.2 Classification analysis

Configuration 1. $R\uparrow\begin{smallmatrix}\xrightarrow{R}\\ \\ \xrightarrow[R]{}\end{smallmatrix}\uparrow R$: In this case, the wave relations read

$$\begin{aligned} &u_1 - u_2 = \varpi_{12}\,, \quad u_4 - u_3 = \varpi_{43}\,, \quad u_1 = u_4\,, \quad u_2 = u_3\,, \\ &v_1 - v_4 = \varpi_{14}\,, \quad v_2 - v_3 = \varpi_{23}\,, \quad v_1 = v_2\,, \quad v_3 = v_4\,, \end{aligned} \qquad \rho_1 > \begin{matrix}\rho_2\\ \rho_4\end{matrix} > \rho_3\,.$$

The system is consistent if and only if the initial data satisfy the compatibility condition $\varpi_{12} = \varpi_{43}$, or equivalently $\varpi_{14} = \varpi_{23}$. Moreover, given any initial data $0 < \rho_3 < \rho_2 < \rho_1$, the value $\rho_4 \in (\rho_3, \rho_1)$ is determined uniquely by the compatibility condition, and thus the system has a three parameter family of solutions.

Configuration 2. $R\downarrow\begin{smallmatrix}\xrightarrow{R}\\ \\ \xleftarrow[R]{}\end{smallmatrix}\uparrow R$: In this case, the wave relations read

$$\begin{aligned} &u_1 - u_2 = \varpi_{12}\,, \quad u_4 - u_3 = -\varpi_{43}\,, \quad u_1 = u_4\,, \quad u_2 = u_3\,, \\ &v_1 - v_4 = \varpi_{14}\,, \quad v_2 - v_3 = -\varpi_{23}\,, \quad v_1 = v_2\,, \quad v_3 = v_4\,, \end{aligned} \qquad \rho_1 > \begin{matrix}\rho_2\\ \rho_4\end{matrix} < \rho_3\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\varpi_{12} + \varpi_{43} = 0$, and $\varpi_{14} + \varpi_{23} = 0$. Upon taking the sum/difference of the two compatibility conditions, we find $\varpi_{13} = \varpi_{24} = 0$, which implies immediately that $\rho_1 = \rho_3$ and $\rho_2 = \rho_4$ are necessary, but otherwise arbitrary. Therefore this configuration is symmetric with respect to the lines $x_2 - x_1 = v_1 - u_1$, and $x_1 + x_2 = u_2 + v_2$.

Configuration 3. $R\uparrow\begin{smallmatrix}\xrightarrow{R}\\ \\ \xleftarrow[R]{}\end{smallmatrix}\uparrow R$: In this case, the wave relations read

$$\begin{aligned} &u_1 - u_2 = \varpi_{12}\,, \quad u_4 - u_3 = -\varpi_{43}\,, \quad u_1 = u_4\,, \quad u_2 = u_3\,, \\ &v_1 - v_4 = \varpi_{14}\,, \quad v_2 - v_3 = \varpi_{23}\,, \quad v_1 = v_2\,, \quad v_3 = v_4\,, \end{aligned} \qquad \rho_1 > \rho_2 > \rho_3 > \rho_4\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\varpi_{12} + \varpi_{43} = 0$, and $\varpi_{14} = \varpi_{23}$. However, $\varpi_{14} > \varpi_{23}$ because $(\rho_3, \rho_2) \subset (\rho_4, \rho_1)$, and so the system is inconsistent.

Configuration 4. $S\downarrow\overset{\leftarrow S}{\underset{\leftarrow S}{\square}}\downarrow S$: In this case, the wave relations read

$$\begin{array}{llll} u_1 - u_2 = -\ell_{12}\,, & u_4 - u_3 = -\ell_{43}\,, & u_1 = u_4\,, & u_2 = u_3\,, \\ v_1 - v_4 = -\ell_{14}\,, & v_2 - v_3 = -\ell_{23}\,, & v_1 = v_2\,, & v_3 = v_4\,, \end{array} \qquad \rho_1 > \begin{matrix}\rho_2\\ \rho_4\end{matrix} > \rho_3\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\ell_{12} = \ell_{43}$, and $\ell_{14} = \ell_{23}$. From these compatibility conditions, it should follow that $\rho_2 = \rho_4$, though the formal proof of this fact remains elusive. For the system of Euler equations, the corresponding result is given by [72, Theorem 1], or see [12, §5].

Configuration 5. $S\uparrow\overset{\leftarrow S}{\underset{\rightarrow S}{\square}}\downarrow S$: In this case, the wave relations read

$$\begin{array}{llll} u_1 - u_2 = -\ell_{12}\,, & u_4 - u_3 = -\ell_{43}\,, & u_1 = u_4\,, & u_2 = u_3\,, \\ v_1 - v_4 = -\ell_{14}\,, & v_2 - v_3 = -\ell_{23}\,, & v_1 = v_2\,, & v_3 = v_4\,, \end{array} \qquad \rho_1 > \begin{matrix}\rho_2\\ \rho_4\end{matrix} < \rho_3\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\ell_{12} = \ell_{43}$, and $\ell_{14} = \ell_{23}$. Upon taking the difference of the compatibility conditions, we obtain $\ell_{12} - \ell_{14} = \ell_{43} - \ell_{23}$, from which it follows that $\rho_2 = \rho_4$ is necessary, otherwise the left and right hand sides would have different signs due to the monotonicity properties of ℓ. Subsequently $\rho_1 = \rho_3$ follows from the first compatibility condition after $\rho_2 = \rho_4$. Therefore this configuration is symmetric with respect to the lines $x_2 - x_1 = v_1 - u_1$, and $x_1 + x_2 = u_2 + v_2$.

Configuration 6. $S\downarrow\overset{\leftarrow S}{\underset{\rightarrow S}{\square}}\downarrow S$: In this case, the wave relations read

$$\begin{array}{llll} u_1 - u_2 = -\ell_{12}\,, & u_4 - u_3 = -\ell_{43}\,, & u_1 = u_4\,, & u_2 = u_3\,, \\ v_1 - v_4 = -\ell_{14}\,, & v_2 - v_3 = -\ell_{23}\,, & v_1 = v_2\,, & v_3 = v_4\,, \end{array} \qquad \rho_1 > \rho_2 > \rho_3 > \rho_4\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\ell_{12} = \ell_{43}$, and $\ell_{14} = \ell_{23}$. However, $\ell_{14} > \ell_{23}$ by the monotonicity properties

of ℓ because $(\rho_3, \rho_2) \subset (\rho_4, \rho_1)$, and so the system is inconsistent.

Configuration 7. $S \downarrow \begin{array}{c} \xrightarrow{R} \\ \\ \xrightarrow{R} \end{array} \downarrow S$: In this case, the wave relations read

$$\begin{array}{llll} u_1 - u_2 = \varpi_{12}\,, & u_4 - u_3 = \varpi_{43}\,, & u_1 = u_4\,, & u_2 = u_3\,, \\ v_1 - v_4 = -\ell_{14}\,, & v_2 - v_3 = -\ell_{23}\,, & v_1 = v_2\,, & v_3 = v_4\,, \end{array} \qquad \rho_1 > \begin{array}{c} \rho_2 \\ \\ \rho_4 \end{array} > \rho_3\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\varpi_{12} = \varpi_{43}$, and $\ell_{14} = \ell_{23}$. We prove that there is no choice of initial data which satisfies this system, except in the case $\gamma = 3$.

Set $x := \frac{\rho_2}{\rho_1}, y := \frac{\rho_4}{\rho_1}$, and $z := \frac{\rho_3}{\rho_1}$, then $0 < z < \begin{array}{c} x \\ \\ y \end{array} < 1$, and the compatibility conditions become

$$\begin{cases} f(x,y,z) := 1 - x^{\frac{\gamma-1}{2}} - y^{\frac{\gamma-1}{2}} + z^{\frac{\gamma-1}{2}} = 0 \\ g(x,y,z) := \ell(1,y) - \ell(x,z) = 0\,. \end{cases}$$

Fix any $y \in (0,1)$, and write $\bar{f}(x,z) := f(x,y,z)$, and $\bar{g}(x,z) := g(x,y,z)$. From equation $\bar{f}(x,z) = 0$, and the requirement $0 < z < x < 1$, we solve uniquely for $z = z(x)$, that is

$$z(x) = (x^{\frac{\gamma-1}{2}} + y^{\frac{\gamma-1}{2}} - 1)^{\frac{2}{\gamma-1}}\,, \qquad \text{for } x \in ((1 - y^{\frac{\gamma-1}{2}})^{\frac{2}{\gamma-1}}, 1)\,.$$

Observe that $z(1) = y$, and hence $\bar{g}(1, z(1)) = g(1,y,y) = 0$ is satisfied at the end point $x = 1$. We also have the derivative

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \Big(\frac{x}{z}\Big)^{\frac{\gamma-3}{2}}\,, \qquad \text{for } x \in ((1 - y^{\frac{\gamma-1}{2}})^{\frac{2}{\gamma-1}}, 1)\,.$$

We calculate further

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\bar{g}(x, z(x)) = -\frac{\mathrm{d}}{\mathrm{d}x}\,\ell(x, z(x)) = \frac{x^{\kappa-1}\tau(1-\tau^\kappa)}{\ell(1,\tau)(1+\tau)^2}\big(\tfrac{1}{\kappa}(\tau^{-\kappa} - \tau^\kappa) - (\tau^{-1} - \tau)\big)\,,$$

where $\tau := \frac{z}{x} \in (0,1)$, and $\kappa := \frac{\gamma-1}{2} > 0$. For fixed $\tau \in (0,1)$, consider briefly the

function $\sigma(\kappa) \coloneqq \frac{1}{\kappa}(\tau^{-\kappa} - \tau^{\kappa})$, which satisfies $\sigma'(\kappa) = -\frac{1}{\kappa}\big(\tau^{-\kappa} + \tau^{\kappa} + \sigma(\tau,\kappa)\big)\log\tau > 0$ for all $\kappa > 0$. Therefore, whenever $\kappa \neq 1$, that is $\gamma \neq 3$, we have

$$g(x,y,z(x)) = \bar{g}(x,z(x)) \neq \bar{g}(1,z(1)) = 0 \qquad \text{for any } x \in ((1-y^{\frac{\gamma-1}{2}})^{\frac{2}{\gamma-1}}, 1)\,,$$

and so the compatibility conditions are inconsistent for any $y \in (0,1)$.

In the case $\gamma = 3$, the system degenerates to the underdetermined linear system

$$\rho_1 - \rho_2 = \rho_4 - \rho_3\,, \qquad \rho_1 > \begin{matrix}\rho_2\\ \rho_4\end{matrix} > \rho_3\,,$$

for which all solutions are given by $(\rho_1,\rho_2,\rho_3,\rho_4) = (a+d,a,b,b+d)$ for any constants $a,b,d \in (0,1)$ satisfying $b < a$ and $a + d < 1$.

Configuration 8. $S\uparrow\begin{matrix}\xrightarrow{R}\\ \xleftarrow{R}\end{matrix}\downarrow S$: In this case, the wave relations read

$$\begin{matrix} u_1 - u_2 = \varpi_{12}\,, & u_4 - u_3 = -\varpi_{43}\,, & u_1 = u_4\,, & u_2 = u_3\,, \\ v_1 - v_4 = -\ell_{14}\,, & v_2 - v_3 = -\ell_{23}\,, & v_1 = v_2\,, & v_3 = v_4\,, \end{matrix} \qquad \rho_1 > \begin{matrix}\rho_2\\ \rho_4\end{matrix} < \rho_3\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\varpi_{12} + \varpi_{43} = 0$, and $\ell_{14} = \ell_{23}$. Clearly $(\rho_1,\rho_2) = (\rho_3,\rho_4)$ is a solution to this system whenever $\rho_1 > \rho_2 > 0$; we prove all solutions are of this form, and hence this configuration is symmetric with respect to the lines $x_2 - x_1 = v_1 - u_1$, and $x_1 + x_2 = u_2 + v_2$.

Indeed, suppose $\rho_1 \neq \rho_3$; without loss of generality, we may assume $\rho_3 < \rho_1$, otherwise we could exchange $(\rho_1,\rho_2) \leftrightarrow (\rho_3,\rho_4)$. Set $x \coloneqq \frac{\rho_2}{\rho_1}$, $y \coloneqq \frac{\rho_4}{\rho_1}$, and $z \coloneqq \frac{\rho_3}{\rho_1}$, then $0 < \begin{matrix}x\\ y\end{matrix} < z < 1$, and the compatibility conditions become

$$\begin{cases} f(x,y,z) \coloneqq 1 - x^{\frac{\gamma-1}{2}} + y^{\frac{\gamma-1}{2}} - z^{\frac{\gamma-1}{2}} = 0 \\ g(x,y,z) \coloneqq \ell(1,y) - \ell(x,z) = 0\,. \end{cases}$$

Fix any $y \in (0,1)$, and write $\bar{f}(x,z) \coloneqq f(x,y,z)$, and $\bar{g}(x,z) \coloneqq g(x,y,z)$. From

equation $\bar{f}(x,z)=0$, and the requirement $0<x<z<1$, we solve uniquely for $x=x(z)$, that is

$$x(z)=(1+y^{\frac{\gamma-1}{2}}-z^{\frac{\gamma-1}{2}})^{\frac{2}{\gamma-1}}\,,\qquad \text{for } z\in((\tfrac{1}{2}(1+y^{\frac{\gamma-1}{2}}))^{\frac{2}{\gamma-1}},1)\,.$$

Observe that $x(1)=y$, and hence $\bar{g}(x(1),1)=g(y,y,1)=0$ is satisfied at the end point $z=1$. We also have the derivative

$$\frac{\mathrm{d}x}{\mathrm{d}z}=-\Big(\frac{z}{x}\Big)^{\frac{\gamma-3}{2}}\,,\qquad \text{for } z\in((\tfrac{1}{2}(1+y^{\frac{\gamma-1}{2}}))^{\frac{2}{\gamma-1}},1)\,.$$

We calculate further

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\bar{g}(x(z),z)=-\frac{\mathrm{d}}{\mathrm{d}z}\,\ell(x(z),z)=\frac{z^{\kappa-1}\tau(1+\tau^{\kappa})}{\ell(1,\tau)(1+\tau)^2}\big(\tfrac{1}{\kappa}(\tau^{-\kappa}-\tau^{\kappa})+(\tau^{-1}-\tau)\big)>0\,,$$

where $\tau\coloneqq\frac{x}{z}\in(0,1)$, and $\kappa\coloneqq\frac{\gamma-1}{2}>0$. Therefore

$$g(x(z),y,z)=\bar{g}(x(z),z)\neq\bar{g}(x(1),1)=0\qquad \text{for any } z\in((\tfrac{1}{2}(1+y^{\frac{\gamma-1}{2}}))^{\frac{2}{\gamma-1}},1)\,,$$

and so the only solutions to the compatibility conditions are of the form $z=1$ and $x=y\in(0,1)$.

Configuration 9. $S\big\downarrow\overset{R}{\underset{R}{\rightleftarrows}}\big\downarrow S$: In this case, the wave relations read

$$\begin{aligned}&u_1-u_2=\varpi_{12}\,,\quad u_4-u_3=-\varpi_{43}\,,\quad u_1=u_4\,,\quad u_2=u_3\,,\\&v_1-v_4=-\ell_{14}\,,\quad v_2-v_3=-\ell_{23}\,,\quad v_1=v_2\,,\quad v_3=v_4\,,\end{aligned}\qquad \rho_1>\rho_2>\rho_3>\rho_4\,.$$

The system is consistent if and only if the initial data satisfy the compatibility conditions $\varpi_{12}+\varpi_{43}=0$, and $\ell_{14}=\ell_{23}$. However, just like Configuration 6, $\ell_{14}>\ell_{23}$ by the monotonicity properties of ℓ because $(\rho_3,\rho_2)\subset(\rho_4,\rho_1)$, and so the system is inconsistent.

2.3.3 Final classification

Through the calculations above, we have determined that there are generically five genuinely different four-wave interactions for the 2-D Riemann problem in four quadrants for potential flow. These are the same five configurations that appear for the system of isentropic Euler equations when one does not consider characteristic discontinuities, though the specific choices of initial data do not coincide due to the different associated Rankine–Hugoniot conditions. In the case $\gamma = 3$, it appears that a sixth configuration is possible, albeit a little spurious, and perhaps in future it might be possible to eliminate this configuration via other considerations. It remains to give the rigorous proof that $\rho_2 = \rho_4$ in Configuration 4.

To the best of my knowledge, numerical experiments similar to [73, 55] have not yet been performed for the case of potential flow, though presumably the resulting interaction patterns will be somewhat similar to those obtained for the system of isentropic Euler equations. In Chapter 4, we give the rigorous analytic proof for the existence of solutions to Configuration 5—one of the two distinct four-shock interaction configurations.

Chapter 3

Critical Angles and Monotonicity Properties

The results contained in this chapter are largely results obtained from my own independent research, which constitute some new, and interesting features of the shock polar for 2-D steady potential flow.

3.1 Properties of the shock polar

In §2.1.3 we gave the construction of the shock polar curve for 2-D steady potential flow as the collection of all velocities for which there exists a choice of unit normal such that the shock relations (2.1.13) are satisfied. Due to the algebraic nonlinearity of the polytropic pressure law, it is unlikely for there to exist an explicit algebraic expression for the steady shock polar curve. Nevertheless, by direct calculation, we find two possible representations for the curve.

Parameterisation by shock angle β. Following the proof of [20, Lemma 7.3.2], the steady shock polar may be parameterised by the angle β which is formed between the straight oblique shock and the vertical axis; see Figure 3.1.

Given a fixed, constant supersonic upstream state $U_\infty = (\rho_\infty, u_\infty, 0)$, that is with $u_\infty > c_\infty := \rho_\infty^{\frac{\gamma-1}{2}}$, or equivalently $M_\infty := \frac{u_\infty}{c_\infty} > 1$, the constant downstream state $U = (\rho, u, v)$ found behind a shock making angle $\beta \in [0, \frac{\pi}{2})$ to the vertical axis

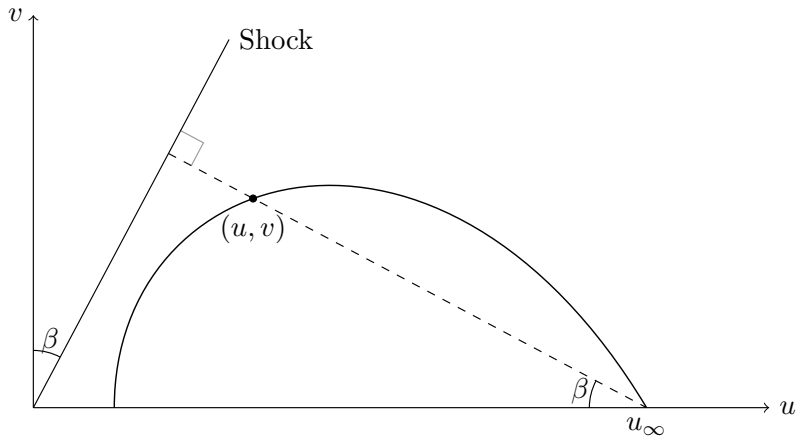


Figure 3.1: The shock polar curve

is given by

$$(u,v) \coloneqq \Big(u_\infty(1-(1-\frac{\rho_\infty}{\rho})\cos^2\beta)\,,\; u_\infty(1-\frac{\rho_\infty}{\rho})\sin\beta\cos\beta\Big)\,, \tag{3.1.1}$$

with

$$\rho \coloneqq \Big(\frac{u_\infty^2\cos^2\beta + 2h(\rho_\infty) + \frac{2}{\gamma-1}}{M_n^2+\frac{2}{\gamma-1}}\Big)^{\frac{1}{\gamma-1}}\,, \tag{3.1.2}$$

and $M_n \in (0,1]$ the unique solution to [1, Eq. (A.6)]; that is

$$M_n^{-\frac{2(\gamma-1)}{\gamma+1}}(M_n^2+\tfrac{2}{\gamma-1}) = M_{\infty,n}^{-\frac{2(\gamma-1)}{\gamma+1}}(M_{\infty,n}^2+\tfrac{2}{\gamma-1})\,, \tag{3.1.3}$$

for $M_{\infty,n} \coloneqq M_\infty\cos\beta$, the normal component of the upstream Mach number. The entropy condition $\rho > \rho_\infty$ leads to the restriction $\beta \in [0, \arccos(M_\infty^{-1}))$, since we require $M_{\infty,n} > 1$. We note that the limiting value $\beta = \arccos(M_\infty^{-1})$ corresponds to $(\rho,u,v) = (\rho_\infty, u_\infty, 0)$, whereby the upstream and downstream states coincide and there is no shock, whilst the limiting value $\beta = 0$ corresponds to a normal shock.

Definition 3.1 (Steady shock polar)**.** Fix $\gamma \geq 1$ and (ρ_∞, u_∞) with $M_\infty \coloneqq \frac{u_\infty}{c_\infty} > 1$. Define $\Upsilon^{(\rho_\infty,u_\infty)}$, the steady shock polar for 2-D potential flow, as the set of all $(u,v)(\beta) \in \mathbf{R}^2$ given by (3.1.1)–(3.1.3) for $\beta \in [0, \arccos(M_\infty^{-1})]$.

Remark 3.2. Direct calculation using (3.1.1) gives the useful estimate

$$\Big|(u,v)-(\tfrac{u_\infty}{2},0)\Big|<\tfrac{u_\infty}{2}\,,\qquad \text{for all } \beta\in[0,\arccos(M_\infty^{-1}))\,,$$

and so $\Upsilon^{(\rho_\infty,u_\infty)}$ is bounded by a circle.

Parameterisation by downstream density. Alternatively, as in [27, §17.2], the steady shock polar may be parameterised by the density of the downstream state. Let the constant supersonic upstream state $U_\infty=(\rho_\infty,u_\infty,0)$ be given as before. Then the possible downstream states $U=(\rho,u,v)$ are given by

$$(u,v)\coloneqq\Big(u_\infty-\frac{2\big(h(\rho)-h(\rho_\infty)\big)}{\rho+\rho_\infty}\frac{\rho}{u_\infty}\,,\sqrt{u_\infty^2-u^2-2\big(h(\rho)-h(\rho_\infty)\big)}\Big)\,, \tag{3.1.4}$$

for $\rho\in(\rho_\infty,\overline{\rho})$. The maximal density $\overline{\rho}>\rho_\infty$ is the unique solution on (ρ_∞,∞) to the equation $v(\overline{\rho})=0$ in (3.1.4), which satisfies

$$M_\infty^2\Big(1-\big(\frac{\rho_\infty}{\overline{\rho}}\big)^2\Big)=\frac{2\big(h(\overline{\rho})-h(\rho_\infty)\big)}{c_\infty^2}\,, \tag{3.1.5}$$

for $M_\infty>1$ the Mach number of the supersonic upstream state.

The following fundamental properties of $\Upsilon^{(\rho_\infty,u_\infty)}$ are obtained; see [20, Lemma 7.3.2].

Proposition 3.3. *Fix $\gamma\geq 1$ and (ρ_∞,u_∞) with $M_\infty\coloneqq\frac{u_\infty}{c_\infty}>1$. Let $\Upsilon^{(\rho_\infty,u_\infty)}$ be given as in* Definition 3.1. *Then*

(i) *$\Upsilon^{(\rho_\infty,u_\infty)}\subset\{v\geq 0\}$ is the graph of a continuous function that is smooth and strictly concave in its relative interior. That is, there exists an open interval $I=(\underline{u},u_\infty)$ and a function $f_{\text{polar}}\in C(\overline{I})\cap C^\infty(I)$ such that*

$$\Upsilon^{(\rho_\infty,u_\infty)}=\{(u,f_{\text{polar}}(u))\,:\,u\in\overline{I}\}\,, \tag{3.1.6}$$

and $f''_{\text{polar}}(u)<0$ for all $u\in I$. The constant $\underline{u}\in(0,u_\infty)$ above is the unique solution on $(0,u_\infty)$ to the equation

$$\big(\rho_\infty^{\gamma-1}+\tfrac{\gamma-1}{2}(u_\infty^2-\underline{u}^2)\big)^{\frac{1}{\gamma-1}}\,\underline{u}=\rho_\infty u_\infty\,, \tag{3.1.7}$$

which depends only on $(\gamma, \rho_\infty, u_\infty)$. *Moreover, the map* $(\rho_\infty, u_\infty) \mapsto \underline{u}$ *is smooth on the domain* $\{(\rho_\infty, u_\infty) : \rho_\infty > 0, u_\infty > c_\infty\}$, *and* $(u, \rho_\infty, u_\infty) \mapsto f_{\rm polar}(u)$ *is smooth on the domain* $\{(u, \rho_\infty, u_\infty) : \rho_\infty > 0, u_\infty > c_\infty, \underline{u} < u < u_\infty\}$.

(ii) *There exists unique* $u_{\rm d} \in (\underline{u}, u_\infty)$, *the detachment value of* u, *such that the line passing through* $(0,0)$ *and* $(u_{\rm d}, f_{\rm polar}(u_{\rm d}))$ *is tangent to* $\Upsilon^{(\rho_\infty, u_\infty)}$. *Moreover,* $(\rho_\infty, u_\infty) \mapsto u_{\rm d}$ *is (at least)* C^1 *on the domain* $\{(\rho_\infty, u_\infty) : \rho_\infty > 0, u_\infty > c_\infty\}$.

(iii) *There exists unique* $u_{\rm s} \in (u_{\rm d}, u_\infty)$, *the sonic value of* u, *such that* $q_{\rm s}^2 = c^2(q_{\rm s}^2)$, *where*

$$q_{\rm s}^2 := u_{\rm s}^2 + (f_{\rm polar}(u_{\rm s}))^2, \qquad \text{and} \qquad c^2(q^2) := \rho_\infty^{\gamma-1} + \tfrac{\gamma-1}{2}(u_\infty^2 - q^2),$$

that is, the downstream flow is exactly sonic. Moreover, the map $(\rho_\infty, u_\infty) \mapsto u_{\rm s}$ *is (at least)* C^1 *on the domain* $\{(\rho_\infty, u_\infty) : \rho_\infty > 0, u_\infty > c_\infty\}$.

3.2 Monotonicity of critical angles

In this section, we present a new monotonicity property of the steady shock polar for 2-D potential flow. Let us first introduce the steady critical angles.

Definition 3.4 (Steady deflection angle and critical angles)**.** Fix $\gamma \geq 1$ and (ρ_∞, u_∞) with $M_\infty := \frac{u_\infty}{c_\infty} > 1$. Let $f_{\rm polar}$, $\underline{u}$, $u_{\rm d}$, and $u_{\rm s}$ be given as in Proposition 3.3. Denote by $\theta_{\rm stdy}$ the steady deflection angle between the upstream and downstream flows, given by

$$\theta_{\rm stdy}(u) := \arctan\big(\frac{f_{\rm polar}(u)}{u}\big), \qquad \text{for } u \in (\underline{u}, u_\infty). \tag{3.2.1}$$

Define the steady detachment angle

$$\theta_{\rm stdy}^{\rm d} := \theta_{\rm stdy}(u_{\rm d}), \tag{3.2.2}$$

and the steady sonic angle

$$\theta^{\mathrm{s}}_{\mathrm{stdy}} \coloneqq \theta_{\mathrm{stdy}}(u_{\mathrm{s}})\,. \tag{3.2.3}$$

It is shown in [20, Lemma 7.3.3] that

$$\lim_{u_\infty \to c_\infty^+} \theta^{\mathrm{d}}_{\mathrm{stdy}}(\rho_\infty, u_\infty) = 0\,,$$

which corresponds to the fact that shocks are not supported within flows having upstream Mach numbers $M_\infty \leq 1$. In the following lemma, we quantify exactly the dependence of the steady critical angles upon the upstream Mach number.

Lemma 3.5. *Fix $\gamma \geq 1$ and (ρ_∞, u_∞) with $M_\infty \coloneqq \frac{u_\infty}{c_\infty} > 1$. Let $\theta^{\mathrm{d}}_{\mathrm{stdy}}$ and $\theta^{\mathrm{s}}_{\mathrm{stdy}}$ denote the steady detachment and sonic angles for the steady potential flow with constant supersonic upstream state $U_\infty = (\rho_\infty, u_\infty, 0)$, as defined by* Definition 3.4. *Then $\theta^{\mathrm{d}}_{\mathrm{stdy}}$ and $\theta^{\mathrm{s}}_{\mathrm{stdy}}$ are smooth, strictly increasing functions of the upstream Mach number $M_\infty > 1$. Moreover, the following limits hold:*

$$\lim_{M_\infty \to 1^+} (\theta^{\mathrm{d}}_{\mathrm{stdy}}, \theta^{\mathrm{s}}_{\mathrm{stdy}}) = (0,0)\,, \qquad \lim_{M_\infty \to \infty} (\theta^{\mathrm{d}}_{\mathrm{stdy}}, \theta^{\mathrm{s}}_{\mathrm{stdy}}) = (\tfrac{\pi}{2}, \arctan\sqrt{\tfrac{2}{\gamma-1}})\,.$$

Proof. The proof of Lemma 3.5 is given in four steps.

1. In the following, we denote by $w \coloneqq \tan\theta_{\mathrm{stdy}}$, where $\theta_{\mathrm{stdy}} \in (0, \frac{\pi}{2})$ is the angle between the velocities of the upstream and the downstream flows given by (3.2.1), and by $\tau \coloneqq \frac{\rho}{\rho_\infty} \in (1, \overline{\tau})$ the ratio between the downstream and upstream densities, with $\overline{\tau} \coloneqq \frac{\overline{\rho}}{\rho_\infty}$ and $\overline{\rho}$ the maximal density from (3.1.5). Then, using the polytropic pressure law with $\gamma \geq 1$, from (3.1.4) and (3.2.1) we have

$$w = \frac{v}{u} = \frac{\big(2h(\tau)\big)^{1/2}\big(M_\infty^2(1-\tau^{-2}) - 2h(\tau)\big)^{1/2}}{M_\infty^2(1+\tau^{-1}) - 2h(\tau)}\,. \tag{3.2.4}$$

From Proposition 3.3, there exist both a unique detachment angle $\theta^{\mathrm{d}}_{\mathrm{stdy}} \in (0, \frac{\pi}{2})$,

and a unique sonic angle $\theta^{\rm s}_{\rm stdy} \in (0, \theta^{\rm d}_{\rm stdy})$, which are characterised by

$$w_{\rm d} := \tan\theta^{\rm d}_{\rm stdy} = \sup\{w(\tau) : \tau \in (1,\overline{\tau})\},$$
$$w_{\rm s} := \tan\theta^{\rm s}_{\rm stdy} = \frac{v}{u} \qquad \text{when } u^2+v^2 = \rho^{\gamma-1}.$$

By differentiation of (3.2.4) with respect to $\tau \in (1,\overline{\tau})$, it follows that $w_{\rm d} = w(\tau_{\rm d})$, where $\tau_{\rm d} \in (1,\overline{\tau})$ satisfies

$$M_\infty^2\big(2h(\tau_{\rm d}) + (\tau_{\rm d}^2-1)h'(\tau_{\rm d})\big) - 2h(\tau_{\rm d})\big(2h(\tau_{\rm d}) + \tau_{\rm d}(\tau_{\rm d}+1)h'(\tau_{\rm d})\big) = 0. \tag{3.2.5}$$

Similarly, by routine calculation, it follows that $w_{\rm s} = w(\tau_{\rm s})$, where $\tau_{\rm s} \in (1,\overline{\tau})$ satisfies

$$M_\infty^2 = 1 + (\gamma+1)h(\tau_{\rm s}). \tag{3.2.6}$$

Therefore, for the polytropic gas, it is clear that the steady detachment angle and sonic angle $\theta^{\rm d}_{\rm stdy}$ and $\theta^{\rm s}_{\rm stdy}$ depend only on γ and M_∞.

Before proceeding, we observe the simple identities which hold for any $\gamma \geq 1$,

$$\tau\, h'(\tau) = 1 + (\gamma-1)h(\tau), \quad h'(\tau) = \tau^{\gamma-2} \qquad \text{for all } \tau \in (0,\infty). \tag{3.2.7}$$

2. For the detachment angle, we combine (3.2.4) with (3.2.5) to obtain a parametric description of the detachment angle and the upstream Mach number in terms of $\tau_{\rm d}$

$$w_{\rm d} = f_w(\tau_{\rm d}) := \frac{1}{2}\frac{\big(\tau_{\rm d}(\tau_{\rm d}^2-1)h' - 2h\big)^{1/2}\big((\tau_{\rm d}^2-1)h' + 2h\big)^{1/2}}{\tau_{\rm d}(\tau_{\rm d}+1)h' + h}, \tag{3.2.8}$$

for $\tau_{\rm d} \in (1,\overline{\tau})$ the unique solution to

$$M_\infty^2 = f_M(\tau_{\rm d}) := \frac{2\big(2h + \tau_{\rm d}(\tau_{\rm d}+1)h'\big)h}{2h + (\tau_{\rm d}^2-1)h'}, \tag{3.2.9}$$

where h and h' are evaluated at $\tau = \tau_{\rm d}$. That (3.2.9) admits a unique solution $\tau_{\rm d} \in (1,\overline{\tau})$ for any $M_\infty > 1$ is guaranteed by Proposition 3.3(i)–(ii), though we can verify this fact directly at the end of this step.

A direct calculation from (3.2.8)–(3.2.9) gives

$$f_w'(\tau) = \frac{(1+\tau)^2 h' f_{\mathrm{d}}(\tau)}{8(h+\tau(\tau+1)h')^3 f_w(\tau)}\,, \qquad f_M'(\tau) = \frac{2h' f_{\mathrm{d}}(\tau)}{(2h+(\tau^2-1)h')^2}\,,$$

where

$$f_{\mathrm{d}}(\tau) \coloneqq 2\big(3+(\tau+1)\frac{h''}{h'}\big)h^2 + (3\tau-5)(\tau+1)hh' + \tau(\tau+1)(\tau^2-1)(h')^2\,.$$

Therefore, it is clear that f_{d} determines the signs of f_w' and f_M'.

Applying (3.2.7), and using $\gamma \geq 1$, $\tau > 1$, and $h > 0$, we can rewrite f_{d} as follows

$$\begin{aligned} f_{\mathrm{d}}(\tau) &= \big(3(\tau-\tau^{-1}) + \big(3(\gamma-1)(\tau-\tau^{-1}) + 2(2-\tau^{-1})\big)h\big)h \\ &\quad + (1+\tau^{-1})\big(\tau^2(\tau^2-1)(h')^2 - 2h\big) \\ &> (1+\tau^{-1})\tilde{f}_{\mathrm{d}}(\tau)\,, \end{aligned}$$

where $\tilde{f}_{\mathrm{d}}(\tau) \coloneqq \tau^2(\tau^2-1)(h')^2 - 2h$. We note that $\tilde{f}_{\mathrm{d}}(1) = 0$ and

$$\tilde{f}_{\mathrm{d}}'(\tau) = 2\big((\gamma-1)\tau(\tau^2-1)h'(\tau) + \tau^{\gamma+1} - 1\big)h'(\tau) > 0 \qquad \text{for any } \tau > 1\,.$$

It follows that $\tilde{f}_{\mathrm{d}}(\tau) > 0$ and $f_{\mathrm{d}}(\tau) > 0$ for all $\tau > 1$, and subsequently that $f_w'(\tau) > 0$ and $f_M'(\tau) > 0$ for all $\tau > 1$. From the chain rule, we conclude that w_{d} and hence $\theta_{\mathrm{stdy}}^{\mathrm{d}}$ are strictly increasing with respect to $M_\infty > 1$.

Returning to the question of the unique existence of a solution to (3.2.9), now that it is known $f_M'(\tau) > 0$ for all $\tau > 1$, it suffices to check the limiting values of $f_M(\tau)$ as $\tau \to 1^+$ and $\tau \to +\infty$, respectively. Indeed, using the polytropic pressure law with $\gamma \geq 1$ it is simple to obtain the limits

$$\lim_{\tau\to 1^+} f_M(\tau) = 1\,, \qquad \lim_{\tau\to+\infty} f_M(\tau) = +\infty\,, \tag{3.2.10}$$

and therefore (3.2.9) admits a unique solution $\tau_{\mathrm{d}} > 1$ for any $M_\infty > 1$. Finally, it can be verified that $\tau_{\mathrm{d}} \in (1, \overline{\tau})$, with $\overline{\tau} \coloneqq \frac{\overline{\rho}}{\rho_\infty}$ and $\overline{\rho}$ the maximal density from (3.1.5). By

inspection of (3.1.5) we find that $\tau_{\rm d} \in (1, \overline{\tau})$ if and only if $\tau_{\rm d} > 1$ and

$$M_\infty^2(1 - \tau_{\rm d}^{-2}) > 2h(\tau_{\rm d})\,.$$

For the polytropic pressure law with $\gamma \geq 1$ this is equivalent to the inequality

$$\tau_{\rm d}^{\gamma+1} - \tau_{\rm d}^{\gamma-1} - 2h(\tau_{\rm d}) > 0\,,$$

which holds automatically for any $\tau_{\rm d} > 1$ by considering the derivative of the left-hand side above.

3. For the sonic angle, we combine (3.2.4) and (3.2.6) to give a parametric description of the sonic angle and upstream Mach number in terms of $\tau_{\rm s}$

$$w_{\rm s} = f_w(\tau_{\rm s}) \coloneqq \frac{\big(2h\big)^{1/2}\big((\tau_{\rm s}^2 - 1)\tau_{\rm s}^{\gamma-1} - 2h\big)^{1/2}}{(\tau_{\rm s} + 1)\tau_{\rm s}^{\gamma-1} + 2h}\,, \tag{3.2.11}$$

for $\tau_{\rm s} \in (1, \tau_{\rm d})$ the unique solution to

$$M_\infty^2 = f_M(\tau_{\rm s}) \coloneqq 1 + (\gamma + 1)h\,, \tag{3.2.12}$$

where h and h' are evaluated at $\tau = \tau_{\rm s}$.

It is clear that (3.2.12) admits a unique solution $\tau_{\rm s} > 1$ since $f_M(1) = 0$, $\lim\limits_{\tau\to+\infty} f_M(\tau) = +\infty$, and $f_M'(\tau) > 0$ for all $\tau > 1$. Furthermore, $\tau_{\rm s} \in (1, \tau_{\rm d})$ follows by combining $u_{\rm s} \in (u_{\rm d}, u_\infty)$ from Proposition 3.3(iii) with the fact that $\frac{{\rm d}u}{{\rm d}\rho} < 0$ for any $\rho > \rho_\infty$, where $u(\rho)$ is given by (3.1.4), which is verified by direct calculation.

For f_w, direct calculation gives

$$f_w'(\tau) = \frac{(\tau + 1)f_{\rm s}(\tau)}{\big((\tau + 1)(1 + (\gamma - 1)h) + 2h\big)^3 f_w(\tau)}\,, \tag{3.2.13}$$

where

$$f_{\rm s}(\tau) \coloneqq 2\big(1 + (\gamma - 1)h\big)\big(1 + (\gamma + 1)h\big)h + (\tau + 1)\big((\tau - 1)(1 + (\gamma - 1)h) - 2h\big)h'\,.$$

Therefore, it is clear that $f_{\rm s}$ determines the sign of f_w'.

Applying (3.2.7), along with $\gamma \geq 1$ and $\tau > 1$, we obtain

$$\begin{aligned} f_{\rm s}(\tau) &= \big(2(\gamma+1)\tau h^2 + \tau(\tau^2-1)h' - 2h\big)h' \\ &> \tilde{f}_{\rm s}(\tau)h' \,, \end{aligned}$$

where $\tilde{f}_{\rm s}(\tau) \coloneqq \tau(\tau^2-1)h' - 2h$. Notice that $\tilde{f}_{\rm s}(1) = 0$ and

$$\tilde{f}_{\rm s}'(\tau) = (\gamma+1)(\tau^2-1)h'(\tau) > 0 \qquad \text{for any } \tau > 1 \,.$$

Then it follows that $\tilde{f}_{\rm s}(\tau) > 0$ and $f_{\rm s}(\tau) > 0$ so that $f_w'(\tau) > 0$ for any $\tau > 1$. Recall also that $f_M'(\tau) > 0$ for any $\tau > 1$. We conclude via the chain rule that $w_{\rm s}$ and hence $\theta^{\rm s}_{\rm stdy}$ are strictly increasing with respect to $M_\infty > 1$.

4. The limiting values stated in the lemma can be checked directly from (3.2.8)–(3.2.12) now that the monotonicity of (f_w, f_M) with respect to τ in (3.2.8)–(3.2.9), and in (3.2.11)–(3.2.12) have been verified. The proof of Lemma 3.5 is complete. □

The monotonicity of the steady critical angles, and the analysis carried out within the proof of Lemma 3.5, have been useful tools in the proofs of Theorem 3.6 and Proposition 4.2, among other things.

The formulae (3.2.8)–(3.2.9) and (3.2.11)–(3.2.12) constitute parametric expressions for the steady detachment and sonic angles for 2-D potential flow in terms of the upstream Mach number. For any $M_\infty > 1$, one must first solve for $\tau > 1$, and then substitute to find the critical angles. Of course, in general, explicitly inverting the relation between M_∞ and τ is likely impossible, though, the cases $\gamma = 2$, and $\gamma = 3$ are, in theory, computable in terms of radicals. Indeed, for $\gamma = 2$, one has

$$\begin{aligned} \tan\theta^{\rm d}_{\rm stdy} &= \frac{(X + M_\infty^2 - 10)\sqrt{3X - M_\infty^2 - 26}}{8(X-8)} \,, \qquad X = \sqrt{68 + 12M_\infty^2 + M_\infty^4} \\ \tan\theta^{\rm s}_{\rm stdy} &= \frac{\sqrt{2(M_\infty^2-1)^3(M_\infty^2+8)}}{M_\infty^4 + 13M_\infty^2 + 4} \,. \end{aligned}$$

whilst the case $\gamma = 3$ is more complicated.

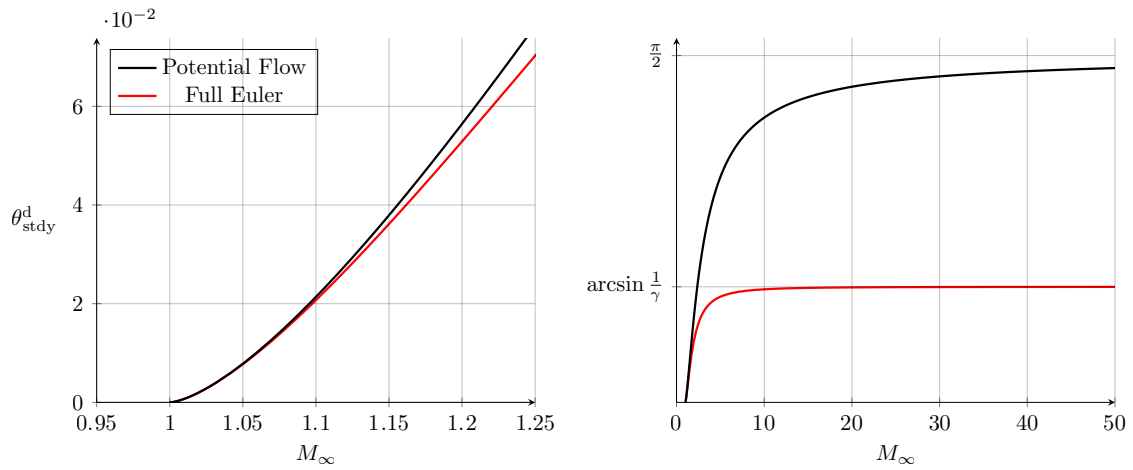


Figure 3.2: The steady detachment angles for potential flow and the full system of Euler equations in the case $\gamma = 2$. Left: small incident Mach numbers; right: large incident Mach numbers.

In comparison, in the context of the full system of Euler equations, there are well-known explicit formulae for the steady detachment angle; see *e.g.* [52, §92] for a derivation. For any $\gamma > 1$, we have

$$\theta^{\mathrm{d}}_{\mathrm{stdy,\,Euler}} = \arctan\left(2\cot\beta\,\frac{M_\infty^2\sin^2\beta - 1}{M_\infty^2(\gamma+\cos 2\beta)+2}\right), \tag{3.2.14}$$

where $\beta := \frac{1}{2}\arccos\left(\frac{4+(\gamma-1)M_\infty^2-\sqrt{(\gamma+1)\left((\gamma+1)M_\infty^4+8(\gamma-1)M_\infty^2+16\right)}}{2\gamma M_\infty^2}\right) \in (0, \frac{\pi}{2})$.

In Figure 3.2 we plot the steady detachment angles for potential flow and the full system of Euler equations in the case $\gamma = 2$. We see that the detachment angles correspond asymptotically in the limit of weak shock strengths $M_\infty \to 1^+$, though the limiting values as $M_\infty \to +\infty$ are distinct. Indeed, for potential flow, we calculated previously that $\theta^{\mathrm{d}}_{\mathrm{stdy}} \to \frac{\pi}{2}$ as $M_\infty \to +\infty$, whilst for the full system of Euler equations, it can be checked directly that $\theta^{\mathrm{d}}_{\mathrm{stdy,\,Euler}} \to \arcsin\left(\frac{1}{\gamma}\right)$ as $M_\infty \to +\infty$.

3.3 Regular reflection

When a planar shock is incident upon an inclined plane, the shock may or may not experience regular reflection, depending on the angle and the strength of the incident

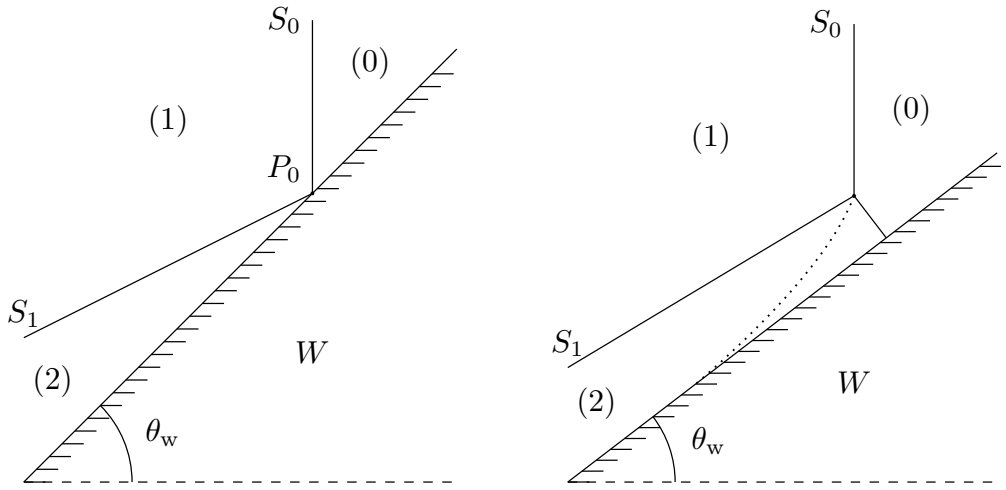


Figure 3.3: Regular reflection (left), and simple Mach reflection (right). The precise mathematical structure of Mach reflections are not currently known, and there must be present at least one additional wave emanating from the triple shock point since pure triple shock configurations are not mathematically consistent [75].

shock. The phenomena of shock reflection were first reported by Mach in 1878 [64], who experimentally observed two types of shock reflection: the regular reflection, and the Mach reflection; see Figure 3.3. Subsequent experimental and numerical work has identified a plethora of intricate reflection types, such as the double Mach reflection and the irregular Mach reflection; see [2] for a systematic review.

For the von Neumann reflection-diffraction problem, it was shown in [20] that the regular reflection of the incident shock is both necessary and sufficient for the existence of a regular reflection-diffraction configuration. Furthermore, the type of regular reflection, either supersonic or subsonic, determines the qualitative structure of the reflection-diffraction configurations. In the context of the full system of Euler equations, the necessary and sufficient conditions for the appearance of regular reflection are known; see [11, 76], and see [74] for the Chaplygin gas. For the equations of potential flow, a partial answer was given by [20, Theorem 7.1.1], however the precise criteria remain unclear. Below, we give the necessary and sufficient conditions for the appearance of regular reflection in potential flow, including the transition between supersonic and subsonic reflections, and their rigorous proofs.

3.3.1 Mathematical formulation of regular reflection

After transformation into an appropriate reference frame, the basic setup for the shock regular reflection problem involves a vertical planar shock, S_0, incident upon an inclined plane of angle $\theta_{\rm w}$. Denote by (0) the constant upstream state ahead of the shock S_0, having density ρ_0 and velocity $\boldsymbol{u}_0 \coloneqq (0,0)$; and by (1) the constant downstream state behind the shock S_0, with density $\rho_1 > \rho_0$ and velocity $\boldsymbol{u}_1 \coloneqq (u_1, 0)$ with $u_1 > 0$. We wish to determine a third constant state, denoted (2), with density $\rho_2 > \rho_1$ and velocity $\boldsymbol{u}_2$, and a reflected shock S_1 such that the Rankine–Hugoniot conditions hold between states (1) and (2), and the slip boundary condition $\boldsymbol{u}_2 \cdot \boldsymbol{\nu}_{\rm w} = 0$ holds along the wedge boundary. The desired regular reflection configuration is shown in Figure 3.3 (left).

Clearly the shock regular reflection problem admits self-similar solutions, and thus it is convenient to analyse the problem using the self-similar formulation introduced in §2.2, whereby the location of the incident and reflected shocks are fixed. Furthermore, without loss of generality, we may always fix $\rho_1 = 1$ through scaling, so that $\rho_0 \in (0,1)$ and $\rho_2 > 1$; see Remark 2.8

In the self-similar coordinates, the states (0) and (1) have pseudo-potentials

$$\varphi_0(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2\,, \qquad \varphi_1(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u_1, 0)\cdot\boldsymbol{\xi} - u_1\xi_{P_0}\,, \tag{3.3.1}$$

where the constants (u_1, ξ_{P_0}) are uniquely determined by the Rankine–Hugoniot conditions across the incident shock $S_0 \coloneqq \{\varphi_0 = \varphi_1\} \equiv \{\xi_1 = \xi_{P_0}\}$. In particular, we have

$$u_1 = \sqrt{\frac{2(1-\rho_0)(1-\rho_0^{\gamma-1})}{(\gamma-1)(1+\rho_0)}}\,, \qquad \xi_{P_0} = \frac{u_1}{1-\rho_0}\,. \tag{3.3.2}$$

Note that $u_1 \in (0, \xi_{P_0})$ because $\rho_0 \in (0,1)$. Here, the point $P_0 \coloneqq (\xi_{P_0}, \xi_{P_0}\tan\theta_{\rm w})$ represents the point of intersection between the incident shock S_0 and the wedge

$W \coloneqq \{\eta < \xi \tan\theta_{\mathrm{w}}\}$. The pseudo-Mach number of state (1) at P_0 is given by

$$M_1^2 \coloneqq |D\varphi_1(P_0)|^2 = (\xi_{P_0} - u_1)^2 + (\xi_{P_0}\tan\theta_{\mathrm{w}})^2 \,. \tag{3.3.3}$$

We seek the regular reflection of S_0 at P_0. Denote the reflected shock by S_1, with downstream state (2) having density $\rho_2 > 1$, and flow velocity $\boldsymbol{u}_2$. The pseudo-potential of state (2) is therefore given by

$$\varphi_2(\boldsymbol{\xi}) \coloneqq -\frac{1}{2}|\boldsymbol{\xi}|^2 + \boldsymbol{u}_2 \cdot \boldsymbol{\xi} + k_2 \,, \tag{3.3.4}$$

for some constant k_2 to be determined.

Following the general idea presented in §2.2, or see [20, §7.4], the Rankine–Hugoniot conditions between states (1) and (2) necessitate that candidate values of $\boldsymbol{u}_2$ should lie on the steady shock polar curve associated with $D\varphi_1(P_0) = \boldsymbol{u}_1 - P_0$, the pseudo-velocity of state (1) at P_0. Furthermore, since state (2) must satisfy the slip boundary condition on the wedge boundary

$$D\varphi_2(\boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\mathrm{w}} = 0 \qquad \text{for } \boldsymbol{\xi} \in \partial W \,,$$

it follows that $\boldsymbol{u}_2 \in \partial W$. Upon combining these two observations, we can rephrase the existence of a regular reflected state (2) as the following geometric statement: there exists a regular reflection state (2) if and only if the shock polar curve associated with $D\varphi_1(P_0)$ intersects the wedge boundary.

The geometric construction of state (2) is shown in Figure 3.4, whereby there may exist at most two solutions for state (2), depending on the relation between (γ, ρ_0) and θ_{w}, which will be clarified exactly in Theorem 3.6 below. In the case where there are two solutions for state (2), we refer to them as the weak and strong reflections, denoted by superscripts "wk" and "sg" respectively, distinguished by $|D\varphi_2^{\mathrm{wk}}(P_0)| > |D\varphi_2^{\mathrm{sg}}(P_0)|$, or equivalently $\rho_2^{\mathrm{wk}} < \rho_2^{\mathrm{sg}}$. The strong reflection state (2) is always pseudo-subsonic at the reflection point: $|D\varphi_2^{\mathrm{sg}}(P_0)| < c_2^{\mathrm{sg}}$; whilst the weak reflection state (2) may be pseudo-supersonic, -sonic or -subsonic at the reflection point. Hereafter, we are

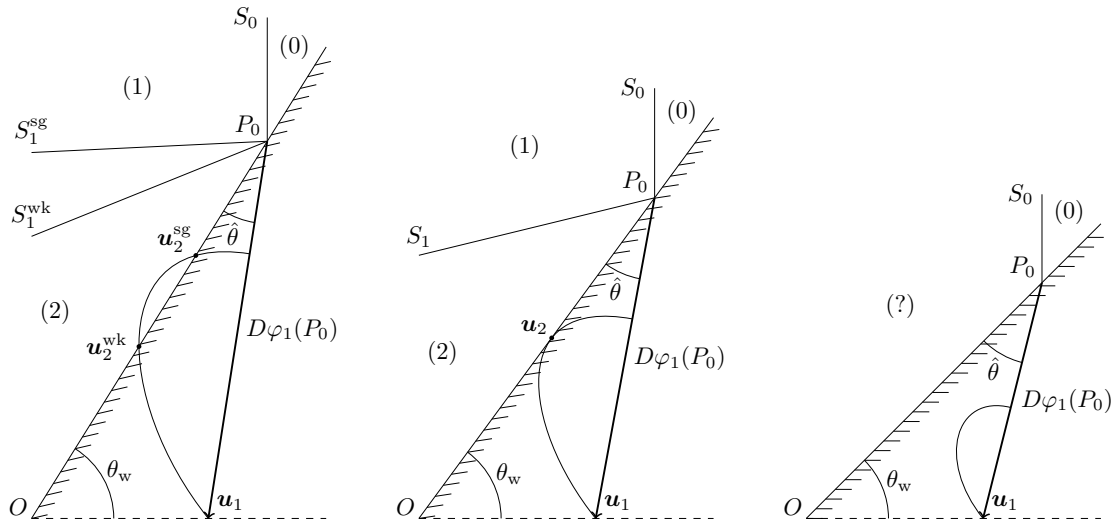


Figure 3.4: The geometric construction of the regular reflection when $\theta_{\rm w} > \theta^{\rm d}$ (left); $\theta_{\rm w} = \theta^{\rm d}$ (centre); and the non-construction when $\theta_{\rm w} < \theta^{\rm d}$ (right).

concerned only with the weak reflection, and therefore we remove the superscripts.

Theorem 3.6. *Fix $\gamma \geq 1$ and $\rho_0 \in (0,1)$. There exists unique $\theta^{\rm d} \in (0, \frac{\pi}{2})$, called the detachment angle, depending only on (γ, ρ_0), such that regular reflection is possible if and only if $\theta_{\rm w} \in [\theta^{\rm d}, \frac{\pi}{2})$. Moreover, there exists unique $\theta^{\rm s} \in (\theta^{\rm d}, \frac{\pi}{2})$, called the sonic angle, depending only on (γ, ρ_0), such that the reflected state (2) is supersonic if and only if $\theta_{\rm w} \in (\theta^{\rm s}, \frac{\pi}{2})$.*

The proof of this theorem will be given within the following two subsections, involving careful analysis of the shock polar curve for 2-D steady potential flow, and making important use of Lemma 3.5. We present the proof in the case $\gamma > 1$, though the case $\gamma = 1$ follows after minor modifications.

3.3.2 Existence and uniqueness of the detachment angle

Introduce $\hat\theta \in (0, \frac{\pi}{2})$ the acute angle between the wedge boundary and the pseudo-velocity of state (1) at P_0, as shown in Figure 3.4, which therefore satisfies

$$\cos\hat\theta = \frac{-D\varphi_1(P_0)\cdot(\cos\theta_{\rm w}, \sin\theta_{\rm w})}{|D\varphi_1(P_0)|}, \qquad \text{and} \qquad \sin\hat\theta = \frac{u_1}{|D\varphi_1(P_0)|}\sin\theta_{\rm w}. \tag{3.3.5}$$

It is clear that the necessary and sufficient conditions for the existence of a regular reflected state (2) are

$$M_1 > 1 \qquad \text{and} \qquad \hat{\theta} \le \theta^{\rm d}_{\rm stdy}(1, |D\varphi_1(P_0)|)\,, \tag{3.3.6}$$

where $\theta^{\rm d}_{\rm stdy}(\rho_\infty, u_\infty)$ is the steady detachment angle for the 2-D steady potential flow with constant supersonic upstream state $U_\infty = (\rho_\infty, u_\infty, 0)$, as given by Definition 3.4. Therefore, in order to prove the first part of Theorem 3.6, it is sufficient to determine the existence of a unique detachment wedge angle $\theta^{\rm d} \in (0, \frac{\pi}{2})$, such that (3.3.6) holds if and only if $\theta_{\rm w} \in (\theta^{\rm d}, \frac{\pi}{2})$. This will be the objective of the subsequent seven steps.

1. We begin with the simple observation that M_1 given by (3.3.3) is strictly increasing as a function of $\theta_{\rm w} \in (0, \frac{\pi}{2})$, and that $M_1 > 1$ if and only if $\theta_{\rm w} \in (\theta^+, \frac{\pi}{2})$, where

$$\tan\theta^+ = \frac{\sqrt{1-(\xi_{P_0}-u_1)^2}}{\xi_{P_0}} > 0\,. \tag{3.3.7}$$

Note that $\theta^+ \in (0, \frac{\pi}{2})$ is well-defined since $1-(\xi_{P_0}-u_1)^2 > 0$ for any $\rho_0 \in (0,1)$. Indeed, from (3.3.2) we have

$$1-(\xi_{P_0}-u_1)^2 = \frac{\gamma(1-\rho_0^2)-\rho_0^2(1-2\rho_0^{\gamma-1})-1}{(\gamma-1)(1-\rho_0^2)} > 0 \qquad \text{for } \rho_0 \in (0,1)\,,$$

because the numerator $f_1(\rho) \coloneqq \gamma(1-\rho^2)-\rho^2(1-2\rho^{\gamma-1})-1$ satisfies $f_1(1)=0$ and $f_1'(\rho) = -2(\gamma+1)\rho(1-\rho^{\gamma-1}) < 0$ for $\rho \in (0,1)$.

From (3.3.5), it is clear that $\hat{\theta} > 0$ on any compact subset of $(0, \frac{\pi}{2})$, whilst from Lemma 3.5 we have that $\theta^{\rm d}_{\rm stdy}(1, |D\varphi_1(P_0)|) \to 0$ as $\theta_{\rm w} \to \theta^+$. Therefore (3.3.6) does not hold for any $\theta_{\rm w}$ in some open neighbourhood of $(0, \theta^+)$. On the other hand, since M_1 grows arbitrarily large as $\theta_{\rm w} \to \frac{\pi}{2}^-$, we see directly from (3.3.5) and Lemma 3.5 that $\hat{\theta} \to 0$ and $\theta^{\rm d}_{\rm stdy}(1, |D\varphi_1(P_0)|) \to \frac{\pi}{2}$ as $\theta_{\rm w} \to \frac{\pi}{2}^-$, and therefore (3.3.6) holds for all $\theta_{\rm w}$ in some open neighbourhood of $\frac{\pi}{2}$. By continuity, there exists at least one value

$\theta^{\mathrm{d}} \in (\theta^+, \frac{\pi}{2})$ satisfying the equality

$$\hat{\theta} = \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_1(P_0)|)\,, \qquad \text{when } \theta_{\mathrm{w}} = \theta^{\mathrm{d}}\,. \tag{3.3.8}$$

It remains to show that any solution θ^{d} of (3.3.8) is unique on the interval $(\theta^+, \frac{\pi}{2})$.

2. From Lemma 3.5 and (3.3.3), the right hand side of (3.3.8) is strictly increasing as a function of $\theta_{\mathrm{w}} \in (\theta^+, \frac{\pi}{2})$. For the left hand side, using (3.3.2)–(3.3.3) and (3.3.5), we may calculate

$$\begin{aligned}
\frac{\mathrm{d}\sin\hat{\theta}}{\mathrm{d}\theta_{\mathrm{w}}} &= \frac{u_1}{M_1}\cos\theta_{\mathrm{w}} - \frac{u_1}{M_1^2}\frac{\mathrm{d}M_1}{\mathrm{d}\theta_{\mathrm{w}}}\sin\theta_{\mathrm{w}} \\
&= \frac{u_1\cos\theta_{\mathrm{w}}}{M_1^3}\Big((\xi_{P_0}-u_1)^2 + (\xi_{P_0}\tan\theta_{\mathrm{w}})^2 - (\xi_{P_0}\tan\theta_{\mathrm{w}}\sec\theta_{\mathrm{w}})^2\Big) \\
&= \frac{u_1\cos\theta_{\mathrm{w}}}{M_1^3}\xi_{P_0}^2\big(\rho_0^2 - \tan^4\theta_{\mathrm{w}}\big)\,.
\end{aligned} \tag{3.3.9}$$

Then it is clear that $\hat{\theta}$ is decreasing on $(\theta^+, \frac{\pi}{2})$ if and only if $\rho_0 \le \tan^2\theta^+$, which, by (3.3.2) and (3.3.7), is equivalent to the following condition on $\rho_0 \in (0,1)$, that

$$f_*(\rho_0) \coloneqq 1 - \frac{2\rho_0(1-\rho_0^{\gamma-1})}{(\gamma-1)(1-\rho_0)} \ge 0\,. \tag{3.3.10}$$

Simple computation gives $f_*(0^+) = 1$, $f_*(1^-) = -1$, and

$$f_*'(\rho) = -\frac{2(1-((\gamma-1)(1-\rho)+1)\rho^{\gamma-1})}{(\gamma-1)(1-\rho)^2} < 0\,, \qquad \text{for } \rho \in (0,1)\,, \tag{3.3.11}$$

because the numerator $f_2(\rho) \coloneqq 1 - ((\gamma-1)(1-\rho)+1)\rho^{\gamma-1}$ satisfies $f_2(1) = 0$, and $f_2'(\rho) = -\gamma(\gamma-1)(1-\rho)\rho^{\gamma-1} < 0$ for $\rho \in (0,1)$. We conclude that, for each $\gamma > 1$, there exists a unique value $\rho_0^*(\gamma) \in (0,1)$ satisfying $f_*(\rho_0^*) = 0$, and hence that the inequality (3.3.10) holds if and only if $\rho_0 \in (0, \rho_0^*]$.

Through the above discussion, we have shown that $\hat{\theta}$ is strictly decreasing as a function of $\theta_{\mathrm{w}} \in (\theta^+, \frac{\pi}{2})$ if and only if $\rho_0 \in (0, \rho_0^*]$. It follows directly that solutions of (3.3.8) are unique whenever $\rho_0 \in (0, \rho_0^*]$; see also Figure 3.5 (left).

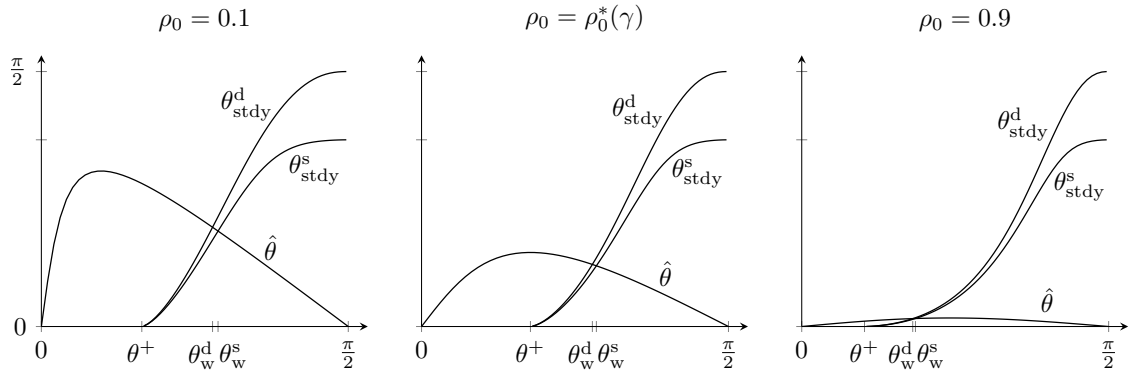


Figure 3.5: Plots of $\hat{\theta}$, $\theta^{\mathrm{d}}_{\mathrm{stdy}}$ and $\theta^{\mathrm{s}}_{\mathrm{stdy}}$ against θ_{w} for three different values of ρ_0, each with fixed $\gamma = \frac{7}{5}$, for which the critical value $\rho_0^*(\frac{7}{5}) \approx 0.39$.

3. We focus our attention now on the remaining case: $\rho_0 \in (\rho_0^*, 1)$. In this case, the uniqueness of solutions to (3.3.8) will be obtained through careful analysis of the derivatives of $\hat{\theta}$ and $\theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_1(P_0)|)$. In particular, for any $\gamma > 1$ and $\rho_0 \in (\rho_0^*, 1)$, we will show that the following property holds: for any $\theta_{\mathrm{w}} \in (\theta^+, \frac{\pi}{2})$

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\theta_{\mathrm{w}}} < \frac{\mathrm{d}\theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_1(P_0)|)}{\mathrm{d}\theta_{\mathrm{w}}} \qquad \text{whenever } \hat{\theta} \le \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_1(P_0)|)\,, \tag{3.3.12}$$

which is clearly sufficient for the uniqueness of solutions to (3.3.8).

The proof of (3.3.12) involves direct computation and careful estimates of the necessary derivatives. For the right hand side, we write $w_{\mathrm{d}} := \tan\theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_1(P_0)|)$ so that

$$\frac{\mathrm{d}\theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_1(P_0)|)}{\mathrm{d}\theta_{\mathrm{w}}} = \frac{1}{1+w_{\mathrm{d}}^2}\frac{\mathrm{d}w_{\mathrm{d}}}{\mathrm{d}\theta_{\mathrm{w}}} = \frac{1}{1+w_{\mathrm{d}}^2}\frac{\mathrm{d}w_{\mathrm{d}}}{\mathrm{d}M_1^2}\frac{\mathrm{d}M_1^2}{\mathrm{d}\theta_{\mathrm{w}}}\,. \tag{3.3.13}$$

For the left hand side, we use (3.3.2)–(3.3.3) and (3.3.5) to rewrite (3.3.9) in a

useful, though somewhat ad hoc manner

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\theta_{\mathrm{w}}} &= \frac{1}{\cos\hat{\theta}}\frac{\mathrm{d}\sin\hat{\theta}}{\mathrm{d}\theta_{\mathrm{w}}} \\
&= \frac{1}{\cos\hat{\theta}}\frac{u_1\cos\theta_{\mathrm{w}}}{M_1^3}\Big((\xi_{P_0}-u_1)^2+(\xi_{P_0}\tan\theta_{\mathrm{w}})^2-(\xi_{P_0}\tan\theta_{\mathrm{w}}\sec\theta_{\mathrm{w}})^2\Big) \\
&= \frac{1}{\cos\hat{\theta}}\frac{u_1\sin\theta_{\mathrm{w}}}{M_1^3}\xi_{P_0}^2\tan\theta_{\mathrm{w}}\sec^2\theta_{\mathrm{w}}\Big(\frac{(\xi_{P_0}-u_1)^2+(\xi_{P_0}\tan\theta_{\mathrm{w}})^2}{(\xi_{P_0}\tan\theta_{\mathrm{w}}\sec\theta_{\mathrm{w}})^2}-1\Big) \\
&= \frac{1}{2M_1^2}\tan\hat{\theta}\,\frac{\mathrm{d}M_1^2}{\mathrm{d}\theta_{\mathrm{w}}}\Big(\frac{\rho_0^2+\tan^2\theta_{\mathrm{w}}}{\sec^2\theta_{\mathrm{w}}}\frac{1}{\tan^2\theta_{\mathrm{w}}}-1\Big)\,.
\end{aligned}
\tag{3.3.14}
$$

On combining (3.3.13)–(3.3.14), and making use of the simple observations that $\tan\hat{\theta}\le w_{\mathrm{d}}$ whenever $\hat{\theta}\le\theta^{\mathrm{d}}_{\mathrm{stdy}}(1,|D\varphi_1(P_0)|)$, and

$$
\frac{\rho_0^2+\tan^2\theta_{\mathrm{w}}}{\sec^2\theta_{\mathrm{w}}}=\frac{\rho_0^2+\tan^2\theta_{\mathrm{w}}}{1+\tan^2\theta_{\mathrm{w}}}<1\,,\qquad\text{for any }\theta_{\mathrm{w}}\in(0,\tfrac{\pi}{2})\,,
$$

we see that, in order to prove (3.3.12), it is sufficient to prove

$$
1+\frac{2M_1^2}{w_{\mathrm{d}}}\frac{1}{1+w_{\mathrm{d}}^2}\frac{\mathrm{d}w_{\mathrm{d}}}{\mathrm{d}M_1^2}-\frac{1}{\tan\theta_{\mathrm{w}}}>0\,,\qquad\text{for all }\theta_{\mathrm{w}}\in(\theta^+,\tfrac{\pi}{2})\,.
$$

Moreover, using (3.3.3) and (3.3.7) to eliminate θ_{w} in terms of M_1, the above inequality is equivalent to showing that

$$
1+\frac{2M_1^2}{w_{\mathrm{d}}}\frac{1}{1+w_{\mathrm{d}}^2}\frac{\mathrm{d}w_{\mathrm{d}}}{\mathrm{d}M_1^2}-\frac{\xi_{P_0}^2}{M_1^2-(\xi_{P_0}-u_1)^2}>0\,,\qquad\text{for all }M_1>1\,.
\tag{3.3.15}
$$

4. Our strategy to prove (3.3.15) will be the direct substitution of equations (3.2.8)–(3.2.9) that were derived in §3.2. Before this substitution, we make one final simplification to the last term on the left hand side of (3.3.15). In Step **2**, we showed that $(\xi_{P_0}-u_1)^2<1$ for all $\rho_0\in(0,1)$. We claim further that

$$
\xi_{P_0}^2=\frac{2(1-\rho_0^{\gamma-1})}{(\gamma-1)(1-\rho_0^2)}<3\,,\qquad\text{for any }\rho_0\in(\rho_0^*,1)\,,
\tag{3.3.16}
$$

after which, in order to prove (3.3.15), it is sufficient to prove the inequality

$$1+\frac{2M_1^2}{w_{\rm d}}\frac{1}{1+w_{\rm d}^2}\frac{{\rm d}w_{\rm d}}{{\rm d}M_1^2}-\frac{3}{M_1^2-1}>0\,,\qquad\text{for any } M_1>1\,. \tag{3.3.17}$$

To prove (3.3.16), we first calculate directly

$$\frac{\partial\xi^2_{P_0}}{\partial\gamma}=-\frac{2(1-\rho_0^{\gamma-1}(1-(\gamma-1)\log\rho_0))}{(\gamma-1)^2(1-\rho_0^2)}<0\,,\qquad\text{for all } \rho\in(0,1)\,, \tag{3.3.18}$$

because the numerator $f_3(\rho)\coloneqq 1-\rho^{\gamma-1}(1-(\gamma-1)\log\rho)$ satisfies $f_3(1)=0$ and $f_3'(\rho)=(\gamma-1)^2\rho^{\gamma-2}\log\rho<0$, for $\rho\in(0,1)$. Therefore, we may estimate

$$\xi^2_{P_0}<\lim_{\gamma\to1^+}\xi^2_{P_0}=-\frac{2\log\rho_0}{1-\rho_0^2}\,,\qquad\text{for all } \rho_0\in(0,1)\,, \tag{3.3.19}$$

Furthermore, using (3.3.10) and the above estimate, we have

$$0=f_*(\rho_0^*)=1-\rho_0^*(1+\rho_0^*)\xi^2_{P_0}\big|_{\rho_0=\rho_0^*}>1+\frac{2\rho_0^*\log\rho_0^*}{1-\rho_0^*}\,. \tag{3.3.20}$$

Consider briefly the function $f_{*,\rm iso}(\rho)\coloneqq 1+\frac{2\rho\log\rho}{1-\rho}$, which satisfies $f_{*,\rm iso}(0^+)=1$, $f_{*,\rm iso}(1^-)=-1$, and $f'_{*,\rm iso}(\rho)<0$ for $\rho\in(0,1)$, and therefore admits a unique solution $\rho^*_{\rm iso}\in(0,1)$ to the equality $f_{*,\rm iso}(\rho^*_{\rm iso})=0$. In fact, the exact value of $\rho^*_{\rm iso}$ can be computed in terms of the product-log function, giving

$$\rho^*_{\rm iso}=-\Big(2{\rm W}_{-1}\big(-\frac{1}{2\sqrt{\rm e}}\big)\Big)^{-1}\approx 0.28467\,, \tag{3.3.21}$$

and it follows from(3.3.20) that $\rho_0^*(\gamma)\in(\rho^*_{\rm iso},1)$ for any $\gamma>1$.

