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ORIGINAL PAPER

On the well-posedness of a nonlocal kinetic model for dilute polymers with anomalous diffusion

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Abstract

In this work, we study a class of nonlocal-in-time kinetic models of incompressible dilute polymeric fluids. The system couples a macroscopic balance of linear momentum equation with a mesoscopic subdiffusive Fokker–Planck equation governing the evolution of the probability density function (PDF) of polymers. The model incorporates nonlocal features to capture subdiffusive and memory-type phenomena. Our main result asserts the existence of global-in-time large-data weak solutions to this nonlocal system. The proof relies on an energy estimate involving a suitable relative entropy, which enables us to handle the critical general non-corotational drag term that couples the two equations. Crucial steps in our analysis are the proof of the nonnegativity of the PDF and establishing strong convergence of the sequence of Galerkin approximations. This involves a novel compactness result for nonlocal PDEs. Lastly, we prove the uniqueness of weak solutions with sufficient regularity.

Keywords Navier–Stokes–Fokker–Planck system · Existence of weak solutions · Nonlocal-in-time PDEs · Fractional derivatives · Entropy estimates

Mathematics Subject Classification 35Q30 · 35Q84 · 35R11 · 60G22 · 82C31 · 82D60

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1 Introduction

Kinetic models of dilute polymeric fluids are extensively described in the monographs [11, 47] and the review article [41]. Such models involve the coupling of the macroscopic Navier–Stokes equations for the description of incompressible fluid flow and the Fokker–Planck equation for mesoscopic processes associated with the statistical properties of polymer molecules immersed in the fluid.

Differential equations exhibiting nonlocal-in-time features have been the focus of considerable attention in the mathematical and engineering literature in recent years; see the textbooks on nonlocal effects with applications in viscoelasticity [45, 55], mathematical finance [20], and mechanical processes [3, 48]. Such equations typically involve fractional derivatives and are of relevance in applications where memory effects are present. The time-fractional Fokker–Planck system, in particular, models subdiffusive behaviour and has been previously studied in [2, 22, 32, 44, 53]. Further developments within the fractional calculus framework can be found in the survey papers [4, 29, 50].

This paper is concerned with the existence and uniqueness of weak solutions to a system of nonlocal-in-time PDEs, whose local counterpart arises in the kinetic theory of dilute solutions of polymeric fluids. In addition to the classical theory, we incorporate memory/history effects and subdiffusive dynamics into the system. In our previous work [26], we considered a Riemann–Liouville time-fractional derivative in the system. Although that model captured essential physical phenomena, our analysis relied on a simplified, corotational, drag term in the Fokker–Planck equation, enabling us to establish energy estimates in a Hilbert space setting and prove the existence of weak solutions. The extension of that analysis to a physically realistic drag term involves significant mathematical obstacles. The fundamental difficulty lies in the singular nature, at initial time, of solutions to the time-fractional Fokker–Planck equation involving the Riemann–Liouville time derivative, which conflicts with the physical requirement that the nonnegativity of the solution as a probability density function should be inherited from the nonnegativity of the initial datum. To address these limitations while preserving the memory effect, we introduce a nonlocal derivative involving a singular kernel of Prabhakar–Caputo type. This approach is, in general, *not* equivalent to the one we considered in our previous work [26].

The paper is organized as follows: Section 2 introduces the notation, the mathematical model, and key results from fractional calculus, e.g., Alikhanov’s inequality, and provides a new compactness result for nonlocal derivatives central to our analysis. In Section 3, we state the weak formulation of our model problem and the assumptions on the data, and prove our main results (Theorems 1, 2, and 3), which establish the existence of global-in-time weak velocity and PDF solutions, the existence of a pressure, and uniqueness under sufficient regularity. The proof of Theorem 1 relies on a spatial Galerkin approximation, compactness argument and the nonnegativity of the PDF. Nonnegativity is critical for the use of a suitable logarithmic relative entropy involving the Maxwellian. Section 4 concludes with a brief summary and some open questions.

2 Preliminaries

In this section, we state our model problem and introduce the function spaces required in our analysis. Most of the analytical tools that we use can be found in standard textbooks such as [12, 13, 21, 49]. Nonlocal derivatives and their properties play a key role in our analysis; standard bibliographical sources on the analysis of time-fractional differential equations include [18, 33, 38].

2.1 Definitions

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domain, $T > 0$ the length of the time interval, and $\Omega_T := \Omega \times (0, T)$ the space-time domain. The conformation space $D \subset \mathbb{R}^{d\times K}$ is a K-fold Cartesian product $D := D^1 \times \cdots \times D^K$, where each D^j, $j \in \{1, \dots, K\}$, is an open bounded d-dimensional ball centred at the origin, and the extended domains are defined as $\mathcal{O} := \Omega \times D$ and $\mathcal{O}_T := \mathcal{O} \times (0, T)$. Moreover, the Maxwellian distribution $M(q) = \prod_{j=1}^{K} M^j(q^j)$ on D is given component-wise on D^j, $j \in \{1, \dots, K\}$, by:

$$M^j(q^j) = \frac{e^{-U^j(\frac{1}{2}|q^j|^2)}}{\int_{D^j} e^{-U^j(\frac{1}{2}|q^j|^2)}\,\mathrm{d}q^j}, \tag{2.1}$$

for a spring potential $q^j \in D^j \mapsto U^j(\frac{1}{2}|q^j|^2)$ contained in $C^1(D^j; \mathbb{R}_{\geq 0})$, which is required to be such that $U^j(\frac{1}{2}|q^j|^2) \to +\infty$ as $\operatorname{dist}(q^j, \partial D^j) \to 0$, $j \in \{1, \dots, K\}$. Consequently, $M \in C_0^1(\overline{D})$.

We use standard notation for Lebesgue, Sobolev and Bochner spaces. For any pair of functions $u \in L^p(O)$ and $v \in L^{p'}(O)$ with $O \subset \mathbb{R}^m$ being a measurable set and $p' \in [1, \infty]$ being the Hölder conjugate of $p \in [1, \infty]$, we shall write $(u, v)_O := \int_O u(z)v(z)\,\mathrm{d}z$. Since we shall be working with Maxwellian-weighted spaces we define, for any nonnegative $w \in C(\overline{O})$ and any $p \in [1, \infty)$, the weighted Lebesgue and Sobolev spaces

$$\begin{aligned} L^p_w(O) &:= \big\{u \in L^p_{\mathrm{loc}}(O) : \|u\|^p_{L^p_w(O)} := (w, |u|^p)_O < \infty\big\}, \\ W^{1,p}_w(O) &:= \big\{u \in W^{1,p}_{\mathrm{loc}}(O) : \|u\|^p_{W^{1,p}_w(O)} := (w, |\nabla u|^p + |u|^p)_O < \infty\big\}. \end{aligned}$$

We refer to [27, 39] for the theory of weighted function spaces. For $p^* := pm/(m-p)$, $1 \leq p < m$, and $O \subset \mathbb{R}^m$ a Lipschitz domain, thanks to the Sobolev embedding theorem, we have the continuous embedding $W^{1,p}(O) \hookrightarrow L^{p^*}(O)$, and by the Rellich–Kondrashov theorem the compact embedding $W^{1,p}(O) \hookrightarrow\hookrightarrow L^r(O)$ for any $r \in [1, p^*)$. Moreover, we define the following spaces of divergence-free d-component vector functions for $p \in [1, \infty)$:

$$\begin{aligned} W^{1,p}_{0,\mathrm{div}}(\Omega) &:= \big\{v \in W^{1,p}_0(\Omega)^d : \operatorname{div} v = 0 \text{ in } \Omega\big\}, \\ L^p_{0,\mathrm{div}}(\Omega) &:= \big\{v \in L^p(\Omega)^d : \operatorname{div} v = 0 \text{ on } \Omega,\ v \cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega\big\}. \end{aligned}$$

Lastly, we introduce a kernel $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ of $\mathcal{PC}$ type; that is, we assume that k is nonnegative and nonincreasing, and there exists a resolvent kernel $\tilde{k} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that the Sonine property $k * \tilde{k} = 1$ is fulfilled, where $*$ denotes the usual half-line convolution. In this case, we say that $(k, \tilde{k})$ is a $\mathcal{PC}$ pair and write $(k, \tilde{k}) \in \mathcal{PC}$. We refer to [56] for the analysis of a nonlocal diffusion equation with kernels of $\mathcal{PC}$ type. The prototypical example is the singular Abel kernel $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $\alpha \in (0, 1)$, which is central to fractional calculus; we refer to [31] for a discussion of regular versus singular kernels in generalised fractional derivatives. From now on k will always be assumed to be a $\mathcal{PC}$ type kernel.

2.2 Modelling

Dilute polymers are viscoelastic fluids, which can be viewed as mixtures of a viscous base liquid, typically a Newtonian fluid, and elastic polymer macromolecules flowing in it. This is reflected in the form of the Cauchy stress tensor, which is the sum of two parts: one part being the classical Newtonian viscous stress tensor, corresponding to the response of the viscous base liquid, and the other, called the elastic extra-stress tensor and denoted by $\mathbb{S}$, corresponding to the elastic response. In a linear bead-spring chain model for dilute polymers, see [6, 15, 36], each polymer chain consists of $K + 1$ beads linearly coupled with K identical elastic springs. Then the elastic extra-stress tensor $\mathbb{S}$ is defined by the Kramers expression (see eq. (2.3) below) through the PDF $\psi = \psi(x, q, t)$, depending on $(x, t) \in Q_T$ and the conformation vector $q = ((q^1)^{\mathrm{T}}, \dots, (q^K)^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{d\times K}$ of the chain, with the column vector $q^j = (q^j_1, \dots, q^j_d)^{\mathrm{T}}$ representing the d-component conformation vector of the j-th spring in the bead-spring chain.

The evolution of the velocity u and the pressure p of the fluid is governed by the following system of partial differential equations that models the balance of linear momentum and incompressibility, respectively:

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) - \operatorname{div}(-p\mathbb{I} + 2\nu\mathbb{D}(u) + \mathbb{S}(\widehat{\psi})) = 0 \quad &\text{in } \Omega_T, \\ \operatorname{div} u = 0 \quad &\text{in } \Omega_T, \end{aligned} \tag{2.2}$$

where $\widehat{\psi} := \psi/M$ is the Maxwellian-scaled PDF, ν is the viscosity and $\mathbb{D}(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$ is the symmetric velocity gradient. The density of the incompressible fluid is assumed to be constant and is, without loss of generality, set to 1. We define the polymeric extra-stress tensor $\mathbb{S}$ by Kramers expression

$$\mathbb{S}(\widehat{\psi}(x, \cdot, t)) := -K \int_D \psi(x, q, t) \, \mathrm{d}q \, \mathbb{I} + \sum_{j=1}^{K} \int_D \psi(x, q, t) q^j (q^j)^{\mathrm{T}} (U^j)'(\tfrac{1}{2}|q^j|^2) \, \mathrm{d}q, \tag{2.3}$$

where $\mathbb{I} \in \mathbb{R}^{d\times d}$ is the identity matrix, and U^j is the spring potential that appears in the definition of the Maxwellian (2.1). We note that

$$M(q)\nabla_{q^j}(M(q))^{-1} = -(M(q))^{-1}\nabla_{q^j} M(q) = \nabla_{q^j} U^j(\tfrac{1}{2}|q^j|^2) = (U^j)'(\tfrac{1}{2}|q^j|^2) q^j.$$

Inserting the equality $(U^j)'(\frac{1}{2}|q^j|^2)q^j = -(M(q))^{-1}\nabla_{q^j} M(q)$ into the second summand in the definition of the extra-stress tensor (2.3), performing partial integration using that $M|_{\partial D} = 0$, we deduce that

$$\mathbb{S}(\widehat{\psi}) = \sum_{j=1}^{K} \int_D M(q)\nabla_{q^j} \widehat{\psi}(x,q,t) \otimes q^j \,\mathrm{d}q. \tag{2.4}$$

In this work, we assume that the evolution of ψ is governed by the nonlocal-in-time Fokker–Planck equation

$$\begin{aligned} &\partial_t \psi + \operatorname{div}(\partial_t(\tilde{k} * (u\psi))) + \textstyle\sum_{j=1}^{K} \operatorname{div}_{q^j}(\partial_t(\tilde{k} * ((\nabla u)q^j \psi))) \\ &\quad = \varepsilon \Delta \partial_t(\tilde{k} * \psi) + \textstyle\sum_{i=1}^{K} \sum_{j=1}^{K} \frac{A_{ij}}{4\lambda} \operatorname{div}_{q^i}(M \nabla_{q^j} \partial_t(\tilde{k} * \widehat{\psi})), \end{aligned} \tag{2.5}$$

which exhibits anomalous diffusion effects and has transport coefficients that depend on the velocity field u. Here, $\varepsilon > 0$ denotes the centre-of-mass diffusion coefficient, λ represents the elastic relaxation parameter, and $\mathbb{A} = (A_{ij})_{i,j=1}^{K} \in \mathbb{R}^{K\times K}_{\mathrm{sym}}$ is a constant positive definite matrix. Permitting $\mathbb{A}$ to depend on $(x,q,t) \in \Omega \times D \times [0,T]$ would not create additional complications in our analysis, provided that the symmetric matrix-function is uniformly positive definite and bounded on $\Omega \times D \times [0,T]$. We emphasize that (2.5) should be viewed as a constitutive assumption introducing memory into the fluxes; it is not, in general, equivalent to applying a Riemann–Liouville derivative to the full generator. We note that choosing $\tilde{k} = 1$ in (2.5) recovers the standard Navier–Stokes–Fokker–Planck system.

For simplicity of exposition, and without loss of generality, we set $\nu = 1$, $\varepsilon = 1$, $\lambda = 1/4$, $\mathbb{A} = \mathbb{I} \in \mathbb{R}^{K\times K}$, because none of our theoretical results depend on the specific values of these parameters. Additionally, we convolve (2.5) with the kernel function k; this then eliminates the nonlocal time derivatives from the spatial derivative terms in the equation thanks to the inverse convolution property $k * \partial_t(\tilde{k} * \psi) = \psi$; see [26, Lemma 3.2]. The first term becomes $k * \partial_t \psi = \partial_t(k * [\psi - \psi_0])$, which would represent the Caputo derivative of order $\alpha \in (0,1)$ if one chooses $k = g_{1-\alpha}$. Furthermore, to obtain an equation in terms of $\widehat{\psi}$, we recall that $\psi = M\widehat{\psi}$ and note that the Maxwellian M is independent of t and x. We shall therefore study in this work the system

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p - \Delta u - \operatorname{div} \mathbb{S}(\widehat{\psi}) &= 0 \quad \text{in } \Omega_T, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega_T, \\ \partial_t(Mk * [\widehat{\psi} - \widehat{\psi}_0]) + \operatorname{div}(M\widehat{\psi}u) + \operatorname{div}_q(M\widehat{\psi}(\nabla u)q)& \\ - \Delta(M\widehat{\psi}) - \operatorname{div}_q(M\nabla_q \widehat{\psi}) &= 0 \quad \text{in } \mathcal{O}_T, \end{aligned} \tag{2.6}$$

which is supplemented by the following initial conditions $u(0) = u_0$ in Ω and $(k * [\widehat{\psi} - \widehat{\psi}_0])(0) = 0$ in $\mathcal{O}$ and boundary conditions

$$\begin{aligned} u\cdot n_{\partial\Omega} &= 0 \quad \text{on } \partial\Omega\times(0,T),\\ M\nabla\widehat{\psi}\cdot n_{\partial\Omega} &= 0 \quad \text{on } \partial\Omega\times D\times(0,T),\\ (M\widehat{\psi}(\nabla u)q^j - M\nabla_q\widehat{\psi})\cdot n_{\partial D_j} &= 0 \quad \text{on } \Omega\times\partial D_j\times(0,T), \end{aligned} \tag{2.7}$$

where $n_{\partial\Omega}$ is the unit outward normal vector to $\partial\Omega$ and $n_{\partial D_j}$ is the unit outward normal vector to ∂D_j, $j = 1,\dots,K$. In $(2.6)_1$ we have used $(2.6)_2$ to replace $\operatorname{div}(2\mathbb{D}(u))$ appearing in (2.2) with Δu exploiting the identity $\operatorname{div}(\mathbb{D}(v)) = \frac{1}{2}(\nabla(\operatorname{div} v) + \Delta v)$.

