

Irreducible polynomials over finite fields produced by composition of quadratics

D.R. Heath-Brown and G. Micheli
Mathematical Institute, Oxford

1 Introduction

If $f(X) = f^{(1)}(X) = X^2 + 1$ then the iterates $f^{(n)}(X) = f^{(n-1)}(f(X))$ are irreducible over \mathbb{Q} for all $n \geq 1$. Indeed the polynomials $f^{(n)}(X)$ are irreducible over \mathbb{F}_3 .

This paper will explore a number of questions suggested by this result. In particular we investigate the situation in which one composes more than one polynomial. We make the following definition:

Definition *Let $f_1(X), \dots, f_r(X)$ be polynomials of positive degree over a finite field \mathbb{F}_q . We say they are “dynamically-irreducible” if all polynomials formed by composition of f_1, \dots, f_r are irreducible over \mathbb{F}_q .*

Other authors have used the term “stable” for the situation $r = 1$ (see also [5]), but we believe that “dynamically-irreducible” is more suggestive in this general context.

We will be particularly interested in the situation in which the f_i are quadratic polynomials. When $r = 1$ a beautiful criterion for a quadratic to be dynamically-irreducible was developed by Boston and Jones [1, Proposition 2.3], but unfortunately the proof included a minor error.¹ After correcting this the criterion says that $f(X) = a\{(X - b)^2 + c\} \in \mathbb{F}_q$ is dynamically-irreducible if and only if f is irreducible, and $f^{(n)}(c) \notin a\mathbb{F}_q^2$ for $n \geq 1$. For example, when $f(X) = X^2 + 1 \in \mathbb{F}_3$ we see that $f^{(n)}(1) = 2 \notin \mathbb{F}_3^2$, for all $n \geq 1$. This confirms the claim made above that this polynomial is dynamically-irreducible (in fact, this is exactly the polynomial needed to build an infinite F-set in [4] for $p = 3$).

¹In the final display on page 1851, the second equality must be adjusted unless $g \circ f^{n-1}$ is monic.

If $f(X) = (X - b)^2 + b - 2 \in \mathbb{F}_q[X]$ we find that $f^{(n)}(b - 2) = b + 2$ for $n \geq 1$. If q is odd we therefore obtain a dynamically-irreducible polynomial whenever $2 - b$ and $2 + b$ are both non-squares in \mathbb{F}_q . It is an elementary exercise to show that a suitable b may always be found.

Since the set of iterates above must eventually produce a cycle it is clear from the above criterion that one can test in finite time whether a given quadratic polynomial over \mathbb{F}_q is dynamically-irreducible. Indeed Ostafe and Shparlinski [3] show that one has a repetition within $O(q^{3/4})$ steps, so that one can test whether a polynomial is dynamically-irreducible in $O(q^{3/4})$ operations.

The focus of this paper will be on large dynamically-irreducible sets of quadratic polynomials. The criterion of Boston and Jones has been extended as follows by Ferraguti, Micheli and Schnyder [2, Theorem 2.4].

Lemma 1 *Let $f_i(X) = (X - b_i)^2 + c_i \in \mathbb{F}_q$ be irreducible polynomials for $1 \leq i \leq r$. Then the set f_1, \dots, f_r is dynamically-irreducible if and only if every iterate*

$$(f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n})(c_j) \tag{1}$$

with $n \geq 1$ and $1 \leq i_1, \dots, i_n, j \leq r$, is a non-square in \mathbb{F}_q .

As an example (taken from [2]) we may consider the polynomials $(X - a)^2 + a$ and $(X - a - 1)^2 + a$ in \mathbb{F}_q , with $q \equiv 1 \pmod{4}$. One then finds that the iterates (1) take only the values a and $a + 1$. Thus, choosing a so that both a and $a + 1$ are non-squares (as we always can) we obtain a dynamically-irreducible set of size 2.

Notice that this criterion above is only formulated for the case in which all the polynomials f_i are monic. However, it has been pointed out by Alina Ostafe that one should be able to extend this criterion to non-monic quadratics, with consequences for the various results in the present paper. Indeed she is able to exhibit large dynamically-irreducible sets of non-monic quadratics over fields of prime cardinality $p \equiv 1 \pmod{4}$.

