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PAPER

Non-stationarity and dissipative time crystals: spectral properties and finite-size effects

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Abstract

We discuss the emergence of non-stationarity in open quantum many-body systems. This leads us to the definition of dissipative time crystals which display experimentally observable, persistent, time-periodic oscillations induced by noisy contact with an environment. We use the Loschmidt echo and local observables to indicate the presence of a finite sized dissipative time crystal. Starting from the closed Hubbard model we then provide examples of dissipation mechanisms that yield experimentally observable quantum periodic dynamics and allow analysis of the emergence of finite sized dissipative time crystals. For a disordered Hubbard model including two-particle loss and gain we find a dark Hamiltonian driving oscillations between GHZ states in the long-time limit. Finally, we discuss how the presented examples could be experimentally realized.

1. Introduction

Non-stationary many-body systems are those which cannot be described in terms of time-independent states or probability distributions. From seasonal weather patterns, across financial markets, to the behaviour of living organisms, such behaviour is ubiquitous in the world around us. Still, we lack a detailed understanding of how non-stationary, complex dynamics emerge from the microscopic laws governing the interactions between constituent quantum particles. This mystery was already noted in the seminal essay on complexity and symmetry 'more is different' [1] by P W Anderson who wrote:

> Keeping on with the attempt to characterize types of broken symmetry which occur in living things, I find that at least one further phenomenon seems to be identifiable and either universal or remarkably common, namely, ordering (regularity or periodicity) in the time domain.

Statistical physics is arguably the most powerful physics approach to studying many-particle systems starting from microscopic quantum mechanical laws. However, even non-equilibrium statistical physics can only deal with probability distributions and quantum states that describe the system *after* it has relaxed to stationarity. Relaxation to stationarity is widely assumed but the validity of this assumption has recently attracted significant theoretical and experimental scrutiny that we now briefly summarize.

1.1. Relaxation to stationarity in many-body quantum systems

For isolated systems the eigenstate thermalization hypothesis (ETH) [2–4] states that relaxation to stationarity happens by eigenstate dephasing [5–7]. A physical quantity $\hat{O}$ at a time t after evolving under the action of a Hamiltonian H has expectation value

$$\langle \hat{O}(t)\rangle = \mathrm{Tr}[\hat{O}\rho(t)] = \sum_{n,m} \mathrm{e}^{-\mathrm{i}\omega_{nm}t} r_{nm} \langle \phi_m|\hat{O}|\phi_n\rangle, \tag{1}$$

where $|\phi_n\rangle$ is an eigenstate of H with energy E_n, and we have taken $\hbar = 1$. The coefficients r_{nm} describe the initial state of the system $\rho(0) = \sum_{nm} r_{nm}|\phi_n\rangle\langle\phi_m|$.

The coherences in equation (1) oscillate with angular frequencies $\omega_{nm} = E_n - E_m$. In a generic many-body system the frequencies ω_{nm} are expected to be dense and incommensurate without any specific structure. The coherences will thus dephase quickly by destructive interference and take the system to stationarity within several characteristic times, e.g. a few hopping times for lattice models [2]. In contrast, in real-world examples, such as those introduced above, non-stationarity persists for extremely long times compared to the characteristic atomic time-scales.

Exceptions to the behaviour predicted by the ETH in physical systems are mostly found in single-body low-temperature effects, such as solitons [8], or other low energy, single particle excitations [9, 10]. Another notable exception are quantum many-body scars [11–19]. Here persistent non-stationarity, dubbed weak ergodicity breaking, has been linked to the existence of special eigenstates of H that are almost equally spaced in energy and have high overlap with specific initial states with low-entanglement. Due to this, they exhibit persistent oscillations at one or few frequencies [11]. These scars are beginning to be understood through dynamical symmetries [16–19]. Persistent oscillations have also been observed and studied in long-range interacting models [20–22], models with confinement [23–28] and *many-body* breathing modes [29–35].

Similarly, in open quantum systems, one typically finds stationary behaviour at late times. Here, eigenstate dephasing is reduced since most modes of the system decay through interactions with the environment. Instead, system-environment interactions generally steer the many-body systems towards a unique, usually mixed, stationary state in the long-time limit. This is corroborated by random matrix theory calculations [36–40] and stationarity is implied when making the assumption of ergodicity.

Non-stationarity may arise in an open quantum system if it contains dark states [41–43] which are eigenstates of the system Hamiltonian that are decoupled from the environment. An initial coherent superposition of dark states will continuously oscillate as if the system were isolated. Identifying dark states in many-body systems is challenging with only a limited number of examples, e.g. in a Heisenberg ring with ultra-local loss [44] and driven dissipative Fermi–Hubbard models [45], known.

1.2. Origins of time-crystals

1.2.1. Closed and periodically driven systems

Ordering in the time domain was studied by Wilczek in [46] where he proposed quantum time crystals, which would be states of matter that spontaneously break time-translation symmetry, much like crystals break space translation symmetry and oscillate periodically. This original proposal was criticised on several grounds [47, 48]. Most notably, Watanabe and Oshikawa [49] formalised time crystalline behaviour as having an extensive observable, $\hat{O}$, which persistently oscillates at equilibrium with a single frequency ω. They defined the equilibrium auto-correlation function,

$$f(t)=\frac{\langle\hat{O}(t)\hat{O}\rangle_{\mathrm{eq}}}{V^2}, \tag{2}$$

where V is the volume of the system. For a time crystal, at late times $f(t)$ would be time periodic,

$$f(t)\underset{t\to\infty}{=}C\ \cos(\omega t+\theta)+D, \tag{3}$$

where C, D are finite constants and θ is some phase. The motivation for this was to probe time-translation symmetry breaking in the same way that one probes space-translation symmetry breaking in crystals: as response functions at equilibrium. They then proved that, provided H is sufficiently local, C vanishes as $V\to\infty$, so $f(t)$ in fact becomes constant at late times [49]. However, even for equal time ($t=0$) correlators at finite temperature and a wide range of equilibrium states, C vanishes in the thermodynamic limit [50]. In one-dimension, this is a consequence of the Mermin–Wanger theorem [51]. The criterion proposed in [49] could, therefore, be interpreted as probing the absence of long-range spatial order rather than the presence of time order.

A natural relaxation of the criterion (3) is to consider a physically measurable correlation function,

$$\frac{\langle\hat{O}(t)\hat{O}\rangle_{\mathrm{eq}}}{\langle\hat{O}^2\rangle_{\mathrm{eq}}}\underset{t\to\infty}{=}C\ \cos(\omega t+\theta)+D, \tag{4}$$

which is by construction normalized to 1 at $t=0$. A finite, non-zero C has been shown to be implied by the existence of a *dynamical* extensive symmetry of the form [16, 44, 50, 52, 53],

$$[H,A]=\omega A, \tag{5}$$

where A is an extensive operator with $\mathrm{Tr}(A^{\dagger}\hat{O})\neq 0$.

Dynamical symmetries can result in the time-translation symmetry breaking of observables in closed systems following a quench, providing another possible definition of time crystalline behaviour. In the presence of a dynamical symmetry we assume the system to evolve towards a time-dependent generalized Gibbs ensemble (GGE)[1] [50, 52] given by

$$\rho(t) \underset{t\to\infty}{=} Z^{-1}\ \exp\left(-\beta H + \sum_j \mu_j^Q Q_j - \mu_j^A(t)A_j - \mu_j^A(t)^* A_j^\dagger\right), \tag{6}$$

where β is the temperature, Z is a partition function, and Q_j are conserved charges with $[H, Q_j] = 0$. The generalized chemical potentials $\mu_j^A(0)$ and μ_j^Q are determined from the initial state. We have allowed multiple orthogonal[2] dynamical symmetries A_j with frequencies ω_j which determine the evolution of $\mu_j^A(t) = \mu_j^A(0)\ \exp(\mathrm{i}\omega_j t)$. Equation (6) is an extension of the well-established time-independent GGE that has been shown by both the ETH and extensive studies in integrable models to generically follow from a quantum quench in the absence of dynamical symmetries [54].