2.3 Nonlocal derivative

The $L^p((0,T))$ norm of a function $u:(0,T)\to\mathbb{R}$ is bounded by its convolution with a $\mathcal{PC}$ kernel k, which can be seen from the following Hardy–Littlewood–Sobolev inequality (recall that k is, by definition, nonnegative and nonincreasing):

$$\|u\|^p_{L^p(0,t)} \le \frac{1}{k(t)}\int_0^t k(t-s)|u(s)|^p\,\mathrm{d}s \le \frac{1}{k(T)}\big(k*|u|^p\big)(t)\ \forall t\in(0,T]. \tag{2.8}$$

Next, we assume that H is a Hilbert space and consider the space of functions $u:(0,T)\to H$ with nonlocal derivatives with respect to $t\in(0,T)$ defined by

$$\mathcal{W}^p_0(0,T;H) := \big\{u\in L^p(0,T;H) : k*u \in W^{1,p}_0(0,T;H)\big\},\quad p\in[1,\infty).$$

We emphasize that $W^{1,p}_0(0,T;H)$ denotes the subspace of $W^{1,p}(0,T;H)$ with zero initial trace at $t=0$ (in the usual Sobolev trace sense), not a boundary condition in space. We note that the function $k*u\in W^{1,p}(0,T;H)$ has a well-defined trace at $t=0$, even when u itself does not have one, thanks to the continuous embedding $k*u\in W^{1,p}(0,T;H)\hookrightarrow AC([0,T];H)$. Furthermore, for given data $u_0\in H$, we define the "translated" space

$$\mathcal{W}^p_{u_0}(0,T;H) := \big\{u\in L^p(0,T;H) : u-u_0\in\mathcal{W}^p_0(0,T;H)\big\}.$$

It is well known that the classical chain rule does not hold for fractional derivatives. However, the so-called *fundamental identity* holds. It is stated in (2.9) and can be seen as the analogue of the chain rule $[G(u)]' = G'(u)u'$; cf. [35, Lemma 6.1]. Suppose that $T>0$ and let U be an open subset of $\mathbb{R}$. Let $k\in W^{1,1}((0,T))$, $G\in C^1(U)$, and $u\in L^1(0,T;U)$. Suppose further that the functions $G(u(\cdot))$, $G'(u(\cdot))u(\cdot)$ and $G'(u(\cdot))(k'*u)(\cdot)$ belong to $L^1((0,T))$ (which is the case if $u\in L^\infty((0,T))$); then, we have the following identity, referred to as the fundamental identity:

$$\begin{aligned} G'(u(t))\frac{\mathrm{d}}{\mathrm{d}t}(k*u)(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(k*G(u))(t) + \big(-G(u(t)) + G'(u(t))u(t)\big)\,k(t)\\ &+ \int_0^t \big(G(u(t-s)) - G(u(t)) - G'(u(t))[u(t-s)-u(t)]\big)\,[-k'(s)]\,\mathrm{d}s. \end{aligned} \tag{2.9}$$

If, in addition, $u_0 \in U$, and G is convex, then

$$G'(u(t))\frac{\mathrm{d}}{\mathrm{dt}}(k * [u - u_0])(t) \geq \frac{\mathrm{d}}{\mathrm{dt}}(k * [G(u) - G(u_0)])(t)$$

for almost all $t \in (0, T)$; cf. Lemma 6.2 in [35]. Suppose that we have $k \in W^{1,1}(0, T)$, $G \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ is convex, $u_0 \in L^1(\mathcal{O}; \mathbb{R}_{\geq 0})$, $G(u_0) \in L^1(\mathcal{O}; \mathbb{R}_{\geq 0})$ $u \in L^1(\mathcal{O}_T; \mathbb{R}_{\geq 0})$, and $G(u(\cdot))$, $G'(u(\cdot))u(\cdot)$, $G'(u(\cdot))(k' * u)(\cdot)$ and $G'(u(\cdot))u_0 k$ belong to $L^1(\mathcal{O}_T; \mathbb{R}_{\geq 0}))$; then,

$$\int_{\mathcal{O}} G'(u(t))\frac{\mathrm{d}}{\mathrm{dt}}(k * [u - u_0])(t) \,\mathrm{d}q \,\mathrm{d}x \geq \frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{O}} (k * [G(u) - G(u_0)])(t) \,\mathrm{d}q \,\mathrm{d}x$$

for almost all $t \in (0, T)$. A result of this type for Hilbert-space-valued functions $u : (0, T) \to H$ is stated in [24, Lemma 3.1] and [25, Prop. 1] for the case $k = g_\alpha$. We note that the proof there can be adapted to a general $\mathcal{PC}$-kernel pair $(k, \tilde{k})$ as it does not require any results other than the Sonine property. Specifically, in the case of $G(s) = \frac{1}{2}s^2$, this inequality is referred to as Alikhanov's inequality, and has the form (see [52, Theorem 2.1])

$$(u, \partial_t(k * [u - u_0]))_H \geq \frac{1}{2}\partial_t(k * [\|u\|_H^2 - \|u_0\|_H^2]) \quad \forall u \in \mathcal{W}^2_{u_0}(0, T; H). \quad (2.10)$$

2.4 A generalisation of the Aubin–Lions lemma with nonlocal derivatives

Since the equations under consideration are nonlinear, the existence of a weak solution obtained via a Galerkin approximation requires compactness arguments to ensure the strong convergence of the approximating sequences. To this end, let (X, Y, Z) be a triple of Banach spaces such that $X \hookrightarrow\hspace{-0.8em}\rightarrow Y \hookrightarrow Z$. It follows from the Aubin–Lions compactness lemma (see [12, Theorem II.5.16]) that

$$\begin{aligned} &L^p(0, T; X) \cap W^{1,1}(0, T; Z) \hookrightarrow\hspace{-0.8em}\rightarrow L^p(0, T; Y), \quad && p \in [1, \infty), \\ &L^\infty(0, T; X) \cap W^{1,r}(0, T; Z) \hookrightarrow\hspace{-0.8em}\rightarrow C([0, T]; Y), \quad && r > 1. \end{aligned} \quad (2.11)$$

In the following, we will require a compactness result with weaker assumptions on the regularity of the time derivative. Specifically, we require the following lemma.

Lemma 1 *Let $X \hookrightarrow\hspace{-0.8em}\rightarrow Y \hookrightarrow Z$ be real Banach spaces, $p \in [1, \infty)$ and $T > 0$. If $\mathcal{G}$ is a bounded subset of $L^p(0, T; X) \cap BV(0, T; Z)$, then $\mathcal{G}$ is relatively compact in $L^p(0, T; Y)$.*

This result is a special case of [34, Proposition 2] with $q = 1$ there. Although the statement of Kalita's Proposition 2 assumes that X is reflexive, a careful reading of the proof shows that this property is not actually used. We have, therefore, not assumed reflexivity in the above lemma.

When Z is a dual Banach space, meaning that Z admits a predual Z' (that is, $Z = [Z']^*$), the space $BV(0, T; Z)$ can be identified with the space of functions $u \in$

$L^1(0,T;Z)$ whose distributional time derivative u_t is a Z-valued finite Radon measure on $(0,T]$, that is, $u_t \in \mathcal{M}(0,T;Z)$. In particular, $\mathcal{M}(0,T;Z)$ can be identified with the dual space $[C([0,T];Z')]^*$; see, e.g., [1, Section 3.1]. In our analysis, we will use Lemma 1 with functions $u \in L^1(0,T;Z)$ whose time derivative u_t satisfies a bound of the form $|\langle u_t, \phi\rangle| \le C\|\phi\|_{C([0,T];Z')}$; such a bound naturally implies that $u_t \in \mathcal{M}(0,T;Z)$, and therefore $u \in BV(0,T;Z)$.

There exist several nonlocal counterparts of the Aubin–Lions lemma in the literature, formulated under various assumptions on the spaces and kernels; see, for instance, [40, 46, 54]. In particular, [54, Theorem 3.2] establishes the compact embedding $\mathcal{F}' \hookrightarrow\!\!\!\rightarrow L^p(0,T;Y)$ with

$$\mathcal{F}' = \big\{u \in L^p(0,T;X) : \partial_t(k * [u-u_0]) \in L^p(0,T;Z),\, (k*[u-u_0])(0) = 0\big\}.$$

for a Hilbert triple with $Z = X'$ and arbitrary $p \in [1,\infty)$, and the proof readily extends to Banach spaces. We emphasize that, in this setting, the nonlocal time derivative must be bounded in an L^p-space with the same exponent p as the bound for u. This requirement marks a key difference from the classical Aubin–Lions lemma, where an L^1-bound on the time derivative suffices. That is, compactness for the nonlocal case comes at the cost of stronger regularity in time, reflecting the weaker smoothing effect of the fractional derivative.

In what follows, we show, in analogy with Lemma 1, that a bound of the nonlocal derivative in the measure space $\mathcal{M}(0,T;Z)$ is already sufficient to guarantee compactness. We note that the new assumption $\tilde{k} \in L^p(0,T)$ may impose a restriction on the fractional exponent in the case of Abel-type kernels. However, in our applications, we will take $p = 1$, in which case the condition $\tilde{k} \in L^1(0,T)$ is easily satisfied for Abel kernels. Moreover, we note that for any $u \in \mathcal{F}$ one has $w = k*(u-u_0) \in BV(0,T;Z)$. Consequently, w admits a canonical BV-representative (still denoted by w), which has one-sided limits at every point, in particular the right trace $w(0+) := \lim_{t\downarrow 0} w(t) \in Z$ is well-defined.

Proposition 1 *Let $X \hookrightarrow\!\!\!\rightarrow Y \hookrightarrow Z$ be real Banach spaces where Z is a dual Banach space. Let $1 \le p < \infty$, $T > 0$, and let $(k,\tilde{k}) \in \mathcal{PC}$ with $\tilde{k} \in L^p(0,T)$. For $u_0 \in Z$, define*

$$\begin{aligned}\mathcal{F} =& \big\{u \in L^p(0,T;X) : \partial_t(k*[u-u_0]) \in \mathcal{M}(0,T;Z),\\ &(k*[u-u_0])(0+) = 0 \text{ in } Z\big\}.\end{aligned}$$

If $\mathcal{F}$ is bounded in $L^p(0,T;X)$ and $\sup_{u\in\mathcal{F}} \|\partial_t(k*[u-u_0])\|_{\mathcal{M}(0,T;Z)} < \infty$, then $\mathcal{F}$ is relatively compact in $L^p(0,T;Y)$.

To prove this result, we require a generalisation of the well-known inverse convolution property of nonlocal derivatives, which follows from the Sonine property of the $\mathcal{PC}$ pairing $(k,\tilde{k})$; see [26, Lemma 3.2]:

$$u = \tilde{k} * \partial_t(k*u) \qquad \forall\, u \in \mathcal{W}_0^p(0,T;H). \tag{2.12}$$

We generalise this property to functions with measure-valued nonlocal derivatives in the following lemma.

Lemma 2 *Let $u \in \mathcal{F}$ as defined in Prop. 1 and let $v := \partial_t(k * [u - u_0])$. Then, the following equality holds:*

$$u(t) - u_0 = \int_0^t \tilde{k}(t-s)\, \mathrm{d}v(s) \ \textit{for a.e. } t \in (0,T),$$

where the integral is a Bochner integral with respect to the vector measure v.

Proof The space $\mathcal{M}(0,T;Z)$ is endowed with the total variation norm $\|v\|_{\mathcal{M}(0,T;Z)} := |v|(0,T]$, where $|v|$ is the total variation measure of v defined for Borel sets $E \subseteq (0,T]$ by

$$|v|(E) = \sup\left\{ \sum_{i=1}^n \|v(E_i)\|_Z : E = \bigcup_{i=1}^n E_i,\ E_i \text{ are pairwise disjoint Borel sets} \right\}.$$

Define the primitive $\widetilde{K}(t) := \int_0^t \tilde{k}(s)\, \mathrm{d}s$. Since, by assumption, $\tilde{k} \in L^p(0,T) \subset L^1(0,T)$, we have $\widetilde{K} \in W^{1,1}(0,T)$. Moreover, by the Sonine property, we obtain

$$\partial_t(k * \widetilde{K}) = k * \tilde{k} = 1 \quad \text{and} \quad (k * \widetilde{K})(0) = 0,$$

hence $(k * \widetilde{K})(t) = (1 * 1)(t) = t$. We now convolve the identity $w := k * (u - u_0)$ with $\tilde{k}$ and use commutativity/associativity of convolution with L^1-kernels and finite Radon measures to get

$$\tilde{k} * w = \tilde{k} * k * (u - u_0) = (k * \tilde{k}) * (u - u_0) = 1 * (u - u_0).$$

On the other hand, using $w = 1 * v$ with $v = \partial_t w$ and $\tilde{k} * 1 = \widetilde{K}$, we have

$$\tilde{k} * w = \tilde{k} * (1 * v) = (\tilde{k} * 1) * v = \widetilde{K} * v.$$

Therefore,

$$1 * (u - u_0) = \widetilde{K} * v \qquad \text{in } L^1(0,T;Z). \tag{2.13}$$

Since $1*(u-u_0) \in W^{1,1}(0,T;Z)$ with $\partial_t(1*(u-u_0)) = u-u_0$, and $\widetilde{K} \in W^{1,1}(0,T)$ with $\widetilde{K}(0) = 0$, differentiating (2.13) in the sense of distributions yields, for a.e. $t \in (0,T)$,

$$u(t) - u_0 = (\partial_t \widetilde{K}) * v(t) = \tilde{k} * v(t) = \int_0^t \tilde{k}(t-s)\, \mathrm{d}v(s).$$

□

Proof of Prop. 1 For any $u \in \mathcal{F}$, define

$$w := k * (u - u_0) \in BV(0,T;Z), \qquad v := \partial_t w \in \mathcal{M}(0,T;Z),$$

Let $|v|$ be the total variation measure of v, so that by assumption

$$|v|(0,T] \leq M := \sup_{u\in\mathcal{F}} \|\partial_t(k*[u-u_0])\|_{\mathcal{M}(0,T;Z)} < \infty.$$

Since $w \in BV(0,T;Z)$ and $w(0+) = 0$, we have (in the sense of BV-representatives) $w(t) = v((0,t])$ for any $t \in [0,T]$, which can be written as $w = 1 * v$ where convolution with a measure is understood in the sense of [1, Definition 2.1]. We will show that $\mathcal{F}$ is relatively compact in $L^p(0,T;Y)$ by verifying the conditions of the Fréchet–Kolmogorov theorem [51, Theorem 1], which are the following:

(i) for every $0 < t_1 < t_2 < T$, the set $G(t_1,t_2) = \{\int_{t_1}^{t_2} u(t)\,\mathrm{d}t : u \in \mathcal{F}\}$ is relatively compact in Y, and

(ii) $\mathcal{F}$ is strongly integrally equicontinuous in $L^p(0,T;Y)$, i.e.,

$$\lim_{h\to 0}\sup_{u\in\mathcal{F}}\int_0^{T-h} \|u(t+h)-u(t)\|_Y^p\,\mathrm{d}t = 0.$$

Step (i). For any $u \in \mathcal{F}$, we have

$$\left\|\int_{t_1}^{t_2} u(t)\,\mathrm{d}t\right\|_X \leq \int_{t_1}^{t_2} \|u(t)\|_X\,\mathrm{d}t \leq (t_2-t_1)^{1-\frac{1}{p}}\|u\|_{L^p(0,T;X)}.$$

Thus $G(t_1,t_2)$ is bounded in X, and the compact embedding $X \hookrightarrow\!\!\!\!\rightarrow Y$ implies its relative compactness in Y.

Step (ii). For $h > 0$, the difference $u(t+h)-u(t)$ decomposes due to Lemma 2 as

$$\begin{aligned} u(t+h)-u(t) &= \int_0^t [\tilde{k}(t+h-s)-\tilde{k}(t-s)]\,\mathrm{d}v(s) + \int_t^{t+h} \tilde{k}(t+h-s)\,\mathrm{d}v(s) \\ &=: (\Delta_h\tilde{k} * v)(t) + B_h(t), \end{aligned} \tag{2.14}$$

where $\Delta_h\tilde{k}(t) := \tilde{k}(t+h)-\tilde{k}(t)$ (with $\tilde{k}(t) = 0$ for $t < 0$). By Minkowski's integral inequality and the definition of total variation, we can bound the first term in (2.14) as

$$\begin{aligned} \|\Delta_h\tilde{k}*v\|_{L^p(0,T-h;Z)} &\leq \int_0^T \|\Delta_h\tilde{k}(\cdot - s)\|_{L^p(0,T)}\,\mathrm{d}|v|(s) \\ &\leq \|\Delta_h\tilde{k}\|_{L^p(0,T)}\,|v|(0,T) \leq M\|\Delta_h\tilde{k}\|_{L^p(0,T)} \xrightarrow[h\to 0]{} 0. \end{aligned} \tag{2.15}$$

Next, we consider the second term in (2.14). By Jensen's inequality for integrals with respect to a finite measure,

$$|B_h(t)|^p \leq |v|(t,t+h]^{p-1}\int_t^{t+h} |\tilde{k}(t+h-s)|^p\,\mathrm{d}|v|(s).$$

Integrating over $t \in (0, T-h)$ and using Tonelli's theorem yields

$$\begin{aligned}\|B_h\|^p_{L^p(0,T-h;Z)} &\le M^{p-1}\int_0^{T-h}\int_t^{t+h} |\tilde{k}(t+h-s)|^p \,\mathrm{d}|v|(s)\,\mathrm{d}t \\ &= M^{p-1}\int_0^T \int_{(s-h)_+}^{\min(s,T-h)} |\tilde{k}(t+h-s)|^p \,\mathrm{d}t\,\mathrm{d}|v|(s).\end{aligned}$$

For fixed s, performing the change of variables $r = t+h-s$ gives

$$\int_{(s-h)_+}^{\min(s,T-h)} |\tilde{k}(t+h-s)|^p \,\mathrm{d}t \le \int_0^h |\tilde{k}(r)|^p \,\mathrm{d}r.$$

Hence

$$\|B_h\|^p_{L^p(0,T-h;Z)} \le M^{p-1}|v|(0,T)\int_0^h |\tilde{k}(r)|^p \,\mathrm{d}r \le M^p \|\tilde{k}\|^p_{L^p(0,h)} \xrightarrow[h\to 0]{} 0. \tag{2.16}$$

Combining the estimates (2.15) and (2.16), and returning to (2.14), we have

$$\sup_{u\in\mathcal{F}} \int_0^{T-h} \|u(t+h)-u(t)\|^p_Z \,\mathrm{d}t \le M^p \left(\|\varDelta_h \tilde{k}\|^p_{L^p(0,T)} + \|\tilde{k}\|^p_{L^p(0,h)}\right) \xrightarrow[h\to 0]{} 0.$$

Thus $\mathcal{F}$ is strongly integrally equicontinuous in $L^p(0,T;Z)$. Finally, applying the so called Ehrling lemma [49, Lemma 7.6] (whose proof, as has been pointed out in the recent paper [30] by Hanche-Olsen & Holden, is in fact due to J.-L. Lions [42, Prop. 4.1, p. 59] and [43, Lemma 5.1, p. 58]; for this reason, Hanche-Olsen & Holden suggest 'Aubin–Lions lemma' instead of the misnomer 'Ehrling's lemma') for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$\|x\|^p_Y \le \varepsilon \|x\|^p_X + C_\varepsilon \|x\|^p_Z \quad \forall x \in X.$$

Hence, for any $u \in \mathcal{F}$ and $h > 0$,

$$\begin{aligned}\int_0^{T-h} \|u(t+h)-u(t)\|^p_Y \,\mathrm{d}t \le{}& \varepsilon \int_0^{T-h} \|u(t+h)-u(t)\|^p_X \,\mathrm{d}t \\ &+ C_\varepsilon \int_0^{T-h} \|u(t+h)-u(t)\|^p_Z \,\mathrm{d}t.\end{aligned}$$

The first term is bounded by εC because $\mathcal{F}$ is bounded in $L^p(0,T;X)$, while the second term tends to 0 uniformly in $u \in \mathcal{F}$ as $h \to 0$. Therefore,

$$\lim_{h\to 0} \sup_{u\in\mathcal{F}} \int_0^{T-h} \|u(t+h)-u(t)\|^p_Y \,\mathrm{d}t = 0.$$

This proves the required strong integral equicontinuity in $L^p(0,T;Y)$. By the Fréchet–Kolmogorov theorem, $\mathcal{F}$ is relatively compact in $L^p(0,T;Y)$. □

3 Analysis of the model problem

We start our theoretical analysis with the definition of weak solution.

