# Zonostrophic instability driven by discrete particle noise

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(Dated: 19 March 2017)

The consequences of discrete particle noise for a system possessing a possibly unstable collective mode are discussed. It is argued that a zonostrophic instability (of homogeneous turbulence to the formation of zonal flows) occurs just below the threshold for linear instability. The scenario provides a new interpretation of the random forcing that is ubiquitously invoked in stochastic models such as the second-order cumulant expansion (CE2) or stochastic structural instability theory (SSST); neither intrinsic turbulence nor coupling to extrinsic turbulence is required. A representative calculation of the zonostrophic neutral curve is made for a simple two-field model of toroidal ion-temperature-gradient-driven modes. To the extent that the damping of zonal flows is controlled by the ion–ion collision rate, the point of zonostrophic instability is independent of that rate.

## I. INTRODUCTION

The zonostrophic instability[1] of a statistically homogeneous steady state to inhomogeneous zonal flows figures importantly in current research on the physics of zonal-flow generation and the interaction of zonal flows with turbulence. The general framework, called statistical state dynamics by Farrell and Ioannou,[2] is intrinsically a statistical formalism — perturbations are made to a statistical ensemble, not a particular realization. Analytical treatment must therefore involve some sort of stochastic model. Considerable attention has been given to low-order cumulant truncations, in particular the CE2 (second-order cumulant expansion) studied by Tobias and Marston[3–5] and the closely related (in some cases mathematically identical) SSST or S3T (stochastic structural stability theory) introduced earlier by Farrell and Ioannou.[6,7] A brief introduction to these topics is given in Sec. 6.3 of reference 8. In CE2 the so-called eddy–eddy nonlinearities of the turbulence are neglected; only the interactions between the turbulence and the zonal flows are retained. Generally the turbulence is driven by a white-noise stochastic forcing $\widetilde{f}$, for which various interpretations have been given. Sometimes, especially in S3T papers, it is taken to represent the effects of the missing eddy–eddy interactions. In strict CE2, it is instead taken to represent the effects of extrinsic fluctuations — e.g., baroclinic instabilities — whose dynamics are not described by the partial differential equation under study. In the present paper we introduce a variant of this latter interpretation; we attribute $\widetilde{f}$ to the discrete particle noise of a weakly coupled many-body plasma. We describe the relationship of this viewpoint to Kadomtsev's classic discussion of the transition from stable to unstable collective modes, and we illustrate with a calculation of the neutral curve for the zonostrophic instability of a simple model of the toroidal ion-temperature-gradient (ITG) instability. New insights about the onset of the Dimits shift follow.

In Kadomtsev's famous review/monograph,[9] he considered the following equation for fluctuation intensity $I$ (we have altered his notation slightly):

$$\frac{dI}{dt} = 2\gamma I - 2\alpha I^2 + 2F. \tag{1}$$

Here $\gamma$ is the linear growth rate of a collective mode, $\alpha$ is a positive mode-coupling coefficient, the $I^2$ term describes the possibility of nonlinear saturation of the linear instability, and $F$ describes the (small) mean-square noise level due to discrete particles. (This equation and many other facets of Kadomtsev's book are discussed in a lengthy tutorial article by

Krommes.[8]) In thermal equilibrium, $\gamma$ can be interpreted as the negative of the Landau-damping rate of a typical fluctuation with $k\lambda_D \ll 1$, where $\lambda_D$ is the Debye length [$\lambda_D^{-2} = \sum_s k_{Ds}^2$, with $k_{Ds} \doteq (4\pi n_s q_s^2/T)^{1/2}$ being the Debye wave number for species $s$]. Out of thermal equilibrium, it is assumed that $F$ remains unchanged while $\gamma$ changes as an order parameter (e.g., the temperature gradient $\kappa$) is varied. For stable plasma ($\gamma < 0$), the steady-state balance is

$$I \approx F/|\gamma|. \tag{2}$$

As $\gamma \to 0_-$, that approximate level diverges. However, as the fluctuations become sufficiently large, the mode-coupling term takes over and permits a smooth transition through the point $\gamma = 0$; for large $\gamma$, the nonlinear saturation level is $I \approx \gamma/\alpha$. Of course, the steady-state solution of the quadratic equation (1) can be found exactly; it is graphed in Fig. 1.

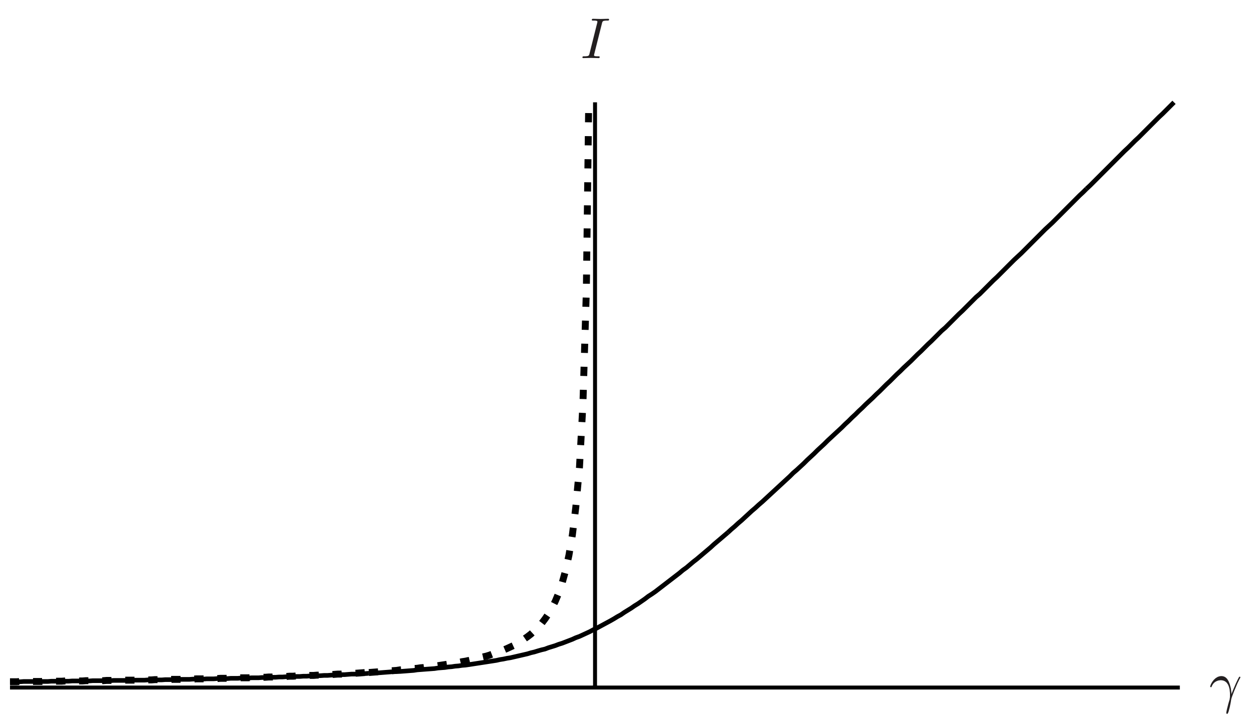


FIG. 1. Solid curve: Steady-state solution of Eq. (1). Dashed curve: Approximate solution when the quadratic mode-coupling term is neglected.

The mode-coupling term $\propto I^2$ in Eq. (1) implicitly assumes turbulence; it represents the net effect of the statistical closure[8,10] of a quadratic nonlinearity. Thus it does not capture the Dimits-shift phenomenon. The Dimits shift was first observed in the computer simulations described in Ref. 11. It refers to the fact that as the background temperature gradient $\kappa$ is varied from below to above the linear threshold $\kappa_c$ for instability (the focus of Dimits *et al.* was on the ITG instability), the ion heat flux $Q(\kappa)$ turns on not for $\kappa > \kappa_c$ but only for a larger value $\kappa > \kappa_* > \kappa_c$. The difference $\kappa_* - \kappa_c$ is called the Dimits shift. It is understood that the suppression of heat flux in the Dimits-shift regime is due to the excitation of zonal flows.[11,12] In that regime, there is no turbulence, so conventional statistical closure does not apply. It is of interest to understand how the transition to turbulence is modified when the physics of the Dimits shift is included.

In the parlance of CE2 or S3T, the mode-coupling considered by Kadomtsev describes only the effect of the turbulent eddy–eddy interactions. In contrast, those stochastic models, which although quite simple have been surprisingly successful in various contexts, neglect those interactions altogether, but do consider the interaction of zonal modes with turbulence. They are also tractable, so they provide a good starting point for further investigations. However, in the regimes of linear stability and the Dimits shift, there is no turbulence (we exclude the possibility of subcritical turbulence in this discussion), so the S3T interpretation of $\widetilde{f}$ as representing the effects of turbulent eddy–eddy interactions is not viable. Instead, we shall use CE2 and interpret $\widetilde{f}$ as being due to discrete particle noise, a forcing that is extrinsic from the point of view of the collective ITG dynamics. We assume that zonal modes are not driven directly by the particle noise (a very weak effect); rather, they arise from the Reynolds stress due to collective modes. The following scenario then pertains. In the absence of zonal flows, the balance (2) between particle noise and modal damping creates a statistically homogeneous steady state (in agreement with Kadomtsev's interpretation). For sufficiently large Landau damping or sufficiently small $\kappa$, the homogeneous fluctuation level is small and the homogeneous state is stable against the formation of inhomogeneous zonal modes (which suffer a small but finite amount of collisional damping). Now imagine increasing $\kappa$, thereby driving the collective modes toward the threshold for linear instability. If zonal modes would never form, then the fluctuation intensity would diverge as $\gamma \to 0_-$, as does the dashed curve in Fig. 1. However, as $\gamma \to 0_-$ the forcing represented by the right-hand side of Eq. (2) becomes relatively large and, on the average, overcomes the damping on the zonal modes; a zonostrophic instability[1] (a supercritical bifurcation) occurs somewhere to the left of the linear threshold $\gamma = 0$ ($\kappa = \kappa_c$). These two thresholds are clearly distinct, with the zonostrophic one involving nonlinear effects. In the context of the ITG problem, the latter threshold defines the left-hand boundary of the Dimits-shift regime.[11] This interpretation of that boundary, as being related to a zonostrophic instability driven by discrete particle noise, is new.

To the extent that the particle noise is very small (as it is in hot, magnetically confined fusion plasmas), it might be thought that the zonostrophic threshold is very close to the linear one and that in the collisionless limit where the noise approaches zero the two thresholds should become coincident. Thus in neither the discussion by Dimits *et al.* in Ref. 11 of collisionless simulations nor the earlier work on the ITG Dimits shift by Kolesnikov and Krommes[13,14] is there any mention of particle noise. In fact, however, a basic scaling with the discreteness parameter $\epsilon_p \doteq (n\lambda_D^3)^{-1}$ cancels out in the competition between the forcing and the zonal damping that defines the zonostrophic transition because the zonal damping rate is proportional to the ion–ion collision rate $\nu_{ii}$, which of course scales with the discrete-ion noise level. However, in gyrokinetics[15] the equilibrium fluctuation level[16,17] also scales with the inverse of the large perpendicular dielectric function $\mathcal{D}_\perp \doteq \omega_{pi}^2/\omega_{ci}^2 = \rho_{\mathrm{s}}^2/\lambda_{De}^2 \gg 1$. [Here $\omega_{pi} \doteq (4\pi n_i q_i^2/m_i)^{1/2}$ is the ion plasma frequency, $\omega_{ci} \doteq q_i B/m_i c$ is the ion gyrofrequency, $\lambda_{De} \doteq k_{De}^{-1}$, and $\rho_{\mathrm{s}} \doteq c_{\mathrm{s}}/\omega_{ci}$ is the sound radius, where $c_{\mathrm{s}} \doteq (ZT_e/m_i)^{1/2}$ is the sound speed. $\mathcal{D}_\perp$ arises because of the shielding effect due to ion polarization.] Because $\nu_{ii}$ does not involve $\mathcal{D}_\perp$, the distance of the zonostrophic threshold to the linear threshold scales with $\mathcal{D}_\perp^{-1}$. That is also small, but not nearly as small as the typical discreteness parameter $\epsilon_p$.