The presence of a large number of dense and incommensurate ω_j can lead to the absence of persistent oscillations and result in noisy dynamics or even dephasing to stationarity. This can be particularly important for free or many-body localized models [7, 55] which are likely to have many A_j. We also remark that if the A_j are not spatially translationally invariant this can lead to the absence of synchronization inside the system [56–58].

The quench-based time crystal definition has led to another particularly fruitful line of research, which is based on extending the notion of time translation symmetry breaking to discrete time translation symmetry breaking for Floquet systems with period T. More specifically, discrete time symmetry breaking is defined for local observables o as a subharmonic response, $\langle\psi|o(t+nT)|\psi\rangle = \langle\psi|o(t)|\psi\rangle$, for some integer $n > 1$, and $\langle\psi|o(t+t_1)|\psi\rangle \neq \langle\psi|o(t)|\psi\rangle$ for $t_1 < nT$ and for some generic enough initial state $|\psi\rangle$. This concept has been explored in both closed [59–67] and open [68–71] quantum systems. Various other extensions of the time crystal concept, including classical models [63, 72–75], introducing non-local Hamiltonians [76], time quasi-crystals [77] and time glass [55] have been proposed.

For a more in depth review of the broad and exciting field of time crystals, the authors direct the reader to the excellent review article of Khemani *et al* [55]

1.2.2. Open systems

The works discussed above have been primarily focused on time crystals in closed and periodically driven systems. Realistically, physical systems are never perfectly isolated from their environment and in general one would expect system-environment interactions to disrupt the time crystalline behaviour of the previously discussed models. In this paper we will study extensions of quench based time crystals to open quantum systems which we label dissipative time crystals. These are characterized by persistent oscillations induced through the coupling with an external noisy environment [44, 78–97].

In [44] a general set of conditions leading to such behaviour was developed. These conditions were shown to be implied by the existence of strong dynamical symmetries. They were used [85] to describe a recent experiment where a two-component Bose–Einstein condensate in a cavity, which is expected to exhibit chaotic behaviour and thermalization in the absence of dissipation, was stabilised up to a persistently oscillating phase by dissipation [83, 84].

Importantly, dissipative time crystals should be distinguished from related *boundary* time crystals [98, 99] which are usually taken to be single-body systems that exhibit persistent oscillations despite dissipation. If these single-particle systems were closed they would exhibit simple Rabi oscillations (see [53]) instead of thermalizing via the ETH.

In this paper, we will illustrate under what conditions coupling to an environment can induce time-crystalline behaviour. We will start from the Hubbard model as a genuine strongly correlated many-body system. In individual steps we will add two-body loss, two body gain and disorder to this system and explore their effect on the spectrum. Unlike in many-body localization-based time crystals, disorder will here not serve to stabilize the time crystal but rather introduce more different eigenfrequencies and increase the opportunity for eigenstate dephasing.

We will investigate the dynamics of these systems starting from a random pure initial state which we assume can be reliably created repeatedly to measure ensemble averages over different histories of interactions with the environment. This can e.g. be achieved in ultracold atom setups by using the same speckle pattern of pseudo-random on-site fields for each run of the experiment [100]. The ensuing time evolution will be probed through the dynamics of local observables and the Loschmidt echo to detect

[1] This should be understood only as the effective state from which to calculate expectation values of local observables, as the actual state of the system remains pure.

[2] In the Hilbert–Schmidt sense, $\langle A, B\rangle = \mathrm{Tr}(A^\dagger B)$.

dissipative time crystalline behaviour. The Loschmidt echo is widely used in many fields, e.g. quantum chaos [101] and more recently dynamical phase transitions [102]. Its periodicity is one of the most stringent definitions for probing a time crystal because even for small systems the Poincare recurrence time diverges very fast with system size. As a result, so too does the period of the Loschmidt echo in the absence of criticality [103, 104]. This should not be confused with revivals [105] for which a value of the Loschmidt echo close to the initial value 1 is restored aperiodically.

We will find that the noisy environment can indeed induce time crystalline behaviour despite the closed Hubbard model dephasing very quickly according to the ETH. We will show that the existence of a dissipative time crystal is closely related to dynamical symmetries but that they are neither necessary nor sufficient. For the case of a disordered Hubbard model with two-body loss, we furthermore derive a dark Hamiltonian [44] describing the long-time coherent dynamics of the system. This non-local effective Hamiltonian is engineered from local physical dissipation and a local physical Hamiltonian rather than assumed outright (see [76]).

The paper is organised as follows. In sections 2–4 we define dissipative time crystals. More precisely, we outline the conditions required for them to arise in finite systems and demonstrate that these conditions are highly compatible with the existence of strong dynamical symmetries. In section 5 we study several non-stationarity probes for small 1D Hubbard models undergoing various types of dissipation in order to illustrate the conditions for which finite sized dissipative time crystals emerge. In section 6 we discuss the thermodynamic limit of our finite sized dissipative time crystals. Finally, we conclude in section 7.

2. Definitions of dissipative time crystals

We follow [44] and define a dissipative time crystal as follows: a many-body quantum system coupled to a noise inducing environment which exhibits non-trivial periodic motion in some observable at late time for generic initial conditions. This naturally extends the notion of a time crystal as a closed quantum system which continuously oscillates periodically in time to the paradigm where the system can be disrupted by its environment.

The key contrast with previous studies of time crystals is that here we are considering a system in the presence of noise, without any time-dependent driving. External noise/decoherence is generally understood as a mechanism for destroying quantum behaviour and inducing relaxation to stationarity, but here we explore the remarkable case where the external noise in fact induces persistent periodic motion within our system. This is of particular interest for uncovering experimentally realisable time crystals since in all practical settings the quantum systems we investigate are subject to some level of external noise from an environment that we cannot completely control.

In this work, we will primarily be interested in dissipative time crystals in finite systems. These are of particular interest since they can be experimentally realised in cold atom experiments, as we will discuss later. We consider initialising a finite-sized, many-body system in a generic state and then require that after some finite transient time period during which parts of the system decay, the observable in question displays persistent, clearly measurable oscillations which do not decay further. We remark that these systems do not adhere to the view of [98] that a time crystal can appear only in the thermodynamic limit, however we will show that they can be easily modified so that the time translational symmetry breaking occurs only in this limit. Furthermore, as we will show, they demonstrate the remarkable phenomenon where external noise can stabilise the dynamics of an otherwise noisy system.

We will discuss the subtleties surrounding definitions of time crystals in open systems in more detail in section 6. Until then, unless stated otherwise, all discussions will be for strictly finite systems.

3. Spectral requirements for dissipative time crystals

Under the assumptions that we have a sufficient handle on insulating our system from the environment so that the two are only weakly coupled and their interactions are Markovian, the natural framework in which to study these systems is the Lindblad master equation [106–108],

$$\dot{\rho}(t) = \mathcal{L}\rho = -\mathrm{i}[H, \rho] + \sum_{\mu} \left(2L_\mu \rho L_\mu^\dagger - L_\mu^\dagger L_\mu \rho - \rho L_\mu^\dagger L_\mu\right). \tag{7}$$

This governs the evolution of a system with Hamiltonian H whose coupling to the environment is described by the set of Lindblad jump operators $\{L_\mu\}$. These operators model the influence of the environment's noise on our system. Such operators may for example be particle creation/annihilation operators to describe random particle gain/loss, or number operators to describe dephasing [108]. Through careful choice of which jump operators to include we can precisely characterise the effects of the system-environment interactions without a detailed description of the exact underlying mechanism.

The evolution of a system observable, $\hat{O}$, can then be described in terms of the eigensystem of the superoperator $\mathcal{L}$. This is defined as the set $\{\rho_k, \sigma_k, \lambda_k\}$, where λ_k are the eigenvalues of $\mathcal{L}$ and ρ_k, σ_k are the corresponding right and left eigenstates respectively. The eigensystem obeys the relations,

$$\mathcal{L}[\rho_k] = \lambda_k \rho_k, \qquad \mathcal{L}^\dagger[\sigma_k] = \lambda_k \sigma_k, \qquad \mathrm{Tr}(\sigma_k^\dagger \rho_{k'}) = \delta_{k,k'}. \tag{8}$$

Note that for physical $\mathcal{L}$ the eigenvalues, λ_k, can only lie in the left half of the complex plane with $\mathrm{Re}(\lambda_k) \leqslant 0$.

