

RECEIVED: February 4, 2026

REVISED: March 23, 2026

ACCEPTED: March 28, 2026

PUBLISHED: May 14, 2026

Extended BMS representations and strings

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ABSTRACT: We construct in detail the irreducible representations of the BMS group with super rotations in three and four dimensions that have the same rest frame momenta as the massive and massless Poincaré point particles. We compare these representations to those of the Poincaré group and also to the analogous representations of global BMS. We argue that these extended BMS representations are carried by a string rather than a point particle. The super rotations play a crucial role in our discussions.

KEYWORDS: Bosonic Strings, Classical Theories of Gravity, Space-Time Symmetries, Global Symmetries

ARXIV EPRINT: [2601.00662](https://arxiv.org/abs/2601.00662)

Contents

1	Introduction	1
2	Irreducible representations of BMS_3	3
2.1	Massive BMS_3 representations	4
2.2	Massless extended BMS_3 representations	7
3	Irreducible representations of Poincaré and BMS_3 at time-like infinity	10
3.1	Massive Poincaré particles at time-like infinity	11
3.2	Massive BMS_3 irreducible representations at time-like infinity	13
4	Extended BMS_4 irreducible representations	15
4.1	Massive extended BMS_4 representations	16
4.2	Massless extended BMS_4 representations	19
4.3	Interpretation of the extended BMS_4 representations	21
5	Irreducible representations of BMS without the super rotations	22
6	Summary and discussion	25

1 Introduction

A remarkable feature about general relativity is that, even far from the source of the gravitational field, the theory does not reduce to special relativity. This fact is particularly transparent in the context of asymptotically flat spacetime, where the asymptotic symmetry group is given by the Bondi, van der Burg, Metzner and Sachs (BMS) group [1, 2], and not simply the Poincaré group. It was subsequently enhanced to include super rotations [3–6], this is called extended BMS. Later the symmetry was further enhanced by incorporating diffeomorphisms on the celestial sphere [7–11, 44].

Although the importance of super rotations for scattering in flat space was understood through their relation with subleading soft graviton theorem [12, 45], several fundamental questions remain. One of them concerns understanding the precise role of the BMS representations in formulating a well-defined theory of scattering amplitudes, with the long-term goal of better understanding the infrared divergences of the S-matrix as well as establishing a holographic description of asymptotically flat spacetime.

The works of McCarthy [13–17] provided a mathematical classification of the unitary irreducible representations of the BMS group in four dimensions in the absence of the super rotations. A similar analysis was performed for the BMS group in three dimensions [18–20] including the super rotations. The irreducible representations of BMS_3 were also discussed in reference [46] from the viewpoint of ultra relativistic limit of $SL(2, R)$ modules. More recently the results of McCarthy were further discussed [21, 22] in relation to the infrared structure of the S-matrix. Despite these results, this mathematical machinery has not yet

found concrete applications in the context of scattering theory and the role of super rotations in the representation theory remains unexplored.

To fill this gap, we take a very pedestrian approach to constructing the wave functions associated with the irreducible representations of the extended BMS group in three and four dimensions that correspond to the massive and massless representations of the Poincaré group. Such representations are of importance for the Carrollian proposal for flat space holography [5, 23, 25–29] which hopes to determine the scattering of the usual particles by using BMS symmetry at infinity.

Our concrete approach consists in extending the standard Wigner construction of Poincaré particles [30] to the infinite-dimensional extended BMS group which has the same semi-direct product structure. In particular we find the wavefunctions in super momentum space and Fourier transform them to find the wavefunctions in position space. We argue that these irreducible representations are carried by extended objects rather than particles and in particular strings. Unlike in the Poincaré case, taking the Fourier transformation requires an infinite number of position coordinates as we need one for each momentum, including the super momentum. These infinite coordinates can be interpreted as modes of a string that lives in the familiar spacetime. The super rotations are crucial for this conclusion as without them the super momenta obey an infinite number of constraints which reduce their number to be that of the usual spacetime. We also discuss the form of the wavefunction in position space as we take the limit to time-like infinity for the massive extended BMS irreducible representations in three dimensions.

The paper is organized as follows. In section two we find the irreducible representations of the extended BMS group in three dimensions that correspond to the massive and massless particles of the Poincaré group. We compare our results with those of references [18–20]. In section three we push the massive irreducible representations of BMS_3 to time-like infinity i^+ . In this section and section two we argue that the representations of extended BMS_3 are carried by a string. In section four we construct the irreducible representations of the extended BMS_4 group, including the super rotations, in four dimensions that correspond to the massive and massless Poincaré particles. In section five we construct the irreducible representations of the BMS groups, without including the super rotations, in three and four dimensions that correspond to the massive and massless Poincaré particles. We compare our results with those of McCarthy. In section 6 we summarise and discuss our results.

The contents of this paper are as follows:

- 1 Introduction
- 2 Irreducible representations of BMS_3
 - 2.1 Massive BMS_3 representations
 - 2.2 Massless BMS_3 representations
- 3 Irreducible representations of Poincaré and BMS_3 at time-like infinity
 - 3.1 Massive Poincaré particles at time-like infinity
 - 3.2 Massive BMS_3 irreducible representations at time-like infinity

4 Extended BMS_4 irreducible representations

4.1 Massive extended BMS_4 representations

4.2 Massless extended BMS_4 representations

4.3 Interpretation of the extended BMS_4 representations

5 Irreducible representations of BMS without the super rotations

6 Summary and Discussion

2 Irreducible representations of BMS_3

In this section, we construct the irreducible representations of BMS_3 which have the same rest frame momentum as those of the irreducible representations of the Poincaré group. The BMS_3 algebra [31, 33] is given by

$$\begin{aligned} [J_m, J_n] &= i(m-n)J_{m+n} + i\frac{Z_1}{12}m(m^2-1)\delta_{n+m,0}, \\ [J_m, P_n] &= i(m-n)P_{m+n} + i\frac{Z_2}{12}m(m^2-1)\delta_{n+m,0}, \quad [P_m, P_n] = 0 \end{aligned} \quad (2.1)$$

where J_n and P_n , $n = 0, \pm 1, \pm 2, \dots$ are the super rotations and the super translations, respectively and Z_1 and Z_2 are the possible central generators. All the generators in the above commutators are related by a minus sign to the commutators in equation (5.7) given in reference [18]. The values of the central charges for three dimensional gravity in asymptotically flat spacetimes is given by $c_1 = 0$ and $c_2 = \frac{3}{G} = -\langle Z_2 \rangle$.

The Poincaré subalgebra in three dimensions has the generators $\mathbf{J}_{\mu\nu}$ and \mathbf{P}_μ , $\mu, \nu = 0, 1, 2$ which are contained in the BMS_3 generators $J_{-1}, J_0, J_1, P_{-1}, P_0, P_1$. The precise identification is given by

$$J_0 = -\mathbf{J}_{12}, \quad J_{\pm 1} = \mathbf{J}_{01} \pm i\mathbf{J}_{02}, \quad \mathbf{P}_0 = P_0, \quad P_{\pm 1} = \mathbf{P}_2 \mp i\mathbf{P}_1 \quad (2.2)$$

We take the following conventions for the Poincaré algebra

$$[\mathbf{J}_{\mu\nu}, \mathbf{J}_{\rho\kappa}] = \eta_{\nu\rho}\mathbf{J}_{\mu\kappa} - \eta_{\mu\rho}\mathbf{J}_{\nu\kappa} - \eta_{\nu\kappa}\mathbf{J}_{\mu\rho} + \eta_{\mu\kappa}\mathbf{J}_{\nu\rho}, \quad [\mathbf{J}_{\mu\nu}, \mathbf{P}_\rho] = \eta_{\nu\rho}\mathbf{P}_\mu - \eta_{\mu\rho}\mathbf{P}_\nu \quad (2.3)$$

Since the super translations P_n commute we can, just as in the case of the Poincaré algebra, take the representations of the algebra to act on super momentum eigenstates $|p_n\rangle$ with $P_n|p_n\rangle = p_n|p_n\rangle$. Acting with a super rotation the momentum eigenstates transform as $|p'_n\rangle = e^{\sum_m \Lambda_{-m} J_m} |p_n\rangle$. As a result we find that

$$P_n|p'_n\rangle = e^{\sum_m \Lambda_{-m} J_m} e^{-\sum_m \Lambda_{-m} J_m} P_n e^{\sum_m \Lambda_{-m} J_m} |p_n\rangle \quad (2.4)$$

Using the above commutators of BMS_3 we find, for infinitesimal Λ_m , that the super momenta transform as

$$\delta p_n = -i \sum_m \Lambda_{-m} (-n+m) p_{n+m} - \frac{ic_2}{12} n(n^2-1) \Lambda_n \quad (2.5)$$

The BMS_3 algebra is of the form of a semi-direct product of the algebra of the super rotations, J_n , with the abelian super translations, P_n . As such it has the same generic form as the Poincaré algebra, which it contains, and as such we can apply the Wigner method to find the irreducible representations. The irreducible representation of BMS_3 were previously studied in [18–20] and we will discuss the relation between our results and those given in that paper.

The first step in the Wigner method is to choose which momenta are non-zero in the rest frame and then find the isotropy group that preserves this choice. Since there are an infinite number of such generators there is a considerable choice. In this paper we will confine our attention to when only the usual momenta \mathbf{p}_μ are non-zero. As such we will make contact with the usual irreducible representation of the Poincaré group. We begin with the massive representation.

2.1 Massive BMS_3 representations

We choose the only non-zero rest frame super momentum to take the values $p_n^{(0)}$ to be $p_0^{(0)} = m = \mathbf{p}_0^{(0)}$, $p_n^{(0)} = 0, |n| \geq 1$. This corresponds to the usual massive particle with $\mathbf{p}_0^{(0)} = m, \mathbf{p}_1^{(0)} = 0 = \mathbf{p}_2^{(0)}$. To begin with we will take the central charge $c_2 = 0 = c_1$ but we will consider c_2 to be non-zero later on which is the case relevant for gravity in asymptotically flat spacetime. Using equation (2.5) we find that this choice of momentum is preserved if

$$\frac{\delta p_n^{(0)}}{m} = 2in\Lambda_n = 0 \tag{2.6}$$

Hence we find that the group which preserves our chosen momentum $p_n^{(0)}$ is just $\mathcal{H}_{m \neq 0}^3 = \{J_0\}$ which is the same isotropy group as for the Poincaré case, namely $SO(2)$ as $J_0 = J_{12}$.

Next we take an irreducible representation of the isotropy group $\mathcal{H}_{m \neq 0}^3$ carried by the states $|p_n^{(0)}, a\rangle$ which obey $P_n |p_n^{(0)}, a\rangle = m\delta_{n,0} |p_n^{(0)}, a\rangle$ where the index transforms under $\mathcal{H}_{m \neq 0}^3 = \{J_0\}$. For example, for spin one this representation is given by

$$J_0 |p_n^{(0)}, 1\rangle = |p_n^{(0)}, 2\rangle, \quad J_0 |p_n^{(0)}, 2\rangle = -|p_n^{(0)}, 1\rangle \tag{2.7}$$

The general state in the irreducible representation is given by

$$|p_n, a\rangle \equiv e^{\sum_{|n| \geq 2} \varphi_{-n} J_n} e^{\varphi_{-1} J_1 + \varphi_1 J_{-1}} |p_n^{(0)}, a\rangle \tag{2.8}$$

We will refer to this representation, which is based on the above choice of rest frame momentum, as the massive irreducible representation of BMS_3 . Similarly in the next section we will refer to the massless BMS_3 irreducible representations as the one that corresponds to the same choice of momenta as for the massless irreducible representation of the Poincaré group.

By acting on the state $|p_n, a\rangle$ with P_n we find its eigenvalue p_n , indeed

$$\frac{p_n}{m} = \delta_{n,0} + 2in\varphi_n + \sum_p \varphi_{p+n} \varphi_{-p} (p^2 - n^2) + \dots \tag{2.9}$$

We recall that $\varphi_0 = 0$ and in deriving this equation we have combined the exponentials in equation (2.8) and the φ_n in the last equation are the new variables. This irreducible representation is in the list given in [18].

It is instructive to consider the restriction of the above to the Poincaré case where the general state is given by

$$|p_n, a\rangle \equiv e^{\varphi_{-1}J_1 + \varphi_{+1}J_{-1}} |p_n^{(0)}, a\rangle = e^{(\varphi_{+1} + \varphi_{-1})J_{01} - i(\varphi_{+1} - \varphi_{-1})J_{02}} |p_n^{(0)}, a\rangle \quad (2.10)$$

We find at lowest level

$$\begin{aligned} \frac{p_0}{m} &= 1 + 2\varphi_{-1}\varphi_{+1} + \dots, & \frac{p_{+1}}{m} &= 2i\varphi_{+1} + \dots, \\ \frac{p_{-1}}{m} &= -2i\varphi_{-1} + \dots \end{aligned} \quad (2.11)$$

The momenta are parameterised by the two degrees of freedom $\varphi_{\pm 1}$ which is the correct count given that $\mathbf{p}_\mu \mathbf{p}^\mu = -p_0^2 + p_1 p_{-1} = -m^2$.