In the remainder of this paper we shall make this idea quantitative by showing how to calculate the neutral curve (the marginality condition for the onset of zonostrophic instability as a function of zonal wave number $q$) associated with an ultra-simple two-field fluid model of the toroidal ITG mode. The purposes of the analysis are primarily to illustrate conceptual principles, to unify the treatment of an important plasma-physics phenomenon with analyses of zonal-flow physics in other fields, and to establish the consistency of our interpretation of the random forcing $\widetilde{f}$. We do not attempt to incorporate all details of the noise sources in the equations for fluctuating potential and temperature (a proper calculation should be kinetic, whereas we use a fluid description), so we cannot be fully quantitative — and of course the model itself lacks many details of importance in practice. Nevertheless, the calculation demonstrates the basic concept of the onset of the noise-driven zonostrophic instability, it shows how to extend to a two-field model the earlier one-field analyses of Srinivasan and Young[1] and Parker and Krommes,[18–20] and it makes a prediction for the wave number of the first zonal mode $q_*$ that is driven unstable.

Even at a qualitative level, it is unlikely that the onset of the zonostrophic instability discussed here will be observable in simulations or experiments because the distance between the zonostrophic threshold and the linear threshold is small. In addition to showing that it is possible to predict $q_*$, the qualitative significance of the calculation is that it "opens up" the transition to the left-hand end of the Dimits-shift regime, connecting the physics to mechanisms that have been recently discussed extensively in other fields,[21] and generalizing part of the classic paradigm of Kadomtsev (see the discussion of Fig. 1). Nevertheless, of much greater practical interest would be a prediction of the size of the Dimits shift and an analysis of the transition from Dimits shift to turbulence. Those are not attempted in this paper; see the concluding Sec. V for further remarks.

The plan of the paper is as follows. In Sec. II we introduce a simple nonlinear model of the ITG mode. In Sec. III we review the CE2 approximation. The heart of the paper is Sec. IV, in which we apply CE2 to our ITG model. We summarize and discuss the work in Sec. V. Finally, several appendices record various calculational details.

## II. TWO-FIELD MODEL OF THE ION-TEMPERATURE-GRADIENT-DRIVEN MODE

The ion-temperature-gradient-driven (ITG) mode is believed to be responsible for the anomalously large ion heat losses in the cores of modern tokamaks. It has both a slab branch and a toroidal branch,[22] and it has been extensively studied both numerically[23–25] and analytically.[12,26–28] In this paper we adopt the simplest possible model that possesses a toroidal ITG mode. Thus we consider a two-dimensional two-field gyrofluid model that retains only the curvature drift, an ion temperature gradient (with a flat density profile), the advective $\boldsymbol{E} \times \boldsymbol{B}$ nonlinearity, and adiabatic electrons with corrected zonal mode ($k_y = 0$) response.[24] We use the usual plasma slab coordinates in which $x$ and $y$ correspond to the radial and poloidal directions, respectively; we neglect all parallel ($z$) dynamics. With time and space being scaled to $a/c_\mathrm{s}$ and $\rho_\mathrm{s}$, respectively ($a$ is the minor radius), the equations are[28]

$$\partial_t \widetilde{\zeta} + \widetilde{\boldsymbol{V}}_{\boldsymbol{E}} \cdot \boldsymbol{\nabla} \widetilde{\zeta} + \epsilon\, \partial_y \widetilde{T} = -\nu_\zeta \widetilde{\zeta} + \widetilde{f}_\zeta, \tag{3a}$$

$$\partial_t \widetilde{T} + \widetilde{\boldsymbol{V}}_{\boldsymbol{E}} \cdot \boldsymbol{\nabla} \widetilde{T} + \kappa\, \partial_y \widetilde{\varphi} = -\nu_T \widetilde{T} + \widetilde{f}_T. \tag{3b}$$

Here tilde denotes a random variable; $\varphi \doteq (e\phi/T_e)(a/\rho_\mathrm{s})$ is the dimensionless electrostatic potential; the definition and properties of the $\boldsymbol{E} \times \boldsymbol{B}$ velocity are

$$\widetilde{\boldsymbol{V}}_{\boldsymbol{E}} \doteq \widehat{\boldsymbol{b}} \times \boldsymbol{\nabla}\widetilde{\varphi} = -\partial_y\widetilde{\varphi}\,\widehat{\boldsymbol{x}} + \partial_x\widetilde{\varphi}\,\widehat{\boldsymbol{y}} \equiv \widetilde{U}\,\widehat{\boldsymbol{x}} + \widetilde{V}\,\widehat{\boldsymbol{y}}, \tag{4a}$$

$$\boldsymbol{V}_E \cdot \boldsymbol{\nabla} A = \{\varphi, A\} \equiv (\partial_x\varphi)(\partial_y A) - (\partial_y\varphi)(\partial_x A); \tag{4b}$$

the generalized vorticity (derived from quasineutrality using the perturbed ion gyrocenter density $\delta\widetilde{N}_i \equiv -\widetilde{\zeta}$) is

$$\zeta \doteq (\nabla_\perp^2 - \widehat{\alpha})\varphi \equiv \overline{\nabla}^2\varphi, \tag{5}$$

where $\widehat{\alpha}$ is an operator that vanishes when acting on zonal modes and is unity otherwise; $\epsilon \doteq 2a/R$ describes the curvature drive; $\kappa \doteq a/L_T$, where $L_T \doteq -d\ln T_i/d\ln x$ is taken to be constant; $\widetilde{T} \doteq (\widetilde{T}_i/T_e)(a/\rho_\mathrm{s})$ ($T_e$ is taken to be constant) describes the deviation of the ion temperature profile from one with constant $\kappa$; $\nu_\zeta$ and $\nu_T$ represent damping operators that in Fourier space are assumed to be even in both $k_x$ and $k_y$; and the $\widetilde{f}$'s represent the random particle noise. A significant approximation is to neglect finite-Larmor-radius (FLR) effects; this precludes detailed comparisons with the equations and results of Ref. 12. In the absence of forcing, damping, and zonal modes, the linearized system

$$\partial_t\Delta\zeta + \epsilon\,\partial_y\Delta T = 0, \quad \partial_t\Delta T + \kappa\,\partial_y\Delta\varphi = 0 \tag{6}$$

implies the dispersion relation

$$\lambda^2 = \Gamma_0^2 \tag{7}$$

(time variations $e^{\lambda t}$ are assumed), where

$$\Gamma_0 \doteq k_y\sqrt{\kappa\epsilon}/\overline{k} \tag{8}$$

and

$$\overline{k}^2 \doteq 1 + k_\perp^2. \tag{9}$$

The eigenvalue $\lambda_+ \approx \Gamma_0$, with unnormalized eigenvector $\boldsymbol{e} \doteq (1, -i\sqrt{\kappa/\epsilon}/\overline{k})^\mathrm{T}$ (T denotes transpose), is the unstable toroidal ITG mode. With small dissipation added, the eigenvalues are

$$\lambda_\pm = \pm[\Gamma_0^2 + (\nu_\zeta - \nu_T)^2/4]^{1/2} - \overline{\nu} \approx \pm\Gamma_0 - \overline{\nu}, \tag{10}$$

where

$$\overline{\nu} \doteq \frac{1}{2}(\nu_\zeta + \nu_T) \tag{11}$$

and the approximation holds for $\Gamma_0 \gg \overline{\nu}$. With

$$\xi \doteq \Gamma_0^2 - \nu_\zeta \nu_T, \tag{12}$$

the transition point $\lambda = 0$ corresponds to $\xi = 0$, with $\xi > 0$ defining the regime of linear instability.

It will be shown in Sec. IV B 3 that consistency of the model requires that $\nu_\zeta = \nu_T = \overline{\nu} = \nu$, where $\nu$ can be interpreted as the Landau-damping rate of the electrostatic potential. Let quantities normalized to $\nu$ be denoted by a hat, e.g.,

$$\widehat{\kappa} \doteq \kappa/\nu, \quad \widehat{\Gamma}_0 \doteq k_y \sqrt{\widehat{\kappa}\widehat{\epsilon}}/\overline{k}, \quad \widehat{\xi} \doteq \xi/\nu^2, \quad \text{etc.}; \tag{13}$$

then the regime of linear stability is $-1 \leq \widehat{\xi} \leq 0$. For now, however, we shall keep $\nu_\zeta$ and $\nu_T$ distinct in order to help one keep track of the origin of various terms.

## III. THE SECOND-ORDER CUMULANT EXPANSION (CE2)

### A. General strategy

We shall treat Eqs. (3) by means of the stochastic model known as the CE2 (second-order cumulant expansion).[3–5] The basic strategy is to decompose the fields into mean and fluctuating parts, e.g., $\widetilde{T} = \overline{T} + \delta T$, where the overline denotes a zonal average, then to ignore products of fluctuating terms (the "eddy–eddy" interactions) in the evolution equation for the fluctuations. An ergodicity assumption is also made, so the barring operation is taken to be equivalent to the ensemble average $\langle \ldots \rangle$ over the microscopic state; we assume that $\langle \widetilde{f} \rangle = 0$. The resulting system that couples the mean and fluctuating fields is known as the quasilinear approximation. Without further approximation, it can be closed exactly by constructing equations for the two-point space-time correlation functions. Because the statistics are constructed from primitive amplitude equations, they are guaranteed to be realizable, i.e., to be compatible with a legitimate probability density functional. For example, the solution of the equations for the two-point correlation matrix is guaranteed to be a positive-semidefinite form. For an introduction to realizability and statistical closure in this context, see Ref. 29.

Although it is possible to close at the level of two-time-point correlation functions, generally closure is done at the level of one-time correlation functions by assuming that the $\widetilde{f}$'s are white noise (delta correlated in time). The resulting equations, which define the standard CE2 approximation and will be written below, are also realizable.

Use of a white-noise approximation may be questionable, since clearly the physical fluctuations are not white. However, this method has a successful track record, not only in the present CE2 context but also in the general theory of statistical closures.[10] Thus the direct-interaction approximation (DIA),[30] which is nonlocal in time and has a nontrivial representation in frequency space,[9] has a Langevin representation[31,32] in which the forcing is not white. However, a related Markovian approximation[29,33] does use white forcing. While such an approximation cannot do complete justice to the details of two-time correlations, it has been shown to be reasonably successful at predicting single-time wave-number spectra. We view the CE2 in the same light.

