

SPLITTINGS AND THE ASYMPTOTIC TOPOLOGY OF THE LAMPLIGHTER GROUP

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ABSTRACT. It is known that splittings of one ended finitely presented groups over 2-ended groups can be characterized geometrically. We show here that this characterization does not extend to all finitely generated groups, by showing that the lamplighter group is coarsely separated by quasi-lines.

It is also known that virtual surface groups are characterized in the class of one-ended finitely presented groups by the property that their Cayley graphs are coarsely separated by quasi-circles. Answering a question of Kleiner we show that the Cayley graph of the lamplighter group is coarsely separated by quasi-circles. It follows that the quasi-circle characterization of virtual surface groups does not extend to the finitely generated case.

1. INTRODUCTION

Stallings [14] showed that a finitely generated group has more than one end if and only if it splits over a finite group. This gives a geometric characterization of finitely generated groups which split over a finite group, where geometric here means in the sense of quasi-isometries.

One has a similar geometric characterization for splittings over 2-ended groups which applies to finitely presented groups. It was shown in [12] (see also [10]) that a one ended finitely presented group, which is not virtually a surface group, splits over a 2-ended group if and only if its Cayley graph is coarsely separated by a quasi-line. Another geometric characterization of splittings of hyperbolic groups over 2-ended groups was given earlier by Bowditch [1]. It is natural to ask whether the characterization given in [12] applies in fact to all one ended finitely generated groups (as it is the case for Stallings' theorem).

We show here that there is a one-ended finitely generated group (the lamplighter group) which is coarsely separated by a quasi-line but does not split over a 2-ended group. It turns out that the same group can be used to answer a question of Kleiner ([13], problem 4.5).

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2. PRELIMINARIES

2.1. Asymptotic Topology. We recall some definitions relating to asymptotic topology. The idea (due to Gromov) behind these definitions is that one can develop a large scale analog of topological notions. It is in a similar vein to large scale geometry except in asymptotic topology one tries to give a ‘rough’ or ‘large scale’ version of topological rather than geometric notions (see for example [9], [4]). Asymptotic topology notions are invariant under quasi-isometries (and one can see this as a large scale version of the fact that topological notions are invariant under isometries).

Let $K \geq 1$, $L \geq 0$ be constants. A (K, L) -*quasi-isometry* between two metric spaces X, Y is a map $f : X \rightarrow Y$ such that the following two properties are satisfied:

- 1) $\frac{1}{K}d(x, y) - L \leq d(f(x), f(y)) \leq Kd(x, y) + L$ for all $x, y \in X$.
- 2) For every $y \in Y$ there is an $x \in X$ such that $d(y, f(x)) \leq K$.

We will usually simply say quasi-isometry instead of (K, L) -quasi-isometry. Two metric spaces X, Y are called quasi-isometric if there is a quasi-isometry $f : X \rightarrow Y$.

A *geodesic metric space* is a metric space in which any two points x, y are joined by a path of length $d(x, y)$. One turns a connected graph into a geodesic metric space by giving each edge length 1.

We call a map $f : X \rightarrow Y$ between metric spaces a *uniform embedding* (see [7]) if the following two conditions are satisfied:

- 1) There are K, L such that for all $x, y \in X$ we have $d(f(x), f(y)) < K(d(x, y)) + L$.
- 2) For every $E > 0$ there is $D > 0$ such that $\text{diam}(A) < E \Rightarrow \text{diam}(f^{-1}A) < D$.

If f satisfies only condition 1 above we say that f is a *coarse lipschitz* map. We remark that a coarse lipschitz map is a uniform embedding if there is a function $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $A \subset Y$, $\text{diam}(f^{-1}A) \leq h(\text{diam}(A))$. We say then that the *distortion* of f is bounded by h . We remark that f is a quasi-isometric embedding if its distortion is bounded by a linear function.

The following lemma ([7]) gives a simple characterization of uniform embeddings which will be used in the sequel.

Lemma 2.1. *Let X, Y be geodesic metric spaces. $f : X \rightarrow Y$ is a uniform embedding if and only if the following condition is satisfied:*

For any two sequences $(x_n), (y_n)$ of X , $d(x_n, y_n) \rightarrow \infty$ if and only if $d(f(x_n), f(y_n)) \rightarrow \infty$.

Proof. We show first that if the condition is satisfied then f is a uniform embedding. To see that f satisfies 1 (ie is coarse lipschitz) we set

$$K = \sup\{d(f(x), f(y)) : d(x, y) \leq 1\}$$

Clearly $K < \infty$ by hypothesis. If x, y are any two distinct points in X we join them by a geodesic γ and we pick consecutive points on γ , $x_0 = x, x_1, \dots, x_n = y$ so that $d(x_{i-1}, x_i) = 1$ if $i < n$ and $d(x_{n-1}, x_n) \leq 1$. It follows that

$$d(f(x), f(y)) \leq Kn \leq Kd(x, y) + K$$

Let $E > 0$ be given. We set

$$D = \sup\{d(x, y) : d(f(x), f(y)) \leq E\}$$

Clearly $D < \infty$. This shows that f satisfies 2. Conversely now, assume that f is a uniform embedding. If $d(f(x_n), f(y_n)) \rightarrow \infty$, since $d(f(x_n), f(y_n)) < K(d(x_n, y_n)) + L$, it follows that $d(x_n, y_n) \rightarrow \infty$. Assume that $d(x_n, y_n) \rightarrow \infty$. If $d(f(x_n), f(y_n))$ does not tend to infinity then there is some $E > 0$ and subsequences x_{n_k}, y_{n_k} so that for all k ,

$$d(f(x_{n_k}), f(y_{n_k})) < E$$

It follows by 2) that there is some $D > 0$ so that for all k , $d(x_{n_k}, y_{n_k}) < D$, which is a contradiction. \square