Our first new results concern the existence of large dynamically-irreducible sets of quadratic polynomials. Of course, without the restriction to quadratic polynomials, one can have infinitely large sets over \mathbb{F}_q , since once f is dynamically-irreducible, the set $f^{(1)}, f^{(2)}, \dots$ is dynamically-irreducible. We write $M(q)$ for the size of the maximal set of monic quadratic polynomials over \mathbb{F}_q which is dynamically-irreducible.

Theorem 1 *Let p be a prime and h a non-zero element of \mathbb{F}_p . Then the polynomial $X^p - X - h$ is irreducible over \mathbb{F}_p . Take ξ to be a root of $X^p - X - h$*

let $K = \mathbb{F}_p(\xi)$, so that K is a finite field with $q = p^p$ elements. Define polynomials

$$f_{b,c}(X) = (X - b - \xi)^2 + c + \xi \in K[X], \text{ for all } (b, c) \in \mathbb{F}_p \times \mathbb{F}_p.$$

If $p \equiv 1 \pmod{4}$, then the polynomials $f_{b,c}(X)$ form a dynamically-irreducible set over K whenever h is a non-square in \mathbb{F}_p .

Theorem 2 *There are infinitely many finite fields for which*

$$M(q) \geq \frac{1}{2}(\log q)^2.$$

The coefficient $1/2$ can be replaced by any constant $C < (\log 4)^{-2}$. There is such a finite field for any sufficiently large characteristic $p \equiv 1 \pmod{4}$.

Notice that the explicit construction in Theorem 1 yields $M(p^p) \geq p^2$ and hence provides examples with $M(q) \geq (\log q)^2(\log \log q)^{-2}$.

Unfortunately Theorems 1 and 2 do not provide fields of prime order in which there are large dynamically-irreducible sets. A little experimentation shows that $M(3) = 1$, $M(5) = 3$, $M(7) = 2$, and $M(11) = 1$. In particular, over \mathbb{F}_5 , the three polynomials

$$(x - 2)^2 + 2, (x - 3)^2 + 2, x^2 + 3$$

form a dynamically-irreducible set. Similarly, over \mathbb{F}_{13} , the three polynomials

$$(x - 1)^2 - 2, (x - 9)^2 - 6, (x - 3)^2 - 5$$

form a dynamically-irreducible set. As mentioned above, we can find a dynamically-irreducible set of size 2 over \mathbb{F}_p whenever $p \equiv 1 \pmod{4}$.

Our next result describes the set of iterates (1).

Theorem 3 *Let f_1, \dots, f_r be a dynamically-irreducible set of monic quadratic polynomials over a finite field \mathbb{F}_q of odd characteristic, and suppose that $r \geq 2$. Then the set of iterates (1), together with the values of the c_j , has size at most $((\log 5)^{-2} + (\log 2)^{-2})(\log q)^2 \sqrt{q}$.*

Notice that the upper bound given by Theorem 3 is uniform in r .

One should compare this to the corresponding result for $r = 1$ given by Ostafe and Shparlinski [3] which we mentioned above, and in which the size is $O(q^{3/4})$. It may seem counter-intuitive that one should have fewer values (1), despite having more polynomials to use (and possibly more starting points). However the requirement that all the elements obtained should be non-squares imposes the additional conditions which compress the size of the set of values of (1).

We have three corollaries to Theorem 3.

Corollary 1 *Let $C = (\log 5)^{-2} + (\log 2)^{-2}$. Let f_1, \dots, f_r be a set of monic dynamically-irreducible polynomials, all with the same value for c_j . Then $r \leq 2C(\log q)^2 \sqrt{q}$.*

Corollary 2 *Let $C = (\log 5)^{-2} + (\log 2)^{-2}$. For any odd prime power q we have $M(q) \leq C^2(\log q)^4 q$.*

Corollary 3 *Let \mathbb{F}_q be a finite field of odd characteristic. There is an algorithm to test whether or not a set f_1, \dots, f_r of monic quadratic polynomials over \mathbb{F}_q is dynamically-irreducible, which takes $O(r(\log q)^3 q^{1/2})$ operations, and requires $O((\log q)^2 q^{1/2})$ storage locations.*

The upper bound in Corollary 2 is disappointingly weak. The proof we give is extremely simple and discards much of the available information. It would be interesting to know whether a more sophisticated approach would lead to an improved estimate for $M(q)$.

