Analysis of Stochastic PDEs arising from large portfolios of Stochastic Volatility models



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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2018

This thesis is mainly dedicated to my mother, Marzena Zalewska, and my father, Vasileios Kolliopoulos, who not only supported me financially during the years of my D.Phil, but have also done everything for me throughout my life. It is also dedicated to all my good friends, including the ones I made during the 4 years I lived in Oxford. Moreover, I would like to dedicate this thesis to professor Michail Loulakis, who introduced me to the area of Mathematical Finance and has helped me a lot during my first steps in mathematical research, and professor Anargyros Fellouris, who has also helped me in many occasions. Finally, this thesis is dedicated to my secondary school and high school mathematics teachers, Mr. Panagiotis Skoufis and Mr. Vangelis Boukouris respectively, who both helped me realize the beauty of this wonderful science and my talent in mathematical problem solving at a relatively young age.

Acknowledgements

First, I would like to thank my mother, Marzena Zalewska, and my father, Vasileios Kolliopoulos, who supported me financially during the years on which I was only partially funded. Next, I would like to thank very much my main supervisor, Professor Ben Hambly, who advised me to study the class of models studied in this thesis. His guidance and support during my D.phil studies have been crucial for the completion of this work. I am very grateful for the generous and extremely helpful financial support I received from the Foundation for Education and European Culture during the final year of my D.Phil, and also for the dedication of this financial support to the memory of Karolos Trivizas, one of the Foundation's benefactors, which was a great honour for me. I would also like to acknowledge the financial support I received both from the Engineering and Physical Sciences Research Council (EPSRC) and from Trinity college for my tuition and college fees for the duration of this D.Phil programme. Certain people from Trinity college, including the accommodation manager, Miss Mandy Giles, and the librarian, Miss Sharon Cure, helped me a lot during the years on which I was not fully funded and I want to thank them very much for this. Moreover, I would like to mention a prize I received during the final year of my D.Phil from the G-Research company for an excerpt of this thesis, and to thank the members of the prize committee for this prestigious award which was also an extra financial support. I am also very grateful to my adviser at Trinity college, Professor Ian Hewitt, who guided and encouraged me during my first two months at Oxford University. Finally, I would like to thank my second supervisor, Professor Zhongmin Qian, the founder and director of the Oxford CDT in PDEs, Professor Gui-Qiang G. Chen, the CDT administrator, Mr Jonathan Whyman, and Professors Samuel Cohen, Jeff Dewynne, Terry Lyons, Michael Monoyios, Harald Oberhauser and Christoph Reisinger, for their guidance and support during the 4 years I was doing my D.phil at Oxford University.

Abstract

The aim of this thesis is to study a large market model of defaultable assets in which the asset price processes are modelled as stochastic volatility models with volatility mean-reversion and default upon hitting a lower boundary. We assume that both the asset prices and their volatilities are correlated through systemic Brownian motions. In the first part of the thesis, we are interested in the loss process that arises in this setting and we prove the existence of a large portfolio limit for the empirical measure process of this system. This limit evolves as a measure valued process and we show that it will have a density given in terms of a solution to a stochastic partial differential equation of filtering type in the two-dimensional half-space, with a Dirichlet boundary condition. Next, we employ Malliavin calculus to establish the existence of a regular density for the volatility component of the SPDE, and an approximation by models of piecewise constant volatilities combined with a kernel smoothing technique to obtain existence and regularity for the full two-dimensional filtering problem. We are able to establish good regularity properties and uniqueness for solutions, except in the CIR case where uniqueness remains an open problem. In the second part of the thesis, we study the convergence of the total mass of a solution to this stochastic initial-boundary value problem when the mean-reversion coefficients of the volatilities are multiples of a parameter which tends to infinity. When the coefficients of the noises of the volatilities are multiples of the square root of the same parameter, the convergence is extremely weak. On the other hand, when the noise coefficients are independent of this exploding parameter, the volatilities converge to their means and we can have much better approximations. Such approximations can be used to improve the accuracy of certain risk-management methods in markets where fast volatility mean-reversion is observed.

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Chapter 1

Introduction

In the study of large portfolios of assets it is common to model correlation through factor models. In this setting the random drivers of individual asset prices come from two independent sources. Firstly there is an idiosyncratic component that reflects the movements due to the asset's individual circumstances. Secondly there are systemic components that reflect the impact of macroscopic events at the whole market or sector level. The motivations for the study of this setting come from developing such models firstly for credit derivatives such as CDOs which are functions of large portfolios of credit risky assets, but also for the evolution of large portfolios which have exposure to a significant proportion of the whole market. The financial crisis of 2008 showed that the correlation between credit risky assets was not adequately modeled, since the default of certain assets had an unexpectedly huge impact on the whole economy with catastrophic results, including more defaults and immense losses to investors who had invested in misspriced CDOs. In this work we will examine the behaviour of a large market when all the individual assets follow classical stochastic volatility models but are correlated through market factors.

Our starting point is a simple structural model for default in a large portfolio, studied in [7]. In this setting there is a market of N credit risky assets in which the i-th asset price A^i for $i = 1, \ldots, N$ is modelled by a geometric Brownian motion with a single systemic risk factor, in that under a risk neutral measure

$$\begin{aligned} dA_t^i &= rA_t^i dt + \sigma A_t^i \left(\sqrt{1-\rho^2} dW_t^i + \rho dW_t^0 \right), \quad 0 \leq t \leq T_i \\ A_t^i &= b^i, \ t > T_i \\ A_0^i &= a^i, \end{aligned}$$

where $T_i = \inf\{t \geq 0 : A_t^i = b^i\}$ for some constant default barrier b^i and the parameters r, σ, ρ, a^i are constants. Here the Brownian motions $W^0, W^1, \ldots$ are all

independent and we see that it is W^0 which captures the macroscopic effects felt by the whole market. We note that the parameters of the geometric Brownian motions, are the same for each asset, it is just the starting point and idiosyncratic noise which cause the differences in asset prices. By rewriting this in terms of a distance to default process and considering the empirical measure restricted on $(0, +\infty)$, it was shown in [7] that the limit empirical measure process (as $N \longrightarrow +\infty$) has a density which is the unique strong solution to an SPDE on the positive half-line. The density takes values in a weighted Sobolev space as the derivatives of the density may not be well behaved at the origin. The exact regularity of the density at the origin was the subject of [35], where it was shown that the regularity is a function of the parameter ρ.

This is a naive model and has the problems that would be expected from such a simple structural default model. The short term credit spreads go to 0 and we see correlation skew when using the model to price the tranches of CDOs. Thus we wish to investigate a model which incorporates more realistic features. In particular we take stochastic volatility models for the underlying assets and allow there to be global volatility factors driving the market volatility as well as idiosyncratic factors for the volatilities of the individual assets. It is also the case that we would like to allow the parameters that describe the volatility and correlation between assets to vary.

In this thesis we consider a large portfolio of N credit risky assets, where now stochastic volatility models are used instead of Black-Scholes models to describe the evolution of the asset values. Each volatility is modeled as a function of a process, which is picked from a large class of mean-reverting processes containing the CIR process and the Ornstein-Uhlenbeck process. We assume the i-th value process A_i satisfies the following system of SDEs

$$\begin{aligned} dA_t^i &= A_t^i \mu_i dt + A_t^i h\left(\sigma_t^i\right)\left(\sqrt{1-\rho_{1,i}^2}dW_t^i + \rho_{1,i}dW_t^0\right),\ 0 \le t \le T_i \\ d\sigma_t^i &= k_i(\theta_i - \sigma_t^i)dt + \xi_i q\left(\sigma_t^i\right)\left(\sqrt{1-\rho_{2,i}^2}dB_t^i + \rho_{2,i}dB_t^0\right),\ t \ge 0 \\ A_t^i &= b^i,\ t > T_i \\ (A_0^i,\, \sigma_0^i) &= (a^i,\, \sigma^i), \end{aligned} \tag{1.0.1}$$

for all $i \in \{1, 2, \ldots, N\}$, where $T_i = \inf\{t \ge 0 : A_t^i = b^i\}$. Here, a^1, a^2, ..., a^N and σ^1, σ^2, ..., σ^N are the initial values of the asset prices and the volatilities respectively, b^i is the constant default barrier for the value of the i-th asset, $C_i = (k_i,\, \theta_i,\, \xi_i,\, r_i,\, \rho_{1,i},\, \rho_{2,i})$ for $i \in \{1,\, 2,\, ...,\, N\}$ are vectors for the various parameters of the model, h and q are functions with enough regularity, and W_t^1, B_t^1, ..., W_t^N, B_t^N

are standard Brownian motions. We will assume that $(a^i,\, \sigma^i)$ and C_i are drawn independently from some distribution and that the Brownian motions are independent from each other and from each a^i, σ^i and C_i, while we will impose the natural condition $a^i \geq b^i$ (the initial value cannot be below the default barrier). Finally, (W_t^0, B_t^0) is a pair of Brownian motions with correlation coefficient ρ_3 (i.e $dW_t^0 \cdot dB_t^0 = \rho_3 dt$), independent of both W^i and B^i as well as a^i, σ^i and C_i for all $i = 1, \ldots, N$, which represents the impact of macroscopic factors on each asset and each volatility respectively.

As is usual in a credit setting we consider the distance to default, or the log asset prices, by setting $X_t^i = (\ln A_t^i - \ln b^i)$ in (1.0.1). Applying Ito's formula, our model becomes

$$\begin{array}{rcl} dX_t^i & = & \left(r_i - \frac{h^2(\sigma_t^i)}{2}\right) dt + h(\sigma_t^i)\left(\sqrt{1-\rho_{1,i}^2}dW_t^i + \rho_{1,i}dW_t^0\right), \quad 0 \leq t \leq T_i \\ d\sigma_t^i & = & k_i(\theta_i - \sigma_t^i)dt + \xi_i q\left(\sigma_t^i\right)\left(\sqrt{1-\rho_{2,i}^2}dB_t^i + \rho_{2,i}dB_t^0\right), \quad t \geq 0 \\ X_t^i & = & 0, \; t > T_i \\ (X_0^i,\, \sigma_0^i) & = & (x^i, \sigma^i), \end{array} \tag{1.0.2}$$

for $i \in \{1,\, 2,\, ...,\, N\}$, where $x^i = (\ln a^i - \ln b^i) \geq 0$ and $T_i = \inf\{t \geq 0 : \; X_t^i = 0\}$, $\forall\, 1 \leq i \leq N$.

An important output from such large portfolio models is the loss process, which gives the proportion of assets that have defaulted by any time t. This can be used to capture some key quantities in risk management, such as the probability of loss from a portfolio and the expected loss given default. In credit derivative pricing, the payoffs of CDO tranches are piecewise linear functions of this loss process.

In our set up the loss process is given by the mass of the two-dimensional empirical measure

$$v_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i, \sigma_t^i} \tag{1.0.3}$$

on $\{0\} \times \mathbb{R}$, which is equal to

$$L_t^N = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}_{\{T_i \leq t\}}, \tag{1.0.4}$$

while the restriction of v_t^N to $(0, \infty) \times \mathbb{R}$ for $t \geq 0$ is given by

$$v_{1,t}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i, \sigma_t^i} \mathbb{I}_{\{T_i > t\}}. \tag{1.0.5}$$

In chapter 2, we are going to establish a convergence result (as N tends to infinity) for the above measure-valued processes, and to study a stochastic initial-boundary value problem in the two-dimensional positive half-space, which describes the behaviour of the limiting processes and has the form of initial-boundary value problems that arise in stochastic filtering. The convergence result we are going to establish is the following: almost surely and for all positive t we have both

$$v_t^N \longrightarrow v_t = \mathbb{P}\left(\left(X_t^1,\, \sigma_t^1\right) \in \cdot \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)$$

and

$$\begin{aligned} v_{1,t}^N \longrightarrow v_{1,t} &= \mathbb{P}\left(\left(X_t^1,\, \sigma_t^1\right) \in \cdot, T_1 > t \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right) \\ &= \mathbb{E}\left[v_{t,\,C_1}\left(\cdot\right) \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right] \end{aligned}$$

weakly as $N \longrightarrow \infty$, for some σ-algebra $\mathcal{G}^0$ contained in the initial data, where we denote by $v_{t,\,C_1}\left(\cdot\right)$ the measure-valued process $\mathbb{P}\left(\left(X_t^1,\, \sigma_t^1\right) \in \cdot, T_1 > t \,|\, W_\cdot^0,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right)$. Then, we will prove that v_{t,C_1} - depending on the information contained in $(W_\cdot^0,\, B_\cdot^0)$, $\mathcal{G}^0$ and the coefficient vector $C_1 = (k_1, \theta_1, \xi_1, r_1, \rho_{1,1}, \rho_{2,1})$ - has a density u_{t,C_1} in a weighted Sobolev-Lebesgue space of the two-dimensional positive half-space, but with no differentiability in the second spatial variable y. Moreover, we will show that given C_1, $u_{t,\,C_1}$ satisfies an SPDE in that function space, along with a Dirichlet boundary condition at $x = 0$. In the last two sections of chapter 2, we will use the SPDE to obtain (weak) differentiability also in y, along with some good integrability for the derivative, and to show that no other regular solutions exist when q satisfies certain conditions. Our SPDE for u_{t,C_1} has the form

$$\begin{aligned} u_{t,C_1}(x,\, y) &= u_0(x,\, y) - \int_0^t \left(r_1 - \frac{1}{2}h^2(y)\right) \left(u_{s,C_1}(x,\, y)\right)_x \, ds \\ &\quad - \int_0^t k_1 \left(\theta_1 - y\right) \left(u_{s,C_1}(x,\, y)\right)_y \, ds + \frac{1}{2}\int_0^t h^2(y) \left(u_{s,C_1}(x,\, y)\right)_{xx} \, ds \\ &\quad + \frac{\xi_1^2}{2} \int_0^t \left(q^2(y) u_{s,C_1}(x,\, y)\right)_{yy} \, ds \\ &\quad + \xi_1 \rho_3 \rho_{1,1} \rho_{2,1} \int_0^t \left(h(y) q(y) u_{s,C_1}(x,\, y)\right)_{xy} \, ds \\ &\quad - \rho_{1,1} \int_0^t h(y) \left(u_{s,C_1}(x,\, y)\right)_x \, dW_s^0 \\ &\quad - \xi_1 \rho_{2,1} \int_0^t \left(q(y) u_{t,C_1}(x,\, y)\right)_y \, dB_s^0, \end{aligned} \tag{1.0.6}$$

where u_0 is the initial density, and the boundary condition $u_{t,C_1}(0, y) = 0$ is satisfied for all $y \in \mathbb{R}$ and $t \geq 0$.

In order to implement the model we could solve the initial-boundary value problem for the SPDE numerically for samples C_i of the parameters. Then, we can approximate the loss process from

$$\begin{aligned}\lim_{N\to\infty} v_t^N(\{0\}\times\mathbb{R}) &= 1-\lim_{N\to\infty} v_{1,t}^N(\mathbb{R}^2) = 1-\int\int_{\mathbb{R}^2} u_t(x,y)dxdy \\ &= 1-\mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}} u_{t,C_1}(x,\,y)dxdy,\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right] \\ &\approx 1-\frac{1}{n}\sum_{i=1}^n\int_0^\infty\int_{\mathbb{R}} u_{t,c_i}(x,\,y)dxdy \qquad (1.0.7)\end{aligned}$$

where $\{c_1,\,c_2,\,...,\,c_n\}$ is a random sample from the distribution of C_1. As the SPDE satisfied by each u_{t,c_i} is driven by the two-dimensional Brownian path $(W_\cdot^0,\,B_\cdot^0)$, we only need to simulate $(W_\cdot^0,\,B_\cdot^0)$ and solve the corresponding SPDEs. This approach is quite efficient when the number of assets N is large, since we do not have to simulate the $2N$ idiosyncratic Brownian paths.

There are other approaches to the modelling of credit risk in large portfolios which lead to stochastic partial differential equations. For example in a reduced form setting, see [20, 41, 40, 42]. However, this is the first structural large portfolio model to incorporate stochastic volatility and also the first to introduce random coefficients in the SDEs describing the evolution of the asset prices. This provides a level of generality which captures many features of asset prices. The constant coefficient case is just a special case of the model we have considered, in which the weak limit $v_{1,t}$ of the empirical process coincides with the measure-valued process v_{t,C_1} whose density u_{t,C_1} satisfies our SPDE. Our aim here is to establish the theoretical background for the general case.

The calibration of the model for its use in the pricing of CDOs would follow a similar approach to that used in [7]. In its simplest form we take the version where the parameters of the model are assumed to be constants. The initial condition would be fitted to the CDS prices of the underlying constituents of the portfolio. The parameters of the model are then determined from the market tranche prices of the CDOs with different maturities. This should be done by solving the model forward from different parameter settings to find model tranche prices and then minimizing the least squares distance between model and market to locate the best fit parameters.

The approach to solving the model forward must be done numerically. This type of model is more computationally intensive than that considered in [7] as the SPDE is in two dimensions. The technique is to generate the two dimensional Brownian

path and then solve the SPDE using a finite element approach. Speed up could be achieved by extending the work of [21] where the multilevel Monte Carlo approach was used for the model of [7]. We will not discuss the numerical analysis for the model as, even in the one-dimensional case, this is challenging.

Of course, a disadvantage in our setting is the introduction of a large number of parameters that need to be simulated or estimated in order to implement the model. Moreover, the random coefficients need assumptions on their joint distributions and, when we use Monte Carlo techniques to estimate expectations in (1.0.7), a very large number of simulations may still be required due to the random parameters. This is not surprising, since the calibration of stochastic volatility models in finance is quite complex in terms of computations, even when we want to compute the prices of simple vanilla option.

A very useful tool for dealing with this complexity has been developed in [17], where the prices of several vanilla and exotic options under a stochastic volatility model are written as series of negative powers of the mean-reversion coefficient of the volatility process. Since volatility mean-reversion is generally observed to be fast, by keeping the first two or three terms of these series we obtain good approximations for the option prices, which are more accurate than the corresponding Black-Scholes prices - which can be obtained by keeping only the first term - while their computation is much simpler than the computation of the exact prices under the stochastic volatility model. Fast mean-reverting processes in mathematical finance have been considered in many different settings for similar purposes, see for example [13, 14, 15, 16, 18, 19, 45]. Ideally, we would like to adapt these ideas to the large portfolio setting in order to obtain an approximate model, which is more accurate than the constant volatility model studied in [7], but which is also easier to handle in terms of both computational cost and the ability to answer key mathematical questions like the uniqueness of solutions to the equations describing it. This is going to be the subject of chapter 3 of this thesis, where we will show that our model converges (as mean-reversion tends to infinity) to a constant volatility model like the one studied in [7].

Unfortunately, we will see that obtaining a higher order correction is not possible, but we are going to obtain the best possible convergence results, along with an upper bound for the rate of convergence when the volatilities of the volatilities (vol-of-vol) do not depend on the mean-reversion.

Remark 1.0.1. (1) We will not discuss the issue of asymptotic arbitrage which can arise when there is a large portfolio limit of assets (see [28, 27]). As we are using the limiting model as an approximation to a large finite model, which will not admit arbitrage, the question is only of theoretical interest.

(2) When calibrating the model for pricing credit derivatives the drift term in each asset's logarithmically scaled value process is replaced by a known interest rate. Including more parameters than that used in [7] should improve the calibration of the model and may allow observed features such as correlation skew in CDOs to be captured. Though we should note that even including jumps in the basic model of [7] still makes it difficult to capture all the observed features of CDO tranche prices [6].

(3) One could view the empirical mean of such a model as a natural model for an index, see [25] for the simple case. Here we would produce a stochastic volatility model for the index and this could be used to price volatility dependent derivatives.

(4) It would be natural to develop central limit theorems and a large deviation analysis in further work, potentially by adapting and extending appropriately the ideas of [43, 44]. For applications in systemic risk it would also be interesting to add a mean field interaction.

(5) The popular Heston model is obtained when the functions h and q in our setting are just the square root function.

Chapter 2

Analysis of the SPDE describing the large portfolio limit

2.1 Introduction

There are significant mathematical challenges in extending large portfolio models to the stochastic volatility setting. A key point is to estimate the boundary behaviour of the empirical measure and, with a non-constant volatility path, this needs a novel approach. The kernel smoothing technique used by [7] also needs alteration to cope with this volatility process, to enable us to obtain the best possible regularity for our two-dimensional density. All these things are going to be the subject of this chapter.

In Section 2.2, we assume that the initial data satisfies some reasonable exchangeability conditions in order to obtain the convergence result for the empirical measure process as $N \longrightarrow \infty$. This is not just a two-dimensional version of the corresponding result in [7], since it gives the convergence of the restriction of the empirical measure process to $(0, \infty) \times \mathbb{R}$, while it also shows that the limiting measure-valued process can be represented as a process of conditional probability laws. It includes thus a law of large numbers which is particularly important for dealing with this two-dimensional version of the large portfolio analysis problem.

In Section 2.3 we extend some existing Malliavin calculus results and techniques, in order to obtain strong norm estimates for the density of our volatility process when a component of the driving Brownian Motion (the market factor) is given. In the CIR case where $q(z) = \sqrt{|z|}$ for all $z \in \mathbb{R}$, we will need to impose a condition on the parameters which is stronger than the standard Feller condition which prevents the process from hitting zero. This is due to the fact that the CIR process does not have Lipschitz coefficients, which means that standard Malliavin calculus techniques for proving the existence of a density of an Ito process are not directly applicable

and approximations with processes having better coefficients are needed. However, we will see that this absence of Lipschitz continuity causes another major problem: it does not allow us to prove uniqueness of solutions to our SPDE (1.0.6). For this reason, we will also consider a class of volatility processes where $q(\cdot)$ is bounded away from zero and has bounded derivatives. Obviously, this class covers the standard Ornstein-Uhlenbeck volatility case as well.

Next, in Section 2.4 we prove a convergence result for a sequence of stopped Ito processes when the sequence of volatility paths decreases pointwise to a continuous and positive path, in order to extend the results of [7] to the case when the volatility path is non-constant. Combining this with the results of Section 3 in a divide-and-conquer approach, we obtain the existence of a regular density for the measure-valued process v_{t,C_1}, for any good enough value of C_1, and also the SPDE and the boundary condition satisfied by that density.

In Section 2.5, we extend the kernel smoothing method developed in [7, 34, 24], by proving that the standard heat kernel maintains its smoothing and convergence properties when it is composed with a regular enough function, and also in certain weighted L^2 spaces. This allows us to obtain differentiability of our density in the y-direction, and also weighted L^2 integrability of the derivative. This improved kernel smoothing method does not work in distribution spaces for our SPDE and thus, the regularity results of the previous two sections are crucial. Finally, in Section 2.6 we use the regularity results obtained earlier to prove uniqueness of solutions to our stochastic initial-boundary value problem, but only when q has good regularity and boundedness properties.

In order to begin the study of our setting, some assumptions need to be made. We assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete fitlered probability space, which can be decomposed as a product of three independent probability spaces representing the three different sources of randomness. That is, we can write

$$\begin{aligned}&(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})\\ &\quad = (\Omega^0 \times \Omega^1 \times \Omega^C, \mathcal{F}^0 \otimes \mathcal{F}^1 \otimes \mathcal{F}^C, \{\mathcal{F}_t^0 \otimes \mathcal{F}_t^1 \otimes \mathcal{F}^C\}_{t\geq 0}, \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^C),\end{aligned} \tag{2.1.1}$$

where the standard Brownian motions W_t^0 and B_t^0 are defined on the complete filtered probability space $(\Omega^0, \mathcal{F}^0, \{\mathcal{F}_t^0\}_{t\geq 0}, \mathbb{P}^0)$, adapted to the filtration $\{\mathcal{F}_t^0\}_{t\geq 0}$, and correlated with $dW_t^0 \cdot dB_t^0 = \rho_3 dt$, while $\{W_t^1, B_t^1, W_t^2, B_t^2, ...\}$ is an infinite sequence of pairwise independent standard Brownian motions which are defined on the complete filtered probability space $(\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t\geq 0}, \mathbb{P}^1)$ and adapted to the filtration

$\{\mathcal{F}_t^1\}_{t\geq 0}$, and finally $C_i = (k_i, \theta_i, \xi_i, r_i, \rho_{1,i}, \rho_{2,i})$ for $i \in \mathbb{N}$ are i.i.d 6-dimensional random vectors defined on the complete probability space $(\Omega^C, \mathcal{F}^C, \mathbb{P}^C)$ such that $\mathbb{P}^C$- almost surely we have $\rho_{1,i}, \rho_{2,i} \in (-1, 1)$ $\forall i \in \mathbb{N}$. All the filtrations are assumed to be complete and right-continuous. Next, we assume that $\{(x^1, \sigma^1), (x^2, \sigma^2), ...\}$ is an exchangeable infinite sequence of two-dimensional random vectors, which are measurable with respect to the σ-algebra $\mathcal{F}_0 = \mathcal{F}_0^0 \otimes \mathcal{F}_0^1 \otimes \mathcal{F}^C$. The exchangeability condition implies that there exists a σ-algebra $\mathcal{G}$ contained in $\mathcal{F}_0$, such that the two-dimensional vectors: $(x^1, \sigma^1), (x^2, \sigma^2), ...$ are i.i.d given $\mathcal{G}$ (see, for example, [1]) and, without loss of generality, we may assume that $\mathcal{G} = \mathcal{G}^0 \otimes \{\emptyset, \Omega^1\} \otimes \{\emptyset, \Omega^C\}$, for some $\mathcal{G}^0 \subset \mathcal{F}_0^0$. The last is clear if we define the random vectors on Ω^0, but allowing some dependence on $\Omega^1 \times \Omega^C$ provides consistency if we decide to restart our processes at some $t > 0$.

2.2 The large portfolio limit and the SPDE

Under the above assumptions and for each $N \in \mathbb{N}$, we consider the interacting particle system described by equations (1.0.2) and the corresponding empirical measure processes v_t^N, and $v_{1,t}^N$ given by (1.0.3) and (1.0.5) respectively. We also define $v_{2,t}^N := v_t^N - v_{1,t}^N$, the restriction of v_t^N to $\{0\} \times \mathbb{R}$, for all $t \geq 0$. In this section, we will study the asymptotic behaviour of these measure-valued processes as $N \to +\infty$.

2.2.1 Existence of a large portfolio limit

We start by proving the following convergence theorem, the proof of which is a simple modification of the convergence theorem for the one-dimensional empirical measure process in [7]. It is stronger than a convergence result for the empirical measure process v_t^N, but we need this stronger form to prove Theorem 2.2.3, a crucial result for establishing the convergence of the $\{0\} \times \mathbb{R}$ - supported component $v_{2,t}^N$ (Theorem 2.2.5).

Theorem 2.2.1. *For each $N \in \mathbb{N}$ and any $t, s \geq 0$, consider the random measure given by*

$$v_{3,t,s}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i, \sigma_t^i, \sigma_s^i}.$$

The sequence $v_{3,t,s}^N$ of three-dimensional empirical measures converges weakly to some measure $v_{3,t,s}$ for all $t, s \geq 0$, $\mathbb{P}$-almost surely. Moreover, the measure-valued process $\{v_{3,t,s} : t, s \geq 0\}$ is $\mathbb{P}$-almost surely continuous in both t and s under the weak topology.

Proof. First, we consider each pair $(X^i_\cdot,\, \sigma^i_\cdot)$ as a random variable taking values in the usual Wiener space $\left(C\left([0,\, T]\,;\, \mathbb{R}^4\right),\, ||\,\cdot\,||_\infty,\, \mathcal{W}_2\right)$, which is the space of continuous $\mathbb{R}^2$-valued functions defined on $[0,\, T]$, equipped with the corresponding supremum norm $||\,\cdot\,||$ and the corresponding Wiener σ-algebra $\mathcal{W}_2$.

Since $(X^i_\cdot,\, \sigma^i_\cdot)$ is obviously a strong solution to (1.0.2), there exists a map

$$S : \left(C\left([0,\, T]\,;\, \mathbb{R}^4\right),\, ||\,\cdot\,||_\infty,\, \mathcal{W}_4\right) \times \left(\mathbb{R}^8,\, \mathcal{B}_8\right) \longrightarrow \left(C\left([0,\, T]\,;\, \mathbb{R}^2\right),\, ||\,\cdot\,||_\infty,\, \mathcal{W}_2\right)$$

such that for each $\omega \in \Omega$ we can write

$$\left(X^i_\cdot(\omega),\, \sigma^i_\cdot(\omega)\right) = S\left(B^i_\cdot(\omega),\, W^i_\cdot(\omega),\, B^0_\cdot(\omega),\, W^0_\cdot(\omega),\, x^i(\omega),\, \sigma^i(\omega),\, C_i(\omega)\right),$$

where $\left(C\left([0,\, T]\,;\, \mathbb{R}^2\right),\, ||\,\cdot\,||_\infty,\, \mathcal{W}_4\right)$ is the usual Wiener space of 4-dimensional continuous paths, while $\mathcal{B}_8$ stands for the Borel σ-algebra in $\mathbb{R}^8$. Moreover, S is a measurable map, since we can approximate it pointwise by solution maps corresponding to a discretized version of (1.0.2) (see [22] for a uniformly converging numerical scheme that is applicable to the CIR process as well), which are continuous.

For a permutation $\pi : \{1, 2, ..., N\} \longrightarrow \{1, 2, ..., N\}$ and a collection $\{G_1, ..., G_N\}$ of $\mathcal{W}_2$-measurable sets, the event

$$\left\{\left(B^{\pi(i)}_\cdot,\, W^{\pi(i)}_\cdot,\, B^0_\cdot,\, W^0_\cdot,\, x^{\pi(i)},\, \sigma^{\pi(i)},\, C_{\pi(i)}\right) \in S^{-1}(G_i),\, \forall\, 1 \leq i \leq N\right\}$$

has a probability which is equal to

$$\mathbb{P}\left(\omega \in \Omega :\, \left(X^{\pi(i)}_\cdot(\omega),\, \sigma^{\pi(i)}_\cdot(\omega)\right) \in G_i,\, \forall\, 1 \leq i \leq N\right).$$

We claim that the law

$$\begin{aligned}
&\mathbb{P}\left(\omega \in \Omega :\, \left(X^{\pi(i)}_\cdot(\omega),\, \sigma^{\pi(i)}_\cdot(\omega)\right) \in G_i,\, \forall\, 1 \leq i \leq N\right) \\
&\quad = \mathbb{P}\left(\left(B^{\pi(i)}_\cdot,\, W^{\pi(i)}_\cdot,\, B^0_\cdot,\, W^0_\cdot,\, x^{\pi(i)},\, \sigma^{\pi(i)},\, C_{\pi(i)}\right) \in S^{-1}\left(G_i\right),\, \forall\, 1 \leq i \leq N\right)
\end{aligned}$$

is independent of the permutation π. Indeed, by a linear inversion it is enough to show that the joint law of $\left\{B^{\pi(i)}_\cdot,\, W^{\pi(i)}_\cdot,\, x^{\pi(i)},\, \sigma^{\pi(i)},\, C_{\pi(i)} :\, 1 \leq i \leq N\right\} \cup \left\{B^0_\cdot,\, W^0_\cdot\right\}$ is independent of the permutation $\pi\,(\cdot)$, which is a consequence of our exchangeability assumptions. As a result the set $\left\{(X^i_\cdot,\, \sigma^i_\cdot) :\, 1 \leq i \leq N\right\}$ is an exchangeable set of $C\left([0,\, T]\,;\, \mathbb{R}^2\right)$-valued random variables. Hence, by de Finetti's Theorem (see Theorem 4.1 in [31], but it can also be found in [1]), we obtain that the sequence of measure-valued processes

$$v^N_* = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^i_\cdot,\, \sigma^i_\cdot}$$

converges weakly to some probability measure v_* (which is defined on $\mathcal{W}_2$), $\mathbb{P}$-almost surely. Thus there is a set $\tilde{\Omega} \subset \Omega$ where the convergence is valid for any $\omega \in \tilde{\Omega}$, where $\mathbb{P}(\tilde{\Omega}) = 1$.

Let $P_{t,s} : \left(C\left([0,\, T]\,;\, \mathbb{R}^2\right),\, ||\,\centerdot\,||_\infty,\, \mathcal{W}_2\right) \longrightarrow \mathbb{R}^3$ be an evaluation functional at some $(t,\, s) \in [0,\, T]^2$, which maps $(f(\cdot),\, g(\cdot))$ to $(f(t),\, g(t),\, g(s))$ and which is obviously continuous. We fix an $\omega \in \tilde{\Omega}$ and we define $v_{3,t,s} = v_* \circ P_{t,s}^{-1}$ for all $(t,\, s) \in [0,\, T]^2$. Then, for this ω and for any Borel set $A \subset \mathbb{R}^3$ we have

$$\begin{aligned} v_{3,t,s}^N(A) &= \frac{1}{N}\#\left\{1 \leq i \leq N :\, \left(X_t^i,\, \sigma_t^i,\, \sigma_s^i\right) \in A\right\} \\ &= \frac{1}{N}\#\left\{1 \leq i \leq N :\, P_{t,s}\left(X_\cdot^i,\, \sigma_\cdot^i\right) \in A\right\} \\ &= \frac{1}{N}\#\left\{1 \leq i \leq N :\, \left(X_\cdot^i,\, \sigma_\cdot^i\right) \in P_{t,s}^{-1}(A)\right\} = v_*^N(P_{t,s}^{-1}(A)) \end{aligned}$$

for all $N \in \mathbb{N}$ and all $(t,\, s) \in [0,\, T]^2$. This means that for this ω and for any $f \in C_b\left(\mathbb{R}^3;\, \mathbb{R}\right)$ we have

$$\int_{\mathbb{R}^2} f dv_{3,t,s}^N = \int_{C([0,\,T];\,\mathbb{R}^2)} f \circ P_{t,s} dv_*^N, \tag{2.2.1}$$

since we can easily show this for a sequence of simple functions approximating f from below and conclude then by the Monotone Convergence Theorem. Taking now $N \longrightarrow \infty$ we find

$$\int_{\mathbb{R}^2} f dv_{3,t,s}^N \to \int_{C([0,\,T];\,\mathbb{R}^2)} f \circ P_{t,s} \cdot dv_* = \int_{\mathbb{R}^2} f dv_{3,t,s}, \tag{2.2.2}$$

for any $f \in C_b\left(\mathbb{R}^3;\, \mathbb{R}\right)$, where the last equality in (2.2.2) is obtained exactly as (2.2.1). Since this holds for any $f \in C_b\left(\mathbb{R}^3;\, \mathbb{R}\right)$, we have the desired convergence.

Finally, to show continuity under the weak topology for a given $\omega \in \tilde{\Omega}$, we shall invoke the Portmanteau Theorem (see, for example, [30]), according to which we only need to show that

$$\liminf_{n\to\infty} v_{3,t_n,s_n}(A) \geq v_{3,t,s}(A)$$

whenever $(t_n,\, s_n) \longrightarrow (t,\, s) \in [0,\, T]^2$ and for any open $A \subset \mathbb{R}^3$. This is obtained by observing that

$$\begin{aligned} v_{3,t,s}(A) &= v_*\left(P_{t,s}^{-1}(A)\right) \\ &= v_*\left(\left\{(Y_\cdot,\, Z_\cdot) \in C\left([0,\, T]\,;\, \mathbb{R}^2\right) :\, (Y_t,\, Z_t,\, Z_s) \in A\right\}\right) \\ &= v_*\left(\cup_{k=1}^\infty \cap_{n=k}^\infty \left\{(Y_\cdot,\, Z_\cdot) \in C\left([0,\, T]\,;\, \mathbb{R}^2\right) :\, (Y_{t_n},\, Z_{t_n},\, Z_{s_n}) \in A\right\}\right), \end{aligned}$$

which holds because $(Y_{t_n},\, Z_{t_n},\, Z_{s_n}) \longrightarrow (Y_t,\, Z_t,\, Z_s)$ by the continuity of the path $(Y_\cdot,\, Z_\cdot)$, and hence $(Y_{t_n},\, Z_{t_n},\, Z_{s_n})$ is finally contained in any open set containing $(Y_t,\, Z_t,\, Z_s)$. Then, the last quantity is equal to

$$\begin{aligned}
&\lim_{k\to\infty} v_*\left(\cap_{n=k}^{\infty}\left\{(Y_\cdot,\, Z_\cdot) \in C\left([0,\, T]\,;\, \mathbb{R}^2\right) :\, (Y_{t_n},\, Z_{t_n},\, Z_{s_n}) \in A\right\}\right) \\
&\quad \leq \lim_{k\to\infty} \inf_{n\geq k} v_*\left(\left\{(Y_\cdot,\, Z_\cdot) \in C\left([0,\, T]\,;\, \mathbb{R}^2\right) :\, (Y_{t_n},\, Z_{t_n},\, Z_{s_n}) \in A\right\}\right) \\
&\quad = \liminf_{n\to\infty} v_*\left(P^{-1}_{t_n,s_n}(A)\right) \\
&\quad = \liminf_{n\to\infty} v_{3,t_n,s_n}(A)
\end{aligned}$$

and the desired continuity has been proven. Since this continuous limit process of measures is obtained almost surely for $(t,\, s) \in [0,\, n]^2$, for any $n \in \mathbb{N}$, with $\mathbb{N}$ being countable, it is actually obtained almost surely for all $t,\, s \geq 0$. The proof of the Theorem is now complete. □

The convergence result for v_t^N is a direct consequence of the above theorem and it is given in the following corollary.

Corollary 2.2.2. *The sequence v_t^N of two-dimensional empirical measures given by* (1.0.3) *converges weakly to some measure v_t for all $t \geq 0$, $\mathbb{P}$-almost surely. Moreover, the path $\{v_t :\, t \geq 0\}$ is $\mathbb{P}$-almost surely continuous under the weak topology. The measure-valued process v_t is the restriction of $v_{3,t,s}$ to the space of functions which are constant in the third variable, for any $t \geq 0$.*

Proof. Since v_t^N is the restriction of $v_{3,t,s}^N$ to the space of functions which are constant in the third variable, the result follows by testing the measure against such functions and by taking $N \longrightarrow \infty$. □

2.2.2 Representing the limit as a conditional probability law

We prove now that the weak limits of the empirical measures $v_{3,t,s}$ can be represented as conditional probability laws. This is given in the following theorem.

Theorem 2.2.3. *There exists an $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for any $\omega \in \Omega'$, we have $\int_{\mathbb{R}^3} f dv_{3,t,s} = \mathbb{E}\left[f\left(X_t^1,\, \sigma_t^1,\, \sigma_s^1\right) | \, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right]$ for any $t,\, s \geq 0$ and any $f \in C_b(\mathbb{R}^3;\, \mathbb{R})$.*

Proof. We know that $\left(B_\cdot^k,\, W_\cdot^k,\, C_k\right)$ for $k \in \mathbb{N}$ are i.i.d and since they are also independent from $(B_\cdot^0,\, W_\cdot^0,\, \mathcal{G}^0)$, they are also i.i.d. under the probability measure $\mathbb{P}(\,\cdot\,|\, W^0,\, B^0,\, \mathcal{G}^0)$. The same holds for the vectors $(x^1,\, \sigma^1)\,,\, (x^2,\, \sigma^2)\,,$ ..., since they are

i.i.d given $\mathcal{G}^0$ and measurable with respect to the bigger σ-algebra $\mathcal{F}_0$, with $(W^0_\cdot,\, B^0_\cdot)$ being independent from $\mathcal{F}_0$. Thus, noting that there is a function g, measurable with respect to the appropriate σ-algebras, such that

$$\left(X^k_t,\, \sigma^k_t,\, \sigma^k_s\right) = g\left(t,\, s,\, B^k_\cdot,\, W^k_\cdot,\, B^0_\cdot,\, W^0_\cdot,\, C_k,\, x^k,\, \sigma^k\right),$$

it follows that $\left(X^k_t, \sigma^k_t, \sigma^k_s\right)$ for $k \in \mathbb{N}$ are also i.i.d. random vectors under the conditional probability measure $\mathbb{P}(\,\cdot\,|W^0, B^0, \mathcal{G}^0)$.

Thus, for any $f \in C_b(\mathbb{R}^3;\mathbb{R})$ we have

$$\begin{aligned}
1 \;&\geq\; \mathbb{P}\left(\int_{\mathbb{R}^2} f dv^N_{3,t,s} \xrightarrow{N\longrightarrow+\infty} \mathbb{E}\left[f\left(X^1_t, \sigma^1_t,\, \sigma^1_s\right)|W^0_\cdot, B^0_\cdot,\, \mathcal{G}^0\right]\;\forall\, t,\, s \in \mathbb{Q}^+\right)\\
&=\; \mathbb{E}\Bigg[\mathbb{P}\Bigg(\int_{\mathbb{R}^2} f dv^N_{3,t,s}\\
&\qquad \xrightarrow{N\longrightarrow+\infty} \mathbb{E}\left[f\left(X^1_t, \sigma^1_t, \sigma^1_s\right)|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right]\;\forall\, t,\, s \in \mathbb{Q}^+ | W^0_\cdot, B^0_\cdot, \mathcal{G}^0\Bigg)\Bigg]\\
&=\; \mathbb{E}\Bigg[1 - \mathbb{P}\Bigg(\cup_{t,s\in\mathbb{Q}^+}\Bigg\{\int_{\mathbb{R}^2} f dv^N_{3,t,s}\\
&\qquad \xrightarrow{N\longrightarrow+\infty} \mathbb{E}\left[f\left(X^1_t, \sigma^1_t, \sigma^1_s\right)|W^0_\cdot, B^0_\cdot, \mathcal{G}^0\right]\Bigg\}^c | W^0_\cdot, B^0_\cdot, \mathcal{G}^0\Bigg)\Bigg]\\
&\geq\; \mathbb{E}\Bigg[1 - \sum_{t,s\in\mathbb{Q}^+}\Bigg(1 - \mathbb{P}\Bigg(\int_{\mathbb{R}^2} f dv^N_{3,t,s}\\
&\qquad \xrightarrow{N\longrightarrow+\infty} \mathbb{E}\left[f\left(X^1_t, \sigma^1_t,\, \sigma^1_s\right)|W^0_\cdot, B^0_\cdot, \mathcal{G}^0\right]|W^0_\cdot, B^0_\cdot, \mathcal{G}^0\Bigg)\Bigg)\Bigg] = 1,
\end{aligned}$$

where, in the last expectation, by the strong law of large numbers, the probability of convergence is one for each t, s. Hence, there is an $\Omega_f \subset \Omega$ (depending on f) with $\mathbb{P}(\Omega_f) = 1$, such that

$$\int_{\mathbb{R}^2} f dv^N_{3,t,s} \longrightarrow \mathbb{E}\left[f\left(X^1_t,\, \sigma^1_t,\, \sigma^1_s\right)|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right]\;\forall\, t,\, s \in \mathbb{Q}^+$$

as $N \longrightarrow \infty$, for all $\omega \in \Omega_f$.

If we denote by Ω'_f the intersection of Ω_f with the set of events for which the results of Theorem 2.2.1 hold, we see that $\mathbb{P}\left(\Omega'_f\right) = 1$ and that for all $\omega \in \Omega'_f$ we have

$$\int_{\mathbb{R}^2} f dv_{3,t,s} = \mathbb{E}\left[f\left(X^1_t,\, \sigma^1_t,\, \sigma^1_s\right)|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right] \tag{2.2.3}$$

for any $t,\, s \in \mathbb{Q}^+$. Since both quantities in (2.2.3) are continuous in $(t,\, s)$ (this follows from Theorem 2.2.1 for the LHS, and by using the dominated convergence theorem for

the RHS) and since they coincide for any t, $s \in \mathbb{Q}^+$, we conclude that they coincide for all t, $s \geq 0$ in Ω'_f.

Finally, taking the intersection of all Ω'_p for all p belonging to a countable and dense subset D of $C_b(\mathbb{R}^3; \mathbb{R})$, we obtain the desired set Ω'. This follows from the fact that both quantities in (2.2.3) are bounded functionals in $C_b(\mathbb{R}^3; \mathbb{R})$ with the supremum norm, where for the LHS this follows by taking limits in the obvious inequality, $\int_{\mathbb{R}^2} f dv^n_{3,t,s} \leq ||f||_\infty$, $\forall\, n \in \mathbb{N}$. Our proof is now complete. □

Corollary 2.2.4. *Let $\{v_t : t \geq 0\}$ be the measure-valued process defined in Corollary 2.2.2. There exists an $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for any $\omega \in \Omega'$, we have $\int_{\mathbb{R}^2} f dv_t = \mathbb{E}[f(X^1_t, \sigma^1_t) \,|\, W^0_\cdot, B^0_\cdot, \mathcal{G}^0]$ for any $t \geq 0$ and for any $f \in C_b(\mathbb{R}^3; \mathbb{R})$.*

Proof. This result follows by testing the measure against functions which are constant in the third variable and by recalling Corollary 2.2.2. □

The above corollary completes the convergence result for v^N_t which was given in Corollary 2.2.2. An alternative way for showing that the limits of the empirical measures can be represented as conditional expectations is provided in [9] (page 954). However, what we need is a similar result for the restriction of v^N_t to $\{0\} \times \mathbb{R}$, that is $v^N_{2,t}$. This is given in the following Theorem, the proof of which is based on the more general convergence result given in Theorem 2.2.1 and on Theorem 2.2.3.

Theorem 2.2.5. *There exists a measure-valued process $\{v_{2,t} : t \geq 0\}$ and an $\Omega'' \subset \Omega'$ with $\mathbb{P}(\Omega'') = 1$, such that for any $\omega \in \Omega''$ we have that $v^N_{2,t} \xrightarrow{N \longrightarrow +\infty} v_{2,t}$ weakly for all $t \geq 0$. Moreover, we have $\int_{\mathbb{R}^2} f dv_{2,t} = \mathbb{E}\left[f(X^1_t, \sigma^1_t)\, \mathbb{I}_{\{T_1 < t\}} | \, W^0_\cdot, B^0_\cdot, \mathcal{G}^0\right]$ for all $t \geq 0$ and for all $f \in C_b(\mathbb{R}^2; \mathbb{R})$.*

Proof. First, observe that when $(W^0, B^0, \mathcal{G}^0)$ is given, T_1 has a continuous distribution, since it is a stopping time for the Gaussian process X^1_t. Moreover, observe that

$$v^N_{2,t} = \delta_0 \sum_{i=1}^{N} \delta_{\sigma^i_t} \mathbb{I}_{\{T_i < t\}}$$

and that

$$\mathbb{E}\left[f\left(X^1_t, \sigma^1_t\right) \mathbb{I}_{\{T_1 < t\}} \,|\, W^0_\cdot, B^0_\cdot, G^0\right] = \mathbb{E}\left[f\left(0, \sigma^1_t\right) \mathbb{I}_{\{T_1 < t\}} \,|\, W^0_\cdot, B^0_\cdot, \mathcal{G}^0\right],$$

for any $t \geq 0$ and $f \in C_b(\mathbb{R}^2; \mathbb{R})$, which means that we only need to work with functions f which are constant in the first variable.

Let now $v_{2,t}(\cdot)$ be the probability law $\mathbb{P}((X^1_t, \sigma^1_t) \in \cdot\,; T_1 \leq t \,|\, W^0, B^0, \mathcal{G}^0)$ and fix a function f in $C_b(\mathbb{R}; \mathbb{R})$ with positive values. Since T_i is adapted to (X^i_t, σ^i_t)

for any $i \in \mathbb{N}$, by the independence obtained in the first paragraph of the proof of Theorem 2.2.3 and a Law of Large Numbers argument similar to the one in that proof, we have that the desired convergence holds for the chosen f, for all $t \in \mathbb{Q}^+$ and for all ω in some $\Omega''_f \subset \Omega$ with $\mathbb{P}\left(\Omega''_f\right) = 1$. By intersecting with the full-probability set Ω' given in Theorem 2.2.3, we can take $\Omega''_f \subset \Omega'$. Now for a $t \geq 0$ and an $\omega \in \Omega''_f$, we pick any two rational numbers t_1, $t_2 \geq 0$ such that $t_1 \leq t \leq t_2$. Then we have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f\left(\sigma_t^i\right) \mathbb{I}_{\{T_i \leq t\}} \\
&\quad = \liminf_{N\to\infty} \frac{1}{N} \left[\sum_{i=1}^{N} f\left(\sigma_{t_1}^i\right) \mathbb{I}_{\{T_i < t\}} + \sum_{i=1}^{N} \left(f\left(\sigma_t^i\right) - f\left(\sigma_{t_1}^i\right) \right) \mathbb{I}_{\{T_i < t\}} \right] \\
&\quad \geq \liminf_{N\to\infty} \frac{1}{N} \left[\sum_{i=1}^{N} f\left(\sigma_{t_1}^i\right) \mathbb{I}_{\{T_i < t\}} - \sum_{i=1}^{N} \left| f\left(\sigma_t^i\right) - f\left(\sigma_{t_1}^i\right) \right| \right] \\
&\quad \geq \liminf_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f\left(\sigma_{t_1}^i\right) \mathbb{I}_{\{T_i < t_1\}} - \limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left| f\left(\sigma_t^i\right) - f\left(\sigma_{t_1}^i\right) \right|,
\end{aligned}$$

where the first term equals $\mathbb{E}\left[f\left(\sigma_{t_1}^1\right) \mathbb{I}_{\{T_1 < t_1\}} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0 \right]$ for each rational time t_1. Next, by recalling Theorem 2.2.3 for $f(x,\, y,\, z) = |f(y) - f(z)|$ and $s = t_1$, we find that the second term equals $\mathbb{E}\left[|f\left(\sigma_t^1\right) - f\left(\sigma_{t_1}^1\right)| \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0 \right]$. Now taking $t_1 \longrightarrow t$ and using the Dominated Convergence Theorem and the fact that the random variable T_1 has a continuous distribution, we obtain

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(\sigma_t^i) \mathbb{I}_{\{T_i \leq t\}} \geq \mathbb{E}\left[f(\sigma_t^1) \mathbb{I}_{\{T_1 < t\}} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0 \right]. \tag{2.2.4}$$

Similarly, we have

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f\left(\sigma_t^i\right) \mathbb{I}_{\{T_i \leq t\}} \\
&\quad = \limsup_{N\to\infty} \frac{1}{N} \left[\sum_{i=1}^{N} f\left(\sigma_{t_2}^i\right) \mathbb{I}_{\{T_i < t\}} + \sum_{i=1}^{N} \left(f\left(\sigma_t^i\right) - f\left(\sigma_{t_2}^i\right) \right) \mathbb{I}_{\{T_i < t\}} \right] \\
&\quad \leq \limsup_{N\to\infty} \frac{1}{N} \left[\sum_{i=1}^{N} f\left(\sigma_{t_2}^i\right) \mathbb{I}_{\{T_i < t\}} + \sum_{i=1}^{N} \left| f\left(\sigma_t^i\right) - f\left(\sigma_{t_2}^i\right) \right| \right] \\
&\quad \leq \limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f\left(\sigma_{t_2}^i\right) \mathbb{I}_{\{T_i < t_2\}} + \limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left| f\left(\sigma_t^i\right) - f\left(\sigma_{t_2}^i\right) \right|
\end{aligned}$$

and by the same argument for the rational number $t_2 \longrightarrow t$, we find

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f\left(\sigma_t^i\right) \mathbb{I}_{\{T_i \leq t\}} \leq \mathbb{E}\left[f\left(\sigma_t^1\right) \mathbb{I}_{\{T_1 < t\}} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0 \right]. \tag{2.2.5}$$

Hence, by (2.2.4) and (2.2.5), the desired convergence holds in $\Omega''_f \subset \Omega'$ for all $t \geq 0$ and any f in $C_b(\mathbb{R};\, \mathbb{R})$ with positive values. By linearity, and since every continuous function can be decomposed into its continuous positive part and its continuous negative part, we can have our convergence result for any $f \in C_b(\mathbb{R};\, \mathbb{R})$. Let now $S = \{p_i : i \in \mathbb{N}\}$ be a countable basis of $C_b(\mathbb{R};\, \mathbb{R})$. Then, by linearity, the desired convergence holds for all $t \geq 0$, all $p \in [S]$ and all $\omega \in \Omega'' = \cap_{i\in\mathbb{N}}\Omega''_{p_i}$, with $\mathbb{P}(\Omega'') = 1$. Now for any $f \in C_b(\mathbb{R};\, \mathbb{R})$ and $\epsilon > 0$, we can pick $p \in [S]$ such that $||f - p||_\infty < \frac{\epsilon}{3}$, so we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} f(y)dv^N_{2,t}(x,y) - \int_{\mathbb{R}^2} f(y)dv_{2,t}(x,y)\right| \\
&\quad\leq \left|\int_{\mathbb{R}^2} (f(y)-p(y))dv^N_{2,t}(x,y)\right| + \left|\int_{\mathbb{R}^2} p(y)dv^N_{2,t}(x,y) - \int_{\mathbb{R}^2} p(y)dv_{2,t}(x,y)\right| \\
&\qquad\qquad + \left|\int_{\mathbb{R}^2} (p(y)-f(y))dv_{2,t}(x,y)\right| \\
&\quad\leq ||f-p||_\infty + \left|\int_{\mathbb{R}^2} p(y)dv^N_{2,t}(x,y) - \int_{\mathbb{R}^2} p(y)dv_{2,t}(x,y)\right| + ||f-p||_\infty \\
&\quad\leq 3 \times \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

for all N sufficiently large. Thus, we have our convergence result for all $t \geq 0$ and all $\omega \in \Omega'' = \cap_{i\in\mathbb{N}}\Omega''_{p_i}$ with $\mathbb{P}(\Omega'') = 1$, so we are done. □

Remark 2.2.6. A different study concerning particle systems and the asymptotic behaviour of weighted empirical measure processes like $v^N_{1,t}$ can be found in [8]

2.2.3 The two-dimensional SPDE

By Corollary 2.2.2 and Theorem 2.2.5, we have that $v^N_{1,t} = v^N_t - v^N_{2,t} \longrightarrow v_t - v_{2,t} =: v_{1,t}$ weakly, for all $t \geq 0$, $\mathbb{P}$-almost surely. Also, it follows from Corollary 2.2.4 and Theorem 2.2.5 that

$$\begin{aligned}
\int_{\mathbb{R}^2} f dv_{1,t} &= \mathbb{E}\left[f\left(X^1_t,\, \sigma^1_t\right)\mathbb{I}_{\{T_1>t\}} \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[f\left(X^1_t,\, \sigma^1_t\right)\mathbb{I}_{\{T_1>t\}} \,|\, W^0_\cdot,\, B^0_\cdot,\, C_1,\, \mathcal{G}^0\right] \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right],
\end{aligned}$$

for any $f \in C_b\left(\mathbb{R}^2;\, \mathbb{R}\right)$ and $t \geq 0$, $\mathbb{P}$-almost surely. Note also that conditional expectations given $(W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0)$ are defined on the component space $(\Omega^0,\, \mathcal{F}^0,\, \{\mathcal{F}^0_t\}_{t\geq 0},\, \mathbb{P}^0)$, while conditional expectations given $(W^0_\cdot,\, B^0_\cdot,\, C_1,\, \mathcal{G}^0)$ are defined on the product space $(\Omega^0,\, \mathcal{F}^0,\, \{\mathcal{F}^0_t\}_{t\geq 0},\, \mathbb{P}^0) \times (\Omega^C,\, \mathcal{F}^C,\, \mathbb{P}^C)$. It is therefore reasonable to study the behaviour of the process of measures $v_{t,C_1}(\cdot)$ defined as

$$v_{t,C_1}(\cdot) = \mathbb{P}\left(\left(X^1_t, \sigma^1_t\right) \in \cdot, T_1 > t \,|W^0_\cdot, B^0_\cdot, C_1, \mathcal{G}^0\right),$$

on the product space $(\Omega^0,\, \mathcal{F}^0,\, \{\mathcal{F}_t^0\}_{t\geq 0},\, \mathbb{P}^0) \times (\Omega^C,\, \mathcal{F}^C,\, \mathbb{P}^C)$. The behaviour of this process of measures is given in the following Theorem.

