Operator Logarithms and Exponentials

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Abstract

Since McIntosh's introduction of the $\mathcal{H}^\infty$-calculus for sectorial operators, the topic has been studied by many authors. Haase has constructed a similar functional calculus for strip-type operators, and has also developed an abstract framework which unifies both of these examples and more. In this thesis we use this abstract functional calculus setting to study two particular problems in operator theory.

The first of these is concerned with operator sums. We ask the question of when the sum $\log A + \log B$ is closed, where A and B are a pair of injective sectorial operators whose resolvents commute. We show that the sum is always closable and, when A and B are invertible, we determine sufficient conditions for the sum to be closed. These conditions are of Kalton-Weis type, and in fact ensure that AB is sectorial and that the identity $\log A + \log B = \log(AB)$ holds. We then identify an interpolation space on which these conditions are automatically satisfied.

Our second problem is connected to the exponential of a strip-type operator B, specifically the question of whether e^B is sectorial. When $-1 \in \rho(e^B)$, the spectrum of e^B lies in a sector, and we obtain an estimate on the resolvent outside this sector. This estimate becomes closer to sectoriality as more restrictions are placed on the resolvents of B itself. This leads us to introduce the ideas of F-sectorial and F-strong strip-type operators, whose spectra are contained in a sector or strip, but which satisfy a different resolvent estimate from that of a sectorial or strong strip-type operator. In some cases it is possible to define the logarithm of an F-sectorial operator or the exponential of an F-strong strip-type operator. We prove resolvent estimates for the resulting logarithms and exponentials, and explore the relationships between the various classes of operators considered.

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Chapter 1

Introduction

1.1 Background

The basic idea behind functional calculus is to provide a reasonable definition for expressions of the form $f(A)$, where A is some closed linear operator on a Banach space X, and f belongs to a suitable class of functions. By a *reasonable definition*, we mean that the family of operators thus defined should in some way exhibit behaviour similar to that of the original functions themselves. One of the earliest examples of such a functional calculus is that developed by Riesz and Dunford for bounded operators [13]. More recently, McIntosh has introduced a holomorphic functional calculus for sectorial operators [35].

This holomorphic functional calculus provides a unified theory for dealing with various operators associated to a sectorial operator A. For example, the fractional powers $(A^z)_{z\in\mathbb{C}}$ and the semigroup $(e^{-tA})_{t\geq 0}$ generated by $-A$, both of which have been extensively researched in their own right [14, 34], can be studied in a functional calculus framework. Another function of interest is the logarithm function. The logarithm of an injective sectorial operator was first defined by Nollau in [38], and has been subsequently studied by Okazawa [39] and Haase [19]. It is well-known that $\log A$ is an example of a *strip-type operator*. The class of strip-type operators also includes generators of C_0-groups, and it is possible to construct a functional calculus for such operators [19].

As well as being an important topic in its own right, functional calculus can be used to provide a new setting for problems in operator theory, often simplifying the situation and sometimes even leading to new proofs of well-known results. For example, Haase has set up a functional calculus for operators whose spectrum lies in a half-plane [24], a class of operators which includes generators of C_0-semigroups. Consequently it is possible to prove some of the classic theorems of semigroup theory (e.g., Hille-Yosida, Trotter-Kato) using fundamental tools of functional calculus such as the Convergence Lemma [35, Section 4]. In [25], Haase has obtained new proofs of the theorems of Dore-Venni and Monniaux (both of which we shall discuss in more detail below), as well as Fattorini's Theorem for cosine functions, originally proved in [15]. This result says that if B generates a cosine function on a UMD space, then the first component of the phase space is essentially equal to the domain of $B^{1/2}$.

The contents of this thesis can broadly be divided into two main areas, in both of which we use functional calculus methods to study a particular problem in operator theory. The first problem we look at concerns operator sums. Let A and B be a pair of sectorial operators with commuting resolvents on a Banach space X. The question of when the sum $A + B$ is closed has a long history, beginning with the paper of Da Prato and Grisvard [8]. It was shown there that the sum is always closable, and that the restriction of the sum to an *interpolation space* between X and the domain of either A or B is actually closed. Sufficient conditions for the sum to be closed on the original space X were first proved in the seminal paper by Dore and Venni [12].