The evolution of $\hat{O}$ given an initial state $\rho(0)$ can then be written as

$$\langle \hat{O} \rangle(t) = \mathrm{Tr}(\hat{O}\rho(t)) = \sum_k O_k\, \mathrm{e}^{\lambda_k t}, \tag{9}$$

where $O_k = \mathrm{Tr}(\sigma_k^\dagger \rho(0))\, \mathrm{Tr}(\hat{O}\rho_k)$. We see immediately that in order to exhibit long time non-stationarity, there must exist some non-zero, purely imaginary eigenvalues, $\lambda_k = \mathrm{i}\omega_k, \omega_k \neq 0$. In order for this non-stationarity to be periodic, the existence of these so-called quantum limit cycles [90, 109, 110] with purely imaginary eigenvalues is not sufficient, and instead we require that they be *commensurable*, that is $\frac{\omega_k}{\omega_l} \in \mathbb{Q}$ for all k, l. While this condition is theoretically sufficient for idealised dissipative time crystal behaviour, we explain later how in practice this must be modified slightly in order to yield a measurable dissipative time crystal.

Further, we can see that the transient time which we must wait before such time crystalline behaviour is visible is determined by the Liouvillian gap, R, defined as

$$R = \min\{\|\mathrm{Re}(\lambda_k)\| : \mathrm{Re}(\lambda_k) < 0\}. \tag{10}$$

This characterises the time scales of the slowest decaying parts of the system.

As a result, it is clear that the study of dissipative time crystals in this regime is intrinsically linked to the study of the eigenvalues, especially the purely imaginary eigenvalues, of the Liouvillian superoperator $\mathcal{L}$. We remark that the existence of commensurable, purely imaginary eigenvalues is highly non-trivial for Liouvillians describing many-body systems due to their exponential size. Consequently developing conditions under which these eigenvalues can exist in realisable Liouvillians is central to understanding dissipative time crystals.

4. Dark states and strong dynamical symmetries

As we have remarked above, in order to study dissipative time crystal we must understand the spectral properties of $\mathcal{L}$. However, for many-body quantum systems, $\mathcal{L}$ in general becomes exponentially complex and out of reach for current computational methods. To analytically find even all purely imaginary eigenvalues remains, in general, an open problem. However in the examples we have studied, we found them to arise only from two key mechanisms: dark states and strong dynamical symmetries.

Dark states [111, 112], defined by,

$$H|\phi_n\rangle = E_n|\phi_n\rangle, \qquad L_\mu|\phi_n\rangle = 0 \quad \forall\ \mu, \tag{11}$$

are eigenstates of the closed system Hamiltonian which are invisible to the external noise. They span a decoherence free subspace [111] which is left unaffected by the dissipation. The resulting dark state coherences, of the form $\rho_{n,m} = |\phi_n\rangle\langle\phi_m|$, are then trivially eigenstates of $\mathcal{L}$ with purely imaginary eigenvalues $\lambda_{n,m} = -\mathrm{i}(E_n - E_m)$. We thus find that coherences between dark states with differing energies, $E_n \neq E_m$, have the non-zero purely imaginary eigenvalues which are necessary for a dissipative time crystal. In a way, they may be understood as quantum scars [11, 12, 16–19], but now selected through the dissipation, instead of being special eigenstates.

In general, for large systems, the computation of these dark states and their energies remains a computationally unfeasible task, unless the system Hamiltonian has some integrability structure from which to find the energies, E_n, and eigenstates, $|\phi_n\rangle$. Additionally, these must be suitably compatible with the jump operators L_μ such that the calculation of $L_\mu|\phi_n\rangle$ possible. We therefore look to more powerful symmetry arguments in order to deal with larger systems.

A strong dynamical symmetry, A, of the system with symmetry-frequency ω is defined by

$$[H, A] = \omega A, \qquad [L_\mu, A] = [L_\mu^\dagger, A] = 0, \quad \omega \in \mathbb{R}. \tag{12}$$

While this trivially implies that A is a strong symmetry of H with $[H, A] = \omega A$ [41, 42, 113, 114], note that the converse does not hold. A theory of these symmetries has been studied and developed in the past [44]. It has been shown that if a system has a strong dynamical symmetry A and ρ_∞ is a non-equilibrium steady state with $\mathcal{L}[\rho_\infty] = 0$ (NESS), then mixed coherences of the form,

$$\rho_{n,m} = (A^\dagger)^n \rho_\infty A^m, \tag{13}$$

are eigenstates of $\mathcal{L}$ with purely imaginary eigenvalues,

$$\lambda_{n,m} = \mathrm{i}(n - m)\omega. \tag{14}$$

We also remark that it is possible for the system to have multiple independent symmetries A_j with different symmetry-frequencies ω_j. In this case we find several 'strings' of imaginary eigenvalues, $\lambda^j_{n,m} = \mathrm{i}(n - m)\omega_j$, which are integer multiples of the different ω_j.

It is important to distinguish these mixed coherences from the decoherence free subspace of dark states discussed above. The dark states are completely decoupled from the dissipation through $L_\mu|\phi_n\rangle = 0$. Contrastingly, for mixed coherences we generally have $L_\mu \rho_{n,m} L_\mu^\dagger \neq 0$ which results in the time evolution being influenced by the dissipation.

While these two mechanisms allow us to show the existence of purely imaginary eigenvalues, unfortunately neither mechanism is in fact necessary or sufficient for time crystalline behaviour. In the case of having both a strong dynamical symmetry and purely imaginary eigenvalues for other reasons (e.g. dark states) the desired eigenvalue commensurability is generically broken. Alternatively, having multiple independent *strong* dynamical symmetries can also break the desired spectral structure. We also demonstrate how commensurability can arise in the absence of any strong dynamical symmetries.

In the next section, we introduce examples that progressively reduce the number of eigenfrequencies in the asymptotic dynamics leading to increasingly more periodic behaviour. The least periodic one will be the closed model with an exponentially large number of eigenfrequencies, and we will reach a case with only a single pair of eigenfrequencies oscillating between two GHZ states. The focus will be on small systems to examine the finite-size effects which demonstrate the key points of the above discussion but we expect some results to be generalisable to larger systems. All examples are for physical local Hamiltonians and dissipation which we believe could in principle be experimentally realisable.

5. Dissipative time crystals in the 1D Hubbard model

We now demonstrate the emergence of dissipative time crystals from the 1D Hubbard model by coupling it to an environment with two body loss and gain processes. These models are naturally realizable in cold atom settings [115]. We find that in fact, due to the exponentially large number of eigenfrequencies, we can illustrate the basic principles while looking at only very small systems.

5.1. Methods

The setup considered will be as follows: we initialise the system in some random pure state and then allow it to evolve while calculating the expectation values of two non-stationarity probes. Such a setup could in principle be realized in ultra-cold atoms by fixing once a choice of some very strong pseudo-random on-site fields, in addition to those already present in the model being studied, and then cooling the system so that it is (at least very nearly) in the ground state. This pure ground state then becomes our random pure initial state when we remove the additional fields and quench the system. Further, when considering a system coupled weakly to the environment, by using the same choice of pseudo-random on-site fields we can repeatedly construct the same initial state in order to construct ensemble averages.

The two non-stationarity probes which we shall consider are the transverse fermion spin on each site and the Loschmidt echo of the initial state. As per the definition in section 2, for the system to be a dissipative time crystal with respect to one of these probes, the probe's signal must persistently oscillate at later times.

The Loschmidt echo, defined as

$$\mathcal{E}_\mathrm{I} = \mathrm{Tr}(\rho(0)^\dagger \rho(t)), \tag{15}$$

is widely used in many fields, e.g. quantum chaos [101] and more recently dynamical phase transitions [102]. For an initially pure state, $\rho(0) = |\psi\rangle\langle\psi|$, it measures the fidelity between the current state of the system and the initial state. The periodicity of this is one of the most stringent definitions and probes of time crystal-like behaviour in our setup because it is evolution will typically be influenced by all the eigenfrequencies of $\mathcal{L}$. Since there are generally a large number of eigenfrequencies in even small many-body systems, the Poincare recurrence time, and thus the period of the Loschmidt echo, can be effectively infinite for experimental purposes. Note, however, that the Loschmidt echo is not a practical probe in larger systems as the amplitude of the late time signal decays exponentially with system size. Nevertheless, in the examples we present below it excellently demonstrates the consequences of our previous discussion about the eigenvalue structure required for dissipative time crystals.