Definition 1 We call a tuple $(u,\widehat{\psi})$ a weak solution to the nonlocal Navier–Stokes–Fokker–Planck system (2.6), (2.7) provided that

$$\begin{aligned} u &\in C_w([0,T];L^2_{0,\mathrm{div}}(\varOmega))\cap L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\varOmega))\\ \partial_t u &\in L^{4/d}(0,T;[W^{1,2}_{0,\mathrm{div}}(\varOmega)]^*),\\ \mathbb{S}(\widehat{\psi}) &\in L^2(0,T;L^2(\varOmega)^{d\times d}),\\ \widehat{\psi} &\in L^\infty(\varOmega_T;L^1_M(D))\\ \text{with } \widehat{\psi}\geq 0 &\text{ a.e. in } \mathcal{O}_T,\\ \nabla\widehat{\psi} &\in L^2(\varOmega_T;L^1_M(D)^{d(K+1)}),\\ Mk*[\widehat{\psi}-\widehat{\psi}_0] &\in L^1(0,T;L^1(\mathcal{O}))\cap BV([0,T];[W^{1,\infty}(\mathcal{O})]^*),\\ \widehat{\psi}\ln\widehat{\psi} &\in L^1_M(\mathcal{O}_T), \end{aligned}$$

satisfying

$$\begin{aligned} &\langle\partial_t u,w\rangle-(u\otimes u,\nabla w)_{\varOmega_T}\\ &\quad+(\nabla u,\nabla w)_{\varOmega_T}+(\mathbb{S}(\widehat{\psi}),\nabla w)_{\varOmega_T}-(f,w)_{\varOmega_T}=0,\\ &\langle\partial_t(Mk*[\widehat{\psi}-\widehat{\psi}_0]),\zeta\rangle+(M\nabla\widehat{\psi},\nabla\zeta)_{\mathcal{O}_T}+(M\nabla_q\widehat{\psi},\nabla_q\zeta)_{\mathcal{O}_T}\\ &\quad-(Mu\widehat{\psi},\nabla\zeta)_{\mathcal{O}_T}-(M\widehat{\psi}(\nabla u)q,\nabla_q\zeta)_{\mathcal{O}_T}=0, \end{aligned} \tag{3.1}$$

with $\mathbb{S}(\widehat{\psi})=\sum_{j=1}^K\int_D M(q)\nabla_{q^j}\widehat{\psi}\otimes q_j\,\mathrm{d}q$, for any $w\in L^{4/(4-d)}(0,T;W^{1,2}_{0,\mathrm{div}}(\varOmega))$ and $\zeta\in C([0,T];W^{1,\infty}(\mathcal{O}))$, and the initial data is attained in the sense

$$\lim_{t\to 0}(u(t)-u_0,w)_\varOmega+(M(k*[\widehat{\psi}-\widehat{\psi}_0])(t),\zeta)_\mathcal{O}=0\quad\forall w\in L^2_{0,\mathrm{div}}(\varOmega),\ \zeta\in W^{1,\infty}(\mathcal{O}).$$

We make the following assumptions on the model parameters and the functions that we require in our proof of the existence of weak solutions.

Assumption 1 We assume the following:

- $K\in\mathbb{N}$ arbitrary, $D^j\subset\mathbb{R}^d$, $d\in\{2,3\}$ and $j=1,\dots,K$, bounded open balls centered at the origin;
- $\varOmega\subset\mathbb{R}^d$ bounded Lipschitz domain;
- $(k,\tilde{k})\in\mathcal{PC}$ with $k,\tilde{k}\in L^1(0,T)$;
- $f\in L^2(0,T;L^2(\varOmega)^d)$;

- $M \in C_0^1(\overline{D})$
  with $M > 0$ on D;
- $u_0 \in L^2_{0,\mathrm{div}}(\Omega)$;
- $\widehat{\psi}_0 \geq 0$ a.e. in $\mathcal{O}$, $\widehat{\psi}_0 \ln \widehat{\psi}_0 \in L^1_M(\mathcal{O})$, $\varrho_0 := \int_D M(q)\widehat{\psi}_0(\cdot, q)\,\mathrm{d}q \in L^\infty(\Omega)$.

The main result of this work regarding the existence of weak solutions to the nonlocal model (2.6) is the following.

Theorem 1 *Let Assumption 1 hold. Then, there exists at least one weak solution* $(u, \widehat{\psi})$ *to the system* (2.6), (2.7) *in the sense of Definition 1.*

Before we begin to prove the theorem, we first comment on the existence of a pressure. We consider divergence-free test functions in the variational form for the Navier–Stokes equations, and thus the pressure is eliminated. However, after having proved the existence of a solution $(u, \widehat{\psi})$ as stated in Theorem 1, we can associate a pressure with the velocity; see [12, Ch. V.1.5].

Theorem 2 *There exists a pressure* $p \in W^{-1,\infty}(0, T; L^2_0(\Omega))$ *such that the triple* $(u, p, \widehat{\psi})$ *solves the nonlocal model* (2.6) *in the distributional sense.*

Proof Theorem 1 guarantees the existence of a weak solution $(u, \widehat{\psi})$. We then define $q := \partial_t u + \operatorname{div}(u \otimes u) - \nu \Delta u - \operatorname{div} \mathbb{S}(\widehat{\psi}) - f$, and from the regularity of the solution, as stated in Definition 1, it directly follows that

$$q \in W^{-1,\infty}(0, T; W^{-1,2}(\Omega)^d) := \mathcal{L}(W_0^{1,1}(0, T); W^{-1,2}(\Omega)^d).$$

Further, for any $\eta \in W_0^{1,1}(0, T)$ and $w \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, we have that

$$\begin{aligned}
&\langle q(\eta), w\rangle_{W^{-1,2} \times W_0^{1,2}} \\
&= \left\langle \int_0^T -u\eta' + \operatorname{div}(u \otimes u)\eta - \nu \Delta u \eta - \operatorname{div} \mathbb{S}(\widehat{\psi})\eta - f\eta \,\mathrm{d}t, w \right\rangle_{W^{-1,2} \times W_0^{1,2}} \\
&= \int_0^T \Big(\frac{\mathrm{d}}{\mathrm{dt}}(u, w)_\Omega - (u \otimes u, \nabla w)_\Omega + (\nabla u, \nabla w) + (\mathbb{S}(\widehat{\psi}), \nabla w)_\Omega - (f, w)_\Omega \Big) \eta(t)\,\mathrm{d}t \\
&= 0.
\end{aligned}$$

By the de Rham lemma, see [12, Theorem IV.2.3], there exists a unique $p \in W^{-1,\infty}(0, T; L^2_0(\Omega))$ such that $\nabla p = -q$. □

Further, we prove a uniqueness result for weak solutions to the nonlocal micro-macro model in the sense of Definition 1 that further fulfil some regularity assumptions. The assumed regularity aligns with the Escauriaza–Seregin–Šverák condition [19] for the backward uniqueness for the Navier–Stokes equations, ensuring smoothness of strong solutions. Furthermore, we require that the radius of each D_j, $j \in \{1, \dots, K\}$, in the configuration space is sufficiently small. This is a justified assumption for the microscopic setting. However, we note that the smallness of the radii can be heavily relaxed if a small final time T is assumed. In fact, we will see in the proof that the radii will be subtracted from $\tilde{k}(T)$, and $\tilde{k}$ is nonincreasing with $\lim_{t \to 0} \tilde{k}(t) = \infty$.

Theorem 3 *Let Assumption 1 hold and further let the volume of D be sufficiently small. Each weak solution $(u, \widehat{\psi})$ with regularity $u \in L^\infty(0, T; W_0^{1,3}(\Omega)^d)$ and $\widehat{\psi} \in L^\infty(0, T; W^{1,\infty}(\mathcal{O}))$ is unique.*

As the proofs of Theorem 1 and Theorem 3 on the well-posedness of the model (2.6) are quite technical, we first provide their structure with five sub-steps.

Step 1 (ℓ, m, n): We start by introducing an ℓ-approximate problem based on truncating $\widehat{\psi}$ in the drag term of the nonlocal Fokker–Planck equation. Additionally, Kramers' expression is truncated to obtain an energy balance. To this truncated problem with solution $(u^\ell, \widehat{\psi}^\ell)$, we then apply a Galerkin–Faedo approximation, yielding a discrete solution $(u^{\ell,m,n}, \widehat{\psi}^{\ell,m,n})$ of the truncated system, where $n \in \mathbb{N}$ stands for the dimension of the approximate velocity subspace, and $m \in \mathbb{N}$ for the dimension of the probability density subspace.

Step 2 ($n \to \infty$): In this step, we derive n-independent a priori estimates allowing us to pass to the limit $n \to \infty$ in the (n, m)-th Galerkin system, i.e., we obtain a semi-discrete solution $(u^{\ell,m}, \widehat{\psi}^{\ell,m})$. The crucial nonnegativity of $\widehat{\psi}^{\ell,m}$ is also established in this step. It is derived by using a technique based on the weak maximum principle for nonlocal diffusion equations.

Step 3 ($m \to \infty$): We derive m-independent a priori bounds on the sequence $(\widehat{\psi}^{\ell,m})_m$, allowing us to pass to the limit $m \to \infty$ in the m-th Galerkin system. This argument involves proving the strong convergence of $\widehat{\psi}^{\ell,m}$ to $\widehat{\psi}^\ell$ in the Maxwellian-weighted space $L^1_M(\mathcal{O}_T)$. We show that $(G(\widehat{\psi}^{\ell,m}))_m$ is bounded in $L^\infty(0, T; L^1_M(\mathcal{O}))$, where $G(s) = s \ln s + e^{-1}$, which we use in combination with our novel nonlocal compactness result to first obtain weak convergence and then strong convergence to $\widehat{\psi}^\ell$ in $L^1_M(\mathcal{O}_T)$.

Step 4 ($\ell \to \infty$): Finally, we collect the necessary weak and strong convergence results from our energy estimates. We pass to the limit $\ell \to \infty$ in the truncation parameter ℓ within the various approximating sequences, which completes the proof of Theorem 1.

Step 5 (Uniqueness): In this last step, we prove Theorem 3, that is, the uniqueness of weak solutions with sufficient regularity. We do this by means of a suitable testing procedure.

Step 1: Truncation and Galerkin approximation

Following the ideas in [14], we truncate the extra-stress tensor $\mathbb{S}$, see (2.4), in the Navier–Stokes equation using a cutoff function $\Gamma \in C_c^\infty(-2, 2)$ such that $\Gamma(s) = 1$ for all $s \in [-1, 1]$. In addition, we define $\Gamma_\ell(s) = \Gamma(s/\ell)$ for an arbitrary $\ell \in \mathbb{N}$. The primitive and a scaled version of Γ_ℓ are denoted by, respectively,

$$T_\ell(s) = \textstyle\int_0^s \Gamma_\ell(\tau)\,\mathrm{d}\tau \quad \text{and} \quad \Lambda_\ell(s) = s\Gamma_\ell(s). \tag{3.2}$$

For a real-valued function Φ defined on D, we define the ℓ-th approximation of $\mathbb{S}$ as $\mathbb{S}_\ell(\Phi) := \mathbb{S}(T_\ell(\Phi))$. The definition of $\mathbb{S}$ given in (2.4) and integration by parts, using that $M \in C_0^1(D)$, yield

$$\mathbb{S}_\ell(\Phi) = \int_D \left[-KM(q)T_\ell(\Phi)\mathbb{I} + \sum_{j=1}^{K} T_\ell(\Phi)\nabla_{q^j} M(q) \otimes q^j \right] \mathrm{d}q, \tag{3.3}$$

see [6, Lemma 3.1] for details. Then, we define the ℓ-approximation $(u^\ell, p^\ell, \widehat{\psi}^\ell)$ of the nonlocal Navier–Stokes–Fokker–Planck system (2.6) as

$$\begin{aligned} \partial_t u^\ell + \operatorname{div}(\Gamma_\ell(|u^\ell|^2)u^\ell \otimes u^\ell) - \nabla u^\ell + \nabla p^\ell - \operatorname{div} \mathbb{S}_\ell(\widehat{\psi}^\ell) = f \quad &\text{in } \Omega_T, \\ \operatorname{div} u^\ell = 0 \quad &\text{in } \Omega_T, \end{aligned} \tag{3.4}$$

$$\begin{aligned} M\partial_t(k * [\widehat{\psi}^\ell - \widehat{\psi}_0^\ell]) - \Delta(M\widehat{\psi}^\ell) - \operatorname{div}_q(M\nabla_q \widehat{\psi}^\ell) \\ + \operatorname{div}(M\widehat{\psi}^\ell u^\ell) + \operatorname{div}_q(M\Lambda_\ell(\widehat{\psi}^\ell)(\nabla u^\ell)q) = 0 \quad \text{in } \mathcal{O}_T, \end{aligned} \tag{3.5}$$

with initial and boundary data as before in (2.7), with u and $\widehat{\psi}$ being replaced by u^ℓ and $\widehat{\psi}^\ell$. In order to avoid technical difficulties, we truncate the initial condition of $\widehat{\psi}^\ell$ as follows: $(M(k * [\widehat{\psi}^\ell - \widehat{\psi}_0^\ell])(0), \zeta)_{\mathcal{O}} = 0$ for any $\zeta \in L^\infty(\mathcal{O})$ with $\widehat{\psi}_0^\ell := T_\ell(\widehat{\psi}_0)$. We note that we also modified the nonlocal Fokker–Planck equation in the drag term by replacing $\widehat{\psi}^\ell$ by $\Lambda_\ell(\widehat{\psi}^\ell)$ to preserve the energy identity following the truncation process resulting in (3.4).

We begin by stating the result concerning the existence of a global-in-time large-data weak solution to the ℓ-truncated system (3.4), (3.5), which will be proved at the end of Step 3. In Step 4 we shall then pass to the limit $\ell \to \infty$ to remove the truncation.