Theorem 1.1.1. *[12, Theorem 2.1] Let A and B be invertible sectorial operators with commuting resolvents on the Banach space X. Suppose that*

1. *A and B have bounded imaginary powers, and there exist constants C, θ_A and θ_B such that*

$$\|A^{is}\| \leq Ce^{\theta_A|s|} \qquad \|B^{is}\| \leq Ce^{\theta_B|s|} \qquad (s \in \mathbb{R}),$$

where $\theta_A + \theta_B < \pi$.

2. X is a UMD space.

Then the sum $A + B$ *is closed, and in fact the operator* $(A + B)^{-1}$ *is bounded.*

Since one of the operators A or B can be taken to be a differential operator, this result has important applications to maximal regularity and solutions of abstract evolution equations [12, Section 3]. Prüss and Sohr showed in [43] that Theorem 1.1.1 remains true if A and B are simply assumed to be injective rather than invertible. Lancien et al. proved in [32] that the sum $A+B$ is closed if the pair (A, B) has a joint bounded $\mathcal{H}^\infty$-calculus, without the assumption that X is a UMD space. Moreover they proved that, in this case, $A + B$ itself also has a bounded $\mathcal{H}^\infty$-calculus. Most recently, Kalton and Weis showed in [30] that the sum is closed if A has a bounded $\mathcal{H}^\infty$-calculus and B is R-sectorial, again without any geometrical assumptions on the Banach space X. There are even some results for sums of operators whose resolvents do not commute [37, 42].

It is easy to formulate the corresponding problem for strip-type operators, in particular for logarithms of sectorial operators. It might be expected that, if the sum $\log A + \log B$ were closed, an identity of the form $\log A + \log B = \log(AB)$ could hold. It seems that this problem has not been studied up to now. Indeed, Nollau writes as a footnote in [38]:

> *Eine vollständige Analogie zu der Beziehung* $\log ab = \log a + \log b$ *ist für zwei Operatoren A und B vom Typ(M) nicht zu erwarten, da der Operator AB i.a. - selbst unter der Voraussetzung der Vertauschbarkeit von A und B - nicht vom Typ(M) ist.* [1]

[1] A complete analogy of the relationship $\log ab = \log a + \log b$ for two operators A and B of type M is not to be expected, since the operator AB is not in general, even under the assumption of commutativity of A and B, of type M.

where the term *Typ(M)* means sectorial. However, this footnote was written some time before the appearance of several results which give sufficient conditions for AB (or more generally $\overline{AB}$) to be sectorial [26, 32, 43].

If the logarithm function provides a means of taking a sectorial operator to a strip-type operator, it seems reasonable that the exponential function might send a strip-type operator to a sectorial operator. Indeed, if B is the logarithm of some sectorial operator A, then e^B is sectorial and $e^B = A$ [22, Corollary 4.2.5]. However, the question of precisely which strip-type operators are logarithms of some sectorial operator (i.e., those strip-type operators B for which e^B is sectorial) remains open, and forms the basis of the second half of this thesis. The theorem of Monniaux [36] seems to be the only substantial result in this direction.

Theorem 1.1.2. *[36, Theorem 4.3] Let B be a strip-type operator on a UMD space X such that iB generates a C_0-group of type strictly less than π. Then e^B is sectorial.*

At the worst end of the scale, there do exist strip-type operators B such that the spectrum of e^B is the whole complex plane [22, Corollary 8.4.6]. Haase has shown that if B is a strip-type operator of height ω such that $\rho(e^B)$ contains some point λ with $|\arg \lambda| > \omega$, then in fact the spectrum of e^B is contained in a sector [22, Lemma 4.4.2]. The question of whether non-emptiness of the resolvent set implies sectoriality of e^B also seems to be open.