We use the discrete Fourier transform (DFT), denoted $F[f(t)]$, as a natural tool to detect oscillatory behaviour in the observables. All DFT spectra have been calculated using Blackman windowing to reduce spectral leakage caused by only measuring over a finite time period which does not necessarily contain an integer number of oscillations [116]. Further, we have computed all DFT's at late times after the transient contributions have vanished.

5.2. Closed Hubbard model

In order to show that the dissipative time crystal behaviour in later examples is not a trivial consequence of the dynamical properties and strong symmetry of the closed system, we first discuss the closed Hubbard model for spin 1/2 fermions on a chain of length L. The system is described by the Hamiltonian

$$H = -\sum_{i=1}^{L-1} \sum_{s\in\{\uparrow,\downarrow\}} (c_{i,s}^{\dagger} c_{i+1,s} + \text{h.c}) + \sum_{j=1}^{L} U_j n_{j,\uparrow} n_{j,\downarrow} + \epsilon_j n_j + \frac{B}{2}(n_{j,\uparrow} - n_{j,\downarrow}) \tag{16}$$

where $c_{j,s}$ annihilates a fermion of spin s on site j and obeys the canonical fermionic anti-commutation relations. The number operators are given by

$$n_{j,s} = c_{j,s}^{\dagger} c_{j,s}, \qquad n_j = n_{j,\uparrow} + n_{j,\downarrow}. \tag{17}$$

The Hamiltonian includes inhomogeneous onsite interactions U_j, an inhomogeneous spin independent potential ϵ_j and a uniform magnetic field B aligned with the z-direction of the fermion spins. In all our numerics we chose the $U_j, \epsilon_j \in [0, 3]$ to be a fixed realisation of i.i.d. uniform random variables. The inhomogeneous interaction and potential break the usual $SU(2)$ η-symmetry of the system [117] so that we are left with only the spin $SU(2)$ symmetry,

$$[H, S^z] = 0, \;\; [H, S^{\pm}] = \pm B S^{\pm}, \tag{18}$$

where

$$S^z = \sum_j \left[n_{j,\uparrow} - n_{j,\downarrow}\right], \qquad S^+ = \sum_j c_{j,\uparrow}^{\dagger} c_{j,\downarrow}, \qquad S^+ = \sum_j c_{j,\downarrow}^{\dagger} c_{j,\uparrow}. \tag{19}$$

The transverse fermion spin is then given by,

$$S_j^x = \frac{1}{2}(c_{j,\uparrow}^{\dagger} c_{j,\downarrow} + c_{j,\downarrow}^{\dagger} c_{j,\uparrow}). \tag{20}$$

While the spectrum of H (and thus $\mathcal{L}$) can be found analytically for constant U_j, ϵ_j [117], here we must resort to numerical calculations. We plot the spectrum of $\mathcal{L}$ in figure 1(a) for a 3 site system. This system size is chosen so that the spectra may be directly compared across the different examples and in the open cases any larger systems are computationally out of reach. However it is reasonable to expect that the spectra of larger systems are qualitatively the same. Importantly for the discussion of periodicity we find that the purely imaginary eigenvalues are *effectively* incommensurable. By this we mean that while the numerically calculated eigenvalues are commensurable, as a result of finite precision computation, the time period over which any oscillations would occur is far longer than any reasonable time scale of an experiment. The immediate consequence of this is that unless we carefully choose our initial state and observable such that the expansion in equation (9) contains only a commensurable subset of frequencies, the signal will never be periodic on experimental timescales. In general, without such specific choices, the many incommensurate frequencies will interfere producing either noise in smaller systems or decay to stationarity by eigenstate dephasing [5] in larger ones.

We can see the manifestation of this by looking at plots of the transverse spin and Loschmidt echo in figures 1(b) and (c) where a 4 site system has been evolved from one instance of a random initial state $|\phi\rangle \propto \sum_n u_n |\phi_n\rangle$. The u_n are i.i.d. uniform random numbers on $[0, 1]$ and $|\phi_n\rangle$ are the eigenstates of H. The calculations were repeated for many different random initial states to check that indeed the resulting behaviour is generic and not a consequence of some particular initial conditions. For both probes the signal is very noisy and far from periodic. The noise is suppressed slightly in the transverse spin signal since some of the eigenstates have no overlap with S_k^x, thus removing the corresponding frequencies in (9). In addition,

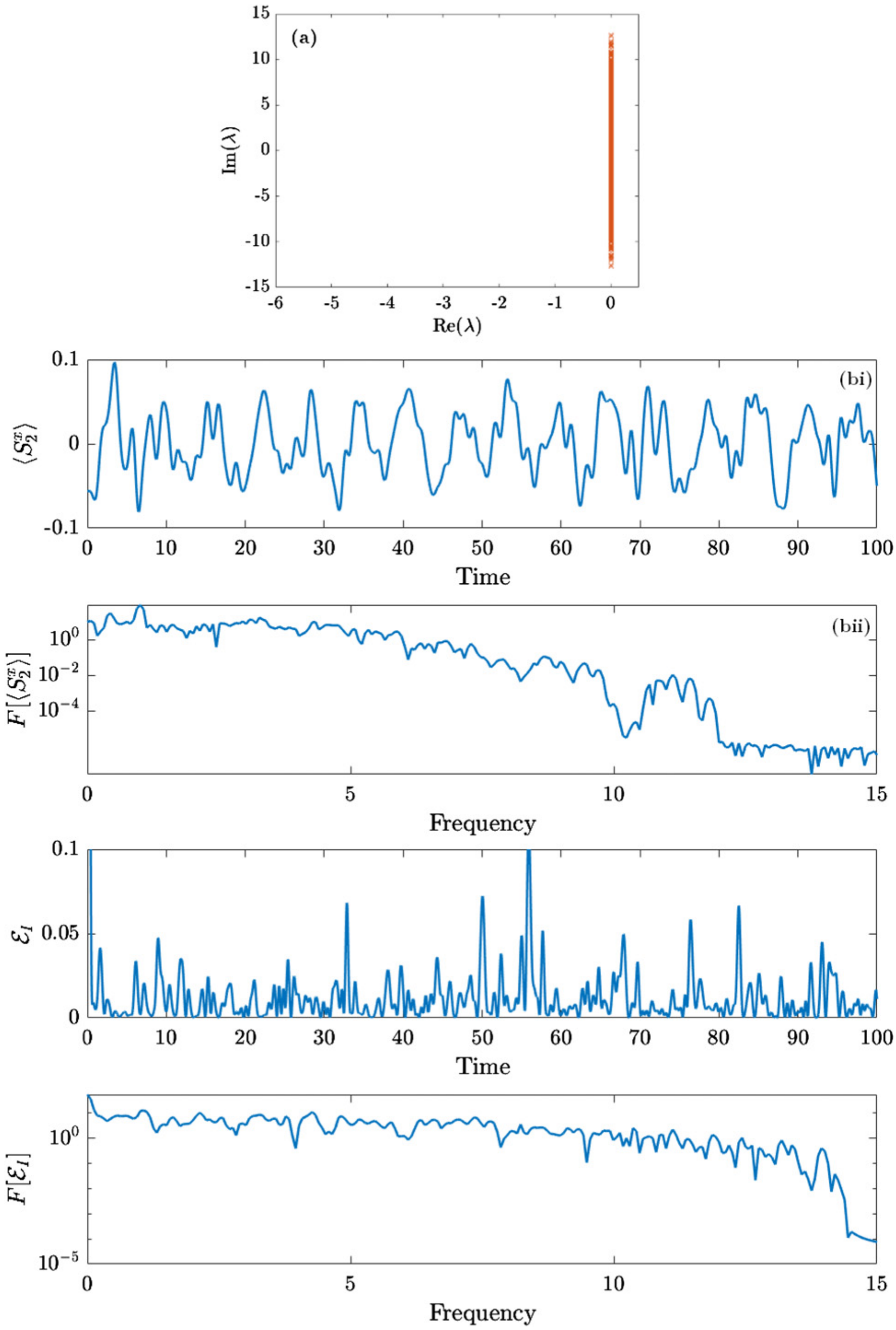

Figure 1. Closed system. (a) The spectrum of a 3 site system, characteristic of larger systems. (b) and (c) The evolution and DFT of the transverse spin on the second site and Loschmidt echo respectively for a 4 site system which is initialised in a random pure state.

these observations are corroborated by the DFT spectra for the signal from both probes, which are roughly flat for the majority of the frequencies, characteristic of noise [118].