Theorem 2.2.7. *Let A be the two-dimensional differential operator mapping any smooth function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ to*

$$\begin{aligned} Af\,(x,\, y) \;&=\; \left(r_1 - \frac{h^2\,(y)}{2}\right) f_x\,(x,\, y) + k_1\,(\theta_1 - y)\, f_y\,(x,\, y) + \frac{1}{2} h^2\,(y)\, f_{xx}\,(x,\, y) \\ &\quad + \frac{1}{2}\xi_1^2 q^2(y) f_{yy}\,(x,\, y) + \xi_1 \rho_3 \rho_{1,1} \rho_{2,1} h(y) q(y) f_{xy}\,(x,\, y) \end{aligned}$$

for all $(x,\, y) \in \mathbb{R}^+ \times \mathbb{R}$. Then, for any $\omega^C \in \Omega^C$ (i.e for any fixed value of $C_1 = (k_1,\, \theta_1,\, \xi_1,\, r_1,\, \rho_{1,1},\, \rho_{2,1})$), the measure-valued stochastic process $v_{t,C_1}\left(\cdot,\, \omega^C\right)$ defined on $(\Omega^0,\, \mathcal{F}^0,\, \{\mathcal{F}_t^0\}_{t\geq 0},\, \mathbb{P}^0)$, satisfies the following weak form SPDE

$$\begin{aligned} \int_{\mathbb{R}^2} f\,(x,\, y)\, dv_{t,C_1}\,(x,\, y) \;&=\; \int_{\mathbb{R}^2} f\,(x,\, y)\, dv_{0,C_1}\,(x,\, y) \\ &\quad + \int_0^t \int_{\mathbb{R}^2} Af\,(x,\, y)\, dv_{s,C_1}\,(x,\, y)\, ds \\ &\quad + \rho_{1,1} \int_0^t \int_{\mathbb{R}^2} h\,(y)\, f_x\,(x,\, y)\, dv_{s,C_1}\,(x,\, y)\, dW_s^0 \\ &\quad + \xi_1 \rho_{2,1} \int_0^t \int_{\mathbb{R}^2} q(y) f_y\,(x,\, y)\, dv_{s,C_1}\,(x,\, y)\, dB_s^0, \end{aligned}$$

for all $t \geq 0$ and any $f \in C_0^{test} = \{g \in C_b^2\,(\mathbb{R}^+ \times \mathbb{R}) :\, g\,(0,\, y) = 0,\, \forall\, y \in \mathbb{R}\}$.

Proof. By using Ito's formula for the stopped two-dimensional stochastic process $\{(X_t^1,\, \sigma_t^1) : t \geq 0\}$ given by (1.0.2) and by recalling that $f\,(0,\, y) = 0$ for all y, we obtain

$$\begin{aligned} &f\left(X_{t\wedge T_1}^1,\, \sigma_t^1\right) \\ &\quad = f\left(x^1,\, \sigma^1\right) \\ &\qquad + \int_0^t \left[f_x\left(X_s^1,\, \sigma_s^1\right) \left(r_1 - \frac{h^2\left(\sigma_s^1\right)}{2}\right) + k_1 f_y\left(X_s^1,\, \sigma_s^1\right)\left(\theta_1 - \sigma_s^1\right) \right] \mathbb{I}_{\{T_1 > s\}} ds \\ &\qquad + \frac{1}{2}\int_0^t \left[f_{xx}\left(X_s^1,\, \sigma_s^1\right) h^2\left(\sigma_s^1\right) + \xi_1^2 f_{yy}\left(X_s^1,\, \sigma_s^1\right) q^2\left(\sigma_s^1\right) \right] \mathbb{I}_{\{T_1 > s\}} ds \\ &\qquad + \xi_1 \rho_3 \rho_{1,1} \rho_{2,1} \int_0^t f_{xy}\left(X_s^1,\, \sigma_s^1\right) h\left(\sigma_s^1\right) q\left(\sigma_s^1\right) \mathbb{I}_{\{T_1 > s\}} ds \\ &\qquad + \int_0^t f_x\left(X_s^1,\, \sigma_s^1\right) \mathbb{I}_{\{T_1 > s\}} h\left(\sigma_s^1\right) \rho_{1,1} dW_s^0 \\ &\qquad + \xi_1 \int_0^t f_y\left(X_s^1,\, \sigma_s^1\right) \mathbb{I}_{\{T_1 > s\}} q\left(\sigma_s^1\right) \rho_{2,1} dB_s^0 \end{aligned}$$

$$+\int_0^t f_x\left(X_s^1,\,\sigma_s^1\right)\mathbb{I}_{\{T_1>s\}}h(\sigma_s^1)\sqrt{1-\rho_{1,1}^2}dW_s^1$$
$$+\xi_1\int_0^t f_y\left(X_s^1,\,\sigma_s^1\right)\mathbb{I}_{\{T_1>s\}}q\left(\sigma_s^1\right)\sqrt{1-\rho_{2,1}^2}dB_s^1$$

and the desired result follows by taking conditional expectations given $(W_\cdot^0,\,B_\cdot^0)$, C_1 and $\mathcal{G}^0$, by noticing that Ito integrals with respect to $B_\cdot^1$ and $W_\cdot^1$ vanish due to the independence between the market and the idiosyncratic Brownian motions, and by taking the given coefficients out of the conditional expectations. □

2.3 Volatility Analysis - A Malliavin Calculus approach

Now that we have connected our problem to the study of the probabilistic solution of an SPDE, we need to establish the best possible regularity result for that solution. Before showing that the measure-valued process $\mathbb{E}\left[\cdot(X_t^1,\,\sigma_t^1)\mathbb{I}_{T_1>t}\,|\,B_\cdot^0,\,W_\cdot^0,\,C_1,\,\mathcal{G}^0\right]$ does indeed have a regular density $\mathbb{P}^0\times\mathbb{P}^C$-almost surely, it is natural (and important as we will see) to ask whether the same holds $\mathbb{P}^0$-almost surely for the 1-dimensional distribution-valued process $\mathbb{E}\left[\cdot(\sigma_t)\,|\,B_\cdot^0,\,\mathcal{G}^0\right]$, whose value at time t maps any suitable function f to $\mathbb{E}\left[f(\sigma_t)\,|\,B_\cdot^0,\,\mathcal{G}^0\right]$, where $\sigma_\cdot$ is a general mean-reverting volatility process driven by a combination of $B_\cdot^0$ and $B_\cdot^1$, that is a process satisfying

$$d\sigma_t=k(\theta-\sigma_t)dt+\xi q\left(\sigma_t\right)\sqrt{1-\rho_2^2}dB_t^1+\xi q\left(\sigma_t\right)\rho_2 dB_t^0,\ t\geq 0. \tag{2.3.1}$$

for some suitable function q, with σ_0 being $F_0^0\times F_0^1$-measurable.

2.3.1 The main results

As we have already mentioned, apart from the well-known CIR model where we take q to be the square root function, we will consider a class of functions q which are bounded away from zero and have bounded derivatives. For the CIR case, we will need to assume that $\frac{k\theta}{\xi^2}>x^*$, where x^* is much larger than $\frac{1}{2}$, i.e the bound given by the usual Feller condition which prevents the volatility process from being absorbed at zero. Then, if we assume that $\rho_2\in(-1,\,1)$, the answer to our question is given in the next two theorems

Theorem 2.3.1. *Let $q(\cdot)$ be a $C^3\left(\mathbb{R}\right)$ function with bounded and $\mathcal{O}\left(\frac{1}{|x|}\right)$ (as $|x|\longrightarrow+\infty$) derivatives up to third order, which is also lower bounded by a positive constant. Suppose also that σ_0 is a random variable in $L^p\left(\Omega^0,\,\mathcal{F}_0^0,\,\mathbb{P}^0\right)$ for all $p\geq 0$. Then*

we have the following result $\mathbb{P}^0$ *- almost surely: the conditional probability measure* $\mathbb{P}(\sigma_t \in \cdot \,|\, B^0_\cdot, \mathcal{G}^0)$ *possesses a continuous density* $p_t(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$*, for all* $t > 0$*. Moreover, for any* $T > 0$ *we have*

$$M(\cdot) := \operatorname*{ess\,sup}_{y \in \mathbb{R},\, \omega^0 \in \Omega^0} p_\cdot(y \,|\, B^0_\cdot, \mathcal{G}^0)\left(\omega^0\right) \in L^1\left([0,\, T]\right)$$

and also

$$M^\alpha_{B^0_\cdot, \mathcal{G}^0}(\cdot) := \sup_{y \in \mathbb{R}} \left(|y|^\alpha p_\cdot(y \,|\, B^0_\cdot, \mathcal{G}^0)\right) \in L^p\left(\Omega^0 \times [0,\, T]\right)$$

for any $\alpha > 0$ *and* $p < 2$*.*

Theorem 2.3.2. *Let* $q(z) = \sqrt{|z|}$ *for all* $z \in \mathbb{R}$*, and suppose that* $\frac{k\theta}{\xi^2} > x^*$*, where* x^* *is the largest root of the equation*

$$16x^3 - 60x^2 + 24x - 3 = 0. \tag{2.3.2}$$

Suppose also that σ_0 *is a positive random variable which is bounded away from zero and infinity. Then we have the following result* $\mathbb{P}^0$ *- almost surely: the conditional probability measure* $\mathbb{P}(\sigma_t \in \cdot \,|\, B^0_\cdot, \mathcal{G}^0)$ *possesses a continuous density* $p_t(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$ *which is supported in* $[0, \infty)$*, for all* $t > 0$*. Moreover, for any* $T > 0$*, any* $\alpha \geq 0$ *and all sufficiently small* $p > 1$*, we have*

$$M^\alpha_{B^0_\cdot, \mathcal{G}^0}(\cdot) := \sup_{y \geq 0} \left(y^\alpha p_\cdot(y \,|\, B^0_\cdot, \mathcal{G}^0)\right) \in L^p\left(\Omega^0 \times [0,\, T]\right)$$

Remark 2.3.3. The class of functions $q(\cdot)$ satisfying the conditions of Theorem 2.3.1 obviously contains all functions which are identically equal to a positive constant, making the Theorem applicable to an Ornstein-Uhlenbeck volatility setting. However, it is actually a much bigger class, which contains positive C^3 functions almost behaving like a positive constant for large x. Indeed, it is not hard to check that for any positive and bounded function $\tilde{q}$ with bounded derivatives, the function defined as $\tilde{q}\left(e^{-\frac{1}{x}}\right)$ for $x > 0$ and $\tilde{q}(0)$ otherwise, satisfies the conditions of Theorem 2.3.1.

To prove the above two Theorems, we need a few results that involve the notion of Malliavin differentiability. The Malliavin derivative of a random variable adapted to a Brownian motion is a stochastic process measuring, in some sense, the rate of change of the random variable when the Brownian path changes at any time t. Extending this to random variables taking values in Banach spaces, we can define the k-th Malliavin derivative as a random function of k time variables (provided that it exists). The existence and behaviour of these derivatives are inextricably connected to the existence of a regular density for the random variable. We refer to [37] for

the basics of Malliavin calculus and, as in there, we denote by $\mathbb{D}^{n,p}(V)$ the space of n-times Malliavin differentiable random variables taking values in the Banach space V, whose k-th Malliavin derivative has an L^2 norm (as a function in k time variables taking values in V) of a finite L^p norm as a random variable, for all $0 \leq k \leq n$.

A generalization of Malliavin calculus which applies to conditional probability measures is the Partial Malliavin Calculus which has been developed in [38]. In this section we are working under the conditional probability measure $\mathbb{P}(\cdot \mid B^0_\cdot, \mathcal{G}^0)$, which means that Partial Malliavin Calculus is the natural tool for proving our results. However, it is not hard to see that in our setting we can $\mathbb{P}^0$-almost surely fix an $\omega^0 \in \Omega^0$, and work under $\mathbb{P}(\sigma_t \in \cdot \mid B^0_\cdot, \mathcal{G}^0) = \mathbb{P}^1(\{\omega^1 \in \Omega^1 : \sigma_t(\omega^1, \omega^0) \in \cdot\})$ for that given ω^0, where ordinary Malliavin calculus can be used with respect to the Brownian motion $B^1_\cdot$. It is not hard to check that the Malliavin derivative with respect to $B^1_\cdot$ for any continuous path of $B^0_\cdot$, coincides with the partial Malliavin derivative in the sense of [38], when the random subspace $K(\omega)$ is the orthogonal complement of the space generated by the Malliavin derivatives of B^0_q with respect to $(B^0_\cdot, B^1_\cdot)$, for $q \in \mathbb{Q}^+$.

For both Theorems, we need a lemma that connects the existence of a regular density to the existence of a regular non-vanishing Malliavin Derivative. There are many results of this kind in the literature, but in this situation we will need something stronger. The next lemma, which is exactly what we are going to need, contains a generalization of Proposition 2.1.1 from page 78 in [37], and its proof can be found in the Appendix.

Lemma 2.3.4. *Let $B_\cdot$ be a Brownian motion defined on $[0,T] \times \Omega$ for some $T > 0$ and some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let F be a random variable which is adapted to the Brownian motion $B_\cdot$. Suppose also that for some $p, p', r, r', \lambda, \lambda' > 1$ and $\alpha \geq 0$, with $p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'} = \frac{pr'}{\lambda'} + \frac{pr'}{\lambda} = 1$, we have $F \in L^{\alpha\lambda \vee p'}(\Omega) \cap \mathbb{D}^{1,2\lambda' \vee \frac{pr'}{pr'-1}}(\Omega) \cap \mathbb{D}^{2,pr}(\Omega)$ and also $\frac{|F|^m}{\|D.F\|^2_{L^2([0,T])}} \in L^{pr'}(\Omega)$ for $m \in \{0, \alpha\}$. Then, the domain of the adjoint of the derivative operator $D : L^{p'}(\Omega) \cap \mathbb{D}^{1,2\vee\frac{pr'}{pr'-1}}(\Omega) \longrightarrow L^{2\vee\frac{pr'}{pr'-1}}(\Omega; L^2([0,T]))$ (which is an extension of the standard Skorokhood integral δ) contains the process $\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}$. Moreover, F possesses a bounded and continuous density f_F which satisfies*

$$\sup_{x \in \mathbb{R}} |x|^\alpha f_F(x) \leq \left\| |F|^\alpha \delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}} \right) \right\|_{L^p(\Omega)}$$

$$\leq (C+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}F\right\|^{pr}_{L^2\left([0,T]^2\right)}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^\alpha}{\|D_\cdot F\|^2_{L^2([0,T])}}\right|^{pr'}\right]$$
$$+C\mathbb{E}^{\frac{1}{pr}}\left[\|D_\cdot F\|^{pr}_{L^2([0,T])}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^\alpha}{\|D_\cdot F\|^2_{L^2([0,T])}}\right|^{pr'}\right], \tag{2.3.3}$$

for some $C>0$, with the RHS of the above being finite by our assumptions.

The finiteness of certain moments of F is a main assumption in the above crucial Lemma, and this shows that we are going to need that our volatility processes are sufficiently integrable. This is given in the following technical lemma, the proof of which has also been put in the Appendix.

Lemma 2.3.5. *Fix a $T>0$. Then, under the assumptions of Theorem 2.3.1 or Theorem 2.3.2, for any $p\geq 0$ we have*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\sigma_t^p\right]<\infty.$$

2.3.2 Proof of Theorem 2.3.1: The regular vol-of-vol case

We can proceed now to the proof of the first Theorem

Proof of Theorem 2.3.1. We will show that σ_t possesses regular first and second order Malliavin derivatives with respect to the Brownian motion $B^1_\cdot$, under the conditional probability measure $\mathbb{P}(\cdot\,|\,B^0_\cdot,\mathcal{G}^0)$, $\mathbb{P}^0$ - almost surely, which will allow us to obtain the desired result by recalling Lemma 2.3.4.

We fix a $T>0$ and we consider the strictly increasing function $Q(y)=\int_0^y\frac{1}{q(z)}dz$. Then, by using Ito's formula on (2.3.1), we find that $v_t=Q(\sigma_t)$ satisfies the SDE

$$dv_t=V\left(v_t\right)dt+\xi\sqrt{1-\rho_2^2}dB^1_t+\xi\rho_2 dB^0_t \tag{2.3.4}$$

for all $t\in[0,\,T]$, where

$$V(x)=\left[k\frac{(\theta-Q^{-1}\left(x\right))}{q\left(Q^{-1}\left(x\right)\right)}-\frac{\xi^2}{2}q'\left(Q^{-1}\left(x\right)\right)\right] \tag{2.3.5}$$

for any $x\in\mathbb{R}$. We will compute first the Malliavin derivative of v_t with respect to the Brownian motion $B^1_\cdot$, under the probability measure $\mathbb{P}(\cdot\,|\,B^0_\cdot,\mathcal{G}^0)$. Since we have $(Q^{-1}(x))'=q\left(Q^{-1}(x)\right)$, we can easily compute

$$V'(x) \;=\; q\left(Q^{-1}(x)\right)$$

$$\times \left[k \frac{-q\left(Q^{-1}(x)\right) - \left(\theta - Q^{-1}(x)\right) q'\left(Q^{-1}(x)\right)}{q^2\left(Q^{-1}(x)\right)} - \frac{\xi^2}{2} q''\left(Q^{-1}(x)\right) \right] \tag{2.3.6}$$

and

$$\begin{aligned} V''(x) &= q\left(Q^{-1}(x)\right) q'\left(Q^{-1}(x)\right) \\ &\quad \times \left[k \frac{-q\left(Q^{-1}(x)\right) - \left(\theta - Q^{-1}(x)\right) q'\left(Q^{-1}(x)\right)}{q^2\left(Q^{-1}(x)\right)} - \frac{\xi^2}{2} q''\left(Q^{-1}(x)\right) \right] \\ &\quad + 2k q'\left(Q^{-1}(x)\right) - k(\theta - Q^{-1}(x)) \left[q''\left(Q^{-1}(x)\right) - 2 \frac{\left(q'\left(Q^{-1}(x)\right)\right)^2}{q\left(Q^{-1}(x)\right)} \right] \\ &\quad - \frac{\xi^2}{2} q^{(3)}\left(Q^{-1}(x)\right) \left(q\left(Q^{-1}(x)\right)\right)^2 \end{aligned} \tag{2.3.7}$$

which are both bounded by our assumptions. This allows us to recall Theorem 2.2.1 from page 102 in [37] and the Remark after its proof, which imply that the desired Malliavin derivative exists under $\mathbb{P}$ and it satisfies

$$D_{t'} v_t = \xi \sqrt{1 - \rho_2^2} - \int_{t'}^{t} V'(v_t) D_{t'} \sigma_s ds \tag{2.3.8}$$

for $t \in [t', T]$. By looking at the proof of that Theorem, we see that the underlying probability measure does not play any role as long as we are differentiating with respect to the path of a Brownian motion, which means that here we can have the same result under the probability measure $\mathbb{P}(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$ as well. Then, (2.3.8) is a linear ODE in $t \in [t', T]$, which can be solved to give

$$D_{t'} v_t = \xi \sqrt{1 - \rho_2^2} e^{-\int_{t'}^{t} V'(v_s) ds} \tag{2.3.9}$$

for any $0 \leq t' \leq t \leq T$, while we have $D_{t'} v_t = 0$ for $0 \leq t < t' \leq T$. By the boundedness of V', the above Malliavin derivative is positive, and for any $t \in [0, T]$, it belongs to $L^\infty\left(\Omega^0 \times \Omega^1 \times [0, T]\right)$. Therefore, we can use the standard Malliavin chain rule (Proposition 1.2.2 on page 29 in [37]) to obtain

$$\begin{aligned} D_{t'} \sigma_t &= D_{t'} Q^{-1}(v_t) \\ &= q\left(Q^{-1}(v_t)\right) D_{t'} v_t \\ &= \xi \sqrt{1 - \rho_2^2} q\left(Q^{-1}(v_t)\right) e^{-\int_{t'}^{t} V'(v_s) ds} \\ &= \xi \sqrt{1 - \rho_2^2} q(\sigma_t) e^{-\int_{t'}^{t} V'(Q(\sigma_s)) ds} \end{aligned} \tag{2.3.10}$$

which belongs to $L^\infty\left(\Omega^0 \times \Omega^1 \times [0, T]\right)$ as well.

To compute the second order derivative of σ_t under the probability measure $\mathbb{P}(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$, we use again the standard Malliavin chain rule to obtain

$$\begin{aligned} D_{t''}V'(v_s) &= V''(v_s)D_{t''}v_s \\ &= \xi\sqrt{1-\rho_2^2}V''(v_s)e^{-\int_{t''}^s V'(v_{s'})ds'} \end{aligned} \tag{2.3.11}$$

for any $s \in [t', t]$ with $0 \le t' \le t \le T$ and $t'' \in [0, T]$. It is not hard to see that the absolute value of this derivative is bounded by some $b_T > 0$ depending only on T. Thus, for any $h_\cdot \in L^2(\Omega^0 \times \Omega^1 \times [0, T])$, if we denote by δ the Skorokhood integral with respect to $B^1_\cdot$, by Fubini's Theorem and duality we have

$$\begin{aligned} \mathbb{E}\left[\int_0^t h_{t''}\int_{t'}^t D_{t''}V'(v_s)dsdt'' \,|\, B^0_\cdot, \mathcal{G}^0\right] &= \int_{t'}^t \mathbb{E}\left[\int_0^t D_{t''}V'(v_s)h_{t''}dt'' \,|\, B^0_\cdot, \mathcal{G}^0\right]ds \\ &= \int_{t'}^t \mathbb{E}\left[V'(v_s)\delta(h_\cdot) \,|\, B^0_\cdot, \mathcal{G}^0\right]ds \\ &= \mathbb{E}\left[\delta(h_\cdot)\int_{t'}^t V'(v_s)ds \,|\, B^0_\cdot, \mathcal{G}^0\right] \end{aligned}$$

Therefore, if we approximate $\int_{t'}^t V'(v_s)ds$ by square integrable random variables which are Malliavin differentiable, we can show that the Malliavin derivatives converge in L^2 to $\int_{t'}^t D_{t''}V'(v_s)ds$, and Lemma 1.2.3 on page 30 in [37] implies then that $\int_{t'}^t V'(v_s)ds$ has a Malliavin derivative which equals $\int_{t'}^t D_{t''}V'(v_s)ds$. Combining the last two results we find that

$$D_{t''}\int_{t'}^t V'(v_s)ds = \xi\sqrt{1-\rho_2^2}\int_{t'}^t V''(v_s)e^{-\int_{t''}^s V'(v_{s'})ds'}ds \tag{2.3.12}$$

which has an absolute value bounded by Tb_T. Then, since $\int_{t'}^t V'(v_s)ds$ is bounded for any t and t', by (2.3.10) we have that

$$D_{t'}\sigma_t = \xi\sqrt{1-\rho_2^2}q\left(Q^{-1}(v_t)\right)e^{-F_{t,t'}\left(\int_{t'}^t V'(v_s)ds\right)}$$

for a smooth and compactly supported function $F_{t,t'}$ which is the identity on a compact set containing all the possible values of $\int_{t'}^t V'(v_s)ds$. Hence, applying the Malliavin chain rule once more, we finally obtain

$$\begin{aligned} D_{t',t''}\sigma_t &= \xi\sqrt{1-\rho_2^2}q'\left(Q^{-1}(v_t)\right)q\left(Q^{-1}(v_t)\right)e^{-\int_{t'}^t V'(v_s)ds}D_{t''}v_t \\ &\quad +\xi\sqrt{1-\rho_2^2}q\left(Q^{-1}(v_t)\right)e^{-\int_{t'}^t V'(v_s)ds}D_{t''}\int_{t'}^t V'(v_s)ds \\ &= \xi^2\left(1-\rho_2^2\right)q'\left(Q^{-1}(v_t)\right)q\left(Q^{-1}(v_t)\right)e^{-\int_{t'}^t V'(v_s)ds}e^{-\int_{t''}^t V'(v_s)ds} \end{aligned}$$

$$
\begin{aligned}
&+\xi^2\left(1-\rho_2^2\right) q\left(Q^{-1}(v_t)\right) e^{-\int_{t'}^t V'(v_s)ds}\int_{t'}^t V''(v_s)e^{-\int_{t''}^s V'(v_{s'})ds'}ds\\
=\;& \xi^2\left(1-\rho_2^2\right) q'\left(\sigma_t\right) q\left(\sigma_t\right) e^{-\int_{t'}^t V'(Q(\sigma_s))ds}e^{-\int_{t''}^t V'(Q(\sigma_s))ds}\\
&+\xi^2\left(1-\rho_2^2\right) q\left(\sigma_t\right) e^{-\int_{t'}^t V'(Q(\sigma_s))ds}\int_{t'}^t V''(Q(\sigma_s))e^{-\int_{t''}^s V'(Q(\sigma_{s'}))ds'}ds
\end{aligned}
\tag{2.3.13}
$$

and by our assumptions, this has also an absolute value which is bounded by some $b'_T > 0$ depending only on T.

By Lemma 2.3.5 now, we have that $\sigma_t \in L^p\left(\Omega^0 \times \Omega^1\right)$ for any $p > 1$ and $t \leq T$, under the probability measure $\mathbb{P}$, which implies that $\sigma_t \in L^p\left(\Omega^1\right)$ holds under the conditional probability measure $\mathbb{P}\left(\cdot \,|\, B^0_\cdot, \mathcal{G}^0\right)$, $\mathbb{P}^0$- almost surely. Combining this with the above results, we deduce that $\sigma_t \in L^p\left(\Omega^1\right) \cap \mathbb{D}^{\infty,2}\left(\Omega^1\right)$ with respect to $B^1_\cdot$ for all $t \leq T$, under the probability measure $\mathbb{P}\left(\cdot \,|\, B^0_\cdot, \mathcal{G}^0\right)$ ($\mathbb{P}^0$- almost surely). Moreover, it is not hard to check that for all $t' < t$, $D_{t'}\sigma_t$ is bounded between the positive quantities $b''_T := \xi\sqrt{1-\rho_2^2}m_q e^{-TM_{|V'|}}$ and $\tilde{b}_T := \xi\sqrt{1-\rho_2^2}M_q e^{TM_{|V'|}}$, where we denote by m_f the minimum of a function f and by M_f the maximum of that function. This allows us to recall Lemma 2.3.4 for $\alpha = 0$ and deduce that the conditional probability measure $\mathbb{P}(\sigma_t \in \cdot \,|\, B^0_\cdot, \mathcal{G}^0)$ posseses a continuous density $p_t(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$, which satisfies

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}} p_t(y \,|\, B^0_\cdot, \mathcal{G}^0)\\
&\quad\leq (C+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}\sigma_t\right\|^{pr}_{L^2\left([0,T]^2\right)} \,|\, B^0_\cdot, \mathcal{G}^0\right]\\
&\qquad\qquad\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{1}{\|D_\cdot\sigma_t\|^2_{L^2([0,T])}}\right|^{pr'} \,|\, B^0_\cdot, \mathcal{G}^0\right]\\
&\qquad\quad+C\mathbb{E}^{\frac{1}{pr}}\left[\|D_\cdot\sigma_t\|^{pr}_{L^2([0,T])} \,|\, B^0_\cdot, \mathcal{G}^0\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{1}{\|D_\cdot\sigma_t\|^2_{L^2([0,T])}}\right|^{pr'} \,|\, B^0_\cdot, \mathcal{G}^0\right],\\
&\quad= (C+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}\sigma_t\right\|^{pr}_{L^2\left([0,t]^2\right)} \,|\, B^0_\cdot, \mathcal{G}^0\right]\\
&\qquad\qquad\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{1}{\|D_\cdot\sigma_t\|^2_{L^2([0,t])}}\right|^{pr'} \,|\, B^0_\cdot, \mathcal{G}^0\right]\\
&\qquad\quad+C\mathbb{E}^{\frac{1}{pr}}\left[\|D_\cdot\sigma_t\|^{pr}_{L^2([0,t])} \,|\, B^0_\cdot, \mathcal{G}^0\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{1}{\|D_\cdot\sigma_t\|^2_{L^2([0,t])}}\right|^{pr'} \,|\, B^0_\cdot, \mathcal{G}^0\right],\\
&\quad\leq (C+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left(\sqrt{t^2 b'_T}\right)^{pr} \,|\, B^0_\cdot, \mathcal{G}^0\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left(\frac{1}{tb''_T}\right)^{pr'} \,|\, B^0_\cdot, \mathcal{G}^0\right]
\end{aligned}
$$

$$+C\mathbb{E}^{\frac{1}{pr}}\left[\left(\sqrt{t\tilde{b}_T}\right)^{pr} \mid B^0_\cdot,\, \mathcal{G}^0\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left(\frac{1}{tb''_T}\right)^{pr'} \mid B^0_\cdot,\, \mathcal{G}^0\right]$$
$$= (C+2)\sqrt{t^2 b'_T} \times \frac{1}{tb''_T} + C\sqrt{t\tilde{b}_T} \times \frac{1}{tb''_T}$$

where C is a deterministic positive constant, p, r can be anything bigger than 1, and $\tilde{b}_T, b'_T, b''_T$ are the deterministic bounds obtained earlier. The above now gives

$$\operatorname*{ess\,sup}_{y\in\mathbb{R},\,\omega^0\in\Omega^0} p_t(y \mid B^0_\cdot,\, \mathcal{G}^0) \leq C' + C''\frac{1}{\sqrt{t}}$$

for some deterministic constants C' and C'', which can be integrated with respect to t in $[0,\, T]$ to give

$$\int_0^T \operatorname*{ess\,sup}_{y\in\mathbb{R},\,\omega^0\in\Omega^0} p_t(y \mid B^0_\cdot,\, \mathcal{G}^0)dt \leq C'T + C''\sqrt{T}.$$

and this completes the proof of the first estimate.

For the second estimate, we recall again Lemma 2.3.4 but for $a > 0$, and we use the same bounds as previously for the Malliavin derivatives to obtain:

$$\begin{aligned}
&\sup_{y\in\mathbb{R}} |y|^\alpha p_t(y \mid B^0_\cdot,\, \mathcal{G}^0) \\
&\qquad \leq (C+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left(\sqrt{t^2 b'_T}\right)^{pr} \mid B^0_\cdot,\, \mathcal{G}^0\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left(\frac{1}{tb''_T}\right)^{pr'} |\sigma_t|^\alpha \mid B^0_\cdot,\, \mathcal{G}^0\right] \\
&\qquad\quad +C\mathbb{E}^{\frac{1}{pr}}\left[\left(\sqrt{t\tilde{b}_T}\right)^{pr} \mid B^0_\cdot,\, \mathcal{G}^0\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left(\frac{1}{tb''_T}\right)^{pr'} |\sigma_t|^\alpha \mid B^0_\cdot,\, \mathcal{G}^0\right].
\end{aligned}$$

Therefore, by Holder's inequality we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{y\in\mathbb{R}} |y|^\alpha p_t(y \mid B^0_\cdot,\, \mathcal{G}^0)\right)^p\right] \\
&\qquad \leq (C+2)\,\mathbb{E}^{\frac{1}{r}}\left[\left(\sqrt{t^2 b'_T}\right)^{pr}\right] \times \mathbb{E}^{\frac{1}{r'}}\left[\left(\frac{1}{tb''_T}\right)^{pr'} |\sigma_t|^\alpha\right] \\
&\qquad\quad +C\mathbb{E}^{\frac{1}{r}}\left[\left(\sqrt{t\tilde{b}_T}\right)^{pr}\right] \times \mathbb{E}^{\frac{1}{r'}}\left[\left(\frac{1}{tb''_T}\right)^{pr'} |\sigma_t|^\alpha\right] \\
&\qquad \leq \left((C+2)\left(\frac{\sqrt{b'_T}}{b''_T}\right)^p + Ct^{-\frac{p}{2}}\left(\frac{\sqrt{\tilde{b}_T}}{b''_T}\right)^p\right)\mathbb{E}^{\frac{1}{r'}}\left[\sup_{0\leq s\leq T} |\sigma_s|^\alpha\right]
\end{aligned}$$

whose integral in t over $[0,\, T]$ is finite by Lemma 2.3.5, since $p < 2$. This completes the proof of the Theorem. □

2.3.3 Proof of Theorem 2.3.2: The CIR volatility case

To prove Theorem 2.3.2 which is about the CIR volatility case, we will follow the same approach. It is proven in [3] that the CIR process has a Malliavin derivative under the probability measure $\mathbb{P}$. In [3], it is also proven that under the strong assumption $k\theta > \frac{3}{4}\xi^2$ (obviously weaker than $\frac{k\theta}{\xi^2} > x^*$), a second Malliavin derivative with some regularity exists as well. The next two lemmas extend these results to the case where the path of the market noise $B^0_\cdot$ is given, and where we differentiate with respect to $B^1_\cdot$. The proofs of these extensions are more or less based on the same ideas as the corresponding initial results (except that Lemma 2.3.5 is also needed at some points) and can be found in the Appendix.

Lemma 2.3.6. *Suppose that the assumptions of Theorem 2.3.2 are satisfied. Then, $\mathbb{P}^0$-almost surely, the process $\{\sigma_t : t \geq 0\}$ has a Malliavin derivative under the probability measure $\mathbb{P}(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$ with respect to the Brownian Motion $B^1_\cdot$, which is given by*

$$D_{t'}\sigma_t = \xi\sqrt{1-\rho_2^2}\, e^{-\int_{t'}^t \left[\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right)\frac{1}{\sigma_s} + \frac{k}{2}\right] ds} \sqrt{\sigma_t}, \tag{2.3.14}$$

for all $t > 0$ and $0 \leq t' \leq t$. This is a process in t' which belongs to $L^r_{B^0_\cdot, \mathcal{G}^0}([0,\, t] \times \Omega^1)$ $\mathbb{P}^0$-almost surely, for any fixed $r \geq 1$ and any $t \geq 0$, where the notation $L^r_{B^0_\cdot, \mathcal{G}^0}$ is used for an L^r space when the probability measure $\mathbb{P}$ is replaced by $\mathbb{P}(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$.

Lemma 2.3.7. *Suppose that the assumptions of Theorem 2.3.2 are satisfied. Then, $\mathbb{P}^0$-almost surely, for any $1 \leq r < \frac{4k\theta}{3\xi^2}$ and $T > 0$, the process $\{\sigma_t : 0 < t \leq T\}$ belongs to the space $\mathbb{D}^{2,r}(\Omega^1)$ under $\mathbb{P}(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$ with respect to the Brownian Motion $B^1_\cdot$. Moreover, the second order Malliavin derivative is given by*

$$D^2_{t',t''}\sigma_t = D_{t'}\sigma_t \times \left[\int_{t'}^t \left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right)\frac{1}{\sigma_s^2} D_{t''}\sigma_s ds + \frac{1}{2\sigma_t} D_{t''}\sigma_t\right], \tag{2.3.15}$$

for all $0 < t \leq T$ and $0 \leq t', t'' \leq t$, where the first order derivatives are given by Lemma 2.3.6. Furthermore we have

$$\mathbb{E}\left[\sup_{0\leq t', t''\leq t\leq T} |D^2_{t',t''}\sigma_t|^r\right] < \infty. \tag{2.3.16}$$

The same holds for the process $v_t = \sqrt{\sigma_t}$, but this time the second Malliavin Derivative is given by

$$D^2_{t',t''}v_t = D_{t'}v_t \times \left[\int_{t'}^t \left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right)\frac{1}{\sigma_s^2} D_{t''}\sigma_s ds\right]. \tag{2.3.17}$$

We are now ready to prove Theorem 2.3.1

Proof of Theorem 2.3.2. By Lemmas 2.3.5, 2.3.6 and 2.3.7, we have that σ_t satisfies the assumptions of Lemma 2.3.4 for some $p, r > 1$ with $pr < \frac{4k\theta}{3\xi^2}$, any $\alpha \geq 0$, and any $\lambda \geq pr'$, under the conditional probability measure $\mathbb{P}(\cdot \,|\, B_\cdot^0, \mathcal{G}^0)$, since we can show that p, r can be chosen such that $\frac{\sigma_t^\alpha}{\|D_\cdot \sigma_t\|^2_{L^2([0,T])}} \in L^{pr'}_{B_\cdot^0, \mathcal{G}^0}(\Omega^1)$, $\mathbb{P}^0$-almost surely. Indeed, using the Cauchy-Schwartz inequality and recalling Theorem 3.1 from [26], we find that

$$\begin{aligned}
&\mathbb{E}\left[\frac{\sigma_t^{pr'\alpha}}{\|D_\cdot \sigma_t\|^{2pr'}_{L^2([0,T])}}\right] \\
&\qquad \leq \frac{\xi\sqrt{1-\rho_2^2}}{t^{2pr'}}\mathbb{E}\left[\sigma_t^{pr'(\alpha-1)}\left(\int_0^t e^{2\int_{t'}^t\left[\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right)\frac{1}{\sigma_s}+\frac{k}{2}\right]ds}\right)^{pr'} dt'\right] \\
&\qquad \leq \frac{\xi\sqrt{1-\rho_2^2}}{t^{pr'}}\mathbb{E}\left[\sigma_t^{pr'(\alpha-1)} e^{2pr'\int_0^t\left[\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right)\frac{1}{\sigma_s}+\frac{k}{2}\right]ds}\right] \\
&\qquad = \frac{\xi\sqrt{1-\rho_2^2}}{t^{pr'}}\mathbb{E}\left[\mathbb{E}\left[\sigma_t^{pr'(\alpha-1)} e^{2pr'\int_0^t\left[\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right)\frac{1}{\sigma_s}+\frac{k}{2}\right]ds} \,|\, \sigma_0\right]\right] \\
&\qquad \leq \frac{\tilde{C}}{t^{pr'}}\gamma_t^{v-pr'(\alpha-1)}\mathbb{E}\left[\sigma_0^v H\left(-\gamma_t\sigma_0 e^{-kt}\right)\right] \qquad (2.3.18)
\end{aligned}$$

for some $c, \tilde{C} > 0$, provided that $2pr'\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right) < \frac{\xi^2}{8}\left(\frac{2k\theta}{\xi^2}-1\right)^2$ and $pr'(\alpha-1) > -\frac{2k\theta}{\xi^2} - v$, where

$$v = \frac{1}{2}\left(-\left(\frac{2k\theta}{\xi^2}-1\right)+\sqrt{\left(\frac{2k\theta}{\xi^2}-1\right)^2 - 2pr'\left(\frac{4k\theta}{\xi^2}-1\right)}\right), \qquad (2.3.19)$$

$\gamma_t = \frac{2k}{\xi^2}\left(1-e^{-kt}\right)^{-1} > \frac{2k}{\xi^2}$ for all $t \geq 0$, and H is a hypergeometric function for which we have the asymptotic estimate of page 17 in [26]. To have $2pr'\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right) < \frac{\xi^2}{8}\left(\frac{2k\theta}{\xi^2}-1\right)^2$, $pr'(\alpha-1) > -\frac{2k\theta}{\xi^2} - v$ and $pr < \frac{4k\theta}{3\xi^2}$ for sufficiently small $p > 1$, it suffices to obtain all these strict inequalities for $p = 1$. Since $r < \frac{4k\theta}{3\xi^2}$ is equivalent to $r' > \frac{4x}{4x-3}$ for $x = \frac{k\theta}{\xi^2}$ with $\frac{4x}{4x-3} > 1$, we can have this inequality along with $2r'\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right) < \frac{\xi^2}{8}\left(\frac{2k\theta}{\xi^2}-1\right)^2 \Leftrightarrow r' < \frac{(2x-1)^2}{2(4x-1)}$ for some $r' > 1$ if and only if $\frac{4x}{4x-3} < \frac{(2x-1)^2}{2(4x-1)}$. The last inequality is satisfied since it is equivalent to $16x^3 - 60x^2 + 24x - 3 > 0$ and we have $x = \frac{k\theta}{\xi^2} > x^*$. Then, $pr'(\alpha-1) > -\frac{2k\theta}{\xi^2} - v$ can be obtained for $p = 1$, for any $\alpha \geq 0$ and r' sufficiently close to its upper bound $\frac{(2x-1)^2}{2(4x-1)}$, provided that it holds for $p = 1, \alpha = 0$ and $r' = \frac{(2x-1)^2}{2(4x-1)}$, i.e when

$$-\frac{(2x-1)^2}{2(4x-1)} > -2x + \frac{1}{2}(2x-1) \qquad (2.3.20)$$

which is also satisfied when $x = \frac{k\theta}{\xi^2} > x^*$ (since $x^* > 1$). Next, since σ_0 is bounded away from zero, the argument of H in (2.3.18) is bounded from below by some $m > 0$, and then the estimate of page 17 in [26] gives $H(-z) \leq K|z|^{-v+pr'(\alpha-1)}$ for all $z \geq m$, for some $K > 0$. Therefore, from (2.3.18) we obtain

$$\mathbb{E}\left[\frac{\sigma_t^{pr'\alpha}}{\|D_\cdot\sigma_t\|^{2pr'}_{L^2([0,T])}}\right] \leq \frac{\tilde{C}}{t^{pr'}} e^{cT}\mathbb{E}\left[\sigma_0^{pr'(\alpha-1)}\right] \tag{2.3.21}$$

for some $c > 0$, with the RHS of the above being finite, and this implies also

$$\mathbb{E}\left[\frac{\sigma_t^{pr'\alpha}}{\|D_\cdot\sigma_t\|^{2pr'}_{L^2([0,T])}} \mid B^0_\cdot,\, \mathcal{G}^0\right] < \infty \tag{2.3.22}$$

$\mathbb{P}^0$- almost surely. The last means that the assumptions of Lemma 2.3.4 are indeed satisfied.

From the above we deduce that under $\mathbb{P}(\cdot \mid B^0_\cdot,\, \mathcal{G}^0)$, σ_t has a density $p_t(y \mid B^0_\cdot,\, \mathcal{G}^0)$ which is supported in $[0,\, +\infty)$ (since this CIR process does not hit zero) and which satisfies

$$\begin{aligned}
&\sup_{y\in\mathbb{R}^+} y^\alpha p_t(y \mid B^0_\cdot,\, \mathcal{G}^0) \\
&\quad\leq (C+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}\sigma_t\right\|^{pr}_{L^2\left([0,T]^2\right)} \mid B^0_\cdot,\, \mathcal{G}^0\right] \\
&\qquad\qquad \times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{\sigma_t^\alpha}{\|D_\cdot\sigma_t\|^2_{L^2([0,T])}}\right|^{pr'} \mid B^0_\cdot,\, \mathcal{G}^0\right] \\
&\qquad\quad +C\mathbb{E}^{\frac{1}{pr}}\left[\|D_\cdot\sigma_t\|^{pr}_{L^2([0,T])} \mid B^0_\cdot,\, \mathcal{G}^0\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{\sigma_t^\alpha}{\|D_\cdot\sigma_t\|^2_{L^2([0,T])}}\right|^{pr'} \mid B^0_\cdot,\, \mathcal{G}^0\right] \\
&\quad= (C+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}\sigma_t\right\|^{pr}_{L^2\left([0,t]^2\right)} \mid B^0_\cdot,\, \mathcal{G}^0\right] \\
&\qquad\qquad \times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{\sigma_t^\alpha}{\|D_\cdot\sigma_t\|^2_{L^2([0,t])}}\right|^{pr'} \mid B^0_\cdot,\, \mathcal{G}^0\right] \\
&\qquad\quad +C\mathbb{E}^{\frac{1}{pr}}\left[\|D_\cdot\sigma_t\|^{pr}_{L^2([0,t])} \mid B^0_\cdot,\, \mathcal{G}^0\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{\sigma_t^\alpha}{\|D_\cdot\sigma_t\|^2_{L^2([0,t])}}\right|^{pr'} \mid B^0_\cdot,\, \mathcal{G}^0\right],
\end{aligned}$$

so raising to the power p, taking expectations and using Holder's inequality we obtain

$$\mathbb{E}\left[\left(\sup_{y\in\mathbb{R}^+} y^\alpha p_t(y \mid B^0_\cdot,\, \mathcal{G}^0)\right)^p\right]$$

$$\leq \tilde{C}\mathbb{E}^{\frac{1}{r}}\left[\left\|D^2_{\cdot,\cdot}\sigma_t\right\|^{pr}_{L^2([0,t]^2)}\right]\times\mathbb{E}^{\frac{1}{r'}}\left[\left|\frac{\sigma_t^\alpha}{\|D_\cdot\sigma_t\|^2_{L^2([0,t])}}\right|^{pr'}\right]$$
$$+\tilde{C}\mathbb{E}^{\frac{1}{r}}\left[\|D_\cdot\sigma_t\|^{pr}_{L^2([0,t])}\right]\times\mathbb{E}^{\frac{1}{r'}}\left[\left|\frac{\sigma_t^\alpha}{\|D_\cdot\sigma_t\|^2_{L^2([0,t])}}\right|^{pr'}\right] \qquad (2.3.23)$$

for some $\tilde{C}>0$. Next, by Lemmas 2.3.5 and 2.3.6 we have

$$\begin{aligned}\mathbb{E}^{\frac{1}{pr}}\left[\|D_\cdot\sigma_t\|^{pr}_{L^2([0,t])}\right] &= \mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^t\xi^2\left(1-\rho_2^2\right)e^{-2\int_{t'}^t\left[\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right)\frac{1}{\sigma_s}+\frac{k}{2}\right]ds}\sigma_t dt'\right)^{\frac{pr}{2}}\right]\\ &\leq \xi\sqrt{1-\rho_2^2}\mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^t\sup_{s\leq T}\sigma_s dt'\right)^{\frac{pr}{2}}\right]\\ &= \xi\sqrt{1-\rho_2^2}\sqrt{t}\mathbb{E}^{\frac{1}{pr}}\left[\left(\sup_{s\leq T}\sigma_s\right)^{\frac{pr}{2}}\right]\\ &= C'\sqrt{t} \qquad (2.3.24)\end{aligned}$$

for some $C'>0$, while by the estimate of Lemma 2.3.7 we have

$$\begin{aligned}\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}\sigma_t\right\|^{pr}_{L^2([0,t])}\right] &\leq \mathbb{E}^{\frac{1}{pr}}\left[t^{pr}\sup_{0\leq t',t''\leq t\leq T}|D^2_{t',t''}\sigma_t|^{pr}\right]\\ &= C''t \qquad (2.3.25)\end{aligned}$$

for some $C''>0$, since $pr<\frac{4k\theta}{3\xi^2}$. Moreover, for our choice of p and r, by (2.3.21) we have

$$\mathbb{E}^{\frac{1}{pr'}}\left[\frac{\sigma_t^{pr'\alpha}}{\|D_\cdot\sigma_t\|^{2pr'}_{L^2([0,T])}}\right]\leq C^{(3)}\frac{1}{t} \qquad (2.3.26)$$

for some $C^{(3)}>0$. Substituting now (2.3.24), (2.3.25) and (2.3.26) in (2.3.23), we obtain

$$\mathbb{E}\left[\left(\sup_{y\in\mathbb{R}^+}y^\alpha p_t(y\,|\,B^0_\cdot,\mathcal{G}^0)\right)^p\right]\leq C^{(4)}+C^{(5)}\frac{1}{\sqrt{t^p}} \qquad (2.3.27)$$

for some $C^{(4)},C^{(5)}>0$. Since the RHS of the last is integrable in t for $t\in[0,T]$ (since we can take $p<2$), the desired result follows. □

2.4 Existence of a regular density for the SPDE

In this section, we combine the results we have obtained in the previous section for the volatility process with the regularity results we have for the constant volatility model

of [7], in order to obtain a regular density for the probabilistic solution of the SPDE obtained in Section 2.2, when the value of C_1 is given. First, for any Hilbert space H, we denote by $L^2(\Omega^0 \times [0,\, T]\,;\, H)$ the space of H- valued stochastic processes, which are L^2-integrable and adapted to the Brownian motion $(W_\cdot^0,\, B_\cdot^0)$.

2.4.1 A one-dimensional density given the volatility path

For our purpose we will need the following useful Theorem which extends the results of [7] to the case where the volatility path is non-constant.

Theorem 2.4.1. *Let $\{X_t : t \geq 0\}$ satisfy the stopped SDE*

$$\begin{cases} dX_t = \left(r - \frac{\sigma_t}{2}\right) dt + \sqrt{\sigma_t}\sqrt{1-\rho_1^2}dW_t^1 + \sqrt{\sigma_t}\rho_1 dW_t^0, & 0 \leq t \leq \tau, \\ \\ X_t = 0, & t \geq \tau, \end{cases} \tag{2.4.1}$$

for $\tau = \inf\{t \geq 0 : X_t = 0\}$, under the initial condition $X_0 = x_0$, where x_0 is a continuous random variable with an L^2 density $u_0 = u_0(\cdot \,|\, \mathcal{G}^0)$ given $\mathcal{G}^0 \subset \sigma(x_0)$, $W_\cdot^0$ and $W_\cdot^1$ are pairwise independent standard Brownian motions which are also independent of x_0, $r > 0$ and $\rho_1 \in (-1,\, 1)$ are given constants, and $\{\sigma_t : t \geq 0\}$ is just a deterministic path which is continuous and positive. Let $\{V_t : t \geq 0\}$ be the measure-valued process given by

$$V_t(A) = \mathbb{P}\left(X_t \in A, \tau \geq t \,|\, W_\cdot^0,\, \mathcal{G}^0\right)$$

for any Borel set $A \subset (0,\, \infty)$. Then, $\mathbb{P}^0$-almost surely, the following are true for all $T > 0$;

1. *V_t possesses a density $u(t,\, \cdot) = u\,(t,\, \cdot\,;\, W_\cdot^0,\, \mathcal{G}^0)$ for all $0 \leq t \leq T$, which is the unique solution to the SPDE*

$$du(t,\, x) = -\left(r - \frac{\sigma_t}{2}\right) u_x(t,\, x)dt + \frac{\sigma_t}{2}u_{xx}(t,\, x)dt - \sqrt{\sigma_t}\rho_1 u_x(t,\, x)dW_t^0 \tag{2.4.2}$$

   in $L^2\,(\Omega^0 \times [0,\, T]\,;\, H_0^1\,(\mathbb{R}^+))$ under the initial condition $u(0, \cdot) = u_0$, where u_0 is the density of x_0 given $\mathcal{G}^0$.

2. *For all $0 \leq t \leq T$, the following identity holds*

$$\|u(t,\, \cdot)\|_{L^2(\mathbb{R}^+)}^2 + \left(1 - \rho_1^2\right) \int_0^t \sigma_s \,\|u_x(s,\, \cdot)\|_{L^2(\mathbb{R}^+)}^2 \, ds = \|u_0\|_{L^2(\mathbb{R}^+)}^2 . \tag{2.4.3}$$

3. *For some $M > 0$ depending only on r and on some compact interval $\mathcal{I} \subset \mathbb{R}^+$ containing the minimum and the maximum of $\sigma_\cdot$, we have the estimate*

$$\mathbb{E}\left[\sup_{0\le t\le T}\|w(\cdot)u_x(t,\,\cdot)\|^2_{L^2(\mathbb{R}^+)}\right] \;\le\; Me^{MT}\mathbb{E}\left[\|w(\cdot)\,(u_0)_x\,(\cdot)\|^2_{L^2(\mathbb{R}^+)}\right] + Me^{MT}\mathbb{E}\left[\|u_0(\cdot)\|^2_{L^2(\mathbb{R}^+)}\right] \tag{2.4.4}$$

where $w(x) = \min\{1,\,\sqrt{x}\}$ for all $x \ge 0$, provided that the RHS is finite. Then, for some $M' > 0$ depending on M and the initial data, we have the maximum principle

$$\mathbb{E}\left[\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^+} u^2(t,\,x)\right] \le M', \tag{2.4.5}$$

To prove the above theorem, we need the following convergence result for a sequence of stopped Ito processes, when the corresponding sequence of volatility paths decreases pointwise to some continuous and positive path.

Lemma 2.4.2. *Let $\{\sigma^0_t : 0 \le t \le T\}$ be a continuous and strictly positive path, which is approximated from above by a pointwise decreasing sequence $\{\{\sigma^m_t : 0 \le t \le T\}\}_{m\in\mathbb{N}}$ of positive and bounded paths. For any $m \in \mathbb{N}\cup\{0\}$, denote by $X^m_\cdot$ the stopped Ito process given by*

$$\begin{cases} dX^m_t = \left(r - \frac{\sigma^m_t}{2}\right)dt + \sqrt{\sigma^m_t}\,dW_t, & 0 \le t \le \tau^m, \\ \\ X^m_t = 0, & t > \tau^m, \end{cases} \tag{2.4.6}$$

where $\tau^m = \inf\{t \ge 0 :\, X^m_t = 0\}$ and $W_\cdot$ is a standard Brownian Motion defined on a filtered probability space $\left(\Omega,\,\mathcal{F},\,\{\mathcal{F}_t\}_{t\in[0,]},\,\mathbb{P}\right)$, with the initial condition

$$X^m_0 = \max\left\{x_0 - l_m,\,\frac{x_0}{2}\right\},$$

for $x_0 \ge 0$ and $l_m = \left(\left\|\sqrt{\sigma^m_\cdot} - \sqrt{\sigma^0_\cdot}\right\|_{L^2[0,\,T]}\right)^{1/2}$. Then, for a sequence $\{m_k : k \in \mathbb{N}\} \subset \mathbb{N}$, we have almost surely: $X^{m_k}_t \longrightarrow X^0_t$ uniformly on any compact interval $[0,\,T]$.

Proof. By using the bounded convergence theorem we can easily obtain $\sigma^m_\cdot \to \sigma^0_\cdot$ and $\sqrt{\sigma^m_\cdot} \longrightarrow \sqrt{\sigma^0_\cdot}$ in L^2, as $m \longrightarrow \infty$.

To prove our claim, we consider the process $Y^m_\cdot$ satisfying the same SDE and initial condition as $X^m_\cdot$ for any $m \in \mathbb{N}\cup\{0\}$, but without being stopped when it hits

zero. We will first show that for a subsequence $\{m_k : k \in \mathbb{N}\} \subset \mathbb{N}$, we have almost surely: $Y_\cdot^{m_k} \longrightarrow Y_\cdot^0$ uniformly on any compact interval $[0, T]$, and there exists a $k_0 \in \mathbb{N}$ such that $Y_t^{m_k} < Y_t^0 \, \forall \, k \geq k_0$ and all $t \leq T$. Indeed, we have

$$\sup_{t \leq T} \left| Y_t^m - Y_t^0 \right| = \sup_{t \leq T} \left| X_0^m - X_0^0 - \frac{1}{2} \int_0^t \left(\sigma_s^m - \sigma_s^0 \right) ds + \int_0^t \left(\sqrt{\sigma_s^m} - \sqrt{\sigma_s^0} \right) dW_s \right|,$$

This is bounded by

$$\begin{aligned} &\left| X_0^m - X_0^0 \right| + \frac{1}{2} \int_0^T \left| \sigma_s^m - \sigma_s^0 \right| ds + \sup_{t \leq T} \left| \int_0^t \left(\sqrt{\sigma_s^m} - \sqrt{\sigma_s^0} \right) dW_s \right| \\ &\quad = \min \left\{ \frac{x_0}{2}, \, l_m \right\} + \frac{1}{2} \left\| \sigma_\cdot^m - \sigma_\cdot^0 \right\|_{L^1[0,T]} + \sup_{t \leq T} \left| \int_0^t \left(\sqrt{\sigma_s^m} - \sqrt{\sigma_s^0} \right) dW_s \right|, \end{aligned}$$

where the first two terms tend obviously to zero, while the last term tends to zero in probability due to Doob's Martingale inequality for $p = 2$ and Ito's isometry, so along a subsequence, the whole quantity tends almost surely to zero. Next, we have

$$\begin{aligned} Y_t^m - Y_t^0 &= -\min \left\{ \frac{x_0}{2}, \, l_m \right\} - \frac{1}{2} \int_0^t \left(\sigma_s^m - \sigma_s^0 \right) ds + \int_0^t \left(\sqrt{\sigma_s^m} - \sqrt{\sigma_s^0} \right) dW_s \\ &\leq -\min \left\{ \frac{x_0}{2}, \, l_m \right\} + \sup_{t \leq T} \left| \int_0^t \left(\sqrt{\sigma_s^m} - \sqrt{\sigma_s^0} \right) dW_s \right|, \end{aligned} \tag{2.4.7}$$

and once more, by Doob's Martingale inequality for $p = 2$ and $l_m^0 = \frac{\min\{\frac{x_0}{2}, l_m\}}{2}$, we have

$$\begin{aligned} &\mathbb{P} \left(\sup_{t \leq T} \left| \int_0^t \left(\sqrt{\sigma_s^m} - \sqrt{\sigma_s^0} \right) dW_s \right| > l_m^0 \, | \, \mathcal{F}_0 \right) \\ &\qquad \leq \frac{1}{(l_m^0)^2} \mathbb{E} \left(\left(\int_0^T \left(\sqrt{\sigma_s^m} - \sqrt{\sigma_s^0} \right) dW_s \right)^2 | \, \mathcal{F}_0 \right) \\ &\qquad = \frac{(l_m)^4}{(l_m^0)^2}, \end{aligned}$$

where $l_m^0 = \frac{l_m}{2} \longrightarrow 0$ for large m. Thus, there exists a subsequence $\{m_k : k \in \mathbb{N}\} \subset \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \frac{(l_{m_k})^4}{(l_{m_k}^0)^2} < \infty$$

which implies that almost surely, $\sup_{t \leq T} \left| \int_0^t \left(\sqrt{\sigma_s^{m_k}} - \sqrt{\sigma_s^0} \right) dW_s \right| < l_{m_k}^0 = \frac{l_{m_k}}{2}$ for all large k (by the Borel-Cantelli lemma). Therefore, by (2.4.7) we obtain

$$\sup_{t \leq T} \left(Y_t^{m_k} - Y_t^0 \right) \leq -l_{m_k} + \frac{l_{m_k}}{2} = -\frac{l_{m_k}}{2} < 0 \tag{2.4.8}$$

almost surely for all large k.