If $K \subset X$ and $R > 0$ we say that a component of $X - N_R(K)$ is *deep* if it is not contained in $N_{R_1}(K)$ for any $R_1 > 0$. We say that K *coarsely separates* X if there is an $R > 0$ such that $X - N_R(K)$ has at least two deep components. For example X has more than one end if and only if X is coarsely separated by a point.

We note that this asymptotic notion of separation was used, for example, to formulate a large scale version of Jordan's curve theorem for the plane. (see [9], [5]).

Let (X, d) be a geodesic metric space. If $R \subset X$ is path connected subset of X , we denote the path metric of R by d_R . We say that R is a *quasi-line* if (R, d_R) is quasi-isometric to \mathbb{R} and the inclusion map $i : (R, d_R) \rightarrow (X, d)$ is a uniform embedding. We say that R is a *quasi-ray* if (R, d_R) is quasi-isometric to \mathbb{R}^+ and the inclusion map $i : (R, d_R) \rightarrow (X, d)$ is a uniform embedding.

Kleiner has suggested the following asymptotic way to generalize the topological notion of a separating simple closed curve. Let X be a geodesic metric space. Consider a sequence of simple closed curves

C_n in X and denote by d_n the path metric on C_n . We say that C_n is a *sequence of quasi-circles* if $\text{diam } C_n$ tends to infinity and there is a distortion function $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$d_n(a, b) \leq h(d(a, b))$$

for any a, b lying on any C_n .

We say that a sequence of quasi-circles C_n *coarsely separates* X if there is a K (independent of n) such that $N_K(C_n)$ separates X and at least two components of $X - N_K(C_n)$ are not contained in $N_n(C_n)$.

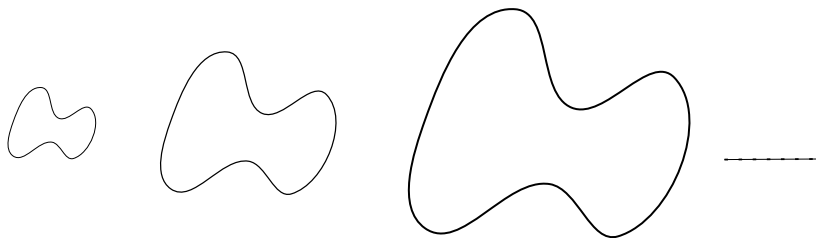


FIGURE 1. A sequence of quasi-circles

We may define similarly a sequence of quasi-intervals I_n which coarsely separates X by taking $I_n = [a_n, b_n]$ to be simple paths, rather than simple closed curves.

2.2. Asymptotic topology and groups. We remark that if H is a finitely generated subgroup of a finitely generated group G then the embedding $H \hookrightarrow G$ is a uniform embedding (where we consider H, G equipped with their word metrics).

Coarse separation is related to group splittings. We have the following easy lemma:

Lemma 2.2. *If a finitely generated group G splits over a finitely generated group H then H coarsely separates the Cayley graph of G .*

Proof. If $G = A *_H B$ or $G = A *_H$ we may pick a finite generating set S of G so that, in the first case, all generators lie in $A \cup B$ or, in the second case, the generators are given by the stable letter t and a finite set of elements of A . To see this take any finite set of generators of G , S' and write each element of S' in normal form with respect to the amalgam or the HNN-extension. Now take as new generating set S of G the set of all elements of A, B (and t in the HNN-extension case) that appear in these normal form expressions. Let Γ be the Cayley graph of G with respect to S and let T be the Bass-Serre tree of G for the splitting $G = A *_H B$ or $G = A *_H$. We consider the barycentric subdivisions