Finally, returning to (3.3.19) and using (3.3.21), we have for any $\rho_0\in(\rho_0^*,1)$, that

$$\xi^2_{P_0}<\sup_{\rho\in(\rho^*_{\rm iso},1)}\frac{-2\log\rho}{1-\rho^2}=-\frac{2\log\rho^*_{\rm iso}}{1-\rho^{*\,2}_{\rm iso}}\approx 2.73445<3\,, \tag{3.3.22}$$

which completes the proof of (3.3.16).

5. In §3.2 it was shown that the steady detachment angle for 2-D steady potential flow has a parametric representation in terms of the upstream Mach number, given

by (3.2.8)–(3.2.9). We substitute (3.2.8)–(3.2.9) directly into (3.3.17) to obtain the following equivalent sufficient condition in terms of a dimensionless variable $\tau > 1$

$$\frac{A_{\rm d}(\tau)B_{\rm d}(\tau)}{(\gamma-1)(\tau+1)\tau^{\gamma-1}C_{\rm d}(\tau)} - \frac{3(\gamma-1)A_{\rm d}(\tau)}{D_{\rm d}(\tau)} > 0\,, \qquad \text{for all } \tau > 1\,, \tag{3.3.23}$$

where

(3.3.24)

$$\begin{aligned}
A_{\rm d}(\tau) &\coloneqq 2\tau(\tau^{\gamma-1}-1) + (\gamma-1)(\tau^2-1)\tau^{\gamma-1}\\
B_{\rm d}(\tau) &\coloneqq \tau^{\gamma-1}((\gamma-1)\tau+\gamma+1) - 2\\
C_{\rm d}(\tau) &\coloneqq 2 + (\gamma-1)\tau^{\gamma+1} - (\gamma+1)\tau^{\gamma-1}\\
D_{\rm d}(\tau) &\coloneqq 2(\gamma+1)\tau(\tau^{\gamma-1}-1)^2 + (\gamma-1)\tau^{\gamma-1}\Big(2\tau^2(\tau^{\gamma-1}-1) - (\gamma-1)(\tau^2-1)\Big)\,.
\end{aligned}$$

Note that $M_1 > 1$ in (3.3.17) is equivalent to $\tau > 1$ in (3.3.23) by (3.2.10).

We can simplify (3.3.23) as follows. First, it is true that $A_{\rm d}(\tau) > 0\,,\, C_{\rm d}(\tau) > 0\,,$ and $D_{\rm d}(\tau) > 0$ for all $\tau > 1\,.$ Indeed, $A_{\rm d}(\tau)$ is clearly positive, whilst for $C_{\rm d}(\tau)\,,$ we have $C_{\rm d}(1) = 0$ and $C_{\rm d}'(\tau) = (\gamma^2-1)(\tau^2-1)\tau^{\gamma-2} > 0\,,$ for $\tau > 1\,.$ For $D_{\rm d}(\tau)\,,$ we have that $d(\tau) \coloneqq 2\tau^2(\tau^{\gamma-1}-1) - (\gamma-1)(\tau^2-1)$ satisfies $d(1) = 0$ and $d'(\tau) = 2(\gamma+1)\tau(\tau^{\gamma-1}-1) > 0$ for $\tau > 1\,.$

Therefore, we are free to cancel terms and rearrange the inequality (3.3.23) to obtain

$$f_{\rm d}(\tau) \coloneqq B_{\rm d}(\tau)D_{\rm d}(\tau) - 3(\gamma-1)^2(\tau+1)\tau^{\gamma-1}C_{\rm d}(\tau) > 0\,, \qquad \text{for all } \tau > 1\,. \tag{3.3.25}$$

The proof of inequality (3.3.25) is split into two cases depending on $\gamma > 1\,.$

6. Case I: $\gamma \geq 2\,.$ We write the full expression for $f_{\rm d}$ in the following way

(3.3.26)

$$f_{\rm d}(\tau) = 8\tau(\tau^{\gamma-1}-1)^3 + 2(\gamma-1)\tau(\tau^{\gamma-1}-1)f_4(\tau) + 2(\gamma-1)(\tau+1)\tau^{\gamma-1}f_5(\tau)\,,$$

where

$$
\begin{aligned}
f_4(\tau) &\coloneqq (\gamma-1)(\tau^2-1)\tau^{\gamma-1} - 2(\tau^{\gamma-1}-1) \\
f_5(\tau) &\coloneqq (\gamma-1)\tau(\tau-1)(\tau^{\gamma-1}-1)^2 + 2f_6(\tau) \\
f_6(\tau) &\coloneqq (\gamma+1)\tau(\tau^{\gamma-1}-1)^2 - (\gamma-1)f_4(\tau)\,.
\end{aligned}
\tag{3.3.27}
$$

First, we observe that $f_4(\tau) > 0$ for any $\tau > 1$ because $f_4(1) = 0$ and $f_4'(\tau) = (\gamma^2-1)(\tau^2-1)\tau^{\gamma-2} > 0$ for $\tau > 1$. Then, in order to prove (3.3.25) in the present case, it is sufficient to show that $f_6(\tau) > 0$ for any $\tau > 1$, whenever $\gamma \geq 2$. Indeed, we have $f_6(1) = 0$, and we calculate directly

$$
f_6'(\tau) = (\gamma+1)(\tau^{\gamma-1}-1)^2 + (\gamma^2-1)\tau^{\gamma-2}\tilde{f}_6(\tau)\,,
\tag{3.3.28}
$$

where $\tilde{f}_6(\tau) \coloneqq 2\tau(\tau^{\gamma-1}-1) - (\gamma-1)(\tau^2-1)$. Then $f_6'(\tau) > 0$ for any $\tau > 1$ because $\tilde{f}_6(1) = 0$, and $\tilde{f}_6'(\tau) = 2(\tau-1) + 2\gamma\tau(\tau^{\gamma-2}-1) > 0$, for $\tau > 1$, so long as $\gamma \geq 2$. Therefore $f_6(\tau) > 0$ for any $\tau > 1$, and subsequently $f_5(\tau)$ and $f_{\mathrm{d}}(\tau) > 0$ for any $\tau > 1$, meaning that (3.3.25) is proved in this case.

7. Case II: $\gamma \in (1,2)$. In this case, we rewrite the full expression for f_{d} in the following way

$$
f_{\mathrm{d}}(\tau) = 8\tau(\tau^{\gamma-1}-1)^3 + 2(\gamma-1)(\tau^{\gamma-1}-1)f_7(\tau) + 2(\gamma-1)(\tau+1)\tau^{\gamma-1}f_8(\tau)\,,
\tag{3.3.29}
$$

where

$$
\begin{aligned}
f_7(\tau) &\coloneqq (\gamma-1)(\tau^2-1)\tau^{\gamma-1} - 2\tau(\tau^{\gamma-1}-1) \\
f_8(\tau) &\coloneqq (\tau^{\gamma-1}-1)(4+(\gamma-1)(\tau+1))f_9(\tau) \\
&\qquad + (\gamma-1)(\tau^{\gamma-1}-1)\big(4\tau + (\tau-1)(4(\tau^{\gamma-1}-1)+\tau-1)\big) \\
&\qquad - (\gamma-1)^2(\tau^2-1)\tau^{\gamma-1}(3-\tau^{\gamma-1}) \\
f_9(\tau) &\coloneqq \tau(\tau^{\gamma-1}-1) - (\gamma-1)(\tau-1)\tau^{\gamma-1}\,.
\end{aligned}
\tag{3.3.30}
$$

First, we observe that $f_7(\tau) > 0$ for any $\tau > 1$ because $f_7(1) = 0$ and $f_7'(\tau) = 2(\gamma-1)(\tau-1)\tau^{\gamma-1} - 2(\tau^{\gamma-1}-1) + (\gamma-1)^2(\tau^2-1)\tau^{\gamma-2} > 2(\gamma-1)(\tau-1)\tau^{\gamma-1} - 2(\tau^{\gamma-1}-1)$ for $\tau > 1$, and furthermore $\tilde{f}_7(\tau) \coloneqq (\gamma-1)(\tau-1)\tau^{\gamma-1} - (\tau^{\gamma-1}-1)$ satisfies $\tilde{f}_7(1) = 0$ and $\tilde{f}_7'(\tau) = \gamma(\gamma-1)(\tau-1)\tau^{\gamma-2} > 0$, for $\tau > 1$.

Next, we check that $f_9(\tau) > 0$ for any $\tau > 1$, whenever $\gamma \in (1,2)$. Indeed, by direct calculation, $f_9(1) = f_9'(1) = 0$, and $f_9''(\tau) = (2-\gamma)(\gamma-1)(1+\gamma(\tau-1))\tau^{\gamma-3} > 0$, for $\tau > 1$, so long as $\gamma \in (1,2)$. It follows immediately that $f_8(\tau) > 0$ whenever $\tau^{\gamma-1} \geq 3$, and so (3.3.25) is proved under this additional restriction.

Hereafter, we assume $\tau^{\gamma-1} \in (1,3)$. Making use of this new upper bound on τ, we rewrite $f_8(\tau)$ as follows

(3.3.31)
$$\begin{aligned} f_8(\tau) &= 2(\gamma-1)^2(\tau-1)(\tau^{\gamma-1}-1)^2 + (\tau^{\gamma-1}-1)\big(4 + (\gamma-1)(\tau+3)\big) f_9(\tau) \\ &\quad + (\gamma-1)(\tau+1)(3-\tau^{\gamma-1}) f_{10}(\tau) + \tfrac{1}{2}(\gamma-1)(\tau-1)(\tau^{\gamma-1}-1) f_{11}(\tau) \end{aligned}$$

where

(3.3.32)
$$\begin{aligned} f_{10}(\tau) &\coloneqq f_9(\tau) - \tfrac{2-\gamma}{2}(\tau-1)(\tau^{\gamma-1}-1) \\ f_{11}(\tau) &\coloneqq (\tau^{\gamma-1}-1)\big(7 + \tau + (\gamma-1)(\tau+1)\big) - 2(\gamma-1)(\tau-1)\,. \end{aligned}$$

Then $f_{10}(\tau) > 0$ for $\tau > 1$ because $f_{10}(1) = 0$ and $f_{10}'(\tau) = \frac{\gamma}{2\tau} f_9(\tau) > 0$, for $\tau > 1$ and $\gamma \in (1,2)$, as before. Finally, we have $f_{11}(\tau) > 0$ because $f_{11}(1) = 0$, and

(3.3.33)
$$f_{11}'(\tau) = \frac{f_9(\tau)}{\tau} + (\gamma-1)\big(\tau^{\gamma-2}(6 + (\gamma-1)(\tau+1)) + 3(\tau^{\gamma-1}-1)\big) > 0\,,$$

for any $\tau > 1$ and $\gamma \in (1,2)$.

This completes the proof of (3.3.25), and therefore the first part of Theorem 3.6 has been proved.

3.3.3 Existence and uniqueness of the sonic angle

Similar to the previous section, the reflected state (2) is pseudo-supersonic at the reflection point P_0 if and only if

$$M_1 > 1 \qquad \text{and} \qquad \hat{\theta} \le \theta^{\rm s}_{\rm stdy}(1, |D\varphi_1(P_0)|)\,, \tag{3.3.34}$$

where $\theta^{\rm s}_{\rm stdy}(\rho_\infty, u_\infty)$ is the steady detachment angle for the 2-D steady potential flow with constant supersonic upstream state $U_\infty = (\rho_\infty, u_\infty, 0)$, as given by Definition 3.4. Therefore, in order to prove the second part of Theorem 3.6, it is sufficient to determine the existence of a unique value $\theta^{\rm s} \in (\theta^{\rm d}, \frac{\pi}{2})$ satisfying

$$\hat{\theta} = \theta^{\rm s}_{\rm stdy}(1, |D\varphi_1(P_0)|)\,, \qquad \text{when } \theta_{\rm w} = \theta^{\rm s}\,. \tag{3.3.35}$$

Repeating a similar argument to Steps **1**–**4** in §3.3.2, with the steady detachment angle replaced by the steady sonic angle, it is sufficient to prove the inequality

$$1 + \frac{2M_1^2}{w_{\rm s}} \frac{1}{1+w_{\rm s}^2} \frac{{\rm d}w_{\rm s}}{{\rm d}M_1^2} - \frac{3}{M_1^2 - 1} > 0\,, \qquad \text{for any } M_1 > 1\,, \tag{3.3.36}$$

where $w_{\rm s} \coloneqq \tan\theta^{\rm s}_{\rm stdy}(1, |D\varphi_1(P_0)|)$. This will be the objective of the subsequent three steps.

1. In §3.2, we showed that the steady sonic angle for 2-D steady potential flow has a parametric representation in terms of the upstream Mach number, given by (3.2.11)–(3.2.12). We substitute (3.2.11)–(3.2.12) directly into (3.3.36) to obtain the following equivalent sufficient condition in terms of a dimensionless variable $\tau > 1$

$$\frac{\big((\gamma+1)\tau^{\gamma-1} - 2\big)A_{\rm s}(\tau)}{(\gamma^2-1)(\tau+1)(\tau^{\gamma-1}-1)\tau^{\gamma-1}B_{\rm s}(\tau)} - \frac{3(\gamma-1)}{(\gamma+1)(\tau^{\gamma-1}-1)} > 0\,, \qquad \text{for all } \tau > 1\,, \tag{3.3.37}$$

where

$$
\begin{aligned}
A_{\mathrm{s}}(\tau) &\coloneqq 2(\gamma-1+(\gamma+1)\tau)-2(3+\gamma+(\gamma-1)\tau)\tau^{\gamma} \\
&\qquad +(1+3\tau+(\tau-1)(\gamma+(\gamma-1)\tau)^2)\tau^{2\gamma-2} \\
B_{\mathrm{s}}(\tau) &\coloneqq 2+(\gamma-1)\tau^{\gamma+1}-(\gamma+1)\tau^{\gamma-1}\,.
\end{aligned}
\tag{3.3.38}
$$

Note that $M_1 > 1$ in (3.3.36) is equivalent to $\tau > 1$ in (3.3.37) by (3.2.12).

We identify $B_{\mathrm{s}}(\tau) \equiv C_{\mathrm{d}}(\tau) > 0$ from (3.3.24), and therefore we are free to multiply through (3.3.37) by the strictly positive factor $(\gamma^2-1)(\tau+1)(\tau^{\gamma-1}-1)\tau^{\gamma-1}B_{\mathrm{s}}(\tau)$, thereby obtaining

$$
\big((\gamma+1)\tau^{\gamma-1}-2\big)A_{\mathrm{s}}(\tau)-3(\gamma-1)^2(\tau+1)\tau^{\gamma-1}B_{\mathrm{s}}(\tau)>0\,,\qquad \text{for all } \tau>1\,.
\tag{3.3.39}
$$

We write the full expression for f_{s} in the following way

$$
f_{\mathrm{s}}(\tau) \coloneqq 4(\gamma+1)\tau(\tau^{\gamma-1}-1)^3+2(\gamma-1)(\tau^{\gamma-1}-1)f_4(\tau)+(\gamma-1)(\tau+1)\tau^{\gamma-1}f_{12}(\tau)\,,
\tag{3.3.40}
$$

where $f_4(\tau) > 0$ is given by (3.3.27), and

$$
\begin{aligned}
f_{12}(\tau) &\coloneqq (\gamma+1)(\gamma+1+(\gamma-1)\tau)(\tau-1)(\tau^{\gamma-1}-1)^2+(\gamma-1)f_{13}(\tau) \\
f_{13}(\tau) &\coloneqq 2(\tau^2+1)(\tau^{\gamma-1}-1)-(\gamma-1)(\tau^2-1)(\tau^{\gamma-1}+1)\,.
\end{aligned}
\tag{3.3.41}
$$

We claim that $f_{12}(\tau) > 0$ for any $\tau > 1$, and hence, subject to this claim, the proof of Theorem 3.6 is complete. The proof of this claim is then split into two cases depending on $\gamma > 1$.

2. Case I: $\gamma \in (1,3)$. In this case, we calculate directly $f_{13}(1) = 0$, and $f_{13}'(\tau) = (\gamma+1)\tau\tilde{f}_{13}(\tau)$, with $\tilde{f}_{13}(\tau) \coloneqq 2(\tau^{\gamma-1}-1)-(\gamma-1)\tau^{\gamma-3}(\tau^2-1)$. Moreover, $\tilde{f}_{13}$ satisfies $\tilde{f}_{13}(1) = 0$, and $\tilde{f}_{13}'(\tau) = (\gamma-1)(3-\gamma)(\tau^2-1)\tau^{\gamma-4} > 0$, for $\tau > 1$, so long as $\gamma \in (1,3)$. Therefore, in this case, $f_{12}(\tau) > 0$ for all $\tau > 1$.

3. Case II: $\gamma > 3$**.** In this case, we rewrite the expression for $f_{12}(\tau)$ as follows

$$f_{12}(\tau) = 2(\gamma+1)(\tau^2+\tau-2)(\tau^{\gamma-1}-1)^2 + f_{14}(\tau)\,, \tag{3.3.42}$$

where

$$f_{14}(\tau) \coloneqq ((\gamma-1)^2-4)(\tau^2-1)(\tau^{\gamma-1}-1)^2 + (\gamma-1)f_{13}(\tau)\,. \tag{3.3.43}$$

Then it remains to show that $f_{14}(\tau) > 0$ for any $\tau > 1$, whenever $\gamma > 3$. Indeed, $f_{14}(1) = 0$, and we calculate directly

$$\begin{aligned} f_{14}'(\tau) = {} & 2(\gamma+1)(\gamma-3)\tau^{-1}(\tau^{\gamma-1}-1)^2\big(1+\gamma(\tau^2-1)\big) \\ & + (\gamma-1)(\gamma+1)\tau^{-1}\big(3f_{15}(\tau) - f_{13}(\tau)\big)\,, \end{aligned}$$

where $f_{15}(\tau) \coloneqq 2(\tau^{\gamma-1}-1) - (\gamma-1)(\tau^2-1)$, and f_{13} is given by (3.3.41). Then $f_{15}(1) = 0$, and $f_{15}'(\tau) = 2(\gamma-1)\tau(\tau^{\gamma-3}-1) > 0$, for $\tau > 1$, so long as $\gamma > 3$.

Finally, from Step **2** above, it is clear that $f_{13}(\tau) < 0$ for any $\tau > 1$, so long as $\gamma > 3$. Therefore $f_{12}(\tau) > 0$ for all $\tau > 1$, and the proof of Theorem 3.6 is complete. □

In Theorem 3.6, by proving the uniqueness of the detachment angle, we have given the precise existence criteria for the regular reflection of a planar shock by a wedge in potential flow. Similarly, by proving the uniqueness of the sonic angle, we have clarified exactly the transition between supersonic and subsonic regular reflections. The values of the critical angles may be computed as follows:

- Substitute (3.3.5) into (3.3.6)/(3.3.34) and use (3.3.3) to eliminate $\tan\theta_{\mathrm{w}}$ in terms of M_1.
- Substitute (3.2.8)–(3.2.9)/(3.2.11)–(3.2.12) into the resulting equation and compute the unique root $\tau > 1$.
- Substitute (3.2.9)/(3.2.12) into (3.3.3) and evaluate at τ to obtain the critical angle.

Of course, in general, finding explicit formulae for the critical angles is likely impossible, though, the case $\gamma = 2$ is, in theory, computable in terms of radicals. Indeed, following the above procedure, direct calculation gives

$$\theta^{\mathrm{d}} = \arctan \sqrt{\frac{(1+\rho_0)\tau^2 + (3+3\rho_0-\rho_0^2)\tau - (2+2\rho_0+3\rho_0^2)}{3+\tau}} \,,$$

where $\tau > 1$ is the unique root on $(1, \infty)$ of the fourth degree polynomial

$$a_4\tau^4 + a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0 = 0 \,,$$

with coefficients $a_4 \coloneqq (1+\rho_0)^2$, $a_3 \coloneqq 2(1+\rho_0)(1+3\rho_0)$, $a_2 \coloneqq -4(3+\rho_0-3\rho_0^2)$, $a_1 \coloneqq -2(3-\rho_0)(3+5\rho_0)$, and $a_0 \coloneqq 11+2\rho_0+3\rho_0^2$.

In comparison, in the context of the full system of Euler equations, a rather beautiful explicit formula for the detachment angle was obtained in [11] for any $\gamma > 1$, based on the earlier works [5, 68, 69]. It reads

$$\theta^{\mathrm{d}}_{\mathrm{Euler}} = \arctan\left(\rho_0 \sqrt{\tfrac{2}{3}\sqrt{b^2+3c}\,\cos\tfrac{\beta}{3} + \tfrac{b}{3} - 1}\right) ,$$

where $a \coloneqq \frac{2}{1-\mu^2}(\frac{1}{\rho_0} - 1)(\frac{1}{\rho_0} - \mu^2)$, $b \coloneqq 1 + \frac{2\mu^2}{1-\mu^2}\,a$, $c \coloneqq (\frac{\rho_0}{1-\rho_0} + \frac{\rho_0\mu^2}{1-\rho_0\mu^2})a^2$, for $\mu^2 \coloneqq \frac{\gamma-1}{\gamma+1}$; and $\beta \coloneqq \arccos\left(\frac{27a^2+2b^2+9bc}{2(b^2+3c)^{3/2}}\right) \in (0, \frac{\pi}{2})$. Later, in [76], an explicit, though comparatively gruesome formula for the sonic angle was obtained.

In Figure 3.6 we plot the detachment angles for potential flow and the full system of Euler equations in the case $\gamma = 2$. We see that the detachment angles correspond asymptotically in the limit of weak incident shocks $\rho_0 \to 1^-$, whilst for stronger incident shocks they remain in reasonable agreement.

3.4 Future directions

In Lemma 3.5 and Theorem 3.6, we have obtained new quantitative properties of the shock polar curve for 2-D steady potential flow which should be useful in the further investigation of 2-D Riemann problems. Throughout the proofs of these results, it was

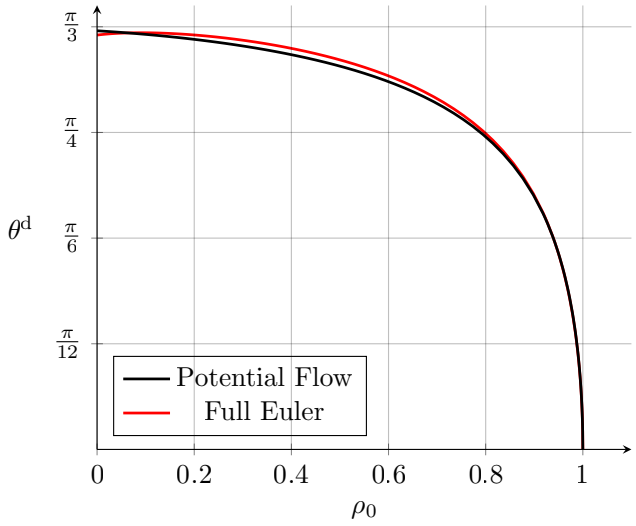


Figure 3.6: The detachment angles for potential flow and the full system of Euler equations in the case $\gamma = 2$.

necessary to use the precise algebraic form of the polytropic pressure law $p(\rho) = \frac{1}{\gamma}\rho^{\gamma}$. In my opinion, obtaining similar results for general pressure laws will not be an easy task, and should likely require a different approach.

Some properties of the shock polar for 2-D potential flow for general pressure laws have been investigated in [34] under the assumptions of hyperbolicity and *convex equations of state*. In particular, it has been shown that the shock polars admit a unique sonic and detachment point in this case.

Chapter 4

Four-shock Interaction Configurations

The work presented in this chapter is based on the preprint [17], to appear in J. Eur. Math. Soc. (JEMS), which is joint work with the authors, for which I made a significant contribution. The main result is Theorem 4.9, the existence of global weak solutions to the four-shock interaction problem.

4.1 General Setting

We are concerned with the 2-D Riemann problem for the Euler equations for potential flow consisting of the conservation law of mass and the Bernoulli law

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\boldsymbol{x}}(\rho \nabla_{\boldsymbol{x}} \Phi) = 0\,, \\ \partial_t \Phi + \frac{1}{2}|\nabla_{\boldsymbol{x}} \Phi|^2 + h(\rho) = B\,. \end{cases} \tag{4.1.1}$$

The mathematical preliminaries for potential flow were covered in §2, including the definition of weak solutions, the self-similar formulation, and the classification of the 2-D Riemann problems in four quadrants.

The 2-D Riemann initial data under consideration takes the following form

$$(\rho, \Phi)(0, \boldsymbol{x}) = (\rho_i, (u_i, v_i) \cdot \boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Lambda_i\,, \tag{4.1.2}$$

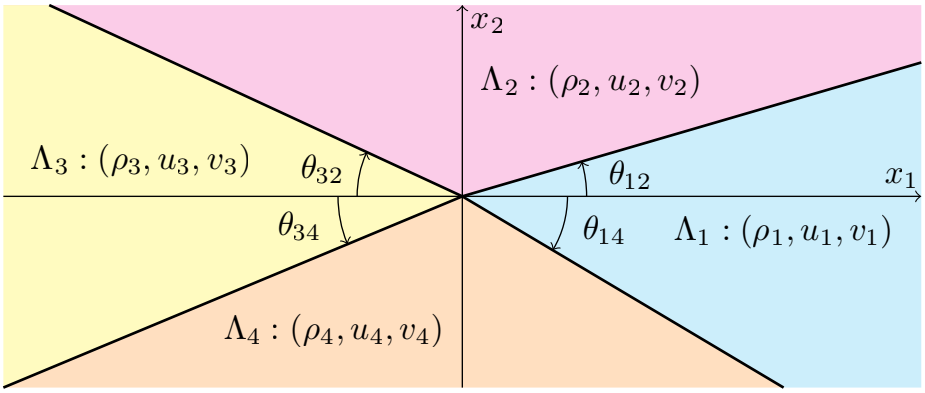


Figure 4.1: The four sectors of the Riemann initial data.

for suitable constant states (i) with values $U_i \coloneqq (\rho_i, u_i, v_i)$, and corresponding scale-invariant domains $\Lambda_i \subseteq \mathbf{R}^2$ for $i = 1, 2, 3, 4$. We choose these domains to be four sectors of $\mathbf{R}^2$, defined by

(4.1.3)

$$\Lambda_1 \coloneqq \left\{\boldsymbol{x} \in \mathbf{R}^2 \,:\, \theta \in (-\theta_{14}, \theta_{12})\right\}, \qquad \Lambda_2 \coloneqq \left\{\boldsymbol{x} \in \mathbf{R}^2 \,:\, \theta \in (\theta_{12}, \pi - \theta_{32})\right\},$$
$$\Lambda_3 \coloneqq \left\{\boldsymbol{x} \in \mathbf{R}^2 \,:\, \theta \in (\pi - \theta_{32}, \pi + \theta_{34})\right\}, \quad \Lambda_4 \coloneqq \left\{\boldsymbol{x} \in \mathbf{R}^2 \,:\, \theta \in (\pi + \theta_{34}, 2\pi - \theta_{14})\right\},$$

where θ is the plane polar angle corresponding to the point $\boldsymbol{x} \in \mathbf{R}^2$, and the four parameters θ_{12}, θ_{32}, θ_{34}, and θ_{14} take values in $(0, \frac{\pi}{2})$. The initial data and discontinuities are depicted in Figure 4.1.

More specifically, the 2-D Riemann initial data (4.1.2) under consideration will satisfy that, for $j = 2, 4$,

$$\begin{cases} \text{one right-moving shock } \overrightarrow{S}_{1j} \text{ is generated between states } (1) \text{ and } (j)\,, \\ \text{one left-moving shock } \overleftarrow{S}_{3j} \text{ is generated between states } (3) \text{ and } (j)\,. \end{cases}$$

Furthermore, we restrict our attention to the case that the angular bisectors at the vertices of Λ_1 and Λ_3 in Figure 4.1 coincide, *i.e.*, without loss of generality, we choose $\theta_{12} = \theta_{14}$ and $\theta_{32} = \theta_{34}$. In this case, we show later that the x_1–axis may be considered as a rigid wall. The above Riemann problem corresponds to Configuration 5 given in §2.3.

The problem considered has a close relation to the shock reflection-diffraction problem, and the Prandtl–Meyer reflection problem that were discussed in §1.3.3. In the most recent three decades, there have been many theoretical results on shock reflection/diffraction problems. Indeed, in the context of potential flow, Chen-Feldman [18] proved the first global existence result for solutions of the shock reflection-diffraction problem when the incident angles are small, and later in [20], they succeeded to establish the global existence of solutions for incident angles up to the detachment angle. Furthermore, Chen-Feldman-Xiang [22] proved the convexity of transonic shocks, which leads to the uniqueness and stability of the admissible solution [23]. More recently, Bae-Chen-Feldman [1] obtained the existence and optimal regularity of admissible solutions of the Prandtl–Meyer reflection problem for the general case of parameters up to the detachment angle, superseding the earlier partial result of Elling-Liu [32].

For the 2-D Riemann problem with the interaction of four shockwaves considered presently, we can reformulate the problem into a shock reflection-diffraction problem with respect to a symmetric line. Compared with the previous results, such as the shock reflection-diffraction problem [20] and the Prandtl–Meyer reflection problem [1], we now have two independent incident angles. Correspondingly, we have two sonic boundaries varying with the choice of incident angles, which leads to more complicated configurations. After introducing the self-similar coordinates, the original problem can be reformulated as a free boundary problem for a second-order quasilinear PDE of mixed elliptic-hyperbolic type. The difficulties come from the degenerate ellipticity of the nonlinear PDE near the sonic boundaries, the nonlinearity of the free boundary conditions, the singularity of the solution near the corners of the domain, and the geometric properties of the free boundary. To solve our problem, we need to analyse the solutions of a quasilinear degenerate elliptic equation by the maximum principle of mixed-boundary value problems and the theory of the oblique derivative boundary value problems [40, 60, 61], as well as the uniform estimates and iteration method from [1, 20, 33].

The organisation of the remainder of this chapter is as follows: in §4.2, we present

the mathematical formulation of the 2-D Riemann problem with four-shock interactions and give the definition of admissible solutions of the free boundary problem, after which we state our main existence theorem. In §4.3, we obtain the directional monotonicity, the strict ellipticity, and the uniform weighted Hölder estimates of admissible solutions. Finally, in §4.4, we prove the existence of admissible solutions by constructing a suitable iteration set and iteration map and applying the Leray–Schauder degree theorem. In Appendix A, we give the details for some calculations involving admissible solutions and, in Appendix B, we present some known results that are needed throughout the proof of the main theorem.

4.2 The Riemann Problem and Main Theorems for Four-Shock Interactions

In this section, we first present the mathematical formulation of the 2-D Riemann problem with four-shock interactions as a free boundary problem, along with the definition of admissible solutions of the free boundary problem. We conclude with the statement of our main existence theorem.

4.2.1 Mathematical formulation

The Riemann problem (4.1.1)–(4.1.2) is invariant under the self-similar scaling (2.2.1). This leads us to consider the self-similar solution $(\rho,\phi)(\boldsymbol{\xi}) \coloneqq (\rho,\frac{\Phi}{t})(t,\boldsymbol{x})$ in the self-similar coordinates $\boldsymbol{\xi} = (\xi,\eta) \coloneqq \frac{\boldsymbol{x}}{t} \in \mathbf{R}^2$, as introduced in §2.2. Recall that the pseudo-potential function $\varphi(\boldsymbol{\xi}) \coloneqq \phi(\boldsymbol{\xi}) - \frac{1}{2}|\boldsymbol{\xi}|^2$ satisfies the pseudo-steady potential flow equation

$$\operatorname{div}(\rho(|D\varphi|,\varphi)D\varphi) + 2\rho(|D\varphi|,\varphi) = 0\,, \tag{4.2.1}$$

where div and D represent the divergence and gradient operators with respect to the self-similar coordinates $\boldsymbol{\xi} \in \mathbf{R}^2$, and the density is determined from the Bernoulli

law as

$$\rho(|D\varphi|,\varphi)=\begin{cases}\big(1+(\gamma-1)(B-\frac{1}{2}|D\varphi|^2-\varphi)\big)^{\frac{1}{\gamma-1}} & \text{for } \gamma>1\,,\\ \exp(B-\frac{1}{2}|D\varphi|^2-\varphi) & \text{for } \gamma=1\,.\end{cases} \tag{4.2.2}$$

Equation (4.2.1) with (4.2.2) is a second-order nonlinear equation of mixed elliptic-hyperbolic type. It is elliptic if and only if $|D\varphi|<c(|D\varphi|,\varphi)$ (pseudo-subsonic), while it is hyperbolic if and only if $|D\varphi|>c(|D\varphi|,\varphi)$ (pseudo-supersonic).

The definition of weak solutions of (4.2.1)–(4.2.2) was given in Definition 2.5, followed by the associated Rankine–Hugoniot conditions (2.2.8), and the physical entropy condition that must be satisfied by shocks in Definition 2.6.

In the self-similar coordinates $\boldsymbol{\xi}=(\xi,\eta)$, the initial conditions (4.1.2) become the following asymptotic boundary conditions at infinity: For any $\theta\in[0,2\pi)$ and $i=1,2,3,4$,

$$\lim_{r\to\infty}\|\varphi-\varphi_i\|_{C^{0,1}(R_\theta\setminus B_r(\mathbf{0}))}=0\,,$$

whenever the ray $R_\theta\coloneqq\{r(\cos\theta,\sin\theta)\,:\,r>0\}$ is contained in the sector Λ_i as defined in (4.1.3), where the pseudo-potential function φ_i corresponding to the uniform state (i) is given by

$$\varphi_i(\boldsymbol{\xi})\coloneqq-\frac{1}{2}|\boldsymbol{\xi}|^2+(u_i,v_i)\cdot\boldsymbol{\xi}+k_i\,, \tag{4.2.3}$$

and the constant k_i is uniquely determined to satisfy (4.2.2) with $(\rho,\varphi)=(\rho_i,\varphi_i)$ for $i=1,2,3,4$. We may simply fix $k_2=0$ in (4.2.3) since the Bernoulli constant can be shifted by the value of k_2.

By the Galilean invariance of the system (4.1.1), *cf.* Remark 2.8, it is convenient to normalise the initial data to satisfy $u_2=v_1=0$. From (4.2.2) and (4.2.3) with $k_2=0$, the Bernoulli constant B is then fixed as

$$B=\frac{1}{2}v_2^2+h(\rho_2)\,. \tag{4.2.4}$$

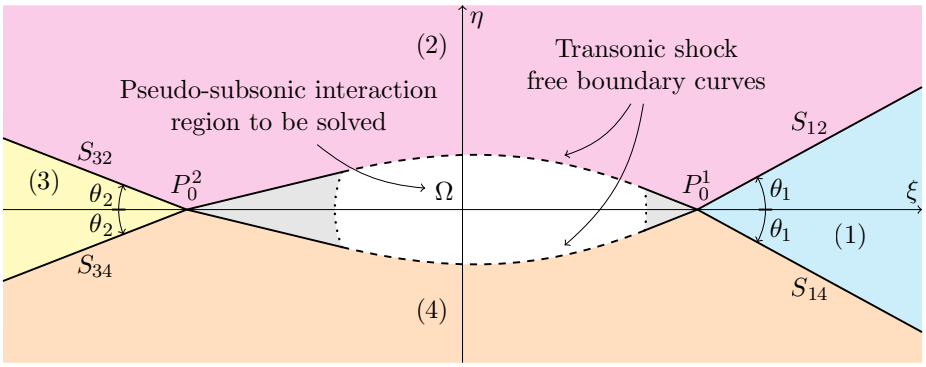


Figure 4.2: Four-shock interaction with symmetric incident shocks.

In §4.1, we stated the primary restrictions on our initial data—the initial data (4.1.2) are chosen such that, in the far field region, any two neighbouring states are connected by exactly one planar shock discontinuity. Additionally, we restrict our attention to the case

$$\max\{\rho_1, \rho_3\} < \min\{\rho_2, \rho_4\}\,, \tag{4.2.5}$$

so that the initial data consist precisely of two right-moving shocks $\{\overrightarrow{S}_{12}, \overrightarrow{S}_{14}\}$, and two left-moving shocks $\{\overleftarrow{S}_{32}, \overleftarrow{S}_{34}\}$, which travel away from the origin in pairs; see Configuration 5 in §2.3. Moreover, to facilitate our analysis, we focus on the symmetric case

$$\theta_{12} = \theta_{14} =: \theta_1\,, \quad \theta_{32} = \theta_{34} =: \theta_2 \qquad \text{with } \theta_1, \theta_2 \in (0, \tfrac{\pi}{2})\,. \tag{4.2.6}$$

We refer to the parameters $\boldsymbol{\theta} := (\theta_1, \theta_2)$ as the incident angles of the shock discontinuities; see Figure 4.2.

4.2.1.1 Compatibility conditions

It is clear that the initial data and parameters $\boldsymbol{\theta}$ cannot be specified arbitrarily. In the following lemma, we give the necessary and sufficient conditions for the initial data $U_i = (\rho_i, u_i, v_i)$ and parameters $\boldsymbol{\theta}$ to generate precisely two right-moving shocks $\{\overrightarrow{S}_{12}, \overrightarrow{S}_{14}\}$ and two left-moving shocks $\{\overleftarrow{S}_{32}, \overleftarrow{S}_{34}\}$.

Lemma 4.1. *Fix $\gamma \geq 1$ and $\rho_2 > 0$. There exist both a constant $v_{\min} \in [-\infty, 0)$ depending on (γ, ρ_2) and a critical angle $\theta^{\mathrm{cr}} \in (0, \frac{\pi}{2}]$ depending on (γ, ρ_2, v_2) such that: there exists a unique choice of constant states $U_i = (\rho_i, u_i, v_i)$, for $i = 1, 3, 4$, satisfying the Rankine–Hugoniot conditions (2.2.8) between any two neighbouring states and the entropy condition (4.2.5), and generating precisely two right-moving shocks $\{\overrightarrow{S}_{12}, \overrightarrow{S}_{14}\}$ and two left-moving shocks $\{\overleftarrow{S}_{32}, \overleftarrow{S}_{34}\}$, if and only if $v_2 \in (v_{\min}, 0)$ and $\boldsymbol{\theta} \in (0, \theta^{\mathrm{cr}})^2$.*

Proof. Throughout this proof, we take $i = 1, 3$, and $j = 2, 4$. Temporarily, we do not impose the symmetry condition (4.2.6), so as to derive the full compatibility conditions for the initial data which may be useful for later endeavours.

In the self-similar coordinates, we denote the shock discontinuity line between state (i) and state (j) by

$$S_{ij} \coloneqq \{\boldsymbol{\xi} \in \mathbf{R}^2 \,:\, \varphi_i(\boldsymbol{\xi}) = \varphi_j(\boldsymbol{\xi})\} = \big\{\boldsymbol{\xi} \in \mathbf{R}^2 \,:\, \eta = (-1)^{\frac{i+j+1}{2}} \xi \tan\theta_{ij} + a_{ij}\big\}\,, \tag{4.2.7}$$

where the expressions of φ_i and φ_j are given in (4.2.3); $\theta_{ij} \in (0, \frac{\pi}{2})$ is the acute angle between the ξ–axis and the shock discontinuity line S_{ij}; and a_{ij} is a constant determined below. We fix the unit normal vector on S_{ij} to be

$$\boldsymbol{\nu}_{ij} = \big((-1)^{\frac{i+1}{2}} \sin\theta_{ij}, (-1)^{\frac{j+2}{2}} \cos\theta_{ij}\big)\,,$$

which points towards the downstream state according to the entropy condition (4.2.5), so that

$$D\varphi_i(\boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{ij} > D\varphi_j(\boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{ij} > 0 \qquad \text{for } \boldsymbol{\xi} \in S_{ij}\,. \tag{4.2.8}$$

Using (2.2.8), (4.2.2)–(4.2.3) and (4.2.7)–(4.2.8), we find

$$(u_i, v_i) - (u_j, v_j) = \ell_{ij} \boldsymbol{\nu}_{ij}\,, \qquad a_{ij} = (-1)^{\frac{j+2}{2}} \frac{\big(\rho_i(u_i, v_i) - \rho_j(u_j, v_j)\big) \cdot \boldsymbol{\nu}_{ij}}{(\rho_i - \rho_j)\cos\theta_{ij}}\,, \tag{4.2.9}$$

where $\ell_{ij} \coloneqq \ell(\rho_i, \rho_j)$ is the function given by (2.3.3), which is a symmetric function and satisfies the strict monotonicity property (2.3.4).

From (4.2.6) and (4.2.9), we deduce that $(\ell_{12}-\ell_{14})\sin\theta_1=(\ell_{34}-\ell_{32})\sin\theta_2$, from which $\rho_4=\rho_2$ must hold by virtue of the entropy condition (4.2.5) and (2.3.4), else the left and right hand sides would have different signs. Thus, we obtain the following relations

$$\ell_{12}\cos\theta_1=\ell_{32}\cos\theta_2\,,\qquad \begin{cases} u_2=u_4=u_1+\ell_{12}\sin\theta_1=u_3-\ell_{32}\sin\theta_2\,,\\ v_1=v_3=v_2+\ell_{12}\cos\theta_1=v_4-\ell_{32}\cos\theta_2\,. \end{cases}$$

Recall that $u_2=v_1=0$ are fixed, so that

(4.2.10)
$$U_1=(\rho_1,-\ell_{12}\sin\theta_1,0)\,,\ \ U_2=(\rho_2,0,v_2)\,,\ \ U_3=(\rho_3,\ell_{32}\sin\theta_2,0)\,,\ \ U_4=(\rho_2,0,-v_2)\,,$$

whenever $\rho_1,\rho_3\in(0,\rho_2)$ satisfy

$$\ell_{12}\cos\theta_1=\ell_{32}\cos\theta_2=-v_2\,. \tag{4.2.11}$$

We refer to (4.2.11) as the compatibility condition for the initial data. We define

$$v_{\min}:=-\ell(0^+,\rho_2)\in[-\infty,0)\,,\qquad \theta^{\rm cr}:=\arccos\big(\frac{-v_2}{\ell(0^+,\rho_2)}\big)\in(0,\tfrac{\pi}{2}]\,. \tag{4.2.12}$$

Then, by (2.3.4), the necessary and sufficient conditions for the existence of ρ_1 and ρ_3 uniquely solving (4.2.11) subject to (4.2.5) are

$$v_2\in(v_{\min},0)\,,\qquad \boldsymbol{\theta}\in(0,\theta^{\rm cr})^2\,.$$

The constant states U_i, $i=1,3,4$, are now uniquely determined by (4.2.10)–(4.2.11) depending only on $(\gamma,\rho_2,v_2,\boldsymbol{\theta})$. □

We call $\theta^{\rm cr}$ the vacuum critical angle, because $\rho_1\to0^+$ as $\theta_1\to\theta^{\rm cr-}$ and $\rho_3\to0^+$ as $\theta_2\to\theta^{\rm cr-}$, according to (4.2.11). Note that, in potential flow, the incident shock with vacuum upstream state is mathematically consistent, which is in contrast to the non-potential Euler flow for which the maximal density ratio across a shock is

bounded.

During the proof of Lemma 4.1 it was determined that $\rho_2 = \rho_4$. Without loss of generality, we fix $\rho_2 = \rho_4 = 1$ via the scale-invariance of equation (4.2.1), *cf.* Remark 2.8. After scaling, the entropy condition (4.2.5) becomes $\max\{\rho_1, \rho_3\} < 1$, and $v_{\min} = -\ell(0^+, 1)$ depends only on $\gamma \geq 1$. For simplicity, we often use the abbreviation $\ell(\cdot) \coloneqq \ell(\cdot, 1)$.

4.2.1.2 The asymptotic boundary value problem in the upper half-plane

Equation (4.1.1) and the initial data in (4.1.2) given by (4.2.10) are invariant under reflection with respect to the x_1–axis. Thus, we look for solutions of (4.1.1)–(4.1.2) in the symmetric form:

$$\Phi(t, x_1, x_2) = \Phi(t, x_1, -x_2) \qquad \text{for all } (t, x_1, x_2) \in (0, \infty) \times \mathbf{R}^2 \,,$$

which is equivalent to

$$\varphi(\xi, \eta) = \varphi(\xi, -\eta) \qquad \text{for all } (\xi, \eta) \in \mathbf{R}^2 \,.$$

For this reason, it suffices to consider the restriction of solutions to the upper half-plane

$$\mathbf{R}^2_+ \coloneqq \left\{ \boldsymbol{\xi} \in \mathbf{R}^2 \,:\, \eta > 0 \right\},$$

so long as we impose the slip boundary condition

$$D\varphi \cdot (0, 1) = 0 \qquad \text{on } L_{\text{sym}} \coloneqq \left\{ \boldsymbol{\xi} \in \mathbf{R}^2 \,:\, \eta = 0 \right\}.$$

This condition means that L_{sym} can be regarded as a solid wall.

Hereafter, the symbols S_{12} and S_{32} will represent only the closure of the rays of the incident shocks S_{12} and S_{32} lying in the upper half-plane. We denote by $P_0^1 = (\xi^{P_0^1}, \eta^{P_0^1})$ the point of intersection between S_{12} and L_{sym}, and by $P_0^2 = (\xi^{P_0^2}, \eta^{P_0^2})$ the point of intersection between S_{32} and L_{sym}. We expect the incident shocks S_{12} and S_{32} to

undergo shock reflection-diffraction at points P_0^1 and P_0^2 respectively; see Figure 4.2. From (4.2.7), (4.2.9), and (4.2.10), we have

(4.2.13)
$$P_0^1 = \big(-\ell(\rho_1)\sin\theta_1 + \frac{1}{1-\rho_1}\frac{\ell(\rho_1)}{\sin\theta_1},\, 0\big)\,, \qquad P_0^2 = \big(\ell(\rho_3)\sin\theta_2 - \frac{1}{1-\rho_3}\frac{\ell(\rho_3)}{\sin\theta_2},\, 0\big)\,.$$

Using (4.2.3), (4.2.10)–(4.2.11), and (4.2.13), the pseudo-potentials of states (1)–(3) are given by

(4.2.14)
$$\varphi_1 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + v_2(\xi - \xi^{P_0^1})\tan\theta_1,\ \varphi_2 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + v_2\eta,\ \varphi_3 = -\frac{1}{2}|\boldsymbol{\xi}|^2 - v_2(\xi - \xi^{P_0^2})\tan\theta_2.$$

In particular, φ_1 is independent of θ_2, while φ_3 is independent of θ_1.

Define $M_2^{(\theta_1)} := |D\varphi_2(P_0^1)|$ the pseudo-Mach number of state (2) at the reflection point P_0^1. In Lemma A.1, we show that $M_2^{(\theta_1)}$ is a strictly increasing function of $\theta_1 \in (0, \theta^{\rm cr})$. Then, since we may write $M_2^{(\theta_1)} = |(-\xi^{P_0^1}, v_2)|$, it follows immediately that $\xi^{P_0^1} > 0$ is a strictly decreasing function of $\theta_1 \in (0, \theta^{\rm cr})$ with $\xi^{P_0^1} \to \infty$ as $\theta_1 \to 0^+$. Similarly, $\xi^{P_0^2} < 0$ is a strictly increasing function of $\theta_2 \in (0, \theta^{\rm cr})$ with $\xi^{P_0^2} \to -\infty$ as $\theta_2 \to 0^+$.

We define the background solution $\bar\varphi \in C^{0,1}_{\rm loc}(\mathbf{R}^2_+)$ as

$$\bar\varphi(\boldsymbol{\xi}) := \min\big\{\varphi_1(\boldsymbol{\xi}), \varphi_2(\boldsymbol{\xi}), \varphi_3(\boldsymbol{\xi})\big\}\,, \tag{4.2.15}$$

and define the three open domains

$$\Omega_i := \operatorname{int}\{\boldsymbol{\xi} \in \mathbf{R}^2_+ \,:\, \varphi_i(\boldsymbol{\xi}) = \bar\varphi(\boldsymbol{\xi})\} \qquad \text{for } i = 1, 2, 3\,, \tag{4.2.16}$$

where $\operatorname{int} G$ denotes the interior of a set $G \subseteq \mathbf{R}^2$. Note that Ω_1 and Ω_3 are wedge-shaped domains with wedge angles θ_1 and θ_2 respectively, and they become empty sets in the limits $\theta_1 \to 0^+$ and $\theta_2 \to 0^+$, respectively.

In light of the above discussion, we seek solutions of the following asymptotic boundary value problem in the self-similar coordinates $\boldsymbol{\xi} = (\xi, \eta)$ in the upper half-plane $\mathbf{R}^2_+$.

Problem A (Asymptotic boundary value problem)**.** *Fix* $\gamma \geq 1$ *and* $v_2 \in (v_{\min}, 0)$. *For any* $\boldsymbol{\theta} \in (0, \theta^{\rm cr})^2$, *determine the existence of a weak solution* $\varphi \in W^{1,\infty}_{\rm loc}(\mathbf{R}^2_+)$ *of equations* (4.2.1)–(4.2.2) *in* $\mathbf{R}^2_+$ *satisfying the following conditions*:

(i) *The asymptotic boundary condition at infinity*

$$\lim_{r\to\infty} \|\varphi - \bar{\varphi}\|_{C^{0,1}(R_\theta \setminus B_r(\mathbf{0}))} = 0 \qquad \text{for any } \theta \in (0,\pi)\,;$$

(ii) *The slip boundary condition on the symmetric boundary*

$$D\varphi \cdot (0,1) = 0 \qquad \text{on } L_{\rm sym} = \left\{\boldsymbol{\xi} \in \mathbf{R}^2 \,:\, \eta = 0\right\}.$$

4.2.1.3 Normal reflection configurations

It is meaningful to extend the range of parameters in Problem A to allow $\boldsymbol{\theta} \in [0, \theta^{\rm cr})^2$. We study the reflection for the case $\boldsymbol{\theta} = \mathbf{0}$, which we call the normal reflection configuration. It is clear that setting $\varphi = \bar{\varphi} \equiv \varphi_2$ would satisfy Problem A(i) at infinity. However, since φ_2 does not satisfy Problem A(ii), a boundary layer must be present. For this reason, we introduce a uniform downstream state (0), determined by a pseudo-potential φ_0 satisfying Problem A(ii), such that a straight reflected shock S_0 is formed between states (2) and (0). In fact, the only possible straight reflected shock that satisfies Problem A(i) is a normal shock $S_0 \coloneqq \{\boldsymbol{\xi} \in \mathbf{R}^2_+ \,:\, \eta = \eta_0\}$ for some $\eta_0 > 0$, which is parallel to $L_{\rm sym}$. The constant density of state (0), given by $\rho_0 \coloneqq \rho(|D\varphi_0|, \varphi_0)$, should satisfy the entropy condition $\rho_0 > 1$. We demonstrate that state (0) described above is uniquely determined by (γ, v_2).

The pseudo-potential φ_0 has form (4.2.3) with $i = 0$, for suitable constants (u_0, v_0, k_0) to be determined. It follows from Problem A(ii) that $v_0 = 0$, whilst $u_0 = 0$ follows from the Rankine–Hugoniot conditions (2.2.8) between states (2) and (0) because S_0 is parallel to $L_{\rm sym}$. The value of $k_0 = v_2\eta_0$ is then obtained by using the continuity of $\varphi = \varphi_0 = \varphi_2$ on S_0.

It remains to determine the constant $\eta_0 > 0$, which fixes the location of S_0. Combining the Bernoulli law (4.2.2) with the Rankine–Hugoniot conditions (2.2.8),

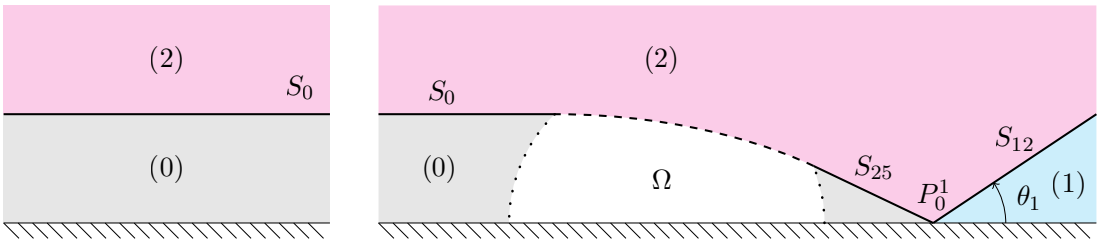


Figure 4.3: Structure of solutions of Problem A involving normal shocks. Left: the normal reflection $(\theta_1, \theta_2) = (0, 0)$. Right: a unilateral normal reflection $\theta_1 > \theta_2 = 0$.

we find the necessary condition

$$\ell(\rho_0) = -v_2 \,. \tag{4.2.17}$$

It is clear that (4.2.17) admits a unique solution $\rho_0 \in (1, \infty)$ depending only on (γ, v_2). Indeed, $\ell(\cdot)$ is strictly increasing on $(1, \infty)$ and $\ell(1) = 0$. The constant η_0 is then uniquely determined by the Rankine–Hugoniot conditions (2.2.8) to be

$$\eta_0 \coloneqq -\frac{v_2}{\rho_0 - 1} > 0 \qquad \text{where } \rho_0 \in (1, \infty) \text{ satisfies (4.2.17).} \tag{4.2.18}$$

Thus, the pseudo-potential of state (0) has been uniquely determined by (γ, v_2) as

$$\varphi_0(\boldsymbol{\xi}) \coloneqq -\frac{1}{2}|\boldsymbol{\xi}|^2 + v_2\eta_0 \,.$$

It is direct to verify that

$$\varphi_{\mathrm{norm}}(\boldsymbol{\xi}) \coloneqq \min\{\varphi_0(\boldsymbol{\xi}), \varphi_2(\boldsymbol{\xi})\} = \begin{cases} \varphi_2(\boldsymbol{\xi}) & \text{for } \eta > \eta_0 \,, \\ \varphi_0(\boldsymbol{\xi}) & \text{for } 0 < \eta < \eta_0 \,, \end{cases} \tag{4.2.19}$$

is a solution to Problem A when $\boldsymbol{\theta} = \mathbf{0}$; see Figure 4.3 (left).

Briefly, we mention the case $\boldsymbol{\theta} \in \big(\{0\} \times (0, \theta^{\mathrm{cr}})\big) \cup \big((0, \theta^{\mathrm{cr}}) \times \{0\}\big)$, which is called the unilateral normal reflection; a normal reflection on one side with respect to the symmetric boundary L_{sym} occurs by the same argument as the normal reflection case $\boldsymbol{\theta} = \mathbf{0}$ above, while the other side undergoes a regular reflection at the reflection point,

of which details will be discussed in §4.2.2 below; see the case: $\boldsymbol{\theta} \in (0, \theta^{\mathrm{cr}}) \times \{0\}$ in Figure 4.3 (right). Moreover, the unilateral normal reflections are essentially the same as the Prandtl–Meyer reflection configurations considered in [1].

4.2.2 Regular reflection configurations

In this section, we introduce the detachment and sonic angles and describe the structure of three genuinely different configurations of the four-shock interaction.

4.2.2.1 Detachment and sonic angles

For $\boldsymbol{\theta} \in (0, \theta^{\mathrm{cr}})^2$, the incident shocks S_{12} and S_{32} intersect the symmetric line L_{sym} at the points P_0^1 and P_0^2 respectively, with coordinates given by (4.2.13). As discussed above, the symmetric line can be regarded as a rigid wall and, similarly to the previous work [20], we expect that, if the incident angle θ_1 is less than some detachment angle θ^{d}, there should exist a regular reflection of S_{12} at the point P_0^1, *i.e.*, there should exist both a uniform state (5) determined by a pseudo-potential φ_5 and a straight reflected shock S_{25} passing through P_0^1 such that S_{25} separates the uniform upstream state (2) from the uniform downstream state (5). Similarly, we expect that there should exist a regular reflection of S_{32} at the point P_0^2 if θ_2 is less than θ^{d}; that is, there should exist both a uniform state (6) determined by a pseudo-potential φ_6 and a straight reflected shock S_{26} passing through P_0^2. The expected structure of shock regular reflection on the upper half-plane can be seen in Figure 4.2. We now determine the detachment angle θ^{d} and the sonic angle θ^{s}.

The regular reflection of S_{12} at the point P_0^1 can be reduced to the study of an algebraic system. By the argument in [20, §7.1], to determine the uniform state (5), it suffices to apply the Rankine–Hugoniot conditions at the reflection point P_0^1 only. The pseudo-potential φ_5 has form (4.2.3) with $i = 5$, and the straight reflected shock S_{25} must have the form

$$S_{25} \coloneqq \big\{\boldsymbol{\xi} \in \overline{\mathbf{R}^2_+} \,:\, \varphi_5(\boldsymbol{\xi}) = \varphi_2(\boldsymbol{\xi})\big\} = \big\{\boldsymbol{\xi} \in \overline{\mathbf{R}^2_+} \,:\, \eta = \xi \tan\theta_{25} + a_{25}\big\}$$

for some constants $(u_5, v_5, k_5, \theta_{25}, a_{25})$ depending on (γ, v_2, θ_1) to be determined, where $\theta_{25} \in (\frac{\pi}{2}, \pi]$ is the angle between the reflected shock S_{25} and the positive ξ–axis, and $a_{25} > 0$ is the η–intercept of the reflected shock S_{25}. Applying the Rankine–Hugoniot conditions (2.2.8) between states (2) and (5) at P_0^1, we have

(4.2.20)
$$\varphi_5 = \varphi_2\,, \quad D\varphi_5 \cdot \boldsymbol{\tau}_{25} = D\varphi_2 \cdot \boldsymbol{\tau}_{25}\,, \quad \rho_5 D\varphi_5 \cdot \boldsymbol{\nu}_{25} = D\varphi_2 \cdot \boldsymbol{\nu}_{25}\,, \qquad \text{at } \boldsymbol{\xi} = P_0^1\,,$$

where $\boldsymbol{\nu}_{25} \coloneqq \frac{D(\varphi_2-\varphi_5)(P_0^1)}{|D(\varphi_2-\varphi_5)(P_0^1)|}$, $\boldsymbol{\tau}_{25} \coloneqq \boldsymbol{\nu}_{25}^{\perp}$, and $\rho_5 \coloneqq \rho(|D\varphi_5|, \varphi_5)$ is given by (4.2.2). Furthermore, the pseudo-potential φ_5 should satisfy the physical entropy condition

(4.2.21)
$$\rho_5 > 1\,, \quad \text{or equivalently}\,, \quad D\varphi_2(P_0^1) \cdot \boldsymbol{\nu}_{25} > D\varphi_5(P_0^1) \cdot \boldsymbol{\nu}_{25} > 0\,;$$

and the slip boundary condition

(4.2.22)
$$D\varphi_5 \cdot (0,1) = 0 \qquad \text{on } L_{\text{sym}}\,.$$

A necessary condition for the regular reflection of S_{12} at P_0^1 is clearly that state (2) must be pseudo-supersonic at P_0^1. From Lemma A.1, there exists a unique $\theta^+ \in (0, \theta^{\text{cr}}]$ such that $M_2^{(\theta_1)} > 1$ if and only if $\theta_1 \in (0, \theta^+)$. Note however that $M_2^{(\theta_1)}|_{\theta_1=\theta^+} = 1$ is *not* necessarily true if $\theta^+ = \theta^{\text{cr}}$, though $M_2^{(\theta_1)}|_{\theta_1=\theta^+} = 1$ whenever $\theta^+ < \theta^{\text{cr}}$.

Introduce the steady detachment angle $\theta^{\text{d}}_{\text{stdy}}(\rho_\infty, u_\infty)$ for the steady potential flow with constant supersonic upstream state $U_\infty = (\rho_\infty, u_\infty, 0)$, as defined in §3.2. Also, for $\theta_1 \in (0, \theta^{\text{cr}})$, denote by $\hat{\theta}_{25}(\theta_1)$ the acute angle between the pseudo-velocity $D\varphi_2(P_0^1) = (-\xi^{P_0^1}, v_2)$ and the symmetric line L_{sym}

(4.2.23)
$$\hat{\theta}_{25}(\theta_1) \coloneqq \arccos\big(\frac{\xi^{P_0^1}}{|D\varphi_2(P_0^1)|}\big) = \arcsin\big(\frac{-v_2}{|D\varphi_2(P_0^1)|}\big) \in (0, \tfrac{\pi}{2})\,.$$

Similarly to §3.3, or see [20, §7.4], whenever $\theta_1 \in (0, \theta^+)$, the existence and multiplicity of solutions of the algebraic system (4.2.20)–(4.2.22) are determined as follows:

(a) If $\hat{\theta}_{25}(\theta_1) < \theta^{\text{d}}_{\text{stdy}}(1, |D\varphi_2(P_0^1)|)$, there are two solutions;

(b) If $\hat{\theta}_{25}(\theta_1) = \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_2(P_0^1)|)$, there is one solution;

(c) If $\hat{\theta}_{25}(\theta_1) > \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_2(P_0^1)|)$, there are no solutions.

With the above discussion, the following proposition gives the necessary and sufficient criterion on the incident angle θ_1 for the existence of a regular reflection of S_{12} at the point P_0^1.

Proposition 4.2 (Local reflection theory)**.** *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. There exists a unique $\theta^{\mathrm{d}} = \theta^{\mathrm{d}}(\gamma, v_2) \in (0, \theta^{\mathrm{cr}}]$, called the detachment angle, such that, for $\theta_1 \in (\theta^{\mathrm{d}}, \theta^{\mathrm{cr}})$, there are no states* (5) *of form* (4.2.3) *satisfying* (4.2.20)–(4.2.22) *and, for each $\theta_1 \in (0, \theta^{\mathrm{d}})$, there are exactly two states* (5) *of form* (4.2.3) *satisfying* (4.2.20)–(4.2.22)*: the weak reflection state φ_5^{wk} and the strong reflection state φ_5^{sg}, distinguished by $1 < \rho_5^{\mathrm{wk}} < \rho_5^{\mathrm{sg}}$, where $\rho_5^{\mathrm{wk}} := \rho(|D\varphi_5^{\mathrm{wk}}|, \varphi_5^{\mathrm{wk}})$ and $\rho_5^{\mathrm{sg}} := \rho(|D\varphi_5^{\mathrm{sg}}|, \varphi_5^{\mathrm{sg}})$ are the constant densities of the weak and strong reflection states* (5) *respectively. Furthermore, denoting by $O_5^{\mathrm{wk}} := (u_5^{\mathrm{wk}}, v_5^{\mathrm{wk}})$ and $O_5^{\mathrm{sg}} := (u_5^{\mathrm{sg}}, v_5^{\mathrm{sg}})$ the sonic centres, and by c_5^{wk} and c_5^{sg} the sonic speeds of the weak and strong reflection states* (5) *respectively, we have*

(i) *$(\rho_5^{\mathrm{wk}}, O_5^{\mathrm{wk}}) \in C([0, \theta^{\mathrm{d}}]) \cap C^\infty([0, \theta^{\mathrm{d}}))$ and $(\rho_5^{\mathrm{sg}}, O_5^{\mathrm{sg}}) \in C((0, \theta^{\mathrm{d}}]) \cap C^\infty((0, \theta^{\mathrm{d}}))$, with $(\rho_5^{\mathrm{wk}}, O_5^{\mathrm{wk}}, u_5^{\mathrm{wk}}\xi^{P_0^1}) \to (\rho_0, \mathbf{0}, -v_2\eta_0)$ as $\theta_1 \to 0^+$, for ρ_0 and η_0 the density and shock location of the normal reflection state* (0) *given in* §4.2.1.3*. Furthermore, if $\theta^{\mathrm{d}} < \theta^{\mathrm{cr}}$, then $(\rho_5^{\mathrm{wk}}, O_5^{\mathrm{wk}})$ and $(\rho_5^{\mathrm{sg}}, O_5^{\mathrm{sg}})$ have a common (finite) limit as $\theta_1 \to \theta^{\mathrm{d}-}$.*

(ii) *For any $\theta_1 \in (0, \theta^{\mathrm{d}})$, we have $|D\varphi_2(P_0^1)| > |D\varphi_2(P_0^1|_{\theta_1=\theta^{\mathrm{d}}})| > 1$, and*

$$0 < u_5^{\mathrm{wk}} < u_5^{\mathrm{sg}} < \xi^{P_0^1}, \qquad v_5^{\mathrm{wk}} = v_5^{\mathrm{sg}} = 0. \tag{4.2.24}$$

(iii) *There exists a unique $\theta^{\mathrm{s}} = \theta^{\mathrm{s}}(\gamma, v_2) \in (0, \theta^{\mathrm{d}}]$, called the sonic angle, such that*

$$\begin{aligned} &|D\varphi_5^{\mathrm{wk}}(P_0^1)| > c_5^{\mathrm{wk}} \qquad \text{for all } \theta_1 \in (0, \theta^{\mathrm{s}}), \\ &|D\varphi_5^{\mathrm{wk}}(P_0^1)| < c_5^{\mathrm{wk}} \qquad \text{for all } \theta_1 \in (\theta^{\mathrm{s}}, \theta^{\mathrm{d}}]. \end{aligned}$$

Furthermore, if $\theta^{\mathrm{s}} < \theta^{\mathrm{cr}}$, then $\theta^{\mathrm{s}} < \theta^{\mathrm{d}}$ and $|D\varphi_5^{\mathrm{wk}}(P_0^1)| = c_5^{\mathrm{wk}}$ when $\theta_1 = \theta^{\mathrm{s}}$.

Proof. The proof follows similarly to [20, Theorem 7.1.1], once θ^{d} and θ^{s} are determined. Thus, it remains to show the existence and uniqueness of θ^{d} and θ^{s}.

For the detachment angle, we demonstrate that there exists at most one value $\theta^{\mathrm{d}} \in (0, \theta^{+})$ such that $\theta_1 = \theta^{\mathrm{d}}$ satisfies the equality in (b) above. Indeed, using (4.2.23) and Lemma A.1, it is clear that $\hat{\theta}_{25}(\theta_1)$ is a strictly increasing function of $\theta_1 \in (0, \theta^{\mathrm{cr}})$, whilst by Lemma 3.5 and Lemma A.1, it is clear that $\theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_2(P_0^1)|)$ is a strictly decreasing function of $\theta_1 \in (0, \theta^{+})$. Furthermore, Lemmas 3.5 and A.1 show that

$$\lim_{\theta_1 \to 0^+} \hat{\theta}_{25}(\theta_1) = 0 \ < \ \tfrac{\pi}{2} = \lim_{\theta_1 \to 0^+} \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_2(P_0^1)|)\,.$$

Then it follows that there exists at most one solution $\vartheta^{\mathrm{d}} \in (0, \theta^{+})$ to the equation

$$\hat{\theta}_{25}(\vartheta^{\mathrm{d}}) = \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_2(P_0^1|_{\theta_1 = \vartheta^{\mathrm{d}}})|)\,. \tag{4.2.25}$$

We define the detachment angle $\theta^{\mathrm{d}} \in (0, \theta^{+}]$ by

$$\theta^{\mathrm{d}} \coloneqq \min\big\{\theta^{+},\, \inf\{\vartheta^{\mathrm{d}} \in (0, \theta^{+}) \,:\, \vartheta^{\mathrm{d}} \text{ satisfies (4.2.25)}\}\big\}\,.$$

In this sense, the symbol θ^{d} may refer to either the conventional detachment angle ϑ^{d} that satisfies equation (4.2.25) when such a solution exists, or to the angle θ^{+} otherwise.

The existence and properties of the sonic angle θ^{s} contained in statement (iii) of Proposition 4.2 follow from a similar argument as above. In particular, the symbol θ^{s} may refer to either the conventional sonic angle $\vartheta^{\mathrm{s}} \in (0, \theta^{\mathrm{d}}]$ which should satisfy

$$\hat{\theta}_{25}(\vartheta^{\mathrm{s}}) = \theta^{\mathrm{s}}_{\mathrm{stdy}}(1, |D\varphi(P_0^1|_{\theta_1 = \vartheta^{\mathrm{s}}})|)\,; \tag{4.2.26}$$

or we simply define $\theta^{\mathrm{s}} = \theta^{\mathrm{d}}$ if no such solution exists. That is, we define

$$\theta^{\mathrm{s}} \coloneqq \min\big\{\theta^{\mathrm{d}},\, \inf\{\vartheta^{\mathrm{s}} \in (0, \theta^{\mathrm{d}}) \,:\, \vartheta^{\mathrm{s}} \text{ satisfies (4.2.26)}\}\big\}\,. \qquad \square$$

Remark 4.3. We have the following remarks about Proposition 4.2:

(i) Fix $\gamma \geq 1$. For any $v_2 \in (v_{\min}, 0)$, let θ^{cr} be the critical angle given by (4.2.12), and let $v_2^{\mathrm{s}}, v_2^{\mathrm{d}} \in (v_{\min}, 0)$ be the constants given by Lemma A.2. Then

$$\begin{aligned} \theta^{\mathrm{s}} = \theta^{\mathrm{d}} = \theta^{\mathrm{cr}} \quad &\Longleftrightarrow \quad v_2 \in (v_{\min}, v_2^{\mathrm{s}}], \\ \theta^{\mathrm{s}} < \theta^{\mathrm{d}} = \theta^{\mathrm{cr}} \quad &\Longleftrightarrow \quad v_2 \in (v_2^{\mathrm{s}}, v_2^{\mathrm{d}}], \\ \theta^{\mathrm{s}} < \theta^{\mathrm{d}} < \theta^{\mathrm{cr}} \quad &\Longleftrightarrow \quad v_2 \in (v_2^{\mathrm{d}}, 0). \end{aligned}$$

(ii) The estimate (4.2.24) follows from Remark 3.2 due to the geometric construction of state (5) as the intersection of a shock polar curve with the ξ–axis.

(iii) By the symmetry of the problem, the conclusions of Proposition 4.2 immediately apply to the regular reflection of S_{32} at P_0^2 after exchanging $(\theta_1, P_0^1, (5))$ with $(\theta_2, P_0^2, (6))$, and reversing the inequalities in (4.2.24). In particular, the same detachment angle $\theta^{\mathrm{d}}(\gamma, v_2)$ and sonic angle $\theta^{\mathrm{s}}(\gamma, v_2)$ apply to θ_2 in the construction of state (6).