We are ready now to prove the uniform convergence of the stopped processes. For a fixed event, the stopping times τ^{m_k}, τ^0 are given and (2.4.8) implies that $\tau^{m_k} \leq \tau^0$ for all large k. Moreover, we have $\tau^{m_k} \longrightarrow \tau^0$ as $k \longrightarrow \infty$. Indeed, for any $\epsilon > 0$, Y^{m_k} is lower bounded by a positive constant in $[0,\, \tau^0 - \epsilon]$ for all k bigger than some $k^0(\epsilon)$ (since the same holds for the continuous process $Y_\cdot^0$ by the definition of τ^0, and since $Y_\cdot^{m_k}$ tends uniformly to $Y_\cdot^0$), which implies that $\tau^0 - \epsilon < \tau^{m_k} \leq \tau^0$ for all $k \geq k^0(\epsilon)$. Now, if we have $\tau^0 > T$, then we have also $\tau^{m_k} > T$ for all big enough k, which gives

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} \left|X_t^{m_k} - X_t^0\right| = \lim_{k\to\infty} \sup_{0\leq t\leq T} \left|Y_t^{m_k} - Y_t^0\right| = 0.$$

On the other hand, if $\tau^0 \leq T$ we have

$$\begin{aligned} &\sup_{t\leq T} \left|X_t^{m_k} - X_t^0\right| \\ &\qquad = \max\left\{ \sup_{0\leq t\leq \tau^{m_k}} \left|X_t^{m_k} - X_t^0\right|,\ \sup_{\tau^{m_k}\leq t\leq \tau^0} \left|X_t^{m_k} - X_t^0\right|,\ \sup_{\tau^0\leq t\leq T} \left|X_t^{m_k} - X_t^0\right| \right\}. \end{aligned} \tag{2.4.9}$$

The first supremum of the RHS of the (2.4.9) is equal to the supremum of $|Y_t^{m_k} - Y_t^0|$ for $t \leq \tau^{m_k}$, which tends to zero since $Y_\cdot^{m_k} \longrightarrow Y_\cdot^0$ uniformly in $[0,\, T]$, while the third one is always equal to 0. Hence, we only need to show that the second supremum of the RHS of (2.4.9) tends also to 0 as $k \longrightarrow \infty$. Indeed, by the continuity of our processes, for some $t_k \in [\tau^{m_k},\, \tau^0]$ (where $X_\cdot^{m_k}$ is identically zero) we have

$$\sup_{\tau^{m_k}\leq t\leq \tau^0} \left|X_t^{m_k} - X_t^0\right| = \left|X_{t_k}^{m_k} - X_{t_k}^0\right| = \left|X_{t_k}^0\right| \longrightarrow \left|X_{\tau^0}^0\right| = 0,$$

as $k \longrightarrow +\infty$ (since $\tau^{m_k} \longrightarrow \tau^0$ implies $t_k \longrightarrow \tau^0$), so the desired result follows. □

We can proceed now to the proof of Theorem 2.4.1

Proof of Theorem 2.4.1. Without loss of generality we can assume $\rho_1 \in [0, 1)$. We will prove first that 2. and 3. hold for the unique solution u of (2.4.2) in the space $L^2\left(\Omega^0 \times [0,\, T]\,;\, H_0^1\left(\mathbb{R}^+\right)\right)$. Since the existence and uniqueness of this solution follow from the results of [33] (pages 18-20), for any fixed volatility path and any square integrable initial density, we do not need to recall 1. at this stage. This means that we may use 2. for proving 1. later.

To prove 2. for that u, we apply Ito's formula for the L^2 norm (Theorem 3.2 in [33] for the triple $H_0^1 \subset L^2 \subset H^{-1}$) on (2.4.2) and we observe that $\int_0^\infty u_{xx} u = -\int_0^\infty u_x^2$

(by the usual definition of the weak second derivative u_{xx} as a distribution in H^{-1} for $u \in H_0^1$, for which the existence and uniqueness theory developed in chapter 2 of [33] is applicable). This way we obtain

$$\left\|u(t,\,\cdot,\,W_\cdot^0,\,\mathcal{G}^0)\right\|_{L^2(\mathbb{R}^+)}^2 + \left(1-\rho_1^2\right)\int_0^t \sigma_s \left\|u_x(s,\,\cdot,\,W_\cdot^0,\,\mathcal{G}^0)\right\|_{L^2(\mathbb{R}^+)}^2 ds = \|u_0\|_{L^2(\mathbb{R}^+)}^2$$

for all $0 \leq t \leq T$, for all $\omega^0 \in \Omega^{0,T} \subset \Omega^0$ with $\mathbb{P}^0\left(\Omega^{0,T}\right) = 1$. The desired result follows since $\mathbb{P}^0\left(\Omega^{0,\infty}\right) = 1$ for $\Omega^{0,\infty} = \cap_{N\in\mathbb{N}}\Omega^{0,N}$.

To prove 3. we assume that $u_0 = u_0(\cdot\,|\,\mathcal{G}^0)$ is differentiable and that $w(\cdot)\,(u_0)_x$ is $L^2\left(\Omega^0 \times \mathbb{R}^+\right)$-integrable. Then, by the theory developed in [32] we have that u coincides with the unique solution to (2.4.2) in a $w^2(\cdot)$ - weighted Sobolev space of higher regularity, and that (2.4.4) is also satisfied, with M depending only on a compact interval $\mathcal{I} \subset \mathbb{R}^+$ which contains both $\min_{0\leq t\leq T}\sigma_t$ and $\max_{0\leq t\leq T}\sigma_t$. Note that even though the constants appearing in the Sobolev estimates obtained in [32] depend also on the modulus of spatial continuity of the coefficients of the SPDE, here this modulus of continuity is always zero since the coefficients do not depend on the spatial variable x. Next, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^+} u^2(t,\,x)\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in(0,\,1)} u^2(t,\,x)\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\geq 1} u^2(t,\,x)\right], \tag{2.4.10}$$

and we can use Morrey's inequality (see [12]) to control the second term in the RHS of the above by

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\int_1^{+\infty} u_x^2(t,\,x)dx\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\int_1^{+\infty} u^2(t,\,x)dx\right] \\
&\qquad \leq \mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^{+\infty} w^2(x)u_x^2(t,\,x)dx\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^{+\infty} u^2(t,\,x)dx\right] \\
&\qquad \leq Me^{MT}\mathbb{E}\left[\left\|w(\cdot)\,(u_0)_x\,(\cdot)\right\|_{L^2(\mathbb{R}^+)}^2\right] + \left(Me^{MT}+1\right)\mathbb{E}\left[\left\|u_0(\cdot)\right\|_{L^2(\mathbb{R}^+)}^2\right],
\end{aligned}$$

where we have also used (2.4.3) and (2.4.4). On the other hand, we can use Theorem 1 from [10] to control the first term in the RHS of (2.4.10) by $\mathbb{E}\left[\sup_{0\leq t\leq T} u^2(t,\,1)\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\geq 1} u^2(t,\,x)\right]$ and by the maximum of the initial density, and this completes the proof of (2.4.5).

We proceed now to the proof of 1. which will be divided into 3 steps. In what follows, we will be working on the set $\Omega^{0,\,\infty}$ which is defined above and which is a set of full probability.

1. *The constant volatility case:*

We assume first that the path σ_t is constant in $t \geq 0$, i.e $\sigma_t = \sigma^2 \ \forall t \geq 0$. In that case, V_t is the limit empirical process studied in [7] and [24] (without the compactly supported initial data restriction), scaled by $\sigma > 0$, so it does have a density $u(t, \cdot) = u(t, \cdot, W_\cdot^0, \mathcal{G}^0)$ which is the unique solution of the SPDE

$$du(t, x) = -\left(r - \frac{\sigma^2}{2}\right) u_x(t, x)dt + \frac{\sigma^2}{2} u_{xx}(t, x)dt - \sigma\rho_1 u_x(t, x)dW_t^0$$

in $L^2\left(\Omega^0 \times [0, T]\,;\, H_0^1\left(\mathbb{R}^+\right)\right)$, under the initial condition $u(0, \cdot) = u_0$, which is actually (2.4.2). It holds also that $xu_{xx}(t, x)$ is square integrable.

2. *The piecewise constant volatility case:*

We assume now that the path $\sigma_\cdot$ is piecewise constant in $0 \leq t \leq T$, i.e $\sigma_t = \sigma_i^2 > 0 \ \forall t \in [t_i, t_{i+1}]$ (almost everywhere), where $0 = t_0 < t_1 < ... < t_n = T$ is a partition of $[0, T]$. We shall prove that 1. holds for T replaced by t_{i+1} for all $i \leq n-1$, by using induction on i. The base case ($i = 0$) follows directly from Step 1 for $T = t_1$. Assume that our desired result holds for $i = j < n-1$ and thus, we have obtained the desired density $u(t, \cdot)$ for $0 \leq t \leq t_{j+1}$. For $i = j+1$ now, by starting our (Markovian) processes at $t = t_{j+1}$ and using Step 1 again, we have that $V_t(A) = \mathbb{P}\left(X_t \in A, \tau \geq t \,|\, W_\cdot^0, \mathcal{G}^0\right)$ has a density $\tilde{u}(t, \cdot) = \tilde{u}\left(t, \cdot\,; W_\cdot^0, \mathcal{G}^0\right)$ for all $t_{j+1} \leq t \leq t_{j+2}$, which is the unique solution of the SPDE

$$d\tilde{u}(t, x) = -\left(r - \frac{\sigma_{j+1}^2}{2}\right) \tilde{u}_x(t, x)dt + \frac{\sigma_{j+1}^2}{2} \tilde{u}_{xx}(t, x)dt - \sigma_{j+1}\rho_1 \tilde{u}_x(t, x)dW_t^0$$

in $L^2\left(\Omega^0 \times [t_{j+1}, t_{j+2}]\,;\, H_0^1\left(\mathbb{R}^+\right)\right)$, i.e an equivalent to (2.4.2) in $[t_{j+1}, t_{j+2}]$, under the initial condition

$$\tilde{u}(t_{j+1}, \cdot) = u\left(t_{j+1}, \cdot\right)$$

Therefore, by defining $u(t, \cdot) = \tilde{u}(t, \cdot)$ in $(t_{j+1}, t_{j+2}]$, we see that $u(t, \cdot)$ is both the density of $V_t(\cdot)$ and the unique solution of (2.4.2) in $L^2\left(\Omega^0 \times [0, t_{j+2}]\,;\, H_0^1\left(\mathbb{R}^+\right)\right)$ under the initial condition $u(0, \cdot) = u_0$. Our induction is now complete.

3. *The general case:*

We assume finally that the path $\sigma_\cdot$ is any continuous and positive path. Recall Lemma 2.4.2 for $\sigma_\cdot^0 = \sigma_\cdot$ and for a pointwise decreasing sequence of piecewise constant paths $\{\sigma_\cdot^m\}_{m\in\mathbb{N}}$ approximating $\sigma_\cdot$ from above. This gives a sequence of drifted Brownian Motions $X_\cdot^{m_k}$ converging $\mathbb{P}^0 \times \mathbb{P}^1$-almost surely to $X_\cdot$, with corresponding volatility paths $\{\sigma_\cdot^{m_k}\}_{m\in\mathbb{N}}$. If we denote by $u^k(t, \cdot; W_\cdot^0, \mathcal{G}^0)$ the density of $X_t^{m_k}$ for

any $t \geq 0$, then Step 2 shows that u^k is the unique solution of (2.4.2) with $\sigma_\cdot$ replaced by $\sigma_\cdot^{m_k}$ and with an initial density given by

$$u_0^k(x) = u_0\left(\min\{2x,\, x + \ell_{m_k}\}\right).$$

Thus, by 2., which has been proven for the unique regular enough solution of (2.4.2) (independently from 1. which we are now proving), we have that

$$\left\|u^k(t,\,\cdot;\, W_\cdot^0,\, \mathcal{G}^0)\right\|_{L^2(\mathbb{R}^+)}^2 + \left(1-\rho_1^2\right)\int_0^t \sigma_s^{m_k}\left\|u_x^k(s,\,\cdot;\, W_\cdot^0,\, \mathcal{G}^0)\right\|_{L^2(\mathbb{R}^+)}^2 ds = \left\|u_0^k\right\|_{L^2(\mathbb{R}^+)}^2,$$

where we can take expectations to obtain

$$\left\|u^k(t,\,\cdot;\, W_\cdot^0,\, \mathcal{G}^0)\right\|_{L^2}^2 + \left(1-\rho_1^2\right)\int_0^t \sigma_s^{m_k}\left\|u_x^k(s,\,\cdot;\, W_\cdot^0,\, \mathcal{G}^0)\right\|_{L^2}^2 ds = \left\|u_0^k\right\|_{L^2}^2 \quad (2.4.11)$$

for all $0 \leq t \leq T$, where L^2 stands for $L^2\left(\Omega^0 \times \mathbb{R}^+\right)$. Then, by the choice of the approximating sequence, we can see that all the volatility paths are bounded below by $m = \min_{0\leq s\leq T}\{\sigma_s\}$, while the sequence of the norms of the u_0^ks is bounded. Thus, we can easily obtain from (2.4.11) that the sequence of $L^2\left(\Omega^0 \times [0,\, T]\,;\, H_0^1\left(\mathbb{R}^+\right)\right)$ norms of the u^ks is bounded. Hence, there exists a $u \in L^2\left(\Omega^0 \times [0,\, T]\,;\, H_0^1\left(\mathbb{R}^+\right)\right)$ such that $u^{n_k} \longrightarrow u$ weakly in that space, for a sequence $\{n_k : k \in \mathbb{N}\}$ of positive integers. Moreover, if we denote by $Y_\cdot$ the process satisfying the same SDE and initial condition as $X_\cdot$, but without the stopping condition at 0, we can easily see that it is a Gaussian process given the path $W_\cdot^0$ and given $\mathcal{G}^0$, which implies that

$$\mathbb{P}\left(X_t \in A \,|\, W_\cdot^0,\, \mathcal{G}^0\right) \leq \mathbb{P}\left(Y_t \in A \,|\, W_\cdot^0,\, \mathcal{G}^0\right) = 0 \quad (2.4.12)$$

for any Borel set $A \subset \mathbb{R}^+$ of zero Lebesgue measure, and thus X_t has a density $\tilde{u}(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0)$ given $(W_\cdot^0,\, \mathcal{G}^0)$ for all t. The last implies also that every open subset of $\mathbb{R}^+$ is a continuity set for X_t given $(W_\cdot^0,\, \mathcal{G}^0)$, and since convergence almost surely implies convergence in distribution, for any smooth function f defined on $[0,\, T]$, any open set $A \subset \mathbb{R}^+$ and any $B \in \sigma\left(W_\cdot^0,\, \mathcal{G}^0\right)$, we have

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\mathbb{I}_B \int_A \tilde{u}(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0)dx\right] f(t)dt \\
&\qquad = \int_0^T \mathbb{E}\left[\mathbb{P}\left(X_t \in A \,|\, W_\cdot^0,\, \mathcal{G}^0\right)\mathbb{I}_B\right] f(t)dt \\
&\qquad = \int_0^T \mathbb{E}\left[\lim_{k\to\infty}\mathbb{P}\left(X_t^{m_{n_k}} \in A \,|\, W_\cdot^0,\, \mathcal{G}^0\right)\mathbb{I}_B\right] f(t)dt \\
&\qquad = \lim_{k\to\infty}\int_0^T \mathbb{E}\left[\mathbb{P}\left(X_t^{m_{n_k}} \in A \,|\, W_\cdot^0,\, \mathcal{G}^0\right)\mathbb{I}_B\right] f(t)dt
\end{aligned}$$

$$= \lim_{k\to\infty} \int_0^T \mathbb{E}\left[\mathbb{I}_B \int_A u^{n_k}(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0)dx\right] f(t)dt$$
$$= \int_0^T \mathbb{E}\left[\mathbb{I}_B \int_A u(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0)dx\right] f(t)dt,$$

which implies that $\tilde{u} = u \in L^2\left(\Omega^0 \times [0,\, T];\, H_0^1\left(\mathbb{R}^+\right)\right)$ is the density process of X_t given $W_\cdot^0$ and $\mathcal{G}^0$. By applying Ito's formula on any smooth function computed at X_t and taking expectations given $W_\cdot^0$ and $\mathcal{G}^0$, we find that u is also a solution of (2.4.2), which also satisfies the initial condition $u(0,\cdot) = u_0$. Thus, u is the unique solution of (2.4.2) in $L^2\left(\Omega^0 \times [0,\, T]\,;\, H_0^1\left(\mathbb{R}^+\right)\right)$, under the initial condition $u(0,\cdot) = u_0$. The proof is now complete. □

2.4.2 The density of the two-dimensional SPDE

In order to prove the existence of a regular density for v_{t,C_1} which was defined in Section 2.2 as a measure-valued process on $(\Omega^0,\, \mathcal{F}^0,\, \{\mathcal{F}_t^0\}_{t\geq 0},\, \mathbb{P}^0) \times (\Omega^C,\, \mathcal{F}^C,\, \mathbb{P}^C)$, we need to define the spaces

$$L_\alpha = L^2\left(\left(\Omega^0,\, \mathcal{F}^0,\, \mathbb{P}^0\right) \times [0,\, T]\,;\, L^2_{|y|^\alpha}\left(\mathbb{R}^+ \times \mathbb{R}\right)\right)$$

and

$$H_\alpha = L^2\left(\left(\Omega^0,\, \mathcal{F}^0,\, \mathbb{P}^0\right) \times [0,\, T]\,;\, H^1_{0,w^2(x)}\left(\mathbb{R}^+\right) \times L^2_{|y|^\alpha}\left(\mathbb{R}\right)\right)$$

for any $\alpha \geq 0$, where we write $L^2_{g(y)}$ for the weighted L^2 space with weight function $\{g(y) : y \in \mathbb{R}\}$, and $H^1_{0,g(x)}\left(\mathbb{R}^+\right)$ for the weighted $H_0^1\left(\mathbb{R}^+\right)$ space with weight function $\{g(x) : x \geq 0\}$. Apart from the integrability conditions, a function u' belonging to the second space has to satisfy $\lim_{x\to 0^+} \|u'(\cdot,\, x,\, \cdot)\|_{L^2_{|y|^\alpha}(\Omega^0\times[0,\,T]\times\mathbb{R})} = 0$. Observe that this definition is not problematic, since $\|u'(\cdot,\, x,\, \cdot)\|_{L^2_{|y|^\alpha}(\Omega^0\times[0,\,T]\times\mathbb{R})}$ has to be continuous in x for $x > 0$ (this follows by applying Morrey's inequality away from $x = 0$), so changing the value of the above limit gives a different function in an $L^2\left(\Omega^0 \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}\right)$ sense. Then, the existence of a density for v_{t,C_1} and it's regularity are given in the next two Theorems

Theorem 2.4.3. *Suppose that h is a continuous function taking values in some compact subset of $\mathbb{R}^+$. Suppose also that given $\mathcal{G}^0$, X_0^1 has an L^2-integrable density $u_0(\cdot|\mathcal{G}^0)$ in $\mathbb{R}^+$ such that $\mathbb{E}\left[\|w(\cdot)\,(u_0)_x\,(\cdot)\|^2_{L^2(\mathbb{R}^+)}\right] < \infty$ and $\mathbb{E}\left[\|u_0\|^2_{L^2(\mathbb{R}^+)} \mid \mathcal{G}^0\right] \in L^{p'}\left(\Omega^0\right)$ for some $p' > 2$. Suppose finally that the assumptions of Theorem 2.3.1 are satisfied for the function q and for $\sigma_0 = \sigma_0^1$. Then, for any value of the coefficient vector C_1, the measure-valued stochastic process v_{t,C_1} has a two-dimensional density $u_{C_1}(t,\, \cdot,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0)$ belonging to the space L_α for all $\alpha \geq 0$, and also to the space H_0.*

Theorem 2.4.4. *Suppose that h is a continuous function taking values in some compact subset of $\mathbb{R}^+$. Suppose also that given $\mathcal{G}^0$, X_0^1 has an L^2-integrable density $u_0(\cdot|\mathcal{G}^0)$ in $\mathbb{R}^+$ such that $\mathbb{E}\left[\|u_0\|^2_{L^2(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right] \in L^{p'}(\Omega^0)$ for any $p' > 1$. Suppose finally that the assumptions of Theorem 2.3.2 are satisfied for the function q, for $\sigma_0 = \sigma_0^1$ and for $(k,\, \theta,\, \xi) = (k_1,\, \theta_1,\, \xi_1)$, for any possible value of $C_1 = (k_1,\, \theta_1,\, \xi_1,\, r_1,\, \rho_{1,1},\, \rho_{2,1})$. Then, for any possible value of C_1, the measure-valued stochastic process v_{t,C_1} has a two-dimensional density $u_{C_1}(t,\, \cdot,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0)$ belonging to the space L_α for any $\alpha \geq 0$. Moreover, when $\rho_3 := \int_0^1 dW_t^0 dB_t^0 = 0$ and $\mathbb{E}\left[\|u_0\|^2_{H^1_{0,w^2(x)}(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right] \in L^{p'}(\Omega^0)$ for any $p' > 1$, the density belongs to H_α as well for any $\alpha \geq 0$.*

Proof of Theorem 2.4.3. Let f be a smooth function, compactly supported in $\mathbb{R}^2$, such that f vanishes on the y - axis. Then by Theorem 2.3.1 we have

$$\begin{aligned} v_{t,C_1}(f) &= \mathbb{E}\left[f\left(X_t^1,\, \sigma_t^1\right)\mathbb{I}_{\{T_1 \geq t\}} \,|W_\cdot^0, B_\cdot^0, C_1, \mathcal{G}^0\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f\left(X_t^1,\, \sigma_t^1\right)\mathbb{I}_{\{T_1 \geq t\}} \,|W_\cdot^0, \sigma_t^1, B_\cdot^0, C_1, \mathcal{G}^0\right] \,|W_\cdot^0, B_\cdot^0, C_1, \mathcal{G}^0\right] \\ &= \int_{\mathbb{R}} \mathbb{E}\left[f\left(X_t^1,\, y\right)\mathbb{I}_{\{T_1 \geq t\}} \,|W_\cdot^0, \sigma_t^1 = y, B_\cdot^0, C_1, \mathcal{G}^0\right] p_t\left(y|B_\cdot^0, \mathcal{G}^0\right) dy. \end{aligned} \tag{2.4.13}$$

Next we have

$$\begin{aligned} &\mathbb{E}\left[f\left(X_t^1,\, y\right)\mathbb{I}_{\{T_1 \geq t\}}|W_\cdot^0, \sigma_t^1 = y, B_\cdot^0, C_1, \mathcal{G}^0\right] \\ &\quad = \mathbb{E}\left[\mathbb{E}\left[f\left(X_t^1,\, y\right)\mathbb{I}_{\{T_1 \geq t\}}|W_\cdot^0, \sigma_\cdot, C_1, \mathcal{G}^0\right] \,|W_\cdot^0, \sigma_t^1 = y, B_\cdot^0, C_1, \mathcal{G}^0\right] \\ &\quad = \mathbb{E}\left[\int_{\mathbb{R}^+} f(x,\, y) u\left(t, x, W_\cdot^0, \mathcal{G}^0, C_1, h(\sigma_\cdot)\right) dx |W_\cdot^0, \sigma_t^1 = y, B_\cdot^0, C_1, \mathcal{G}^0\right] \\ &\quad = \int_{\mathbb{R}^+} f(x,\, y) \mathbb{E}\left[u\left(t, x, W_\cdot^0, \mathcal{G}^0, C_1, h(\sigma_\cdot)\right) |W_\cdot^0, \sigma_t^1 = y, B_\cdot^0, C_1, \mathcal{G}^0\right] dx, \end{aligned} \tag{2.4.14}$$

where $u(t, x, W_\cdot^0, C_1, \mathcal{G}^0, h(\sigma_\cdot))$ is the $L^2(\Omega^0 \times [0,\, T];\, H_0^1(\mathbb{R}^+))$ density given by Theorem 2.4.1, when the coefficient vector C_1 is given and the volatility path is $h(\sigma_\cdot)$. By (2.4.13) and (2.4.14) we obtain that the desired density exists and is given by

$$\begin{aligned} &u_{C_1}\left(t, x, y, W_\cdot^0, B_\cdot^0, \mathcal{G}^0\right) \\ &\qquad = p_t\left(y|B_\cdot^0, \mathcal{G}^0\right) \mathbb{E}\left[u\left(t, x, W_\cdot^0, \mathcal{G}^0, C_1, h(\sigma_\cdot)\right) \,|W_\cdot^0, \sigma_t^1 = y, B_\cdot^0, C_1, \mathcal{G}^0\right] \end{aligned} \tag{2.4.15}$$

which is obviously supported in $\mathbb{R}^+ \times \mathbb{R}$. Then, by the Cauchy-Schwarz inequality we have

$$w^2(x)\left(\frac{\partial u_{C_1}}{\partial x}\left(t,\, x,\, y,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)^2$$

$$
\begin{aligned}
&\leq M(t) p_t\left(y \,|\, B^0_\cdot,\, \mathcal{G}^0\right) \\
&\qquad \times w^2(x) \mathbb{E}^2\left[u_x\left(t,\, x,\, W^0_\cdot,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right) \,|\, W^0_\cdot,\, \sigma^1_t = y,\, B^0_\cdot,\, C_1,\, \mathcal{G}^0\right] \\
&\leq M(t) p_t\left(y \,|\, B^0_\cdot,\, \mathcal{G}^0\right) \\
&\qquad \times \mathbb{E}\left[w^2(x) u_x^2\left(t,\, x,\, W^0_\cdot,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right) \,|\, W^0_\cdot,\, \sigma^1_t = y,\, B^0_\cdot,\, C_1,\, \mathcal{G}^0\right]
\end{aligned}
$$

where $M(t) = \operatorname*{ess\,sup}_{y \in \mathbb{R},\, \omega^0 \in \Omega^0} p_\cdot(y \,|\, B^0_\cdot,\, \mathcal{G}^0)\left(\omega^0\right) \in L^1\left([0,\, T]\right)$ (defined in Theorem 2.3.1). Integrating the above in y and using the law of total expectation, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} w^2(x) \left(\frac{\partial u_{C_1}}{\partial x}\left(t,\, x,\, y,\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right)\right)^2 dy \\
&\qquad \leq M(t) \times \mathbb{E}\left[w^2(x) u_x^2\left(t,\, x,\, W^0_\cdot,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right) \,|\, W^0_\cdot,\, B^0_\cdot,\, C_1,\, \mathcal{G}^0\right]
\end{aligned}
$$

Thus, writing $\mathbb{E}_{C_1}$ for the expectation given C_1, we have

$$
\begin{aligned}
&\mathbb{E}_{C_1}\left[\int_0^T \int_{\mathbb{R}^+} \int_{\mathbb{R}} w^2(x) \left(\frac{\partial u_{C_1}}{\partial x}\left(t,\, x,\, y,\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right)\right)^2 dy dx dt\right] \\
&\leq \mathbb{E}_{C_1}\Big[\int_0^T M(t) \\
&\qquad \times \int_{\mathbb{R}^+} \mathbb{E}\left[w^2(x) u_x^2\left(t, x, W^0_\cdot, \mathcal{G}^0, C_1, h\left(\sigma_\cdot\right)\right) \,|W^0_\cdot, B^0_\cdot, C_1, \mathcal{G}^0\right] dx dt\Big] \\
&= \int_0^T M(t) \mathbb{E}_{C_1}\left[\int_{\mathbb{R}^+} w^2(x) u_x^2\left(t, x, W^0_\cdot, \mathcal{G}^0, C_1, h\left(\sigma_\cdot\right)\right) dx\right] dt \\
&\leq \int_0^T M(t) dt \mathbb{E}_{C_1}\left[\sup_{0 \leq s \leq T} \int_{\mathbb{R}^+} w^2(x) u_x^2\left(s, x, W^0_\cdot, \mathcal{G}^0, C_1, h\left(\sigma_\cdot\right)\right) dx\right] \\
&\leq M e^{MT} \int_0^T M(t) dt \mathbb{E}\left[\left\| w(\cdot)\left(u_0\right)_x(\cdot)\right\|^2_{L^2(\mathbb{R}^+)}\right] \\
&< \infty
\end{aligned}
$$

by Tonelli's Theorem, Theorem 2.3.1 and 3. of Theorem 2.4.1. The reason we can take M to be deterministic is that h has a deterministic (positive) lower bound and a deterministic upper bound. This is the estimate for the x-derivative of u_{C_1}. To obtain the weighted integrability for the density, we use the Cauchy-Schwarz inequality, Tonelli's Theorem and 2. of Theorem 2.4.1, so for any $\alpha \geq 0$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |y|^a \left(u_{C_1}\left(t,\, x,\, y,\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}^0\right)\right)^2 dy dx \\
&\qquad \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} M^\alpha_{B^0_\cdot, \mathcal{G}^0}(t) p_t\left(y \,|\, B^0_\cdot,\, \mathcal{G}^0\right) \\
&\qquad\qquad \times \mathbb{E}^2\left[u\left(t,\, x,\, W^0_\cdot,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right) \,|\, W^0_\cdot,\, \sigma^1_t = y,\, B^0_\cdot,\, C_1,\, \mathcal{G}^0\right] dy dx \\
&\qquad \leq M^\alpha_{B^0_\cdot, \mathcal{G}^0}(t) \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} p_t\left(y \,|\, B^0_\cdot,\, \mathcal{G}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\mathbb{E}\left[u^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\,|\, W_\cdot^0,\, \sigma_t^1 = y,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] dydx \\
&= M^{\alpha}_{B_\cdot^0,\,\mathcal{G}^0}(t)\int_{\mathbb{R}^+}\mathbb{E}\left[u^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\,|\, W_\cdot^0,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] dx \\
&= M^{\alpha}_{B_\cdot^0,\,\mathcal{G}^0}(t)\mathbb{E}\left[\int_{\mathbb{R}^+} u^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right) dx\,|\, W_\cdot^0,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] \\
&\leq M^{\alpha}_{B_\cdot^0,\,\mathcal{G}^0}(t)\mathbb{E}\left[\left\|u_0(\cdot)\right\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}\right]
\end{aligned}
$$

where $M^{\alpha}_{B_\cdot^0,\,\mathcal{G}^0}(\cdot) = \sup\limits_{y\in\mathbb{R}}\left(|y|^{\alpha} p_\cdot(y\,|\,B_\cdot^0,\,\mathcal{G}^0)\right) \in L^p\left(\Omega^0\times[0,\,T]\right)$ for any $p<2$ (by Theorem 2.3.1). Taking then $p'>2$ sufficiently close to 2 and $p<2$ such that $\frac{1}{p}+\frac{1}{p'}=1$, by the above and by Holder's inequality we have

$$
\begin{aligned}
&\mathbb{E}_{C_1}\left[\int_0^T\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}|y|^a\left(u_{C_1}\left(t,\, x,\, y,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)^2 dydxdt\right] \\
&\quad\leq \mathbb{E}_{C_1}\left[\int_0^T M^{\alpha}_{B_\cdot^0,\,\mathcal{G}^0}(t)\mathbb{E}\left[\left\|u_0(\cdot)\right\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}^0\right] dt\right] \\
&\quad\leq \mathbb{E}^{\frac{1}{p}}_{C_1}\left[\left(\int_0^T M^{\alpha}_{B_\cdot^0,\,\mathcal{G}^0}(t)dt\right)^p\right]\mathbb{E}^{\frac{1}{p'}}_{C_1}\left[\mathbb{E}^{p'}\left[\left\|u_0(\cdot)\right\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}^0\right]\right] \\
&\quad\leq T^{\frac{1}{p'}}\mathbb{E}^{\frac{1}{p}}_{C_1}\left[\int_0^T\left(M^{\alpha}_{B_\cdot^0,\,\mathcal{G}^0}(t)\right)^p dt\right]\mathbb{E}^{\frac{1}{p'}}_{C_1}\left[\mathbb{E}^{p'}\left[\left\|u_0(\cdot)\right\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}^0\right]\right] \\
&\quad< \infty
\end{aligned}
$$

which implies that the density belongs to the space L_α for any $\alpha\geq 0$. Finally, we need to show that $\lim\limits_{x\to 0^+}\left\|u_{C_1}(\cdot,\, x,\, \cdot)\right\|_{L^2(\Omega\times[0,\,T]\times\mathbb{R})}=0$, which follows from the estimate

$$
\begin{aligned}
&\mathbb{E}_{C_1}\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\left(u_{C_1}\left(t,\, x,\, y,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)^2 dydt\right] \\
&\quad\leq \mathbb{E}_{C_1}\Bigg[\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} M(t)p_t\left(y\,|\,B_\cdot^0,\,\mathcal{G}^0\right) \\
&\qquad\times\mathbb{E}^2\left[u\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\,|\, W_\cdot^0,\, \sigma_t^1 = y,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] dydt\Bigg] \\
&\quad\leq \mathbb{E}_{C_1}\Bigg[\int_{\mathbb{R}^+} M(t)\int_{\mathbb{R}^+} p_t\left(y\,|\,B_\cdot^0,\,\mathcal{G}^0\right) \\
&\qquad\times\mathbb{E}\left[u^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\,|\, W_\cdot^0,\, \sigma_t^1 = y,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] dydt\Bigg] \\
&\quad= \mathbb{E}_{C_1}\left[\int_{\mathbb{R}^+} M(t)\mathbb{E}\left[u^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\,|\, W_\cdot^0,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] dt\right] \\
&\quad= \int_{\mathbb{R}^+} M(t)\mathbb{E}_{C_1}\left[u^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\right] dt
\end{aligned}
$$

since we can use the maximum principle given in 3. of Theorem 2.4.1, the integrability of $M(\cdot)$ and the Dominated Convergence Theorem, to show that the RHS of the last tends to zero as $x \longrightarrow 0^+$. The proof of the Theorem is now complete. □

Proof of Theorem 2.4.4. Working as in the proof of Theorem 2.4.3 we find that the desired density exists and is given by

$$\begin{aligned} &u_{C_1}\left(t,x,y,W_\cdot^0,B_\cdot^0,\mathcal{G}^0\right) \\ &\qquad = p_t\left(y|B_\cdot^0,\mathcal{G}^0\right)\mathbb{E}\left[u\left(t,x,W_\cdot^0,\mathcal{G}^0,C_1,h\left(\sigma_\cdot\right)\right)\,|W_\cdot^0,\sigma_t^1=y,B_\cdot^0,C_1,\mathcal{G}^0\right], \end{aligned} \tag{2.4.16}$$

which is supported in $\mathbb{R}^+\times\mathbb{R}^+$ this time, while for any $\alpha\geq 0$ we have

$$\begin{aligned} &\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} y^a\left(u_{C_1}\left(t,\,x,\,y,\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}^0\right)\right)^2 dydx \\ &\qquad \leq M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\mathbb{E}\left[\|u_0(\cdot)\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}^0\right], \end{aligned}$$

where this time we have $M^\alpha_{B_\cdot^0,\mathcal{G}^0}(\cdot)=\sup\limits_{y\geq 0}\left(y^\alpha p_\cdot(y\,|\,B_\cdot^0,\,\mathcal{G}^0)\right)\in L^p\left(\Omega^0\times[0,\,T]\right)$ for a small enough $p>1$ (by Theorem 2.3.2). Taking $p'>1$ such that $\frac{1}{p}+\frac{1}{p'}=1$, by the above and by Holder's inequality we obtain

$$\begin{aligned} &\mathbb{E}_{C_1}\left[\int_0^T\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} y^a\left(u_{C_1}\left(t,\,x,\,y,\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}^0\right)\right)^2 dydxdt\right] \\ &\quad\leq \mathbb{E}_{C_1}\left[\int_0^T M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\mathbb{E}\left[\|u_0(\cdot)\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}^0\right]dt\right] \\ &\quad\leq \mathbb{E}^{\frac{1}{p}}_{C_1}\left[\left(\int_0^T M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)dt\right)^p\right]\mathbb{E}^{\frac{1}{p'}}_{C_1}\left[\mathbb{E}^{p'}\left[\|u_0(\cdot)\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}^0\right]\right] \\ &\quad\leq T^{\frac{1}{p'}}\mathbb{E}^{\frac{1}{p}}_{C_1}\left[\int_0^T\left(M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\right)^p dt\right]\mathbb{E}^{\frac{1}{p'}}_{C_1}\left[\mathbb{E}^{p'}\left[\|u_0(\cdot)\|^2_{L^2(\mathbb{R}^+)}\,|\,\mathcal{G}^0\right]\right] \\ &\quad< \infty, \end{aligned}$$

which shows that the density belongs to the space L_α for any $\alpha\geq 0$. Moreover, repeating the above computations but for the derivative multiplied by $w(x)$, we find that

$$\begin{aligned} &\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} w^2(x)y^a\left(\frac{\partial u_{C_1}}{\partial x}\left(t,\,x,\,y,\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}^0\right)\right)^2 dydx \\ &\qquad\leq M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\mathbb{E}\left[\int_{\mathbb{R}^+} w^2(x)u_x^2\left(t,\,x,\,W_\cdot^0,\,\mathcal{G}^0,\,C_1,\,h\left(\sigma_\cdot\right)\right)dx\,|\,W_\cdot^0,\,B_\cdot^0,\,C_1,\,\mathcal{G}^0\right], \end{aligned}$$

so when $\rho_3 := \int_0^1 dW_t^0 dB_t^0 = 0$, writing $\mathbb{E}^{B_\cdot^0}_{C_1,\mathcal{G}^0}$ for the expectation given C_1, $\mathcal{G}^0$ and $B_\cdot^0$ and using 3. of Theorem 2.4.1, we obtain

$$\begin{aligned}
&\mathbb{E}^{B_\cdot^0}_{C_1,\mathcal{G}^0}\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} w^2(x)y^a\left(\frac{\partial u_{C_1}}{\partial x}\left(t,\, x,\, y,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)^2 dydx\right] \\
&\quad\leq M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\mathbb{E}\left[\int_{\mathbb{R}^+} w^2(x)u_x^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)dx \,|\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] \\
&\quad\leq M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\mathbb{E}\left[\sup_{0\leq s\leq T}\int_{\mathbb{R}^+} w^2(x)u_x^2\left(s,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)dx \,|\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] \\
&\quad\leq Me^{MT}M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\left(\mathbb{E}\left[\left\|w(\cdot)\left(u_0\right)_x(\cdot)\right\|^2_{L^2(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right] + \mathbb{E}\left[\left\|u_0(\cdot)\right\|^2_{L^2(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right]\right).
\end{aligned}$$

The last implies that

$$\begin{aligned}
&\mathbb{E}_{C_1}\left[\int_0^T\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} w^2(x)y^a\left(\frac{\partial u_{C_1}}{\partial x}\left(t,\, x,\, y,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)^2 dydxdt\right] \\
&\quad\leq Me^{MT}\mathbb{E}_{C_1}\left[\int_0^T M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)dt\mathbb{E}\left[\left\|w(\cdot)\left(u_0\right)_x(\cdot)\right\|^2_{L^2(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right]\right] \\
&\qquad + Me^{MT}\mathbb{E}_{C_1}\left[\int_0^T M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)dt\mathbb{E}\left[\left\|u_0(\cdot)\right\|^2_{L^2(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right]\right] \\
&\quad\leq MT^{\frac{1}{p'}}e^{MT}\mathbb{E}^{\frac{1}{p}}_{C_1}\left[\int_0^T\left(M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\right)^p dt\right] \\
&\qquad\qquad \times\mathbb{E}^{\frac{1}{p'}}_{C_1}\left[\mathbb{E}^{p'}\left[\left\|w(\cdot)\left(u_0\right)_x(\cdot)\right\|^2_{L^2(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right]\right] \\
&\qquad + MT^{\frac{1}{p'}}e^{MT}\mathbb{E}^{\frac{1}{p}}_{C_1}\left[\int_0^T\left(M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\right)^p dt\right]\mathbb{E}^{\frac{1}{p'}}_{C_1}\left[\mathbb{E}^{p'}\left[\left\|u_0(\cdot)\right\|^2_{L^2(\mathbb{R}^+)} \,|\, \mathcal{G}^0\right]\right] \\
&\quad< \infty
\end{aligned}$$

which gives the weighted integrability of the derivative. To obtain the boundary condition when $\rho_3 = 0$, we work as in the proof of the previous Theorem, i.e

$$\begin{aligned}
&\mathbb{E}_{C_1}\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}|y|^\alpha\left(u_{C_1}\left(t,\, x,\, y,\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)^2 dydt\right] \\
&\quad\leq \mathbb{E}_{C_1}\Bigg[\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)p_t\left(y \,|\, B_\cdot^0,\, \mathcal{G}^0\right) \\
&\qquad\qquad \times\mathbb{E}^2\left[u\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right) \,|\, W_\cdot^0,\, \sigma_t^1 = y,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] dydt\Bigg] \\
&\quad\leq \mathbb{E}_{C_1}\Bigg[\int_{\mathbb{R}^+} M^\alpha_{B_\cdot^0,\mathcal{G}^0}(t)\int_{\mathbb{R}^+} p_t\left(y \,|\, B_\cdot^0,\, \mathcal{G}^0\right) \\
&\qquad\qquad \times\mathbb{E}\left[u^2\left(t,\, x,\, W_\cdot^0,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right) \,|\, W_\cdot^0,\, \sigma_t^1 = y,\, B_\cdot^0,\, C_1,\, \mathcal{G}^0\right] dydt\Bigg]
\end{aligned}$$

$$= \mathbb{E}_{C_1}\left[\int_{\mathbb{R}^+} M^{\alpha}_{B^0_\cdot,\mathcal{G}^0}(t)\mathbb{E}\left[u^2\left(t,\, x,\, W^0_\cdot,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\,|\, W^0_\cdot,\, B^0_\cdot,\, C_1,\, \mathcal{G}^0\right] dt\right]$$
$$= \mathbb{E}_{C_1}\left[\int_{\mathbb{R}^+} M^{\alpha}_{B^0_\cdot,\mathcal{G}^0}(t)\mathbb{E}_{C_1}\left[u^2\left(t,\, x,\, W^0_\cdot,\, \mathcal{G}^0,\, C_1,\, h\left(\sigma_\cdot\right)\right)\,|\, B^0_\cdot,\, C_1,\, \mathcal{G}^0\right] dt\right],$$

where we can use the maximum principle given in 3. of Theorem 2.4.1, the integrability of $M^{\alpha}_{B^0_\cdot,\mathcal{G}^0}(\cdot)$ and the Dominated Convergence Theorem, to show that the RHS of the last tends to zero as $x \longrightarrow 0^+$. This completes the proof of the Theorem. □

If C_1 has a nice distribution such that the RHS in each of the norm estimates obtained above has a finite expectation, we can deduce the existence of a regular density for the limiting empirical measure process, justifying the validity of the approximate computation in (1.0.7). Substituting now $\int_{\mathbb{R}^2} f \cdot dv_{t,C_1} = \int_{\mathbb{R}^2} f(x,y)u_{C_1}(t,\, x,\, y)dxdy$ in the distributional SPDE of Theorem 2.2.7 and integrating by parts, we obtain the SPDE for the density of v_{t,C_1}:

$$\begin{aligned}
u_{C_1}(t,\, x,\, y) \;=\; & U_0(x,\, y\,|\,\mathcal{G}^0,\, C_1) - r_1\int_0^t \left(u_{C_1}(s,\, x,\, y)\right)_x ds \\
& + \frac{1}{2}\int_0^t h^2(y)\left(u_{C_1}(s,\, x,\, y)\right)_x ds - k_1\theta_1\int_0^t \left(u_{C_1}(s,\, x,\, y)\right)_y ds \\
& + k_1\int_0^t \left(y u_{C_1}(s,\, x,\, y)\right)_y ds + \frac{1}{2}\int_0^t h^2(y)\left(u_{C_1}(s,\, x,\, y)\right)_{xx} ds \\
& + \xi_1\rho_3\rho_{1,1}\rho_{2,1}\int_0^t \left(h\left(y\right)q(y)u_{C_1}(s,\, x,\, y)\right)_{xy} ds \\
& + \frac{\xi_1^2}{2}\int_0^t \left(q^2(y)u_{C_1}(s,\, x,\, y)\right)_{yy} ds \\
& - \rho_{1,1}\int_0^t h(y)\left(u_{C_1}(s,\, x,\, y)\right)_x dW^0_s \\
& - \xi_1\rho_{2,1}\int_0^t \left(q(y)u_{C_1}(s,\, x,\, y)\right)_y dB^0_s, \qquad (2.4.17)
\end{aligned}$$

where the second derivative in x and the derivatives in y are considered in the distributional sense (over the test space C_0^{test} defined in Theorem 2.2.7), while $U_0(x,\, y\,|\,\mathcal{G}^0,\, C_1)$ stands for the initial density, if it exists.

2.5 Using the SPDE to improve the regularity

In this section, we are going to exploit the initial-boundary value problem satisfied by u_{C_1}, in order to establish the best possible regularity for our density. When $q(z) = \sqrt{z}$, we will need to work in weighted L^2 spaces with polynomial weights, and we will also

need to approximate the solution to our problem by something smooth which can take negative values as well. However, $|y|^\alpha$ has no good regularity for $y < 0$, which means that we need to work with polynomial weights supported in $\mathbb{R}^+$. Thus, we denote by $(y^\alpha)^+$ a weight function which is everywhere 1 for $\alpha = 0$, while for $\alpha \neq 0$ it equals y^α for $y > 0$ and 0 for $y \leq 0$, and then we define $\tilde{L}^2_\delta := L^2_{\left(y^\delta\right)^+}(\mathbb{R}^+ \times \mathbb{R})$ and $\tilde{L}^2_{\delta,w} := L^2_{\left(y^\delta\right)^+ w^2(x)}(\mathbb{R}^+ \times \mathbb{R})$. For the rest of this section, we will fix the value of the coefficient vector C_1, which means that we are going to work in the probability space $(\Omega^0, \mathcal{F}^0, \{\mathcal{F}^0_t\}_{t\geq 0}, \mathbb{P}^0)$ for a fixed event $\omega^C \in \Omega^C$.

2.5.1 The improved regularity result

In order to study the initial-boundary value problem which is solved by u_{C_1}, we need first to define it explicitly. We give the following definition of an (α, β)-solution to our problem for any $\alpha \geq 0$ and $\beta \geq 0$, the properties of which are all satisfied by the density function u_{C_1} for certain values of α and β, as we have shown in the previous section.

Definition 2.5.1. For a given domain $\mathcal{D} \subset \mathbb{R}$, a real number ρ and a given value of the coefficient vector C_1, let $q : \mathcal{D} \longrightarrow \mathbb{R}^+$ be a continuous function, $h : \mathcal{D} \longrightarrow \mathbb{R}^+$ be a function having polynomial growth, and U_0 be a random function which is extended to be zero outside $\mathcal{D}$, such that $U_0 \in L^2\left(\Omega^0;\, \tilde{L}^2_\alpha\right)$ and $(U_0)_x \in L^2\left(\Omega^0;\, \tilde{L}^2_{\beta,w}\right)$ for some $\alpha,\, \beta > 0$. Given $\mathcal{D}$, C_1, ρ, α, β and the functions q, h and U_0, we say that u is an $(\alpha,\, \beta)$-solution to our problem when the following are satisfied;

1. u is adapted to the filtration $\{\sigma\left(\mathcal{G}^0,\, W^0_t,\, B^0_t\right) :\, t \geq 0\}$ and belongs to the space $H_\alpha \cap L_\beta$, where L_α and H_β are defined in section 1.4.

2. u is supported in the domain $\mathcal{D}$ and satisfies the SPDE

$$\begin{aligned} u(t,x,y) \;=\; & U_0(x,\, y) - r_1 \int_0^t \left(u(s,x,y)\right)_x ds \\ & + \frac{1}{2}\int_0^t h^2(y)\left(u(s,x,y)\right)_x ds - k_1\theta_1 \int_0^t \left(u(s,x,y)\right)_y ds \\ & + k_1 \int_0^t \left(yu(s,\, x,\, y)\right)_y ds + \frac{1}{2}\int_0^t h^2(y)\left(u(s,x,y)\right)_{xx} ds \\ & + \rho \int_0^t \left(h\left(y\right) q(y) u(s,\, x,\, y)\right)_{xy} ds \\ & + \frac{\xi_1^2}{2}\int_0^t \left(q^2(y)u(s,x,y)\right)_{yy} ds - \rho_{1,1}\int_0^t h(y)\left(u(s,x,y)\right)_x dW^0_s \end{aligned}$$

$$-\xi_1\rho_{2,1}\int_0^t \left(q(y)u(s,x,y)\right)_y dB_s^0, \tag{2.5.1}$$

for all $x \geq 0$ and $y \in \mathcal{D}$, where u_y, u_{yy} and u_{xx} are considered in the distributional sense over the space of test functions

$$C_0^{test} = \{g \in C_b^2(\mathbb{R}^+ \times \mathbb{R}) : g(0,y) = 0, \ \forall y \in \mathbb{R}\}.$$

Observe that for $\rho = \xi_1\rho_3\rho_{1,1}\rho_{2,1}$, where ρ_3 is the correlation between W^0 and B^0 (i.e $dW_t^0 \cdot dB_t^0 = \rho_3 dt$), we obtain the SPDE obtained in the previous section. The main result of this section is given in the following theorem.

Theorem 2.5.2. *Fix the value of the coefficient vector C_1, the function h, the real number ρ and the initial data function U_0. Then we have the following results;*

1. *Let u be an (α, α)-solution to our problem for $\mathcal{D} = \mathbb{R}^+$ and $q(z) = \sqrt{z}$ for all $z \geq 0$, for all $\alpha \geq 0$. Then, the weak derivative u_y of u exists and we have*

$$u_y \in L^2\left([0,\, T] \times \Omega^0;\, \tilde{L}^2_{\alpha,w}\right)$$

for all $\alpha \geq 2$.

2. *Let u be an $(\alpha, 0)$-solution to our problem for $\mathcal{D} = \mathbb{R}$, for all $\alpha \geq 0$, where $q(\cdot)$ is a bounded $C^3(\mathbb{R})$ function with bounded and $\mathcal{O}\left(\frac{1}{|x|}\right)$ (as $|x| \longrightarrow +\infty$) derivatives up to third order, and with $q(z) \geq m$ for some $m > 0$. Then, the weak derivative u_y of u exists and we have*

$$u_y \in L^2\left([0,\, T] \times \Omega^0;\, \tilde{L}^2_{0,w}\right)$$

Remark 2.5.3. The above result tells us that any u satisfying the conditions of definition 2.5.1 for certain values of α and β, is also weakly differentiable in y, and u_y has good integrability properties just like u_x. It is possible that higher order derivatives belonging to certain weighted L^2 spaces exist as well, but this is something we will not investigate in this thesis

2.5.2 The extended kernel smoothing method

To prove the above Theorem, we need to modify appropriately the kernel smoothing method which has been developed in [7, 34, 24]. The idea is to test our SPDE against

$$\phi_\epsilon(z,\, y) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(Q(z)-y)^2}{2\epsilon}}, \ y,\, z \in \mathbb{R},$$

for $Q(z) = \int_0^z \frac{1}{q(z')} dz'$, in order to obtain a smoothed version of it. Keep in mind that we do not have to integrate outside $\mathcal{D}$, where the $q(\cdot)$ is not defined, since by definition our solution vanishes there. From the smoothed version of our SPDE, we can obtain for appropriate values of $\delta \geq 0$, an identity involving some finite $\tilde{L}^2_\delta$, $\tilde{L}^2_{\delta-1}$ and $\tilde{L}^2_{\delta-2}$ norms and inner products of smoothed quantities involving the solution and its derivatives (we shall refer to this as the δ-identity). The finiteness of the $\tilde{L}^2$ norms appearing in the δ-identity for any δ, follows from the good global regularity of our functions when they are smoothed with $\phi_\epsilon(z,\, y)$. This good smoothing property of $\phi_\epsilon(z,\, y)$, which is not obvious and has to be established concretely, follows from the fact that $\phi_\epsilon(z,\, y)$ is just the standard heat kernel (used in the standard kernel smoothing method) composed with a regular enough bijection. Then, we can obtain the desired result by manipulating appropriately the δ-identity for different values of $\delta \geq 0$ and by taking $\epsilon \longrightarrow 0^+$, provided that $\phi_\epsilon(z,\, y)$ has the same convergence properties as the standard heat kernel. The composition with Q leads to the elimination of some bad terms in our δ-identities (terms that could explode as $\epsilon \longrightarrow 0^+$ under our weak regularity assumptions) which would appear if we used the standard kernel smoothing method. The intuition behind the choice of this composition is that our solution is expected to be the density of a law describing a volatility process in the y-direction, which is transformed by the mapping $z \longrightarrow \sqrt{z}$ into a process of a constant volatility (like the Brownian motion with drift in the constant volatility model, where the standard kernel smoothing method works).

As we have already mentioned, we need to show that $\phi_\epsilon(z,\, y)$ introduced above possesses all the nice smoothing and convergence (as $\epsilon \longrightarrow 0^+$) properties of the standard heat kernel, and also for many different weighted L^2 norms. Below, we state and prove two lemmas which contain these natural extensions.

Lemma 2.5.4. *Let $(\Lambda,\, \mu)$ be a measure space. For any function u supported in $\Lambda \times \mathcal{D}$ we define the functions*

$$J_{u,\epsilon}(\lambda,\, y) = \int_{\mathcal{D}} u(\lambda,\, z)\phi_\epsilon(z,\, y)dz$$

and

$$J_u(\lambda,\, y) = \begin{cases} q\left(Q^{-1}(y)\right) u(\lambda,\, Q^{-1}(y)) & , \qquad y \in Q\left(\mathcal{D}\right) \\ \\ 0 & , \qquad y \notin Q\left(\mathcal{D}\right) \end{cases}$$

Suppose that $J_u \in L^2\left(\Lambda;\, L^2\left(\mathbb{R}\right)\right) \cap L^2\left(\Lambda;\, L^2_{|y|^{\delta'}}\left(\mathbb{R}\right)\right)$ for some $\delta' > -1$. Then we have the following regularity and convergence results;

1. *$J_{u,\epsilon}(\cdot,\,\cdot)$ is smooth and for all $n \in \mathbb{N}$ it holds that*

$$\frac{\partial^n}{\partial y^n} J_{u,\epsilon}(\cdot,\,\cdot) \in L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$$

2. *$J_{u,\epsilon}(\cdot,\,\cdot) \to J_u(\cdot,\,\cdot)$ strongly in $L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$, as $\epsilon \to 0^+$.*

The same results hold if we replace $\left(y^{\delta'}\right)^+$ by $|y|^{\delta'}$.

