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SIMILARITY SOLUTIONS FOR THE GENERALIZED EQUATION OF STEADY TRANSONIC GAS FLOW WITH A SINGULAR SOURCE

By

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Abstract. In this brief paper, we consider the generalized equation of steady transonic gas flow with the addition of a singular source term. While the addition of a source term often destroys self-similarity of such flows, we demonstrate that a self-similar solution can still exist in the case of a singular source. We first reduce the governing nonlinear partial differential equation into an ordinary differential equation for a class of similarity solutions. Then, we study the existence of solutions for this similarity equation. After that, several explicit solution forms are given. In constructing exact solutions analytically, we demonstrate that dual solution branches may exist for some parameter regimes. For those parameter regimes where exact or analytical solutions are not possible, we obtain numerical solutions. The results demonstrate interesting properties of the solutions which warrant further study.

1. Introduction. The standard generalized equation of steady transonic gas flow reads

$$\frac{\partial^2 w}{\partial y^2} + \frac{\alpha}{y}\frac{\partial w}{\partial y} + \beta \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} = 0\,, \tag{1}$$

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where $w = w(x, y)$ is the potential of the velocity field, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are space variables, and both α and β are constant model parameters. Solutions to (1) have appeared in the literature; see [1–3] for more on the background and solution methods for this type of equation. Note that the $\alpha = 0$ case is sometimes also considered, depending on the flow geometry. The $\alpha = 0$ case would correspond to the steady Lin-Reissner-Tsien (LRT) equation [4]. It is also possible to consider the unsteady problem [4–7]. Some solution properties of the LRT problem were previously discussed [8–10]. In particular, source terms were considered, which replace the right hand side of (1) with a function of x and y. Such a function can model a variety of sources, such as jets or obstacles to the flow.

In the present paper, we shall be interested in the modification of (1) to take into account a singular source positioned at $y = 0$. Physically, this corresponds to an impenetrable barrier. The modified equation reads

$$\frac{\partial^2 w}{\partial y^2} + \frac{\alpha}{y}\frac{\partial w}{\partial y} + \beta \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} = \frac{\gamma}{y^2}\,, \tag{2}$$

where γ is a parameter describing the shape of the singularity. From the form of the source term, we see that the equation is singular for all x along $y = 0$ in the xy-plane. A reasonable problem domain is therefore $x \geq 0$ and $y \in \mathbb{R}$. Physically, this problem represents a generalization of the LTR theory in order to include a singular source term. Such a source can be caused by a small jet of infinitesimal width at $y = 0$ or by a small impenetrable surface aligned at $y = 0$.

A direct solution of (2) would be very complicated. In order to reduce the complexity of the problem, we make a similarity transformation

$$w(x, y) = \frac{9\gamma}{4} f(\eta) \quad \text{where} \quad \eta = \left(\frac{16}{81\beta\gamma}\right)^{1/3} \frac{x}{y^{2/3}}\,, \tag{3}$$

which puts equation (2) into the form

$$\eta^2 f'' + f' f'' + \alpha\eta f' = 1\,, \tag{4}$$

which is valid for $\eta \geq 0$. The solution of equation (4) with any reasonable initial or boundary conditions will be interesting.

Before we get started, note that the linear equation

$$\eta^2 f'' + \alpha\eta f' = 1\,, \tag{5}$$

which is closely related to equation (4), has the exact solution form

$$f(\eta) = \begin{cases} A_1 + A_2\eta^{1-\alpha} - (1-\alpha)^{-1}\ln\eta & \text{for } \alpha \neq 1\,, \\ A_1 + A_2\ln\eta + \frac{1}{2}(\ln\eta)^2 & \text{for } \alpha = 1\,, \end{cases} \tag{6}$$

where A_1 and A_2 are real-valued constants. However, as we shall show later, the inclusion of the nonlinear term $f'f''$ drastically alters the behavior of the solutions, and hence the linearized model solution is not representative of the behavior to the full nonlinear model. Hence, (4) poses an interesting challenge, as it is both singular and strongly nonlinear.