(iv) The main difference between Proposition 4.2 and [20, Theorem 7.1.1] is that we assert the uniqueness of solutions to (4.2.25) and (4.2.26), although we cannot in general assert the existence of such solutions due to the restriction $\theta_1 \in (0, \theta^{\mathrm{cr}})$; whereas [20, Theorem 7.1.1] asserts the existence, but not the uniqueness of such solutions. Of course, in light of Theorem 3.6, the results of [20, Theorem 7.1.1] can now be strengthened to include uniqueness.

Definition 4.4 (Admissible parameters and states (5) and (6))**.** Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. Let $\theta^{\mathrm{s}} := \theta^{\mathrm{s}}(\gamma, v_2)$ and $\theta^{\mathrm{d}} := \theta^{\mathrm{d}}(\gamma, v_2)$ be given by Proposition 4.2. Denote the set of admissible parameters

$$\Theta := [0, \theta^{\mathrm{d}})^2 \setminus \{\mathbf{0}\} \subseteq [0, \theta^{\mathrm{cr}}]^2 .$$

Hereafter, for $\boldsymbol{\theta} \in \overline{\Theta}$, states (5) and (6) refer to the weak reflection states (5) and (6) from Proposition 4.2. For $j = 5, 6$, we write $(\rho_j, c_j, u_j, v_j) := (\rho_j^{\mathrm{wk}}, c_j^{\mathrm{wk}}, u_j^{\mathrm{wk}}, v_j^{\mathrm{wk}})$, $O_j := (u_j, v_j)$, and $\varphi_j := \varphi_j^{\mathrm{wk}}$. Note that $u_5 \geq 0$, $u_6 \leq 0$, and $v_5 = v_6 = 0$ by (4.2.24).

The pseudo-potential of state (j) is given by

$$\varphi_j(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u_j, v_j)\cdot\boldsymbol{\xi} + v_2 a_{2j}\,, \tag{4.2.27}$$

where $a_{2j} \coloneqq -\frac{1}{v_2}\xi^{P_0^{j-4}} u_j$ for $\theta_{j-4} > 0$, and $a_{2j} \coloneqq \lim_{\theta_{j-4}\to 0^+} a_{2j} = \eta_0$ when $\theta_{j-4} = 0$; *cf.* Proposition 4.2(i).

Define the reflected shocks S_{25} and S_{26} by

$$S_{2j} \coloneqq \big\{\boldsymbol{\xi} \in \overline{\mathbf{R}^2_+} \,:\, \varphi_j(\boldsymbol{\xi}) = \varphi_2(\boldsymbol{\xi})\big\} = \big\{\boldsymbol{\xi} \in \overline{\mathbf{R}^2_+} \,:\, \eta = \xi\tan\theta_{2j} + a_{2j}\big\} \qquad \text{for } \ j = 5, 6\,,$$

where θ_{25} and θ_{26} are the angles between the reflected shocks S_{25} and S_{26} and the positive ξ–axis, given respectively by

$$\theta_{25} \coloneqq \arctan\big(\frac{u_5}{v_2}\big) \in (\tfrac{\pi}{2}, \pi]\,, \qquad \theta_{26} \coloneqq \arctan\big(\frac{u_6}{v_2}\big) \in [0, \tfrac{\pi}{2})\,. \tag{4.2.28}$$

The unit tangent vectors to S_{25} and S_{26} are given respectively by

$$\boldsymbol{e}_{S_{25}} \coloneqq (\cos\theta_{25}, \sin\theta_{25})\,, \qquad \boldsymbol{e}_{S_{26}} \coloneqq (\cos\theta_{26}, \sin\theta_{26})\,. \tag{4.2.29}$$

It follows from Proposition 4.2(i) and Definition 4.4 that $\varphi_j = \varphi_0$, and $\theta_{2j} = (6-j)\pi$ when $\theta_{j-4} = 0$ for $j = 5, 6$.

Lemma 4.5 (Further properties of states (5) and (6))**.** *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. Let ρ_5 and ρ_6 be the constant densities of states* (5) *and* (6) *given by* Proposition 4.2, *and let θ_{25} and θ_{26} be the reflected shock angles given by* (4.2.28)*. Then, for $j = 5, 6$, there exists a constant $\delta_{5,6} > 0$ depending only on (γ, v_2) such that, for any $\boldsymbol{\theta} \in \overline{\Theta}$,*

$$1 + \delta_{5,6} \leq \rho_j \leq \delta_{5,6}^{-1}\,, \tag{4.2.30}$$

$$\delta_{5,6} \leq |\theta_{2j} - \tfrac{\pi}{2}|\,. \tag{4.2.31}$$

Moreover, for any $\bar\theta \in (0, \theta^{\mathrm{d}})$, there exists a constant $\delta_{5,6}^{(\bar\theta)} > 0$ depending only on

$(\gamma, v_2, \bar{\theta})$ *such that*

$$\delta_{5,6}^{(\bar{\theta})} < |\theta_{2j} - (6-j)\pi| \qquad \textit{whenever } \boldsymbol{\theta} \in \overline{\Theta} \cap \{\theta_{j-4} > \bar{\theta}\}\,. \tag{4.2.32}$$

Proof. We prove the results only for state (5), since the proof for state (6) is similar.

Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. By Proposition 4.2(i), we see that $\rho_5 \in C(\overline{\Theta})$, so that the uniform upper bound in (4.2.30) is clear. For the uniform lower bound, the entropy condition (4.2.21) gives $\rho_5 > 1$. Furthermore, the Rankine–Hugoniot conditions between states (2) and (5) imply that

$$\ell(\rho_5) = \big|(u_2, v_2) - (u_5, v_5)\big| = \sqrt{v_2^2 + u_5^2} \geq |v_2| > 0 \qquad \text{for any } \boldsymbol{\theta} \in \overline{\Theta}\,, \tag{4.2.33}$$

with $\ell(\cdot)$ given by (2.3.3). In particular, $\ell(\cdot)$ is an increasing function on $[1, \infty)$ with $\ell(1) = 0$, and hence (4.2.33) implies the existence of a small constant $\delta_{5,6}^{(1)} > 0$, depending only on (γ, v_2), such that $\rho_5 > 1 + \delta_{5,6}^{(1)}$ for any $\boldsymbol{\theta} \in \overline{\Theta}$.

Next, we prove (4.2.31). From (4.2.28) and the geometric properties of the shock polar curve for 2-D steady potential flow, we have

$$\tfrac{\pi}{2} + \hat{\theta}_{25} < \theta_{25} \leq \pi \qquad \text{for any } \boldsymbol{\theta} \in \overline{\Theta}\,, \tag{4.2.34}$$

with $\hat{\theta}_{25}$ given by (4.2.23). From Proposition 4.2(i) and (4.2.28), we see that $\theta_{25} \to \pi^-$ as $\theta_1 \to 0^+$. By continuity, there exists a small constant $\delta_{5,6}^{(2)} > 0$ depending only on (γ, v_2) such that $\theta_{25} > \frac{3\pi}{4}$ whenever $\theta_1 \in [0, \delta_{5,6}^{(2)})$. On the other hand, by (4.2.23) and Lemma A.1, there exists a constant $\delta_{5,6}^{(3)} > 0$ depending only on (γ, v_2) such that $\hat{\theta}_{25} > \delta_{5,6}^{(3)}$ whenever $\theta_1 \in [\delta_{5,6}^{(2)}, \theta^{\rm d}]$. Together with (4.2.34), we conclude that $\theta_{25} > \min\{\frac{3\pi}{4}, \frac{\pi}{2} + \delta_{5,6}^{(3)}\}$ for all $\theta_1 \in [0, \theta^{\rm d}]$, which leads to (4.2.31).

Finally, the bound (4.2.32) follows directly from (4.2.34) and Lemma A.3, which states that θ_{25} is a strictly decreasing function of $\theta_1 \in [0, \theta^{\rm d}]$. □

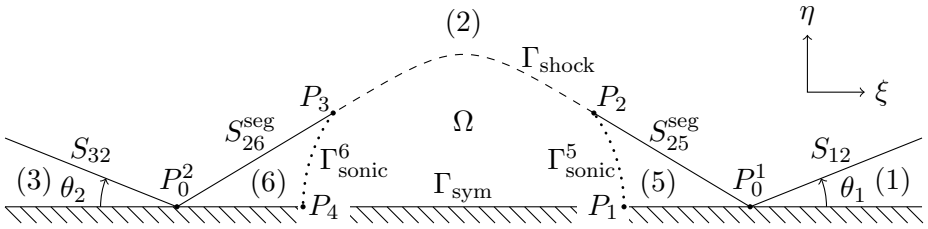


Figure 4.4: **Case I.** The supersonic-supersonic reflection configuration.

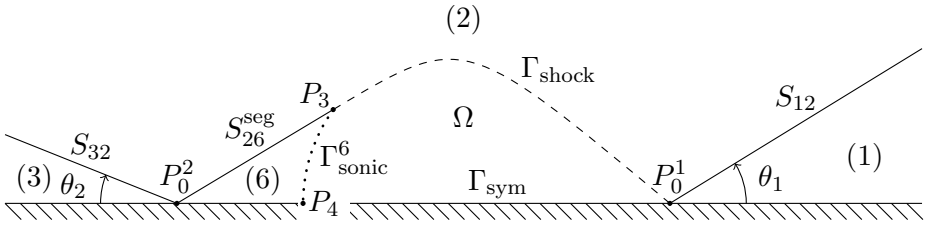


Figure 4.5: **Case II.** The supersonic-subsonic reflection configuration in the case $\boldsymbol{\theta} \in [\theta^{\text{s}}, \theta^{\text{d}}) \times (0, \theta^{\text{s}})$, *i.e.*, state (5) is pseudo-sonic/pseudo-subsonic at P_0^1.

4.2.2.2 Configurations for four-shock interactions

The four-shock interaction configurations are split into three genuinely different cases depending on parameters $\boldsymbol{\theta} \in \Theta$ and the critical angles θ^{s} and θ^{d}. We briefly describe the expected structure of each configuration and introduce some important notation.

Case I. *Supersonic-supersonic reflection configuration.* When $\boldsymbol{\theta} \in (0, \theta^{\text{s}})^2$, the supersonic-supersonic reflection configuration is expected; see Figure 4.4. In this case, both states (5) and (6) are pseudo-supersonic at the reflection points P_0^1 and P_0^2, respectively.

Case II. *Supersonic-subsonic reflection configuration.* When $\boldsymbol{\theta} \in [\theta^{\text{s}}, \theta^{\text{d}}) \times (0, \theta^{\text{s}})$, or $\boldsymbol{\theta} \in (0, \theta^{\text{s}}) \times [\theta^{\text{s}}, \theta^{\text{d}})$, the supersonic-subsonic reflection configuration is expected; see Figure 4.5 for the case $\boldsymbol{\theta} \in [\theta^{\text{s}}, \theta^{\text{d}}) \times (0, \theta^{\text{s}})$. In this case, either state (5) is pseudo-sonic/pseudo-subsonic at P_0^1 and state (6) is pseudo-supersonic at P_0^2, or vice versa.

Case III. *Subsonic-subsonic reflection configuration.* When $\boldsymbol{\theta} \in [\theta^{\text{s}}, \theta^{\text{d}})^2$, the subsonic-subsonic reflection configuration is expected; see Figure 4.6. In this case, both states (5) and (6) are pseudo-subsonic at the reflection points P_0^1 and P_0^2, respectively.

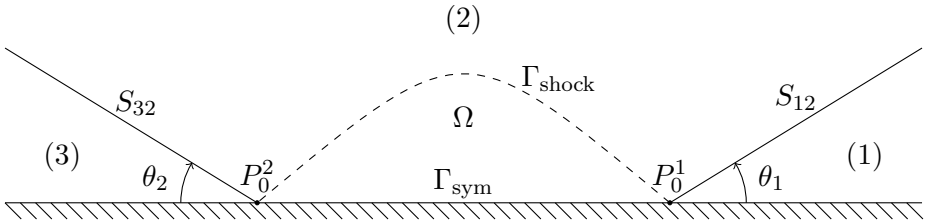


Figure 4.6: **Case III.** The subsonic-subsonic reflection configuration.

It follows from Remark 4.3(i) that the configurations of **Case II** and **Case III** are not possible unless $v_2 \in (v_2^{\text{s}}, 0)$, for the constant v_2^{s} from Lemma A.2.

Definition 4.6 (The sonic boundaries and intersection points)**.** Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For any $\boldsymbol{\theta} \in \overline{\Theta}$, let Q_5 and Q_6 be the orthogonal projections of O_5 and O_6 onto S_{25} and S_{26}, respectively. When $\theta_1 \in (0, \theta^{\text{s}})$, the right sonic boundary and intersection points are given by

$$\begin{aligned}
&\Gamma^5_{\text{sonic}} \coloneqq \left\{\boldsymbol{\xi} \in \partial B_{c_5}(O_5) \cap \overline{\mathbf{R}^2_+} \,:\, \xi^{Q_5} \leq \xi \leq \xi^{P_0^1},\, \varphi_5(\boldsymbol{\xi}) \leq \varphi_2(\boldsymbol{\xi})\right\},\\
&\{P_1\} \coloneqq \Gamma^5_{\text{sonic}} \cap L_{\text{sym}}\,, \quad \{P_2\} \coloneqq \Gamma^5_{\text{sonic}} \cap S_{25}\,,
\end{aligned}$$

whilst, for $\theta_1 \in [\theta^{\text{s}}, \theta^{\text{d}})$,

$$\Gamma^5_{\text{sonic}} \equiv \{P_1\} \equiv \{P_2\} \coloneqq \{P_0^1\}\,.$$

Similarly, when $\theta_2 \in (0, \theta^{\text{s}})$, the left sonic boundary and intersection points are given by

$$\begin{aligned}
&\Gamma^6_{\text{sonic}} \coloneqq \left\{\boldsymbol{\xi} \in \partial B_{c_6}(O_6) \cap \overline{\mathbf{R}^2_+} \,:\, \xi^{P_0^2} \leq \xi \leq \xi^{Q_6},\, \varphi_6(\boldsymbol{\xi}) \leq \varphi_2(\boldsymbol{\xi})\right\},\\
&\{P_3\} \coloneqq \Gamma^6_{\text{sonic}} \cap S_{26}\,, \quad \{P_4\} \coloneqq \Gamma^6_{\text{sonic}} \cap L_{\text{sym}}\,,
\end{aligned}$$

whilst, for $\theta_2 \in [\theta^{\text{s}}, \theta^{\text{d}})$,

$$\Gamma^6_{\text{sonic}} \equiv \{P_3\} \equiv \{P_4\} \coloneqq \{P_0^2\}\,.$$

The open regions are given by

$$\Omega_5 \coloneqq \left\{\boldsymbol{\xi} \in \mathbf{R}^2_+ \setminus \overline{B_{c_5}(O_5)} \,:\, \varphi_5(\boldsymbol{\xi}) < \varphi_2(\boldsymbol{\xi}),\, \xi > \xi^{P_2}\right\},$$
$$\Omega_6 \coloneqq \left\{\boldsymbol{\xi} \in \mathbf{R}^2_+ \setminus \overline{B_{c_6}(O_6)} \,:\, \varphi_6(\boldsymbol{\xi}) < \varphi_2(\boldsymbol{\xi}),\, \xi < \xi^{P_3}\right\}.$$

Finally, define the symmetric boundary

$$\Gamma_{\text{sym}} \coloneqq \left\{(\xi, 0) \in L_{\text{sym}} \,:\, \xi^{P_4} < \xi < \xi^{P_1}\right\},$$

and the line segments

$$S^{\text{seg}}_{25} \coloneqq S_{25} \cap \left\{\xi^{P_2} \le \xi \le \xi^{P_0^1}\right\}, \qquad S^{\text{seg}}_{26} \coloneqq S_{26} \cap \left\{\xi^{P_0^2} \le \xi \le \xi^{P_3}\right\}.$$

Note that the curves Γ^5_{sonic}, Γ^6_{sonic}, and $\overline{\Gamma_{\text{sym}}}$ as defined in Definition 4.6 above are fixed by $(\gamma, v_2, \boldsymbol{\theta})$ and do not share any common points except their endpoints $\{P_1, P_4\}$. Also, note that $\Omega_j = \varnothing$ if $\theta_{j-4} \in [\theta^{\text{s}}, \theta^{\text{d}})$ for $j = 5, 6$. For a suitable simple open curve segment Γ_{shock} (yet to be determined) with endpoints P_2 and P_3, we use Ω to denote the open bounded domain enclosed by the simple closed curve $\Gamma_{\text{shock}} \cup \Gamma^5_{\text{sonic}} \cup \Gamma_{\text{sym}} \cup \Gamma^6_{\text{sonic}}$; see Figures 4.4–4.6.

4.2.3 Admissible solutions and main theorems

We reformulate the asymptotic boundary value problem, Problem A, into a free boundary problem. Our goal is to find a suitable curved free boundary Γ_{shock} between the points P_2 and P_3, and a solution φ of Problem A which satisfies (4.2.1)–(4.2.2) in the open region Ω (as depicted in Figures 4.4–4.6) and the corresponding boundary conditions on $\partial\Omega \coloneqq \Gamma_{\text{shock}} \cup \Gamma^5_{\text{sonic}} \cup \Gamma_{\text{sym}} \cup \Gamma^6_{\text{sonic}}$. Furthermore, equation (4.2.1) should be elliptic in Ω, and the free boundary Γ_{shock} should be a transonic shock which separates the hyperbolic and elliptic regions of the solution.

Problem B (Free boundary problem)**.** *Fix $\gamma \ge 1$ and $v_2 \in (v_{\min}, 0)$. For any $\boldsymbol{\theta} \in \overline{\Theta}$, find a free boundary (curved reflected-diffracted shock) Γ_{shock} in the region $\mathbf{R}^2_+ \cap \{\xi^{P_3} < \xi < \xi^{P_2}\}$, and a function φ defined in the region Ω, as shown in*

Figures 4.4–4.6, *such that* φ *satisfies the following properties*:

(i) φ *is a solution of* (4.2.1) *in* Ω *and* (4.2.1) *is elliptic inside* Ω;

(ii) $\varphi = \varphi_2$ *and* $\rho D\varphi \cdot \boldsymbol{\nu} = \rho_2 D\varphi_2 \cdot \boldsymbol{\nu}$ *on the free boundary* Γ_{shock};

(iii) $\varphi = \varphi_j$ *and* $D\varphi = D\varphi_j$ *on the sonic boundary* Γ^j_{sonic} *for* $j = 5, 6$;

(iv) $D\varphi \cdot (0,1) = 0$ *on the symmetric boundary* Γ_{sym},

where $\boldsymbol{\nu}$ *is the unit normal vector on* Γ_{shock} *pointing into* Ω.

Definition 4.7 (Admissible solutions)**.** Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For any $\boldsymbol{\theta} \in \overline{\Theta}$, a function $\varphi \in W^{1,\infty}_{\text{loc}}(\mathbf{R}^2_+)$ is called an admissible solution to the four-shock interaction problem if φ is a solution of Problem B and satisfies the following properties:

(i) There exists a relatively open curve segment Γ_{shock} (without self-intersection) with endpoints $P_2 = (\xi^{P_2}, \eta^{P_2})$ and $P_3 = (\xi^{P_3}, \eta^{P_3})$ given by Definition 4.6 such that Γ_{shock} is C^2 in its relative interior, $\Gamma^{\text{ext}}_{\text{shock}} := S^{\text{seg}}_{25} \cup \Gamma_{\text{shock}} \cup S^{\text{seg}}_{26}$ is a C^1–curve including at P_2 and P_3, with S^{seg}_{25} and S^{seg}_{26} given in Definition 4.6,

$$\Gamma_{\text{shock}} \subseteq \left\{ \boldsymbol{\xi} \in \mathbf{R}^2_+ \setminus \overline{B_{c_2}(O_2)} \,:\, \xi^{P_3} < \xi < \xi^{P_2} \right\},$$

with $\partial B_{c_2}(O_2)$ as the sonic circle of state (2), and the curves $\overline{\Gamma_{\text{shock}}}$ and $\Gamma^5_{\text{sonic}} \cup \Gamma_{\text{sym}} \cup \Gamma^6_{\text{sonic}}$ share no common points except their endpoints $\{P_2, P_3\}$ so that $\Gamma_{\text{shock}} \cup \Gamma^5_{\text{sonic}} \cup \Gamma_{\text{sym}} \cup \Gamma^6_{\text{sonic}}$ is a closed curve without self-intersection.

(ii) For the open bounded domain $\Omega \subseteq \mathbf{R}^2_+$ enclosed by $\Gamma_{\text{shock}} \cup \Gamma^5_{\text{sonic}} \cup \Gamma_{\text{sym}} \cup \Gamma^6_{\text{sonic}}$, the solution φ is in $C^3(\Omega) \cap C^2\big(\overline{\Omega} \setminus (\Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}})\big) \cap C^1(\overline{\Omega})$ and further satisfies $\varphi \in C^{0,1}_{\text{loc}}(\mathbf{R}^2_+) \cap C^1_{\text{loc}}(\overline{\mathbf{R}^2_+} \setminus \overline{\Gamma^{\text{ext}}_{\text{shock}} \cup S_{12} \cup S_{32}})$ and

$$\varphi = \begin{cases} \max\{\varphi_5, \varphi_6\} & \text{in } \Omega_5 \cup \Omega_6\,, \\ \bar{\varphi} & \text{in } \mathbf{R}^2_+ \setminus (\Omega \cup \Omega_5 \cup \Omega_6)\,, \end{cases} \tag{4.2.35}$$

where Ω_5 and Ω_6 are given in Definition 4.6, and $\bar{\varphi}$ is given by (4.2.15).

(iii) Equation (4.2.1) is strictly elliptic in $\overline{\Omega} \setminus (\Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}})$, *i.e.*, the solution φ satisfies

$$|D\varphi(\boldsymbol{\xi})| < c(|D\varphi(\boldsymbol{\xi})|, \varphi(\boldsymbol{\xi})) \qquad \text{for all } \boldsymbol{\xi} \in \overline{\Omega} \setminus (\Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) \,.$$

(iv) In Ω, the solution φ satisfies

$$\max\{\varphi_5, \varphi_6\} \le \varphi \le \varphi_2 \,, \tag{4.2.36}$$

$$D(\varphi_2 - \varphi) \cdot (\cos\theta, \sin\theta) \le 0 \qquad \text{for all } \theta \in [\theta_{26}, \theta_{25}] \,. \tag{4.2.37}$$

Remark 4.8. From the definition of admissible solutions above, we have

(i) The requirement that φ is a solution of Problem B includes the condition that $\varphi = \varphi_j$ and $D\varphi = D\varphi_j$ on Γ^j_{sonic} for $j = 5, 6$, which becomes a one-point boundary condition in the case $\theta_{j-4} \in (\theta^{\text{s}}, \theta^{\text{d}}]$, according to the definition of Γ^j_{sonic} given by Definition 4.6.

(ii) Let φ be an admissible solution in the sense of Definition 4.7, and let $\boldsymbol{\nu}$ be the unit normal vector on Γ_{shock} pointing into Ω. Similarly to [1, Lemma 2.26], it can be shown that Γ_{shock} is a transonic shock, since φ satisfies

$$D\varphi_2 \cdot \boldsymbol{\nu} > D\varphi \cdot \boldsymbol{\nu} > 0 \,, \quad 0 < \frac{D\varphi \cdot \boldsymbol{\nu}}{c(|D\varphi|, \varphi)} < 1 < D\varphi_2 \cdot \boldsymbol{\nu} \qquad \text{on } \Gamma_{\text{shock}} \,.$$

(iii) It follows from Proposition 4.2(i) that φ_j depends continuously on $\theta_{j-4} \in [0, \theta^{\text{d}}]$ for $j = 5, 6$. Moreover, for any $r > 0$,

$$\lim_{\theta_{j-4} \to 0^+} \|\varphi_j - \varphi_0\|_{C^{0,1}(B_r(\mathbf{0}) \cap \mathbf{R}^2_+)} = 0 \qquad \text{for } j = 5, 6 \,,$$

i.e., the weak reflection state (j) coincides with the normal reflection state (0) when $\theta_{j-4} = 0$. One can verify directly that the normal reflection solution $\varphi_{\text{norm}} \in C^{0,1}_{\text{loc}}(\mathbf{R}^2_+)$ as defined by (4.2.19) is an admissible solution corresponding to $\boldsymbol{\theta} = \mathbf{0}$ in the sense of Definition 4.7.

The main theorem of this chapter is to provide the global existence of such admissible solutions in the sense of Definition 4.7. The proof is given in §4.4.4.

Theorem 4.9 (Existence of admissible solutions)**.** *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For any $\boldsymbol{\theta} \in \Theta$, there exists an admissible solution φ of the four-shock interaction problem corresponding to the parameters $\boldsymbol{\theta}$ in the sense of* Definition 4.7*. Moreover, the global solution φ converges to the normal reflection solution φ_{norm} in $W^{1,1}_{\text{loc}}(\mathbf{R}^2_+)$ as $\boldsymbol{\theta} \in \Theta$ converges to $\mathbf{0}$, where φ_{norm} is given by* (4.2.19)*.*

We mention briefly that, in addition to the existence of admissible solutions, the optimal regularity of admissible solutions and the convexity of the free boundary are also obtained in [17, Theorems 2.2–2.3], though we will not discuss these results any further here.

4.3 Uniform Estimates for Admissible Solutions

To establish the main existence theorem, Theorem 4.9, we apply the Leray–Schauder degree theorem to a suitable iteration map, similar to [1, 20]. To achieve this, in this section, we make several essential uniform estimates of the admissible solutions. Later, in §4.4, we construct a suitable iteration map between two function spaces equipped with weighted Hölder norms, where any fixed point of the iteration map is an admissible solution in the sense of Definition 4.7.

Our problem shares many similarities with the Prandtl–Meyer shock reflection problem [1], and the von Neumann shock reflection-diffraction problem [20]. Many of the following proofs are similar to those within [1, 20], but have been adapted to the current setting with careful estimates.

The main results we establish in this section are the following:

(a) The strict directional monotonicity of $\varphi_2 - \varphi$, and the directional monotonicity of $\varphi - \varphi_5$ and $\varphi - \varphi_6$;

(b) The uniform positive lower bound of $\text{dist}(\Gamma_{\text{shock}}, \partial B_1(O_2))$;

(c) The uniform estimate of the ellipticity of equation (4.2.1) in Ω;

(d) The uniform estimates of admissible solutions under some suitable weighted Hölder norms.

4.3.1 Monotonicity and ellipticity

4.3.1.1 Directional monotonicity and uniform bounds

Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. Let φ be an admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$ in the sense of Definition 4.7, and let the constant states φ_2 and φ_5, φ_6 be given by (4.2.14) and (4.2.27) respectively. We aim to establish directional monotonicity properties for a function ϕ that satisfies

$$(c^2 - \varphi_\xi^2)\phi_{\xi\xi} - 2\varphi_\xi\varphi_\eta\phi_{\xi\eta} + (c^2 - \varphi_\eta^2)\phi_{\eta\eta} = 0 \qquad \text{in } \Omega\,, \tag{4.3.1}$$

where ϕ can be taken as either $\varphi_2 - \varphi$ or $\varphi - \varphi_j$, for $j = 5, 6$, and

$$c^2(|D\varphi|, \varphi) \coloneqq 1 + (\gamma - 1)\big(\tfrac{1}{2}v_2^2 - \tfrac{1}{2}|D\varphi|^2 - \varphi\big)\,. \tag{4.3.2}$$

Lemma 4.10. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For $\boldsymbol{\theta} \in \Theta$, let φ be an admissible solution corresponding to $\boldsymbol{\theta}$. Then φ satisfies that, for $j = 5, 6$,*

(i) *$\partial_{\boldsymbol{e}}(\varphi - \varphi_j)$ is not constant in Ω for any unit vector $\boldsymbol{e} \in \mathbf{R}^2$;*

(ii) *For the vector $\boldsymbol{e}_{S_{2j}}$ given by (4.2.29), $\partial_{\boldsymbol{e}_{S_{2j}}}(\varphi_2 - \varphi) < 0$ in $\overline{\Omega} \setminus \Gamma^j_{\text{sonic}}$;*

(iii) *$\partial_{\boldsymbol{e}_{S_{2j}}}(\varphi - \varphi_j) \geq 0$, and $\partial_\eta(\varphi - \varphi_j) \leq 0$ in $\overline{\Omega}$.*

Proof. The proof of (i) is given by a slight modification of [1, Lemma 3.1]. Indeed, for $\phi \coloneqq \varphi - \varphi_5$, if $\partial_{\boldsymbol{e}}\phi$ is constant in Ω, then $\partial_{\boldsymbol{e}}\phi = 0$ in Ω since $D\phi = \mathbf{0}$ at P_1. Furthermore, $\partial_{\boldsymbol{e}}\phi = \boldsymbol{e} \cdot (u_6 - u_5, 0)$ at P_4 so that $\boldsymbol{e}$ must be parallel to $(0, 1)$, since $u_5 - u_6 > 0$ when $\boldsymbol{\theta} \neq \mathbf{0}$. Thus, $\partial_\eta\phi \equiv 0$ in Ω and hence $\partial_{\xi\eta}\phi = \partial_{\eta\eta}\phi \equiv 0$ in Ω. By the strict ellipticity of equation (4.3.1) in Ω, $\partial_{\xi\xi}\phi \equiv 0$ in Ω, which implies that there exist constants (u, v, k) such that $\phi = u\xi + v\eta + k$ in Ω. We note that $v = 0$ since $\partial_\eta\phi \equiv 0$.

Applying the boundary conditions: $\phi = 0$ and $D\phi = \mathbf{0}$ at P_1, we find that $u = k = 0$, so that $\phi \equiv 0$ in Ω. However, $D\phi(P_4) = D\varphi_6(P_4) - D\varphi_5(P_4) = (u_6 - u_5, 0) \neq \mathbf{0}$, which is a contradiction. The proof is similar for $\phi = \varphi - \varphi_6$. This concludes the proof of (i).

The proofs of (ii) and (iii) are similar to [1, Lemma 3.2] and [1, Lemma 3.6], respectively. □

For any $\boldsymbol{\theta} \in \overline{\Theta}$, define the following fan-shaped set by

$$\mathrm{Cone}(\boldsymbol{e}_{S_{25}}, \boldsymbol{e}_{S_{26}}) \coloneqq \left\{ r(\cos\theta, \sin\theta) \,:\, \theta_{26} \le \theta \le \theta_{25},\, r \ge 0 \right\}.$$

Let $\mathrm{Cone}^0(\boldsymbol{e}_{S_{25}}, \boldsymbol{e}_{S_{26}})$ be the interior of $\mathrm{Cone}(\boldsymbol{e}_{S_{25}}, \boldsymbol{e}_{S_{26}})$. It follows from Lemma 4.10 that, if φ is an admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$, then φ satisfies

$$\partial_{\boldsymbol{e}}(\varphi_2 - \varphi) < 0 \qquad \text{in } \overline{\Omega} \text{ for any } \boldsymbol{e} \in \mathrm{Cone}^0(\boldsymbol{e}_{S_{25}}, \boldsymbol{e}_{S_{26}}), \tag{4.3.3}$$

and $-\boldsymbol{\nu}(P) \in \left\{ (\cos\theta, \sin\theta) \,:\, \theta_{25} - \frac{\pi}{2} < \theta < \theta_{26} + \frac{\pi}{2} \right\}$ for all $P \in \Gamma_{\mathrm{shock}}$, where $\boldsymbol{\nu}$ is the unit normal vector on Γ_{shock} pointing into the interior of Ω.

Lemma 4.11. *Fix $\gamma \ge 1$ and $v_2 \in (v_{\min}, 0)$. For $\boldsymbol{\theta} \in \Theta$, let φ be an admissible solution corresponding to $\boldsymbol{\theta}$. Then there exists a C^1–function f_{sh} such that*

$$\Gamma_{\mathrm{shock}} = \{ \boldsymbol{\xi} \in \mathbf{R}^2_+ \,:\, \eta = f_{\mathrm{sh}}(\xi),\, \xi^{P_3} < \xi < \xi^{P_2} \},$$

and $f'_{\mathrm{sh}}(\xi) \in (\tan\theta_{25}, \tan\theta_{26})$ for $\xi \in (\xi^{P_3}, \xi^{P_2})$ with $f'_{\mathrm{sh}}(\xi^{P_{j-3}}) = \tan\theta_{2j}$, for $j = 5, 6$.

Proof. By Lemma 4.5, $\boldsymbol{e}_\eta = (0, 1) \in \mathrm{Cone}^0(\boldsymbol{e}_{S_{25}}, \boldsymbol{e}_{S_{26}})$. From the strict directional monotonicity (4.3.3), we see that $\partial_{\boldsymbol{e}_\eta}(\varphi_2 - \varphi) < 0$ on $\overline{\Gamma_{\mathrm{shock}}}$, which implies the existence of a C^1–function f_{sh} that solves the implicit relation $\varphi_2(\xi, f_{\mathrm{sh}}(\xi)) = \varphi(\xi, f_{\mathrm{sh}}(\xi))$. Taking the derivative, we have

$$f'_{\mathrm{sh}}(\xi) = -\frac{\partial_{\boldsymbol{e}_\xi}(\varphi_2 - \varphi)(\xi, f_{\mathrm{sh}}(\xi))}{\partial_{\boldsymbol{e}_\eta}(\varphi_2 - \varphi)(\xi, f_{\mathrm{sh}}(\xi))}.$$

From Definition 4.7(i), we obtain that $f'_{\mathrm{sh}}(\xi^{P_2}) = \tan\theta_{25}$ and $f'_{\mathrm{sh}}(\xi^{P_3}) = \tan\theta_{26}$.

Denoting by $\boldsymbol{\nu}(P)$ the unit normal vector on $\Gamma_{\rm shock}$ at a point $P \in \Gamma_{\rm shock}$, pointing into Ω, we have

$$\boldsymbol{\nu}(P) = \frac{D(\varphi_2 - \varphi)(P)}{|D(\varphi_2 - \varphi)(P)|} = \frac{(f'_{\rm sh}(\xi), -1)}{\sqrt{1 + \big(f'_{\rm sh}(\xi)\big)^2}}\,.$$

Combining the above with property (ii) in Lemma 4.10, for any $(a_1, a_2) \in [0,\infty)^2$ with $(a_1, a_2) \neq (0,0)$,

$$\begin{aligned} &a_1\big(f'_{\rm sh}(\xi)\cos\theta_{25} - \sin\theta_{25}\big) + a_2\big(f'_{\rm sh}(\xi)\cos\theta_{26} - \sin\theta_{26}\big) \\ &\quad = \sqrt{1 + \big(f'_{\rm sh}(\xi)\big)^2}\,\boldsymbol{\nu}(P)\cdot(a_1\boldsymbol{e}_{S_{25}} + a_2\boldsymbol{e}_{S_{26}}) < 0 \qquad \text{for any } \xi \in (\xi^{P_3}, \xi^{P_2})\,. \end{aligned}$$

Choosing (a_1, a_2) as $(1,0)$ and $(0,1)$ in turn, we obtain that $f'_{\rm sh}(\xi) > \tan\theta_{25}$ and $f'_{\rm sh}(\xi) < \tan\theta_{26}$, respectively, where we have used $\theta_{25} \in (\frac{\pi}{2}, \pi]$ and $\theta_{26} \in [0, \frac{\pi}{2})$. □

Lemma 4.12. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For any $\boldsymbol{\theta} \in \Theta$, let φ be an admissible solution corresponding to $\boldsymbol{\theta}$. Then there exists a constant $C_{\rm ub} > 0$ depending only on (γ, v_2) such that*

$$\overline{\Omega} \subseteq B_{C_{\rm ub}}(\mathbf{0})\,, \tag{4.3.4}$$

$$\|\varphi\|_{C^{0,1}(\overline{\Omega})} \leq C_{\rm ub}\,, \tag{4.3.5}$$

$$\rho^*(\gamma) \leq \rho \leq C_{\rm ub} \quad \text{in } \Omega\,, \qquad 1 < \rho \leq C_{\rm ub} \quad \text{on } \Gamma_{\rm shock}\,, \tag{4.3.6}$$

where $\rho^*(\gamma) := (\frac{2}{\gamma+1})^{\frac{1}{\gamma-1}}$ when $\gamma > 1$, and $\rho^*(1) := \lim\limits_{\gamma\to 1+}\rho^*(\gamma) = \frac{1}{\sqrt{e}}$.

Proof. We divide the proof into three steps.

1. To prove (4.3.4), we use the expression of $\Gamma_{\rm shock}$ from Lemma 4.11. In particular, it follows from Definition 4.7(i) and Lemma 4.11 that $0 < f_{\rm sh}(\xi) \leq \min\{\xi\tan\theta_{25} + a_{25}, \xi\tan\theta_{26} + a_{26}\}$ whenever $\xi^{P_3} < \xi < \xi^{P_2}$, which implies that

$$\Omega \subseteq \big\{\boldsymbol{\xi} : u_6 - c_6 < \xi < u_5 + c_5\,,\ 0 < \eta \leq \max\{a_{25}, a_{26}\}\big\}\,.$$

From (4.2.13) and Proposition 4.2(i) and Definition 4.4, we know that (ρ_5, u_5, a_{25}) depend continuously on $\theta_1 \in [0, \theta^{\rm d}]$, while (ρ_6, u_6, a_{26}) depend continuously on $\theta_2 \in$

$[0, \theta^{\rm d}]$, from which (4.3.4) follows directly.

2. From Definition 4.7(iv), $\inf_{\overline{\Omega}} \max\{\varphi_5, \varphi_6\} \le \varphi \le \sup_{\overline{\Omega}} \varphi_2$. By (4.3.4) and the expressions of $(\varphi_2, \varphi_5, \varphi_6)$ given in (4.2.14) and (4.2.27), there exists a constant $C_1 > 0$ depending only on (γ, v_2) such that $-C_1 \le \min_{\overline{\Omega}} \max\{\varphi_5, \varphi_6\} \le \max_{\overline{\Omega}} \varphi_2 \le C_1$, which implies that $\max_{\overline{\Omega}} |\varphi| \le C_1$. Then the uniform bound (4.3.5) follows from

$$|D\varphi|^2 < c^2(|D\varphi|, \varphi) = \frac{2}{\gamma+1}\big(1 + (\gamma-1)(B-\varphi)\big)\,,$$

where we have used Definition 4.7(iii) and (4.3.2).

3. The upper bounds in (4.3.6) follow directly from the expression of the density (4.2.2) along with (4.3.5). For the lower bound in Ω, we combine the Bernoulli law (4.2.2) with Definition 4.7(iv) to obtain

$$\frac{1}{2}|D\varphi|^2 + h(\rho) = \frac{1}{2}|D\varphi_2|^2 + (\varphi_2 - \varphi) + h(\rho_2) \ge \frac{1}{2}|D\varphi_2|^2 \ge 0\,.$$

Then, from Definition 4.7(iii), we have

$$\frac{1}{\gamma-1}\big(\frac{1}{2}(\gamma+1)\rho^{\gamma-1} - 1\big) = \frac{1}{2}c^2 + h(\rho) \ge \frac{1}{2}|D\varphi|^2 + h(\rho) \ge 0\,,$$

so that the lower bound of ρ in Ω is obtained. For the lower bound of ρ on $\Gamma_{\rm shock}$, we combine Problem B(ii) and Remark 4.8(iii) to obtain

$$\rho = \frac{D\varphi_2 \cdot \boldsymbol{\nu}}{D\varphi \cdot \boldsymbol{\nu}} > 1 \qquad \text{on } \Gamma_{\rm shock}\,. \qquad \square$$

4.3.1.2 Uniform positive lower bound of $\mathrm{dist}(\Gamma_{\rm shock}, \partial B_1(O_2))$

We now obtain a uniform estimate for the positive lower bound of $\mathrm{dist}(\Gamma_{\rm shock}, \partial B_1(O_2))$ for any admissible solution. This allows us to make the uniform estimate for the ellipticity of equation (4.2.1) within the pseudo-subsonic domain of admissible solutions.

Proposition 4.13. *Fix $\gamma \ge 1$ and $v_2 \in (v_{\min}, 0)$. There exists a constant $C_{\rm sh} > 0$ depending only on (γ, v_2) such that any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$*

satisfies

$$\operatorname{dist}\big(\Gamma_{\rm shock},\, \partial B_1(O_2)\big) \geq C_{\rm sh}^{-1}\,. \tag{4.3.7}$$

Proof. The proof of (4.3.7) follows the same argument as for [1, Proposition 3.7] except the case that $v_2 \leq -1$. In the following, we focus on the case that $v_2 \leq -1$ and give the proof in three steps.

1. We first rewrite equation (4.2.1) as

$$\operatorname{div}\mathcal{A}(D\varphi, \varphi) + \mathcal{B}(D\varphi, \varphi) = 0\,,$$

where $\mathcal{A}(\boldsymbol{p}, z) \coloneqq \rho(|\boldsymbol{p}|, z)\boldsymbol{p}$ and $\mathcal{B}(|\boldsymbol{p}|, z) \coloneqq 2\rho(|\boldsymbol{p}|, z)$ with $\boldsymbol{p} = (p_1, p_2) \in \mathbf{R}^2$, $z \in \mathbf{R}$, and $\rho(|\boldsymbol{p}|, z)$ given by (4.2.2) with $B = \frac{1}{2}v_2^2$. The sonic speed is given by $c(|\boldsymbol{p}|, z) \coloneqq \rho^{\frac{\gamma-1}{2}}$ for any $\gamma \geq 1$. For a constant $R > 1$, define

(4.3.8)
$$\mathcal{K}_R \coloneqq \Big\{(\boldsymbol{p}, z) \in \mathbf{R}^2 \times \mathbf{R} \,:\, |\boldsymbol{p}| + |z| \leq R,\, \rho(|\boldsymbol{p}|, z) \geq R^{-1},\, \frac{|\boldsymbol{p}|^2}{c^2(|\boldsymbol{p}|, z)} \leq 1 - R^{-1}\Big\}\,.$$

For each $R > 1$, there exists a constant $\lambda_R > 0$ depending only on (γ, v_2, R) such that, for any $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbf{R}^2$,

$$\sum_{i,j=1}^{2} \partial_{p_j}\mathcal{A}_i(\boldsymbol{p}, z)\kappa_i\kappa_j \geq \lambda_R|\boldsymbol{\kappa}|^2 \qquad \text{for any } (\boldsymbol{p}, z) \in \mathcal{K}_R\,.$$

The next two lemmas follow directly from [1, Lemmas 3.8–3.9].

Lemma 4.14. *For $R > 2$, let $\mathcal{K}_R$ be given by* (4.3.8). *Then there exist functions $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\boldsymbol{p}, z)$ on $\mathbf{R}^2 \times \mathbf{R}$ satisfying the following properties*:

(i) *For any $(\boldsymbol{p}_0, z_0) \in \mathcal{K}_R$, $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\boldsymbol{p}, z) = (\mathcal{A}, \mathcal{B})(\boldsymbol{p}, z)$ for all $(\boldsymbol{p}, z) \in B_\varepsilon((\boldsymbol{p}_0, z_0))$;*

(ii) *For any $(\boldsymbol{p}, z) \in \mathbf{R}^2 \times \mathbf{R}$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbf{R}^2$, $\sum\limits_{i,j=1}^{2} \partial_{p_j}\tilde{\mathcal{A}}_i(\boldsymbol{p}, z)\kappa_i\kappa_j \geq \lambda_R|\boldsymbol{\kappa}|^2$;*

(iii) *$|\tilde{\mathcal{B}}(\boldsymbol{p}, z)| \leq C_0$ and $|D^m_{(\boldsymbol{p},z)}(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\boldsymbol{p}, z)| \leq C_m$ on $\mathbf{R}^2 \times \mathbf{R}$ for each $m = 1, 2, \cdots$,*

where the positive constants ε and λ_R depend only on (γ, v_2, R), and C_m depends only on (γ, v_2, R, m).

Lemma 4.15. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For any given constants $\alpha \in (0,1)$, $m \in \mathbf{N}$, and $r > 0$, there exist constants $C, C_m > 0$ depending only on (γ, v_2, α, r), with C_m depending additionally on m, such that any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$ satisfies the following estimates*:

(i) *For any $B_{4r}(P) \subseteq \Omega$,*

$$\|\varphi\|_{2,\alpha,\overline{B_{2r}(P)}} \leq C\,, \qquad \|\varphi\|_{m,\overline{B_r(P)}} \leq C_m\,;$$

(ii) *If $P \in \Gamma_{\text{sym}}$, and $B_{4r}(P) \cap \Omega$ is the half-ball $B_{4r}^+(P) \coloneqq B_{4r}(P) \cap \{\eta > 0\}$, then*

$$\|\varphi\|_{2,\alpha,\overline{B_{2r}(P)\cap\Omega}} \leq C\,, \qquad \|\varphi\|_{m,\overline{B_r(P)}\cap\Omega} \leq C_m\,.$$

2. Fix $\varepsilon > 0$ small enough such that $2\xi^{P_3} < \xi^{P_0}$ for all $\theta_2 \in [0, \varepsilon)$, where $\xi^{P_0} < 0$ is the ξ–coordinate of $P_0 \coloneqq \lim_{\theta_2 \to 0^+} P_3$, which depends only on (γ, v_2). We define

$$d_{\text{sep}} \coloneqq \min\big\{-\tfrac{\xi^{P_0}}{2}, \inf_{\theta_2 \in [\varepsilon, \theta^{\text{d}}]} |u_6|\big\}\,. \tag{4.3.9}$$

It is direct to verify that $d_{\text{sep}} > 0$ by using $u_6 = v_2 \tan\theta_{26}$ and property (4.2.32), and that d_{sep} depends only on (γ, v_2). Then, for any $\boldsymbol{\theta} \in \Theta$, $\operatorname{dist}(P_2, P_3) \geq 2d_{\text{sep}}$. We also define

$$r_1 \coloneqq \frac{1}{2} \inf_{\boldsymbol{\theta} \in \Theta} \big\{|D\varphi_5(P_0^1)|, |D\varphi_6(P_0^2)|\big\}\,. \tag{4.3.10}$$

By (4.2.20) and Lemma 4.5, it is direct to verify that $r_1 > 0$ depends only on (γ, v_2).

For any $r \in (0, r_1)$ and for $i = 1, 2$, we define the two sets

$$\mathscr{A} \coloneqq \big\{\theta_i \in (0, \theta^{\text{d}}] \,:\, |P_0^i - P_{i+1}| \geq \frac{r}{20}\big\} \cup \{0\}\,,$$
$$\mathscr{B} \coloneqq \big\{\theta_i \in [0, \theta^{\text{d}}] \,:\, |P_{3i-2} - P_{i+1}| \geq \frac{r}{20}\big\}\,.$$

From the continuous dependence of P_0^i, P_{3i-2}, and P_{i+1} on $\theta_i \in (0, \theta^{\mathrm{d}}]$, we know that $\mathscr{A}$ and $\mathscr{B}$ are both closed sets. Then there exists a constant $C_1 > 0$ such that

$$\operatorname{dist}(P_{i+1}, \Gamma_{\mathrm{sym}}) \geq \frac{2}{C_1} \qquad \text{for all } \theta_i \in \mathscr{A} \cup \mathscr{B}\,.$$

If $\operatorname{dist}(P_{i+1}, \Gamma_{\mathrm{sym}}) \leq \frac{1}{C_1}$, then $\theta_i \notin \mathscr{A} \cup \mathscr{B}$, which implies that

$$\max\{|P_0^i - P_{i+1}|, |P_{3i-2} - P_{i+1}|\} < \frac{r}{20}\,.$$

We now seek a constant $C_r > 0$ depending only on (γ, v_2, r) such that, for any admissible solution of Problem B corresponding to $\boldsymbol{\theta} \in \Theta$,

$$\sup_{P \in \Gamma_{\mathrm{shock}} \cap B_{r/2}(P_{3i-2})} \operatorname{dist}(P, \Gamma_{\mathrm{sym}}) > C_r^{-1} \qquad \text{if } |P_{3i-2} - P_{i+1}| \leq \frac{r}{10}\,. \tag{4.3.11}$$

To obtain the above result, we follow the proof as in [20, Lemma 9.4.2], in which we need to take a limit of parameters $\boldsymbol{\theta}^{(n)}$. In order to take the limit, we use a compactness lemma below. For concreteness, in the following, we use the notation defined in Definition 4.6, and append the superscripts (n) and $*$ to indicate that the corresponding quantities are related to $\boldsymbol{\theta}^{(n)}$ and $\boldsymbol{\theta}^*$, respectively.

Lemma 4.16. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. Let $\{\boldsymbol{\theta}^{(n)}\}_{n \in \mathbf{N}} \subseteq \Theta$ be any sequence of parameters satisfying $\lim_{n\to\infty} \boldsymbol{\theta}^{(n)} = \boldsymbol{\theta}^*$ for some $\boldsymbol{\theta}^* \in \overline{\Theta}$. For each $n \in \mathbf{N}$, let $\varphi^{(n)}$ be an admissible solution corresponding to $\boldsymbol{\theta}^{(n)}$. Then there exists a subsequence $\{\varphi^{(n_j)}\}_{j \in \mathbf{N}}$ converging uniformly on any compact subset of $\mathbf{R}^2_+$ to a function $\varphi^* \in C^{0,1}_{\mathrm{loc}}(\overline{\mathbf{R}^2_+})$. Moreover, φ^* is a weak solution of equation (4.2.1) in $\mathbf{R}^2_+$ and satisfies the following properties*:

(i) *For the corner points P_m, $m = 1, 2, 3, 4$, and the reflection points P_0^i, $i = 1, 2$,*

$$\lim_{j\to\infty} P_m^{(n_j)} = P_m^*\,, \qquad \lim_{j\to\infty} P_0^{i,(n_j)} = P_0^{i,*}\,.$$

Note that $\xi^{P_0^{1,*}} = \infty$ in the case that $\theta_1^* = 0$, and $\xi^{P_0^{2,*}} = -\infty$ in the case that $\theta_2^* = 0$.

(ii) *Let* $\eta = f_{2l}^{(n_j)}(\xi)$, $l = 5, 6$, *and* $\eta = f_{\rm sh}^{(n_j)}(\xi)$ *be the expressions of* $S_{2l}^{(n_j)}$ *and* $\Gamma_{\rm shock}^{(n_j)}$ *from* Definition 4.4 *and* Lemma 4.11, *respectively. Extend* $f_{\rm sh}^{(n_j)}$ *by*

$$f_{\rm sh}^{(n_j)}(\xi) = \begin{cases} f_{25}^{(n_j)}(\xi) & \text{for } \xi^{P_2^{(n_j)}} < \xi < \xi^{P_0^{1,(n_j)}}, \\ f_{26}^{(n_j)}(\xi) & \text{for } \xi^{P_0^{2,(n_j)}} < \xi < \xi^{P_3^{(n_j)}}. \end{cases}$$

Then $\{f_{\rm sh}^{(n_j)}\}_{j\in\mathbf{N}}$ *converges uniformly to a function* $f_{\rm sh}^* \in C^{0,1}([\xi^{P_0^{2,*}}, \xi^{P_0^{1,*}}])$.

(iii) *Let the smooth functions* $\eta = g_{l,{\rm so}}^{(n_j)}(\xi)$ *be the expressions of* $\Gamma_{\rm sonic}^{l,(n_j)}$ *for* $l = 5, 6$. *Then* $\{g_{5,{\rm so}}^{(n_j)}\}_{j\in\mathbf{N}}$ *converges pointwise to* $g_{5,{\rm so}}^*$ *on* $(\xi^{P_2^*}, \xi^{P_1^*})$, *and* $\{g_{6,{\rm so}}^{(n_j)}\}_{j\in\mathbf{N}}$ *converges pointwise to* $g_{6,{\rm so}}^*$ *on* $(\xi^{P_4^*}, \xi^{P_3^*})$.

(iv) *Denote* $\widehat{\Omega^*} := \big\{\boldsymbol{\xi} \in [\xi^{P_4^*}, \xi^{P_1^*}] \times \overline{\mathbf{R}_+} : 0 \le \eta \le f_{\rm bd}^*(\xi)\big\}$, *where* $f_{\rm bd}^*(\cdot)$ *is given by*

$$f_{\rm bd}^*(\xi) = \begin{cases} g_{6,{\rm so}}^*(\xi) & \text{for } \xi^{P_4^*} \le \xi \le \xi^{P_3^*}, \\ f_{\rm sh}^*(\xi) & \text{for } \xi^{P_3^*} < \xi \le \xi^{P_2}, \\ g_{5,{\rm so}}^*(\xi) & \text{for } \xi^{P_2^*} < \xi \le \xi^{P_1^*}. \end{cases}$$

Denote by Ω^* *the interior of* $\widehat{\Omega^*}$, $\Gamma_{\rm shock}^* := \{(\xi, f_{\rm sh}^*(\xi)) : \xi \in (\xi^{P_3^*}, \xi^{P_2^*})\}$, *and* $\Gamma_{\rm sym}^* := \{(\xi, 0) : \xi \in (\xi^{P_4^*}, \xi^{P_1^*})\}$, *and denote by* $\Gamma_{\rm sym}^{*,0}$ *the relative interior of* $\Gamma_{\rm sym}^* \setminus \Gamma_{\rm shock}^*$. *Then* $\varphi^* \in C^\infty(\Omega^* \cup \Gamma_{\rm sym}^{*,0})$, *and*

(iv.1) $\lim_{j\to\infty} \|\varphi^{(n_j)} - \varphi^*\|_{C^2(K)} = 0$ *for any* $K \Subset \Omega^* \cup \Gamma_{\rm sym}^{*,0}$,

(iv.2) $D(\varphi_2 - \varphi^*) \cdot (\cos\theta, \sin\theta) \le 0$ *for all* $\theta \in (\theta_{26}^*, \theta_{25}^*)$,

(iv.3) *Equation* (4.2.1) *is strictly elliptic in* $\Omega^* \cup \Gamma_{\rm sym}^{*,0}$,

(iv.4) *In* $\mathbf{R}_+^2 \setminus \Omega^*$, φ^* *is equal to the constant states* $\bar\varphi^*$, φ_5^*, *and* φ_6^* *in their respective domains as given in* Definition 4.7(ii), *where* $\bar\varphi^*$ *is given by* (4.2.15) *with* $\boldsymbol{\theta} = \boldsymbol{\theta}^*$.

(v) $\Gamma_{\rm sym}^{*,0} = \{\varphi_2 > \varphi^*\} \cap \Gamma_{\rm sym}^*$ *is dense in* $\Gamma_{\rm sym}^*$.

Using Lemmas 4.12 and 4.15, properties (i)–(iv) in Lemma 4.16 can be proved by similar arguments as for [1, Corollary 3.10].

For property (v), if it were not true, we could choose $\xi_1, \xi_2 \in (\xi^{P_3^*}, \xi^{P_2^*})$ such that $\{(\xi, 0) : \xi_1 < \xi < \xi_2\} \subseteq \{\varphi_2 \le \varphi^*\} \cap \Gamma^*_{\rm sym}$. By Definition 4.7(iv), $\varphi^{(n_j)} \le \varphi_2$ in $\Omega^{(n_j)}$ so that $\varphi^* \le \varphi_2$ in $\overline{\Omega^*}$. Combining with property (iv.4), we obtain that $\varphi^*(\boldsymbol{\xi}) = \varphi_2(\boldsymbol{\xi})$ on $\{\boldsymbol{\xi} \in \overline{\mathbf{R}^2_+} : \xi_1 < \xi_2\}$. We take any non-negative function $\zeta \in C^\infty_{\rm c}(\overline{\mathbf{R}^2_+})$ satisfying that $\operatorname{supp}\zeta \subseteq \{\boldsymbol{\xi} \in \overline{\mathbf{R}^2_+} : \xi_1 < \xi < \xi_2\}$ and $\zeta > 0$ on some subset of $\{(\xi, 0) : \xi_1 < \xi < \xi_2\}$ with positive measure. Then

$$\begin{aligned}
&\int_{\mathbf{R}^2_+} (\rho(|D\varphi^*|, \varphi^*) D\varphi^* \cdot D\zeta(\boldsymbol{\xi}) - 2\rho(|D\varphi^*|, \varphi^*)\zeta(\boldsymbol{\xi}))\, \mathrm{d}\boldsymbol{\xi} \\
&\quad = \int_{\mathbf{R}^2_+} (\rho_2 D\varphi_2 \cdot D\zeta(\boldsymbol{\xi}) - 2\rho_2\zeta(\boldsymbol{\xi}))\, \mathrm{d}\boldsymbol{\xi} = -v_2 \int_{\xi_1}^{\xi_2} \zeta(\xi, 0)\, \mathrm{d}\xi > 0\,,
\end{aligned}$$

which contradicts the fact that φ^* is a weak solution of equation (4.2.1) in $\mathbf{R}^2_+$. Then Lemma 4.16(v) is verified.

We now prove (4.3.11) by contradiction. If it were not true, we could choose a sequence of parameters $\{\boldsymbol{\theta}^{(n)}\}_{n\in\mathbf{N}}$ such that

$$\sup_{P \in \Gamma^{(n)}_{\rm shock} \cap B_{r/2}(P^{(n)}_{3i-2})} \operatorname{dist}(P, \Gamma^{(n)}_{\rm sym}) \le \frac{1}{n} \qquad \text{if } |P^{(n)}_{3i-2} - P^{(n)}_{i+1}| \le \frac{r}{10}\,.$$

Then, by Lemma 4.16, we could further choose a subsequence $\{\boldsymbol{\theta}^{(n_j)}\}_{j\in\mathbf{N}}$ and take a limit to some $\boldsymbol{\theta}^* \in \overline{\Theta}$, which provides a contradiction to Lemma 4.16(v).

From (4.3.11), we can choose $\hat{Q}_i \in \overline{\Gamma_{\rm shock} \cap B_{r/2}(P_{3i-2})}$ with $\operatorname{dist}(\hat{Q}_i, \Gamma_{\rm sym}) \ge C_r^{-1}$, so that $\hat{Q}_i \in \overline{\Gamma_{\rm shock} \cap B_{3r/4}(P^i_0)}$. For $i = 1, 2$, we take $Q_i = P_{i+1} \in \overline{\Gamma_{\rm shock}}$ if $\operatorname{dist}(P_{i+1}, \Gamma_{\rm sym}) > C_1^{-1}$; otherwise, we take $Q_i = \hat{Q}_i \in \overline{\Gamma_{\rm shock} \cap B_{3r/4}(P^i_0)}$ as chosen above. Then it follows that

$$\min\{\operatorname{dist}(Q_1, Q_2)\,,\, \operatorname{dist}(Q_1, \Gamma_{\rm sym})\,,\, \operatorname{dist}(Q_2, \Gamma_{\rm sym})\} \ge \min\{d_{\rm sep}\,,\, C_1^{-1},\, C_r^{-1}\} =: a\,.$$

Combining with Lemma 4.16 and following a similar proof as for [20, Lemmas 9.4.1 and 15.4.1], we obtain that, for $a > 0$ defined directly above, there exists a constant $C_2 > 0$ depending only on (γ, v_2, a) such that, for any admissible solution corresponding

to $\boldsymbol{\theta} \in \Theta$,

$$\operatorname{dist}(\Gamma_{\text{shock}}[Q_1, Q_2]\,,\, \Gamma_{\text{sym}}) \geq C_2^{-1}\,,$$

where $\Gamma_{\text{shock}}[Q_1, Q_2]$ represents the segment on Γ_{shock} between points Q_1 and Q_2. Therefore,

$$\operatorname{dist}(\Gamma_{\text{shock}} \setminus (B_r(P_0^1) \cup B_r(P_0^2)),\, \Gamma_{\text{sym}}) > C_2^{-1}\,. \tag{4.3.12}$$

3. Since $v_2 \leq -1$, $B_1(O_2) \subseteq \mathbf{R} \times (-\infty, 0)$. By Lemma A.1 and property (ii) of Proposition 4.2, $d_{\text{ref}} \coloneqq |D\varphi_2(P_0^1|_{\theta_1=\theta^{\text{d}}})| - 1 > 0$ depending only on (γ, v_2) satisfies

$$\operatorname{dist}(P_0^1,\, B_1(O_2)) = |D\varphi_2(P_0^1)| - 1 \geq d_{\text{ref}}\,.$$

By symmetry, we also have $\operatorname{dist}(P_0^2,\, B_1(O_2)) \geq d_{\text{ref}}$. Denote $\bar{r} \coloneqq \frac{1}{2}\min\{r_1, d_{\text{ref}}\} > 0$ with r_1 defined by (4.3.10). Then

$$\operatorname{dist}(\Gamma_{\text{shock}} \cap (B_{\bar{r}}(P_0^1) \cup B_{\bar{r}}(P_0^2)),\, B_1(O_2)) \geq \bar{r}\,.$$

Meanwhile, from the argument in Step **2**, there exists a constant $C_2 > 0$ depending only on $(\gamma, v_2, \bar{r})$ such that (4.3.12) holds with r replaced by $\bar{r}$. Thus, for any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$, we have

$$\operatorname{dist}(\Gamma_{\text{shock}},\, B_1(O_2)) \geq \min\{\bar{r},\, C_2^{-1}\} =: C_{\text{sh}}^{-1}\,.$$

This completes the proof of (4.3.7). □

4.3.1.3 Uniform estimate for the ellipticity of equation (4.3.1)

We now show a uniform estimate for the ellipticity of equation (4.3.1), which may be degenerate near $\Gamma^5_{\mathrm{sonic}} \cup \Gamma^6_{\mathrm{sonic}}$. Fix a function $h^* \in C^\infty(\mathbf{R})$ such that

$$h^*(s) = \begin{cases} s & \text{for } s \le \frac{1}{2}\,, \\ 1 & \text{for } s \ge 1\,, \end{cases}$$

satisfying $0 \le (h^*)'(\cdot) \le 2$ on $\mathbf{R}$. For each $\boldsymbol{\theta} \in \overline{\Theta}$, define

$$\hat{c}_j \coloneqq \min\{c_j,\, |D\varphi_j(P_0^{j-4})|\}\,, \quad q_j \coloneqq \operatorname{dist}(O_j,\, S_{2j}) \qquad \text{for } j = 5, 6\,. \tag{4.3.13}$$

By Lemma 4.5 and [20, Lemma 6.1.2], we see that $\hat{c}_j > q_j$ for every $\boldsymbol{\theta} \in \overline{\Theta}$, so that there exists a constant $\hat{\delta} > 0$ depending only on (γ, v_2) such that

$$\inf_{\boldsymbol{\theta} \in \overline{\Theta}} \min\{\hat{c}_5 - q_5,\, \hat{c}_6 - q_6\} \ge \hat{\delta}\,. \tag{4.3.14}$$

For $j = 5, 6$, define $g_j^* : \mathbf{R}^2 \to \mathbf{R}_+$ by

$$g_j^*(\boldsymbol{\xi}) \coloneqq \frac{1}{2}(\hat{c}_j - q_j) h^*\Big(\frac{\operatorname{dist}(\boldsymbol{\xi},\, \partial B_{\hat{c}_j}(O_j))}{\hat{c}_j - q_j}\Big) + \Big(1 - \frac{\hat{c}_j^2}{c_j^2}\Big)\,.$$

We choose a function $\chi \in C^\infty(\mathbf{R})$ such that

$$\chi(\xi) = \begin{cases} 1 & \text{for } \xi \le -\frac{d_{\mathrm{sep}}}{4}\,, \\ 0 & \text{for } \xi \ge \frac{d_{\mathrm{sep}}}{4}\,, \end{cases}$$

satisfying $-\frac{4}{d_{\mathrm{sep}}} \le \chi'(\cdot) \le 0$ on $\mathbf{R}$, where $d_{\mathrm{sep}} > 0$ is given by (4.3.9) depending only on (γ, v_2). Finally, define a function $g : \mathbf{R}^2 \to \mathbf{R}_+$ by

$$g(\boldsymbol{\xi}) \coloneqq \chi(\xi) g_6^*(\boldsymbol{\xi}) + (1 - \chi(\xi)) g_5^*(\boldsymbol{\xi})\,. \tag{4.3.15}$$

By Definition 4.7 and Lemma 4.12, there exist constants $d > 0$ and $C > 1$

depending only on (γ, v_2) such that, if φ is an admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$, then g given by (4.3.15) satisfies the following properties:

(a) For $\boldsymbol{\xi} \in \Omega$ satisfying $\operatorname{dist}(\boldsymbol{\xi}, \Gamma^j_{\text{sonic}}) < d$,

$$C^{-1}\operatorname{dist}_j(\boldsymbol{\xi},\, \Gamma^j_{\text{sonic}}) \le g(\boldsymbol{\xi}) \le C\operatorname{dist}_j(\boldsymbol{\xi},\, \Gamma^j_{\text{sonic}})\,,$$

where $\operatorname{dist}_j(\boldsymbol{\xi},\, \Gamma^j_{\text{sonic}}) := \operatorname{dist}(\boldsymbol{\xi},\, \Gamma^j_{\text{sonic}}) + (c_j - \hat{c}_j)$ for $j = 5, 6$.

(b) For each $\varepsilon > 0$, there exists a constant $C_\varepsilon > 1$ depending only on $(\gamma, v_2, \varepsilon)$ such that, if a point $\boldsymbol{\xi} \in \overline{\Omega}$ satisfies $\operatorname{dist}(\boldsymbol{\xi},\, \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) > \varepsilon$, then $C_\varepsilon^{-1} \le g(\boldsymbol{\xi}) \le C_\varepsilon$.

In fact, for property (a) above, it suffices to choose $d := \min\{\frac{d_{\text{sep}}}{2}, \hat{\delta}\}$, with constants d_{sep} and $\hat{\delta}$ given by (4.3.9) and (4.3.14) respectively. Then, choosing $\varepsilon := \frac{d}{2}$ in property (b), we deduce that there exists a constant $C_\flat > 1$ depending only on (γ, v_2) such that, for any $\boldsymbol{\theta} \in \Theta$,

(4.3.16)
$$C_\flat^{-1}\operatorname{dist}^\flat(\boldsymbol{\xi},\, \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) \le g(\boldsymbol{\xi}) \le C_\flat \operatorname{dist}^\flat(\boldsymbol{\xi},\, \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) \qquad \text{for all } \boldsymbol{\xi} \in \overline{\Omega}\,,$$

where

$$\operatorname{dist}^\flat(\boldsymbol{\xi},\, \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) := \min\Big\{\frac{d}{2},\, \operatorname{dist}_5(\boldsymbol{\xi},\, \Gamma^5_{\text{sonic}}),\, \operatorname{dist}_6(\boldsymbol{\xi},\, \Gamma^6_{\text{sonic}})\Big\}\,. \tag{4.3.17}$$

Proposition 4.17. *There exists a constant $\mu > 0$ depending only on (γ, v_2) such that, for any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$, with pseudo-subsonic region Ω, the pseudo-Mach number*

$$M(\boldsymbol{\xi}) := \frac{|D\varphi(\boldsymbol{\xi})|}{c(|D\varphi(\boldsymbol{\xi})|, \varphi(\boldsymbol{\xi}))}$$

satisfies

$$M^2(\boldsymbol{\xi}) \le 1 - \mu g(\boldsymbol{\xi}) \qquad \text{in } \overline{\Omega}\,, \tag{4.3.18}$$

and there exists a constant $C > 1$ depending only on (γ, v_2) such that

$$C^{-1}\mathrm{dist}^\flat(\boldsymbol{\xi}, \Gamma^5_{\rm sonic} \cup \Gamma^6_{\rm sonic})|\boldsymbol{\kappa}|^2 \le \sum_{i,j=1}^{2} \mathcal{A}^i_{p_j}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}))\kappa_i\kappa_j \le C|\boldsymbol{\kappa}|^2 \tag{4.3.19}$$

for all $\boldsymbol{\xi} \in \overline{\Omega}$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbf{R}^2$, where $\mathcal{A}(\boldsymbol{p}, z) = \rho(|\boldsymbol{p}|, z)\boldsymbol{p}$, and $\mathrm{dist}^\flat(\,\cdot\,,\,\cdot\,)$ is given by (4.3.17)*.*

Proof. Let φ be any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$ with pseudo-subsonic region Ω and curved transonic shock $\Gamma_{\rm shock}$.

1. From Lemma 4.12, there exists a constant $R > 1$ depending only on (γ, v_2) such that

$$\Omega \subseteq B_R(\mathbf{0}), \qquad \|c(|D\varphi|, \varphi)\|_{C^0(\overline{\Omega})} + \|g\|_{C^2(\overline{\Omega})} \le R,$$

where $g(\cdot)$ is given by (4.3.15). Since $O_5, O_6 \in \Gamma_{\rm sym}$, $\partial_\eta g = 0$ on $\Gamma_{\rm sym}$. Then, by [20, Theorem 5.3.1], we can choose the constants $C_0 > 1$, $\delta \in (0, \frac{3}{4C_0})$, and $\mu_1 \in (0, 1)$ depending only on (γ, v_2) such that, for any $\mu \in (0, \mu_1]$, either the inequality $M^2 + \mu g \le C_0\delta < 1$ holds in Ω, or the maximum of $M^2 + \mu g$ over $\overline{\Omega}$ cannot be attained in $\Omega \cup \Gamma_{\rm sym}$.

In the first case, (4.3.18) follows immediately. Thus, we focus on the second case that the maximum of $M^2 + \mu g$ must be attained on $\partial\Omega \setminus \Gamma_{\rm sym}$.

2. We claim that there exists a constant $\mu_2 \in (0, \mu_1]$ depending only on (γ, v_2) such that, whenever $\mu \in (0, \mu_2]$, $(M^2 + \mu g) \le 1$ holds in Ω, from which (4.3.18) follows.

Let the maximum of $M^2 + \mu g$ be attained at $P_{\max} \in \partial\Omega \setminus \Gamma_{\rm sym}$, and suppose that $(M^2 + \mu g)(P_{\max}) > 1$. Then, using Proposition 4.13, we may proceed as in Steps **2**–**3** of the proof of [1, Proposition 3.15] to find $\mu_2 \in (0, \mu_1]$ depending only on (γ, v_2) such that $P_{\max} \notin \Gamma_{\rm shock}$ when $\mu \in (0, \mu_2]$. Therefore, $P_{\max} \in \Gamma^5_{\rm sonic} \cup \Gamma^6_{\rm sonic}$. However, reducing μ_2 depending only on (γ, v_2) if necessary, we have

$$(M^2 + \mu g)(P_{\max}) = \sup_{\boldsymbol{\xi} \in \Gamma^5_{\rm sonic} \cup \Gamma^6_{\rm sonic}} (M^2 + \mu g)(\boldsymbol{\xi}) \le 1,$$

which is a contradiction. In the final inequality above, we have used that $g|_{\Gamma^j_{\rm sonic}} = 0$ for

$\theta_{j-4} \in [0, \theta^{\rm s})$, whilst $g|_{\Gamma^j_{\rm sonic}} \leq C_\flat(c_j - |D\varphi_j(P_0^{j-4})|)$ for $\theta_{j-4} \in [\theta^{\rm s}, \theta^{\rm d})$, for $j = 5, 6$. $\square$

4.3.2 Hölder estimates for the solution and free boundary away from the sonic boundaries

For any set $U \subseteq \mathbf{R}^2$ and any constant $\varepsilon > 0$, define the ε-neighbourhood of U as

$$\mathcal{N}_\varepsilon(U) \coloneqq \{\boldsymbol{\xi} \in \mathbf{R}^2 \,:\, \mathrm{dist}(\boldsymbol{\xi}, U) < \varepsilon\}\,. \tag{4.3.20}$$

From Proposition 4.13, there exists a constant $C_{\rm sh} > 0$ depending only on (γ, v_2) such that

$$\mathrm{dist}(\Gamma_{\rm shock}, \partial B_1(O_2)) \geq C_{\rm sh}^{-1}$$

for any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$. It follows that

$$|D\varphi_2(\boldsymbol{\xi})|^2 \geq 1 + d_0 \qquad \text{for any } \boldsymbol{\xi} \in \mathcal{N}_{\frac{1}{2C_{\rm sh}}}(\Gamma_{\rm shock})$$

for some constant $d_0 > 0$ depending only on (γ, v_2). Subsequently, we claim that there exist constants $d_0' \in (0, d_0)$ and $\varepsilon \in (0, \frac{1}{2C_{\rm sh}})$ depending only on (γ, v_2) such that, for any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$,

$$|D\varphi_2(\boldsymbol{\xi})| - |D\varphi(\boldsymbol{\xi})| \geq d_0' \qquad \text{for any } \boldsymbol{\xi} \in \overline{\Omega} \cap \mathcal{N}_\varepsilon(\Gamma_{\rm shock})\,. \tag{4.3.21}$$

Indeed, in the case $\gamma > 1$, from Definition 4.7(iii)–(iv) and the Bernoulli law

$$\frac{1}{2}|D\varphi|^2 + \varphi + \frac{\rho^{\gamma-1} - 1}{\gamma - 1} = B = \frac{1}{2}|D\varphi_2|^2 + \varphi_2\,,$$

we have

$$\begin{aligned}\frac{\gamma+1}{2}\left(\rho^{\gamma-1} - 1\right) &\geq \left(\rho^{\gamma-1} - 1\right) + \frac{\gamma-1}{2}\left(|D\varphi|^2 - 1\right)\\ &= (\gamma-1)(\varphi_2 - \varphi) + \frac{\gamma-1}{2}\left(|D\varphi_2|^2 - 1\right) \geq \frac{\gamma-1}{2} d_0\,,\end{aligned}$$

and

$$|D\varphi_2|^2 - |D\varphi|^2 = \frac{2\,(\rho^{\gamma-1}-1)}{\gamma-1} + (\varphi - \varphi_2) \geq \frac{2d_0}{\gamma+1} + (\varphi - \varphi_2)\,.$$

Since φ is Lipschitz continuous in $\overline{\Omega}$ from Lemma 4.12, and $\varphi = \varphi_2$ on $\Gamma_{\rm shock}$, we obtain

$$|D\varphi_2(\boldsymbol{\xi})|^2 - |D\varphi(\boldsymbol{\xi})|^2 \geq \frac{d_0}{\gamma+1} \qquad \text{for any } \boldsymbol{\xi} \in \overline{\Omega} \cap \mathcal{N}_\varepsilon(\Gamma_{\rm shock})\,,$$

with some $\varepsilon \in (0, \frac{1}{2C_{\rm sh}})$ sufficiently small. Then (4.3.21) follows from the above inequality and the uniform boundedness property in Lemma 4.12. In the case $\gamma = 1$, it is straightforward to obtain (4.3.21) by using Definition 4.7(iii): for any admissible solution φ, $|D\varphi|^2 \leq c^2(|D\varphi|, \varphi) \equiv 1$ in $\overline{\Omega}$, since the sonic speed is a constant.