### B. CE2 equations for the ITG model

For the 2D ITG model, we define the zonal mean of an arbitrary quantity $A(\boldsymbol{x})$ as[34]

$$\overline{A}(x) \doteq \frac{1}{L_y}\int_0^{L_y} dy\, A(x,y). \tag{14}$$

Upon dropping the overlines on the zonally averaged fields, the equations for the mean fields are

$$\partial_t U(x,t) = -\partial_x(\overline{\delta u\,\delta v}) - \nu_\zeta^{\mathrm{Z}} U, \tag{15a}$$

$$\partial_t T(x,t) = -\partial_x(\overline{\delta u\,\delta T}) - \nu_T^{\mathrm{Z}} T \tag{15b}$$

where $U(x,t) \doteq \partial_x\overline{\varphi}$ is the $y$-directed zonal velocity. In arriving at Eq. (15a), we integrated the equation for $\overline{\zeta}$ once in $x$. The fluctuations obey

$$\partial_t\delta\zeta = -U\,\partial_y\delta\zeta + (\partial_x^2 U)\partial_y\delta\varphi - \epsilon\,\partial_y\delta T - \nu_\zeta\,\delta\zeta + \delta f_\zeta, \tag{16a}$$

$$\partial_t\delta T = -U\,\partial_y\delta T - (\kappa - \partial_x T)\partial_y\delta\varphi - \nu_T\,\delta T + \delta f_T. \tag{16b}$$

To construct the CE2 equations, we define the two-space-point covariance tensor

$$C_{AB}(x_1,x_2,t;k_y) \doteq \int_{-\infty}^{\infty} dr\, e^{-ik_y r}\langle\delta A(x_1,y_2+r,t)\delta B(x_2,y_2,t)\rangle; \tag{17}$$

this quantity is independent of $y_2$ by virtue of statistical homogeneity (translational invariance) in $y$. We shall drop the $k_y$ argument when there is no possibility of confusion. Similarly, we assume that the forcing is stationary, homogeneous white noise and write

$$\int_{-\infty}^{\infty} dr\, e^{-ik_y r}\langle \delta f_A(\boldsymbol{x}_1, t)\delta f_B(\boldsymbol{x}_2, t')\rangle$$

$$\doteq F_{AB}(x_1, t; x_2, t'; k_y) \tag{18a}$$

$$= 2D_{AB}(x_1 - x_2; k_y)\delta(t - t'). \tag{18b}$$

In terms of $U_i \doteq U(x_i)$, $T_i \doteq T(x_i)$, $\overline{\nabla}_i^2 \doteq \partial_{x_i}^2 - k_y^2 - 1$, and $\Lambda_i \doteq \kappa - \partial_{x_i} T_i$, one then finds

$$\partial_t \overline{\nabla}_1^2 \overline{\nabla}_2^2 C_{\varphi\varphi} = -ik_y(U_1 - U_2)\overline{\nabla}_1^2\overline{\nabla}_2^2 C_{\varphi\varphi} + ik_y(U_1''\overline{\nabla}_2^2 - U_2''\overline{\nabla}_1^2)C_{\varphi\varphi} - ik_y\epsilon(\overline{\nabla}_2^2 C_{T\varphi} - \overline{\nabla}_1^2 C_{\varphi T})$$
$$- 2\nu_\zeta \overline{\nabla}_1^2\overline{\nabla}_2^2 C_{\varphi\varphi} + 2D_{\zeta\zeta}, \tag{19a}$$

$$\partial_t \overline{\nabla}_1^2 C_{\varphi T} = -ik_y[(U_1 - U_2)\overline{\nabla}_1^2 - (\partial_{x_1}^2 U_1)]C_{\varphi T} + ik_y\Lambda_2\overline{\nabla}_1^2 C_{\varphi\varphi}$$
$$- ik_y\epsilon C_{TT} - 2\overline{\nu}\overline{\nabla}_1^2 C_{\varphi T} + 2D_{\zeta T}, \tag{19b}$$

$$\partial_t C_{TT} = -ik_y(U_1 - U_2)C_{TT} - ik_y(\Lambda_1 C_{\varphi T} - \Lambda_2 C_{T\varphi}) - 2\nu_T C_{TT} + 2D_{TT}, \tag{19c}$$

together with $C_{T\varphi}(x_1, x_2) = C_{\varphi T}^*(x_2, x_1)$.

More informative coordinates are the sum and difference variables

$$\overline{x} \doteq \frac{1}{2}(x_1 + x_2), \quad x \doteq x_1 - x_2, \tag{20}$$

which isolate any dependence on statistical inhomogeneity in $\overline{x}$. The inverse transformation is

$$x_1 = \overline{x} + \frac{1}{2}x, \quad x_2 = \overline{x} - \frac{1}{2}x. \tag{21}$$

The use of $x$ and $\overline{x}$ enable one to make contact with the theory of Wigner–Moyal transforms.[35] Also define the modified Laplacian

$$\overline{\nabla}_\pm^2 \doteq \overline{\nabla}_x^2 \pm \partial_x\partial_{\overline{x}} + \frac{1}{4}\partial_{\overline{x}}^2, \tag{22}$$

where $\overline{\nabla}_x^2 \doteq \partial_x^2 - k_y^2 - 1$, as well as

$$U_\pm \doteq U\left(\overline{x} \pm \tfrac{1}{2}x\right), \quad T_\pm \doteq T\left(\overline{x} \pm \tfrac{1}{2}x\right), \quad \Lambda_\pm \doteq \kappa - T_\pm', \tag{23}$$

and write $C(x_1, x_2, k_y, t) \equiv C(x, k_y \mid \overline{x}, t)$. The transcription of Eqs. (19) is then immediate:

$$\partial_t \overline{\nabla}_+^2 \overline{\nabla}_-^2 C_{\varphi\varphi} = -ik_y(U_+ - U_-)\overline{\nabla}_+^2 \overline{\nabla}_-^2 C_{\varphi\varphi} + ik_y(U_+'' \overline{\nabla}_-^2 - U_-'' \overline{\nabla}_+^2) C_{\varphi\varphi}$$
$$- ik_y\epsilon(\overline{\nabla}_-^2 C_{T\varphi} - \overline{\nabla}_+^2 C_{\varphi T}) - 2\nu_\zeta \overline{\nabla}_+^2 \overline{\nabla}_-^2 C_{\varphi\varphi} + 2D_{\zeta\zeta}, \tag{24a}$$
$$\partial_t \overline{\nabla}_+^2 C_{\varphi T} = -ik_y[(U_+ - U_-)\overline{\nabla}_+^2 - U_+''] C_{\varphi T} + ik_y \Lambda_- \overline{\nabla}_+^2 C_{\varphi\varphi} - ik_y\epsilon C_{TT}$$
$$- 2\overline{\nu}\overline{\nabla}_+^2 C_{\varphi T} + 2D_{\zeta T}, \tag{24b}$$
$$\partial_t C_{TT} = -ik_y(U_+ - U_-)C_{TT} - ik_y(\Lambda_+ C_{\varphi T} - \Lambda_- C_{T\varphi}) - 2\nu_T C_{TT} + 2D_{TT}, \tag{24c}$$

together with $C_{T\varphi}(x \mid \overline{x}) = C_{\varphi T}^*(-x \mid \overline{x}) = C_{\varphi T}(x \mid -\overline{x})$. The mean equations are

$$\left(\frac{\partial}{\partial t} + \nu_\zeta\right) U(\overline{x}, t) = -i\frac{\partial}{\partial \overline{x}} \int \frac{dk_y}{2\pi}\, k_y (\partial_x C_{\varphi\varphi})_{|x=0}, \tag{25a}$$
$$\left(\frac{\partial}{\partial t} + \nu_T\right) T(\overline{x}, t) = \frac{1}{2} i \frac{\partial}{\partial \overline{x}} \int \frac{dk_y}{2\pi}\, k_y (C_{\varphi T} - C_{T\varphi})_{|x=0}. \tag{25b}$$

In the derivation of Eq. (25a), a term $\propto k_y \partial_{\overline{x}} C_{\varphi\varphi}(0, k_y \mid \overline{x}, t)$ vanished by symmetry under the $k_y$ integration. The result (25b) has been written in a convenient symmetrized form.

## IV. CE2 ANALYSIS OF THE ITG MODEL

### A. Homogeneous steady states of the ITG model

We denote statistically steady ($\partial_t = 0$) and homogeneous ($\partial_{\overline{x}} = 0$) solutions by a superscript (0). One obtains $U^{(0)} = T^{(0)} = 0$, which permits a ready Fourier transformation in $x$. Thus $\overline{\nabla}_\pm^2 \to -\overline{k}^2$, where $\overline{k}^2 \doteq 1 + k_\perp^2$, and the components of the equilibrium covariance matrix $\mathsf{C}^{(0)}$ obey

$$0 = 2k_y\epsilon\overline{k}^2 \operatorname{Im} C_{\varphi T}^{(0)} - 2\nu_\zeta \overline{k}^4 C_{\varphi\varphi}^{(0)} + 2D_{\zeta\zeta}, \tag{26a}$$
$$0 = -ik_y\kappa\overline{k}^2 C_{\varphi\varphi}^{(0)} - ik_y\epsilon C_{TT}^{(0)} + 2\overline{\nu}\overline{k}^2 C_{\varphi T}^{(0)} + 2D_{\zeta T}, \tag{26b}$$
$$0 = 2k_y\kappa \operatorname{Im} C_{\varphi T}^{(0)} - 2\nu_T C_{TT}^{(0)} + 2D_{TT}. \tag{26c}$$

The real part of $C_{\varphi T}^{(0)}$ follows from the real part of Eq. (26b):

$$\operatorname{Re} C_{\varphi T}^{(0)} \equiv C_{\varphi T}^{(0)\prime} = -D_{\zeta T}'/\overline{\nu}\overline{k}^2, \tag{27}$$

Equations (26a), (26c), and the imaginary part of Eq. (26b) then define a 3D linear algebraic system that can be solved for $C_{\varphi\varphi}^{(0)}$, $\operatorname{Im} C_{\varphi T}^{(0)} \equiv C_{\varphi T}^{(0)\prime\prime}$, and $C_{TT}^{(0)}$ in terms of the as-yet-unspecified noise sources. The algebra is tractable by hand; one finds

$$\begin{aligned}\begin{pmatrix} C^{(0)}_{\varphi\varphi} \\ C^{(0)\prime\prime}_{\varphi T} \\ C^{(0)}_{TT} \end{pmatrix} &= \frac{1}{2\overline{\nu}\overline{k}^4\xi} \begin{pmatrix} \xi-\nu_T^2 & k_y\epsilon\nu_T & -k_y^2\epsilon^2 \\ -k_y\kappa\nu_T & \overline{k}^2\nu_\zeta\nu_T & -k_y\overline{k}^2\epsilon\nu_\zeta \\ -k_y^2\kappa^2 & k_y\kappa\overline{k}^2\nu_\zeta & \overline{k}^4(\xi-\nu_\zeta^2) \end{pmatrix} \begin{pmatrix} D_{\zeta\zeta} \\ 2D''_{\zeta T} \\ D_{TT} \end{pmatrix} \\ &\to \frac{1}{2\overline{k}^4\widehat{\xi}} \begin{pmatrix} \widehat{\xi}-1 & k_y\widehat{\epsilon} & -k_y^2\widehat{\epsilon}^2 \\ -k_y\widehat{\kappa} & \overline{k}^2 & -k_y\overline{k}^2\widehat{\epsilon} \\ -k_y^2\widehat{\kappa}^2 & k_y\widehat{\kappa}\overline{k}^2 & \overline{k}^4(\widehat{\xi}-1) \end{pmatrix} \frac{1}{\overline{\nu}} \begin{pmatrix} D_{\zeta\zeta} \\ 2D''_{\zeta T} \\ D_{TT} \end{pmatrix}, \end{aligned} \tag{28}$$

where the last result follows by setting all of the $\nu$'s equal. We shall show in Sec. IV B 3 that for discrete particle noise $\widehat{\mathsf{D}} = \overline{\nu}\mathsf{C}^{(\mathrm{eq})}$ [Eq. (58)], so Eq. (28) describes the amplification of the equilibrium fluctuations by the factor $|\widehat{\xi}|^{-1}$, where $\widehat{\xi} \to 0_-$ as the linear threshold is approached. As a check, when $\widehat{\kappa} = \widehat{\epsilon} = 0$ ($\widehat{\xi} = -1$), the matrix in Eq. (28) becomes diagonal and correctly recovers the equilibrium $C$'s (with the $\overline{k}^2$ converting between $\varphi$ and $\zeta$).