Proof. Observe that by setting $z = Q^{-1}(v), v \in \mathbb{R}^+$, any integration against ϕ_ϵ can be written as an integration against the standard heat kernel, thus

$$\begin{aligned} J_{u,\epsilon}(\lambda,\, y) &= \int_{\mathcal{D}} u(\lambda,\, z) \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(Q(z)-y)^2}{2\epsilon}} dz \\ &= \int_{\mathbb{R}} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \mathbb{I}_{Q(\mathcal{D})}(v) \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(v-y)^2}{2\epsilon}} dv \end{aligned}$$

We are going to prove 1. first. Observe that by our regularity assumptions and the properties of the standard heat kernel, $J_{u,\epsilon}(\lambda,\, y)$ is smooth and it's n-th derivative in y equals

$$\int_{Q(\mathcal{D})} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{1}{\sqrt{2\pi\epsilon}} P(v-y) e^{-\frac{(v-y)^2}{2\epsilon}} dv$$

where P is some polynomial of degree n. Thus we need to show that for any $n \in \mathbb{N}$, the quantity

$$\int_\Lambda \int_{\mathbb{R}} \left(y^{\delta'}\right)^+ \left(\int_{Q(\mathcal{D})} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) (v-y)^n \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv \right)^2 dy d\mu(\lambda)$$

is finite. By Cauchy-Schwartz, this quantity is bounded by:

$$\begin{aligned} &\int_\Lambda \int_{\mathbb{R}} \left(y^{\delta'}\right)^+ \left(\int_{Q(\mathcal{D})} q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right) (v-y)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv \right) \\ &\qquad \times \left(\int_{\mathbb{R}} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv \right) dy d\mu(\lambda) \\ &= \int_\Lambda \int_{\mathbb{R}} \int_{Q(\mathcal{D})} \left(y^{\delta'}\right)^+ q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right) (y-v)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv dy d\mu(\lambda) \end{aligned} \tag{2.5.2}$$

and thus, by Fubini's Theorem, we only need to show that the integral

$$\int_\Lambda \int_{Q(\mathcal{D})} q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right) \left(\int_{\mathbb{R}} \left(y^{\delta'}\right)^+ (y-v)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \right) dv d\mu(\lambda)$$

is finite. Due to our integrability assumptions on $J_u\left(\lambda,\, v\right) = q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right)$, the last follows if we can show that

$$\int_{\mathbb{R}} \left(y^{\delta'}\right)^+ (y-v)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy = \mathcal{O}\left(|v|^{\delta'} + 1\right).$$

For $\delta' \geq 0$, we use the well-known estimate $|a+b|^{\delta'} \leq C\left(|a|^{\delta'} + |b|^{\delta'}\right)$ to obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \left(y^{\delta'}\right)^+ (y-v)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\quad \leq \int_{\mathbb{R}} |y|^{\delta'} (y-v)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\quad \leq C|v|^{\delta'} \int_{\mathbb{R}} (y-v)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy + C \int_{\mathbb{R}} (|y-v|)^{2n+\delta'} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\quad \leq C\epsilon^n |v|^{\delta'} \int_{\mathbb{R}} w^{2n} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw + C\epsilon^{n+\frac{\delta'}{2}} \int_{\mathbb{R}} (|w|)^{2n+\delta'} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw
\end{aligned} \tag{2.5.3}$$

which is exactly what we wanted.

On the other hand, for $\delta' \in (-1, 0)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left(y^{\delta'}\right)^+ (y-v)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\quad \leq \int_{|y| \leq \frac{|v|}{2}} |y|^{\delta'} (v-y)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy + \int_{|y| > \frac{|v|}{2}} |y|^{\delta'} (v-y)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\quad \leq \frac{(2\epsilon)^n}{\sqrt{\pi}} \int_{|y| \leq \frac{|v|}{2}} |y|^{\delta'} \left(\frac{(v-y)^2}{2\epsilon}\right)^{n+\frac{1}{2}} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{|v-y|} dy \\
&\qquad + \left(\frac{|v|}{2}\right)^{\delta'} \int_{|y| > \left|\frac{v}{2}\right|} (v-y)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\quad \leq C(n)(2\epsilon)^n \int_{|y| \leq \left|\frac{v}{2}\right|} |y|^{\delta'} \frac{1}{|v-y|} dy + \left(\frac{|v|}{2}\right)^{\delta'} \int_{\mathbb{R}} (v-y)^{2n} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\quad \leq C(n)(2\epsilon)^n \left(\frac{|v|}{2}\right)^{-1} \int_{|y| \leq \left|\frac{v}{2}\right|} |y|^{\delta'} dy + \left(\frac{|v|}{2}\right)^{\delta'} \epsilon^n \int_{\mathbb{R}} w^{2n} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw
\end{aligned}$$

$$\leq \frac{2C(n)(2\epsilon)^n}{\delta'+1}\left(\frac{|v|}{2}\right)^{\delta'} + \left(\frac{|v|}{2}\right)^{\delta'}\epsilon^n \int_{\mathbb{R}} w^{2n}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dy \tag{2.5.4}$$

which is again what we needed and thus the proof of 1. is complete. We proceed now to the proof of 2..

By the Cauchy-Schwarz inquality and Fubini's Theorem we have

$$\begin{aligned}
&\left\|J_{u,\epsilon}(\cdot,\,\cdot)\right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)} \\
&= \int_\Lambda \int_{\mathbb{R}} \left(y^{\delta'}\right)^+ \\
&\qquad \times \left(\int_{Q(\mathcal{D})} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{e^{-\frac{(v-y)^2}{4\epsilon}}}{\sqrt[4]{2\pi\epsilon}} \frac{e^{-\frac{(v-y)^2}{4\epsilon}}}{\sqrt[4]{2\pi\epsilon}} dv\right)^2 dyd\mu(\lambda) \\
&\leq \int_\Lambda \int_{\mathbb{R}} \left(y^{\delta'}\right)^+ \left(\int_{Q(\mathcal{D})} q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right) \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv\right) \\
&\qquad \times \left(\int_{\mathbb{R}} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv\right) dyd\mu(\lambda) \\
&= \int_\Lambda \int_{Q(\mathcal{D})} q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right) \int_{\mathbb{R}} \left(y^{\delta'}\right)^+ \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dydvd\mu(\lambda)
\end{aligned} \tag{2.5.5}$$

Next, we see that

$$\begin{aligned}
&q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right) \int_{\mathbb{R}} \left(y^{\delta'}\right)^+ \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \\
&\longrightarrow \left(v^{\delta'}\right)^+ q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right)
\end{aligned}$$

as $\epsilon \longrightarrow 0^+$ for all $v \in Q(\mathcal{D})$, and it can also be bounded by something integrable, uniformly in $\epsilon > 0$ (this can be seen by recalling (2.5.3) and (2.5.4) for $n = 0$). Thus, by the Dominated Convergence Theorem, the RHS of (2.5.5) converges to

$$\left\|J_u(\cdot,\,\cdot)\right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)} = \int_\Lambda \int_{Q(\mathcal{D})} \left(v^{\delta'}\right)^+ q^2\left(Q^{-1}(v)\right) u^2\left(\lambda,\, Q^{-1}(v)\right) dvd\mu(\lambda)$$

as $\epsilon \longrightarrow 0^+$. Therefore, we obtain

$$\limsup_{\epsilon\to 0^+} \left\|J_{u,\epsilon}(\cdot,\,\cdot)\right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)} \leq \left\|J_u(\cdot,\,\cdot)\right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)}. \tag{2.5.6}$$

Next, fix a measurable $A \subset \Lambda$ with $\mu(A) < +\infty$ and a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ supported in some interval $[-b,\, b]$, with $b > 0$. Then it holds that

$$\int_{\mathbb{R}} f(y) \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \longrightarrow f(v)$$

pointwise as $\epsilon \longrightarrow 0^+$. Moreover, we can easily find a $C > 0$ such that

$$\int_{\mathbb{R}} f(y) \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \leq C \sup_{y\in\mathbb{R}} |f(y)| \times \begin{cases} 1\,, & v \in [-2b,\, 2b] \\ \frac{2b}{|v-b|}\,, & v > 2b \\ \frac{2b}{|v+b|}\,, & v < -2b \end{cases}$$

for all $\epsilon > 0$ and then, by applying the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} &\int_{\Lambda} \int_{[2b,\, +\infty] \cap Q(\mathcal{D})} \mathbb{I}_A(\lambda) q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{2b}{|v-b|} dv d\mu(\lambda) \\ &\quad \leq 2b \left\| J_u(\cdot,\, \cdot) \right\|_{L^2\left(\Lambda;\, L^2_{|y|^{\delta'}}(\mathbb{R})\right)} \left(\mu(A) \int_{2b}^{+\infty} \frac{1}{|v|^{\delta'} \, |v-b|^2} dv \right)^{1/2} \\ &\quad < \infty \end{aligned}$$

and also

$$\begin{aligned} &\int_{\Lambda} \int_{[-\infty,\, -2b] \cap Q(\mathcal{D})} \mathbb{I}_A(\lambda) q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{2b}{|v+b|} dv d\mu(\lambda) \\ &\quad \leq 2b \left\| J_u(\cdot,\, \cdot) \right\|_{L^2\left(\Lambda;\, L^2_{|y|^{\delta'}}(\mathbb{R})\right)} \left(\mu(A) \int_{-\infty}^{-2b} \frac{1}{|v|^{\delta'} \, |v+b|^2} dv \right)^{1/2} \\ &\quad < \infty. \end{aligned}$$

This means that we can apply the Dominated Convergence Theorem to obtain

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \int_{\Lambda} \int_{\mathbb{R}} J_{u,\epsilon}(\lambda,\, y) f(y) \mathbb{I}_A(\lambda) dy d\mu(\lambda) \\ &\quad = \lim_{\epsilon\to 0^+} \int_{\Lambda} \int_{\mathbb{R}} f(y) \left(\int_{\mathcal{D}} u(\lambda,\, z) \frac{e^{-\frac{(Q(z)-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} \mathbb{I}_A(\lambda) dz \right) dy d\mu(\lambda) \\ &\quad = \lim_{\epsilon\to 0^+} \int_{\Lambda} \int_{\mathbb{R}} f(y) \\ &\qquad \times \left(\int_{Q(\mathcal{D})} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} \mathbb{I}_A(\lambda) dv \right) dy d\mu(\lambda) \\ &\quad = \lim_{\epsilon\to 0^+} \int_{\Lambda} \int_{Q(\mathcal{D})} \mathbb{I}_A(\lambda) q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \left(\int_{\mathbb{R}} f(y) \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy \right) dv d\mu(\lambda) \end{aligned}$$

$$= \int_{\Lambda} \int_{Q(\mathcal{D})} \mathbb{I}_A(\lambda) q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) f(v) dv d\mu(\lambda) \tag{2.5.7}$$

so we deduce that $J_{u,\epsilon}(\cdot,\,\cdot) \longrightarrow J_u(\cdot,\,\cdot)$ weakly in the space $L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$ (since A and f are arbitrary). We recall now that in any uniformly convex Banach space $(\mathcal{X},\, \|\cdot\|)$, whenever a sequence x_n converges weakly to some x as $n \to +\infty$ and $\limsup_{n\to+\infty} \|x_n\| \leq \|x\|$, we have also strong convergence of x_n to x (see Proposition III.30 in [5]). As a Hilbert space, $L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$ is a uniformly convex space, and for any sequence ϵ_m tending to zero from above, from (2.5.6) we can easily see that

$$\limsup_{m\to+\infty} \|J_{u,\epsilon_m}\|_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)} \leq \|J_u\|_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)},$$

so the above result implies that the weak convergence of J_{u,ϵ_m} to J_u is also strong. Since we have this for an arbitrary sequence ϵ_m tending to zero from above, we deduce 2.. Finally, both 1. and 2. with $|y|^{\delta'}$ replaced by $\left(y^{\delta'}\right)^+$ can be obtained by following exactly the same steps (the only difference is that inequalities like $\left(y^{\delta'}\right)^+ \leq |y|^{\delta'}$ are not needed). □

Lemma 2.5.5. *In the notation of lemma 2.5.4, assume that for some $\delta' > 0$, there exists a constant $C > 0$ and an $n \in \mathbb{N}$ such that for any $\epsilon > 0$ we have*

$$\left\| \frac{\partial^l}{\partial y^l} J_{u,\epsilon}\,(s,\,\cdot) \right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)} \leq C \tag{2.5.8}$$

for some function u supported in $\Lambda \times \mathcal{D}$ and all $l \in \{1,\, 2,\, ...,\, n\}$. Then we have $\frac{\partial^l}{\partial y^l} J_u \in L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$ and also $\frac{\partial^l}{\partial y^l} J_{u,\epsilon} \longrightarrow \frac{\partial^l}{\partial y^l} J_u$ strongly in $L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$ as $\epsilon \longrightarrow 0^+$, for all $l \in \{1,\, 2,\, ...,\, n\}$.

Proof. First, by our boundedness assumption we have that for any sequence $\{\epsilon_k : k \in \mathbb{N}\}$ tending to 0^+, there exists a decreasing subsequence $\{\epsilon_{k_m}\}_{m\in\mathbb{N}}$ and an element $J^l_u \in L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$ for all $l \in \{1,\, 2,\, ...,\, n\}$, such that

$$\frac{\partial^l}{\partial y^l} J_{u,\epsilon_{k_m}} \longrightarrow J^l_u,$$

for all $l \in \{1,\, 2,\, ...,\, n\}$, weakly in $L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)$ as $m \longrightarrow +\infty$. Then, for any measurable $A \subset \Lambda$ with $\mu(A) < +\infty$ and any smooth and compactly supported

function $f(z)$, we have

$$\int_{\mathbb{R}} \frac{\partial^l}{\partial z^l} f(z) \frac{e^{-\frac{(z-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dz \longrightarrow \frac{\partial^l}{\partial y^l} f(y)$$

pointwise as $\epsilon \longrightarrow 0^+$. Hence, we can use Fubini's Theorem and the Dominated Convergence Theorem as we did in the proof of Lemma 2.5.4 (but for a partial derivative of f) to obtain

$$\begin{aligned}
&\lim_{m\to\infty} \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) J_{u,\epsilon_{k_m}}(\lambda,\, y) \frac{\partial^l}{\partial y^l} f(y) dy d\mu(\lambda) \\
&\quad = \lim_{m\to\infty} \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) \frac{\partial^l}{\partial y^l} f(y) \left(\int_{\mathcal{D}} u(\lambda,\, z) \frac{e^{-\frac{(Q(z)-y)^2}{2\epsilon_{k_m}}}}{\sqrt{2\pi\epsilon_{k_m}}} dz \right) dy d\mu(\lambda) \\
&\quad = \lim_{m\to\infty} \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) \frac{\partial^l}{\partial y^l} f(y) \\
&\qquad \times \left(\int_{Q(\mathcal{D})} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{e^{-\frac{(v-y)^2}{2\epsilon_{k_m}}}}{\sqrt{2\pi\epsilon_{k_m}}} dv \right) dy d\mu(\lambda) \\
&\quad = \lim_{m\to\infty} \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) J_u(\lambda,\, v) \left(\int_{\mathbb{R}} \frac{\partial^l}{\partial y^l} f(y) \frac{e^{-\frac{(v-y)^2}{2\epsilon_{k_m}}}}{\sqrt{2\pi\epsilon_{k_m}}} dy \right) dv d\mu(\lambda) \\
&\quad = \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) J_u(\lambda,\, v) \frac{\partial^l}{\partial y^l} f(v) dv d\mu(\lambda)
\end{aligned}$$

and thus we have

$$\begin{aligned}
&\int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) J_u^l(\lambda,\, y) f(y) dy d\mu(\lambda) \\
&\qquad = \lim_{m\to\infty} \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) \frac{\partial^l}{\partial y^l} J_{u,\epsilon_{k_m}}(\lambda,\, y) f(y) dy d\mu(\lambda) \\
&\qquad = \lim_{m\to\infty} (-1)^l \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) J_{u,\epsilon_{k_m}}(\lambda,\, y) \frac{\partial^l}{\partial y^l} f(y) dy d\mu(\lambda) \\
&\qquad = (-1)^l \int_\Lambda \int_{\mathbb{R}} \mathbb{I}_A(\lambda) J_u(\lambda,\, v) \frac{\partial^l}{\partial v^l} f(v) dv d\mu(\lambda),
\end{aligned}$$

which means that J_u^l is the l-th weak derivative of J_u. Next, for any $\epsilon > 0$ and $l \leq n$ we have

$$\left\| \frac{\partial^l}{\partial y^l} J_{u,\epsilon}(\cdot,\, \cdot) \right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)}$$

$$
\begin{aligned}
&= \int_{\Lambda}\int_{\mathbb{R}} \left(y^{\delta'}\right)^{+} \left(\frac{\partial^l}{\partial y^l} \int_{Q(\mathcal{D})} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv \right)^2 dy d\mu(\lambda) \\
&= \int_{\Lambda}\int_{\mathbb{R}} \left(y^{\delta'}\right)^{+} \\
&\qquad \times \left(\int_{Q(\mathcal{D})} q\left(Q^{-1}(v)\right) u\left(\lambda,\, Q^{-1}(v)\right) \frac{\partial^l}{\partial y^l} \left(\frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} \right) dv \right)^2 dy d\mu(\lambda) \\
&= \int_{\Lambda}\int_{\mathbb{R}} \left(y^{\delta'}\right)^{+} \left(\int_{\mathbb{R}} J_u^l\left(\lambda,\, v\right) \frac{e^{-\frac{(v-y)^2}{4\epsilon}}}{\sqrt[4]{2\pi\epsilon}} \frac{e^{-\frac{(v-y)^2}{4\epsilon}}}{\sqrt[4]{2\pi\epsilon}} dv \right)^2 dy d\mu(\lambda) \\
&\leq \int_{\Lambda}\int_{\mathbb{R}} \left(y^{\delta'}\right)^{+} \left(\int_{\mathbb{R}} \left(J_u^l\left(\lambda,\, v\right)\right)^2 \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv \right) \left(\int_{\mathbb{R}} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dv \right) dy d\mu(\lambda) \\
&= \int_{\Lambda}\int_{\mathbb{R}} \left(J_u^l\left(\lambda,\, v\right)\right)^2 \int_{\mathbb{R}} \left(y^{\delta'}\right)^{+} \frac{e^{-\frac{(v-y)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} dy dv d\mu(\lambda)
\end{aligned}
\tag{2.5.9}
$$

which converges (by the same argument as in (2.5.5) in the proof of Lemma 2.5.4) to

$$
\int_{\Lambda}\int_{\mathbb{R}} \left(v^{\delta'}\right)^{+} \left(J_u^l\left(\lambda,\, v\right)\right)^2 dv d\mu(\lambda) = \left\| J_u^l(\cdot,\, \cdot) \right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^{+}}(\mathbb{R})\right)}
$$

as $\epsilon \to 0^+$, and thus we have

$$
\limsup_{m\to+\infty} \left\| \frac{\partial^l}{\partial y^l} J_{u,\epsilon_{k_m}}(\cdot,\, \cdot) \right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^{+}}(\mathbb{R})\right)} \leq \left\| J_u^l(\cdot,\, \cdot) \right\|^2_{L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^{+}}(\mathbb{R})\right)}. \tag{2.5.10}
$$

Hence, by recalling Proposition III.30 from [5] as in the proof of Lemma 2.5.4, we can deduce that the weak convergence $\frac{\partial^l}{\partial z^l} J_{u,\epsilon_{k_m}} \longrightarrow J_u^l$ in $L^2\left(\Lambda;\, L^2_{\left(y^{\delta'}\right)^{+}}(\mathbb{R})\right)$ as $m \longrightarrow +\infty$, which was obtained earlier for all $l \in \{1,\, 2,\, ...,\, n\}$, is also a strong convergence. The desired result follows since the sequence $\{\epsilon_m\}_{m\in\mathbb{N}}$ is arbitrary and since a weak derivative is always unique. □

We will use the above lemmas for $\Lambda = \Omega^0 \times \mathbb{R}^+$ and $\Lambda = [0,\, t] \times \Omega^0 \times \mathbb{R}^+$ for $t \geq 0$, with the corresponding product of measures (where Ω^0 is equipped with the measure $\mathbb{P}^0(\cdot)$, $[0,\, t]$ is equipped with the standard Lebesgue measure, and $\mathbb{R}^+$ is equipped with the weighted Lebesgue measure with weight 1 or $w^2(x)$). This means that, in the notation we introduced at the beginning of this section, the two lemmas will be used for functions in the spaces

$$
L^2\left(\Omega^0 \times \mathbb{R}^+;\, L^2_{\left(y^{\delta'}\right)^{+}}(\mathbb{R})\right) = L^2_{w^2(x)}\left(\Omega^0;\, \tilde{L}^2_{\delta'}\right),
$$

$$L^2_{w^2(x)}\left(\Omega^0\times\mathbb{R}^+;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)=L^2_{w^2(x)}\left(\Omega^0;\,\tilde{L}^2_{\delta',w}\right),$$

$$L^2\left([0,\,t]\times\Omega^0\times\mathbb{R}^+;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)=L^2\left([0,\,t]\times\Omega^0;\,\tilde{L}^2_{\delta'}\right),$$

and

$$L^2_{w^2(x)}\left([0,\,t]\times\Omega^0\times\mathbb{R}^+;\, L^2_{\left(y^{\delta'}\right)^+}(\mathbb{R})\right)=L_2\left([0,\,t]\times\Omega^0;\,\tilde{L}^2_{\delta',w}\right).$$

2.5.3 Smoothing the SPDE: The δ-identity

We fix now a function u which is an $(\alpha,\,\beta)$-solution to our problem for all $\alpha\geq 0$ and some $\beta\geq 0$, and we set:

$$I_{\epsilon,g(z)}(s,\,x,\,y)=\int_{\mathcal{D}}g(z)u(s,\,x,\,z)\phi_\epsilon(z,\,y)dz$$

for any function g of z and $\epsilon>0$. Under the above notation, the δ-identity for our solution u is given in the following lemma:

Lemma 2.5.6 (**the δ-identity**). *The following estimate holds for $\delta=0$ when $q(\cdot)$ is bounded from below and has bounded derivative, and for any $\delta\geq 1$ when $q(z)=\sqrt{z}$ for all $z\geq 0$,*

$$\begin{aligned}
\|I_{\epsilon,1}(t,\,\cdot)\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)} &= \left\|\int_{\mathcal{D}}U_0(\cdot,\,z)\phi_\epsilon(z,\,\cdot)dz\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}\\
&\quad+r_1\int_0^t\left\|\mathbb{I}_{[0,\,1]\times\mathbb{R}}(\cdot)I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta}\right)}ds\\
&\quad+\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&\quad+2\delta k_1\theta_1\int_0^t\left\langle I_{\epsilon,Q'(z)}(s,\,\cdot),\,I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)}ds\\
&\quad+2k_1\theta_1\int_0^t\left\langle I_{\epsilon,Q'(z)}(s,\,\cdot),\,\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&\quad-\delta\xi_1^2\int_0^t\left\langle I_{\epsilon,q'(z)}(s,\,\cdot),\,I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)}ds\\
&\quad-\xi_1^2\int_0^t\left\langle I_{\epsilon,q'(z)}(s,\,\cdot),\,\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&\quad-2\delta k_1\int_0^t\left\langle I_{\epsilon,zQ'(z)}(s,\,\cdot),\,I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)}ds
\end{aligned}$$

$$
\begin{aligned}
&-2k_1 \int_0^t \left\langle I_{\epsilon,zQ'(z)}(s,\,\cdot),\, \frac{\partial}{\partial y} I_{\epsilon,1}(s,\,\cdot) \right\rangle_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta,w}\right)} ds \\
&- \int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h^2(z)}(s,\,\cdot),\, \frac{\partial}{\partial x} I_{\epsilon,1}(s,\,\cdot) \right\rangle_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta,w}\right)} ds \\
&- \int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h^2(z)}(s,\,\cdot),\, \mathbb{I}_{[0,\,1]\times\mathbb{R}}(\cdot) I_{\epsilon,1}(s,\,\cdot) \right\rangle_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta}\right)} ds. \\
&-2\delta\rho \int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h(z)}(s,\cdot),\, I_{\epsilon,1}(s,\cdot) \right\rangle_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta-1,w}\right)} ds \\
&+\rho_{1,1}^2 \int_0^t \left\| \frac{\partial}{\partial x} I_{\epsilon,h(z)}(s,\,\cdot) \right\|^2_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta,w}\right)} ds \\
&+\delta(\delta-1)\frac{\xi_1^2}{2} \int_0^t \left\| I_{\epsilon,1}(s,\,\cdot) \right\|^2_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta-2,w}\right)} ds \\
&-\xi_1^2 \left(1-\rho_{2,1}^2\right) \int_0^t \left\| \frac{\partial}{\partial y} I_{\epsilon,1}(s,\,\cdot) \right\|^2_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta,w}\right)} ds. \\
&-2\left(\rho - \xi_1\rho_3\rho_{1,1}\rho_{2,1}\right) \\
&\qquad \times \int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h(z)}(s,\cdot),\, \frac{\partial}{\partial y} I_{\epsilon,1}(s,\cdot) \right\rangle_{L^2\left(\Omega^0;\, \tilde{L}^2_{\delta,w}\right)} ds.
\end{aligned}
\tag{2.5.11}
$$

where the $\tilde{L}^2_{\delta-2}$ and $\tilde{L}^2_{\delta-1}$ norm terms do not appear when $\delta \in \{0,\,1\}$ and $\delta = 0$ respectively. All the terms in the above identity are finite.

Proof. The finiteness of all the terms in the identity we are proving is a consequence of Lemma 2.5.4 and the assumed weighted integrability of u and u_x. Next, we observe that by definition of ϕ_ϵ we have

$$
\begin{aligned}
\int_{\mathcal{D}} g(z)u(s,\,x,\,z)\frac{\partial}{\partial z}\phi_\epsilon(z,\,y)dz &= -\int_{\mathcal{D}} Q'(z)g(z)u(s,\,x,\,z)\frac{\partial}{\partial y}\phi_\epsilon(z,\,y)dz \\
&= -\frac{\partial}{\partial y} I_{\epsilon,g(z)Q'(z)}(s,\,x,\,y)
\end{aligned}
\tag{2.5.12}
$$

and also

$$
\begin{aligned}
&\int_{\mathcal{D}} g(z)u(s,\,x,\,z)\frac{\partial^2}{\partial z^2}\phi_\epsilon(z,\,y)dz \\
&\quad = -\int_{\mathcal{D}} g(z)u(s,\,x,\,z)\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\phi_\epsilon(z,\,y)Q'(z)\right)dz \\
&\quad = -\int_{\mathcal{D}} g(z)u(s,\,x,\,z)\frac{\partial}{\partial y}\frac{\partial}{\partial z}\phi_\epsilon(z,\,y)Q'(z)dz \\
&\qquad -\int_{\mathcal{D}} g(z)u(s,\,x,\,z)\frac{\partial}{\partial y}\phi_\epsilon(z,\,y)Q''(z)dz
\end{aligned}
$$

$$
\begin{aligned}
&= -\left(\int_{\mathcal{D}} g(z)Q'(z)u(s,\,x,\,z)\frac{\partial}{\partial z}\phi_\epsilon(z,\,y)dz\right)_y \\
&\quad -\int_{\mathcal{D}} g(z)Q''(z)u(s,\,x,\,z)\frac{\partial}{\partial y}\left(\phi_\epsilon(z,\,y)\right)dz \\
&= \left(I_{\epsilon,g(z)(Q'(z))^2}(s,\,x,\,y)\right)_{yy} - \left(I_{\epsilon,g(z)Q''(z)}(s,\,x,\,y)\right)_y, \qquad (2.5.13)
\end{aligned}
$$

for any $\epsilon > 0$ and function g. Thus, after testing (2.5.1) against ϕ_ϵ, by substituting from (2.5.12) and (2.5.13), and by interchanging the x-derivatives with the integrals, we obtain

$$
\begin{aligned}
I_{\epsilon,1}(t,\,x,\,y) \;=\; & \int_{\mathcal{D}} U_0(x,\,z)\phi_\epsilon(z,\,y)dz - r_1\int_0^t \frac{\partial}{\partial x}I_{\epsilon,1}(s,\,x,\,y)ds \\
& +\frac{1}{2}\int_0^t \frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,x,\,y)ds - k_1\theta_1\int_0^t \frac{\partial}{\partial y}I_{\epsilon,Q'(z)}(s,\,x,\,y)ds \\
& +k_1\int_0^t \frac{\partial}{\partial y}I_{\epsilon,zQ'(z)}(s,\,x,\,y)ds + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}I_{\epsilon,h^2(z)}(s,\,x,\,y)ds \\
& +\frac{\xi_1^2}{2}\int_0^t \frac{\partial^2}{\partial y^2}I_{\epsilon,1}(s,\,x,\,y)ds + \frac{\xi_1^2}{2}\int_0^t \frac{\partial}{\partial y}I_{\epsilon,q'(z)}(s,\,x,\,y)ds \\
& +\rho\int_0^t \frac{\partial^2}{\partial x\partial y}I_{\epsilon,h(z)}(s,\,x,\,y)ds - \rho_{1,1}\int_0^t \frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\,x,\,y)dW_s^0 \\
& -\xi_1\rho_{2,1}\int_0^t \frac{\partial}{\partial y}I_{\epsilon,1}(s,\,x,\,y)dB_s^0. \qquad (2.5.14)
\end{aligned}
$$

Multiplying (2.5.14) by $w^2(x)\left(y^\delta\right)^+$, applying Ito's formula for the $L^2(\mathbb{R}^+)$ norm (Theorem 3.1 from [33] for the triple $H_0^1 \subset L^2 \subset H^{-1}$, with $\Lambda(u) = w(\cdot)u$), and then integrating in y over $\mathbb{R}^+$, we obtain the equality

$$
\begin{aligned}
\|I_{\epsilon,1}(t,\,\cdot)\|^2_{\tilde{L}^2_{\delta,w}} &= \left\|\int_{\mathcal{D}} U_0(\cdot,z)\phi_\epsilon(z,\cdot)dz\right\|^2_{\tilde{L}^2_{\delta,w}} \\
&-2r_1\int_0^t \left\langle \frac{\partial}{\partial x}I_{\epsilon,1}(s,\cdot), I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}} ds + \int_0^t \left\langle \frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\cdot), I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}} ds \\
&-2k_1\theta_1\int_0^t \left\langle \frac{\partial}{\partial y}I_{\epsilon,Q'(z)}(s,\cdot), I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}} ds \\
&+2k_1\int_0^t \left\langle \frac{\partial}{\partial y}I_{\epsilon,zQ'(z)}(s,\cdot), I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}} ds \\
&+\int_0^t \left\langle \frac{\partial^2}{\partial x^2}I_{\epsilon,h^2(z)}(s,\cdot),\, I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}} ds \\
&+\xi_1^2\int_0^t \left\langle \frac{\partial^2}{\partial y^2}I_{\epsilon,1}(s,\cdot), I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}} ds
\end{aligned}
$$

$$
\begin{aligned}
&+2\rho\int_0^t\left\langle\frac{\partial^2}{\partial x\partial y}I_{\epsilon,h(z)}(s,\cdot),\,I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}}ds\\
&+2\xi_1\rho_3\rho_{1,1}\rho_{2,1}\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\cdot),\,\frac{\partial}{\partial y}I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}}ds\\
&+\xi_1^2\int_0^t\left\langle\frac{\partial}{\partial y}I_{\epsilon,q'(z)}(s,\cdot),I_{\epsilon,1}(s,\,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}}ds+\rho_{1,1}^2\int_0^t\left\|\frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\cdot)\right\|^2_{\tilde{L}^2_{\delta,w}}ds\\
&+\xi_1^2\rho_{2,1}^2\int_0^t\left\|\frac{\partial}{\partial y}I_{\epsilon,1}(s,\cdot)\right\|^2_{\tilde{L}^2_{\delta,w}}ds\\
&-2\rho_{1,1}\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\cdot),I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}}dW_s^0\\
&-2\xi_1\rho_{2,1}\int_0^t\left\langle\frac{\partial}{\partial y}I_{\epsilon,1}(s,\cdot),I_{\epsilon,1}(s,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}}dB_s^0.
\end{aligned}
\tag{2.5.15}
$$

Observe now that by the definition of u_{xx} in our SPDE, we have

$$
\begin{aligned}
\int_{\mathbb{R}^+}\int_{\mathbb{R}}u_{xx}(s,\,x,\,z)\phi_\epsilon(z,\,y)f(x)dzdx&=\int_{\mathbb{R}^+}\int_{\mathbb{R}}u(s,\,x,\,z)\phi_\epsilon(z,\,y)f_{xx}(x)dzdx\\
&=-\int_{\mathbb{R}^+}\int_{\mathbb{R}}u_x(s,\,x,\,z)\phi_\epsilon(z,\,y)f_x(x)dzdx
\end{aligned}
\tag{2.5.16}
$$

for any smooth f vanishing at zero. Since $u\in H_0^1$, this mapping over all such functions f defines a distribution in H^{-1}, and since those test functions are dense in H_0^1, we have that (2.5.16) holds for any $f\in H_0^1$. In particular, for $f=w^2(\cdot)I_{\epsilon,1}(s,\,\cdot,\,y)$, multiplying (2.5.16) by $\left(y^\delta\right)^+$ and then integrating in (y,t) over $\mathbb{R}\times\mathbb{R}^+$, we obtain

$$
\begin{aligned}
&\int_0^t\left\langle\frac{\partial^2}{\partial x^2}I_{\epsilon,h^2(z)}(s,\,\cdot),\,I_{\epsilon,1}(s,\,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}}ds\\
&\quad=\int_0^t\left\langle\frac{\partial^2}{\partial x^2}I_{\epsilon,h^2(z)}(s,\,\cdot),\,w^2(\cdot)I_{\epsilon,1}(s,\,\cdot)\right\rangle_{\tilde{L}^2_{\delta}}ds\\
&\quad=-\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,w^2(\cdot)\frac{\partial}{\partial x}I_{\epsilon,1}(s,\,\cdot)\right\rangle_{\tilde{L}^2_{\delta}}ds.\\
&\qquad-\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,\mathbb{I}_{[0,\,1]}(\cdot)I_{\epsilon,1}(s,\,\cdot)\right\rangle_{\tilde{L}^2_{\delta}}ds.\\
&\quad=-\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,\frac{\partial}{\partial x}I_{\epsilon,1}(s,\,\cdot)\right\rangle_{\tilde{L}^2_{\delta,w}}ds.\\
&\qquad-\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,\mathbb{I}_{[0,\,1]\times\mathbb{R}}(\cdot)I_{\epsilon,1}(s,\,\cdot)\right\rangle_{\tilde{L}^2_{\delta}}ds.
\end{aligned}
\tag{2.5.17}
$$

Next, integration by parts for $\delta > 1$ implies

$$\int_0^t \left\langle \frac{\partial^2}{\partial y^2} I_{\epsilon,1}(s,\,\cdot),\, I_{\epsilon,1}(s,\,\cdot) \right\rangle_{\tilde{L}^2_{\delta,w}} ds$$
$$= \delta(\delta-1)\frac{1}{2}\int_0^t \|I_{\epsilon,1}(s,\,\cdot)\|^2_{\tilde{L}^2_{\delta-2,w}}\, ds - \int_0^t \left\| \frac{\partial}{\partial y} I_{\epsilon,1}(s,\,\cdot) \right\|^2_{\tilde{L}^2_{\delta,w}} ds \tag{2.5.18}$$

and

$$\int_0^t \left\langle \frac{\partial^2}{\partial x \partial y} I_{\epsilon,h(z)}(s,\cdot),\, I_{\epsilon,1}(s,\cdot) \right\rangle_{\tilde{L}^2_{\delta,w}} ds$$
$$= -\int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h(z)}(s,\cdot),\, \frac{\partial}{\partial y} I_{\epsilon,1}(s,\cdot) \right\rangle_{\tilde{L}^2_{\delta,w}} ds$$
$$-\delta \int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h(z)}(s,\cdot),\, I_{\epsilon,1}(s,\cdot) \right\rangle_{\tilde{L}^2_{\delta-1,w}} ds \tag{2.5.19}$$

and also

$$\int_0^t \left\langle \frac{\partial}{\partial y} I_{\epsilon,g(z)}(s,\,\cdot),\, I_{\epsilon,1}(s,\,\cdot) \right\rangle_{\tilde{L}^2_{\delta,w}} ds$$
$$= -\delta \int_0^t \left\langle I_{\epsilon,g(z)}(s,\,\cdot),\, I_{\epsilon,1}(s,\,\cdot) \right\rangle_{\tilde{L}^2_{\delta-1,w}} ds$$
$$- \int_0^t \left\langle I_{\epsilon,g(z)}(s,\,\cdot),\, \frac{\partial}{\partial y} I_{\epsilon,1}(s,\,\cdot) \right\rangle_{\tilde{L}^2_{\delta,w}} ds, \tag{2.5.20}$$

for $g(z) \in \{Q'(z),\, zQ'(z),\, q'(z)\}$. Moreover, for $\delta = 1$, integration by parts gives the same identities without the $\tilde{L}^2_{\delta-2}$ norm term, while for $\delta = 0$ (and $q(\cdot)$ lower bounded with bounded derivative) it gives the same identities without both the $\tilde{L}^2_{\delta-1}$ and $\tilde{L}^2_{\delta-2}$ norm terms. Note that integrating by parts in the y-direction is possible without leaving any boundary term at infinity, since all the terms inside the inner products which do not involve u_x are rapidly decreasing in y. This is also a consequence of the applicability of Lemma 2.5.4 for large δ', since for any large $n \in \mathbb{N}$ and any function f having derivatives in polynomially weighted L^2 spaces, by Morrey's inequality we have:

$$y^n f(y) \leq \frac{1}{y} \sup_{z\in\mathbb{R}} |z^{n+1} f(z)|$$
$$\leq \frac{1}{y}\left(\int_{\mathbb{R}} z^{2(n+1)} f^2(z) dz + \int_{\mathbb{R}} z^{2n} f^2(z) dz + \int_{\mathbb{R}} z^{2(n+1)} (f'(z))^2 dz \right)^{\frac{1}{2}} \longrightarrow 0$$

as $y \to \infty$. Of course, for $\delta \geq 1$ we do not have boundary terms at zero either, due to the weight function $\left(y^{\delta}\right)^{+}$.

We will use (2.5.17) - (2.5.19) to get rid of second order derivative terms in our estimate. Here, it becomes clear why we have chosen to compose the standard heat kernel with $Q(z)$: In (2.5.15), substituting the term in the sixth row from (2.5.18) gives again the term of the tenth row but with a negative coefficient of a bigger absolute value, which allows us to control y-derivative terms. It is not hard to check that that this wouldn't have been the case if we had composed the standard heat kernel with another function, when the existence of u_y is not assumed (as in our case). Using now integration by parts on the first inner product in the RHS of (2.5.15), substituting (2.5.17), (2.5.19) and (2.5.20) for $g(z) \in \{Q'(z),\, zQ'(z),\, q'(z)\}$ in (2.5.15), and finally taking expectations, we obtain the desired identity. □

2.5.4 Proof of the improved regularity result

Now that we have obtained the δ-identity, we can proceed to the proof of our main Theorem. Our strategy is to establish the regularity result by controlling the derivative terms in the δ-identity for appropriate values of δ, by taking $\epsilon \to 0^{+}$ and by using Lemma 2.5.4 and Lemma 2.5.5 (which gives the regularity of the limits).

Proof of Theorem 2.5.2. For all the inner products in the δ-identity except the first and the tenth, we can use the Cauchy-Schwartz inequality in the form

$$\langle u_1,\, u_2 \rangle_{\tilde{L}^2_{\delta,w}} \leq \|u_1\|_{\tilde{L}^2_{2\delta_1,w}} \|u_2\|_{\tilde{L}^2_{2\delta_2,w}}$$

for $\delta = \delta_1 + \delta_2$, and then the AM-GM inequality ($ab \leq \frac{a^2}{4C} + Cb^2$) for the products of norms to obtain

$$\begin{aligned}
\|I_{\epsilon,1}(t,\,\cdot)\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)} \;\leq\; & \left\| \int_{\mathcal{D}} U_0(\cdot,\,z)\phi_{\epsilon}(z,\,\cdot)dz \right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)} \\
& + r_1 \int_0^t \left\| \mathbb{I}_{[0,\,1]\times\mathbb{R}}(\cdot) I_{\epsilon,1}(s,\,\cdot) \right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta}\right)} ds \\
& + \int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h^2(z)}(s,\,\cdot),\, I_{\epsilon,1}(s,\,\cdot) \right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)} ds \\
& - \delta\rho \int_0^t \left\langle \frac{\partial}{\partial x} I_{\epsilon,h(z)}(s,\cdot),\, I_{\epsilon,1}(s,\cdot) \right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)} ds \\
& + k_1\theta_1 \int_0^t C' \left\| I_{\epsilon,Q'(z)}(s,\,\cdot) \right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)} ds
\end{aligned}$$

$$
\begin{aligned}
&+k_1\theta_1\int_0^t \frac{1}{C'}\left\|\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\delta k_1\theta_1\int_0^t C''\left\|I_{\epsilon,Q'(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\delta k_1\theta_1\int_0^t \frac{1}{C''}\left\|I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-2,w}\right)}ds\\
&+\xi_1^2\int_0^t C'\left\|I_{\epsilon,q'(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\xi_1^2\int_0^t \frac{1}{4C'}\left\|\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\delta\xi_1^2\int_0^t C''\left\|I_{\epsilon,q'(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\delta\xi_1^2\int_0^t \frac{1}{4C''}\left\|I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-2,w}\right)}ds\\
&+k_1\int_0^t C'\left\|I_{\epsilon,zQ'(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+k_1\int_0^t \frac{1}{C'}\left\|\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\delta k_1\int_0^t C''\left\|I_{\epsilon,zQ'(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)}ds\\
&+\delta k_1\int_0^t \frac{1}{C''}\left\|I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)}ds\\
&-\int_0^t \left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,\frac{\partial}{\partial x}I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\rho_{1,1}^2\int_0^t \left\|\frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&-\int_0^t \left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,\mathbb{I}_{[0,\,1]\times\mathbb{R}}(\cdot)I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta}\right)}ds\\
&+\delta(\delta-1)\frac{\xi_1^2}{2}\int_0^t \left\|I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-2,w}\right)}ds\\
&-\xi_1^2\left(1-\rho_{2,1}^2\right)\int_0^t \left\|\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+C'\left|\rho-\xi_1\rho_3\rho_{1,1}\rho_{2,1}\right|\int_0^t \left\|\frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&+\frac{1}{C'}\left|\rho-\xi_1\rho_3\rho_{1,1}\rho_{2,1}\right|\int_0^t \left\|\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds
\end{aligned}
$$

(2.5.21)

and this for any C', $C''>0$. If we choose $C''=1$ and a large enough C'' such that

$$\left(k_1\theta_1+\frac{\xi_1^2}{4}+|\rho-\xi_1\rho_3\rho_{1,1}\rho_{2,1}|+k_1\right)\frac{1}{C'}<\xi_1^2\left(1-\rho_{2,1}^2\right),$$

then from (2.5.21) we can obtain the following estimate

$$\begin{aligned}
&\|I_{\epsilon,1}(t,\,\cdot)\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}+M_1\int_0^t\left\|\frac{\partial}{\partial y}I_{\epsilon,1}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&\quad\leq\left\|\int_{\mathcal{D}}U_0(\cdot,\,z)\phi_\epsilon(z,\,\cdot)dz\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}\\
&\qquad+M_2\sum_{g(z)\in\{Q'(z),zQ'(z),q'(z)\}}\int_0^t\left\|I_{\epsilon,g(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&\qquad+\delta M_2\sum_{g(z)\in\{1,Q'(z),zQ'(z),q'(z)\}}\int_0^t\left\|I_{\epsilon,g(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)}ds\\
&\qquad+M_2\int_0^t\left\|\frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&\qquad+\delta(\delta-1)\frac{\xi_1^2}{2}\int_0^t\|I_{\epsilon,1}(s,\,\cdot)\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-2,w}\right)}ds\\
&\qquad+r_1\int_0^t\left\|\mathbb{I}_{[0,\,1]\times\mathbb{R}}(\cdot)I_{\epsilon,1}(s,\cdot)\right\|^2_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta}\right)}ds\\
&\qquad-\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,\mathbb{I}_{[0,\,1]\times\mathbb{R}}(\cdot)I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta}\right)}ds\\
&\qquad+\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds\\
&\qquad-\delta\rho\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h(z)}(s,\cdot),\,I_{\epsilon,1}(s,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta-1,w}\right)}ds\\
&\qquad-\int_0^t\left\langle\frac{\partial}{\partial x}I_{\epsilon,h^2(z)}(s,\,\cdot),\,\frac{\partial}{\partial x}I_{\epsilon,1}(s,\,\cdot)\right\rangle_{L^2\left(\Omega^0;\,\tilde{L}^2_{\delta,w}\right)}ds,
\end{aligned}\tag{2.5.22}$$

for some positive constants M_1 and M_2. Now, for any function g, it is easy to check that with the notation of Lemma 2.5.4 we have $I_{\epsilon,g(z)}=J_{g\cdot u,\epsilon}$ and $\frac{\partial}{\partial x}I_{\epsilon,g(z)}=J_{g\cdot u_x,\epsilon}$. Then, for any $\delta'\in\{\delta,\,\delta-1,\,\delta-2\}$, by 2. of Lemma 2.5.4 we can compute the limits of these quantities in $L^2\left([0,\,t]\times\Omega^0;\,\tilde{L}^2_{\delta,w}\right)$, which are equal to

$$J_{g\cdot u}(s,\,x,\,v)=g\left(Q^{-1}(v)\right)q\left(Q^{-1}(v)\right)u\left(s,\,x,\,Q^{-1}(v)\right)$$

and

$$J_{g\cdot u_x}(s,\, x,\, v) = g\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u_x\left(s,\, x,\, Q^{-1}(v)\right)$$

respectively, provided that they belong to $L^2\left([0,\, t]\times\Omega^0;\, \tilde{L}^2_{\delta,w}\right)$. This can be verified by computing their norms in that space. This computation (after setting $v \longrightarrow Q(v)$) gives

$$\begin{aligned}
&\left\|J_{g\cdot u}\right\|^2_{L^2_{y^{\delta'}w^2(x)}([0,t]\times\Omega^0\times\mathbb{R}^+\times\mathbb{R})} \\
&\quad = \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^+}\int_{\mathcal{D}} Q(y)^{\delta'} w^2(x) g^2\left(y\right) q\left(y\right) u^2\left(s,\, x,\, y\right) dvdx\right] ds
\end{aligned} \tag{2.5.23}$$

and

$$\begin{aligned}
&\left\|J_{g\cdot u_x}\right\|^2_{L^2_{y^{\delta'}w^2(x)}([0,t]\times\Omega^0\times\mathbb{R}^+\times\mathbb{R})} \\
&\quad = \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^+}\int_{\mathcal{D}} Q(y)^{\delta'} w^2(x) g^2\left(y\right) q\left(y\right) u_x^2\left(s,\, x,\, y\right) dvdx\right] ds
\end{aligned} \tag{2.5.24}$$

which are both finite for all the combinations of g and δ' appearing in the norm terms of the RHS of (2.5.22) (with $\delta \geq 1$ when q is the square root function, and $\delta = 0$ when q is lower bounded with bounded derivative), as we can easily check. For the last, it is crucial that when q is bounded from below, Q has a linear growth as $z \longrightarrow \infty$. Hence, 2. of Lemma 2.5.4 and the continuity of the inner products imply that all the terms in the RHS of (2.5.22) are convergent as $\epsilon \longrightarrow 0^+$. Therefore, the RHS of (2.5.22) is also bounded in ϵ and thus, Lemma 2.5.5 applied on the y-derivative term in the LHS of that estimate implies that $\frac{\partial}{\partial v} J_u = \frac{\partial}{\partial v} q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)$ exists in $L^2\left([0,\, t]\times\Omega^0;\, \tilde{L}^2_{\delta,w}\right)$ and that in this space we have $\frac{\partial}{\partial v} I_{\epsilon,1} = \frac{\partial}{\partial v} J_{u,\epsilon} \longrightarrow \frac{\partial}{\partial v} J_u$ as $\epsilon \longrightarrow 0^+$.

From the above we can easily deduce that the weak derivative $\frac{\partial}{\partial y} u\left(s,\, x,\, y\right)$ exists, and by using the product rule and the inequality $(a-b)^2 \leq 2a^2 + 2b^2$ we have

$$\begin{aligned}
&\int_0^t \mathbb{E}\left[\int_{\mathbb{R}^+}\int_{\mathcal{D}} \left(Q(y)^{\delta}\right)^+ u_y^2(s,\, x,\, y) dydx\right] ds \\
&\quad = \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} \left(v^{\delta}\right)^+ q\left(Q^{-1}(v)\right) u_y^2\left(s,\, x,\, Q^{-1}(v)\right) dvdx\right] ds \\
&\quad = \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} \left(v^{\delta}\right)^+ q^{-1}\left(Q^{-1}(v)\right) \left(\left(u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right)^2 dvdx\right] ds
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} \left(v^\delta\right)^+ \\
&\qquad \times \mathbb{E}\left[q^{-3}\left(Q^{-1}(v)\right)\left(\left(q\left(Q^{-1}(v)\right)u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right)^2\right] dvdxds \\
&\quad +2\int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} \left(v^\delta\right)^+ \\
&\qquad \times \mathbb{E}\left[q^{-1}\left(Q^{-1}(v)\right)\left(q'\left(Q^{-1}(v)\right)u\left(s,\, x,\, Q^{-1}(v)\right)\right)^2\right] dvdxds.
\end{aligned}
$$

Since $\frac{\partial}{\partial v} J_u \in L^2\left([0,\, t] \times \Omega^0;\, \tilde{L}^2_{\delta,w}\right)$, the RHS of the above is finite for $\delta = 0$ when q is bounded from below with q' bounded, so we obtain $u_y \in L^2\left([0,\, T] \times \Omega^0;\, \tilde{L}^2_{0,w}\right)$. On the other hand, when $q(z) = \sqrt{|z|}$, we have $Q(z) = z^{\frac{\delta+3}{2}}$, and we need $\delta \geq 1$ for the last estimate to have a finite RHS. This implies that $u_y \in L^2\left([0,\, T] \times \Omega^0;\, \tilde{L}^2_{\alpha,w}\right)$ for $\alpha = \frac{\delta+3}{2} \in [2, +\infty)$ □

Remark 2.5.7. The flexibility in the choice of ρ allows us to extend our results to the case where the idiosyncratic noises have nonzero correlation. Indeed, suppose that for any $i \geq 1$ we have $W^i_t = w_i \tilde{W}^i_t + \sqrt{1 - w_i^2} Z^i_t$ and $B^i_t = b_i \tilde{B}^i_t + \sqrt{1 - b_i^2} Z^i_t$, where $\tilde{W}^i_\cdot$, $\tilde{B}^i_\cdot$ and $Z^i_\cdot$ are pairwise independent standard Brownian Motions, and $w_i, b_i \in [-1, 0) \cup (0, 1]$. Then, we can obtain the convergence results of section 2 in exactly the same way, and the SPDE we obtain is the one treated in the previous section with

$$
\rho = \xi_1 \rho_3 \rho_{1,1} \rho_{2,1} + \xi_1 \sqrt{1 - \rho^2_{1,1}} \sqrt{1 - \rho^2_{2,1}} \sqrt{1 - w_1^2} \sqrt{1 - b_1^2} \tag{2.5.25}
$$

The extension will be complete if we manage to embed the measure-valued process v_{t,C_1} in $L_\alpha \cap H_\beta$ for appropriate values of $\alpha,\, \beta \geq 0$ and for a given value of C_1, as we have done in Section 4 for the zero correlation case. Since v_{t,C_1} can be expressed as a conditional law of the pair $(X^1_\cdot,\, \sigma^1_\cdot)$ as in the zero correlation case, this embedding can be done by conditioning on $Z^1_\cdot$ to reduce the problem to the zero correlation case, with $\sqrt{1 - \rho^2_{1,1}} W^1_\cdot$, $\sqrt{1 - \rho^2_{2,1}} B^1_\cdot$, $\rho_{1,1} W^0_\cdot$ and $\rho_{2,1} B^0_\cdot$ replaced by $w_1 \sqrt{1 - \rho^2_{1,1}} \tilde{W}^1_\cdot$, $b_1 \sqrt{1 - \rho^2_{2,1}} \tilde{B}^1_\cdot$, $\rho_{1,1} W^0_\cdot + \sqrt{1 - w_1^2} Z^1_\cdot$ and $\rho_{2,1} B^0_\cdot + \sqrt{1 - b_1^2} Z^1_\cdot$ respectively. This approach obviously fails when $w_1 = 0$ or $b_1 = 0$.

2.6 Discussion of uniqueness

The previous sections have established existence and regularity results for a class of SPDEs arising from stochastic volatility models for large portfolios. Now we would

like to investigate whether each of these SPDEs has always a unique solution for given initial data. Having a uniqueness result is particularly important from an application point of view, since it implies that a numerical solution to our stochastic initial-boundary value problem will always approximate the solution arising from the corresponding large portfolio model.

We are only able to prove uniqueness of solutions to our problem (definition 2.5.1) under the assumptions of 2. of Theorem 2.5.2. Therefore, in this section we assume that the function q in our SPDE (2.4.17), which drives the vol-of-vols (volatilities of the volatilities) of the asset prices in the corresponding large portfolio model, is a $C^3(\mathbb{R})$ function with bounded and $\mathcal{O}\left(\frac{1}{|x|}\right)$ (as $x \to +\infty$) derivatives up to third order, and also bounded from below by a positive constant. As we have mentioned in Remark 2.3.1, these functions are positive smooth functions behaving almost like a positive constant for large x, with the simplest examples being the constant functions which correspond to a model with Ornstein-Uhlenbeck volatility processes. Moreover, we will need to assume that the coefficients of (2.4.17) satisfy the condition

$$|\rho - \xi_1 \rho_3 \rho_{1,1} \rho_{2,1}| \leq \xi_1 \sqrt{1-\rho_{1,1}^2}\sqrt{1-\rho_{2,1}^2}, \tag{2.6.1}$$

which is always satisfied when the SPDE arises from a large portfolio model (even when the idiosyncratic noises are correlated, by Remark 2.5.7).