Lemma 3 *Let Assumption 1 hold. Then, there exists a weak solution* $(u^\ell, \widehat{\psi}^\ell)$ *to the* ℓ*-approximation* (3.4), (3.5) *such that*

$$\begin{aligned} &u^\ell \in L^\infty(0,T; L^2_{0,\mathrm{div}}(\Omega)) \cap L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\Omega)) \cap W^{1,2}(0,T; [W^{1,2}_{0,\mathrm{div}}(\Omega)]^*), \\ &\mathbb{S}_\ell(\widehat{\psi}^\ell) \in L^\infty(0,T; L^\infty(\Omega)^{d\times d}), \\ &\widehat{\psi}^\ell \in L^\infty(\Omega_T; L^1_M(D)) \text{ with } \widehat{\psi}^\ell \geq 0 \text{ a.e. in } \mathcal{O}_T, \\ &\nabla_{x,q}\widehat{\psi}^\ell \in L^2(\Omega_T; L^1_M(D)^{d(K+1)}), \\ &Mk * \widehat{\psi}^\ell \in L^1(0,T; L^1(\mathcal{O})) \cap BV(0,T; [W^{1,\infty}(\mathcal{O})]^*), \text{ with} \end{aligned}$$

$$\begin{aligned} &\langle \partial_t u^\ell, w\rangle - (\Gamma_\ell(|u^\ell|^2)u^\ell \otimes u^\ell, \nabla w)_{\Omega_T} \\ &\quad + (\nabla u^\ell, \nabla w)_{\Omega_T} + (\mathbb{S}_\ell(\widehat{\psi}^\ell), \nabla w)_{\Omega_T} = (f, w)_{\Omega_T}, \\ &\langle M\partial_t(k * [\widehat{\psi}^\ell - \widehat{\psi}_0^\ell]), \zeta\rangle + (M\nabla\widehat{\psi}^\ell, \nabla\zeta)_{\mathcal{O}_T} + (M\nabla_q\widehat{\psi}^\ell, \nabla_q\zeta)_{\mathcal{O}_T} \\ &\quad = (Mu^\ell\widehat{\psi}^\ell, \nabla\zeta)_{\mathcal{O}_T} + (M\Lambda_\ell(\widehat{\psi})(\nabla u^\ell)q, \nabla_q\zeta)_{\mathcal{O}_T}, \end{aligned} \tag{3.6}$$

for any $w \in L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\Omega))$ *and* $\zeta \in C([0,T]; W^{1,\infty}(\mathcal{O}))$*, and the initial data is attained in the sense*

$$\lim_{t\to 0} (u^\ell(t) - u_0, w)_\Omega + (M(k * [\widehat{\psi}^\ell - \widehat{\psi}_0^\ell])(t), \zeta)_{\mathcal{O}} = 0 \quad \forall w \in L^2_{0,\mathrm{div}}(\Omega),\ \zeta \in L^\infty(\mathcal{O}).$$

Moreover, the tuple $(u^\ell, \widehat{\psi}^\ell)$ *satisfies the energy estimate*

$$\begin{aligned}&\|u^\ell\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^\ell\|^2_{L^2(\Omega_T)} + \|G(\widehat{\psi}^\ell)\|_{L^1_M(\mathcal{O}_T)} + \left\|\nabla_{x,q}\sqrt{\widehat{\psi}^\ell}\right\|^2_{L^2_M(\mathcal{O}_T)}\\ &\quad \leq C\Big(\|u^\ell_0\|^2_{L^2(\Omega)} + \|G(\widehat{\psi}^\ell_0)\|_{L^1_M(\mathcal{O})} + \|f\|^2_{L^2(\Omega_T)}\Big) \leq C(\ell).\end{aligned} \tag{3.7}$$

We point out that Lemma 3 is formulated on a divergence-free velocity space, and thus the pressure is absent. Although (3.4) also involves the pressure p^ℓ acting as a Lagrange multiplier, it does not play a role in the Fokker–Planck equation; this motivates us to consider the weak formulation (3.6).

For simplicity of notation, we temporarily omit the superscript ℓ in $\widehat{\psi}^\ell$ and u^ℓ and simply write $\widehat{\psi}$ and u. Analogously, in terms of our abbreviated notation $(u^{\ell,m,n}, \widehat{\psi}^{\ell,m,n})$ reads $(u^{m,n}, \widehat{\psi}^{m,n})$. The omitted superscript ℓ will be reinstated later, in Step 4, when we consider passing to the limit $\ell \to \infty$. We emphasize that by doing so we will arrive at upper bounds where the index ℓ appears explicitly on the right-hand sides of inequalities, but on the left-hand sides of inequalities it is suppressed in our notation. However, it is of utmost importance to carefully track all ℓ-dependencies in the constants because in Step 4 we have to pass to the limit $\ell \to \infty$, and thus uniformity in ℓ of our bounds has to be shown.

To prove Lemma 3, we introduce a Galerkin approximation of the ℓ-approximated system (3.6). To this end we define an approximated Maxwellian M^m, $m \in \mathbb{N}$, by

$$M^m := M + 1/m. \tag{3.8}$$

We choose the same Galerkin subspaces as in [14]. In fact, there exists a countable set $\{w_i\}_{i\in\mathbb{N}}$ in $W^{1,2}_{0,\mathrm{div}}(\Omega)\cap W^{d+1,2}(\Omega)^d$, whose linear span is dense in $L^2_{0,\mathrm{div}}(\Omega)$, such that w_i, $i \in \mathbb{N}$, are orthogonal in the inner product of $W^{d+1,2}(\Omega)^d$ and orthonormal in the inner product of $L^2(\Omega)^d$. Similarly, there is a countable set $\{\zeta^m_i\}_{i\in\mathbb{N}}$ of eigenfunctions in $W^{1,2}(\mathcal{O})$ that are orthogonal in $W^{1,2}_{M^m}(\mathcal{O})$ and orthonormal in $L^2_{M^m}(\mathcal{O})$. Finally, we fix $m, n \in \mathbb{N}$ and look for a tuple $(u^{m,n}, \widehat{\psi}^{m,n})$ given by

$$u^{m,n}(x,t) := \sum_{i=1}^{m} c^{m,n}_i(t) w_i(x), \qquad \widehat{\psi}^{m,n}(x,q,t) := \sum_{i=1}^{n} d^{m,n}_i(t)\zeta^m_i(x,q),$$

that solves the Galerkin system

$$\begin{aligned}&\langle \partial_t u^{m,n}, w_i\rangle - (\Gamma_\ell(|u^{m,n}|^2)u^{m,n}\otimes u^{m,n}, \nabla w_i)_\Omega + (\nabla u^{m,n}, \nabla w_i)_\Omega\\ &\quad + (\mathbb{S}_\ell(\widehat{\psi}^{m,n}), \nabla w_i)_\Omega = (f, w_i)_\Omega,\\ &\langle M^m\partial_t(k * [\widehat{\psi}^{m,n} - \widehat{\psi}^{m,n}_0]), \zeta^m_i\rangle + (M^m\nabla\widehat{\psi}^{m,n}, \nabla\zeta^m_i)_{\mathcal{O}} + (M^m\nabla_q\widehat{\psi}^{m,n}, \nabla_q\zeta^m_i)_{\mathcal{O}}\\ &\quad = (M^m u^{m,n}\widehat{\psi}^{m,n}, \nabla\zeta^m_i)_{\mathcal{O}} + (M\Lambda_\ell(\widehat{\psi}^{m,n})(\nabla u^{m,n})q, \nabla_q\zeta^m_i)_{\mathcal{O}},\end{aligned} \tag{3.9}$$

for a.e. $t \in (0, T)$, with initial conditions

$$u^{m,n}(0) = u_0^m := \sum_{i=1}^{m} (u_0, w_i)_\Omega w_i, \quad (k * [\widehat{\psi}^{m,n} - \widehat{\psi}_0^{m,n}])(0) = 0, \tag{3.10}$$

where we define $\widehat{\psi}_0^{m,n}$ by

$$\widehat{\psi}_0^{m,n} := \sum_{i=1}^{n} (T_\ell(\widehat{\psi}_0^m), \zeta_i^m)_{\mathcal{O}} \zeta_i^m, \tag{3.11}$$

with $\widehat{\psi}_0^m := \widehat{\psi}_0 M / M^m$. We stress that in (3.9) the original Maxwellian M is used in combination with $\Lambda_\ell(\cdot)$, while at all other locations the approximated one, M^m, is present.

For fixed (m, n), denote by $C(t) \in \mathbb{R}^m$ and $D(t) \in \mathbb{R}^n$ the vectors of coefficients $c_i^{m,n}(t)$ and $d_i^{m,n}(t)$. Testing (3.9) with the basis functions yields a finite-dimensional system with invertible mass matrices; hence it can be rewritten as

$$C'(t) = \mathcal{F}_1\big(t, C(t), D(t)\big), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(k * [D - D_0]\Big)(t) = \mathcal{F}_2\big(t, C(t), D(t)\big),$$

for a.e. $t \in (0, T)$ where $\mathcal{F}_1, \mathcal{F}_2$ are measurable in t and locally Lipschitz in (C, D) because the truncations $\Gamma_\ell, T_\ell, \Lambda_\ell$ guarantee boundedness and Lipschitz continuity of the nonlinearities on bounded sets. Convolving the second equation with $\tilde{k}$ and using the condition $(k * [D - D_0])(0) = 0$ yields an equivalent Volterra integral equation. Thus, (C, D) solves a coupled system consisting of an ODE for C and a Volterra equation of the second kind for D. Standard Volterra theory therefore yields a unique local solution; cf. [28, Ch. 9] (see also the analogous Galerkin transformation in [23, 57]). In particular, on any interval $[0, T']$ of existence we have $C, D \in C([0, T'])$, and thus $\mathcal{F}_2(\cdot, C, D) \in L^\infty(0, T'; \mathbb{R}^n)$. Consequently, $C \in W^{1,\infty}(0, T'; \mathbb{R}^m)$ and $k * [D - D_0] \in W^{1,\infty}(0, T'; \mathbb{R}^n)$. Since $u^{m,n}(t) \in \operatorname{span}\{w_i\}_{i=1}^m$ and $\hat{\psi}^{m,n}(t) \in \operatorname{span}\{\zeta_i^m\}_{i=1}^n$ for all t, it follows that

$$\begin{aligned} u^{m,n} &\in W^{1,\infty}\big(0, T'; \operatorname{span}\{w_i\}_{i=1}^m\big) \subset W^{1,\infty}\big(0, T'; W^{1,2}_{0,\mathrm{div}}(\Omega)\big), \\ \hat{\psi}^{m,n} &\in C\big([0, T']; \operatorname{span}\{\zeta_i^m\}_{i=1}^n\big) \subset C\big([0, T']; W^{1,2}_{M^m}(\mathcal{O})\big), \\ k * (\hat{\psi}^{m,n} - \hat{\psi}_0^{m,n}) &\in W^{1,\infty}\big(0, T'; W^{1,2}_{M^m}(\mathcal{O})\big). \end{aligned}$$

Step 2: Uniform a priori estimates in *n* and the limit process *n* → ∞

First, we state and prove a lemma that provides us with an n-independent energy estimate for the tuple $(u^{m,n}, \widehat{\psi}^{m,n})$. Thereafter, we pass to the limit $n \to \infty$ in the Galerkin system (3.9), (3.10).

Lemma 4 *Let Assumption 1 hold. For the Galerkin solution* $(u^{m,n}, \widehat{\psi}^{m,n})$ *of* (3.9)*, the following* n*-independent a priori estimate holds:*

$$\|u^{m,n}\|^2_{L^\infty(\Omega_T)} + \|\nabla u^{m,n}\|^2_{L^\infty(\Omega_T)} + \|\widehat{\psi}^{m,n}\|^2_{L^\infty(0,T;L^2(\mathcal{O}))} + \|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2(\mathcal{O}_T)} \\ \leq C(m,\ell,T). \tag{3.12}$$

Proof Thanks to our assumptions on M, M^m and T_ℓ, see Assumption 1, (3.2) and (3.8), we have that

$$\|\mathbb{S}_\ell(\widehat{\psi}^{m,n})\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\ell.$$

We multiply the i-th equation in $(3.9)_1$ by $c_i^{m,n}(t)$ and sum with respect to $i = 1,\dots,m$ to deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u^{m,n}\|^2_{L^2(\Omega)} + \|\nabla u^{m,n}\|^2_{L^2(\Omega)} = -(\mathbb{S}_\ell(\widehat{\psi}^{m,n}), \nabla u^{m,n})_\Omega + (f, u^{m,n})_\Omega,$$

where we have used that the convective term vanishes thanks to $\operatorname{div} u^{m,n} = 0$. Then, using Young's inequality, we have after integration over $(0,T)$ and applying the supremum over all $t \in (0,T)$ that

$$\sup_{t\in(0,T)} \|u^{m,n}\|^2_{L^2(\Omega)} + \|\nabla u^{m,n}\|^2_{L^2(\Omega_T)} \leq C(M,\ell,u_0,f). \tag{3.13}$$

Using the definition of $u^{m,n}$ via its expansion in terms of the divergence-free Galerkin basis functions for the velocity field, we further obtain that

$$\sup_{t,i}\left(|c_i^{m,n}(t)| + \left|\frac{\mathrm{d}}{\mathrm{dt}}c_i^{m,n}(t)\right|\right) \leq C(m,\ell,u_0,f,M).$$

Similarly, multiplying the i-th equation in $(3.9)_2$ by $d_i^{m,n}(t)$ and summing with respect to $i = 1,\dots,n$, we deduce that

$$(M^m\partial_t(k*[\widehat{\psi}^{m,n} - \widehat{\psi}_0^{m,n}]), \widehat{\psi}^{m,n})_{\mathcal{O}} + \|\nabla\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} + (M^m, |\nabla_q\widehat{\psi}^{m,n}|^2)_{\mathcal{O}} \\ = (M\Lambda_\ell(\widehat{\psi}^{m,n})(\nabla u^{m,n})q, \nabla_q\widehat{\psi}^{m,n})_{\mathcal{O}}, \tag{3.14}$$

where we have used again that $\operatorname{div} u^{m,n} = 0$ to eliminate the convective term. The first term in (3.14) can be estimated using Alikhanov's inequality, see (2.10), as follows:

$$\big(M^m\partial_t(k*[\widehat{\psi}^{m,n} - \widehat{\psi}_0^{m,n}]), \widehat{\psi}^{m,n}\big)_{\mathcal{O}} \geq \frac{1}{2}\partial_t\Big(k*\Big[\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} - \|\widehat{\psi}_0^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}\Big]\Big).$$

Using Hölder's inequality and Young's inequality, the right-hand side of (3.14) can be bounded by

$$
\begin{aligned}
(M\Lambda_\ell(\widehat{\psi}^{m,n})(\nabla u^{m,n})q, \nabla_q \widehat{\psi}^{m,n})_{\mathcal{O}} \leq \frac{1}{2}\|\nabla_q \widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} \\
+ C(\ell)\|\nabla u^{m,n}\|^2_{L^\infty(\Omega)}\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})},
\end{aligned}
$$

where we have used that M/M^m is nonnegative and bounded from above by 1 to obtain the M^m-weighted norms on the right-hand side. Using the smoothness of the basis functions and noting the Sobolev embedding $W^{d+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and the bound (3.13) on the functions $u^{m,n}$, it follows by norm-equivalence in finite-dimensional spaces that

$$
\|\nabla u^{m,n}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(m,\ell,T). \tag{3.15}
$$

Therefore, we deduce that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\Big(k * \Big[\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} - \|\widehat{\psi}_0^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}\Big]\Big) + \|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} \\
&\quad \leq C(m,\ell)\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}.
\end{aligned}
$$

At this point, we convolve this inequality with the resolvent kernel $\tilde{k}$ and apply the inverse convolution property $k * \partial_t(\tilde{k} * \psi) = \psi$, see [26, Lemma 3.2], to deduce for the leading term on the left-hand side of the inequality that

$$
\tilde{k} * \partial_t\Big(k * \Big[\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} - \|\widehat{\psi}_0^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}\Big]\Big) = \|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} - \|\widehat{\psi}_0^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}.
$$

Together with the other terms in the inequality, we obtain

$$
\begin{aligned}
\|\widehat{\psi}^{m,n}(t)\|^2_{L^2_{M^m}(\mathcal{O})} - \|\widehat{\psi}_0^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} + \Big(\tilde{k} * \|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}\Big)(t) \\
\leq C(m,\ell)\Big(\tilde{k} * \|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}\Big)(t).
\end{aligned} \tag{3.16}
$$

Using the fact that $\tilde{k}$ is nonincreasing, the second term on the left can be bounded by

$$
\begin{aligned}
\big(\tilde{k} * \|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}\big)(t) &\geq \tilde{k}(t)\|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2(0,t;L^2_{M^m}(\mathcal{O}))} \\
&\geq \tilde{k}(T)\|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2(0,t;L^2_{M^m}(\mathcal{O}))}.
\end{aligned} \tag{3.17}
$$

Therefore, we deduce that

$$
\begin{aligned}
&\|\widehat{\psi}^{m,n}(t)\|^2_{L^2_{M^m}(\mathcal{O})} + \tilde{k}(T)\|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2(0,t;L^2_{M^m}(\mathcal{O}))} \\
&\quad \leq \|\widehat{\psi}_0^{m}\|^2_{L^2_{M^m}(\mathcal{O})} + C(m,\ell)\Big(\tilde{k} * \|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}\Big)(t),
\end{aligned}
$$

where we have also estimated $\|\widehat{\psi}_0^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}$ by $\|\widehat{\psi}_0^{m}\|^2_{L^2_{M^m}(\mathcal{O})}$ using the initial condition (3.11). Then, we apply the Henry–Gronwall inequality [25, Corollary 1] to absorb the second term on the right-hand side into the left-hand side of the inequality. Thus, we obtain the n-uniform estimate

$$\|\widehat{\psi}^{m,n}(t)\|^2_{L^2_{M^m}(\mathcal{O})} + \|\nabla_{x,q}\widehat{\psi}^{m,n}\|^2_{L^2(0,T;L^2_{M^m}(\mathcal{O}))} \le C(m,\ell,T)\|\widehat{\psi}_0^{m}\|^2_{L^2_{M^m}(\mathcal{O})}.$$

At this point, we use that $M^m \ge 1/m$; see (3.8) for the definition of the approximated Maxwellian M^m. That completes the proof of the lemma. ◻

The n-uniform estimates yield weakly converging subsequences. As we want to pass to the limit $n \to \infty$ in the Galerkin system (3.9) which involves nonlinearities, we require a strong convergence result, and we shall apply the compact embedding stated in Proposition 1 to prove this. To this end, we begin by bounding the nonlocal time derivative of $\widehat{\psi}^{m,n}$.