As mentioned above Ostafe and Shparlinski [3] show that one can test whether a quadratic polynomial is dynamically-irreducible in $O(q^{3/4})$ operations. It is pleasing to see that Corollary 3 produces a faster algorithm for $r = 2$, say, than one has for $r = 1$. However when $r = 1$ the space requirement is $O(1)$, while our algorithm needs non-trivial amounts of memory since it uses a tree structure.

Acknowledgments. Giacomo Micheli was supported by the Swiss National Science Foundation grant number 161757 and 171249.

We would also like to record our thanks to Professor Edith Elkind, who suggested to us the use of Red-Black trees in the proof of Corollary 3, and to the anonymous referee, who suggested improvements to the mathematics and the exposition of this paper.

2 Proof of Theorems 1 and 2

To prove that $T(X) = X^p - X - h$ is irreducible over \mathbb{F}_p we begin by observing that if ξ is a root of $T(X)$ then so is $\xi + a$ for any $a \in \mathbb{F}_p$. Thus the roots of $T(X)$ are precisely the values $\xi + a$ as a runs over \mathbb{F}_p . Let $G(X) \in \mathbb{F}_p[X]$ be a factor of $T(X)$ of degree d , with roots $\xi + a_1, \dots, \xi + a_d$ say. Then the sum of the roots is given by the coefficient of X^{d-1} , and hence lies in \mathbb{F}_p . However the sum of the roots will be $d\xi + a_1 + \dots + a_d$. If this is in \mathbb{F}_p then either $p \mid d$ or $\xi \in \mathbb{F}_p$. However $T(n) = n^p - n - h = -h \neq 0$ for every $n \in \mathbb{F}_p$, whence G must have degree 0 or p . The proof of this fact (which is the basis for Artin-Schreier theory) can also be found in [6, p. 325].

We now come to the key idea for the two theorems. Suppose now that we have an extension M/L of finite fields of odd characteristic and $\xi \in M$. For $b, c \in L$, the polynomials

$$f_{b,c}(X) = (X - b - \xi)^2 + c + \xi.$$

are irreducible over M if $-c - \xi$ is not a square in M . Let $g_{b,c}(X) = (X - b)^2 + c$, $G = g_{b_1, c_1} \circ \dots \circ g_{b_n, c_n}$ and $F = f_{b_1, c_1} \circ \dots \circ f_{b_n, c_n}$. Observe that $F(X) = G(X - \xi) + \xi$. For any $c \in L$ we have $F(c + \xi) = G(c) + \xi$. However $G(c) \in L$. It therefore follows from Lemma 1 that the polynomials $f_{b,c}$ form a dynamically-irreducible set over M , provided that every element of the additive cosets $L + \xi$ and $L - \xi$ is a non-square in M . If -1 is a square in L it will be enough to consider $L + \xi$.

For Theorem 1 we take $L = \mathbb{F}_p$ and $M = K$. If $a + \xi$ were a square in K , then its norm $N_{K/\mathbb{F}_p}(a + \xi)$ would be a square in \mathbb{F}_p . However this norm is simply the product of the roots of $F(X) = X^p - X - h$, which is h . Thus if h is a non-square in \mathbb{F}_p , then every element $a + \xi$ is a non-square in K . This completes the proof of Theorem 1.

For Theorem 2, we let q be a power p^e of a prime $p \equiv 1 \pmod{4}$, and apply the previous ideas with $L = \mathbb{F}_p$ and $M = \mathbb{F}_q$. We therefore hope to find an element $\alpha \in \mathbb{F}_q$ such that $a + \alpha$ is a non-square in \mathbb{F}_q for every $a \in \mathbb{F}_p$. For this we will use estimates for character sums.

Let χ be the quadratic character for \mathbb{F}_q , and consider

$$S = \sum_{\alpha \in \mathbb{F}_q} \prod_{a \in \mathbb{F}_p} \{1 - \chi(a + \alpha)\}. \quad (2)$$

Our goal is therefore to show that $S > 0$, so that there must be some $\alpha \in \mathbb{F}_q \setminus \mathbb{F}_p$ such that $a + \alpha$ is a non-square for every a .