In momentum space the wavefunction of equation (2.10) is a function of $\varphi_{\pm 1}$, or equivalently \mathbf{p}_μ , $\mu = 0, 1, 2$ subject to $\mathbf{p}_\mu \mathbf{p}^\mu = -m^2$. We denote the Cartesian coordinates of spacetime to be given by X^μ , $\mu = 0, 1, 2$ and the Fourier transform to x space is given by

$$\begin{aligned} \Psi(X^\mu, a) &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \delta(\mathbf{p}^\mu \mathbf{p}_\mu + m^2) e^{i\mathbf{p}^\mu X_\mu} \Theta(\mathbf{p}^0) \psi(\mathbf{p}, a) = \int \frac{d^2 \mathbf{p}}{(2\pi)^2 \mathbf{p}^0} e^{i\mathbf{p}^\mu X_\mu} \psi(\mathbf{p}, a) \\ &= \int \frac{d\varphi_1 \varphi_{-1}}{(2\pi)^2 2} \frac{\sinh \varphi (\varphi_1 - \varphi_{-1})^2}{\varphi^3} e^{i(-x_0 p_0 + x_1 p_{-1} + x_{-1} p_1)} \psi(\mathbf{p}, a) \\ &= \int \frac{d\varphi_1 \varphi_{-1}}{(2\pi)^2 2} \frac{\sinh \varphi (\varphi_1 - \varphi_{-1})^2}{\varphi^3} e^{i(x_0 p_0 - 2ix_1 \varphi_{-1} + 2ix_{-1} \varphi_1)} \psi(\mathbf{p}, a) \end{aligned} \quad (2.12)$$

where $\psi(p, a) = \langle p | p_n, a \rangle$. In going between the first and second lines we have identified x_0 and $x_{\pm 1}$ to be given by

$$x_0 = X_0, \quad x_{\pm 1} \equiv \frac{1}{2}(X^2 \mp iX^1) \quad (2.13)$$

Hence we find the usual result, a function $\Psi(x)$ subject to $(\partial^\mu \partial_\mu - m^2)\Psi = 0$.

The situation for BMS_3 is rather different. Equation (2.8) shows that the wavefunction in momentum space depends on φ_n , $n \neq 0$, or equivalently p_n for all n , subject to the condition $(-P_0^2 + P_1 P_{-1} + m^2) |p_n^{(0)}, a\rangle = 0$. As a result we find that

$$\begin{aligned} e^{\sum_{|n| \geq 2} \varphi_{-n} J_n} e^{\varphi_{-1} J_1 + \varphi_1 J_{-1}} (-P_0^2 + P_1 P_{-1} + m^2) e^{-(\varphi_{-1} J_1 + \varphi_1 J_{-1})} e^{-\sum_{|n| \geq 2} \varphi_{-n} J_n} \\ e^{\sum_{|n| \geq 2} \varphi_{-n} J_n} e^{\varphi_{-1} J_1 + \varphi_1 J_{-1}} |p_n^{(0)}\rangle = C(p_n) |p_n, a\rangle = 0 \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} c(P_n) &\equiv e^{\sum_{|n| \geq 2} \varphi_{-n} J_n} e^{\varphi_{-1} J_1 + \varphi_1 J_{-1}} (-P_0^2 + P_1 P_{-1} + m^2) e^{-(\varphi_{-1} J_1 + \varphi_1 J_{-1})} e^{-\sum_{|n| \geq 2} \varphi_{-n} J_n} \quad (2.15) \\ &= -P_0^2 + P_1 P_{-1} + m^2 + \sum_{n, |n| \geq 2} i\varphi_{-n} (-2nP_n P_0 + (n-1)P_{n+1} P_{-1} + (n+1)P_{n-1} P_1 + m^2) + \dots \\ &= -P_0^2 + P_1 P_{-1} + m^2 + \sum_{n, |n| \geq 2} \frac{P_{-n}}{2n} (2nP_n P_0 - (n-1)P_{n+1} P_{-1} - (n+1)P_{n-1} P_1) + \dots \end{aligned}$$

where we have used equation (2.9) and $+\dots$ means terms of order P^4 and higher. At very lowest order $p_0 = m + \dots$ and so at lowest order $C(p) = -p_0^2 + \sum_{n, n \neq 0} p_n p_{-n} + m^2 = 0$. Thus

the super momenta of the states obey the constraint $C(p_n) = 0$ and hence the wavefunction in momentum space depends on an infinite number of variables. We note that the above general states of equation (2.8) do not include those with only the usual massless momentum of the Poincaré group, that is, they cannot have $\mathbf{p}^\mu \mathbf{p}_\mu = 0$ with all higher super momenta being zero. This justifies our division of irreducible representations into massive and massless.

We can take the Fourier transform to x space for the above massive BMS_3 irreducible representation. However, now we must introduce a space that is parameterised by an infinite number of variables which we denote by x_n , $n = 0, \pm 1, \pm 2, \dots$ as we need one coordinate for each momentum p_n . The Fourier transform has the generic form

$$\Psi(x_n, a) = \int \prod_n d\varphi_n J e^{i \sum_n p_n x_n} \psi(p_n, a) \tag{2.16}$$

where J is a function of φ_n which is required for an invariant measure. The wavefunction $\Psi(x_n, a)$ is subject to one equation, which is the x -space version of equation (2.14), namely $C(-i \frac{\partial}{\partial x_n}) \Psi(x_n, a) = 0$.

We choose the x_n to be coordinates on the coset space constructed from BMS_3 with the subgroup being the group generated by the super rotations J_n . This mimics the case for the Poincaré group where the usual spacetime is the coset space of the Poincaré group divided by the Lorentz group. The coordinates x_n then transform as

$$\sum_n x'_n P_n = e^{-\sum_m \Lambda_{-m} J_m} \sum_n x_{-n} P_n e^{\sum_m \Lambda_{-m} J_m} \Rightarrow \delta x_n = -i \sum_m (2m + n) \Lambda_{-m} x_{n+m} \tag{2.17}$$

At first sight it looks as if we have a particle moving in an infinite spacetime but this is not consistent with the fact that the BMS_3 algebra is derived in three dimensional spacetime. To better understand the meaning of the many coordinates we will reformulate the above by introducing $z = e^{i\theta}$ and the quantities

$$J(z) = \sum_m J_m z^m, \quad \Lambda(z) = \sum_m \Lambda_m z^m, \quad X(z) = \sum_m x_m z^m \tag{2.18}$$

In which case we can write the transformation of the coordinates of equation (2.17) as

$$\delta X(z) = -i \Lambda(z) z \frac{dX(z)}{dz} + iz \frac{d\Lambda(z)}{dz} X(z) \tag{2.19}$$

Thus $X(z)$ transforms under diffeomorphisms of z as a weighted scalar which is consistent with interpreting z as a parameter labelling a one dimensional object. We can then write the wavefunction in x -space as $\Psi(X(z), a)$. It follows that the massive irreducible representation of BMS_3 should be viewed as an object of dimension one, that is, a string which is parameterised by z . We will discuss this string interpretation in greater detail in section 3.2.

We now find the massive irreducible representation when the central charge $c_2 \neq 0$. In this case the momenta vary under the super rotations as given in equation (2.5). Taking into account that the rest frame momentum is given by $p_n^{(0)} = m \delta_{n,0}$ we find that the condition to preserve this choice of momentum is given by

$$\frac{\delta p_n}{m} = ni \left(2 - \frac{c_2}{12} (n^2 - 1) \right) \Lambda_n = 0 \tag{2.20}$$

In the generic case, the isotropy group is the same as above, namely the isotropy group just contains the generator J_0 . However, if the central charge is such that $n = \sqrt{1 + 24/c_2}$ is an integer, then the isotropy group is $\mathcal{H}_{\text{mass}}^c = \{J_0, J_n, J_{-n}\}$ which is the algebra of $\text{SL}(2, \mathbb{R})$. These irreducible representations are also in the list of [18]. There are only a limited number of solutions if c_2 is positive as it should be. We note that if c_2 is negative then there is only one solution, that is, $c_2 = -24$ with $n = 0$ but this does not lead to a larger isotropy group. The situation is a bit like that for supersymmetry where for certain values of the central charge the irreducible representations of supersymmetry are smaller.

If the central charge does not have one of these special values the general momentum state is given by

$$|p_n, a\rangle \equiv e^{\sum_{|n|\geq 1} \varphi_{-n} J_n} |p_n^{(0)}, a\rangle \tag{2.21}$$

While if $n = \pm\sqrt{1 + 24/c_2}$ is an integer then we have the above state but must set $\varphi_n = 0 = \varphi_{-n}$.

The momenta of this state are given in terms of the φ_n by

$$\frac{p_n}{m} = \delta_{n,0} + 2in\varphi_n \left(1 - \frac{c_2}{24}(n^2 - 1)\right) + \sum_p (p^2 - n^2)\varphi_{-p}\varphi_{n+p} \left(1 - \frac{c_2}{24}((n+p)^2 - 1)\right) + \dots \tag{2.22}$$

We note that $-p_0^2 + \sum_n p_n p_{-n} + \dots = 0$ where $+\dots$ indicates higher terms in powers of φ_n and the central charge.

To construct the wavefunction in x space we must introduce a coordinate for each independent momenta and so we introduce $x_n, n \geq 1$ as well as a coordinate y corresponding to the presence of the central charge generator Z_2 . The transformation of these coordinates follows from demanding that

$$\sum_n x'_n P_n + y' Z_2 = e^{-\sum_m \Lambda_{-m} J_m} \left(\sum_n x_n P_n + y Z_2 \right) e^{\sum_m \Lambda_{-m} J_m} \tag{2.23}$$

We find that

$$\delta x_n = -i \sum_m (2m + n) \Lambda_{-m} x_{n+m}, \quad \delta y = i \sum_n x_n \Lambda_n n(n^2 - 1) \tag{2.24}$$

We can rewrite this in terms of the quantities of equation (2.19) as

$$\delta X(z) = -i\Lambda(z)z \frac{dX(z)}{dz} + iz \frac{d\Lambda(z)}{dz} X(z), \quad \delta y = i \int dz z \Lambda(z) \frac{d^3}{dz^3} (zX(z)) \tag{2.25}$$

The wavefunction in x space then has the form $\Psi(X(z), y, a)$. The dependence on the coordinate y could be rather trivial and it may be related to the central charge the string may carry.

2.2 Massless extended BMS_3 representations

We choose the rest frame momentum to be the momentum one usually takes for the massless irreducible representation of the Poincaré group, namely $\mathbf{p}^{(0)+} \equiv \frac{1}{\sqrt{2}}(\mathbf{p}^{(0)2} + \mathbf{p}^{(0)0}) = 1$,

$\mathbf{p}^{(0)-} \equiv \frac{1}{\sqrt{2}}(\mathbf{p}^{(0)2} - \mathbf{p}^{(0)0}) = 0$ and $\mathbf{p}^{(0)1} = 0$, which has $\mathbf{p}^\mu \mathbf{p}_\mu = 0$, and all other super momenta vanishing. This is equivalent to

$$p_0^{(0)} = -\frac{1}{\sqrt{2}}, \quad p_1^{(0)} = p_{-1}^{(0)} = \frac{1}{\sqrt{2}}, \quad p_n^{(0)} = 0, \quad |n| \geq 2 \quad (2.26)$$

The first step is to find the isotropy group, which preserves this momentum. The variation of the super momenta P_n under the super rotations is given in equation (2.5) and so the super rotations preserving this choice of super momenta satisfy

$$\delta p_n^{(0)} = 0 = -\frac{i}{\sqrt{2}} \left(2n\Lambda_n \left(1 + \frac{\sqrt{2}c_2}{24}(n^2 - 1) \right) + (1 - 2n)\Lambda_{n-1} - (1 + 2n)\Lambda_{1+n} \right) \quad (2.27)$$

At lowest orders we find the conditions on the parameters

$$\begin{aligned} \Lambda_1 - \Lambda_{-1} &= 0, & 2\Lambda_1 - \Lambda_0 - 3\Lambda_2 &= 0, & 2\Lambda_{-1} - \Lambda_0 - 3\Lambda_{-2} &= 0, & (2.28) \\ 4\Lambda_2 \left(1 + \frac{\sqrt{2}c_2}{8} \right) - 3\Lambda_1 - 5\Lambda_3 &= 0, & 4\Lambda_{-2} \left(1 + \frac{\sqrt{2}c_2}{8} \right) - 3\Lambda_{-1} - 5\Lambda_{-3} &= 0, \dots \end{aligned}$$

Taking $n \rightarrow -n$ in equation (2.27) it is easy to realise that $\Lambda_n = \Lambda_{-n}$. In the BMS_3 case we find that equations (2.28) can be solved in terms of two real parameters a and b as follows

$$\begin{aligned} \Lambda_1 &= a = \Lambda_{-1}, & \Lambda_0 &= 2b, & \Lambda_2 &= \frac{2}{3}(a - b) = \Lambda_{-2}, \\ \Lambda_3 &= -\frac{(a + 8b - \sqrt{2}c_2(a - b))}{15} = \Lambda_{-3}, & \dots & \end{aligned} \quad (2.29)$$

As a result the isotropy group for the massless BMS_3 is $\mathcal{H}_{m=0}^3 = \{H_1, H_2\}$ where

$$\begin{aligned} H_1 &= J_0 - \frac{1}{3}(J_2 + J_{-2}) - \frac{4}{15} \left(1 + \frac{\sqrt{2}c_2}{8} \right) (J_3 + J_{-3}) + \dots, \\ H_2 &= (J_1 + J_{-1}) + \frac{2}{3}(J_2 + J_{-2}) - \frac{1}{15}(1 - \sqrt{2}c_2)(J_3 + J_{-3}) + \dots \end{aligned} \quad (2.30)$$

As such the isotropy group has two elements and this conclusion appears to be independent of whether c_2 is zero or not.

It will be instructive, following [18], to rephrase the above steps in terms of the quantities.

$$P(z) = \sum_m P_m z^m, \quad \Lambda(z) = \sum_n \Lambda_n z^n, \quad J(z) = \sum_m J_m z^m \quad (2.31)$$

where $z = e^{i\theta}$. Under a super rotation $\int \frac{dz}{z} \Lambda(z) J(z)$ the super momentum transformation of equation (2.5) can be written as [18–20]

$$\delta p(z) = iz \left(\Lambda(z) \frac{d}{dz} p(z) + 2p(z) \frac{d}{dz} \Lambda(z) - \frac{c_2}{12} \frac{d^3}{dz^3} (z\Lambda(z)) \right) \quad (2.32)$$

Thus our rest frame momenta will be preserved if

$$\delta p(z) = 0 = iz \left(\Lambda(z) \frac{d}{dz} p^{(0)}(z) + 2p^{(0)}(z) \frac{d}{dz} \Lambda(z) - \frac{c_2}{12} \frac{d^3}{dz^3} (z\Lambda(z)) \right) \quad (2.33)$$

which we can rewrite as

$$\frac{d}{dz}(\Lambda(z)^2 p^{(0)}(z) - \frac{c_2}{12} \frac{d^2}{dz^2}(z\Lambda(z))) = 0 \tag{2.34}$$

As we now illustrate one has to handle this equation with care. For simplicity we will illustrate this for $c_2 = 0$. In this case it might appear that one can rewrite equation (2.34) as

$$\Lambda(z)^2 p^{(0)}(z) = e^2 \quad \text{or} \quad \Lambda(z) \sqrt{|p^{(0)}(z)|} = \pm e \tag{2.35}$$

where e is a constant.