We should verify that our formula for $\mathsf{C}^{(0)}$ is realizable. We shall be interested in forcing such that $D''_{\zeta T} = 0$. Then, since $\xi < 0$ defines the regime of linear stability, one can see that the diagonal elements $C^{(0)}_{\varphi\varphi}$ and $C^{(0)}_{TT}$ are realizable (positive) up to the linear threshold where $\xi = 0$. At that point all quantities on the left-hand side of Eq. (28) diverge to infinity. One must also check that $\mathsf{C}$ is a positive-definite form for all realizable forcings. This is shown in Sec. A to be true up to the linear threshold.

## B. Zonostrophic instability of the ITG model

### *1. The general dispersion relation*

To determine the stability of the homogeneous equilibrium, we add to each equilibrium quantity $Q^{(0)}$ a perturbation of the form $\Delta Q e^{\widehat{\lambda}t} e^{iq\overline{x}}$, linearize Eqs. (24) and (the already linear) Eqs. (25), Fourier transform in the difference variable $x$, and ultimately derive a dispersion relation. Define

$$\overline{h}^2_\pm \doteq \left(k_x \pm \frac{1}{2}q\right)^2 + k_y^2 + 1 = \overline{k}^2 \pm k_x q + \frac{1}{4}q^2 > 0. \tag{29}$$

Then upon defining $C^{(\pm)} \doteq C^{(0)}(k_x \pm \frac{1}{2}q, k_y)$, the perturbed equations become

$$(\lambda + 2\nu_\zeta)\overline{h}_+^2\overline{h}_-^2\Delta C_{\varphi\varphi} = -ik_y[\overline{h}_-^2(\overline{h}_-^2 - q^2)C_{\varphi\varphi}^{(-)} - \overline{h}_+^2(\overline{h}_+^2 - q^2)C_{\varphi\varphi}^{(+)}]\Delta U - ik_y\epsilon(\overline{h}_+^2\Delta C_{\varphi T} - \overline{h}_-^2\Delta C_{T\varphi}), \tag{30a}$$

$$-(\lambda + 2\overline{\nu})\overline{h}_+^2\Delta C_{\varphi T} = ik_y[(\overline{h}_-^2 - q^2)C_{\varphi T}^{(-)} - \overline{h}_+^2 C_{\varphi T}^{(+)}]\Delta U - ik_y\kappa\overline{h}_+^2\Delta C_{\varphi\varphi} - k_y q\overline{h}_+^2 C_{\varphi\varphi}^{(+)}\Delta T - ik_y\epsilon\,\Delta C_{TT}, \tag{30b}$$

$$-(\lambda + 2\overline{\nu})\overline{h}_-^2\Delta C_{T\varphi} = -ik_y[(\overline{h}_+^2 - q^2)C_{T\varphi}^{(+)} - \overline{h}_-^2 C_{T\varphi}^{(-)}]\Delta U + ik_y\kappa\overline{h}_-^2\Delta C_{\varphi\varphi} + k_y q\overline{h}_-^2 C_{\varphi\varphi}^{(-)}\Delta T + ik_y\epsilon\,\Delta C_{TT}, \tag{30c}$$

$$(\lambda + 2\nu_T)\Delta C_{TT} = -ik_y(C_{TT}^{(-)} - C_{TT}^{(+)})\Delta U - ik_y\kappa(\Delta C_{\varphi T} - \Delta C_{T\varphi}) - k_y q(C_{\varphi T}^{(-)} - C_{T\varphi}^{(+)})\Delta T. \tag{30d}$$

If the perturbed variances are arranged as the column vector

$$\Delta\boldsymbol{C} \doteq (\Delta C_{\varphi\varphi},\, \Delta C_{\varphi T},\, \Delta C_{T\varphi},\, \Delta C_{TT})^{\mathrm{T}}, \tag{31}$$

then after dividing each of Eqs. (30) by $\overline{\nu}$ one can write the system (30) as

$$\widehat{\mathsf{M}} \cdot \Delta\boldsymbol{C} = \widehat{\boldsymbol{s}}_U\Delta U + \widehat{\boldsymbol{s}}_T\Delta T, \tag{32}$$

where the elements of $\widehat{\mathsf{M}}$, $\widehat{\boldsymbol{s}}_U$, and $\widehat{\boldsymbol{s}}_T$ are easily identified from Eqs. (30). (Again, the hats denote normalization with respect to $\overline{\nu}$.) The solution of Eq. (32),

$$\Delta\boldsymbol{C} = (\widehat{\mathsf{M}}^{-1} \cdot \widehat{\boldsymbol{s}}_U)\Delta U + (\widehat{\mathsf{M}}^{-1} \cdot \widehat{\boldsymbol{s}}_T)\Delta T, \tag{33}$$

then provides the components needed to evaluate the Reynolds stresses in the perturbed Eqs. (25), which can be written as

$$(\lambda + \nu_\zeta^{\mathrm{Z}})\Delta U = iq\int\frac{d\boldsymbol{k}}{(2\pi)^2}\,k_x k_y\Delta C_{\varphi\varphi}, \tag{34a}$$

$$(\lambda + \nu_T^{\mathrm{Z}})\Delta T = -\frac{1}{2}q\int\frac{d\boldsymbol{k}}{(2\pi)^2}\,k_y(\Delta C_{\varphi T} - \Delta C_{T\varphi}). \tag{34b}$$

From Eq. (32), the right-hand sides of Eqs. (34) can be written as the negative of a $2 \times 2$ matrix $\mathsf{m}$ operating on $(\Delta U,\, \Delta T)^{\mathrm{T}}$. The dispersion relation is then

$$\det(\mathsf{A}) = 0, \tag{35}$$

where

$$\mathsf{A} \doteq \lambda\mathsf{I} + \nu^{\mathrm{Z}} + \mathsf{m}(\lambda) \tag{36}$$

and $\nu^{\rm Z} = {\rm diag}(\nu_\zeta^{\rm Z},\, \nu_T^{\rm Z})$. The details of $\mathsf{m}$ are recorded in Appendix B.

We now introduce the concept of the neutral curve, which for fixed zonal damping describes the forcing strength for which the zonostrophic growth rate vanishes. In the simpler contexts of the barotropic vorticity equation[1] and the generalized Hasegawa–Mima equation,[18–20] the bifurcation is known to be of Type $\rm I_s$ in the language of pattern formation,[36] meaning that the onset of instability occurs at nonzero $q$ and ${\rm Im}\,\widehat{\lambda} = 0$. The same type of bifurcation can be shown to obtain in the present problem, and a cartoon of such a neutral curve is shown in Fig. 2. Note that in those earlier calculations with a scalar field, a single dimensionless parameter describes the effective forcing. Here we have multiple zonal dampings, but one can introduce a common scaling parameter $\eta$ that multiplies both $\nu_\zeta^{\rm Z}$ and $\nu_T^{\rm Z}$ and sets their nominal sizes. Furthermore, the forcing matrix has a common scaling with the level $\varepsilon$ of particle noise. We hold that level fixed (the relationship between the various elements of $\mathsf{D}$ will be discussed below). Then the effective forcing is controlled by the modal instability parameter $\xi$ [Eq. (12)]. Because the level of the homogeneous equilibrium diverges to $\infty$ as $\xi \to 0_-$, it is intuitively clear, and will be made more precise below, that the zonostrophic bifurcation will occur just to the left of the linear stability threshold. Given the crude nature of the model, quantitative precision is unimportant, although the calculation can certainly be done numerically and we shall display a representative neutral curve in Sec. IV C. More importantly, it is of interest to understand in principle how to calculate the bifurcation point and the value of the zonal wave number at onset.

To be more precise, observe from Fig. 2 that at the point of bifurcation the neutral curve N has a minimum at a point $(q_*, \xi_*)$. By definition, N obeys

$$\widehat{\lambda}(q,\xi) = 0, \tag{37}$$

which defines N as a function $\xi(q)$. From

$$\left(\frac{\partial\widehat{\lambda}}{\partial q}\right)_{\xi} dq + \left(\frac{\partial\widehat{\lambda}}{\partial \xi}\right)_{q} d\xi = 0, \tag{38}$$

one finds that on N one has

$$\frac{d\xi}{dq} = -\frac{(\partial\widehat{\lambda}/\partial q)_{|\xi}}{(\partial\widehat{\lambda}/\partial \xi)_{|q}}, \tag{39}$$