To prove our uniqueness result, we recall that $\frac{\partial}{\partial v} J_u = \frac{\partial}{\partial v} q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)$ exists in $L^2\left([0,\, t] \times \Omega^0;\, \tilde{L}^2_{\delta,w}\right)$ and that in this space we have $\frac{\partial}{\partial v} I_{\epsilon,1} = \frac{\partial}{\partial v} J_{u,\epsilon} \longrightarrow \frac{\partial}{\partial v} J_u$ as $\epsilon \longrightarrow 0^+$, as we saw in the proof of Theorem 2.5.2. Taking now $\epsilon \longrightarrow 0^+$ on the δ-identity for $\delta = 0$ and using the above convergence along with Lemma 2.5.4 and the continuity of the L^2 inner product, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} w^2(x)\mathbb{E}\left[q^2\left(Q^{-1}(v)\right)u^2\left(t,\, x,\, Q^{-1}(v)\right)\right]dxdv \\
&\quad = \int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} w^2(x)\mathbb{E}\left[q^2\left(Q^{-1}(v)\right)U_0^2\left(x,\, Q^{-1}(v)\right)\right]dxdv \\
&\qquad + r_1\int_0^t\int_0^1\int_{Q(\mathcal{D})} \mathbb{E}\left[q^2\left(Q^{-1}(v)\right)u^2\left(t,\, x,\, Q^{-1}(v)\right)\right]dxdvds \\
&\qquad + \int_0^t\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} w^2(x)\mathbb{E}\Big[\left(h^2\left(Q^{-1}(v)\right)q\left(Q^{-1}(v)\right)u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x \\
&\qquad\quad \times q\left(Q^{-1}(v)\right)u\left(s,\, x,\, Q^{-1}(v)\right)\Big]dxdvds \\
&\qquad + 2k_1\theta_1\int_0^t\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} w^2(x)\mathbb{E}\Big[Q'\left(Q^{-1}(v)\right)q\left(Q^{-1}(v)\right)u\left(s,\, x,\, Q^{-1}(v)\right) \\
&\qquad\quad \times \left(q\left(Q^{-1}(v)\right)u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\Big]dxdvds
\end{aligned}$$

$$
\begin{aligned}
&-\xi_1^2 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[q'\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right. \\
&\qquad \left.\times \left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right] dxdvds \\
&-2k_1 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[vQ'\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right. \\
&\qquad \left.\times \left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right] dxdvds \\
&-\int_0^t \int_0^1 \int_{Q(\mathcal{D})} \mathbb{E}\left[\left(h^2\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x\right. \\
&\qquad \left.\times\, q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right] dxdvds \\
&-\int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[\left(h^2\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x\right. \\
&\qquad \left.\times \left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x\right] dxdvds \\
&+\rho_{1,1}^2 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \\
&\qquad \times \mathbb{E}\left[\left(\left(h\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x\right)^2\right] dxdvds \\
&-\xi_1^2\left(1-\rho_{2,1}^2\right) \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \\
&\qquad \times \mathbb{E}\left[\left(\left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right)^2\right] dxdvds \\
&-2\left(\rho - \xi_1 \rho_3 \rho_{1,1} \rho_{2,1}\right) \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \\
&\qquad \times \mathbb{E}\left[\left(h\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x\right. \\
&\qquad \left.\times \left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right] dxdvds.
\end{aligned}
\tag{2.6.2}
$$

and since our SPDE is linear and the difference of two solutions with the same initial data is a solution with zero initial data, the uniqueness problem is reduced to showing that the above identity implies that u vanishes everywhere when $U_0 = 0$. To do this, observe first that if we fix a $T > 0$ and work for $t \in [0, T]$, by using Girsanov's Theorem we can transform $W_\cdot^0$ into a drifted Brownian Motion for any drift $r < 0$, and then r_1 is replaced by $r_1 - r\rho_{1,1}h(y)$. Thus, since h is lower bounded, we can make this term negative. Moreover, by using (2.6.1) and the standard inequality $2ab \leq a^2 + b^2$ on the last summand of (2.6.2), we can bound it by something that cancels with the three summands above it. Furthermore, we have

$$
\begin{aligned}
&-\int_0^t \int_0^1 \int_{Q(\mathcal{D})} \mathbb{E}\left[\left(h^2\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x\right. \\
&\qquad \left.\times\, q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right] dxdvds
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^t \int_0^1 \int_{Q(\mathcal{D})} \frac{h^2\left(Q^{-1}(v)\right)}{2} \left(\mathbb{E}\left[q^2\left(Q^{-1}(v)\right) u^2\left(s,\, x,\, Q^{-1}(v)\right)\right]\right)_x dxdvds \\
&= -\int_0^t \int_{Q(\mathcal{D})} \frac{h^2\left(Q^{-1}(v)\right)}{2} \mathbb{E}\left[q^2\left(Q^{-1}(v)\right) u^2\left(s,\, 1,\, Q^{-1}(v)\right)\right] dvds \\
&\leq 0
\end{aligned} \tag{2.6.3}
$$

and also

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[\left(h^2\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_x \right. \\
&\qquad \left. \times\, q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right] dxdvds \\
&= \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \frac{h^2\left(Q^{-1}(v)\right)}{2} \left(q^2\left(Q^{-1}(v)\right) u^2\left(s,\, x,\, Q^{-1}(v)\right)\right)_x dxdvds\right] \\
&= -\mathbb{E}\left[\int_0^t \int_0^1 \int_{Q(\mathcal{D})} \frac{h^2\left(Q^{-1}(v)\right)}{2} q^2\left(Q^{-1}(v)\right) u^2\left(s,\, x,\, Q^{-1}(v)\right) dxdvds\right] \\
&\leq 0
\end{aligned} \tag{2.6.4}
$$

Therefore, for $U_0 = 0$, from (2.6.2) we can obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[q^2\left(Q^{-1}(v)\right) u^2\left(t,\, x,\, Q^{-1}(v)\right)\right] dxdv \\
&\quad \leq 2k_1\theta_1 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[Q'\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right. \\
&\qquad\qquad \left. \times \left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right] dxdvds \\
&\qquad -\xi_1^2 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[q'\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right. \\
&\qquad\qquad \left. \times \left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right] dxdvds \\
&\qquad -2k_1 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \mathbb{E}\left[vQ'\left(Q^{-1}(v)\right) q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right. \\
&\qquad\qquad \left. \times \left(q\left(Q^{-1}(v)\right) u\left(s,\, x,\, Q^{-1}(v)\right)\right)_v\right] dxdvds \\
&\quad = 2k_1\theta_1 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \frac{Q'\left(Q^{-1}(v)\right)}{2} \\
&\qquad\qquad \times \left(\mathbb{E}\left[q^2\left(Q^{-1}(v)\right) u^2\left(s,\, x,\, Q^{-1}(v)\right)\right]\right)_v dxdvds \\
&\qquad -\xi_1^2 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \frac{q'\left(Q^{-1}(v)\right)}{2} \\
&\qquad\qquad \times \left(\mathbb{E}\left[q^2\left(Q^{-1}(v)\right) u^2\left(s,\, x,\, Q^{-1}(v)\right)\right]\right)_v dxdvds \\
&\qquad -2k_1 \int_0^t \int_{\mathbb{R}^+} \int_{Q(\mathcal{D})} w^2(x) \frac{vQ'\left(Q^{-1}(v)\right)}{2} \\
&\qquad\qquad \times \left(\mathbb{E}\left[q^2\left(Q^{-1}(v)\right) u^2\left(s,\, x,\, Q^{-1}(v)\right)\right]\right)_v dxdvds
\end{aligned}
$$

(2.6.5)

where we can use integration by parts and the boundedness of the derivatives of $Q'\left(Q^{-1}(v)\right)$, $q'\left(Q^{-1}(v)\right)$ and $vQ'\left(Q^{-1}(v)\right)$ (follows easily when q is bounded away from 0 and infinity, and when its derivatives are $\mathcal{O}\left(\frac{1}{|x|}\right)$ as $x \longrightarrow \infty$) to obtain

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} w^2(x)\mathbb{E}\left[q^2\left(Q^{-1}(v)\right)u^2\left(t,\, x,\, Q^{-1}(v)\right)\right]dxdv \\ \leq K\int_0^t\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} w^2(x)\mathbb{E}\left[q^2\left(Q^{-1}(v)\right)u^2\left(s,\, x,\, Q^{-1}(v)\right)\right]dxdvds\end{aligned} \tag{2.6.6}$$

for some $K > 0$ and all $t \geq 0$. By Gronwall's inequality, the above can only hold when the nonnegative quantity $\int_{\mathbb{R}^+}\int_{Q(\mathcal{D})} w^2(x)\mathbb{E}\left[q^2\left(Q^{-1}(v)\right)u^2\left(t,\, x,\, Q^{-1}(v)\right)\right]dxdv$ is zero for all $t \leq T$, which can only hold when u is zero in an $L^2\left(\Omega^0 \times \mathbb{R}^+ \times \mathcal{D}\right)$ sense for all $t \leq T$. We have thus proven the following Theorem

Theorem 2.6.1. *Fix the value of the coefficient vector C_1, the function h, the real number ρ and the initial data function U_0, and suppose that $|\rho - \xi_1\rho_3\rho_{1,1}\rho_{2,1}| \leq \xi_1\sqrt{1-\rho_{1,1}^2}\sqrt{1-\rho_{2,1}^2}$. Suppose then that $q(\cdot)$ is a bounded $C^3(\mathbb{R})$ function with bounded and $\mathcal{O}\left(\frac{1}{|x|}\right)$ (as $|x| \longrightarrow +\infty$) derivatives up to third order, with $q(z) \geq m$ for some $m > 0$. Then, there exists a unique u which is an $(\alpha, 0)$-solution to our problem for all $\alpha \geq 0$, where uniqueness holds in an $L^2\left(\Omega^0 \times \mathbb{R}^+ \times \mathcal{D}\right)$ sense for all $t \leq T$. For this u, the weak derivative u_y exists and we have*

$$u_y \in L^2\left([0,\, T] \times \Omega^0;\, \tilde{L}^2_{0,w}\right)$$

Remark 2.6.2. It is not hard to check that the above approach does not work under the assumptions of 1. of Theorem 2.5.2, i.e when q is a square root function. The reason is that we end up with an estimate involving several non-equivalent weighted norms, where Gronwall's inequality is inapplicable. Therefore, at this stage we are unable to obtain a uniqueness result when our SPDE arises from a portfolio of defaultable assets with volatilities modeled as CIR processes. However, as we have already mentioned, the above Theorem covers the case in which the volatilities are modeled as Ornstein-Uhlenbeck processes.

Remark 2.6.3. The best possible result would be to have our solution in the smaller space $L^2\left([0,\, T] \times \Omega^0; H_0^1\left(\mathbb{R}^+ \times \mathbb{R}\right)\right)$ (without the weight function $w^2(\cdot)$), which would be the case if we had an existence result in that space. However, standard Theorems

that could give such an existence result are not applicable to our SPDE due to the unboundedness of the term $k_1(\theta_1 - y)$ (see [33]). This is also the reason we cannot deduce higher regularity at this stage, as we have mentioned in Remark 2.5.3. A possible solution to these problems could be to introduce mean-reverting volatility processes σ_t^i having a bounded drift, i.e $q_2(\theta_1 - \sigma_t^i)$ for all $i \in \mathbb{N}$, for some increasing and bounded function q_2 vanishing at zero.

Chapter 3

Fast mean-reversion asymptotics

3.1 Introduction

As we have mentioned in the introduction of the thesis, due to a large number of parameters that need to be simulated, a big computational effort is needed for implementing the large portfolio setting when stochastic volatility is incorporated. Moreover, the uniqueness of solutions to the SPDE arising from large portfolios of CIR volatility models remains an open question, which means that numerical solutions to this SPDE could approximate solutions not necessarily equal to the solution describing the large portfolio loss. The subject of this chapter is to try to adapt the ideas of fast mean-reversion analysis from [17] to the large portfolio setting, in order to deal with these issues.

We have already mentioned that the main applications of large portfolio modelling arise in risk management and in the pricing of derivatives like CDO tranches. In risk management applications we need to estimate the probability that the total loss L_t within the portfolio at some time $t > 0$ exceeds a certain proportion, while probabilities of this form appear also in the prices of CDO tranches, but under a risk-neutral probability measure which makes the drifts of the logarithmically scaled asset price processes equal to a known interest rate. Following now the notation and the results from Chapter 2, we have that $L_t = 1 - \mathbb{P}\left(X_t^1 > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)$, where $X_\cdot^i$ stands for the i-th logarithmically scaled price process which satisfies the system of SDEs

$$\begin{aligned} dX_t^i &= \left(r_i - \tfrac{h^2(\sigma_t^i)}{2}\right) dt + h(\sigma_t^i)\left(\sqrt{1-\rho_{1,i}^2}\, dW_t^i + \rho_{1,i} dW_t^0\right), \quad 0 \le t \le T_i \\ d\sigma_t^i &= k_i(\theta_i - \sigma_t^i)dt + \xi_i q\left(\sigma_t^i\right)\left(\sqrt{1-\rho_{2,i}^2}\, dB_t^i + \rho_{2,i} dB_t^0\right), \quad t \ge 0 \\ X_t^i &= 0, \; t > T_i \\ (X_0^i,\, \sigma_0^i) &= (x^i, \sigma^i), \end{aligned} \tag{3.1.1}$$

for $i \in \{1, 2, ...\}$, with T_i being the first time $X_\cdot^i$ hits 0, $C_i = (k_1, \theta_1, \xi_1, r_1, \rho_{1,1}, \rho_{2,1})$ for $i = 1, 2, ...$ being i.i.d random vectors, $W_\cdot^1$, $B_\cdot^1$, $W_\cdot^2$, $B_\cdot^2$, ... being pairwise independent standard Brownian Motions representing idiosyncratic factors that affect each asset's price, $W_\cdot^0$, $B_\cdot^0$ being two (possibly correlated) standard Brownian Motions describing the interdependence among the asset prices, and (x^i, σ^i) for $i = 1, 2, ...$ being random vectors which are pairwise independent given $\mathcal{G}^0$. The function q is chosen such that the volatility processes possess certain crucial properties, as we will see later. Therefore, under the above notation, our aim is to estimate probabilities of the form:

$$\mathbb{P}^0 \left(L_t \in (1-b, 1-a) \right) = \mathbb{P}^0 \left(\mathbb{P} \left(X_t^1 > 0 \,|\, W_\cdot^0, B_\cdot^0, \mathcal{G}^0 \right) \in (a, b) \right) \tag{3.1.2}$$

for some $0 \leq a < b \leq 1$.

The natural way to adapt the ideas from [17] to our setting is to set $k_i = \frac{\kappa_i}{\epsilon}$ and $\xi_i = \frac{v_i}{\sqrt{\epsilon}}$ for all $i = 1, 2, ...$, with $\{\kappa_i : i \in \{1, 2, ...\}\}$ and $\{v_i : i \in \{1, 2, ...\}\}$ being sequences of non-negative i.i.d random variables that do not depend on ϵ, and then try to approximate the quantity in (3.1.2) by something more accurate than its limit as $\epsilon \to 0^+$. However, we will see that in this fast mean-reversion - large vol-of-vol setting, the convergence of the system as $\epsilon \to 0^+$ is very weak, which does not allow us to hope for such an approximation. For this reason, we will also consider a fast mean-reversion - small vol-of-vol setting, where again we have $k_i = \frac{\kappa_i}{\epsilon}$ for all $i = 1, 2, ...$, but this time $\{\xi_i : i \in \{1, 2, ...\}\}$ is a sequence of non-negative i.i.d random variables which does not depend on ϵ. We will see that in this small vol-of-vol setting, we can also estimate the rate of convergence as $\epsilon \to 0^+$, when a certain regularity condition is satisfied at both a and b.

3.2 Fast mean-reversion - large vol-of-vol

We begin with the study of the fast mean-reversion - large vol-of-vol setting. For this study, we will need to assume that $W_\cdot^0$ and $B_\cdot^0$ are uncorrelated.

3.2.1 A first approach to the approximation problem

It has been proven in Chapter 1 that

$$\mathbb{P} \left(X_t^1 > 0 \,|\, W_\cdot^0, B_\cdot^0, \mathcal{G}^0 \right) = \mathbb{E} \left[\int_0^{+\infty} \int_0^{+\infty} u_{C_1} (t, x, y) \, dxdy \,|\, W_\cdot^0, B_\cdot^0, \mathcal{G}^0 \right] \tag{3.2.1}$$

where u_{C_1} is a regular solution to the SPDE

$$\begin{aligned} u_{C_1}(t,\,x,\,y) \;=\; & U_0(x,\,y\,|\,\mathcal{G}^0) - r_1\int_0^t \left(u_{C_1}(s,\,x,\,y)\right)_x ds \\ & +\frac{1}{2}\int_0^t h^2(y)\left(u_{C_1}(s,\,x,\,y)\right)_x ds - k_1\theta_1\int_0^t \left(u_{C_1}(s,\,x,\,y)\right)_y ds \\ & +k_1\int_0^t \left(y u_{C_1}(s,\,x,\,y)\right)_y ds + \frac{1}{2}\int_0^t h^2(y)\left(u_{C_1}(s,\,x,\,y)\right)_{xx} ds \\ & +\xi_1\rho_3\rho_{1,1}\rho_{2,1}\int_0^t \left(h\,(y)\,q\,(y)\,u_{C_1}(s,\,x,\,y)\right)_{xy} ds \\ & +\frac{\xi_1^2}{2}\int_0^t \left(q^2\,(y)\,u_{C_1}(s,\,x,\,y)\right)_{yy} ds \\ & -\rho_{1,1}\int_0^t h(y)\left(u_{C_1}(s,\,x,\,y)\right)_x dW_s^0 \\ & -\xi_1\rho_{2,1}\int_0^t \left(q\,(y)\,u_{C_1}(s,\,x,\,y)\right)_y dB_s^0, \end{aligned} \tag{3.2.2}$$

in the half-space $\mathbb{R}^+ \times \mathbb{R}$, and expectation in (3.2.1) is taken to average over all the possible values of the coefficient vector C_1. Substituting from $k_1 = \frac{\kappa_1}{\epsilon}$ and $\xi_1 = \frac{v_1}{\sqrt{\epsilon}}$ and writing C_1^ϵ for C_1 to mention the dependence on ϵ, the above SPDE can be written as

$$\begin{aligned} u_{C_1^\epsilon}(t,\,x,\,y) \;=\; & U_0(x,\,y\,|\,G) - r_1\int_0^t \left(u_{C_1^\epsilon}(s,\,x,\,y)\right)_x ds \\ & +\frac{1}{2}\int_0^t h^2(y)\left(u_{C_1^\epsilon}(s,\,x,\,y)\right)_x ds - \frac{\kappa_1\theta_1}{\epsilon}\int_0^t \left(u_{C_1^\epsilon}(s,\,x,\,y)\right)_y ds \\ & +\frac{\kappa_1}{\epsilon}\int_0^t \left(y u_{C_1^\epsilon}(s,\,x,\,y)\right)_y ds + \frac{1}{2}\int_0^t h^2(y)\left(u_{C_1^\epsilon}(s,\,x,\,y)\right)_{xx} ds \\ & +\frac{v_1}{\sqrt{\epsilon}}\rho_3\rho_{1,1}\rho_{2,1}\int_0^t \left(h\,(y)\,q\,(y)\,u_{C_1^\epsilon}(s,\,x,\,y)\right)_{xy} ds \\ & +\frac{v_1^2}{2\epsilon}\int_0^t \left(q^2\,(y)\,u_{C_1^\epsilon}(s,\,x,\,y)\right)_{yy} ds \\ & -\rho_{1,1}\int_0^t h(y)\left(u_{C_1^\epsilon}(s,\,x,\,y)\right)_x dW_s^0 \\ & -\frac{v_1}{\sqrt{\epsilon}}\rho_{2,1}\int_0^t \left(q\,(y)\,u_{C_1^\epsilon}(s,\,x,\,y)\right)_y dB_s^0, \end{aligned} \tag{3.2.3}$$

A reasonable approach to our problem is to try to expand $u_{C_1}^\epsilon$ as a series of natural powers of ϵ^p for some $p > 0$, with the coefficients being random functions of t, x and y, substitute in (3.2.3), and solve the stochastic equations arising by equating coefficients. However, due to the presence of the stochastic integral with respect to

$B^0_\cdot$, we obtain two equations whenever we equate the coefficients of two terms of the form ϵ^{np}, and this renders the system of stochastic equations arising this way unsolvable. This difficulty we face has also a second explanation: In Chapter 2, $u_{C_1^\epsilon}$ has been computed explicitly and it is equal to

$$p_t^\epsilon\left(y|B^0_\cdot,\mathcal{G}^0\right)\mathbb{E}\left[u\left(t,x,W^0_\cdot,\mathcal{G}^0,C_1^\epsilon,h\left(\sigma^{1,\epsilon}_\cdot\right)\right)\,|W^0_\cdot,\sigma_t^{1,\epsilon}=y,B^0_\cdot,C_1^\epsilon,\mathcal{G}^0\right]$$

where we write $\sigma^{1,\epsilon}_\cdot$ for $\sigma^1_\cdot$ to mention again the dependence on ϵ, and p_t^ϵ for the density of the volatility process when the path of $B^0_\cdot$ is given. Under appropriate restrictions on the function q, we can extend the above formula to the more general case (model (3.1.1)). Obviously, p_t^ϵ contains information for the path of $B^0_\cdot$ and, intuitively, this means that we need pathwise convergence of $\sigma^{1,\epsilon}_\cdot$ as $\epsilon\to 0^+$ to have convergence of this density. On the other hand, the volatility processes we consider here converge weakly when we take the mean-reversion coefficient tending to infinity. However, since the above problem is mainly caused by p_t^ϵ, we hope that we may be able to obtain some good results by controlling appropriately p_t^ϵ and by trying to approximate the second factor of $u_{C_1^\epsilon}$, i.e the term

$$\mathbb{E}\left[u\left(t,x,W^0_\cdot,\mathcal{G}^0,C_1^\epsilon,h\left(\sigma^{1,\epsilon}_\cdot\right)\right)\,|W^0_\cdot,\sigma_t^{1,\epsilon}=y,B^0_\cdot,C_1^\epsilon,\mathcal{G}^0\right].$$

We observe now that since we are interested in the probability of an event concerning the loss process, which depends on the conditional density p_t^ϵ and the above conditional expectation, we can replace $(W^1_\cdot,\,W^0_\cdot,\,B^1_\cdot,\,B^0_\cdot,\,C_1^\epsilon)$ by anything having the same law for each $\epsilon>0$. We look thus at the SDE satisfied by the process $\sigma^{1,\epsilon}_\cdot$, i.e

$$\sigma_t^{1,\epsilon}=\sigma_0^{1,\epsilon}+\frac{\kappa_1}{\epsilon}\int_0^t(\theta_1-\sigma_s^{1,\epsilon})ds+\frac{v_1}{\sqrt{\epsilon}}\int_0^t q\left(\sigma_s^{1,\epsilon}\right)d\left(\sqrt{1-\rho_{2,1}^2}B_s^1+\rho_{2,1}B_s^0\right). \quad (3.2.4)$$

and we observe that if we substitute $t=\epsilon t'$ and $s=\epsilon s'$ for $0\le s'\le t'$, and then we replace $(W^1_\cdot,\,W^0_\cdot,\,B^1_\cdot,\,B^0_\cdot,\,C_1^\epsilon)$ by $\left(W^1_\cdot,\,W^0_\cdot,\,\sqrt{\epsilon}B^1_{\frac{\cdot}{\epsilon}},\,\sqrt{\epsilon}B^0_{\frac{\cdot}{\epsilon}},\,C_1^\epsilon\right)$ which has the same law, the above SDE becomes

$$\sigma_{\epsilon t'}^{1,\epsilon}=\sigma_0^{1,\epsilon}+\kappa_1\int_0^{t'}(\theta_1-\sigma_{\epsilon s'}^{1,\epsilon})ds'+v_1\int_0^{t'}q\left(\sigma_{\epsilon s'}^{1,\epsilon}\right)d\left(\sqrt{1-\rho_{2,1}^2}B_{s'}^1+\rho_{2,1}B_{s'}^0\right). \quad (3.2.5)$$

This shows that $\sigma^{1,\epsilon}_{\epsilon\cdot}$ can be replaced by the volatility process of our model when the coefficient vector C_1 is replaced by $C_1'=(\kappa_1,\,\theta_1,\,v_1,\,r_1,\,\rho_{1,1},\,\rho_{2,1})$, which is just $\sigma^{1,1}_\cdot$ (the volatility process when $\epsilon=1$). Thus, we can replace $\sigma_t^{1,\epsilon}$ by $\sigma^{1,1}_{\frac{t}{\epsilon}}$ for all $t\ge 0$, which allows us to replace

$$\mathbb{E}\left[u\left(t,x,W^0_\cdot,\mathcal{G}^0,C_1^\epsilon,h\left(\sigma^{1,\epsilon}_\cdot\right)\right)\,|W^0_\cdot,\sigma_t^{1,\epsilon}=y,B^0_\cdot,C_1^\epsilon,\mathcal{G}^0\right]$$

with the conditional expectation

$$\mathbb{E}\left[u\left(t,x,W_{\cdot}^{0},\mathcal{G}^{0},C_{1}',h\left(\sigma_{\frac{\cdot}{\epsilon}}^{1,1}\right)\right)\,|W_{\cdot}^{0},\sigma_{\frac{t}{\epsilon}}^{1,1}=y,B_{\cdot}^{0},C_{1}',\mathcal{G}^{0}\right].$$

The above quantity is what we need to approximate now. Of course, a first step is to show the convergence of that conditional expectation as $\epsilon \to 0^+$, which motivates us to look for some kind of convergence for the random function $u^{\epsilon}(t,x) := u\left(t,x,W_{\cdot}^{0},\mathcal{G}^{0},C_{1}',h\left(\sigma_{\frac{\cdot}{\epsilon}}^{1,1}\right)\right)$ as $\epsilon \longrightarrow 0^+$, when the volatility path $\sigma_{\cdot}^{1,1}$ and the coefficient vector C_1' are given.

The approach explained above is the subject of the next subsection, but we will also see a different approach to the same problem later. The two approaches are going to give different limits, and thus it will become clear that the convergence of our system is so weak that no good approximations should be expected. In both approaches, we need to assume that our function q is chosen such that every pair of volatility processes has a nice ergodic behaviour. This property is defined below

Definition 3.2.1 (**Positive recurrence property).** Fix the distribution from which each C_i is chosen. Then, we say that q has the strong positive recurrence property when the two-dimensional process $(\sigma_{\cdot}^{i,1},\,\sigma_{\cdot}^{j,1})$ is a positive recurrent diffusion for any two $i,j \in \mathbb{N}$. This means that there exists a two-dimensional random variable $(\sigma^{i,j,1,*},\,\sigma^{i,j,2,*})$ whose distribution is stationary for $(\sigma_{\cdot}^{i,1},\,\sigma_{\cdot}^{j,1})$, and whenever $\mathbb{E}\left[\left|F\left(\sigma^{i,j,1,*},\,\sigma^{i,j,2,*}\right)\right|\,|\,\mathcal{F}^{C}\right]$ exists and is finite for some function $F:\mathbb{R}^2\to\mathbb{R}$, we also have:

$$\lim_{T\to+\infty}\frac{1}{T}\int_{0}^{T}F\left(\sigma_{s}^{i,1},\,\sigma_{s}^{j,1}\right)ds=\mathbb{E}\left[F\left(\sigma^{i,j,1,*},\,\sigma^{i,j,2,*}\right)\,|\,\mathcal{F}^{C}\right] \tag{3.2.6}$$

$\mathbb{P}$-almost surely.

Remark 3.2.2. By a change of variables, we can easily verify that when we have (3.2.6), we also have

$$\lim_{\epsilon\to0^{+}}\frac{1}{t}\int_{0}^{t}F\left(\sigma_{\frac{s}{\epsilon}}^{i,1},\,\sigma_{\frac{s}{\epsilon}}^{j,1}\right)ds=\mathbb{E}\left[F\left(\sigma^{i,j,1,*},\,\sigma^{i,j,2,*}\right)\,|\,\mathcal{F}^{C}\right] \tag{3.2.7}$$

$\mathbb{P}$-almost surely, for any $t>0$, and this shows why we have chosen to change the timescale of our volatility processes by replacing $(B_{\cdot}^{1},\,B_{\cdot}^{0},\,C_{1}^{\epsilon})$ by $\left(\sqrt{\epsilon}B_{\frac{\cdot}{\epsilon}}^{1},\,\sqrt{\epsilon}B_{\frac{\cdot}{\epsilon}}^{0},\,C_{1}^{\epsilon}\right)$

Before proceeding to the convergence result we have for u^{ϵ} as $\epsilon \to 0^+$, we will mention two Theorems which give us a few classes of models for which the positive recurrence property is satisfied. The first theorem shows that for the Ornstein-Uhlenbeck model ($q(x)=1$ for all $x\in\mathbb{R}$) we always have the positive recurrence property. The

second theorem shows that for the CIR model ($q(x) = \sqrt{|x|}$ for all $x \in \mathbb{R}$) we have the positive recurrence property provided that the random coefficients of the volatilities satisfy certain conditions.

Theorem 3.2.3. *Suppose that q is a differentiable function, bounded from below by some $c_q > 0$. Suppose also that $q'(x)\kappa_i(\theta_i - x) < \kappa_i q(x) + \frac{v_i}{2} q''(x)q^2(x)$ for all $x \in \mathbb{R}$ and $i \in \mathbb{N}$, for all possible values of C_i. Then q has the positive recurrence property.*

Theorem 3.2.4. *Suppose that $q(x) = \sqrt{|x|}\tilde{q}(x)$, where the function $\tilde{q}$ is a continuously differentiable, strictly positive and increasing function taking values in $[c_q, 1]$ for some $c_q > 0$. Then, there exists an $\eta > 0$ such that q has the positive recurrence property when $\|C_i - C_j\|_{L^\infty(\mathbb{R}^6)} < \eta$ and $\frac{\kappa_i}{v_j^2} > \frac{1}{4} + \frac{1}{\sqrt{2}}$ for all $i, j \in \mathbb{N}$, $\mathbb{P}^C$ - almost surely.*

The proofs of the above two theorems can be found in the Appendix.

3.2.2 Weak L^2 convergence of the density given the volatility path

We fix now the volatility path $\sigma_\cdot^{1,1}$ and the coefficient vector C_1', which means that all expectations are taken given $\sigma_\cdot^{1,1}$ and C_1'. We will write $\mathbb{E}_{\sigma,\mathcal{F}^C}$ to denote these expectations, and $L^2_{\sigma,\mathcal{F}^C}$ to denote the corresponding L^2 norms. Given the pair $(\sigma_\cdot^{1,1}, C_1')$, we prove that u^ϵ does indeed converge as $\epsilon \longrightarrow 0^+$. We can only prove weak L^2 convergence, but we are able to characterize the limit provided that the function h satisfies a few boundedness conditions.

We start by recalling Theorem 2.4.1, according to which, u^ϵ is the unique solution to the following SPDE:

$$\begin{aligned} u^\epsilon(t,\, x) &= u_0(x) - \int_0^t \left(r_1 - \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} \right) u_x^\epsilon(s,\, x)ds \\ &\quad + \int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} u_{xx}^\epsilon(s,\, x)ds - \rho_{1,1}\int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) u_x^\epsilon(s,\, x)dW_s^0 \end{aligned} \tag{3.2.8}$$

for which we have also the identity

$$\|u^\epsilon(t,\, \cdot)\|^2_{L^2(\mathbb{R}^+)} + \left(1 - \rho_{1,1}^2\right)\int_0^t h^2\left(\sigma^{1,1}_{\frac{t}{\epsilon}}\right)\|u_x^\epsilon(s,\, \cdot)\|^2_{L^2(\mathbb{R}^+)}\, ds = \|u_0\|^2_{L^2(\mathbb{R}^+)}\,. \tag{3.2.9}$$

where u_0 stands for the common probability density of the asset prices at $t = 0$, given the information of the σ-algebra $\mathcal{G}^0$. The last identity shows that the $L^2(\mathbb{R}^+)$ norms of the solutions u^ϵ, and their $L^2([0,\, T] \times \mathbb{R}^+)$ norms as well (for any $T > 0$), are all uniformly bounded by a random variable which has a finite $L^2_{\sigma,\mathcal{F}^C}(\Omega^0)$ norm. It follows that in a subsequence of any given sequence of values of ϵ which tends to zero, we have weak convergence to some element u, and we can have this both in $L^2_{\sigma,\mathcal{F}^C}([0,\, T] \times \mathbb{R}^+ \times \Omega^0)$ and $\mathbb{P}^0$-almost surely in $L^2([0,\, T] \times \mathbb{R}^+)$. The characterization of the weak limits u is given in the following theorem.

Theorem 3.2.5. *Suppose that q has the positive recurrence property and that for some $C > 0$ we have $|h(x)| \leq C$ for all $x \geq 0$. Any weak limit u of u^ϵ in $L^2_{\sigma,\mathcal{F}^C}([0,\, T] \times \mathbb{R}^+ \times \Omega^0)$ solves the following SPDE*

$$\begin{aligned} u(t,\, x) &= u_0(x) - \left(r - \frac{\sigma_{1,1}^2}{2}\right) \int_0^t u_x(s,\, x) ds \\ &\quad + \frac{\sigma_{1,1}^2}{2} \int_0^t u_{xx}(s,\, x) ds - \rho_{1,1}\sigma_{2,1} \int_0^t u_x(s,\, x) dW_s^0 \end{aligned} \tag{3.2.10}$$

for $\sigma_{1,1} = \sqrt{\mathbb{E}\left[h^2\left(\sigma^{1,*}\right) \mid \mathcal{F}^C\right]}$ and $\sigma_{2,1} = \mathbb{E}\left[h\left(\sigma^{1,*}\right) \mid \mathcal{F}^C\right]$, where $\sigma^{1,*}$ is a random variable following the stationary distribution of our volatility process $\sigma_\cdot^{1,1}$. If h is bounded from below by a positive constant $c > 0$, the same weak convergence holds also in $H_0^1(\mathbb{R}^+) \times L^2_{\sigma,\mathcal{F}^C}(\Omega^0 \times [0,\, T])$, and u is then the unique solution to (3.2.10) *in that space. The last means that there is a unique subsequential weak limit, and thus we have weak convergence as $\epsilon \longrightarrow 0^+$.*

Proof. Let $\mathbb{V}$ be the set of $W_\cdot^0$-adapted, square-integrable semimartingales on $[0,\, T]$. This means that for any $\{V_t : 0 \leq t \leq T\} \in \mathbb{V}$, there exist two $W_\cdot^0$-adapted and square-integrable stochastic processes $\{v_{1,t} : 0 \leq t \leq T\}$ and $\{v_{2,t} : 0 \leq t \leq T\}$, such that

$$V_t = V_0 + \int_0^t v_{1,s} ds + \int_0^t v_{2,s} dW_s^0 \tag{3.2.11}$$

for all $t \geq 0$. The processes of the above form for which $\{v_{1,t} : 0 \leq t \leq T\}$ and $\{v_{2,t} : 0 \leq t \leq T\}$ are simple processes, i.e

$$v_{i,t} = F_i \mathbb{I}_{[t_1,\, t_2]}(t) \tag{3.2.12}$$

for all $0 \leq t \leq T$ and $i \in \{1,\, 2\}$, with each F_i being $\mathcal{F}_{t_1}^{W^0}$-measurable, span a linear subspace $\tilde{\mathbb{V}}$ which is dense in $\mathbb{V}$ under the L^2 norm. By using the estimate (3.2.9),

for any $p > 0$ and any $T > 0$ we can easily obtain

$$\int_0^T \left\| h^p\left(\sigma^{1,1}_{\frac{t}{\epsilon}}\right) u^\epsilon(t,\,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\mathbb{R}^+\times\Omega^0)} dt \leq TC^{2p}\left\|u_0\right\|^2_{L^2(\mathbb{R}^+)} \tag{3.2.13}$$

It follows that for any sequence $\epsilon_n \longrightarrow 0^+$, there exists a subsequence $\{\epsilon_{k_n} : n \in \mathbb{N}\}$, such that $h^p\left(\sigma^{1,1}_{\frac{\cdot}{\epsilon}}\right) u^\epsilon(\cdot,\,\cdot)$ converges weakly to some $u_p(\cdot,\cdot)$ in $L^2_{\sigma,\mathcal{F}^C}([0,\,T]\times\mathbb{R}^+\times\Omega^0)$ for all $p \in \{1,\,2\}$. Testing (3.2.8) against an arbitrary smooth and compactly supported function f of $x \in \mathbb{R}^+$, using Ito's formula for the product of $\int_{\mathbb{R}^+} u^\epsilon\left(\cdot,\,x\right) f(x)dx$ with a process $V_\cdot \in \tilde{\mathbb{V}}$ having the form (3.2.11) - (3.2.12), and finally taking expectations, we find that:

$$\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_t\int_{\mathbb{R}^+} u^\epsilon(t,\,x)f(x)dx\right]\\
&\quad= \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_0\int_{\mathbb{R}^+} u_0(x)f(x)dx\right] + r\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} u^\epsilon(s,\,x)f'(x)dx\right]ds\\
&\qquad- \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} u^\epsilon(s,\,x)f'(x)dx\right]ds\\
&\qquad+ \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} u^\epsilon(s,\,x)f''(x)dx\right]ds\\
&\qquad+ \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{1,s}\int_{\mathbb{R}^+} u^\epsilon(s,\,x)f(x)dx\right]ds\\
&\qquad+\rho_{1,1}\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s}\int_{\mathbb{R}^+} h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) u^\epsilon(s,\,x)f'(x)dx\right]ds
\end{aligned} \tag{3.2.14}$$

for all $t \leq T$. Thus, setting $\epsilon = \epsilon_{k_n}$ and taking $n \longrightarrow +\infty$, by the weak convergence results mentioned above we obtain

$$\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_t\int_{\mathbb{R}^+} u(t,\,x)f(x)dx\right]\\
&\quad= \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_0\int_{\mathbb{R}^+} u_0(x)f(x)dx\right] - r\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} u(s,\,x)f'(x)dx\right]ds\\
&\qquad+\frac{1}{2}\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} u_2(s,\,x)f'(x)dx\right]ds\\
&\qquad+\frac{1}{2}\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} u_2(s,\,x)f''(x)dx\right]ds\\
&\qquad+ \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{1,s}\int_{\mathbb{R}^+} u(s,\,x)f(x)dx\right]ds\\
&\qquad+\rho_{1,1}\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s}\int_{\mathbb{R}^+} u_1(s,\,x)f'(x)dx\right]ds
\end{aligned} \tag{3.2.15}$$

for all $0 \leq t \leq T$. The convergence of the terms in the RHS of (3.2.14) holds pointwise in t, while the one term in the LHS converges weakly. Since we can easily find uniform bounds for all the terms in (3.2.14) (by using (3.2.13)), the dominated convergence theorem implies that all the weak limits coincide with the corresponding pointwise limits, which gives (3.2.15) as a limit of (3.2.14) both weakly and pointwise in t. It is clear then that $\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_t \int_{\mathbb{R}^+} u(t,\, x)f(x)dx\right]$ is differentiable in t (in a $W^{1,1}$ sense). Next, we can also check that $\mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{i,t} \int_{\mathbb{R}^+} u^{\epsilon_{k_n}}(t,\, x)f(x)dx\right]$ converges to $\mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{i,t} \int_{\mathbb{R}^+} u(t,\, x)f(x)dx\right]$ for $i \in \{1,\, 2\}$, both weakly and pointwise in $t \in [0,\, T]$, while the limits are also differentiable in t everywhere except the two jump points t_1 and t_2. This follows from the fact that everything is zero outside $[t_1,\, t_2]$, while both $v_{1,\cdot}$ and $v_{2,\cdot}$ are constant in t and thus of the form (3.2.11) - (3.2.12) if we restrict on that interval. Observe now that we can write (3.2.14) as

$$
\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_t \int_{\mathbb{R}^+} u^{\epsilon}(t,\, x)f(x)dx\right] \\
&\quad = \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_0 \int_{\mathbb{R}^+} u_0(x)f(x)dx\right] + r\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u^{\epsilon}(s,\, x)f'(x)dx\right] ds \\
&\quad - \int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} \\
&\qquad \times \left(\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u^{\epsilon}(s,\, x)f'(x)dx\right] - \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x)f'(x)dx\right]\right) ds \\
&\quad - \int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x)f'(x)dx\right] ds \\
&\quad + \int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} \\
&\qquad \times \left(\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s \int_{\mathbb{R}^+} u^{\epsilon}(s,\, x)f''(x)dx\right] - \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x)f''(x)dx\right]\right) ds \\
&\quad + \int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x)f''(x)dx\right] ds \\
&\quad + \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{1,s} \int_{\mathbb{R}^+} u^{\epsilon}(s,\, x)f(x)dx\right] ds \\
&\quad + \rho_{1,1} \int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) \\
&\qquad \times \left(\mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u^{\epsilon}(s,\, x)f'(x)dx\right] - \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u(s,\, x)f'(x)dx\right]\right) ds \\
&\quad + \rho_{1,1} \int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u(s,\, x)f'(x)dx\right] ds
\end{aligned}
$$

(3.2.16)

Then we have

$$\begin{aligned}
&\Bigg| \int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) \\
&\qquad \times \left(\mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u^\epsilon(s,\, x) f'(x) dx \right] - \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u(s,\, x) f'(x) dx \right] \right) ds \Bigg| \\
&\leq C \int_0^t \left| \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u^\epsilon(s,\, x) f'(x) dx \right] - \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u(s,\, x) f'(x) dx \right] \right| ds
\end{aligned}$$

which tends to zero (when $\epsilon = \epsilon_{k_n}$ and $n \longrightarrow +\infty$) by the dominated convergence theorem, since the quantity inside the last integral converges pointwise to zero as we have mentioned earlier, while it can be dominated by using (3.2.13). The same argument is used to show that 4th and 6th terms in (3.2.16) tend also to zero in the same subsequence. Finally, for any term of the form

$$\int_0^t h^p\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x) f^{(m)}(x) dx \right] ds$$

for $p, m \in \{0,\, 1,\, 2\}$, we can recall the differentiability of the second factor inside the integral (which was mentioned earlier) and then use integration by parts to write it as:

$$\begin{aligned}
&\int_0^t h^p\left(\sigma^{1,1}_{\frac{w}{\epsilon}}\right) dw \left(\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(t,\, x) f^{(m)}(x) dx \right] \right) \\
&\qquad - \int_0^t \int_0^s h^p\left(\sigma^{1,1}_{\frac{w}{\epsilon}}\right) dw \left(\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x) f^{(m)}(x) dx \right] \right)' ds
\end{aligned}$$

which converges, by the positive recurrence property, to the quantity

$$\begin{aligned}
&t\mathbb{E}\left[h^p\left(\sigma^{1,*}\right) \mid \mathcal{F}^C \right] \left(\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(t,\, x) f^{(m)}(x) dx \right] \right) \\
&\qquad - \int_0^t s\mathbb{E}\left[h^p\left(\sigma^{1,*}\right) \mid \mathcal{F}^C \right] \left(\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x) f^{(m)}(x) dx \right] \right)' ds.
\end{aligned}$$

By using integration by parts once more, the last is equal to

$$\mathbb{E}\left[h^p\left(\sigma^{1,*}\right) \mid \mathcal{F}^C \right] \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x) f^{(m)}(x) dx \right] ds \tag{3.2.17}$$

The last convergence result holds also if we replace $V_\cdot$ by $v_{1,\cdot}$ or $v_{2,\cdot}$, as we can show by following exactly the same steps in the subinterval $[t_1,\, t_2]$ (where $v_{i,\cdot}$ is supported for $i \in \{1,\, 2\}$ and where we have differentiability that allows integration by parts).

If we set now $\epsilon = \epsilon_{k_n}$ in (3.2.16), take $n \longrightarrow +\infty$, and substitute all the above convergence results, we obtain

$$\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_t \int_{\mathbb{R}^+} u(t,\, x) f(x) dx\right] \\
&\quad = \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_0 \int_{\mathbb{R}^+} u_0(x) f(x) dx\right] \\
&\qquad + \left(r - \frac{\sigma_{1,1}^2}{2}\right) \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x) f'(x) dx\right] ds \\
&\qquad + \frac{\sigma_{1,1}^2}{2} \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} u(s,\, x) f''(x) dx\right] ds \\
&\qquad + \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{1,s} \int_{\mathbb{R}^+} u(s,\, x) f(x) dx\right] ds \\
&\qquad + \rho_{1,1}\sigma_{2,1} \int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[v_{2,s} \int_{\mathbb{R}^+} u(s,\, x) f'(x) dx\right] ds. \qquad (3.2.18)
\end{aligned}$$

Since $\tilde{\mathbb{V}}$ is dense in $\mathbb{V}$, for a fixed $t \leq T$, we can have (3.2.18) for any square-integrable martingale $\{V_s : 0 \leq s \leq t\}$, for which we obviously have $v_{1,s} = 0$ for all $0 \leq s \leq t$. Next, we denote by $R_u(t,\, x)$ the RHS of (3.2.10). Using then Ito's formula for the product of $\int_{\mathbb{R}^+} R_u(t,\, x) f(x) dx$ with V_t, subtracting $V_t \int_{\mathbb{R}^+} u(t,\, x) f(x) dx$ from both sides, taking expectations and finally substituting from (3.2.18), we find that

$$\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_t \left(\int_{\mathbb{R}^+} R_u(t,\, x) f(x) dx - \int_{\mathbb{R}^+} u(t,\, x) f(x) dx\right)\right] = 0$$

for any $t \leq T$. By the Martingale Representation Theorem, for that fixed $t \leq T$, V_t can be taken equal to the indicator $\mathbb{I}_{\mathcal{E}_t}$, where we define

$$\mathcal{E}_t = \left\{\omega^0 \in \Omega^0 : \int_{\mathbb{R}^+} R_u(t,\, x) f(x) dx > \int_{\mathbb{R}^+} u(t,\, x) f(x) dx\right\} \qquad (3.2.19)$$

and this allows us to write

$$\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\mathbb{I}_{\mathcal{E}_t} \left(\int_{\mathbb{R}^+} R_u(t,\, x) f(x) dx - \int_{\mathbb{R}^+} u(t,\, x) f(x) dx\right)\right] = 0$$

for all $0 \leq t \leq T$. If we integrate the above for $t \in [0,\, T]$ we obtain that

$$\int_0^T \mathbb{E}_{\sigma,\mathcal{F}^C}\left[\mathbb{I}_{\mathcal{E}_t} \left(\int_{\mathbb{R}^+} R_u(t,\, x) f(x) dx - \int_{\mathbb{R}^+} u(t,\, x) f(x) dx\right)\right] dt = 0$$

where the quantity inside the expectation is always non-negative and becomes zero only when $\mathbb{I}_{\mathcal{E}_t} = 0$. This implies that $\int_{\mathbb{R}^+} R_u(t,\, x) f(x) dx \leq \int_{\mathbb{R}^+} u(t,\, x) f(x) dx$ almost everywhere, and working in the same way with the indicator of the complement $\mathbb{I}_{\mathcal{E}_t^c}$ we can deduce the opposite inequality as well. Thus, we must have

$\int_{\mathbb{R}^+} R_u(t,\, x)f(x)dx = \int_{\mathbb{R}^+} u(t,\, x)f(x)dx$ almost everywhere, and since the function f is an arbitrary smooth function with compact support, we can deduce that R_u coincides with u almost everywhere, which gives (3.2.10).

If h is bounded from below, we can use (3.2.9) to obtain a uniform (independent from ϵ) bound for the $H_0^1(\mathbb{R}^+) \times L^2_{\sigma,\mathcal{F}^C}(\Omega^0 \times [0,\, T])$ norm of $u^{\epsilon_{k_n}}$, which implies that in a further subsequence, the weak convergence to u holds also in that Sobolev space, in which (3.2.10) has a unique solution (see [7]). This implies convergence of u^ϵ to the unique solution of (3.2.10) in $H_0^1(\mathbb{R}^+) \times L^2_{\sigma,\mathcal{F}^C}(\Omega^0 \times [0,\, T])$, as $\epsilon \longrightarrow 0^+$ $\square$

3.2.3 Convergence in distribution to a different limit

As we have already mentioned, our final goal is to find good approximations for mass probabilities of the form (3.1.2), which means that first we need to prove a convergence result of the form

$$\begin{aligned}&\mathbb{P}^0\left(\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right) \in (a,\, b)\right)\\ &\qquad \longrightarrow \mathbb{P}^0\left(\mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right) \in (a,\, b)\right)\end{aligned} \tag{3.2.20}$$

as $\epsilon \longrightarrow 0^+$, where $X_\cdot^{1,\epsilon}$ stands for the first asset price process with volatility path $\sigma^{1,1}_{\frac{\cdot}{\epsilon}}$ and coefficient vector C_1' (as defined in the previous subsection), while $X_\cdot^{1,*}$ stands for some other stochastic process. This is by definition the convergence in distribution of the random mass over $\mathbb{R}^+$, i.e $\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)$, to $\mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)$. In this section we are going to prove the more general

$$\begin{aligned}&\mathbb{P}^0\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right) \in (a,\, b)\right)\\ &\qquad \longrightarrow \mathbb{P}^0\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right) \in (a,\, b)\right)\end{aligned} \tag{3.2.21}$$

for some process $X_\cdot^{1,*}$ and any interval $\mathcal{I} = (0, U]$ with $U \in (0, +\infty]$. However, what is $X_\cdot^{1,*}$ going to be? Obviously, if $X_\cdot^{1,*}$ coincides (in distribution) with the process $X_\cdot^{1,w}$ whose density given $(W_\cdot^0,\, \sigma_\cdot^{1,1},\, C_1',\, \mathcal{G}^0)$ (on the set of positive real numbers) is the weak limit u of u^ϵ derived in the previous subsection, we have a very stable result that allows us to hope for better approximations. Otherwise, we cannot expect any convergence of the random mass over $\mathcal{I}$, better than convergence in distribution. Indeed, a limit in probability has to coincide with the limit in distribution, and in this case it has to be also a strong L^2 limit in some sequence $\epsilon_n \downarrow 0$ (because it will be a $\mathbb{P}^0$ - almost surely limit in some sequence, and then we can apply the

Dominated Convergence Theorem), so it has to be a weak L^2 limit as well. Then, for $\Xi = \mathbb{P}\left(X_t^{1,w} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right) - \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right)$, assuming that the coefficient vector is the same constant vector for all the assets we have

$$
\begin{aligned}
0 &= \lim_{n\to+\infty} \mathbb{E}\left[\Xi\left(\mathbb{P}\left(X_t^{1,\epsilon_n} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right) - \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] \\
&= \lim_{n\to+\infty} \mathbb{E}\left[\Xi\left(\mathbb{P}\left(X_t^{1,\epsilon_n} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right) - \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right)\right)\right] \\
&= \lim_{n\to+\infty} \mathbb{E}\left[\int_0^{+\infty} \Xi \mathbb{I}_{\mathcal{I}}(x) u^{\epsilon_n}(t,x)dx - \Xi\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right)\right] \\
&= \mathbb{E}\left[\int_0^{+\infty} \Xi \mathbb{I}_{\mathcal{I}}(x) u(t,x)dx - \Xi\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right)\right] \\
&= \mathbb{E}\left[\Xi\left(\mathbb{P}\left(X_t^{1,w} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right) - \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right)\right)\right] \\
&= \mathbb{E}\left[\left(\mathbb{P}\left(X_t^{1,w} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right) - \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}^0\right)\right)^2\right] \\
&> 0
\end{aligned}
$$

for any bounded interval $\mathcal{I}$, which is a contradiction.

When $X_\cdot^{1,*}$ and $X_\cdot^{1,w}$ do not coincide and we only have weak convrgence, good approximations of the RHS of (3.2.21) can only be obtained by studying the asymptotic behaviour of the distribution of the random mass over $\mathcal{I}$, and not the asymptotic behaviour of that mass itself, which makes our problem really challenging since we have no idea how this distribution looks like. Unfortunately, we will see that $X_\cdot^{1,*}$ is only equal to $X_\cdot^{1,w}$ when we have no market noise in our model, which means that we can only hope for nice convergence results when the most important feature of the model we are studying is removed! Moreover, the convergence in distribution result of this subsection can only be obtained when the volatility coefficients are deterministic and the same for each asset, i.e $(\kappa_i,\, \theta_i,\, v_i,\, \rho_{2,i}) = (\kappa,\, \theta,\, v,\, \rho_2)$ for all $i \in \mathbb{N}$. In the general case of i.i.d vectors $(\kappa_i,\, \theta_i,\, v_i,\, \rho_{2,i})$ for $i \in \mathbb{N}$, we will see that we cannot even hope for the existence of a process $X_\cdot^{1,*}$ satisfying (3.2.21). The main result of this subsection is given in the following theorem

Theorem 3.2.6. *Suppose that $(\kappa_i,\, \theta_i,\, v_i,\, \rho_{2,i}) = (\kappa,\, \theta,\, v,\, \rho_2)$ for all $i \in \mathbb{N}$, which is a deterministic 4-dimensional vector, and consider the stochastic process $Y_\cdot^{1,*}$ which is given by $Y_t^{1,*} = X_0^1 + \left(r_1 - \frac{\sigma_{1,1}^2}{2}\right)t + \tilde{\rho}_{1,1}\sigma_{1,1}W_t^0 + \sqrt{1-\tilde{\rho}_{1,1}^2}\sigma_{1,1}W_t^1$ for all $t \geq 0$, with $\tilde{\rho}_{1,1} = \rho_{1,1}\frac{\tilde{\sigma}}{\sigma_{1,1}}$ for some $\tilde{\sigma} \in [\sigma_{2,1},\, \sigma_{1,1}]$, where $\sigma_{2,1},\, \sigma_{1,1}$ are defined in Theorem 3.2.5. Define then $X_t^{1,*} = Y_{t\wedge\tau_{1,*}}^{1,*}$, for $\tau_{1,*} = \inf\{s \geq 0 :\, Y_s^{1,*} \leq 0\}$. Then, if the function h is bounded and the function q has the positive recurrence property, we have that $\mathbb{P}\left(X_t^{1,\epsilon} \in (0,U] \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)$ converges in distribution to the random conditional probability $\mathbb{P}\left(X_t^{1,*} \in (0,U] \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)$ as $\epsilon \longrightarrow 0^+$, for any $U \in (0,+\infty]$.*

Proof. To have the desired convergence in distribution, we need to show that for every bounded and continuous function $G : \mathbb{R} \longrightarrow \mathbb{R}$ we have:

$$\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] \longrightarrow \mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] \tag{3.2.22}$$

as $\epsilon \longrightarrow 0^+$, where $\mathcal{I} = (0, +\infty]$. Observe now that since the conditional probabilities describing the default mass take values in the compact interval $[0, 1]$, it is equivalent to have the above for all continuous $G : [0, 1] \longrightarrow \mathbb{R}$. We actually need to have this only when G is a polynomial, since in that case, for an arbitrary continuous function G and for a polynomial P such that $|P(x) - G(x)| < \frac{\eta}{2}$ for all $x \in [0, 1]$, we have:

$$\begin{aligned}
&\left|\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] - \mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right]\right| \\
&\quad \leq \left|\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] - \mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right]\right| \\
&\qquad + \left|\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] - \mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right]\right| \\
&\qquad + \left|\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] - \mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right]\right| \\
&\quad \leq \eta + \left|\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] - \mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right]\right|
\end{aligned}$$

where η can be taken as small as we want, and for that fixed η the last difference tends to 0 as $\epsilon \longrightarrow 0^+$. Finally, by linearity we only need to have (3.2.22) when $G(x) = x^m$ for all $x \in [0, 1]$, for some $m \in \mathbb{N}$.

For a given $m \in \mathbb{N}$ now, let $\{X_\cdot^{i,\epsilon} : 1 < i \leq m\}$ be a collection of $m - 1$ processes with the same distribution as $X_\cdot^{1,\epsilon}$, which are driven by the 4-dimensional Brownian Motions $(W_\cdot^0, W_\cdot^2, B_\cdot^0, B_\cdot^2)$, $(W_\cdot^0, W_\cdot^3, B_\cdot^0, B_\cdot^3)$, ..., $(W_\cdot^0, W_\cdot^m, B_\cdot^0, B_\cdot^m)$. That is, for $1 \leq i \leq m$ we define:

$$Y_t^{i,\epsilon} = Y_0^i + \int_0^t \left(r_i - \frac{h^2\left(\sigma_{\frac{s}{\epsilon}}^{i,1}\right)}{2}\right) ds + \int_0^t h(\sigma_{\frac{s}{\epsilon}}^{i,1}) \left(\sqrt{1 - \rho_{1,i}^2} dW_s^i + \rho_{1,i} dW_s^0\right)$$

with

$$\sigma_t^{i,1} = \sigma_0^{i,1} + \kappa \int_0^t \left(\theta - \sigma_s^{i,1}\right) ds + v \int_0^t q\left(\sigma_s^{i,1}\right) \rho_2 dB_s^0 + v \int_0^t q\left(\sigma_s^{i,1}\right) \sqrt{1 - \rho_2^2} dB_s^i$$

for all $t \geq 0$, and then we define $X_t^{i,\epsilon} = Y_{t \wedge \tau^{i,\epsilon}}^{i,\epsilon}$ for $\tau^{i,\epsilon} = \inf\{s \geq 0 : Y_s^{i,\epsilon} \leq 0\}$. The m processes $\{X_\cdot^{i,\epsilon} : 1 \leq i \leq m\}$ are obviously pairwise i.i.d when the information contained in $W_\cdot^0, B_\cdot^0$ and $\mathcal{G}^0$ is given. Therefore we can write:

$$\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbb{P}^m\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I},\, X_t^{2,\epsilon} \in \mathcal{I},\, ...,\, X_t^{m,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right] \\
&= \mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I},\, X_t^{2,\epsilon} \in \mathcal{I},\, ...,\, X_t^{m,\epsilon} \in \mathcal{I}\right) \\
&= \mathbb{P}\left(\left(\min_{1\leq i\leq m}\min_{0\leq s\leq t} Y_s^{i,\epsilon},\, \max_{1\leq i\leq m} Y_t^{i,\epsilon}\right) \in (0,\, +\infty)\times(-\infty, U]\right)
\end{aligned}
\tag{3.2.23}
$$

Next, we consider the collection of processes $\{X_\cdot^{i,*} : 1 \leq i \leq m\}$, which are defined for $i > 1$ exactly as for $i = 1$, i.e $X_t^{i,*} = Y_{t\wedge\tau_{i,*}}^{i,*}$ for $\tau_{i,*} = \inf\{s \geq 0 : Y_s^{i,*} \leq 0\}$, where each $Y_\cdot^{i,*}$ is defined as

$$
Y_t^{i,*} = X_0^i + \left(r_i - \frac{\sigma_{1,1}^2}{2}\right)t + \tilde{\rho}_{1,i}\sigma_{1,1}W_t^0 + \sqrt{1-\tilde{\rho}_{1,i}^2}\sigma_{1,1}W_t^i
$$

for all $t \geq 0$, with $\tilde{\rho}_{1,i} = \rho_{1,i}\frac{\tilde{\sigma}}{\sigma_{1,1}}$ for some $\tilde{\sigma} \in [\sigma_{2,1},\, \sigma_{1,1}]$ (which will be chosen later) and all $i \in \{2,\, ...,\, m\}$. Again, it is easy to check that the above processes are pairwise i.i.d processes when the information contained in $W_\cdot^0, B_\cdot^0$ and $\mathcal{G}^0$ is given. Thus, we can write

$$
\begin{aligned}
&\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right)\right] \\
&\qquad = \mathbb{E}\left[\mathbb{P}^m\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right] \\
&\qquad = \mathbb{E}\left[\mathbb{P}\left(X_t^{1,*} \in \mathcal{I},\, X_t^{2,*} \in \mathcal{I},\, ...,\, X_t^{m,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)\right] \\
&\qquad = \mathbb{P}\left(X_t^{1,*} \in \mathcal{I},\, X_t^{2,*} \in \mathcal{I},\, ...,\, X_t^{m,*} \in \mathcal{I}\right) \\
&\qquad = \mathbb{P}\left(\left(\min_{1\leq i\leq m}\min_{0\leq s\leq t} Y_s^{i,*},\, \max_{1\leq i\leq m} Y_t^{i,*}\right) \in (0,\, +\infty)\times(-\infty, U]\right).
\end{aligned}
\tag{3.2.24}
$$

Then, (3.2.23) and (3.2.24) show that the result we want to prove has been reduced to the convergence of $\left(\min_{1\leq i\leq m}\min_{0\leq s\leq t} Y_s^{i,\epsilon},\, \max_{1\leq i\leq m} Y_t^{i,\epsilon}\right)$ to $\left(\min_{1\leq i\leq m}\min_{0\leq s\leq t} Y_s^{i,*},\, \max_{1\leq i\leq m} Y_t^{i,*}\right)$, in distribution as $\epsilon \longrightarrow 0^+$ (since the probability that any of the m minimums equals zero is zero, as the minimum of any Gaussian process is always continuously distributed, while $Y_\cdot^{i,\epsilon}$ is obviously Gaussian for any given path of $\sigma_\cdot^{i,1}$).