First, we shall address the issue of the existence of a solution to (4) under suitable boundary data of the form $f'(0) = 0$ and $f(0)$ arbitrary in Section 2. The condition

$f'(0) = 0$ is equivalent to the physical boundary condition $\frac{\partial w}{\partial x}(x = 0) = 0$, which implies no flow at the boundary. In Section 3, we construct an exact solution to (4) in the case of $\alpha = 2$ and an implicit solution in the case of $\alpha = 0$. Furthermore, we discuss the possibility of allowing $f'(0) \neq 0$, and we show that the limit $f'(0) \to 0$ case is actually a degenerate limit. Indeed, when $f'(0) \neq 0$, a Taylor series solution is possible locally near $\eta = 0$, while for $f'(0) = 0$ no such analytic solution is possible. In Section 4, we consider perturbation solutions in the $\alpha \approx 2$ regime. For both the exact solution of Section 3 and the perturbation solution of Section 4, we demonstrate the existence of dual solutions. One solution is monotone increasing, whereas the other is monotone decreasing. Finally, in Section 5, we obtain numerical solutions for the problem (4) for various values of α. We find that the solutions corresponding to $\alpha < 0$ are qualitatively distinct from the $\alpha > 0$ solutions, with the difference appearing in the behavior of the function f'.

2. Existence of solutions.

We are interested in the initial value problem

$$\eta^2 f'' + f' f'' + \alpha \eta f' = 1 \,, \tag{7}$$

$$f'(0) = 0 \,. \tag{8}$$

We start by setting $u = f'$ so that the problem above can be written as a first order problem of the form

$$\eta^2 u' + u u' + \alpha \eta u = 1 \,, \tag{9}$$

$$u(0) = 0 \,. \tag{10}$$

The solution u can then be integrated in order to determine f.

Now we leave this problem aside for a while and instead consider the initial value problem

$$\eta' = \frac{\eta^2 + u}{1 - \alpha \eta u} \,, \tag{11}$$

$$\eta(0) = 0 \,. \tag{12}$$

Here η is our unknown function and u is our variable. That is, η is a function of u. This type of mapping between $u = u(\eta)$ and $\eta = \eta(u)$ is possible only when the two variables are related in a one-to-one manner, which we shall demonstrate later.

Note that by Picard's theorem the problem (11)-(12) has a unique local solution $\eta(u)$ defined for $u \in (-\theta, \theta)$, where θ is some positive number. By Picard's theorem we have that $\eta(u)$ is at least a C^1 function; therefore for u in a small enough interval $(-\varepsilon, \varepsilon)$ we have that $|\eta(u)| \leq c|u|$. From this, we deduce that for $u \in (-\varepsilon, \varepsilon)$, $\dfrac{\eta^2 + u}{1 - \alpha \eta u}$ has the same sign as u.

Differentiating (11) with respect to u and setting $u = 0$ we find that $\eta''(0) = 1$. It follows that η is a convex function of u near $u = 0$. Consequently we have that η is increasing in the interval $0 \leq \eta \leq \dfrac{1}{\alpha u}$ and also that η is a convex function of u while $0 \leq \eta \leq \dfrac{1}{\alpha u}$.

Let u^* be the upper bound on the existence region of u, that is, $u \in (0, u^*)$ such that $0 \leq \eta(u) \leq \dfrac{1}{2u^* \alpha}$. Then η is growing on the interval $(0, u^*)$ and is convex. Therefore, η

is invertible on the interval $(0, u^*)$, and hence $u(\eta)$ is well defined and invertible on the interval $\eta \in (0, \eta(u^*))$.