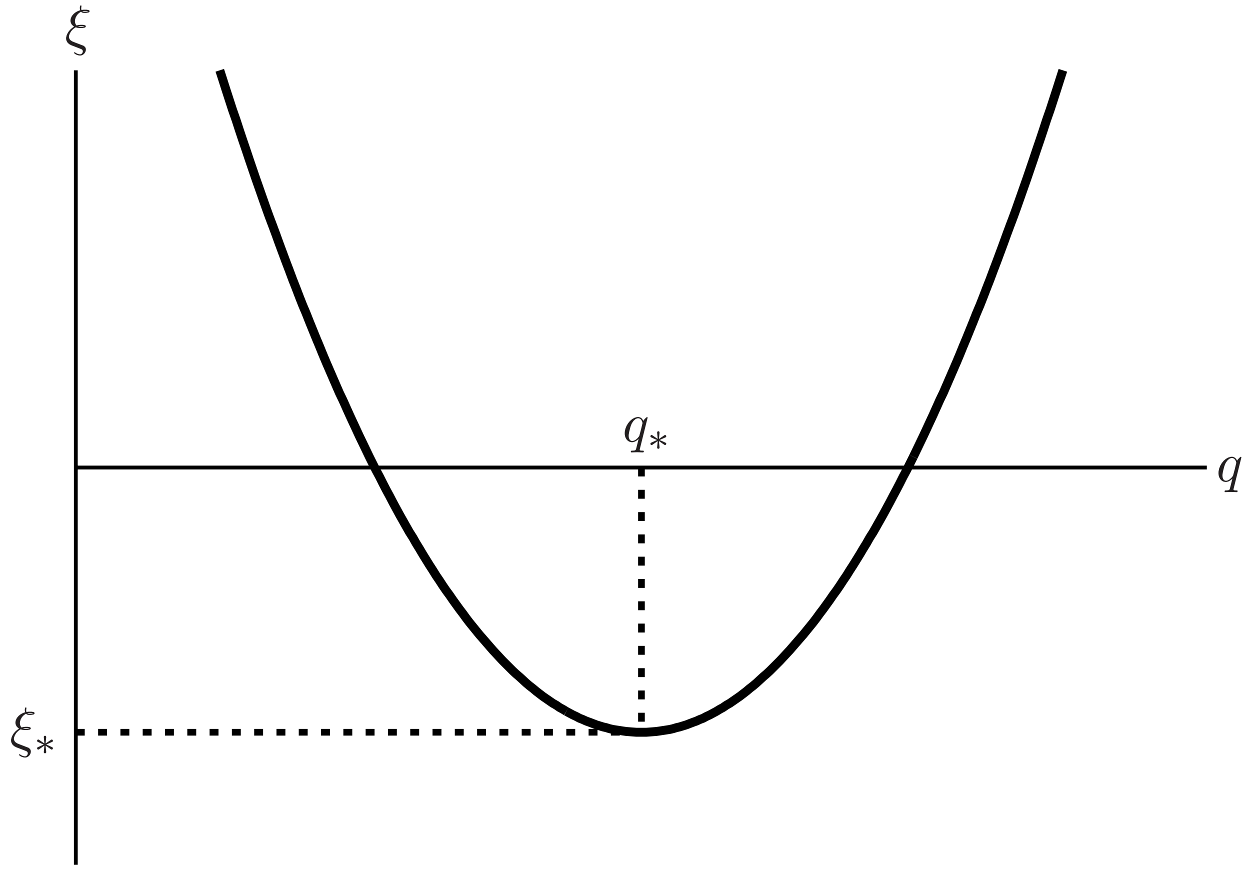


FIG. 2. A cartoon of a typical neutral curve, showing the bifurcation point ($q_*$, $\xi_*$). For the ITG model, $\xi = \Gamma^2$ [Eq. (12)], where $\xi = 0$ is the linear threshold. For gyrokinetic particle noise, $\widehat{\xi}_* < 0$ with $|\widehat{\xi}_*| = O(\mathcal{D}_\perp^{-1}) \ll 1$.

so at the bifurcation point where $N$ has a minimum one has

$$\left(\frac{\partial\widehat{\lambda}}{\partial q}\right)_\xi = 0. \tag{40}$$

This condition and Eq. (37) provide two simultaneous equations to be solved for $q_*$ and $\xi_*$. Equation (37) is just

$$\det[\nu^{\mathrm{Z}} + \mathsf{m}(0)] = 0, \tag{41}$$

where $\mathsf{m}(0) \equiv \mathsf{m}(\widehat{\lambda}{=}0, q, \xi)$. To simplify the condition (40), one can appeal to the Jacobi formula for the derivative of the determinant of a matrix $\mathsf{A}$:

$$d\det(\mathsf{A}) = \mathrm{tr}[\mathrm{adj}(\mathsf{A}) \cdot d\mathsf{A}], \tag{42}$$

where tr denotes the trace and adj denotes the adjugate, i.e., the transpose of the cofactor matrix. Upon differentiating Eq. (35) and enforcing Eq. (40), one obtains

$$0 = \mathrm{tr}\{\mathrm{adj}[\nu^{\mathrm{Z}} + \mathsf{m}(0)] \cdot \partial q\, \mathsf{m}(0)\}. \tag{43}$$

Solution of the simultaneous nonlinear equations Eq. (41) and (43) determine the bifurcation point $(q_*,\, \xi_*)$.

The elements of $\mathsf{m}$ are linear functions of the elements of $\mathsf{C}$, and each of those elements is a linear function of the elements of the noise matrix $\mathsf{D}$, which we take to scale with the common level parameter $\varepsilon$: $\mathsf{D} = \varepsilon\overline{\nu}\widehat{\mathsf{D}}$, where $\widehat{\mathsf{D}} = O(1)$. Also let $\nu^{\mathrm{Z}} = \eta\overline{\nu}\widehat{\nu}^{\mathrm{Z}}$, where $\eta$ sets the common size of $\nu^{\mathrm{Z}}_{\zeta}$ and $\nu^{\mathrm{Z}}_{T}$ such that $\widehat{\nu}^{\mathrm{Z}} = O(1)$. Since the elements of $\mathsf{m}$ are linearly proportional to those of $\mathsf{D}$, it can be seen that the solution will depend only on the ratio $\overline{\varepsilon} \doteq \varepsilon/\eta$. Upon factoring $\eta^n$ from each of Eqs. (41) and (43), where $n = 2$ for the ITG model, those equations become

$$0 = \det[\widehat{\nu}^{\mathrm{Z}} + \mathsf{m}(0, q_*, \widehat{\xi}_*; \overline{\varepsilon}, \widehat{\mathsf{D}})], \tag{44a}$$

$$0 = \mathrm{tr}\{\mathrm{adj}[\widehat{\nu}^{\mathrm{Z}} + \mathsf{m}(0, q_*, \widehat{\xi}_*; \overline{\varepsilon}, \widehat{\mathsf{D}})] \cdot \partial_{q_*} \mathsf{m}(0, q_*, \widehat{\xi}_*; \overline{\varepsilon}, \widehat{\mathsf{D}}). \tag{44b}$$

From Eq. (28), we see that near linear threshold the divergent elements of $\mathsf{C}^{(0)}$ scale as $\mathsf{D}/\overline{\nu}\widehat{\xi} = \eta(\overline{\varepsilon}/\widehat{\xi})\widehat{\mathsf{D}}$. Therefore, one has

$$\mathsf{m}(0, q_*, \widehat{\xi}_*; \overline{\varepsilon}, \widehat{\mathsf{D}}) \approx \mathsf{m}(0, q_*, \widehat{\xi}_*/\overline{\varepsilon}; \widehat{\mathsf{D}}). \tag{45}$$

In the absence of additional small parameters in $\mathsf{m}$, it is then clear that the solution for $(q_*,\, \widehat{\xi}_*/\overline{\varepsilon})$ is $O(1)$, from which it follows that $\widehat{\xi}_* = O(\overline{\varepsilon})$. In cases in which the forcing is due to turbulence and the zonal damping is weak, $\overline{\varepsilon}$ is large. Some analytical progress can be made by using isotropic ring forcing,[1] $\widehat{\mathsf{D}}(k_x, k_y) \propto \delta(k - k_{\mathrm{f}})$. (Parker[20] has used such forcing to show how the zonostrophic instability is a generalization of the modulational instability.) However, the situation is different when the forcing is due to discrete particle noise, as discussed next.

### *2. Forcing due to particle discreteness*

In a tokamak, the zonal damping rates are expected to be proportional to the ion–ion collision rate[37]: $\eta \sim \nu_{ii}$. Built into that rate is the noise level $\epsilon$ due to ion discreteness. Therefore $\overline{\varepsilon}$ *is independent of the basic noise level, specifically the plasma parameter* $\epsilon_p$.

However, it has been shown[16,17] that in magnetized, gyrokinetic plasma the thermal fluctuation level is reduced by the strong shielding effect of ion polarization, i.e., that level scales with the inverse of the perpendicular dielectric constant $\mathcal{D}_\perp$. Thus $\bar{\varepsilon} = O(\mathcal{D}_\perp^{-1}) \ll 1$. The particle-noise-driven zonostrophic bifurcation therefore occurs at

$$\widehat{\Gamma}_0^2 = \widehat{\nu}_\zeta \widehat{\nu}_T - a\bar{\varepsilon}, \tag{46}$$

where $a$ is a constant. Recall that all of $\widehat{\Gamma}_0^2$, $\widehat{\nu}_\zeta$, and $\widehat{\nu}_T$ are positive. To the extent that $\bar{\varepsilon} \ll \widehat{\nu}_\zeta \widehat{\nu}_T$, the bifurcation occurs essentially at the linear threshold. We understand this to define the onset of the Dimits-shift regime. For $\bar{\varepsilon} \gg \widehat{\nu}_\zeta \widehat{\nu}_T$, no zonostrophic instability occurs to the left of the linear threshold.

For a quantitative calculation, we recall the theory of gyrokinetic noise.[16,17] The general theory of statistical fluctuations in the presence of both particle discreteness and turbulence is complicated[38]; some discussion is given in Ref. 39. There are at least two qualitative issues. Most fundamentally, common statistical closures such as the DIA[40] are incorrect because they make an assumption about Gaussian initial statistics that is incorrect in the presence of particle discreteness[38] (this can easily be seen from the form of the thermal-equilibrium Gibbs distribution). Also, the details of discrete particle noise depend on the shape of the one-particle distribution function $f$. Fluid descriptions of the kind pursued in the present article assume that $f$ is a local Maxwellian. That is not unreasonable for the level of description to which we aspire here, particularly since there is no turbulence in the regimes of either linear stability or the Dimits shift, but it should be revisited in the regime where the particle noise is strongly amplified close to linear threshold. That is left for future work.

Thus we shall proceed as follows.

(i) For $\kappa = 0$, we determine the equilibrium gyrokinetic noise level using the noise calculations of Krommes[16] and Nevins *et al.*,[17] based on the Rostoker superposition principle. The latter calculations are more appropriate for the present case since they were done using the assumption of adiabatic electron response.

(ii) We infer the forcing functions by balancing the forcing against the Landau damping rates assumed in the basic ITG fluid mode, for a given noise level. Thus, if a scalar random variable $\widetilde{\psi}$ obeys

$$\frac{d\widetilde{\psi}}{dt} + \nu\widetilde{\psi} = \widetilde{f}(t), \tag{47}$$

where $\widetilde{f}(t)$ is Gaussian white noise with covariance $\langle \delta f(t)\delta f(t')\rangle = 2D\delta(t-t')$, the steady-state solution for the second-order statistics of the Langevin equation (47) is[41]

$$\langle \delta\psi^2 \rangle = D/\nu. \tag{48}$$

Given $\langle \delta\widetilde{\psi}^2 \rangle$, $D$ is determined as $\nu\langle \delta\widetilde{\psi}^2 \rangle$. The determination of the $n$-dimensional forcing matrix $\mathsf{D}$ is a simple generalization of this result, as we shall explicate below.

(iii) Finally, now knowing $\mathsf{D}$, we turn on $\kappa$ and proceed as in the earlier part of this article to calculate the noise-driven, $\kappa$-dependent fluctuation level and then the onset of the zonostrophic instability.

### 3. *Constraints on the damping coefficients; the forcing matrix*

To carry out the above program, we first calculate the equilibrium correlation matrix $\mathsf{C}^{(\mathrm{eq})} \doteq \langle \delta\boldsymbol{\psi}\,\delta\boldsymbol{\psi}^\dagger \rangle^{(\mathrm{eq})}$, where $\widetilde{\boldsymbol{\psi}} \doteq (\widetilde{\zeta},\, \widetilde{T})^{\mathrm{T}}$. (We omit the dependence on $\boldsymbol{k}$ of this and the other quantities in the following discussion.) Because we work in the electrostatic approximation, both components of $\delta\boldsymbol{\psi}$ are driven by the random potential $\delta\varphi$. The vorticity is linearly related to $\delta\varphi$: $\delta\zeta = \mathcal{Z}\delta\varphi$, where $\mathcal{Z} \doteq 1 + \overline{k}^2$. The fluctuating temperature could in principle contain nonlinear contributions, but since we restrict ourselves to weak coupling, the linear approximation is adequate: $\delta T = \mathcal{T}\delta\varphi$, where we shall calculate $\mathcal{T}$ from the equilibrium gyrokinetic equation below.