For any $\boldsymbol{\xi} = (\xi, \eta) \in \mathbf{R}^2 \setminus \{O_2\}$, define the polar coordinates (r, θ) centred at the sonic centre $O_2 = (u_2, v_2)$ by

$$r(\cos\theta, \sin\theta) \coloneqq (\xi, \eta) - O_2 \qquad \text{with } r > 0\,,\ \theta \in [0, 2\pi)\,,$$

It follows from (4.3.21) that

$$\partial_r(\varphi_2 - \varphi) \leq -(|D\varphi_2| - |D\varphi|) \leq -d_0' \qquad \text{in } \overline{\Omega} \cap \mathcal{N}_\varepsilon(\Gamma_{\rm shock})\,. \tag{4.3.22}$$

Therefore, we may apply the implicit function theorem to express $\Gamma_{\rm shock}$ under these polar coordinates as

$$\Gamma_{\rm shock} = \{\boldsymbol{\xi}(r,\theta) \,:\, r = f_{O_2,{\rm sh}}(\theta),\, \theta_{P_2} < \theta < \theta_{P_3}\}\,,$$

where $(f_{O_2,{\rm sh}}(\theta_{P_i}), \theta_{P_i})$ are the polar coordinates of the points P_i, for $i = 2, 3$. By Lemma 4.12 and (4.3.22), there exists a constant $C_1 > 0$ depending only on (γ, v_2) such that

$$\|f_{O_2,{\rm sh}}\|_{C^{0,1}([\theta_{P_2}, \theta_{P_3}])} \leq C_1\,.$$

From the above, with the help of Lemma 4.14, we deduce the following properties; the proof is similar to that of [1, Lemmas 3.17–3.18].

Lemma 4.18. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. Let φ be an admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$. Then*

(i) *There exists a constant $\delta' > 0$ depending only on (γ, v_2) such that*

$$\partial_{\boldsymbol{\nu}} \varphi_2 - \delta' > \partial_{\boldsymbol{\nu}} \varphi \geq \delta' \qquad \text{on } \overline{\Gamma_{\text{shock}}},$$

where $\boldsymbol{\nu} = \frac{D(\varphi_2 - \varphi)}{|D(\varphi_2 - \varphi)|}$ is the unit normal vector on Γ_{shock} pointing into Ω.

(ii) *For each $d > 0$ and $m = 2, 3, \cdots$, there exist positive constants s and C_m depending only on (γ, v_2, d) such that, whenever $P = \boldsymbol{\xi}(r_P, \theta_P) \in \Gamma_{\text{shock}}$ satisfies $\text{dist}(P, \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) \geq d$,*

$$|f^{(m)}_{O_2,\text{sh}}(\theta_P)| + \sup_{B_s(P) \cap \Omega} |D^m \varphi| \leq C_m .$$

From Lemmas 4.15, 4.17, and 4.18(ii), we have the following *a priori* interior estimates.

Corollary 4.19. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For each $d > 0$ and $m = 2, 3, \cdots$, there exists a constant $C_{m,d} > 0$ depending only on (γ, v_2, m, d) such that any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$ satisfies*

$$\|\varphi\|_{m, \overline{\Omega \cap \{\text{dist}(\boldsymbol{\xi}, \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) > d\}}} \leq C_{m,d} .$$

4.3.3 Weighted Hölder estimates near the sonic boundaries

From Definition 4.6, the sonic boundary Γ^5_{sonic} depends continuously on $\theta_1 \in [0, \theta^{\text{d}})$ and shrinks into a single point $P^1_0 = P_1 = P_2$ when $\theta_1 \in [0, \theta^{\text{s}})$ tends to θ^{s} from below (at least in the case that $v_2 \in [v_2^{\text{s}}, 0)$). Moreover, from Proposition 4.17, the ellipticity of equation (4.3.1) degenerates on the sonic boundary Γ^5_{sonic} for $\theta_1 \in [0, \theta^{\text{s}})$, while equation (4.3.1) is uniformly elliptic in the case that $\theta_1 \in (\theta^{\text{s}}, \theta^{\text{d}})$ up to Γ^5_{sonic} and away from Γ^6_{sonic}.

The above reasons lead us to consider the following four cases separately for the

weighted Hölder estimates of the admissible solutions near $\Gamma^5_{\rm sonic}$. Since it is possible that $\theta^{\rm s}$ can be equal to $\theta^{\rm cr}$ depending on (γ, v_2), some of the following cases may not occur; *cf.* Remark 4.3(i):

Case 1. $\theta_1 \in [0, \theta^{\rm s})$ away from $\theta^{\rm s}$;

Case 2. $\theta_1 \in [0, \theta^{\rm s})$ close to $\theta^{\rm s}$;

Case 3. $\theta_1 \in [\theta^{\rm s}, \theta^{\rm d})$ close to $\theta^{\rm s}$;

Case 4. $\theta_1 \in [\theta^{\rm s}, \theta^{\rm d})$ away from $\theta^{\rm s}$.

Corresponding weighted $C^{2,\alpha}$–estimates near $\Gamma^6_{\rm sonic}$ for the four cases with respect to $\theta_2 \in [0, \theta^{\rm d})$ are obtained immediately due to the symmetry of our problem.

For any $\boldsymbol{\theta} \in \Theta$, define an open set $\Omega^5 \subseteq \Omega$ by

$$\Omega^5 := \big(\Omega \cap \{\boldsymbol{\xi} \in \mathbf{R}^2_+ \,:\, \xi > u_5 + q_5 \sin\theta_{25}\}\big) \setminus \overline{B_{c_5^*}(O_5)}\,,$$

where $q_5 = \operatorname{dist}(O_5, S_{25})$ is given by (4.3.13), and $c_5^* := \frac{1}{2}\big(|O_5P_2| + \operatorname{dist}(O_5, S_{25})\big)$. For $\boldsymbol{\xi} \in \overline{\Omega^5}$, we introduce the modified polar coordinates (x, y) centred at O_5 defined by (4.3.23)

$$(x, y) := \mathcal{R}(\xi, \eta) \quad \Longleftrightarrow \quad (\xi, \eta) - O_5 = (c_5 - x)(\cos y, \sin y) \qquad \text{with } y \in [0, \tfrac{\pi}{2}]\,.$$

It is clear that $\mathcal{R}(\Omega^5) \subseteq \{(x, y) \,:\, 0 < x < c_5 - c_5^*\}$. For the pseudo-potential φ_5 given by (4.2.27), we have

$$\varphi_5 \circ \mathcal{R}^{-1}(x, y) = -\frac{1}{2}(c_5 - x)^2 + \frac{1}{2}u_5^2 + v_2 a_{25} \qquad \text{for } (x, y) \in \mathcal{R}(\Omega^5)\,,$$

where a_{25} is a continuous function of $\theta_1 \in [0, \theta^{\rm d})$; see Definition 4.4.

For any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$, let ψ be given by

$$\psi(x, y) := (\varphi - \varphi_5) \circ \mathcal{R}^{-1}(x, y) \qquad \text{for } (x, y) \in \mathcal{R}(\Omega^5)\,.$$

Then ψ satisfies the following free boundary problem for an elliptic equation:

(i) *Equation for ψ in $\mathcal{R}(\Omega^5)$*: Direct calculation gives

(4.3.24)
$$\mathcal{N}(\psi) = \big(2x - (\gamma+1)\psi_x + \mathcal{O}_1\big)\psi_{xx} + \mathcal{O}_2\psi_{xy} + \big(\tfrac{1}{c_5} + \mathcal{O}_3\big)\psi_{yy} - \big(1 + \mathcal{O}_4\big)\psi_x + \mathcal{O}_5\psi_y = 0\,,$$

with $\mathcal{O}_j = \mathcal{O}_j(\psi_x, \psi_y, \psi, x, c_5)$ for $j = 1, \cdots, 5$, defined as follows:

(4.3.25)
$$\begin{aligned}
\mathcal{O}_1(p_x, p_y, z, x, c) &= -\frac{x^2}{c} + \frac{\gamma+1}{2c}(2x - p_x)p_x - \frac{\gamma-1}{c}\Big(z + \frac{p_y^2}{2(c-x)^2}\Big)\,,\\
\mathcal{O}_2(p_x, p_y, z, x, c) &= -\frac{2(p_x + c - x)p_y}{c(c-x)^2}\,,\\
\mathcal{O}_3(p_x, p_y, z, x, c) &= \frac{1}{c(c-x)^2}\Big((2c-x)x - (\gamma-1)\big(z + (c-x)p_x - \frac{p_x^2}{2}\big) + \frac{(\gamma-1)p_y^2}{2(c-x)^2}\Big)\,,\\
\mathcal{O}_4(p_x, p_y, z, x, c) &= \frac{1}{c-x}\Big(x - \frac{\gamma-1}{c}\big(z + (c-x)p_x + \frac{p_x^2}{2} + \frac{(\gamma+1)p_y^2}{2(\gamma-1)(c-x)^2}\big)\Big)\,,\\
\mathcal{O}_5(p_x, p_y, z, x, c) &= -\frac{2(p_x + c - x)p_y}{c(c-x)^3}\,.
\end{aligned}$$

(ii) *Boundary condition for ψ on $\mathcal{R}(\Gamma_{\mathrm{shock}} \cap \partial\Omega^5)$*: Denote the Rankine–Hugoniot condition for φ on Γ_{shock} as

(4.3.26)
$$g^{\mathrm{sh}}(D\varphi, \varphi, \boldsymbol{\xi}) \coloneqq \rho(|D\varphi|, \varphi)D\varphi \cdot \boldsymbol{\nu} - D\varphi_2 \cdot \boldsymbol{\nu} = (\mathcal{A}(D\varphi, \varphi) - D\varphi_2) \cdot \boldsymbol{\nu} = 0\,,$$

where $\mathcal{A}(\boldsymbol{p}, z) = \rho(|\boldsymbol{p}|, z)\boldsymbol{p}$, and $\boldsymbol{\nu} = \frac{D(\varphi_2 - \varphi)}{|D(\varphi_2 - \varphi)|}$ is the unit normal vector on Γ_{shock} pointing into Ω. For constant $\delta' > 0$ given in Lemma 4.18, we define a function $\zeta \in C^\infty(\mathbf{R})$ by

$$\zeta(t) = \begin{cases} t & \text{for } t \geq \frac{3\delta'}{4}\,,\\ \frac{\delta'}{2} & \text{for } t < \frac{\delta'}{2}\,, \end{cases}$$

satisfying $\zeta'(t) \geq 0$ on $\mathbf{R}$. Also, we define an extension of $g^{\mathrm{sh}}(\boldsymbol{p}, z, \boldsymbol{\xi})$ onto $\mathbf{R}^2 \times \mathbf{R} \times \overline{\Omega^5}$ by

$$g^{\mathrm{sh}}_{\mathrm{mod}}(\boldsymbol{p}, z, \boldsymbol{\xi}) \coloneqq \big(\tilde{\mathcal{A}}(\boldsymbol{p}, z) - D\varphi_2(\boldsymbol{\xi})\big) \cdot \frac{D\varphi_2(\boldsymbol{\xi}) - \boldsymbol{p}}{\zeta(|D\varphi_2(\boldsymbol{\xi}) - \boldsymbol{p}|)}\,, \tag{4.3.27}$$

where $\tilde{\mathcal{A}}(\boldsymbol{p}, z)$ is given by Lemma 4.14, and $g^{\mathrm{sh}}_{\mathrm{mod}}(\boldsymbol{p}, z, \boldsymbol{\xi}) = g^{\mathrm{sh}}(\boldsymbol{p}, z, \boldsymbol{\xi})$ whenever

$|D\varphi_2(\boldsymbol{\xi}) - \boldsymbol{p}| \geq \frac{3\delta_1}{4}$. Then we define

$$M^{(\theta_1)}(\boldsymbol{p}, z, \xi) \coloneqq g^{\mathrm{sh}}_{\mathrm{mod}}(\boldsymbol{p} + D\varphi_5, z + \varphi_5, \xi, \xi \tan\theta_{25} + a_{25} + \tfrac{z}{v_2})$$

with $(D\varphi_5, \varphi_5)$ evaluated at $\boldsymbol{\xi} = (\xi,\, \xi\tan\theta_{25} + a_{25} + \frac{z}{v_2})$. Note that, by Proposition 4.2(i) and Definition 4.4, $(\theta_{25},\, a_{25})$ depend continuously on $\theta_1 \in [0, \theta^{\mathrm{d}}]$ with the limits $\lim_{\theta_1 \to 0^+} (\theta_{25}, a_{25}) = (\pi, \eta_0)$, where $\eta_0 > 0$ is given by (4.2.18).

We prescribe the free boundary condition for ψ to be

$$\mathcal{B}^{(\theta_1)}(\psi_x, \psi_y, \psi, x, y) = 0 \qquad \text{on } \mathcal{R}(\Gamma_{\mathrm{shock}} \cap \partial\Omega^5)\,,$$

where $\mathcal{B}^{(\theta_1)}$ is given by

$$\mathcal{B}^{(\theta_1)}(p_x, p_y, z, x, y) \coloneqq |D\bar{\phi}_{25} - (p_1, p_2)| M^{(\theta_1)}(p_1, p_2, z, \xi)\,, \tag{4.3.28}$$

with the function $\bar{\phi}_{25}(\boldsymbol{\xi}) \coloneqq (\varphi_2 - \varphi_5)(\boldsymbol{\xi}) = v_2(\eta - \xi\tan\theta_{25} - a_{25})$, and

$$\xi = v_2 \tan\theta_{25} + (c_5 - x)\cos y\,, \qquad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \cos y & \sin y \\ \sin y & -\cos y \end{pmatrix} \begin{pmatrix} -p_x \\ -\frac{p_y}{c_5 - x} \end{pmatrix}.$$

(iii) *Other conditions for* ψ: From Problem B(iii)–(iv) and Definition 4.7(iv), ψ satisfies

$$\psi \geq 0 \ \text{ in } \mathcal{R}(\Omega^5)\,, \qquad \psi = 0 \ \text{ on } \mathcal{R}(\Gamma^5_{\mathrm{sonic}})\,, \qquad \psi_y = 0 \ \text{ on } \mathcal{R}(\Gamma_{\mathrm{sym}} \cap \partial\Omega^5)\,.$$

Definition 4.20 (Parabolic norm)**.** Fix $\alpha \in (0,1)$. Define the α-th parabolic distance between points $\boldsymbol{x} = (x, y),\ \tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}) \in (0,\infty) \times \mathbf{R}$ to be

$$\delta^{(\mathrm{par})}_{\alpha}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) \coloneqq \left(|x - \tilde{x}|^2 + \max\{x, \tilde{x}\}|y - \tilde{y}|^2\right)^{\frac{\alpha}{2}}.$$

Then, given constants $\sigma > 0$ and $m \in \mathbf{N}$, the parabolic norms are defined as follows:

(i) For any open set $\mathcal{D} \subseteq (0,\infty) \times \mathbf{R}$ and any function $u \in C^2(\mathcal{D})$ in the (x, y)–

coordinates, define

$$\|u\|^{(\sigma),(\mathrm{par})}_{m,0,\mathcal{D}} \coloneqq \sum_{0\le k+l\le m} \sup_{\boldsymbol{x}\in\mathcal{D}} \left(x^{k+\frac{l}{2}-\sigma} |\partial_x^k \partial_y^l u(\boldsymbol{x})| \right),$$

$$[u]^{(\sigma),(\mathrm{par})}_{m,\alpha,\mathcal{D}} \coloneqq \sum_{k+l=m} \sup_{\substack{\boldsymbol{x},\tilde{\boldsymbol{x}}\in\mathcal{D}\\ \boldsymbol{x}\neq\tilde{\boldsymbol{x}}}} \left(\min\left\{ x^{\alpha+k+\frac{l}{2}-\sigma}, \tilde{x}^{\alpha+k+\frac{l}{2}-\sigma} \right\} \frac{|\partial_x^k \partial_y^l u(\boldsymbol{x}) - \partial_x^k \partial_y^l u(\tilde{\boldsymbol{x}})|}{\delta^{(\mathrm{par})}_\alpha(\boldsymbol{x},\tilde{\boldsymbol{x}})} \right),$$

$$\|u\|^{(\sigma),(\mathrm{par})}_{m,\alpha,\mathcal{D}} \coloneqq \|u\|^{(\sigma),(\mathrm{par})}_{m,0,\mathcal{D}} + [u]^{(\sigma),(\mathrm{par})}_{m,\alpha,\mathcal{D}}.$$

(ii) For any $a > 0$, fix an open interval $I \coloneqq (0, a)$. For a function $f \in C^2(I)$, define

$$\|f\|^{(\sigma),(\mathrm{par})}_{m,0,I} \coloneqq \sum_{k=0}^{m} \sup_{x\in I} \left(x^{k-\sigma} |\partial_x^k f(x)| \right),$$

$$[f]^{(\sigma),(\mathrm{par})}_{m,\alpha,I} \coloneqq \sup_{\substack{x,\tilde{x}\in I\\ x\neq\tilde{x}}} \left(\min\left\{ x^{\alpha+m-\sigma}, \tilde{x}^{\alpha+m-\sigma} \right\} \frac{|\partial_x^m f(x) - \partial_x^m f(\tilde{x})|}{|x-\tilde{x}|^\alpha} \right),$$

$$\|f\|^{(\sigma),(\mathrm{par})}_{m,\alpha,I} \coloneqq \|f\|^{(\sigma),(\mathrm{par})}_{m,0,I} + [f]^{(\sigma),(\mathrm{par})}_{m,\alpha,I}.$$

(iii) In the case $\sigma = 2$, write $\|u\|^{(\mathrm{par})}_{2,\alpha,\mathcal{D}} \coloneqq \|u\|^{(2),(\mathrm{par})}_{2,\alpha,\mathcal{D}}$ and $\|f\|^{(\mathrm{par})}_{2,\alpha,I} \coloneqq \|f\|^{(2),(\mathrm{par})}_{2,\alpha,I}$.

Case 1. **Admissible solutions for $\theta_1 < \theta^{\mathrm{s}}$ away from θ^{s}.**

Proposition 4.21. *Fix $\gamma \ge 1$ and $v_2 \in (v_{\min}, 0)$. For some constant $\bar{\varepsilon} > 0$ depending only on (γ, v_2), any $\sigma \in (0, \theta^{\mathrm{s}})$, and any $\alpha \in (0,1)$, there exist both $\varepsilon \in (0, \bar{\varepsilon}]$ depending only on (γ, v_2, σ) and $C > 0$ depending only on (γ, v_2, α) such that any admissible solution $\varphi = \psi \circ \mathcal{R} + \varphi_5$ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{0 \le \theta_1 \le \theta^{\mathrm{s}} - \sigma\}$ satisfies*

$$\|\psi\|^{(\mathrm{par})}_{2,\alpha,\mathcal{R}(\Omega^5_\varepsilon)} + \|f_{5,\mathrm{sh}} - f_{5,0}\|^{(\mathrm{par})}_{2,\alpha,(0,\varepsilon)} \le C, \tag{4.3.29}$$

where $\Omega^5_\varepsilon \subseteq \Omega$ is a neighbourhood of $\Gamma^5_{\mathrm{sonic}}$, and the functions $f_{5,\mathrm{sh}}$ and $f_{5,0}$ represent Γ_{shock} and S_{25} respectively to be defined in Step **1** *below.*

The proof of this proposition is similar to that of [1, Propositions 3.26 and 3.30]. We sketch the main steps of the proof, while omitting most details.

1. *The depiction and properties of the reflected shock S_{25} and the free boundary $\Gamma_{\rm shock}$ in a small neighbourhood of the sonic boundary $\Gamma^5_{\rm sonic}$.* It follows from the geometric properties of $\overline{\Omega^5}$ that there exist positive constants $(\varepsilon_1, \varepsilon_0, \omega_0)$ depending only on (γ, v_2), with $\varepsilon_0 < \varepsilon_1$ and $\omega_0 \in (0,1)$, and a unique function $f_{5,0} \in C^\infty([-\varepsilon_0, \varepsilon_0])$ satisfying

$$\mathcal{R}\big(S_{25} \cap \mathcal{N}_{\varepsilon_1}(\Gamma^5_{\rm sonic})\big) \cap \{|x| < \varepsilon_0\} = \big\{(x,y) \,:\, |x| < \varepsilon_0,\, y = f_{5,0}(x)\big\}\,,$$

where $\mathcal{N}_{\varepsilon_1}(\Gamma^5_{\rm sonic})$ is defined by (4.3.20), and $2\omega_0 \le f'_{5,0}(x) \le \omega_0^{-1}$ for $x \in (-\varepsilon_0, \varepsilon_0)$. Then, for any $\varepsilon \in (0, \varepsilon_1]$, we define a set $\Omega^5_\varepsilon \subseteq \Omega$ by

$$\Omega^5_\varepsilon := \Omega \cap \mathcal{N}_{\varepsilon_1}(\Gamma^5_{\rm sonic}) \cap \mathcal{R}^{-1}\big(\{x < \varepsilon\}\big). \tag{4.3.30}$$

Using (4.3.19) and (4.3.24), we can show that there exist $\bar\varepsilon \in (0, \varepsilon_0]$, $L \ge 1$, $\delta \in (0, \frac12)$, and $\omega \in (0, \omega_0)$ depending only on (γ, v_2) such that, for all $(x,y) \in \mathcal{R}(\Omega^5_{\bar\varepsilon})$,

$$0 \le \psi(x,y) \le Lx^2\,, \qquad 0 \le \psi_x \le \frac{2-\delta}{1+\gamma}x \le Lx\,, \qquad |\psi_y(x,y)| \le Lx\,, \tag{4.3.31}$$

and there exists a unique function $f_{5,\rm sh} \in C^1([0,\bar\varepsilon])$ such that

$$\begin{aligned}
&\mathcal{R}(\Omega^5_{\bar\varepsilon}) = \big\{(x,y) \,:\, 0 < x < \bar\varepsilon,\, 0 < y < f_{5,\rm sh}(x)\big\}\,,\\
&\mathcal{R}(\Gamma_{\rm shock} \cap \partial\Omega^5_{\bar\varepsilon}) = \big\{(x,y) \,:\, 0 < x < \bar\varepsilon,\, y = f_{5,\rm sh}(x)\big\}\,,
\end{aligned}$$

and $\omega \le f'_{5,\rm sh}(x) \le L$ for any $x \in (0, \bar\varepsilon)$.

2. *Ellipticity for the equation and the free boundary condition.* Equation (4.3.24) for ψ in $\mathcal{R}(\Omega^5)$ can be written as

$$\sum_{i,j=1}^2 \hat A_{ij}^{(\theta_1)}(\psi_x, \psi_y, \psi, x)\partial_i\partial_j\psi + \sum_{i,j=1}^2 \hat A_i^{(\theta_1)}\partial_i\psi = 0\,, \tag{4.3.32}$$

with $(\partial_1, \partial_2) = (\partial_x, \partial_y)$ and $\hat A_{12}^{(\theta_1)} = \hat A_{21}^{(\theta_1)}$. Then there exist $\varepsilon_5 \in (0, \frac{\bar\varepsilon}{4}]$ and $\lambda_5 > 0$

depending only on (γ, v_2) such that, for all $(x,y) \in \mathcal{R}(\overline{\Omega^5_{4\varepsilon_5}})$,

$$\frac{\lambda_5}{2}|\boldsymbol{\kappa}|^2 \le \sum_{i,j=1}^{2} \hat{A}_{ij}^{(\theta_1)}(\psi_x(x,y), \psi_y(x,y), \psi(x,y), x)\frac{\kappa_i\kappa_j}{x^{2-\frac{i+j}{2}}} \le \frac{2}{\lambda_5}|\kappa|^2 \qquad \text{for all } \boldsymbol{\kappa} \in \mathbf{R}^2\,.$$

Moreover, $\mathcal{B}^{(\theta_1)}$ defined by (4.3.28) satisfies $\mathcal{B}^{(\theta_1)}(\mathbf{0}, 0, x, y) = 0$ for all $(x,y) \in \mathbf{R}^2$. Let y_{P_2} be the y–coordinate of the point P_2 defined in Definition 4.6. For each $m = 2, 3, \cdots$, there exist constants $\delta_{\rm bc} > 0$ and $C > 1$ depending only on (γ, v_2, m) such that, whenever $|(p_x, p_y, z, x)| \le \delta_{\rm bc}$ and $|y - y_{P_2}| \le \delta_{\rm bc}$,

$$\left|D^m \mathcal{B}^{(\theta_1)}(p_x, p_y, z, x, y)\right| \le C\,.$$

Furthermore, there exist constants $\hat{\delta}_{\rm bc} > 0$, $C > 1$, and $\varepsilon' > 0$ depending only on (γ, v_2) such that, whenever $|(p_x, p_y, z, x)| \le \hat{\delta}_{\rm bc}$ and $|y - y_{P_2}| \le \hat{\delta}_{\rm bc}$,

$$\begin{aligned} &\partial_a \mathcal{B}^{(\theta_1)}(p_x, p_y, z, x, y) \le -C^{-1} && \text{for } \partial_a = \partial_{p_x}, \partial_{p_y}, \text{ or } \partial_z\,,\\ &D_{(p_x,p_y)}\mathcal{B}^{(\theta_1)}(p_x, p_y, z, x, y) \cdot \boldsymbol{\nu}_{\rm sh}^{(x,y)} \ge C^{-1} && \text{on } \mathcal{R}(\Gamma_{\rm shock} \cap \partial\Omega^5_{\varepsilon'})\,, \end{aligned}$$

where the vector field $\boldsymbol{\nu}_{\rm sh}^{(x,y)}$ is the unit normal vector to $\mathcal{R}(\Gamma_{\rm shock})$ pointing into $\mathcal{R}(\Omega)$, expressed in the (x,y)–coordinates.

3. *Extension of the domain for the coefficients of equation* (4.3.32). Let $\varepsilon_0 > 0$ and $L \ge 1$ be the constants defined as above. Then there exist constants $\varepsilon \in (0, \frac{\varepsilon_0}{2}]$ and $C > 0$ depending only on (γ, v_2) such that, for any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$, the function $\psi = (\varphi - \varphi_5) \circ \mathcal{R}^{-1}$ satisfies

(4.3.33)
$$\sum_{i,j=1}^{2} \hat{A}_{ij}^{(\rm mod)}(\psi_x, \psi_y, \psi, x)\partial_i\partial_j\psi + \sum_{i,j=1}^{2} \hat{A}_{i}^{(\rm mod)}(\psi_x, \psi_y, \psi, x)\partial_i\psi = 0 \qquad \text{in } \mathcal{R}(\Omega^5_\varepsilon)\,,$$

with $(\partial_1, \partial_2) \coloneqq (\partial_x, \partial_y)$ and coefficients $(\hat{A}_{ij}^{(\rm mod)}, \hat{A}_{i}^{(\rm mod)})$ satisfying

$$(\hat{A}_{ij}^{(\rm mod)}, \hat{A}_{i}^{(\rm mod)}) = (\hat{A}_{ij}^{(\theta_1)}, \hat{A}_{i}^{(\theta_1)}) \quad \text{in } \{(p_x, p_y, z, x) \,:\, |(p_x, p_y)| \le Lx,\, |z| \le Lx^2,\, 0 < x < \varepsilon\}\,.$$

Moreover, the modified coefficients satisfy

$$\begin{aligned}
&\big|(\hat{A}_{11}^{(\mathrm{mod})}, \hat{A}_{12}^{(\mathrm{mod})}, \hat{A}_{2}^{(\mathrm{mod})})(p_x, p_y, z, x)\big| \le Cx \qquad \text{in } \mathbf{R}^2 \times \mathbf{R} \times (0, \varepsilon)\,, \\
&\big\|(\hat{A}_{22}^{(\mathrm{mod})}, \hat{A}_{1}^{(\mathrm{mod})})\big\|_{0, \mathbf{R}^2 \times \mathbf{R} \times (0,\varepsilon)} + \big\|D_{(p_x,p_y,z,x)}(\hat{A}_{ij}^{(\mathrm{mod})}, \hat{A}_{i}^{(\mathrm{mod})})\big\|_{0, \mathbf{R}^2 \times \mathbf{R} \times (0,\varepsilon)} \le C\,.
\end{aligned}$$

4. *The rescaling method in the local domain to obtain weighted Hölder estimates.* Using the property that η^{P_2} is strictly decreasing with respect to $\theta_1 \in [0, \theta^{\mathrm{s}}]$, as shown in Lemma A.3(ii), there exists a constant $l_{\mathrm{so}} > 0$ depending only on (γ, v_2, σ) such that, for any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{0 \le \theta_1 \le \theta^{\mathrm{s}} - \sigma\}$,

$$y_P = f_{5,\mathrm{sh}}(x_P) \ge l_{\mathrm{so}} \qquad \text{for any } x_P \in [0, \bar{\varepsilon}]\,.$$

Subsequently, we define $\varepsilon_* := \min\{\frac{\bar{\varepsilon}}{2}, l_{\mathrm{so}}^2\}$.

For fixed $\varepsilon \in (0, \varepsilon_*]$ and any $z_0 = (x_0, y_0) \in \mathcal{R}(\overline{\Omega_\varepsilon^5} \setminus \overline{\Gamma_{\mathrm{sonic}}^5})$, define the rescaled coordinates (S, T) by $(x, y) =: z_0 + \frac{1}{4}(x_0 S, \sqrt{x_0} T)$, which take values in a local domain

$$Q_r^{(z_0)} := \big\{(S,T) \in (-1,1)^2 \,:\, z = z_0 + \frac{r}{4}(x_0 S, \sqrt{x_0}T) \in \mathcal{R}(\Omega_{2\varepsilon}^5)\big\} \qquad \text{for any } r \in (0,1]\,.$$

Considering property (4.3.31), we define a rescaled function $\psi^{(z_0)}$ by

$$\psi^{(z_0)}(S,T) := \frac{1}{x_0^2}\psi(x_0 + \frac{x_0}{4}S, y_0 + \frac{\sqrt{x_0}}{4}T) \qquad \text{for } (S,T) \in Q_1^{(z_0)}\,.$$

With a suitable rescaling of the coefficients of the modified equation (4.3.33), we obtain a uniformly elliptic equation for $\psi^{(z_0)}$ under the (S,T)–coordinates.

Note that, for $z_0 \in \mathcal{R}(\overline{\Omega_\varepsilon^5} \cap \Gamma_{\mathrm{shock}})$, we can express Γ_{shock} locally under the rescaled coordinates as $\Gamma_{\mathrm{shock}}^{(z_0)} := \big\{(S,T) \in (-1,1)^2 \,:\, T = F^{(z_0)}(S)\big\} \subseteq \partial Q_1^{(z_0)}$ with $F^{(z_0)}(S)$ defined by

$$F^{(z_0)}(S) := \frac{4}{\sqrt{x_0}}\big(f_{5,\mathrm{sh}}(x_0 + \frac{x_0}{4}S) - f_{5,\mathrm{sh}}(x_0)\big) \qquad \text{for } S \in (-1,1)\,.$$

Then, in the local domain $Q_1^{(z_0)}$, we apply [20, Theorem 4.2.3] for $z_0 \in \mathcal{R}(\Omega_\varepsilon^5)$,

[20, Theorem 4.2.8] for $z_0 \in \mathcal{R}(\Gamma_{\text{shock}} \cap \partial\Omega^5_\varepsilon)$, and [20, Theorem 4.2.10] for $z_0 \in \mathcal{R}(\Gamma_{\text{sym}} \cap \partial\Omega^5_\varepsilon)$, respectively, to obtain the uniform bound

$$\sup_{z_0 \in \mathcal{R}(\Omega^5_\varepsilon)} \big\|\psi^{(z_0)}\big\|_{2,\alpha,\overline{Q^{(z_0)}_{1/4}}} + \sup_{z_0 \in \mathcal{R}(\Gamma_{\text{shock}} \cap \partial\Omega^5_\varepsilon)} \frac{1}{\sqrt{x_0}} \big\|F^{(z_0)}\big\|_{2,\alpha,[-\frac{1}{4},\frac{1}{4}]} \leq C\,.$$

Scaling back into the (x,y)–coordinates, we obtain the weighted Hölder estimates (4.3.29).

Case 2. **Admissible solutions for $\theta_1 < \theta^{\text{s}}$ close to θ^{s}.**

Proposition 4.22. *Fix $\gamma \geq 1$ and $v_2 \in [v_2^{\text{s}}, 0)$. Let $\bar{\varepsilon} > 0$ be the constant from* Proposition 4.21. *For any $\alpha \in (0,1)$, there exist constants $\varepsilon \in (0, \bar{\varepsilon}]$ and $\sigma_1 \in (0, \theta^{\text{s}})$ depending only on (γ, v_2), and $C > 0$ depending only on (γ, v_2, α) such that any admissible solution $\varphi = \psi \circ \mathcal{R} + \varphi_5$ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\text{s}} - \sigma_1 \leq \theta_1 < \theta^{\text{s}}\}$ satisfies*

$$\|\psi\|^{(\text{par})}_{2,\alpha,\mathcal{R}(\Omega^5_\varepsilon)} + \|f_{5,\text{sh}} - f_{5,0}\|^{(\text{par})}_{2,\alpha,(0,\varepsilon)} \leq C\,,$$

where $\Omega^5_\varepsilon \subseteq \Omega$, and the functions $f_{5,\text{sh}}$ and $f_{5,0}$ are defined in Step **1** *of* Case 1 *above.*

The proof is similar to that of [1, Proposition 3.32]. The admissible solutions for Case 2 share the same properties as in Steps **1**–**3** of Case 1. We omit the details of the proof and only draw the main differences compared with the proof of Proposition 4.21 here.

1. *Weighted Hölder estimate near $\mathcal{R}(P_0^1)$.* Because $v_2 \in [v_2^{\text{s}}, 0)$, $y_{P_2} > 0$ tends to zero as $\theta_1 \to \theta^{\text{s}-}$. Therefore, for the same rescaled coordinates (S,T) and local domain $Q_r^{(z_0)}$ with $z_0 = (x_0, y_0) \in \mathcal{R}(\Omega^5_\varepsilon)$ as given in Step **4** of Case 1 above, it is possible that "$Q_r^{(z_0)}$ does not fit into Ω^5_ε", meaning that the image of $Q_r^{(z_0)}$ under the transform $(S,T) \mapsto (x,y)$ intersects simultaneously the boundaries $\mathcal{R}(\Gamma_{\text{shock}} \cap \partial\Omega^5_\varepsilon)$ and $\mathcal{R}(\Gamma_{\text{sym}} \cap \partial\Omega^5_\varepsilon)$.

For that reason, we define $\Omega^5_{\text{fc}} \coloneqq \Omega^5_{\bar{\varepsilon}} \cap \mathcal{R}^{-1}(\{x < y_{P_2}^2\})$. It follows that $Q_r^{(z_0)}$ fits into Ω^5_{fc} for any $z_0 \in \mathcal{R}(\Omega^5_{\text{fc}})$ due to similar considerations as for the choice of ε_* above. In this case, we follow the same idea as for Proposition 4.21 to obtain that there exists a constant $C > 0$ depending only on (γ, v_2, α) such that, for $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\text{s}} - \sigma^* \leq \theta_1 < \theta^{\text{s}}\}$,

$\|\psi\|^{(\mathrm{par})}_{2,\alpha,\mathcal{R}(\Omega^5_{\mathrm{fc}})} \le C$ for some small constant $\sigma^* > 0$ depending only on (γ, v_2).

2. *Weighted Hölder estimate away from* $\mathcal{R}(P_0^1)$. Note that there exists sufficiently large $k > 1$ depending only on (γ, v_2) such that the following geometric relations hold

$$\{0 < x < 2\bar{\varepsilon},\, 0 < y < y_{P_2} + \frac{x}{k}\} \;\subseteq\; \mathcal{R}(\Omega^5_{2\bar{\varepsilon}}) \;\subseteq\; \{0 < x < 2\bar{\varepsilon},\, 0 < y < y_{P_2} + kx\}\,.$$

In the rest of the domain, there exist constants $\hat{\varepsilon} \in (0, \frac{\bar{\varepsilon}}{2}]$, $\sigma' \in (0, \sigma^*]$, and $C^* > 0$ depending only on (γ, v_2) such that, for $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\mathrm{s}} - \sigma' \le \theta_1 < \theta^{\mathrm{s}}\}$,

$$0 \le \psi(x,y) \le C^* x^4 \qquad \text{in } \mathcal{R}(\Omega^5_{2\hat{\varepsilon}}) \cap \{x > \tfrac{1}{10} y^2_{P_2}\}\,. \tag{4.3.34}$$

Then, for any $z_0 = (x_0, y_0) \in \mathcal{R}(\overline{\Omega^5_{\hat{\varepsilon}}}) \cap \{x > \frac{1}{5} y^2_{P_2}\}$, we introduce the rescaled coordinates (S,T) as $(x,y) =: z_0 + \frac{\sqrt{x_0}}{10k}(x_0 S, \sqrt{x_0}T)$, which take values in a local domain $Q_r^{(z_0)}$ defined by

$$Q_r^{(z_0)} \coloneqq \big\{(S,T) \in (-1,1)^2 \,:\, z = z_0 + \frac{r\sqrt{x_0}}{10k}(x_0 S, \sqrt{x_0}T) \in \mathcal{R}(\Omega^5_{2\hat{\varepsilon}})\big\} \qquad \text{for any } r \in (0,1]\,.$$

Thus, the local domain $Q_r^{(z_0)}$ fits into $\Omega^5_{2\hat{\varepsilon}}$, meaning that the image of $Q_r^{(z_0)}$ under transform $(S,T) \mapsto (x,y)$ does not intersect both $\mathcal{R}(\Gamma_{\mathrm{shock}})$ and $\mathcal{R}(\Gamma_{\mathrm{sym}})$ simultaneously. Considering property (4.3.34), we define a rescaled function $\psi^{(z_0)}$ by

$$\psi^{(z_0)}(S,T) \coloneqq \frac{1}{x_0^4}\psi(x_0 + \frac{x_0^{3/2}}{10k}S, y_0 + \frac{x_0}{10k}T) \qquad \text{for } (S,T) \in Q_1^{(z_0)}\,.$$

The rescaled function $\psi^{(z_0)}$ satisfies a uniformly elliptic equation in the (S,T)–coordinates after a suitable rescaling of the coefficients of the modified equation (4.3.33). The rest of the proof follows similarly to Step **4** of the proof of Proposition 4.21.

Case 3. **Admissible solutions for $\theta_1 \ge \theta^{\mathrm{s}}$ close to θ^{s}.**

Proposition 4.23. *Fix $\gamma \ge 1$ and $v_2 \in (v_2^{\mathrm{s}}, 0)$. Let $\bar{\varepsilon} > 0$ be the constant from* Proposition 4.21. *For each $\alpha \in (0,1)$, there exist constants $\varepsilon \in (0, \bar{\varepsilon}]$ and $\sigma_3 \in (0, \theta^{\mathrm{d}} - \theta^{\mathrm{s}})$ depending only on (γ, v_2), and a constant $C > 0$ depending only on (γ, v_2, α)*

such that any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\rm s} \le \theta_1 < \theta^{\rm s} + \sigma_3\}$ satisfies

$$
\begin{aligned}
&\|\varphi - \varphi_5\|_{2,\alpha,\overline{\Omega_\varepsilon^5}} + \|f_{5,\rm sh} - f_{5,0}\|_{2,\alpha,(0,\varepsilon)} \le C\,,\\
&|D_{\boldsymbol{\xi}}^m(\varphi - \varphi_5)(P_0^1)| = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(f_{5,\rm sh} - f_{5,0}\right)(0) = 0 \qquad \text{for } m = 0,1,2\,,
\end{aligned}
\tag{4.3.35}
$$

where $\Omega_\varepsilon^5 \subseteq \Omega$, and the functions $f_{5,\rm sh}$ and $f_{5,0}$ are defined in Step **1** *below.* □

The proof is similar to that of [1, Proposition 3.39], so we omit most details here and only emphasise the difference compared with the previous two cases.

1. *The depiction and properties of the reflected shock S_{25} and the free boundary $\Gamma_{\rm shock}$ in a small neighbourhood of the sonic boundary $\Gamma^5_{\rm sonic}$.* Let ε_1, ε_0, $\omega_0 > 0$ be chosen as in Step **1** of the proof of Proposition 4.21. For the coordinates $(x,y) = \mathcal{R}(\boldsymbol{\xi})$ given by (4.3.23), we define $\hat{x} := x - x_{P_2}$. Note that $x_{P_2} = 0$ if $\theta_1 \in [0, \theta^{\rm s}]$ and $x_{P_2} > 0$ if $\theta_1 \in (\theta^{\rm s}, \theta^{\rm s} + \sigma_2]$. Then there exists a unique function $f_{5,0} \in C^\infty([-\varepsilon_0, \varepsilon_0])$ satisfying

$$
\mathcal{R}\big(S_{25} \cap \mathcal{N}_{\varepsilon_1}(\Gamma^5_{\rm sonic})\big) \cap \{(x,y) \,:\, |\hat{x}| < \varepsilon_0\} = \big\{(x,y) \,:\, |\hat{x}| < \varepsilon_0,\, y = f_{5,0}(\hat{x})\big\}\,,
$$

and $2\omega_0 \le f_{5,0}'(\hat{x}) \le \omega_0^{-1}$ for $\hat{x} \in (-\varepsilon_0, \varepsilon_0)$.

Reducing $\bar{\varepsilon} \in (0, \varepsilon_0]$ further, we can choose $\sigma_2 \in (0, \theta^{\rm d} - \theta^{\rm s})$ depending only on $(\gamma, v_2, \bar{\varepsilon})$ such that, for any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{0 \le \theta_1 \le \theta^{\rm s} + \sigma_2\}$, the set $\Omega^5_{\bar{\varepsilon}}$ defined by (4.3.30) is nonempty due to the monotonicity of the pseudo-Mach number in Lemma A.1, from which we can generalise the definition of $\Omega^5_{\bar{\varepsilon}}$ to

$$
\Omega^5_{\bar{\varepsilon}} := \Omega \cap \mathcal{N}_{\varepsilon_1}(\Gamma^5_{\rm sonic}) \cap \mathcal{R}^{-1}\big(\{(x,y) \,:\, 0 < \hat{x} < \bar{\varepsilon}\}\big)\,.
\tag{4.3.36}
$$

There exist constants $L \ge 1$, $\delta \in (0, \frac{1}{2})$, and $\omega \in (0, \omega_0)$ depending only on (γ, v_2) such that, for all $(x,y) \in \mathcal{R}(\Omega^5_{\bar{\varepsilon}})$,

$$
0 \le \psi(x,y) \le Lx^2\,, \qquad 0 \le \psi_x \le \frac{2-\delta}{1+\gamma}x \le Lx\,, \qquad |\psi_y(x,y)| \le Lx\,,
\tag{4.3.37}
$$

for $\psi := (\varphi - \varphi_5) \circ \mathcal{R}^{-1}$, and there exists a unique function $f_{5,\mathrm{sh}} \in C^1([0, \bar{\varepsilon}])$ such that

$$\mathcal{R}(\Omega^5_{\bar{\varepsilon}}) = \{(x, y) \,:\, 0 < \hat{x} < \bar{\varepsilon},\, 0 < y < f_{5,\mathrm{sh}}(\hat{x})\}\,,$$
$$\mathcal{R}(\Gamma_{\mathrm{shock}} \cap \partial\Omega^5_{\bar{\varepsilon}}) = \{(x, y) \,:\, 0 < \hat{x} < \bar{\varepsilon},\, y = f_{5,\mathrm{sh}}(\hat{x})\}\,,$$

and $\omega \le f'_{5,\mathrm{sh}}(\hat{x}) \le L$ for any $\hat{x} \in (0, \bar{\varepsilon})$.

2. *The free boundary condition and the modification of equation* (4.3.32). Similarly to [1, Lemma 3.37 and Corollary 3.38], it follows that, for $\sigma_2 \in (0, \theta^{\mathrm{d}} - \theta^{\mathrm{s}})$ defined in Step **1**, there exists a constant $\mu_0 > 0$ depending only on (γ, v_2, σ_2) such that, for any $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\mathrm{s}} \le \theta_1 \le \theta^{\mathrm{d}} - \sigma_2\}$, $g^{\mathrm{sh}}_{\mathrm{mod}}$ defined by (4.3.27) satisfies the following properties

$$\partial_a g^{\mathrm{sh}}_{\mathrm{mod}}(D\varphi_5(P_2), \varphi_5(P_2), P_2) \le -\mu_0\,,$$

where $\partial_a = \partial_{p_1}, \partial_{p_2}$, or ∂_z. From the properties above, we can show that $\mathcal{B}^{(\theta_1)}$ defined by (4.3.28) satisfies the same properties as in Step **2** of Case 1. The extension of the domain for the coefficients of equation (4.3.32) is the same as Step **3** of Case 1.

3. *The rescaling method in the local domain to obtain weighted Hölder estimates.* Note that, from the properties of $f_{5,0}$ and $f_{5,\mathrm{sh}}$ in Step **1**, there exists some $k > 1$ depending only on (γ, v_2) such that

$$\{0 < \hat{x} < \bar{\varepsilon},\, 0 < y < \frac{\hat{x}}{k}\} \;\subseteq\; \mathcal{R}(\Omega^5_{\bar{\varepsilon}}) \;\subseteq\; \{0 < \hat{x} < \bar{\varepsilon},\, 0 < y < k\hat{x}\}\,.$$

Moreover, there exist constants $\varepsilon \in (0, \frac{\bar{\varepsilon}}{2}], \sigma_3 \in (0, \sigma_2]$, and $C > 1$ depending only on (γ, v_2) such that, for $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\mathrm{s}} \le \theta_1 \le \theta^{\mathrm{s}} + \sigma_3\}$,

$$x_{P_2} \le \frac{\varepsilon}{10}\,, \qquad 0 \le \psi(x, y) \le C(x - x_{P_2})^5 \quad \text{in } \mathcal{R}(\Omega^5_{2\varepsilon})\,. \tag{4.3.38}$$

Then, for any $z_0 = (x_0, y_0) \in \mathcal{R}(\overline{\Omega^5_{\varepsilon}} \setminus \{P_2\})$, we introduce the rescaled coordinates (S, T) as $(x, y) =: z_0 + \frac{x_0 - x_{P_2}}{10k}(\sqrt{x_0}S, T)$, which take values in a local domain $Q_r^{(z_0)}$

defined by

$$Q_r^{(z_0)} \coloneqq \big\{(S,T)\in(-1,1)^2 \,:\, z = z_0 + \frac{r(x_0-x_{P_2})}{10k}(\sqrt{x_0}S,T)\in\mathcal{R}(\Omega_{2\varepsilon}^5)\big\} \qquad \text{for any } r\in(0,1]\,.$$

Then the local domain $Q_r^{(z_0)}$ fits into $\Omega_{2\varepsilon}^5$, meaning that the image of $Q_r^{(z_0)}$ under transform $(S,T)\mapsto(x,y)$ does not intersect both $\mathcal{R}(\Gamma_{\text{shock}})$ and $\mathcal{R}(\Gamma_{\text{sym}})$ simultaneously. Considering property (4.3.38), we define a rescaled function $\psi^{(z_0)}$ by

$$\psi^{(z_0)}(S,T) \coloneqq \frac{1}{(x_0-x_{P_2})^5}\psi(x_0+\frac{x_0-x_{P_2}}{10k}\sqrt{x_0}S,\, y_0+\frac{x_0-x_{P_2}}{10k}T) \qquad \text{for } (S,T)\in Q_1^{(z_0)}.$$

The rescaled function $\psi^{(z_0)}$ satisfies a uniformly elliptic equation in the (S,T)–coordinates after a suitable rescaling of the coefficients of the modified equation (4.3.33). The rest of the proof follows the same idea as Step **4** of Case 1.

Case 4. **Admissible solutions for $\theta_1\geq\theta^{\text{s}}$ away from θ^{s}.**

Let $\sigma_3>0$ be from Proposition 4.23. By Proposition 4.17, there exists a constant $\delta\in(0,1)$ depending only on (γ,v_2) such that any admissible solution φ corresponding to $\boldsymbol{\theta}\in\Theta\cap\{\theta^{\text{s}}+\frac{\sigma_3}{2}\leq\theta_1<\theta^{\text{d}}\}$ satisfies

$$\frac{|D\varphi|^2}{c^2(|D\varphi|,\varphi)}\leq 1-\delta \qquad \text{in } \overline{\Omega}\cap\{\xi\geq 0\}\,, \tag{4.3.39}$$

where $c(|D\varphi|,\varphi)$ is defined by (4.3.2). By Lemma 4.12 and (4.3.39), $(D\varphi(\boldsymbol{\xi}),\varphi(\boldsymbol{\xi}))\in\mathcal{K}_{R_*}$ for some constant $R_*\geq 2$ depending only on (γ,v_2), where $\mathcal{K}_R$ is defined by (4.3.8). In particular, there exist $\lambda_*>0$ and $r_*>0$, depending only on (γ,v_2), such that any admissible solution φ corresponding to $\boldsymbol{\theta}\in\Theta\cap\{\theta^{\text{s}}+\frac{\sigma_3}{2}\leq\theta_1<\theta^{\text{d}}\}$ satisfies

$$\lambda_*|\boldsymbol{\kappa}|^2\leq\sum_{i,j=1}^{2}\partial_{p_j}\mathcal{A}_i(D\varphi(\boldsymbol{\xi}),\varphi(\boldsymbol{\xi}))\kappa_i\kappa_j\leq\lambda_*^{-1}|\boldsymbol{\kappa}|^2 \tag{4.3.40}$$

for any $\boldsymbol{\xi}\in\overline{\Omega}\cap B_{2r_*}(P_0^1)$ and any $\boldsymbol{\kappa}=(\kappa_1,\kappa_2)\in\mathbf{R}^2$.

Set $\bar{\phi}(\boldsymbol{\xi})\coloneqq\varphi_2(\boldsymbol{\xi})-\varphi(\boldsymbol{\xi})$. Recalling equation (4.3.1) for the function $\phi=\varphi-\varphi_5$, the free boundary condition (4.3.26) on Γ_{shock}, and the slip boundary condition defined

in Problem B(iv) on $\Gamma_{\rm sym}$, it follows that $\bar\phi$ satisfies the system

$$\begin{cases} \sum\limits_{i,j=1}^{2} \bar A_{ij}(D\bar\phi,\bar\phi,\boldsymbol{\xi})\partial_i\partial_j\bar\phi = 0 & \text{in } \Omega\cap B_{r_*}(P_0^1)\,,\\ \bar g^{\rm sym}(D\bar\phi,\bar\phi,\boldsymbol{\xi}) = 0 & \text{on } \Gamma_{\rm sym}\,,\\ \bar g^{\rm sh}(D\bar\phi,\bar\phi,\boldsymbol{\xi}) = 0 & \text{on } \Gamma_{\rm shock}\,, \end{cases} \tag{4.3.41}$$

where the coefficients $\bar A_{ij}$ and the boundary functions $\bar g^{\rm sym}$ and $\bar g^{\rm sh}$ are defined by

$$\begin{aligned} &\bar A_{ij}(\boldsymbol{p},z,\boldsymbol{\xi}) := c^2(|D\varphi_2-\boldsymbol{p}|,\varphi_2-z)\delta_{ij} - (\partial_i\varphi_2-p_i)(\partial_j\varphi_2-p_j)\,,\\ &\bar g^{\rm sym}(\boldsymbol{p},z,\boldsymbol{\xi}) := v_2-p_2-\eta\,,\\ &\bar g^{\rm sh}(\boldsymbol{p},z,\boldsymbol{\xi}) := -g^{\rm sh}(D\varphi_2-\boldsymbol{p},\varphi_2-z,\boldsymbol{\xi})\,, \end{aligned} \tag{4.3.42}$$

for $g^{\rm sh}$ given by (4.3.26), and where $(\partial_1,\partial_2)=(\partial_\xi,\partial_\eta)$.

We define a vector $\boldsymbol{e}$ as

$$\boldsymbol{e} := \frac{D\varphi_2(P_0^1)}{|D\varphi_2(P_0^1)|} = \frac{O_2-P_0^1}{|O_2-P_0^1|}\,,$$

and choose $\boldsymbol{e}^\perp$ to be the clockwise rotation of $\boldsymbol{e}$ by $\frac{\pi}{2}$. Note that $\boldsymbol{e}$ depends only on (γ,v_2,θ_1) and satisfies that, for any $\boldsymbol{\xi}\in\Gamma_{\rm shock}\cap B_{r_*}(P_0^1)$,

$$\partial_{\boldsymbol{e}}(\varphi_2-\varphi)(\boldsymbol{\xi}) = -\partial_r(\varphi_2-\varphi) + \Big(\boldsymbol{e}-\frac{D\varphi(\boldsymbol{\xi})}{|D\varphi(\boldsymbol{\xi})|}\Big)\cdot D(\varphi_2-\varphi) \ge \frac{d_0'}{2}\,, \tag{4.3.43}$$

where $d_0'>0$ is the constant from (4.3.22), depending only on (γ,v_2). Then we define the new coordinates (S,T) by

$$\boldsymbol{\xi} =: P_0^1 + S\boldsymbol{e} + T\boldsymbol{e}^\perp\,. \tag{4.3.44}$$

Under these (S,T)–coordinates, we can express $S_{25},\Gamma_{\rm shock},\Gamma_{\rm sym}$, and the domain near

the corner point P_0^1 as

(4.3.45)

$$S_{25} \cap B_{r_*}(P_0^1) = \{\boldsymbol{\xi}(S,T) \,:\, S = a_{\mathrm{ref}} T,\, T > 0,\, (S,T) \in B_{r_*}(\mathbf{0})\}\,,$$

$$\Gamma_{\mathrm{shock}} \cap B_{r_*}(P_0^1) = \{\boldsymbol{\xi}(S,T) \,:\, S = f_{\boldsymbol{e}}(T),\, T > 0,\, (S,T) \in B_{r_*}(\mathbf{0})\}\,,$$

$$\Gamma_{\mathrm{sym}} \cap B_{r_*}(P_0^1) = \{\boldsymbol{\xi}(S,T) \,:\, S = a_{\mathrm{sym}} T,\, T > 0,\, (S,T) \in B_{r_*}(\mathbf{0})\}\,,$$

$$\Omega \cap B_{r_*}(P_0^1) = \{\boldsymbol{\xi}(S,T) \,:\, a_{\mathrm{ref}} T \le f_{\boldsymbol{e}}(T) < S < a_{\mathrm{sym}} T,\, T > 0,\, (S,T) \in B_{r_*}(\mathbf{0})\}\,,$$

where the positive constants $a_{\mathrm{ref}} = \tan(\theta_{25} - \hat{\theta}_{25} - \frac{\pi}{2})$ and $a_{\mathrm{sym}} = \cot\hat{\theta}_{25}$ depend continuously on $\theta_1 \in (0, \theta^{\mathrm{d}})$, and there exists a constant $C > 0$ depending only on (γ, v_2) such that $C^{-1} \le a_{\mathrm{ref}} < a_{\mathrm{sym}} \le C$ for all $\theta_1 \in [\theta^{\mathrm{s}}, \theta^{\mathrm{d}})$. Note that $\hat{\theta}_{25}$ is defined by (4.2.23), and we have used Proposition 4.2, including (4.2.34).

Definition 4.24 (Weighted Hölder norm)**.** For any $\sigma \in \mathbf{R}$, $\alpha \in (0,1)$, and $m \in \mathbf{N}$, the weighted Hölder norms are defined as follows.

(i) For an open bounded connected set $U \subseteq \mathbf{R}^2$, let Γ be a closed portion of ∂U. Write

$$\delta_{\boldsymbol{\xi}_1} \coloneqq \mathrm{dist}(\boldsymbol{\xi}_1, \Gamma)\,, \quad \delta_{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2} \coloneqq \min\{\delta_{\boldsymbol{\xi}_1}, \delta_{\boldsymbol{\xi}_2}\} \qquad \text{for any } \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in U\,.$$

For any $u \in C^m(U)$, define

$$\|u\|_{m,0,U}^{(\sigma),\Gamma} \coloneqq \sum_{0 \le |\boldsymbol{\beta}| \le m} \sup_{\boldsymbol{\xi}_1 \in U} \left(\delta_{\boldsymbol{\xi}_1}^{\max\{|\boldsymbol{\beta}|+\sigma,0\}} \left| D^{\boldsymbol{\beta}} u(\boldsymbol{\xi}_1) \right| \right),$$

$$[u]_{m,\alpha,U}^{(\sigma),\Gamma} \coloneqq \sum_{|\boldsymbol{\beta}|=m} \sup_{\substack{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in U \\ \boldsymbol{\xi}_1 \ne \boldsymbol{\xi}_2}} \left(\delta_{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2}^{\max\{m+\alpha+\sigma,0\}} \frac{|D^{\boldsymbol{\beta}} u(\boldsymbol{\xi}_1) - D^{\boldsymbol{\beta}} u(\boldsymbol{\xi}_2)|}{|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{\alpha}} \right),$$

$$\|u\|_{m,\alpha,U}^{(\sigma),\Gamma} \coloneqq \|u\|_{m,0,U}^{(\sigma),\Gamma} + [u]_{m,\alpha,U}^{(\sigma),\Gamma}\,,$$

where $D^{\boldsymbol{\beta}} \coloneqq \partial_\xi^{\beta_1} \partial_\eta^{\beta_2}$ for $\boldsymbol{\beta} \coloneqq (\beta_1, \beta_2)$ with $\beta_1, \beta_2 \in \mathbf{N}$ and $|\boldsymbol{\beta}| = \beta_1 + \beta_2$.

(ii) For an open bounded interval $I \subseteq \mathbf{R}$, let $x_0 \in \partial I$ be an endpoint of I. For any

$f \in C^m(I)$, define

$$\|f\|_{m,0,I}^{(\sigma),\{x_0\}} := \sum_{k=0}^{m} \sup_{x\in I} \left(|x-x_0|^{\max\{k+\sigma,0\}} \left| f^{(k)}(x) \right| \right),$$

$$[f]_{m,\alpha,I}^{(\sigma),\{x_0\}} := \sup_{\substack{x_1,x_2\in I \\ x_1\neq x_2}} \left(\min\left\{ |x_1-x_0|, |x_2-x_0| \right\}^{\max\{m+\alpha+\sigma,0\}} \frac{|f^{(m)}(x_1) - f^{(m)}(x_2)|}{|x_1-x_2|^\alpha} \right),$$

$$\|f\|_{m,\alpha,I}^{(\sigma),\{x_0\}} := \|f\|_{m,0,I}^{(\sigma),\{x_0\}} + [f]_{m,\alpha,I}^{(\sigma),\{x_0\}}.$$

Proposition 4.25. *Fix $\gamma \geq 1$ and $v_2 \in (v_2^{\mathrm{s}}, 0)$. Let $\sigma_3 > 0$ be the constant from* Proposition 4.23. *For sufficiently small constants $\sigma_{\mathrm{s}} \in (0, \frac{\sigma_3}{2}]$ and $\sigma_{\mathrm{d}} \in (0, \frac{\theta^{\mathrm{d}}}{10})$, there exist $r \in (0, r_*)$, $\alpha \in (0,1)$, and $C > 0$ depending only on $(\gamma, v_2, \sigma_{\mathrm{s}}, \sigma_{\mathrm{d}})$ such that any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\mathrm{s}} + \sigma_{\mathrm{s}} \leq \theta_1 \leq \theta^{\mathrm{d}} - \sigma_{\mathrm{d}}\}$ satisfies the estimates*

(4.3.46)
$$\left\|\varphi\right\|_{2,\alpha,\Omega\cap B_r(P_0^1)}^{(-1-\alpha),\{P_0^1\}} + \left\|f_{\boldsymbol{e}}\right\|_{2,\alpha,(0,r)}^{(-1-\alpha),\{0\}} \leq C, \qquad |D_{\boldsymbol{\xi}}^m(\varphi-\varphi_5)(P_0^1)| = 0 \ \text{ for } m = 0,1.$$

The proof is similar to the proof of [1, Proposition 3.42]. We omit the details of the proof and sketch only the main ideas here.

1. *The hodograph transform.* Define $\bar\phi^{(h)}(S,T) := \bar\phi(\boldsymbol{\xi}(S,T))$ for the function $\bar\phi = \varphi_2 - \varphi$ and the (S,T)–coordinates given by (4.3.44). By property (4.3.43), we can apply the hodograph transform $\boldsymbol{y} = (y_1, y_2) = (\bar\phi^{(h)}(S,T), T)$ to $\bar\phi$ and the original system (4.3.41). We write $S =: v(y_1, y_2)$ if it satisfies $\bar\phi^{(h)}(S, y_2) = y_1$ for any

$$\boldsymbol{y} \in \mathcal{D}_{r_*}^{(\theta_1)} := \left\{ (\bar\phi^{(h)}(S,T), T) \,:\, \boldsymbol{\xi}(S,T) \in \Omega \cap B_{r_*}(P_0^1) \right\},$$

from which we obtain

$$\partial_{y_1} v = \frac{1}{\partial_S \bar\phi^{(h)}}, \qquad \partial_{y_2} v = -\frac{\partial_T \bar\phi^{(h)}}{\partial_S \bar\phi^{(h)}}.$$

It follows from Lemma 4.12 and property (4.3.43) that there exists a constant $K > 1$

depending only on (γ, v_2) such that

$$\frac{1}{K} \le \partial_{y_1} v \le \frac{2}{d_0'}\,, \qquad |v| + |D_{\boldsymbol{y}} v| \le 2K\,. \tag{4.3.47}$$

Under the hodograph transform, the system (4.3.41) is equivalent to

$$\begin{cases} \sum_{i,j=1}^{2} a_{ij}(D_{\boldsymbol{y}} v, v, \boldsymbol{y})\partial_{y_i}\partial_{y_j} v = 0 & \text{in } \mathcal{D}_{r_*}^{(\theta_1)}\,, \\ g_h^{\mathrm{sym}}(D_{\boldsymbol{y}} v, v, \boldsymbol{y}) = 0 & \text{on } \Gamma_{\mathrm{sym}}^{(h)}\,, \\ g_h^{\mathrm{sh}}(D_{\boldsymbol{y}} v, v, \boldsymbol{y}) = 0 & \text{on } \Gamma_{\mathrm{shock}}^{(h)}\,, \end{cases} \tag{4.3.48}$$

for suitable functions $(a_{ij}, g_h^{\mathrm{sym}}, g_h^{\mathrm{sh}})$, with the boundary sets given by

$$\begin{aligned} \Gamma_{\mathrm{sym}}^{(h)} &\coloneqq \big\{(\bar{\phi}^{(h)}(S,T), T) \,:\, \boldsymbol{\xi}(S,T) \in \Gamma_{\mathrm{sym}} \cap B_{r_*}(P_0^1)\big\}\,, \\ \Gamma_{\mathrm{shock}}^{(h)} &\coloneqq \big\{(0,T) \,:\, \boldsymbol{\xi}(S,T) \in \Gamma_{\mathrm{shock}} \cap B_{r_*}(P_0^1)\big\}\,. \end{aligned}$$

Due to (4.3.47), we extend the domain of functions $(a_{ij}, g_h^{\mathrm{sym}}, g_h^{\mathrm{sh}})$ to the set

$$U \coloneqq \big\{(\boldsymbol{p}, z, \boldsymbol{y}) \in \mathbf{R}^2 \times \mathbf{R} \times \mathcal{D}_{r_*}^{(\theta_1)} \,:\, |\boldsymbol{p}| + |z| \le 2K\big\}\,,$$

and continue to use the same notation as in (4.3.48) for this modified system.

2. *Hölder continuity of g_h^{sym} at the corner.* From Lemma 4.12, there exists a constant $\tau_0 > 0$ depending only on (γ, v_2) such that

$$\Gamma_{\mathrm{sym}}^{(h)} \cup \Gamma_{\mathrm{shock}}^{(h)} \subseteq \{\boldsymbol{y} \in \mathbf{R}^2 \,:\, y_2 > \tau_0 |y_1|\}\,.$$

Moreover, $\Gamma_{\mathrm{shock}}^{(h)}$ is in C^2 up to endpoint $\{\mathbf{0}\}$.

From the definition of admissible solutions and the hodograph transform, we obtain that $v \in C^1(\overline{\mathcal{D}_{r_*}^{(\theta_1)}}) \cap C^2(\mathcal{D}_{r_*}^{(\theta_1)} \cup \Gamma_{\mathrm{shock}}^{(h)}) \cap C^3(\mathcal{D}_{r_*}^{(\theta_1)})$ satisfies (4.3.47). From (4.3.42) and the notation defined in Step **1**, we can obtain the boundedness of the coefficients $(a_{ij}, g_h^{\mathrm{sym}}, g_h^{\mathrm{sh}})$ in the modified system (4.3.48). For any $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\mathrm{s}} + \sigma_{\mathrm{s}} \le \theta_1 < \theta^{\mathrm{d}}\}$ with $\sigma_{\mathrm{s}} \in (0, \frac{\sigma_3}{2}]$, we may reduce $\lambda_* > 0$ depending only on $(\gamma, v_2, \sigma_{\mathrm{s}})$ in (4.3.40) to

obtain the uniform ellipticity of the modified system (4.3.48) in $\mathcal{D}_{r_*}^{(\theta_1)}$ by using (4.3.47) and the hodograph transform. It follows from Lemma 4.12, (4.3.40), and (4.3.47) that there exists a constant $\lambda_1 > 0$ depending only on $(\gamma, v_2, \sigma_{\rm s})$ such that, for any $y \in \Gamma_{\rm shock}^{(h)}$,

$$\boldsymbol{\nu} \cdot D_{\boldsymbol{p}} g_h^{\rm sh}(D_{\boldsymbol{y}} v, v, \boldsymbol{y}) = \frac{1}{v_{y_1}} D\bar{\phi} \cdot D_{\boldsymbol{q}} g^{\rm sh}(D\varphi, \varphi, \boldsymbol{\xi}) \geq \lambda_1 \,,$$

where we have used the expression of $D_{\boldsymbol{q}} g^{\rm sh}(D\varphi, \varphi, \boldsymbol{\xi})$ on $\Gamma_{\rm shock}^{(h)}$, and $\boldsymbol{\nu} = (1, 0)$ is the unit normal vector to $\Gamma_{\rm shock}^{(h)}$. From (4.3.47), $|D_{\boldsymbol{p}} g_h^{\rm sym}(D_{\boldsymbol{y}} v, v, \boldsymbol{y})| \geq \frac{d_0'}{2} > 0$.

Similar to the proof of [1, Lemma 3.40], one can show that, for small $\sigma_{\rm d} \in (0, \frac{\theta^{\rm d}}{10})$, there exist constants $M > 0$ and $\bar{r} \in (0, r_*)$ depending only on $(\gamma, v_2, \sigma_{\rm d})$ such that any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta^{\rm s} \leq \theta_1 < \theta^{\rm d} - \sigma_{\rm d}\}$ satisfies

$$\partial_{q_1} g_{\rm mod}^{\rm sh}(D\varphi, \varphi, \boldsymbol{\xi}) \leq -M^{-1} \qquad \text{for all } \boldsymbol{\xi} \in \Gamma_{\rm shock} \cap B_{\bar{r}}(P_0^1) \,.$$

Then, for any $\boldsymbol{y} \in \Gamma_{\rm shock}^{(h)}$, $|{\rm det} D_{\boldsymbol{p}}(g_h^{\rm sym}, g_h^{\rm sh})| > 0$ holds.