Using the fact that $\dfrac{du}{d\eta} = \dfrac{1}{\eta'}$, we find from equation (11) that u satisfies the differential equation

$$u' = \frac{1 - \alpha\eta u}{\eta^2 + u}\,, \tag{13}$$

$$u(\eta^*) = u^* > 0\,. \tag{14}$$

We already know that $u(0) = 0$ and that u is increasing over the interval $\eta \in (0, \eta^*)$. Now it follows from the analysis from equation (13) that u increases until we have $u = \dfrac{1}{\alpha\eta}$ and it decreases after that.

We wish to prove rigorously the statement above. We already established that for some $\epsilon > 0$ the solution $\eta(u)$ of the problem (11)-(12) is increasing and convex on the interval $(0, \epsilon)$. The solution can be extended for all $u \in (0, \infty)$ as an increasing positive solution unless there exists U_b such that $\lim_{u \to U_b} 1 - \alpha\eta u = 0$.

Assume that there is no such finite U_b. It then follows that there exists $\gamma > 0$ such that

$$1 - \alpha\eta u \geq \gamma \quad \text{for all} \quad u\,. \tag{15}$$

It then follows that for all $n > 0$ the problem (11)-(12) has a solution $\eta(u)$ defined on the interval $u \in (0, n)$. Furthermore, from (12), we have that $\eta(0) = 0$, and from (15) we have that $\eta(n) \leq \frac{1}{\alpha n}$. We also proved earlier that $\eta(u) \geq 0$. Now $\eta(u)$ will have a positive maximum over the interval $(0, n)$ and $\eta(u)$ is increasing over the interval $(0, n)$. But $\eta(n)$ goes to zero as n goes to zero. This would imply that the function η is identically zero over the interval $(0, \infty)$. This will be inconsistent with equation (11). This then proves that there exists a finite U_b such that the solution $\eta(u)$ of the problem (11)-(12) satisfies

$$\lim_{u \to U_b} 1 - \eta u = 0\,. \tag{16}$$

By taking the inverse of the function η it then follows that the problem (13)-(14) has a unique positive increasing solution $u(\eta)$ on the interval $\eta \in (0, \eta_m)$ such that $u(\eta_m) = \frac{1}{\alpha\eta_m}$. Furthermore

$$u(\eta) \leq \frac{1}{\alpha\eta} \quad \text{for all} \quad \eta \in (0, \eta_m) \tag{17}$$

and $u'(\eta_m) = 0$.

Now we set $w(\eta) = u(\eta) - \frac{1}{\alpha\eta}$. We have that w is continuously differentiable on the interval $\eta \geq \eta_m$ and that $w(\eta_m) = 0$ and $w'(\eta_m) > 0$. We wish to prove that $w(\eta) \geq 0$ for all $\eta \geq \eta_m$. Assume that this is not the case. Then there exists a smallest η_z such that $w(\eta_z) = 0$ and $w(\eta) \geq 0$ for all $\eta \in (\eta_m, \eta_z)$. It then follows from equation (13) that $w'(\eta_z) = \frac{1}{\alpha\eta^2}$ and that $w(\eta_z) = 0$; this will contradict the fact that $w(\eta) \geq 0$ for all $\eta \in (\eta_m, \eta_z)$. Indeed, if w is increasing at η_z it cannot be going from positive to zero. Therefore, we must have $w(\eta) \geq 0$ for all $\eta \geq \eta_m$.

3. Exact and closed-form solutions. Consider the substitution $g(\eta) = \eta^2 + f'(\eta)$ so that (4) becomes

$$g(g-\eta^2)' + \alpha\eta(g-\eta^2) = 1 \tag{18}$$

or, equivalently,

$$gg' + (\alpha-2)\eta g = 1 + \alpha\eta^3\,. \tag{19}$$

3.1. *The case of* $\alpha = 2$. In the case where $\alpha = 2$,

$$gg' = 1 + 2\eta^3\,. \tag{20}$$

A first integral of (20) yields

$$g^2 = g(0)^2 + 2\eta + \eta^4 = 2\eta + \eta^4\,, \tag{21}$$

hence

$$f'(\eta) = \pm\sqrt{2\eta+\eta^4} - \eta^2\,. \tag{22}$$

Integrating once, we have the dual exact solutions

$$f(\eta) = f(0) \pm \int_0^\eta \sqrt{2\tau+\tau^4}\,d\tau - \frac{1}{3}\eta^3\,. \tag{23}$$

With this, we have constructed a monotone increasing exact solution on $\eta > 0$ (taking $+$) and a separate monotone decreasing exact solution on $\eta > 0$ (taking $-$).