Γ' of Γ and T' of T . We define now a simplicial map $p : \Gamma' \rightarrow T'$. Let e be the edge of T with stabilizer H . Let v be the midpoint of e . So v is a vertex of T' . We recall that the vertices of Γ are the elements of G . If g is a vertex of Γ define $p(g) = gv$. If (g, gs) is an edge of Γ (so $g \in G, s \in S$) then $d(sv, v)$ is either 2 or 0. So $d(gv, gsv)$ is 2 or 0 and we can extend p to (g, gs) either by mapping it to the 2 consecutive edges joining gv, gsv or by collapsing it to the vertex gv . This shows that the map p can be extended from the set of vertices of Γ to a simplicial map $p : \Gamma' \rightarrow T'$. Since the map p' is simplicial we have that $d(p(a), p(b)) \leq d(a, b)$ for all vertices of Γ . Further p is clearly onto. By our choice of v , $p^{-1}(v) = H$. Let $n \in \mathbb{N}$ and let v_1, v_2 be vertices of $T' - T$ lying in distinct connected components of $T' - v$ such that $d(v_1, v) \geq n$, $d(v_2, v) \geq n$. Let $g_1 \in p^{-1}(v_1)$, $g_2 \in p^{-1}(v_2)$. Then if α is a path in Γ' joining g_1 to g_2 , v lies in $p(\alpha)$. It follows that α intersects $p^{-1}(v) = H$. Further if h is the first vertex of α lying in H and $\alpha_1 = [g_1, h]$ the subpath of α with endpoints g_1, h , then $p(\alpha_1)$ joins v_1 to v . It follows that $\text{length}(p(\alpha_1)) \geq n$. Since p is distance non increasing we conclude that $d(g_1, H) \geq n$. Similarly $d(g_2, H) \geq n$. Since this is true for any n we conclude that H coarsely separates Γ . \square

In the language of asymptotic topology Stallings' ends theorem ([14]) says that a finitely generated group G splits over a finite group if and only if the Cayley graph of G is coarsely separated by a point.

Clearly if a finitely generated group splits over a 2-ended group then a quasi-line coarsely separates the Cayley graph of G . It was shown in [12] that the converse also holds when G is 1-ended, finitely presented and not virtually a surface group. We will see in the next section that this characterization does not generalize to finitely generated groups.

Geometric characterizations of surface groups among finitely presented groups or hyperbolic groups were used in [12] and [1] to show that splittings over 2-ended groups are invariant under quasi-isometries. Such a characterization was a crucial ingredient also in the work of Mess [11] on the Seifert conjecture. It follows from [2] that a sequence of quasi-circles coarsely separates the Cayley graph of a one-ended finitely presented group G , if and only if G is virtually a surface group. Kleiner asked whether this characterization of virtually surface groups extends to all finitely generated groups. We show in the next section that the Cayley graph of the lamplighter group L is coarsely separated by quasi-circles, showing that the answer to Kleiner's question is negative.

3. THE ASYMPTOTIC TOPOLOGY OF THE LAMPLIGHTER GROUP

The lamplighter group L is defined as

$$L = (\oplus_{\mathbb{Z}} \mathbb{Z}_2) \rtimes \mathbb{Z}$$

where \mathbb{Z} acts on the infinite direct sum $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ by sending the generator of the i -th factor, x_i , to the generator of the next factor x_{i+1} ($i \in \mathbb{Z}$). It is easy to see that L is in fact finitely generated and is given by the presentation

$$L = \langle a, t | a^2, [t^i a t^{-i}, t^j a t^{-j}], i, j \in \mathbb{Z} \rangle$$

We will denote by X the Cayley graph of L with respect to the generators a, t .

3.1. Elements of the lamplighter group and metric properties.

It is convenient to represent elements of L geometrically. We explain briefly how is this done. We see L as a semi-direct product

$$L = (\oplus_{\mathbb{Z}} \mathbb{Z}_2) \rtimes \mathbb{Z}$$

so an element g of L is of the form $g = (h, k)$ where $h \in \oplus_{\mathbb{Z}} \mathbb{Z}_2$, $k \in \mathbb{Z}$. Now since h is an element of a direct sum of \mathbb{Z}_2 's it can be seen as a bi-infinite string of 0's and 1's with at most finitely many 1's. One thinks of these strings of 0, 1's as lamps at the integers on the real line. A 0 indicates a lamp which is off, while an 1 indicates a lamp which is lit. We encode $g = (h, k)$ by the infinite string of lamps corresponding to h with a cursor at the position k . If we denote by e the identity element of $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ and by x_0 the element of $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ given by 1 at the position 0 and 0 everywhere else then clearly $t = (e, 1)$ and $a = (x_0, 0)$ generate L . By the representation of L as a semi-direct product $L = (\oplus_{\mathbb{Z}} \mathbb{Z}_2) \rtimes \mathbb{Z}$ one obtains the presentation

$$L = \langle a, t | a^2, [t^i a t^{-i}, t^j a t^{-j}], i, j \in \mathbb{Z} \rangle$$

for the generators a, t .

We explain now how the generators a, t act on the geometric representation of the elements of the lamplighter group. Clearly

$$(h, k) \cdot t = (h, k) \cdot (e, 1) = (h, k + 1)$$

So multiplication by t has the effect of moving the cursor by 1 to the right. Also

$$(h, k) \cdot a = (h, k) \cdot (x_0, 0) = (h \cdot x_k, k)$$

where x_k is the element of $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ given by 1 at the k -th position and 0 everywhere else. So the t generator corresponds to a move of the cursor to the right while the a generator corresponds to switching a lamp at the position of the cursor.

One calls the cursor ‘lamplighter’. A word on the a, t generators corresponds to a sequence of ‘moves’ of the lamplighter, the t ’s correspond to a change of position by 1 while the a ’s to switching the lamp at the position of the lamplighter.