The fact that the vector $\delta\boldsymbol{\psi}$ is linearly proportional to the scalar $\delta\varphi$ has important consequences for the properties of the equilibrium correlation matrix. Let $\delta\boldsymbol{\psi} = \boldsymbol{\mathcal{S}}\,\delta\varphi$, where $\boldsymbol{\mathcal{S}}$ is an $n$-dimensional, non-zero scaling vector. [For our specific problem, we have $\boldsymbol{\mathcal{S}} = (\mathcal{Z},\, \mathcal{T})^{\mathrm{T}}$.] Thus $\mathsf{C}^{(\mathrm{eq})} = \boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{S}}^{\mathrm{T}} C^{(\mathrm{eq})}_{\varphi\varphi}$. This matrix has one positive eigenvalue, $\lambda_+ = \|\boldsymbol{\mathcal{S}}\|^2 \doteq \boldsymbol{\mathcal{S}}^\dagger \cdot \boldsymbol{\mathcal{S}}$, with associated eigenvector $\boldsymbol{e}_+ = \widehat{\boldsymbol{\mathcal{S}}} \doteq \boldsymbol{\mathcal{S}}/\|\boldsymbol{\mathcal{S}}\|$. The remaining $n-1$ eigenvalues vanish. This follows since $\widehat{\boldsymbol{\mathcal{S}}}$ can be taken to define the normal to an $(n-1)$-dimensional hyperplane; thus one can find $n-1$ vectors $\widehat{\boldsymbol{q}}_i$ $(i = 2, \ldots, n)$ such that $\widehat{\boldsymbol{\mathcal{S}}}^\dagger \cdot \widehat{\boldsymbol{q}}_i = 0$. It follows that $\mathsf{C}^{(\mathrm{eq})}$ is diagonalized by the unitary matrix $\mathsf{U} \doteq (\boldsymbol{e}_+,\, \widehat{\boldsymbol{q}}_2, \ldots, \widehat{\boldsymbol{q}}_n)$ according to

$$\overline{\mathsf{C}} \doteq \mathsf{U}^\dagger \cdot \mathsf{C} \cdot \mathsf{U} = \|\boldsymbol{\mathcal{S}}\|^2 \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix} C^{(\mathrm{eq})}_{\varphi\varphi}. \tag{49}$$

We now want to work backwards and determine the associated forcing matrix $\mathsf{D}$. The equilibrium Langevin equation

$$\frac{d\delta\widetilde{\boldsymbol{\psi}}}{dt} + \mathsf{V}\cdot\delta\widetilde{\boldsymbol{\psi}} = \delta\boldsymbol{f}, \tag{50}$$

where $\mathsf{V}$ contains the linear dissipation, transforms to

$$\frac{d\delta\overline{\boldsymbol{\psi}}}{dt} + \overline{\mathsf{V}}\cdot\delta\overline{\boldsymbol{\psi}} = \delta\overline{\boldsymbol{f}}, \tag{51}$$

where

$$\delta\overline{\boldsymbol{\psi}} \doteq \|\boldsymbol{\mathcal{S}}\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \delta\varphi, \quad \delta\overline{\boldsymbol{f}} \doteq \|\boldsymbol{\mathcal{S}}\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \delta f_\varphi \tag{52}$$

and $\overline{\mathsf{V}} \doteq \mathsf{U}^\dagger\cdot\mathsf{V}\cdot\mathsf{U}$. The fact that the time derivative and forcing appear only in the first component of Eq. (51) places restrictions on the form of the transformed dissipation matrix $\overline{\mathsf{V}}$, and thus on the original $\mathsf{V}$. Specifically, if $\mathsf{V}$ is taken to be diagonal, as we did in writing the model system Eq. (3), then one can show that the diagonal elements must be equal. To see this explicitly for $n = 2$, suppose that

$$\overline{\mathsf{V}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{53}$$

The unitary transformation matrix (whose columns are the eigenvectors) can be chosen to be

$$\mathsf{U} = \frac{1}{\Delta^{1/2}} \begin{pmatrix} \mathcal{Z} & -\mathcal{T}^* \\ \mathcal{T} & \mathcal{Z}^* \end{pmatrix}, \tag{54}$$

where $\Delta \doteq |\mathcal{Z}|^2 + |\mathcal{T}|^2 = \|\boldsymbol{\mathcal{S}}\|^2$. With $\delta\overline{\boldsymbol{\psi}} = \mathsf{U}^\dagger\cdot\delta\boldsymbol{\psi}$, this leads to the transformed Langevin equations

$$\frac{d\delta\varphi}{dt} + a\,\delta\varphi = \delta f_\varphi, \tag{55a}$$

$$0 + c\,\delta\varphi = 0. \tag{55b}$$

Thus $c = 0$, with $b$ and $d$ being arbitrary at this point. Upon transforming back, one finds

$$\mathsf{V} = \begin{pmatrix} |\mathcal{Z}|^2 a - \mathcal{Z}\mathcal{T} b + |\mathcal{T}|^2 d & \mathcal{Z}\mathcal{T}^*(a-d) + \mathcal{Z}^2 b \\ \mathcal{Z}^*\mathcal{T}(a-d) - \mathcal{T}^2 b & |\mathcal{T}|^2 a + \mathcal{Z}\mathcal{T} b + |\mathcal{Z}|^2 d \end{pmatrix}. \tag{56}$$

The requirement that the off-diagonal elements vanish is easily seen to imply $b = 0$ and $d = a$; thus $\mathsf{V} = \overline{\mathsf{V}} = a\mathsf{I}$.

Upon replacing $a$ by the common damping rate $\nu = \widehat{\nu}_\zeta = \widehat{\nu}_T = \overline{\nu}$ and applying the result (48) to Eq. (55a), one determines

$$D_{\zeta\zeta}^{(\mathrm{eq})} = \nu \langle |\delta\varphi|^2 \rangle^{(\mathrm{eq})}. \tag{57}$$

The back transformation $\delta \boldsymbol{f} = \mathsf{U} \cdot \delta \overline{\boldsymbol{f}}$ leads to

$$\mathsf{D}^{(\mathrm{eq})} = \nu \mathsf{C}^{(\mathrm{eq})}, \tag{58}$$

where

$$\mathsf{C}^{(\mathrm{eq})} = \begin{pmatrix} |\mathcal{Z}|^2 & \mathcal{Z}\mathcal{T}^* \\ \mathcal{Z}^*\mathcal{T} & |\mathcal{T}|^2 \end{pmatrix} C_{\varphi\varphi}^{(\mathrm{eq})}. \tag{59}$$

### *4. The scaling coefficients*

To determine the scaling coefficient $\mathcal{T}$, we solve the gyrokinetic equation linearized (denoted by $\Delta$) around thermal equilibrium,

$$\frac{\partial \Delta F}{\partial t} + v_\parallel \nabla_\parallel \Delta F + \frac{q}{m} \Delta E_\parallel \frac{\partial F_{\mathrm{M}}}{\partial v_\parallel} = 0 \tag{60}$$

to find

$$\Delta F_i(\boldsymbol{k}, \omega) = Z\tau^{-1} \left( \frac{k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel + i\epsilon} \right) \Delta\varphi(\boldsymbol{k}, \omega) F_{\mathrm{M}i}, \tag{61}$$

where $Z$ is the atomic number and $\tau \doteq T_i/T_e$. The fluctuating temperature is then

$$\Delta T_i = \int d\boldsymbol{v} \left( \frac{1}{2} m_i v^2 - \frac{3}{2} T_i \right) \Delta F_i. \tag{62}$$

Since we ignore FLR effects, the perpendicular part of $\frac{1}{2}v^2 - \frac{3}{2} = (\frac{1}{2}v_\parallel^2 - \frac{1}{2}) + (\frac{1}{2}v_\perp^2 - 1)$ integrates away. For the parallel part, it is conventional in standard drift-wave theory to look for modes with $\omega \gg k_\parallel v_{ti}$. Upon expanding the denominator for small $k_\parallel v_\parallel/\omega$, one finds

$$\mathcal{T}(\boldsymbol{k}, \omega) \doteq \Delta\varphi^{-1}(\boldsymbol{k}, \omega) \left( \frac{\Delta T_i(\boldsymbol{k}, \omega)}{T_i} \right) \approx Z\tau^{-1} \frac{k_\parallel^2 v_{ti}^2}{\omega^2} \ll 1. \tag{63}$$

Unfortunately, this expansion is not well justified for the ITG mode. However, one can already see another issue, which is that any such $\mathcal{T}$, calculated with or without expansion, will depend on $\omega$. This goes beyond the level of detail assumed in the above white-noise calculations, which assume that the scaling coefficients depended only on $\boldsymbol{k}$. In problems for which the linear eigenfrequency is dominantly real, this difficulty can be justifiably surmounted by replacing $\omega$ by the real mode frequency $\Omega_{\boldsymbol{k}}$. When $\omega$ is purely imaginary, this procedure is less justified, and to properly deal with the fact that $|\omega| \sim k_\parallel v_{ti}$, one should do a kinetic analysis. Alternatively, qualitatively correct results should obtain by ignoring $\mathcal{T}$ altogether, thus forcing only the vorticity.

#### 5. *The gyrokinetic noise level*

As a trivial modification of the work of Nevins *et al.*,[17] one finds for the case of ITG modes with adiabatic electrons[39] (consistent with the constructs of our model given in Sec. II) the fluctuating noise level

$$\frac{\langle\delta\phi\,\delta\phi\rangle(\boldsymbol{k})}{8\pi} = \frac{T_{\mathrm{i}0}/2}{\epsilon_{\mathrm{GV}}(\boldsymbol{k})}\left(\frac{k_{Di}^2\Gamma(k_\perp^2\rho_i^2)/k_D^2}{k^2(1+\epsilon_{\mathrm{GV}}^{-1}k_{De}^2/k^2)(1+k^2\lambda_D^2)}\right). \tag{64}$$

Here $k_{Ds}$ is the Debye length for species $s$, $k_D^2 \doteq k_{De}^2 + k_{Di}^2$, $\Gamma(b) \doteq I_0(b)e^{-b}$, and the dielectric permittivity of the gyrokinetic vacuum $\epsilon_{\mathrm{GV}}$ is given by

$$\epsilon_{\mathrm{GV}} \approx 1 + \left(\frac{k_{Di}^2}{k^2}\right)(1-\Gamma). \tag{65}$$

In our normalized variables we have

$$\begin{aligned}\mathsf{C}^{(\mathrm{eq})}_{\varphi\varphi}(k_x,k_y) = &\int \frac{dk_z}{2\pi}\frac{1}{n_i}\left(\frac{\tau\mathcal{D}_\perp}{\epsilon_{\mathrm{GV}}}\right)\\ &\times\left(\frac{\Gamma(\tau k_\perp^2)}{\rho_*^2 k^2(1+\epsilon_{\mathrm{GV}}^{-1}\mathcal{D}_\perp/k^2)(1+\tau+\tau k^2/\mathcal{D}_\perp)}\right)\end{aligned} \tag{66}$$

with

$$\epsilon_{\mathrm{GV}} \approx 1 + \left(\frac{\mathcal{D}_\perp}{\tau k^2}\right)[1-\Gamma(\tau k^2)]. \tag{67}$$

Here, $\rho_* \doteq \rho_{\mathrm{s}}/a$, $\tau \doteq T_{\mathrm{i}0}/T_{\mathrm{e}0}$, $\mathcal{D}_\perp \doteq \omega_{pi}^2/\omega_{ci}^2$, and $n_i$ denotes the ion density scaled by $\rho_{\mathrm{s}}^3$. The zonostrophic instability problem is thus specified in terms of the parameters $n_i$, $\rho_*$, $\tau$ and $\mathcal{D}_\perp$. In the gyrokinetic regime[16] $\mathcal{D}_\perp \gg 1$, so in the cold-ion limit one has $\lim_{\tau\to 0}\epsilon_{\mathrm{GV}} \approx \mathcal{D}_\perp$.