On expanding the product in (2) we see that

$$S = q + \sum_H \varepsilon(H) \sum_{\alpha \in \mathbb{F}_q} \chi(H(\alpha)),$$

where H runs over polynomials of the form

$$H(X) = \prod_{a \in A} (X + a)$$

for the various non-empty subsets $A \subseteq \mathbb{F}_p$, and $\varepsilon(H) = (-1)^{\#A}$. It follows from Weil's bound for character sums that

$$\left| \sum_{\alpha \in \mathbb{F}_q} \chi(H(\alpha)) \right| \leq \deg(H) q^{1/2},$$

whence

$$S \geq q - q^{1/2} \sum_{k=1}^p \binom{p}{k} k = q - q^{1/2} p 2^{p-1}.$$

It follows that a suitable α exists as soon as $q \geq p^2 4^p$. In particular, it suffices to take e as the first integer greater than equal to $2 + p(\log 4)/(\log p)$. With this choice we have

$$e \leq 3 + p(\log 4)/(\log p) \leq \sqrt{2} \frac{p}{\log p}$$

provided that we take p large enough. As a result we have $\log q \leq \sqrt{2}p$, whence $M(q) \geq p^2 \geq \frac{1}{2}(\log q)^2$, as claimed. Moreover we may replace $\sqrt{2}$ by $C^{1/2}$ for any constant $C < (\log 4)^{-2}$.

3 Proof of Theorem 3

For $q = 3$ the theorem is clearly true, so we can restrict to $q \geq 5$. First, we need an ancillary result

Lemma 2 *Let f_1, f_2 be distinct monic quadratic polynomials over \mathbb{F}_q , with q odd. Suppose that*

$$f_{i_1} \circ \dots \circ f_{i_n} = f_{j_1} \circ \dots \circ f_{j_m} \tag{3}$$

with $i_1, \dots, i_n, j_1, \dots, j_m \in \{1, 2\}$. Then $m = n$ and $i_h = j_h$ for every index h .

Proof. If (3) holds, the two sides have degrees 2^n and 2^m so that we must have $n = m$. We now argue by contradiction, supposing that we have a non-trivial relation (3) in which n is minimal. Then $f_i \circ F = f_j \circ G$, say, in which either $f_i \neq f_j$, or $f_i = f_j$ but $F \neq G$. Let $f_i(X) = (X - a)^2 + b$ and $f_j(X) = (X - c)^2 + d$. Then

$$\begin{aligned} b - d &= (G(X) - c)^2 - (F(X) - a)^2 \\ &= \{G(X) + F(X) - a - c\} \{G(X) - F(X) + a - c\}. \end{aligned}$$

Since F and G are monic, and \mathbb{F}_q has odd characteristic, the polynomial $G(X) + F(X) - a - c$ has positive degree. We therefore see that $b = d$, and that $G(X) - F(X) + a - c = 0$. If $a = c$ we would have $f_i = f_j$ and $F = G$, giving us a contradiction. Hence $a \neq c$ so that f_1 and f_2 are $(X - a)^2 + b$ and $(X - c)^2 + b$, in some order. Moreover $F \neq G$, so that we must have $n \geq 2$.

Now let $F = f_r \circ U$ and $G = f_s \circ V$, say, with $f_r(X) = (X - e)^2 + b$ and $f_s(X) = (X - f)^2 + b$ for $\{e, f\} = \{a, c\}$. Then

$$\begin{aligned} 0 &\neq a - c \\ &= F(X) - G(X) \\ &= (U(X) - e)^2 - (V(X) - f)^2 \\ &= \{U(X) + V(X) - e - f\}\{U(X) - V(X) + f - e\}. \end{aligned}$$

This however is impossible, since the factor $U(X) + V(X) - e - f$ has positive degree, its leading coefficient being $2 \neq 0$. This contradiction completes the proof of the lemma. \square

Let now $\mathcal{J} \subseteq \mathbb{F}_q$ denote the set of values (1), together with the values of the c_j . Assuming that f_1, \dots, f_r form a dynamically-irreducible set, one sees that $(f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n})(\beta)$ must be a non-square in \mathbb{F}_q for every $\beta \in \mathcal{J}$. We shall work with a finite subset \mathcal{F} of distinct polynomials given as compositions $f_{i_1} \circ \dots \circ f_{i_n}$ for some fixed $n \geq 1$. Thus \mathcal{F} consists of distinct irreducible monic polynomials of degree 2^n .