We can thus conclude that in the absence of interactions with an environment, the 1D Hubbard model does not display periodic behaviour for generic initial conditions on experimental timescales even for very small systems. This can be understood by eigenstate dephasing and demonstrates that the clearly periodic behaviour we find in the open models is not a trivial consequence of the closed system's dynamics.

5.3. Two-body loss

The first dissipative case we consider is when the system leaks eta-pairs, each composed of a spin-up and a spin-down fermion, to the environment from every site at a possibly inhomogeneous but strictly positive rate, $\gamma_j > 0$. This is described by introducing the set of jump operators $L_j = \gamma_j \eta_j^- = \gamma_j c_{j,\downarrow} c_{j,\uparrow}$ in equation (7) while keeping the same Hamiltonian as in equation (16).

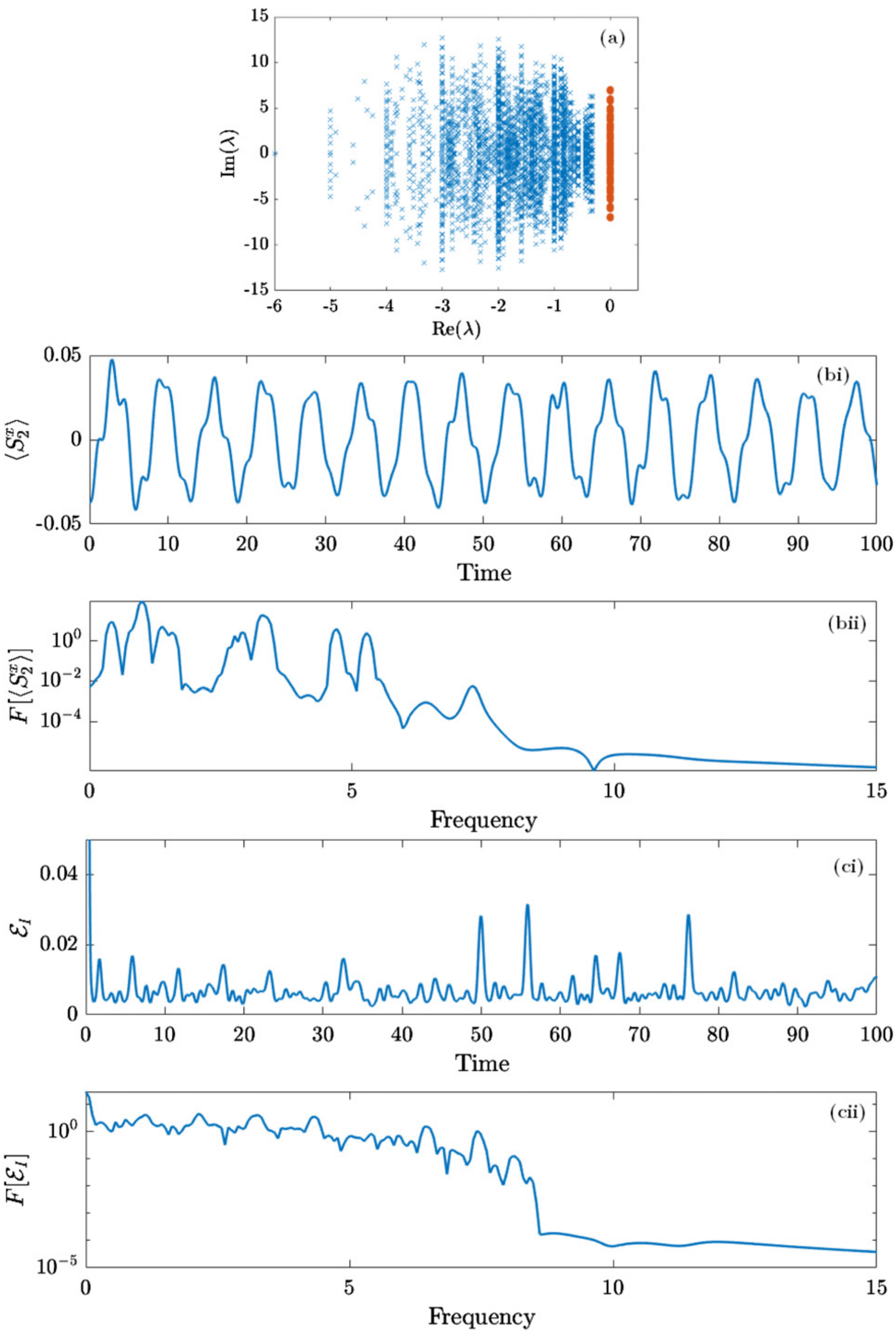

Figure 2. Open system with pure loss. (a) The Spectrum of a 3 site system, characteristic of larger systems which are computationally out of reach. The purely imaginary eigenvalue have been highlighted in red. (b) and (c) The evolution and DFT of the transverse spin on the second site and Loschmidt echo respectively for a 4 site system which is initialised in a random pure state.

In this system we find numerically that the only purely imaginary eigenvalues correspond to coherences between dark states which are left unaffected by the loss. These dark states have no overlap with any eta pairs. We expect that one could prove the non-existence of other purely imaginary eigenvalues as a consequence of supplementary theorem 2 of [119], but this is beyond the scope of the current work. There exists a strong dynamical symmetry, generated by S^+ with $[H, S^\pm] = \pm BS^\pm$. However, we do not find any mixed coherences here as all NESSs are pure dark states and so the states of the form $(S^+)^n \rho_\infty (S^-)^m$ are coherences between dark states. All states with purely imaginary eigenvalues are independent of the γ_j, and so the only relevance of these coupling strengths for our discussion is to determine the time period of the transient dynamics. Thus in what follows we will take $\gamma_j = 1$ for all sites.

In figure 2(a) we plot the spectrum for a 3 site system, but again this is expected to be representative of larger systems. Comparing with the spectrum of the closed system, we see that there are significantly fewer