It is tempting to conclude that, if $p^{(0)}(z)$ does not have any zeros, then we can take $\Lambda(z) = \frac{e}{\sqrt{|p^{(0)}(z)|}}$. In this case the isotropy group has one element $\int \frac{dz}{z} \Lambda(z) J(z)$ and so is a group of dimension one. This is the case of the massive particle as $p^{(0)}(z) = 1$, and as a result $\Lambda = \pm e$, and so the isotropy group contains the one generator J_0 in agreement with what we found in section 2.1. In the massless case $p^{(0)}(z) = -\frac{1}{\sqrt{2}}(1 - (z + z^{-1}))$ which has two zeros as $z = \frac{1}{2}(1 \pm \sqrt{3}i)$. Taking this value in equation (2.35), we find that $e = 0$ and so $\Lambda = 0$. This implies that the isotropy group has no elements which is not in agreement with the above two dimensional isotropy group.

We note that

$$\begin{aligned} 2^{\frac{1}{4}} \sqrt{|p^{(0)}|} &= (1 - (z + z^{-1}))^{\frac{1}{2}} = 1 - \frac{1}{2}(z + z^{-1}) - \frac{1}{4}(z + z^{-1})^2 - \frac{3}{8}(z + z^{-1})^3 - \frac{15}{24}(z + z^{-1})^4 + \dots \\ &= 1 - \frac{1}{2} - \frac{90}{2^3} + \dots + z \text{ dependent terms} \end{aligned} \tag{2.36}$$

The problem is that this is not a convergent series as the constant term goes as 2^{2n} for large n . As a result one cannot take the square root as in equation (2.35). This problem is generic for any rest frame momenta which, when written as $p^{(0)}(z)$, possess positive and negative powers of z .

The $\Lambda(z)$ corresponding to the isotropy group of equation (2.30) is given by

$$\Lambda(z) = 2b + a(z^1 + z^{-1}) + \frac{2}{3}(a - b)(z^2 + z^{-2}) - \frac{(a + 8b)}{15}(z^3 + z^{-3}) + \dots \tag{2.37}$$

One can verify that this does satisfy equation (2.33) at lowest orders when taking $p^{(0)}(z) = -\frac{1}{\sqrt{2}}(1 - (z + z^{-1}))$. In fact the same divergence problem occurs when we take $(\Lambda(z))^2$ and so one cannot reliably use equation (2.34), or indeed the first equation in (2.35). We note that equation (2.33) does not involve any divergent expressions and so can be used to find the isotropy group.

It is instructive to consider the restriction of the above to the Poincaré group. In this case $\Lambda_n = 0$, $|n| \geq 2$ and we only consider the action of J_0 and $J_{\pm 1}$ and p_0 and $p_{\pm 1}$. Looking at the first three constraints of equation (2.28) we find that $\Lambda_0 = 2\Lambda_1 = 2\Lambda_{-1}$ which is equivalent to taking $a = b$. As such the isotropy group has the one generator $(J_0 + \frac{1}{2}(J_1 + J_{-1})) = -\sqrt{2}J_{1+}$ where J_{1+} is the Lorentz generator in light-cone notation. The remaining Lorentz generators are $\sqrt{2}J_{1-} = (-J_0 + \frac{1}{2}(J_1 + J_{-1}))$ and $J_{+-} = J_{02} = -\frac{i}{2}(J_1 - J_{-1})$. In the massless irreducible representation of the Poincaré group the generator J_{1+} is trivially realised on the rest frame

states and as such the effective isotropy group is trivial. As such the general state for the massless Poincaré irreducible representation is given by

$$|\mathbf{p}_\mu\rangle = e^{\varphi J_{1-}} e^{\phi J_{+-}} |\mathbf{p}_\mu^{(0)}\rangle \tag{2.38}$$

The boost in effect contains the generators J_1 and J_{-1} .

Returning to the BMS_3 irreducible representation, the general state in the massless irreducible representation of BMS_3 can be written as

$$|p_n\rangle = e^{\sum_{n,n\neq 0} \varphi_{-n} J_n} |p_n^{(0)}, a\rangle = e^{\varphi_{-1}(J_1 - J_{-1}) + \sum_{|n|\geq 2} \varphi_{-n} J_n} |p_n^{(0)}, a\rangle \tag{2.39}$$

where $\varphi_0 = 0$ and $\varphi_1 = -\varphi_{-1}$. Even when restricted to just the Poincaré group, that is, $\varphi_n = 0$ for $|n| \geq 2$, the massless extended BMS_3 general state is different to that for the Poincaré case in that even restricted to these generators the latter has an additional boost $J_1 + J_{-1}$.

The momenta of the general state of equation (2.39) are given, when $c_2 = 0$, by

$$\begin{aligned} p_m = & p_m^{(0)} - \frac{i}{\sqrt{2}} (2m\varphi_m + (1 - 2m)\varphi_{m-1} - (1 + 2m)\varphi_{1+m}) \\ & - \frac{1}{2\sqrt{2}} \sum_n \varphi_{-n} \{ (n^2 - m^2)\varphi_{n+m} + (n - m)(1 - 2(n + m))\varphi_{n+m-1} \\ & - (n - m)(1 + 2(n + m))\varphi_{1+n+m} \} + \dots \end{aligned} \tag{2.40}$$

subject to the above condition, $\varphi_0 = 0$ and $\varphi_1 = -\varphi_{-1}$.

In the Poincaré case we have the momentum \mathbf{p}_μ which is subject to the one condition $\mathbf{p}_\mu \mathbf{p}^\mu = 0$ leaving two independent components. Calculating the momentum of the general state of equation (2.38) we find that it is expressed in terms of φ and ϕ , which is in agreement. In the massless BMS_3 case we have an infinite number of momenta which can be expressed in terms of φ_{-1} and φ_n , $|n| \geq 2$ but not φ_0 or $\varphi_1 + \varphi_{-1}$ as these vanish. This suggests that the super momenta are subject to two constraints, one of which will be the analogue of the Poincaré condition.

There is, however, a problem with the above isotropy group $\mathcal{H}_{m=0}^3$: computing the commutator of the two generators appears to give a divergent result. One could take one of the generators to vanish on the rest frame state leaving the other to have a non-trivial action. In particular one could take the latter generator to be the one with $a = b$ which agrees with the isotropy group for the massless Poincaré group when restricted to this case. It could be that this step is required by taking the super momenta to satisfy certain topologies.

3 Irreducible representations of Poincaré and BMS_3 at time-like infinity

In this section we push the massive representations of the Poincaré group in any dimension to time-like infinity i^+ . For the massless Poincaré particle this was carried out in [34] and we follow this paper for the massive Poincaré particles. We then carry out the analogous construction for the massive irreducible representation of BMS_3 to find it is described by a string.

3.1 Massive Poincaré particles at time-like infinity

We will first look at how the irreducible representations of the Poincaré group behave at time-like infinity. This has already been discussed in [35, 47], but we will revisit it using the techniques of section 5 of reference [34] which were used to find the behaviour of the massless irreducible representations of the Poincaré group at null infinity. We will carry out this calculation in a spacetime with D dimensions. This approach uses only group theory manipulations starting from the irreducible representations.

Let us choose the rest frame momentum to be given by

$$\mathbf{p}_\mu^{(0)} = m(1, 0, \dots, 0) \tag{3.1}$$

such that $\mathbf{p}_\mu^{(0)} \mathbf{p}^{(0)\mu} = -m^2$. The isotropy group is given by

$$\mathcal{H}_{m \neq 0}^{\text{Poincare}} = \{J_{ij}, \quad i, j = 1, \dots, D-1\} \tag{3.2}$$

which form the $SO(D-1)$ algebra. The irreducible representation of the Poincaré group is built from an irreducible representation of $\mathcal{H}_{m \neq 0}^{\text{Poincare}}$ carried by a wavefunction of the momentum $\mathbf{p}_\mu^{(0)}$, namely $\psi(\mathbf{p}^{(0)}, a)$, which satisfies $\mathbf{P}_\mu \psi(\mathbf{p}^{(0)}, a) = \mathbf{p}_\mu^{(0)} \psi(\mathbf{p}^{(0)}, a)$. This wave function carries an index a which transforms in an irreducible representation of $\mathcal{H}_{m \neq 0}^{\text{Poincare}}$.

The wave function associated with a generic momentum \mathbf{p}^μ in the orbit of $\mathbf{p}^{(0)}$ is by definition given by

$$\psi(\mathbf{p}, a) = e^{\sum_j \hat{\varphi}_j J_{j0}} \psi(\mathbf{p}^{(0)}, a) \tag{3.3}$$

and satisfies $\mathbf{P}_\mu \psi(p, a) = \mathbf{p}_\mu \psi(\mathbf{p}, a)$.

The value of the momenta is determined by the action of the momentum operator on the general state and we find that

$$\mathbf{p}_\mu = m(\cosh \hat{\varphi}, \frac{\hat{\varphi}_i}{\hat{\varphi}} \sinh \hat{\varphi}), \quad i = 1, \dots, D-1 \tag{3.4}$$

where $\hat{\varphi} = \sqrt{\sum_{i=1}^{D-1} \hat{\varphi}_i^2}$. Rather than take the momentum to be a function of the $D-1$ $\hat{\varphi}_i$ we will take it to be a function of $\eta_i = \frac{\hat{\varphi}_i}{\hat{\varphi}}, i = 1, \dots, D-2$ and $\hat{\varphi}$. We note that $\eta^i \eta_i = 1$.

The wave function in position space is given by

$$\Psi(X, a) = \int \frac{d^D \mathbf{p}}{(2\pi)^{D-1}} \delta(\mathbf{p}^\mu \mathbf{p}_\mu + m^2) e^{i\mathbf{p}^\mu X_\mu} \Theta(\mathbf{p}^0) \psi(\mathbf{p}, a) = \int \frac{d^{D-1} \mathbf{p}}{(2\pi)^{D-1} \mathbf{p}^0} e^{i\mathbf{p}^\mu X_\mu} \psi(\mathbf{p}, a) \tag{3.5}$$

where the second integral is over the $D-1$ spatial components of the momentum. Rather than integrate over the momenta we can integrate over η_i and $\hat{\varphi}$. The corresponding Jacobian J is given by

$$J = \frac{(\sinh \hat{\varphi})^{D-2} \cosh \hat{\varphi}}{\eta_{D-1}} \tag{3.6}$$

where it is understood that $(\eta_{D-1})^2 = 1 - \sum_{i=1}^{D-2} \eta_i^2$. As a result we find that

$$\Psi(X, a) = \int d\hat{\varphi} \prod_{i=1}^{D-2} d\eta_i \frac{(\sinh \hat{\varphi})^{D-2}}{2(2\pi)^{D-1} \eta_{D-1}} e^{i\mathbf{p}^\mu X_\mu} \psi(\mathbf{p}, a) \tag{3.7}$$

We now wish to push the associated position space wave function to time-like infinity. We will focus on future time-like infinity i^+ , but the analogue procedure can be repeated for past time-like infinity i^- . Let us introduce the spacetime coordinates τ and y^μ as follows

$$X^\mu = \tau y^\mu, \quad y^\mu \eta_{\mu\nu} y^\nu = -1 \quad (3.8)$$

In this parametrization, the flat space metric reads as

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = -d\tau^2 + \tau^2 \eta_{\mu\nu} dy^\mu dy^\nu \quad (3.9)$$

Each $\tau = \text{constant}$ leaf in the foliation is a hyperboloid, and taking $\tau \rightarrow +\infty$ leads us to i^+ . A convenient parametrization on the hyperboloid is given by

$$y^\mu = (\cosh \zeta, \frac{\zeta^i}{\zeta} \sinh \zeta), \quad \text{where} \quad \zeta = \sqrt{\sum_{i,j}^{D-1} \zeta^i \zeta^j \delta_{ij}} \quad (3.10)$$

with $i = 1, \dots, D-1$. While one could take the variables to be ζ^i , $i = 1, \dots, D-1$ in what follows we will take them to be $\xi^i = \frac{\zeta^i}{\zeta}$, $i = 1, \dots, D-2$ and ζ . In terms of these coordinates the metric reads as

$$ds^2 = -d\tau^2 + \tau^2 (d\xi^2 + \xi^2 \sinh^2 \zeta d\Omega_{D-2}^2) \quad (3.11)$$

where $d\Omega_{D-2}^2$ is the unit round sphere metric on S^{D-2} .