We remark that the second derivatives of our solutions become singular like $\pm 1/\sqrt{2\eta}$ as $\eta \to 0^+$. Indeed,

$$f''(\eta) = \pm\frac{1+2\eta^3}{\sqrt{2\eta+\eta^4}} - 2\eta\,, \tag{24}$$

so

$$\begin{aligned}\lim_{\eta\to 0^+}\left(f''(\eta) \mp \frac{1}{\sqrt{2\eta}}\right) &= \pm\lim_{\eta\to 0^+}\left(\frac{1}{\sqrt{2\eta+\eta^4}} - \frac{1}{\sqrt{2\eta}}\right)\\ &= \mp\lim_{\eta\to 0^+}\frac{\eta^{5/2}}{\sqrt{2}\sqrt{2+\eta^3}(\sqrt{2}+\sqrt{2+\eta^3})} = 0\,.\end{aligned} \tag{25}$$

Regarding the asymptotic limit $\eta \to \infty$, note that the $-$ branch solution has derivative f' which tends to negative infinity. Meanwhile, the $+$ branch solution's derivative tends to zero:

$$\lim_{\eta\to\infty} f'(\eta) = \lim_{\eta\to\infty}\frac{2\eta}{\sqrt{2\eta+\eta^4}+\eta^2} = 0\,. \tag{26}$$

However, the solution itself does not tend to a finite limit, as

$$\begin{aligned}\lim_{\eta\to\infty}(f(\eta)-f(0)) &= \lim_{\eta\to\infty}\int_0^\eta\left\{\sqrt{2\tau+\tau^4}-\tau^2\right\}d\tau\\ &= \lim_{\eta\to\infty}\int_0^\eta\left\{\frac{2\tau}{\sqrt{2\tau+\tau^4}+\tau^2}\right\}d\tau\\ &> \lim_{\eta\to\infty}\int_1^\eta\frac{2\tau}{\sqrt{3\tau^2}+\tau^2}d\tau\\ &= \infty\,,\end{aligned} \tag{27}$$

i.e. $f(\eta) \to \infty$ as $\eta \to \infty$ for the $+$ branch solution.

3.2. *The case of* $\alpha = 0$. Meanwhile, in the case of $\alpha = 0$, we have $\eta^2 u' + uu' = 1$, where again $u = f'$. Reversing the independent and dependent variables locally near $\eta = 0$, $u = 0$, we may write $\eta' = \eta^2 + u$. This is a Riccati equation for η. We may solve this equation, subject to $u = 0$ and $\eta = 0$, obtaining the relation

$$\eta(u) = -\frac{d}{du}\ln\left\{\sqrt{3}\mathrm{Ai}(-u) + \mathrm{Bi}(-u)\right\}. \tag{28}$$

Here, Ai and Bi denote Airy functions of first and second kind. We may verify that the relation is one-to-one locally. In fact, for each $\eta \geq 0$, there exists a value of $u \in [0, u^*]$ where u^* is the first positive root of $\sqrt{3}\mathrm{Ai}(-u) + \mathrm{Bi}(-u)$. Numerically, one may show that $u^* = 1.986352707$ to ten digits. Therefore, we may invert this relation to solve for f'.

We see that $f'(0) = 0$ is satisfied. Furthermore, as $\eta \to \infty$, we must have that $f' \to u^*$. The function $f'(\eta)$ is monotone increasing on $\eta > 0$.