Lemma 5 *Let Assumption 1 hold. For the nonlocal time derivative of the Galerkin solution* $\widehat{\psi}^{m}$ *of* (3.9), *the following estimate holds:*

$$\|M^m\partial_t(k*[\widehat{\psi}^{m,n}-\widehat{\psi}_0^{m,n}])\|_{L^2(0,T;[W^{1,2}(\mathcal{O})]^*)} \le C(m,\ell,T). \tag{3.18}$$

Proof We consider an arbitrary element $\zeta \in L^2(0,T;W^{1,2}(\mathcal{O}))$ and test the Galerkin equation of $\widehat{\psi}^{m,n}$, see $(3.9)_2$, with the orthogonal projection of ζ onto the space that is spanned by the first m eigenfunctions $\{\zeta_i^m\}$, that is, $\widehat{\Pi}_n\zeta := \sum_{i=1}^n (T_\ell(\zeta),\zeta_i^m)_{\mathcal{O}}\zeta_i^m$. We apply Hölder's inequality to the inner products to deduce that

$$\begin{aligned}
&\langle M^m\partial_t(k*[\widehat{\psi}^{m,n}-\widehat{\psi}_0^{m,n}]), \Pi_n\zeta\rangle \\
&= -(M^m\nabla\widehat{\psi}^{m,n},\nabla\Pi_n\zeta)_{\mathcal{O}_T} + (M^m u^{m,n}\widehat{\psi}^{m,n},\nabla\Pi_n\zeta)_{\mathcal{O}_T} \\
&\quad -(M^m\nabla_q\widehat{\psi}^{m,n},\nabla_q\Pi_n\zeta)_{\mathcal{O}_T} + (M\Lambda_\ell(\widehat{\psi}^{m,n})(\nabla u^{m,n})q,\nabla_q\Pi_n\zeta)_{\mathcal{O}_T} \\
&\le \|M^m\widehat{\psi}^{m,n}\|_{L^2(0,T;W^{1,2}(\mathcal{O}))}\|\zeta\|_{L^2(0,T;W^{1,2}(\mathcal{O}))} \\
&\quad + \|u^{m,n}\|_{L^\infty(\Omega_T)}\|M^m\widehat{\psi}^{m,n}\|_{L^2(\mathcal{O}_T)}\|\zeta\|_{L^2(0,T;W^{1,2}(\mathcal{O}))} \\
&\quad + C(\ell)\|q\|_{L^\infty(D)}\|\nabla u^{m,n}\|_{L^\infty(\Omega_T)}\|\widehat{\psi}^{m,n}\|_{L^2(0,T;L^2_{M^m}(\mathcal{O}))}\|\zeta\|_{L^2(0,T;W^{1,2}(\mathcal{O}))}.
\end{aligned}$$

Then, using the n-independent estimates of Lemma 4 and the Lipschitz continuity of M^m, we obtain

$$\langle M^m\partial_t(k*[\widehat{\psi}^{m,n}-\widehat{\psi}_0^{m,n}]), \Pi_n\zeta\rangle \le C(m,\ell,T)\|\zeta\|_{L^2(0,T;W^{1,2}(\mathcal{O}))}.$$

Thus, taking the supremum over all functions $\zeta \in L^2(0,T;W^{1,2}(\mathcal{O}))$, we arrive at the desired bound on $M^m\partial_t(k*[\widehat{\psi}^{m,n}-\widehat{\psi}_0^{m,n}])$. ◻

Having proved the n-independent a priori estimates stated in Lemma 4 and Lemma 5, we are now able to pass to the limit $n \to \infty$ in the Galerkin system (3.9), (3.10) in the next lemma.

Lemma 6 *Let Assumption 1 hold. Then, as* $n \to \infty$*, the limit functions* $(u^m, \widehat{\psi}^m)$ *satisfy the system*

$$\begin{aligned}
&\langle \partial_t u^m, w_i\rangle - (\Gamma_\ell(|u^m|^2)u^m \otimes u^m, \nabla w_i)_\Omega \\
&\quad + (\nabla u^m, \nabla w_i)_\Omega + (\mathbb{S}_\ell(\widehat{\psi}^m), \nabla w_i)_\Omega = (f, w_i)_\Omega, \\
&\langle M^m \partial_t(k * [\widehat{\psi}^m - \widehat{\psi}_0^m]), \zeta\rangle - (M^m u^m \widehat{\psi}^m, \nabla\zeta)_{\mathcal{O}} + (\nabla(M^m\widehat{\psi}^m), \nabla\zeta)_{\mathcal{O}} \\
&\quad + (M^m \nabla_q \widehat{\psi}^m, \nabla_q\zeta)_{\mathcal{O}} = (M\Lambda_\ell(\widehat{\psi}^m)(\nabla u^m)q, \nabla_q\zeta)_{\mathcal{O}},
\end{aligned} \tag{3.19}$$

for any $i = 1, \ldots, n$ *and* $\zeta \in W^{1,2}(\mathcal{O})$*, together with the initial condition* $u^m(0) = u_0^m$ *for the velocity and* $(k * [\widehat{\psi}^m - \widehat{\psi}_0^m])(0) = 0$ *for the density.*

Proof Using the n-independent bounds, see Lemma 4 and Lemma 5, we obtain the weak/weak-$*$ convergences

$$\begin{aligned}
c_i^{m,n} &\overset{*}{\rightharpoonup} c_i^m && \text{weakly-}* \text{ in } W^{1,\infty}(0,T), \\
\widehat{\psi}^{m,n} &\rightharpoonup \widehat{\psi}^m && \text{weakly in } L^2(0,T; W^{1,2}(\mathcal{O})), \\
\widehat{\psi}^{m,n} &\overset{*}{\rightharpoonup} \widehat{\psi}^m && \text{weakly-}* \text{ in } L^\infty(0,T; L^2(\mathcal{O})), \\
\mathbb{S}_\ell(\widehat{\psi}^{m,n}) &\overset{*}{\rightharpoonup} \mathbb{S}_\ell(\widehat{\psi}^m) && \text{weakly-}* \text{ in } L^\infty(0,T; L^\infty(\Omega)^{d\times d}), \\
M^m \partial_t(k * d\widehat{\psi}^{m,n}) &\rightharpoonup M^m \partial_t(k * d\widehat{\psi}^m) && \text{weakly in } L^2(0,T; [W^{1,2}(\mathcal{O})]^*),
\end{aligned}$$

where $d\widehat{\psi}^{m,n} := \widehat{\psi}^{m,n} - \widehat{\psi}_0^{m,n}$ and $d\widehat{\psi}^m := \widehat{\psi}^m - \widehat{\psi}_0^m$. We note that the limit of the nonlocal derivative is indeed of this form because of linearity of the convolution operator and using integration by parts, see [40, Proposition 3.5]. Then, we use the Aubin–Lions compactness lemma (cf. (2.11)) and its nonlocal counterpart stated in Prop. 1 to infer the strong convergences

$$\begin{aligned}
c_i^{m,n} &\to c_i^m && \text{strongly in } C([0,T]), \\
u^{m,n} &\to u^m && \text{strongly in } C([0,T]; W_{0,\mathrm{div}}^{1,2}(\Omega) \cap W^{d+1,2}(\Omega)^d), \\
\widehat{\psi}^{m,n} &\to \widehat{\psi}^m && \text{strongly in } L^2(\mathcal{O}_T).
\end{aligned}$$

Taking the limit $n \to \infty$ in the Galerkin equations (3.9) then gives the system (3.19) stated in the lemma. Regarding the attainment of the initial conditions, we argue as follows. By the strong convergence of $u^{m,n}$ stated above, we have $u^{m,n}(0) \to u^m(0)$ in $W_{0,\mathrm{div}}^{1,2}(\Omega) \cap W^{d+1,2}(\Omega)^d$, from which it follows that $u^{m,n}(0) = u_0^m$ and thus $u^m(0) = u_0^m$ by the uniqueness of the limits. Moreover, by the previous convergences, we have

$$\begin{aligned}
M^m k * [\widehat{\psi}^{m,n} - \widehat{\psi}_0^{m,n}] \overset{*}{\rightharpoonup} M^m k * [\widehat{\psi}^m - \widehat{\psi}_0^m] \ \text{weakly-}* \text{ in } &L^\infty(0,T; L^2(\mathcal{O})) \\
\cap\, &H^1(0,T; [W^{1,2}(\mathcal{O})]^*).
\end{aligned}$$

As the embedding operator from $W^{1,2}(\mathcal{O})$ into $L^2(\mathcal{O})$ is compact, its transpose, mapping $[L^2(\mathcal{O})]^*$ (which is isometrically isomorphic to $L^2(\mathcal{O})$) into $[W^{1,2}(\mathcal{O})]^*$

is a compact linear operator. Hence, applying the compact embedding $(2.11)_2$ with $X = L^2(\mathcal{O})$, $Z = [W^{1,2}(\mathcal{O})]^*$ and $Y = Z$, it follows that

$$M^m k * [\widehat{\psi}^{m,n} - \widehat{\psi}_0^{m,n}] \to M^m k * [\widehat{\psi}^m - \widehat{\psi}_0^m] \text{ strongly in } C([0,T];[W^{1,2}(\mathcal{O})]^*),$$

which gives, inserting $t = 0$, that $M^m(k * [\widehat{\psi}^m - \widehat{\psi}_0^m])(0) = 0$ in $[W^{1,2}(\mathcal{O})]^*$. □

Step 3: Uniform a priori estimates in *m* and the limit process $m \to \infty$

In Step 2 we passed to the limit $n \to \infty$ in the (ℓ, n, m)-th Galerkin system (3.9), which resulted in the (ℓ, m)-th approximation (3.19). Next, we derive m-uniform bounds and take the limit $m \to \infty$ in (3.19). A key step in the argument is to prove the nonnegativity of $\widehat{\psi}^m$. While a probability density function is by definition nonnegative, it does not automatically follow that $\widehat{\psi}^m$ satisfies this property; instead, the nonnegativity of $\widehat{\psi}^m$ has to be inferred from the nonnegativity of the initial datum $\widehat{\psi}_0^m$. Having shown the nonnegativity of $\widehat{\psi}^m$, we shall be able to test with the logarithm of $\widehat{\psi}^m + \delta$ for some small $\delta > 0$.

Lemma 7 *Let Assumption 1 hold. Then* $\widehat{\psi}^m$ *is nonnegative.*

Proof The nonnegative part of a function w is defined by $[w]_+ := \max\{0, w\}$. As $[\cdot]_+$ is a Lipschitz-continuous function over $\mathbb{R}$, we deduce that $-[-\widehat{\psi}^m]_+$ preserves the spatial $W^{1,2}$-regularity and the integrability-in-time of $\widehat{\psi}^m$. Therefore, $-[-\widehat{\psi}^m]_+$ is a valid test function in $(3.19)_2$. Indeed, testing $(3.19)_2$ with $-[-\widehat{\psi}^m]_+$ gives

$$\begin{aligned}
&\langle M^m \partial_t(k * [-\widehat{\psi}^m + \widehat{\psi}_0^m]), [-\widehat{\psi}^m]_+\rangle - (M^m u^m(-\widehat{\psi}^m), \nabla[-\widehat{\psi}^m]_+)_{\mathcal{O}} \\
&\quad + (M^m \nabla(-\widehat{\psi}^m), \nabla[-\widehat{\psi}^m]_+)_{\mathcal{O}} + (M^m \nabla_q(-\widehat{\psi}^m), \nabla_q[-\widehat{\psi}^m]_+)_{\mathcal{O}} \\
&= (M^m \Lambda_\ell(-\widehat{\psi}^m)(\nabla u^m)q, \nabla_q[-\widehat{\psi}^m]_+)_{\mathcal{O}}.
\end{aligned} \tag{3.20}$$

Regarding the first term of (3.20), we note that the function $H(y) = \frac{1}{2}(y_+)^2$ is convex with derivative $H'(y) = y_+$. Thus, we infer from Alikhanov's inequality (2.10) that

$$\begin{aligned}
&\langle M^m \partial_t(k * [-\widehat{\psi}^m + \widehat{\psi}_0^m]), [-\widehat{\psi}^m]_+\rangle \\
&\geq \frac{1}{2}\partial_t\Big(k * \Big[\|[-\widehat{\psi}^m]_+\|^2_{L^2_{M^m}(\mathcal{O})} - \|[-\widehat{\psi}_0^m]_+\|^2_{L^2_{M^m}(\mathcal{O})}\Big]\Big) \\
&= \frac{1}{2}\partial_t(k * \|[-\widehat{\psi}^m]_+\|^2_{L^2_{M^m}(\mathcal{O})}),
\end{aligned}$$

where we have used that $[-\widehat{\psi}_0^m]_+ = 0$; this equality follows from the definition of $\widehat{\psi}_0^m$ (recall that $\widehat{\psi}_0^m := \widehat{\psi}_0 M/M^m$), the assumed nonnegativity of $\widehat{\psi}_0$ (see Assumption 1), the nonnegativity of M and the positivity of M^m. The remaining terms on the left-hand side of (3.20) are dealt with using a standard argument, resulting in

$$\begin{aligned}
(M^m \nabla(-\widehat{\psi}^m), \nabla[-\widehat{\psi}^m]_+)_{\mathcal{O}} &= \|\nabla[-\widehat{\psi}^m]_+\|^2_{L^2_{M^m}(\mathcal{O})}, \\
((u^m \cdot \nabla)(-\widehat{\psi}^m), [-\widehat{\psi}^m]_+)_{\mathcal{O}} &= 0.
\end{aligned}$$

It remains to bound the term on the right-hand side of (3.20). To this end, we use that $\|\nabla u^m\|_{L^\infty(\Omega_T)} \leq C(m, \ell)$ in conjunction with the definition of Λ_ℓ, see (3.2), to deduce that

$$\begin{aligned}&(M^m \Lambda_\ell(-\widehat{\psi}^m)(\nabla u^m)q, \nabla_q[-\widehat{\psi}^m]_+)_{\mathcal{O}}\\&\quad \leq C(m, \ell)\|[-\widehat{\psi}^m]_+\|^2_{L^2_{M^m}(\mathcal{O})} + \frac{1}{2}\|\nabla_q[-\widehat{\psi}^m]_+\|^2_{L^2_{M^m}(\mathcal{O})}.\end{aligned}$$

Putting everything together, we arrive at the following inequality for $\Xi := [-\widehat{\psi}^m]_+$:

$$\frac{1}{2}\partial_t(k * \|\Xi\|^2_{L^2_{M^m}(\mathcal{O})}) + \frac{1}{2}\|\nabla_{x,q}\Xi\|^2_{L^2_{M^m}(\mathcal{O})} \leq C(m, \ell)\|\Xi\|^2_{L^2_{M^m}(\mathcal{O})}.$$

We convolve this inequality with $\tilde{k}$ and use the inverse convolution property (2.12). Lastly, we apply the Henry–Gronwall inequality [25, Corollary 1] to infer that $\|\Xi\|^2_{L^2_{M^m}(\mathcal{O})} = 0$, which implies that $\widehat{\psi}^m \geq 0$ almost everywhere in $\mathcal{O}_T$. □

Our first m-independent estimate is for $\varrho^m := \int_D M^m \widehat{\psi}^m \,\mathrm{d}q$ over the space-time domain Ω_T. Note that, thanks to Lemma 7, $\varrho^m \geq 0$ on Ω_T.

Lemma 8 *Let Assumption 1 hold. Then ϱ^m satisfies the uniform bound*

$$\|\varrho^m\|_{L^\infty(\Omega_T)} \leq C\|\varrho_0\|_{L^\infty(\Omega)}. \tag{3.21}$$

Proof By choosing a q-independent test function $\bar{\zeta}(x, t) \in W^{1,2}(\Omega)$ in the m-th approximation $(3.19)_2$, we have the following nonlocal convection-diffusion equation that governs the evolution of $\varrho^m = (M^m, \widehat{\psi}^m)_D$:

$$\begin{aligned}&\langle \partial_t(k * [\varrho^m - \varrho^m_0]), \bar{\zeta}\rangle - (u^m \varrho^m, \nabla\bar{\zeta})_\Omega + (\nabla\varrho^m, \nabla\bar{\zeta})_\Omega = 0\\&\forall \bar{\zeta} \in W^{1,2}(\Omega),\end{aligned}$$

supplemented with the initial condition $(k * (\varrho^m - \varrho^m_0))(0) = 0$ where we define ϱ^m_0 by

$$0 \leq \varrho^m_0(x) := (M^m, T_\ell(\psi^m_0(x, \cdot)))_D \leq (M, \psi_0(x, \cdot))_D = \varrho_0(x).$$

Since $\operatorname{div} u^m = 0$ and $\varrho_0 \in L^\infty(\Omega)$ thanks to Assumption 1, we deduce the desired uniform bound (3.21) by the maximum principle for nonlocal-in-time differential equations, see [56, Theorem 3.2]. □

Next, we derive a crucial m-uniform bound on the tuple $(u^m, \widehat{\psi}^m)$ in suitable norms and then pass to the limit $m \to \infty$ in the m-th approximation (3.19) to the ℓ-truncated system (3.4), (3.5). In contrast with the previous energy estimate, we derive at this step an entropy estimate by choosing an appropriate convex test function. This is essential for the final energy (in)equality and the handling of the nonlinearities present in the system. We stress that the key technical difference between the corotational model considered in [26] and the general noncorotational model studied here is that in the case of the noncorotational model the absence of an m independent bound on $\|\nabla u^m\|_{L^\infty(\Omega_T)}$ poses significant technical difficulties; in contrast, in the case of the corotional model, where the gradient of the velocity field in the Fokker–Planck equation is replaced by its skew-symmetric part, such a bound is not required in the proof of the existence of a global weak solution, and that then greatly simplifies the analysis.