We want to consider the representation of a massive particle at time-like infinity in equation (3.7). The argument of the exponential is given by

$$\begin{aligned} \mathbf{p}^\mu X_\mu &= \tau m g(\eta_i, \zeta, \xi_i, \hat{\varphi}, \eta_i), \\ g(\eta_i, \zeta, \xi_i, \hat{\varphi}, \eta_i) &= \cosh(\hat{\varphi} - \zeta) + \frac{1}{2} \sum_{i=1}^{D-1} (\xi^i + \eta^i)^2 \sinh \zeta \sinh \hat{\varphi} \end{aligned} \quad (3.12)$$

We will now take the limit of $\tau \rightarrow +\infty$ of $\Psi_\sigma(X)$ to obtain the states at the boundary i^+ by using the stationary phase approximation formula

$$\lim_{\tau \rightarrow \infty} \int_a^b dx f(x) e^{i\tau g(x)} = \lim_{\tau \rightarrow \infty} f(z) e^{i\tau g(z)} \sqrt{\frac{2\pi i}{\tau g''(z)}}, \quad (3.13)$$

where z is a point in the interval $[a, b]$ where $g'(z) = 0$ and $g''(x) = \frac{d^2 g(x)}{dx^2}$. Setting the derivative of g to zero we find that

$$\eta_i = -\xi_i, \quad \text{and} \quad \hat{\varphi} = \zeta \quad \text{or equivalently} \quad \hat{\varphi}_i = -\zeta_i \quad (3.14)$$

Putting everything together, in the saddle point approximation, we find that

$$\Psi(\zeta, \xi^i, a) \equiv \lim_{\tau \rightarrow \infty} \tau^{\frac{D-1}{2}} \psi_\sigma(X) = - \lim_{\tau \rightarrow \infty} \left(\frac{i}{2\pi} \right)^{\frac{D-1}{2}} \times \frac{m^{D-2}}{2\zeta^{D-1}} \times e^{im\tau} \psi(\zeta, \xi^i, a) \quad (3.15)$$

The massive irreducible representation of the Poincaré group in momentum space is carried by $\psi(\mathbf{p}, a)$ in equation (3.3) which depends on \mathbf{p}_i , or equivalently η_i and $\hat{\varphi}$, that is, $\psi(\hat{\varphi}, \eta_i, a)$.

The wave function in position space depends on X^μ , namely $\Psi(X, a)$ but at time-like infinity which corresponds to taking $\tau \rightarrow \infty$ it depends on hypersphere with parameters ξ_i and η . Equation (3.14) tells us that the coordinates in momentum space and in position space are, up to a sign, equal. Furthermore equation (3.15) tells us that the wave function in momentum space is essentially equal to the wave function in position space when we take $\tau \rightarrow \infty$.

Hence the massive irreducible representations of the Poincaré group are completely described by functions on the hypersphere at $\tau \rightarrow \infty$ which carry a representation of $\mathcal{H}_{m \neq 0}^{\text{Poincare}}$. As such at the kinematic level, massive particles can be completely described by the behaviour of wave functions at i^+ , the final hyperboloid in the foliation and so in terms of local operators on this surface.

3.2 Massive BMS_3 irreducible representations at time-like infinity

We now wish to generalise the above construction and push the massive irreducible BMS_3 representations, found in section 3.1, to time-like infinity. As we discussed before, massive in this context means they are the analogous representations to the massive irreducible representations of the Poincaré group. As a first step let us review the calculation given in previous section, but in three dimensions and using the notation of the BMS_3 formulation. The general state of equation (3.3) can be written as

$$\psi(\mathbf{p}, a) = e^{\sum_j \hat{\varphi}_j J_{j0}} \psi(\mathbf{p}^{(0)}, a) = e^{\varphi_{-1} J_1 + \varphi_{+1} J_{-1}} \psi(\mathbf{p}^{(0)}, a) \quad (3.16)$$

Looking at equation (2.10), or the above equation, and using equation (2.2) we identify

$$\hat{\varphi}_1 = -(\varphi_{+1} + \varphi_{-1}), \quad \hat{\varphi}_2 = i(\varphi_{+1} - \varphi_{-1}) \quad \text{or equivalently} \quad \varphi_{\pm 1} = -\frac{1}{2}(\hat{\varphi}_1 \pm i\hat{\varphi}_2) \quad (3.17)$$

Recalling equation (2.13), namely $x_0 = X_0$, $x_{\pm 1} \equiv \frac{1}{2}(X^2 \mp iX^1)$, we can write equations (3.8) and (3.10) as

$$x_0 = \tau y_0 = \tau \cosh \zeta, \quad x_{\pm 1} = \tau y_{\pm 1} = \tau \zeta_{\pm 1} \frac{\sinh \zeta}{\zeta} \quad (3.18)$$

where $\zeta_{\pm 1} = \frac{1}{2}(\zeta^2 \mp i\zeta^1)$ and $\zeta = 2\sqrt{\zeta_{+1}\zeta_{-1}} = \sqrt{(\zeta^1)^2 + (\zeta^2)^2}$. While, using equation (3.17), equation (3.4) can be written as

$$\mathbf{p}_0 = m \cosh \varphi, \quad p_{\pm 1} = \mathbf{p}_2 \mp i\mathbf{p}_1 = \pm 2im\varphi_{\pm 1} \frac{\sinh \varphi}{\varphi} \quad (3.19)$$

where $\varphi \equiv \hat{\varphi} = \sqrt{\hat{\varphi}_1^2 + \hat{\varphi}_2^2} = 2\sqrt{\varphi_{+1}\varphi_{-1}}$. The reader can verify that for small φ this agrees with equation (2.11).

In terms of these variables

$$X^\mu \mathbf{p}_\mu = \tau m g = m\tau \left(\cosh \varphi \cosh \zeta + 2i \frac{\sinh \varphi}{\varphi} \frac{\sinh \zeta}{\zeta} (-\zeta_1 \varphi_{-1} + \zeta_{-1} \varphi_{+1}) \right) \quad (3.20)$$

It is straightforward to take the derivatives of g , as we did above, and find that the contribution to the wave function in position space as $\tau \rightarrow \infty$ is given when $\zeta_{\pm 1} = \mp 2i\varphi_{\pm 1}$.

Let us now discuss how to push the irreducible representation of BMS_3 to time-like infinity. In section 2.1, we introduced a coordinate x_n for each of the super momenta p_n and

the corresponding wavefunction was given in equation (2.16). Examining equation (3.18) the above discussion of the Poincaré case can be naturally generalised to BMS_3 by taking

$$x_n = \tau y_n, \quad y^0 = \cosh \zeta, \quad y_n = \zeta_n \frac{\sinh \zeta}{\zeta}, \quad n = \pm 1, \pm 2, \dots \quad (3.21)$$

where $\zeta = 2\sqrt{\sum_{n=1}^{\infty} \zeta_n \zeta_{-n}}$. As $(x_n)^* = x_{-n}$ we find that $(\zeta_n)^* = \zeta_{-n}$ for all n , although $\zeta_0 = 0$. By abuse of notation we use the same symbol for ζ .

The expression for the wavefunction in position space given in equation (2.16) contains the quantity $e^{i \sum_n x_{-n} p_n}$ which is crucial for the behaviour at infinity and which we will now study. Using equation (3.21) we can write

$$\tau g(x_n, p_n) \equiv \sum_n x_{-n} p_n = x_0 p_0 + \sum_{n, n \neq 0} x_{-n} p_n = \tau \left(p_0 \cosh \zeta + \frac{\sinh \zeta}{\zeta} \sum_{n, n \neq 0} \zeta_{-n} p_n \right) \quad (3.22)$$

The next step in pushing the Poincaré wavefunction to time-like infinity was to use the parameterisation of \mathbf{p}_μ of equation (3.4), or equivalently (3.19). To do the same for BMS_3 we would need the analogue of these latter equations. Calculating the super momenta using equation (2.4) would give a parametrisation of p_n in terms of φ_n ($\varphi_0 = 0$) which would also solve equation (2.14). The lowest order solution was given in equation (2.9) but the all orders solution is not known.

As such we can write $p_n(\varphi_n)$ and $x_n(\zeta_n)$ and then equation (2.16) takes the form

$$\Psi(x_n, a) = \int \prod_n d\varphi_n J(\varphi_m) e^{i\tau g(\zeta_n, \varphi_n)} \psi(p_n(\varphi_m), a) \quad (3.23)$$

where $g(\zeta_n, \varphi_n) = g(x_n(\zeta_n), p_n(\varphi_m))$. The contribution as $\tau \rightarrow \infty$ is then given by taking

$$\frac{\partial g(\zeta_n, \varphi_n)}{\partial \varphi_n} = 0 \quad (3.24)$$

We will now explicitly carry out this calculation to lowest order in φ_n , which at lowest order is related to the super momenta by $\frac{p_n}{m} = \delta_{n,0} + 2in\varphi_n$, and the coordinates ζ_n . In this case, setting $m = 1$, we find that

$$g(\zeta_n, p_n) = p_0 \left(1 + \frac{\zeta^2}{2} \right) + \sum_{n, n \neq 0} \zeta_{-n} p_n = 1 + \frac{1}{2} \sum_{n, n \neq 0} (\zeta_n + p_n)(\zeta_{-n} + p_{-n}) \quad (3.25)$$

where we have used the fact, noted below equation (2.15), that at lowest order

$$p_0 = \sqrt{1 + \sum_{n, n \neq 0} p_{-n} p_n} + \dots = 1 + \frac{1}{2} \sum_{n, n \neq 0} p_{-n} p_n + \dots \quad (3.26)$$

The contribution as $\tau \rightarrow \infty$ is given by $\zeta_n = -p_n = -2in\varphi_n$ which agrees when restricted to the Poincaré case with the result of equation (3.14).

Thus at lowest order we find that the coordinates at infinity, ζ_n , are essentially the same as the parameters, φ_n , which describe the momenta. It would seem inevitable that the solution of equation (3.24) will result in a one to one map between the spaces parameterised by φ_n

and ζ_n . As such the massive BMS_3 irreducible representation in position space is described by a wavefunction that depends on the infinite number of coordinates ζ_n at time-like infinity.

In order to get a better understanding of what is going on we introduced, in section 2.1, a parameter z in equation (2.18). In terms of the above variables we introduce

$$X(z) = \sum_m x_m z^m, \quad \zeta(z) = \sum_{n, n \neq 0} \zeta_n z^n, \quad p(z) = \sum_{n, n \neq 0} p_n z^n, \quad \varphi(z) = \sum_{n, n \neq 0} \varphi_n z^n \quad (3.27)$$

We note that $(X(z))^* = X(z)$, $\zeta(z)^* = \zeta(z)$ and similarly for the other quantities.

The lowest order relation between the coordinates ζ_n at time-like infinity and the φ_n which parameterise the super momenta was given below equation (3.26) and this can be written as

$$\zeta(z) = -p(z) = -2iz \frac{d}{dz} \varphi(z) \quad (3.28)$$

although the relation at higher orders may be more complicated.

In terms of these variables the wave function in x space of the massive BMS_3 irreducible representation at time-like infinity depends on $\zeta(z)$, that is, $\Psi(\zeta(z), a)$. To clarify the meaning of this dependence it is useful to introduce the variables X_n^1, X_n^2 , or equivalently ζ_n^1, ζ_n^2 , as well as p_{1n}, p_{2n} for $n = 1, 2, \dots$. These are related to the above variables by

$$X_{\pm n} = \frac{1}{2}(X_n^2 \mp iX_n^1), \quad p_{\pm n} = p_{2n} \mp ip_{1n}, \quad \zeta_{\pm n} = \frac{1}{2}(\zeta_n^2 \mp i\zeta_n^1) \quad (3.29)$$

We identify the coordinates of the original spacetime as $X^1 = X_1^1, X^2 = X_1^2$ or $\zeta^1 = \zeta_1^1$ and $\zeta^2 = \zeta_1^2$. In terms of these variables we have

$$\zeta(z) = \sum_{n=1}^{\infty} (\zeta_n^2 \cos n\theta + \zeta_n^1 \sin n\theta) \equiv \zeta^2(z) + \zeta^1(z) = \zeta^2 \cos \theta + \zeta^1 \sin \theta + \dots \quad (3.30)$$

where $z = e^{i\theta}$. We can think of z , or θ , as a parameter of a closed string. As we take θ from 0 to 2π we move in spacetime in a circle in the ζ^1, ζ^2 original space with higher modes in the higher coordinate ζ_n^1, ζ_n^2 space.

We can interpret the massive extended BMS_3 irreducible representation at time-like infinity as a string living in the hyperboloid at time-like infinity which has coordinates ζ^1 and ζ^2 with θ being the parameter on the string. The higher order coordinates ζ_n^1 and ζ_n^2 describe the higher modes of the string. Equation (2.9) tells us that the super rotations act so as to change the parameterisation of the string. Thus the massive BMS_3 irreducible representation is described at time-like infinity by a one dimensional extended object, a string, living in the two dimensional hyperboloid at time-like infinity.

4 Extended BMS_4 irreducible representations

In this section we will find irreducible representations of extended BMS_4 which has the super rotation generators $J_n, \bar{J}_n, n \in Z$, and the supertranslations generators $P_{r,s}$ and $\bar{P}_{r,s}, r, s \in Z + \frac{1}{2}$. Their algebra was written in [5] and we use the conventions of [24, 38].

$$[J_n, J_m] = (n - m)J_{n+m}, \quad [\bar{J}_n, \bar{J}_m] = (n - m)\bar{J}_{n+m}, \quad [J_n, \bar{J}_m] = 0 \quad (4.1)$$

and

$$[J_n, P_{k,l}] = \left(\frac{n}{2} - k\right) P_{k+n,l}, \quad [\bar{J}_n, P_{k,l}] = \left(\frac{n}{2} - l\right) P_{k,l+n}, \quad [P_{k,l}, P_{r,s}] = 0 \quad (4.2)$$

The Poincaré subalgebra $\mathbf{P}_\mu, \mathbf{J}_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3, 4$ is generated by

$$\begin{aligned} \mathbf{J}_{13} &= -\frac{1}{2}(J_1 + J_{-1} + \bar{J}_1 + \bar{J}_{-1}), & \mathbf{J}_{23} &= \frac{i}{2}(-J_1 + J_{-1} + \bar{J}_1 - \bar{J}_{-1}), & \mathbf{J}_{12} &= i(J_0 - \bar{J}_0) \\ \mathbf{J}_{01} &= \frac{1}{2}(-J_1 + J_{-1} - \bar{J}_1 + \bar{J}_{-1}), & \mathbf{J}_{02} &= -\frac{i}{2}(J_1 + J_{-1} - \bar{J}_1 - \bar{J}_{-1}), & \mathbf{J}_{03} &= J_0 + \bar{J}_0 \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \mathbf{P}_0 &= P_{\frac{1}{2}, \frac{1}{2}} + P_{-\frac{1}{2}, -\frac{1}{2}}, & \mathbf{P}_1 &= -(P_{-\frac{1}{2}, \frac{1}{2}} + P_{\frac{1}{2}, -\frac{1}{2}}), \\ \mathbf{P}_2 &= -i(P_{\frac{1}{2}, -\frac{1}{2}} - P_{-\frac{1}{2}, \frac{1}{2}}), & \mathbf{P}_3 &= -(P_{\frac{1}{2}, \frac{1}{2}} - P_{-\frac{1}{2}, -\frac{1}{2}}) \end{aligned} \quad (4.4)$$

The momentum eigenvalues $p_{k,l}$ transform as

$$\delta p_{k,l} = -\sum_n \left(\frac{n}{2} - k\right) \Lambda_{-n} p_{n+k,l} - \sum_n \left(\frac{n}{2} - l\right) \bar{\Lambda}_{-n} p_{k,n+l} \quad (4.5)$$

under the super rotation $e^{\sum_m (\Lambda_{-m} J_m + \bar{\Lambda}_{-m} \bar{J}_m)}$.