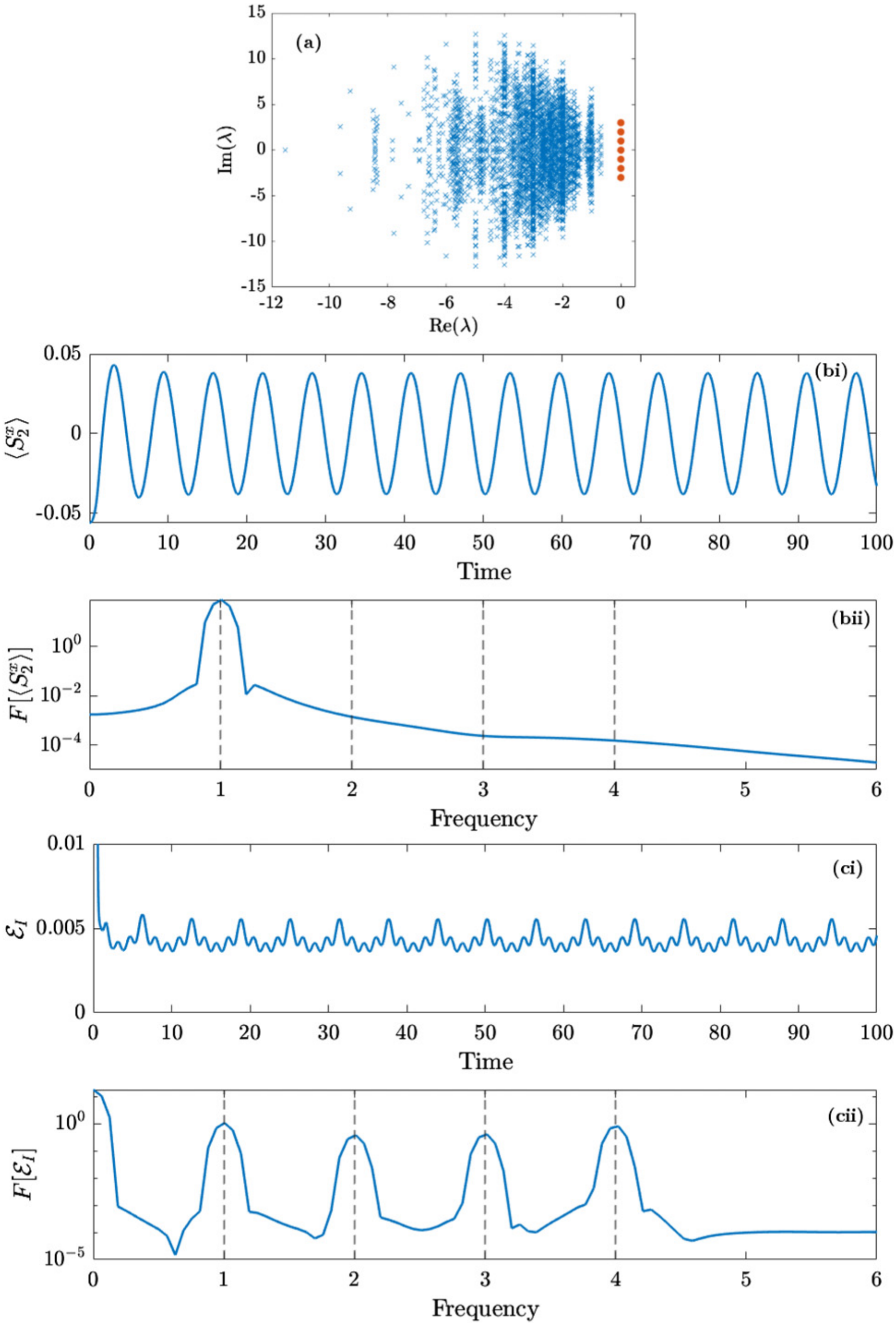

Figure 3. Open system with both loss and gain. (a) The Spectrum of a 3 site system, characteristic of larger systems which are computationally out of reach. The purely imaginary eigenvalues have been highlighted in red. (b) and (c) The evolution and DFT of the transverse spin on the second site and Loschmidt echo respectively for a 4 site system which is initialised in a random pure state. We highlight the non-zero imaginary eigenvalues by dashed vertical lines on the spectra.

imaginary eigenvalues. Again, however, we find that the remaining imaginary eigenvalues are effectively incommensurate.

We examine the time dynamics of our two stationarity probes in figures 2(b) and (c) for a 4 site system in a random initial state. Starting with the echo we see that the signal is extremely noisy, an observation which is corroborated by the DFT being roughly flat for much of the spectrum.

Looking next at the transverse spin signal we see that, while clearly not periodic, the signal is noticeably less noisy than for the closed case. As before this is a result of many coherences between dark states not having any overlap with S_k^x and thus the series in equation (9) having significantly fewer terms. This is clear from the DFT spectrum where we can now see the emergence of roughly 8–10 pronounced peaks at effectively incommensurable frequencies. While this is not strictly time crystalline behaviour, it can reasonably be called quasi-time crystalline behaviour, in analogy to quasi-periodicity created by a small number of incommensurable frequencies [77, 120].

Through exploring this example, we have demonstrated that in the presence of only two body losses, our 1D Hubbard model does not exhibit experimentally observable dissipative time crystalline behaviour owing again to the effectively incommensurable eigenfrequencies. This shows how the existence of a strong dynamical symmetry is not sufficient for such behaviour as this does not guarantee commensurable purely imaginary eigenvalues. We have however found evidence of quasi-time crystalline behaviour, at least with respect to the transverse spin, S_k^x. This weaker condition requires only that there are few frequencies present in the evolution, not that they are commensurable. We do not explore quasi-time crystal behaviour any further here, however we envisage this to be an interesting area for further study.

5.4. Two-body loss and gain

To finally uncover truly dissipative time crystalline behaviour, we now additionally introduce two body gain terms into the Liouvillian (7), as $L_j = \Gamma_j \eta_j^+ = \Gamma_j c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger$, again for $\Gamma_j > 0$. This now represents a system where eta-pairs are both gained and lost on each site at various rates. Importantly these additional jump operators do not break the strong dynamical S^+ symmetry.

With both two body loss and gain, we can prove that the space of dark states is restricted such that they are exactly the states

$$|\phi_n\rangle = (S^+)^n |\downarrow\downarrow \ldots \downarrow\rangle. \tag{21}$$

Coherences between these states, $|\phi_n\rangle\langle\phi_m|$, are eigenstates of $\mathcal{L}$ with purely imaginary eigenvalues $\lambda_{n,m} = -\mathrm{i}(n-m)B$. We see that these are in fact generated by the strong dynamical symmetry since they can all be written as

$$|\phi_n\rangle\langle\phi_m| = (S^+)^n \rho_{\mathrm{down}} (S^-)^m, \tag{22}$$

where $\rho_{\mathrm{down}} = |\downarrow\downarrow \ldots \downarrow\rangle\langle\downarrow\downarrow \ldots \downarrow|$ is a NESS of $\mathcal{L}$. We additionally find that this system also has mixed NESSs which are not trivial linear combinations of the $|\phi_n\rangle\langle\phi_n|$. Consequently the strong dynamical symmetry also generates true mixed coherences which are distinct from the space of dark states but also have eigenvalues $\lambda_{n,m} = -\mathrm{i}(n-m)B$. Numerically, it was found that there are no further states with purely imaginary eigenvalues, and so for this system *all* non-decaying eigenstates are generated by the strong dynamical S^+ symmetry.

As a consequence of the strong dynamical S^+ symmetry generating all the purely imaginary eigenvalues, they are all integer multiples of B and thus easily seen to be commensurable. We show this in figure 3(a) for $\gamma_j = \Gamma_j = 1, \forall j$. This is an example of the eigenvalue structure that we require for time crystalline behaviour.

Indeed when we now look at the evolution of our probes (figures 3(b) and (c)) we see that both the transverse spin and Loschmidt echo rapidly decay into persistent oscillations. For the echo, the DTF spectrum shows that all purely imaginary eigenvalues are present whereas for the transverse spin only the eigenstates with imaginary eigenvalue $\pm\mathrm{i}B$ contribute.

Here we have found the emergence of a dissipative time crystal. Critically the purely imaginary eigenvalues have the necessary structure of commensurability that is required for periodic behaviour. We have further seen that this structure has been provided by the strong dynamical symmetry generating *all* non-decaying eigenstates.

We also note that, counterintuitively, as we can see in figure 4, each individual quantum trajectory [121] for the same initial state (obtained through stochastic unravelling of the master equations) is different, even at very late times after transient behaviour has vanished. This essentially embodies the stochastic nature of the quantum bath that acts on the system (see with [70]). This may also be understood as a non-stationary quantum stochastic process [122].

5.5. Inhomogeneous magnetic field

Finally, we take the system above with both two-body loss and gain but introduce a weakly inhomogeneous magnetic field, B_j, in the Hamiltonian (16). This breaks the strong dynamical S^+ symmetry in addition to the structure of the dark states. As a result, the two remaining dark states are the all-spin-up and all-spin-down states and further we find that coherences between these two states are the only eigenstates of $\mathcal{L}$ with non-zero, purely imaginary eigenvalues, given by $\pm\mathrm{i}\sum_j B_j$. At late times the system therefore can only oscillate at a single frequency. In figure 5(a) we also see that there are several eigenvalues with very small real parts, corresponding to states which would have purely imaginary eigenvalues if the disorder in B_j were removed, resulting in a very small Liouvillian gap.