The idea of an *almost sectorial* operator, namely an operator whose spectrum is contained in a sector but whose resolvent need not satisfy the required estimate for sectoriality, has been studied by various authors. DeLaubenfels [9] constructed a functional calculus for operators whose resolvent is polynomially bounded outside a sector. Such operators have also been studied by Periago and Straub [40, 41], who constructed an alternative functional calculus based on that of McIntosh for sectorial operators. Gorodniĭ and Chaĭkovskiĭ [17] have recently considered the more general idea of an operator A such that $\|R(\lambda, A)\| \leq CG(|\lambda|)$ for λ outside some sector, where G is a non-increasing function on $[0, \infty)$, such that $G(t) \to 0$ as $t \to \infty$ and such that

$1/G$ is Lipschitz. They studied the fractional powers $(A^z)_{z\in\mathbb{C}}$ of such an operator, as well as the associated semigroup $(e^{-tA})_{t\geq 0}$.

We have already mentioned the idea of an interpolation space, in connection with the work of Da Prato and Grisvard on operator sums. The study of interpolation theory in Banach spaces began in the late 1950s, and was developed by various authors, including Lions, Peetre and Calderón. There are several comprehensive accounts of the subject [33, 44]. Among the various methods for constructing interpolation spaces, the most important are those which give rise to the so-called *real* interpolation spaces, and those which give rise to the *complex* interpolation spaces. In this thesis we shall only be concerned with real interpolation spaces, which have an interesting relationship with the functional calculus; roughly speaking it seems that certain functional calculus properties of an operator actually improve if one looks at the part of the operator in some suitable real interpolation space [10, 11, 23]. We will study real interpolation spaces in connection with both of the topics mentioned above.

1.2 Overview of Thesis

As functional calculus is such a central theme to this thesis, we wish to have all of the required definitions and results available for reference. We present all of the necessary background information in Chapter 2. We begin by outlining the abstract methods developed by Haase, then explain how these methods can be used to construct the familiar functional calculus of a sectorial or a strip-type operator, including modifications to cover certain spectral properties such as invertibility. We also show how it is possible to construct functional calculi based on operator-valued functions, or functions of two variables.

In Chapter 3 we focus on the first of the two main problems identified above, namely the question of when the sum $\log A + \log B$ is closed, where A and B are a pair of injective sectorial operators whose resolvents commute. It turns out that this sum is always closable, and that its closure can be identified in terms of the logarithm of the product of A and B (whenever this product is sectorial). When A and B are invertible, we show that the sum is actually closed under the same conditions as the Kalton-Weis Theorem of [30], i.e., when one of the operators A and B has a bounded $\mathcal{H}^\infty$-calculus, and the other is R-sectorial. Our methods also show that, in this case, the identity $\log A + \log B = \log(AB)$ holds. In Chapter 4, after providing the required background on the theory of real interpolation, we identify an interpolation space on which these conditions are automatically satisfied.

In Chapter 5 we turn our attention from operator sums to exponentials. If B is a strip-type operator with $-1 \in \rho(e^B)$, then we show that it is possible to obtain an estimate on the norm of $\lambda R(\lambda, e^B)$ for λ outside some sector. The precise nature of this estimate depends on the growth of the norms of the resolvents of B, and whilst we may not be able to prove that e^B is sectorial, we do obtain some logarithmic estimates which are very close to sectoriality. We also prove a necessary and sufficient condition for e^B to be sectorial when B is a strip-type operator whose resolvents decay to 0 sufficiently quickly. Using known Fourier multiplier theorems, we show that $-1 \in \rho(e^B)$ when B is the derivative on the real line, considered in some Besov space. We conclude the chapter by applying our norm estimates to this example.

In Chapter 6 we consider operators which are *almost sectorial*, in the sense described above. In particular we introduce the concept of an *F-sectorial operator*, based on the work in [17]. A functional calculus can be constructed for such operators and, inspired by the results of Chapter 5, we look at particular examples whose logarithm can be defined. We investigate how the properties of this logarithm compare with those of the logarithm of a sectorial operator. Specifically, we prove a representation of the resolvent outside the horizontal strip of height π, and use this to estimate the norm of the resolvent outside this strip.