It follows from [20, Proposition 4.3.7] that there exist constants $\alpha_1 \in (0, 1)$, $C > 0$, and $r_0 \in (0, \bar{r}]$ depending only on $(\gamma, v_2, \sigma_{\rm s}, \sigma_{\rm d})$ such that, for any $\boldsymbol{y} \in \overline{\mathcal{D}_{\bar{r}}^{(\theta_1)} \cap B_{r_0}(\mathbf{0})}$,

$$\left| g_h^{\rm sym}(D_{\boldsymbol{y}} v(\boldsymbol{y}), v(\boldsymbol{y}), \boldsymbol{y}) - g_h^{\rm sym}(D_{\boldsymbol{y}} v(\mathbf{0}), v(\mathbf{0}), \mathbf{0}) \right| \leq C|\boldsymbol{y}|^{\alpha_1} \,. \tag{4.3.49}$$

3. *Hölder continuity of $v(\boldsymbol{y})$ at the corner.* Notice that both $g_h^{\rm sym}$ and $g_h^{\rm sh}$ are Lipschitz continuous in $(p, z, \boldsymbol{y})$. Since $v \in C^1(\overline{\mathcal{D}_{\bar{r}}^{(\theta_1)}})$, the inequality (4.3.49) also holds for any $\boldsymbol{y} \in \Gamma_{\rm shock}^{(h)}$ after replacing $g_h^{\rm sym}$ by $g_h^{\rm sh}$. Moreover, the functions $(g_h^{\rm sym}, g_h^{\rm sh})(\cdot, v(\mathbf{0}), \mathbf{0})$ are bounded in $C^{1,\alpha_1}(\overline{B_{2K}(D_{\boldsymbol{y}} v(\mathbf{0}))})$, and $|{\rm det} D_{\boldsymbol{p}}(g_h^{\rm sym}, g_h^{\rm sh})(D_{\boldsymbol{y}} v(\mathbf{0}), v(\mathbf{0}), \mathbf{0})| > 0$ follows from Step **2**. Then, from [20, Proposition 4.3.9], there exist constants $r_1 \in (0, r_0)$ and $C_1 > 0$ depending only on $(\gamma, v_2, \sigma_{\rm s}, \sigma_{\rm d})$ such that

$$|D_{\boldsymbol{y}} v(\boldsymbol{y}) - D_{\boldsymbol{y}} v(\mathbf{0})| \leq C_1 |\boldsymbol{y}|^{\alpha_1} \qquad \text{for any } \boldsymbol{y} \in \Gamma_{\rm shock}^{(h)} \cap B_{r_1}(\mathbf{0}) \,. \tag{4.3.50}$$

By the hodograph transform and the geometric properties of the domain near the

corner P_0^1, it follows from (4.3.49)–(4.3.50) that

$$
\begin{aligned}
&|\partial_\eta\varphi(\boldsymbol{\xi}) - \partial_\eta\varphi(P_0^1)| \le C|\boldsymbol{\xi} - P_0^1|^{\alpha_1} \qquad \text{for any } \boldsymbol{\xi} \in \overline{\Omega \cap B_{r_1}(P_0^1)}\,,\\
&|D\varphi(\boldsymbol{\xi}) - D\varphi(P_0^1)| \le C|\boldsymbol{\xi} - P_0^1|^{\alpha_1} \qquad \text{for any } \boldsymbol{\xi} \in \Gamma_{\rm shock} \cap B_{r_1}(P_0^1)\,.
\end{aligned} \tag{4.3.51}
$$

The open domain $\Omega_{r_1} := \Omega \cap B_{r_1}(P_0^1)$ contains two Lipschitz boundaries $\Gamma^1 := \Gamma_{\rm shock} \cap B_{r_1}(P_0^1)$ and $\Gamma^2 := \Gamma_{\rm sym} \cap B_{r_1}(P_0^1)$ which intersect at the corner P_0^1. We also take

$$
B^{(1)}(\boldsymbol{p}, z, \boldsymbol{\xi}) := (1,0)\cdot(\boldsymbol{p} - D\bar\phi(P_0^1))\,, \qquad B^{(2)}(\boldsymbol{p}, z, \boldsymbol{\xi}) := \bar g_{\rm sym}(\boldsymbol{p}, z, \boldsymbol{\xi})\,.
$$

Now we update the system (4.3.41) for $\bar\phi \in C^1(\overline{\Omega_{r_1}}) \cap C^2(\Omega_{r_1} \cup \Gamma^1) \cap C^3(\Omega_{r_1})$ into the domain Ω_{r_1}, replacing the boundary conditions by

$$
B^{(1)}(D\bar\phi, \bar\phi, \boldsymbol{\xi}) = \bar h(\boldsymbol{\xi}) \ \text{ on } \Gamma^1\,, \qquad B^{(2)}(D\bar\phi, \bar\phi, \boldsymbol{\xi}) = 0 \ \text{ on } \Gamma^2\,,
$$

where $\bar h(\boldsymbol{\xi}) := (1,0)\cdot(D\bar\phi(\boldsymbol{\xi}) - D\bar\phi(P_0^1))$, which satisfies

$$
|\bar h(\boldsymbol{\xi}) - \bar h(P_0^1)| \le C|\boldsymbol{\xi} - P_0^1|^{\alpha_1} \qquad \text{for any } \boldsymbol{\xi} \in \Gamma_{\rm shock} \cap B_{r_1}(P_0^1)\,.
$$

We can also extend the domain of the functions $(\bar A_{ij}, B^{(1)}, B^{(2)})$ for the updated system to the set

$$
V := \{(\boldsymbol{p}, z, \boldsymbol{\xi}) \in \mathbf{R}^2 \times \mathbf{R} \times \overline{\Omega_{r_1}} \,:\, |\boldsymbol{p}| + |z| \le 2K_1\}\,.
$$

Using [20, Proposition 4.3.7] again, we obtain that there exist $\alpha \in (0, \alpha_1]$, $r_2 \in (0, r_1]$, and $C > 0$ depending only on $(\gamma, v_2, \sigma_{\rm s}, \sigma_{\rm d})$ such that, for any $\boldsymbol{\xi} \in \overline{\Omega \cap B_{r_2}(P_0^1)}$,

$$
\big|B^{(1)}(D\bar\phi(\boldsymbol{\xi}), \bar\phi(\boldsymbol{\xi}), \boldsymbol{\xi}) - B^{(1)}(D\bar\phi(P_0^1), \bar\phi(P_0^1), P_0^1)\big| \le C\big|\boldsymbol{\xi} - P_0^1\big|^{\alpha}\,,
$$

which, combined with (4.3.51), implies that

$$
\big|D\varphi(\boldsymbol{\xi}) - D\varphi(P_0^1)\big| \le C\big|\boldsymbol{\xi} - P_0^1\big|^{\alpha} \qquad \text{for any } \boldsymbol{\xi} \in \overline{\Omega \cap B_{r_2}(P_0^1)}\,. \tag{4.3.52}
$$

4. *Weighted Hölder estimate near the corner P_0^1.* From Lemma 4.11, Remark 4.8(i), and (4.3.52), there exists a constant $\varepsilon_{\rm bd} > 0$ depending only on (γ, v_2) such that, for any $\boldsymbol{\xi}_0 \in \Gamma_{\rm bd} \cap B_{r_3}(P_0^1)$ for some $r_3 \in (0, r_2]$ with either $\Gamma_{\rm bd} \coloneqq \Gamma_{\rm shock}$ or $\Gamma_{\rm bd} \coloneqq \Gamma_{\rm sym}$,

$$\left\{\boldsymbol{\xi} \in \partial\Omega \,:\, |\boldsymbol{\xi} - \boldsymbol{\xi}_0| < \varepsilon_{\rm bd}|\boldsymbol{\xi}_0 - P_0^1|\right\} \subseteq \Gamma_{\rm bd}\,.$$

We now consider the following problem for $\varphi \in C^1(\overline{\Omega \cap B_{r_3}(P_0^1)}) \cap C^3(\Omega \cap B_{r_3}(P_0^1))$

$$\begin{cases} {\rm div}\tilde{\mathcal{A}}(D\varphi, \varphi) + \tilde{\mathcal{B}}(D\varphi, \varphi) = 0 & \text{in } \Omega \cap B_{r_3}(P_0^1)\,, \\ g_{\rm mod}^{\rm sh}(D\varphi, \varphi, \boldsymbol{\xi}) = 0 & \text{on } \Gamma_{\rm shock} \cap B_{r_3}(P_0^1)\,, \\ g_{\rm sym}(D\varphi, \varphi, \boldsymbol{\xi}) = 0 & \text{on } \Gamma_{\rm sym} \cap B_{r_3}(P_0^1)\,, \end{cases}$$

where $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\boldsymbol{q}, w)$ are defined on $\mathbf{R}^2 \times \mathbf{R}$ by Lemma 4.14, $g_{\rm mod}^{\rm sh}(\boldsymbol{q}, w, \boldsymbol{\xi})$ is defined on $\mathbf{R}^2 \times \mathbf{R} \times \overline{\Omega \cap B_{r_3}(P_0^1)}$ by (4.3.27), and $g_{\rm sym}(\boldsymbol{q}, w, \boldsymbol{\xi}) \coloneqq (0,1) \cdot \boldsymbol{q}$. Combining the above with property (4.3.52) and using [20, Proposition 4.3.11(i)], we obtain that there exist constants $\tilde{\alpha} \in (0, \alpha]$ and $C > 0$ depending only on $(\gamma, v_2, \sigma_{\rm s}, \sigma_{\rm d})$ such that $\varphi \in C^{1,\tilde{\alpha}}(\Omega \cap B_r(P_0^1))$ satisfies $\|\varphi\|_{1,\tilde{\alpha},\Omega\cap B_r(P_0^1)} \le C$ and, for $\Gamma_{\rm shock}$ given by (4.3.45), $\|f_{\boldsymbol{e}}\|_{1,\tilde{\alpha},(0,r)} \le C$. Finally, using [20, Proposition 4.3.11(ii)], we obtain (4.3.46).

4.3.4 Compactness of the set of admissible solutions

Fix $\gamma \ge 1$ and $v_2 \in (v_{\rm min}, 0)$. For any $\theta_* \in (0, \theta^{\rm d})$, the arguments in §4.3.3 are also valid for the weighted $C^{2,\alpha}$–estimates near $\Gamma_{\rm sonic}^6$ with respect to $\theta_2 \in [0, \theta^{\rm d})$. According to all the *a priori* estimates obtained in Lemma 4.18, Corollary 4.19, and Propositions 4.21–4.23 and 4.25, there exists a constant $\bar{\alpha} \in (0,1)$ depending only on (γ, v_2, θ_*) such that the set

$$\left\{ \|\varphi\|_{1,\bar{\alpha},\overline{\Omega}} + \|f_{\rm sh}\|_{1,\bar{\alpha},[\xi^{P_3},\xi^{P_2}]} \,:\, \begin{array}{l} \varphi \text{ is an admissible solution} \\ \text{corresponding to } \boldsymbol{\theta} \in \Theta \cap [0,\theta_*]^2 \end{array} \right\}$$

is bounded, where $\eta = f_{\rm sh}(\xi)$ is the expression of $\Gamma_{\rm shock}$ as a graph, given by Lemma 4.11. For each admissible solution, the associated pseudo-subsonic region Ω

is a bounded domain enclosed by $\Gamma^5_{\rm sonic}$, $\Gamma_{\rm shock}$, $\Gamma^6_{\rm sonic}$, and $\Gamma_{\rm sym}$. These four curves intersect only at P_i for $i = 1, 2, 3, 4$. Note that $\Gamma^5_{\rm sonic}$, O_5, P_1, and P_2 depend continuously on $\theta_1 \in [0, \theta^{\rm d}]$, while $\Gamma^6_{\rm sonic}$, O_6, P_3, and P_4 depend continuously on $\theta_2 \in [0, \theta^{\rm d}]$. Combining the above results with (4.3.12) and Lemma 4.16, we have the following compactness result; the details of the proof can be found in [20, Proposition 11.6.1].

Lemma 4.26. *Fix $\gamma \geq 1$, $v_2 \in (v_{\min}, 0)$, and $\theta_* \in (0, \theta^{\rm d})$. Let $\{\boldsymbol{\theta}^{(j)}\}_{j\in\mathbf{N}} \subseteq \Theta \cap [0, \theta_*]^2$ be a sequence of parameters satisfying that $\lim_{j\to\infty} \boldsymbol{\theta}^{(j)} = \boldsymbol{\theta}^{(\infty)}$ for some $\boldsymbol{\theta}^{(\infty)} \in [0, \theta_*]^2$. For each $j \in \mathbf{N}$, let $\varphi^{(j)}$ be an admissible solution corresponding to $\boldsymbol{\theta}^{(j)}$, with pseudo-subsonic region $\Omega^{(j)}$ and curved transonic shock $\Gamma^{(j)}_{\rm shock}$. Then there exists a subsequence $\{\varphi^{(j_k)}\}_{k\in\mathbf{N}}$ such that the following properties hold:*

(i) *$\{\varphi^{(j_k)}\}_{k\in\mathbf{N}}$ converges uniformly on any compact subset of $\overline{\mathbf{R}^2_+}$ to a function $\varphi^{(\infty)} \in C^{0,1}_{\rm loc}\big(\overline{\mathbf{R}^2_+}\big)$, and the limit function $\varphi^{(\infty)}$ is an admissible solution corresponding to $\boldsymbol{\theta}^{(\infty)}$;*

(ii) *$\Omega^{(j_k)} \to \Omega^{(\infty)}$ in the Hausdorff metric;*

(iii) *If $\boldsymbol{\xi}^{(j_k)} \in \overline{\Omega^{(j_k)}}$ for each $k \in \mathbf{N}$, and $\{\boldsymbol{\xi}^{(j_k)}\}_{k\in\mathbf{N}}$ converges to $\boldsymbol{\xi}^{(\infty)} \in \overline{\Omega^{(\infty)}}$, then*

$$\varphi^{(j_k)}(\boldsymbol{\xi}^{(j_k)}) \to \varphi^{(\infty)}(\boldsymbol{\xi}^{(\infty)}), \quad D\varphi^{(j_k)}(\boldsymbol{\xi}^{(j_k)}) \to D\varphi^{(\infty)}(\boldsymbol{\xi}^{(\infty)}) \qquad \text{as } k \to \infty,$$

where, in the case that $\boldsymbol{\xi}^{(j_k)} \in \Gamma^{(j_k)}_{\rm shock}$, the derivative of $\varphi^{(j_k)}$ at $\boldsymbol{\xi}^{(j_k)}$ is defined as the limit of the derivative from the interior of $\Omega^{(j_k)}$

$$D\varphi^{(j_k)}(\boldsymbol{\xi}^{(j_k)}) := \lim_{\boldsymbol{\xi}\to\boldsymbol{\xi}^{(j_k)},\, \boldsymbol{\xi}\in\Omega^{(j_k)}} D\varphi^{(j_k)}(\boldsymbol{\xi}),$$

and, similarly, for $\boldsymbol{\xi}^{(\infty)} \in \Gamma^{(\infty)}_{\rm shock}$,

$$D\varphi^{(\infty)}(\boldsymbol{\xi}^{(\infty)}) := \lim_{\boldsymbol{\xi}\to\boldsymbol{\xi}^{(\infty)},\, \boldsymbol{\xi}\in\Omega^{(\infty)}} D\varphi^{(\infty)}(\boldsymbol{\xi}).$$

4.4 Iteration Method and Existence of Admissible Solutions

In this section, we carefully construct the iteration set and the iteration map based on the uniform estimates obtained in Section 4.3. Then we apply the Leray–Schauder degree theorem to show the existence of admissible solutions. The construction of the iteration set follows closely the process described in [1, Chapter 4] with the main difference that both sonic arcs Γ^5_{sonic} and Γ^6_{sonic} may degenerate into single points. In this section, we always fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$.

4.4.1 Mapping between the elliptic and standard domains

4.4.1.1 Mapping into the standard domain

Define the standard domain to be the rectangle $\mathcal{Q}^{\text{iter}} := (-1,1) \times (0,1)$. Before constructing a map between Ω and $\mathcal{Q}^{\text{iter}}$, we introduce some useful notation and basic geometric properties. Let φ_2 be given by (4.2.14), and let (φ_5, φ_6) be given by (4.2.27). For any $\delta_0 > 0$, we define $S^{\delta_0}_{2j} := \{\boldsymbol{\xi} \in \mathbf{R}^2 : (\varphi_2 - \varphi_j)(\boldsymbol{\xi}) = -\delta_0\}$ and $q_j^{\delta_0} := \operatorname{dist}(O_j, S^{\delta_0}_{2j})$ for $j = 5, 6$, and also write

$$u_5^{\delta_0} := u_5 + q_5^{\delta_0} \sin\theta_{25} \geq 0\,, \qquad u_6^{\delta_0} := u_6 - q_6^{\delta_0} \sin\theta_{26} \leq 0\,.$$

For the constant $\hat{\delta} > 0$ from (4.3.14), we set $\tilde{\delta}_0 := \frac{\hat{\delta}}{4}$, which depends only on (γ, v_2). Then, for $\hat{c}_j$ given by (4.3.13), a direct calculation gives that, for any $\boldsymbol{\theta} \in \overline{\Theta}$,

$$\hat{c}_j - q_j^{\delta_0} \geq \hat{\delta} - \frac{\delta_0}{\ell(\rho_j)} \geq 2\tilde{\delta}_0\,,$$

after choosing $\delta_0 > 0$ small enough, depending only on (γ, v_2), and using Lemma 4.5. Therefore, $S^{\delta_0}_{2j}$ and $\partial B_{\hat{c}_j}(O_j)$ intersect at two distinct points for any $\boldsymbol{\theta} \in \overline{\Theta}$.

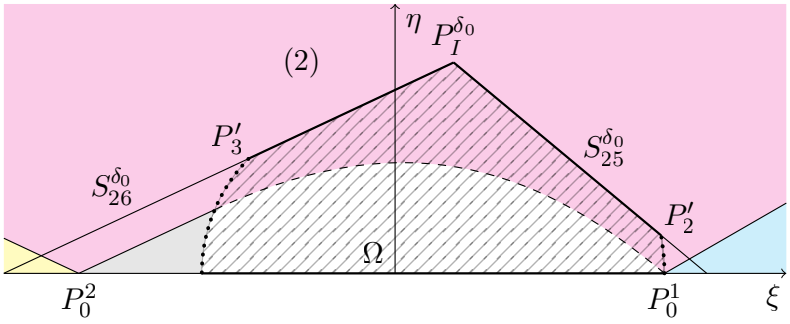


Figure 4.7: The extended domain $Q^{\boldsymbol{\theta}}$, indicated by grey stripes, along with the translated shocks $S_{25}^{\delta_0}$ and $S_{26}^{\delta_0}$ and extended sonic circles $\Gamma_{\text{sonic}}^{5,\delta_0}$ and $\Gamma_{\text{sonic}}^{6,\delta_0}$ for the case: $\boldsymbol{\theta} \in [\theta^{\text{s}}, \theta^{\text{d}}) \times (0, \theta^{\text{s}})$.

Definition 4.27 (Extended domain $Q^{\boldsymbol{\theta}}$)**.** Let $\delta_0 > 0$ be chosen as above. Define the new intersection points P_2' and P_3' by

$$\{P_2'\} := S_{25}^{\delta_0} \cap \partial B_{\hat{c}_5}(O_5) \cap \{\xi \geq u_5^{\delta_0}\}\,, \qquad \{P_3'\} := S_{26}^{\delta_0} \cap \partial B_{\hat{c}_6}(O_6) \cap \{\xi \leq u_6^{\delta_0}\}\,.$$

Denote by $\Gamma_{\text{sonic}}^{5,\delta_0}$ the arc of $\partial B_{\hat{c}_5}(O_5) \cap \{\xi \geq u_5^{\delta_0}\}$ with endpoints $\{P_1, P_2'\}$, and denote by $\Gamma_{\text{sonic}}^{6,\delta_0}$ the arc of $\partial B_{\hat{c}_6}(O_6) \cap \{\xi \leq u_6^{\delta_0}\}$ with endpoints $\{P_4, P_3'\}$. Define the extended domain $Q^{\boldsymbol{\theta}}$ as the open bounded region enclosed by $\Gamma_{\text{sonic}}^{5,\delta_0}$, $S_{25}^{\delta_0}$, $S_{26}^{\delta_0}$, $\Gamma_{\text{sonic}}^{6,\delta_0}$, and Γ_{sym}; see Figure 4.7.

For any $\boldsymbol{\xi} = (\xi, \eta) \in (\mathbf{R} \setminus [u_6^{\delta_0}, u_5^{\delta_0}]) \times [0, \infty)$, define the (x, y)–coordinates by $(x, y) := \mathcal{R}(\boldsymbol{\xi})$ if and only if

$$\boldsymbol{\xi} = \begin{cases} O_6 + (c_6 - x)\,(\cos(\pi - y),\, \sin(\pi - y)) & \text{if } \xi < u_6^{\delta_0}\,, \\ O_5 + (c_5 - x)\,(\cos y,\, \sin y) & \text{if } \xi > u_5^{\delta_0}\,, \end{cases} \tag{4.4.1}$$

with $y \in (0, \pi)$. For any constant $\epsilon > 0$, we define two sets $\mathcal{D}_\epsilon^5, \mathcal{D}_\epsilon^6 \subseteq Q^{\boldsymbol{\theta}}$ by

(4.4.2)

$$\mathcal{D}_\epsilon^5 := \left\{\boldsymbol{\xi} \in Q^{\boldsymbol{\theta}} \,:\, \xi > u_5^{\delta_0}\right\} \setminus \overline{B_{\hat{c}_5 - \epsilon}(O_5)}\,, \qquad \mathcal{D}_\epsilon^6 := \left\{\boldsymbol{\xi} \in Q^{\boldsymbol{\theta}} \,:\, \xi < u_6^{\delta_0}\right\} \setminus \overline{B_{\hat{c}_6 - \epsilon}(O_6)}\,.$$

Let $\omega_0 \in (0, \frac{\pi}{2})$ be the solution of $(q^{\delta_0} + \tilde{\delta}_0)\cos\omega_0 = q^{\delta_0}$, where $q^{\delta_0} := \max\{q_5^{\delta_0}, q_6^{\delta_0}\}$.

Then

$$\tilde{\omega}_0 \coloneqq \frac{1}{2}\inf_{\boldsymbol{\theta}\in\overline{\Theta}}\left\{\theta_{25}-\tfrac{\pi}{2},\,\tfrac{\pi}{2}-\theta_{26},\,\omega_0\right\}>0$$

by (4.3.13)–(4.3.14) and Lemma 4.5. A direct computation shows that, for any $\epsilon\in(0,\tilde{\delta}_0)$,

(4.4.3)

$$\mathcal{D}^5_\epsilon \subseteq \left\{\boldsymbol{\xi}\,:\, x_{P_2}<x<x_{P_2}+\epsilon,\,0<y<\theta_{25}-\tfrac{\pi}{2}-\tilde{\omega}_0\right\}\cap\left\{\xi>u_5^{\delta_0}+\tilde{\delta}_0\sin\tilde{\omega}_0\right\},$$
$$\mathcal{D}^6_\epsilon \subseteq \left\{\boldsymbol{\xi}\,:\, x_{P_3}<x<x_{P_3}+\epsilon,\,0<y<\tfrac{\pi}{2}-\theta_{26}-\tilde{\omega}_0\right\}\cap\left\{\xi<u_6^{\delta_0}-\tilde{\delta}_0\sin\tilde{\omega}_0\right\}.$$

In particular, we fix a constant $k>4$ sufficiently large, depending only on (γ,v_2), such that $\max\{\hat{c}_5,\hat{c}_6\}\le\frac{1}{4}k\tilde{\delta}_0\sin\tilde{\omega}_0$ for any $\boldsymbol{\theta}\in\overline{\Theta}$ and

$$\mathcal{D}^5_{\frac{3\hat{c}_5}{k}} \subseteq \left\{\xi>u_5^{\delta_0}+\frac{3\hat{c}_5}{k}\right\},\qquad \mathcal{D}^6_{\frac{3\hat{c}_6}{k}} \subseteq \left\{\xi<u_6^{\delta_0}-\tfrac{3\hat{c}_6}{k}\right\}. \tag{4.4.4}$$

For each $\boldsymbol{\theta}\in\overline{\Theta}$, we define an invertible map $\mathcal{G}_1^{\boldsymbol{\theta}}:\overline{Q^{\boldsymbol{\theta}}}\to[-1,1]\times[0,\infty)$, which flattens the extended sonic arcs $\Gamma^{5,\delta_0}_{\rm sonic}$ and $\Gamma^{6,\delta_0}_{\rm sonic}$. For each admissible solution φ corresponding to $\boldsymbol{\theta}\in\overline{\Theta}$, we further construct a map $G_{2,\mathfrak{g}_{\rm sh}}$ from $\mathcal{G}_1^{\boldsymbol{\theta}}(\Omega)$ to the standard domain $\mathcal{Q}^{\rm iter}$, where Ω is the pseudo-subsonic domain associated with φ. Finally, we present the *a priori* estimates in the standard domain for admissible solutions under the map $G_{2,\mathfrak{g}_{\rm sh}}\circ\mathcal{G}_1^{\boldsymbol{\theta}}$.

Construction of $\mathcal{G}_1^{\boldsymbol{\theta}}$. For $r_0\in\mathbf{R}$ and $r_{\rm d}>0$, we fix a basic cut-off function $\tilde{\zeta}(\,\cdot\,;r_0,r_{\rm d})\in C^4(\mathbf{R})$ such that $\tilde{\zeta}'(\,\cdot\,;r_0,r_{\rm d})\in[0,\frac{2}{r_{\rm d}}]$ and

$$\tilde{\zeta}(r;\,r_0,r_{\rm d}) = \begin{cases} 0 & \text{for } r\le r_0\,,\\ 1 & \text{for } r\ge r_0+r_{\rm d}\,.\end{cases} \tag{4.4.5}$$

For $j=5,6$, let the cut-off functions (ζ_j,χ_j) be defined by

$$\zeta_j(r)\coloneqq\tilde{\zeta}(r;\,(1-\tfrac{3}{k})\hat{c}_j,\,\tfrac{\hat{c}_j}{k}),\qquad \chi_j(\xi)\coloneqq\tilde{\zeta}((-1)^{j-1}\xi;\,(-1)^{j-1}u_j^{\delta_0},\,\tfrac{2\hat{c}_j}{k}).$$

Then we define $F_1 : \overline{Q^{\boldsymbol{\theta}}} \to \mathbf{R}^2$ by $F_1(\xi, \eta) \coloneqq (h_1(\xi, \eta), \eta)$, where

$$h_1(\boldsymbol{\xi}) \coloneqq \zeta_{\mathrm{d}}(\boldsymbol{\xi}) u_r + \big(1 - \zeta_{\mathrm{d}}(\boldsymbol{\xi})\big)\xi\,,$$

and ζ_{d} and u_r are given by

$$\begin{aligned}
\zeta_{\mathrm{d}}(\boldsymbol{\xi}) &\coloneqq \chi_5(\xi)\zeta_5(|\boldsymbol{\xi} - O_5|) + \chi_6(\xi)\zeta_6(|\boldsymbol{\xi} - O_6|)\,,\\
u_r &\coloneqq (u_5 + |\boldsymbol{\xi} - O_5|)\chi_5(\xi) + (u_6 - |\boldsymbol{\xi} - O_6|)\chi_6(\xi)\,.
\end{aligned}$$

For each $\boldsymbol{\theta} \in \overline{\Theta}$, we write $s_5 \coloneqq u_5 + \hat{c}_5 > 0$ and $s_6 \coloneqq u_6 - \hat{c}_6 < 0$, which depend continuously on $\boldsymbol{\theta} \in \overline{\Theta}$. Let the ε-neighbourhood $\mathcal{N}_\varepsilon(\Gamma)$ be defined by (4.3.20). By construction of the map F_1, we observe that

$$h_1(\boldsymbol{\xi}) = u_j + (-1)^{j-1}\mathrm{dist}(\boldsymbol{\xi}, O_j) \qquad \text{for any } \boldsymbol{\xi} \in Q^{\boldsymbol{\theta}} \cap \mathcal{N}_{\frac{2\hat{c}_j}{k}}(\Gamma^{j,\delta_0}_{\mathrm{sonic}})\,,$$

and $\overline{F_1(Q^{\boldsymbol{\theta}})} \subseteq [s_6, s_5] \times [0, \infty)$. Let the linear map $L_{\boldsymbol{\theta}} : [s_6, s_5] \to [-1, 1]$ be given by

$$L_{\boldsymbol{\theta}}(s') \coloneqq d_L(s' - s_6) - 1 \qquad \text{with } d_L \coloneqq \frac{2}{s_5 - s_6}\,, \tag{4.4.6}$$

which maps the interval $[s_6, s_5]$ to the standard interval $[-1, 1]$. For $k > 4$ given by (4.4.4), define the cut-off functions $\tilde{\chi}_j$ by

$$\tilde{\chi}_j(s') \coloneqq \tilde{\zeta}((-1)^{j-1}s';\, (-1)^{j-1}u_j + (1 - \tfrac{1}{k})\hat{c}_j,\, \tfrac{\hat{c}_j}{2k}) \qquad \text{for } j = 5, 6\,.$$

We define a function $h_2 : \overline{F_1(Q^{\boldsymbol{\theta}})} \to [0, \infty)$ by

$$h_2(s', t'') \coloneqq \tilde{\chi}_5 \arcsin\big(\frac{t''}{s' - u_5}\big) + \tilde{\chi}_6 \arcsin\big(\frac{t''}{u_6 - s'}\big) + (1 - \tilde{\chi}_5 - \tilde{\chi}_6)\, t''\,,$$

where $\tilde{\chi}_5$ and $\tilde{\chi}_6$ are evaluated at s'. For the maps F_1 and h_2 constructed above, we see that

$$h_2 \circ F_1(\boldsymbol{\xi}) = y \qquad \text{for any } \boldsymbol{\xi} \in Q^{\boldsymbol{\theta}} \cap \big(\mathcal{N}_{\frac{\hat{c}_5}{2k}}(\Gamma^{5,\delta_0}_{\mathrm{sonic}}) \cup \mathcal{N}_{\frac{\hat{c}_6}{2k}}(\Gamma^{6,\delta_0}_{\mathrm{sonic}})\big)\,,$$

where the (x, y)–coordinates are given by (4.4.1).

Finally, we define the map $\mathcal{G}_1^{\boldsymbol{\theta}} : \overline{Q^{\boldsymbol{\theta}}} \to [-1, 1] \times [0, \infty)$ by

$$\mathcal{G}_1^{\boldsymbol{\theta}}(\boldsymbol{\xi}) \coloneqq (L_{\boldsymbol{\theta}} \circ h_1(\boldsymbol{\xi}),\, h_2 \circ F_1(\boldsymbol{\xi})) \qquad \text{for any } \boldsymbol{\xi} \in \overline{Q^{\boldsymbol{\theta}}}\,, \tag{4.4.7}$$

and write the coordinates $(s, t') \coloneqq \mathcal{G}_1^{\boldsymbol{\theta}}(\boldsymbol{\xi})$.

It can be verified directly that $\mathcal{G}_1^{\boldsymbol{\theta}}(\Gamma_{\mathrm{sym}}) = \{(s, 0) \,:\, -1 < s < 1\}$. Moreover, the map $\mathcal{G}_1^{\boldsymbol{\theta}} : \overline{Q^{\boldsymbol{\theta}}} \to [-1, 1] \times [0, \infty)$ constructed above satisfies the following properties by using (4.3.3), (4.4.5), and Lemma 4.26; the proof is similar to [1, Lemmas 4.3 and 4.5].

Lemma 4.28. *There exists a constant $\mu_{\mathcal{G}_1} \in (0, 1)$ depending only on (γ, v_2) such that, for each $\boldsymbol{\theta} \in \overline{\Theta}$, the map $\mathcal{G}_1^{\boldsymbol{\theta}}$ defined by* (4.4.7) *satisfies the following*:

(i) $\|\mathcal{G}_1^{\boldsymbol{\theta}}\|_{C^4(\overline{Q^{\boldsymbol{\theta}}})} + \|(\mathcal{G}_1^{\boldsymbol{\theta}})^{-1}\|_{C^4(\overline{\mathcal{G}_1^{\boldsymbol{\theta}}(Q^{\boldsymbol{\theta}})})} \le \mu_{\mathcal{G}_1}^{-1}$, *and* $|\det(D\mathcal{G}_1^{\boldsymbol{\theta}})| \ge \mu_{\mathcal{G}_1}$ *in* $\overline{Q^{\boldsymbol{\theta}}}$;

(ii) *For $j = 5, 6$, and for any $(s, t') \in \overline{\mathcal{G}_1^{\boldsymbol{\theta}}(Q^{\boldsymbol{\theta}})}$,* $\partial_{t'}\big((\varphi_2 - \varphi_j) \circ (\mathcal{G}_1^{\boldsymbol{\theta}})^{-1}\big)(s, t') \le -\mu_{\mathcal{G}_1}$;

(iii) *For any $\theta_* \in (0, \theta^{\mathrm{d}})$, there exists a constant $\mu_* > 0$ depending only on (γ, v_2, θ_*) such that any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta_1, \theta_2 \le \theta_*\}$ satisfies*

$$\partial_{t'}\big((\varphi_2 - \varphi) \circ (\mathcal{G}_1^{\boldsymbol{\theta}})^{-1}\big)(s, t') \le -\mu_* < 0 \qquad \textit{for all } (s, t') \in \overline{\mathcal{G}_1^{\boldsymbol{\theta}}(\Omega)}\,.$$

Using Lemma 4.28(ii), there exists a unique function $f_{\boldsymbol{\theta}} \in C^{0,1}([-1, 1])$ such that

$$\mathcal{G}_1^{\boldsymbol{\theta}}\big(\{\boldsymbol{\xi} \in \mathbf{R}^2_+ \,:\, \varphi_2 = \max\{\varphi_5, \varphi_6\} - \delta_0\}\big) = \{(s, f_{\boldsymbol{\theta}}(s)) \,:\, -1 < s < 1\}\,,$$

which follows from the implicit function theorem, and

$$\mathcal{G}_1^{\boldsymbol{\theta}}(Q^{\boldsymbol{\theta}}) = \big\{(s, t') \in \mathbf{R}^2_+ \,:\, -1 < s < 1,\, 0 < t' < f_{\boldsymbol{\theta}}(s)\big\}\,. \tag{4.4.8}$$

Using (4.3.12), Definition 4.7(ii), Lemma 4.18(ii), Corollary 4.19, Propositions 4.21–4.23 and 4.25, and Lemma 4.28, we introduce a function $\mathfrak{g}_{\mathrm{sh}} : [-1, 1] \to [0, \infty)$

representing Γ_{shock} in the (s,t')–coordinates and state its important properties; the proof is similar to [1, Proposition 4.6].

Proposition 4.29. *For each admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$, there exists a unique function $\mathfrak{g}_{\text{sh}} : [-1,1] \to [0,\infty)$ satisfying the following properties*:

(i) *In the (s,t')–coordinates*:

$$\mathcal{G}_1^{\boldsymbol{\theta}}(\Omega) = \big\{(s,t') \,:\, -1 < s < 1,\, 0 < t' < \mathfrak{g}_{\text{sh}}(s)\big\}\,,$$
$$\mathcal{G}_1^{\boldsymbol{\theta}}(\Gamma_{\text{shock}}) = \big\{(s,\mathfrak{g}_{\text{sh}}(s)) \,:\, -1 < s < 1,\, t' = \mathfrak{g}_{\text{sh}}(s)\big\}\,,$$

and, for any interval $I_1 \Subset (-1,1)$, $\|\mathfrak{g}_{\text{sh}}\|_{C^3(\overline{I_1})} \le C_{I_1}$ for some constant $C_{I_1} > 0$ depending only on (γ, v_2, I_1).

(ii) *Let $Q_0^{\boldsymbol{\theta}}$ be the bounded region enclosed by Γ^5_{sonic}, S_{25}, S_{26}, Γ^6_{sonic}, and Γ_{sym}, which satisfies $\Omega \subseteq Q_0^{\boldsymbol{\theta}} \subseteq Q^{\boldsymbol{\theta}}$. For $\mathcal{D}_\epsilon^5$ and $\mathcal{D}_\epsilon^6$ given by* (4.4.2), *there exist unique functions $\mathfrak{g}_5$ and $\mathfrak{g}_6$ with*

$$\mathcal{G}_1^{\boldsymbol{\theta}}(Q_0^{\boldsymbol{\theta}} \cap \mathcal{D}_\epsilon^j) = \big\{(s,t') \in \mathcal{Q}^{\text{iter}} \,:\, 1 + (-1)^j s < d_L\epsilon,\, 0 < t' < \mathfrak{g}_j(s)\big\} \qquad \text{for } j = 5,6\,.$$

Let $\epsilon_0^* > 0$ be the smallest constant ε_0 in Step **1** *of the proofs of* Propositions 4.21 *and* 4.23. *For $d_L > 0$ given by* (4.4.6), *set $\hat{\epsilon}_0^* := d_L\epsilon_0^*$. There exists a constant $C > 0$ depending only on (γ, v_2) such that*

$$\|\mathfrak{g}_5\|_{C^3([1-\hat{\epsilon}_0^*,1])} + \|\mathfrak{g}_6\|_{C^3([-1,-1+\hat{\epsilon}_0^*])} \le C\,.$$

(iii) *For each $\theta_* \in (0,\theta^{\text{d}})$, there exist constants $\bar{\alpha} \in (0,1)$ and $C_{\theta_*} > 0$ depending only on (γ, v_2, θ_*) such that, for any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta_1, \theta_2 \le \theta_*\}$,*

$$\|\mathfrak{g}_{\text{sh}}\|^{(-1-\bar{\alpha}),\{-1\}}_{2,\bar{\alpha},(-1,-1+\hat{\epsilon}_0^*)} + \|\mathfrak{g}_{\text{sh}}\|^{(-1-\bar{\alpha}),\{1\}}_{2,\bar{\alpha},(1-\hat{\epsilon}_0^*,1)} \le C_{\theta_*}\,,$$
$$\frac{\mathrm{d}^m}{\mathrm{d}s^m}(\mathfrak{g}_{\text{sh}} - \mathfrak{g}_j)((-1)^{j-1}) = 0 \qquad \text{for } m = 0,1, \text{ and } j = 5,6\,.$$

where $\|\cdot\|_{2,\alpha,I}^{(\sigma),\{x_0\}}$ *is given by* Definition 4.24(ii). *Note that the properties above are equivalent to*

$$\|\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_5\|_{2,\bar{\alpha},(1-\hat{\epsilon}_0^*,1)}^{(1+\bar{\alpha}),(\mathrm{par})} + \|\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_6\|_{2,\bar{\alpha},(-1,-1+\hat{\epsilon}_0^*)}^{(1+\bar{\alpha}),(\mathrm{par})} \le C'_{\theta_*}$$

for some constant $C'_{\theta_*} > 0$ *depending only on* (γ, v_2, θ_*), *where* $\|\cdot\|_{2,\alpha,I}^{(1+\alpha),(\mathrm{par})}$ *is given by* Definition 4.20(ii) *with replacement of* x *by* $1 - |s|$.

(iv) *For each* $\theta_* \in (0, \theta^{\mathrm{d}})$, *there exists a constant* $\hat{k} > 1$ *depending only on* (γ, v_2, θ_*) *such that, for any admissible solution* φ *corresponding to* $\boldsymbol{\theta} \in \Theta \cap \{\theta_1, \theta_2 \le \theta_*\}$ *and any* $s \in [-1, 1]$,

$$\begin{aligned}\min\{\mathfrak{g}_{\mathrm{sh}}(-1) + \frac{s+1}{\hat{k}},\, \mathfrak{g}_{\mathrm{sh}}(1) + \frac{1-s}{\hat{k}},\, \frac{1}{\hat{k}}\} &\le \mathfrak{g}_{\mathrm{sh}}(s) \\ &\le \min\{\mathfrak{g}_{\mathrm{sh}}(-1) + \hat{k}(s+1),\, \mathfrak{g}_{\mathrm{sh}}(1) + \hat{k}(1-s),\, f_{\boldsymbol{\theta}}(s) - \frac{1}{\hat{k}}\}\,.\end{aligned}$$

Remark 4.30. By Propositions 4.21–4.22, for each $\alpha \in (0,1)$, there exist $\hat{\epsilon}_3 > 0$ and $C_\alpha > 0$ depending only on (γ, v_2, α) such that, for any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$,

$$\|\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_j\|_{2,\alpha,I_j}^{(\mathrm{par})} \le C_\alpha \qquad \text{whenever } \theta_{j-4} \le \theta^{\mathrm{s}}\,,$$

for $j = 5, 6$, where $I_j \coloneqq \{s \in (-1,1) \,:\, 1 + (-1)^j s < \hat{\epsilon}_3\}$, and $\|\cdot\|_{2,\alpha,I_j}^{(\mathrm{par})}$ is given by Definition 4.20(ii) with replacement of x by $1 - |s|$.

By Proposition 4.23, for each $\alpha \in (0,1)$, there exist constants $\hat{\epsilon}_4 > 0$ and $C'_\alpha > 0$ depending only on (γ, v_2, α) such that, for any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta$,

$$\|\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_j\|_{2,\alpha,\overline{I'_j}} \le C'_\alpha\,, \qquad \frac{\mathrm{d}^m}{\mathrm{d}s^m}(\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_j)((-1)^{j-1}) = 0 \quad \text{for } m = 0, 1, 2\,,$$

whenever $\theta_{j-4} \in [\theta^{\mathrm{s}}, \theta^{\mathrm{s}} + \sigma_3]$ for $j = 5, 6$, where $I'_j \coloneqq \{s \in (-1,1) \,:\, 1 + (-1)^j s < \hat{\epsilon}_4\}$.

Construction of $u^{(\varphi,\boldsymbol{\theta})}$. For each $\boldsymbol{\theta} \in \overline{\Theta} \setminus \{\mathbf{0}\}$, it can be verified directly that S_{25} and S_{26} given by Definition 4.4 intersect at the unique point $P_I \coloneqq (\xi^{P_I}, \eta^{P_I})$ with

$$\xi^{P_I} \coloneqq \frac{u_5 \xi_0^{P_0^1} - u_6 \xi_0^{P_0^2}}{u_5 - u_6}, \qquad \eta^{P_I} \coloneqq \frac{u_5 u_6}{u_5 - u_6} \frac{\xi_0^{P_0^1} - \xi_0^{P_0^2}}{v_2}. \tag{4.4.9}$$

Similarly, $S_{25}^{\delta_0}$ and $S_{26}^{\delta_0}$ intersect at the unique point $P_I^{\delta_0} \coloneqq (\xi^{P_I}, \eta^{P_I^{\delta_0}})$ where $\eta^{P_I^{\delta_0}} \coloneqq \eta^{P_I} - \frac{\delta_0}{v_2} > \eta^{P_I}$. In Lemma A.4, we show the limits

$$\lim_{\boldsymbol{\theta} \to \mathbf{0},\, \boldsymbol{\theta} \in \overline{\Theta} \setminus \{\mathbf{0}\}} (\xi^{P_I}, \eta^{P_I}) = (0, \eta_0),$$

where η_0 is given by (4.2.18). Then we define $P_I|_{\boldsymbol{\theta}=\mathbf{0}} \coloneqq (0, \eta_0)$ such that P_I is $C(\overline{\Theta})$. By (4.4.9) and Lemma A.3(ii), $\xi^{P_I} \geq 0$ for any $\boldsymbol{\theta} \in \overline{\Theta} \cap \{\theta_2 = 0\}$. Therefore, there exists a constant $\sigma_{\mathrm{sep}} > 0$ small enough, depending only on (γ, v_2), such that

$$\xi^{P_3} \leq -d_{\mathrm{sep}} < -\tfrac{d_{\mathrm{sep}}}{2} \leq \xi^{P_I} \qquad \text{for any } \boldsymbol{\theta} \in \overline{\Theta} \cap \{0 \leq \theta_2 \leq \sigma_{\mathrm{sep}}\},$$

where $d_{\mathrm{sep}} > 0$ is given by (4.3.9). Also, by Lemma A.3(ii), there exists a small constant $\delta_{\mathrm{sep}} > 0$ depending only on $(\gamma, v_2, \sigma_{\mathrm{sep}})$ such that

$$\eta^{P_3} \leq \eta_0 - \delta_{\mathrm{sep}} < \eta_0 \leq \min\{a_{25}, a_{26}\} \leq \eta^{P_I} \qquad \text{for any } \boldsymbol{\theta} \in \overline{\Theta} \cap \{\sigma_{\mathrm{sep}} \leq \theta_2 \leq \theta^{\mathrm{d}}\}.$$

From the above two inequalities, using $P_3, P_I \in S_{26}$ and Lemma 4.5, we obtain the uniform positive lower bound

$$\inf_{\boldsymbol{\theta} \in \overline{\Theta}} (\xi^{P_I} - \xi^{P_3}) > 0.$$

Definition 4.31. Let $\tilde{\zeta}(r; r_0, r_{\mathrm{d}})$ be given by (4.4.5), and let ξ^{P_I} be given by (4.4.9). For $\boldsymbol{\theta} \in \overline{\Theta}$, define

$$\chi^*_{\boldsymbol{\theta}}(\xi) \coloneqq \tilde{\zeta}\big(-\xi; -\xi^{P_I}, \frac{\xi^{P_I} - \xi^{P_3}}{5C_{\mathrm{bg}}}\big)$$

for some constant $C_{\mathrm{bg}} > 1$. The smooth background function $\varphi^*_{\boldsymbol{\theta}}(\boldsymbol{\xi})$ is defined by

$$\varphi^*_{\boldsymbol{\theta}}(\boldsymbol{\xi}) \coloneqq \varphi_5(\boldsymbol{\xi})\,(1 - \chi^*_{\boldsymbol{\theta}}(\xi)) + \varphi_6(\boldsymbol{\xi})\,\chi^*_{\boldsymbol{\theta}}(\xi)\,.$$

For the constant $\delta_0 > 0$ from Definition 4.27, the constant C_{bg} is fixed sufficiently large, depending only on (γ, v_2), such that $\max\{\varphi_5, \varphi_6\} - \varphi^*_{\boldsymbol{\theta}} \le \frac{\delta_0}{2}$.

From the construction above, it can be checked that $\varphi^*_{\boldsymbol{\theta}} \le \max\{\varphi_5, \varphi_6\}$ in $\mathbf{R}^2$ and there exists a constant $\bar{k} > 1$ sufficiently large, depending only on (γ, v_2), such that, for $j = 5, 6$,

$$\varphi^*_{\boldsymbol{\theta}} = \varphi_{\boldsymbol{\theta}} = \varphi_j \qquad \text{in } \mathcal{D}^j_{\hat{c}_j/\bar{k}}\,, \tag{4.4.10}$$

where $\hat{c}_j$ is given by (4.3.13). Then

$$\big\{\boldsymbol{\xi} \in \mathbf{R}^2_+ \,:\, \xi^{P_3} < \xi < \xi^{P_2},\, (\varphi_2 - \varphi^*_{\boldsymbol{\theta}})(\boldsymbol{\xi}) = 0\big\} \subseteq Q^{\boldsymbol{\theta}}\,. \tag{4.4.11}$$

Moreover, using Lemma 4.28(ii) and Definition 4.31, we can directly obtain that, for any $\boldsymbol{\theta} \in \overline{\Theta}$, there exists a constant $\bar{\mu} > 0$ depending only on (γ, v_2) such that

$$\partial_{t'}\big((\varphi_2 - \varphi^*_{\boldsymbol{\theta}}) \circ (\mathcal{G}_1^{\boldsymbol{\theta}})^{-1}\big)(s, t') \le -\bar{\mu} \qquad \text{for any } (s, t') \in \overline{\mathcal{G}_1^{\boldsymbol{\theta}}(Q^{\boldsymbol{\theta}})}\,. \tag{4.4.12}$$

For any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta$, the map $G_{2,\mathfrak{g}_{\mathrm{sh}}} : \mathcal{G}_1^{\boldsymbol{\theta}}(Q^{\boldsymbol{\theta}}) \to \mathbf{R}^2$ is defined by

$$(s, t) = G_{2,\mathfrak{g}_{\mathrm{sh}}}(s, t') \coloneqq (s, \frac{t'}{\mathfrak{g}_{\mathrm{sh}}(s)}) \qquad \text{for any } (s, t') \in \mathcal{G}_1^{\boldsymbol{\theta}}(Q^{\boldsymbol{\theta}})\,, \tag{4.4.13}$$

where $\mathfrak{g}_{\mathrm{sh}} : [-1, 1] \to [0, \infty)$ is the unique function given by Proposition 4.29. By Proposition 4.29(iv), the map $G_{2,\mathfrak{g}_{\mathrm{sh}}}$ is well defined and invertible with $G^{-1}_{2,\mathfrak{g}_{\mathrm{sh}}}(s, t) = (s, t\mathfrak{g}_{\mathrm{sh}}(s))$. Then $G_{2,\mathfrak{g}_{\mathrm{sh}}} \circ \mathcal{G}_1^{\boldsymbol{\theta}}(\Omega) = \mathcal{Q}^{\mathrm{iter}} = (-1, 1) \times (0, 1)$ by Proposition 4.29(i).

Therefore, $u^{(\varphi,\boldsymbol{\theta})}$ defined as

$$u^{(\varphi,\boldsymbol{\theta})}(s,t) \coloneqq (\varphi - \varphi^*_{\boldsymbol{\theta}}) \circ (G_{2,\mathfrak{g}_{\mathrm{sh}}} \circ \mathcal{G}_1^{\boldsymbol{\theta}})^{-1}(s,t)\,, \tag{4.4.14}$$

is well defined for all $(s,t) \in \mathcal{Q}^{\mathrm{iter}}$.

A priori estimates for $u^{(\varphi,\boldsymbol{\theta})}$. For $\epsilon_0^* > 0$ from Proposition 4.29(ii), define a positive constant ϵ_0' depending only on (γ, v_2) as

$$\epsilon_0' \coloneqq \inf_{\boldsymbol{\theta}\in\overline{\Theta}} d_L \epsilon_0^* \;\le\; \hat{\epsilon}_0^*\,, \tag{4.4.15}$$

where $d_L > 0$ is given by (4.4.6). For $r \in (0,1)$ and for $j = 5, 6$, we define

(4.4.16)

$$\mathcal{Q}_r^{\mathrm{int}} \coloneqq \big\{(s,t) \in \mathcal{Q}^{\mathrm{iter}} \,:\, |s| < 1 - r\big\}\,, \qquad \mathcal{Q}_r^j \coloneqq \big\{(s,t) \in \mathcal{Q}^{\mathrm{iter}} \,:\, 1 + (-1)^j\, s < r\big\}\,.$$

Definition 4.32 (Weighted Hölder norms in the standard domain)**.** For $\alpha \in (0,1)$ and any $u \in C^2(\mathcal{Q}^{\mathrm{iter}})$, introduce the norm

$$\|u\|^{(*)}_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}} \coloneqq \|u\|_{2,\alpha,\overline{\mathcal{Q}^{\mathrm{int}}_{\epsilon_0'/4}}} + \sum_{j=5,6} \Big(\|u\|^{(1+\alpha),(\mathrm{par})}_{2,\alpha,\mathcal{Q}^j_{\epsilon_0'}} + \|u\|^{(1+\alpha),(\mathrm{subs})}_{1,\alpha,\mathcal{Q}^j_{\epsilon_0'}} \Big)\,,$$

where $\|\cdot\|_{2,\alpha,U}$ is the standard Hölder norm, and the weighted Hölder norms $\|\cdot\|^{(\sigma),(\mathrm{subs})}_{m,\alpha,U}$ and $\|\cdot\|^{(\sigma),(\mathrm{par})}_{m,\alpha,U}$, for an open set $U \subseteq \mathcal{Q}^{\mathrm{iter}}$, $\sigma > 0$, and $m \in \mathbf{N}$, are given as follows.

(i) For any $\boldsymbol{s} = (s,t)$, $\tilde{\boldsymbol{s}} = (\tilde{s},\tilde{t}) \in \mathcal{Q}^{\mathrm{iter}}$, define

$$\delta^{(\mathrm{subs})}_{\alpha}(\boldsymbol{s},\tilde{\boldsymbol{s}}) \coloneqq \big((s-\tilde{s})^2 + (\max\{1-|s|, 1-|\tilde{s}|\})^2 (t-\tilde{t})^2\big)^{\frac{\alpha}{2}}\,.$$

The Hölder norms with subsonic scaling are given by

$$\|u\|^{(\sigma),(\mathrm{subs})}_{m,0,U} \coloneqq \sum_{0\le k+l\le m} \sup_{\boldsymbol{s}\in U} \left((1-|s|)^{k-\sigma}|\partial_s^k\partial_t^l u(\boldsymbol{s})|\right),$$

$$[u]^{(\sigma),(\mathrm{subs})}_{m,\alpha,U} \coloneqq \sum_{k+l=m} \sup_{\substack{\boldsymbol{s},\tilde{\boldsymbol{s}}\in U\\ \boldsymbol{s}\neq\tilde{\boldsymbol{s}}}} \left(\min\left\{(1-|s|)^{\alpha+k-\sigma},(1-|\tilde{s}|)^{\alpha+k-\sigma}\right\}\frac{|\partial_s^k\partial_t^l u(\boldsymbol{s})-\partial_s^k\partial_t^l u(\tilde{\boldsymbol{s}})|}{\delta^{(\mathrm{subs})}_\alpha(\boldsymbol{s},\tilde{\boldsymbol{s}})}\right),$$

$$\|u\|^{(\sigma),(\mathrm{subs})}_{m,\alpha,U} = \|u\|^{(\sigma),(\mathrm{subs})}_{m,0,U} + [u]^{(\sigma),(\mathrm{subs})}_{m,\alpha,U}.$$

(ii) For any $\boldsymbol{s}=(s,t),\tilde{\boldsymbol{s}}=(\tilde{s},\tilde{t})\in\mathcal{Q}^{\mathrm{iter}}$, define

$$\delta^{(\mathrm{par})}_\alpha(\boldsymbol{s},\tilde{\boldsymbol{s}}) \coloneqq \left((s-\tilde{s})^2+\max\{1-|s|,1-|\tilde{s}|\}(t-\tilde{t})^2\right)^{\frac{\alpha}{2}}.$$

The Hölder norms with parabolic scaling are given by

$$\|u\|^{(\sigma),(\mathrm{par})}_{m,0,U} \coloneqq \sum_{0\le k+l\le m} \sup_{\boldsymbol{s}\in U} \left((1-|s|)^{k+\frac{l}{2}-\sigma}|\partial_s^k\partial_t^l u(\boldsymbol{s})|\right),$$

$$[u]^{(\sigma),(\mathrm{par})}_{m,\alpha,U} \coloneqq \sum_{k+l=m} \sup_{\substack{\boldsymbol{s},\tilde{\boldsymbol{s}}\in U\\ \boldsymbol{s}\neq\tilde{\boldsymbol{s}}}} \left(\min\left\{(1-|s|)^{\alpha+k+\frac{l}{2}-\sigma},(1-|\tilde{s}|)^{\alpha+k+\frac{l}{2}-\sigma}\right\}\frac{|\partial_s^k\partial_t^l u(\boldsymbol{s})-\partial_s^k\partial_t^l u(\tilde{\boldsymbol{s}})|}{\delta^{(\mathrm{par})}_\alpha(\boldsymbol{s},\tilde{\boldsymbol{s}})}\right),$$

$$\|u\|^{(\sigma),(\mathrm{par})}_{m,\alpha,U} = \|u\|^{(\sigma),(\mathrm{par})}_{m,0,U} + [u]^{(\sigma),(\mathrm{par})}_{m,\alpha,U}.$$

Denote by $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ the set of all functions in $C^2(\mathcal{Q}^{\mathrm{iter}})$ whose $\|\cdot\|^{(*)}_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}}$–norms are finite.

Note that $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ is compactly embedded into $C^{2,\tilde{\alpha}}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ whenever the constants $(\alpha,\tilde{\alpha})$ satisfy $0\le\tilde{\alpha}<\alpha<1$, which follows from the properties given in [20, Lemma 4.6.3 and Corollary 17.2.7].

Using Corollary 4.19 and Propositions 4.21–4.23 and 4.25, we have the following estimate for admissible solutions φ after mapping the pseudo-subsonic domain Ω to the standard domain $\mathcal{Q}^{\mathrm{iter}}$; the proof is similar to [1, Proposition 4.12].

Proposition 4.33. *For each $\theta_* \in (0, \theta^{\mathrm{d}})$, there exist constants $M > 0$ and $\bar{\alpha} \in (0, \frac{1}{3}]$ depending only on (γ, v_2, θ_*) such that, for any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta_1, \theta_2 \leq \theta_*\}$, $u = u^{(\varphi,\boldsymbol{\theta})} : \mathcal{Q}^{\mathrm{iter}} \to \mathbf{R}$, defined by (4.4.14), satisfies $u \in C^{2,\bar{\alpha}}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ and*

$$\|u\|^{(*)}_{2,\bar{\alpha},\mathcal{Q}^{\mathrm{iter}}} = \|u\|_{2,\bar{\alpha},\mathcal{Q}^{\mathrm{int}}_{\epsilon'_0/4}} + \sum_{j=5,6} \left(\|u\|^{(1+\bar{\alpha}),(\mathrm{par})}_{2,\bar{\alpha},\mathcal{Q}^j_{\epsilon'_0}} + \|u\|^{(1+\bar{\alpha}),(\mathrm{subs})}_{1,\bar{\alpha},\mathcal{Q}^j_{\epsilon'_0}} \right) \leq M\,.$$

4.4.1.2 Mapping to approximate admissible solutions

For each $\boldsymbol{\theta} \in \overline{\Theta}$, let the extended domain $Q^{\boldsymbol{\theta}}$ be given by Definition 4.27, and let $\varphi^*_{\boldsymbol{\theta}}$ be given by Definition 4.31. For each $s^* \in (-1, 1)$, define the set $Q^{\boldsymbol{\theta}}(s^*) \coloneqq Q^{\boldsymbol{\theta}} \cap (\mathcal{G}^{\boldsymbol{\theta}}_1)^{-1}(\{s = s^*\})$. Then, for $i = 1, 2$,

$$\inf_{Q^{\boldsymbol{\theta}}((-1)^{i-1})} (\varphi_2 - \varphi^*_{\boldsymbol{\theta}}) < 0 \leq \sup_{Q^{\boldsymbol{\theta}}((-1)^{i-1})} (\varphi_2 - \varphi^*_{\boldsymbol{\theta}})\,,$$

where the inequality on the right above becomes a strict inequality when $\theta_i \in [0, \theta^{\mathrm{s}})$ and an equality when $\theta_i \in [\theta^{\mathrm{s}}, \theta^{\mathrm{d}}]$. Moreover, by the choice of $C_{\mathrm{bg}} > 0$ in Definition 4.31,

$$\sup_{Q^{\boldsymbol{\theta}}(s)} (\varphi_2 - \varphi^*_{\boldsymbol{\theta}}) - \inf_{Q^{\boldsymbol{\theta}}(s)} (\varphi_2 - \varphi^*_{\boldsymbol{\theta}}) \geq \tfrac{\delta_0}{2} > 0 \qquad \text{for any } s \in (-1, 1)\,.$$

Definition 4.34. Fix $\alpha \in (0, 1)$, $\theta_* \in (0, \theta^{\mathrm{d}})$, and $\boldsymbol{\theta} \in \Theta \cap \{\theta_1, \theta_2 \leq \theta_*\}$. Let $u \in C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}})$ be any function satisfying that

$$\inf_{Q^{\boldsymbol{\theta}}(s)} (\varphi_2 - \varphi^*_{\boldsymbol{\theta}}) < u(s, 1) < \sup_{Q^{\boldsymbol{\theta}}(s)} (\varphi_2 - \varphi^*_{\boldsymbol{\theta}}) \qquad \text{for any } s \in (-1, 1)\,. \tag{4.4.17}$$

(i) For any $s \in (-1, 1)$, let $\bar{t}' > 0$ be given as the unique solution on $(0, \infty)$ of the equation $(\varphi_2 - \varphi^*_{\boldsymbol{\theta}}) \circ (\mathcal{G}^{\boldsymbol{\theta}}_1)^{-1}(s, \bar{t}') = u(s, 1)$, which exists by (4.4.12) and (4.4.17). Define $\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}} : (-1, 1) \to (0, \infty)$ by $\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}(s) \coloneqq \bar{t}'$.

(ii) For $\mathcal{G}^{\boldsymbol{\theta}}_1$ and $G_{2,\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}}$ given by (4.4.7) and (4.4.13) respectively, define the map $\mathfrak{F}_{(u,\boldsymbol{\theta})} : \mathcal{Q}^{\mathrm{iter}} \to Q^{\boldsymbol{\theta}}$ by $\mathfrak{F}_{(u,\boldsymbol{\theta})} \coloneqq (\mathcal{G}^{\boldsymbol{\theta}}_1)^{-1} \circ G^{-1}_{2,\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}}$.

(iii) Define sets $\Omega(u,\boldsymbol{\theta}) \coloneqq \mathfrak{F}_{(u,\boldsymbol{\theta})}(\mathcal{Q}^{\mathrm{iter}})$ and $\Gamma_{\mathrm{shock}}(u,\boldsymbol{\theta}) \coloneqq \mathfrak{F}_{(u,\boldsymbol{\theta})}((-1,1)\times\{1\})$. Finally, define the function $\varphi^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}) : \Omega(u,\boldsymbol{\theta}) \to \mathbf{R}$ by

$$\varphi^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}) \coloneqq (u \circ \mathfrak{F}^{-1}_{(u,\boldsymbol{\theta})})(\boldsymbol{\xi}) + \varphi^{*}_{\boldsymbol{\theta}}(\boldsymbol{\xi}) \qquad \text{for all } \boldsymbol{\xi} \in \Omega(u,\boldsymbol{\theta})\,.$$

We call $\varphi^{(u,\boldsymbol{\theta})}$ above an approximate admissible solution if it satisfies the same boundary conditions on $\Gamma^5_{\mathrm{sonic}} \cup \Gamma^6_{\mathrm{sonic}}$ as the admissible solutions in Definition 4.7. For $\alpha \in (0,1)$ and $\theta_* \in (0,\theta^{\mathrm{d}})$, define the space of such approximate admissible solutions

$$\mathfrak{G}^{\theta_*}_{\alpha} \coloneqq \left\{ (u,\boldsymbol{\theta}) \in C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}}) \times [0,\theta_*]^2 \,:\, \begin{array}{l} (u,\boldsymbol{\theta}) \text{ satisfies (4.4.17) and} \\ (u, Du)(\pm 1, \cdot\,) = (0,\mathbf{0}) \end{array} \right\}. \tag{4.4.18}$$

By construction, for any $(u,\boldsymbol{\theta}) \in \mathfrak{G}^{\theta_*}_{\alpha}$, the map $\mathfrak{F}_{(u,\boldsymbol{\theta})}$ satisfies that $P_1 = \mathfrak{F}_{(u,\boldsymbol{\theta})}(1,0)$, $P_2 = \mathfrak{F}_{(u,\boldsymbol{\theta})}(1,1)$, $P_3 = \mathfrak{F}_{(u,\boldsymbol{\theta})}(-1,1)$, and $P_4 = \mathfrak{F}_{(u,\boldsymbol{\theta})}(-1,0)$ for the points P_i, for $i = 1,2,3,4$, given by Definition 4.6. Furthermore, $\varphi^{(u,\boldsymbol{\theta})} = \varphi_2$ on $\Gamma_{\mathrm{shock}}(u,\boldsymbol{\theta})$. Other properties of the set $\mathfrak{G}^{\theta_*}_{\alpha}$ are given below; the proof is similar to [20, Lemmas 12.2.7 and 17.2.13].

Lemma 4.35. *Fix $\alpha \in (0,1)$ and $\theta_* \in (0,\theta^{\mathrm{d}})$. For each $(u,\boldsymbol{\theta}) \in \mathfrak{G}^{\theta_*}_{\alpha}$, the function $\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}} \in C^{1,\alpha}([-1,1])$ and the following properties hold.*

(i) *$\Omega(u,\boldsymbol{\theta}) \cup \Gamma_{\mathrm{shock}}(u,\boldsymbol{\theta}) \subseteq Q^{\boldsymbol{\theta}} \subseteq \mathbf{R}^2_+$. Moreover, $\Gamma_{\mathrm{shock}}(u,\boldsymbol{\theta})$ is a $C^{1,\alpha}$–curve up to its endpoints P_2 and P_3, and is tangential to S_{25} and S_{26} at P_2 and P_3 respectively. For the functions $f_{j,0}$, for $j = 5,6$, given as in* Step **1** *of the proofs of* Propositions 4.21 *and* 4.23, *we have*

$$\big(\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}, \frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}\big)\big|_{s=(-1)^{j-1}} = \big(f_{j,0}(0)\,, \frac{(-1)^j}{d_L} f'_{j,0}(0)\big) \qquad \textit{for } j = 5,6\,,$$

where d_L is given by (4.4.6).

(ii) *Let $\mathcal{N}_{\varepsilon}(\Gamma)$ denote the open ε-neighbourhood of a set Γ, as given by* (4.3.20). *Let δ_0 be from* Definition 4.27, *and let the coordinates $(x,y) = \mathcal{R}(\boldsymbol{\xi})$ be given by* (4.4.1). *Then there exists a constant $\epsilon_0 > 0$ depending only on (γ, v_2) such*

that, for $j = 5, 6$, *and any* $\epsilon \in (0, \epsilon_0)$,

$$\begin{aligned}
\hat{\Omega}^j_\epsilon &\coloneqq \mathcal{N}_{\epsilon_0}(\Gamma^{j,\delta_0}_{\text{sonic}}) \cap \left\{\boldsymbol{\xi} = \mathcal{R}^{-1}(x,y) \,:\, x_{P_{j-3}} < x < x_{P_{j-3}} + \epsilon\right\} \cap \Omega(u,\boldsymbol{\theta}) \\
&= \left\{\boldsymbol{\xi} = \mathcal{R}^{-1}(x,y) \,:\, x_{P_{j-3}} < x < x_{P_{j-3}} + \epsilon,\, 0 < y < \hat{f}_{j,\text{sh}}(x)\right\}, \\
\Gamma_{\text{shock}}(u,\boldsymbol{\theta}) \cap \partial\hat{\Omega}^j_\epsilon &= \left\{\boldsymbol{\xi} = \mathcal{R}^{-1}(x, \hat{f}_{j,\text{sh}}(x)) \,:\, x_{P_{j-3}} < x < x_{P_{j-3}} + \epsilon\right\}, \\
\Gamma^j_{\text{sonic}} \cap \partial\hat{\Omega}^j_\epsilon &= \left\{\boldsymbol{\xi} = \mathcal{R}^{-1}(x_{P_{j-3}}, y) \,:\, 0 < y < \hat{f}_{j,\text{sh}}(x_{P_{j-3}})\right\} = \Gamma^j_{\text{sonic}}, \\
\Gamma_{\text{sym}} \cap \partial\hat{\Omega}^j_\epsilon &= \left\{\boldsymbol{\xi} = \mathcal{R}^{-1}(x,0) \,:\, x_{P_{j-3}} < x < x_{P_{j-3}} + \epsilon\right\},
\end{aligned}$$

where $\hat{f}_{j,\text{sh}}(x) \coloneqq \mathfrak{g}^{(u,\boldsymbol{\theta})}_{\text{sh}} \circ L_{\boldsymbol{\theta}}(u_j + (-1)^{j-1}(c_j - x))$ *and* $L_{\boldsymbol{\theta}}$ *is given by* (4.4.6).

(iii) *Let* $(u,\boldsymbol{\theta}), (\tilde{u},\tilde{\boldsymbol{\theta}}) \in \mathfrak{S}^{\theta_*}_\alpha$ *satisfy* $\|(u,\tilde{u})\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} < M$ *for some constant* $M > 0$. *Then there exists a constant* $C > 0$ *depending only on* $(\gamma, v_2, \theta_*, M, \alpha)$ *such that*

$$\begin{aligned}
&\|\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\text{sh}}\|_{C^{1,\alpha}([-1,1])} + \|\mathfrak{F}_{(u,\boldsymbol{\theta})}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} \le C, \\
&\|\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\text{sh}} - \mathfrak{g}^{(\tilde{u},\tilde{\boldsymbol{\theta}})}_{\text{sh}}\|_{C^{1,\alpha}([-1,1])} + \|\mathfrak{F}_{(u,\boldsymbol{\theta})} - \mathfrak{F}_{(\tilde{u},\tilde{\boldsymbol{\theta}})}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} \\
&\quad + \|\varphi^{(u,\boldsymbol{\theta})} \circ \mathfrak{F}_{(u,\boldsymbol{\theta})} - \varphi^{(\tilde{u},\tilde{\boldsymbol{\theta}})} \circ \mathfrak{F}_{(\tilde{u},\tilde{\boldsymbol{\theta}})}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} + \|\varphi^*_{\boldsymbol{\theta}} \circ \mathfrak{F}_{(u,\boldsymbol{\theta})} - \varphi^*_{\tilde{\boldsymbol{\theta}}} \circ \mathfrak{F}_{(\tilde{u},\tilde{\boldsymbol{\theta}})}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} \\
&\qquad\qquad \le C\big(\|u - \tilde{u}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} + |\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}|\big).
\end{aligned}$$

(iv) *Let* $\epsilon_0 > 0$ *be the constant from* (ii), *and let* $\hat{\epsilon}_0 \coloneqq d_L\epsilon_0$ *with* $d_L > 0$ *given by* (4.4.6). *Assume that, for constants* $\alpha \in (0,1)$, $\sigma \in (1,2]$, *and* $M > 0$,

$$\|u\|_{2,\alpha,\mathcal{Q}^{\text{iter}} \cap \{|s| < 1 - \frac{\hat{\epsilon}_0}{10}\}} + \|u\|^{(\sigma),(\text{par})}_{2,\alpha,\mathcal{Q}^{\text{iter}} \cap \{|s| > 1 - \hat{\epsilon}_0\}} \le M. \tag{4.4.19}$$

Then there exists a constant $C > 0$ *depending only on* $(\gamma, v_2, \theta_*, \alpha, \sigma)$ *such that*

$$\|\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\text{sh}}\|_{2,\alpha,[-1+\frac{\hat{\epsilon}_0}{10}, 1-\frac{\hat{\epsilon}_0}{10}]} + \|\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\text{sh}} - \mathfrak{g}_5\|^{(\sigma),(\text{par})}_{2,\alpha,(1,1-\hat{\epsilon}_0)} + \|\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\text{sh}} - \mathfrak{g}_6\|^{(\sigma),(\text{par})}_{2,\alpha,(-1,-1+\hat{\epsilon}_0)} \le CM,$$

where $\mathfrak{g}_5$ *and* $\mathfrak{g}_6$ *are defined in* Proposition 4.29(ii).

Furthermore, for $I_{\text{so}} \coloneqq (-1, -1+\hat{\epsilon}_0) \cup (1-\hat{\epsilon}_0, 1)$, *define* $\mathfrak{F}_{(0,\boldsymbol{\theta})} : I_{\text{so}} \times (0,\infty) \to \mathbf{R}^2$

by

$$\mathfrak{F}_{(0,\boldsymbol{\theta})}(s,t') \coloneqq \begin{cases} (G_{2,\mathfrak{g}_5} \circ \mathcal{G}_1^{\boldsymbol{\theta}})^{-1}(s,t') & \text{if } s \in I_{\mathrm{so}} \cap \{s > 0\}\,, \\ (G_{2,\mathfrak{g}_6} \circ \mathcal{G}_1^{\boldsymbol{\theta}})^{-1}(s,t') & \text{if } s \in I_{\mathrm{so}} \cap \{s < 0\}\,. \end{cases}$$

Then there exists a constant $C_0 > 0$ depending only on (γ, v_2, θ_*) such that

$$\|\mathfrak{F}_{(0,\boldsymbol{\theta})}\|_{C^3(\overline{I_{\mathrm{so}}}\times[0,1])} + \|\mathfrak{F}_{(u,\boldsymbol{\theta})}\|_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}\cap\{|s|<1-\frac{\hat{\epsilon}_0}{10}\}} + \|\mathfrak{F}_{(u,\boldsymbol{\theta})} - \mathfrak{F}_{(0,\boldsymbol{\theta})}\|^{(\sigma),(\mathrm{par})}_{2,\alpha,I_{\mathrm{so}}\times(0,1)} \le C_0\,.$$

(v) *Let $f_{\boldsymbol{\theta}}$ be given by* (4.4.8)*. For some constants $M > 0$ and $\delta_{\mathrm{sh}} > 0$, assume that $(u,\boldsymbol{\theta}) \in \mathfrak{G}_\alpha^{\theta_*}$ satisfies* (4.4.19) *and, for any $s \in [-1,1]$,*

$$\begin{aligned} \min\big\{\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}(-1) + \frac{s+1}{M},\, \mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}(1) + \frac{1-s}{M},\, \delta_{\mathrm{sh}}\big\} &\le \mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}(s) \\ &\le \min\big\{\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}(-1) + M(s+1),\, \mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}(1) + M(1-s),\, f_{\boldsymbol{\theta}}(s) - \frac{1}{M}\big\}\,. \end{aligned}$$

Then, for any $\epsilon \in (0, \frac{1}{4}\min\{s_5, -s_6\})$, there exists a constant $C_\epsilon > 0$ depending only on $(\gamma, v_2, \theta_*, \alpha, \delta_{\mathrm{sh}}, \epsilon, M)$ such that

$$\|(\mathfrak{F}^{-1}_{(u,\boldsymbol{\theta})}, \varphi - \varphi^*_{\boldsymbol{\theta}})\|_{2,\alpha,\Omega(u,\boldsymbol{\theta})\setminus(\overline{\hat{\Omega}^5_\epsilon\cup\hat{\Omega}^6_\epsilon})} \le C_\epsilon\,.$$

(vi) *Let $(u,\boldsymbol{\theta}), (\tilde{u},\tilde{\boldsymbol{\theta}}) \in \mathfrak{G}_\alpha^{\theta_*}$ satisfy $\|(u,\tilde{u})\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}})} < M$ for some constant $M > 0$. For any open set $K \Subset \mathcal{Q}^{\mathrm{iter}}$ with $\delta = \mathrm{dist}(K, \overline{\mathcal{Q}^{\mathrm{iter}}} \cap \{|s| = 1\})$, there exists a constant $C_\delta > 0$ depending only on $(\gamma, v_2, \theta_*, \alpha, \sigma, \delta, M)$ such that*

$$\begin{aligned} &\|\mathfrak{F}_{(u,\boldsymbol{\theta})} - \mathfrak{F}_{(\tilde{u},\tilde{\boldsymbol{\theta}})}\|_{C^{2,\alpha}(\overline{K})} \le C_\delta\big(\|(u-\tilde{u})(\cdot,1)\|_{C^{2,\alpha}([-1+\delta,1-\delta])} + |\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}|\big)\,, \\ &\|(\varphi^{(u,\boldsymbol{\theta})} \circ \mathfrak{F}_{(u,\boldsymbol{\theta})} - \varphi^{(\tilde{u},\tilde{\boldsymbol{\theta}})} \circ \mathfrak{F}_{(\tilde{u},\tilde{\boldsymbol{\theta}})},\, \psi^{(u,\boldsymbol{\theta})} \circ \mathfrak{F}_{(u,\boldsymbol{\theta})} - \psi^{(\tilde{u},\tilde{\boldsymbol{\theta}})} \circ \mathfrak{F}_{(\tilde{u},\tilde{\boldsymbol{\theta}})})\|_{C^{2,\alpha}(\overline{K})} \\ &\qquad\qquad \le C_\delta\big(\|u-\tilde{u}\|_{C^{2,\alpha}(\overline{K})} + |\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}|\big)\,, \end{aligned}$$

where $\psi^{(u,\boldsymbol{\theta})}$ is given by $\psi^{(u,\boldsymbol{\theta})} \coloneqq \varphi^{(u,\boldsymbol{\theta})} - \varphi^*_{\boldsymbol{\theta}}$ for each $(u,\boldsymbol{\theta}) \in \mathfrak{G}_\alpha^{\theta_*}$.