If we think that it takes 1 unit of time for the lamplighter to move from the position n to the position $n + 1$ (or $n - 1$) and 1 unit of time to turn the switch on or off, then the distance between the identity configuration (lamps off, lamplighter at 0) to a given configuration is given by the time it takes the lamplighter to turn on the lamps and to move to the position of the given configuration. For a more detailed exposition of normal forms and the geometric representation of the elements of L we refer to [3].

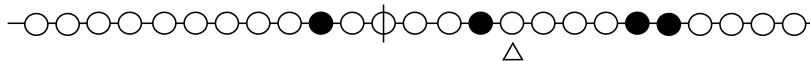


FIGURE 2. Here we give an example of a geometric representation of an element of L . The vertical line stands for the origin, lamps which are on are filled-in circles and lamps which are off are circles with no filling. The lamplighter position is indicated by a triangle. The lamplighter in this example is at the position 4.

We introduce now some notation that will be useful in the sequel.

We associate to each configuration c of 0, 1’s on \mathbb{Z} a binary integer c^+ , where c^+ is the binary number that we obtain from the digits of c which are right from 0 starting at 0 (where we read the number in reverse, from right to left). Similarly we associate to c the binary number c^- which is the number which we read from c if we start at -1 and move left. So for example if c is the configuration:

$$c = \dots 0, 0, 1, 0, S0, 1, 1, 0, 0, \dots$$

where we denote by S the origin, then $c^+ = 011 = 0 \cdot 1 + 1 \cdot 2 + 1 \cdot 2^2 = 6$ and $c^- = 10 = 0 \cdot 1 + 1 \cdot 2 = 2$ (see figure 2). If c is a configuration we denote by $c(k)$ the digit at the position k . So for instance in the example above we have $c(0) = 0, c(-2) = 1, c(1) = 1$. With this notation we have

$$c^+ = \sum_{k=0}^{\infty} c(k)2^k, \quad c^- = \sum_{k=-1}^{-\infty} c(k)2^{-k-1}$$



FIGURE 3. In this configuration $c^+ = 011 = 0 \cdot 1 + 1 \cdot 2 + 1 \cdot 2^2 = 6$ and $c^- = 10 = 0 \cdot 1 + 1 \cdot 2 = 2$. The position of the lamplighter does not affect c^+ and c^- .

3.2. A separating quasi-line in the lamplighter group. We show now that there is a quasi-ray which coarsely separates the Cayley graph of L (problem 4.2 in [13]). This contrasts with the case of finitely presented groups where it is impossible for a quasi-ray to coarsely separate the Cayley graph of a 1-ended group (this can be shown using the results and methods of [12]).

Before proceeding to the formal proof we explain the idea of the construction of the separating quasi-ray. Consider the set of all configurations c such that c^- is 0 and the lamplighter is at the position 0. Let's denote by A the subset of L corresponding to these configurations. If X is the Cayley graph of L , it is not very hard to see that A separates X . Indeed let x_n be the configuration with 0 everywhere and the lamplighter at n and let y_n be the configuration with 0 everywhere and the lamplighter at $-n$. Then to move from x_n to y_n the lamplighter has to pass by the position 0. However no matter how the lamplighter moves, starting from x_n , the first time that he reaches 0 the resulting configuration is an element of A . This shows that A separates x_n from y_n .

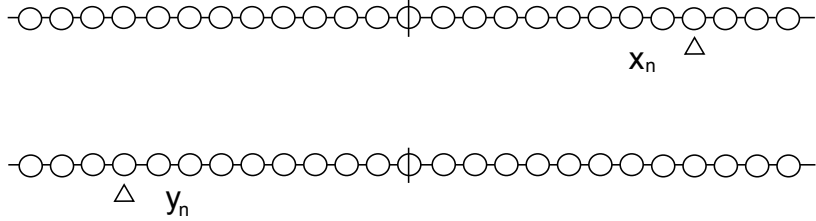


FIGURE 4. A separates x_n, y_n

Now the idea is to create a quasi-ray that contains A . The main technical issue while doing this is to control distortion. Of course there are many ways to do this, so our construction is, in a sense, quite arbitrary.

Theorem 3.1. *There is a quasi-ray N which coarsely separates the Cayley graph of L , X .*

Proof. We describe N using the geometric representation of the elements of L described earlier. We split the proof in 3 parts. In the first part we define N . In the second part we show that N is uniformly embedded. In the third part we show that it coarsely separates X .

(1) N will be defined as a graph isomorphic to a half-line given by a sequence of vertices of X . Using our geometric representation we may define N as an infinite ‘trip’ of the lamplighter which moves on the line, turning switches on and off. It will be convenient to describe this trip in stages.