3.3. *On the condition* $f'(0) = 0$. Let us relax the condition $f'(0) = 0$ momentarily in order to demonstrate a property of the solutions. Suppose that $f'(0) = m$ for some nonzero constant m. A Taylor series solution g to (19) then reads

$$g(\eta) = m + \frac{\eta}{m} - \left\{\frac{\alpha-2}{2} + \frac{1}{2m^3}\right\}\eta^2 + \left\{\frac{\alpha-2}{6m^3} + \frac{1}{2m^5}\right\}\eta^3 + O(\eta^4). \tag{29}$$

What this shows is that the limit $m \to 0$ is a singular limit. So, the solution corresponding to $f'(0) = 0$ is qualitatively distinct from the solutions corresponding to $f'(0) \neq 0$. The reason for this should be clear upon inspecting (19). Indeed, forcing $g(0) = 0$ implies that $g' \to \infty$ as $\eta \to \infty$; otherwise we would have $0 = 1$.

Locally, the Taylor series can adequately describe the solutions to (19) provided $f'(0) \neq 0$, while for the $f'(0) = 0$ case, we need another approach. Hence, there is some difficulty in describing the $f'(0) = 0$ solutions, which makes this an interesting mathematical problem.

4. Iterative solution for the $|\alpha - 2| < 1$ case. For the case where α is close to two, note that (19) can be integrated once, and we obtain

$$g^2(\eta) + 2(\alpha-2)\int_0^\eta \tau g(\tau)d\tau = 2\eta + \frac{\alpha}{2}\eta^4. \tag{30}$$

If α is close to two, we can write $\alpha = 2 + \epsilon$ for some small ϵ. Assume that g may be written as

$$g(\eta) = g_0(\eta) + \epsilon g_1(\eta) + \epsilon^2 g_2(\eta) + O(\epsilon^3). \tag{31}$$

Then (30) gives

$$\begin{aligned} g_0^2 &= 2\eta + \eta^4, \\ 2g_0g_1 + 2\int_0^\eta \tau g_0(\tau)d\tau &= \frac{1}{2}\eta^4, \\ 2g_0g_2 + g_1^2 + 2\int_0^\eta \tau g_0(\tau)d\tau &= 0. \end{aligned} \tag{32}$$

Solving these equations successively, we find that

$$\begin{aligned} g_0(\eta) &= \pm\sqrt{2\eta+\eta^4}\,, \\ g_1(\eta) &= \pm\frac{\frac{1}{4}\eta^4 - \int_0^\eta \tau\sqrt{2\tau+\tau^4}d\tau}{\sqrt{2\eta+\eta^4}}\,, \\ g_2(\eta) &= \mp\frac{\left(\frac{1}{4}\eta^4 - \int_0^\eta \tau\sqrt{2\tau+\tau^4}d\tau\right)^2}{2(2\eta+\eta^4)^{3/2}} - \frac{1}{4\sqrt{2\eta+\eta^4}}\int_0^\eta \frac{\tau^5 d\tau}{\sqrt{2\tau+\tau^4}} \\ &\quad + \frac{1}{\sqrt{2\eta+\eta^4}}\int_0^\eta \frac{\tau}{\sqrt{2\tau+\tau^4}}\int_0^\tau \sigma\sqrt{2\sigma+\sigma^4}d\sigma d\tau\,. \end{aligned} \tag{33}$$