4.1 Massive extended BMS_4 representations

We will now construct the massive extended BMS_4 irreducible representations meaning the ones that correspond to the massive irreducible representation of the Poincaré group. In the latter case we can choose to take the rest frame momentum to be $\mathbf{p}_0^{(0)} = m$, with all other components zero. In the extended BMS_4 case we take only the corresponding super momentum to be non-zero which is

$$p_{\frac{1}{2}, \frac{1}{2}}^{(0)} = \frac{m}{2} = p_{-\frac{1}{2}, -\frac{1}{2}}^{(0)} \quad (4.6)$$

Taking the super rotations to be of the form $e^{\sum_m (\Lambda_{-m} J_m + \bar{\Lambda}_{-m} \bar{J}_m)}$ we find, using equation (4.5), that this choice is preserved if

$$\begin{aligned} \delta p_{k,l} &= -\frac{m}{2} \left(\delta_{l, \frac{1}{2}} \frac{(1-6k)}{4} \Lambda_{k-\frac{1}{2}} - \delta_{l, -\frac{1}{2}} \frac{(1+6k)}{4} \Lambda_{k+\frac{1}{2}} \right. \\ &\quad \left. + \delta_{k, \frac{1}{2}} \frac{(1-6l)}{4} \bar{\Lambda}_{l-\frac{1}{2}} - \delta_{k, -\frac{1}{2}} \frac{(1+6l)}{4} \bar{\Lambda}_{l+\frac{1}{2}} \right) = 0 \end{aligned} \quad (4.7)$$

It is straightforward to show that this implies that Λ_p and $\bar{\Lambda}_p$ vanish for all p except for $p = 0$ whose parameters obey the condition

$$\Lambda_0 + \bar{\Lambda}_0 = 0 \quad (4.8)$$

Thus the isotropy group contains the one generator $\mathcal{H}_{m \neq 0}^4 = i(J_0 - \bar{J}_0) = J_{12}$. The irreducible representation is built out of a single state which is an eigenstate of J_{12} . The most general state in the irreducible representation is given by boosting this state with $e^{\sum_m (\varphi_{-m} J_m + \bar{\varphi}_{-m} \bar{J}_m)}$ with $\varphi_0 = \bar{\varphi}_0$

This is quite different from what one might expect if one was thinking about the massive irreducible representation of the Poincaré subgroup. To understand more clearly what is going on we will repeat the above calculation but for just the Poincaré group. As such we take only the parameters $\Lambda_0, \bar{\Lambda}_0, \Lambda_{\pm 1}, \bar{\Lambda}_{\pm 1}$ to be non-zero as well as considering only the momenta $p_{\pm\frac{1}{2}, \pm\frac{1}{2}}$ and $p_{\pm\frac{1}{2}, \mp\frac{1}{2}}$. Putting these restrictions into equation (4.7) we find the three conditions

$$\Lambda_0 + \bar{\Lambda}_0 = 0, \quad \Lambda_{-1} - \bar{\Lambda}_{-1} = 0, \quad \Lambda_1 - \bar{\Lambda}_{-1} = 0 \quad (4.9)$$

Thus the isotropy group has the generators $J_0 - \bar{J}_0, J_{-1} + \bar{J}_1$ and $J_1 + \bar{J}_{-1}$ which we can also write as

$$J_{12} = i(J_0 - \bar{J}_0), \quad J_{13} = -\frac{1}{2}(J_1 + \bar{J}_{-1} + J_{-1} + \bar{J}_1), \quad J_{23} = -\frac{i}{2}(J_1 + \bar{J}_{-1} - J_{-1} - \bar{J}_1) \quad (4.10)$$

These generate $SO(3)$ as indeed had to be the case as this is the isotropy group for the massive particle of the Poincaré group.

The irreducible representation of the massive Poincaré group is constructed from a state $|p^{(0)}, a\rangle$ which carries a representation of $SO(3)$. The general state is boosted by the generators J_{01}, J_{02} and J_{03} which, in terms of the above notation, can be written as

$$|p, a\rangle = g_1 |p^{(0)}, a\rangle \quad \text{where} \quad g_1(\chi) = e^{\chi(J_1 - \bar{J}_{-1}) + \bar{\chi}(J_{-1} - \bar{J}_1) + \chi_0(J_0 + \bar{J}_0)} \quad (4.11)$$

We get a smaller isotropy group for extended BMS_4 as even the usual Lorentz rotations act on some of the higher super momenta to give additional constraints, for example

$$\delta P_{-\frac{3}{2}, -\frac{1}{2}} = m\Lambda_{-1} = 0 \quad (4.12)$$

Thus the massive irreducible representation of the Poincaré group is constructed from a representation of $SO(3)$ while the massive irreducible representations of extended BMS_4 is constructed out of a representation of the much smaller $SO(2)$. As such these latter irreducible representations do not contain the massive irreducible representations of the Poincaré group!

The general state in the irreducible massive representation of extended BMS_4 is given by

$$|p_{r,s}, a\rangle = g_2(\varphi) g_1(\chi) g_3(\phi) |p^{(0)}, \lambda\rangle \quad \text{where} \quad (J_0 - \bar{J}_0) |p^{(0)}, \lambda\rangle = \lambda |p^{(0)}, \lambda\rangle \quad (4.13)$$

with

$$g_2(\varphi) = e^{\sum_{m, |m| \geq 2} (\varphi_{-m} J_m + \bar{\varphi}_{-m} \bar{J}_m)}, \quad g_3(\phi) = e^{\phi(J_1 + \bar{J}_{-1}) + \bar{\phi}(J_{-1} + \bar{J}_1)} \quad (4.14)$$

We recognise $g_1(\chi)$ as the boost for the Poincaré group, $g_3(\phi)$ as the boost arising because the isotropy group for extended BMS_4 is smaller and g_2 as the boost involving generators not in the Poincaré group. The group element $g_2 g_1 g_3$ belongs to the coset of extended BMS_4 with subgroup $J_0 - \bar{J}_0$.

Examining equation (4.13) we see that the general extended BMS_4 state depends on φ_m and $\bar{\varphi}_m, |m| \geq 2$, and $\chi, \bar{\chi}, \chi_0$, which correspond to the same boosts as for the Poincaré group, it also depends on $\phi, \bar{\phi}$ which correspond to Poincaré generators which were in the $SO(3)$ isotropy group of the massive Poincaré irreducible representation. As such the general extended BMS_4 state has an additional dependence, that is on ϕ and $\bar{\phi}$ to that found in

the case of the Poincaré group in addition to the φ and $\bar{\varphi}$ dependence arising from the generators not in the Poincaré group.

Although the massive irreducible representations of extended BMS_4 does not contain the massive irreducible representations of the Poincaré group, we can construct reducible representations of extended BMS_4 that do contain the irreducible representations of the Poincaré group. We take a representation of the $SO(3)$ group, which was the isotropy group of the Poincaré group, to be carried by the states $|p_{r,s}^{(0)}, a\rangle$ and boost it by the extended BMS_4 transformation to find the general state as follows

$$|p_{r,s}, a\rangle = g_2(\varphi)g_1(\chi)|p_{r,s}^{(0)}, a\rangle \quad (4.15)$$

We note that this boost is now missing the Poincaré group element $g_3(\phi)$ which is now in the new $SO(3)$ “isotropy group”. The group element $g_2(\varphi)g_1(\chi)$ belongs to the coset of extended BMS_4 with subgroup $SO(3)$.

Under a super rotation the general state transforms as

$$e^{\sum_m(\Lambda_{-m}J_m + \bar{\Lambda}_{-m}\bar{J}_m)}|p_{r,s}, a\rangle = g_2(\varphi')g_1(\chi')h(\varphi, \chi)|p_{r,s}^{(0)}, a\rangle \quad (4.16)$$

where $h(\varphi, \chi)$ is an element of $SO(3)$ which acting on the rest frame states leads to a matrix transformation.

Let us now compute the momentum of the general state of the irreducible representation which we can rewrite in the generic form as

$$|p_{k,l}\rangle = e^{\sum_n \varphi_{-n} J_n} e^{\sum_n \bar{\varphi}_{-n} \bar{J}_n} |p_{k,l}^{(0)}\rangle \quad (4.17)$$

The φ_n and $\bar{\varphi}_n$ are subject to certain constraints, which for the massive case is $\varphi_0 = \bar{\varphi}_0$, but we will do the calculation without initially specifying the constraints so that it applies to the general case. The momenta are then given by

$$\begin{aligned} p_{k,l} = & p_{k,l}^{(0)} - \sum_n \varphi_{-n} \left(\frac{n}{2} - k\right) p_{k+n,l}^{(0)} \\ & + \frac{1}{2} \sum_{n,m} \varphi_{-n} \varphi_{-m} \left(\frac{n}{2} - k\right) \left(\frac{m}{2} - k - n\right) p_{k+n+m,l}^{(0)} + \dots + (\varphi_n \rightarrow \bar{\varphi}_n, k \leftrightarrow l) \end{aligned} \quad (4.18)$$

Thus we find an expression for the super momenta $p_{k,l}$ in terms of φ_n and $\bar{\varphi}_n$, but there are many more of the former than the latter and so there must be an infinite number of constraints on the super momenta.

For the massive case the rest frame momentum is given by equation (4.6) and substituting this into the above equation we immediately find that $p_{k,l} = 0$ unless either k or l takes the values $\pm\frac{1}{2}$. One finds at lowest order that the non-zero super momenta are given by

$$\begin{aligned} p_{\frac{1}{2}, \frac{1}{2}} &= \frac{m}{2} \left(1 + \frac{1}{2}(\varphi_0 + \bar{\varphi}_0)\right), & p_{\frac{1}{2}, -\frac{1}{2}} &= \frac{m}{2}(\varphi_1 - \bar{\varphi}_{-1}), & p_{\frac{1}{2}, l} &= -\frac{m}{2} \frac{(1-6l)}{4} \bar{\varphi}_{-\frac{1}{2}+l}, l \neq \pm\frac{1}{2} \\ p_{-\frac{1}{2}, -\frac{1}{2}} &= \frac{m}{2} \left(1 - \frac{1}{2}(\varphi_0 + \bar{\varphi}_0)\right), & p_{-\frac{1}{2}, \frac{1}{2}} &= \frac{m}{2}(-\varphi_{-1} + \bar{\varphi}_1), \\ p_{-\frac{1}{2}, l} &= \frac{m}{2} \frac{(1+6l)}{4} \bar{\varphi}_{\frac{1}{2}+l}, & & & l &\neq \pm\frac{1}{2} \end{aligned} \quad (4.19)$$

All the super momenta of the form $p_{\pm\frac{1}{2},l}$ are determined by $\bar{\varphi}_n$. We observe that up to this order $p_{\frac{1}{2},l+1} = \frac{(5+6l)}{(1+6l)}p_{-\frac{1}{2},l}$. The equivalent results for $p_{\pm\frac{1}{2},l}$ follow from the above if we swap k and l , $\varphi_n \leftrightarrow \bar{\varphi}_n$ and take $p_{k,l} \leftrightarrow p_{l,k}$ and we conclude that $p_{k,\pm\frac{1}{2}}$ are determined by φ_n . It is clear that the super momenta must satisfy an infinite number of constraints whose form at lowest order could be computed from the above expressions when extended to higher orders.

4.2 Massless extended BMS_4 representations

We now construct the massless irreducible representations of extended BMS_4 and so choose our rest frame momentum to be

$$\mathbf{p}_-^{(0)} \equiv \frac{1}{\sqrt{2}}(p_3^{(0)} - p_0^{(0)}) = 1, \text{ or equivalently } p_{\frac{1}{2},\frac{1}{2}}^{(0)} = -\frac{1}{\sqrt{2}} \quad (4.20)$$

all other super momenta being zero. Using equation (4.5) we find this is preserved provided

$$\delta p_{k,l} = \frac{1}{\sqrt{2}} \left(\delta_{l,\frac{1}{2}} \Lambda_{k-\frac{1}{2}} \frac{(1-6k)}{4} + \delta_{k,\frac{1}{2}} \bar{\Lambda}_{l-\frac{1}{2}} \frac{(1-6l)}{4} \right) = 0 \quad (4.21)$$

From this we find that $\Lambda_p = 0$ and $\bar{\Lambda}_p = 0$ for $p \neq 0$ as well as the condition

$$\Lambda_0 + \bar{\Lambda}_0 = 0 \quad (4.22)$$

Thus the isotropy group $\mathcal{H}_{m=0}^4$ contains one generator, namely $J_0 - \bar{J}_0$. This is the same isotropy group as for the massive case.