Let now $C\left([0,\, t]\,;\mathbb{R}^m\right)$ be the classical Wiener space of continuous functions defined on $[0,\, t]$ and taking values in $\mathbb{R}^k$ (i.e the space of these functions equipped with the supremum norm and the Wiener probability measure), and observe that $\min_{1\leq i\leq m} p_i\left(\min_{0\leq s\leq t}\cdot(s)\right)$ defined on $C\left([0,\, t]\,;\mathbb{R}^m\right)$, where p_i stands for the projection on

the i-th axis, is a continuous functional. Indeed, for any two continuous functions $f_1,\, f_2$ in $C\left([0,\, t]\,;\mathbb{R}^m\right)$, we have:

$$\left|\min_{1\leq i\leq m} p_i\left(\min_{0\leq s\leq t} f_1(s)\right) - \min_{1\leq i\leq m} p_i\left(\min_{0\leq s\leq t} f_2(s)\right)\right| = \left|p_{i_1}\left(f_1(s_1)\right) - p_{i_2}\left(f_2(s_2)\right)\right|$$

for some $s_1, s_2 \in [0,\, t]$ and $1 \leq i_1, i_2 \leq m$, and without loss of generality we may assume that the difference inside the last absolute value is nonnegative. Moreover we have:

$$p_{i_1}\left(f_1(s_1)\right) = \min_{1\leq i\leq m} p_i\left(\min_{0\leq s\leq t} f_1(s)\right) \leq p_{i_2}\left(f_1(s_2)\right)$$

and thus we have:

$$\begin{aligned}\left|\min_{1\leq i\leq m} p_i\left(\min_{0\leq s\leq t} f_1(s)\right) - \min_{1\leq i\leq m} p_i\left(\min_{0\leq s\leq t} f_2(s)\right)\right| &= p_{i_1}\left(f_1(s_1)\right) - p_{i_2}\left(f_2(s_2)\right)\\ &\leq p_{i_2}\left(f_1(s_2)\right) - p_{i_2}\left(f_2(s_2)\right)\\ &\leq \left|p_{i_2}\left(f_1(s_2)\right) - p_{i_2}\left(f_2(s_2)\right)\right|\\ &\leq \left\|f_1 - f_2\right\|_{C([0,\, t];\mathbb{R}^m)}\end{aligned}$$

Obviously, $\max\limits_{1\leq i\leq m} p_i\left(\cdot\,(t)\right)$ defined on $C\left([0,\, t]\,;\mathbb{R}^m\right)$ is also continuous (as the maximum of finitely many evaluation functionals). Therefore, our problem is finally reduced to showing that $(Y_{\cdot}^{1,\epsilon},\, Y_{\cdot}^{2,\epsilon},\, ...,\, Y_{\cdot}^{m,\epsilon})$ converges in distribution to $(Y_{\cdot}^{1,*},\, Y_{\cdot}^{2,*},\, ...,\, Y_{\cdot}^{m,*})$ in the space $C\left([0,\, t]\,;\mathbb{R}^m\right)$, as $\epsilon \longrightarrow 0^+$.

We will now follow the standard way for showing convergence in distribution results like the above: We will show first that a limit in distribution exists as as $\epsilon \longrightarrow 0^+$ by using a tightness argument, and then we will try to characterize the limits of the finite dimensional distributions. To show tightness of the laws of $(Y_{\cdot}^{1,\epsilon},\, Y_{\cdot}^{2,\epsilon},\, ...,\, Y_{\cdot}^{m,\epsilon})$ for $\epsilon \in \mathbb{R}^+$, which implies the desired convergence in distribution, we recall a special case of Theorem 7.2 in [11] for continuous processes, according to which it suffices to prove that for a given $\eta > 0$, there exist some $\delta > 0$ and $N > 0$ such that:

$$\mathbb{P}\left(\left\|\left(Y_0^{1,\epsilon},\, Y_0^{2,\epsilon},\, ...,\, Y_0^{m,\epsilon}\right)\right\|_{\mathbb{R}^m} > N\right) \leq \eta \tag{3.2.25}$$

and

$$\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta} \left\|\left(Y_{s_1}^{1,\epsilon},\, Y_{s_1}^{2,\epsilon},\, ...,\, Y_{s_1}^{m,\epsilon}\right) - \left(Y_{s_2}^{1,\epsilon},\, Y_{s_2}^{2,\epsilon},\, ...,\, Y_{s_2}^{m,\epsilon}\right)\right\|_{\mathbb{R}^m} > \eta\right) \leq \eta \tag{3.2.26}$$

for all $\epsilon > 0$. (3.2.25) can easily be achieved for some N since $\left(Y_0^{1,\epsilon},\, Y_0^{2,\epsilon},\, ...,\, Y_0^{m,\epsilon}\right) = (X_0^1,\, X_0^2,\, ...,\, X_0^m)$, which is independent of ϵ and almost surely finite (the sum of

the probabilities that the norm of this vector belongs to $[n,\, n+1]$ over $n \in \mathbb{N}$ is a convergent series and thus, by Cauchy criteria, the same sum but for $n \geq N$ tends to zero as N tends to infinity). For (3.2.26) now, observe that $\|\cdot\|_{\mathbb{R}^m}$ can be any of the standard equivalent L^p norms of $\mathbb{R}^m$, and we choose it to be L^∞. Then we have:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\left\|\left(Y^{1,\epsilon}_{s_1},\, Y^{2,\epsilon}_{s_1},\, ...,\, Y^{m,\epsilon}_{s_1}\right)-\left(Y^{1,\epsilon}_{s_2},\, Y^{2,\epsilon}_{s_2},\, ...,\, Y^{m,\epsilon}_{s_2}\right)\right\|_{\mathbb{R}^m}>\eta\right)\\
&\qquad=\mathbb{P}\left(\cup_{i=1}^m\{\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\left|Y^{i,\epsilon}_{s_1}-Y^{i,\epsilon}_{s_2}\right|>\eta\}\right)\\
&\qquad\leq\sum_{i=1}^m\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\left|Y^{i,\epsilon}_{s_1}-Y^{i,\epsilon}_{s_2}\right|>\eta\right)\\
&\qquad=m\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\left|Y^{1,\epsilon}_{s_1}-Y^{1,\epsilon}_{s_2}\right|>\eta\right)
\end{aligned}\tag{3.2.27}$$

and since the Ito integral $\int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)\left(\sqrt{1-\rho_1^2}dW^1_s+\rho_1 dW^0_s\right)$ can be written as $\tilde{W}_{\int_0^t h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)ds}$, where $\tilde{W}_\cdot$ is another standard Brownian motion, denoting the maximum of h by M we also have:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\left|Y^{1,\epsilon}_{s_1}-Y^{1,\epsilon}_{s_2}\right|>\eta\right)\\
&\quad=\mathbb{P}\Bigg(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\Bigg|\int_{s_2}^{s_1}\left(r-\frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2}\right)ds\\
&\qquad\qquad+\left(\tilde{W}_{\int_0^{s_1}h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)ds}-\tilde{W}_{\int_0^{s_2}h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}\right)\Bigg|>\eta\Bigg)\\
&\quad\leq\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\left|\int_{s_2}^{s_1}\left(r-\frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2}\right)ds\right|>\frac{\eta}{2}\right)\\
&\qquad+\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\,|s_1-s_2|\leq\delta}\left|\tilde{W}_{\int_0^{s_1}h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)ds}-\tilde{W}_{\int_0^{s_2}h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)ds}\right|>\frac{\eta}{2}\right)\\
&\quad\leq\mathbb{P}\left(\delta\left(r+M\right)>\frac{\eta}{2}\right)\\
&\qquad+\mathbb{P}\left(\sup_{0\leq s_3,s_4\leq M^2t,\,|s_3-s_4|\leq M^2\delta}\left|\left(\tilde{W}_{s_3}-\tilde{W}_{s_4}\right)\right|>\frac{\eta}{2}\right)
\end{aligned}\tag{3.2.28}$$

since $\left|\int_a^b h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)ds\right|\leq M^2\left|a-b\right|$ for all $a,b\in\mathbb{R}^+$. The first of the last two probabilities is clearly zero for $\delta<\frac{\eta}{2(r+M)}$, while the second one can also be made arbitrarily

small for small enough δ, since by a well known result about the modulus of continuity of a Brownian motion (see [36]) the supremum within that probability converges almost surely (and thus also in probability) to 0 as fast as $M\sqrt{2\delta\log\left(\frac{1}{M^2\delta}\right)}$. Plugging these in (3.2.27) we deduce that (3.2.26) is also satisfied and we have the desired tightness result, which implies that $(Y_\cdot^{1,\epsilon},\, Y_\cdot^{2,\epsilon},\, ...,\, Y_\cdot^{m,\epsilon})$ converges in distribution to some limit $(Y_\cdot^{1,0},\, Y_\cdot^{2,0},\, ...,\, Y_\cdot^{m,0})$.

To conclude our proof, we need to show that $(Y_\cdot^{1,0},\, Y_\cdot^{2,0},\, ...,\, Y_\cdot^{m,0})$ coincides with $(Y_\cdot^{1,*},\, Y_\cdot^{2,*},\, ...,\, Y_\cdot^{m,*})$. Since both m-dimensional processes are uniquely determined by their finite-dimensional distributions, and since evaluation functionals on the space $C\left([0,\, t]\,;\mathbb{R}^m\right)$ preserve convergences in distribution (as continuous functionals), we only need to show that for any fixed $(i_1,\, i_2,\, ...,\, i_\ell) \in \{1,\, 2,\, ...,\, m\}^\ell$, any fixed $(t_1,\, t_2,\, ...,\, t_\ell) \in (0,\, +\infty)^\ell$, and any fixed continuous and bounded function $g : \mathbb{R}^\ell \to \mathbb{R}$, for an arbitrary $\ell \in \mathbb{N}$, we have

$$\mathbb{E}\left[g\left(Y_{t_1}^{i_1,\epsilon},\, Y_{t_2}^{i_2,\epsilon},\, ...,\, Y_{t_\ell}^{i_\ell,\epsilon}\right)\right] \longrightarrow \mathbb{E}\left[g\left(Y_{t_1}^{i_1,*},\, Y_{t_2}^{i_2,*},\, ...,\, Y_{t_\ell}^{i_\ell,*}\right)\right]$$

as $\epsilon \to 0^+$. Due to the Dominated Convergence Theorem, the above follows if we are able to show that

$$\begin{aligned}\lim_{\epsilon\to 0^+} &\mathbb{E}\left[g\left(Y_{t_1}^{i_1,\epsilon},\, Y_{t_2}^{i_2,\epsilon},\, ...,\, Y_{t_\ell}^{i_\ell,\epsilon}\right)\,|\sigma_\cdot^{i_1,1},\, \sigma_\cdot^{i_2,1},\, ...,\, \sigma_\cdot^{i_\ell,1},\, \mathcal{F}^C\right] \\ &= \lim_{\epsilon\to 0^+} \mathbb{E}\left[g\left(Y_{t_1}^{i_1,*},\, Y_{t_2}^{i_2,*},\, ...,\, Y_{t_\ell}^{i_\ell,*}\right)\,|\sigma_\cdot^{i_1,1},\, \sigma_\cdot^{i_2,1},\, ...,\, \sigma_\cdot^{i_\ell,1},\, \mathcal{F}^C\right]\end{aligned}$$

$\mathbb{P}$- almost surely. However, when the information contained in $\sigma_\cdot^{i_1,1},\, \sigma_\cdot^{i_2,1},\, ...,\, \sigma_\cdot^{i_\ell,1}$ and $\mathcal{F}^C$ is given, both $\left(Y_{t_1}^{i_1,\epsilon},\, Y_{t_2}^{i_2,\epsilon},\, ...,\, Y_{t_\ell}^{i_\ell,\epsilon}\right)$ and $\left(Y_{t_1}^{i_1,*},\, Y_{t_2}^{i_2,*},\, ...,\, Y_{t_\ell}^{i_\ell,*}\right)$ have an ℓ-dimensional normal distribution. This means that given $(\sigma_\cdot^{i_1,1},\, \sigma_\cdot^{i_2,1},\, ...,\, \sigma_\cdot^{i_\ell,1})$ and $\mathcal{F}^C$, we only need to show that as $\epsilon \to 0^+$, the mean vector and the covariance matrix of $\left(Y_{t_1}^{i_1,\epsilon},\, Y_{t_2}^{i_2,\epsilon},\, ...,\, Y_{t_\ell}^{i_\ell,\epsilon}\right)$ converge to the mean vector and the covariance matrix of $\left(Y_{t_1}^{i_1,*},\, Y_{t_2}^{i_2,*},\, ...,\, Y_{t_\ell}^{i_\ell,*}\right)$ respectively. Given $(\sigma_\cdot^{i_1,1},\, \sigma_\cdot^{i_2,1},\, ...,\, \sigma_\cdot^{i_\ell,1})$, the information contained in $\mathcal{F}^C$, and a $k \in \{1,\, 2,\, ...,\, \ell\}$, the k-th coordinate of the mean vector of $\left(Y_{t_1}^{i_1,\epsilon},\, Y_{t_2}^{i_2,\epsilon},\, ...,\, Y_{t_\ell}^{i_\ell,\epsilon}\right)$ is equal to $X_0^{i_k} + \int_0^{t_k}\left(r_{i_k} - \frac{h^2\left(\sigma_{\frac{s}{\epsilon}}^{i_k,1}\right)}{2}\right)ds$, and by the positive recurrence property it converges as $\epsilon \to 0^+$ to $X_0^{i_k} + \left(r_{i_k} - \frac{\sigma_{1,1}^2}{2}\right)t_k$ (since the volatility processes have all the same coefficients and thus the same stationary distributions), which is the k-th coordinate of the mean vector of $\left(Y_{t_1}^{i_1,*},\, Y_{t_2}^{i_2,*},\, ...,\, Y_{t_\ell}^{i_\ell,*}\right)$. Now we only need to obtain the corresponding convergence result for the covariance matrices of our processes. For some $1 \le p, q \le \ell$, given $(\sigma_\cdot^{i_1,1},\, \sigma_\cdot^{i_2,1},\, ...,\, \sigma_\cdot^{i_\ell,1})$ and the

information contained in $\mathcal{F}^C$, the covariance of $Y_{t_p}^{i_p,\epsilon}$ and $Y_{t_q}^{i_q,\epsilon}$ is equal to

$$\left(\rho_{1,i_p}\rho_{1,i_q}+\delta_{i_p,i_q}\sqrt{1-\rho_{1,i_p}}\sqrt{1-\rho_{1,i_q}}\right)\int_0^{t_p\wedge t_q} h\left(\sigma_{\frac{s}{\epsilon}}^{i_p,1}\right)h\left(\sigma_{\frac{s}{\epsilon}}^{i_q,1}\right)ds$$

while the covariance of $Y_{t_p}^{i_p,*}$ and $Y_{t_q}^{i_q,*}$ is equal to

$$\left(\tilde{\rho}_{1,i_p}\tilde{\rho}_{1,i_q}+\delta_{i_p,i_q}\sqrt{1-\tilde{\rho}_{1,i_p}^2}\sqrt{1-\tilde{\rho}_{1,i_q}^2}\right)\sigma_{1,1}^2 t_p\wedge t_q.$$

This means that for $i_p=i_q=i\in\{1,\,2,\,...,\,m\}$ we need to show that

$$\int_0^{t_p\wedge t_q} h^2\left(\sigma_{\frac{s}{\epsilon}}^{i,1}\right)ds\longrightarrow\sigma_{1,1}^2 t_p\wedge t_q$$

as $\epsilon\longrightarrow 0^+$, while for $i_p\neq i_q$ we need to show that:

$$\rho_{1,i_p}\rho_{1,i_q}\int_0^{t_p\wedge t_q} h\left(\sigma_{\frac{s}{\epsilon}}^{i_p,1}\right)h\left(\sigma_{\frac{s}{\epsilon}}^{i_q,1}\right)ds\longrightarrow\tilde{\rho}_{1,i_p}\tilde{\rho}_{1,i_q}\sigma_{1,1}^2 t_p\wedge t_q$$

as $\epsilon\longrightarrow 0^+$, where $\tilde{\rho}_{1,i}\sigma_{1,1}=\rho_{1,i}\tilde{\sigma}$ for all $i\leq m$. Both convergence results follow from the positive recurrence property for $\tilde{\sigma}=\sqrt{\mathbb{E}\left[h\left(\sigma^{i_p,i_q,1,*}\right)h\left(\sigma^{i_p,i_q,2,*}\right)\right]}$, which does not depend on i_p and i_q since the volatility processes have all the same coefficients and thus the same joint stationary distributions.

It remains to show that $\tilde{\sigma}\in[\sigma_{2,1},\,\sigma_{1,1}]$. The upper bound can be obtained by a simple Cauchy-Schwartz inequality, i.e

$$\begin{aligned}\tilde{\sigma} &= \sqrt{\mathbb{E}\left[h\left(\sigma^{1,2,1,*}\right)h\left(\sigma^{1,2,2,*}\right)\right]}\\ &\leq \sqrt{\sqrt{\mathbb{E}\left[h^2\left(\sigma^{1,2,1,*}\right)\right]}\sqrt{\mathbb{E}\left[h^2\left(\sigma^{1,2,2,*}\right)\right]}}\\ &= \sqrt{\sigma_{1,1}\times\sigma_{1,1}}\\ &= \sigma_{1,1}\end{aligned}\tag{3.2.29}$$

For the lower bound, considering our volatility processes for $i=1$ and $i=2$ started from their 1-dimensional stationary distributions independently, we have for any $t,\epsilon\geq 0$

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{t}\int_0^t h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)ds\right]\\ &\quad=\frac{1}{t}\int_0^t\mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)\right]ds\\ &\quad=\frac{1}{t}\int_0^t\mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)\right]\mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)\right]ds\\ &\qquad+\frac{1}{t}\int_0^t\mathbb{E}\left[\left(h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)-\mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)\right]\right)\left(h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)-\mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)\right]\right)\right]ds\end{aligned}$$

$$
\begin{aligned}
&= \sigma_{2,1}^2 \\
&\quad + \frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) - \mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)\right]\right)\right. \\
&\qquad \left.\times\left(h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right) - \mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)\right]\right) \mid B_\cdot^0\right]\right] ds \\
&= \sigma_{2,1}^2 + \frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) - \sigma_{2,1}\right) \mid B_\cdot^0\right]^2\right] ds \\
&\geq \sigma_{2,1}^2
\end{aligned} \tag{3.2.30}
$$

since $\sigma_\cdot^{1,1}$ and $\sigma_\cdot^{2,1}$ are identically distributed, and also i.i.d when $B_\cdot^0$ is given. Taking $\epsilon \longrightarrow 0^+$ on (3.2.30) and recalling the positive recurrence property, the definition of $\tilde{\sigma}$, and the Dominated Convergence Theorem on the LHS (since the quantity inside the expectation there is bounded by the square of an upper bound of h), we obtain the desired. The proof of the Theorem is now complete. □

Remark 3.2.7. A few comments need to be made about the bounds we have derived for $\tilde{\sigma}$:

1. The upper bound $\tilde{\sigma} \leq \sigma_{1,1}$ is needed to ensure that $\tilde{\rho}_{1,1} \leq 1$, so $X_\cdot^{1,*}$ is a real-valued stochastic process. The above proof shows that this bound is only attainable when $\sigma^{i,j,1,*} = \sigma^{i,j,2,*}$ for all i and j with $i \neq j$, which happens only when all the assets share a common stochastic volatility (i.e $\rho_2 = 1$).

2. The lower bound $\tilde{\sigma} \geq \sigma_{2,1}$ can also be shown to be unattainable in general, which means that the conditional (given $(W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0)$) density of $X_t^{1,*}$ on $(0,\, +\infty)$ does not coincide with the weak limit derived in the previous section, and as we mentioned earlier, this shows that convergence of $\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right)$ as $\epsilon \longrightarrow 0^+$ better than in distribution cannot be expected. Indeed, if we choose h such that its composition $\tilde{h}$ with the square function is strictly increasing and convex, and if q is chosen to be a square root function (thus we are in the CIR volatility case, which is the most common), for any $\alpha > 0$ we have

$$
\begin{aligned}
&\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) - \sigma_{2,1}\right) \mid B_\cdot^0\right]^2\right] ds \\
&\qquad = \mathbb{E}\left[\frac{1}{t}\int_0^t \left(\mathbb{E}\left[\tilde{h}\left(\sqrt{\sigma_{\frac{s}{\epsilon}}^{1,1}}\right) \mid B_\cdot^0\right] - \sigma_{2,1}\right)^2 ds\right] \\
&\qquad \geq \alpha^2 \mathbb{E}\left[\frac{1}{t}\int_0^t \mathbb{I}_{\sigma_{\frac{s}{\epsilon}}^{B^0,h} \geq \alpha + \sigma_{2,1}} ds\right]
\end{aligned} \tag{3.2.31}
$$

where $\sigma_s^{B^0,h} := \mathbb{E}\left[\tilde{h}\left(\sqrt{\sigma_s^{1,1}}\right) \mid B_\cdot^0\right] \geq \tilde{h}\left(\sigma_s^{B^0}\right)$ for $\sigma_s^{B^0} := \mathbb{E}\left[\sqrt{\sigma_s^{1,1}} \mid B_\cdot^0\right]$. Thus, (3.2.31) implies

$$\begin{aligned}\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) - \sigma_{2,1}\right) \mid B_\cdot^0\right]^2\right] ds \\ \geq \alpha^2 \mathbb{E}\left[\frac{1}{t}\int_0^t \mathbb{I}_{\sigma_{\frac{s}{\epsilon}}^{B^0} \geq \tilde{h}^{-1}(\alpha+\sigma_{2,1})} ds\right]\end{aligned} \tag{3.2.32}$$

Let now σ_t^ρ be the solution to the SDE

$$\sigma_t^\rho = \sigma_0^{B^0} + \frac{1}{2}\int_0^t \left(\kappa\theta - \frac{v^2}{4}\right)\frac{1}{\sigma_s^\rho} ds + \frac{\kappa}{2}\int_0^t \sigma_s^\rho ds + \frac{\rho_2 v}{2} B_s^0$$

The above can easily be shown to be the square root of a CIR process having the same mean-reversion and volatility as $\sigma^{1,1}$ and a different stationary mean, which however satisfies the Feller condition for not hitting zero. If for some $t_1 > 0$ we have $\sigma_{t_1}^\rho > \sigma_{t_1}^{B^0}$, we consider $t_0 = \inf\{s \leq t_1 : \sigma_s^\rho = \sigma_s^{B^0}\}$ which is obviously non-negative. Then, since $\mathbb{E}\left[\frac{1}{\sqrt{\sigma_s^{1,1}}} \mid B_\cdot^0\right] \geq \frac{1}{\mathbb{E}\left[\sqrt{\sigma_s^{1,1}} \mid B_\cdot^0\right]} = \frac{1}{\sigma_s^{B^0}}$ we have

$$\begin{aligned}\sigma_{t_1}^{B^0} &= \sigma_{t_0}^{B^0} + \frac{1}{2}\int_0^{t_1}\left(\kappa\theta - \frac{v^2}{4}\right)\mathbb{E}\left[\frac{1}{\sqrt{\sigma_s^{1,1}}} \mid B_\cdot^0\right] ds \\ &\quad -\frac{\kappa}{2}\int_0^{t_1}\sigma_s^{B^0} ds + \frac{\rho_2 v}{2}\left(B_{t_1}^0 - B_{t_0}^0\right) \\ &\geq \sigma_{t_0}^{B^0} + \frac{1}{2}\int_0^{t_1}\left(\kappa\theta - \frac{v^2}{4}\right)\frac{1}{\sigma_s^{B^0}} ds \\ &\quad -\frac{\kappa}{2}\int_0^{t_1}\sigma_s^{B^0} ds + \frac{\rho_2 v}{2}\left(B_{t_1}^0 - B_{t_0}^0\right) \\ &\geq \sigma_{t_0}^{\rho} + \frac{1}{2}\int_0^{t_1}\left(\kappa\theta - \frac{v^2}{4}\right)\frac{1}{\sigma_s^{\rho}} ds \\ &\quad -\frac{\kappa}{2}\int_0^{t_1}\sigma_s^{B^\rho} ds + \frac{\rho_2 v}{2}\left(B_{t_1}^0 - B_{t_0}^0\right) \\ &= \sigma_{t_1}^\rho\end{aligned}$$

which is a contradiction. Thus $\sigma_s^\rho \leq \sigma_s^{B^0}$ for all $s \geq 0$, which can be plugged in (3.2.32) to give

$$\begin{aligned}\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) - \sigma_{2,1}\right) \mid B_\cdot^0\right]^2\right] ds \\ \geq \alpha^2 \mathbb{E}\left[\frac{1}{t}\int_0^t \mathbb{I}_{\sigma_{\frac{s}{\epsilon}}^{\rho} \geq \tilde{h}^{-1}(\alpha+\sigma_{2,1})} ds\right]\end{aligned} \tag{3.2.33}$$

Finally, by the positive recurrence of $\sigma^{\rho}_{\cdot}$ (which is the root of a CIR process, the ergodicity of which has been discussed in [17]), the RHS of the above converges to $\alpha^2\mathbb{P}\left(\sigma^{\rho,*} \geq \tilde{h}^{-1}\left(\alpha + \sigma_{2,1}\right)\right)$ as $\epsilon \longrightarrow 0^+$, where $\sigma^{\rho,*}$ has the stationary distribution of $\sigma^{\rho}_{\cdot}$. Thus, since the square of $\sigma^{\rho}_{\cdot}$ satisfies Feller's boundary condition, the RHS of (3.2.33) converges to something strictly positive as $\epsilon \longrightarrow 0^+$, which proves our claim. Of course, the above argument fails if $\rho_2 = 0$ ($\sigma^{\rho}_{\cdot}$ becomes deterministic), and in this case we can easily check that the LHS of (3.2.33) tends to zero, which implies $\tilde{\sigma} = \sigma_{2,1}$. Then we can hope for better convergence results for our model. However, $\rho_2 = 0$ means that we assume uncorrelated volatilities, while interdependence is the main feature that makes our model realistic.

Remark 3.2.8. In the case where the coefficients are non-constant but independently chosen from some distribution for each assset price, for each pair $(i,\, j)$ with $i \neq j$, the correlation between the i-th and the j-th asset prices will converge (as $\epsilon \longrightarrow 0^+$) to something containing $\tilde{\sigma}_{i,j} := \sqrt{\mathbb{E}\left[h\left(\sigma^{i,j,1,*}\right) h\left(\sigma^{i,j,2,*}\right) \mid \mathcal{F}^C\right]}$, which will be a random quantity depending on both i and j. However, since we assume correlated volatilities, we are unable to express this quantity as $\tilde{\sigma}_i\tilde{\sigma}_j$ for some $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$. This means that we are unable to achieve a limit of the desired form (as in (3.2.21), with each $X^i_{\cdot}$ being a logarithmicaly scaled Black-Scholes price process, driven by $W^i_{\cdot}$ and $W^0_{\cdot}$, and killed when it hits zero), since the correlation between $X^{i,*}_{\cdot}$ and $X^{j,*}_{\cdot}$ in that setting has always the form c_ic_j for some $c_i, c_i \in [0,\, 1]$, for any i and j with $i \neq j$.

Remark 3.2.9. To obtain our convergence result we have assumed that the function h is bounded, and that q must satisfy the positive recurrence property. This means that the result we have at this stage may not be extendable to the classic Heston model, or to models that are based on CIR volatility processes which do not satisfy the conditions of Theorem 3.2.4. However, it covers a very wide range of stochastic volatility models which can capture many important features of large portfolios that constant volatility models cannot.

Remark 3.2.10. Even though we do not have any better approximation at this stage, which would have been great for the practical implementation of our model, the convergence result we have just proven is an important first step. Our convergence result shows also that quite accurate results can be obtained in risk management of large portfolios by using a very simple constant volatility model (like the one studied in [7], but with random coefficients), provided that the volatilities of the assets are fast mean-reverting (which is frequently observed in markets). The coefficients can

be estimated by solving the corresponding stochastic boundary-initial value problem backwards, and by using regression against certain known quantities.

3.3 Fast mean-reversion - small vol-of-vol

In order to deal with the disadvantages of the model studied in the previous section, we will now study a different setting, where the volatility of each volatility process is allowed to be small compared to the square root of the mean-reversion coefficient. This time, the i-th logarithmically scaled asset price is assumed to evolve according to the system

$$\begin{aligned} dX_t^{i,\epsilon} &= \left(r_i - \tfrac{h^2(\sigma_t^{i,\epsilon})}{2}\right) dt + h(\sigma_t^{i,\epsilon}) \left(\sqrt{1-\rho_{1,i}^2} dW_t^i + \rho_{1,i} dW_t^0\right), \quad 0 \le t \le T_i \\ d\sigma_t^{i,\epsilon} &= \tfrac{\kappa_i}{\epsilon}(\theta_i - \sigma_t^{i,\epsilon})dt + \xi_i q\left(\sigma_t^{i,\epsilon}\right) \left(\sqrt{1-\rho_{2,i}^2} dB_t^i + \rho_{2,i} dB_t^0\right), \quad t \ge 0 \\ X_t^{i,\epsilon} &= 0, \; t > T_i := \inf\{s \ge 0 : X_s^{i,\epsilon} \le 0\} \\ (X_0^{i,\epsilon},\, \sigma_0^{i,\epsilon}) &= (x^i, \sigma^i), \end{aligned} \tag{3.3.1}$$

where the function q is assumed to be continuous with at most linear growth (to include both the Ornstein-Uhlenbeck and the CIR volatility case). We also assume that $\sigma^i, \xi_i, \theta_i, \kappa_i$ are bounded random variables with $0 < \kappa_i \le 1$ (for simplicity, since the size of the mean-reversion coefficients is measured by ϵ), for every $i \in \mathbb{N}$.

3.3.1 Advantages of the small vol-of-vol setting

Under the above assumptions, we will obtain a convergence result which is stronger than the one we had in the fast mean-reversion - large vol-of-vol setting, and also when the Brownian motions $W_\cdot^0$ and $B_\cdot^0$ describing the impact of the market on each asset are allowed to be correlated. Moreover, when $B_\cdot^0$ and $W_\cdot^0$ are uncorrelated and q has good regularity and boundedness properties, we will see that we are able to obtain a correction of order $O\left(\sqrt{\epsilon}\right)$ in a weak sense. The usefulness of this setting will become more clear in the last subsection, where we will discuss the rate of convergence of probabilities of the form (3.1.2).

The main feature of the model we are now studying is that each volatility process $\sigma_\cdot^{i,\epsilon}$ converges in L^p to the constant value θ_i as $\epsilon \longrightarrow 0^+$, for any $p > 0$, which is a major advantage. Indeed, the reason for having very weak convergence results in the fast mean-reversion - large vol-of-vol setting was the fact that the limiting quantities $\sigma_{1,1}$, $\sigma_{2,1}$ and $\tilde{\sigma}$ did not coincide, while the corresponding limits do coincide when the volatilities converge in some strong sense to constant values, as we can easily check.

3.3.2 Convergence of the volatilities: two useful lemmas

We start by establishing the convergence of each volatility to its mean as $\epsilon \longrightarrow 0^+$. This is given in the following technical lemma

Lemma 3.3.1. *Suppose that q is continuous and satisfies $|q(z)| \leq |z| + c_q$ for some $c_q > 0$ and for all $z \in \mathbb{R}$, and that $\sigma^i, \xi_i, \theta_i, \kappa_i$ are bounded random variables. Then, for any $t \geq 0$ and $p \geq 1$, it holds that $\sigma_s^{i,\epsilon} \longrightarrow \theta_i$ in $L^p(\Omega \times [0,\, t])$, as $\epsilon \longrightarrow 0^+$.*

Proof. First, we will show that each volatility process has a finite $2p$-moment for any $p \in \mathbb{N}$. Indeed, we fix a $p \in \mathbb{N}$ and we consider the sequence of stopping times $\{\tau_{n,\epsilon} : n \in \mathbb{N}\}$, where $\tau_{n,\epsilon} = \inf\{t \geq 0 : \sigma_t^{i,\epsilon} > n\}$. Setting $\sigma_t^{i,n,\epsilon} = \sigma_{t\wedge\tau_{n,\epsilon}}^{i,\epsilon}$, by Ito's formula we have

$$\begin{aligned}\left(\sigma_t^{i,n,\epsilon} - \theta_i\right)^{2p} &= \left(\sigma_0^{i,n,\epsilon} - \theta_i\right)^{2p} - \frac{2p\kappa_i}{\epsilon}\int_0^t \mathbb{I}_{[0,\tau_{n,\epsilon}]}(s)\left(\sigma_s^{i,n,\epsilon} - \theta_i\right)^{2p} ds \\ &\quad + 2p\xi_i \int_0^t \mathbb{I}_{[0,\tau_{n,\epsilon}]}(s)\left(\sigma_s^{i,n,\epsilon} - \theta_i\right)^{2p-1} q\left(\sigma_s^{i,n,\epsilon}\right) d\tilde{B}_s^i \\ &\quad + p(2p-1)\xi_i^2 \int_0^t \mathbb{I}_{[0,\tau_{n,\epsilon}]}(s)\left(\sigma_s^{i,n,\epsilon} - \theta_i\right)^{2p-2} q^2\left(\sigma_s^{i,n,\epsilon}\right) ds \end{aligned} \tag{3.3.2}$$

for $\tilde{B}_s^i = \sqrt{1-\rho_{2,i}^2}B_t^i + \rho_{2,i}B_t^0$, where the stochastic integral is a Martingale. Taking expectations, setting $f(t,n,p,\epsilon) = \mathbb{E}\left[\left(\sigma_t^{i,n,\epsilon} - \theta_i\right)^{2p}\right]$ and using the growth condition of g and simple inequalities, we can easily obtain

$$f(t,n,p,\epsilon) \leq M + M'\int_0^t f(s,n,p,\epsilon)ds$$

with M, M' depending only on p, c_q and the bounds of $\sigma^i, \xi_i, \theta_i$. Thus, using Gronwall's inequality we get a uniform (in n) estimate for $f(t,n,p,\epsilon)$, and then by Fatou's lemma we obtain the desired finiteness of $f(t,p,\epsilon) := \mathbb{E}\left[\left(\sigma_t^{i,\epsilon} - \theta_i\right)^{2p}\right]$. This implies the almost sure finiteness of $f_C(t,p,\epsilon) := \mathbb{E}\left[\left(\sigma_t^{i,\epsilon} - \theta_i\right)^{2p} \mid \mathcal{F}^C\right]$ as well. Taking then expectations given $\mathcal{F}^C$ and $n \longrightarrow +\infty$ on (3.3.2), and using the Monotone Convergence Theorem (all quantities are monotone for large enough n) and the growth condition on q, we find that

$$f_C(t,p,\epsilon) \leq M + \left(M' - \frac{2\kappa_i}{\epsilon}\right)\int_0^t f_C(s,p,\epsilon)ds$$

where again, M, M' depend only on p, c_q and the bounds of $\sigma^i, \xi_i, \theta_i$. Using Grownwall's inequality again on the above, we finally have

$$\int_0^t f_C(s,p,\epsilon)ds \leq M\int_0^t e^{\left(M' - \frac{2\kappa_i}{\epsilon}\right)(t-s)}ds \tag{3.3.3}$$

where the integral on the RHS of the above is bounded and tends to 0 as $\epsilon \longrightarrow 0^+$. Thus, by the Dominated Convergence Theorem we have that $f(t,p,\epsilon) = \mathbb{E}\left[f_C(t,p,\epsilon)\right]$ tends also to zero, and this gives the desired convergence result. □

The above is a nice convergence result, but in order to obtain a correction of some order, we need some result about the rate of convergence. This is given in the next lemma

Lemma 3.3.2. *Let q be a C^1 function such that both q and $(q^2)'$ are bounded, h be an analytic function, and $\sigma^1, \xi_1, \theta_1, \kappa_1$ be bounded random variables. Moreover, suppose that the sequence $\left\{h^{(n)}(\theta_1) : n \in \mathbb{N}\right\}$ is bounded by some $M_h > 0$ for all the possible values of θ_1, and that $\kappa_1 > c_\kappa > 0$ $\mathbb{P}^C$-almost surely, where M_h and c_κ are deterministic constants. Let also $\{Z_s : s \geq 0\}$ be a C^1 path such that for any $t > 0$, both Z and Z' are bounded in $[0,\, t]$ by some deterministic constant $M_{z,t} > 0$. Then, for any sequence $\epsilon_n \longrightarrow 0^+$, there exists a subsequence ϵ_{k_n} such that for almost all $t \geq 0$ we have*

$$\begin{aligned}\frac{1}{\epsilon_{k_n}}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h\left(\theta_1\right)\right)^2 Z_s ds \;\longrightarrow\; & \frac{Z_0}{\kappa_1}\int_{\theta_1}^{\sigma^1} \frac{\left(h\left(y\right) - h\left(\theta_1\right)\right)^2}{y-\theta_1} dy \\ & + \frac{\xi_1^2}{2\kappa_1}\left(h'\left(\theta_1\right) q\left(\theta_1\right)\right)^2 \int_0^t Z_s ds \quad (3.3.4)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\epsilon_{k_n}}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h\left(\theta_1\right)\right) Z_s ds \;\longrightarrow\; & \frac{Z_0}{\kappa_1}\int_{\theta_1}^{\sigma^1} \frac{h\left(y\right) - h\left(\theta_1\right)}{y-\theta_1} dy \\ & + \frac{\xi_1}{\kappa_1} h'\left(\theta_1\right) q\left(\theta_1\right) \int_0^t Z_s d\tilde{B}_s^1 \\ & + \frac{\xi_1^2}{4\kappa_1} h''\left(\theta_1\right) q^2\left(\theta_1\right) \int_0^t Z_s ds \quad (3.3.5)\end{aligned}$$

in $L^2(\Omega)$ as $n \longrightarrow +\infty$, where $\tilde{B}_s^1$ stands for the Brownian Motion $\sqrt{1-\rho_{2,1}^2}B_t^1 + \rho_{2,1}B_t^0$. If we replace the boundedness of q by linear growth, the same results hold when h is a polynomial.

Proof. From the hypothesis, we can write $Z_t = Z_0 + \int_0^t z_s ds$ with $Z_\cdot$, $z_\cdot$ bounded in $[0,\, t]$ by $M_{z,t}$. For any $j > 0$, since q has a linear growth (in the worst case), there exist some $a_j, b_j > 0$ such that $q^{2j}(z) \leq a_j(z-\theta_1)^{2j} + b_j$ for all $z \in \mathbb{R}$. Then, for a given sequence $\epsilon_n \longrightarrow 0^+$, by using Ito's formula we have

$$-\frac{\kappa_1}{\epsilon_n}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_n}\right) - h\left(\theta_1\right)\right)^2 Z_s ds$$

$$
\begin{aligned}
&= -\frac{\kappa_1}{\epsilon_n}\int_0^t \left(\sum_{m=1}^{+\infty}\frac{1}{m!}h^{(m)}(\theta_1)\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^m\right)^2 Z_s ds \\
&= -\frac{\kappa_1}{\epsilon_n}\int_0^t \sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2} Z_s ds \\
&= -\frac{\kappa_1}{\epsilon_n}\int_0^t \sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}\int_0^t\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2} Z_s ds \\
&= \sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)}\left(\sigma_t^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2} Z_t \\
&\quad -\sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)}\left(\sigma^1-\theta_1\right)^{m_1+m_2} Z_0 \\
&\quad -\sum_{m_1,m_2=1}^{+\infty}\xi_1\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}\int_0^t\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2-1} q\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1 \\
&\quad -\sum_{m_1,m_2=1}^{+\infty}\frac{\xi_1^2}{2}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)(m_1+m_2-1)}{m_1!m_2!} \\
&\qquad\qquad \times\int_0^t\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2-2} q^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds \\
&\quad -\sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)}\int_0^t\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2} z_s ds. \qquad (3.3.6)
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
&\sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)}\left(\sigma^1-\theta_1\right)^{m_1+m_2} Z_0 \\
&\quad = \sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}\int_{\theta_1}^{\sigma^1}(y-\theta_1)^{m_1+m_2-1}dy Z_0 \\
&\quad = \int_{\theta_1}^{\sigma^1}\frac{1}{(y-\theta_1)}\sum_{m_1,m_2=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}(y-\theta_1)^{m_1+m_2}dy Z_0 \\
&\quad = \int_{\theta_1}^{\sigma^1}\frac{1}{(y-\theta_1)}\left(\sum_{m_1=1}^{+\infty}\frac{h^{(m_1)}(\theta_1)}{m_1!}(y-\theta_1)^{m_1}\right)^2 dy Z_0 \\
&\quad = Z_0\int_{\theta_1}^{\sigma^1}\frac{1}{(y-\theta_1)}\left(h(y)-h(\theta_1)\right)^2 dy \qquad (3.3.7)
\end{aligned}
$$

while for $m_1 = m_2 = 1$ we have

$$
-\frac{\xi_1^2}{2}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)(m_1+m_2-1)}{m_1!m_2!}
$$

$$\times \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2-2} q^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds$$
$$= -\frac{\xi_1^2}{2}\left(h'(\theta_1)\right)^2 \int_0^t q\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds. \tag{3.3.8}$$

which converges to $-\frac{\xi_1^2}{2}\left(h'(\theta_1)\, q^2(\theta_1)\right)^2 \int_0^t Z_s ds$ in L^2 for all $t \geq 0$, as $n \longrightarrow +\infty$. To show this convergence, we compute the squared L^2 norm of the difference between the sequence and the desired limit, which equals

$$\mathbb{E}\left[\frac{\xi_1^4}{4}\left(h'(\theta_1)\right)^4 \left(\int_0^t \left(q^2\left(\sigma_s^{1,\epsilon_n}\right) - q^2(\theta_1)\right) Z_s ds\right)^2\right]$$
$$\leq \frac{\|\xi_1\|_\infty^4}{4} M_h^4 t M_{z,t}^2 \mathbb{E}\left[\int_0^t \left(q^2\left(\sigma_s^{1,\epsilon_n}\right) - q^2(\theta_1)\right)^2 ds\right],$$

and since the derivative of q^2 is bounded by some $M_q > 0$, by using the mean value theorem we have

$$\mathbb{E}\left[\int_0^t \left(q^2\left(\sigma_s^{1,\epsilon_n}\right) - q^2(\theta_1)\right)^2 ds\right] \;\leq\; M_q^2 \mathbb{E}\left[\int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^2 ds\right]$$

which tends to zero as $n \longrightarrow +\infty$ (by Lemma 3.3.1).

We denote now by S_n the sum of all the other terms in (3.3.6), so next we need to show that $\|S_n\|_{L^2(\Omega)}$ tends to zero in a subsequence $\{\epsilon_{k_n} : n \in \mathbb{N}\}$ of $\{\epsilon_n : n \in \mathbb{N}\}$, for almost all $t \geq 0$ as $n \longrightarrow +\infty$. Writing $\mathbb{E}_{\mathcal{F}^C}$ for the expectation given the coefficients and $L^2_{\mathcal{F}^C}$ for the corresponding L^2 norm, we obviously have $\|S_n\|_{L^2(\Omega)} = \sqrt{\mathbb{E}\left[\|S_n\|^2_{L^2_{\mathcal{F}^C}(\Omega)}\right]}$, and by the triangle inequality we have that $\|S_n\|_{L^2_{\mathcal{F}^C}(\Omega)}$ is less than the sum of the $L^2_{\mathcal{F}^C}(\Omega)$ norms of the terms of S_n, which equals

$$\sum_{m_1,m_2=1}^{+\infty} \frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)\right|}{m_1!m_2!(m_1+m_2)} \sqrt{\mathbb{E}\left[\left(\sigma_t^{1,\epsilon_n} - \theta_1\right)^{2(m_1+m_2)} Z_t^2 \,|\, \mathcal{F}^C\right]}$$
$$+ \sum_{m_1,m_2=1}^{+\infty} \xi_1 \frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)\right|}{m_1!m_2!}$$
$$\times \sqrt{\mathbb{E}\left[\left(\int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2-1} q\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1\right)^2 \,|\, \mathcal{F}^C\right]}$$
$$+ \sum_{m_1+m_2\geq 3}^{+\infty} \frac{\xi_1^2}{2} \frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)(m_1+m_2-1)\right|}{m_1!m_2!}$$
$$\times \sqrt{\mathbb{E}\left[\left(\int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2-2} q^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds\right)^2 \,|\, \mathcal{F}^C\right]}$$

$$+\sum_{m_1,m_2=1}^{+\infty}\frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)\right|}{m_1!m_2!(m_1+m_2)}\sqrt{\mathbb{E}\left[\left(\int_0^t\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2}z_s ds\right)^2\mid\mathcal{F}^C\right]}. \tag{3.3.9}$$

Then, by using Ito's isometry, the boundedness of $Z_\cdot$ and $z_\cdot$ in $[0,t]$ by $M_{z,t}$, the linear growth of q combined with the triangle inequality, the boundedness of the sequence $\left\{h^{(n)}(\theta_1):n\in\mathbb{N}\right\}$ by the deterministic constant M_h, and finally the Cauchy-Schwartz inequality, we can bound the above by a multiple of

$$\begin{aligned}
&\sum_{m_1,m_2=1}^{+\infty}\frac{1}{m_1!m_2!(m_1+m_2)}\sqrt{\mathbb{E}\left[\left(\sigma_t^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)}\mid\mathcal{F}^C\right]}\\
&\quad+\sum_{m_1,m_2=1}^{+\infty}\frac{1}{m_1!m_2!}\sqrt{\int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)}\mid\mathcal{F}^C\right]ds}\\
&\quad+\sum_{m_1,m_2=1}^{+\infty}\frac{1}{m_1!m_2!}\sqrt{\int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2-1)}\mid\mathcal{F}^C\right]ds}\\
&\quad+\sum_{m_1+m_2\geq 3}^{+\infty}\frac{(m_1+m_2-1)}{m_1!m_2!}\sqrt{\int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2-1)}\mid\mathcal{F}^C\right]ds}\\
&\quad+\sum_{m_1+m_2\geq 3}^{+\infty}\frac{(m_1+m_2-1)}{m_1!m_2!}\sqrt{\int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2-2)}\mid\mathcal{F}^C\right]ds}\\
&\quad+\sum_{m_1,m_2=1}^{+\infty}\frac{1}{m_1!m_2!(m_1+m_2)}\sqrt{\int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)}\mid\mathcal{F}^C\right]ds}.
\end{aligned} \tag{3.3.10}$$

Using now the inequality $x<\frac{1}{\eta^2}x^2+\eta^2$ to get rid of the square roots, then Tonelli's Theorem to interchange the sums with the integrals and the expectations, and finally a few trivial inequalities like $m_1+m_2-1\leq m_1m_2$, we find that each of the above six sums is less than

$$\begin{aligned}
&\mathbb{E}\left[\sum_{m_1+m_2\geq 1}^{+\infty}\frac{\frac{1}{\eta^2}\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)}+\eta^2}{m_1!m_2!}\mid\mathcal{F}^C\right]\\
&\quad=\frac{2}{\eta^2}\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^2\mid\mathcal{F}^C\right]+\frac{1}{\eta^2}\mathbb{E}\left(\left[\sum_{m=1}^{+\infty}\frac{\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2m}}{m!}\right)^2\mid\mathcal{F}^C\right]\\
&\qquad+2\eta^2+\eta^2\left(\sum_{m=1}^{+\infty}\frac{1}{m!}\right)^2
\end{aligned}$$

$$= \frac{1}{\eta^2}\left(2\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^2 \mid \mathcal{F}^C\right] + \mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^2}-1\right)^2 \mid \mathcal{F}^C\right]\right)$$
$$+\eta^2\left(2+(e-1)^2\right)$$

computed at $s=t$, or its integral for $s\in[0,t]$. In both cases, the desired result follows because in a subsequence $\{\epsilon_{k_n}\}$ of $\{\epsilon_n\}$, for very small $\eta>0$ and very large n, the $L^2\left(\Omega^C\right)$ norm of both quantities can be made arbitrarily small, for almost all $t>0$. Indeed, for any $t>0$ we have

$$\begin{aligned}
&\sqrt{\mathbb{E}\left[\left(\int_0^t\left(2\mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2 \mid \mathcal{F}^C\right]+\mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}-1\right)^2 \mid \mathcal{F}^C\right]\right)ds\right)^2\right]} \\
&\leq 2\sqrt{\mathbb{E}\left[\left(\int_0^t\left(\mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2 \mid \mathcal{F}^C\right]\right)ds\right)^2\right]} \\
&\quad+\sqrt{\mathbb{E}\left[\left(\int_0^t\left(\mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}-1\right)^2 \mid \mathcal{F}^C\right]\right)ds\right)^2\right]} \\
&\leq 2\sqrt{\int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^4\right]ds}+\sqrt{\int_0^t\mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}-1\right)^4\right]ds}
\end{aligned} \tag{3.3.11}$$

and in a similar way

$$\begin{aligned}
&\sqrt{\mathbb{E}\left[\left(2\mathbb{E}\left[\left(\sigma_t^{1,\epsilon}-\theta_1\right)^2 \mid \mathcal{F}^C\right]+\mathbb{E}\left[\left(e^{\left(\sigma_t^{1,\epsilon}-\theta_1\right)^2}-1\right)^2 \mid \mathcal{F}^C\right]\right)^2\right]} \\
&\leq 2\sqrt{\mathbb{E}\left[\left(\sigma_t^{1,\epsilon}-\theta_1\right)^4\right]}+\sqrt{\mathbb{E}\left[\left(e^{\left(\sigma_t^{1,\epsilon}-\theta_1\right)^2}-1\right)^4\right]}
\end{aligned} \tag{3.3.12}$$

and we will show that the RHS of (3.3.11) tends to zero as $\epsilon\longrightarrow 0^+$ for all $t>0$, which implies also that the RHS of (3.3.12) tends to zero in subsequences for almost all $t>0$. For the first term of the RHS of (3.3.11), convergence to zero follows by Lemma 3.3.1, while for second term we can use the inequality $e^x-1\leq xe^x$ for $x\geq 0$ to obtain

$$\begin{aligned}
\int_0^t\mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}-1\right)^4\right]ds &\leq \int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^4 e^{4\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}\right]ds \\
&\leq \sqrt{\int_0^t\mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^8\right]ds}\sqrt{\int_0^t\mathbb{E}\left[e^{8\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}\right]ds}
\end{aligned}$$

with $\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^8\right] ds$ tending to zero as $\epsilon \longrightarrow 0^+$, and $\int_0^t \mathbb{E}\left[e^{8\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}\right] ds \leq 1$ for small enough ϵ. The last is true because

$$\begin{aligned}\left(\sigma_t^{1,\epsilon}-\theta_1\right)^2 &= \left(\sigma_t^{1,\epsilon}-\theta_1\right)^2 - \frac{2\kappa_1}{\epsilon}\int_0^t \left(\sigma_t^{1,\epsilon}-\theta_1\right)^2 ds \\ &\quad +2\xi_1 \int_0^t \left(\sigma_t^{1,\epsilon}-\theta_1\right) q\left(\sigma_t^{1,\epsilon}\right) d\tilde{B}_s^1 \\ &\quad +\xi_1^2 \int_0^t g^2\left(\sigma_t^{1,\epsilon}\right) ds \end{aligned} \tag{3.3.13}$$

which means that when $\epsilon < \epsilon_0$ for ϵ_0 small enough, we can use the boundedness of q, κ_1 and ξ_1 to control $e^{8\left(\sigma_t^{1,\epsilon}-\theta_1\right)^2}$ by the Doleans exponential of the local Martingale $4\xi_1 \int_0^t \left(\sigma_t^{1,\epsilon}-\theta_1\right) q\left(\sigma_t^{1,\epsilon}\right) d\tilde{B}_s^1$, whose expectation is less than 1. Observe that ϵ_0 should not depend on the coefficients, and this is why a deterministic positive lower bound for κ_1 is needed. This completes the proof of the first convergence result.

If h is a polynomial, there are only finitely many terms in (3.3.9), so we can replace the infinite sums in (3.3.10) with finite ones. In that case, we can just use the boundedness of the coefficients and 3.3.1 to show that the finitely many terms in these sums tend all to zero in $L^2\left(\Omega^C\right)$, without the need to assume that q is bounded.