The logarithms of these F-sectorial operators can be thought of as *almost strip-type* operators, in that their spectrum lies in a horizontal strip, but the resolvent need not be bounded outside this strip. This leads us to define the idea of an F-*strong strip-type operator* in Chapter 7. We show that a functional calculus can be constructed for such an operator, and that in certain cases it is possible to define the exponential. Using the same techniques as in Chapter 5, we show that the exponential of certain F-strong strip-type operators is again an F-sectorial operator. Finally we summarise the relationships between all of the various classes of operators considered.

1.3 Notation and Definitions

Linear Operators and Banach Spaces

Given a linear operator A on a Banach space X, we denote its domain and range by $D(A)$ and $R(A)$ respectively. The **graph** $G(A)$ of A is the subset of $X \times X$ defined by

$$G(A) := \{ (x, Ax) : x \in D(A) \}.$$

If $G(A)$ is a closed subset of $X \times X$ then we say that A is **closed**, and we write $\mathcal{C}(X)$ to denote the space of all closed linear operators on X. If furthermore there exists a constant C such that

$$\|Ax\| \leq C \|x\| \qquad (x \in X),$$

where $\|\cdot\|$ denotes the norm on X, then we say that A is **bounded**, and write $\mathcal{L}(X)$ for the Banach algebra of bounded operators on X.

If the operator A is injective we define its inverse A^{-1} by

$$D(A^{-1}) = R(A) \qquad A^{-1}x = y,$$

where $y \in D(A)$ is the unique element such that $Ay = x$. If A^{-1} is bounded then we say that A is **invertible**. If $\lambda \in \mathbb{C}$ is such that $\lambda - A$ is invertible, then we say that λ belongs to the **resolvent set** $\rho(A)$ of A, and we write $R(\lambda, A)$ to denote the **resolvent operator** $(\lambda - A)^{-1}$ of A at λ. The **spectrum** of A is defined to be the set $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

For $A, B \in \mathcal{C}(X)$ we write $A \subset B$ if $D(A) \subset D(B)$ and $Bx = Ax$ for all $x \in D(A)$. Thus $A = B$ if and only if both inclusions $A \subset B$ and $B \subset A$ hold. We define the sum $A + B$ and product AB as follows:

$$\begin{aligned} D(A+B) &= D(A) \cap D(B) & (A+B)x &= Ax + Bx, \\ D(AB) &= \{ x \in D(B) : Bx \in D(A) \} & (AB)x &= A(Bx). \end{aligned}$$

Two closed operators A and B are said to be **resolvent commuting** if

$$R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$$

for all $\lambda \in \rho(A)$ and $\mu \in \rho(B)$. A closed operator A and a bounded operator T are said to **commute** if $TA \subset AT$. Note that T commutes with A if and only if T commutes with the resolvents of A [2, Proposition B.7]. If Y is another Banach space which is continuously embedded in X, then the operator A_Y defined by

$$D(A_Y) = \{ y \in D(A) \cap Y : Ay \in Y \} \qquad A_Y y = Ay$$

is said to be the **part** of A in Y

Let (Ω, μ) be a σ-finite measure space. For $1 \leq p \leq \infty$ we denote by $L^p(\Omega; X)$ the space of all measurable, p-integrable, X-valued functions on Ω. $C(\Omega; X)$ denotes the space of continuous functions from X to Ω, $C^\infty(\Omega; X)$ the space of infinitely differentiable continuous functions, and $C_0^\infty(\Omega; X)$ the space of infinitely differentiable functions which vanish at infinity.

A Banach space X is said to be a **UMD space** if the **Hilbert transform** H, defined by

$$(Hf)(t) = \frac{i}{\pi} \lim_{\varepsilon \to 0} \int_{|s| \geq \varepsilon} \frac{f(t-s)}{s} \, ds \qquad (f \in C_0^\infty(\mathbb{R}; X),\ t \in \mathbb{R})$$

extends to a bounded linear operator on $L^p(\mathbb{R}; X)$ for some (equivalently all) $p \in (1, \infty)$. All Hilbert spaces are examples of UMD spaces. Also, if X is a UMD space and (Ω, μ) is a σ-finite measure space then $L^p(\Omega; X)$ is a UMD space for $p \in (1, \infty)$ (see [12, Remark 2.7]).