The irreducible representation is built out of a single state $|p_{k,l}^{(0)}, \lambda\rangle$ which is an eigenstate of this generator, that is, $(J_0 - \bar{J}_0)|p_{k,l}^{(0)}, \lambda\rangle = \lambda|p_{k,l}^{(0)}, \lambda\rangle$. The general state is found by boosting with the generators J_m, \bar{J}_m $|m| \geq 1$ and $J_0 + \bar{J}_0$. We can write this as

$$\begin{aligned} |p_{k,l}, \lambda\rangle &= e^{\sum_{m,|m|\geq 2} (\varphi_{-m} J_m + \bar{\varphi}_{-m} \bar{J}_m)} e^{\chi(J_1 + \bar{J}_1) + \bar{\chi}(J_1 - \bar{J}_1) + \chi_0(J_0 + \bar{J}_0)} \\ &e^{\phi(J_{-1} + \bar{J}_{-1}) + \bar{\phi}(J_{-1} - \bar{J}_{-1})} |p_{k,l}^{(0)}, \lambda\rangle \equiv g_2(\varphi) g_1(\chi) g_3(\phi) |p_{k,l}^{(0)}, \lambda\rangle \end{aligned} \quad (4.23)$$

In order to better understand what is going on it is useful to analyse the situation for the Poincaré group in the same notation. In this case we only consider $p_{\pm\frac{1}{2},\pm\frac{1}{2}}$ and $p_{\pm\frac{1}{2},\mp\frac{1}{2}}$ and the generators $J_0, J_{\pm 1}, \bar{J}_0$ and $\bar{J}_{\pm 1}$. As such we only take the parameters $\Lambda_0, \bar{\Lambda}_0, \Lambda_{\pm}, \bar{\Lambda}_{\pm}$ to be non-zero. Equation (4.21) then implies that

$$\Lambda_0 + \bar{\Lambda}_0 = 0, \Lambda_{-1} = 0 = \bar{\Lambda}_{-1} \quad (4.24)$$

Hence the isotropy group $\mathcal{H}_{m \neq 0}^{4dPoincare} = SO(2) \otimes_s T^2$ contains the generators $J_0 - \bar{J}_0, J_{-1}$ and \bar{J}_{-1} which we can write as

$$J_{12} = i(J_0 - \bar{J}_0), J_{-1} + \bar{J}_{-1} = -\sqrt{2}J_{1+}, J_{-1} - \bar{J}_{-1} = -i\sqrt{2}J_{2+} \quad (4.25)$$

leaving the generators

$$J_0 + \bar{J}_0 = J_{03} = J_{+-}, J_1 + \bar{J}_1 = -\sqrt{2}J_{1-}, J_1 - \bar{J}_1 = i\sqrt{2}J_{2-} \quad (4.26)$$

from which the boosts will be constructed.

The usual irreducible representations of the Poincaré group which correspond to conventional particles with discrete spin are constructed from the state $|p^{(0)}, \lambda\rangle$ which obeys

$$(J_0 - \bar{J}_0)|p^{(0)}, \lambda\rangle = \lambda|p^{(0)}, \lambda\rangle, \quad J_{-1}|p^{(0)}, \lambda\rangle = 0, \quad \bar{J}_{-1}|p^{(0)}, \lambda\rangle = 0 \quad (4.27)$$

Adopting the latter constraints means that the effective isotropy group is generated by $J_{12} = i(J_0 - \bar{J}_0)$ and so is $SO(2)$ which is the same as for the massless extended BMS_4 irreducible representation. The general state is given by

$$|p, \lambda\rangle = g_1(\chi)|p^{(0)}, \lambda\rangle \quad (4.28)$$

Returning to the extended BMS_4 massless irreducible representation we observe that if we were to take the rest state to obey

$$J_{-1}|p_{r,s}^{(0)}, \lambda\rangle = 0, \quad \bar{J}_{-1}|p_{r,s}^{(0)}, \lambda\rangle = 0 \quad (4.29)$$

then the general state of equation (4.23) takes the form of $|p_{r,s}, \lambda\rangle = g_2(\varphi)g_1(\chi)|p_{r,s}^{(0)}, \lambda\rangle$. Since g_1 is the boost of the Poincaré massless particle we can think of this as a Poincaré state which is boosted by the higher mode generators of extended BMS_4 contained in $g_2(\varphi)$. As such the massless extended BMS_4 irreducible representations do contain the massless irreducible representations of the Poincaré group.

There are other irreducible representations of the Poincaré group called the continuous spin representations. These do not satisfy the last two conditions in equation (4.27).

Given the general state can be written in the form of equation (4.17), subject to the constraint $\varphi_0 = \bar{\varphi}_0$, and that the rest frame momenta given in equation (4.20), we can compute its super momenta to find that

$$p_{k,l} = -\frac{1}{\sqrt{2}} \left(\delta_{k,\frac{1}{2}} \delta_{l,\frac{1}{2}} - \delta_{l,\frac{1}{2}} \frac{(1-6k)}{4} \varphi_{-\frac{1}{2}+k} - \delta_{k,\frac{1}{2}} \frac{(1-6l)}{4} \bar{\varphi}_{-\frac{1}{2}+l} + \dots \right) \quad (4.30)$$

Hence $p_{k,l}$ is zero unless either k or l is equal to $\frac{1}{2}$. The non-zero components at lowest order are

$$\begin{aligned} p_{\frac{1}{2},l} &= -\frac{1}{\sqrt{2}} \left(\delta_{l,\frac{1}{2}} \left(1 + \frac{1}{2} \varphi_0 \right) - \frac{(1-6l)}{4} \bar{\varphi}_{-\frac{1}{2}+l} \right), \\ p_{k,\frac{1}{2}} &= -\frac{1}{\sqrt{2}} \left(\delta_{k,\frac{1}{2}} \left(1 + \frac{1}{2} \bar{\varphi}_0 \right) - \frac{(1-6k)}{4} \varphi_{-\frac{1}{2}+k} \right) \end{aligned} \quad (4.31)$$

As such $p_{\frac{1}{2},l}$ and $p_{k,\frac{1}{2}}$ are determined by $\bar{\varphi}_n$ and φ_n respectively.

It would be interesting to find the isotropy group for other momenta that are not in the orbit of the massive and massless cases. An interesting case is to take $p_{-\frac{3}{2},-\frac{3}{2}} = 1$. Looking at equation (4.5) we find three group elements with non-zero Λ 's, $\bar{\Lambda}$'s given by $\Lambda_1 = 1$, $\bar{\Lambda}_1 = 1$ and $\Lambda_0 = -\bar{\Lambda}_0 = 1$, that is, the algebra $SO(3)$. The corresponding parameters can also be written as $\Lambda(z) = z$, $\bar{\Lambda}(z) = 0$, $\Lambda(z) = 0$, $\bar{\Lambda}(z) = \bar{z}$, and $\Lambda(z) = 1 = -\bar{\Lambda}(\bar{z})$ and the reader can verify that they provide three solutions for the isotropy algebra by using equation (4.35). We note that this is not one of the momenta in the orbit of the massive case, see equation (4.40).¹

¹We thank Blagoje Oblak for drawing our attention to this case.

4.3 Interpretation of the extended BMS_4 representations

The above results are summarised at the beginning of section 6. At the end of section 2.1 and in section 3.2 we argued that the massive irreducible representations of BMS_3 were not carried by a particle but by an extended one dimensional object. In this section we give the analogous discussion for the massive irreducible representations of extended BMS_4 . Given the wave function in super momentum space $\psi(p_{r,s})$ we can find the wave function in x space. To do this we must introduce x space coordinates $x_{r,s}$ for each of the super momenta $p_{r,s}$. The wave function in x space has the generic form

$$\int d\varphi J(\varphi) e^{i \sum_{r,s} p_{r,s} x_{-r,-s}} \psi(p_{r,s}) \quad (4.32)$$

The coordinates $x_{r,s}$ transform as

$$\begin{aligned} \sum_{r,s} x'_{-r,-s} P_{r,s} &= e^{-\sum_m \Lambda_{-m} J_m + \bar{\Lambda}_{-m} \bar{J}_m} \sum_n x_{-r,-s} P_{r,s} e^{\sum_m \Lambda_{-m} J_m + \bar{\Lambda}_{-m} \bar{J}_m} \Rightarrow \\ \delta x_{r,s} &= -\sum_m \left(\frac{3}{2} m + r \right) \Lambda_{-m} x_{r+m,s} - \sum_m \left(\frac{3}{2} m + s \right) \bar{\Lambda}_{-m} x_{r,s+m} \end{aligned} \quad (4.33)$$

Like in the three dimensional case we introduce parameters $z = e^{i\theta}$ and $\bar{z} = e^{i\bar{\theta}}$ and define

$$p(z, \bar{z}) = \sum_{r,s} p_{r,s} z^r \bar{z}^s, \quad \Lambda(z) = \sum_m \Lambda_m z^m, \quad \bar{\Lambda}(\bar{z}) = \sum_m \bar{\Lambda}_m \bar{z}^m, \quad x(z, \bar{z}) = \sum_{r,s} x_{r,s} z^r \bar{z}^s \quad (4.34)$$

The transformation of the super momenta of equation (4.5) can be written as

$$\delta p(z, \bar{z}) = \frac{3}{2} z \frac{d\Lambda(z)}{dz} p(z, \bar{z}) + \Lambda(z) z \frac{dp(z, \bar{z})}{dz} + \frac{3}{2} \bar{z} \frac{d\bar{\Lambda}(\bar{z})}{d\bar{z}} p(z, \bar{z}) + \bar{\Lambda}(\bar{z}) \bar{z} \frac{dp(z, \bar{z})}{d\bar{z}} \quad (4.35)$$

While the transformation of the coordinates is given by

$$\delta x(z, \bar{z}) = \frac{1}{2} z \frac{d\Lambda(z)}{dz} x(z, \bar{z}) - z \frac{dx(z, \bar{z})}{dz} \Lambda(z) + \frac{1}{2} \bar{z} \frac{d\bar{\Lambda}(\bar{z})}{d\bar{z}} x(z, \bar{z}) - \bar{z} \frac{dx(z, \bar{z})}{d\bar{z}} \bar{\Lambda}(\bar{z}) \quad (4.36)$$

Thus the super rotations of extended BMS_4 are a reparameterisation $x(z, \bar{z})$ under the parameters of z and \bar{z} but separately in z and \bar{z} .

In section 3.1 we argued that the massive irreducible representation of extended BMS_3 was described by a string and we will give the analogous discussion here for extended BMS_4 . What the above very generic discussion does not take account of is that one must introduce a position coordinate for every independent momenta, or equivalently every φ_n and $\bar{\varphi}_n$. For both the above massive and massless representations the momenta are very highly constrained as is apparent from equations (4.19) and (4.31) respectively and as such we must only introduce the corresponding position space coordinates. In particular for the massive case the momenta $p_{r,s}$ are only non-zero if either r or s are equal to $\pm \frac{1}{2}$. As such, and to make contact with our usual spacetime, we introduce the quantities

$$\begin{aligned} p_{\frac{1}{2},r} &= \frac{1}{2} (p_{0,r} - p_{3,r}), & p_{-\frac{1}{2},-r} &= \frac{1}{2} (p_{0,r} + p_{3,r}), \\ p_{r,-\frac{1}{2}} &= \frac{1}{2} (-p_{1,r} + ip_{2,r}), & p_{-r,\frac{1}{2}} &= -\frac{1}{2} (-p_{1,r} + ip_{2,r}), \quad r \geq \frac{1}{2} \end{aligned} \quad (4.37)$$

where we recognise the momenta of our usual spacetime as

$$p_{0,\frac{1}{2}} = p_0, \quad p_{3,-\frac{1}{2}} = p_3, \quad p_{1,\frac{1}{2}} = -p_1, \quad p_{2,-\frac{1}{2}} = p_2 \quad (4.38)$$

We also introduce

$$\begin{aligned} x_{-\frac{1}{2},-r} &= (x_r^0 - X_r^3), & x_{\frac{1}{2},r} &= (x_r^0 + x_r^3), \\ x_{-r,\frac{1}{2}} &= (x_r^1 - iX_r^2), & x_{r,-\frac{1}{2}} &= (-x_r^1 + ix_r^2), \end{aligned} \quad r \geq \frac{1}{2} \quad (4.39)$$

Then

$$\sum_{r,s} p_{r,s} x_{-r,-s} = \sum_{r \geq \frac{1}{2}} (p_{0,r} x_r^0 + p_{3,r} x_r^3 + p_{2,r} x_r^2 + p_{1,r} x_r^1) \quad (4.40)$$

$$x_{\frac{1}{2}}^0 = X^0, \quad x_{\frac{1}{2}}^3 = X^3, \quad x_{\frac{1}{2}}^1 = X^1, \quad x_{\frac{1}{2}}^2 = X^2, \quad (4.41)$$

The above quantities can be naturally encoded in two quantities if we introduce parameters z and \bar{z} , namely

$$\bar{p}(\bar{z}) = \sum_{r \geq \frac{1}{2}} (p_{\frac{1}{2},r} \bar{z}^r + p_{-\frac{1}{2},-r} \bar{z}^r), \quad p(z) = \sum_{r \geq \frac{1}{2}} (p_{r,-\frac{1}{2}} z^r + p_{-r,\frac{1}{2}} z^r) \quad (4.42)$$

and

$$\bar{X}(\bar{z}) = \sum_{r \geq \frac{1}{2}} (x_{\frac{1}{2},r} \bar{z}^r + x_{-\frac{1}{2},-r} \bar{z}^r), \quad X(z) = \sum_{r \geq \frac{1}{2}} (x_{r,-\frac{1}{2}} z^r + x_{-r,\frac{1}{2}} z^r) \quad (4.43)$$

Then

$$\sum_{r,s} x_{-r,-s} p_{r,s} = \int \frac{dz}{z} p(z) X\left(\frac{1}{z}\right) + \int \frac{d\bar{z}}{\bar{z}} \bar{p}(\bar{z}) \bar{X}\left(\frac{1}{\bar{z}}\right) \quad (4.44)$$

These expansions are guided by the solutions of the momenta in terms of φ_n and $\bar{\varphi}_n$ given in equation (4.19).

Taking the Fourier transformation we find the massive irreducible of extended BMS_4 is a functional which depends on $X(z)$ and $\bar{X}(\bar{z})$. As such this representation is carried by a string.

It would be interesting to push this representation to time-like infinity as we did in three dimensions. To do this we should express equation (4.44) as a square, like in equation (3.25) and then differentiate to find the relationship between the independent momenta and the coordinates on the hyperboloid at time-like infinity. One should find that the above representation is carried by a string living on the three dimensional hyperboloid at time-like infinity.