If χ is the quadratic character for \mathbb{F}_q we have

$$\prod_{F \in \mathcal{F}} \{1 - \chi(F(\beta))\} \begin{cases} = 2^{\#\mathcal{F}}, & \beta \in \mathcal{J}, \\ \geq 0, & \beta \notin \mathcal{J}, \end{cases}$$

and it follows that

$$\#\mathcal{J} \leq 2^{-\#\mathcal{F}} \sum_{\beta \in \mathbb{F}_q} \prod_{F \in \mathcal{F}} \{1 - \chi(F(\beta))\}. \quad (4)$$

We proceed to expand the product and to use the Weil bound to estimate the resulting character sums

$$\sum_{j \in \mathbb{F}_q} \chi(F_1(\beta) \dots F_m(\beta)).$$

The polynomial $F_1 \dots F_m$ will be square-free, with degree $2^m m \leq 2^n \#\mathcal{F}$, whence the Weil bound produces

$$\left| \sum_{\beta \in \mathbb{F}_q} \chi(F_1(\beta) \dots F_m(\beta)) \right| \leq 2^n (\#\mathcal{F}) q^{1/2}$$

as long as $m \geq 1$. The sum corresponding to $m = 0$ is just q , and there are $2^{\#\mathcal{F}} - 1$ other sums, so that (4) yields

$$\#\mathcal{J} \leq 2^{-\#\mathcal{F}} q + 2^n (\#\mathcal{F}) q^{1/2}. \quad (5)$$

However at this point we encounter a potential difficulty. If it were true that all compositions $f_{i_1} \circ \dots \circ f_{i_n}$ were different we could take $\#\mathcal{F} = 2^n$. Unfortunately this is not the case. For example if $f_1(X) = (X+1)^2$, $f_2(X) = X^2$ and $f_3(X) = X^2+1$ then $f_1 \circ f_2 = f_2 \circ f_3$. Of course this problem does not arise when $r = 1$ since in this situation the iterates will have different degrees. Lemma 2 come now into help: we take \mathcal{F} to consist of all compositions of n polynomials each of which is either f_1 or f_2 . By Lemma 2 we obtain 2^n distinct polynomials this way, and (5) becomes

$$\#\mathcal{J} \leq 2^{-2^n} q + 4^n q^{1/2}.$$

Finally, for $q \geq 5$ we choose $n \geq 1$ so that

$$\frac{\log q}{\log 4} \leq 2^n < 2 \frac{\log q}{\log 4},$$

whence

$$\#\mathcal{J} \leq \sqrt{q} + \left(\frac{\log q}{\log 2} \right)^2 \sqrt{q} \leq (1/(\log 5)^2 + 1/(\log 2)^2)(\log q)^2 \sqrt{q},$$

as claimed.

4 Proof of the Corollaries

We now prove Corollary 1. By the assumption, we have that $c_j = u \in \mathbb{F}_q$ for all $j \in \{1, \dots, r\}$. Consider the map

$$\begin{aligned} \Gamma : \{f_1, \dots, f_r\} &\longrightarrow \mathbb{F}_q \\ f_i &\mapsto f_i(u). \end{aligned}$$

Suppose that $f_i(u) = f_j(u) = f_k(u)$ for distinct $i, j, k \in \{1, \dots, r\}$, then

$$(u - a)^2 = (u - b)^2 = (u - c)^2$$

for some $a, b, c \in \mathbb{F}_q$. After reordering a, b, c if necessary, this implies $u = (a + b)/2 = (a + c)/2$, which forces $b = c$, but then $f_j = f_k$. This shows that the map Γ is at most 2-to-1. Since the image of Γ is clearly contained in the set of iterates (1), the claim follows by applying directly Theorem 3.

We now prove Corollary 2. Since the bound of Corollary 1 is stronger, without loss of generality we can assume that c_1 and c_2 are distinct (in the notation of Lemma 1). Consider now the map

$$\Gamma' : \{f_1, \dots, f_r\} \longrightarrow \mathbb{F}_q \times \mathbb{F}_q$$

$$f_i \mapsto (f_i(c_1), f_i(c_2)).$$

Since Γ' is injective, Theorem 3 leads directly to $r \leq (C(\log q)^2 \sqrt{q})^2$.