The first vertex of N is the identity configuration: the lamplighter at 0 and all lamps at the 0 state (off). This is the stage 0 of the trip. We describe now how the lamplighter moves. At stage n ($n \in \mathbb{N}$) we are at vertex given by the configuration c_n where the lamplighter is at the origin, c_n^+ is the number n written in binary and there are only 0’s left of the lamplighter. Now to go from the stage n to the stage $n+1$ the lamplighter does the following: If $c_n(0) = 0$ then he simply switches 0 to 1. The configuration obtained in this way is the configuration of stage $n+1$. If $c_n(i) = 1$ for all i , $0 \leq i \leq k-1$ and $c_n(k) = 0$ then the lamplighter moves left to $-k$, switches it to 1 and moves back to the origin turning the switch at position $-k+r$ so that it is the same as the switch at position $k+r$ (for all $r \leq k-1$). We denote the configuration obtained after this by c_{n0} . After that the lamplighter moves to the right switching as he moves all lamps that are on to off until he reaches the k -th lamp which is off and switches it on. If we denote by c_{n1} the configuration that results from this, it is easy to see that

$$c_{n1}^+ = n + 1$$

After that the lamplighter moves again left until he reaches the origin, after the origin he goes left turning off all lamps which are on, and after turning off the last lamp which is on, at position $-k$, returns to the origin. In this way we move from configuration c_n to c_{n+1} . It is clear that N does not meet the same configuration twice, so N is indeed a half line.

(2) We show now that N is uniformly embedded. Let’s denote by $N(i)$ the i -th vertex of N .

It suffices to show the following: If $a = N(i)$, $b = N(j)$ ($i < j$) and $d(a, b) \leq M$ then $j - i$ is bounded by a constant $D = D(M)$ which depends only on M . Let’s denote by $l(a), l(b)$ the lamplighter positions

for a, b respectively. We remark that

$$|l(a) - l(b)| \leq d(a, b) \leq M$$

We distinguish now some cases:

Case 1 : $l(a), l(b) \in [-M, M]$. Then the configurations of a, b are identical in $[2M, \infty]$. This implies that when we move from $N(i)$ to $N(j)$ the lamplighter moves in the interval $[-2M, 2M]$. But then there is a bounded number of new configurations that can be created so $j - i$ is bounded in this case by a constant $D_1 = D_1(M)$.

Case 2 : At least one of $l(a), l(b)$ lies in $[M + 1, \infty)$. Then both $l(a), l(b)$ lie in $[1, \infty)$. It is clear that if there is some n so that both a, b are configurations that appear as we move from c_n to c_{n+1} then $j - i \leq M$. We assume therefore that a is a configuration that appears in the stage c_k to c_{k+1} , while b is a configuration that appears in the stage c_n to c_{n+1} for some $n > k$. Clearly $|l(a) - l(b)| \leq M$. We remark that since $l(a), l(b) > 0$, a is a configuration that appears at the stage c_{k0} to c_k and b is a configuration that appears at the stage c_{n0} to c_n . We recall now that as the lamplighter moves from c_{k0} to c_{k1} he erases a string of consecutive 1's. Let's say that the initial string of 1's of c_k consists of d_k 1's and the initial string of 1's of c_n consists of d_n 1's. By the definition of c_{k0} , c_{n0} the lamps at $-d_k - 1$, $-d_n - 1$ are respectively on for these two configurations. Recall that, by the definition of N , to move from configuration, say, c_k to c_{k0} the lamplighter moves left to $-d_k - 1$ and turns the lamp on before returning to the origin. If $d_k \neq d_n$ then $d(a, b) > M$. This is because, by the way they were defined c_{k0}^- and c_{n0}^- differ at the position $\min(-d_k - 1, -d_n - 1)$ (and it takes the lamplighter more than M steps to change this, since one of $l(a), l(b)$ is in $[M + 1, \infty)$).

We assume therefore that $d_k = d_n = d > M$. Since $d(a, b) \leq M$ we have that $c_{k0}^- = c_{n0}^-$. This however implies that the configurations c_k, c_n are identical in all positions in $[0, 2d]$. Indeed recall that, by the definition of N , as the lamplighter moves from c_k to c_{k0} he goes to $-d_k - 1$, switches the lamp on, and as he returns to 0 turns lamps on in the same way as in the interval $[d, 2d]$. In other words the configuration c_{k0}^- in the interval $[-d - 1, -1]$ is identical to the part of configuration c_k in the interval $[d, 2d]$. Since $d(a, b) \leq M$ and the lamplighter for a, b lies in the interval $[0, d]$ we have also that c_k, c_n are identical in $[d + M, \infty)$. But $d > M$, so $c_n = c_k$, which is a contradiction since we assumed that $n > k$.

Case 3: At least one of $l(a), l(b)$ lies in $(\infty, -M - 1]$. As in case 2 we see that a is a configuration that appears in the stage c_k to c_{k+1} ,

while b is a configuration that appears in the stage c_n to c_{n+1} for some $n > k$. However since $k \neq n$ we have that $c_k^+ \neq c_n^+$. On the other hand one of $l(a), l(b)$ lies in $(\infty, -M - 1]$. This implies that $d(a, b) > M$.

This finishes the proof that N is uniformly embedded.