5 Irreducible representations of BMS without the super rotations

In this section we will construct the irreducible representations of BMS symmetries without including the super rotations. We begin with BMS_3 . In this case we have the Lorentz generators J_0 and $J_{\pm 1}$ and all the super translations P_n . As such only the parameters Λ_0 and $\Lambda_{\pm 1}$ are non-zero. Clearly the isotropy group can only be the same or smaller than that for the BMS group.

For the **massive** irreducible representation, considered for BMS_3 in section 2.1, we find, examining equation (2.6), that only Λ_0 is non-zero and so the isotropy group just contains

the generator J_0 . This is the same isotropy group as we had when the super rotations were present and indeed we also had for the massive Poincaré particle.

The general state can be taken to be

$$|p_n, a\rangle \equiv e^{\varphi_{+1}J_{-1}} e^{\varphi_{-1}J_1} |p_n^{(0)}, a\rangle \quad (5.1)$$

One finds that the super momenta p_n are given by

$$p_n = (n+1)m(i\varphi_{+1})^n(1-2\varphi_{+1}\varphi_{-1}), n \geq 0, \quad p_{-1} = -2im\varphi_{-1} \quad (5.2)$$

The result for the remaining super momenta is more complicated, one finds that

$$p_{-n} = m \frac{(-\varphi_{+1}\varphi_{-1})^n}{(n-2)!} \sum_{p=0}^{\infty} \frac{(n+p+2)!}{m!} (n+p+1)(-i\varphi_{-1})^p, \quad n \geq 2 \quad (5.3)$$

Since we have an infinite number of super momenta but the general state only depends on the two variables, $\varphi_{\pm 1}$, the super momenta must obey an infinite number of constraints that can be found from the above expressions for the momenta. Indeed we can express all the super momenta in terms of $p_{\pm 1}$. As a result the irreducible representation is rather similar to that for the Poincaré group.

The **massless** BMS_3 irreducible representation was the subject of section 2.2 and looking at equations (2.27) and (2.28) we conclude that $\Lambda_p = 0$ for all $p = 0, \pm 1$. In particular we find that $\delta p_2 = -\frac{3im}{2}\Lambda_1 = 0$ and $\delta p_{-2} = -\frac{3im}{2}\Lambda_{-1} = 0$ and so $\Lambda_{\pm 1} = 0$ which, with the variations of $p_{\pm 1}$, lead to the result that there is no isotropy group in contrast to the case when the super rotations were present which has an isotropy group with two generators while the massless Poincaré particle which has an isotropy group with the one generator J_{1+} . However, in the usual representations this latter generator is taken to vanish on the rest frame state. We can write the general state for BMS_3 without super rotations as

$$|p\rangle = e^{\varphi_{-}J_{1+}} e^{\varphi_{+}J_{1-}} e^{\phi J_{+-}} |p^{(0)}\rangle \quad (5.4)$$

where $J_{1-} = -\frac{1}{\sqrt{2}}(J_0 + \frac{1}{2}(J_1 + J_{-1}))$, $J_{1+} = \frac{1}{\sqrt{2}}(-J_0 + \frac{1}{2}(J_1 + J_{-1}))$ and $J_{+-} = J_{02} = -\frac{i}{2}(J_1 - J_{-1})$ which are Lorentz generators in light-cone notation. It will coincide with the Poincaré case if we also take J_{1+} to vanish on the rest frame state.

We now consider the same calculations for extended **BMS₄**, but without the super rotations. We now have only the Lorentz generators $J_0, J_{\pm 1}$ and $\bar{J}_0, \bar{J}_{\pm 1}$ and so only the corresponding parameters in the group transformations are now non-zero. The **massive** irreducible representations of BMS_4 were given in section 4.1. Examining equation (4.7) we find that $\Lambda_{\pm 1} = 0$ and $\bar{\Lambda}_{\pm 1} = 0$ leaving only Λ_0 and $\bar{\Lambda}_0$ which obey the condition $\Lambda_0 + \bar{\Lambda}_0 = 0$. Thus the isotropy group is generated by $J_0 - \bar{J}_0$. Thus we find the same isotropy group as when the super rotations were present but different to the isotropy group $SO(3)$ of the massive particle. The general state is given by equation (4.13) provided we take $\varphi_m = 0 = \bar{\varphi}_m$ for $|m| \geq 2$. It therefore depends on $\phi, \bar{\phi}, \chi, \bar{\chi}$ and χ_0 . As such the infinite number of super momenta satisfy an infinite number of constraints.

The **massless** irreducible representations of BMS_4 were given in section 4.2 and taking only parameters that correspond to the Lorentz group we find that $\Lambda_{\pm 1} = 0$ and $\bar{\Lambda}_{\pm 1} = 0$

leaving only Λ_0 and $\bar{\Lambda}_0$ which obey the condition $\Lambda_0 + \bar{\Lambda}_0 = 0$. Thus the isotropy group has only one generator $J_0 - \bar{J}_0$ which is the same as when the super rotations were present and in effect agrees with that of the massless Poincaré particle when we take account of the fact that J_{1+} and J_{2+} are trivially realised. The general state is given by equation (4.23) provided we set φ_m and $\bar{\varphi}_m$ to zero for $|m| \geq 2$. Again the momenta obey an infinite number of constraints.

For the orbits with the same representative as in the Poincaré case that we are considering here, the irreducible representations of BMS, when super rotations are absent, are much more trivial than when super rotations are present, since the boosts contain only the generators of the Lorentz group that lie outside the isotropy group. As a result even though there are an infinite number of super momenta they are trivial in the sense that they satisfy an infinite number of constraints that express them in terms of the usual momenta. This is a general feature of representations of BMS when the super rotations are absent.

Above we truncated the super rotations in the extended BMS_4 algebra and found the irreducible representations. However this is not the same as the global BMS_4 algebra of Bondi, Metzner and Sachs that also has just Lorentz rotations, but using a spherical harmonic decomposition for the super translations. This algebra has momenta $Z_{j,m}$ where $j = 0, 1, 2, \dots$, $m \leq |j|$ and it is given by [38]

$$\begin{aligned}
 [J_m, J_n] &= (m-n)J_{m+n} & [\bar{J}_m, \bar{J}_n] &= (m-n)\bar{J}_{m+n} & [J_m, \bar{J}_n] &= 0 & (5.5) \\
 [J_{-1}, Z_{j,m}] &= \frac{(j+2)(j+m)(j+m-1)}{4(2j+1)(2j-1)} Z_{j-1,m-1} - \frac{j+m}{2} Z_{j,m-1} + (j-1) Z_{j+1,m-1} \\
 [J_0, Z_{j,m}] &= -\frac{(j+2)(j+m)(j-m)}{4(2j+1)(2j-1)} Z_{j-1,m} - \frac{m}{2} Z_{j,m} + (j-1) Z_{j+1,m} \\
 [J_1, Z_{j,m}] &= \frac{(j+2)(j-m)(j-m-1)}{4(2j+1)(2j-1)} Z_{j-1,m+1} + \frac{j-m}{2} Z_{j,m+1} + (j-1) Z_{j+1,m+1} \\
 [\bar{J}_{-1}, Z_{j,m}] &= -\frac{(j+2)(j-m)(j-m-1)}{4(2j+1)(2j-1)} Z_{j-1,m+1} + \frac{j-m}{2} Z_{j,m+1} - (j-1) Z_{j+1,m+1} \\
 [\bar{J}_0, Z_{j,m}] &= -\frac{(j+2)(j+m)(j-m)}{4(2j+1)(2j-1)} Z_{j-1,m} + \frac{m}{2} Z_{j,m} + (j-1) Z_{j+1,m} \\
 [\bar{J}_1, Z_{j,m}] &= -\frac{(j+2)(j+m)(j+m-1)}{4(2j+1)(2j-1)} Z_{j-1,m-1} - \frac{j+m}{2} Z_{j,m-1} - (j-1) Z_{j+1,m-1} \\
 [Z_{j,m}, Z_{j',m'}] &= 0 & & & & & (5.6)
 \end{aligned}$$

We identify the usual Poincaré algebra as the identification of equation (4.3) as well as

$$\mathbf{P}_0 = Z_{0,0}, \quad \mathbf{P}_1 = Z_{1,1} - Z_{1,-1}, \quad \mathbf{P}_2 = i(Z_{1,1} + Z_{1,-1}), \quad \mathbf{P}_3 = -2Z_{1,0}. \quad (5.7)$$

In the **massive** case we take $\mathbf{p}_0^{(0)} = z_{0,0}^{(0)} = m$ where we have used the symbol $z_{j,m}$ to denote the momenta in this basis. Inserting this value on the right-hand side of the commutators of equation (5.6) we find that this choice is preserved if

$$\frac{\delta z_{1,1}}{m} = \frac{1}{2}(\Lambda_1 - \bar{\Lambda}_{-1}) = 0, \quad \frac{\delta z_{1,0}}{m} = -\frac{1}{4}(\Lambda_0 + \bar{\Lambda}_0) = 0, \quad \frac{\delta z_{1,-1}}{m} = \frac{1}{2}(\Lambda_{-1} - \bar{\Lambda}_1) = 0, \quad (5.8)$$

all other variations being trivially satisfied. Thus we find the conditions of equation (4.9) and so the isotropy group $SO(3)$, which is the same isotropy group as the massive irreducible

representation as the Poincaré group but not the same as for extended BMS_4 which was $SO(2)$. The general state has the form of equation (4.15) provided we set $\varphi_n = 0$.

Now we will find the **massless** irreducible representations for which we take $p_0^{(0)} = -\frac{1}{\sqrt{2}} = -p_3^{(0)}$, or equivalently $z_{1,0}^{(0)} = -\frac{1}{2\sqrt{2}}$ and $z_{0,0}^{(0)} = -\frac{1}{\sqrt{2}}$. One finds, using commutators of equation (5.6), that the only non-trivial results are

$$\begin{aligned} \sqrt{2}\delta z_{1,1} = \bar{\Lambda}_{-1} = 0, \quad \sqrt{2}\delta z_{1,0} = \frac{1}{4}(\Lambda_0 + \bar{\Lambda}_0) = 0, \quad \sqrt{2}\delta z_{1,-1} = -\Lambda_{-1} = 0, \\ \sqrt{2}\delta z_{0,0} = \frac{1}{2}(\Lambda_0 + \bar{\Lambda}_0) = 0 \end{aligned} \tag{5.9}$$

as well as

$$\begin{aligned} \sqrt{2}\delta z_{2,1} = -\frac{1}{5}(\Lambda_1 - \bar{\Lambda}_{-1}) = 0, \quad \sqrt{2}\delta z_{2,-1} = -\frac{1}{5}(\Lambda_{-1} - \bar{\Lambda}_1) = 0, \\ \sqrt{2}\delta z_{2,0} = \frac{4}{15}(\Lambda_0 + \bar{\Lambda}_0) = 0 \end{aligned} \tag{5.10}$$

The conditions of equation (5.9) are the same as those of the massless Poincaré particle, as must be the case, but the variation of the super momentum leads to the additional conditions of equation (5.10) that leave only Λ_0 and $\bar{\Lambda}_0$ subject to $\Lambda_0 + \bar{\Lambda}_0 = 0$. As such we have the same isotropy group as for extended BMS_4 . We recall that in the Poincaré case the isotropy group is, in effect, reduced to $SO(2)$ and so to this extent it is the same.

Since we only have at most three and five boosts for the massive and massless cases the general state of the massive and massless irreducible representations respectively, the super momenta satisfy an infinite number of constraints as they did for global BMS_4 . As such these representations are rather trivial compared to extended BMS_4 .

The little groups discussed above for the massive and massless cases all appear in the classification of McCarthy [14], and one can check that the corresponding Poincaré representatives considered here are in agreement with those discussed in that reference. It is worth mentioning that additional little groups have been discussed later, for example in [32], and arise in a different functional-analytic setting, namely by changing the topology on the supertranslation space and allowing distributional representatives. In particular, the $ISO(2)$ little group, which is the same as the massless Poincaré little group before imposing additional constraints, can be obtained in that way.

6 Summary and discussion

In this paper we have constructed the irreducible representations of the BMS group in three and four dimensions that correspond to the massive and massless irreducible representations of the Poincaré group, or put another way the Poincaré particle. More precisely we took the BMS representations to have the same rest frame momenta as the Poincaré particles. We found that

- The massive irreducible representation of BMS_3 has the isotropy group $SO(2)$. This agrees with that of the massive Poincaré particle in three dimensions and it contains this representation. While if we included a central charge with a very precise value the isotropy group was enhanced to $SL(2, \mathbb{R})$. These isotropy groups agree with those found in references [18–20]

- The massless irreducible representation of BMS_3 has an isotropy group that has two generators even if the central charge c_2 is non-zero. This contrasts with that of the massless Poincaré particle in three dimensions, whose isotropy group has only one generator. However, the commutator of the two generators is ill defined and so it is difficult to make a comparison with the massless Poincaré particle. To properly understand this result requires further study.
- The massive irreducible representation of extended BMS_4 has isotropy group $SO(2)$ while the isotropy group of the massive Poincaré particle in four dimensions is $SO(3)$. As such the former does not contain the latter representation, but we showed that there is a reducible representation of extended BMS_4 that does contain that of the massive Poincaré particle.
- The massless irreducible representation of extended BMS_4 has isotropy group $SO(2)$ while the isotropy group of the irreducible massless Poincaré particle in four dimensions is $SO(2) \otimes_s T^2$. However for the usual representations of the massless Poincaré particle the generators of the T^2 , that is J_{1+} and J_{2+} , are trivially realised and in this case the isotropy groups coincide and the former representations contains the latter.

The isotropy group of representations of BMS is generally smaller than that of the Poincaré group as the variation of some of the higher momenta, the super momenta, can transform under a Lorentz transformation into a non-zero frame momenta.