4.4.2 Definition and properties of the iteration set

For any $\theta_* \in (0, \theta^{\mathrm{d}})$, we now define the iteration set $\mathcal{K} \subseteq C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}}) \times [0, \theta_*]^2$ for some $\alpha \in (0,1)$, as a set of functions satisfying properties similar to those of the admissible solutions. In the definition of $\mathcal{K}$, we construct a nonlinear boundary value problem in a fixed domain and obtain the well-posedness and *a priori* estimates for the solutions.

4.4.2.1 Definition of the iteration set

Before giving the definition of the iteration set, we collect some further useful estimates for admissible solutions.

Lemma 4.36. *Fix $\theta_* \in (0, \theta^{\mathrm{d}})$. For any admissible solution φ corresponding to $\boldsymbol{\theta} \in \overline{\Theta} \cap \{\theta_1, \theta_2 \le \theta_*\}$, denote $\psi := (\varphi - \max\{\varphi_5, \varphi_6\}) \circ \mathcal{R}^{-1}$. Then there exist constants C_1, $\bar{\epsilon} > 0$ depending only on (γ, v_2), and $C_2^* > 0$ depending on (γ, v_2, θ_*) such that, for any $\epsilon \in (0, \bar{\epsilon}]$,*

$$\begin{aligned} &\|\varphi - \varphi_5\|_{C^{0,1}(\overline{\Omega})} + \|\varphi - \varphi_6\|_{C^{0,1}(\overline{\Omega})} \le C_1\,, \\ &|D_{(x,y)}\psi(x,y)| \le C_2^* x \qquad \text{for } (x,y) \in \mathcal{R}\big(\overline{\Omega} \cap (\mathcal{D}_\epsilon^5 \cup \mathcal{D}_\epsilon^6)\big)\,. \end{aligned}$$

Proof. The first result follows directly from (4.2.27), Proposition 4.2, and Lemma 4.12.

Without loss of generality, we assume that $\theta_* \in (\theta^{\mathrm{s}}, \theta^{\mathrm{d}})$. Then, for sufficiently small $\sigma > 0$ and any $\boldsymbol{\theta} \in [0, \theta^{\mathrm{s}} + \sigma] \times [0, \theta_*]$,

$$|D_{(x,y)}\psi(x,y)| \le C_2 x \qquad \text{for } (x,y) \in \mathcal{R}\big(\overline{\Omega} \cap \mathcal{D}_\epsilon^5\big)\,,$$

whenever $\epsilon \in (0, \bar{\epsilon}]$, where $\bar{\epsilon} > 0$ is the minimum of the constants, $\varepsilon > 0$, in Propositions 4.21–4.23.

From the relations $\ell(\rho_5)\cos\theta_{25} = v_2$ and $M_2^{(\theta_1)} = |(-\xi^{P_0^1}, v_2)|$, and Lemmas A.1 and A.3(i)

$$\frac{\mathrm{d}x_{P_0^1}}{\mathrm{d}\theta_1} = \frac{\mathrm{d}}{\mathrm{d}\theta_1}\big(c_5 - (\xi^{P_0^1} - u_5)\big) = \frac{\mathrm{d}}{\mathrm{d}\theta_1}\big(c_5 - \xi^{P_0^1} + v_2 \tan\theta_{25}\big) > 0\,. \tag{4.4.20}$$

Then there exists a positive constant C_2' depending only on (γ, v_2, θ_*) such that, for any $\boldsymbol{\theta} \in [\theta^{\mathrm{s}} + \frac{\sigma}{2}, \theta_*] \times [0, \theta_*]$,

$$|D_{(x,y)}\psi(x,y)| \le C_2' x^{\bar{\alpha}} \le C_2' \big(x_{P_0^1}|_{\theta_1=\theta^{\mathrm{s}}+\frac{\sigma}{2}}\big)^{\bar{\alpha}-1} x \qquad \text{for } (x,y) \in \mathcal{R}\big(\overline{\Omega} \cap \mathcal{D}_\epsilon^5\big)\,,$$

after possibly reducing $\bar{\epsilon}$ further so that $\bar{\epsilon} < r$, for $r > 0$ from Proposition 4.25 and $\bar{\alpha} \in (0, \frac{1}{3}]$ from Proposition 4.33. From the symmetry of the four-shock interaction problem, we can repeat the above discussion for $\psi(x,y)$ in $\mathcal{R}(\overline{\Omega} \cap \mathcal{D}_\epsilon^6)$. □

Define $u^{(\mathrm{norm})} \in C^3(\overline{\mathcal{Q}^{\mathrm{iter}}})$ by (4.4.14) with $\boldsymbol{\theta} = \mathbf{0}$ and $\varphi = \varphi_0$ from §4.2.1.3. Note that $\varphi_{\boldsymbol{\theta}}^* \equiv \varphi_0$ in $Q^{\boldsymbol{\theta}}$ by Definition 4.31 because $\varphi_5 = \varphi_6 = \varphi_0$ when $\boldsymbol{\theta} = \mathbf{0}$, which yields that

$$u^{(\mathrm{norm})} \equiv 0 \qquad \text{in } \mathcal{Q}^{\mathrm{iter}}\,.$$

Let $\bar{\alpha} \in (0, \frac{1}{3}]$ be the constant from Proposition 4.33, $\epsilon_0 > 0$ from Lemma 4.35(ii), and $\bar{\epsilon} > 0$ from Lemma 4.36. For constants $\alpha \in (0, \frac{\bar{\alpha}}{2}]$, δ_1, δ_2, δ_3, $\epsilon \in (0, \frac{1}{2}\min\{\epsilon_0, \bar{\epsilon}\})$, and $N_1 > 1$ to be specified later, we now define the iteration set $\mathcal{K} \subseteq C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}}) \times [0,\theta_*]^2$, where the space $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ is given by Definition 4.32. For simplicity, in the following, we use $\big(\mathfrak{g}_{\mathrm{sh}}, \varphi, \Omega, \Gamma_{\mathrm{shock}}\big)$ to denote $\big(\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}, \varphi^{(u,\boldsymbol{\theta})}, \Omega(u,\boldsymbol{\theta}), \Gamma_{\mathrm{shock}}(u,\boldsymbol{\theta})\big)$ that are given by Definition 4.34.

Definition 4.37 (Iteration set)**.** Fix $\theta_* \in (0, \theta^{\mathrm{d}})$. The iteration set $\mathcal{K} \subseteq C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}}) \times [0,\theta_*]^2$ is the set of all $(u, \boldsymbol{\theta})$ satisfying the following properties.

(i) $\|u - u^{(\mathrm{norm})}\|^{(*)}_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}} < \mathscr{K}_1(\max\{\theta_1,\theta_2\})$ for $\mathscr{K}_1 \in C^{0,1}(\mathbf{R})$ given by

$$\mathscr{K}_1(\theta) = \begin{cases} \delta_1 & \text{if } \theta \le \frac{\delta_1}{N_1}\,, \\ N_0 & \text{if } \theta \ge \frac{2\delta_1}{N_1}\,, \\ \text{linear} & \text{otherwise}\,, \end{cases}$$

with $N_0 \coloneqq \max\{10M, 1\}$ for $M = M(\gamma, v_2, \theta_*) > 0$ from Proposition 4.33.

(ii) $(u, \boldsymbol{\theta}) \in \mathfrak{G}_\alpha^{\theta_*}$, where $\mathfrak{G}_\alpha^{\theta_*}$ is defined by (4.4.18).

(iii) Γ_{shock} satisfies $\operatorname{dist}(\Gamma_{\text{shock}}, B_1(O_2)) > N_2^{-1}$, whilst $\mathfrak{g}_{\text{sh}}$ satisfies $\mathfrak{g}_{\text{sh}}(\pm 1) \geq 0$ and

$$\begin{aligned} &\min_{i=1,2} \big\{ \mathfrak{g}_{\text{sh}}((-1)^{i-1}) + \frac{1+(-1)^i s}{N_3},\ \frac{1}{N_3} \big\} \leq \mathfrak{g}_{\text{sh}}(s) \\ &\qquad \leq \min_{i=1,2} \big\{ \mathfrak{g}_{\text{sh}}((-1)^{i-1}) + N_3(1+(-1)^i s),\ f_{\boldsymbol{\theta}}(s) - \frac{1}{N_3} \big\} \end{aligned} \qquad \text{for all } s \in (-1,1)\,,$$

with $N_2 \coloneqq 2C_{\text{sh}}$ for $C_{\text{sh}} = C_{\text{sh}}(\gamma, v_2) > 0$ from Proposition 4.13, $N_3 \coloneqq 2\hat{k}$ for $\hat{k} = \hat{k}(\gamma, v_2, \theta_*) > 0$ from Proposition 4.29(iv), and $f_{\boldsymbol{\theta}}$ given by (4.4.8).

(iv) Let the coordinates $(x, y) = \mathcal{R}(\boldsymbol{\xi})$ be given by (4.4.1) and, for $r > 0$, let $\mathcal{D}_r^5$ and $\mathcal{D}_r^6$ be given by (4.4.2). Then φ and $\psi \coloneqq (\varphi - \max\{\varphi_5, \varphi_6\}) \circ \mathcal{R}^{-1}$ satisfy

$$\begin{aligned} &\varphi - \max\{\varphi_5, \varphi_6\} > \mathscr{K}_2(\max\{\theta_1, \theta_2\}) && \text{in } \overline{\Omega} \setminus (\mathcal{D}_{\epsilon/10}^5 \cup \mathcal{D}_{\epsilon/10}^6)\,, \\ &\partial_{\boldsymbol{e}_{S_{2j}}}(\varphi_2 - \varphi) < -\mathscr{K}_2(\max\{\theta_1, \theta_2\}) && \text{in } \overline{\Omega} \setminus \mathcal{D}_{\epsilon/10}^j \text{ for } j = 5, 6\,, \end{aligned} \tag{4.4.21}$$

and also

(4.4.22)

$$\begin{aligned} &|\partial_x \psi(x,y)| < \mathscr{K}_3(\theta_{j-4})x && \text{in } \mathcal{R}\big(\overline{\Omega} \cap (\mathcal{D}_{\epsilon_0}^j \setminus \mathcal{D}_{\epsilon/10}^j)\big) \text{ for } j = 5, 6\,, \\ &|\partial_y \psi(x,y)| < N_4 x && \text{in } \mathcal{R}\big(\overline{\Omega} \cap \big((\mathcal{D}_{\epsilon_0}^5 \setminus \mathcal{D}_{\epsilon/10}^5) \cup (\mathcal{D}_{\epsilon_0}^6 \setminus \mathcal{D}_{\epsilon/10}^6)\big)\big)\,, \\ &|D_{(x,y)}\psi| < N_4 \epsilon && \text{in } \mathcal{R}\big(\overline{\Omega} \cap (\overline{\mathcal{D}_\epsilon^5} \cup \overline{\mathcal{D}_\epsilon^6})\big)\,, \\ &\min\{\partial_{\boldsymbol{\nu}}(\varphi_2 - \varphi),\ \partial_{\boldsymbol{\nu}}\varphi\} > \mu_1 && \text{on } \overline{\Gamma_{\text{shock}}}\,, \\ &\|\varphi - \varphi_j\|_{C^{0,1}(\overline{\Omega})} < N_5 && \text{for } j = 5, 6\,, \end{aligned}$$

for $\boldsymbol{e}_{S_{25}}$ and $\boldsymbol{e}_{S_{26}}$ given by (4.2.29), and the unit normal vector $\boldsymbol{\nu}$ on Γ_{shock} pointing into Ω. In the above conditions, the functions $\mathscr{K}_2, \mathscr{K}_3 \in C^{0,1}(\mathbf{R})$ are defined by

$$\mathscr{K}_2(\theta) \coloneqq \delta_2 \min\big\{\theta - \frac{\delta_1}{N_1^2},\ \frac{\delta_1}{N_1^2}\big\}\,, \qquad \mathscr{K}_3(\theta) \coloneqq \begin{cases} \frac{2-\mu_0}{1+\gamma} & \text{if } 0 \leq \theta \leq \theta^{\text{s}} + \frac{\sigma_2}{2}\,, \\ N_4 & \text{if } \theta \geq \theta^{\text{s}} + \sigma_2\,, \\ \text{linear} & \text{otherwise}\,, \end{cases}$$

for the constants μ_1, σ_2, μ_0, ϵ_0, N_4, and N_5 chosen as follows:

- Let $\delta' = \delta'(\gamma, v_2) > 0$ be from Lemma 4.18. Choose $\mu_1 \coloneqq \frac{\delta'}{2}$.
- Let $\sigma_2 = \sigma_2(\gamma, v_2) > 0$ and $\delta = \delta(\gamma, v_2) > 0$ be from Step **1** of the proof of Proposition 4.23. Choose $\mu_0 \coloneqq \frac{\delta}{2}$.
- Let $\epsilon_0 = \epsilon_0(\gamma, v_2) > 0$ be from Lemma 4.35(ii).
- Let $C_1 = C_1(\gamma, v_2) > 0$ and $C_2^* = C_2^*(\gamma, v_2, \theta_*) > 0$ be from Lemma 4.36. Choose $N_4 \coloneqq 10C_2^*$ and $N_5 \coloneqq 10C_1$.

(v) The density function $\rho(|D\varphi|, \varphi)$ defined by (4.2.2) with $B = \frac{1}{2}v_2^2$ satisfies

$$\frac{\rho^*(\gamma)}{2} < \rho(|D\varphi|, \varphi) < 2C_{\text{ub}} \qquad \text{in } \overline{\Omega} \setminus (\mathcal{D}^5_{\epsilon/10} \cup \mathcal{D}^6_{\epsilon/10})$$

for constants $\rho^*(\gamma) > 0$ and $C_{\text{ub}} = C_{\text{ub}}(\gamma, v_2) > 0$ given by (4.3.6) in Lemma 4.12.

(vi) For the sound speed $c(|D\varphi|, \varphi)$ as defined in (4.3.2), the function φ satisfies

$$\frac{|D\varphi(\boldsymbol{\xi})|^2}{c^2(|D\varphi(\boldsymbol{\xi})|, \varphi(\boldsymbol{\xi}))} < 1 - \tilde{\mu}\, \text{dist}^\flat(\boldsymbol{\xi}, \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}) \qquad \text{for } \boldsymbol{\xi} \in \overline{\Omega} \setminus \left(\mathcal{D}^5_{\epsilon/10} \cup \mathcal{D}^6_{\epsilon/10}\right),$$

where $\tilde{\mu} \coloneqq \frac{\mu_{\text{el}}}{2}$, for $\mu_{\text{el}} \coloneqq \frac{\mu}{C_\flat} > 0$ with $C_\flat = C_\flat(\gamma, v_2) > 0$ from (4.3.16) and $\mu = \mu(\gamma, v_2) > 0$ from Proposition 4.17; and $\text{dist}^\flat(\cdot, \cdot)$ is defined in (4.3.17).

(vii) The iteration boundary value problem

$$(4.4.23) \quad \begin{cases} \mathcal{N}_{(u,\boldsymbol{\theta})}(\hat{\phi}) = A_{11}\hat{\phi}_{\xi\xi} + 2A_{12}\hat{\phi}_{\xi\eta} + A_{22}\hat{\phi}_{\eta\eta} = 0 & \text{in } \Omega\,, \\ \mathcal{M}_{(u,\boldsymbol{\theta})}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi}) = 0 & \text{on } \Gamma_{\text{shock}}\,, \\ \hat{\phi} = \max\{\varphi_5, \varphi_6\} + \frac{1}{2}|\boldsymbol{\xi}|^2 & \text{on } \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}\,, \\ \partial_\eta \hat{\phi} = 0 & \text{on } \Gamma_{\text{wedge}}\,, \end{cases}$$

has a unique solution $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega})$, where the nonlinear operators $\mathcal{N}_{(u,\boldsymbol{\theta})}$

and $\mathcal{M}_{(u,\boldsymbol{\theta})}$ are determined later. Moreover, under the transformation

$$\hat{u} \coloneqq (\hat{\phi} - \frac{1}{2}|\boldsymbol{\xi}|^2 - \varphi^*_{\boldsymbol{\theta}}) \circ \mathfrak{F}_{(u,\boldsymbol{\theta})} \qquad \text{in } \mathcal{Q}^{\text{iter}}\,, \tag{4.4.24}$$

the solution $\hat{\phi}$ satisfies the estimate

$$\|\hat{u} - u\|^{(*)}_{2,\alpha,\mathcal{Q}^{\text{iter}}} < \delta_3\,. \tag{4.4.25}$$

Definition 4.38 (Extended iteration set)**.** Define the extended iteration set $\mathcal{K}^{\text{ext}}$ as

$$\mathcal{K}^{\text{ext}} \coloneqq \big\{(u,\boldsymbol{\theta}) \in C^{2,\alpha}_{(*)}(\mathcal{Q}^{\text{iter}}) \times [0,\theta_*]^2 \,:\, (u,\boldsymbol{\theta}) \text{ satisfies Definition 4.37(i)–(vi)}\big\}\,.$$

Write $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}^{\text{ext}}}$ for the closures of $\mathcal{K}$ and $\mathcal{K}^{\text{ext}}$ in $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\text{iter}}) \times [0,\theta_*]^2$ respectively.

To complete Definition 4.37, it remains to construct a suitable iteration boundary value problem (4.4.23) through the definition of the nonlinear differential operators $\mathcal{N}_{(u,\boldsymbol{\theta})}$ and $\mathcal{M}_{(u,\boldsymbol{\theta})}$ for each $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\text{ext}}}$. The construction of these operators is similar to the construction in [1, §4.4], so we present only the main details.

In the following, we write $\Gamma_{\text{sonic}} \coloneqq \Gamma^5_{\text{sonic}} \cup \Gamma^6_{\text{sonic}}$ and $\mathcal{D}_r \coloneqq \mathcal{D}^5_r \cup \mathcal{D}^6_r$ for any $r > 0$ with $\mathcal{D}^5_r$ and $\mathcal{D}^6_r$ given by (4.4.2), respectively. Moreover, for any $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\text{ext}}}$, we define

$$\phi \coloneqq \varphi + \frac{1}{2}|\boldsymbol{\xi}|^2\,, \qquad \psi \coloneqq (\varphi - \varphi^*_{\boldsymbol{\theta}}) \circ \mathcal{R}^{-1} \equiv (\phi - \phi^*_{\boldsymbol{\theta}}) \circ \mathcal{R}^{-1}\,,$$

where $\varphi = \varphi^{(u,\boldsymbol{\theta})}$ is given by Definition 4.34, $\varphi^*_{\boldsymbol{\theta}}$ is given by Definition 4.31, $\phi^*_{\boldsymbol{\theta}} \coloneqq \varphi^*_{\boldsymbol{\theta}} + \frac{1}{2}|\boldsymbol{\xi}|^2$, and the coordinates $(x,y) = \mathcal{R}(\boldsymbol{\xi})$ are given by (4.4.1).

Construction of $\mathcal{N}_{(u,\boldsymbol{\theta})}$ in (4.4.23). For $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\text{ext}}}$, the nonlinear operator $\mathcal{N}_{(u,\boldsymbol{\theta})}$ in (4.4.23) is defined by

$$\mathcal{N}_{(u,\boldsymbol{\theta})}(\hat{\phi}) \coloneqq \sum_{i,j=1}^{2} A_{ij}(D\hat{\phi},\boldsymbol{\xi})\partial_i\partial_j\hat{\phi} \qquad \text{for any } \hat{\phi} \in C^2(\Omega)\,, \tag{4.4.26}$$

with $(\partial_1,\partial_2) \coloneqq (\partial_\xi,\partial_\eta)$ and the coefficients $A_{ij}(\boldsymbol{p},\boldsymbol{\xi})$, for $i,j = 1,2$, determined

below such that the equation $\mathcal{N}_{(u,\boldsymbol{\theta})}(\hat{\phi}) = 0$ is strictly elliptic in $\overline{\Omega} \setminus \Gamma_{\text{sonic}}$ and, if ϕ solves (4.4.23), then $\mathcal{N}_{(u,\boldsymbol{\theta})}(\phi) = 0$ coincides with (4.3.1). The coefficients $A_{ij}(\boldsymbol{p}, \boldsymbol{\xi})$ are constructed similarly to [1, §4.4.1], the main difference being that both sonic boundaries may now degenerate into single points. We briefly describe the construction in the following four steps and highlight where the differences arise.

1. Away from Γ_{sonic}. Let $\epsilon_0 > 0$ be the constant from Lemma 4.35(ii), and let $\epsilon_{\text{eq}} \in (0, \frac{\epsilon_0}{2})$ be chosen later. Away from Γ_{sonic}, the coefficients $A_{ij}(\boldsymbol{p}, \boldsymbol{\xi})$ are independent of $\boldsymbol{p}$. Indeed, for $i, j = 1, 2$, and $\boldsymbol{\xi} \in \Omega \setminus \mathcal{D}_{\epsilon_{\text{eq}}/10}$, we define $A_{ij}^{(1)}(\boldsymbol{\xi})$ by

$$A_{ij}^{(1)}(\boldsymbol{\xi}) \coloneqq A_{ij}^{\text{pot}}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \,, \tag{4.4.27}$$

where $A_{ij}^{\text{pot}}(\boldsymbol{p}, z, \boldsymbol{\xi}) \coloneqq \delta_{ij} c^2 - (p_i - \boldsymbol{\xi} \cdot \boldsymbol{e}_i)(p_j - \boldsymbol{\xi} \cdot \boldsymbol{e}_j)$ for $c = c(|\boldsymbol{p} - \boldsymbol{\xi}|, z - \frac{1}{2}|\boldsymbol{\xi}|^2)$ given in Definition 4.37(vi), where $\boldsymbol{e}_1 \coloneqq (1, 0)$ and $\boldsymbol{e}_2 \coloneqq (0, 1)$.

2. Supersonic near Γ_{sonic}. For the constant $\mu_0 > 0$ from Definition 4.37(iv), fix an odd function of a single variable $\zeta_1 \in C^3(\mathbf{R})$ such that, for all $s \in \mathbf{R}$,

$$\zeta_1(s) = \begin{cases} s & \text{if } |s| \le \frac{2 - \frac{\mu_0}{5}}{1+\gamma} \,, \\ \frac{(2 - \frac{\mu_0}{10}) \operatorname{sgn}(s)}{1+\gamma} & \text{if } |s| > \frac{2}{1+\gamma} \,, \end{cases} \quad 0 \le \zeta_1'(s) \le 10 \,, \quad -\frac{20(1+\gamma)}{\mu_0} \le \zeta_1''(|s|) \le 0 \,.$$

For any $\hat{\phi} \in C^2(\Omega)$, set

$$\hat{\psi} \coloneqq \big(\hat{\phi} - (\varphi_{\boldsymbol{\theta}}^* + \frac{1}{2}|\boldsymbol{\xi}|^2)\big) \circ \mathcal{R}^{-1} \equiv \big(\hat{\phi} - \phi_{\boldsymbol{\theta}}^*\big) \circ \mathcal{R}^{-1} \qquad \text{in } \mathcal{R}(\Omega \cap \mathcal{D}_{2\epsilon_{\text{eq}}}) \,,$$

and define the nonlinear operator $\mathcal{N}_{(u,\boldsymbol{\theta})}^{\text{polar}}(\hat{\phi})$ in $\Omega \cap \mathcal{D}_{2\epsilon_{\text{eq}}}$ by

$$\mathcal{N}_{(u,\boldsymbol{\theta})}^{\text{polar}}(\hat{\phi}) \coloneqq \Big(\big(2x - (\gamma+1)x\, \zeta_1(\frac{\hat{\psi}_x}{x}) + \mathcal{O}_1^{\text{mod}}\big)\hat{\psi}_{xx} + \mathcal{O}_2^{\text{mod}} \hat{\psi}_{xy} + \big(\frac{1}{c_{\boldsymbol{\theta}}} + \mathcal{O}_3^{\text{mod}}\big)\hat{\psi}_{yy} \\ - \big(1 + \mathcal{O}_4^{\text{mod}}\big)\hat{\psi}_x + \mathcal{O}_5^{\text{mod}} \hat{\psi}_y \Big) \circ \mathcal{R} \,,$$

with the modified remainder terms $\mathcal{O}_j^{\rm mod}$, for $j = 1, \cdots, 5$, given by

$$\mathcal{O}_j^{\rm mod} := \mathcal{O}_j\big(x^{3/4}\zeta_1\big(\frac{\hat{\psi}_x}{x^{3/4}}\big), N_4(\gamma+1)x\,\zeta_1\big(\frac{\hat{\psi}_y}{N_4(\gamma+1)x}\big), \psi(x,y), x, c_{\boldsymbol{\theta}}\big)\,,$$

where $\mathcal{O}_j(p_x, p_y, z, x, c)$ are given by (4.3.25), the constant $N_4 > 0$ is from Definition 4.37(iv), and $c_{\boldsymbol{\theta}} := c_5(1-\chi^*_{\boldsymbol{\theta}}) + c_6\chi^*_{\boldsymbol{\theta}}$ with the function $\chi^*_{\boldsymbol{\theta}}$ from Definition 4.31.

For $(\boldsymbol{p}, \boldsymbol{\xi}) \in \mathbf{R}^2 \times (\Omega \cap \mathcal{D}_{2\epsilon_{\rm eq}})$, the coefficients $A_{ij}^{(2)}(\boldsymbol{p}, \boldsymbol{\xi})$, for $i, j = 1, 2$, are defined to be the second-order coefficients of the operator $c_{\boldsymbol{\theta}}\mathcal{N}^{\rm polar}_{(u,\boldsymbol{\theta})}(\hat{\phi})$ in the $\boldsymbol{\xi}$–coordinates, that is,

$$c_{\boldsymbol{\theta}}\mathcal{N}^{\rm polar}_{(u,\boldsymbol{\theta})}(\hat{\phi}) =: \sum_{i,j=1}^{2} A_{ij}^{(2)}(D_{\boldsymbol{\xi}}\hat{\phi}, \boldsymbol{\xi})\partial_i\partial_j\hat{\phi} \qquad \text{in } \Omega \cap \mathcal{D}_{2\epsilon_{\rm eq}}\,.$$

In particular, there are no lower order terms in the above expression, since they vanish identically in $\Omega \cap \mathcal{D}_{2\epsilon_{\rm eq}}$, which can be verified by direct computation.

3. Subsonic cut-off near $\Gamma_{\rm sonic}$. For any $\sigma \in (0,1)$, there exists a constant $C_\sigma > 0$ depending only on $(\gamma, v_2, \theta_*, \sigma)$ such that, for any $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\rm ext}}$, there is a function $v_\sigma^{(u,\boldsymbol{\theta})} \in C^4(\overline{\Omega})$ with

(a) $\|v_\sigma^{(u,\boldsymbol{\theta})} - \phi\|_{C^1(\overline{\Omega})} \le \sigma^2$ and $\|v_\sigma^{(u,\boldsymbol{\theta})}\|_{C^4(\overline{\Omega})} \le C_\sigma$;

(b) For any sequence $\{(u_k, \boldsymbol{\theta}_k)\} \subset \overline{\mathcal{K}^{\rm ext}}$ converging to $(u, \boldsymbol{\theta})$ in $C^{1,\alpha}(\overline{\mathcal{Q}^{\rm iter}}) \times [0, \theta_*]^2$,

$$v_\sigma^{(u_k,\boldsymbol{\theta}_k)} \circ \mathfrak{F}_{(u_k,\boldsymbol{\theta}_k)} \to v_\sigma^{(u,\boldsymbol{\theta})} \circ \mathfrak{F}_{(u,\boldsymbol{\theta})} \qquad \text{in } C^{1,\alpha}(\overline{\mathcal{Q}^{\rm iter}})\,.$$

The proof of this statement can be found in [1, Lemma 4.26].

For any $\sigma > 0$, fix a family of cut-off functions $\varsigma_\sigma \in C^\infty(\mathbf{R})$ satisfying

$$\varsigma_\sigma(t) := \varsigma_1\big(\frac{t}{\sigma}\big) \qquad \text{with } \varsigma_1(t) = \begin{cases} 1 & \text{for } t < 1\,, \\ \\ 0 & \text{for } t > 2\,, \end{cases} \quad \text{and } 0 \le \varsigma_1 \le 1 \text{ on } \mathbf{R}\,. \tag{4.4.28}$$

For a constant $\sigma_{\rm cf} \in (0,1)$ to be specified later, we define the subsonic coefficients

$$A_{ij}^{\rm subs}(\boldsymbol{p},\boldsymbol{\xi}) \coloneqq \varsigma_{\sigma_{\rm cf}} A_{ij}^{\rm pot}(\boldsymbol{p},\phi(\boldsymbol{\xi}),\boldsymbol{\xi}) + (1-\varsigma_{\sigma_{\rm cf}}) A_{ij}^{\rm pot}(Dv_{\sigma_{\rm cf}}^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}),\phi(\boldsymbol{\xi}),\boldsymbol{\xi}) \quad \text{for } i,j=1,2\,,$$

with $\varsigma_{\sigma_{\rm cf}} = \varsigma_{\sigma_{\rm cf}}(|\boldsymbol{p} - Dv_{\sigma_{\rm cf}}^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi})|)$. In particular, from now on, we choose $\sigma_{\rm cf} \coloneqq \sqrt{\delta_1}$.

Fix a cut-off function $\chi_{\rm eq} \in C^\infty(\mathbf{R})$ satisfying

$$\chi_{\rm eq}(\theta) = \begin{cases} 1 & \text{for } \theta \le \theta^{\rm s} + \frac{\sigma_3}{4}\,, \\ 0 & \text{for } \theta \ge \theta^{\rm s} + \frac{\sigma_3}{2}\,, \end{cases} \qquad \chi_{\rm eq}'(\theta) \le 0 \text{ on } \mathbf{R}\,.$$

Then, for $(\boldsymbol{p},\boldsymbol{\xi}) \in \mathbf{R}^2 \times (\Omega \cap \mathcal{D}_{2\epsilon_{\rm eq}})$, the coefficients $A_{ij}^{(3)}(\boldsymbol{p},\boldsymbol{\xi})$, for $i,j=1,2$, are defined by

$$A_{ij}^{(3)}(\boldsymbol{p},\boldsymbol{\xi}) \coloneqq \sum_{l=5,6} \mathbf{1}_{\boldsymbol{\xi}\in\Omega\cap\mathcal{D}^l_{2\epsilon_{\rm eq}}} \left(\chi_{\rm eq}(\theta_{l-4}) A_{ij}^{(2)}(\boldsymbol{p},\boldsymbol{\xi}) + (1-\chi_{\rm eq}(\theta_{l-4})) A_{ij}^{\rm subs}(\boldsymbol{p},\boldsymbol{\xi})\right).$$

4. Coefficients $A_{ij}(\boldsymbol{p},\boldsymbol{\xi})$. We now complete the definition of the coefficients $A_{ij}(\boldsymbol{p},\boldsymbol{\xi})$ for the nonlinear differential operator $\mathcal{N}_{(u,\boldsymbol{\theta})}$ given by (4.4.26). Let ϵ_0 be from Lemma 4.35(ii). For each $\epsilon \in (0, \frac{\epsilon_0}{2})$, fix a family of cut-off functions $\zeta_2^{(\epsilon,\boldsymbol{\theta})} \in C^4(\overline{Q^{\boldsymbol{\theta}}})$ with the following properties:

(a) The function $\zeta_2^{(\epsilon,\cdot)}(\cdot) : (\boldsymbol{\xi},\boldsymbol{\theta}) \mapsto \zeta_2^{(\epsilon,\boldsymbol{\theta})}(\boldsymbol{\xi})$ is continuous on $\bigcup_{\boldsymbol{\theta}\in[0,\theta_*]^2} Q^{\boldsymbol{\theta}} \times \{\boldsymbol{\theta}\}$;

(b) $\|\zeta_2^{(\epsilon,\boldsymbol{\theta})}\|_{C^4(\overline{Q^{\boldsymbol{\theta}}})} \le C_\epsilon$ for some positive constant C_ϵ depending only on (γ, v_2, ϵ);

(c) $\zeta_2^{(\epsilon,\boldsymbol{\theta})}(\boldsymbol{\xi}) = \begin{cases} 1 & \text{for } \boldsymbol{\xi} \in \Omega \setminus \mathcal{D}_\epsilon\,, \\ 0 & \text{for } \boldsymbol{\xi} \in \Omega \cap \mathcal{D}_{\epsilon/2}\,. \end{cases}$

Such a family of cut-off functions is constructed in [1, Definition 4.28].

The coefficients $A_{ij}(\boldsymbol{p},\boldsymbol{\xi})$, for $i,j=1,2$, are defined by

(4.4.29)
$$A_{ij}(\boldsymbol{p},\boldsymbol{\xi}) \coloneqq \zeta_2^{(\epsilon_{\rm eq},\boldsymbol{\theta})}(\boldsymbol{\xi}) A_{ij}^{(1)}(\boldsymbol{\xi}) + (1-\zeta_2^{(\epsilon_{\rm eq},\boldsymbol{\theta})}(\boldsymbol{\xi})) A_{ij}^{(3)}(\boldsymbol{p},\boldsymbol{\xi}) \qquad \text{for } (\boldsymbol{p},\boldsymbol{\xi}) \in \mathbf{R}^2 \times \Omega\,.$$

We have the following lemma concerning the properties of the coefficients A_{ij}; see [1, Lemma 4.30].

Lemma 4.39. *There exist positive constants* $\epsilon^{(1)}$, $\delta_1^{(1)}$, $\epsilon_{\rm eq} \in (0, \frac{\epsilon_0}{2})$, $\lambda_0 \in (0,1)$, $N_{\rm eq} \geq 1$, *and* C *with* $(\epsilon^{(1)}, \delta_1^{(1)}, \lambda_0)$ *depending only on* (γ, v_2), $(\epsilon_{\rm eq}, N_{\rm eq})$ *depending only on* (γ, v_2, θ_*), *and* C *depending only on* $(\gamma, v_2, \theta_*, \alpha)$ *such that, whenever* (ϵ, δ_1) *from* Definition 4.37 *are chosen from* $(0, \epsilon^{(1)}] \times (0, \delta_1^{(1)}]$, *for each* $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\rm ext}}$, *the coefficients* $A_{ij}(\boldsymbol{p}, \boldsymbol{\xi})$ *of* $\mathcal{N}_{(u,\boldsymbol{\theta})}$ *given by* (4.4.29) *satisfy the following properties.*

(i) $A_{12}(\boldsymbol{p}, \boldsymbol{\xi}) = A_{21}(\boldsymbol{p}, \boldsymbol{\xi})$ *for all* $(\boldsymbol{p}, \boldsymbol{\xi}) \in \mathbf{R}^2 \times \Omega$, *and* $A_{ij}(\boldsymbol{p}, \boldsymbol{\xi}) = A_{ij}^{(1)}(\boldsymbol{\xi})$ *if* $\boldsymbol{\xi} \in \Omega \setminus \mathcal{D}_{\epsilon_{\rm eq}}$. *Furthermore,* $D_{\boldsymbol{p}}^m A_{ij} \in C^{1,\alpha}(\mathbf{R}^2 \times (\overline{\Omega} \setminus \Gamma_{\rm sonic}))$ *for* $m = 0, 1, 2$,

$$\lambda_0 \operatorname{dist}(\boldsymbol{\xi}, \Gamma_{\rm sonic}) |\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\boldsymbol{p}, \boldsymbol{\xi}) \kappa_i \kappa_j \leq \lambda_0^{-1} |\boldsymbol{\kappa}|^2$$

for any $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbf{R}^2$ *and for any* $(\boldsymbol{p}, \boldsymbol{\xi}) \in \mathbf{R}^2 \times \Omega$, *and*

$$\|A_{ij}\|_{L^\infty(\mathbf{R}^2 \times \Omega)} + \|A_{ij}(\boldsymbol{p}, \cdot)\|_{C^{3/4}(\overline{\Omega})} + \|D_{\boldsymbol{p}} A_{ij}(\boldsymbol{p}, \cdot)\|_{L^\infty(\Omega)} \leq N_{\rm eq} \qquad \text{for any } \boldsymbol{p} \in \mathbf{R}^2,$$
$$\|A_{ij}\|_{C^{1,\alpha}(\mathbf{R}^2 \times \overline{\Omega \setminus \mathcal{D}_{\epsilon_{\rm eq}}})} + s^5 \|D_{\boldsymbol{p}}^m A_{ij}\|_{C^{1,\alpha}(\mathbf{R}^2 \times (\overline{\Omega} \setminus \mathcal{N}_s(\Gamma_{\rm sonic})))} \leq C \qquad \text{for each } s \in (0, \frac{\epsilon_0}{2}).$$

(ii) $A_{ij}(\boldsymbol{p}, \boldsymbol{\xi}) = A_{ij}^{(3)}(\boldsymbol{p}, \boldsymbol{\xi})$ *for any* $(\boldsymbol{p}, \boldsymbol{\xi}) \in \mathbf{R}^2 \times (\Omega \cap \mathcal{D}^l_{\epsilon_{\rm eq}/2})$, *and* $l = 5, 6$. *Furthermore, if* $\theta_{l-4} \leq \theta^{\rm s} + \frac{\sigma_3}{4}$, *then* $A_{ij}(\boldsymbol{p}, \boldsymbol{\xi}) = A_{ij}^{(2)}(\boldsymbol{p}, \boldsymbol{\xi})$; *whereas, if* $\theta_{l-4} \in [\theta^{\rm s} + \delta, \theta_*]$ *for some* $\delta \in (0, \frac{\sigma_3}{2})$, *then*

$$\lambda_0 (\operatorname{dist}(\boldsymbol{\xi}, \Gamma_{\rm sonic}) + \delta) |\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\boldsymbol{p}, \boldsymbol{\xi}) \kappa_i \kappa_j \leq \lambda_0^{-1} |\boldsymbol{\kappa}|^2 \qquad \text{for any } \boldsymbol{\xi} \in \Omega \cap \mathcal{D}^l_{\epsilon_{\rm eq}/2},$$
$$\sup_{\boldsymbol{p} \in \mathbf{R}^2} \|D_{\boldsymbol{p}}^m A_{ij}(\boldsymbol{p}, \cdot)\|_{C^{1,\alpha}(\overline{\Omega \cap \mathcal{D}^l_{\epsilon_{\rm eq}/2}})} \leq C \qquad \text{for } m = 0, 1, 2.$$

(iii) *Suppose that* ϵ *from* Definition 4.37 *satisfies* $\epsilon \in (0, \frac{\epsilon_{\rm eq}}{2})$. *Then the equation* $\mathcal{N}_{(u,\boldsymbol{\theta})}(\phi) = 0$ *coincides with* (4.3.1) *in* $\Omega \setminus \mathcal{D}_{\epsilon/10}$. *In addition, for* $l = 5, 6$, *if* $x_{P_0^{l-4}} \geq \frac{\epsilon}{10}$ *or* $\theta_{l-4} \geq \theta^{\rm s} + \frac{\sigma_3}{2}$, *then the equation* $\mathcal{N}_{(u,\boldsymbol{\theta})}(\phi) = 0$ *coincides with* (4.3.1) *in* $\Omega \setminus \mathcal{D}^{11-l}_{\epsilon/10}$.

Construction of $\mathcal{M}_{(u,\boldsymbol{\theta})}$ in (4.4.23). We turn our attention to the definition of the nonlinear linear operator $\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p}, z, \boldsymbol{\xi})$ in (4.4.23). The construction is presented in the following four steps that follow closely [1, §4.4.2].

1. For g^{sh} given by (4.3.26), set

$$\mathcal{M}_0(\boldsymbol{p}, z, \boldsymbol{\xi}) \coloneqq g^{\mathrm{sh}}(\boldsymbol{p} - \boldsymbol{\xi}, z - \frac{1}{2}|\boldsymbol{\xi}|^2, \boldsymbol{\xi}), \qquad \mathcal{M}_1(\boldsymbol{p}, z, \xi) \coloneqq \mathcal{M}_0(\boldsymbol{p}, z, (\xi, \frac{z}{v_2})).$$

For $j = 5, 6$, write $\phi_j \coloneqq \varphi_j + \frac{1}{2}|\boldsymbol{\xi}|^2$, which is independent of the η–coordinate since $O_j \in \{\eta = 0\}$. Then $(\xi, \frac{1}{v_2}\phi_j(\boldsymbol{\xi})) \in \{\varphi_j = \varphi_2\}$ for any $\boldsymbol{\xi} \in \mathbf{R}^2$, and $\mathcal{M}_1$ is homogeneous in the sense that

$$\mathcal{M}_1(D\phi_j(\boldsymbol{\xi}), \phi_j(\boldsymbol{\xi}), \xi) = 0 \qquad \text{for all } \boldsymbol{\xi} = (\xi, \eta) \in \mathbf{R}^2 \text{ and } j = 5, 6.$$

Furthermore, observe that $\boldsymbol{\xi} = (\xi, \eta) \in \Gamma_{\mathrm{shock}}$ if and only if $\boldsymbol{\xi} \in \overline{\Omega}$ with $\eta = \frac{1}{v_2}\phi(\boldsymbol{\xi})$, and hence

$$\text{(4.4.30)} \qquad \begin{cases} \mathcal{M}_1(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \xi) = \mathcal{M}_0(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}), \\ D_{\boldsymbol{p}}\mathcal{M}_1(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \xi) = D_{\boldsymbol{p}}\mathcal{M}_0(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}), \end{cases} \qquad \text{for } \boldsymbol{\xi} \in \Gamma_{\mathrm{shock}}.$$

2. For the cut-off function ς_σ given by (4.4.28) and a constant $\sigma_{\mathrm{bc}} > 0$ to be determined later, we set

$$\text{(4.4.31)} \qquad \begin{aligned} \mathcal{M}(\boldsymbol{p}, z, \boldsymbol{\xi}) \coloneqq \big(\varsigma^{(5)}_{\sigma_{\mathrm{bc}}}\varsigma^{(6)}_{\sigma_{\mathrm{bc}}} + (1 - \varsigma^{(5)}_{\sigma_{\mathrm{bc}}})\varsigma^{(6)}_{\sigma_{\mathrm{bc}}} + \varsigma^{(5)}_{\sigma_{\mathrm{bc}}}(1 - \varsigma^{(6)}_{\sigma_{\mathrm{bc}}})\big)\mathcal{M}_1(\boldsymbol{p}, z, \xi) \\ + (1 - \varsigma^{(5)}_{\sigma_{\mathrm{bc}}})(1 - \varsigma^{(6)}_{\sigma_{\mathrm{bc}}})\mathcal{M}_0(\boldsymbol{p}, z, \boldsymbol{\xi}), \end{aligned}$$

where $\varsigma^{(j)}_{\sigma_{\mathrm{bc}}} \coloneqq \varsigma_{\sigma_{\mathrm{bc}}}(|(\boldsymbol{p}, z) - (D\phi_j, \phi_j(\boldsymbol{\xi}))|)$ for $j = 5, 6$. There exist positive constants σ_{bc}, δ_{bc}, and $\bar{\epsilon}_{\mathrm{bc}}$ depending only on (γ, v_2) such that, if ϵ from Definition 4.37 satisfies $\epsilon \in (0, \bar{\epsilon}_{\mathrm{bc}})$, then $\mathcal{M}(\boldsymbol{p}, z, \boldsymbol{\xi})$ satisfies that, for all $\boldsymbol{\xi} \in \Gamma_{\mathrm{shock}}$,

$$\delta_{\mathrm{bc}} \le D_{\boldsymbol{p}}\mathcal{M}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}) \le \delta_{\mathrm{bc}}^{-1}, \qquad D_z\mathcal{M}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \le -\delta_{\mathrm{bc}},$$

where $\boldsymbol{\nu}_{\rm sh}$ is the unit normal vector to $\Gamma_{\rm shock}$ pointing into Ω. These results follow from the calculations in [1, Lemma 4.32]. Indeed, the first follows from a direct calculation by using (4.4.30), [1, Eq. A.18], the fact that $\mathcal{M}_0$ is C^1, and the properties in Definition 4.37(v)–(vi). For the second, direct computation gives

$$D_z\mathcal{M}_0(D\phi(\boldsymbol{\xi}),\phi(\boldsymbol{\xi}),\boldsymbol{\xi}) = -\rho^{2-\gamma}D\varphi\cdot\boldsymbol{\nu}_{\rm sh}(\boldsymbol{\xi}) < 0 \qquad \text{for } \boldsymbol{\xi}\in\Gamma_{\rm shock}\,,$$

whilst, for $j=5,6$, and any $\boldsymbol{\xi}=(\xi,\eta)\in Q^{\boldsymbol{\theta}}$, we have

$$D_z\mathcal{M}_1(D\phi_j(\boldsymbol{\xi}),\phi_j(\boldsymbol{\xi}),\xi) = -\rho_j^{2-\gamma}\,\mathrm{dist}(O_j,S_{2j}) + (-1)^j\frac{\rho_j-1}{v_2}\cos\theta_{2j} < 0\,,$$

after which an argument similar to [1, Lemma 4.32] applies. The constant $\sigma_{\rm bc}>0$ is now fixed.

3. For $j=5,6$, set

$$\mathcal{M}^{(j)}(\boldsymbol{q},z,\boldsymbol{\xi}) \coloneqq \mathcal{M}(\boldsymbol{q}+D\phi_j, z+\phi_j(\boldsymbol{\xi}),\boldsymbol{\xi})\,. \tag{4.4.32}$$

Then, for the coordinates $(x,y)=\boldsymbol{x}=\mathcal{R}(\boldsymbol{\xi})$ given by (4.4.1), define

(4.4.33)
$$\hat{\mathcal{M}}^{(j)}(q_x,q_y,z,\boldsymbol{x}) \coloneqq \mathcal{M}^{(j)}\big((-1)^j(q_x\cos y+\frac{q_y\sin y}{c_j-x}), -q_x\sin y+\frac{q_y\cos y}{c_j-x}, z, \mathcal{R}^{-1}(\boldsymbol{x})\big)\,.$$

4. Finally, we extend the definition of $\mathcal{M}$ in (4.4.31) to all $(\boldsymbol{p},z,\boldsymbol{\xi})\in\mathbf{R}^2\times\mathbf{R}\times\overline{\Omega}$. For each $(u,\boldsymbol{\theta})\in\overline{\mathcal{K}^{\rm ext}}$ and a constant $\sigma>0$, let $v_\sigma^{(u,\boldsymbol{\theta})}\in C^4(\overline{\Omega})$ be given as in Step **3** in the construction of $\mathcal{N}_{(u,\boldsymbol{\theta})}$ above. We define a linear operator $\mathcal{L}_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi})$ by

$$\begin{aligned}\mathcal{L}_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi}) \coloneqq\ & \mathcal{M}(Dv_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}), v_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}),\boldsymbol{\xi}) + D_{\boldsymbol{p}}\mathcal{M}(Dv_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}), v_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}),\boldsymbol{\xi})\cdot\boldsymbol{p}\\ &+ D_z\mathcal{M}(Dv_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}), v_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}),\boldsymbol{\xi})z\,,\end{aligned}$$

The nonlinear operator $\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi})$ in (4.4.23) is then defined by

(4.4.34)
$$\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi}) \coloneqq \varsigma_\sigma \mathcal{M}(\boldsymbol{p},z,\boldsymbol{\xi}) + (1-\varsigma_\sigma)\mathcal{L}_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{p} - Dv_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}), z - v_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}), \boldsymbol{\xi})$$

for $\varsigma_\sigma \coloneqq \varsigma_\sigma(|(\boldsymbol{p},z) - (Dv_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}), v_\sigma^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}))|)$ given by (4.4.28). We have the following lemma concerning the properties of the nonlinear operator $\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi})$; see [1, Lemma 4.34].

Lemma 4.40. *Let $\bar{\epsilon}_{\rm bc}$ be the constant from* Step **2** *above. Then there exist positive constants $\epsilon_{\rm bc} \in (0,\bar{\epsilon}_{\rm bc})$, $\delta_1^{(2)}$, $N_1^{(1)}$, $\delta_{\rm bc}$, C, C_{θ_*}, and $\epsilon_{\mathcal{M}} \in (0,\epsilon_{\rm bc}]$ with $(\epsilon_{\rm bc},\delta_1^{(2)},N_1^{(1)},\delta_{\rm bc},C)$ depending only on (γ,v_2), $\epsilon_{\mathcal{M}}$ depending on (γ,v_2,θ_*), and C_{θ_*} depending on $(\gamma,v_2,\theta_*,\alpha)$ such that, whenever the parameters (ϵ,δ_1,N_1) from* Definition 4.37 *are chosen from $(0,\bar{\epsilon}_{\rm bc}]\times(0,\delta_1^{(2)}]\times[N_1^{(1)},\infty)$, the following statements hold: For each $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\rm ext}}$, the operator $\mathcal{M}_{(u,\boldsymbol{\theta})} : \mathbf{R}^2\times\mathbf{R}\times\overline{\Omega}\to\mathbf{R}$ given by (4.4.34) with $\sigma=\sigma_{\rm cf}\equiv\sqrt{\delta_1}$ satisfies*

(i) *$\mathcal{M}_{(u,\boldsymbol{\theta})} \in C^3(\mathbf{R}^2\times\mathbf{R}\times\overline{\Omega})$ and, for all $(\boldsymbol{p},z,\boldsymbol{\xi}) \in \mathbf{R}^2\times\mathbf{R}\times\overline{\Omega}$ and $\boldsymbol{\xi}_{\rm sh} \in \overline{\Gamma_{\rm shock}}$,*

$$\begin{aligned}
&\|(\mathcal{M}_{(u,\boldsymbol{\theta})}(\mathbf{0},0,\cdot),\, D^m_{(\boldsymbol{p},z)}\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\cdot))\|_{C^3(\overline{\Omega})} \le C_{\theta_*} \qquad \text{for } m=1,2,3\,,\\
&\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi}) = \mathcal{M}(\boldsymbol{p},z,\boldsymbol{\xi}) \qquad \text{for } |(\boldsymbol{p},z)-(D\phi(\boldsymbol{\xi}),\phi(\boldsymbol{\xi}))| < \tfrac{1}{2}\sqrt{\delta_1}\,,\\
&|D_{(\boldsymbol{p},z)}\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi}) - D_{(\boldsymbol{p},z)}\mathcal{M}(D\phi(\boldsymbol{\xi}),\phi(\boldsymbol{\xi}),\boldsymbol{\xi})| \le C\sqrt{\delta_1}\,,\\
&\delta_{\rm bc} \le D_{\boldsymbol{p}}\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi}_{\rm sh})\cdot\boldsymbol{\nu}_{\rm sh} \le \frac{1}{\delta_{\rm bc}}\,, \qquad D_z\mathcal{M}_{(u,\boldsymbol{\theta})}(\boldsymbol{p},z,\boldsymbol{\xi}_{\rm sh}) \le -\delta_{\rm bc}\,,
\end{aligned}$$

where $\mathcal{M}(\boldsymbol{p},z,\boldsymbol{\xi})$ is defined by (4.4.31), *and $\boldsymbol{\nu}_{\rm sh}$ is the unit normal vector to $\Gamma_{\rm shock}$ pointing into Ω.*

(ii) *Denote $\mathcal{B}^{(u,\boldsymbol{\theta})}_{\sigma,\Gamma_{\rm shock}}(\boldsymbol{p},z,\boldsymbol{\xi}) \coloneqq \mathcal{L}^{(u,\boldsymbol{\theta})}_\sigma(\boldsymbol{p}-Dv^{(u,\boldsymbol{\theta})}_\sigma(\boldsymbol{\xi}), z-v^{(u,\boldsymbol{\theta})}_\sigma(\boldsymbol{\xi}),\boldsymbol{\xi})$ and set*

$$\mathcal{B}^{(u,\boldsymbol{\theta})}_{\sigma,\Gamma_{\rm shock}}(\boldsymbol{p},z,\boldsymbol{\xi}) =: (b_1^{\rm sh}(\boldsymbol{\xi}),b_2^{\rm sh}(\boldsymbol{\xi}),b_0^{\rm sh}(\boldsymbol{\xi}),h^{\rm sh}(\boldsymbol{\xi}))\cdot(p_1,p_2,z,1)\,.$$

Then $\|(b_i^{\rm sh}, h^{\rm sh})\|_{C^3(\overline{\Gamma_{\rm shock}})} \le C_{\theta_*}$ *for* $i = 0, 1, 2$, *and for all* $(\boldsymbol{p}, z, \boldsymbol{\xi}) \in \mathbf{R}^2 \times \mathbf{R} \times \overline{\Omega}$,

$$\|\big(\mathcal{M}_{(u,\boldsymbol{\theta})} - \mathcal{B}_{\sigma_{\rm cf},\Gamma_{\rm shock}}\big)(\boldsymbol{p}, z, \boldsymbol{\xi})\| \le C\sqrt{\delta_1}\left|(\boldsymbol{p}, z) - \big(Dv_{\sigma_{\rm cf}}^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}), v_{\sigma_{\rm cf}}^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi})\big)\right|,$$
$$\left|D_{(\boldsymbol{p},z)}\big(\mathcal{M}_{(u,\boldsymbol{\theta})} - \mathcal{B}_{\sigma_{\rm cf},\Gamma_{\rm shock}}\big)(\boldsymbol{p}, z, \boldsymbol{\xi})\right| \le C\sqrt{\delta_1}\,.$$

(iii) $\mathcal{M}_{(u,\boldsymbol{\theta})}$ *is homogeneous in the sense that, for* $j = 5, 6$,

$$\mathcal{M}_{(u,\boldsymbol{\theta})}(D\phi_j(\boldsymbol{\xi}), \phi_j(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0 \quad \text{for all} \begin{cases} \boldsymbol{\xi} \in \Gamma_{\rm shock} \cap \mathcal{D}_{\epsilon_{\mathcal{M}}} & \text{for any } \boldsymbol{\theta} \in [0, \theta_*]^2\,, \\ \boldsymbol{\xi} \in \Gamma_{\rm shock} & \text{when } \max\{\theta_1, \theta_2\} \in [0, \frac{\delta_1}{N_1}]\,. \end{cases}$$

(iv) *For the coordinates* $(x, y) = \mathcal{R}(\boldsymbol{\xi})$ *given by* (4.4.1), *and* $\boldsymbol{\xi} \in \overline{\Gamma_{\rm shock} \cap \mathcal{D}^j_{\epsilon_{\rm bc}}}$, *for* $j = 5, 6$, *define* $\hat{\mathcal{M}}^{(j)}_{(u,\boldsymbol{\theta})}(q_x, q_y, z, x, y)$ *by* (4.4.33) *with* $\mathcal{M}$ *replaced by* $\mathcal{M}_{(u,\boldsymbol{\theta})}$ *in* (4.4.32). *Then, for* $j = 5, 6$, $\hat{\mathcal{M}}^{(j)}_{(u,\boldsymbol{\theta})}$ *satisfies the following properties when* $\Gamma_{\rm shock} \cap \mathcal{D}^j_{\epsilon_{\rm bc}}$ *is non-empty*:

$$\|\hat{\mathcal{M}}^{(j)}_{(u,\boldsymbol{\theta})}\|_{C^3(\mathbf{R}^2\times\mathbf{R}\times\mathcal{R}(\overline{\Gamma_{\rm shock}\cap\mathcal{D}^j_{\epsilon_{\rm bc}}}))} \le C_{\theta_*}\,,$$
$$\hat{\mathcal{M}}^{(j)}_{(u,\boldsymbol{\theta})}(\boldsymbol{q}, z, x, y) = \hat{\mathcal{M}}^{(j)}(\boldsymbol{q}, z, x, y) \qquad \text{in } \mathcal{R}(\Gamma_{\rm shock} \cap \mathcal{D}^j_{\epsilon_{\rm bc}}) \text{ for any } |(\boldsymbol{q}, z)| \le \tfrac{\delta_{\rm bc}}{C}\,,$$
$$\partial_a\hat{\mathcal{M}}^{(j)}_{(u,\boldsymbol{\theta})}(\boldsymbol{q}, z, x, y) \le -\delta_{\rm bc} \qquad \text{in } \mathcal{R}(\Gamma_{\rm shock} \cap \mathcal{D}^j_{\epsilon_{\mathcal{M}}}) \text{ for any } (\boldsymbol{q}, z) \in \mathbf{R}^2 \times \mathbf{R}\,,$$

with $\partial_a = \partial_{q_x}$, ∂_{q_y} *or* ∂_z, *provided that* $\Gamma_{\rm shock} \cap \mathcal{D}^j_{\epsilon_{\mathcal{M}}}$ *is non-empty.*

(v) $\mathcal{M}_{(u,\boldsymbol{\theta})}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0$ *on* $\Gamma_{\rm shock}$ *if and only if* $\varphi = \phi - \frac{1}{2}|\boldsymbol{\xi}|^2$ *satisfies the Rankine–Hugoniot jump condition* (4.3.26) *on* $\Gamma_{\rm shock} = \{\varphi = \varphi_2\}$.

4.4.2.2 Well-posedness of the boundary value problem (4.4.23)

Since the complete definition of the iteration boundary value problem (4.4.23) has been given, we now consider its well-posedness.

Lemma 4.41. *Fix* $\gamma \ge 1$, $v_2 \in (v_{\min}, 0)$, *and* $\theta_* \in (0, \theta^{\rm d})$. *Let the constant* σ_2 *be from* Step **1** *of the proof of* Proposition 4.23. *Let* $\bar{\alpha} \in (0, \frac{1}{3}]$ *and* $\epsilon_0 > 0$ *be from* Proposition 4.33 *and* Lemma 4.35(ii) *respectively, and let* $\alpha \in (0, \frac{\bar{\alpha}}{2}]$ *be from*

Definition 4.37. *Then there exist constants* $\epsilon^{(\mathrm{w})} \in (0, \epsilon_0]$, $\delta_1^{(\mathrm{w})} \in (0,1)$, $N_1^{(\mathrm{w})} \geq 1$, *and* $\alpha_1^* \in (0, \bar{\alpha}]$ *depending only on* (γ, v_2, θ_*) *such that, whenever the parameters* $(\epsilon, \delta_1, N_1)$ *in* Definition 4.37 *are chosen with* $\epsilon \in (0, \epsilon^{(\mathrm{w})}]$, $\delta_1 \in (0, \delta_1^{(\mathrm{w})}]$, *and* $N_1 \geq N_1^{(\mathrm{w})}$, *the following statements hold*:

Case (i). *If* $\max\{\theta_1, \theta_2\} \leq \theta^{\mathrm{s}} + \sigma_2$, *then the boundary value problem* (4.4.23) *associated with* $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}} \cap \{\max\{\theta_1, \theta_2\} \leq \theta^{\mathrm{s}} + \sigma_2\}$ *admits a unique solution* $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus (\Gamma^5_{\mathrm{sonic}} \cup \Gamma^6_{\mathrm{sonic}})) \cap C^0(\overline{\Omega})$. *Moreover, there exists a constant* $C > 0$ *depending only on* $(\gamma, v_2, \theta_*, \alpha)$ *such that the solution* $\hat{\phi}$ *satisfies*

$$\|\hat{\phi}\|_{L^\infty(\Omega)} \leq C\,, \qquad |\hat{\phi}(\boldsymbol{\xi}) - \phi_{\boldsymbol{\theta}}(\boldsymbol{\xi})| \leq C\,\mathrm{dist}(\boldsymbol{\xi}, \Gamma^5_{\mathrm{sonic}} \cup \Gamma^6_{\mathrm{sonic}}) \quad \text{in } \Omega\,, \tag{4.4.35}$$

for $\phi_{\boldsymbol{\theta}} \coloneqq \max\{\varphi_5, \varphi_6\} + \frac{1}{2}|\boldsymbol{\xi}|^2$. *Furthermore, for each* $d \in (0, \epsilon_0)$, *there exists a constant* $C_d > 0$ *depending only on* $(\gamma, v_2, \theta_*, \alpha, d)$ *such that*

$$\|\hat{\phi}\|_{2, \alpha_1^*, \Omega \setminus \mathcal{D}_d} \leq C_d\,. \tag{4.4.36}$$

Case (ii). *For each* $\delta \in (0, \frac{\sigma_2}{2})$, *whenever* $\max\{\theta_1, \theta_2\} \in [\theta^{\mathrm{s}} + \delta, \theta_*]$, *the boundary value problem* (4.4.23) *associated with* $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}} \cap \{\max\{\theta_1, \theta_2\} \geq \theta^{\mathrm{s}} + \delta\}$ *admits a unique solution* $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus (\Gamma^5_{\mathrm{sonic}} \cup \Gamma^6_{\mathrm{sonic}})) \cap C^0(\overline{\Omega})$. *Moreover, there exists a constant* $C' > 0$ *depending only on* $(\gamma, v_2, \theta_*, \alpha, \delta)$ *such that the solution* $\hat{\phi}$ *satisfies* (4.4.35), *while there exists a constant* $C'_d > 0$ *depending only on* $(\gamma, v_2, \theta_*, \alpha, \delta, d)$ *such that* (4.4.36) *holds.*

Proof. For Case (i), we fix $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}} \cap \{\max\{\theta_1, \theta_2\} \leq \theta^{\mathrm{s}} + \sigma_2\}$ and fix a constant $h > 0$. Using $\mathcal{G}_1^{\boldsymbol{\theta}}$ defined by (4.4.7), we rewrite the boundary value problem (4.4.23) associated with $(u, \boldsymbol{\theta})$ as a new boundary value problem for the unknown function

$$v(\boldsymbol{x}) \coloneqq \hat{\phi} \circ (L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}})^{-1}(\boldsymbol{x}) \qquad \text{for any } \boldsymbol{x} \in L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}}(\Omega(u, \boldsymbol{\theta}))\,,$$

where $\boldsymbol{x} \coloneqq L_h(s, t') = (\frac{s+1}{2}h, t')$ and $(s, t') = \mathcal{G}_1^{\boldsymbol{\theta}}(\boldsymbol{\xi})$ for any $\boldsymbol{\xi} \in \Omega(u, \boldsymbol{\theta})$. We write

$$\Gamma_0 \coloneqq L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}}(\Gamma^6_{\mathrm{sonic}}),\ \ \Gamma_1 \coloneqq L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}}(\Gamma_{\mathrm{shock}}(u, \boldsymbol{\theta})),\ \ \Gamma_2 \coloneqq L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}}(\Gamma^5_{\mathrm{sonic}}),\ \ \Gamma_3 \coloneqq L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}}(\Gamma_{\mathrm{sym}})\,.$$

Moreover, we take

$$g_{\rm so}(\boldsymbol{x}) \coloneqq (1-\varsigma_h)\phi_5 \circ (L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}})^{-1}(\boldsymbol{x}) + \varsigma_h \phi_6 \circ (L_h \circ \mathcal{G}_1^{\boldsymbol{\theta}})^{-1}(\boldsymbol{x}),$$

where $\varsigma_h \coloneqq \varsigma_1(\frac{1}{2} + \frac{2}{h}x_1)$, and the cut-off function $\varsigma_1(\cdot) \in C^\infty(\mathbf{R})$ is given by (4.4.28). Using Lemmas 4.28 and 4.39–4.40, we can choose suitable constants $\epsilon^{(\rm w)} \in (0, \epsilon]$, $\delta_1^{(\rm w)} > 0$, and $N_1^{(\rm w)} \geq 1$ such that the new boundary value problem for v satisfies the conditions in Proposition B.1 when $\epsilon \in (0, \epsilon^{(\rm w)}]$, $\delta_1 \in (0, \delta_1^{(\rm w)}]$, and $N_1 \geq N_1^{(\rm w)}$. Therefore, Proposition B.1 gives the existence of a unique solution $\hat{\phi} \in C(\overline{\Omega}) \cap C^{2,\alpha_1}(\overline{\Omega} \setminus (\Gamma^5_{\rm sonic} \cup \Gamma^6_{\rm sonic}))$ of the boundary value problem (4.4.23) satisfying (4.4.35)–(4.4.36).

For Case (ii), fix any $\delta \in (0, \frac{\sigma_2}{2})$. In the subcase $\min\{\theta_1, \theta_2\} < \theta^{\rm s} + \delta \leq \max\{\theta_1, \theta_2\}$ (without loss of generality, we consider the case: $\theta_1 < \theta^{\rm s} + \delta \leq \theta_2$), one can follow the same analysis as above to check all the conditions for Proposition B.2, from which we obtain the existence of a unique solution $\hat{\phi}$ satisfying (4.4.35)–(4.4.36) for any $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\rm ext}} \cap \{\theta_1 < \theta^{\rm s} + \delta \leq \theta_2\}$. In the other subcase $\min\{\theta_1, \theta_2\} \in [\theta^{\rm s} + \delta, \theta_*]$, we can apply Proposition B.3 to prove the same results. □

For each $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\rm ext}}$, the corresponding pseudo-subsonic region $\Omega = \Omega(u, \boldsymbol{\theta})$ depends continuously on $(u, \boldsymbol{\theta})$. We rewrite (4.4.23) as a boundary value problem for

$$\hat{u}(s,t) \coloneqq (\hat{\phi} - \frac{1}{2}|\boldsymbol{\xi}|^2 - \varphi^*_{\boldsymbol{\theta}}) \circ \mathfrak{F}_{(u,\boldsymbol{\theta})}(s,t) \qquad \text{in } \mathcal{Q}^{\rm iter}, \tag{4.4.37}$$

where $\varphi^*_{\boldsymbol{\theta}}$ is given by Definition 4.31, and $\mathfrak{F} = \mathfrak{F}_{(u,\boldsymbol{\theta})}$ is given by Definition 4.34(ii). We obtain

(4.4.38)

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} \mathcal{A}_{ij}^{(u,\boldsymbol{\theta})}(D_{(s,t)}\hat{u}, s, t)\partial_i\partial_j\hat{u} + \sum_{i=1}^{2} \mathcal{A}_i^{(u,\boldsymbol{\theta})}(D_{(s,t)}\hat{u}, s, t)\partial_i\hat{u} = f^{(u,\boldsymbol{\theta})} & \text{in } \mathcal{Q}^{\rm iter}, \\ \mathscr{M}_{(u,\boldsymbol{\theta})}(D_{(s,t)}\hat{u}, \hat{u}, s) = 0 & \text{on } \partial_{\rm sh}\mathcal{Q}^{\rm iter}, \\ \hat{u} = 0 & \text{on } \partial_{\rm so}\mathcal{Q}^{\rm iter}, \\ \mathscr{B}^{(\rm w)}_{(u,\boldsymbol{\theta})}(D_{(s,t)}\hat{u}, s) \coloneqq \displaystyle\sum_{i=1}^{2} b_i^{(\rm w)}(s)\partial_i\hat{u} = 0 & \text{on } \partial_{\rm w}\mathcal{Q}^{\rm iter}, \end{cases}$$

where $(\partial_1, \partial_2) \coloneqq (\partial_s, \partial_t)$ and

$$\partial_{\mathrm{sh}}\mathcal{Q}^{\mathrm{iter}} \coloneqq (-1,1)\times\{1\}\,, \quad \partial_{\mathrm{so}}\mathcal{Q}^{\mathrm{iter}} \coloneqq \{-1,1\}\times(0,1)\,, \quad \partial_{\mathrm{w}}\mathcal{Q}^{\mathrm{iter}} \coloneqq (-1,1)\times\{0\}\,.$$

From Lemmas 4.35 and 4.39–4.41, we can obtain the following results; see [1, Lemma 4.36].

Lemma 4.42. *For every $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, let $\mathcal{A}_{ij}^{(u,\boldsymbol{\theta})}$, $\mathcal{A}_{i}^{(u,\boldsymbol{\theta})}$, $f^{(u,\boldsymbol{\theta})}$, $\mathscr{M}_{(u,\boldsymbol{\theta})}$, $\mathscr{B}_{(u,\boldsymbol{\theta})}^{(\mathrm{w})}$, and $b_i^{(\mathrm{w})}$ be given as in* (4.4.38)*. Then the following properties hold*:

(i) $\mathcal{A}_{ij}^{(u,\boldsymbol{\theta})}$, $\mathcal{A}_{i}^{(u,\boldsymbol{\theta})} \in C(\mathbf{R}^2\times\mathcal{Q}^{\mathrm{iter}})$, $f^{(u,\boldsymbol{\theta})} \in C(\mathcal{Q}^{\mathrm{iter}})$, $\mathscr{M}_{(u,\boldsymbol{\theta})} \in C(\mathbf{R}^2\times\mathbf{R}\times\partial_{\mathrm{sh}}\mathcal{Q}^{\mathrm{iter}})$, *and* $\mathscr{B}_{(u,\boldsymbol{\theta})}^{(\mathrm{w})} \in C(\mathbf{R}^2\times\mathbf{R}\times\partial_{\mathrm{w}}\mathcal{Q}^{\mathrm{iter}})$;

(ii) *If a sequence $\{(u_k,\boldsymbol{\theta}_k)\}_{k\in\mathbf{N}} \subseteq \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ in $C_{(*)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})\times[0,\boldsymbol{\theta}_*]^2$, then the following sequences converge uniformly*:

- $(\mathcal{A}_{ij}^{(u_k,\boldsymbol{\theta}_k)}, \mathcal{A}_{i}^{(u_k,\boldsymbol{\theta}_k)}) \to (\mathcal{A}_{ij}^{(u,\boldsymbol{\theta})}, \mathcal{A}_{i}^{(u,\boldsymbol{\theta})})$ *on compact subsets of* $\mathbf{R}^2\times\mathcal{Q}^{\mathrm{iter}}$,
- $f^{(u_k,\boldsymbol{\theta}_k)} \to f^{(u,\boldsymbol{\theta})}$ *on compact subsets of* $\mathcal{Q}^{\mathrm{iter}}$,
- $\mathscr{M}_{(u_k,\boldsymbol{\theta}_k)} \to \mathscr{M}_{(u,\boldsymbol{\theta})}$ *on compact subsets of* $\mathbf{R}^2\times\mathbf{R}\times\partial_{\mathrm{sh}}\mathcal{Q}^{\mathrm{iter}}$,
- $\mathscr{B}_{(u_k,\boldsymbol{\theta}_k)}^{(\mathrm{w})} \to \mathscr{B}_{(u,\boldsymbol{\theta})}^{(\mathrm{w})}$ *on compact subsets of* $\mathbf{R}^2\times\mathbf{R}\times\partial_{\mathrm{w}}\mathcal{Q}^{\mathrm{iter}}$.