The consequence of the very small Liouvillian gap is immediately seen in the signals of our two probes. It is easy to see that the coherences between all-spin-up and all-spin-down states have no overlap with S_k^x for

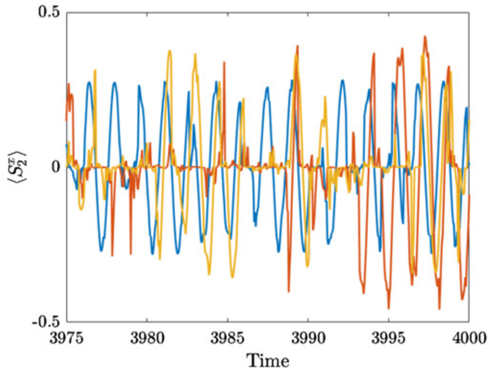

Figure 4. Three quantum trajectories of the same initial pure state for the system in section 5.4 with both two body loss and gain. In a 4 site system, we calculate the transverse spin on site 2 at very late times after transient dynamics have vanished to demonstrate that each trajectory is different.

any k and so after the transient period we have $S_k^x = \text{const}$. Thus comparing figure 5(b) with the previous systems we find that here it takes at least an order of magnitude longer for the transient behaviour to die away. We observe the same behaviour in the echo (figure 5(c)), and the inset demonstrates that at sufficiently late times, once the transient modes have become almost insignificant, the system reaches persistent oscillations at the single frequency we expect.

After the transient period, the dynamics of the system on top of the steady states can be described by the effective, non-local Hamiltonian,

$$H_{\text{eff}} = \frac{\mathcal{B}}{2}\left(|\uparrow\dots\uparrow\rangle\langle\uparrow\dots\uparrow| - |\downarrow\dots\downarrow\rangle\langle\downarrow\dots\downarrow|\right) \tag{23}$$

where $\mathcal{B} = \sum_j B_j$. This drives coherent oscillations between two GHZ states,

$$|\text{GHZ}\rangle_\pm = \frac{|\uparrow\dots\uparrow\rangle \pm |\downarrow\dots\downarrow\rangle}{\sqrt{2}}. \tag{24}$$

Note that here we have engineered an effective non-local dark Hamiltonian [44] through a local and physical Hamiltonian in the presence of noisy dissipation, rather than assuming a closed non-local Hamiltonian with unclear stability properties [76, 123, 124]. However, the amplitude of the resulting dynamics depends on the overlap between the initial states and the GHZ states, which for arbitrary initial states will decrease exponentially with system size.

We see from this model dissipative time crystalline behaviour in the absence of any strong dynamical symmetries and thus conclude that such symmetries are indeed neither necessary nor sufficient.

5.6. Discussion

These examples demonstrate how dissipative time crystalline behaviour can be induced in a finite-sized system, starting from the 1D Hubbard model through noisy contact with an external environment. They also show how purely imaginary eigenvalues of $\mathcal{L}$ are not alone sufficient for such behaviour to emerge. Instead we see that these eigenvalues must at least be commensurable.

In fact, commensurability alone is not sufficient spectral structure in general. For a robust dissipative time crystal in a practical setting we desire persistent oscillations that are *clearly* measurable. We discussed in section 5.2 that as a result of finite precision all frequencies in both numerical calculations and experimental measurements will be commensurable. However, we may find the corresponding time periods are significantly longer than any experimental time scales and so the motion is effectively aperiodic. We thus introduce the notion of *effective* incommensurability as commensurable frequencies with a resultant time period longer than the experimental time scale. Another way in which commensurable frequencies can become effectively incommensurable is if they are too dense with respect to the inverse of the experimental timescale since then the time period can grow exponentially in the number of frequencies. In such a case, the spectral peaks of the signals DFT could also become merged and indistinguishable, rendering the commensurability of the frequencies and hence the presence of persistent periodic motion possibly undetectable. While the examples studied numerically here do not exhibit such behaviour, since quantum many-body systems have exponentially many eigenvalues it is in principal feasible for this to occur and thus should be considered in future works.

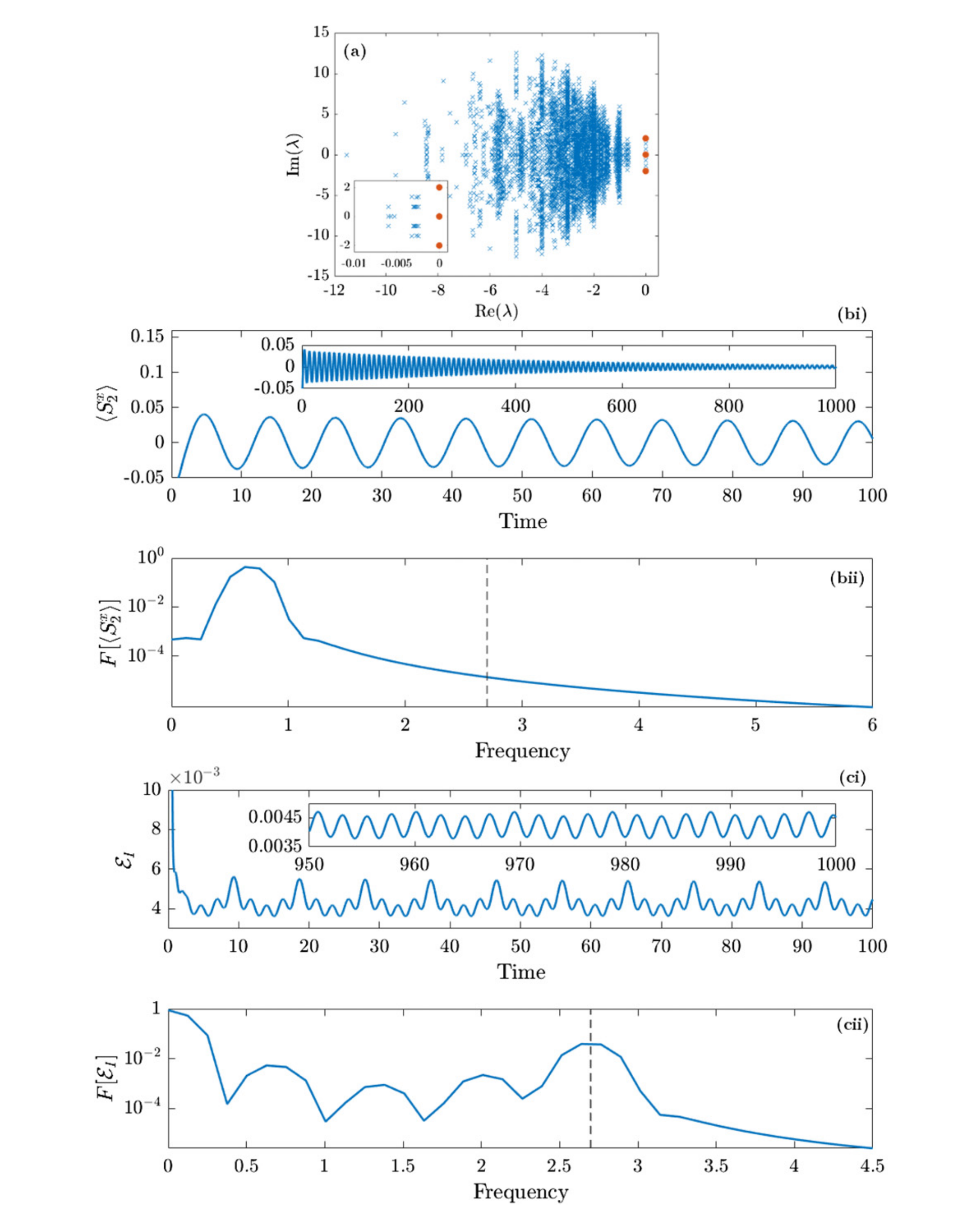

Figure 5. Open system with disordered magnetic field with both loss and gain. (a) The Spectrum of a 3 site system, characteristic of larger systems which are computationally out of reach. The purely imaginary eigenvalues have been highlighted in red. The inset focuses on the eigenvalues with very small but non-zero real part. (b) and (c) The evolution and DFT of the transverse spin on the second site and Loschmidt echo respectively for a 4 site system which is initialised in a random pure state. We highlight the non-zero purely imaginary eigenvalue $\mathcal{B}$ with a dashed vertical line on the spectra. In this case we have calculated both DFTs for the range $950 < t < 1000$ as shown in the inset of (ci).