The second convergence result can be obtained in the same way, since we can compute

$$\begin{aligned}&-\frac{\kappa_1}{\epsilon_n}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_n}\right)-h\left(\theta_1\right)\right) Z_s ds \\ &\quad = -\frac{\kappa_1}{\epsilon_n}\int_0^t \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!}\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^m Z_s ds \\ &\quad = \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m}\left(\sigma_t^{1,\epsilon_n}-\theta_1\right)^m Z_t \\ &\qquad -\sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m}\left(\sigma^1-\theta_1\right)^m Z_0 \\ &\qquad -\sum_{m=1}^{+\infty} \xi_1 \frac{h^{(m)}(\theta_1)}{m!}\int_0^t \left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m-1} q\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1 \\ &\qquad -\sum_{m=1}^{+\infty} \frac{\xi_1^2}{2}\frac{h^{(m)}(\theta_1)(m-1)}{m!} \\ &\qquad\qquad \times \int_0^t \left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m-2} q^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds \\ &\qquad -\sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m}\int_0^t \left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^m z_s ds. \end{aligned} \tag{3.3.14}$$

where

$$\sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m} \left(\sigma^1 - \theta_1\right)^m Z_0 = \int_{\theta_1}^{\sigma^1} \frac{1}{y-\theta_1} \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!} (y-\theta_1)^m \, dy Z_0$$
$$= Z_0 \int_{\theta_1}^{\sigma^1} \frac{h(y) - h(\theta_1)}{y-\theta_1} dy$$

while for $m = 1$ and $m = 2$ we have

$$\xi_1 \frac{h^{(m)}(\theta_1)}{m!} \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m-1} q\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1$$
$$= \xi_1 h'(\theta_1) \int_0^t q\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1$$

and

$$\frac{\xi_1^2}{2} \frac{h^{(m)}(\theta_1)(m-1)}{m!} \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m-2} q^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds$$
$$= \frac{\xi_1^2}{2} h''(\theta_1) \int_0^t q^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds, \tag{3.3.15}$$

whose L^2 distances from $\xi_1 h'(\theta_1) q(\theta_1) \int_0^t Z_s d\tilde{B}_s^1$ and $\frac{\xi_1^2}{4} h''(\theta_1) q^2(\theta_1) \int_0^t Z_s ds$ respectively are bounded by multiples of $\mathbb{E}\left[\int_0^t \left(q^2\left(\sigma_s^{1,\epsilon_n}\right) - q^2(\theta_1)\right)^2 ds\right]$ which tends to zero, as we have shown earlier. To show that the L^2 norm of the sum of all the other terms tends to zero, we need to follow the same steps we followed for showing the corresponding result in the proof of the first convergence result. The only difference is that this time the computations involve controlling certain sums by exponential Taylor series, and not by the squares of these series. The proof of the Lemma is now complete. □

3.3.3 Strong convergence of the mass given the volatility path

We proceed now to our first main result, which is the convergence of our system as $\epsilon \longrightarrow 0^+$. As in previous sections, we denote by C_i^ϵ the coefficient vector of the i-th asset price process, by $\mathbb{E}_{\sigma,\mathcal{F}^C}$ the expectation given the volatility paths $\{\sigma_\cdot^{1,\epsilon}\}_{\epsilon>0}$ and the coefficient vectors, and by $L^2_{\sigma,\mathcal{F}^C}$ the corresponding L^2 norm. Then, as in the large vol-of-vol setting, we need to approximate the random mass of non-defaulted assets when the market factors are given, i.e

$$\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right) = \mathbb{E}\left[\int_0^{+\infty} \int_0^{+\infty} u_{C_1^\epsilon}(t,\, x,\, y)\, dxdy \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0\right]$$

where we have

$$u_{C_1^\epsilon}(t,\,x,\,y) = p_t^\epsilon\left(y|B_\cdot^0,\mathcal{G}^0\right)\mathbb{E}\left[u\left(t,x,W_\cdot^0,\mathcal{G}^0,C_1^\epsilon,h\left(\sigma_\cdot^{1,\epsilon}\right)\right)\,|W_\cdot^0,\sigma_t^{1,\epsilon}=y,B_\cdot^0,C_1^\epsilon,\mathcal{G}^0\right]$$

and where $u^\epsilon(t,\,x) := u\left(t,x,W_\cdot^0,\mathcal{G}^0,C_1^\epsilon,h\left(\sigma_\cdot^{1,\epsilon}\right)\right)$ solves the SPDE

$$\begin{aligned} u^\epsilon(t,\,x) &= u_0(x) - \int_0^t\left(r-\frac{h^2\left(\sigma_s^{1,\epsilon}\right)}{2}\right)u_x^\epsilon(s,\,x)ds \\ &\quad + \int_0^t\frac{h^2\left(\sigma_s^{1,\epsilon}\right)}{2}u_{xx}^\epsilon(s,\,x)ds - \rho_{1,1}\int_0^t h\left(\sigma_s^{1,\epsilon}\right)u_x^\epsilon(s,\,x)dW_s^0 \end{aligned} \tag{3.3.16}$$

and satisfies also the identity

$$\left\|u^\epsilon(t,\,\cdot)\right\|_{L^2(\mathbb{R}^+)}^2 + \left(1-\rho_{1,1}^2\right)\int_0^t h^2\left(\sigma_t^{1,\epsilon}\right)\left\|u_x^\epsilon(s,\,\cdot)\right\|_{L^2(\mathbb{R}^+)}^2 ds = \left\|u_0\right\|_{L^2(\mathbb{R}^+)}^2. \tag{3.3.17}$$

for all $t\geq 0$, $\mathbb{P}^0\times\mathbb{P}^C$ - almost surely. Assuming again that $W_\cdot^0$ and $B_\cdot^0$ are uncorrelated and that $c<h^2(x)<C$ for some $C,c>0$ and all $x>0$, we can use (3.3.17) to show that the $L^2_{\sigma,\mathcal{F}^C}$ and $L^2_{\sigma,\mathcal{F}^C}\left(\Omega^0\times[0,\,T]\,;H_0^1\left(\mathbb{R}^+\right)\right)$ norms of u^ϵ are bounded in ϵ. This implies the existence of a weak limit u of $u^{\epsilon_{k_n}}$ for some subsequence $\{\epsilon_{k_n}\,:\,n\in\mathbb{N}\}$ of an arbitrary sequence $\epsilon_n\longrightarrow 0^+$, in all these reflexive spaces. Then, by following exactly the same steps as in the proof of Theorem 3.2.5, but with $\int_0^t\sigma_{\frac{s}{\epsilon}}^{1,1}ds$ under the large vol-of-vol setting replaced by $\int_0^t\sigma_s^{1,\epsilon}ds$ under the small vol-of-vol setting, we can obtain a similar characterization for the weak limits u

Theorem 3.3.3. *Any weak limit u of u^ϵ in $L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times[0,\,T]\,;H_0^1\left(\mathbb{R}^+\right))$, in any sequence $\epsilon_n\longrightarrow 0^+$ for which we have convergence, is equal to the unique solution to the SPDE*

$$\begin{aligned} u(t,\,x) &= u_0(x) - \left(r-\frac{h^2\left(\theta_1\right)}{2}\right)\int_0^t u_x(s,\,x)ds \\ &\quad + \frac{h^2\left(\theta_1\right)}{2}\int_0^t u_{xx}(s,\,x)ds - \rho_{1,1}h\left(\theta_1\right)\int_0^t u_x(s,\,x)dW_s^0 \end{aligned} \tag{3.3.18}$$

in that space. Thus, the uniqueness of solutions implies that we have convergence to u as $\epsilon\longrightarrow 0^+$.

Again, there are some difficulties (mainly the lack of uniform convergence of $\sigma_\cdot^{1,\epsilon}$ to θ_1) which do not allow us to obtain strong convergence of u^ϵ in $L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times[0,\,T]\,;H_0^1\left(\mathbb{R}^+\right))$, while the independence between $W_\cdot^0$ and $B_\cdot^0$ is a non-realistic assumption we would like to get rid of. Fortunately, under this better setting, we can

show strong convergence of the mass of u^ϵ over $\mathbb{R}^+$ in L^2, just by assuming that h possesses a few nice properties, without the need to assume that $W_\cdot^0$ and $B_\cdot^0$ are uncorrelated. This is possible because when h has these properties, we are able to show strong convergence of the antiderivative $v^{0,\epsilon} := \int_\cdot^{+\infty} u^\epsilon(y)dy$ in $L^2(\Omega\times[0,\,T]\,;H^1\,(\mathbb{R}^+))$. This result is given in the following Theorem

Theorem 3.3.4. *Suppose that h is differentiable and that both h and h' have a polynomial growth. Then, $v^{0,\epsilon}$ converges strongly to v^0 in $L^2(\Omega\times[0,\,T]\,;H^1\,(\mathbb{R}^+))$ as $\epsilon\longrightarrow 0^+$, for any $T>0$, where v^0 is defined as $v^0(t,x)=\int_x^{+\infty}u(t,y)dy$ for all $t,x\geq 0$*

Proof. We can easily check that for any $\omega^C\in\Omega^C$ (i.e for any value of the coefficient vector) and given the volatility path, $v^{0,\epsilon}$ and v^0 are the unique solutions to the SPDEs (3.3.16) and (3.3.18) respectively in the space $L^2(\Omega^0\times[0,\,T]\,;H^2\,(\mathbb{R}^+))$, under the boundary conditions $v_x^{0,\epsilon}\in L^2(\Omega^0\times[0,\,T]\,;H_0^1\,(\mathbb{R}^+))$ and $v_x^0\in L^2(\Omega^0\times[0,\,T]\,;H_0^1\,(\mathbb{R}^+))$ respectively. Subtracting the SPDEs satisfied by $v^{0,\epsilon}$ and v^0 and setting $v^{d,\epsilon}=v^0-v^{0,\epsilon}$, we can easily verify that

$$\begin{aligned}
v^{d,\epsilon}\,(t,\,x) &= -\frac{1}{2}\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\,(\theta_1)\right)v_x^{0,\epsilon}\,(s,\,x)\,ds\\
&\quad+\int_0^t\left(r-\frac{h^2\,(\theta_1)}{2}\right)v_x^{d,\epsilon}\,(s,\,x)\,ds\\
&\quad+\frac{1}{2}\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\,(\theta_1)\right)v_{xx}^{0,\epsilon}\,(s,\,x)\,ds\\
&\quad+\int_0^t\frac{h^2\,(\theta_1)}{2}v_{xx}^{d,\epsilon}\,(s,\,x)\,ds\\
&\quad+\rho_{1,1}\int_0^t\left(h\left(\sigma_s^{1,\epsilon}\right)-h\,(\theta_1)\right)v_x^{0,\epsilon}\,(s,\,x)\,dW_s^0\\
&\quad+\rho_{1,1}\int_0^t h\,(\theta_1)\,v_x^{d,\epsilon}\,(s,\,x)\,dW_s^0
\end{aligned}$$

and we can use Ito's formula for the L^2 norm (see [33]) on the above to obtain

$$\begin{aligned}
\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+}\left(v^{d,\epsilon}\,(t,\,x)\right)^2dx\right] &= -\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\,(\theta_1)\right)\\
&\qquad\times\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+}v_x^{0,\epsilon}\,(s,\,x)\,v^{d,\epsilon}\,(t,\,x)\,dx\right]ds\\
&\quad+2\left(r-\frac{h^2\,(\theta_1)}{2}\right)\\
&\qquad\times\int_0^t\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+}v_x^{d,\epsilon}\,(s,\,x)\,v^{d,\epsilon}\,(t,\,x)\,dx\right]ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)\\
&\qquad\times\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right)v_x^{d,\epsilon}\left(t,\,x\right)dx\right]ds\\
&-\int_0^t h^2\left(\theta_1\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+}\left(v_x^{d,\epsilon}\left(s,\,x\right)\right)^2dx\right]ds\\
&+\rho_{1,1}^2\int_0^t\left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2\\
&\qquad\times\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+}\left(v_x^{0,\epsilon}\left(s,\,x\right)\right)^2dx\right]ds\\
&+2\rho_{1,1}^2h\left(\theta_1\right)\int_0^t\left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)\\
&\qquad\times\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right)v_x^{d,\epsilon}\left(t,\,x\right)dx\right]ds\\
&+\rho_{1,1}^2\int_0^t h^2\left(\theta_1\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+}\left(v_x^{d,\epsilon}\left(s,\,x\right)\right)^2dx\right]ds\\
&+N(t,\epsilon)
\end{aligned}
\tag{3.3.19}
$$

where $N(t,\epsilon)$ is a noise due to the correlation between $B_\cdot^0$ and $W_\cdot^0$, with $\mathbb{E}\left[N(t,\epsilon)\right]=0$. Next, by (3.3.17) we have $\|v_x^{0,\epsilon}(s,\cdot)\|_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}\le\|u_0(\cdot)\|_{L^2(\Omega^0\times\mathbb{R}^+)}$ for all $s\ge 0$. Using this, we can obtain the following estimate

$$
\begin{aligned}
&\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right)v^{d,\epsilon}\left(t,\,x\right)dx\right]ds\\
&\quad\le\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)\left\|v_x^{0,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}\left\|v^{d,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}ds\\
&\quad\le\|u_0(\cdot)\|_{L^2(\Omega^0\times\mathbb{R}^+)}\sqrt{\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)^2ds}\\
&\qquad\qquad\times\sqrt{\int_0^t\|v^{d,\epsilon}(s,\cdot)\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}ds}\\
&\quad\le\frac{1}{2}\|u_0(\cdot)\|^2_{L^2(\Omega^0\times\mathbb{R}^+)}\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)^2ds\\
&\qquad+\frac{1}{2}\int_0^t\left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}ds
\end{aligned}
\tag{3.3.20}
$$

and in the same way

$$
\begin{aligned}
&\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right)v_x^{d,\epsilon}\left(t,\,x\right)dx\right]ds\\
&\qquad\le\|u_0(\cdot)\|_{L^2(\Omega^0\times\mathbb{R}^+)}\sqrt{\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)^2ds}
\end{aligned}
$$

$$\times\sqrt{\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds}$$
$$\leq \frac{1}{2\eta}\left\|u_0(\cdot)\right\|^2_{L^2(\Omega^0\times\mathbb{R}^+)}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right)-h^2\left(\theta_1\right)\right)^2 ds$$
$$+\frac{\eta}{2}\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds \tag{3.3.21}$$

and

$$\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right)v_x^{d,\epsilon}\left(t,\,x\right)dx\right]ds$$
$$\leq \left\|u_0(\cdot)\right\|_{L^2(\Omega^0\times\mathbb{R}^+)}\sqrt{\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2 ds}$$
$$\times\sqrt{\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds}$$
$$\leq \frac{1}{2\eta}\left\|u_0(\cdot)\right\|^2_{L^2(\Omega^0\times\mathbb{R}^+)}\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2 ds$$
$$+\frac{\eta}{2}\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds \tag{3.3.22}$$

for some $\eta>0$. Moreover, we have the estimate

$$\int_0^t \mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+} v_x^{d,\epsilon}\left(s,\,x\right)v^{d,\epsilon}\left(t,\,x\right)dx\right]ds$$
$$\leq \int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}\left\|v^{d,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds$$
$$\leq \sqrt{\int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds}\sqrt{\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds}$$
$$\leq \frac{1}{2\eta}\int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds+\frac{\eta}{2}\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds \tag{3.3.23}$$

and by using $\left\|v_x^{0,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}\leq\left\|u_0(\cdot)\right\|_{L^2(\Omega^0\times\mathbb{R}^+)}$ again, we also obtain

$$\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2\mathbb{E}_{\sigma,\mathcal{F}^C}\left[\int_{\mathbb{R}^+}\left(v_x^{0,\epsilon}\left(s,\,x\right)\right)^2 dx\right]ds$$
$$\leq \left\|u_0(\cdot)\right\|^2_{L^2(\Omega^0\times\mathbb{R}^+)}\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2 ds. \tag{3.3.24}$$

Plugging now (3.3.20), (3.3.21), (3.3.22), (3.3.23) and (3.3.24) in (3.3.19), and taking then η to be sufficiently small, we get the estimate

$$\left\|v^{d,\epsilon}(t,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)}+m\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds$$

$$\leq M \int_0^t \left\| v^{d,\epsilon}(s,\cdot) \right\|^2_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0 \times \mathbb{R}^+)} ds + N(t,\epsilon) + MH(\epsilon) \qquad (3.3.25)$$

for all $t \in [0,\, T]$, where $H(\epsilon) = \int_0^T \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds + \int_0^T \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right)^2 ds$, and where $M, m > 0$ are constants independent of the fixed volatility path. Taking expectations on the above to average over all possible volatility paths and all possible values of the coefficient vectors, we find that

$$\begin{aligned} &\left\| v^{d,\epsilon}(t,\cdot) \right\|^2_{L^2(\Omega \times \mathbb{R}^+)} + m \int_0^t \left\| v_x^{d,\epsilon}(s,\cdot) \right\|^2_{L^2(\Omega \times \mathbb{R}^+)} ds \\ &\qquad \leq M \int_0^t \left\| v^{d,\epsilon}(s,\cdot) \right\|^2_{L^2(\Omega \times \mathbb{R}^+)} ds + M\mathbb{E}\left[H(\epsilon)\right] \end{aligned} \qquad (3.3.26)$$

and using Gronwall's inequality on the above, we finally obtain

$$\left\| v^{d,\epsilon}(t,\cdot) \right\|^2_{L^2(\Omega \times \mathbb{R}^+)} + m \int_0^t \left\| v_x^{d,\epsilon}(s,\cdot) \right\|^2_{L^2(\Omega \times \mathbb{R}^+)} ds \leq M'\mathbb{E}\left[H(\epsilon)\right] \qquad (3.3.27)$$

for some $M' > 0$, with $\mathbb{E}\left[H(\epsilon)\right] \to 0^+$ as $\epsilon \to 0^+$ as we can easily show. Indeed, if $h(x) \leq x^m + c_{h,1}$ and $h'(x) \leq x^m + c_{h,2}$ for all $x \geq 0$, by the mean value theorem we have

$$\begin{aligned} \mathbb{E}\left[\int_0^T \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right)^2 ds\right] &= \mathbb{E}\left[\int_0^T h'\left(\sigma_s^{*,\epsilon}\right)\left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 ds\right] \\ &\leq c_{h,2}\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 ds\right] \\ &\quad + \mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 \left(\sigma_s^{1,\epsilon}\right)^m ds\right] \\ &\leq c_{h,2}\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 ds\right] \\ &\quad + \sqrt{\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon}\right)^{2m} ds\right]} \\ &\quad \times \sqrt{\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^4 ds\right]} \end{aligned}$$

with the RHS of the last tending to zero by Lemma 3.3.1, and in the same way we can show that $\mathbb{E}\left[\int_0^T \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds\right]$ tends also to zero (since $\left(h^2\right)' = 2hh'$ has also a polynomial growth). The proof of the Theorem is now complete. □

Remark 3.3.5. By Morrey's inequality in dimension 1 (see [12]) and the above result, we can easily obtain the convergence of $v^{0,\epsilon}(\cdot, 0)$ to $v^0(\cdot, 0)$ in $L^2\left(\Omega \times [0,\, T]\right)$ as $\epsilon \longrightarrow$

0^+. Taking expectations given $(W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}^0)$ (i.e averaging over the volatility path and the vector C_1'), we get the strong convergence result for the masses of non-defaulted assets. Also, under the appropriate regularity conditions on h and g, by Lemma 3.3.2 we have that $\mathbb{E}\left[H(\epsilon)\right] = \mathcal{O}(\epsilon)$, so (3.3.27) implies that the rate of the above convergence is $\sqrt{\epsilon}$.

3.3.4 An attempt to derive a higher order correction

We continue now with the study of the existence of a correction of some order for u^ϵ. For this purpose, we study the asymptotic behaviour of $v^{1,\epsilon} := \frac{v^{d,\epsilon}}{\sqrt{\epsilon}} = \frac{v^{0,\epsilon}-v^0}{\sqrt{\epsilon}}$. Dividing the estimate (3.3.27) in the previous proof by ϵ, since Lemma 3.3.2 implies that $\mathbb{E}\left[H(\epsilon)\right] = \mathcal{O}(\epsilon)$ (under the appropriate regularity conditions on h and g), we find that

$$\left\|v^{1,\epsilon}(t,\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} + m\int_0^t \left\|v_x^{1,\epsilon}(s,\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} ds \leq M'' \tag{3.3.28}$$

for some $M'' > 0$ and all $t, \epsilon > 0$. This implies that for any sequence $\{\epsilon_n : n \in \mathbb{N}\}$, there exists a subsequence $\{\epsilon_{k_n} : n \in \mathbb{N}\}$ such that $v^{1,\epsilon_{k_n}}$ converges weakly to some v^1 in $L^2([0,\,T]\times\Omega; H^1(\mathbb{R}^+))$ as $n \longrightarrow +\infty$, for any $T > 0$. Below we will show that the weak limit v^1 is always a solution to an SPDE, but since we will not prove convergence of second order derivatives, the Neumann boundary condition satisfied by $v^{1,\epsilon_{k_n}}$ will not be established in the limit. This means that we are currently unable to show uniqueness of weak solutions to the limiting SPDE, and thus uniqueness of weak limits v^1. Therefore, we see that even in this small vol-of-vol setting, at this point we cannot hope for good high order corrections, like the ones established in [17] for the prices of vanilla options. Moreover, to prove the above weak convergence result, we need to assume again that $W_\cdot^0$ and $B_\cdot^0$ are uncorrelated, which is not a reallistic assumption as we have already mentioned. However, we will see in the next section that under certain circumstances, by using 3.3.5 we are able to estimate the exact rate of convergence of probabilities of the form (3.1.2), even when $W_\cdot^0$ and $B_\cdot^0$ are correlated, something we couldn't do in the fast mean-reversion - large vol-of-vol setting. We close this section by proving the Theorem which characterizes the weak limits v^1

Theorem 3.3.6. *Suppose that q is a C^1 function such that $(q^2)'$ is bounded, and that h is an analytic function such that both $\left\{h^{(n)}(\theta_1) : n \in \mathbb{N}\right\}$ and $\left\{(h^2)^{(n)}(\theta_1) : n \in \mathbb{N}\right\}$ are bounded by a deterministic constant $M_h > 0$. Suppose also that either q is bounded*

or h is a polynomial, and that $W_\cdot^0$ and $B_\cdot^0$ are uncorrelated. Then, the weak limit v^1 of $v^{1,\epsilon_{k_n}}$ in $L^2([0,\,T]\times\Omega;H^1(\mathbb{R}^+))$ is a weak solution to the SPDE

$$\begin{aligned} v^1(t,\,x) &= -\int_0^t \left(r-\frac{h^2(\theta_1)}{2}\right) v_x^1(s,\,x)\,ds \\ &\quad +\int_0^t \frac{h^2(\theta_1)}{2} v_{xx}^1(s,\,x)\,ds \\ &\quad -\int_0^t \rho_{1,1} h(\theta_1)\, v_x^1(s,\,x)\,dW_s^0 \end{aligned}$$

in $L^2([0,\,T]\times\Omega^0;H^1(\mathbb{R}^+))$, for any $\omega^C\in\Omega^C$ and any realization of the volatility path $\sigma_\cdot^{1,\epsilon}$ for all $\epsilon>0$.

Proof. The regularity we have assumed here allows us to apply Lemma 3.3.2 whenever we need it (including the proof of estimate (3.3.28) earlier, which gives the existence of weak limits). As in the proof of Theorem 3.3.4, we have that $v^{d,\epsilon_{k_n}}=v^{0,\epsilon_{k_n}}-v^0$ satisfies

$$\begin{aligned} v^{d,\epsilon_{k_n}}(t,\,x) &= -\frac{1}{2}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2(\theta_1)\right) v_x^{0,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\int_0^t \left(r-\frac{h^2(\theta_1)}{2}\right) v_x^{d,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\frac{1}{2}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2(\theta_1)\right) v_{xx}^{0,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\int_0^t \frac{h^2(\theta_1)}{2} v_{xx}^{d,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\rho_{1,1}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h(\theta_1)\right) v_x^{0,\epsilon_{k_n}}(s,\,x)\,dW_s^0 \\ &\quad +\rho_{1,1}\int_0^t h(\theta_1)\, v_x^{d,\epsilon_{k_n}}(s,\,x)\,dW_s^0 \end{aligned}$$

so dividing by $\sqrt{\epsilon_{k_n}}$ we find

$$\begin{aligned} v^{1,\epsilon_{k_n}}(t,\,x) &= -\frac{1}{2\sqrt{\epsilon_{k_n}}}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2(\theta_1)\right) v_x^{0,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\int_0^t \left(r-\frac{h^2(\theta_1)}{2}\right) v_x^{1,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\frac{1}{2\sqrt{\epsilon_{k_n}}}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2(\theta_1)\right) v_{xx}^{0,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\int_0^t \frac{h^2(\theta_1)}{2} v_{xx}^{1,\epsilon_{k_n}}(s,\,x)\,ds \\ &\quad +\frac{\rho_{1,1}}{\sqrt{\epsilon_{k_n}}}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h(\theta_1)\right) v_x^{0,\epsilon_{k_n}}(s,\,x)\,dW_s^0 \end{aligned}$$

$$+\rho_{1,1}\int_0^t h\left(\theta_1\right) v_x^{1,\epsilon_{k_n}}\left(s,\, x\right) dW_s^0 \tag{3.3.29}$$

for any $t \in [0,\, T]$. We test now the above against a smooth and compactly supported function f, and a $W_\cdot^0$-measurable random variable $Z \in L^2\left(\Omega^0\right)$, with $Z = \mathbb{E}\left[Z\right] + \int_0^T z_s dW_s^0$ for some smooth process $z_\cdot$ whose derivatives have bounded moments in $[0,\, T]$. This way we obtain

$$\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}\left(t,\, x\right) f(x)dx\right] \\
&\quad= \frac{1}{2\sqrt{\epsilon_{k_n}}}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h^2\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}\left(s,\, x\right) f'(x)dx\right] ds \\
&\qquad -\int_0^t \left(r - \frac{h^2\left(\theta_1\right)}{2}\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}\left(s,\, x\right) f'(x)dx\right] ds \\
&\qquad +\frac{1}{2\sqrt{\epsilon_{k_n}}}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h^2\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}\left(s,\, x\right) f''(x)dx\right] ds \\
&\qquad +\int_0^t \frac{h^2\left(\theta_1\right)}{2}\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}\left(s,\, x\right) f''(x)dx\right] ds \\
&\qquad -\frac{\rho_{1,1}}{\sqrt{\epsilon_{k_n}}}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[z_s\int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}\left(s,\, x\right) f'(x)dx\right] ds \\
&\qquad -\rho_{1,1}\int_0^t h\left(\theta_1\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[z_s\int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}\left(s,\, x\right) f'(x)dx\right] ds
\end{aligned} \tag{3.3.30}$$

for any $t \in [0,\, T]$.

Observe now that for $g \in \{h,\, h^2\}$, $V_\cdot \in \{Z, z_\cdot\}$ and $k \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\frac{1}{\sqrt{\epsilon_{k_n}}}\int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}\left(s,\, x\right) f^{(k)}(x)dx\right] ds \\
&\quad= \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}\left(s,\, x\right) f^{(k)}(x)dx\right] ds \\
&\qquad +\frac{1}{\sqrt{\epsilon_{k_n}}}\int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} v^{0}\left(s,\, x\right) f^{(k)}(x)dx\right] ds
\end{aligned} \tag{3.3.31}$$

where for any bounded random variable U, by using standard norm estimates and (3.3.28) we have

$$\begin{aligned}
&\left|\mathbb{E}\left[U\int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s\int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}\left(s,\, x\right) f^{(k)}(x)dx\right] ds\right]\right| \\
&\quad\leq \left\|f^{(k)}\right\|_{L^2(\mathbb{R}^+)}\sup_{0\leq s\leq T}\left\|V_s\right\|_{L^2(\Omega^0)} \\
&\qquad\times\mathbb{E}\left[\int_0^t \left|U\right|\left|g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right|\left\|v^{1,\epsilon_{k_n}}\left(s,\cdot\right)\right\|_{L^2_{\sigma,\mathcal{F}^C}(\Omega^0\times\mathbb{R}^+)} ds\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| f^{(k)} \right\|_{L^2(\mathbb{R}^+)} \sup_{0\leq s\leq T} \|V_s\|_{L^2(\Omega^0)} \\
&\quad \times \|U\|_{L^\infty(\Omega)} \sqrt{\mathbb{E}\left[\int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right)^2 ds\right]} \\
&\quad \times \sqrt{\int_0^t \left\| v^{1,\epsilon_{k_n}}\left(s,\cdot\right)\right\|^2_{L^2(\Omega^0\times\mathbb{R}^+)} ds} \\
&\leq \sqrt{tM''} \|U\|_{L^\infty(\Omega)} \left\| f^{(k)} \right\|_{L^2(\mathbb{R}^+)} \sup_{0\leq s\leq T} \|V_s\|_{L^2(\Omega^0)} \\
&\quad \times \sqrt{\mathbb{E}\left[\int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right)^2 ds\right]}
\end{aligned}
$$

which tends to zero by Lemma 3.3.1, in a subsequence $\{\epsilon_{k'_n} : n \in \mathbb{N}\}$ of $\{\epsilon_{k_n} : n \in \mathbb{N}\}$. Thus, since bounded random variables are dense in $L^2(\Omega)$, we have that the first term in the RHS of (3.3.31) tends to zero weakly in $L^2(\Omega)$ (where we average over all the volatility paths) and for almost all $t > 0$. Moreover, since v^0 is bounded by 1 we have

$$
\mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} v^0\left(s,\, x\right) f^{(k)}(x)dx\right] \leq \left\| f^{(k)} \right\|_{L^1(\mathbb{R}^+)} \sup_{0\leq s\leq T} \|V_s\|_{L^1(\Omega^0)}
$$

for any $s \in [0,\, T]$, while we can test the SPDE satisfied by v^0 against $f^{(k)}$ and $V.$ to show that the derivative of the LHS of the above is a linear combination of terms of the same form, which means that this derivative is also bounded. Thus, by Lemma 3.3.2 we have that the second term in the RHS of (3.3.31), i.e

$$
\begin{aligned}
&\frac{1}{\sqrt{\epsilon_{k_n}}} \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} v^0\left(s,\, x\right) f^{(k)}(x)dx\right] ds \\
&\quad = \sqrt{\epsilon_{k_n}} \frac{1}{\epsilon_{k_n}} \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g\left(\theta_1\right)\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} v^0\left(s,\, x\right) f^{(k)}(x)dx\right] ds,
\end{aligned}
$$

tends also to zero in $L^2(\Omega)$, for almost all $t > 0$, in a further subsequence $\{\epsilon_{k''_n} : n \in \mathbb{N}\}$ of $\{\epsilon_{k'_n} : n \in \mathbb{N}\}$. Therefore, in this subsequence, the LHS of (3.3.31) tends to zero weakly in $L^2(\Omega)$, for almost all $t > 0$.

Next, by our weak convergence result, for any $g \in \{h,\, h^2\}$, $V. \in \{Z, z.\}$ and $k \in \mathbb{N}$, it holds that

$$
\begin{aligned}
&\int_0^t g\left(\theta_1\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}\left(s,\, x\right) f^{(k)}(x)dx\right] ds \\
&\qquad \longrightarrow \int_0^t g\left(\theta_1\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} v^1\left(s,\, x\right) f^{(k)}(x)dx\right] ds
\end{aligned}
$$

weakly in $L^2(\Omega)$ and for all $t \in [0,\, T]$, as $n \longrightarrow +\infty$. Thus, by recalling the convergence of the LHS of (3.3.31) to zero as well, we find that the RHS of (3.3.30)

converges weakly in $L^2(\Omega)$ and for all $t \in [0,\, T]$ to

$$\begin{aligned} R(t) &= -\int_0^t \left(r - \frac{h^2(\theta_1)}{2}\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z \int_{\mathbb{R}^+} v^1(s,\, x)\, f'(x)dx\right] ds \\ &\quad + \int_0^t \frac{h^2(\theta_1)}{2} \mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z \int_{\mathbb{R}^+} v^1(s,\, x)\, f''(x)dx\right] ds \\ &\quad -\rho_{1,1} \int_0^t h(\theta_1)\, \mathbb{E}_{\sigma,\mathcal{F}^C}\left[z_s \int_{\mathbb{R}^+} v^1(s,\, x)\, f'(x)dx\right] ds \end{aligned} \tag{3.3.32}$$

in the subsequence $\{\epsilon_{k''_n} : n \in \mathbb{N}\}$ of $\{\epsilon_{k_n} : n \in \mathbb{N}\}$.

We will prove now that the above convergence holds also weakly in $L^2(\Omega \times [0,\, T])$. If we test all the terms in the RHS of (3.3.30) against some $U \in L^2(\Omega)$, we have that the occurring quantities converge in the subsequence $\{\epsilon_{k''_n} : n \in \mathbb{N}\}$ for almost all $t \in [0,\, T]$, and we need to show that this convergence holds also weakly in $L^2([0,\, T])$. This can be shown easily by using the Dominated Convergence Theorem, since we can show that these quantities are uniformly bounded in $t \in [0,\, T]$. Indeed, since $v^{0,\epsilon_{k_n}}$ is bounded by 1, by using the independence of $V_\cdot$ from the volatility paths (since $W^0_\cdot$ and $B^0_\cdot$ are uncorrelated) and from $\mathcal{F}^C$, the Cauchy-Schwartz inequality and other standard norm estimates, we have

$$\begin{aligned} &\mathbb{E}\left[U \frac{1}{\sqrt{\epsilon_{k''_n}}} \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k''_n}}\right) - g(\theta_1)\right) \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} v^{0,\epsilon_{k''_n}}(s,\, x)\, f^{(k)}(x)dx\right] ds\right] \\ &\quad \leq \left\|f^{(k)}\right\|_{L^1(\mathbb{R}^+)} \sup_{0\leq s\leq T} \|V_s\|_{L^1(\Omega^0)}\, \mathbb{E}\left[|U| \frac{1}{\sqrt{\epsilon_{k''_n}}} \int_0^t \left|g\left(\sigma_s^{1,\epsilon_{k''_n}}\right) - g(\theta_1)\right| ds\right] \\ &\quad \leq \left\|f^{(k)}\right\|_{L^1(\mathbb{R}^+)} \sup_{0\leq s\leq T} \|V_s\|_{L^1(\Omega^0)}\, \|U\|_{L^2(\Omega)} \\ &\qquad \times \mathbb{E}\left[\frac{T}{\epsilon_{k''_n}} \int_0^T \left(g\left(\sigma_s^{1,\epsilon_{k''_n}}\right) - g(\theta_1)\right)^2 ds\right], \end{aligned}$$

and by using also estimate (3.3.28)

$$\begin{aligned} &\mathbb{E}\left[U \int_0^t g(\theta_1)\, \mathbb{E}_{\sigma,\mathcal{F}^C}\left[V_s \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}(s,\, x)\, f^{(k)}(x)dx\right] ds\right] \\ &\qquad \leq M_h M'' \left\|f^{(k)}\right\|_{L^2(\mathbb{R}^+)} \sup_{0\leq s\leq T} \|V_s\|_{L^1(\Omega^0)}\, \|U\|_{L^1(\Omega)} \end{aligned}$$

for any $g \in \{h,\, h^2\}$, $V_\cdot \in \{Z, z_\cdot\}$ and $k \in \mathbb{N}$, with the RHS of the first being independent of t and convergent as $n \longrightarrow +\infty$ (by Lemma 3.3.2), and the RHS of the second being a nice uniform bound.

Finally, since we have shown that the RHS of (3.3.30) converges (in a subsequence of $\{\epsilon_{k_n} : n \in \mathbb{N}\}$) to $R(\cdot)$, weakly in $L^2(\Omega \times [0,\, T])$, and since the LHS of (3.3.30)

converges in the same topology to

$$\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+}v^1\left(t,\,x\right)f(x)dx\right],$$

by the uniqueness of weak limits and (3.3.32) we must have

$$\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+}v^1\left(t,\,x\right)f(x)dx\right]\\
&\quad=-\int_0^t\left(r-\frac{h^2\left(\theta_1\right)}{2}\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f'(x)dx\right]ds\\
&\qquad+\int_0^t\frac{h^2\left(\theta_1\right)}{2}\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f''(x)dx\right]ds\\
&\qquad-\rho_{1,1}\int_0^t h\left(\theta_1\right)\mathbb{E}_{\sigma,\mathcal{F}^C}\left[z_s\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f'(x)dx\right]ds\\
&\quad=-\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_0^t\left(r-\frac{h^2\left(\theta_1\right)}{2}\right)\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f'(x)dxds\right]\\
&\qquad+\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_0^t\frac{h^2\left(\theta_1\right)}{2}\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f''(x)dxds\right]\\
&\qquad-\mathbb{E}_{\sigma,\mathcal{F}^C}\left[Z\int_0^t\rho_{1,1}h\left(\theta_1\right)\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f'(x)dxdW_s^0\right].
\end{aligned}$$

The desired result follows, since the space of random variables $Z=\mathbb{E}\left[Z\right]+\int_0^T z_s dW_s^0$ for which $z_\cdot$ has derivatives with bounded moments in $[0,\,T]$ is a dense subspace of $L^2_{\sigma,\mathcal{F}^C}\left(\Omega^0\right)$, no matter what the volatility paths and the coefficient vectors are. □

3.3.5 Discussion of the rate of convergence

What we have now is the strong convergence of our system as $\epsilon\longrightarrow 0^+$, and a weak result for characterizing a possible correction of order $\mathcal{O}\left(\sqrt{\epsilon}\right)$. While the last weak result seems to be the strongest possible, it doesn't seem to be really useful, mainly due to the lack of uniqueness of solutions to the SPDE which characterizes possible corrections. It seems thus that both in this small vol-of-vol setting and the fast mean-reversion - large vol-of-vol setting studied in the previous section, we can only have convergence of probabilities of the form (3.1.2), which means that the two settings are equally good. However, we will see that this is not the case, since in the large vol-of-vol setting we have shown that we have only weak convergence of default masses, while in this small vol-of-vol setting we have established strong convergence, which will allow us to estimate the exact rate of convergence of probabilities of the form (3.1.2). When a certain regularity condition is satisfied at both a and b, this rate is going to be of order $\mathcal{O}\left(\sqrt[3]{\epsilon}\right)$.

To compute the rate of convergence mentioned above, we define first the limiting model, where the i-th logarithmicaly scaled asset price $X^{i,*}_{\cdot}$ evolves in time according to

$$\begin{aligned} dX^{i,*}_t &= \left(r_i - \tfrac{h^2(\theta_i)}{2}\right) dt + h(\theta_i)\left(\sqrt{1-\rho^2_{1,i}}dW^i_t + \rho_{1,i}dW^0_t\right), \quad 0 \le t \le T_i \\ X^{i,*}_t &= 0, \; t > T_i := \inf\{s \ge 0 : X^{i,*}_s \le 0\} \\ X^{i,*}_0 &= x^i \end{aligned} \tag{3.3.33}$$

for all $i \in \mathbb{N}$. Under this model, the random mass of non-defaulted assets equals $\mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right)$, with $\mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, C'_1, \mathcal{G}^0\right) = \int_0^{+\infty} u(t,x)\,dx$, where u is the unique solution to the SPDE (3.3.18) in $L^2(\Omega^0 \times [0,\,T]\,; H^1_0(\mathbb{R}^+))$. We consider now the approximation error for the probabilities we are approximating, i.e

$$\begin{aligned} E(x,T) &= \int_0^T \Big|\mathbb{P}^0\left(\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_{\cdot}, B^0_{\cdot}, \mathcal{G}^0\right) > x\right) \\ &\qquad -\mathbb{P}^0\left(\mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right) > x\right)\Big| dt \end{aligned}$$

for $x \in [0,\,1]$, and we will show that this error is expected to be of order $\mathcal{O}\left(\sqrt[3]{\epsilon}\right)$, in the worst case. Indeed, let $\mathcal{E}_{t,\,\epsilon} = \{\omega^0 \in \Omega^0 : \mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_{\cdot}, B^0_{\cdot}, \mathcal{G}^0\right) > x\}$ for $\epsilon > 0$, $\mathcal{E}_{t,0} = \{\omega^0 \in \Omega^0 : \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right) > x\}$, and observe that

$$\begin{aligned} E(x,T) &= \int_0^T \left|\mathbb{P}^0\left(\mathcal{E}_{t,\,\epsilon}\right) - \mathbb{P}^0\left(\mathcal{E}_{t,0}\right)\right| dt, \\ &= \int_0^T \left|\mathbb{P}^0\left(\mathcal{E}_{t,\,\epsilon} \cap \mathcal{E}^c_{t,0}\right) - \mathbb{P}^0\left(\mathcal{E}_{t,0} \cap \mathcal{E}^c_{t,\,\epsilon}\right)\right| dt, \\ &\le \int_0^T \mathbb{P}^0\left(\mathcal{E}_{t,\,\epsilon} \cap \mathcal{E}^c_{t,0}\right) dt + \int_0^T \mathbb{P}^0\left(\mathcal{E}_{t,0} \cap \mathcal{E}^c_{t,\,\epsilon}\right) dt, \end{aligned}$$

Next, for any $\eta > 0$ we have

$$\begin{aligned} &\mathbb{P}^0\left(\mathcal{E}_{t,\,\epsilon} \cap \mathcal{E}^c_{t,0}\right) \\ &\quad = \mathbb{P}^0\left(\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_{\cdot}, B^0_{\cdot}, \mathcal{G}^0\right) > x > \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right)\right) \\ &\quad = \mathbb{P}^0\left(\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_{\cdot}, B^0_{\cdot}, \mathcal{G}^0\right) > x > x - \eta > \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right)\right) \\ &\qquad +\mathbb{P}^0\left(\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_{\cdot}, B^0_{\cdot}, \mathcal{G}^0\right) > x > \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right) > x - \eta\right) \\ &\quad \le \mathbb{P}^0\left(\left|\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_{\cdot}, B^0_{\cdot}, \mathcal{G}^0\right) - \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right)\right| > \eta\right) \\ &\qquad +\mathbb{P}^0\left(x > \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right) > x - \eta\right) \\ &\quad \le \frac{1}{\eta^2}\mathbb{E}\left[\left(\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_{\cdot}, B^0_{\cdot}, \mathcal{G}^0\right) - \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right)\right)^2\right] \\ &\qquad +\mathbb{P}^0\left(x > \mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_{\cdot}, \mathcal{G}^0\right) > x - \eta\right) \end{aligned} \tag{3.3.34}$$

and if we denote by $\mathcal{S}$ the σ-algebra generated by the volatility paths, since $X_t^{1,*}$ is independent of $\mathcal{S}$ and the path of $B_\cdot^0$, by Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathbb{P}\left(X_t^{1,\epsilon}>0\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}^0\right)-\mathbb{P}\left(X_t^{1,*}>0\,|\,W_\cdot^0,\,\mathcal{G}^0\right)\right)^2\right]\\
&\quad=\mathbb{E}\left[\left(\mathbb{E}\left[\mathbb{P}\left(X_t^{1,\epsilon}>0\,|\,W_\cdot^0,\,C_1',\,\mathcal{S},\,\mathcal{G}^0\right)\right.\right.\right.\\
&\qquad\left.\left.\left.-\;\mathbb{P}\left(X_t^{1,*}>0\,|\,W_\cdot^0,\,C_1',\,\mathcal{G}^0\right)\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}^0\right]\right)^2\right]\\
&\quad\leq\mathbb{E}\left[\mathbb{E}\left(\left[\mathbb{P}\left(X_t^{1,\epsilon}>0\,|\,W_\cdot^0,\,C_1',\,\mathcal{S},\,\mathcal{G}^0\right)\right.\right.\right.\\
&\qquad\left.\left.\left.-\mathbb{P}\left(X_t^{1,*}>0\,|\,W_\cdot^0,\,C_1',\,\mathcal{G}^0\right)\right)^2\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}^0\right]\right]\\
&\quad=\mathbb{E}\left[\left(\mathbb{P}\left(X_t^{1,\epsilon}>0\,|\,W_\cdot^0,\,C_1',\,\mathcal{S},\,\mathcal{G}^0\right)-\mathbb{P}\left(X_t^{1,*}>0\,|\,W_\cdot^0,\,C_1',\,\mathcal{G}^0\right)\right)^2\right]\\
&\quad=\left\|v^{0,\epsilon}(t,0)-v^0(t,0)\right\|_{L^2(\Omega)}^2.
\end{aligned}\tag{3.3.35}$$

We assume now that $\mathbb{P}\left(X_t^{1,*}>0\,|\,W_\cdot^0,\,\mathcal{G}^0\right)$ has a bounded density near x. This is something we are not going to prove here, but we expect it to hold for almost all (if not all) x. Then we have that

$$\mathbb{P}^0\left(x>\mathbb{P}\left(X_t^{1,*}>0\,|\,W_\cdot^0,\,\mathcal{G}^0\right)>x-\eta\right)=\mathcal{O}(\eta).\tag{3.3.36}$$

Thus, we can plug (3.3.35) and (3.3.36) in (3.3.34) to obtain

$$\mathbb{P}^0\left(\mathcal{E}_{t,\epsilon}\cap\mathcal{E}_{t,0}^c\right)\leq\left\|v^{0,\epsilon}(t,0)-v^0(t,0)\right\|_{L^2(\Omega)}^2+\mathcal{O}(\eta)\tag{3.3.37}$$

for any $\eta>0$, and in a similar way we can obtain

$$\mathbb{P}^0\left(\mathcal{E}_{t,0}\cap\mathcal{E}_{t,\epsilon}^c\right)\leq\left\|v^{0,\epsilon}(t,0)-v^0(t,0)\right\|_{L^2(\Omega)}^2+\mathcal{O}(\eta)\tag{3.3.38}$$

Finally, plugging the above two estimates in (3.3.34), taking $\eta=\epsilon^p$ for some $p>0$, and recalling Remark 3.3.5, we find that

$$E(x,T)\leq\mathcal{O}(\epsilon^p)+\mathcal{O}\left(\epsilon^{1-2p}\right)\tag{3.3.39}$$

which becomes optimal as $\epsilon\longrightarrow 0^+$ when $1-2p=p\Leftrightarrow p=\frac{1}{3}$. This gives $E(x,T)=\mathcal{O}\left(\sqrt[3]{\epsilon}\right)$ as $\epsilon\longrightarrow 0^+$.

Appendix A

Proofs of standard and technical results

Proof of Lemma 2.3.4. Let $\{F_n : n \in \mathbb{N}\}$ be a sequence of regular enough (in terms of Malliavin differentiability) random variables which are adapted to the Brownian motion $B_\cdot$ in $[0, T]$, with $F_n \longrightarrow F$ in $L^{\alpha\lambda\vee p'}(\Omega) \cap \mathbb{D}^{1,2\lambda'\vee\frac{pr'}{pr'-1}}(\Omega) \cap \mathbb{D}^{2,pr}(\Omega)$. Then, $\frac{D_\cdot F_n}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon}$ belongs to the domain of the standard Skorokhod integral δ for any $\epsilon > 0$, and for any $m \leq \alpha$, by a well-known property of δ (see property (4) on page 40 in [37]) we have the following relation

$$\begin{aligned}
&|F|^m \delta\left(\frac{D_\cdot F_n}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon}\right) \\
&\quad= \frac{|F|^m \delta\left(D_\cdot F_n\right)}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon} + |F|^m \int_0^T D_s F_n D_s\left(\frac{1}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon}\right) ds \\
&\quad= -\frac{|F|^m L F_n}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon} + |F|^m \int_0^T D_s F_n \frac{-\int_0^T 2 D_{s'} F_n \cdot D^2_{s',s} F_n ds'}{\left(\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon\right)^2} ds \\
&\quad= -\frac{|F|^m L F_n}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon} - 2|F|^m \frac{\int_0^T \int_0^T D_s F_n \cdot D_{s'} F_n \cdot D^2_{s',s} F_n ds' ds}{\left(\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon\right)^2}
\end{aligned} \tag{A.0.1}$$

Thus, by the triangle inequality, a boundedness property of the operator $L = \delta(D_\cdot(\cdot))$ (see Proposition 1.5.4 on page 69 in [37]) and Holder's inequality, we have that

$$\begin{aligned}
&\left\| |F|^m \delta\left(\frac{D_\cdot F_n}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon}\right)\right\|_{L^p(\Omega)} \\
&\quad\leq \mathbb{E}^{\frac{1}{p}}\left[\left|\frac{|F|^m L F_n}{\|D_\cdot F_n\|^2_{L^2([0,T])}+\epsilon}\right|^p\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad +2\mathbb{E}^{\frac{1}{p}}\left[\left||F|^m\frac{\int_0^T\int_0^T D_sF_n\cdot D_{s'}F_n\cdot D^2_{s',s}F_nds'ds}{\left(\|D.F_n\|^2_{L^2([0,T])}+\epsilon\right)^2}\right|^p\right]\\
&\leq \mathbb{E}^{\frac{1}{pr}}\left[|LF_n|^{pr}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\quad +2\mathbb{E}^{\frac{1}{p}}\left[\left||F|^m\frac{\int_0^T\int_0^T D_sF_n\cdot D_{s'}F_n\cdot D^2_{s',s}F_nds'ds}{\left(\|D.F_n\|^2_{L^2([0,T])}+\epsilon\right)^2}\right|^p\right]\\
&\leq C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^T\int_0^T\left|D^2_{s',s}F_n\right|^2ds'ds\right)^{\frac{pr}{2}}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\quad +C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^T\left|D_{s'}F_n\right|^2ds'\right)^{\frac{pr}{2}}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\quad +2\mathbb{E}^{\frac{1}{p}}\left[|F|^m\left(\frac{\|D.F_n\|^2_{L^2([0,T])}\left(\int_0^T\int_0^T\left(D^2_{s',s}F_n\right)^2ds'ds\right)^{\frac{1}{2}}}{\left(\|D.F_n\|^2_{L^2([0,T])}+\epsilon\right)^2}\right)^p\right]\\
&\leq C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^T\int_0^T\left|D^2_{s',s}F_n\right|^2ds'ds\right)^{\frac{pr}{2}}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\quad +C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^T\left|D_{s'}F_n\right|^2ds'\right)^{\frac{pr}{2}}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\quad +2\mathbb{E}^{\frac{1}{p}}\left[|F|^m\left(\frac{\left(\int_0^T\int_0^T\left(D^2_{s',s}F_n\right)^2ds'ds\right)^{\frac{1}{2}}}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right)^p\right]\\
&\leq (C_{pr}+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^T\int_0^T\left|D^2_{s',s}F_n\right|^2ds'ds\right)^{\frac{pr}{2}}\right]\\
&\qquad \times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\quad +C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\left(\int_0^T\left|D_{s'}F_n\right|^2ds'\right)^{\frac{pr}{2}}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&= (C_{pr}+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}F_n\right\|^{pr}_{L^2\left([0,T]^2\right)}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_n\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]
\end{aligned}
$$

$$+C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\left\|D.F_n\right\|^{pr}_{L^2([0,T])}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\left\|D.F_n\right\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right] \tag{A.0.2}$$

Then, for a fixed $\epsilon>0$, we can use the Lipschitz continuity of $\frac{1}{\epsilon+x^2}$, Holder's inequality and our assumptions, to show that the last converges as $n\longrightarrow+\infty$ to the finite quantity

$$\begin{aligned}(C_{pr}+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}F\right\|^{pr}_{L^2\left([0,T]^2\right)}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\left\|D.F\right\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\ +C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\left\|D.F\right\|^{pr}_{L^2([0,T])}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\left\|D.F\right\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right],\end{aligned} \tag{A.0.3}$$

which implies that for a subsequence $\{\epsilon_{k_n}:n\in\mathbb{N}\}$ we have also

$$|F|^m\delta\left(\frac{D.F_{k_n}}{\left\|D.F_{k_n}\right\|^2_{L^2([0,T])}+\epsilon}\right)\longrightarrow\delta^m_{F,\epsilon} \tag{A.0.4}$$

weakly in $L^p(\Omega)$ as $n\longrightarrow+\infty$, for some $\delta^m_{F,\epsilon}$. Moreover, when $m=0$, for any $F'\in L^{p'}(\Omega)\cap\mathbb{D}^{1,2\vee\frac{pr'}{pr'-1}}(\Omega)$ we have

$$\begin{aligned}\mathbb{E}\left[F'\delta^0_{F,\epsilon}\right] &= \lim_{n\to+\infty}\mathbb{E}\left[F'\delta\left(\frac{D.F_{k_n}}{\left\|D.F_{k_n}\right\|^2_{L^2([0,T])}+\epsilon}\right)\right]\\ &= \lim_{n\to+\infty}\mathbb{E}\left[\int_0^T D_sF'\frac{D_sF_{k_n}}{\left\|D.F_{k_n}\right\|^2_{L^2([0,T])}+\epsilon}ds\right]\\ &= \mathbb{E}\left[\int_0^T D_sF'\frac{D_sF}{\left\|D.F\right\|^2_{L^2([0,T])}+\epsilon}ds\right]\end{aligned} \tag{A.0.5}$$

where the last convergence holds because

$$\begin{aligned}&\left|\mathbb{E}\left[\int_0^T D_sF'\left(\frac{D_sF_{k_n}}{\left\|D.F_{k_n}\right\|^2_{L^2([0,T])}+\epsilon}-\frac{D_sF}{\left\|D.F\right\|^2_{L^2([0,T])}+\epsilon}\right)ds\right]\right|\\ &\quad\leq\mathbb{E}\left[\left(\frac{1}{\left\|D.F_{k_n}\right\|^2_{L^2([0,T])}+\epsilon}\right)\int_0^T|D_sF'|\,|D_sF-D_sF_{k_n}|\,ds\right]\\ &\qquad+\mathbb{E}\left[\left|\frac{1}{\left\|D.F_{k_n}\right\|^2_{L^2([0,T])}+\epsilon}-\frac{1}{\left\|D.F\right\|^2_{L^2([0,T])}+\epsilon}\right|\int_0^T D_sF'D_sFds\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\epsilon}\mathbb{E}^{1-\frac{1}{pr'}}\left[\|D.F'\|_{L^2([0,T])}^{\frac{pr'}{pr'-1}}\right]\mathbb{E}^{\frac{1}{pr'}}\left[\|D_sF-D_sF_{k_n}\|_{L^2([0,T])}^{pr'}\right]\\
&\quad+\mathbb{E}\left[\frac{\left|\|D.F_{k_n}\|_{L^2([0,T])}^2-\|D.F\|_{L^2([0,T])}^2\right|\|D.F'\|_{L^2([0,T])}\|D.F\|_{L^2([0,T])}}{\left(\|D.F_{k_n}\|_{L^2([0,T])}^2+\epsilon\right)\left(\|D.F\|_{L^2([0,T])}^2+\epsilon\right)}\right]\\
&\leq \frac{1}{\epsilon}\mathbb{E}^{1-\frac{1}{pr'}}\left[\|D.F'\|_{L^2([0,T])}^{\frac{pr'}{pr'-1}}\right]\mathbb{E}^{\frac{1}{pr'}}\left[\|D_sF-D_sF_{k_n}\|_{L^2([0,T])}^{pr'}\right]\\
&\quad+C_\epsilon\mathbb{E}\left[\left|\|D.F_{k_n}\|_{L^2([0,T])}-\|D.F\|_{L^2([0,T])}\right|\|D.F'\|_{L^2([0,T])}\right]\\
&\leq \left(\frac{1}{\epsilon}+C_\epsilon\right)\mathbb{E}^{1-\frac{1}{pr'}}\left[\|D.F'\|_{L^2([0,T])}^{\frac{pr'}{pr'-1}}\right]\mathbb{E}^{\frac{1}{pr'}}\left[\|D_sF-D_sF_{k_n}\|_{L^2([0,T])}^{pr'}\right]
\end{aligned}
\tag{A.0.6}
$$

for some $C_\epsilon>0$, which obviously tends to zero. Furthermore, by a density argument, we can easily show that there is a unique weak limit $\delta^0_{F,\epsilon}$ and, since the subsequence $\{\epsilon_{k_n}:n\in\mathbb{N}\}$ can be taken to be the same for both $m=0$ and $m=\alpha$, we can also show that $\delta^\alpha_{F,\epsilon}=|F|^\alpha\delta^0_{F,\epsilon}$. Hence, we can define $\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}+\epsilon}\right):=\delta^0_{F,\epsilon}$ and then have $\delta^\alpha_{F,\epsilon}=|F|^\alpha\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}+\epsilon}\right)$ as well. Thus, we have also

$$
\begin{aligned}
&\left\|\,|F|^m\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}+\epsilon}\right)\right\|_{L^p(\Omega)}\\
&\quad\leq\liminf_{n\to+\infty}\left\|\,|F|^m\delta\left(\frac{D.F_{k_n}}{\|D.F_{k_n}\|^2_{L^2([0,T])}+\epsilon}\right)\right\|_{L^p(\Omega)}\\
&\quad\leq(C_{pr}+2)\liminf_{n\to+\infty}\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}F_{k_n}\right\|^{pr}_{L^2\left([0,T]^2\right)}\right]\\
&\qquad\qquad\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_{k_n}\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\qquad+C_{pr}\liminf_{n\to+\infty}\mathbb{E}^{\frac{1}{pr}}\left[\|D.F_{k_n}\|^{pr}_{L^2([0,T])}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F_{k_n}\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\quad=(C_{pr}+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}F\right\|^{pr}_{L^2\left([0,T]^2\right)}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]\\
&\qquad+C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\|D.F\|^{pr}_{L^2([0,T])}\right]\times\mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F\|^2_{L^2([0,T])}+\epsilon}\right|^{pr'}\right]
\end{aligned}
\tag{A.0.7}
$$

for both $m = 0$ and $m = \alpha$, which means that we can take $\epsilon \downarrow 0$ and repeat the previous argument (where this time, we use the Monotone Convergence Theorem to compute the limits) to deduce that $\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)$ can be defined such that

$$\mathbb{E}\left[F'\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right] = \mathbb{E}\left[\int_0^T D_t F' \frac{D.F}{\|D.F\|^2_{L^2([0,T])}} dt\right] \tag{A.0.8}$$

for any $F' \in L^{p'}(\Omega) \cap \mathbb{D}^{1,2\vee\frac{pr'}{pr'-1}}(\Omega)$ and

$$\begin{aligned}
&\left\| |F|^m \delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right\|_{L^p(\Omega)} \\
&\quad \leq (C_{pr}+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}F\right\|^{pr}_{L^2\left([0,T]^2\right)}\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F\|^2_{L^2([0,T])}}\right|^{pr'}\right] \\
&\qquad + C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\|D.F\|^{pr}_{L^2([0,T])}\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^m}{\|D.F\|^2_{L^2([0,T])}}\right|^{pr'}\right].
\end{aligned} \tag{A.0.9}$$

for both $m = 0$ and $m = \alpha$. The finiteness of the RHS in both (A.0.8) and (A.0.9), which allows us to obtain these relations by using the Monotone Convergence Theorem, follows easily by a few simple inequalities and our assumptions.