### C. Solution of the neutral curve equation

A representative, numerically calculated neutral curve is displayed in Fig. 3. It displays the expected qualitative features, including a critical zonal wave number $q_*$, a supercritical bifurcation, and a parabolic shape near onset. Notice that the zonostrophic instability sets in for $q_* \rho_s = O(1)$, not $q_*/k \ll 1$.

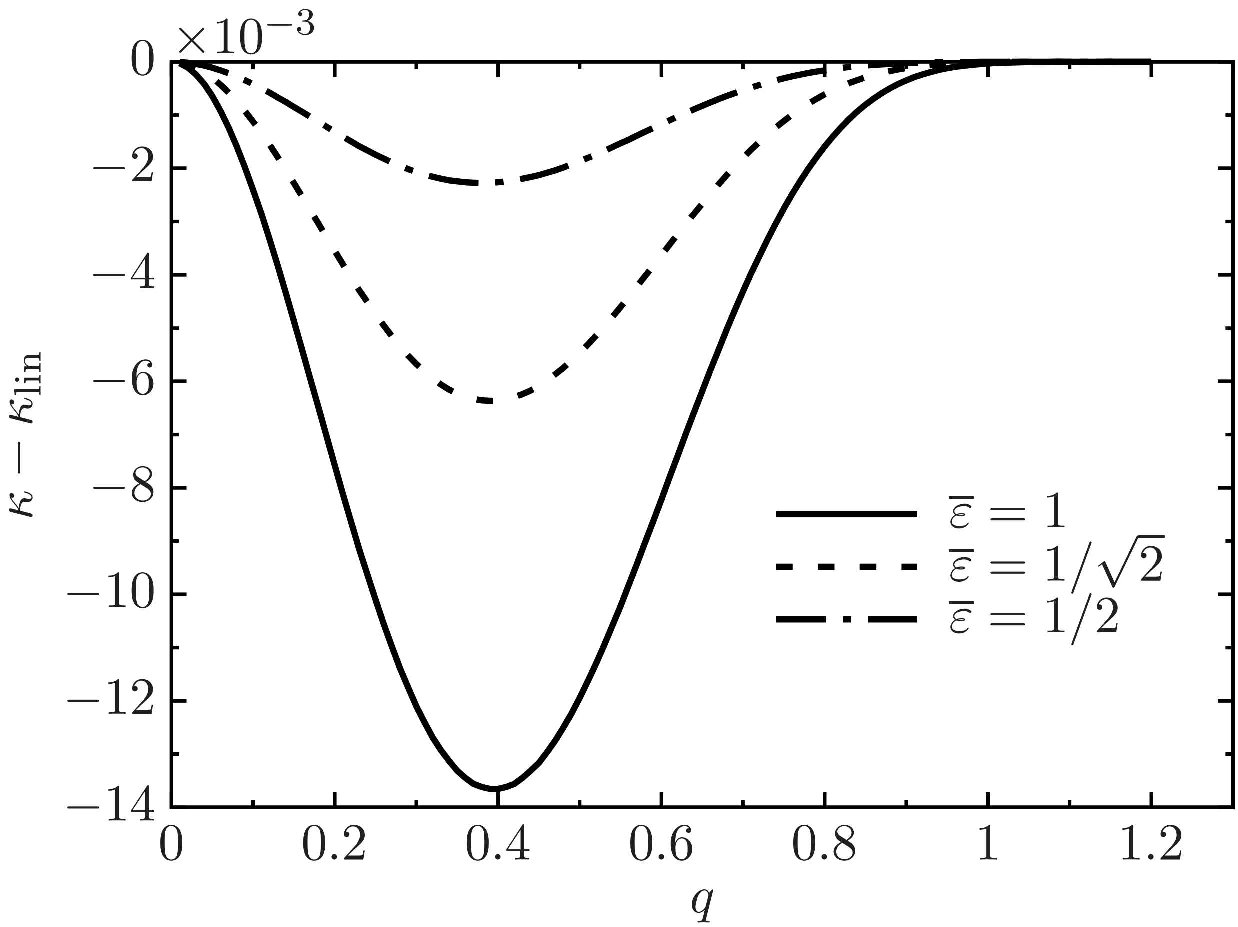


FIG. 3. A a series of representative neutral curves for various values of $\bar{\varepsilon}$, calculated numerically for the gyrokinetic noise spectrum given by Nevins *et al.*[17] for $\tau = 1$, $\rho_* = 0.3$, $n_i = 1$, $\mathcal{D}_\perp = 100$, $\mathcal{T} = 0$, and $\nu = |k_y|$ (representing damping in a Landau-fluid closure.) The curves are quantitatively similar for $\mathcal{T} = -0.25$.

Our principal purpose in displaying a numerically calculated neutral curve is to demonstrate that our analytically derived dispersion relation is robust and that no unexpected pathologies arise. Clearly the entire model is extremely crude, so our results cannot be quantitatively compared with experiments or fully kinetic simulations.

Missing from Fig. 3 is an inner, secondary stability curve for the steady zonal solutions that emerge above the point of zonostrophic instability. The existence of that boundary is known from previous work, including most recently that of Parker and Krommes.[18,19]

Very close to the zonostrophic threshold, the shape of the secondary stability curve can be obtained from analysis of a Ginzburg–Landau equation.[18,19] More generally, it must be calculated numerically, as Parker and Krommes did for the case of the modified Hasegawa–Mima equation.[18,19] Such calculations for the present model are beyond the scope of this paper.

## V. DISCUSSION

In summary, we have used the CE2 stochastic model to derive the dispersion relation of the noise-driven zonostrophic instability for a simple two-field model of the ion-temperature-gradient-driven mode, and we have numerically calculated the neutral curve and found the first unstable zonal mode for a representative noise spectrum.

The principal goal of this work is to present a new interpretation of the zonostrophic instability as being driven by discrete particle noise instead of the more conventional interpretation as being due to coupling to extrinsic turbulence. While it is obvious that in realistic tokamak microturbulence there is a plethora of modes in addition to the ITG mode, coupling to those modes should not be necessary for a self-consistent description of the behavior of the ITG mode itself as $\kappa$ (the normalized magnitude of the temperature gradient) is increased. We have shown that such a self-consistent description is possible when discrete particle noise is included. By introducing that noise, one is able to "open up" the zonostrophic bifurcation that introduces the onset of the Dimits-shift regime, which we have shown occurs just slightly below the linear threshold.

Left undone is the extension of these results through the right-hand boundary of the Dimits shift. This requires addressing the secondary stability boundary of the steady zonal flows that emerge from the zonostrophic bifurcation. If our interpretation is to be consistent with the known behavior observed in the simulations, that stability curve must close off for sufficiently large $\kappa$. Such behavior — the so-called Busse balloon — is known to occur for the closely analogous problem of Rayleigh–Benard thermal convection.[42,43] Large-$\kappa$ analysis of the stability curve requires numerical work that we have not yet attempted. However, it is clear that pursuing this investigation would augment current understanding of the Dimits shift and would provide a bridge between the sometimes arcane specialty of plasma physics and a large and broad literature on bifurcation phenomena in physical systems.

## ACKNOWLEDGMENTS

We would like to thank J. Parker and G. Hammett for many fruitful discussions. This work was supported by a NSERC PGS-D scholarship and by U.S. DoE contract DE-AC02-09CH11466.

## Appendix A: Realizability of the homogeneous solution

The CE2 closure deals only with first- and second-order statistics. Assuming Gaussian forcing, the multivariate PDF of $\widetilde{\varphi}$ and $\widetilde{T}$ is a 2D Gaussian. Realizability of the steady-state solution for a nonsingular PDF requires that

$$\mathsf{C}^{(0)} \doteq \begin{pmatrix} C^{(0)}_{\varphi\varphi} & C^{(0)}_{\varphi T} \\ C^{(0)*}_{\varphi T} & C^{(0)}_{TT} \end{pmatrix} \tag{A1}$$

is positive definite[44] for all realizable forcings. For a matrix to be positive definite, Sylvester's criterion states that its leading principal minor determinants must be positive. Thus in the absence of any constraints on the forcing, one must satisfy

$$C_{\varphi\varphi} > 0, \quad \Delta \doteq C^{(0)}_{\varphi\varphi} C^{(0)}_{TT} - |C^{(0)}_{\varphi T}|^2 > 0. \tag{A2}$$

If no mistakes have been made, realizability is guaranteed up to the linear threshold because the covariance matrix has been derived from a set of primitive amplitude equations driven by realizable random forcings; above threshold, a homogeneous steady state does not exist. As a partial check, we consider the case with $D''_{\zeta T} = 0$. Then from Eq. (28) one finds that

$$C^{(0)}_{\varphi\varphi} = (2\overline{\nu}\overline{k}^4 \widehat{\xi})^{-1} [(\widehat{\xi}^2 - \widehat{\nu}_T^2) D_{\zeta\zeta} - k_y^2 \widehat{\epsilon}^2 D_{TT}]. \tag{A3}$$

For $\xi < 0$ (linear stability), this is easily seen to be positive.

In terms of the real vector $\boldsymbol{D} \doteq (D_{\zeta\zeta},\, D_{TT},\, D'_{\zeta T})^{\mathrm{T}}$, the evaluation of $\Delta$ for $D''_{\zeta T} = 0$ leads to the quadratic form $F_3 = \boldsymbol{D}^{\mathrm{T}} \cdot \mathsf{S}_3 \cdot \boldsymbol{D}$, where for this case

$$\mathsf{S}_3 = \begin{pmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & f \end{pmatrix} ; \tag{A4}$$

thus $F_3 = aD_{\zeta\zeta}^2 + 2bD_{\zeta\zeta}D_{TT} + d\,D_{TT}^2 + f D_{\zeta T}'^2$ and the coefficients $a$, $b$, $d$, and $f$ can be obtained from Eqs. (27) and (28). After some algebra, one finds

$$a = C(\overline{\nu}|\xi|)^{-1}(-k_y^2\kappa^2\xi), \tag{A5a}$$

$$b = C(\overline{\nu}|\xi|)^{-1}\overline{k}^4\left[\xi - \tfrac{1}{2}(\widehat{\nu}_\zeta^2 + \widehat{\nu}_T^2)\right]\xi, \tag{A5b}$$

$$d = C(\overline{\nu}|\xi|)^{-1}\overline{k}^4(-k_y^2\epsilon^2\xi), \tag{A5c}$$

$$f = -(\overline{\nu}\overline{k}^2)^{-1}, \tag{A5d}$$

where $C \doteq (2\overline{k}^4)^{-1}$. In the region of linear stability ($\xi < 0$), we observe that $a$, $b$, and $d$ are positive, while $f < 0$. The criteria that $F_3$ be positive definite are

$$a > 0, \quad \Delta_2 \doteq ad - b^2 > 0, \quad \Delta_3 \doteq \Delta_2 f > 0. \tag{A6}$$

After more algebra, one finds

$$\Delta_2 = C^2[\widehat{\xi} - (\widehat{\nu}_\zeta - \widehat{\nu}_T)^2] < 0, \tag{A7}$$

and then $\Delta_3 > 0$. Thus the submatrix related to $D_{\zeta\zeta}$ and $D_{TT}$ violates unconstrained positive definiteness. However, realizabiity constrains $D_{\zeta\zeta}$ and $D_{TT}$ to be positive. Since all of $a$, $b$ and $d$ are positive, the subform $F_2$ is positive. We have thus verified that $\mathsf{C}$ is realizable for realizable forcing (for the special case $D_{\zeta T}'' = 0$).