(3) We show finally that N coarsely separates X . To show this it is enough to show that there are sequences of vertices x_n, y_n in X such that

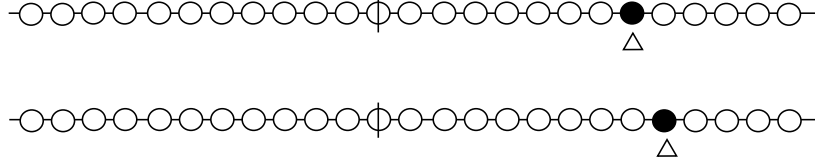
$$d(x_n, N) \geq n, \quad d(y_n, N) \geq n$$

and x_n, y_n lie in distinct components of $X - N$. We define x_n to be the following configuration: x_n^+ consists of $2n$ consecutive 1's, x_n^- consists of 0's and the lamplighter is at the position n . y_n is the configuration with 0's everywhere and the lamplighter at $-n$. It is clear now that to move from x_n to N the lamplighter has to erase all 1's at the interval $[0, n]$, so $d(x_n, N) \geq n$. Similarly to move from y_n to N the lamplighter has to move from position $-n$ to 0 so $d(y_n, N) \geq n$.

We show now that x_n, y_n are separated by N . To move from x_n to y_n the lamplighter has to go through 0. But no matter how the lamplighter switches lamps on and off before reaching 0, when the lamplighter moves to 0 for the first time we are exactly at the beginning of the stage c_k , where k is the integer whose binary expression we see on the right side. This is because we assume that the 'trip' of the lamplighter starts at x_n , so all lamps at the left side of 0 are off. So the first time the lamplighter arrives at 0 all lamps at the left side of 0 are off. We conclude that any path joining x_n, y_n in X intersects N . \square

Some examples: As the definition of N in the previous proof is quite ad hoc we give now some examples to illustrate why more naive ways to define N wouldn't produce a uniformly embedded set. Let's say that we define N to be the path obtained as the lamplighter writes successively the numbers 1, 2, 3, ... in binary (that is N is defined as before but the lamplighter never goes left of the position 0, in other words we eliminate the stage c_{n0} to c_{n1} of our definition). Then when the lamplighter writes the number 2^m we have the following configuration: lamplighter at m and 0's at all positions except at m where we have 1. Let's call the vertex of X represented by this configuration by $v(m)$.

We see then that $v(m)$ and $v(m+1)$ are at distance 3 in X while to move from $v(m)$ to $v(m+1)$ it takes more than 2^m steps in N . This produces unbounded distortion. To avoid this problem we introduced the stage c_{n0} to c_{n1} in our definition of N which gives a 'marker' for the stage we are in. A more naive 'marker' wouldn't work either. Say

FIGURE 5. Far away in N but close in the Cayley graph.

for example that in the stage c_{n0} to c_{n1} the lamplighter just moves left and only turns on the lamp at position $-k$ and no other lamp on his way back to 0. Then again we will have unbounded distortion. Indeed consider the steps of the procedure in which the lamplighter writes the numbers $2^{n+2} + 2^n$ and $2^{n+3} + 2^n$. The ‘markers’ for these two configurations are the same (a lamp lit at position $-n$) however these configuration correspond to points that are close in X but far away in N .

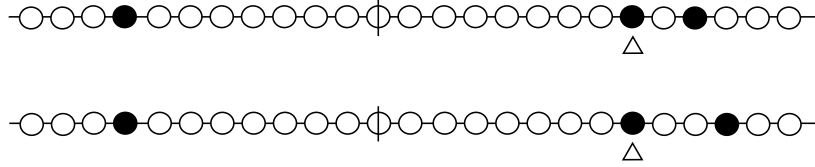


FIGURE 6. The more naive ‘marker’ depicted here does not suffice to bound distortion.

It is not very hard to see that X is separated also by a quasi-line. Indeed we define a quasi-line R by $R(i) = N(i)$ for all $i \in \mathbb{N}$ and we define $R(-i)$ to be the lamplighter at $-i$ and the lamps in $[-i, 0]$ are on (and all other lamps off). One sees easily that R is uniformly embedded in X . It is clear that if we take x_n, y_n as in the previous proof we have that $d(x_n, R) \geq n$, $d(y_n, R) \geq n$ so R coarsely separates X . That is we have the following:

Corollary 3.2. *There is a quasi-line R which coarsely separates the Cayley graph of L , X .*

Proposition 3.3. *L is one-ended and does not split over a 2-ended group.*

Proof. By Stallings’ theorem to show that L is one-ended it suffices to show that L does not split over a finite group. Assume that L splits over a finite or 2-ended subgroup C and consider the Bass-Serre tree

T of this splitting. Then if we write L as a semi-direct product

$$L = (\oplus_{\mathbb{Z}} \mathbb{Z}_2) \rtimes \mathbb{Z}$$

we claim that the direct sum $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ fixes some vertex v of the tree T . Indeed a finite group acting on a tree without inversions fixes a vertex of the tree. $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ is an ascending union of the finite groups

$$F_n = \langle x_i : i = -n, \dots, n \rangle$$

where x_i are the generators of the \mathbb{Z}_2 summands. If $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ does not fix a vertex of T , for arbitrarily big n F_n fixes a vertex v_n , which is not fixed by F_{n-1} . If u_n is a vertex fixed by F_{n-1} then the path joining u_n and v_n is fixed by F_{n-1} . This however contradicts our assumption that the edge stabilizers of T are all conjugate to a fixed finite or 2-ended group and hence have bounded torsion.