Therefore, for α close to two, we have two solution branches which are given by the representation

$$\begin{aligned} g(\eta) = &\pm\sqrt{2\eta+\eta^4} \pm (\alpha-2)\frac{\frac{1}{4}\eta^4 - \int_0^\eta \tau\sqrt{2\tau+\tau^4}d\tau}{\sqrt{2\eta+\eta^4}} \\ &\mp (\alpha-2)^2\frac{\left(\frac{1}{4}\eta^4 - \int_0^\eta \tau\sqrt{2\tau+\tau^4}d\tau\right)^2}{2(2\eta+\eta^4)^{3/2}} - (\alpha-2)^2\frac{1}{4\sqrt{2\eta+\eta^4}}\int_0^\eta \frac{\tau^5 d\tau}{\sqrt{2\tau+\tau^4}} \\ &+ (\alpha-2)^2\frac{1}{\sqrt{2\eta+\eta^4}}\int_0^\eta \frac{\tau}{\sqrt{2\tau+\tau^4}}\int_0^\tau \sigma\sqrt{2\sigma+\sigma^4}d\sigma d\tau + O\left((\alpha-2)^3\right)\,. \end{aligned} \tag{34}$$

Performing one more integral, we may recover $f(\eta)$ by

$$f(\eta) = f(0) - \frac{\eta^3}{3} + \int_0^\eta g(\tau)d\tau\,. \tag{35}$$

5. Numerical simulation. Following the method outlined in Section 2, we may numerically solve (9) for $u = f'$ by instead solving equation (11) for η as a function of u and then inverting the relation (on the interval where this relation is one-to-one). This is depicted in Figure 1. For $\alpha < 0$, this relation remains one-to-one. Therefore, when $\alpha < 0$, the complete solution can be described in this manner. On the other hand, for $\alpha > 0$, we find that there exists a turning point. In particular, the solution starts out at $(u,\eta) = (0,0)$ and increases in both variables to $(u,\eta) = (u^*,\eta^*)$. The solution then decreases monotonically for all $\eta > \eta^*$. Only the branch of the solution between $(u,\eta) = (0,0)$ and $(u,\eta) = (u^*,\eta^*)$ can be described by (11).

Assume a solution $\eta(u)$ is known on $(0,u^*)$. Then, define $\eta^* = \eta(u^*)$. To determine the remaining part of the solution, we solve the initial value problem

$$(\eta^2+u)u' + \alpha\eta u = 1\,, \quad u(\eta^*) = u^*\,. \tag{36}$$

This initial value problem is well defined for all $\eta \geq \eta^*$. Furthermore, since $1-\alpha\eta u < 0$ for all $\eta > \eta^*$, we have that

$$u' = \frac{1-\alpha\eta u}{\eta^2+u} < 0 \tag{37}$$

for all $\eta \in (\eta^*,\infty)$. Therefore, when $\alpha > 0$, the function f' increases on $\eta \in (0,\eta^*)$, attains a maximal value of u^* at $\eta = \eta^*$, and then decreases for all $\eta \in (\eta^*,\infty)$. This behavior of the solutions corresponding to $\alpha > 0$ is shown in Figure 2. Note that this

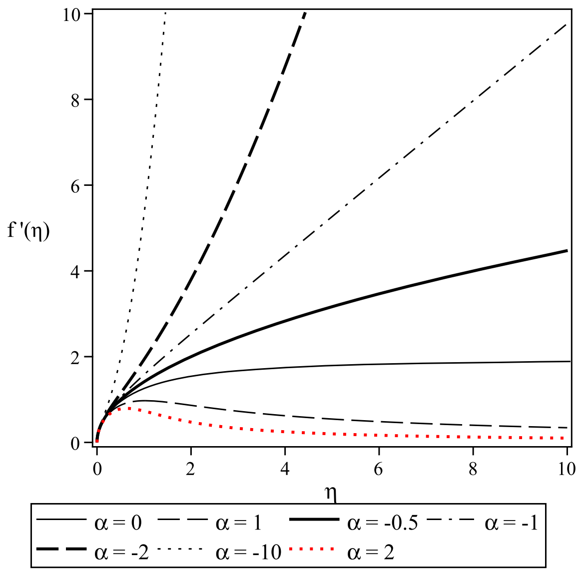

FIG. 1. Plots of the function $f'(\eta)$ versus η for various values of the parameter α. When $\alpha < 0$, the profiles are monotonically increasing. On the other hand, when $\alpha > 0$, the profiles of $f'(\eta)$ increase, attain a maximal value, and then decrease asymptotically toward $f' \to 0$ as $\eta \to \infty$. The case of $\alpha = 2$ corresponds to the exact solution obtained in Section 3, while all other curves are obtained by numerical solutions.