To interpret the fact that $\Delta_2 < 0$, note that the eigenvalues of $\mathsf{S}_2$ are $\lambda = \Sigma \pm (\Sigma^2 - 4\Delta_2)^{1/2}$, where $\Sigma \doteq a + d > 0$. Thus one eigenvalue is negative. The associated eigenvector satisfies $D_{TT} = -b^{-1}(a - \lambda_-)D_{\zeta\zeta} < 0$. Such an unrealizable forcing would violate the condition that the cross-correlation coefficient between $\delta\varphi$ and $\delta T$ must be less than 1 in absolute value.

**Appendix B: Details of the dispersion relation**

For all $\nu$'s equal, the matrix $\widehat{\mathsf{M}}$ and source vectors introduced in Eq. (32) have the form

$$\widehat{\mathsf{M}} \doteq \begin{pmatrix} -(\widehat{\lambda}+2)\overline{h}_+^2\overline{h}_-^2 & -ik_y\widehat{\epsilon}\overline{h}_+^2 & ik_y\widehat{\epsilon}\overline{h}_-^2 & 0 \\ -ik_y\widehat{\kappa}\overline{h}_+^2 & (\widehat{\lambda}+2)\overline{h}_+^2 & 0 & -ik_y\widehat{\epsilon} \\ ik_y\widehat{\kappa}\overline{h}_-^2 & 0 & (\widehat{\lambda}+2\overline{\nu})\overline{h}_-^2 & ik_y\widehat{\epsilon} \\ 0 & -ik_y\widehat{\kappa} & ik_y\widehat{\kappa} & -(\widehat{\lambda}+2) \end{pmatrix}, \tag{B1}$$

$$\widehat{\boldsymbol{s}}_U \doteq \frac{ik_y}{\nu}\begin{pmatrix} \overline{h}_+^2(\overline{h}_+^2 - q^2)C_{\varphi\varphi}^{(+)} - \overline{h}_-^2(\overline{h}_-^2 - q^2)C_{\varphi\varphi}^{(-)} \\ -\overline{h}_+^2 C_{\varphi T}^{(+)} + (\overline{h}_-^2 - q^2)C_{\varphi T}^{(-)} \\ -[-\overline{h}_-^2 C_{T\varphi}^{(-)} + (\overline{h}_+^2 - q^2)C_{T\varphi}^{(+)}] \\ C_{TT}^{(+)} - C_{TT}^{(-)} \end{pmatrix}, \quad \widehat{\boldsymbol{s}}_T \doteq \frac{k_y q}{\nu}\begin{pmatrix} 0 \\ -\overline{h}_+^2 C_{\varphi\varphi}^{(+)} \\ \overline{h}_-^2 C_{\varphi\varphi}^{(-)} \\ -(C_{\varphi T}^{(-)} - C_{T\varphi}^{(+)}) \end{pmatrix}. \quad \text{(B2)}$$

For the formalism to make sense, $\widehat{\mathsf{M}}$ must be invertible, i.e., its determinant must not vanish. One finds

$$\Delta \doteq \det \widehat{\mathsf{M}} = \overline{h}_+^4\overline{h}_-^4(\widehat{\lambda}+2)^4 - 2\overline{h}_+^2\overline{h}_-^2(\overline{h}_+^2 + \overline{h}_-^2)(\widehat{\xi}+1)(\widehat{\lambda}+2)^2 + (\overline{h}_+^2 - \overline{h}_-^2)^2(\widehat{\xi}+1)^2. \quad \text{(B3)}$$

There is no requirement that $\Delta$ be positive definite; one needs only that it not vanish at the point of zonostrophic bifurcation $\widehat{\lambda} = 0$. This is not expected since $\widehat{\xi}_*$ depends on the zonal-flow damping rates, which do not appear in $\widehat{\mathsf{D}}$. Some general properties of $\Delta(\widehat{\xi}, q) \equiv \Delta(\widehat{\lambda} = 0, \widehat{\xi}, q)$ are easy to determine. It can be shown to depend only on $q^2$. For $\widehat{\xi} = -1$, one has

$$\Delta(-1, q) = 16\overline{h}_+^4\overline{h}_-^4 > 0. \quad \text{(B4)}$$

$\Delta$ has a minimum with respect to $\widehat{\xi}$ at

$$\widehat{\xi} = \frac{4\overline{h}_+^2 + \overline{h}_-^2(\overline{h}_+^2 + \overline{h}_-^2)}{\overline{k}^2(\overline{h}_+^2 - \overline{h}_-^2)^4} > 0. \quad \text{(B5)}$$

Its derivative at the linear threshold $\widehat{\xi} = 0$ is

$$\frac{\partial\Delta(\widehat{\xi}, q)}{\partial\widehat{\xi}}\Big|_{\widehat{\xi}=0} = -i\overline{k}^2\overline{h}_+^2\overline{h}_-^2(\overline{h}_+^2 + \overline{h}_-^2) < 0 \quad \text{(B6)}$$

and its second derivative is

$$\frac{\partial^2\Delta}{\partial\widehat{\xi}^2} = 2\overline{k}^4(\overline{h}_+^2 - \overline{h}_-^2)^2 > 0. \quad \text{(B7)}$$

$\Delta(0, q)$ vanishes for $q = 0$ ($\overline{h}_+^2 = \overline{h}_-^2 = \overline{k}^2$), and its derivative with respect to $q^2$ is

$$\frac{\partial\Delta(0, q)}{\partial q^2} = -12k_x^2\overline{k}^4 < 0. \quad \text{(B8)}$$

Thus except for $q = 0$ the determinant is negative at the linear threshold. Given that the second $\widehat{\xi}$ derivative is uniformly positive, one concludes that $\Delta$ must change sign somewhere in the stable region.

The dispersion relation can be simplified by using the transformation $C_{ij}^{(\pm)}(k_x,k_y) = C_{ji}^{(\mp)}(-k_x,-k_y)$. Once all equilibria are expressed at a single point $(k_x - q/2, k_y)$, the transformation $k_x \to k_x + q/2$ is then performed. After defining

$$\begin{aligned}
\overline{h} &\doteq \overline{k}^2 + 2k_x q + q^2, \\
\Omega &\doteq k_y^2 \epsilon\kappa(\overline{k}^2 + \overline{h}^2) - (\lambda + 2\nu)^2 \overline{k}^2 \overline{h}^2, \\
\Gamma_k &\doteq \Omega - 2k_y^2 \kappa\epsilon \overline{k}^2, \\
\Gamma_h &\doteq \Omega - 2k_y^2 \kappa\epsilon \overline{h}^2,
\end{aligned}$$

one finds that the determinant can be written as

$$\Delta' \doteq \Omega^2 - 4k_y^2 \epsilon^2 \kappa^2 \overline{k}^2 \overline{h}^2 \tag{B9}$$

and the dispersion relation becomes

$$(\lambda + \nu_\zeta^{\mathrm{Z}} - iqB)(\lambda + \nu_T^{\mathrm{Z}} - qD) - iq^2 AC = 0, \tag{B10}$$

where

$$A \doteq iq\epsilon \int \frac{dk_x\, dk_y}{(2\pi)^2}\,(k_x + q/2)k_y^3 \overline{k}^2 \Delta' [\Gamma_h C_{\varphi\varphi}^{(0)} + 2i\epsilon k_y \overline{h}^2 (\lambda + 2\nu) C_{\varphi T}^{(0)}], \tag{B11}$$

$$\begin{aligned}
B \doteq 2i \int \frac{dk_x\, dk_y}{(2\pi)^2}\,(k_x + q/2)k_y^2 \Delta' [&\Omega(\lambda + 2\nu)\overline{k}^2(\overline{k}^2 - q^2) C_{\varphi\varphi}^{(0)} - 2k_y^2\epsilon^2(\lambda + 2\nu)\overline{k}^2\overline{h}^2 C_{TT}^{(0)} \\
&- ik_y\epsilon\Gamma_k(\overline{k}^2 - q^2)C_{\varphi T}^{(0)} + ik_y\epsilon\overline{k}^2\Gamma_h C_{T\varphi}^{(0)}],
\end{aligned} \tag{B12}$$

$$\begin{aligned}
C \doteq 2 \int \frac{dk_x\, dk_y}{(2\pi)^2}\, k_y^2 \Delta' [&-2\kappa(\lambda + 2\nu)^2 k_y \overline{k}^4 \overline{h}^2 (\overline{k}^2 - q^2) C_{\varphi\varphi}^{(0)} + k_y\epsilon(\overline{h}^2\Gamma_k + \overline{k}^2\Gamma_h) C_{TT}^{(0)} \\
&+ i(\lambda + 2\nu)\overline{k}^2\overline{h}^2\Gamma_k C_{T\varphi}^{(0)} - i(\lambda + 2\nu)\overline{k}^2(\overline{k}^2 - q^2)\Gamma_h C_{\varphi T}^{(0)}],
\end{aligned} \tag{B13}$$

$$D \doteq q \int \frac{dk_x\, dk_y}{(2\pi)^2}\, k_y^2 \Delta' [(\lambda + 2\nu)\overline{k}^2\overline{h}^2\Gamma_k C_{\varphi\varphi}^{(0)} - ik_y\epsilon(\overline{h}^2\Gamma_k + \overline{k}^2\Gamma_h) C_{\varphi T}^{(0)}]. \tag{B14}$$

For $\mathcal{Z}\mathcal{T}^* = \pm\mathcal{Z}^*\mathcal{T}$ and $D_{ij}(k_x,k_y) = D_{ji}(k_x,-k_y)$, the dispersion relation is real valued. If either $\mathcal{Z}$ or $\mathcal{T}$ are set to zero, then $A = C = 0$ identically. In order for the dispersion relation to be satisfied, one must then solve

$$(\lambda + \nu_\zeta^{\mathrm{Z}} - iqB)(\lambda + \nu_T^{\mathrm{Z}} - qD) = 0. \tag{B15}$$

One can recover the dispersion relation found by Parker[19] in the flat-density limit ($\beta = 0$ in that reference) by forcing only the vorticity ($\mathcal{T} = 0$), turning off the linear coupling terms ($\kappa = \epsilon = 0$), and solving for the first branch $\lambda + \nu_\zeta^{\mathrm{Z}} - iqB = 0$.

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