The relationship between the irreducible representation of extended BMS algebra and its Poincaré subalgebra is more complicated than one might naively have expected. However, one can always find a representation, perhaps reducible, of the former that contains the usual massive and massless irreducible representations of the latter.

We also constructed the equivalent irreducible representations of BMS_3 and extended BMS_4 in the absence of the super rotations. We found the super momenta are subject to an infinite number of constraints to leave in effect only the momenta of the Poincaré group and as such the representations are rather trivial compared to those when the super rotations are present.

In this paper we have constructed the above irreducible representations in detail paying particular attention to the role of the super rotations. The general state is found by boosting the rest frame state $|p^{(0)}\rangle$ by those rotations that are not in the isotropy group. As the isotropy group is very small this contains an infinite number of generators which are of the generic form $e^{\varphi \cdot J}$. The momenta of the general state is determined in terms of the φ and so the number of independent momenta is the same as the number of φ . As such the number of constraints the momenta satisfy is the difference in the number of momenta and the number of φ .

This is just the same as for the Poincaré case, for example for a massive particle in four dimensions there are three boosts ($\mathbf{J}_{0i}, i = 1, 2, 3$) and so three φ leading to only three independent momenta. Put another way \mathbf{p}_μ is subject to the one constraint $\mathbf{p}^\mu \mathbf{p}_\mu + m^2 = 0$.

To find the wave function in position space we took the Fourier transform of the general state of the irreducible representation with a factor $e^{ip \cdot x}$ and integrate over the independent components of the momenta, or equivalently the φ . To do this one must introduce one position space coordinate for every independent momentum. Since there are an infinite

number of the latter we must introduce an infinite number of position coordinates. As the BMS symmetries were deduced in the usual spacetime it is not immediately clear what is the meaning of these infinite number of additional coordinates.

We showed that the position space coordinates can generically be encoded in an object $X(z)$ which at its lowest level contains the usual coordinates of spacetime. The BMS symmetry is then realised as a reparameterisation of z . As such in position space the irreducible representations of BMS are carried by functionals of $X(z)$, that is $\Psi(X(z))$. Thus one can think of this as a string with parameter z . Although this result has been discussed in the context of the representations discussed in this paper, it is very likely to apply to any representations in which the isotropy group is of finite dimension as in this case the boost has an infinite number of parameters and so there are an infinite number of momenta and as a result an infinite number of position coordinates.

It would be interesting to find the dynamics that this string obeys. The dynamics of this string is BMS_3 invariant and so it must be invariant under the reparameterisation of the world line of equation (2.25) which is that for an object with an upper world index. In a Hamiltonian approach one would expect that such a local symmetry would lead to first class constraints and it would be interesting to see how these are realised on the string states.

We showed in section 3.1 that the irreducible representation for the massive Poincaré particle can be described in position space by an unconstrained function at time-like infinity i^+ which carried a representation of the isotropy group. We then argue that this result can be generalised to representations of extended BMS in three and four dimensions in that these can be described by a string, more precisely a functional of a string living on i^+ .

One obvious way to extend the results of this paper is to compute the extended BMS variations of the general state of the representation as was done, for example in reference [34] for the massless Poincaré particle. One could also try to find a covariant representation as one does for the Poincaré group. In this regard it would be interesting to find what equations of motion the irreducible representations satisfy and in particular what equations the string fields satisfy. The latter would be strongly constrained by the BMS symmetries which manifest themselves as reparameterisations. Once one had the equation of motion one could quantise the string and it would be very interesting to know what is the physical meaning of the quantum excitations and if they had something to do with the infrared effects and, in particular, particle dressing.

That a duality connects objects of different dimensions is not new as the AdS/CFT correspondence relates strings to point particles. It would be of interest to see what is the significance of the strings we have identified in the context of the Carrollian proposal for flat space holography. It would seem that the scattering of particles in Minkowski space should be related to the scattering of strings at infinity. It would also be interesting to investigate if there is a connection to the work of references [36] and [37] which consider a world sheet action at null infinity. This construction is based on the ambitwistor string [39] which is related to the null string obtained by taking the tensionless limit of a relativistic string [40–42].

Acknowledgments

We would like to thank Glenn Barnich for discussions and, in particular, on the role of topology of the super momenta in determining the isotropy group. We also thank Blagoje Oblak for a number of very helpful comments on the first version of this paper and in particular for drawing our attention to the role played by the central charge in representation theory in three dimensions. RR is supported by the European Union's Horizon Europe research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 101104845 (UniFlatHolo), hosted at Harvard University and Ecole Polytechnique. PW would like to thank STFC research council for support over many years.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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References

- [1] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond. A* **269** (1962) 21 [[INSPIRE](#)].
- [2] R. Sachs, *Asymptotic symmetries in gravitational theory*, *Phys. Rev.* **128** (1962) 2851 [[INSPIRE](#)].
- [3] G. Barnich and C. Troessaert, *Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited*, *Phys. Rev. Lett.* **105** (2010) 111103 [[arXiv:0909.2617](#)] [[INSPIRE](#)].
- [4] G. Barnich and C. Troessaert, *Supertranslations call for superrotations*, *PoS CNCFG2010* (2010) 010 [[arXiv:1102.4632](#)] [[INSPIRE](#)].
- [5] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, *JHEP* **05** (2010) 062 [[arXiv:1001.1541](#)] [[INSPIRE](#)].
- [6] G. Barnich and C. Troessaert, *BMS charge algebra*, *JHEP* **12** (2011) 105 [[arXiv:1106.0213](#)] [[INSPIRE](#)].
- [7] M. Campiglia and A. Laddha, *Asymptotic symmetries and subleading soft graviton theorem*, *Phys. Rev. D* **90** (2014) 124028 [[arXiv:1408.2228](#)] [[INSPIRE](#)].
- [8] M. Campiglia and A. Laddha, *New symmetries for the gravitational S-matrix*, *JHEP* **04** (2015) 076 [[arXiv:1502.02318](#)] [[INSPIRE](#)].
- [9] G. Compère, A. Fiorucci and R. Ruzziconi, *Superboost transitions, refraction memory and super-Lorentz charge algebra*, *JHEP* **11** (2018) 200 [Erratum *ibid.* **04** (2020) 172] [[arXiv:1810.00377](#)] [[INSPIRE](#)].
- [10] É.É. Flanagan, K. Prabhu and I. Shehzad, *Extensions of the asymptotic symmetry algebra of general relativity*, *JHEP* **01** (2020) 002 [[arXiv:1910.04557](#)] [[INSPIRE](#)].
- [11] M. Campiglia and J. Peraza, *Generalized BMS charge algebra*, *Phys. Rev. D* **101** (2020) 104039 [[arXiv:2002.06691](#)] [[INSPIRE](#)].

- [12] A. Strominger, *Lectures on the infrared structure of gravity and gauge theory*, Princeton University Press, Princeton, NJ, U.S.A. (2018) [INSPIRE].
- [13] P.J.M. McCarthy, *Asymptotically flat space-times and elementary particles*, *Phys. Rev. Lett.* **29** (1972) 817 [INSPIRE].
- [14] P.J. McCarthy, *Representations of the Bondi-Metzner-Sachs group — I. Determination of the representations*, *Proc. Roy. Soc. Lond. A* **330** (1972) 517.
- [15] P.J. McCarthy, *Representations of the Bondi-Metzner-Sachs group — II. Properties and classification of the representations*, *Proc. Roy. Soc. Lond. A* **333** (1973) 317.
- [16] P.J. McCarthy and M. Crampin, *Representations of the Bondi-Metzner-Sachs group — III. Poincaré spin multiplicities and irreducibility*, *Proc. Roy. Soc. Lond. A* **335** (1973) 301.
- [17] M. Crampin and P.J. McCarthy, *Representations of the Bondi-Metzner-Sachs group — IV. Cantoni representations are induced*, *Proc. Roy. Soc. Lond. A* **351** (1976) 55.
- [18] G. Barnich and B. Oblak, *Notes on the BMS group in three dimensions: I. Induced representations*, *JHEP* **06** (2014) 129 [arXiv:1403.5803] [INSPIRE].
- [19] G. Barnich and B. Oblak, *Notes on the BMS group in three dimensions: II. Coadjoint representation*, *JHEP* **03** (2015) 033 [arXiv:1502.00010] [INSPIRE].
- [20] B. Oblak, *BMS particles in three dimensions*, Ph.D. thesis, U. Brussels, Brussels, Belgium (2016) [arXiv:1610.08526] [INSPIRE].
- [21] X. Bekaert, L. Donnay and Y. Herfray, *BMS particles*, *Phys. Rev. Lett.* **135** (2025) 131602 [arXiv:2412.06002] [INSPIRE].
- [22] X. Bekaert and Y. Herfray, *BMS representations for generic supermomentum*, *Commun. Math. Phys.* **407** (2026) 35 [arXiv:2505.05368] [INSPIRE].
- [23] G. Arcioni and C. Dappiaggi, *Exploring the holographic principle in asymptotically flat space-times via the BMS group*, *Nucl. Phys. B* **674** (2003) 553 [hep-th/0306142] [INSPIRE].
- [24] G. Barnich, *Centrally extended BMS4 Lie algebroid*, *JHEP* **06** (2017) 007 [arXiv:1703.08704] [INSPIRE].
- [25] C. Duval, G.W. Gibbons and P.A. Horvathy, *Conformal Carroll groups and BMS symmetry*, *Class. Quant. Grav.* **31** (2014) 092001 [arXiv:1402.5894] [INSPIRE].
- [26] A. Bagchi, R. Basu, A. Kakkar and A. Mehra, *Flat holography: aspects of the dual field theory*, *JHEP* **12** (2016) 147 [arXiv:1609.06203] [INSPIRE].
- [27] L. Ciambelli et al., *Flat holography and Carrollian fluids*, *JHEP* **07** (2018) 165 [arXiv:1802.06809] [INSPIRE].
- [28] L. Donnay, A. Fiorucci, Y. Herfray and R. Ruzziconi, *Carrollian perspective on celestial holography*, *Phys. Rev. Lett.* **129** (2022) 071602 [arXiv:2202.04702] [INSPIRE].
- [29] L. Donnay, A. Fiorucci, Y. Herfray and R. Ruzziconi, *Bridging Carrollian and celestial holography*, *Phys. Rev. D* **107** (2023) 126027 [arXiv:2212.12553] [INSPIRE].
- [30] E.P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, *Annals Math.* **40** (1939) 149 [INSPIRE].
- [31] A. Ashtekar, J. Bicak and B.G. Schmidt, *Asymptotic structure of symmetry reduced general relativity*, *Phys. Rev. D* **55** (1997) 669 [gr-qc/9608042] [INSPIRE].
- [32] P.J. McCarthy, *The Bondi-Metzner-Sachs group in the nuclear topology*, *Proc. Roy. Soc. Lond. A* **343** (1975) 489.

- [33] G. Barnich and G. Compere, *Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions*, *Class. Quant. Grav.* **24** (2007) F15 [[gr-qc/0610130](#)] [[INSPIRE](#)].
- [34] K. Nguyen and P. West, *Carrollian conformal fields and flat holography*, *Universe* **9** (2023) 385 [[arXiv:2305.02884](#)] [[INSPIRE](#)].
- [35] E. Have, K. Nguyen, S. Prohazka and J. Salzer, *Massive carrollian fields at timelike infinity*, *JHEP* **07** (2024) 054 [[arXiv:2402.05190](#)] [[INSPIRE](#)].
- [36] T. Adamo, E. Casali and D. Skinner, *Perturbative gravity at null infinity*, *Class. Quant. Grav.* **31** (2014) 225008 [[arXiv:1405.5122](#)] [[INSPIRE](#)].
- [37] T. Adamo and E. Casali, *Perturbative gauge theory at null infinity*, *Phys. Rev. D* **91** (2015) 125022 [[arXiv:1504.02304](#)] [[INSPIRE](#)].
- [38] G. Barnich and R. Ruzziconi, *Coadjoint representation of the BMS group on celestial Riemann surfaces*, *JHEP* **06** (2021) 079 [[arXiv:2103.11253](#)] [[INSPIRE](#)].
- [39] L. Mason and D. Skinner, *Ambitwistor strings and the scattering equations*, *JHEP* **07** (2014) 048 [[arXiv:1311.2564](#)] [[INSPIRE](#)].
- [40] A. Schild, *Classical null strings*, *Phys. Rev. D* **16** (1977) 1722 [[INSPIRE](#)].
- [41] J. Isberg, U. Lindstrom, B. Sundborg and G. Theodoridis, *Classical and quantized tensionless strings*, *Nucl. Phys. B* **411** (1994) 122 [[hep-th/9307108](#)] [[INSPIRE](#)].
- [42] A. Bagchi, S. Chakraborty and P. Parekh, *Tensionless strings from worldsheet symmetries*, *JHEP* **01** (2016) 158 [[arXiv:1507.04361](#)] [[INSPIRE](#)].
- [43] A. Bagchi et al., *A tale of three — tensionless strings and vacuum structure*, *JHEP* **04** (2020) 061 [[arXiv:2001.00354](#)] [[INSPIRE](#)].
- [44] W. Donnelly, L. Freidel, S.F. Moosavian and A.J. Speranza, *Gravitational edge modes, coadjoint orbits, and hydrodynamics*, *JHEP* **09** (2021) 008 [[arXiv:2012.10367](#)] [[INSPIRE](#)].
- [45] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, *Semiclassical Virasoro symmetry of the quantum gravity \mathcal{S} -matrix*, *JHEP* **08** (2014) 058 [[arXiv:1406.3312](#)] [[INSPIRE](#)].
- [46] A. Campoleoni, H.A. Gonzalez, B. Oblak and M. Riegler, *BMS modules in three dimensions*, *Int. J. Mod. Phys. A* **31** (2016) 1650068 [[arXiv:1603.03812](#)] [[INSPIRE](#)].
- [47] M. Campiglia and A. Laddha, *Asymptotic symmetries of gravity and soft theorems for massive particles*, *JHEP* **12** (2015) 094 [[arXiv:1509.01406](#)] [[INSPIRE](#)].