Lemma 9 (m-uniform estimates) *Let Assumption 1 hold. For the tuple* $(u^m, \widehat{\psi}^m)$*, the following a priori estimate holds:*

$$\begin{aligned}&\|u^m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^m\|^2_{L^2(\Omega_T)} + \|\partial_t u^m\|^2_{L^2(0,T;[W^{1,2}_{0,\mathrm{div}}(\Omega)]^*)}\\ &\quad + \|u^m\|^2_{L^{2(d+2)/d}(\Omega_T)} + \|\mathbb{S}_\ell(\widehat{\psi}^m)\|_{L^\infty(0,T;L^\infty(\Omega))}\\ &\quad + \|\widehat{\psi}^m \ln(\widehat{\psi}^m)\|_{L^1(0,T;L^1_{M^m}(\mathcal{O}))} + \left\|\nabla_{x,q}\sqrt{\widehat{\psi}^m}\right\|^2_{L^2(0,T;L^2_{M^m}(\mathcal{O}))} \le C(\ell).\end{aligned}$$

Proof First, we test the approximated Navier–Stokes equation $(3.19)_1$ with u^m, which gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u^m\|^2_{L^2(\Omega)} + \|\nabla u^m\|^2_{L^2(\Omega)} = (f, u^m)_\Omega - (\mathbb{S}_\ell(\widehat{\psi}^m), \nabla u^m)_\Omega.$$

We shall bound the first term on the right-hand side; however, following the ideas in [6] and [14], we leave the second term on the right-hand side intact because its current form plays a crucial role in cancelling the noncorotational drag term in the Fokker–Planck equation. We integrate the inequality over $(0, t)$ to deduce that

$$\begin{aligned}&\frac{1}{2}\|u^m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u^m\|^2_{L^2(0,t;L^2(\Omega))}\\ &\le \frac{1}{2}\|u^m_0(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(0,T;W^{-1,2}(\Omega)^d)} - \int_0^t (\mathbb{S}_\ell(\widehat{\psi}^m), \nabla u^m)_\Omega \,\mathrm{ds}.\end{aligned} \tag{3.22}$$

Before focusing on the m-th approximation $(3.19)_2$ of the nonlocal Fokker–Planck equation, we first define the convex function $G_\delta \in C^\infty((-\delta,\infty)) \cap C^0([-\delta,\infty))$ by

$$G_\delta(s) := (s+\delta)\ln(s+\delta) + \tfrac{1}{e} \ge 0,$$

where $\delta > 0$ is arbitrary and we set $G(\widehat{\psi}^m) := G_0(\widehat{\psi}^m)$. Further, we define the function $T_{\delta,\ell}(s) := \int_0^s \frac{\Lambda_\ell(t)}{t+\delta}\,\mathrm{d}t$ that converges to T_ℓ in $C([0,\infty))$ as $\delta \to 0$. We test the approximated Fokker–Planck equation $(3.19)_2$ with $G'_\delta(\widehat{\psi}^m) = \ln(\widehat{\psi}^m + \delta) + 1$, which gives

$$\begin{aligned}&(M^m\partial_t(k * [\widehat{\psi}^m - \widehat{\psi}^m_0]), G'_\delta(\widehat{\psi}^m))_{\mathcal{O}} + (M^m\nabla\widehat{\psi}^m, \nabla G'_\delta(\widehat{\psi}^m))_{\mathcal{O}}\\ &\quad + (M^m\nabla_q\widehat{\psi}^m, \nabla_q G'_\delta(\widehat{\psi}^m))_{\mathcal{O}}\\ &= (M^m u^m\widehat{\psi}^m, \nabla G'_\delta(\widehat{\psi}^m))_{\mathcal{O}} + (M\Lambda_\ell(\widehat{\psi}^m)(\nabla u^m)q, \nabla_q G'_\delta(\widehat{\psi}^m))_{\mathcal{O}}.\end{aligned} \tag{3.23}$$

Regarding the first term of the left-hand side of (3.23), we apply Alikhanov's inequality for convex functions, see (2.10), to estimate it from below as follows:

$$\begin{aligned}(M^m\partial_t(k*[\widehat{\psi}^m-\widehat{\psi}^m_0]), G'_\delta(\widehat{\psi}^m))_{\mathcal{O}} &\geq \partial_t\Big(k*\int_{\mathcal{O}} M^m(G_\delta(\widehat{\psi}^m)-G_\delta(\widehat{\psi}^m_0))\,\mathrm{d}(x,q)\Big)\\ &= \partial_t\Big(k*\Big[\|G_\delta(\widehat{\psi}^m)\|_{L^1_{M^m}(\mathcal{O})}-\|G_\delta(\widehat{\psi}^m_0)\|_{L^1_{M^m}(\mathcal{O})}\Big]\Big).\end{aligned}$$

For the second term on the left-hand side of (3.23), we have

$$\lim_{\delta\to 0}(M^m\nabla\widehat{\psi}^m,\nabla G'_\delta(\widehat{\psi}^m))_{\mathcal{O}}=\lim_{\delta\to 0}(M^m G''_\delta(\widehat{\psi}^m), |\nabla\widehat{\psi}^m|^2)_{\mathcal{O}}=4\Big\|\nabla\sqrt{\widehat{\psi}^m}\Big\|^2_{L^2_{M^m}(\mathcal{O})},$$

and in the same manner we handle the third term on the left-hand side of (3.23). The first term on the right-hand side of (3.23) is zero after taking the limit $\delta\to 0$, because

$$\nabla G'_\delta(\widehat{\psi}^m)=G''_\delta(\widehat{\psi}^m)\nabla\widehat{\psi}^m=\frac{\nabla\widehat{\psi}^m}{\widehat{\psi}^m+\delta},$$

and thus, by the monotone convergence theorem, we deduce that

$$-\lim_{\delta\to 0}(u^m, M^m\widehat{\psi}^m\nabla G'_\delta(\widehat{\psi}^m))_{\mathcal{O}}=-(u^m, M^m\nabla\widehat{\psi}^m)_{\mathcal{O}}=(\operatorname{div}u^m, M^m\widehat{\psi}^m)_{\mathcal{O}}=0.$$

Furthermore, for the remaining term on the right-hand side of (3.23), we compute the limit $\delta\to 0$ as follows:

$$\begin{aligned}\lim_{\delta\to 0}(M\Lambda_\ell(\widehat{\psi}^m)(\nabla u^m)q, \nabla_q G'_\delta(\widehat{\psi}^m))_{\mathcal{O}} &= \lim_{\delta\to 0}(Mq\nabla u^m, \nabla_q T_{\delta,\ell}(\widehat{\psi}^m))_{\mathcal{O}}\\ &= -\sum_{j=1}^K(\nabla_{q^j}(M\otimes q^j)\nabla u^m, T_\ell(\widehat{\psi}^m))_{\mathcal{O}}\\ &= (\mathbb{S}_\ell(\widehat{\psi}^m), \nabla u^m)_{\mathcal{O}},\end{aligned}$$

where we used integration by parts in the second step and the boundary term vanishes thanks to fact that $M|_{\partial D}=0$. Taking the limit $\delta\to 0$ in (3.23) and using the above considerations, we obtain the upper bound

$$\begin{aligned}&\partial_t\Big(k*\Big[\|G(\widehat{\psi}^m)\|_{L^1_{M^m}(\mathcal{O})}-\|G(\widehat{\psi}^m_0)\|_{L^1_{M^m}(\mathcal{O})}\Big]\Big)+4\Big\|\nabla_{x,q}\sqrt{\widehat{\psi}^m}\Big\|^2_{L^2_{M^m}(\mathcal{O})}\\ &\quad\leq(\mathbb{S}_\ell(\widehat{\psi}^m), \nabla u^m)_{\mathcal{O}}.\end{aligned}$$

At a first glance, the next natural step might seem to be to convolve this estimate with $\tilde{k}$ and use the inverse convolution property (2.12) to obtain that $G(\psi^m)$ is bounded in $L^\infty(0,T;L^1_{M^m}(\mathcal{O}))$. However, we have to take care of the term on the right-hand side of the inequality. It is essential that it cancels with the right-hand side of (3.22) as otherwise the term cannot be controlled in terms of the available bounds. Therefore, we do not convolve but integrate. As a result, we only obtain that $G(\psi^m)$ is bounded in $L^1_{M^m}(\mathcal{O}_T)$ but not necessarily in $L^\infty(0,T;L^1_{M^m}(\mathcal{O}))$; i.e.,

$$\begin{aligned}&\Big(k * \|G(\widehat{\psi}^m)\|_{L^1_{M^m}(\mathcal{O})}\Big)(t) + 4\Big\|\nabla_{x,q}\sqrt{\widehat{\psi}^m}\Big\|^2_{L^2(0,t;L^2_{M^m}(\mathcal{O}))}\\ &\quad\le \|k\|_{L^1(0,T)}\|G(\widehat{\psi}^m_0)\|_{L^1_{M^m}(\mathcal{O})} + \int_0^t (\mathbb{S}_\ell(\widehat{\psi}^m), \nabla u^m)_{\mathcal{O}}\,\mathrm{d}s,\end{aligned} \tag{3.24}$$

where we further used that $(k*1)(t) = \|k\|_{L^1(0,t)} \le \|k\|_{L^1(0,T)}$. Since k is nonincreasing, its infimum is given by $k(T)$ and thus we can estimate the first term on the left-hand side of (3.24) from below by

$$\Big(k * \|G(\widehat{\psi}^m)\|_{L^1_{M^m}(\mathcal{O})}\Big)(t) \ge k(T)\|G(\widehat{\psi}^m)\|_{L^1(0,t;L^1_{M^m}(\mathcal{O}))}.$$

We add the estimate (3.24) to the integrated inequality (3.22) that we obtained from testing the Navier–Stokes equations. We observe that the nonlinear terms involving the ℓ-truncated extra-stress tensor on the right-hand sides of the two inequalities cancel each other. More precisely, we obtain the m-uniform bound

$$\begin{aligned}&\frac{1}{2}\|u^m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u^m\|^2_{L^2(\Omega_T)} + \|G(\widehat{\psi}^m)\|_{L^1_{M^m}(\mathcal{O}_T)}\\ &\quad + 4\Big\|\nabla_{x,q}\sqrt{\widehat{\psi}^m}\Big\|^2_{L^2_{M^m}(\mathcal{O}_T)}\\ &\le \frac{1}{2}\|u_0^m\|^2_{L^2(\Omega)} + \|k\|_{L^1(0,T)}\|G(\widehat{\psi}^m_0)\|_{L^1_{M^m}(\mathcal{O})} + \frac{1}{2}\|f\|^2_{L^2(0,T;W^{-1,2}(\Omega)^d)}\\ &\le C(\ell),\end{aligned} \tag{3.25}$$

where we additionally used that u_0^m is the $L^2_{0,\mathrm{div}}(\Omega)$ orthogonal projection of u_0 in the final inequality. We also recall that $\widehat{\psi}^m_0$ was defined as $\widehat{\psi}^m_0 = \widehat{\psi}_0 M/M^m$, see (3.10), to remove the m-dependence on the right-hand side in the transition to the right-hand side of the final inequality. By standard L^p-L^q interpolation inequalities, we conclude that

$$\|u^m\|_{L^{2(d+2)/d}(\Omega_T)} \le C(\ell). \tag{3.26}$$

By the definition of the ℓ-truncated stress tensor $\mathbb{S}_\ell$, see (3.3), and the presence of the truncation operator, we can directly deduce that

$$\|\mathbb{S}_\ell(\widehat{\psi}^m)\|_{L^\infty(0,T;L^\infty(\Omega))} \le C(\ell).$$

It remains to derive the bound on $\partial_t u^m$, stated in the lemma. To this end, we consider an arbitrary element $w \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))$. Thanks to the truncation in the convection term of u^m, see $(3.19)_1$, and the boundedness of $\mathbb{S}_\ell(\widehat{\psi}^m)$, we obtain the following inequality

$$
\begin{aligned}
&|\langle \partial_t u^m, \Pi_m w\rangle| \\
&\quad = \left|(\Gamma_\ell(|u^m|^2)u^m \otimes u^m - \nabla u^m - S_\ell(\widehat{\psi}^m), \nabla \Pi_m w)_{\Omega_T} + (f, \Pi_m w)_{\Omega_T}\right| \\
&\quad \le C(\ell)\|w\|_{L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))},
\end{aligned}
$$

where we have bound each of the terms appearing on the right-hand side by means of Hölder's inequality and the m-uniform energy estimate (3.25). By taking the supremum over all functions in $L^2(0, T; W^{1,2}_{0,\mathrm{div}}(\Omega))$, we obtain that $\partial_t u^m$ is bounded in $L^2(0, T; [W^{1,2}_{0,\mathrm{div}}(\Omega)]^*)$. □

Next, we consider the limit process $m \to \infty$ in the m-th approximation (3.19) to prove that the limit tuple $(u^\ell, \widehat{\psi}^\ell)$ of $(u^{\ell,m}, \widehat{\psi}^{\ell,m})$ satisfies the ℓ-approximation (3.6). Hence, we prove Lemma 3 as already stated above. By the m-uniform bounds for $(\widehat{\psi}^m, u^m)$ as stated in Lemma 9, we can already extract a weakly/weakly-$*$ subsequence that. Together with the Aubin–Lions lemma to obtain strong convergence, this allows us to pass to the limit in the m-approximated Navier–Stokes equations. This is shown in the next lemma.

Lemma 10 *Let Assumption 1 hold. Then, the tuple* $(u^\ell, \widehat{\psi}^\ell)$ *solves the truncated Navier–Stokes equations* $(3.19)_1$.

Proof We begin by reiterating our notational conventions that we are suppressing the index ℓ associated with the truncation process, so in the proof below we shall write $(u^m, \widehat{\psi}^m)$ instead of $(u^{m,\ell}, \widehat{\psi}^{m,\ell})$ and $(u, \widehat{\psi})$ instead of $(u^\ell, \widehat{\psi}^\ell)$.

By means of the m-uniform energy estimate in Lemma 9 and the classical Aubin–Lions lemma (2.11), we deduce the existence of limit functions u and $\widehat{\psi}$ with the convergences

$$
u^m \overset{*}{\rightharpoonup} u \text{weakly-}* \text{ in } L^\infty(0, T; L^2_{0,\mathrm{div}}(\Omega)) \cap L^2(0, T; W^{1,2}_{0,\mathrm{div}}(\Omega))
$$

where $r < 2^* := 2d/(d-2)$ is less than the Sobolev conjugate 2^* of 2, ensuring the compactness of the first of the two embeddings in the Gelfand triple $W^{1,2}_{0,\mathrm{div}}(\Omega) \hookrightarrow\!\!\!\rightarrow L^r(\Omega) \subset [W^{1,2}_{0,\mathrm{div}}(\Omega)]^*$ involved in the application of the Aubin–Lions lemma. Then it is standard to let $m \to \infty$ in the m-th Galerkin system of the Navier–Stokes equations $(3.19)_1$ to deduce that u solves the truncated Navier–Stokes equations. Lastly, the strong convergence $u^m(0) \to u(0)$ in $[W^{1,2}_{0,\mathrm{div}}(\Omega)]^*$ guarantees that u satisfies the initial condition. □

Having passed the limit $m \to \infty$ in the approximated Navier–Stokes equations $(3.19)_1$ back in Lemma 10, in order to prove that $(u^\ell, \widehat{\psi}^\ell)$ satisfies the ℓ-truncated system (3.4), (3.5) it remains to pass to the limit $m \to \infty$ in the m-th approximated nonlocal Fokker–Planck $(3.19)_2$ equation. To do this, we begin by noting that, thanks to (3.25),

$$
\|M^m G(\widehat{\psi}^m)\|_{L^1(\mathcal{O}_T)} \le C(\ell), \tag{3.27}
$$

where $G(\widehat{\psi}^m(t)) = \widehat{\psi}^m \ln(\widehat{\psi}^m)+1/e$. In [14] this bound was used in conjunction with a compensated compactness argument based on the div-curl lemma, see [14, Lemma A.1], to infer strong convergence of $\widehat{\psi}^m$. In our setting, the presence of the nonlocal time derivative in the Fokker–Planck equation obstructs the application of the div-curl lemma. We are therefore forced to develop a different approach. We shall, instead, bound the nonlocal time derivative and apply the compactness result from Prop. 1 that we derived in Section 2. Using this procedure, we obtain the following strong convergences for the sequences $\{\widehat{\psi}^m\}_{m\in\mathbb{N}}$ and $\{\psi^m\}_{m\in\mathbb{N}}$, where $\psi^m := M^m\widehat{\psi}^m$, as $m \to \infty$.