Corollary 4.43. *Let $\bar{\alpha} \in (0,1)$ be from* Proposition 4.33, *and let $\alpha_1^* \in (0,\bar{\alpha}]$, $\epsilon^{(\mathrm{w})}$, $\delta_1^{(\mathrm{w})}$, and $N_1^{(\mathrm{w})}$ be from* Lemma 4.41. *Let the parameters (ϵ,δ_1,N_1) from* Definition 4.37 *satisfy $\epsilon \in (0,\epsilon^{(\mathrm{w})}]$, $\delta_1 \in (0,\delta_1^{(\mathrm{w})}]$, and $N_1 \geq N_1^{(\mathrm{w})}$.*

(i) *For each $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, $\hat{\phi}$ solves the boundary value problem* (4.4.23) *if and only if $\hat{u}$ given by* (4.4.37) *solves the boundary value problem* (4.4.38). *Thus,* (4.4.38) *has a unique solution $\hat{u} \in C^2(\mathcal{Q}^{\mathrm{iter}})\cap C^1(\overline{\mathcal{Q}^{\mathrm{iter}}}\setminus\overline{\partial_{\mathrm{so}}\mathcal{Q}^{\mathrm{iter}}})\cap C(\overline{\mathcal{Q}^{\mathrm{iter}}})$. Furthermore, there exists a constant $C \geq 1$ depending on $(\gamma,v_2,\theta_*,\alpha)$ such that*

$$|\hat{u}(s,t)| \leq C(1-|s|) \qquad \text{in } \mathcal{Q}^{\mathrm{iter}}\,.$$

For each $\hat{d} \in (0,\frac{1}{2})$, there exists $C_{\hat{d}} > 0$ depending on $(\gamma,v_2,\theta_*,\hat{d},\alpha)$ such that

$$\|\hat{u}\|_{2,\alpha_1^*,\mathcal{Q}^{\mathrm{iter}}\cap\{1-|s|>\hat{d}\}} \leq C_{\hat{d}}\,.$$

(ii) *Let* $\{(u_k, \boldsymbol{\theta}_k)\}_{k\in\mathbf{N}} \subseteq \overline{\mathcal{K}^{\mathrm{ext}}}$ *converge to* $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ *in* $C^1(\overline{\mathcal{Q}^{\mathrm{iter}}}) \times [0, \theta_*]^2$, *and let* $\hat{u}_k$ *be the solution of the boundary value problem* (4.4.38) *associated with* $(u_k, \boldsymbol{\theta}_k)$. *Then there exists a unique solution* $\hat{u} \in C^2(\mathcal{Q}^{\mathrm{iter}}) \cap C^1(\overline{\mathcal{Q}^{\mathrm{iter}}} \setminus \overline{\partial_{\mathrm{so}}\mathcal{Q}^{\mathrm{iter}}}) \cap C(\overline{\mathcal{Q}^{\mathrm{iter}}})$ *of the boundary value problem* (4.4.38) *associated with* $(u, \boldsymbol{\theta})$. *Moreover,* $\hat{u}_k$ *converges to* $\hat{u}$ *uniformly in* $\overline{\mathcal{Q}^{\mathrm{iter}}}$ *and, for any* $\alpha' \in [0, \alpha_1^*)$,

- $\hat{u}_k \to \hat{u}$ *in* $C^{1,\alpha'}(K)$ *for any compact subset* $K \subseteq \overline{\mathcal{Q}^{\mathrm{iter}}} \setminus \overline{\partial_{\mathrm{so}}\mathcal{Q}^{\mathrm{iter}}}$,
- $\hat{u}_k \to \hat{u}$ *in* $C^{2,\alpha'}(K)$ *for any compact subset* $K \subseteq \mathcal{Q}^{\mathrm{iter}}$.

(iii) *If* $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}}$, *then* $(u, \boldsymbol{\theta})$ *satisfies* Definition 4.37(vii) *with nonstrict inequality in* (4.4.25).

Remark 4.44. For a constant $M > 0$, define the set $\mathcal{K}_M^E$ by

$$\mathcal{K}_M^E \coloneqq \left\{ (u, \boldsymbol{\theta}) \in C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \boldsymbol{\theta}_*]^2 \ : \ \begin{array}{l} (u, \boldsymbol{\theta}) \text{ satisfies Definition 4.37(ii)–(vi)} \\ \text{and } \|u\|^{(*)}_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}} \le M \end{array} \right\}.$$

Let $\overline{\mathcal{K}_M^E}$ be the closure of $\mathcal{K}_M^E$ in $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \boldsymbol{\theta}_*]^2$. Then Lemma 4.42 and Corollary 4.43 still hold when $\overline{\mathcal{K}^{\mathrm{ext}}}$ is replaced by $\overline{\mathcal{K}_M^E}$ for some $M > 0$.

4.4.2.3 Properties of the iteration set

From the well-posedness of the iteration boundary value problem (4.4.23), we can obtain the following *a priori* estimates on the iteration set. Let $\bar{\alpha} \in (0, \frac{1}{3}]$ be the constant from Proposition 4.33, and let $(\epsilon^{(\mathrm{w})}, \delta_1^{(\mathrm{w})}, N_1^{(\mathrm{w})})$ be the constants from Lemma 4.41.

Lemma 4.45 (*A priori* estimates)**.** *There exist positive constants* $\alpha^{(\mathrm{ap})} \in (0, \frac{\bar{\alpha}}{2})$, $\epsilon^{(\mathrm{ap})} \in (0, \epsilon^{(\mathrm{w})}]$, $\delta_1^{(\mathrm{ap})} \in (0, \delta_1^{(\mathrm{w})}]$, $N_1^{(\mathrm{adm})} \ge N_1^{(\mathrm{w})}$, *and* $\delta_3^{(\mathrm{ap})}$ *with* $(\alpha^{(\mathrm{ap})}, \epsilon^{(\mathrm{ap})}, \delta_1^{(\mathrm{ap})})$ *depending only on* (γ, v_2, θ_*), $N_1^{(\mathrm{adm})}$ *depending only on* $(\gamma, v_2, \theta_*, \delta_1)$, *and* $\delta_3^{(\mathrm{ap})}$ *depending only on* $(\gamma, v_2, \theta_*, \delta_1, \delta_2, N_1)$ *such that, whenever the parameters* $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ *in* Definition 4.37 *are chosen according to*

- $(\alpha, \epsilon, \delta_1) \in (0, \alpha^{(\mathrm{ap})}] \times (0, \epsilon^{(\mathrm{ap})}] \times (0, \delta_1^{(\mathrm{ap})}]$,
- $N_1 \in [N_1^{(\mathrm{adm})}(\delta_1), \infty)$,

- $\delta_3 \in (0, \delta_3^{(\mathrm{ap})}(\delta_1, \delta_2, N_1)]$,

with the parameter $\delta_2 > 0$ to be determined later in §4.4.3, we have

(i) *For any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap \{\theta_1, \theta_2 \le \theta_*\}$, the function $u = u^{(\varphi,\boldsymbol{\theta})}$ given by (4.4.14) satisfies that $(u, \boldsymbol{\theta}) \in \mathcal{K}$;*

(ii) *There exists a constant $C > 0$ depending only on (γ, v_2, θ_*) such that, for each $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, the unique solution $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \Gamma_{\mathrm{sonic}}) \cap C^0(\overline{\Omega})$ of the boundary value problem (4.4.23) associated with $(u, \boldsymbol{\theta})$ satisfies*

$$\|\hat{u}\|^{(*)}_{2,2\alpha,\mathcal{Q}^{\mathrm{iter}}} \le C\,, \tag{4.4.39}$$

for $\hat{u} : \overline{\mathcal{Q}^{\mathrm{iter}}} \to \mathbf{R}$ given by (4.4.24), whenever $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ satisfies

$$\|u^\sharp - u\|_{C^1(\overline{\mathcal{Q}^{\mathrm{iter}}})} + |\boldsymbol{\theta}^\sharp - \boldsymbol{\theta}| \le \delta^\sharp \tag{4.4.40}$$

for some $(u^\sharp, \boldsymbol{\theta}^\sharp) \in \overline{\mathcal{K}}$ and some sufficiently small positive constant $\delta^\sharp$ depending only on $(\gamma, v_2, \theta_*, \delta_2, \delta_3, u^\sharp, \boldsymbol{\theta}^\sharp)$.

Proof. We describe the proof in four steps, which follow [1, Corollary 4.40, Lemma 4.42, Lemma 4.44, Corollary 4.45] closely.

1. The proof of statement (i) above follows similarly to [1, Corollary 4.40]. Indeed, let $u = u^{(\varphi,\boldsymbol{\theta})}$ be given by (4.4.14) for any admissible solution φ corresponding to $\boldsymbol{\theta} \in \Theta \cap [0, \theta_*]^2$. Then $(u, \boldsymbol{\theta})$ automatically satisfies properties (ii)–(vi) of Definition 4.37 through the choices of the constants $\{N_i\}_{i=2}^5, \sigma_2, \mu_0, \mu_1, \tilde{\mu}, \rho^*(\gamma)$, and C_{ub} in Definition 4.37(iii)–(vi).

Property (i) of Definition 4.37 follows from the choice of N_0 in Definition 4.37(i) and the choice of N_1 below, in which we use an argument similar to [1, Lemma 4.39]. Indeed, by Lemma 4.42, Corollary 4.43, and Remark 4.44, for any $\alpha \in (0, \frac{\bar{\alpha}}{2}]$, we have the following continuity property of admissible solutions:

Continuity property. For any sequence $\{\boldsymbol{\theta}_k\}_{k\in\mathbf{N}} \subseteq \Theta$ with $\boldsymbol{\theta}_k \to \mathbf{0}$ as $k \to \infty$, let $\varphi^{(\boldsymbol{\theta}_k)}$ be any admissible solution corresponding to $\boldsymbol{\theta}_k$, and let $u^{(\boldsymbol{\theta}_k)}$ be given by (4.4.14) with $\varphi = \varphi^{(\boldsymbol{\theta}_k)}$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_k$. Then there exists a subsequence of $\{u^{(\boldsymbol{\theta}_k)}\}_{k\in\mathbf{N}}$ converging in $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter})$ to $u^{(\rm norm)} = 0$.

Using this continuity property, for any $\delta_1 \in (0, \delta_1^{(\rm w)}]$, there exists $N_1^{(\rm adm)} \geq N_1^{(\rm w)}$ depending on $(\gamma, v_2, \theta_*, \delta_1)$ such that $\|u - u^{(\rm norm)}\|^{(*)}_{2,\alpha,\mathcal{Q}^{\rm iter}} < \frac{\delta_1}{2}$ whenever $\max\{\theta_1, \theta_2\} \in (0, \frac{2\delta_1}{N_1^{(\rm adm)}}]$. Therefore, we have shown that $(u, \boldsymbol{\theta}) \in \mathcal{K}^{\rm ext}$.

Finally, property (vii) in Definition 4.37 follows from Corollary 4.43 because $(u, \boldsymbol{\theta}) \in \mathcal{K}^{\rm ext}$. We conclude that $(u, \boldsymbol{\theta}) \in \mathcal{K}$.

2. It remains to prove statement (ii). Fix any $(u^\sharp, \boldsymbol{\theta}^\sharp) \in \overline{\mathcal{K}}$. We claim that there exist constants $\epsilon^{(\rm lb)} \in (0, \epsilon^{(\rm w)})$ depending only on (γ, v_2, θ_*), $\delta_3^{(\rm ap)} > 0$ depending only on $(\gamma, v_2, \theta_*, \delta_1, \delta_2, N_1)$, and $\delta^\sharp$ depending only on $(\gamma, v_2, \theta_*, \delta_2, \delta_3, u^\sharp, \boldsymbol{\theta}^\sharp)$ such that $\hat{\phi} - (\varphi^*_{\boldsymbol{\theta}} + \frac{1}{2}|\boldsymbol{\xi}|^2) > 0$ in Ω, whenever $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}^{\rm ext}}$ satisfies (4.4.40). The proof of this claim follows from that for [1, Lemma 4.42], which is split into two cases: (i) $\max\{\theta_1, \theta_2\} \in [\frac{2\delta_1}{N_1^2}, \theta_*]$; and (ii) $\max\{\theta_1, \theta_2\} \in [0, \frac{2\delta_1}{N_1^2}]$. In particular, we can choose $\delta_3^{(\rm ap)} = \frac{\delta_1\delta_2}{2N_1^2}$ and $\epsilon^{(\rm lb)} = \bar{k}^{-1} \inf_{\boldsymbol{\theta}\in\overline{\Theta}}\{\hat{c}_5, \hat{c}_6\}$ for the constant $\bar{k} > 1$ that has been fixed after Definition 4.31 such that (4.4.10) holds.

3. We obtain the *a priori* estimates for $\hat{\phi}$ near $\Gamma_{\rm sonic}$. Let the coordinates $(x, y) = \mathcal{R}(\boldsymbol{\xi})$ be given by (4.4.1). For $j = 5, 6$, we have the following estimates (with estimates (b)–(c) below only valid when $\theta_* \in (\theta^{\rm s}, \theta^{\rm d})$):

(a) Whenever $\theta_{j-4} \in [0, \theta^{\rm s})$, for each $\alpha' \in (0, 1)$, there exist positive constants $\epsilon_{\rm p} \in (0, \epsilon_0]$ and $C_{\alpha'}$ depending only on $(\gamma, v_2, \theta_*, \alpha')$ with

$$\|(\hat{\phi} - \phi_j) \circ \mathcal{R}^{-1}\|^{(2),(\rm par)}_{2,\alpha',\mathcal{R}(\Omega\cap\mathcal{D}^j_{\epsilon_{\rm p}})} \leq C_{\alpha'};$$

(b) There exists a constant $\delta_{\rm p} \in (0, \theta_* - \theta^{\rm s})$ depending only on (γ, v_2, θ_*) such that, whenever $\theta_{j-4} \in [\theta^{\rm s}, \theta^{\rm s} + \delta_{\rm p}]$, for each $\alpha' \in (0, 1)$, there exist positive constants $\epsilon_{\rm p} \in (0, \epsilon_0]$ depending only on (γ, v_2, θ_*) and $C_{\alpha'}$ depending only on $(\gamma, v_2, \theta_*, \alpha')$

so that

$$\|\hat{\phi} - \phi_j\|_{C^{2,\alpha'}(\Omega \cap \mathcal{D}^j_{\epsilon_{\rm p}})} \leq C_{\alpha'}\,, \qquad D^m(\hat{\phi} - \phi_j)(P_0^{j-4}) = 0 \quad \text{for } m = 0, 1, 2\,;$$

(c) For $\delta_{\rm p} > 0$ as above, there exist constants $\hat{\alpha} \in (0, \frac{1}{3}]$ and $C > 0$ depending only on (γ, v_2, θ_*) such that, whenever $\theta_{j-4} \in [\theta^{\rm s} + \frac{1}{2}\delta_{\rm p}, \theta_*]$,

$$\|\hat{\phi} - \phi_j\|^{(-1-\hat{\alpha}),\{P_0^{j-4}\}}_{2,\hat{\alpha},\Omega\cap\mathcal{D}^j_{\epsilon_0}} \leq C\,, \qquad D^m(\hat{\phi} - \phi_j)(P_0^{j-4}) = 0 \quad \text{for } m = 0, 1\,.$$

Using Step **2**, the proof of these *a priori* estimates follows from [1, Lemma 4.44], whereby it is necessary to further reduce the constants ϵ and δ_1 depending only on (γ, v_2, θ_*), and we define $\alpha^{(\rm ap)} := \frac{1}{2}\min\{\alpha_1^*, \hat{\alpha}\}$ with α_1^* given in Lemma 4.41. In particular, the estimate in (a) is similar to that for Propositions 4.21–4.22, with the free boundary replaced by a fixed boundary. The estimate in (b) is similar to that for Proposition 4.23, with the free boundary replaced again by a fixed boundary. Finally, the estimate in (c) is similar to that for Proposition 4.25.

4. Following the method from [1, Proposition 4.12], we are able to combine the estimates from Step **3** with the interior estimates from Lemma 4.41 to obtain (4.4.39). □

Let the parameters $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ in Definition 4.37 be chosen as in Lemma 4.45. It is simple to verify that $\mathcal{K}^{\rm ext} \subseteq C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter}) \times [0, \theta_*]^2$ is relatively open. Indeed, the argument is similar to that for [20, Lemmas 12.8.1 and 17.5.1] and [1, Lemma 4.41]. Furthermore, following the proofs of [20, Propositions 12.8.2 and 17.5.3] and [1, Lemma 4.46], and applying the *a priori* estimates from Lemma 4.45, we conclude that the iteration set $\mathcal{K} \subseteq C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter}) \times [0, \theta_*]^2$ is relatively open.

Proposition 4.46 (Openness of the iteration set)**.** *For the choice of parameters* $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ *given in* Lemma 4.45, *the iteration set* $\mathcal{K}$ *defined by* Definition 4.37 *is a relatively open subset of* $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter}) \times [0, \theta_*]^2$.

4.4.3 Definition of the iteration map

Fix $\theta_* \in (0, \theta^{\mathrm{d}})$. For the iteration set $\mathcal{K}$ given by Definition 4.37, let the parameters $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ be chosen as in Lemma 4.45. Define

$$\mathcal{K}(\boldsymbol{\theta}) \coloneqq \left\{ u \in C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}}) \,:\, (u, \boldsymbol{\theta}) \in \mathcal{K} \right\} \qquad \text{for each } \boldsymbol{\theta} \in [0, \theta_*]^2 \,,$$

and similarly for $\overline{\mathcal{K}}(\boldsymbol{\theta})$. We now define an iteration map $\mathcal{I} : \overline{\mathcal{K}} \to C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ satisfying the following properties:

(a) For each $\boldsymbol{\theta} \in [0, \theta_*]^2$, there exists $u \in \mathcal{K}(\boldsymbol{\theta})$ satisfying $\mathcal{I}(u, \boldsymbol{\theta}) = u$;

(b) If u satisfies $\mathcal{I}(u, \boldsymbol{\theta}) = u$, then $\varphi^{(u,\boldsymbol{\theta})}$ defined in Definition 4.34(iii) is an admissible solution corresponding to $\boldsymbol{\theta}$.

Let $\hat{\phi} = \hat{\varphi}^{(u,\boldsymbol{\theta})} + \frac{1}{2}|\boldsymbol{\xi}|^2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve the iteration boundary value problem (4.4.23) associated with $(u, \boldsymbol{\theta})$ in Definition 4.37. Accordingly, from the definition of $\hat{u} : \overline{\mathcal{Q}^{\mathrm{iter}}} \to \mathbf{R}$ given by (4.4.24), we have

$$\hat{\varphi}^{(u,\boldsymbol{\theta})}(\boldsymbol{\xi}) = \hat{u} \circ (\mathfrak{F}_{(u,\boldsymbol{\theta})})^{-1}(\boldsymbol{\xi}) + \varphi^*_{\boldsymbol{\theta}}(\boldsymbol{\xi}) \,,$$

where $\varphi^*_{\boldsymbol{\theta}}(\boldsymbol{\xi})$ is given by Definition 4.31. We also define the functions $(w, w_2, \hat{w})$ by

$$(w, w_2, \hat{w})(s, t') \coloneqq (\varphi - \varphi^*_{\boldsymbol{\theta}}, \varphi_2 - \varphi^*_{\boldsymbol{\theta}}, \hat{\varphi}^{(u,\boldsymbol{\theta})} - \varphi^*_{\boldsymbol{\theta}}) \circ (\mathcal{G}^{\boldsymbol{\theta}}_1)^{-1}(s, t') \,. \tag{4.4.41}$$

From (4.4.12) and the implicit function theorem, for any $\boldsymbol{\theta} \in \overline{\Theta}$, there exists a unique function $\mathfrak{g}_2 : [-1, 1] \to \overline{\mathbf{R}_+}$ satisfying $w_2(s, \mathfrak{g}_2(s)) = 0$ on $[-1, 1]$. Then, for $Q^{\boldsymbol{\theta}}$ defined by Definition 4.27,

$$\{(s, t') \,:\, -1 < s < 1,\, t' = \mathfrak{g}_2(s)\} \subseteq \mathcal{G}^{\boldsymbol{\theta}}_1(Q^{\boldsymbol{\theta}}) \,.$$

It follows from (4.4.11), Lemma 4.28, and Definition 4.31 that $\|\mathfrak{g}_2\|_{C^3([-1,1])} \leq C$ for some constant $C > 0$ depending only on (γ, v_2).

For any $g \in C^{0,1}([-1,1])$ satisfying $g(s) > 0$ for all $s \in (-1,1)$, introduce the following sets: $R_\infty \coloneqq (-1,1) \times \mathbf{R}_+$ and

$$
\begin{aligned}
R_g &\coloneqq \big\{(s,t') \in \mathbf{R}^2_+ \,:\, -1 < s < 1,\, 0 < t' < g(s)\big\}\,, \\
\Sigma_g &\coloneqq \big\{(s,t') \in \mathbf{R}^2_+ \,:\, -1 < s < 1,\, t' = g(s)\big\}\,.
\end{aligned}
\tag{4.4.42}
$$

For each $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}}$, $\mathfrak{g}_{\mathrm{sh}} = \mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})} : [-1,1] \to \overline{\mathbf{R}_+}$ given by Definition 4.34 is a Lipschitz function on $[-1,1]$ and is positive on $(-1,1)$. Then we define the sets $R_{\mathfrak{g}_{\mathrm{sh}}}$ and $\Sigma_{\mathfrak{g}_{\mathrm{sh}}}$ with $g = \mathfrak{g}_{\mathrm{sh}}$ in (4.4.42). Note that w and $\hat{w}$ are defined on $R_{\mathfrak{g}_{\mathrm{sh}}}$, while w_2 is defined on R_∞.

The *regularised distance* function $\delta_{\mathfrak{g}_{\mathrm{sh}}} \in C^\infty(\overline{R_\infty} \setminus \overline{R_{\mathfrak{g}_{\mathrm{sh}}}})$ is given by Lemma B.4. Let $C_{\mathrm{rd}} > 0$ be the constant from Lemma B.4, which depends only on $\mathrm{Lip}[\mathfrak{g}_{\mathrm{sh}}]$. We define

$$
\delta^*_{\mathfrak{g}_{\mathrm{sh}}}(s,t') \coloneqq C_{\mathrm{rd}}^{-1}\delta_{\mathfrak{g}_{\mathrm{sh}}}(s,t') \qquad \text{for any } (s,t') \in \overline{R_\infty} \setminus \overline{R_{\mathfrak{g}_{\mathrm{sh}}}}\,.
\tag{4.4.43}
$$

Then, for all $(s,t') \in R_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}} \setminus \overline{R_{\mathfrak{g}_{\mathrm{sh}}}}$ and all $\lambda \in [1,2]$,

$$
(s, t' - \lambda\delta^*_{\mathfrak{g}_{\mathrm{sh}}}(s,t')) \in \{s\} \times \big[\, \frac{\mathfrak{g}_{\mathrm{sh}}(s)}{3},\, \mathfrak{g}_{\mathrm{sh}}(s) - (t' - \mathfrak{g}_{\mathrm{sh}}(s))\big] \Subset R_{\mathfrak{g}_{\mathrm{sh}}}\,,
\tag{4.4.44}
$$

where the constant $\kappa \in (0, \frac{1}{3}]$ depends only on $\mathrm{Lip}[\mathfrak{g}_{\mathrm{sh}}]$.

Let $\{(u_k,\boldsymbol{\theta}_k)\}_{k\in\mathbf{N}} \subseteq \overline{\mathcal{K}^{\mathrm{ext}}}$ converge to $(u,\boldsymbol{\theta})$ in $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}}) \times [0,\theta_*]^2$. Note that

$$
\|\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}\|^{(-1-\alpha),\{-1,1\}}_{2,\alpha,(-1,1)} \le CN_0\,,
$$

where $N_0 > 0$ is from Definition 4.37(i), and $C > 0$ depends only on (γ, v_2, α). Moreover, for $\mathfrak{g}_5$ and $\mathfrak{g}_6$ defined in Proposition 4.29(ii),

$$
\frac{\mathrm{d}^m}{\mathrm{d}s^m}(\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})} - \mathfrak{g}_5)(1) = \frac{\mathrm{d}^m}{\mathrm{d}s^m}(\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})} - \mathfrak{g}_6)(-1) = 0 \qquad \text{for } m = 0,1\,.
$$

Using Lemma 4.35(iii) and (v)–(vi), we see that $\mathfrak{g}_{\mathrm{sh}}^{(u_k,\boldsymbol{\theta}_k)}$ converges to $\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}$ in $C^{0,1}([-1,1])$

and

$$\lim_{k\to\infty} \|\delta_{\mathfrak{g}_{\mathrm{sh}}^{(u_k,\boldsymbol{\theta}_k)}} - \delta_{\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}}\|_{C^m(K)} = 0$$

for any compact set $K \subseteq \overline{R_\infty} \setminus \overline{R_{\mathfrak{g}_{\mathrm{sh}}}}$ and $m \in \mathbf{N} \cup \{0\}$, which follows by Lemma B.4.

From [20, Lemma 13.9.2], there exists a function $\Psi \in C_{\mathrm{c}}^\infty(\mathbf{R})$ satisfying

$$\operatorname{supp}\Psi \subseteq [1,2]\,, \qquad \int_{-\infty}^{\infty} \lambda^m \Psi(\lambda)\,\mathrm{d}\lambda = 1 - \operatorname{sgn}(m) \quad \text{for } m = 0,1,2\,.$$

Definition 4.47 (Extension operator)**.** For each $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, write $\mathfrak{g}_{\mathrm{sh}} = \mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}$, and let $\delta_{\mathfrak{g}_{\mathrm{sh}}}^*$ be defined by (4.4.43) satisfying (4.4.44) with the constants $C_{\mathrm{rd}} > 0$ and $\kappa \in (0, \frac{1}{3}]$ depending only on (γ, v_2, θ_*). For any $v \in C^0(\overline{R_{\mathfrak{g}_{\mathrm{sh}}}}) \cap C^2(R_{\mathfrak{g}_{\mathrm{sh}}} \cup \Sigma_{\mathfrak{g}_{\mathrm{sh}}})$, define the extension of $v(s,t')$ by

$$\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(v)(s,t') \coloneqq \begin{cases} v(s,t') & \text{for } (s,t') \in \overline{R_{\mathfrak{g}_{\mathrm{sh}}}}\,, \\ \displaystyle\int_{-\infty}^{\infty} v(s, t' - \lambda\delta_{\mathfrak{g}_{\mathrm{sh}}}^*(s,t'))\Psi(\lambda)\,\mathrm{d}\lambda & \text{for } (s,t') \in R_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}} \setminus \overline{R_{\mathfrak{g}_{\mathrm{sh}}}}\,. \end{cases}$$

Combining the above with Lemma 4.35(iii) and the property that the uniform bound of $\mathrm{Lip}[\mathfrak{g}_{\mathrm{sh}}]$ depends only on (γ, v_2, θ_*), we obtain the following proposition, for which the details of the proof can be found in [20, Lemma 13.9.6, and Theorems 13.9.5 and 13.9.8] for each case, respectively. For simplicity, we use the following notation below: For any $a, b \in [-1,1]$, denote

$$R_{\mathfrak{g}_{\mathrm{sh}}}^{(u,\boldsymbol{\theta})}[a,b] \coloneqq \left\{(s,t') \in \mathbf{R}_+^2 \,:\, a < s < b\,, 0 < t' < \mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})}(s)\right\},$$

and write $R_{\mathfrak{g}_{\mathrm{sh}}}$ to represent $R_{\mathfrak{g}_{\mathrm{sh}}}^{(u,\boldsymbol{\theta})}$ when the context is clear.

Proposition 4.48 (Properties of the extension operator $\mathcal{E}$)**.** *Fix $\alpha \in (0,1)$. For each $(u,\boldsymbol{\theta}) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, let the extension operator $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}} : C^2(R_{\mathfrak{g}_{\mathrm{sh}}} \cup \Sigma_{\mathfrak{g}_{\mathrm{sh}}}) \to C^2(R_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}})$ be given by* Definition 4.47. *We introduce the following three different cases*:

(a) *Fix any (b_1, b_2) with $-1 < b_1 < b_2 < 1$, and $C = C_{\mathrm{int}} > 0$ depending only on*

$(\gamma, v_2, \theta_*, \alpha)$. *For any* $\alpha' \in (0, \alpha)$, *denote the function spaces*

$$X \coloneqq C^{2,\alpha}(R^{(u,\boldsymbol{\theta})}_{\mathfrak{g}_{\mathrm{sh}}}[b_1, b_2]), \quad Y \coloneqq C^{2,\alpha}(R^{(u,\boldsymbol{\theta})}_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}}[b_1, b_2]),$$
$$Y^- \coloneqq C^{2,\alpha'}(R^{(u,\boldsymbol{\theta})}_{(1+\frac{\kappa}{2})\mathfrak{g}_{\mathrm{sh}}}[b_1, b_2]);$$

(b) *Fix* $\sigma > 0$ *and* $\epsilon \in (0, \frac{1}{4}]$. *Fix* $(b_1, b_2) = (-1, -1+\epsilon)$ *or* $(1-\epsilon, 1)$, *and* $C = C_{\mathrm{par}} > 0$ *depending only on* $(\gamma, v_2, \theta_*, \alpha, \sigma)$. *For any* $\alpha' \in (0, \alpha)$ *and* $\sigma' \in (0, \sigma)$, *denote the function spaces*

$$X \coloneqq C^{2,\alpha}_{(\sigma),(\mathrm{par})}(R^{(u,\boldsymbol{\theta})}_{\mathfrak{g}_{\mathrm{sh}}}[b_1, b_2]), \qquad Y \coloneqq C^{2,\alpha}_{(\sigma),(\mathrm{par})}(R^{(u,\boldsymbol{\theta})}_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}}[b_1, b_2]),$$
$$Y^- \coloneqq C^{2,\alpha'}_{(\sigma'),(\mathrm{par})}(R^{(u,\boldsymbol{\theta})}_{(1+\frac{\kappa}{2})\mathfrak{g}_{\mathrm{sh}}}[b_1, b_2]);$$

(c) *Fix* (b_1, b_2) *with either* $(b_1, b_2) = (b_1^{(1)}, b_2^{(1)}) = (\frac{1}{2}, 1)$ *or* $(b_1, b_2) = (b_1^{(2)}, b_2^{(2)}) = (-1, -\frac{1}{2})$, *and* $C = C_{\mathrm{sub}} > 0$ *depending only on* $(\gamma, v_2, \theta_*, \alpha)$. *For any* $\alpha' \in (0, \alpha)$, *denote the function spaces*

$$X \coloneqq C^{2,\alpha}_{(-1-\alpha),\{s=(-1)^{i-1}\}}(R^{(u,\boldsymbol{\theta})}_{\mathfrak{g}_{\mathrm{sh}}}[b_1^{(i)}, b_2^{(i)}]),$$
$$Y \coloneqq C^{2,\alpha}_{(-1-\alpha),\{s=(-1)^{i-1}\}}(R^{(u,\boldsymbol{\theta})}_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}}[b_1^{(i)}, b_2^{(i)}]),$$
$$Y^- \coloneqq C^{2,\alpha'}_{(-1-\alpha'),\{s=(-1)^{i-1}\}}(R^{(u,\boldsymbol{\theta})}_{(1+\frac{\kappa}{2})\mathfrak{g}_{\mathrm{sh}}}[b_1^{(i)}, b_2^{(i)}]).$$

Then the extension operator $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}$ *satisfies the following*:

(i) *There exists a constant* $C > 0$ *for each case above such that* $\|\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(v)\|_Y \le C\|v\|_X$. *Furthermore, for case* (c), *if* $(v, Dv) = (0, \mathbf{0})$ *on* $\overline{R^{(u,\boldsymbol{\theta})}_{\mathfrak{g}_{\mathrm{sh}}}} \cap \{x_1 = (-1)^{i-1}\}$ *for* $i = 1, 2$, *then*

$$(\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(v), D\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(v)) = (0, \mathbf{0}) \qquad \text{on } \overline{R^{(u,\boldsymbol{\theta})}_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}}} \cap \{x_1 = (-1)^{i-1}\}.$$

(ii) *For each case above,* $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}} : X \to Y$ *is linear and continuous.*

(iii) *Suppose that* $\{(u_k, \boldsymbol{\theta}_k)\}_{k\in\mathbf{N}} \subseteq \overline{\mathcal{K}^{\mathrm{ext}}}$ *converges to* $(u, \boldsymbol{\theta})$ *in* $C^{2,\tilde{\alpha}}_{(*)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \theta_*]^2$ *for some* $\tilde{\alpha} \in (0, 1)$. *Write* X_k *for the function space* X *with* $(u, \boldsymbol{\theta})$ *replaced by*

$(u_k, \boldsymbol{\theta}_k)$. *If the sequence* $\{v_k\}_{k\in\mathbf{N}} \subseteq X_k$ *satisfies* $\|v_k\|_{X_k} \le M$ *for all* $k \in \mathbf{N}$ *for some constant* $M > 0$ *and* $\{v_k\}_{k\in\mathbf{N}}$ *converges uniformly to* v *on any compact set* $K \Subset R^{(u,\boldsymbol{\theta})}_{\mathfrak{g}_{\mathrm{sh}}}$ *for some* $v \in X$, *then*

$$\mathcal{E}_{\mathfrak{g}^{(u_k,\boldsymbol{\theta}_k)}_{\mathrm{sh}}}(v_k) \to \mathcal{E}_{\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}}(v) \qquad \text{in } Y^- ,$$

where $\mathcal{E}_{\mathfrak{g}^{(u_k,\boldsymbol{\theta}_k)}_{\mathrm{sh}}}(v_k)$ *is well defined on* $\overline{R^{(u,\boldsymbol{\theta})}_{(1+\frac{\kappa}{2})\mathfrak{g}_{\mathrm{sh}}}[b_1, b_2]}$ *for large enough* k.

The proof of the following result is similar to [1, Lemma 5.5].

Lemma 4.49. *Let the parameters* $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ *in* Definition 4.37 *be chosen as in* Lemma 4.45, *with* $\delta_2 > 0$ *to be specified later. Then there exists a constant* $\delta_3^{(\mathrm{imp})} > 0$ *depending only on* $(\gamma, v_2, \theta_*, \delta_2)$ *such that, if* $\delta_3 \in (0, \delta_3^{(\mathrm{imp})})$, *then, for each* $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}}$, *there is a unique function* $\hat{\mathfrak{g}}_{\mathrm{sh}} : [-1, 1] \to \overline{\mathbf{R}_+}$ *such that*

$$\big(w_2 - \mathcal{E}_{\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}}(\hat{w})\big)(s, \hat{\mathfrak{g}}_{\mathrm{sh}}(s)) = 0 \qquad \text{for all } s \in [-1, 1] . \tag{4.4.45}$$

Furthermore, there exists a constant $C > 0$ *depending only on* (γ, v_2, θ_*) *such that* $\hat{\mathfrak{g}}_{\mathrm{sh}}$ *satisfies*

$$\begin{cases} \|\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_2\|^{(-1-2\alpha),\{-1,1\}}_{2,2\alpha,(-1,1)} \le C , \qquad \|\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}\|_{2,\frac{\alpha}{2},(-1,1)} \le C\delta_3 , \\ \dfrac{\mathrm{d}^m}{\mathrm{d}s^m}(\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_2)(\pm 1) = \dfrac{\mathrm{d}^m}{\mathrm{d}s^m}(\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}})(\pm 1) = 0 \qquad \text{for } m = 0, 1 . \end{cases} \tag{4.4.46}$$

Definition 4.50 (Iteration map)**.** Let the parameters $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ in Definition 4.37 be chosen as in Lemma 4.45, with $\delta_2 > 0$ to be specified later. Furthermore, let $\delta_3 \in (0, \delta_3^{(\mathrm{imp})})$ for the constant $\delta_3^{(\mathrm{imp})} > 0$ from Lemma 4.49. For each $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}}$, let $\tilde{u} : \overline{\mathcal{Q}^{\mathrm{iter}}} \to \mathbf{R}$ be given by

$$\tilde{u} \coloneqq \mathcal{E}_{\mathfrak{g}^{(u,\boldsymbol{\theta})}_{\mathrm{sh}}}(\hat{w}) \circ (G_{2,\hat{\mathfrak{g}}_{\mathrm{sh}}})^{-1} ,$$

where $G_{2,\hat{\mathfrak{g}}_{\mathrm{sh}}}$ is given by (4.4.13) with $\hat{\mathfrak{g}}_{\mathrm{sh}} : [-1, 1] \to \mathbf{R}_+$ from (4.4.45). Then define

the iteration map $\mathcal{I} : \overline{\mathcal{K}} \to C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ by

$$\mathcal{I}(u, \boldsymbol{\theta}) = \tilde{u}\,.$$

Note that the map $\mathcal{I} : \overline{\mathcal{K}} \to C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ defined as above is reasonable, since

$$\|\mathcal{I}(u, \boldsymbol{\theta})\|^{(*)}_{2,2\alpha,\mathcal{Q}^{\mathrm{iter}}} \le C \qquad \text{for all } (u, \boldsymbol{\theta}) \in \mathcal{K} \tag{4.4.47}$$

for $C > 0$ depending only on (γ, v_2, θ_*), which follows from Propositions 4.46 and 4.48, and Lemma 4.49.

4.4.4 Proof of Theorem 4.9: Existence of admissible solutions

Fix $\theta_* \in (0, \theta^{\mathrm{d}})$. For the iteration map $\mathcal{I}$ defined in Definition 4.50, we now prove that, for any fixed point $u \in \overline{\mathcal{K}}(\boldsymbol{\theta})$ of $\mathcal{I}(\cdot, \boldsymbol{\theta})$ for some $\boldsymbol{\theta} \in \Theta \cap [0, \theta_*]^2$, we have that $\varphi = \varphi^{(u,\boldsymbol{\theta})}$ defined in Definition 4.34(iii) is an admissible solution corresponding to $\boldsymbol{\theta}$ in the sense of Definition 4.7.

Proposition 4.51 (Fixed points of the map $\mathcal{I}(\cdot, \boldsymbol{\theta})$)**.** *Let the parameters* $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ *in* Definition 4.37 *be chosen as in* Definition 4.50*. Then the parameters* (ϵ, δ_1) *can be chosen even smaller, depending only on* (γ, v_2, θ_*)*, such that, for each* $\boldsymbol{\theta} \in \Theta \cap [0, \theta_*]^2$*,* $u \in \mathcal{K}(\boldsymbol{\theta})$ *is a fixed point of* $\mathcal{I}(\cdot, \boldsymbol{\theta}) : \overline{\mathcal{K}}(\boldsymbol{\theta}) \to C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ *if and only if* $\varphi = \varphi^{(u,\boldsymbol{\theta})}$ *defined in* Definition 4.34(iii) *is an admissible solution corresponding to* $\boldsymbol{\theta}$ *in the sense of* Definition 4.7 *after extending* φ *to* $\mathbf{R}^2_+$ *via* (4.2.35)*.*

Proof. Since the proof is similar to that for [1, Proposition 5.8], we omit the details of the proof here and sketch only the main ideas in the following four steps.

1. For $\boldsymbol{\theta} \in \Theta \cap [0, \theta_*]^2$, let φ be an admissible solution corresponding to $\boldsymbol{\theta}$, and let $u \coloneqq u^{(\varphi,\boldsymbol{\theta})}$ be given by (4.4.14), which satisfies $u \in \mathcal{K}(\boldsymbol{\theta})$ by Lemma 4.45(i). It follows directly from Definition 4.34(i) and the boundary condition on Γ_{shock} that $\mathfrak{g}_{\mathrm{sh}}^{(u,\boldsymbol{\theta})} = \mathfrak{g}_{\mathrm{sh}}$. It is clear that $\hat{\phi} \coloneqq \phi = \varphi + \frac{1}{2}|\boldsymbol{\xi}|^2$ solves the iteration boundary value problem (4.4.23), which indicates that $\hat{w} = w$ in (4.4.41). Then $\hat{\mathfrak{g}}_{\mathrm{sh}} = \mathfrak{g}_{\mathrm{sh}}$ by solving (4.4.45). We

conclude that

$$\mathcal{I}(u,\boldsymbol{\theta}) = \mathcal{E}_{\mathfrak{g}_{\rm sh}^{(u,\boldsymbol{\theta})}}(\hat{w}) \circ (G_{2,\hat{\mathfrak{g}}_{\rm sh}})^{-1} = \mathcal{E}_{\mathfrak{g}_{\rm sh}}(w) \circ (G_{2,\mathfrak{g}_{\rm sh}})^{-1} = u\,,$$

so $u \in \mathcal{K}(\boldsymbol{\theta})$ is a fixed point of $\mathcal{I}(\cdot,\boldsymbol{\theta})$.

2. On the other hand, for any fixed point u of map $\mathcal{I}(\cdot,\boldsymbol{\theta}) : \overline{\mathcal{K}}(\boldsymbol{\theta}) \to C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter})$, let $\varphi^{(u,\boldsymbol{\theta})}$ be given by Definition 4.34(iii). We now show that $\varphi^{(u,\boldsymbol{\theta})}$ is an admissible solution corresponding to $\boldsymbol{\theta}$ in the sense of Definition 4.7. Using Definition 4.34(i), (4.4.45), and Definition 4.50, we have

$$w_2(s,\mathfrak{g}_{\rm sh}^{(u,\boldsymbol{\theta})}) = u(s,1) = \mathcal{E}_{\mathfrak{g}_{\rm sh}^{(u,\boldsymbol{\theta})}}(\hat{w})(s,\hat{\mathfrak{g}}_{\rm sh}) = w_2(s,\hat{\mathfrak{g}}_{\rm sh})\,,$$

from which we obtain that $\hat{\mathfrak{g}}_{\rm sh} = \mathfrak{g}_{\rm sh}^{(u,\boldsymbol{\theta})}$, $\hat{u} = u$, and $\hat{\varphi}^{(u,\boldsymbol{\theta})} = \varphi^{(u,\boldsymbol{\theta})}$.

Define $v \coloneqq \hat{\varphi}^{(u,\boldsymbol{\theta})} - \varphi_2$. Then v solves the following strictly elliptic equation

$$\mathcal{L}_{(u,\boldsymbol{\theta})}(v) = \sum_{i,j=1}^{2} A_{ij}(D\hat{\phi}(\boldsymbol{\xi}),\boldsymbol{\xi})\partial_i\partial_j v = 0\,.$$

From the boundary conditions of the iteration boundary value problem (4.4.23) and $\hat{u} = u$ above, we see that $v \le 0$ on $\Gamma_{\rm shock} \cup \Gamma_{\rm sonic}$ and $\partial_\eta v = -v_2 > 0$ on $\Gamma_{\rm sym}$. From the maximum principle and Hopf's lemma, we obtain that $\hat{\varphi}^{(u,\boldsymbol{\theta})} \le \varphi_2$ on $\overline{\Omega}$.

When $\max\{\theta_1,\theta_2\} \in [\frac{2\delta_1}{N_1^2},\theta_*]$, we see that $\hat{\varphi}^{(u,\boldsymbol{\theta})} \ge \max\{\varphi_5,\varphi_6\}$ in Ω, which follows directly from (4.4.21) in Definition 4.37 and Step **1** of Lemma 4.45.

When $\max\{\theta_1,\theta_2\} \in (0,\frac{2\delta_1}{N_1^2}]$, we can show the same result by applying the maximum principle as in the proof of [1, Lemma 4.42]. Then the inequality (4.2.36) in Definition 4.7(iv) is proved.

Similarly, for the inequality (4.2.37) in Definition 4.7(iv), we also split into two cases. Denote $w \coloneqq \partial_{\boldsymbol{e}_{S_{25}}}(\varphi_2 - \varphi^{(u,\boldsymbol{\theta})})$. We introduce the (X_1,X_2)–coordinates by $\boldsymbol{\xi} =: X_1\boldsymbol{e}_{S_{25}} + X_2\boldsymbol{e}_{S_{25}}^{\perp}$. From the equation for v, one can derive the following strictly

elliptic equation for $w = -\partial_{X_1} v$

$$\hat{\mathcal{L}}_{(u,\boldsymbol{\theta})}(w) \coloneqq \sum_{i,j=1}^{2} \hat{A}_{ij} \partial_{X_i X_j} w + \sum_{i=1}^{2} \hat{A}_i \partial_{X_i} w = 0 \qquad \text{in } \Omega\,.$$

When $\max\{\theta_1, \theta_2\} \in [\frac{2\delta_1}{N_1^2}, \theta_*]$, it follows from (4.4.21) in Definition 4.37 that $w < 0$ in $\overline{\Omega} \setminus \mathcal{D}^5_{\epsilon/10}$. In the domain $\Omega \cap \mathcal{D}^5_{\epsilon}$, one can check

$$\text{(4.4.48)} \qquad \begin{cases} w = 0 & \text{on } \Gamma^5_{\text{sonic}}\,, \\ \boldsymbol{b}_{\text{w}} \cdot Dw = 0 & \text{with } \boldsymbol{b}_{\text{w}} \cdot \boldsymbol{\nu} > 0 \text{ on } \Gamma_{\text{sym}}\,, \\ \boldsymbol{b}_{\text{sh}} \cdot Dw = 0 & \text{with } \boldsymbol{b}_{\text{sh}} \cdot \boldsymbol{\nu} > 0 \text{ on } \Gamma_{\text{shock}}\,, \end{cases}$$

where $\boldsymbol{\nu}$ is the unit normal vector on $\Gamma_{\text{sym}} \cup \Gamma_{\text{shock}}$ pointing into Ω. Note that we have used [20, Lemma 13.4.5] to derive the boundary condition on Γ_{shock}. By the maximum principle and Hopf's lemma, we obtain that $w \le 0$ in $\Omega \cap \mathcal{D}^5_{\epsilon}$ when $\epsilon \in (0, \epsilon_{\text{sh}}]$, for some sufficiently small $\epsilon_{\text{sh}} > 0$ depending only on (γ, v_2, θ_*).

When $\max\{\theta_1, \theta_2\} \in (0, \frac{2\delta_1}{N_1^2}]$, using property (i) of Definition 4.37, one can follow the same procedure as the first case to derive that $w \le 0$ in Ω when $\delta_1 \in (0, \delta_{\text{sh}}]$, for some $\delta_{\text{sh}} > 0$ depending only on (γ, v_2, θ_*). Therefore, $\hat{\varphi}^{(u,\boldsymbol{\theta})}$ satisfies (4.2.37) when $\epsilon \in (0, \epsilon_{\text{sh}}]$ and $\delta_1 \in (0, \delta_{\text{sh}}]$, since the argument above also works for $\partial_{\boldsymbol{e}_{S_{26}}}(\varphi_2 - \varphi^{(u,\boldsymbol{\theta})})$.

3. Now we check that $\mathcal{N}_{(u,\boldsymbol{\theta})}(\phi) = 0$ in Definition 4.37(vii) coincides with (4.3.1) in Ω. From Lemma 4.39(iii), this property holds in $\Omega \setminus \mathcal{D}_{\epsilon/10}$, so it remains to show that this property holds in $\Omega \cap \mathcal{D}_{\epsilon/2}$. We only consider the domain $\Omega \cap \mathcal{D}^5_{\epsilon/2}$, since the proof is the same for the domain $\Omega \cap \mathcal{D}^6_{\epsilon/2}$ due to the symmetry between θ_1 and θ_2 in our problem.

For any $\boldsymbol{\xi} \in \Omega \cap \mathcal{D}_{\epsilon}$, let the coordinates $(x, y) = \mathcal{R}(\boldsymbol{\xi})$ be given by (4.4.1). Using the monotone property (4.4.20), we choose a constant $\epsilon'_{\text{sh}} \in (0, \epsilon_{\text{sh}}]$ sufficiently small, depending only on (γ, v_2, θ_*), such that $\theta_1 \le \theta^{\text{s}} + \min\{\frac{\sigma_3}{2}, \delta_{\text{p}}\}$ whenever $x_{P_0^1} < \frac{\epsilon'_{\text{sh}}}{10}$, for the constant $\delta_{\text{p}} > 0$ from Step **3**(b) in the proof of Lemma 4.45.

Choose $\epsilon \in (0, \epsilon'_{\text{sh}}]$, and suppose $x_{P_0^1} < \frac{\epsilon}{10}$. Denote $\tilde{w} \coloneqq Ax - \partial_x \psi(x, y)$ for $A \coloneqq \frac{1}{1+\gamma}(2 - \frac{\mu_0}{5})$ and $\psi \coloneqq (\varphi^{(u,\boldsymbol{\theta})} - \varphi_5) \circ \mathcal{R}^{-1}$. It is clear that $\tilde{w} = 0$ on $\mathcal{R}(\Gamma^5_{\text{sonic}})$,

and $\tilde{w}_y = 0$ on $\mathcal{R}(\Gamma_{\rm sym} \cap \partial\mathcal{D}^5_{\epsilon/2})$. From Lemma 4.40(i) and (iii)–(iv), the boundary condition $\mathcal{M}_{(u,\boldsymbol{\theta})}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi}) = 0$ in (4.4.23) can be written as

$$(b_1, b_2, b_0) \cdot (D\hat{\phi}, \hat{\phi}) = 0 \qquad \text{on } \Gamma_{\rm shock} \cap \mathcal{D}^5_\epsilon\,,$$

for some $b_0, b_1, b_2 \in [-\frac{1}{\delta_{\rm bc}}, -\delta_{\rm bc}]$. Using the *a priori* estimates (4.4.39) in the case $\theta_1 \le \theta^{\rm s} + \delta_{\rm p}$, we see that $|\psi_x| \le \frac{1}{\delta^2_{\rm bc}}(|\psi_y| + |\psi|) \le Cx^{\frac{3}{2}}$ for any $(x, y) \in \mathcal{R}(\Gamma_{\rm shock} \cap \mathcal{D}^5_\epsilon)$. Reducing $\epsilon'_{\rm sh} > 0$ further, depending only on (γ, v_2), we obtain that $\tilde{w} > 0$ on $\mathcal{R}(\Gamma_{\rm shock} \cap \mathcal{D}^5_\epsilon)$ for any $\epsilon \in (0, \epsilon'_{\rm sh}]$.

By Lemma 4.39(ii) and the fact that $\hat{\phi} = \phi$, we have

$$\sum_{i,j=1}^{2} A_{ij}(D\hat{\phi}, \boldsymbol{\xi})\partial_i\partial_j\hat{\phi} = \sum_{i,j=1}^{2} A^{(3)}_{ij}(D\hat{\phi}, \boldsymbol{\xi})\partial_i\partial_j\hat{\phi} = c_{\boldsymbol{\theta}}\mathcal{N}^{\rm polar}_{(u,\boldsymbol{\theta})}(\hat{\phi})\,,$$

from where we can derive the following equation for $\tilde{w}$

$$a_{11}\tilde{w}_{xx}+2a_{12}\tilde{w}_{xy}+a_{22}\tilde{w}_{yy}+a_1\tilde{w}_x+a_2\tilde{w}_y = -A((\gamma+1)A-1)+E(x,y) \qquad \text{in } \mathcal{R}(\Omega \cap \mathcal{D}^5_{\epsilon/2})\,.$$

From the expression (4.3.25), and the *a priori* estimates (4.4.39) in Lemma 4.45, reducing $\epsilon'_{\rm sh} > 0$ further if necessary, we obtain directly that $E(x, y) < A((\gamma+1)A-1)$ in $\mathcal{R}(\Omega \cap \mathcal{D}^5_{\epsilon/2})$ for any $\epsilon \in (0, \epsilon'_{\rm sh}]$. Therefore, the property $\tilde{w} = Ax - \partial_x\psi(x, y) \ge 0$ holds in $\mathcal{R}(\Omega \cap \mathcal{D}^5_{\epsilon/2})$, which follows from Lemma 4.39(i), the maximum principle, and Hopf's lemma.

Now we show the inequality $\partial_x\psi(x, y) \ge -Ax$. From Step **2**, we have

$$\big(\partial_{e_{S_{25}}}(\varphi_2 - \varphi)\big) \circ \mathcal{R}^{-1} = \psi_x \cos(\theta_{25} - y) - \frac{\sin(\theta_{25} - y)}{c_5 - x}\psi_y \le 0 \qquad \text{in } \mathcal{R}(\Omega \cap \mathcal{D}^5_{\epsilon/2})\,.$$

This implies that

$$\psi_x \ge -\frac{\tan(\frac{\pi}{2} - \tilde{\omega}_0)}{c_5 - x}\psi_y \ge -Cx^{\frac{3}{2}} \qquad \text{in } \mathcal{R}(\Omega \cap \mathcal{D}^5_{\epsilon/2})\,,$$

by using property (4.4.3) and the *a priori* estimates (4.4.39) in Lemma 4.45. Then

$\partial_x\psi(x,y) \geq -Ax$ after further reducing $\epsilon'_{\rm sh} > 0$. Thus, $|\psi_x| \leq Ax$ is proved, which indicates that $\mathcal{N}_{(u,\boldsymbol{\theta})}(\phi) = 0$ in Definition 4.37(vii) coincides with (4.3.1) in $\Omega \cap \mathcal{D}^5_{\epsilon/2}$. Moreover, if $x_{P_0^1} \geq \frac{\epsilon}{10}$, the same conclusion holds in $\Omega \cap \mathcal{D}^5_{\epsilon/2} \subseteq \Omega \setminus \mathcal{D}^6_{\epsilon/10}$ by Lemma 4.39(iii).

4. Definition 4.7(iii) follows directly from Lemma 4.39(i) and the fact proved above in Step **3**. Let the sound speed $c(|D\varphi|,\varphi)$ be given by (4.3.2). The strict ellipticity implies that

$$M := \frac{|\partial_{\boldsymbol{\nu}}\varphi(\boldsymbol{\xi})|}{c(|D\varphi(\boldsymbol{\xi})|,\varphi(\boldsymbol{\xi}))} < 1 \qquad \text{on } \Gamma_{\rm shock}\,.$$

Define $M_2 := |\partial_{\boldsymbol{\nu}}\varphi_2(\boldsymbol{\xi})|$. Similar to [1, Eq. (2.4.9)], we have the following equality

$$(1+\tfrac{\gamma-1}{2}M^2)M^{-\frac{2(\gamma-1)}{\gamma+1}} = (1+\tfrac{\gamma-1}{2}M_2^2)M_2^{-\frac{2(\gamma-1)}{\gamma+1}}\,.$$

Then $M_2 > 1$, and so $|D\varphi_2| > 1 = c(|D\varphi_2|,\varphi_2)$ holds on $\Gamma_{\rm shock}$, *i.e.*, $\Gamma_{\rm shock} \subseteq \mathbf{R}^2_+ \setminus \overline{B_{c_2}(O_2)}$. Finally, for the property $\Gamma_{\rm shock} \subseteq \{\xi^{P_3} < \xi < \xi^{P_2}\}$, we follow the same proof as in Lemma 4.11 by using all the properties above. Therefore, all parts of Definition 4.7 have been verified, which implies that $\varphi^{(u,\boldsymbol{\theta})}$ is an admissible solution in the sense of Definition 4.7. □

Proof of Theorem 4.9. We now complete the proof of Theorem 4.9 in five steps.

1. We prove that the iteration map $\mathcal{I} : \overline{\mathcal{K}} \to C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter})$ is a compact map in the sense of Definition B.5. Let $\{(u_k,\boldsymbol{\theta}_k)\}_{k\in\mathbf{N}} \subseteq \mathcal{K}$ converge to $(u,\boldsymbol{\theta})$ in $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter})\times[0,\theta_*]^2$. For each $k \in \mathbf{N}$, write $\Omega^{(k)} := \Omega(u_k,\boldsymbol{\theta}_k)$ and $\mathfrak{g}^{(k)}_{\rm sh} := \mathfrak{g}^{(u_k,\boldsymbol{\theta}_k)}_{\rm sh}$, as given by Definition 4.34, and $\Gamma^{(k)}_{\rm sonic} := \Gamma^5_{\rm sonic} \cup \Gamma^6_{\rm sonic}$ for the sonic arcs related to $\boldsymbol{\theta}_k$. By Lemma 4.41, there exists a unique solution $\hat{\phi}^{(k)} \in C^2(\Omega^{(k)}) \cap C^1(\overline{\Omega^{(k)}} \setminus \Gamma^{(k)}_{\rm sonic}) \cap C^0(\overline{\Omega^{(k)}})$ of the iteration boundary value problem (4.4.23) related to $(u_k,\boldsymbol{\theta}_k)$. For any $(s,t') \in R_{\mathfrak{g}^{(k)}_{\rm sh}}$, define

$$\hat{w}^{(k)}(s,t') := \big(\hat{\phi}^{(k)} + \frac{1}{2}|\boldsymbol{\xi}|^2 - \varphi^*_{\boldsymbol{\theta}_k}\big) \circ (\mathcal{G}_1^{\boldsymbol{\theta}_k})^{-1}(s,t')\,,$$

where $R_{\mathfrak{g}^{(k)}_{\rm sh}}$, $\varphi^*_{\boldsymbol{\theta}_k}$, and $\mathcal{G}_1^{\boldsymbol{\theta}_k}$ are given by (4.4.42), Definition 4.31, and (4.4.7), respectively.

Then

$$\hat{w}^{(k)} = \hat{u}^{(k)} \circ G_{2,\mathfrak{g}_{\mathrm{sh}}^{(k)}}$$

for $\hat{u}^{(k)} : \overline{\mathcal{Q}^{\mathrm{iter}}} \to \mathbf{R}$ given by (4.4.24) with $\hat{\phi} = \hat{\phi}^{(k)}$, and $G_{2,\mathfrak{g}_{\mathrm{sh}}^{(k)}}$ given by (4.4.13). Subsequently, we obtain the functions $\hat{\mathfrak{g}}_{\mathrm{sh}}^{(k)} : [-1,1] \to \overline{\mathbf{R}_+}$ via Lemma 4.49.

Since $(u_k, \boldsymbol{\theta}_k)$ converges to $(u, \boldsymbol{\theta})$ in $C^{1,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0,\theta_*]^2$, we find that $\mathfrak{g}_{\mathrm{sh}}^{(k)} \to \mathfrak{g}_{\mathrm{sh}}$ in $C^{1,\alpha}([-1,1])$ by Lemma 4.35(iii). Fix any compact subset $K \subseteq R_{\mathfrak{g}_{\mathrm{sh}}}$. From (4.4.13), Definition 4.37(iii), and the convergence of $\{\mathfrak{g}_{\mathrm{sh}}^{(k)}\}_{k\in\mathbf{N}}$, we see that $G_{2,\mathfrak{g}_{\mathrm{sh}}^{(k)}} \to G_{2,\mathfrak{g}_{\mathrm{sh}}}$ in $C^{1,\alpha}(K)$. Then, by Corollary 4.43 and the convergence of $\{\hat{u}^{(k)}\}_{k\in\mathbf{N}}$, we obtain that $\hat{w}^{(k)} \to \hat{w}$ in $C^{1,\alpha}(K)$.

By Lemma 4.45, Proposition 4.48(iii), and the local convergence of $\{\hat{w}^{(k)}\}_{k\in\mathbf{N}}$, for any $b_1, b_2 \in [-1,1]$ chosen as in each case of Proposition 4.48(a)–(c), $\{\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(k)}}(\hat{w}^{(k)})\}_{k\in\mathbf{N}}$ converges to $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w})$ in the corresponding function spaces. Combining the statement above with the convergence of $\{\hat{\mathfrak{g}}_{\mathrm{sh}}^{(k)}\}_{k\in\mathbf{N}}$ from Lemma 4.49, we conclude that

$$\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(k)}}(\hat{w}^{(k)}) \circ (G_{2,\hat{\mathfrak{g}}_{\mathrm{sh}}^{(k)}})^{-1} \to \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}) \circ (G_{2,\hat{\mathfrak{g}}_{\mathrm{sh}}})^{-1} \qquad \text{in } C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})\,.$$

Since $C^{2,2\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ is compactly embedded into $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$, we obtain from (4.4.47) that $\mathcal{I}(U)$ is precompact in $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$ for any bounded subset $U \subseteq \overline{\mathcal{K}}$ so that the iteration map $\mathcal{I}$ is compact.

2. Next, we show that the constants $N_1, \delta_2 > 0$ in Definition 4.37 can be chosen, depending only on (γ, v_2, θ_*), such that, for any $\boldsymbol{\theta} \in \Theta \cap [0,\theta_*]^2$, no fixed point of $\mathcal{I}(\cdot, \boldsymbol{\theta})$ lies on boundary $\partial\mathcal{K}(\boldsymbol{\theta})$, where $\partial\mathcal{K}(\boldsymbol{\theta})$ is considered relative to $C^{2,\alpha}_{(*)}(\mathcal{Q}^{\mathrm{iter}})$.

Suppose that $(u, \boldsymbol{\theta}) \in \overline{\mathcal{K}}$ satisfies $\mathcal{I}(u, \boldsymbol{\theta}) = u$. For $\varphi = \varphi^{(u,\boldsymbol{\theta})}$ defined in $\Omega = \Omega(u, \boldsymbol{\theta})$ by Definition 4.34(iii), extend φ to $\mathbf{R}^2_+$ via (4.2.35). Then φ is an admissible solution corresponding to $\boldsymbol{\theta} \in \Theta \cap [0,\theta_*]^2$ by Proposition 4.51.

From (4.3.16), Propositions 4.13, 4.17, and 4.29(iv), and Lemma 4.12, the constants $(N_2, N_3, \rho^*(\gamma), C_{\mathrm{ub}}, \tilde{\mu})$ are fixed so that any admissible solution satisfies the strict inequalities in Definition 4.37(iii) and (v)–(vi). Moreover, from Lemma 4.45(i), u satisfies the strict inequality given in Definition 4.37(i) when $N_1 \geq N_1^{(\mathrm{adm})}$.

For condition (iv) of Definition 4.37, the constants $(\mu_0, \mu_1, N_4, N_5, \sigma_2)$ are fixed such

that any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta \cap [0, \theta_*]^2$ satisfies the strict inequalities (4.4.22). Next, we focus on the strict inequalities (4.4.21). It is direct to see that the strict inequalities (4.4.21) hold when $\max\{\theta_1, \theta_2\} < \frac{\delta_1}{N_1^2}$, since $\mathscr{K}_2(\max\{\theta_1, \theta_2\}) < 0$ and the inequality (4.2.36) in Definition 4.7(iv) and Lemma 4.10(ii) hold for any admissible solution corresponding to $\boldsymbol{\theta} \in \Theta \cap [0, \theta_*]^2$. When $\max\{\theta_1, \theta_2\} \geq \frac{\delta_1}{N_1^2}$, we similarly have

$$\begin{aligned} &\varphi - \max\{\varphi_5, \varphi_6\} > 0 \qquad && \text{in } \overline{\Omega} \setminus (\mathcal{D}^5_{\epsilon/10} \cup \mathcal{D}^6_{\epsilon/10}), \\ &\partial_{e_{S_{2j}}}(\varphi_2 - \varphi) < 0 && \text{in } \overline{\Omega} \setminus \mathcal{D}^j_{\epsilon/10} \text{ for } j = 5, 6. \end{aligned}$$

By Corollary 4.19, and Propositions 4.21–4.23 and 4.25, we obtain that $\varphi \in C^{1,\alpha}(\Omega)$. Combining this with Lemma 4.26, there exist both $\tau_0 > 0$ depending only on (γ, v_2, θ_*) and $\tau_1 > 0$ depending only on $(\gamma, v_2, \theta_*, \delta_1, N_1)$ such that, for all $\boldsymbol{\theta} \in \Theta \cap \{\frac{\delta_1}{N_1^2} \leq \max\{\theta_1, \theta_2\} \leq \theta_*\}$,

$$\begin{aligned} &\psi = \varphi - \max\{\varphi_5, \varphi_6\} \geq \tau_0 \qquad && \text{in } \overline{\Omega} \setminus (\mathcal{D}^5_{\epsilon/10} \cup \mathcal{D}^6_{\epsilon/10}), \\ &\partial_{e_{S_{2j}}}(\varphi_2 - \varphi) \leq -\tau_1 && \text{in } \overline{\Omega} \setminus \mathcal{D}^j_{\epsilon/10} \text{ for } j = 5, 6. \end{aligned}$$

For any fixed $\delta_1 > 0$, we can choose $\delta_2 > 0$ small enough such that

$$\mathscr{K}_2(\max\{\theta_1, \theta_2\}) \leq \frac{\delta_1 \delta_2}{N_1^2} < \min\{\tau_0, \tau_1\} \qquad \text{for all } \boldsymbol{\theta} \in \Theta \cap \{\frac{\delta_1}{N_1^2} \leq \max\{\theta_1, \theta_2\} \leq \theta_*\},$$

which implies that the strict inequalities (4.4.21) hold when $\max\{\theta_1, \theta_2\} \geq \frac{\delta_1}{N_1^2}$.

Finally, let $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ be chosen as in Lemma 4.45 with δ_2 chosen as above. Then the strict inequality (4.4.25) of Definition 4.37(vii) holds by Proposition 4.46. Therefore, $u \in \mathcal{K}(\boldsymbol{\theta})$ for any $\boldsymbol{\theta} \in \Theta \cap [0, \theta_*]^2$, which completes this step.

3. Now, we can fix the constants $(\alpha, \epsilon, \delta_1, \delta_2, \delta_3, N_1)$ in Definition 4.37 as follows:

- Choose the constants $(\alpha, \epsilon, \delta_1, \delta_3, N_1)$ as in Lemma 4.45 and reduce (ϵ, δ_1) as in Proposition 4.51, depending only on (γ, v_2, θ_*);
- Fix δ_2 as in Step **2**, depending only on $(\gamma, v_2, \theta_*, \delta_1, N_1)$;

- Adjust $\delta_3 > 0$ as in Definition 4.50, depending only on $(\gamma, v_2, \theta_*, \delta_2)$.

4. *Leray–Schauder degree theory.* When $\boldsymbol{\theta} = \mathbf{0}$, $\varphi_5 = \varphi_6 = \varphi_0$ with φ_0 defined in §4.2.1.3, and the iteration boundary value problem in the (s, t')–coordinates is a homogeneous problem. From Lemma 4.41, the iteration boundary value problem (4.4.23) corresponding to any $(u, \mathbf{0}) \in \overline{\mathcal{K}}$ has a unique solution $\hat{\phi} = \frac{1}{2}|\boldsymbol{\xi}|^2 + \varphi_{\mathbf{0}}^* = v_2\eta_0$, where $\eta_0 > 0$ is defined in §4.2.1.3. It follows from Definition 4.37 that the unique fixed point $u \equiv 0$ of $\mathcal{I}(\cdot, \mathbf{0})$ lies in $\mathcal{K}(\mathbf{0})$, and $\mathbf{Ind}(\mathcal{I}(\cdot, \mathbf{0}), \mathcal{K}(\mathbf{0})) = 1$. From the arguments above, the map $\mathcal{I}(\cdot, \boldsymbol{\theta}) : \overline{\mathcal{K}(\boldsymbol{\theta})} \to C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter})$ satisfies $\mathcal{I}(\cdot, \boldsymbol{\theta}) \in V(\mathcal{K}(\boldsymbol{\theta}), C^{2,\alpha}_{(*)}(\mathcal{Q}^{\rm iter}))$ from Definition B.6 in Appendix B. Then for any $t \in [0, 1]$, by Theorem B.9 we have

$$\mathbf{Ind}(\mathcal{I}(\cdot, t\boldsymbol{\theta}), \mathcal{K}(t\boldsymbol{\theta})) = \mathbf{Ind}(\mathcal{I}(\cdot, \mathbf{0}), \mathcal{K}(\mathbf{0})) \qquad \text{for all } \boldsymbol{\theta} \in [0, \theta_*]^2 .$$

We obtain the existence of a fixed point of the map $\mathcal{I}(\cdot, \boldsymbol{\theta})$, which is equivalent to the existence of an admissible solution corresponding to $\boldsymbol{\theta} \in [0, \theta_*]^2$ in the sense of Definition 4.7 by Proposition 4.51. The arbitrary choice of $\theta_* \in (0, \theta^{\rm d})$ completes the proof of the existence of admissible solutions for any $\boldsymbol{\theta} \in [0, \theta^{\rm d})^2$.

5. *Proof of the continuity at* $\boldsymbol{\theta} = \mathbf{0}$. Let $\varphi_{\rm norm}$ be the admissible solution for $\boldsymbol{\theta} = \mathbf{0}$, which is given by (4.2.19), and let $\Omega^{(\mathbf{0})}$ be the corresponding pseudo-subsonic domain. For any sequence $\{\boldsymbol{\theta}_k\}_{k\in\mathbf{N}} \subseteq \Theta$ with $\boldsymbol{\theta}_k \to \mathbf{0}$ as $k \to \infty$, let $\varphi^{(\boldsymbol{\theta}_k)}$ be an admissible solution corresponding to $\boldsymbol{\theta}_k$, and let $\Omega^{(k)}$ be the corresponding pseudo-subsonic domain.

Let $G \subseteq \mathbf{R}^2_+$ be any open set satisfying $\overline{G} \Subset \mathbf{R}^2_+$. Choose any compact subset $K \Subset G \setminus \partial\Omega^{(\mathbf{0})}$. Then, by Lemma 4.26(ii), passing to a subsequence (still denoted) k, $K \cap \Omega^{(\mathbf{0})} \Subset \Omega^{(k)}$ and $K \setminus \Omega^{(\mathbf{0})} \Subset G \setminus \Omega^{(k)}$. Using Proposition 4.29(iv), Proposition 4.33, Lemma 4.35(v), and the continuity property of admissible solutions given in Step **1** of the proof of Lemma 4.45, we have

$$\|\varphi^{(\boldsymbol{\theta}_k)} - \varphi_{\rm norm}\|_{1,\alpha,K\cap\Omega^{(\mathbf{0})}} \to 0 \qquad \text{as } k \to \infty .$$

Similarly, using (4.2.19), Proposition 4.2(i), and (4.2.35), we have

$$\|\varphi^{(\boldsymbol{\theta}_k)} - \varphi_{\rm norm}\|_{W^{1,1}(K\setminus\Omega^{(\mathbf{0})})} \to 0 \qquad \text{as } k \to \infty\,.$$

Finally, by (4.2.35) and the uniform bound in Lemma 4.12, there exists a constant $C > 0$ depending only on (γ, v_2, G) such that $\|\varphi^{(\boldsymbol{\theta}_k)} - \varphi_{\rm norm}\|_{0,1,G} \le C$ for all $k \in \mathbf{N}$, which leads to

$$\|\varphi^{(\boldsymbol{\theta}_k)} - \varphi_{\rm norm}\|_{W^{1,1}(G)} \le \|\varphi^{(\boldsymbol{\theta}_k)} - \varphi_{\rm norm}\|_{W^{1,1}(K)} + 2C\,\mathrm{meas}(G\setminus K)\,.$$

This completes the proof of Theorem 4.9. □

4.5 Future directions

In my opinion, there are two primary directions for future investigation:

1. **Alternative symmetric case.** Determine the existence (or not!) of solutions to the four-shock interaction problem when $\theta_{12} = \theta_{32}$, and $\theta_{14} = \theta_{34}$; see Figure 4.8. In this alternative symmetric case, one can show from the compatibility conditions (4.2.9) that $\rho_1 = \rho_3$ and the reflection configuration becomes symmetric around the vertical axis, after which we obtain a free boundary problem with two free boundaries. After both symmetric cases have been solved, one may consider the fully non-symmetric problem, which may have additional difficulties.