Taking now $\psi(y) = \mathbb{I}_{[a,\,b]}(y)$ for some $a, b \in \mathbb{R}$ with $a < b$ and $\phi(y) = \int_{-\infty}^{y} \psi(z)dz$, we can easily show that $\mathbb{P}$- almost surely we have $|\phi(F)| \leq b - a$, and also $D.\phi(F) = \psi(F)D.F$ by the comment after the proof of Proposition 1.2.3 on page 31 in [37] (since by our assumptions $D.F$ can never be identically zero, so we can use Theorem 2.1.2 from page 86 in the same book to obtain absolute continuity). Thus, by the boundedness of $\psi(F)$ and our assumptions we have $\phi(F) \in L^\infty(\Omega) \cap \mathbb{D}^{1,2\lambda'\vee\frac{pr'}{pr'-1}}(\Omega)$, which is a subspace of $L^{p'}(\Omega) \cap \mathbb{D}^{1,2\vee\frac{pr'}{pr'-1}}(\Omega)$. Then, we can work as in the proof of Proposition 2.1.1 on page 78 in [37] to deduce that

$$\mathbb{E}\left[\psi(F)\right] = \mathbb{E}\left[\phi(F)\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right], \tag{A.0.10}$$

where now δ is the adjoint of the derivative operator $D : L^{p'}(\Omega) \cap \mathbb{D}^{1,2\vee\frac{pr'}{pr'-1}}(\Omega) \longrightarrow L^{2\vee\frac{pr'}{pr'-1}}(\Omega; L^2([0,\,T]))$, the domain of which contains the process $\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}$ as we

have shown above. Since Holder's inequality implies that

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{I}_{F>x}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right|\right] &\leq \|\mathbb{I}_{F>x}\|_{L^{p'}(\Omega)}\left\|\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right\|_{L^p(\Omega)} \\ &\leq \left\|\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right\|_{L^p(\Omega)} \qquad \text{(A.0.11)}\end{aligned}$$

which is finite, applying Fubini's Theorem on (A.0.10) we find that

$$\mathbb{P}\left(a \leq F \leq b\right) = \int_a^b \mathbb{E}\left[\mathbb{I}_{F>x}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right]dx. \qquad \text{(A.0.12)}$$

This implies that F has a density f_F given by

$$f_F(x) = \mathbb{E}\left[\mathbb{I}_{F>x}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right]$$

for all $x \in \mathbb{R}$, and by using the boundedness of the indicator function and the Dominated Convergence Theorem, we can show that this density is continuous.

Finally, by recalling that the Skorokhod integral has always a zero expectation (this holds also for its extension by a density argument), for $x \leq 0$ we have

$$\begin{aligned}&|x|^\alpha f_F(x) \\ &\quad= \mathbb{E}\left[|x|^\alpha\mathbb{I}_{F>x}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right] \\ &\quad= -\mathbb{E}\left[|x|^\alpha\mathbb{I}_{F<x}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right] \\ &\quad= \mathbb{E}\left[|x|^\alpha\mathbb{I}_{F<-|x|}\left(-\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right)\right] \\ &\quad\leq \mathbb{E}\left[|x|^\alpha\mathbb{I}_{|F|>|x|}\left|\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right|\right]\end{aligned} \qquad \text{(A.0.13)}$$

and for $x \geq 0$ obviously

$$\begin{aligned}&|x|^\alpha f_F(x) \\ &\quad= \mathbb{E}\left[|x|^\alpha\mathbb{I}_{F>x}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right] \\ &\quad= \mathbb{E}\left[|x|^\alpha\mathbb{I}_{F>|x|}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right]\end{aligned}$$

$$\leq \mathbb{E}\left[|x|^{\alpha}\mathbb{I}_{|F|>|x|}\left|\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right|\right] \tag{A.0.14}$$

so by (A.0.13), (A.0.14) and (A.0.9) we obtain the estimate

$$\begin{aligned}
&|x|^{\alpha} f_F(x) \\
&\quad \leq \mathbb{E}\left[|F|^{\alpha}\mathbb{I}_{|F|>|x|}\left|\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right|\right] \\
&\quad \leq \mathbb{E}\left[\left||F|^{\alpha}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right|\right] \\
&\quad \leq \left\||F|^{\alpha}\delta\left(\frac{D.F}{\|D.F\|^2_{L^2([0,T])}}\right)\right\|_{L^p(\Omega)} \\
&\quad \leq (C_{pr}+2)\,\mathbb{E}^{\frac{1}{pr}}\left[\left\|D^2_{\cdot,\cdot}F\right\|^{pr}_{L^2\left([0,T]^2\right)}\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^{\alpha}}{\|D.F\|^2_{L^2([0,T])}}\right|^{pr'}\right] \\
&\qquad + C_{pr}\mathbb{E}^{\frac{1}{pr}}\left[\|D.F\|^{pr}_{L^2([0,T])}\right] \times \mathbb{E}^{\frac{1}{pr'}}\left[\left|\frac{|F|^{\alpha}}{\|D.F\|^2_{L^2([0,T])}}\right|^{pr'}\right]
\end{aligned}$$

for all $x \in \mathbb{R}$. This completes the proof of the Lemma. □

Proof of Lemma 2.3.5. Observe that in all cases, the function q satisfies the growth condition $q^2(z) \leq C_q^2|z|^r$ for some $r \in \{0,\, 1\}$ and some $C_q > 0$. Obviously, we only need to consider the case $p = 2n$ with $n \in \mathbb{N}$. By Ito's formula we have

$$\begin{aligned}
\sigma_t^{2n} &= \sigma_0^{2n} + \int_0^t n\left(\sigma_s^{2n-1}2k\left(\theta-\sigma_s\right) + \xi^2\left(2n-1\right)\sigma_s^{2n-2}q^2\left(\sigma_s\right)\right)ds \\
&\qquad + 2n\xi\int_0^t \sigma_s^{2n-1}q\left(\sigma_s\right)d\left(\sqrt{1-\rho_2^2}B_s^1 + \rho_2 B_s^0\right) \\
&\leq \sigma_0^{2n} + C_n T + 2n\xi\int_0^t \sigma_s^{2n-1}q\left(\sigma_s\right)d\left(\sqrt{1-\rho_2^2}B_s^1 + \rho_2 B_s^0\right) \qquad \text{(A.0.15)}
\end{aligned}$$

for some $C_n > 0$ and any $t \leq T$, since we can use the at most linear growth of q^2 to show that the quantity within the Riemannian integral can be bounded from above by $P\left(|\sigma_s|\right)$, where P is a polynomial of an even degree and a negative leading coefficient. Taking expectations on (A.0.15), we find that for any $t \leq T$, the $2n$-th moment of σ_t is bounded by $C_n T + \mathbb{E}\left[\sigma_0^{2n}\right]$. Moreover, by using this result and the inequality $2a \leq a^2 + 1$ for $a \geq 0$, we can prove that there is a uniform bound in $t \leq T$ for odd moments of σ_t as well. Thus, if we set $M_T = \sup_{t\leq T}\sigma_t$, taking supremum on (A.0.15)

for $t \leq T$, then taking expectations and finally using Cauchy-Schwartz and Doob's inequalities, we obtain:

$$\begin{aligned}
\mathbb{E}\left[M_T^{2n}\right] &\leq \mathbb{E}\left[\sigma_0^{2n}\right] + C_n T \\
&\quad + 2n\xi\mathbb{E}^{\frac{1}{2}}\left[\sup_{0\leq t\leq T}\left(\int_0^t \sigma_s^{2n-1} q\left(\sigma_s\right) d\left(\sqrt{1-\rho_2^2}B_s^1 + \rho_2 B_s^0\right)\right)^2\right] \\
&\leq \mathbb{E}\left[\sigma_0^{2n}\right] + C_n T + 4n\xi\mathbb{E}^{\frac{1}{2}}\left[\int_0^T \sigma_s^{4n-2} q^2\left(\sigma_s\right) ds\right] \\
&\leq \mathbb{E}\left[\sigma_0^{2n}\right] + C_n T + 4n\xi C_q\mathbb{E}^{\frac{1}{2}}\left[\int_0^T \sigma_s^{4n-2}\left|\sigma_s\right|^r ds\right] \\
&= \mathbb{E}\left[\sigma_0^{2n}\right] + C_n T + 4n\xi C_q\sqrt{\int_0^T \mathbb{E}\left[\left|\sigma_s\right|^{4n-2+r}\right] ds} \\
&< \infty
\end{aligned}$$

and this completes the proof of the Lemma. □

Proof of Lemma 2.3.6. First we set $v_t = \sqrt{\sigma_t}$ and, as our assumptions ensure that $\sigma_\cdot$ does not hit 0, we can apply Ito's formula on equation (2.3.1) to obtain

$$dv_t = \left[\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right)\frac{1}{v_t} - \frac{k}{2}v_t\right] dt + \frac{\xi}{2} dB_t, \tag{A.0.16}$$

where $B_t := \sqrt{1-\rho_2^2}B_t^1 + \rho_2 B_t^0$ is a standard Brownian Motion. Since the CIR process is an L^1-integrable process (this follows from Lemma 2.3.5), v_t is an L^2-integrable process. Consider now for any $\epsilon > 0$, a twice continuously differentiable and increasing cut-off function $\Phi^\epsilon(x)$ satisfying

$$\Phi^\epsilon(x) = \begin{cases} 1 & \text{if } x \geq 2\epsilon, \\ 0 & \text{if } x < \epsilon. \end{cases}$$

Then the derivative satisfies

$$\frac{\partial}{\partial x}\Phi^\epsilon(x) = \begin{cases} 0 & \text{if } x \geq 2\epsilon, \\ 0 & \text{if } x < \epsilon. \end{cases}$$

Moreover, we define: $J^\epsilon(x) = \frac{\Phi^\epsilon(x)}{x}$ for $x > 0$ and $J^\epsilon(0) = 0$, and we observe that this function is bounded and continuously differentiable with

$$\frac{\partial}{\partial x}J^\epsilon(x) = \begin{cases} -\frac{1}{x^2} & x \geq 2\epsilon, \\ 0 & x < \epsilon, \end{cases}$$

which is also bounded and non-positive for any $\epsilon > 0$.

Let $\{v_t^\epsilon : t \geq 0\}$ be the unique solution to the SDE

$$dv_t^\epsilon = \left[\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right) J^\epsilon(v_t^\epsilon) - \frac{k}{2} v_t^\epsilon\right] dt + \frac{\xi}{2} dB_t \tag{A.0.17}$$

for an arbitrary $\epsilon > 0$, where $B_\cdot$ is the same Brownian motion as in (A.0.16). For any $\epsilon > 0$, Theorem 2.2.1 from page 102 of [37] implies that v_t^ϵ is Malliavin differentiable with respect to the Brownian motion $B_\cdot^1$, and by the remark after the proof of that Theorem we have that the Malliavin derivative of v_t^ϵ satisfies the integral equation

$$D_{t'} v_t^\epsilon = \frac{\xi\sqrt{1-\rho_2^2}}{2} + \int_{t'}^t \left[\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right) \frac{\partial}{\partial x} J^\epsilon(v_s^\epsilon) - \frac{k}{2}\right] D_{t'} v_s^\epsilon ds, \ \forall\, t \geq t' \geq 0,$$

This can be solved in t to give

$$D_{t'} v_t^\epsilon = \frac{\xi\sqrt{1-\rho_2^2}}{2} e^{\int_{t'}^t \left[\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right) \frac{\partial}{\partial x} J^\epsilon(v_s^\epsilon) - \frac{k}{2}\right] ds}, \ \forall\, t \geq t' \geq 0. \tag{A.0.18}$$

As mentioned in [2], the stopping time $\tau_\epsilon = \inf\{t > 0 : v_t \leq \epsilon\}$ tends to ∞ as $\epsilon \longrightarrow 0$ and we also have $v_t^\epsilon = v_t^{\tau_{2\epsilon}} = v_t \geq 2\epsilon$, $\forall\, t \leq \tau_{2\epsilon}$, $\mathbb{P}^0 \times \mathbb{P}^1$-almost surely (this can be seen by observing that when we stop (A.0.16) at $\tau_{2\epsilon}$, we can substitute the $\frac{1}{v_t^{\tau_{2\epsilon}}}$ term by the equal $J^\epsilon\left(v_t^{\tau_{2\epsilon}}\right)$ and obtain exactly (A.0.17) stopped at $\tau_{2\epsilon}$ but for v_t instead of v_t^ϵ, so since this stopped SDE has a pathwise unique solution, v_t and v_t^ϵ must coincide up to time $\tau_{2\epsilon}$). It follows then that $v_t^\epsilon \longrightarrow v_t$ and also $\int_{t'}^t \frac{\partial}{\partial x} J^\epsilon(v_s^\epsilon) ds \longrightarrow -\int_{t'}^t \frac{1}{v_s^2} ds$ for all $t \geq t' \geq 0$ as $\epsilon \longrightarrow 0^+$, $\mathbb{P}^0 \times \mathbb{P}^1$-almost surely. Hence we have

$$\mathbb{E}\left[\mathbb{P}\left(v_t^\epsilon \xrightarrow{\epsilon \longrightarrow 0^+} v_t, \ \forall\, t \geq 0 \,|\, B_\cdot^0, \mathcal{G}^0\right)\right] = \mathbb{P}\left(v_t^\epsilon \xrightarrow{\epsilon \longrightarrow 0^+} v_t, \ \forall\, t \geq 0\right) = 1,$$

which implies that

$$\mathbb{P}\left(v_t^\epsilon \xrightarrow{\epsilon \longrightarrow 0^+} v_t, \ \forall\, t \geq 0 \,|\, B_\cdot^0, \mathcal{G}^0\right) = 1 \tag{A.0.19}$$

$\mathbb{P}^0$-almost surely. Similarly, we can deduce that

$$\mathbb{P}\left(\int_{t'}^t \frac{\partial}{\partial x} J^\epsilon(v_s^\epsilon) ds \xrightarrow{\epsilon \longrightarrow 0^+} -\int_{t'}^t \frac{1}{v_s^2} ds, \ \forall\, t \geq 0 \,|\, B_\cdot^0, \mathcal{G}^0\right) = 1 \tag{A.0.20}$$

$\mathbb{P}^0$-almost surely.

Furthermore, it is shown in [2] that

$$\mathbb{E}\left[\mathbb{P}\left(|v_t^\epsilon| \leq |u_t| + |v_t|, \ \forall\, t \geq 0 \,|\, B_\cdot^0, \mathcal{G}^0\right)\right] = \mathbb{P}\left(|v_t^\epsilon| \leq |v_t| + |u_t|, \ \forall\, t \geq 0\right) = 1,$$

where u_t is an Ornstein - Uhlenbeck process, while we also have

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \mathbb{E}\left[(|v_t| + |u_t|)^2 \,|\, B_\cdot^0, \mathcal{G}^0\right]\right] \ \leq \ \mathbb{E}\left[\sup_{0 \leq t \leq T} (|v_t| + |u_t|)^2\right]$$

$$\leq \quad 2(||\sup_{0\leq t\leq T} v_t||^2_{L^2(\Omega^0\times\Omega^1)} + ||\sup_{0\leq t\leq T} u_t||^2_{L^2(\Omega^0\times\Omega^1)}),$$

which is finite by Lemma 2.3.5. This means that $\mathbb{P}^0$-almost surely we have also

$$\mathbb{P}\left(|v^\epsilon_t| \leq |u_t| + |v_t| \,\forall\, t \geq 0 \,|\, B^0_\cdot,\, \mathcal{G}^0\right) = 1 \tag{A.0.21}$$

and

$$\mathbb{E}\left[(|v_t| + |u_t|)^2 |\, B^0_\cdot,\, \mathcal{G}^0\right] < \infty \tag{A.0.22}$$

for all $0 \leq t \leq T$. By (A.0.19), (A.0.20), (A.0.21) and (A.0.22), we have that $\mathbb{P}^0$-almost surely and for all $0 \leq t' \leq t \leq T$, both v^ϵ_t and $D_{t'}v^\epsilon_t$ converge $\mathbb{P}(\cdot \,|\, B^0_\cdot,\, \mathcal{G}^0)$-almost surely to v_t and

$$V_{t,t'} = \frac{\xi\sqrt{1-\rho_2^2}}{2} e^{-\int_{t'}^t \left[\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right)\frac{1}{v_s^2} + \frac{k}{2}\right] ds}$$

respectively as $\epsilon \longrightarrow 0$, while $\{v^\epsilon_t : t \geq 0\}$ is dominated by an $L^2_{B^0_\cdot,\mathcal{G}^0}$-integrable process and $D_{t'}v^\epsilon_t \leq \frac{\xi\sqrt{1-\rho_2^2}}{2}$ for all $\epsilon > 0$ and all $0 \leq t' \leq t \leq T$. Thus, we can apply the Dominated Convergence Theorem to deduce that the last two convergences hold also in $L^2_{B^0_\cdot,\mathcal{G}^0}$ and $L^2_{B^0_\cdot,\mathcal{G}^0}([0,\, t] \times \Omega^1)$ respectively, for all $0 \leq t \leq T$ and $\mathbb{P}^0$-almost surely. Then, by Lemma 1.2.3 from [37] (page 30) we obtain that $D_{t'}v_t$ exists and is equal to $V_{t,t'}$, i.e

$$D_{t'}v_t = \frac{\xi\sqrt{1-\rho_2^2}}{2} e^{-\int_{t'}^t \left[\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right)\frac{1}{\sigma_s^1} - \frac{k}{2}\right] ds} \tag{A.0.23}$$

for all $0 \leq t' \leq t \leq T$ and $\mathbb{P}^0$-almost surely.

Finally, for any $n \in \mathbb{N}$, let f_n be a smooth and compactly supported function such that $f_n(x) = x$ for all $x \leq n$ and $\left\|\frac{\partial}{\partial x} f_n\right\|_\infty = 1$. Then, by Lemma 2.3.5, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \mathbb{E}\left[\sigma_t^2 \,|\, B^0_\cdot,\, \mathcal{G}^0\right]\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} \sigma_t^2\right] < \infty,$$

which implies that $\mathbb{P}^0$-almost surely, $\sigma_t \in L^2_{B^0_\cdot,\mathcal{G}^0}$ for all $0 \leq t \leq T$. Thus, $\mathbb{P}^0$-almost surely and for all $0 \leq t \leq T$, the Dominated Convergence Theorem implies that $f_n^2(v_t) \longrightarrow \sigma_t$ in $L^2_{B^0_\cdot,\mathcal{G}^0}$ as $n \longrightarrow \infty$ (since we obviously have $\mathbb{P}(\cdot \,|\, B^0_\cdot,\, \mathcal{G}^0)$-almost sure convergence and domination by $v_t^2 = \sigma_t$). Moreover, $\mathbb{P}^0$-almost surely and for all $0 \leq t' \leq t \leq T$, the standard Malliavin chain rule implies that

$$D_{t'}f_n^2(v_t) = 2f_n(v_t)f_n'(v_t)D_{t'}v_t \longrightarrow 2v_tD_{t'}v_t$$

$\mathbb{P}(\cdot \,|\, B^0_\cdot,\, \mathcal{G}^0)$ - almost surely as $n \longrightarrow \infty$, while we have also domination by $2v_tD_{t'}v_t \leq \xi\rho_2 v_t \in L^2_{B^0_\cdot,\mathcal{G}^0}$. Thus, we can use the Dominated Convergence Theorem once more to

see that the last convergence holds also in $L^2_{B^0_\cdot,\mathcal{G}^0}([0,\,t]\times\Omega^1)$, $\mathbb{P}^0$-almost surely and for all $0\leq t\leq T$. Recalling now Lemma 1.2.3 from [37] again, we deduce that $D_{t'}\sigma_t$ exists in $L^2_{B^0_\cdot,\mathcal{G}^0}([0,\,t]\times\Omega^1)$ and it is equal to $2v_tD_{t'}v_t$, i.e

$$D_{t'}\sigma_t=\xi\sqrt{1-\rho_2^2}e^{-\int_{t'}^t\left[\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right)\frac{1}{\sigma_s^{\frac{1}{2}}}-\frac{k}{2}\right]ds}\sqrt{\sigma_t}$$

which is exactly (2.3.14). The higher integrability of this derivative is a consequence of the boundedness of the exponential term and Lemma 2.3.5. The proof is now complete. □

Proof of Lemma 2.3.7. Fix $t>0$. Consider the sequence of stochastic processes (in $t'\in[0,\,t]$)

$$v^n_{t,t'}=\xi\sqrt{1-\rho_2^2}e^{-\int_{t'}^t\left[\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right)\frac{1}{\sigma_s+\frac{1}{n}}+\frac{k}{2}\right]ds}g_n\left(\sqrt{\sigma_t+\frac{1}{n}}\right),\ \ \forall\, n\in\mathbb{N},$$

where the smooth and increasing cut-off function g_n satisfies

$$g_n(x)=\begin{cases}x, & 0\leq x\leq n,\\ -1, & x\leq -2,\\ n+1, & x\geq n+2,\end{cases}$$

and has a derivative which is bounded by 1. This process is uniformly bounded by $\xi\sqrt{1-\rho_2^2}\sqrt{\sigma_s+1}\in L^p\left(\Omega^0\times\Omega^1\right),\ \forall\, p\geq 1$, since

$$\sqrt{\sigma_s+1}\leq\frac{1}{2}\sup_{0\leq s\leq T}\left(\sigma_s+2\right),$$

which has finite moments by Lemma 2.3.5. Thus we have also $\xi\sqrt{1-\rho_2^2}\sqrt{\sigma_s+1}\in L^p_{B^0_\cdot,\mathcal{G}^0}\left(\Omega^1\right)$ for all $p>1$, $\mathbb{P}^0$-almost surely. Moreover, by the Monotone Convergence Theorem, $v^n_{t,t'}$ converges pointwise and $\mathbb{P}^0\times\mathbb{P}^1$-almost surely to $D_{t'}\sigma_t$ as $n\longrightarrow\infty$. Thus, by the Dominated Convergence Theorem we deduce that this convergence holds also in $L^p_{B^0_\cdot,\mathcal{G}^0}\left(\Omega^1\right)$ for any $t'<t$, and also in $L^p_{B^0_\cdot,\mathcal{G}^0}\left(\Omega^1;\,L^2\left([0,\,t]\right)\right)$, for any $p\geq 1$, $\mathbb{P}^0$-almost surely.

Observe now that

$$v^n_{t,t'}=f\left(\int_{t'}^t h_n\left(\sigma_s\right)ds\right)g_n\left(\sqrt{\sigma_t+\frac{1}{n}}\right),\tag{A.0.24}$$

where f, h_n are sufficiently smooth functions with bounded first derivatives, such that $f(x)=\sqrt{1-\rho_2^2}\xi e^{-x}$ and $h_n(x)=\frac{k}{2}+\left(\frac{k\theta}{2}-\frac{\xi^2}{8}\right)\frac{1}{x+\frac{1}{n}}$ for $x>0$ and $n\in\mathbb{N}$, and

$f(x) = h_n(x) = 0$ for $x < -1$ and $n \in \mathbb{N}$. Then, the standard Malliavin chain rule implies that $\mathbb{P}^0$- almost surely, under the probability measure $\mathbb{P}(\cdot \,|\, B^0_\cdot, \mathcal{G}^0)$, we have

$$D_{t''} h_n(\sigma_s) = h_n'(\sigma_s) D_{t''}\sigma_s,$$

which is bounded for any $n \in \mathbb{N}$, so we can integrate in s and intechange the integral with the derivative to obtain

$$D_{t''} \int_{t'}^t h_n(\sigma_s)\, ds = \int_{t'}^t h_n'(\sigma_s) D_{t''}\sigma_s ds. \tag{A.0.25}$$

Next, observe that all the arguments in (A.0.24) are positive, so by applying the same Malliavin chain rule and by substituting from (A.0.25), we obtain

$$\begin{aligned} D_{t''} v^n_{t,t'} &= f'\left(\int_{t'}^t h_n(\sigma_s)\, ds\right) \int_{t'}^t h_n'(\sigma_s) D_{t''}\sigma_s ds g_n\left(\sqrt{\sigma_t + \frac{1}{n}}\right) \\ &\quad + f\left(\int_{t'}^t h_n(\sigma_s)\, ds\right) g_n'\left(\sqrt{\sigma_t + \frac{1}{n}}\right) \frac{D_{t''}\sigma_t}{2\sqrt{\sigma_t + \frac{1}{n}}} \\ &= \sqrt{1-\rho_2^2}\xi e^{-\int_{t'}^t \left(\frac{k}{2} + \left(\frac{k\theta}{2} - \frac{\xi^2}{2}\right)\frac{1}{\sigma_s + \frac{1}{n}}\right) ds} \int_{t'}^t \frac{\left(\frac{k\theta}{2} - \frac{\xi^2}{8}\right) D_{t''}\sigma_s ds}{\left(\sigma_s + \frac{1}{n}\right)^2} g_n\left(\sqrt{\sigma_t + \frac{1}{n}}\right) \\ &\quad + \sqrt{1-\rho_2^2}\xi e^{-\int_{t'}^t \left(\frac{k}{2} + \left(\frac{k\theta}{2} - \frac{\xi^2}{2}\right)\frac{1}{\sigma_s + \frac{1}{n}}\right) ds} g_n'\left(\sqrt{\sigma_t + \frac{1}{n}}\right) \frac{D_{t''}\sigma_t}{2\sqrt{\sigma_t + \frac{1}{n}}}. \end{aligned} \tag{A.0.26}$$

Now we want to bound the above by some process in $L^r_{B^0_\cdot, \mathcal{G}^0}\left(\Omega^1; L^2\left[[0,t]^2\right]\right)$, uniformly in $n \in \mathbb{N}$, so we can apply again the Dominated Convergence Theorem, for some $r > 1$. Observe that $D_{t''}\sigma_s \leq \xi\sqrt{1-\rho_2^2}\sqrt{\sigma_s}$ (by (2.3.14)) and that $0 \leq \frac{d}{dx} g_n(x) \leq 1 \Rightarrow g_n(x) \leq x$ for all $n \in \mathbb{N}$, so if we drop the summand $-\int_{t'}^t \frac{k}{2}$ from the exponents in (A.0.26) we obtain

$$\begin{aligned} D_{t''} v^n_{t,t'} &\leq \xi\sqrt{1-\rho_2^2} e^{-\int_{t'}^t \left(\frac{k\theta}{2} - \frac{\xi^2}{2}\right)\frac{1}{\sigma_s + \frac{1}{n}} ds} \int_{t'}^t \frac{\sqrt{\sigma_s} ds}{\left(\sigma_s + \frac{1}{n}\right)^2} \sqrt{\sigma_t + \frac{1}{n}} \\ &\quad + \xi\sqrt{1-\rho_2^2} e^{-\int_{t'}^t \left(\frac{k\theta}{2} - \frac{\xi^2}{2}\right)\frac{1}{\sigma_s + \frac{1}{n}} ds} \frac{\sqrt{\sigma_t}}{2\sqrt{\sigma_t + \frac{1}{n}}} \\ &< \xi\sqrt{1-{\rho_2}^2}\left[\int_0^T \frac{ds}{\sigma_s^{\frac{3}{2}}} \sup_{0 \leq s \leq T} \sqrt{\sigma_s + 1} + 1\right] \end{aligned}$$

whose $L^r\left(\Omega^0\times\Omega^1;\, L^2\left[[0,\,t]^2\right]\right)$ norm is bounded by

$$\begin{aligned}
&\xi\sqrt{1-\rho_2^2}t\left(\mathbb{E}\left[\left(\int_0^T\frac{ds}{\sigma_s^{\frac{3}{2}}}\sup_{0<s\leq T}\sqrt{\sigma_s+1}\right)^r\right]+1\right)^{\frac{1}{r}}\\
&\qquad\leq\xi\sqrt{1-\rho_2^2}t\left(\mathbb{E}^{\frac{r}{p}}\left[\left(\int_0^T\frac{ds}{\sigma_s^{\frac{3}{2}}}\right)^p\right]\mathbb{E}^{\frac{r}{p'}}\left[\left(\sup_{0<s\leq T}\sqrt{\sigma_s+1}\right)^{p'}\right]+1\right)^{\frac{1}{r}}\\
&\qquad\leq\xi\sqrt{1-\rho_2^2}t\left(T^{\frac{r}{p'}}\mathbb{E}^{\frac{r}{p}}\left[\int_0^T\frac{ds}{\sigma_s^{\frac{3p}{2}}}\right]\mathbb{E}^{\frac{r}{p'}}\left[\left(\sup_{0<s\leq T}\sqrt{\sigma_s+1}\right)^{p'}\right]+1\right)^{\frac{1}{r}},
\end{aligned}\tag{A.0.27}$$

where $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}$.

The second expectation of (A.0.27) is finite for all $p'<\infty\Leftrightarrow p>r$ because of the estimate $(\sigma_s+1)^{\frac{p'}{2}}\leq C_1\left(\sigma_s^{p'}+1\right)$ for some $C_1>0$ and Lemma 2.3.5. On the other hand, if $\frac{2k\theta}{\xi^2}>\frac{3p}{2}$, the first expectation of (A.0.27) can be computed by recalling Theorem 3.1 from [26] as follows

$$\mathbb{E}\left[\int_0^T\frac{ds}{\sigma_s^{\frac{3p}{2}}}\right]=\mathbb{E}\left[\int_0^T\mathbb{E}\left[\sigma_s^{-\frac{3p}{2}}\mid\sigma_0\right]ds\right]=\lambda_1\mathbb{E}\left[\int_0^T\gamma_s^{\frac{3p}{2}}H\left(-\gamma_s\sigma_0e^{-ks}\right)ds\right]\tag{A.0.28}$$

where $\lambda_1>0$, $\gamma_s=\frac{2k}{\xi^2}\left(1-e^{-ks}\right)^{-1}>\frac{2k}{\xi^2}$ for all $s\geq 0$, and H is a hypergeometric function for which we have the asymptotic estimate of page 17 in [26]. That estimate (for $N=0$) easily gives $H(-z)\leq\lambda_2|z|^{-\frac{3p}{2}}$ for some $\lambda_2>0$ and all $z\geq 0$. Thus, by (A.0.28) we find

$$\mathbb{E}\left[\int_0^T\frac{ds}{\sigma_s^{\frac{3p}{2}}}\right]\leq\lambda_1\lambda_2\mathbb{E}\left[\int_0^Te^{\frac{3kps}{2}}\sigma_0^{-\frac{3p}{2}}\right]ds=\lambda_1\lambda_2\int_0^Te^{\frac{3kps}{s}}ds\mathbb{E}\left[\sigma_0^{-\frac{3p}{2}}\right]$$

which is finite by our initial data assumptions if and only if $\frac{2k\theta}{\xi^2}>\frac{3p}{2}$. Thus, the RHS of (A.0.27) is finite iff $\frac{2k\theta}{\xi^2}>\frac{3p}{2}$. This can be achieved by making p sufficiently close to r, provided that $\frac{2k\theta}{\xi^2}>\frac{3r}{2}$, which is equivalent to $r<\frac{4k\theta}{3\xi^2}$. We can choose such an $r>1$ since we have $\frac{4k\theta}{3\xi^2}>1$. Observe also that the same condition is assumed in [3] to obtain L^1 regularity, but for our purpose, we are going to need this L^r regularity for some r strictly bigger than 1. Moreover, we need to have a finite $L^r_{B^0_\cdot,\mathcal{G}^0}\left(\Omega^1;\, L^2\left[[0,\,t]^2\right]\right)$ norm, $\mathbb{P}^0$-almost surely, and this is obtained by the law of total expectation as follows

$$\mathbb{E}\left[\mathbb{E}\left[\left(\int_0^T\frac{ds}{\sigma_s^{\frac{3}{2}}}\sup_{0<s\leq T}\sqrt{\sigma_s+1}\right)^r+1\,|\,B^0_\cdot,\,\mathcal{G}^0\right]\right]$$

$$= \mathbb{E}\left[\left(\int_0^T \frac{ds}{\sigma_s^{\frac{3}{2}}} \sup_{0<s\leq T}\sqrt{\sigma_s+1}\right)^r + 1\right], \tag{A.0.29}$$

so we have

$$\mathbb{E}\left[\left(\int_0^T \frac{ds}{\sigma_s^{\frac{3}{2}}} \sup_{0<s\leq T}\sqrt{\sigma_s+1}\right)^r + 1 \,|\, B_\cdot^0,\, \mathcal{G}^0\right] < \infty$$

$\mathbb{P}^0$-almost surely. Thus, the pointwise convergence of $D_{t''}v_{t,t'}^n$ to the RHS of (2.3.15) and the Dominated Convergence Theorem imply that we have the same convergence in $L^r_{B_\cdot^0,G^0}\left(\Omega^1;\, L^2\left[[0,\, t]^2\right]\right)$, $\mathbb{P}^0$-almost surely. Then, since $v_{t,t'}^n$ converges to $D_{t'}\sigma_t$ in $L^r_{B_\cdot^0,\,\mathcal{G}^0}\left(\Omega^1\right)$ for any $t' < t$ and also in $L^r_{B_\cdot^0,\,\mathcal{G}^0}\left(\Omega^1;\, L^2\left([0,\, t]\right)\right)$, $\mathbb{P}^0$-almost surely, we deduce that $\sigma_t \in \mathbb{D}^{2,r}$ under the probability measure $\mathbb{P}(\cdot \,|\, B_\cdot^0,\, \mathcal{G}^0))$ with respect to $B_\cdot^1$, with the second Malliavin Derivative being given by (2.3.15), $\mathbb{P}^0 \times \mathbb{P}^1$-almost surely. It follows also that (2.3.16) holds, since the sequence converging pointwise to the RHS of (2.3.15) is dominated by a random quantity of finite positive moments, uniformly in $t,\, t',\, t'' \in [0,\, T]$. The proof for $v_t = \sqrt{\sigma_t}$ is similar, the only difference is the absence of the functions g_n and the terminal value term. □

Proof of Theorem 3.2.3. It suffices to show that the two-dimensional continuous Markov chain $(\sigma_\cdot^{1,1},\, \sigma_\cdot^{2,1})$ is positive recurrent. To do this, we set $H^i(x) = \int_0^x \frac{1}{v_i q(y)} dy$ which is a strictly increasing bijection from $\mathbb{R}$ to itself, and $Z_\cdot^i = H^i\left(\sigma_\cdot^{i,1}\right)$ for $i \in \{1,\, 2\}$. Then we need to show that $(Z_\cdot^1,\, Z_\cdot^2)$ is a positive recurrent diffusion. It is easy to verify now that the infinitesimal generator L_Z of $Z = (Z_\cdot^1,\, Z_\cdot^2)$, maps any smooth function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ to

$$L_Z F(x,\, y) = V^1(x)F_x(x,\, y) + V^2(y)F_y(x,\, y) + \frac{1}{2}\left(F_{xx}(x,\, y) + F_{yy}(x,\, y)\right) + \lambda F_{xy}(x,\, y)$$

for $\lambda = \rho_{2,1}\rho_{2,2} < 1$ and $V^i(x) = \frac{\kappa_i\left(\theta_i - \left(H^i\right)^{-1}(x)\right)}{v_i q\left(\left(H^i\right)^{-1}(x)\right)} - \frac{v_i}{2} q'\left(\left(H^i\right)^{-1}(x)\right)$ for $i \in \{1,\, 2\}$, which are two continuous and strictly decreasing bijections from $\mathbb{R}$ to itself. We shall use now Theorem 2.5 from [4]. Under the notation of that paper, we can easily compute

$$\begin{aligned} A_{(z,\,w)}(s,\,(x,\,y)) &= \frac{1}{2} + \lambda\frac{(x-z)(y-w)}{(x-z)^2+(y-w)^2} \\ &\geq \frac{1}{2} + \lambda\frac{-\frac{1}{2}\left((x-z)^2+(y-w)^2\right)}{(x-z)^2+(y-w)^2} \\ &= \frac{1}{2}(1-\lambda) > 0 \end{aligned} \tag{A.0.30}$$

and also $B(s,\,(x,\,y)) = 1$ and

$$C_{(z,\,w)}(s,\,(x,\,y)) = 2\left(V^1(x)(x-z) + V^2(y)(y-w)\right) \tag{A.0.31}$$

for all $(x,\, y), (z,\, w) \in \mathbb{R}^2$. Since the coefficients of L_Z are continuous, with the higher order ones being constant, we can easily verify condition $A_1 - A_2$ from [4]. Moreover, since B and $C_{(z,\,w)}$ are continuous and $A_{(z,\,w)}$ lower-bounded by $\frac{1}{2}(1-\lambda) > 0$, we have A_5 as well. Next, we choose z and w to be the unique roots of $V^1(x)$ and $V^2(y)$ respectively, and with the notation of [4] we have

$$\underline{\alpha}(r; (z,\, w), 0) = \inf_{(x-z)^2+(y-w)^2=r^2} A_{(z,\,w)}(s,\, (x,\, y)) \geq \frac{1}{2}(1-\lambda) > 0 \tag{A.0.32}$$

and

$$\begin{aligned} \overline{\beta}(r; (z,\, w), 0) &= \sup_{(x-z)^2+(y-w)^2=r^2} \frac{B(s,\, (x,\, y)) - A_{(z,\,w)}(s,\, (x,\, y)) + C_{(z,\,w)}(s,\, (x,\, y))}{A_{(z,\,w)}(s,\, (x,\, y))} \\ &\leq \frac{2}{1-\lambda} - 1 + \frac{2}{1+\lambda} \sup_{(x-z)^2+(y-w)^2=r^2} C_{(z,\,w)}(s,\, (x,\, y)) \end{aligned} \tag{A.0.33}$$

since $C_{(z,\,w)}(s,\, (x,\, y))$ is never greater than zero and

$$\begin{aligned} A_{(z,\,w)}(s,\, (x,\, y)) &= \frac{1}{2} + \lambda \frac{(x-z)(y-w)}{(x-z)^2 + (y-w)^2} \\ &\leq \frac{1}{2} + \lambda \frac{\frac{1}{2}\left((x-z)^2 + (y-w)^2\right)}{(x-z)^2 + (y-w)^2} \\ &= \frac{1}{2}(1+\lambda) \end{aligned}$$

Fix now an $r_0 > 0$ and take any $r > r_0$. For the pair $(x,\, y)$ for which the supremum of $C_{(z,\,w)}(s,\, (x,\, y))$ is attained when $(x-z)^2 + (y-w)^2 = r^2$, we have $x = z + r\cos(\phi_r)$ and $y = w + r\sin(\phi_r)$ for some angle ϕ_r. Then, we have either $|\cos(\phi_r)| \geq \frac{\sqrt{2}}{2}$ or $|\sin(\phi_r)| \geq \frac{\sqrt{2}}{2}$. If $\cos(\phi_r) \geq \frac{\sqrt{2}}{2}$ holds, we can estimate

$$\begin{aligned} C_{(z,\,w)}(s,\, (x,\, y)) &= 2r\cos(\phi_r) V^1(z + r\cos(\phi_r)) + 2r\sin(\phi_r) V^2(w + r\sin(\phi_r)) \\ &\leq 2r\cos(\phi_r) V^1(z + r\cos(\phi_r)) \\ &\leq c_1 r \end{aligned}$$

with $c_1 = \sqrt{2} V^1(z + r_0 \frac{\sqrt{2}}{2}) < 0$. In a similar way, by using the fact that both V^1 and V^2 are strictly decreasing, we can find constants $c_2, c_3, c_4 < 0$ such that $C_{(z,\,w)}(s,\, (x,\, y)) < c_2 r$, $C_{(z,\,w)}(s,\, (x,\, y)) < c_3 r$ and $C_{(z,\,w)}(s,\, (x,\, y)) < c_4 r$, when $\cos(\phi_r) \leq -\frac{\sqrt{2}}{2}$, $\sin(\phi_r) \geq \frac{\sqrt{2}}{2}$ and $\sin(\phi_r) \leq -\frac{\sqrt{2}}{2}$ respectively. Thus, for $c^* = \max\{c_1, c_2, c_3, c_4\} < 0$ we have $C_{(z,\,w)}(s,\, (x,\, y)) < c_* r$, which can be plugged in (A.0.33) to give the estimate

$$\overline{\beta}(r; (z,\, w), 0) \leq \frac{2}{1-\lambda} - 1 + \frac{2c^*}{1+\lambda} r \tag{A.0.34}$$

for all $r \geq r_0$. This means that for r_0 large enough, with the notation of [4] we have

$$\begin{aligned}\overline{I}_{(z,w),r_0}(r) &\leq \int_{r_0}^{r} \frac{1}{r'}\left(\frac{2}{1-\lambda} - 1 + \frac{2c^*}{1+\lambda}r'\right)dr' \\ &\leq c^{**}(r - r_0)\end{aligned}$$

for some $c^{**} < 0$ and all $r \geq r_0$. This implies that

$$\int_{r_0}^{+\infty} e^{-\overline{I}_{(z,\,w),r_0}(r)}dr = +\infty$$

and combined with (A.0.32), it also gives

$$\int_{r_0}^{+\infty} \frac{1}{\underline{\alpha}(r;(z,\,w),0)} e^{\overline{I}_{(z,\,w),r_0}(r)}dr \leq \frac{2}{1-\lambda}\int_{r_0}^{+\infty} e^{c^{**}r}dr < +\infty$$

Therefore, we have that all the assumptions of Theorem 2.5 from [4] are satisfied for $Z = (Z^1_\cdot,\, Z^2_\cdot)$, which means that $(Z^1_\cdot,\, Z^2_\cdot)$ is a positive recurrent diffusion, and thus $(\sigma^{1,1}_\cdot,\, \sigma^{2,1}_\cdot)$ is positive recurrent as well. □

Proof of Theorem 3.2.4. We will show first that each volatility process does never hit zero. Recalling the standard properties of the scale function $S(x)$ of $\sigma^{1,1}_\cdot$ (see [39]), we have that

$$S(x) = \int_{\theta_1}^{x} e^{-\int_{\theta_1}^{y} \frac{2\kappa_1(\theta_1 - z)}{v_1^2 z\tilde{q}^2(z)}dz}dy \tag{A.0.35}$$

and we need to show that $\lim_{n\to+\infty} S\left(\frac{1}{n}\right) = -\infty$. Since $\sup_{x\in\mathbb{R}} \tilde{q}^2(x) \leq 1 < \frac{2\kappa_1\theta_1}{v_1^2}$, for $n \geq \frac{1}{\theta_1}$ we have

$$\begin{aligned}S\left(\frac{1}{n}\right) &= -\int_{\frac{1}{n}}^{\theta_1} e^{\int_y^{\theta_1} \frac{2\kappa_1(\theta_1 - z)}{v_1^2 z\tilde{q}^2(z)}dz}dy \\ &\leq -\int_{\frac{1}{n}}^{\theta_1} e^{\int_y^{\theta_1} \frac{(\theta_1 - z)}{\theta_1 z}dz}dy \\ &\leq -\int_{\frac{1}{n}}^{\theta_1} e^{\int_y^{\theta_1} \frac{1}{z}dz - \int_y^{\theta_1} \frac{1}{\theta_1}dz}dy \\ &\leq -\frac{1}{e}\int_{\frac{1}{n}}^{\theta_1} \frac{\theta_1}{y}dy = -\frac{\theta_1}{e}(\ln(n) + \ln(\theta_1))\end{aligned}$$

where the last tends to $-\infty$ as $n \longrightarrow +\infty$. This shows that our volatility processes remain positive forever.

Having now that our volatility processes are positive, we can set $Z^i_\cdot = \ln(\sigma^{i,1}_\cdot)$ for $i \in \{1,\, 2\}$, and we need to show that $(Z^1_\cdot,\, Z^2_\cdot)$ is a positive recurrent diffusion. We

will use the same techniques as in the proof of Theorem 3.2.4. Again, we can easily determine the infinitesimal generator L_Z of $Z = (Z^1_\cdot, Z^2_\cdot)$, which this time maps any smooth function $F : \mathbb{R}^2 \to \mathbb{R}$ to

$$\begin{aligned} L_Z F(x, y) &= V^1(x)F_x(x, y) + V^2(y)F_y(x, y) \\ &\quad + \frac{v_1^2 e^{-x}\tilde{q}^2(e^x)}{2}F_{xx}(x, y) + \frac{v_2^2 e^{-y}\tilde{q}^2(e^y)}{2}F_{yy}(x, y) \\ &\quad + \lambda v_1 v_2 e^{-\frac{x+y}{2}}\tilde{q}(e^y)\tilde{q}(e^y)F_{xy}(x, y) \end{aligned}$$

for $\lambda = \rho_{2,1}\rho_{2,2} < 1$ and $V^i(x) = e^{-x}\left(\kappa_i\theta_i - \frac{v_i^2}{2}\tilde{q}^2(e^x)\right) - \kappa_i$ for $i \in \{1, 2\}$, which are again two continuous and strictly decreasing bijections from $\mathbb{R}$ to itself (this can be shown by using the fact that $\tilde{q}$ is increasing and upper-bounded by 1). Under the notation of [4], by using also the inequality $ab \geq -\frac{a^2+b^2}{2}$ we have

$$\begin{aligned} A_{(z, w)}(s, (x, y)) &= \frac{1}{2}\left(\frac{v_1^2 e^{-x}\tilde{q}^2(e^x)(x - z)^2}{(x - z)^2 + (y - w)^2} + \frac{v_2^2 e^{-y}\tilde{q}^2(e^y)(y - w)^2}{(x - z)^2 + (y - w)^2}\right) \\ &\quad + \lambda\frac{v_1 v_2 e^{-\frac{x+y}{2}}\tilde{q}(e^x)\tilde{q}(e^y)(x - z)(y - w)}{(x - z)^2 + (y - w)^2} \\ &\geq \frac{1 - \lambda}{2}\left(\frac{v_1^2 e^{-x}\tilde{q}^2(e^x)(x - z)^2}{(x - z)^2 + (y - w)^2} + \frac{v_2^2 e^{-y}\tilde{q}^2(e^y)(y - w)^2}{(x - z)^2 + (y - w)^2}\right) \\ &\geq \frac{(1 - \lambda)\min\{v_1^2, v_2^2\}}{2}\min\{e^{-x}\tilde{q}^2(e^x),\, e^{-y}\tilde{q}^2(e^y)\} \end{aligned} \tag{A.0.36}$$

which is strictly positive. Moreover, we can compute

$$B(s, (x, y)) = \frac{v_1^2 e^{-x}\tilde{q}^2(e^x)}{2} + \frac{v_2^2 e^{-y}\tilde{q}^2(e^y)}{2} \tag{A.0.37}$$

and

$$C_{(z, w)}(s, (x, y)) = 2\left(V^1(x)(x - z) + V^2(y)(y - w)\right) \tag{A.0.38}$$

for all $(x, y), (z, w) \in \mathbb{R}^2$. Since the coefficients of L_Z, $A_{(z, w)}(s, (x, y))$, $B(s, (x, y))$ and $C_{(z, w)}(s, (x, y))$ are all continuous, with $A_{(z, w)}(s, (x, y))$ being strictly positive, we can easily verify conditions A_1 and A_5 from [4]. In order to verify A_2, we pick a $N > 0$ and $x, y, \bar{x}, \bar{y} \in [-N, N]$, we set

$$M(x, y) = \begin{pmatrix} \frac{v_1^2 e^{-x}\tilde{q}^2(e^x)}{2} & \frac{\lambda v_1 v_2 e^{-\frac{x+y}{2}}\tilde{q}(e^x)\tilde{q}(e^y)}{2} \\ \frac{\lambda v_1 v_2 e^{-\frac{x+y}{2}}\tilde{q}(e^x)\tilde{q}(e^y)}{2} & \frac{v_2^2 e^{-y}\tilde{q}^2(e^y)}{2} \end{pmatrix}$$

and we compute

$$\|M(x, y) - M(\bar{x}, \bar{y})\|_{L^2}^2 = \left(\frac{v_1^2 e^{-x}\tilde{q}^2(e^x)}{2} - \frac{v_1^2 e^{-\bar{x}}\tilde{q}^2(e^{\bar{x}})}{2}\right)^2$$

$$
\begin{aligned}
&+\left(\frac{v_2^2 e^{-y}\tilde{q}^2(e^y)}{2}-\frac{v_2^2 e^{-\bar{y}}\tilde{q}^2(e^{\bar{y}})}{2}\right)^2\\
&+\left(\frac{\lambda v_1 v_2 e^{-\frac{x+y}{2}}\tilde{q}(e^x)\tilde{q}(e^y)}{2}-\frac{\lambda v_1 v_2 e^{-\frac{\bar{x}+\bar{y}}{2}}\tilde{q}(e^{\bar{x}})\tilde{q}(e^{\bar{y}})}{2}\right)^2\\
\leq\ & C_N\left\|(x,\,y)-(\bar{x},\,\bar{y})\right\|^2_{L^2(\mathbb{R}^2)} \qquad \text{(A.0.39)}
\end{aligned}
$$

where we have used the two-dimensional Mean Value Theorem on each of the three terms, and the fact all the involved functions have a bounded gradient in $[-N,\,N]^2$ (since $\tilde{g}$ has continuous derivatives). Thus, for $\delta_N(r)=C_N r$, we obtain A_2 as well. Next, for some $r_0>0$ and all $r\geq r_0$, we compute

$$
\begin{aligned}
\underline{\alpha}(r;(z,\,w),0) &= \inf_{(x-z)^2+(y-w)^2=r^2} A_{(z,\,w)}(s,\,(x,\,y))\\
&\geq \frac{(1-\lambda)\min\{v_1^2,v_2^2\}}{2}e^{-\max\{z,\,w\}-r}\tilde{q}^2(e^{\max\{z,\,w\}+r})
\end{aligned}
\qquad \text{(A.0.40)}
$$

and

$$
\begin{aligned}
\overline{\beta}(r;(z,\,w),0) &= \sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))-A_{(z,\,w)}(s,\,(x,\,y))+C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}\\
&= -1+\sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))+C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))} \qquad \text{(A.0.41)}
\end{aligned}
$$

where again we choose z and w to be the unique roots of $V^1(x)$ and $V^2(y)$ respectively. Then, by setting $x=z+r\cos(\phi_r)$ and $y=w+r\sin(\phi_r)$ with $\phi_r\in[0,\,2\pi]$ for the $(x,\,y)$ for which the above supremum is attained, since $\tilde{q}$ is increasing we have

$$
\begin{aligned}
C_{(z,\,w)}(s,\,(x,\,y)) &= 2\left(e^{-x}\left(\kappa_1\theta_1-\frac{v_1^2}{2}\tilde{q}^2(e^x)\right)-\kappa_1\right)(x-z)\\
&\quad +2\left(e^{-y}\left(\kappa_2\theta_2-\frac{v_2^2}{2}\tilde{q}^2(e^y)\right)-\kappa_2\right)(y-w)\\
&\leq 2\left(e^{-x}\left(\kappa_1\theta_1-\frac{v_1^2}{2}\tilde{q}^2(e^z)\right)-\kappa_1\right)(x-z)\\
&\quad +2\left(e^{-y}\left(\kappa_2\theta_2-\frac{v_2^2}{2}\tilde{q}^2(e^w)\right)-\kappa_2\right)(y-w)\\
&= 2\kappa_1\left(e^{z-x}-1\right)(x-z)+2\kappa_2\left(e^{w-y}-1\right)(y-w)\\
&\leq 2\min\{\kappa_1,\kappa_2\}\left(\left(e^{z-x}-1\right)(x-z)+\left(e^{w-y}-1\right)(y-w)\right)\\
&= \kappa r\left(\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r)+\left(e^{-r\sin(\phi_r)}-1\right)\sin(\phi_r)\right)
\end{aligned}
\qquad \text{(A.0.42)}
$$

for $\kappa = 2\min\{\kappa_1, \kappa_2\}$, and since $\tilde{q}$ is bounded, for $\xi = \max\left\{\frac{v_1^2}{2}, \frac{v_2^2}{2}\right\}\sup_{x\in\mathbb{R}}\tilde{q}(x)$ we can also show that

$$\begin{aligned}
A_{(z,\,w)}(s,\,(x,\,y)) &\leq \xi\left(e^{-x}\frac{(x-z)^2}{(x-z)^2+(y-w)^2} + e^{-y}\frac{(y-w)^2}{(x-z)^2+(y-w)^2}\right)\\
&= \xi\left(e^{-z-r\cos(\phi_r)}\cos^2(\phi_r) + e^{-w-r\sin(\phi_r)}\sin^2(\phi_r)\right)\\
&= \xi\left(e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos^2(\phi_r) + e^{-w}\left(e^{-r\sin(\phi_r)}-1\right)\sin^2(\phi_r)\right)\\
&\quad +\xi\left(e^{-z}\cos^2(\phi_r) + e^{-w}\sin^2(\phi_r)\right)\\
&\leq -\xi\left(e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r) + e^{-w}\left(e^{-r\sin(\phi_r)}-1\right)\sin(\phi_r)\right)\\
&\quad +\xi\left(e^{-z}+e^{-w}\right) \qquad\qquad (A.0.43)
\end{aligned}$$

where we have also used the elementary inequality $(e^{ab}-1)a^2 \leq -(e^{ab}-1)a$ for $|a|\leq 1$ and $b<0$. By using (A.0.42) and (A.0.43) now we obtain

$$\frac{C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))} \leq -r\frac{\kappa}{\xi}\frac{\ell(r)}{\ell(r)+\xi\left(e^{-z}+e^{-w}\right)} \qquad (A.0.44)$$

where

$$\begin{aligned}
\ell(r) &= -\xi\left(e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r) + e^{-w}\left(e^{-r\sin(\phi_r)}-1\right)\sin(\phi_r)\right)\\
&\geq -\xi e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r)\\
&= \xi e^{-z}\left|e^{-r\cos(\phi_r)}-1\right|\,|\cos(\phi_r)|\\
&\geq \xi\min\{e^{-z},e^{-w}\}\frac{\sqrt{2}}{2}\min\left\{\left|e^{-r_0\frac{\sqrt{2}}{2}}-1\right|,\,\left|e^{r_0\frac{\sqrt{2}}{2}}-1\right|\right\} \qquad (A.0.45)
\end{aligned}$$

since we take $r\geq r_0$, and without loss of generality we can assume that $|\cos(\phi_r)|\geq\frac{\sqrt{2}}{2}$. Thus, (A.0.44) implies that there is a universal $c^*<0$ such that

$$\frac{C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))} \leq c^* r \qquad (A.0.46)$$

when $r\geq r_0$. Plugging the last in (A.0.41) we obtain

$$\begin{aligned}
\overline{\beta}(r;(z,\,w),0) &= -1 + \sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))+C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}\\
&\leq -1+pc^*r\\
&\qquad + \sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))+(1-p)C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}
\end{aligned} \qquad (A.0.47)$$

for all $r\geq r_0$ and a $p\in[0,\,1]$ which will be chosen later. We will show now that the last term in the RHS of the above is negative for r_0 large enough (depending on

p). Indeed, by using (A.0.42), the definition of $B(s,\,(x,\,y))$, and the fact that $\tilde{q}$ is upper-bounded, we can obtain the estimate

$$\begin{aligned}&\sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))+(1-p)C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}\\&\leq\sup_{(x-z)^2+(y-w)^2=r^2}\frac{\kappa^*\left((e^{z-x}-1)\,(x-z)+(e^{w-y}-1)\,(y-w)\right)+\xi(e^{-x}+e^{-y})}{A_{(z,\,w)}(s,\,(x,\,y))}\end{aligned}\tag{A.0.48}$$

where as before, we have $\xi=\max\left\{\frac{v_1^2}{2},\,\frac{v_2^2}{2}\right\}\sup_{x\in\mathbb{R}}\tilde{q}(x)$, and $\kappa^*=(1-p)\kappa$. The numerator of the last quantity can be easily be show to tend to $-\infty$ when x or y tends to $\pm\infty$, which happens when $r\longrightarrow+\infty$. Thus, for $r\geq r_0$ with r_0 large enough, the RHS of (A.0.48) is negative, which can be plugged in (A.0.46) to give

$$\overline{\beta}(r;(z,\,w),0)\;\;\leq\;\;-1+pc^*r$$

for all $r\geq r_0$, with $c^*<0$. This means that as in the proof of Theorem 3.2.3, with the notation of [4], for r_0 large enough we have

$$\overline{I}_{(z,\,w),r_0}(r)\;\;\leq\;\;\int_{r_0}^{r}\frac{1}{r'}\left(-1+pc^*r'\right)dr'\leq pc^*(r-r_0)\tag{A.0.49}$$

with $c^*<0$, for all $r\geq r_0$. This implies that

$$\int_{r_0}^{+\infty}e^{-\overline{I}_{(z,\,w),r_0}(r)}dr=+\infty$$

which means that our two-dimensional process is recurrent (see Theorem 2.4 in [4]). However, what we need is positive recurrence, and thus we need to show that

$$\int_{r_0}^{+\infty}\frac{1}{\underline{\alpha}(r;(z,\,w),0)}e^{\overline{I}_{(z,\,w),r_0}(r)}dr<+\infty.$$

By using (A.0.49), (A.0.40) and the fact that $\tilde{q}$ is lower-bounded by something positive, we can bound the last integral by a positive multiple of

$$\int_{r_0}^{+\infty}e^{(1+pc^*)r}dr$$

which is finite when $pc^*<-1$. Taking η small enough and r_0 big enough, we can force e^{-w} and e^{-z} to be arbitrarily close to each other, and the lower bound of ℓ (given by (A.0.45)) to be arbitrarily close to $\frac{\sqrt{2}\xi}{2}e^{-z}$. This makes c^* (obtained in (A.0.44)) arbitrarily close to $-\frac{\kappa}{\xi}\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}+2}<-1$ by the assumptions of the Theorem and the definitions of κ and ξ. Thus, if p is chosen to be very close to 1, we can achieve $pc^*<-1$ as well, which gives the desired result. □

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