Lemma 11 *Let Assumption 1 hold. Then, the following convergence results hold for any $s \in [1,\infty)$:*

$$\begin{aligned}
&\psi^m \to \psi \quad \textit{strongly in } L^s(\Omega_T; L^1(D)),\\
&\mathbb{S}_\ell(\widehat{\psi}^m) \to \mathbb{S}_\ell(\widehat{\psi}) \textit{strongly in } L^s(\Omega_T)^{d\times d},\\
&M^m\nabla_{x,q}\widehat{\psi}^m \rightharpoonup M\nabla_{x,q}\widehat{\psi} \textit{weakly in } L^1(\mathcal{O}_T)^{d(K+1)}.
\end{aligned}$$

Proof The estimate (3.27) and the bound $M^m \le C$ yield $\|G(\psi^m)\|_{L^1(\mathcal{O}_T)} \le C(\ell)$. Noting that the function $G(s) = s\ln(1+s) + 1/e$ is nonnegative, convex and has superlinear growth at infinity, de la Vallée Poussin's criterion (see [16, II.22]) implies the uniform integrability of $(\psi^m)_m$ in $L^1(\mathcal{O}_T)$, that is,

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \forall m \in \mathbb{N} \quad \forall U \subset \mathcal{O}_T : |U| \le \delta \implies \|\psi^m\|_{L^1(U)} \le \varepsilon. \tag{3.28}$$

Thus, by the Dunford–Pettis theorem, there exists a subsequence (not relabelled) and $\psi \in L^1(\mathcal{O}_T)$ such that

$$\psi^m \rightharpoonup \psi \quad \text{weakly in } L^1(\mathcal{O}_T).$$

Using the identity $\nabla_{x,q}\widehat{\psi}^m = 2\sqrt{\widehat{\psi}^m}\nabla_{x,q}\sqrt{\widehat{\psi}^m}$ and Hölder's inequality, we deduce

$$\begin{aligned}
\|M^m\nabla_{x,q}\widehat{\psi}^m\|_{L^1(\mathcal{O}_T)} &= \left\|2M^m\sqrt{\widehat{\psi}^m}\nabla_{x,q}\sqrt{\widehat{\psi}^m}\right\|_{L^1(\mathcal{O}_T)}\\
&\le 2\left\|\sqrt{M^m}\sqrt{\widehat{\psi}^m}\right\|_{L^2(\mathcal{O}_T)}\left\|\sqrt{M^m}\nabla_{x,q}\sqrt{\widehat{\psi}^m}\right\|_{L^2(\mathcal{O}_T)} \le C(\ell),
\end{aligned} \tag{3.29}$$

where we used Lemma 8 and the bound (3.25) in the last step. With this estimate, we are able to further bound the nonlocal derivative of $\widehat{\psi}^m$ and apply the compactness result Prop. 1 to obtain a strong convergence result for $\widehat{\psi}^m$. We start by considering an arbitrary element $\zeta \in C([0,T]; W^{1,\infty}(\mathcal{O}))$ as time-dependent test function. The reasoning behind this step is that we can multiply the m-approximated Fokker–Planck equation $(3.19)_2$ by a function $\eta \in C_c^1(0,T)$ and integrate the equation from 0 to T; then, the density of $C_c^1(0,T) \otimes W^{1,2}(\mathcal{O})$ in $L^2(0,T; W^{1,2}(\mathcal{O}))(\supset C([0,T]; W^{1,\infty}(\mathcal{O})))$

allows us to use ζ as a test function. We proceed as follows:

$$\begin{aligned}
&\langle M^m \partial_t (k * [\widehat{\psi}^m - \widehat{\psi}_0^m]), \zeta \rangle \\
&\quad = (M^m u^m \widehat{\psi}^m, \nabla \zeta)_{\mathcal{O}_T} - (M^m \nabla_{x,q} \widehat{\psi}^m, \nabla_{x,q} \zeta)_{\mathcal{O}_T} \\
&\qquad + (M \Lambda_\ell(\widehat{\psi}^m)(\nabla u^m) q, \nabla_q \zeta)_{\mathcal{O}_T} \\
&\quad \le \|M^m\|_{L^\infty(\mathcal{O})} \|u^m\|_{L^\infty(0,T;L^2(\Omega))} \|\widehat{\psi}^m\|_{L^1(\mathcal{O}_T)} \|\nabla \zeta\|_{C([0,T];L^\infty(\mathcal{O}))} \\
&\qquad + \|M^m \nabla_{x,q} \widehat{\psi}^m\|_{L^1(\mathcal{O}_T)} \|\nabla_{x,q} \zeta\|_{C([0,T];L^\infty(\mathcal{O}))} \\
&\qquad + \|M\|_{L^\infty(\mathcal{O})} \|\Lambda_\ell(\widehat{\psi}^m)\|_{L^\infty(\mathcal{O}_T)} \|\nabla u^m\|_{L^2(\Omega_T)} \|q\|_{L^\infty(D)} \|\nabla_q \zeta\|_{L^2(\mathcal{O}_T)} \\
&\quad \le C(\ell) \|\zeta\|_{C([0,T];W^{1,\infty}(\mathcal{O}))}.
\end{aligned}$$

Since M^m is time-independent,

$$M^m \partial_t (k * [\widehat{\psi}^m - \widehat{\psi}_0^m]) = \partial_t (k * [M^m \widehat{\psi}^m - M^m \widehat{\psi}_0^m]) = \partial_t (k * [\psi^m - \psi_0^m]),$$

and therefore,

$$\langle \partial_t (k * [\psi^m - \psi_0^m]), \zeta \rangle \le C(\ell) \|\zeta\|_{C([0,T];W^{1,\infty}(\mathcal{O}))}.$$

Hence, $\partial_t (k * [\psi^m - \psi_0^m])$ is bounded, uniformly in m, in $[C([0,T];W^{1,\infty}(\mathcal{O}))]^* = \mathcal{M}(0,T;[W^{1,\infty}(\Omega)]^*)$, the space of regular Borel measures on $[0,T]$ with values in $[W^{1,\infty}(\mathcal{O})]^*$ (cf. Thm. 2 in Ch. VI, Sec. 1 of Diestel and Uhl [17]). Furthermore, ψ^m is bounded in $L^1(0,T;W^{1,1}(\mathcal{O}))$ thanks to the equality

$$\nabla_{x,q} \psi^m = \nabla_{x,q} (M^m \widehat{\psi}^m) = M^m \nabla_{x,q} \widehat{\psi}^m + \widehat{\psi}^m \nabla_{x,q} M^m,$$

where the first term is bounded because of (3.29). As $M^m \in C^1(\overline{D})$ (since $M \in C^1(\overline{D})$), see also (3.8), and $\|\widehat{\psi}^m\|_{L^1(\mathcal{O}_T)} \le C(\ell)$, see (3.25) and the definition of G, the second term is also bounded uniformly in m. We apply the compactness result Prop. 1 with the Gelfand triple:

$$X = W^{1,1}(\mathcal{O}) \quad \hookrightarrow\!\!\!\rightarrow \quad Y = L^1(\mathcal{O}) \quad \hookrightarrow \quad Z = [W^{1,\infty}(\mathcal{O})]^*.$$

Therefore, $(\psi^m)_m$ is relatively compact in $L^1(0,T;L^1(\mathcal{O}))$, which implies the existence of a subsequence (not relabelled) such that:

$$\psi^m \to \psi \quad \text{strongly in } L^1(\mathcal{O}_T).$$

As, by hypothesis, $\psi_0 \ge 0$, and $\psi^m = M^m \widehat{\psi}^m$ and, according to Lemma 7, $\widehat{\psi}^m \ge 0$, the inequality (8) stated in Lemma 8 implies that

$$\|\psi^m\|_{L^\infty(\Omega_T;L^1(D))} \le C \|\psi_0\|_{L^\infty(\Omega;L^1(D))}.$$

Thus, ψ^m is bounded in $L^\infty(\Omega_T; L^1(D))$ uniformly in m. This then, together with the strong convergence in $L^1(\mathcal{O}_T)$, implies that

$$\int_{\Omega_T} \|\psi^m(t) - \psi(t)\|^s_{L^1(D)}\,\mathrm{d}x\,\mathrm{d}t \le \|\psi^m - \psi\|^{s-1}_{L^\infty(\Omega_T;L^1(D))}\|\psi^m - \psi\|_{L^1(\mathcal{O}_T)} \to 0.$$

Thus, $\psi^m \to \psi$ strongly in $L^s(\Omega_T; L^1(D))$ as $m \to \infty$ for any $s \in [1,\infty)$. Since $\widehat{\psi}^m = \psi^m/M^m$ and $M^m \to M$ uniformly, the strong convergence of ψ^m implies the strong convergence

$$\widehat{\psi}^m \to \widehat{\psi} \quad \text{strongly in } L^s(\Omega_T; L^1(D)) \tag{3.30}$$

for all $s \in [1,\infty)$. By the properties of the truncation operator T_ℓ and the stress tensor $\mathbb{S}$, and using Lebesgue's dominated convergence theorem, we obtain:

$$\mathbb{S}_\ell(\widehat{\psi}^m) = \mathbb{S}(T_\ell(\widehat{\psi}^m)) \to \mathbb{S}(T_\ell(\widehat{\psi})) = \mathbb{S}_\ell(\widehat{\psi}) \quad \text{strongly in } L^1(\Omega_T)^{d\times d}.$$

Interpolating between this strong convergence result and the $L^\infty(\Omega_T)$ bound on $\mathbb{S}_\ell(\widehat{\psi}^m)$, we deduce the strong convergence of $\mathbb{S}_\ell(\widehat{\psi}^m)$ in $L^s(\Omega_T)^{d\times d}$ to $\mathbb{S}_\ell(\widehat{\psi})$ as $m \to \infty$ for all $s \in [1,\infty)$.

By the same reasoning as in (3.29), we obtain for any $U \subset \mathcal{O}_T$ with $|U| \le \delta$ that

$$\|M^m\nabla_{x,q}\widehat{\psi}^m\|_{L^1(U)} \le 2\|M^m\|_\infty \left\|\widehat{\psi}^m\right\|^{1/2}_{L^1(U)} \left\|\sqrt{M^m}\nabla_{x,q}\sqrt{\widehat{\psi}^m}\right\|_{L^2(U)} \le C(\ell)\varepsilon^{1/2}, \tag{3.31}$$

where ε is the same parameter as in (3.28). Thus, $(M^m\nabla_{x,q}\widehat{\psi}^m)_m$ is uniformly integrable in $L^1(\mathcal{O}_T)^{d(K+1)}$, and thus we can extract a subsequence (not relabelled) such that

$$M^m\nabla_{x,q}\widehat{\psi}^m \rightharpoonup \phi \quad \text{weakly in } L^1(\mathcal{O}_T)^{d(K+1)}.$$

It remains to identify the weak limit ϕ; we will show that $\phi = M\nabla_{x,q}\widehat{\psi}$. For any test function $\varphi \in C^\infty_c(\mathcal{O}_T)^{d(K+1)}$, we have by partial integration

$$\begin{aligned}\int_{\mathcal{O}_T} M^m\nabla_{x,q}\widehat{\psi}^m\cdot\varphi\,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t &= -\int_{\mathcal{O}_T}\widehat{\psi}^m\,\nabla_{x,q}\cdot(M^m\varphi)\,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t\\ &= -\int_{\mathcal{O}_T}\widehat{\psi}^m M^m\nabla_{x,q}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t\\ &\quad -\int_{\mathcal{O}_T}\widehat{\psi}^m\varphi\cdot\nabla_{x,q}M^m\,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t.\end{aligned}$$

Now we pass to the limit $m \to \infty$. Since $M^m\nabla_{x,q}\widehat{\psi}^m \rightharpoonup \phi$ weakly in $L^1(\mathcal{O}_T)$, φ is bounded, $M^m \to M$ uniformly, $\widehat{\psi}^m \to \widehat{\psi}$ strongly in $L^1(\mathcal{O}_T)$, we can easily pass to the limit on the left-hand side and in the first term on the right-hand side. It remains to deal with the second term on the right-hand side. Since φ has compact support in $\mathcal{O}_T$, its q-support is contained in some compact set $D' \subset D$. On D', we have

$\nabla_q M^m = \nabla_q M \in C(\overline{D'})$. Also, trivially, $\nabla_x M^m = \nabla_x M = 0$. Combined with the strong convergence $\widehat{\psi}^m \to \widehat{\psi}$ in $L^1(\mathcal{O}_T)$, we obtain:

$$\int_{\mathcal{O}_T} \widehat{\psi}^m \varphi \cdot \nabla_{x,q} M^m \,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t \to \int_{\mathcal{O}_T} \widehat{\psi}\varphi \cdot \nabla_{x,q} M \,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t.$$

Thus, in the limit $m \to \infty$ we have:

$$\begin{aligned}\int_{\mathcal{O}_T} \phi \cdot \varphi \,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t &= -\int_{\mathcal{O}_T} \widehat{\psi} M \nabla_{x,q} \cdot \varphi \,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t - \int_{\mathcal{O}_T} \widehat{\psi}\varphi \cdot \nabla_{x,q} M \,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t \\ &= \int_{\mathcal{O}_T} M \nabla_{x,q} \widehat{\psi} \cdot \varphi \,\mathrm{d}x\,\mathrm{d}q\,\mathrm{d}t,\end{aligned}$$

for any $\varphi \in C_c^\infty(\mathcal{O}_T)^{d(K+1)}$, which implies that $\phi = M\nabla_{x,q}\widehat{\psi}$ almost everywhere on $\mathcal{O}_T$. Hence,

$$M^m \nabla_{x,q} \widehat{\psi}^m \rightharpoonup M \nabla_{x,q} \widehat{\psi} \quad \text{weakly in } L^1(\mathcal{O}_T)^{d(K+1)}.$$

□

At this point, we have derived the necessary convergence properties of $\widehat{\psi}^m$ and u^m to pass to the limit the m-th approximated Fokker–Planck equation $(3.19)_2$, and thereby deduce the existence of a weak solution to the ℓ-truncated system (3.4), (3.5) as stated in Lemma 3.

Proof of Lemma 3 We shall pass to the limit $m \to \infty$ in $(3.19)_2$ to deduce that the nonlocal Fokker–Planck equation (3.5) (with cutoff) is satisfied. Passage to the limit $m \to \infty$ in the third and fourth term on the left-hand side of $(3.19)_2$ is straightforward, using the last weak convergence results stated in Lemma 11. It remains to pass to the limit $m \to \infty$ in the first and second terms on the left-hand side of $(3.19)_2$ and the term on the right-hand side of $(3.19)_2$. We note to this end that by interpolating between the weak convergence of u^m in $L^{2d/(d-2)}(\Omega_T)^d$ and its strong convergence in $L^2(0,T;L^r(\Omega)^d)$ for any $r \in [1, 2d/(d-2))$ we have

$$u^m \to u \quad \text{strongly in } L^r(\Omega_T)^d$$

for all $r \in [1, 2d/(d-2))$. Combining this result and the strong convergence of ψ^m in $L^s(\Omega_T; L^1(D))$ for any $s \in [1, \infty)$, see Lemma 11, we deduce that

$$\begin{aligned} M^m \widehat{\psi}^m u^m = \psi^m u^m &\to M\psi u &&\text{strongly in } L^r(\Omega_T; L^1(D)^{d(K+1)}), \\ \Lambda_\ell(\widehat{\psi}^m)\nabla u^m &\rightharpoonup \Lambda_\ell(\widehat{\psi})\nabla u &&\text{weakly in } L^2(\mathcal{O}_T)^{d\times d}, \end{aligned}$$

where we used the strong convergence of $\widehat{\psi}^m$, see (3.30), and the Lebesgue dominated convergence theorem for the second convergence result.

Since all terms in $(3.19)_2$ besides the first converge, we pass to the limit and obtain

$$\lim_{m\to\infty} \langle M^m \partial_t (k * [\widehat{\psi}^m - \widehat{\psi}_0^m]), \zeta \rangle = L(\zeta)$$

for all $\zeta \in C([0,T]; W^{1,\infty}(\mathcal{O}))$, where $L \in [C([0,T]; W^{1,\infty}(\mathcal{O}))]^*$ is to be identified. To this end, consider the function $Mk * [\widehat{\psi} - \widehat{\psi}_0]$. Since $\widehat{\psi}^m \to \widehat{\psi}$ strongly in $L^1(\mathcal{O}_T)$ and $k \in L^1(0,T)$, we have $Mk * [\widehat{\psi}^m - \widehat{\psi}_0^m] \to Mk * [\widehat{\psi} - \widehat{\psi}_0]$ strongly in $L^1(\mathcal{O}_T)$. For any $\eta \in C_c^1(0,T; W^{1,\infty}(\mathcal{O}))$, integration by parts yields

$$\begin{aligned}\langle \partial_t (Mk * [\widehat{\psi} - \widehat{\psi}_0]), \eta\rangle &= -\langle Mk * [\widehat{\psi} - \widehat{\psi}_0], \partial_t \eta\rangle \\ &= \lim_{m\to\infty} -\langle M^m k * [\widehat{\psi}^m - \widehat{\psi}_0^m], \partial_t \eta\rangle \\ &= \lim_{m\to\infty} \langle M^m \partial_t (k * [\widehat{\psi}^m - \widehat{\psi}_0^m]), \eta\rangle.\end{aligned}$$

As the right-hand side of this equality equals $L(\eta)$ and $C_c^1(0,T; W^{1,\infty}(\mathcal{O}))$ is dense in $C([0,T]; W^{1,\infty}(\mathcal{O}))$, it follows that $\langle \partial_t (Mk * [\widehat{\psi} - \widehat{\psi}_0]), \zeta\rangle = L(\zeta)$ for all ζ, which identifies the limit. Now it is standard to pass to the limit $m \to \infty$ in (3.19) to deduce that the limit $(u^\ell, \widehat{\psi}^\ell)$ (with the previously suppressed superscript $\cdot^\ell$ reinstated in our notation) satisfies the ℓ-truncated system (3.4), (3.5).