behavior is seen in the exact solution found for $\alpha = 2$. Indeed, for this solution f' increases from zero to a maximal value of $u^* = 2^{1/3}$, which is attained at $\eta^* = 2^{-2/3}$. The function f' then decreases in value monotonically until $f' \to 0$ as $\eta \to \infty$.

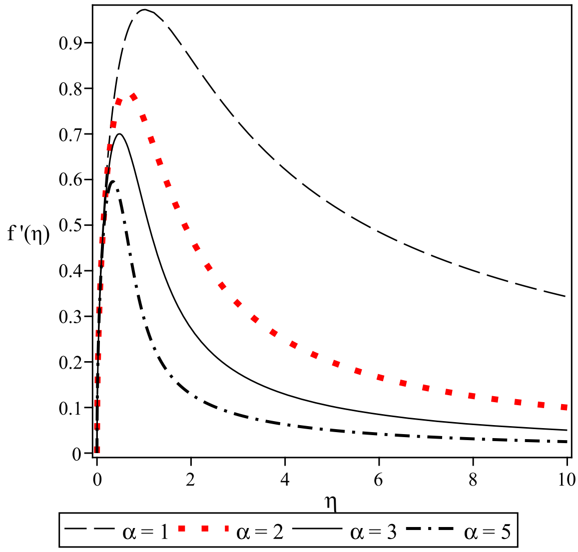

FIG. 2. Plots of the function $f'(\eta)$ versus η for various values of the parameter α in the case where $\alpha > 0$. In the case where $\alpha > 0$, the profiles of $f'(\eta)$ increase, attain a maximal value, and then decrease asymptotically toward $f' \to 0$ as $\eta \to \infty$. The value of η corresponding to this maximal value decreases as α is increased. The case of $\alpha = 2$ corresponds to the exact solution obtained in Section 3, while all other curves are obtained by numerical solutions.

6. Conclusions. Starting with a generalized equation for the velocity potential of steady transonic gas flow (which is a type of LTR equation) with the addition of a singular source, we obtained an equation governing the self-similar flow of such a steady transonic gas. This equation is compact yet complicated due to both its nonlinearity and nonautonomy. Hence, the solution of this equation is nontrivial.

We first obtained existence results for the solution of this equation. We have rigorously determined the local existence of solutions near $\eta = 0$ for the initial data $f'(0) = 0$. Numerical and analytical results have been found in order to verify these results. In the case where $\alpha = 2$, an exact solution is possible in closed-form. In the case of $|\alpha-2| < 1$, we may therefore construct a perturbation solution around the exact solution. Meanwhile, when $\alpha = 0$, an implicit solution is possible. This implicit relation may be investigated near $\eta = 0$ in order to recover the relevant solution.

The numerical solutions obtained demonstrate that for $\alpha < 0$ the profiles of f' are monotone increasing. On the other hand, for $\alpha > 0$, we obtain a class of solutions which increase, attain a maximal value at finite η, and then decrease monotonically toward zero as $\eta \to \infty$. It is worth noting, however, that in the case of $\alpha > 0$, dual exact solutions are obtained. The second solution found decreases monotonically for all $\eta > 0$. Hence, it is possible that for each of the numerical solutions a dual solution exists.

Another interesting property of the solution profiles is that the solutions depend strongly on the initial condition $f'(0)$. Indeed, when $f'(0) \neq 0$, we have shown that a Taylor series representation is possible. In contrast, when $f'(0) = 0$ (corresponding to the physical condition $\frac{\partial w}{\partial x}(x = 0) = 0$, which implies no flow at the boundary) we have that the Taylor series becomes singular in $f'(0)$. So, this $f'(0) = 0$ case is, in a sense, a singular limit of the problem and hence it must be studied with caution.