2. **Configuration 4.** In §2.3 we found two genuinely distinct types of four-shock interaction, Configuration 4 and Configuration 5. Theorem 4.9 gives the existence of solutions for Configuration 5, which leaves Configuration 4 as an open question. The numerical simulations shown in Figure 1.2 suggest that the structures of solutions for Configuration 4 are much more complicated than those of Configuration 5, likely containing intricate networks of Mach reflections. Presently, the rigorous mathematical understanding of Mach reflections is not sufficient to tackle this case. Furthermore, the presumed role of vortex sheets

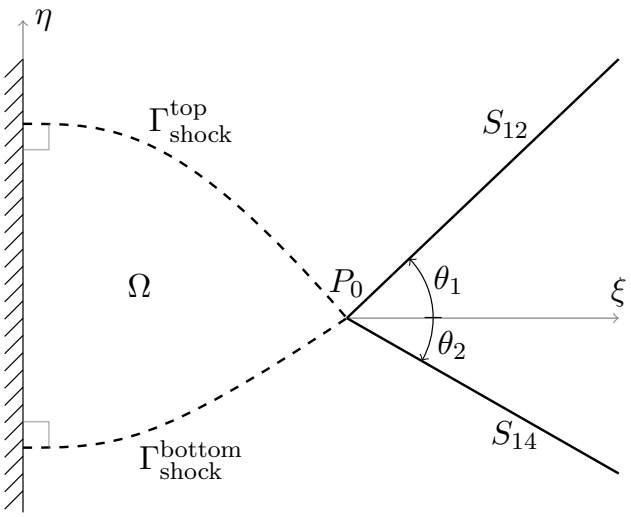


Figure 4.8: The alternative symmetric case for the four-shock interaction problem.

within Mach reflection indicate that the equations of potential flow are not a suitable model in this case, rather the isentropic or full system of Euler equations are perhaps a better choice.

Appendix A

Monotonic Properties of Admissible Solutions

In this appendix, we present some detailed calculations on various monotonic properties of admissible solutions which are important throughout Chapter 4. The calculations were carried out independently.

We begin by proving the strict monotonicity of $M_2^{(\theta_1)}$, the pseudo-Mach number of state (2) at the reflection point P_0^1, with respect to the incident angle $\theta_1 \in (0, \theta^{\rm cr})$, which is fundamental to many of our arguments in §4.

Lemma A.1. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$. For any incident angle $\theta_1 \in (0, \theta^{\rm cr})$, the pseudo-Mach number of state (2) at the reflection point P_0^1 is given by*

$$M_2^{(\theta_1)} := |D\varphi_2(P_0^1)| = \frac{\ell(\rho_1)}{1-\rho_1}\big(\rho_1^2 + \cot^2\theta_1\big)^{\frac{1}{2}}, \tag{A.1}$$

with $\rho_1 = \rho_1(\theta_1; v_2) \in (0,1)$ given by (4.2.11). Moreover, the pseudo-Mach number $M_2^{(\theta_1)}$ is a strictly decreasing function of the incident angle θ_1 and satisfies that $M_2^{(\theta_1)} \to \infty$ as $\theta_1 \to 0^+$.

Proof. Formula (A.1) follows directly from (4.2.11) and (4.2.13)–(4.2.14).

For the second property, the limit value is clear because (4.2.11) implies that $\ell(\rho_1) \to -v_2 > 0$ as $\theta_1 \to 0^+$, whilst $\cot^2\theta_1 \to \infty$ as $\theta_1 \to 0^+$. To determine the strict

monotonicity of $M_2^{(\theta_1)}$, we differentiate both (4.2.11) and (A.1) with respect to θ_1 and re-arrange the resulting expression to obtain

$$\text{(A.2)} \qquad \frac{(1-\rho_1)^2\cot\theta_1}{2(\rho_1+\cot^2\theta_1)(\ell(\rho_1))^2}\frac{\mathrm{d}(M_2^{(\theta_1)})^2}{\mathrm{d}\theta_1} = \rho_1 + \frac{1}{1-\rho_1}\frac{\ell(\rho_1)}{\ell'(\rho_1)} - \cot^2\theta_1\,.$$

Furthermore, from the definition of $\ell(\rho)$ in (2.3.3), we directly compute

$$(1-\rho)\frac{\ell'(\rho)}{\ell(\rho)} = -\frac{(\ell(\rho))^2+(1-\rho)^2h'(\rho)}{(\ell(\rho))^2(1+\rho)}\,.$$

Substituting the above expression into the right-hand side of (A.2) gives

$$\text{RHS of (A.2)} = \frac{\rho_1(1-\rho_1)^2h'(\rho_1)-(\ell(\rho_1))^2}{(\ell(\rho_1))^2+(1-\rho_1)^2h'(\rho_1)} - \cot^2\theta_1\,.$$

Denote the numerator of the first term above by

$$f(\rho) \coloneqq \rho(1-\rho)^2h'(\rho)-(\ell(\rho))^2 \equiv \frac{1-\rho}{1+\rho}\,\tilde{f}(\rho)\,,$$

where $\tilde{f}(\rho) \coloneqq (\gamma+1)h(\rho)-(\gamma-1)\rho^2h(\rho)+(1-\rho^2)$. Observe that $\tilde{f}(1)=0$ and

$$\tilde{f}'(\rho) = (\gamma+1)(1-\rho^2)h'(\rho) > 0 \qquad \text{for any } \rho\in(0,1)\,.$$

Thus, $\tilde{f}(\rho)<0$ for all $\rho\in(0,1)$. It follows directly that the RHS of (A.2) is negative. We conclude that $M_2^{(\theta_1)}$ is a strictly decreasing function of $\theta_1\in(0,\theta^{\mathrm{cr}})$. □

In Lemma A.1, we have shown that the pseudo-Mach number of state (2) at the reflection point P_0^1 is a strictly decreasing function of $\theta_1\in(0,\theta^{\mathrm{cr}})$. Subsequently, in Proposition 4.2, we have used Lemmas 3.5 and A.1 to prove the existence of the unique detachment angle θ^{d} and sonic angle θ^{s}. In Remark 4.3(i), we have stated the relation between θ^{d}, θ^{s}, and θ^{cr}, which depends on the choice of (γ,v_2). The following lemma provides the proof of this statement.

Lemma A.2. *For any $\gamma > 1$, there exist constants v_2^{d} and v_2^{s}, with $v_{\min} < v_2^{\mathrm{s}} < v_2^{\mathrm{d}} < 0$, uniquely determined by γ such that*

$$\mathrm{sgn}\left(\hat{\theta}_{25}(\theta^{\mathrm{cr}}; v_2) - \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, |D\varphi_2(P^1_{0,\mathrm{cr}})|)\right) = \mathrm{sgn}(v_2 - v_2^{\mathrm{d}})\,, \tag{A.3}$$

$$\mathrm{sgn}\left(\hat{\theta}_{25}(\theta^{\mathrm{cr}}; v_2) - \theta^{\mathrm{s}}_{\mathrm{stdy}}(1, |D\varphi_2(P^1_{0,\mathrm{cr}})|)\right) = \mathrm{sgn}(v_2 - v_2^{\mathrm{s}})\,, \tag{A.4}$$

where $P^1_{0,\mathrm{cr}} := P^1_0|_{\theta_1=\theta^{\mathrm{cr}}}$, and we define $\theta^{\mathrm{d}}_{\mathrm{stdy}}(1, M_\infty) \equiv \theta^{\mathrm{s}}_{\mathrm{stdy}}(1, M_\infty) := 0$ for any $0 \le M_\infty \le 1$. Furthermore, when $\gamma = 1$, we define $v_{\min} = v_2^{\mathrm{s}} = v_2^{\mathrm{d}} = -\infty$.

Proof. We divide the proof into two steps.

1. For $\gamma > 1$ and $v_2 \in (v_{\min}, 0)$, a direct calculation by using (4.2.12), (A.1), and Lemma A.1 gives

$$\begin{aligned} &M_{2,\min} := \inf_{\theta_1 \in (0,\theta^{\mathrm{cr}})} |D\varphi_2(P^1_0)| = |D\varphi_2(P^1_{0,\mathrm{cr}})| = \frac{v_2 v_{\min}}{\sqrt{v_{\min}^2 - v_2^2}}\,, \\ &\hat{\theta}_{25}(\theta^{\mathrm{cr}}; v_2) = \arcsin\left(\frac{-v_2}{M_{2,\min}}\right) = \arccos\left(\left|\frac{v_2}{v_{\min}}\right|\right) > 0\,. \end{aligned} \tag{A.5}$$

Then $\hat{\theta}_{25}(\theta^{\mathrm{cr}}; v_2)$ is strictly increasing with respect to $v_2 \in (v_{\min}, 0)$. We also calculate

$$\lim_{v_2 \to v_{\min}^+} M_{2,\min} = \infty\,, \quad \lim_{v_2 \to 0^-} M_{2,\min} = 0\,, \quad \frac{\mathrm{d}M^2_{2,\min}}{\mathrm{d}v_2} = \frac{2v_{\min}^4 v_2}{(v_{\min}^2 - v_2^2)^2} < 0\,, \tag{A.6}$$

so that $M_{2,\min}$ is strictly decreasing with respect to $v_2 \in (v_{\min}, 0)$. Also observe that $M_{2,\min} > 1$ if and only if $v_2 \in (v_{\min}, v_{\mathrm{mid}})$, for $v_{\mathrm{mid}} := \frac{v_{\min}}{\sqrt{v_{\min}^2+1}}$. Then, by Lemma 3.5, $\theta^{\mathrm{d}}_{\mathrm{stdy}}(1, M_{2,\min})$ is strictly decreasing with respect to $v_2 \in (v_{\min}, v_{\mathrm{mid}})$. In particular, by (A.5)–(A.6) and Lemma 3.5, we have

$$\begin{aligned} &(\hat{\theta}_{25}(\theta^{\mathrm{cr}}; v_2), \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, M_{2,\min})) \to (\arctan(|v_{\min}|), 0) && \text{as } v_2 \to v_{\mathrm{mid}}^-\,, \\ &(\hat{\theta}_{25}(\theta^{\mathrm{cr}}; v_2), \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, M_{2,\min})) \to (0, \tfrac{\pi}{2}) && \text{as } v_2 \to v_{\min}^+\,. \end{aligned}$$

Then we conclude from the monotonicity properties of $(\hat{\theta}_{25}(\theta^{\mathrm{cr}}; v_2)\,, \theta^{\mathrm{d}}_{\mathrm{stdy}}(1, M_{2,\min}))$ with respect to $v_2 \in (v_{\min}, v_{\mathrm{mid}})$ that there exists a unique value $v_2^{\mathrm{d}} \in (v_{\min}, v_{\mathrm{mid}})$, depending only on γ, such that (A.3) holds.

The proof of (A.4) follows via a similar argument using again Lemma 3.5, and the fact that $\theta^{\rm s}_{\rm stdy} \in (0, \theta^{\rm d}_{\rm stdy})$, which implies

$$\begin{aligned} &\theta^{\rm s}_{\rm stdy}(1, M_{2,\min}) \to 0 && \text{as } \ v_2 \to v^-_{\rm mid}\,, \\ &\theta^{\rm s}_{\rm stdy}(1, M_{2,\min}) \to \arctan\left(\tfrac{2}{\gamma-1}\right)^{1/2} && \text{as } \ v_2 \to v^+_{\rm min}\,. \end{aligned}$$

2. For $\gamma = 1$, from (4.2.12), we see that $v_{\min} = -\infty$ and $\theta^{\rm cr} = \frac{\pi}{2}$. Taking the limit $v_{\min} \to -\infty$ in Step **1** above, we obtain that $M_{2,\min} = |v_2|$ and $\hat{\theta}_{25}(\frac{\pi}{2}; v_2) = \frac{\pi}{2}$. From Lemma 3.5, we know that the sonic and detachment angles for 2-D steady potential flow satisfy $0 < \theta^{\rm s}(1, M_\infty) < \theta^{\rm d}(1, M_\infty) < \frac{\pi}{2}$ for any $M_\infty > 1$, which implies that (A.3)–(A.4) hold with $v_2^{\rm d} = v_2^{\rm s} = -\infty$. □

Lemma A.1 is also useful to determine the monotonicity of other quantities related to the four-shock interaction problem. In the following lemma, we show that the reflected shock angles θ_{25} and θ_{26} are monotonic with respect to the incident shock angles θ_1 and θ_2 respectively and, subsequently, that certain intersection points of the reflected shocks are also monotonic functions of the incident angles.

Lemma A.3. *Fix $\gamma \geq 1$ and $v_2 \in (v_{\min}, 0)$.*

(i) *For any incident angle $\theta_1 \in [0, \theta^{\rm d})$, let the reflected shock angle $\theta_{25} \in (\frac{\pi}{2}, \pi]$ be given by* (4.2.28). *Then θ_{25} is a continuous, strictly decreasing function of θ_1, and the map $\theta_1 \mapsto \theta_{25}$ is C^∞–smooth for $\theta_1 \in [0, \theta^{\rm d})$.*

(ii) *For the right-most intersection $P_2 = (\xi^{P_2}, \eta^{P_2})$ of $\partial B_{c_5}(O_5)$ and S_{25} as given in* Definition 4.6, *and for the η–intercept of S_{25} which is denoted here by a_{25}: η^{P_2} is strictly decreasing with respect to $\theta_1 \in (0, \theta^{\rm s})$, and a_{25} is strictly increasing with respect to $\theta_1 \in (0, \theta^{\rm d})$.*

Proof. For (i), the continuity and smoothness of the map $\theta_1 \mapsto \theta_{25}$ follows due to Proposition 4.2 and (4.2.28), whilst the strict monotonicity follows directly from Lemma A.1 and [1, Lemma 2.17]. Indeed, suppose that $\theta_1, \tilde{\theta}_1 \in [0, \theta^{\rm d})$ are such that $\theta_{25}(\theta_1) = \theta_{25}(\tilde{\theta}_1)$. Then, by [1, Lemma 2.17], the uniform state (5) is uniquely determined by $(v_2, \gamma, \theta_{25})$ so that the location of the reflected shock S_{25} must coincide

for both θ_1 and $\tilde{\theta}_1$. In particular, for the pseudo-Mach number of state (2) at the reflection point P_0^1, as defined by (A.1), we see that $M_2^{(\theta_1)} = M_2^{(\tilde{\theta}_1)}$ so that $\theta_1 = \tilde{\theta}_1$ by Lemma A.1.

We have shown that the map $\theta_1 \mapsto \theta_{25}$ is continuous and injective, and therefore strictly monotonic. In particular, by property (i) of Proposition 4.2 and (4.2.28), $\theta_{25} \to \pi^-$ as $\theta_1 \to 0^+$; whilst, by property (ii) of Proposition 4.2 and (4.2.28), $\theta_{25} < \pi$ for any $\theta_1 \in (0, \theta^{\rm d}]$. It follows that the map $\theta_1 \mapsto \theta_{25}$ is monotonic decreasing.

For (ii), the first result follows directly from part (i) combined with [1, Lemma 2.22], whilst the second result follows directly from part (i) combined with [1, Eqs. (2.4.14) and (2.4.42)]. □

Finally, we give the proof that the intersection point P_I of the reflected shocks S_{25} and S_{26} can be extended to a $C(\overline{\Omega})$–function.

Lemma A.4. *For $\boldsymbol{\theta} \in \overline{\Theta} \setminus \{\mathbf{0}\}$, denote by $P_I = (\xi^{P_I}, \eta^{P_I})$ the unique intersection point of the reflected shocks S_{25} and S_{26}, as given by* (4.4.9)*. Then*

$$\lim_{\boldsymbol{\theta} \to \mathbf{0},\, \boldsymbol{\theta} \in \overline{\Theta} \setminus \{\mathbf{0}\}} (\xi^{P_I}, \eta^{P_I}) = (0, \eta_0)\,, \tag{A.7}$$

where η_0 is given by (4.2.18)*.*

Proof. By Proposition 4.2 and Lemma A.3(i), the limit (A.7) is equivalent to the limit

$$\lim_{\substack{(\theta_{25},\theta_{26}) \to (\pi,0) \\ (\theta_{25},\theta_{26}) \in \overline{\Theta_{\rm refl}} \setminus \{(\pi,0)\}}} (\xi^{P_I}, \eta^{P_I}) = (0, \eta_0)\,,$$

where $\Theta_{\rm refl} \coloneqq \{(\theta_{25}, \theta_{26}) : \boldsymbol{\theta} \in \Theta\}$. The proof of the above limit consists of three steps.

1. We first consider the unilateral normal reflection case $\boldsymbol{\theta} \in \big(\{0\} \times (0, \theta^{\rm d})\big) \cup \big((0, \theta^{\rm d}) \times \{0\}\big)$. Without loss of generality, we restrict our attention to the case $\boldsymbol{\theta} \in (0, \theta^{\rm d}) \times \{0\}$, and denote $(v_\infty, \beta) \coloneqq (-v_2, \pi - \theta_{25})$, where θ_{25} depends only on (v_2, γ, θ_1). We consider the Prandtl–Meyer reflection configuration associated with parameters (v_∞, β), as defined in [1], and reflect this configuration in the vertical axis so that it corresponds with our choice $\boldsymbol{\theta} \in (0, \theta^{\rm d}) \times \{0\}$. By [1, Lemma 2.17], the

location of the reflected shock S_{25} is uniquely determined by $(v_2, \gamma, \theta_{25})$, and thus coincides with the Prandtl–Meyer oblique shock $S_{\mathcal{O}}$. Similarly, the (normal) reflected shock $S_{26} = S_0$ coincides with the Prandtl–Meyer normal shock $S_{\mathcal{N}}$. Therefore, the intersection point P_I of S_{25} and S_{26}, as given by (4.4.9), coincides with the intersection point of $S_{\mathcal{O}}$ and $S_{\mathcal{N}}$, as given by [1, Eq. (4.1.33)]. That is,

$$(\xi^{P_I}, \eta^{P_I}) = \big(-\frac{a_{25} - \eta_0}{\tan\theta_{25}}, \eta_0 \big) ,$$

where $a_{25} \coloneqq -\xi^{P_0^1} \tan\theta_{25}$ and η_0 are the η–intercepts of S_{25} and the normal reflection S_0, respectively, as given in §4.2. By Proposition 4.2(i) and Lemma A.3(ii), $a_{25} \to \eta_0$ as $\theta_{25} \to \pi^-$. Then we may apply L'Hôpital's rule to give

$$\lim_{\theta_{25}\to\pi^-} \xi^{P_I} = -\lim_{\theta_{25}\to\pi^-} \frac{\mathrm{d}a_{25}}{\mathrm{d}\theta_{25}} .$$

From [1, Eq. (2.4.14)], $a_{25} = v_2 - q_2 \sec\theta_{25}$, for $q_2 \coloneqq \mathrm{dist}(S_{25}, O_2)$. By taking the derivative with respect to θ_{25}, we have

$$\frac{\mathrm{d}a_{25}}{\mathrm{d}\theta_{25}} = -\sec\theta_{25} \frac{\mathrm{d}q_2}{\mathrm{d}\theta_{25}} - q_2 \sec\theta_{25} \tan\theta_{25} .$$

Using [20, Lemma 6.1.2] and (4.2.18), $1 < \lim\limits_{\theta_{25}\to\pi^-} q_2 = \eta_0 - v_2 < \infty$ so that

$$\lim_{\theta_{25}\to\pi^-} \frac{\mathrm{d}a_{25}}{\mathrm{d}\theta_{25}} = \lim_{\theta_{25}\to\pi^-} \frac{\mathrm{d}q_2}{\mathrm{d}\theta_{25}} .$$

By [1, Eq. (2.4.12)], the relation between the pseudo-Mach numbers $(M_2, M_5) \coloneqq (\frac{q_2}{c_2}, \frac{q_5}{c_5})$ for $q_5 \coloneqq \mathrm{dist}(S_{25}, O_5)$ and the given parameters $(\gamma, v_2, \theta_{25})$ is given by

$$M_2^{\frac{\gamma-1}{\gamma+1}} (M_2^{\frac{2}{\gamma+1}} - M_5^{\frac{2}{\gamma+1}}) = v_2 \sec\theta_{25} .$$

Taking the derivative with respect to θ_{25},

$$0 = \lim_{\theta_{25}\to\pi^-} \frac{1}{\gamma+1} \Big((\gamma-1) M_2^{-1} v_2 \sec\theta_{25} + 2 M_2^{\frac{\gamma-1}{\gamma+1}} \big(M_2^{\frac{1-\gamma}{1+\gamma}} - M_5^{\frac{1-\gamma}{1+\gamma}} \frac{\mathrm{d}M_5}{\mathrm{d}M_2} \big) \Big) \frac{\mathrm{d}M_2}{\mathrm{d}\theta_{25}} .$$

By [1, Eqs. (2.4.9)–(2.4.10)], we see that $\frac{\mathrm{d}M_5}{\mathrm{d}M_2} < 0$ for all $M_2 \in (0,\infty)\setminus\{1\}$. Therefore, the bracketed term above is uniformly positive for $\theta_{25} \in (\frac{\pi}{2},\pi)$ so that

$$\lim_{\theta_{25}\to\pi^-} \frac{\mathrm{d}q_2}{\mathrm{d}\theta_{25}} = \lim_{\theta_{25}\to\pi^-} \frac{\mathrm{d}M_2}{\mathrm{d}\theta_{25}} = 0\,.$$

2. We now consider the general case $\boldsymbol{\theta} \in \overline{\Theta}\setminus\{\mathbf{0}\}$. By direct calculation, we have

$$\xi^{P_I} = \frac{a_{25}-a_{26}}{\tan\theta_{26}-\tan\theta_{25}}\,,$$

where $a_{2j} > 0$ represents the η–intercept of S_{2j} for $j = 5, 6$. Note that a_{25} depends only on (v_2,γ,θ_{25}), whilst a_{26} depends only on (v_2,γ,θ_{26}). In particular, from Step **1**,

$$0 = \lim_{\theta_{25}\to\pi^-} \frac{a_{25}-\eta_0}{\tan\theta_{25}} = \lim_{\theta_{26}\to 0^+} \frac{a_{26}-\eta_0}{\tan\theta_{26}}\,. \tag{A.8}$$

Let $\{\boldsymbol{\theta}^{(k)}\}_{k\in\mathbf{N}} \subseteq \overline{\Theta}\setminus\{\mathbf{0}\}$ be any sequence of parameters with $\boldsymbol{\theta}^{(k)} \to \mathbf{0}$ as $k\to\infty$. By moving to a subsequence, we may assume one of the following cases: (i) $\theta_1^{(k)} = 0$ for all $k\in\mathbf{N}$; (ii) $\theta_2^{(k)} = 0$ for all $k\in\mathbf{N}$; (iii) both $\theta_1^{(k)} > 0$ and $\theta_2^{(k)} > 0$ for all $k\in\mathbf{N}$. Denote by $\xi_{(k)}^{P_I}$ the ξ–coordinates of the intersection points P_I associated with parameters $\boldsymbol{\theta}^{(k)}$. In cases (i) and (ii), it is clear from the previous step that $\xi_{(k)}^{P_I} \to 0$ as $k\to\infty$, so we focus on case (iii). Using the triangle inequality, the fact that $\tan\theta_{25}$ and $\tan\theta_{26}$ have opposite signs, and (A.8), we obtain

$$|\xi_{(k)}^{P_I}| = \Big|\frac{a_{25}^{(k)}-a_{26}^{(k)}}{\tan\theta_{26}^{(k)}-\tan\theta_{25}^{(k)}}\Big| \le \Big|\frac{a_{25}^{(k)}-\eta_0}{\tan\theta_{25}^{(k)}}\Big| + \Big|\frac{a_{26}^{(k)}-\eta_0}{\tan\theta_{26}^{(k)}}\Big| \to 0 \qquad \text{as } k\to\infty\,.$$

3. Finally, we show that $\eta^{P_I} \to \eta_0$ as $\boldsymbol{\theta}\to\mathbf{0}$ with $\boldsymbol{\theta}\in\overline{\Theta}\setminus\{\mathbf{0}\}$. Indeed, it is clear from (A.8) that $a_{25}\to\eta_0$ as $\theta_1\to 0^+$, and $a_{26}\to\eta_0$ as $\theta_2\to 0^+$. Furthermore, it follows from Step **1** of the proof of Lemma 4.12 that $\min\{a_{25},a_{26}\} \le \eta^{P_I} \le \max\{a_{25},a_{26}\}$. □

Appendix B

Collection of Known Results

In this appendix, for the convenience of the reader, we present some known results that are used during the proof of Theorem 4.9. Some of the results stated have been adapted slightly to the current context.

B.1 Well-posedness of the iteration boundary value problem

For a constant $h > 0$ and a function $f_{\mathrm{bd}} : [0, h] \to \mathbf{R}_+$, fix a bounded domain $\Omega \subseteq \mathbf{R}^2$

$$\Omega \coloneqq \left\{ \boldsymbol{x} \in \mathbf{R}^2 \,:\, 0 < x_1 < h,\, 0 < x_2 < f_{\mathrm{bd}}(x_1) \right\}, \tag{B.1}$$

where f_{bd} satisfies that, for constants $t_0 \geq 0$, t_h, t_1, t_2, t_3, $M \in (0, \infty)$, and $\alpha \in (0, 1)$,

$$\begin{aligned} &f_{\mathrm{bd}} \in C^1([0, h])\,, \quad f_{\mathrm{bd}}(0) = t_0\,, \quad f_{\mathrm{bd}}(h) = t_h\,, \\ &f_{\mathrm{bd}}(x_1) \geq \min\{t_0 + t_1 x_1,\, t_2,\, t_h - t_3(x_1 - h)\} \qquad \text{for all } x_1 \in (0, h)\,, \\ &\|f_{\mathrm{bd}}\|_{2,\alpha,(0,h)}^{(-1-\alpha),\{0,h\}} \leq M\,. \end{aligned} \tag{B.2}$$

The boundary, $\partial\Omega \coloneqq \bigcup_{k=0}^{3} (\Gamma_k \cup \{P_{k+1}\})$, has vertices and segments

$$
\begin{aligned}
&P_1 = (h, 0)\,, \quad P_2 = (h, f_{\mathrm{bd}}(h))\,, \quad P_3 = (0, f_{\mathrm{bd}}(0))\,, \quad P_4 = (0, 0)\,, \\
&\overline{\Gamma_0} = \partial\Omega \cap \{x_1 = 0\}\,, \quad \overline{\Gamma_1} = \partial\Omega \cap \{x_2 = f_{\mathrm{bd}}(x_1)\}\,, \\
&\overline{\Gamma_2} = \partial\Omega \cap \{x_1 = h\}\,, \quad \overline{\Gamma_3} = \partial\Omega \cap \{x_2 = 0\}\,,
\end{aligned} \tag{B.3}
$$

and Γ_k, for $k = 0, 1, 2, 3$, are the relative interiors of the segments defined above (which may be empty).

Let $g_{\mathrm{so}} \in C^\infty(\mathbf{R}^2 \setminus \{\frac{h}{3} < x_1 < \frac{2h}{3}\})$ be a piecewise smooth function defined on $\mathbf{R}^2$ such that

$$
\begin{aligned}
&\|g_{\mathrm{so}}\|_{C^3(\overline{\Omega \setminus \{\frac{h}{3} < x_1 < \frac{2h}{3}\}})} \le C_{\mathrm{so}}\,, \\
&g_{\mathrm{so}}(\cdot, x_2) \quad \text{is linear in } x_1 \text{ on } \{x_1 \le \tfrac{h}{4}\} \cup \{x_1 \ge \tfrac{3h}{4}\}\,, \\
&\partial_{x_2} g_{\mathrm{so}} = 0 \quad \text{on } \Gamma_3\,.
\end{aligned} \tag{B.4}
$$

We consider the following nonlinear problem

$$
\begin{cases}
\displaystyle\sum_{i,j=1}^{2} \tilde{A}_{ij}(Du, \boldsymbol{x}) D_{ij} u + \sum_{i=1}^{2} \tilde{A}_i(Du, \boldsymbol{x}) D_i u = 0 & \text{in } \Omega\,, \\
\tilde{B}(Du, u, \boldsymbol{x}) = 0 & \text{on } \Gamma_1\,, \\
u = g_{\mathrm{so}}(\boldsymbol{x}) & \text{on } \Gamma_2 \cup \Gamma_0\,, \\
\boldsymbol{b}^{(\mathrm{w})}(\boldsymbol{x}) \cdot Du = 0 & \text{on } \Gamma_3\,.
\end{cases} \tag{B.5}
$$

For constants $\lambda \in (0, 1)$, $M < \infty$, $\alpha \in (0, 1)$, $\beta \in [\frac{1}{2}, 1)$, $\delta \in (0, 1)$, $\sigma \in (0, 1)$, and $\varepsilon \in (0, \frac{h}{10})$, assume that the following conditions are satisfied

(i) For any $\boldsymbol{x} \in \Omega$ and $\boldsymbol{p}, \boldsymbol{\mu} \in \mathbf{R}^2$, $\lambda \operatorname{dist}(\boldsymbol{x}, \Gamma_2 \cup \Gamma_0)|\boldsymbol{\mu}|^2 \le \sum_{i,j=1}^{2} \tilde{A}_{ij}(\boldsymbol{p}, \boldsymbol{x}) \mu_i \mu_j \le \lambda^{-1}|\boldsymbol{\mu}|^2$. Moreover, for any $\boldsymbol{x} \in \Omega \setminus \{\frac{\varepsilon}{2} \le x_1 \le h - \frac{\varepsilon}{2}\}$,

$$
\lambda|\boldsymbol{\mu}|^2 \le \sum_{i,j=1}^{2} \frac{\tilde{A}_{ij}(\boldsymbol{p}, \boldsymbol{x}) \mu_i \mu_j}{(\min\{x_1, h - x_1, \delta\})^{2 - \frac{i+j}{2}}} \le \lambda^{-1}|\boldsymbol{\mu}|^2\,.
$$

(ii) $(\tilde{A}_{ij}, \tilde{A})(\boldsymbol{p}, \boldsymbol{x})$ are independent of $\boldsymbol{p} \in \mathbf{R}^2$ on $\Omega \cap \{\varepsilon \le x_1 \le h - \varepsilon\}$ with

$$\|\tilde{A}_{ij}\|_{L^\infty(\Omega \cap \{\varepsilon \le x_1 \le h-\varepsilon\})} \le \lambda^{-1}\,, \qquad \|(\tilde{A}_{ij}, \tilde{A})\|_{1,\alpha,\Omega\cap\{\varepsilon\le x_1\le h-\varepsilon\}} \le M\,.$$

(iii) For any $\boldsymbol{p} \in \mathbf{R}^2$,

$$\|(\tilde{A}_{ij}, \tilde{A}_i)(\boldsymbol{p}, \cdot)\|_{C^\beta(\overline{\Omega\setminus\{2\varepsilon\le x_1\le h-2\varepsilon\}})} + \|D_{\boldsymbol{p}}(\tilde{A}_{ij}, \tilde{A}_i)(\boldsymbol{p}, \cdot)\|_{L^\infty(\Omega\setminus\{2\varepsilon\le x_1\le h-2\varepsilon\})} \le M\,.$$

(iv) $(\tilde{A}_{ij}, \tilde{A}_i) \in C^{1,\alpha}(\mathbf{R}^2 \times (\overline{\Omega} \setminus (\overline{\Gamma_2} \cup \overline{\Gamma_0})))$ and, for any $s \in (0, \frac{h}{4})$,

$$\|(\tilde{A}_{ij}, \tilde{A}_i)\|_{1,\alpha,\mathbf{R}^2\times(\overline{\Omega}\cap\{s\le x_1\le h-s\})} \le M\Big(\frac{h}{s}\Big)^M.$$

(v) For each $(\boldsymbol{p}, \boldsymbol{x}) \in \mathbf{R}^2 \times (\overline{\Omega} \setminus \{\varepsilon < x_1 < h - \varepsilon\})$ and $i, j = 1, 2$, define

$$\hat{\boldsymbol{p}} \coloneqq \boldsymbol{p} - Dg_{\mathrm{so}}(\boldsymbol{x})\,, \qquad (\tilde{a}_{ij}, \tilde{a}_i)(\hat{\boldsymbol{p}}, \boldsymbol{x}) \coloneqq (\tilde{A}_{ij}, \tilde{A}_i)(\hat{\boldsymbol{p}} + Dg_{\mathrm{so}}(\boldsymbol{x}), \boldsymbol{x})\,.$$

For any $(\boldsymbol{p}, (x_1, 0)) \in \mathbf{R}^2 \times (\Gamma_3 \setminus \{\varepsilon \le x_1 \le h - \varepsilon\})$,

$$(\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_1)((\hat{p}_1, -\hat{p}_2), (x_1, 0)) = (\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_1)((\hat{p}_1, \hat{p}_2), (x_1, 0))\,.$$

For any $(\boldsymbol{p}, \boldsymbol{x}) \in \mathbf{R}^2 \times (\Omega \setminus \{\varepsilon \le x_1 \le h - \varepsilon\})$ and $i = 1, 2$,

$$|\tilde{a}_{ii}(\hat{\boldsymbol{p}}, \boldsymbol{x}) - \tilde{a}_{ii}(Dg_{\mathrm{so}}(\boldsymbol{x}_{\mathrm{e}}), \boldsymbol{x}_{\mathrm{e}})| \le M|\boldsymbol{x} - \boldsymbol{x}_{\mathrm{e}}|^\beta\,, \qquad (\tilde{A}_{12}, \tilde{A}_{21})(\boldsymbol{p}, \boldsymbol{x}_{\mathrm{e}}) = 0\,,$$

either for the point $\boldsymbol{x}_{\mathrm{e}} \coloneqq (0, x_2)$, or the point $\boldsymbol{x}_{\mathrm{e}} \coloneqq (h, x_2)$.

(vi) For any $\boldsymbol{p} \in \mathbf{R}^2$ and $\boldsymbol{x} \in \Omega \setminus \{\frac{\varepsilon}{2} \le x_1 \le h - \frac{\varepsilon}{2}\}$, $\tilde{A}_1(\boldsymbol{p}, \boldsymbol{x}) \le -\lambda\,.$

(vii) The nonlinear boundary condition $(\mathrm{B}.5)_2$ satisfies the following conditions.

(vii–a) For any $(\boldsymbol{p}, z, \boldsymbol{x}) \in \mathbf{R}^2 \times \mathbf{R} \times \Gamma_1$, $D_{\boldsymbol{p}}\tilde{B}(\boldsymbol{p}, z, \boldsymbol{x}) \cdot \boldsymbol{\nu}^{(1)}(\boldsymbol{x}) \ge \lambda$, where $\boldsymbol{\nu}^{(1)}$ is the unit normal vector on Γ_1 pointing into Ω.

(vii–b) For any $(\boldsymbol{p}, z) \in \mathbf{R}^2 \times \mathbf{R}$ and $k = 1, 2, 3$,

$$\begin{aligned}
&\|\tilde{B}(Dg_{\mathrm{so}}, g_{\mathrm{so}}, \cdot)\|_{C^3(\overline{\Omega}\setminus\{\frac{h}{3} \le x_1 \le \frac{2h}{3}\})} + \|D^k_{(\boldsymbol{p},z)}\tilde{B}(\boldsymbol{p}, z, \cdot)\|_{C^3(\overline{\Omega})} \le M\,,\\
&\|D_{\boldsymbol{p}}\tilde{B}(\boldsymbol{p}, z, \cdot)\|_{C^0(\overline{\Omega})} \le \lambda^{-1}\,,\\
&D_z\tilde{B}(\boldsymbol{p}, z, \boldsymbol{x}) \le -\lambda \qquad \text{for all } \boldsymbol{x} \in \Gamma_1\,,\\
&D_{p_1}\tilde{B}(\boldsymbol{p}, z, \boldsymbol{x}) \le -\lambda \qquad \text{for all } \boldsymbol{x} \in \Gamma_1 \setminus \{\varepsilon \le x_1 \le h - \varepsilon\}\,.
\end{aligned}$$

(vii–c) There exist $v \in C^3(\overline{\Omega})$ and an inhomogeneous linear operator

$$L(\boldsymbol{p}, z, \boldsymbol{x}) = \boldsymbol{b}^{(1)}(\boldsymbol{x}) \cdot \boldsymbol{p} + b_0^{(1)}(\boldsymbol{x})z + g_1(\boldsymbol{x})$$

defined for $\boldsymbol{x} \in \Gamma_1$ and $(\boldsymbol{p}, z) \in \mathbf{R}^2 \times \mathbf{R}$, satisfying

$$\|v\|_{C^3(\Omega)} + \|(\boldsymbol{b}^{(1)}, b_0^{(1)}, g_1)\|_{C^3(\overline{\Gamma_1})} \le M\,,$$

such that, for any $(\boldsymbol{p}, z, \boldsymbol{x}) \in \mathbf{R}^2 \times \mathbf{R} \times \Gamma_1$,

$$\begin{aligned}
&\big|\tilde{B}(\boldsymbol{p}, z, \boldsymbol{x}) - L(\boldsymbol{p}, z, \boldsymbol{x})\big| \le \sigma\big(|\boldsymbol{p} - Dv(\boldsymbol{x})| + |z - v(\boldsymbol{x})|\big)\,,\\
&\big|D_{\boldsymbol{p}}\tilde{B}(\boldsymbol{p}, z, \boldsymbol{x}) - \boldsymbol{b}^{(1)}(\boldsymbol{x})\big| + \big|D_z\tilde{B}(\boldsymbol{p}, z, \boldsymbol{x}) - b_0^{(1)}(\boldsymbol{x})\big| \le \sigma\,.
\end{aligned}$$

(vii–d) *Obliqueness requirements*: For the unit normal vector $\boldsymbol{\nu}^{(\mathrm{w})}$ on Γ_3 pointing into Ω,

$$\begin{aligned}
&\boldsymbol{b}^{(\mathrm{w})} \cdot \boldsymbol{\nu}^{(\mathrm{w})} \ge \lambda\,, \quad b_1^{(\mathrm{w})} \le 0 \qquad \text{on } \Gamma_3\,,\\
&\boldsymbol{b}^{(\mathrm{w})} = (0, 1) \ \text{ on } \Gamma_3 \setminus \{\varepsilon \le x_1 \le h - \varepsilon\}\,, \qquad \|\boldsymbol{b}^{(\mathrm{w})}\|_{C^3(\overline{\Gamma_3})} \le M\,.
\end{aligned}$$

(vii–e) $\tilde{B}(\mathbf{0}, 0, \cdot) \equiv 0 \quad$ on $\Gamma_1 \setminus \{\varepsilon \le x_1 \le h - \varepsilon\}$.

Proposition B.1 ([20, Proposition 4.7.2])**.** *Fix the constants h, t_1, t_2, $t_3 \in (0, \infty)$ and t_0, $t_h \in [0, \infty)$. For some constants $\lambda > 0$, $M < \infty$, $\alpha \in (0, 1)$, and $C_{\mathrm{so}} > 0$, assume that the domain $\Omega \subseteq \mathbf{R}^2$ satisfies conditions* (B.1)–(B.4)*. For some constants $\beta \in [\frac{1}{2}, 1)$, $\delta \in (0, 1)$, $\sigma \in (0, 1)$, and $\varepsilon \in (0, \frac{h}{10})$, assume that the nonlinear boundary*

value problem (B.5) *satisfies conditions* (i)–(vii).

Then there exist constants $\alpha_1 \in (0, \frac{1}{2})$ *depending only on* λ, *and* δ_0, $\sigma \in (0,1)$ *depending only on* $(\lambda, M, C_{\rm so}, \alpha, \beta, \varepsilon)$ *such that, under the further requirement that* $\delta \in [0, \delta_0)$, *the boundary value problem* (B.5) *has a unique solution* $u \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus (\overline{\Gamma_2} \cup \overline{\Gamma_0})) \cap C^2(\Omega)$. *Moreover, the solution* u *satisfies*

$$\|u\|_{0,\overline{\Omega}} \le C\,, \qquad |u(\boldsymbol{x}) - g_{\rm so}(\boldsymbol{x})| \le C \min\{x_1, h - x_1\} \quad \text{in } \Omega\,, \tag{B.6}$$

where $C > 0$ *depends only on* $(\lambda, M, C_{\rm so}, \varepsilon)$. *Furthermore,* $u \in C(\overline{\Omega}) \cap C^{2,\alpha_1}(\overline{\Omega} \setminus \overline{\Gamma_2} \cup \overline{\Gamma_0})$ *with*

$$\|u\|_{2,\alpha_1,\overline{\Omega \cap \{s < x_1 < h-s\}}} \le C_s \tag{B.7}$$

for each $s \in (0, \frac{h}{10})$, *where the constant* $C_s > 0$ *depends only on* $(\lambda, M, C_{\rm so}, \alpha, \beta, \varepsilon, s)$.

Proposition B.2 ([20, Proposition 4.8.7])**.** *Fix the constants* h, t_1, t_2, $t_3 \in (0, \infty)$, $t_0 = 0$, *and* $t_h \ge 0$. *For some constants* $\lambda > 0$, $M < \infty$, $\alpha \in (0,1)$, *and* $C_{\rm so} > 0$, *assume that the domain* $\Omega \subseteq \mathbf{R}^2$ *satisfies conditions* (B.1)–(B.4) *with the following changes*: $P_3 = P_4 = (0,0)$, *and* $\overline{\Gamma_0} = \{(0,0)\}$. *For* $\beta \in [\frac{1}{2}, 1)$, $\delta \in (0,1)$, $\sigma \in (0,1)$, *and* $\varepsilon \in (0, \frac{h}{10})$, *assume that the nonlinear boundary value problem* (B.5) *satisfies conditions* (ii), (iv), *and* (vii) *above, and* (i*), (iii*), (v*), *and* (vi*) *below*:

(i*) *For any* $\boldsymbol{x} \in \Omega$ *and* $\boldsymbol{p}, \boldsymbol{\kappa} = (\kappa_1 \kappa_2) \in \mathbf{R}^2$,

$$\min\left\{\lambda \operatorname{dist}(\boldsymbol{x}, \overline{\Gamma_0}) + \delta, \lambda \operatorname{dist}(x, \Gamma_2)\right\} |\boldsymbol{\kappa}|^2 \le \sum_{i,j=1}^{2} \tilde{A}_{ij}(\boldsymbol{p}, \boldsymbol{x}) \kappa_i \kappa_j \le \lambda^{-1} |\boldsymbol{\kappa}|^2\,,$$

$$\left\| ((\tilde{A}_{ij}, \tilde{A}_i)(Dg_{\rm so}, \cdot), D^m_{\boldsymbol{p}}(\tilde{A}_{ij}, \tilde{A}_i)(\boldsymbol{p}, \cdot)) \right\|^{(-\alpha),\{P_4\}}_{1,\alpha,\Omega \cap \{x_1 < 2\varepsilon\}} \le M \qquad \text{for } m = 1, 2\,.$$

Moreover, for any $\boldsymbol{x} \in \Omega \cap \{h - \frac{\varepsilon}{2} < x_1 < h\}$,

$$\lambda |\boldsymbol{\mu}|^2 \le \sum_{i,j=1}^{2} \frac{\tilde{A}_{ij}(\boldsymbol{p}, \boldsymbol{x}) \mu_i \mu_j}{(\min\{h - x_1, \delta\})^{2 - \frac{i+j}{2}}} \le \lambda^{-1} |\boldsymbol{\mu}|^2\,.$$

(iii*) *For any* $\boldsymbol{p} \in \mathbf{R}^2$,

$$\|(\tilde{A}_{ij}, \tilde{A}_i)(\boldsymbol{p}, \cdot)\|_{C^\beta(\overline{\Omega \cap \{h-2\varepsilon < x_1 < h\}})} + \|D_{\boldsymbol{p}}(\tilde{A}_{ij}, \tilde{A}_i)(\boldsymbol{p}, \cdot)\|_{L^\infty(\Omega \cap \{h-2\varepsilon < x_1 < h\})} \le M\,.$$

(v*) *For each* $(\boldsymbol{p}, \boldsymbol{x}) \in \mathbf{R}^2 \times (\overline{\Omega} \cap \{h - \varepsilon < x_1 < h\})$ *and* $i, j = 1, 2$, *define*

$$\hat{\boldsymbol{p}} \coloneqq \boldsymbol{p} - Dg_{\mathrm{so}}(\boldsymbol{x})\,, \qquad (\tilde{a}_{ij}, \tilde{a}_i)(\hat{\boldsymbol{p}}, \boldsymbol{x}) \coloneqq (\tilde{A}_{ij}, \tilde{A}_i)(\hat{\boldsymbol{p}} + Dg_{\mathrm{so}}(\boldsymbol{x}), \boldsymbol{x})\,.$$

For any $(\boldsymbol{p}, (x_1, 0)) \in \mathbf{R}^2 \times (\Gamma_3 \cap \{h - \varepsilon < x_1 < h\})$,

$$(\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_1)((\hat{p}_1, -\hat{p}_2), (x_1, 0)) = (\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_1)((\hat{p}_1, \hat{p}_2), (x_1, 0))\,.$$

For $\boldsymbol{x}_{\mathrm{e}} \coloneqq (h, x_2)$, *any* $(\boldsymbol{p}, \boldsymbol{x}) \in \mathbf{R}^2 \times (\Omega \cap \{h - \varepsilon < x_1 < h\})$, *and* $i = 1, 2$,

$$|\tilde{a}_{ii}(\hat{\boldsymbol{p}}, \boldsymbol{x}) - \tilde{a}_{ii}(Dg_{\mathrm{so}}(\boldsymbol{x}_{\mathrm{e}}), \boldsymbol{x}_{\mathrm{e}})| \le M|\boldsymbol{x} - \boldsymbol{x}_{\mathrm{e}}|^\beta, \quad (\tilde{A}_{12}, \tilde{A}_{21})(\boldsymbol{p}, \boldsymbol{x}_{\mathrm{e}}) = 0\,.$$

(vi*) *For any* $\boldsymbol{p} \in \mathbf{R}^2$ *and* $\boldsymbol{x} \in \Omega \cap \{h - \frac{\varepsilon}{2} < x_1 < h\}$, $\tilde{A}_1(\boldsymbol{p}, \boldsymbol{x}) \le -\lambda\,.$

Then there exist both $\alpha_1 \in (0, \frac{1}{2})$ *depending only on* (λ, δ), *and* $\sigma \in (0, 1)$ *depending only on* $(\lambda, \delta, M, C_{\mathrm{so}}, \alpha, \beta, \varepsilon)$ *such that, under the further requirement that* $\delta \in [0, \delta_0)$, *the boundary value problem* (B.5) *has a unique solution* $u \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus (\overline{\Gamma_2} \cup \overline{\Gamma_0})) \cap C^2(\Omega)$. *Moreover, the solution* u *satisfies*

$$\|u\|_{0,\overline{\Omega}} \le C\,, \qquad |u(\boldsymbol{x}) - g_{\mathrm{so}}(\boldsymbol{x})| \le C \min\{x_1, h - x_1\} \quad \text{in } \Omega\,, \tag{B.8}$$

where $C > 0$ *depends only on* $(\lambda, \delta, M, C_{\mathrm{so}}, \varepsilon)$. *Furthermore,* $u \in C(\overline{\Omega}) \cap C^{2,\alpha_1}(\overline{\Omega} \setminus \overline{\Gamma_2} \cup \overline{\Gamma_0})$ *with*

$$\|u\|_{2,\alpha_1,\overline{\Omega \cap \{s < x_1 < h-s\}}} \le C_s\,, \qquad \|u\|^{(-1-\alpha_1),\{P_4\}}_{2,\alpha_1,\Omega \cap \{x_1 < \frac{h}{4}\}} < \hat{C}\,, \tag{B.9}$$

for each $s \in (0, \frac{h}{10})$, *where the constant* $C_s > 0$ *depends only on* $(\lambda, \delta, M, C_{\mathrm{so}}, \alpha, \beta, \varepsilon, s)$, *and* $\hat{C} > 0$ *depends only on* $(\lambda, \delta, M, \alpha, \varepsilon)$.

Proposition B.3. *Fix the constants* h, t_1, t_2, $t_3 \in (0,\infty)$ *and* $t_0 = t_h = 0$. *For some constants* $\lambda > 0$, $M < \infty$, $\alpha \in (0,1)$, *and* $C_{\rm so} > 0$, *assume that the domain* $\Omega \subseteq \mathbf{R}^2$ *satisfies conditions* (B.1)–(B.4) *with the following changes*:

$$P_3 = P_4 = (0,0)\,, \quad \overline{\Gamma_0} = \{(0,0)\}\,; \qquad P_2 = P_1 = (h,0)\,, \quad \overline{\Gamma_2} = \{(h,0)\}\,.$$

For $\beta \in [\frac{1}{2},1)$, $\delta \in (0,1)$, $\sigma \in (0,1)$, *and* $\varepsilon \in (0,\frac{h}{10})$, *assume that the nonlinear boundary value problem* (B.5) *satisfies conditions* (ii), (iv), *and* (vii) *above, and* (i**) *below.*

(i**) *For any* $\boldsymbol{x} \in \Omega$ *and* $\boldsymbol{p}, \boldsymbol{\kappa} = (\kappa_1,\kappa_2) \in \mathbf{R}^2$,

$$\min\big\{\lambda\,\mathrm{dist}(\boldsymbol{x},\overline{\Gamma_0}) + \delta, \lambda\,\mathrm{dist}(x,\Gamma_2) + \delta\big\}|\boldsymbol{\kappa}|^2 \le \sum_{i,j=1}^{2} \tilde{A}_{ij}(\boldsymbol{p},\boldsymbol{x})\kappa_i\kappa_j \le \lambda^{-1}|\boldsymbol{\kappa}|^2\,,$$
$$\big\|((\tilde{A}_{ij},\tilde{A}_i)(Dg_{\rm so},\cdot), D^m_{\boldsymbol{p}}(\tilde{A}_{ij},\tilde{A}_i)(\boldsymbol{p},\cdot))\big\|^{(-\alpha),\{P_1,P_4\}}_{1,\alpha,\Omega\setminus\{2\varepsilon<x_1<h-2\varepsilon\}} \le M \qquad \text{for } m = 1,2\,.$$

Then there exist constants $\alpha_1 \in (0,\frac{1}{2})$ *depending only on* (λ,δ), *and* $\sigma \in (0,1)$ *depending only on* $(\lambda,\delta,M,C_{\rm so},\alpha,\beta,\varepsilon)$ *such that, under the further requirement that* $\delta \in [0,\delta_0)$, *the boundary value problem* (B.5) *has a unique solution* $u \in C(\overline{\Omega}) \cap C^1(\overline{\Omega}\setminus(\overline{\Gamma_2}\cup\overline{\Gamma_0})) \cap C^2(\Omega)$. *Moreover, the solution* u *satisfies*

$$\|u\|_{0,\overline{\Omega}} \le C\,, \qquad |u(\boldsymbol{x}) - g_{\rm so}(\boldsymbol{x})| \le C\min\{x_1, h-x_1\} \quad \text{in } \Omega\,, \tag{B.10}$$

where $C > 0$ *depends only on* $(\lambda,\delta,M,C_{\rm so},\varepsilon)$. *Furthermore,* $u \in C(\overline{\Omega}) \cap C^{2,\alpha_1}(\overline{\Omega}\setminus\overline{\Gamma_2}\cup\overline{\Gamma_0})$ *with*

$$\|u\|_{2,\alpha_1,\overline{\Omega\cap\{s<x_1<h-s\}}} \le C_s\,, \qquad \|u\|^{(-1-\alpha_1),\{P_1,P_4\}}_{2,\alpha_1,\Omega\setminus\{\frac{h}{4}\le x_1\le\frac{3h}{4}\}} < \hat{C}\,, \tag{B.11}$$

for each $s \in (0,\frac{h}{10})$, *where the constant* $C_s > 0$ *depends only on* $(\lambda,\delta,M,C_{\rm so},\alpha,\beta,\varepsilon,s)$, *and* $\hat{C} > 0$ *depends only on* $(\lambda,\delta,M,\alpha,\varepsilon)$.

B.2 Regularised distance function

Lemma B.4 ([20, Lemmas 13.9.1])**.** *For any $g \in C^{0,1}([-1,1])$ that is positive on $(-1,1)$, there exists a function $\delta_g \in C^\infty(\overline{R_\infty} \setminus \overline{R_g})$, the regularised distance, such that*

(i) $|\delta_g(s,t') - \mathrm{dist}((s,t'),\Sigma_g)| \leq \frac{1}{2}\mathrm{dist}(\mathbf{x},\Sigma_g)$ *for all* $(s,t') \in \mathbf{R}^2 \setminus \overline{\Sigma_g}$;

(ii) $|D^m\delta_g(s,t')| \leq C_m\big(\mathrm{dist}((s,t'),\Sigma_g)\big)^{1-m}$ *for all* $(s,t') \in \mathbf{R}^2\setminus\overline{\Sigma_g}$ *and* $m = 1,2,\cdots,$ *where* $C_m > 0$ *depends only on* m;

(iii) $\delta_g(s,t') \geq C_{\mathrm{rd}}(t' - g(s))$ *for all* $(s,t') \in \overline{R_g} \setminus \overline{\Sigma_g}$, *where* $C_{\mathrm{rd}} > 0$ *depends only on* $\mathrm{Lip}[g]$;

(iv) *Let* $g_k, g \in C^{0,1}([-1,1])$ *be uniformly bounded Lipschitz functions for all* $k \in \mathbf{N}$, *with* g_k *converging to* g *uniformly on* $[-1,1]$. *Then* $\|\delta_{g_k} - \delta_g\|_{C^m(K)} \to 0$ *as* $k \to \infty$ *for any compact set* $K \subseteq \overline{R_\infty} \setminus \overline{R_g}$, *and* $m = 0,1,2,\cdots$.

B.3 Leray–Schauder degree theorem

Definition B.5 (Compact map)**.** Let X and Y be Banach spaces. Fix any open subset $G \subseteq X$. A map $\mathbf{f} : \overline{G} \to Y$ is called compact if

(i) $\mathbf{f}$ is continuous;

(ii) $\mathbf{f}(U)$ is precompact in Y for any bounded subset $U \subseteq \overline{G}$.

Definition B.6. Let X be a Banach space, and let $G \subseteq X$ be a bounded open subset of X. Write $V(G,X)$ to denote the set of all maps $\mathbf{f} : \overline{G} \to X$ satisfying

(i) $\mathbf{f}$ is a compact map in the sense of Definition B.5;

(ii) $\mathbf{f}$ has no fixed point on the boundary of ∂G.

Definition B.7. Let X be a Banach space, and let $G \subseteq X$ be a bounded open subset of X. Two maps $\mathbf{f}, \mathbf{g} \in V(G,X)$ are called compactly homotopic on ∂G if there exists a map $\mathbf{H}$ with the following properties:

(i) $\mathbf{H} : \overline{G} \times [0,1] \to X$ is a compact map;

(ii) $\mathbf{H}(\mathbf{x}, \tau) \neq \mathbf{x}$ for all $(\mathbf{x}, \tau) \in \partial G \times [0,1]$;

(iii) $\mathbf{H}(\mathbf{x}, 0) = \mathbf{f}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x}, 1) = \mathbf{g}(\mathbf{x})$ for all $\mathbf{x} \in \overline{G}$.

We write $\partial G : \mathbf{f} \cong \mathbf{g}$ if $\mathbf{f}$ and $\mathbf{g}$ are compactly homotopic on ∂G in the sense of Definition B.7, and we call $\mathbf{H}$ a compact homotopy.

Theorem B.8 (Leray–Schauder degree theorem)**.** *Let X be a Banach space, and let $G \subseteq X$ be a bounded open subset of X. Then, for each map $\mathbf{f} \in V(G, X)$, there exists a unique integer $\mathbf{Ind}(\mathbf{f}, G)$, called the fixed point index of $\mathbf{f}$ over G, such that*

(i) $\mathbf{Ind}(\mathbf{f}, G) = 1$ *whenever* $\mathbf{f}$ *is a constant map on* $\overline{G}$;

(ii) $\mathbf{Ind}(\mathbf{f}, G) = 0$ *whenever* $\mathbf{f}$ *has no fixed point in* G;

(iii) $\mathbf{Ind}(\mathbf{f}, G) = \sum_{j=1}^{n} \mathbf{Ind}(\mathbf{f}, G_j)$ *whenever* $\mathbf{f} \in V(G, X) \cap (\bigcap_{j=1}^{n} V(G_j, X))$, *for any finite collection of pairwise disjoint open subsets* $G_j \subseteq G$ *satisfying* $\overline{G} = \bigcup_{j=1}^{n} \overline{G_j}$;

(iv) $\mathbf{Ind}(\mathbf{f}, G) = \mathbf{Ind}(\mathbf{g}, G)$ *whenever* $\mathbf{g} \in V(G, X)$ *satisfies* $\partial G : \mathbf{f} \cong \mathbf{g}$.

The following theorem gives a generalised homotopy invariance of the fixed point index.

Theorem B.9 ([82, §13.6, A4*])**.** *Let X be a Banach space, and let $t_1 < t_2$. Let $U \subseteq X \times [t_1, t_2]$, and let $U_t = \{\mathbf{x} : (\mathbf{x}, t) \in U\}$. Then, for any compact map $\mathbf{h} : \overline{U} \to X$ with $\mathbf{h}(\mathbf{x}, t) \neq \mathbf{x}$ on ∂U,*

$$\mathbf{Ind}(\mathbf{h}(\cdot, t), U_t) = \text{const.} \qquad \textit{for all } t \in [t_1, t_2]\,,$$

provided that U is a bounded open set in $X \times [t_1, t_2]$.

Bibliography

[1] M. Bae, G.-Q. G. Chen, and M. Feldman. "Prandtl–Meyer reflection configurations, transonic shocks, and free boundary problems". *Mem. Amer. Math. Soc.* (2023). arXiv: `1901.05916 [math.AP]`. In press.

[2] G. Ben-Dor. *Shock wave reflection phenomena*. 2nd ed. Shock Wave and High Pressure Phenomena. Springer, Berlin, 2007, pp. xiv+342.

[3] D. Bernoulli. *Hydrodynamica: sive de viribus et motibus fluidorum commentarii*. Sumptibus Johannis Reinholdi Dulseckeri, 1738.

[4] S. Bianchini and A. Bressan. "Vanishing viscosity solutions of nonlinear hyperbolic systems". *Ann. of Math. (2)* 161.1 (2005), pp. 223–342.

[5] W. Bleakney and A. H. Taub. "Interaction of shock waves". *Rev. Modern Physics* 21 (1949), pp. 584–605.

[6] A. Bressan, G. Crasta, and B. Piccoli. "Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws". *Mem. Amer. Math. Soc.* 146.694 (2000), pp. viii+134.

[7] A. Bressan and P. Goatin. "Oleĭnik type estimates and uniqueness for $n \times n$ conservation laws". *J. Differential Equations* 156.1 (1999), pp. 26–49.

[8] A. Bressan and P. LeFloch. "Uniqueness of weak solutions to systems of conservation laws". *Arch. Rational Mech. Anal.* 140.4 (1997), pp. 301–317.

[9] A. Bressan and M. Lewicka. "A uniqueness condition for hyperbolic systems of conservation laws". *Discrete Contin. Dynam. Systems* 6.3 (2000), pp. 673–682.

[10] A. Bressan, T.-P. Liu, and T. Yang. "L^1 stability estimates for $n \times n$ conservation laws". *Arch. Ration. Mech. Anal.* 149.1 (1999), pp. 1–22.

[11] T. Chang and G.-Q. G. Chen. "Diffraction of planar shock along compressive corner". *Acta Math. Sci.* 6.3 (1986), pp. 241–257.

[12] T. Chang, G. Q. G. Chen, and S. L. Yang. "On the 2-D Riemann problem for the compressible Euler equations. I. Interaction of shocks and rarefaction waves". *Discrete Contin. Dynam. Systems* 1.4 (1995), pp. 555–584.

[13] T. Chang, G.-Q. G. Chen, and S. Yang. "On the 2-D Riemann problem for the compressible Euler equations. II. Interaction of contact discontinuities". *Discrete Contin. Dynam. Systems* 6.2 (2000), pp. 419–430.

[14] G. Chen, M.-J. Kang, and A. F. Vasseur. "From Navier-Stokes to BV solutions of the barotropic Euler equations" (2024). arXiv: 2401.09305 [math.AP].

[15] G. Chen, S. G. Krupa, and A. F. Vasseur. "Uniqueness and weak-BV stability for 2×2 conservation laws". *Arch. Ration. Mech. Anal.* 246.1 (2022), pp. 299–332.

[16] G.-Q. G. Chen. "Partial differential equations of mixed type—analysis and applications". *Notices Amer. Math. Soc.* 70.1 (2023), pp. 8–23.

[17] G.-Q. G. Chen, A. J. Cliffe, F. Huang, S. Liu, and Q. Wang. "Global solutions of the two-dimensional Riemann problem with four-shock interactions for the Euler equations for potential flow" (2023). arXiv: 2305.15224 [math.AP].

[18] G.-Q. G. Chen and M. Feldman. "Global solutions of shock reflection by large-angle wedges for potential flow". *Ann. of Math. (2)* 171.2 (2010), pp. 1067–1182.

[19] G.-Q. G. Chen and M. Feldman. "Shock reflection-diffraction phenomena and multidimensional conservation laws". *Hyperbolic problems: theory, numerics and applications*. Vol. 67, Part 1. Proc. Sympos. Appl. Math. Amer. Math. Soc., Providence, RI, 2009, pp. 25–51.

[20] G.-Q. G. Chen and M. Feldman. *The mathematics of shock reflection-diffraction and von Neumann's conjectures*. Vol. 197. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2018, pp. xiv+814.

[21] G.-Q. G. Chen, M. Feldman, J. Hu, and W. Xiang. "Loss of regularity of solutions of the Lighthill problem for shock diffraction for potential flow". *SIAM J. Math. Anal.* 52.2 (2020), pp. 1096–1114.

[22] G.-Q. G. Chen, M. Feldman, and W. Xiang. "Convexity of self-similar transonic shocks and free boundaries for the Euler equations for potential flow". *Arch. Ration. Mech. Anal.* 238.1 (2020), pp. 47–124.

[23] G.-Q. G. Chen, M. Feldman, and W. Xiang. "Uniqueness and stability for the shock reflection-diffraction problem for potential flow". *Hyperbolic problems: theory, numerics, applications*. Vol. 10. AIMS Ser. Appl. Math. Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2020, pp. 2–24.

[24] G.-Q. G. Chen, Q. Wang, and S. Zhu. "Global solutions of a two-dimensional Riemann problem for the pressure gradient system". *Commun. Pure Appl. Anal.* 20.7-8 (2021), pp. 2475–2503.

[25] E. Chiodaroli and O. Kreml. "On the energy dissipation rate of solutions to the compressible isentropic Euler system". *Arch. Ration. Mech. Anal.* 214.3 (2014), pp. 1019–1049.

[26] R. Courant and K. O. Friedrichs. *Supersonic Flow and Shock Waves*. Interscience Publishers, Inc., New York, 1948, pp. xvi+464.

[27] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*. 4th ed. Vol. 325. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 2016, pp. xxxviii+826.

[28] C. M. Dafermos. "The entropy rate admissibility criterion for solutions of hyperbolic conservation laws". *J. Differential Equations* 14 (1973), pp. 202–212.

[29] C. De Lellis and L. Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". *Arch. Ration. Mech. Anal.* 195.1 (2010), pp. 225–260.

[30] C. De Lellis and L. Székelyhidi Jr. "The Euler equations as a differential inclusion". *Ann. of Math. (2)* 170.3 (2009), pp. 1417–1436.

[31] R. J. DiPerna. "Convergence of approximate solutions to conservation laws". *Arch. Rational Mech. Anal.* 82.1 (1983), pp. 27–70.

[32] V. Elling and T.-P. Liu. "Supersonic flow onto a solid wedge". *Comm. Pure Appl. Math.* 61.10 (2008), pp. 1347–1448.

[33] V. Elling and T.-P. Liu. "The ellipticity principle for self-similar potential flows". *J. Hyperbolic Differ. Equ.* 2.4 (2005), pp. 909–917.

[34] V. W. Elling. "Shock polars for non-polytropic compressible potential flow". *Commun. Pure Appl. Anal.* 21.5 (2022), pp. 1581–1594.

[35] L. Euler. "Principes généraux du mouvement des fluides". *Mémoires de l'Académie des Sciences de Berlin* (1757), pp. 274–315.

[36] L. Euler. "Sur le mouvement de l'eau par des tuyaux de conduite". *Mémoires de l'Académie des Sciences de Berlin* (1754), pp. 111–148.

[37] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Revised ed. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299.

[38] K. O. Friedrichs. "Nonlinear hyperbolic differential equations for functions of two independent variables". *Amer. J. Math.* 70 (1948), pp. 555–589.

[39] J. Ge and W. Sheng. "The expansion of a wedge of gas into vacuum with small angle in two-dimensional isothermal flow". *Chinese Ann. Math. Ser. B* 37.3 (2016), pp. 395–404.

[40] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. 2nd ed. Vol. 224. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 1983, pp. xiii+513.

[41] J. Glimm. "Solutions in the large for nonlinear hyperbolic systems of equations". *Comm. Pure Appl. Math.* 18 (1965), pp. 697–715.

[42] M. Gromov. *Partial differential relations*. Vol. 9. Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, Berlin, 1986, pp. x+363.

[43] J. Hadamard. *Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique*. Paris: A. Hermann, 1903.

[44] J. Hu. "Loss of Regularity of Solutions to Shock Reflection Problems by a Non-symmetric Convex Wedge with Potential Flow Equations" (2021). arXiv: `2103.16459 [math.AP]`.

[45] Y. Hu, J. Li, and W. Sheng. "Degenerate Goursat-type boundary value problems arising from the study of two-dimensional isothermal Euler equations". *Z. Angew. Math. Phys.* 63.6 (2012), pp. 1021–1046.

[46] M. Kamlet. *NASA captures supersonic shock interaction*. Ed. by M. Conner. Mar. 2019. URL: `https://www.nasa.gov/centers/armstrong/features/supersonic-shockwave-interaction.html`. (accessed: 31 August 2021).

[47] M.-J. Kang and A. Vasseur. "Contraction property for large perturbations of shocks of the barotropic Navier-Stokes system". *J. Eur. Math. Soc. (JEMS)* 23.2 (2021), pp. 585–638.

[48] M.-J. Kang and A. F. Vasseur. "Uniqueness and stability of entropy shocks to the isentropic Euler system in a class of inviscid limits from a large family of Navier-Stokes systems". *Invent. Math.* 224.1 (2021), pp. 55–146.

[49] M.-J. Kang and A. F. Vasseur. "Well-posedness of the Riemann problem with two shocks for the isentropic Euler system in a class of vanishing physical viscosity limits". *J. Differential Equations* 338 (2022), pp. 128–226.

[50] C. Klingenberg, O. Kreml, V. Mácha, and S. Markfelder. "Shocks make the Riemann problem for the full Euler system in multiple space dimensions ill-posed". *Nonlinearity* 33.12 (2020), pp. 6517–6540.

[51] S. N. Kružkov. "First order quasilinear equations with several independent variables". *Mat. Sb. (N.S.)* 81(123) (1970), pp. 228–255.

[52] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics*. 2nd ed. Pergamon Press, 1987, pp. xiii+539.

[53] P. D. Lax. "Hyperbolic systems of conservation laws. II". *Comm. Pure Appl. Math.* 10 (1957), pp. 537–566.

[54] P. D. Lax. "Development of singularities of solutions of nonlinear hyperbolic partial differential equations". *J. Mathematical Phys.* 5 (1964), pp. 611–613.

[55] P. D. Lax and X.-D. Liu. "Solution of two-dimensional Riemann problems of gas dynamics by positive schemes". *SIAM J. Sci. Comput.* 19.2 (1998), pp. 319–340.

[56] B. van Leer. "Towards the ultimate conservative difference scheme. V. A second-order sequel to Godunov's method". *J. Comput. Phys.* 32.1 (1979), pp. 101–136.

[57] J. Li and S.-L. Yang. "Two-dimensional Riemann problem for gas dynamics system in three pieces". *J. Comput. Math.* 17.3 (1999), pp. 327–336.

[58] J. Li, Z. Yang, and Y. Zheng. "Characteristic decompositions and interactions of rarefaction waves of 2-D Euler equations". *J. Differential Equations* 250.2 (2011), pp. 782–798.

[59] J. Li and Y. Zheng. "Interaction of four rarefaction waves in the bi-symmetric class of the two-dimensional Euler equations". *Comm. Math. Phys.* 296.2 (2010), pp. 303–321.

[60] G. M. Lieberman. "Mixed boundary value problems for elliptic and parabolic differential equations of second order". *J. Math. Anal. Appl.* 113.2 (1986), pp. 422–440.

[61] G. M. Lieberman. *Oblique derivative problems for elliptic equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013, pp. xvi+509.

[62] T.-P. Liu. "Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations". *J. Differential Equations* 33.1 (1979), pp. 92–111.

[63] T.-P. Liu and T. Yang. "L^1 stability for 2×2 systems of hyperbolic conservation laws". *J. Amer. Math. Soc.* 12.3 (1999), pp. 729–774.

[64] E. Mach. "Über den Verlauf von Funkenwellen in der Ebene und im Raume". *Sitzungsbr. Akad. Wiss. Wien.* 78 (1878), pp. 819–838.

[65] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*. Vol. 53. Applied Mathematical Sciences. Springer-Verlag, New York, 1984, pp. viii+159.

[66] S. Markfelder and C. Klingenberg. "The Riemann problem for the multidimensional isentropic system of gas dynamics is ill-posed if it contains a shock". *Arch. Ration. Mech. Anal.* 227.3 (2018), pp. 967–994.

[67] S. Müller and V. Šverák. "Convex integration for Lipschitz mappings and counterexamples to regularity". *Ann. of Math. (2)* 157.3 (2003), pp. 715–742.

[68] J. von Neumann. "Oblique reflection of shocks". *Navy Dept., Bureau of Ordinance* Explos. Res. Rep. 12 (1943).

[69] J. von Neumann. "Refraction, intersection and reflection of shock waves". *Navy Dept., Bureau of Ordinance* NAVROD Rep. 203–45 (1945).

[70] O. A. Oleĭnik. "Discontinuous solutions of non-linear differential equations". *Usp. Mat. Nauk (N.S.)* 12.3(75) (1957), pp. 3–73.

[71] B. Riemann. "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite". *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen* (1860), pp. 43–66.

[72] C. W. Schulz-Rinne. "Classification of the Riemann problem for two-dimensional gas dynamics". *SIAM J. Math. Anal.* 24.1 (1993), pp. 76–88.

[73] C. W. Schulz-Rinne, J. P. Collins, and H. M. Glaz. "Numerical solution of the Riemann problem for two-dimensional gas dynamics". *SIAM J. Sci. Comput.* 14.6 (1993), pp. 1394–1414.

[74] D. Serre. "Multidimensional shock interaction for a Chaplygin gas". *Arch. Ration. Mech. Anal.* 191.3 (2009), pp. 539–577.

[75] D. Serre. "Shock reflections in gas dynamics". *Handbook of mathematical fluid dynamics. Vol. IV*. Elsevier/North-Holland, Amsterdam, 2007, pp. 39–122.

[76] W. Sheng and G. Yin. "Transonic shock and supersonic shock in the regular reflection of a planar shock". *Z. Angew. Math. Phys.* 60.3 (2009), pp. 438–449.

[77] W. Sheng and T. Zhang. *The Riemann problem for the transportation equations in gas dynamics*. Vol. 137. 654. 1999, pp. viii+77.

[78] T. C. Sideris. "Formation of singularities in solutions to nonlinear hyperbolic equations". *Arch. Rational Mech. Anal.* 86.4 (1984), pp. 369–381.

[79] T. C. Sideris. "Formation of singularities in three-dimensional compressible fluids". *Comm. Math. Phys.* 101.4 (1985), pp. 475–485.

[80] J. Smoller. *Shock waves and reaction-diffusion equations*. 2nd ed. Vol. 258. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York, 1994, pp. xxiv+632.

[81] G. G. Stokes. "On a difficulty in the theory of sound". *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 33.223 (1848), pp. 349–356.

[82] E. Zeidler. *Nonlinear functional analysis and its applications. I. Fixed-point theorems*. Translated from the German by Peter R. Wadsack. Springer-Verlag, New York, 1986, pp. xxi+897.

[83] T. Zhang and Y. X. Zheng. "Conjecture on the structure of solutions of the Riemann problem for two-dimensional gas dynamics systems". *SIAM J. Math. Anal.* 21.3 (1990), pp. 593–630.

[84] W. Zhao. "The expansion of gas from a wedge with small angle into a vacuum". *Commun. Pure Appl. Anal.* 12.5 (2013), pp. 2319–2330.