A collection $\mathcal{T}$ of bounded operators is said to be **R-bounded** if there exists a constant C such that

$$\left\| \sum_{j=1}^{n} r_j T_j x_j \right\|_{L^2([0,1];X)} \leq C \left\| \sum_{j=1}^{n} r_j x_j \right\|_{L^2([0,1];X)}$$

for all $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{T}$ and $x_1, \dots, x_n \in X$, where $(r_j)_{j \in \mathbb{N}}$ is the sequence of **Rademacher functions**, defined by

$$r_j(t) = \operatorname{sgn} \sin(2^j \pi t) \qquad (t \in [0,1], j \in \mathbb{N}).$$

Let X^* denote the dual space of X. $\mathcal{T}$ is said to be **U-bounded** if there exists a constant C such that

$$\sum_{j=1}^{n} |\langle T_j x_j, x_j^* \rangle| \leq C \max_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \max_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \varepsilon_j x_j^* \right\|$$

for all $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{T}$, $x_1, \dots, x_n \in X$ and $x_1^*, \dots, x_n^* \in X^*$. If $\mathcal{T}$ is R-bounded then it is U-bounded [30, Lemma 3.1].

$\mathbf{C_0}$-Semigroups

The background material on C_0-semigroups and C_0-groups can be found for example in [2] and [14]. A family $T = (T(t))_{t \geq 0}$ of bounded operators on a Banach space X is said to be a **$\mathbf{C_0}$-semigroup** if

1. $T(0) = I$
2. $T(t+s) = T(t)T(s)$ for all $s, t \in \mathbb{R}$
3. The map $t \mapsto T(t)x : [0, \infty) \to X$ is continuous for every $x \in X$.

We define the **generator** A of the C_0-semigroup T as follows:

$$\begin{aligned} D(A) &= \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\} \\ Ax &= \lim_{t \to 0} \frac{T(t)x - x}{t}. \end{aligned}$$

The generator is always closed and densely defined. For every C_0-semigroup T there exist constants ω and M such that

$$\|T(t)\| \leq Me^{\omega t} \qquad (t \geq 0).$$

The infimum of all such ω is called the **growth bound** or **type** of the semigroup T.

For $\omega \in [0, \pi]$ we define

$$S_\omega = \begin{cases} \{\lambda \in \mathbb{C} : \lambda \neq 0, \ |\arg \lambda| < \omega\} & (\omega \in (0, \pi]), \\ (0, \infty) & (\omega = 0). \end{cases} \tag{1.1}$$

Thus, for $\omega > 0$, S_ω is an open sector, symmetric about the positive real axis. The semigroup T is said to be **holomorphic** of angle $\theta \in (0, \frac{\pi}{2}]$ if it has a holomorphic extension to S_θ which is bounded on $S_\varphi \cap \{ z \in \mathbb{C} : |z| \leq 1 \}$ for each $\varphi \in (0, \theta)$. T is said to be a **bounded holomorphic semigroup** of angle $\theta \in (0, \frac{\pi}{2}]$ if it has a bounded holomorphic extension to S_φ for each $\varphi \in (0, \theta)$.

Suppose that both A and $-A$ generate C_0-semigroups T_+ and T_- respectively. Then we say that A generates the **C_0-group** $U = (U(s))_{s \in \mathbb{R}}$, where

$$U(s) = \begin{cases} T_+(s) & (s \geq 0), \\ T_-(-s) & (s < 0). \end{cases}$$

For a C_0-group U there exist constants ω and M such that

$$\|U(s)\| \leq Me^{\omega |s|} \qquad (s \in \mathbb{R}).$$

Again, the infimum of all such ω is called the **growth bound** or **type** of U.