This condition of nowhere-dense, commensurable, purely imaginary eigenvalues which are not effectively incommensurable is in general highly non-trivial for Liouvillians of many-body systems. Nevertheless it can be guaranteed by the existence of a single strong dynamical symmetry [44] which generates all continuously oscillating modes. Further, we can easily see that the imaginary eigenvalues will not be closely spaced provided the symmetry-frequency is not too small with respect to the inverse of the experimental timescale. In experiment, this should be avoidable as the symmetry frequency can generally be tuned by experimental parameters, but it remains an important consideration nonetheless. If instead there are purely imaginary eigenvalues beyond those guaranteed by the strong dynamical symmetry, e.g. coming from dark states, we see that having a strong dynamical symmetry is not sufficient for time crystalline behaviour. This is illustrated by the example having two-body loss in section 5.3. We can further envisage the existence of two distinct strong dynamical symmetries with effectively incommensurate frequencies and easily see that again such a system would not be periodic for generic initial conditions and observables on experimental timescales. In contrast, the example in section 5.5 with an inhomogeneous magnetic field demonstrates that strong dynamical symmetries are not necessary for dissipative time crystals.

While these examples have been focused on finite-sized systems, the qualitative properties can be expected to hold more generally in certain larger systems. More specifically, the persistent oscillations in an observable $\hat{O}$ are a result of the expansion

$$\langle \hat{O} \rangle(t) = \sum_k O_k \, \mathrm{e}^{-\mathrm{i}\omega_k t}, \quad O_k = \mathrm{Tr}(\sigma_k^\dagger \rho(0))\, \mathrm{Tr}(\hat{O}\rho_k), \tag{25}$$

containing only commensurable frequencies ω_k. For the Loschmidt echo in our examples, we find that the coefficients O_k decay exponentially with system size. However, by conservation of the total spin, $S^2 = (S^x)^2 + (S^y)^2 + (S^z)^2$ we see that the same cannot occur for S_k^x and so we expect that these persistent oscillations would be present in larger systems.

6. From finite to infinite dissipative time crystals

When traditionally discussing time crystals in closed systems with unitary dynamics, the time translational symmetry breaking must occur in the thermodynamic limit in order to exclude the trivial oscillations of finite quantum systems. As a consequence of this, it is sometimes felt that even in the open case the spontaneous time translational symmetry breaking should occur *only* in the thermodynamic limit [98]. This means that in a finite system the symmetry breaking will only be transient and meta-stable. This requirement is rather restrictive since it excludes many interesting situations where non-trivial time translational symmetry breaking can occur in open systems. In particular it excludes the highly non-trivial finite size dissipative time crystals discussed above, where the noisy dynamics of a many-body system can be remarkably stabilised by noisy interactions with an external bath. It also excludes many experimental platforms, such as ultracold atoms, where only finite systems can be studied. It is for these reasons that we use the term dissipative time crystal in its most general sense of *any* dissipation induced non-trivial time translational symmetry breaking.

However, one can modify our example in section 5.4 with both loss and gain so that the time translational symmetry breaking occurs strictly in the limit of infinite system size. To do so we simply add an additional local Lindblad operator which breaks the strong dynamical symmetry, say $L = \mu n_1$, to the Liouvillian. For any finite system size there are now no non-zero purely imaginary eigenvalues due to lack of a strong dynamical symmetry. However, as a consequence of finite velocity of propagation [125] the effect of the ultra-local term becomes irrelevant in the bulk of the thermodynamically large system (see also [126]). Therefore, only the thermodynamically large system has purely imaginary eigenvalues and exhibits time-translational symmetry breaking. More generally, this argument applies to any finite size dissipative time crystal where the purely imaginary eigenvalues are the consequence of a strong dynamical symmetry.

This provides additional interest for the dissipative time crystals which we have discussed in detail. Not only are they remarkable in their own right as finite system which exhibit intriguing, experimentally realisable physics, they can also be easily extended to the more restrictive notion of a time crystal as a thermodynamic phenomenon. We propose that this motivates further work to classify and define the plethora of interesting and contrasting non-trivial time translational symmetry breaking phenomena which can occur in these types of open quantum systems and can be encompassed by the general term dissipative time crystal.

7. Conclusion and outlook

We have reviewed the existing works on time crystals and extended the concept in order to define the dissipative time crystal in open quantum systems. Such systems are of interest as they exhibit persistent oscillations due to being in contact with a noisy environment. This is of particular interest for realisable time crystals where the system will never be completely isolated from noisy interactions with its surroundings.

In the Markovian master equation framework, we have demonstrated the importance of structure within the purely imaginary eigenvalues of $\mathcal{L}$ when looking to observe the breaking of time symmetry in open, many-body quantum systems. In particular the purely imaginary eigenvalues must form a nowhere-dense commensurable set which for robustness must not be effectively incommensurable. This is a highly non-trivial condition for general many-body Liouvillians which have exponentially many eigenvalues.

We have also introduced a strict measure of time-crystalline behaviour based on the periodicity of the Loschmidt echo of a generic initial state. This is a very stringent measure because even for very small many-body systems the time of periodicity (Poincare recurrence time) is expo-exponentially large in system size due to the exponential number of eigenfrequencies [103] in particular in the absence of criticality [104]. We also consider *random* initial states which will generally be expected to excite an exponentially large number of eigenfrequencies for a many-body system. Thus the time dissipative time crystal behaviour will be generally observed in the system and not a consequence of precisely engineered initial conditions and observables. We remark however that this can only probe finite-size effects on time-crystalization, as for a random initial state the amplitude of the Loschmidt echo is generally expected to decrease with system size

We looked at finite size examples based on the Hubbard model with two-body loss and gain, naturally realizable in cold atom platforms [115]. These allowed us to demonstrate the requirements of eigenvalue structure for dissipative time crystals and to explore how this non-trivial structure is related to mechanisms of dark states and strong dynamical symmetries. While the Loschmidt echo is not practically applicable to significantly larger systems, the dynamics of certain other observables are expected to remain qualitatively the same.

By considering first the isolated system we showed that the existence of purely imaginary eigenvalues of $\mathcal{L}$ is not sufficient for persistent oscillations for generic initial conditions and observables. However, further examples illustrated that such behaviour can emerge provided the purely imaginary eigenvalues of the Liouvillian are at least commensurable. The example with both two-body loss and gain demonstrated that the condition of commensurability is guaranteed by the existence of a single strong dynamical symmetry provided it generates *all* the continuously oscillating modes. In general, however, the existence of a strong dynamical symmetry is not sufficient for time crystalline behaviour, as illustrated by the example with only two-body loss. This example also allowed us to define a dissipative quasi-time crystal. Finally, the example with an inhomogeneous magnetic field provided an example of a dissipative time crystal without any dynamical symmetries. This model also intriguingly realizes an effective non-local *dark Hamiltonian* at late times giving coherent dynamics between two GHZ states. Remarkably, such dynamics are here achieved with a purely local physical Hamiltonian and realistic dissipation and further are robust to variations in many of the model parameters.

We concluded by showing how our example system can be modified to only become a time crystal in the thermodynamic limit. Open questions still remain regarding a more general theory of time crystalline behaviour in finite open systems described by Lindblad master equations. In particular in exploring whether realistic Liouvillians of this type, describing many-body systems, can have additional non-zero, purely imaginary eigenvalues which are not described by dark states or strong symmetries. Additionally the example in section 5.3 suggests the study of dissipative *quasi*-time crystals, where one can have only a few incommensurate frequencies in analogy with discrete time quasi-crystals [77, 120]. Also, as discussed in section 6, much work is required to examine the subtleties that arise in the thermodynamic limit, and to carefully classify the range of dissipative time crystalline behaviour which can arise.

In future work we expect to study linear response theory of dissipative time crystals, behaviour under periodic driving, the formulation of a semiclassical limit [127], and the influence of dissipation on persistent oscillations at quantum critical points [128], in gauge theories [24, 129], or low energy Luttinger liquid descriptions [130, 131]. Additionally the relations between our dynamical symmetry requirement under dissipation and those for quantum many-body scars will be explored [16–19].

Acknowledgments

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