It remains to prove the energy inequality stated in Lemma 3. We shall do so by passing to the limit in all terms on the left-hand side of the m-uniform energy estimate (3.25), using the weak lower semicontinuity of the norms and Fatou's lemma for the nonnegative function G. Furthermore, for the term involving the initial datum on the right-hand side of the inequality, we notice that

$$\|M^m G(T_\ell(\widehat{\psi}_0^m))\|_{L^1(\mathcal{O})} \to \|MG(T_\ell(\widehat{\psi}_0))\|_{L^1(\mathcal{O})} \le C.$$

Thus, the energy inequality (3.7) holds, and we have proved Lemma 3. ◻

Step 4: Uniform a priori estimates with respect to the truncation parameter ℓ and the final limit process $\ell \to \infty$

In this last step, we shall derive ℓ-independent bounds, which will enable us to pass to the limit $\ell \to \infty$ to prove Theorem 1, that is, the existence of a weak solution to the nonlocal Navier–Stokes–Fokker–Planck system (2.6). To this end, we reinstate the omitted index ℓ.

Proof of Theorem 1 We begin by recalling the uniform estimate (3.7) stated in Lemma 3, according to which

$$\begin{aligned}&\|u^\ell\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^\ell\|^2_{L^2(\Omega_T)} + \|\widehat{\psi}^\ell \ln \widehat{\psi}^\ell\|_{L^1_M(\mathcal{O}_T)} + \left\|\sqrt{\widehat{\psi}^\ell}\right\|^2_{L^2(0,T;H^1_M(\mathcal{O}))} \\ &\quad + \|\varrho^\ell\|_{L^\infty(\Omega_T)} + \|\mathbb{S}_\ell(\widehat{\psi}^\ell)\|^2_{L^2(\Omega_T)} \le C;\end{aligned} \tag{3.32}$$

recall that $\varrho^\ell := (M, \widehat{\psi}^\ell)_D$. As before, we obtain by function space interpolation that u^ℓ is bounded in $L^{2(d+2)/d}(\Omega_T)$ uniformly in ℓ. We consider an arbitrary function $w \in L^{4/(4-d)}(0,T; W^{1,2}_{0,\mathrm{div}}(\Omega))$; such a w is a valid test function in the ℓ-truncated Navier–Stokes system (3.4). Thus, testing with w we find that

$$
\begin{aligned}
&|\langle \partial_t u^\ell, w\rangle| = \left|(\Gamma_\ell(|u^\ell|^2)u^\ell \otimes u^\ell - \nabla u^\ell - S_\ell(\widehat{\psi}^\ell), \nabla w)_{\Omega_T} + (f, w)_{\Omega_T}\right| \\
&\le C\|\Gamma_\ell(|u^\ell|^2)\|_\infty \|u^\ell(s)\|^{2-d/2}_{L^\infty(0,T;L^2(\Omega))} \|\nabla u^\ell(s)\|^{d/2}_{L^2(\Omega_T)} \|\nabla w\|_{L^{4/(4-d)}(0,T;L^2(\Omega))} \\
&+(\|\nabla u^\ell\|_{L^2(\Omega_T)} + \|\mathbb{S}_\ell(\widehat{\psi}^\ell)\|_{L^2(\Omega_T)} + \|f\|_{L^2(0,T;[W^{1,2}_{0,\mathrm{div}}(\Omega)]^*)})\|w\|_{L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))} \\
&\le C\|w\|_{L^{4/(4-d)}(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))},
\end{aligned}
$$

where we have used the ℓ-uniform bounds together with the uniform bound $\|\Gamma_\ell(|u^\ell|^2)\|_\infty \le 1$ in the last step. Thus, taking the supremum over all functions in $L^{4/(4-d)}(0, T; W^{1,2}_{0,\mathrm{div}}(\Omega))$ gives

$$\|\partial_t u^\ell\|_{L^{4/d}(0,T;[W^{1,2}_{0,\mathrm{div}}(\Omega)]^*)} \le C.$$

Using the ℓ-uniform bounds and applying the Banach–Alaoglu theorem and (2.11), we obtain the existence of a subsequence $(u^\ell, \widehat{\psi}^\ell)$ (not relabelled) and limit functions $(u, \widehat{\psi})$ that satisfy the following convergences:

$$
\begin{aligned}
&u^\ell \overset{*}{\rightharpoonup} u \text{ weakly in } L^\infty(0, T; L^2_{0,\mathrm{div}}(\Omega)) \cap L^2(0, T; W^{1,2}_{0,\mathrm{div}}(\Omega)) \\
&\qquad \cap L^{2(d+2)/d}(\Omega_T)^d, \\
&\partial_t u^\ell \rightharpoonup \partial_t u \text{weakly in } L^{4/d}(0, T; [W^{1,2}_{0,\mathrm{div}}(\Omega)]^*), \\
&u^\ell \to u \text{ strongly in } L^2(0, T; L^r(\Omega)^d) \cap C([0, T]; [W^{1,2}_{0,\mathrm{div}}(\Omega)]^*), \\
&\mathbb{S}_\ell(\widehat{\psi}^\ell) \rightharpoonup \mathbb{S}(\widehat{\psi}) \quad \text{weakly in } L^2(\Omega_T)^{d\times d},
\end{aligned}
$$

where $r < 2^*$ is less than the Sobolev conjugate 2^* of 2, ensuring the compactness of the first of the two embeddings involved in the Gelfand triple $W^{1,2}_{0,\mathrm{div}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega) \subset [W^{1,2}_{0,\mathrm{div}}(\Omega)]^*$ in the application of the Aubin–Lions lemma. Now, we can pass to the limit $\ell \to \infty$ in the ℓ-truncated Navier–Stokes equation (3.4) to deduce that (u, ψ) solves the variational form of the Navier–Stokes equation $(3.1)_1$ as stated in Definition 1. It remains to prove that (u, ψ) also satisfies the nonlocal Fokker–Planck equation $(3.1)_2$.

To deduce an essential strong convergence result for $\widehat{\psi}^\ell$, we follow the proof of Lemma 11. In fact, we begin by considering the nonlocal Fokker–Planck equation (3.1) for $M\partial_t(k * [\widehat{\psi}^\ell - \widehat{\psi}^\ell_0])$, and integrate in time from 0 to T to infer for every test function $\zeta \in C([0, T]; W^{1,\infty}(\mathcal{O}))$ that

$$
\begin{aligned}
&\langle M\partial_t(k * [\widehat{\psi}^\ell - \widehat{\psi}^\ell_0]), \zeta\rangle \\
&= (Mu^\ell \widehat{\psi}^\ell, \nabla\zeta)_{\mathcal{O}_T} - (M\nabla\widehat{\psi}^\ell, \nabla\zeta)_{\mathcal{O}_T} - (M\nabla_q\widehat{\psi}^\ell, \nabla_q\zeta)_{\mathcal{O}_T} \\
&\quad + (M\Lambda_\ell(\widehat{\psi}^\ell)(\nabla u^\ell)q, \nabla_q\zeta)_{\mathcal{O}_T} \\
&\le \|M\|_\infty \|u^\ell\|_{L^\infty(0,T;L^2(\Omega))} \|\widehat{\psi}^\ell\|_{L^\infty(\Omega_T;L^1(D))} \|\nabla\zeta\|_{L^1(0,T;L^\infty(\mathcal{O}))} \\
&\quad + \|M\|_\infty \|\nabla_{x,q}\widehat{\psi}^\ell\|_{L^1(\mathcal{O}_T)} \|\nabla_{x,q}\zeta\|_{C([0,T];L^\infty(\mathcal{O}))} \\
&\quad + \|M\|_\infty \|\Lambda_\ell(\widehat{\psi}^\ell)\|_{L^\infty(\Omega_T;L^1(D))} \|\nabla u^\ell\|_{L^2(\Omega_T)} \|q\|_\infty \|\nabla_q\zeta\|_{C([0,T];L^\infty(\mathcal{O}))}.
\end{aligned}
$$

Using the above ℓ-uniform bounds and the definition of Λ_ℓ, we can take the supremum over all test functions and obtain that

$$\|M\partial_t(k * [\widehat{\psi}^\ell - \widehat{\psi}_0^\ell])\|_{[C([0,T];W^{1,\infty}(\mathcal{O}))]^*} \leq C.$$

Hence, because M is independent of t and $M(\widehat{\psi}^\ell - \widehat{\psi}_0^\ell) = \psi^\ell - \psi_0^\ell$, we find

$$\|\partial_t(k * [\psi^\ell - \psi_0^\ell])\|_{[C([0,T];W^{1,\infty}(\mathcal{O}))]^*} \leq C.$$

Because $[C([0,T]; W^{1,\infty}(\mathcal{O}))]^* = \mathcal{M}(0,T; [W^{1,\infty}(\Omega)]^*)$, we can apply the compactness result Prop. 1 to infer the strong convergence

$$\psi^\ell \to \psi \quad \text{strongly in } L^1(\mathcal{O}_T).$$

Reapplying arguments from the previous section, we deduce the following convergence results by interpolating similarly as in the proof of Lemma 11:

$$\begin{aligned}
\widehat{\psi}^\ell &\to \widehat{\psi} && \text{strongly in } L^s(\Omega_T; L^1_M(D)),\\
\nabla_{x,q}\widehat{\psi}^\ell &\rightharpoonup \nabla_{x,q}\widehat{\psi} && \text{weakly in } L^1(\Omega_T; L^1_M(D)^{d(K+1)}),\\
\widehat{\psi}^\ell u^\ell &\rightharpoonup \widehat{\psi} u && \text{strongly in } L^r(\Omega_T; L^1_M(D)^{d(K+1)}),\\
\Lambda_\ell(\widehat{\psi}^\ell)\nabla u^\ell &\rightharpoonup \widehat{\psi}^\ell \nabla u && \text{weakly in } L^2(\Omega_T; L^1_M(D)^{d\times d}),\\
\partial_t(k * [\widehat{\psi}^\ell - \widehat{\psi}_0^\ell]) &\overset{*}{\rightharpoonup} \partial_t(k * [\widehat{\psi} - \widehat{\psi}_0]) && \text{weakly-}* \text{ in } \mathcal{M}(0,T; [W^{1,\infty}(\mathcal{O})]^*),
\end{aligned}$$

for any $s \in [1,\infty)$ and any $r \in [1, 2^*)$, where $2^* := 2d/(d-2)$. We can now pass to the limit $\ell \to \infty$ in (3.4), (3.5) to conclude that (u, ψ) solves the variational form of the Navier–Stokes–Fokker–Planck system as stated in Definition 1. The verification of the attainment of the initial conditions follows the same line of argument as before. In particular, the uniform bound on $k*[\widehat{\psi}^\ell - \widehat{\psi}_0^\ell]$ in $BV([0,T]; [W^{1,\infty}(\mathcal{O})]^*)$ implies (by Helly's selection principle) the existence of a representative with well-defined trace at $t = 0$, and since $(k*[\widehat{\psi}^\ell - \widehat{\psi}_0^\ell])(0) = 0$ for all ℓ, the limit satisfies $(k*[\widehat{\psi} - \widehat{\psi}_0])(0) = 0$ in $[W^{1,\infty}(\mathcal{O})]^*$. □

Step 5: Uniqueness

Proof of Theorem 3 Let us consider two weak solutions $(u_1, \widehat{\psi}_1)$ and $(u_2, \widehat{\psi}_2)$ in the sense of Definition 1 that satisfy the regularity assumptions stated in Theorem 3. We consider their difference $(u, \widehat{\psi}) := (u_1 - u_2, \widehat{\psi}_1 - \widehat{\psi}_2)$ and note that this pair of functions satisfies the following variational problem:

$$\begin{aligned}
&\langle \partial_t u, w\rangle - (u_1 \otimes u, \nabla w)_\Omega - (u \otimes u_2, \nabla w)_\Omega + (\nabla u, \nabla w)_\Omega + (S\widehat{\psi}, \nabla w)_\Omega = 0,\\
&\langle M\partial_t(k * \widehat{\psi}), \zeta\rangle + (M\nabla\widehat{\psi}, \nabla\zeta)_\mathcal{O} + (M\nabla_q\widehat{\psi}, \nabla_q\zeta)_\mathcal{O} - k * (Mu_1\widehat{\psi}, \nabla\zeta)_\mathcal{O}\\
&\quad - k * (Mu\widehat{\psi}_2, \nabla\zeta)_\mathcal{O} - k * (M\widehat{\psi}_1(\nabla u)q, \nabla_q\zeta)_\mathcal{O} - k * (M\widehat{\psi}(\nabla u_2)q, \nabla_q\zeta)_\mathcal{O} = 0,
\end{aligned} \tag{3.33}$$

for the same choice of test functions w and ζ as in Definition 1. Setting $w = u$ in $(3.33)_1$ yields that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} = -((u\cdot\nabla)u_2, u)_\Omega - (\mathbb{S}(\widehat{\psi}), \nabla u)_\Omega \\
&\quad = ((u\cdot\nabla)u, u_2)_\Omega - (\mathbb{S}(\widehat{\psi}), \nabla u)_\Omega \\
&\quad \le \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + C\|u\|^2_{L^2(\Omega)}\|u_2\|^8_{L^4(\Omega)} + Cq_\infty^2\|\nabla_q\widehat{\psi}\|^2_{L^2_M(\mathcal{O})},
\end{aligned} \tag{3.34}$$

where we made use of the specific form of the extra-stress tensor S, see (2.4). Furthermore, choosing $\zeta = \widehat{\psi}$ in $(3.33)_2$ gives with Alikhanov's, Hölder's, Young's inequalities

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(k * \|\widehat{\psi}\|^2_{L^2_M(\mathcal{O})}\right) + \|\nabla_{x,q}\widehat{\psi}\|^2_{L^2_M(\mathcal{O})} \\
&\quad \le (Mu\widehat{\psi}_2, \nabla\widehat{\psi})_\mathcal{O} + (M\widehat{\psi}_1(\nabla u)q, \nabla_q\widehat{\psi})_\mathcal{O} + (M\widehat{\psi}(\nabla u_2)q, \nabla_q\widehat{\psi})_\mathcal{O} \\
&\quad \le C\|\widehat{\psi}_2\|^2_\infty\|u\|^2_{L^2(\Omega)} + \tfrac{\tilde{k}(T)}{2}\|\nabla\widehat{\psi}\|^2_{L^2_M(\mathcal{O})} + q_\infty\|\widehat{\psi}_1\|^2_\infty\|\nabla u\|^2_{L^2(\Omega)} \\
&\qquad + q_\infty\|\nabla_q\widehat{\psi}\|^2_{L^2_M(\mathcal{O})} + q_\infty\|\widehat{\psi}\|_{L^6(\mathcal{O})}\|\nabla u_2\|_{L^3(\Omega)}\|\nabla_q\widehat{\psi}\|_{L^2_M(\mathcal{O})}.
\end{aligned} \tag{3.35}$$

At this point, we integrate (3.34) over $(0, t)$ and convolve (3.35) with the resolvent kernel $\tilde{k}$, use the inverse convolution property (2.12) and add the inequalities to obtain

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - q_\infty\|\widehat{\psi}_1\|^2_{L^\infty(0,T;L^\infty_M(\mathcal{O}))}\right)\|\nabla u\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2}\|\widehat{\psi}(t)\|^2_{L^2_M(\mathcal{O})} \\
&\quad + \left(\frac{\tilde{k}(T)}{2} - q_\infty - Cq_\infty^2 - Cq_\infty\|\nabla u_2\|_{L^\infty(0,T;L^3(\Omega))}\right)\|\nabla_{x,q}\widehat{\psi}\|^2_{L^2(0,t;L^2_M(\mathcal{O}))} \\
&\le C\int_0^t \left(\|u_2(s)\|^8_{L^4(\Omega)} + \|\widehat{\psi}_2\|^2_\infty\right)\|u(s)\|^2_{L^2(\Omega)}\,\mathrm{ds}.
\end{aligned}$$

Since q_∞ is assumed to be sufficiently small, Gronwall's inequality with an integrable prefactor in the integral on the right yields the desired result. □

4 Conclusion and outlook

We have established the existence of large-data global-in-time weak solutions to a nonlocal Navier–Stokes–Fokker–Planck system modeling dilute polymeric fluids with anomalous diffusion. The analysis overcomes three main challenges: (i) the nonlocal temporal coupling between the macroscopic momentum equation and the mesoscopic polymer configuration equation, handled via energy–entropy estimates; (ii) preservation of the physical structure through a maximum-principle argument ensuring nonnegativity of the PDF and yielding an energy inequality; and (iii) a compactness argument based on a novel nonlocal Aubin–Lions-type result. Uniqueness of sufficiently regular weak solutions completes the well-posedness theory. Our theoretical results provide a rigorous analytical framework for subdiffusive polymeric fluid models with memory effects.

Several directions for future research arise. A key challenge is the development of structure-preserving numerical schemes for nonlocal operators, in particular discretizations that retain entropy dissipation and the finite extensibility of polymer molecules in FENE-type models, as known in the integer-order setting [5, 7, 8]. In addition, probabilistic and Monte–Carlo approaches for fractional PDEs [37] offer promising numerical frameworks for systems with memory. Finally, the impact of the choice of temporal fractional derivative on solution behavior requires further analytical and numerical studies: while existing works focus on Hookean models [9, 10] as it allows for a static condensation and a reduced moment based macroscopic formulation, our primary interest lies in the physically more relevant class of FENE-type models.

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Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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