Finally we tackle Corollary 3. The algorithm uses a “Red-Black tree”, for a description of which we refer the reader to [7, Chapter 13]. In brief, a red-black tree is a binary search tree which allows rapid detections of duplicates. We assume that the elements of \mathbb{F}_q are described in such a way as to enable us to impose an ordering on them. For example, if we have $q = p^d$ and take $\mathbb{F}_q = \mathbb{F}_p[\theta]$ where θ is the root of an irreducible polynomial of degree d , we might arrange the elements $a_0 + a_1\theta + \dots + a_{d-1}\theta^{d-1}$ using lexicographic order. Our algorithm inputs a set of irreducible quadratic polynomials f_1, \dots, f_r . It then uses the following steps.

1. Build a Red-Black tree T , starting with the empty set and successively adding the values c_1, \dots, c_r , using the ordering mentioned above, and discarding any duplicate values.
2. Set $j = 0$ and form a list L_j of the elements in T .
3. Set $L = \emptyset$. Then, for each $i = 1, \dots, r$ and each $t \in L_j$
 - i) Check whether $f_i(t)$ is a square. If it is, STOP (the polynomials do not form a dynamically-irreducible set).
 - ii) Otherwise check whether $f_i(t)$ is already in T . If it is, move on to the next pair i, t .
 - iii) If $f_i(t)$ is not yet in T insert it into the tree, and add it to the list L .
4. If L is empty, STOP (the polynomials form a dynamically-irreducible set).
5. If L is non-empty, add one to the value of j and set $L_j = L$. Return to Step 3.

If the algorithm stops at Step 3(i) it has found a square iterate, and so it correctly reports that we do not have a dynamically-irreducible set. Otherwise, when we begin Step 3 the tree T contains all distinct values (1) with $n \leq j$, and the list L_j will consist of all such values with $n = j$ which cannot be obtained from any smaller n . If we stop at Step 4 then there are no new values with $n = j + 1$, so that further iteration will always produce results already contained in T . In this case all iterates will be non-squares

and the algorithm correctly reports that we have a dynamically-irreducible set of quadratics.

One readily sees that, whenever we begin Step 3, the size of T is $\#L_0 + \dots + \#L_j$. On the other hand, Theorem 3 shows that T can have at most $B = \lceil 4(\log q)^2 \sqrt{q} \rceil$ elements. Since the algorithm will terminate at Step 4 unless L is non-empty, it is clear that we must stop after at most B loops.

To analyze the running time of the algorithm we note that we may test an element of \mathbb{F}_q to check whether it is a square using $O(\log q)$ field operations. Moreover, since T has size at most B , we can check whether an element belongs to T , and if not add it to T , in $O(\log q)$ operations. Step 3 requires $r(\#L_j)$ tests of this type, so that the total number of operations needed is

$$\ll r(\#L_0 + \#L_1 + \dots) \log q \ll rB \log q \ll r(\log q)^3 q^{1/2},$$

as claimed. As to the memory requirement, the tree T and the lists L_j will need space $O(B)$. This completes the proof of the corollary.

The use of a Red-Black tree, into which one may insert new elements in order, was suggested to us by Professor Edith Elkind. It is a pleasure to record our thanks for this.

References

- [1] R. Jones and N. Boston, Settled polynomials over finite fields, *Proc. Amer. Math. Soc.* 140 (2012), 1849–1863.
- [2] A. Ferraguti, G. Micheli and R. Schnyder, On sets of irreducible polynomials closed by composition, to appear in *Lecture Notes in Computer Science*, arXiv:1604.05223.
- [3] A. Ostafe and I.E. Shparlinski, On the length of critical orbits of stable quadratic polynomials, *Proc. Amer. Math. Soc.* 138 (2010), 2653–2656.
- [4] A. Ferraguti and G. Micheli, On the existence of infinite, non-trivial F-sets, *Journal of Number Theory*, 168, (2016), 1–12.
- [5] D. Gómez-Pérez, A. P. Nicolás, A. Ostafe and D. Sadornil Stable Polynomials over finite fields *Revista Matemática Iberoamericana*, 30, Vol. 2 (2014) 523–535.
- [6] S. Lang Algebra, Graduate texts in mathematics, vol 211 (2002), Springer-Verlag, New York.

- [7] T. H. Cormen, C. E. Leiserson, R. Rivest, C. Stein Introduction to algorithms, second edition, 2001, The MIT Press.

Mathematical Institute,
Radcliffe Observatory Quarter,
Woodstock Road,
Oxford
OX2 6GG
UK

`rhb@maths.ox.ac.uk` and `giacomo.micheli@maths.ox.ac.uk`