If the generator t of \mathbb{Z} sends v to $t(v) \neq v$ then, since $tx_it^{-1} = x_{i+1}$ (where x_i is the generator of the i -th \mathbb{Z}_2 factor) we have that $\oplus_{\mathbb{Z}} \mathbb{Z}_2$ fixes also $t(v)$, so there is an edge fixed by $\oplus_{\mathbb{Z}} \mathbb{Z}_2$. Clearly if $t(v) = v$ then L fixes a vertex of the tree. In both cases we see that there is no action with finite or 2-ended edge stabilizers.

□

3.3. Quasi-circles. We show now that there is a sequence of quasi-circles that coarsely separates the Cayley graph, X of L . In fact we show that a sequence of quasi-intervals I_n coarsely separates X . We then show that one can ‘close up’ these quasi-intervals to quasi-circles that coarsely separate X .

The construction of the quasi-intervals is similar to the construction of N .

We fix $n \in \mathbb{N}$. We define now a simple path I_n and a simple closed path C_n such that there are $x_n, y_n \in X$ which are separated by I_n (C_n) and such that

$$d(x_n, I_n) > n, d(y_n, I_n) > n, d(x_n, C_n) > n, d(y_n, C_n) > n$$

The interval I_n is defined as follows: We think of I_n as a path $I_n(i)$, $i \in \mathbb{N}$. If N is the quasi-ray defined in the previous section we set $I_n(i) = N(i)$ for all i until we reach the configuration consisting of 1’s in the positions $0, 1, \dots, 2n$, the lamplighter at the origin and 0 everywhere else. After this the lamplighter moves from position 0 to position $2n$ switching off the lamps as he goes. Let’s say that $I_n(k)$ is the vertex of X that we get when the lamplighter reaches $2n$.

When the lamplighter reaches $2n$ he repeats exactly the steps that he did at the beginning except he walks in the reverse direction. More formally, let’s denote by $q(i)$ the walk of the lamplighter, where for

any i , $q(i)$ is either a move by one to the right denoted by $+1$ or a move to the left, denoted by -1 or a change of switch denoted by 0 . Then if $q(k)$ is the lamplighter at $I_n(k)$ we define $q(k+i) = -q(i)$. The lamplighter continues until he reaches the following configuration: lamplighter at $2n$ and configuration consisting of 1 's in the positions $0, 1, \dots, 2n$, and 0 everywhere else. We will show now that the sequence I_n is a sequence of quasi-intervals. In order to analyse distortion we think of I_n as constructed in 3 stages. So we write

$$I_n = I_{n1} \cup I_{n2} \cup I_{n3}$$

where I_{n1}, I_{n2}, I_{n3} are defined as follows: I_{n1} is the initial path, so $I_{n1} = I_n \cap N$. I_{n3} is the last stage where the lamplighter moves exactly as in I_{n1} but in the reverse direction and starting from $2n$ rather than 0 . Finally I_{n2} is the path of I_n joining I_{n1}, I_{n3} . In order to show that I_n is a sequence of quasi-intervals it is enough to show that there are no sequences $a_n, b_n \in I_n$ such that

$$a_n = I_n(i_n), b_n = I_n(j_n)$$

with

$$\limsup_{n \rightarrow \infty} |i_n - j_n| = \infty$$

and $d(a_n, b_n) = M$ for some fixed $M \in \mathbb{N}$. Since N is uniformly embedded in X it is clear that it is not possible that for infinitely many n , a_n, b_n are contained both in I_{n1} or in I_{n3} . Since I_{n2} is a geodesic path it follows similarly that a_n, b_n can not be contained both in I_{n2} for infinitely many n . Finally since $d(I_{n1}, I_{n3}) > n$ it is not possible that, for infinitely many n one of the a_n, b_n is contained in I_{n1} and the other in I_{n3} . So at least one of the following two holds:

Case 1 : For infinitely many n , $a_n \in I_{n1}, b_n \in I_{n2}$.

Case 2 : For infinitely many n , $a_n \in I_{n2}, b_n \in I_{n3}$.

Assume now that we are in case 1 and consider the geometric representation of a_n, b_n . If the lamplighter for b_n is in $[0, M]$ then, since $d(a_n, b_n) \leq M$ the configurations for a_n, b_n are identical in $[2M, \infty)$. But, by the definition of I_n , this implies that $j_n - i_n$ is bounded by some constant $D(M)$ depending only on M . If the lamplighter for b_n lies in $[M+1, 2n]$ we remark that either the lamplighter for a_n is at 0 , so $d(a_n, b_n) > M$, or the configuration of a_n has some lamp on in $[-2n, 1]$. However in this case too $d(a_n, b_n) > M$.