For the exact solution obtained, we may always put the similarity solution back into a solution to the original equation (2) governing the velocity potential by use of equation (3). Using (23), we obtain the dual exact solutions (in the xy-coordinates)

$$w(x, y) = \frac{9\gamma}{4} \pm \frac{9\gamma}{4} \int_0^{[16/81\beta\gamma]^{1/3} x y^{-2/3}} \sqrt{2\tau + \tau^4}\, d\tau - \frac{4}{27\beta} \frac{x^3}{y^2} \,. \qquad (38)$$

This solution is valid when $\alpha = 2$. For other values of α, such a solution can be studied through perturbation (near $|\alpha - 2| < 1$) as in (34) or numerically for any choice of α (as was done in Section 5).

References

[1] S. S. Titov, A method of finite dimensional rings for solving nonlinear equations of mathematical physics [in Russian]. In: *Aerodynamics* (Editor T. P. Ivanova), Saratov Univ., Saratov, pp. 104-110, 1988.

[2] S. R. Svirshchevskiĭ, *Lie-Bäcklund symmetries of linear ODEs and generalized separation of variables in nonlinear equations*, Phys. Lett. A **199** (1995), no. 5-6, 344–348, DOI 10.1016/0375-9601(95)00136-Q. MR1327680 (95m:35008)

[3] Andrei D. Polyanin and Valentin F. Zaitsev, *Handbook of nonlinear partial differential equations*, Chapman & Hall/CRC, Boca Raton, FL, 2004. MR2042347 (2004m:35001)

[4] C. C. Lin, E. Reissner, and H. S. Tsien, *On two-dimensional non-steady motion of a slender body in a compressible fluid*, J. Math. Physics. **27** (1948), 220–231. MR0026499 (10,162e)

[5] W. F. Ames and M. C. Nucci, *Analysis of fluid equations by group methods*, J. Engrg. Math. **20** (1986), no. 2, 181–187, DOI 10.1007/BF00042776. MR846957 (87m:76022)

[6] A. G. Kuz'min and A. V. Ivanova, *The structural instability of transonic flow associated with amalgamation/splitting of supersonic regions*, Fluid Dynamics **18** (2004), 335-344.

[7] J. Haussermann, K. Vajravelu, and R. A. Van Gorder, *Self-similar solutions to Lin-Reissner-Tsien equation*, Appl. Math. Mech. (English Ed.) **32** (2011), no. 11, 1447–1456, DOI 10.1007/s10483-011-1514-6. MR2896096 (2012j:76068)

[8] A. N. Bogdanov and V. N. Diesperov, *On the distinguishing features of asymptotic expansions for the parameters of unsteady transonic flows with axial symmetry* (Russian, with Russian summary), Prikl. Mat. Mekh. **72** (2008), no. 1, 54–57, DOI 10.1016/j.jappmathmech.2008.03.009; English transl., J. Appl. Math. Mech. **72** (2008), no. 1, 33–35. MR2423436

[9] Yu. V. Bibik, V. N. Duesperov, and S. P. Popov, *Structure of time-dependent transonic flows in plane channels* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Mekh. Zhidk. Gaza **2** (2005), 168–179, DOI 10.1007/s10697-005-0071-y; English transl., Fluid Dynam. **40** (2005), no. 2, 315–325. MR2151851

[10] S. N. Glazatov, *On the solvability of a spatially periodic problem for the Lin-Reissner-Tsien equation of transonic gas dynamics* (Russian), Mat. Zametki **87** (2010), no. 1, 137–140, DOI 10.1134/S0001434610010177; English transl., Math. Notes **87** (2010), no. 1-2, 130–134. MR2730392