We assume now that we are in case 2. If the lamplighter for a_n lies in $[0, 2n-M-1]$ then clearly $d(a_n, b_n) > M$. If the lamplighter for a_n lies in $[2n-M, 2n]$ then the configuration of b_n has to be identical with the configuration of a_n in $[0, 2n-M]$ (since $d(a_n, b_n) = M$). However

this implies that $j_n - i_n$ is smaller than a constant $D(M)$ that depends only on M . We see in both cases that it is impossible to have

$$\limsup_{n \rightarrow \infty} |i_n - j_n| = \infty$$

so I_n is indeed a sequence of quasi-intervals.

We show now that there are x_n, y_n in X such that

$$d(x_n, I_n) \geq n, d(y_n, I_n) \geq n$$

and x_n, y_n lie in distinct components of $X - I_n$. We take x_n to be the following configuration: lamplighter at n , 1's at the interval $[0, 2n]$ and 0's everywhere else. We take y_n to be the lamplighter at $-n$ and 0 everywhere. By the way they are defined x_n, y_n satisfy $d(x_n, I_n) \geq n, d(y_n, I_n) \geq n$. We remark now that I_{n1} contains all the configurations of the following form: lamplighter at 0, and 0's outside the interval $[0, 2n]$. Put it differently I_{n1} contains all possible configurations of 0, 1's in $[0, 2n]$ with the lamplighter at 0 and 0's elsewhere. Similarly I_{n3} contains all possible configurations in $[0, 2n]$ with the lamplighter at $2n$ and 0's elsewhere. It is clear that to move from x_n to y_n the lamplighter has to pass through 0. There are two cases.

Case 1 : The lamplighter while moving from x_n to y_n passes through the position 0 before passing through the position $2n$ (of course the lamplighter might not pass through the position $2n$ at all; then we are also in this case). In this case the first time the lamplighter arrives at 0, every lamp is off outside the interval $[0, 2n]$. So the lamplighter is at a point of I_{n1} .

Case 2 : The lamplighter while moving from x_n to y_n passes through the position $2n$ before passing through the position 0. In this case the first time the lamplighter arrives at $2n$, every lamp is off outside the interval $[0, 2n]$. So the lamplighter is at a point of I_{n3} .

It follows that $I_{n1} \cup I_{n3}$ separates x_n from y_n . This shows that the sequence of quasi-intervals I_n coarsely separates X .

It is easy to define a sequence of quasi-circles C_n which coarsely separates X , using the quasi-intervals I_n . Namely we join the endpoint of I_n to the initial point as follows: The lamplighter moves from $2n$ back to 0 turning off as he goes all lamps in the interval $[0, 2n]$. Let's call this last interval I_{n4} . So

$$C_n = I_{n1} \cup I_{n2} \cup I_{n3} \cup I_{n4}$$

It suffices to check that, given $M > 0$, that there are no sequences $a_n, b_n \in C_n$ with the following properties:

- 1) $d(a_n, b_n) = M$.

2) If l_n is the length of the shortest path contained in C_n joining a_n, b_n then $l_n \rightarrow \infty$.

Since $I_n = I_{n1} \cup I_{n2} \cup I_{n3}$ is uniformly embedded and since I_{n4} is a geodesic segment it follows that, for infinitely many n , one of a_n, b_n lies in I_n and the other lies in I_{n4} . However one sees easily by the definition of I_{n4} that if $l_n \rightarrow \infty$ then $d(a_n, b_n) \rightarrow \infty$. It follows that the C_n 's are uniformly embedded (i.e. it is a sequence of quasi-circles).

If we pick x_n, y_n as before we see that

$$d(x_n, C_n) \geq n, d(y_n, C_n) \geq n$$

and x_n, y_n are separated by C_n since they are already separated by $I_n \subset C_n$.

4. DISCUSSION

The idea behind the coarse separation properties of the lamplighter group comes from the fact that the asymptotic dimension of the lamplighter group is 1 ([6], [8]). We remark that if the asymptotic dimension of a finitely presented group is 1 then the group is virtually free ([6]). So one can think of the lamplighter group as a group theoretic (or ‘asymptotic’) analog of the Menger curve. As the topological dimension of the Menger curve is 1 the Menger curve is separated by a totally disconnected set (if one uses the standard construction of the Menger curve drilling holes in a cube, a separating totally disconnected set is obtained by a plane cutting the cube in two halves). It turns out that one can find a simple path containing this totally disconnected separating subset. Our construction is in a way a large scale version of this: If we try to visualize the set A defined before the proof of theorem 3.1 we see that as we move further and further from the identity and we look at larger scales the points in A are quite sparse. This is because even when two configurations are quite close the lamplighter has to move a big distance to transform one configuration to the other. Although we don’t prove this here, one could state this property of A formally by saying that A is a set of asymptotic dimension 0 that separates X . In the proof of the theorem 3.1 we show that there is a quasi-ray that contains A .

This remark prompts us to reformulate some of the questions treated in this paper.

Question. *Let G be a one-ended finitely generated group and let X be the Cayley graph of G . Assume that no subset of asymptotic dimension 0 of X coarsely separates X . Is it true that no sequence of quasi-circles coarsely separates X ? Is it true that no quasi-ray coarsely separates X ?*

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