Chapter 2

Functional Calculus

As mentioned in the introduction, the basic idea behind functional calculus is to give meaning to the expression $f(A)$, where A is some closed operator and f is a suitable holomorphic function. One of the earliest examples of functional calculus is the Riesz-Dunford calculus for bounded operators [13, Chapter VII.3], which is a special case of a more general functional calculus for elements of Banach algebras [6, Chapter VII.4]. We shall primarily be interested in operators A whose spectrum is contained in either a sector or a strip. The holomorphic functional calculus of a sectorial operator was introduced by McIntosh in [35], and developed by Cowling et al. in [7]. A functional calculus for vertical strip-type operators was first considered by Bade in [3], and more recently for horizontal strip-type operators by Haase in [18, 19]. Functional calculi for operators with spectrum in half-planes (corresponding to semigroup generators) and parabolas (corresponding to generators of cosine families) have been considered in [24]. All of the examples mentioned above, and many more, can be unified by the abstract methods developed by Haase [20, 22].

In this chapter we present all of the necessary background material on functional calculus which shall be needed throughout this thesis. We begin in Section 2.1 with the abstract theory, and continue in Section 2.2 by giving some examples of the various functional calculi which we shall need later on. We present the standard constructions of the functional calculi for sectorial and strip-type operators, and also include extensions to invertible sectorial operators and so-called *half-strip-type operators.* As well as these examples, we consider the joint functional calculus of a pair of resolvent commuting operators (Section 2.3), and functional calculi based on operator-valued functions (Section 2.4). In Section 2.5 we conclude with several examples of *composition rules.* These are some of the most important tools in the theory of functional calculus, as they allow calculations to be performed within a functional calculus framework. Much more detailed information can be found in the survey article by Kunstmann and Weis [31], and in the recent book by Haase [22].

2.1 The Abstract Approach

We adopt the terminology used by Haase in [20]. An **abstract functional calculus** (AFC) over the Banach space X is a triple $(\mathcal{E}, \mathcal{F}, \Lambda)$ where

1. $\mathcal{F}$ is a commutative unital algebra of functions with unit $\mathbf{1}$,
2. $\mathcal{E}$ is a subalgebra of $\mathcal{F}$, and
3. $\Lambda : \mathcal{E} \longrightarrow \mathcal{L}(X)$ is an algebra homomorphism.

We say that the AFC is **proper** if the set $\{\, e \in \mathcal{E} : \Lambda(e) \text{ is injective} \,\}$ is non-empty. We say that $f \in \mathcal{F}$ is **regularisable** if there exists $e \in \mathcal{E}$ such that $ef \in \mathcal{E}$ and $\Lambda(e)$ is injective. In this case we call e a **regulariser** for f and define $\Lambda(f) := \Lambda(e)^{-1}\Lambda(ef)$. By [20, Lemma 3.2], $\Lambda(f)$ is a closed operator on X and is independent of the choice of regulariser e. The function $\mathbf{1}$ is regularisable if and only if the AFC is proper, and in this case,

$$\mathcal{F}_r := \{\, f \in \mathcal{F} : f \text{ is regularisable} \,\}$$

is a subalgebra of $\mathcal{F}$ containing $\mathcal{E}$. We have thus extended the original map $\Lambda : \mathcal{E} \rightarrow \mathcal{L}(X)$, (often referred to as the **primary functional calculus**, or PFC), to a map

$\widehat{\Lambda} : \mathcal{F}_r \to \mathcal{C}(X)$, which we shall also denote by Λ. Finally we define

$$\mathcal{F}_b := \{ f \in \mathcal{F}_r : \Lambda(f) \in \mathcal{L}(X) \},$$

the set of all those regularisable functions which give rise to bounded operators. The following result summarises the most fundamental properties of any proper AFC $(\mathcal{E}, \mathcal{F}, \Lambda)$ over a Banach space X. The proofs can all be found in [20, Section 3].

Theorem 2.1.1. *Let $(\mathcal{E}, \mathcal{F}, \Lambda)$ be a proper AFC over a Banach space X, and let $f \in \mathcal{F}_r$. Then the following hold:*

(a) If $T \in \mathcal{L}(X)$ commutes with $\Lambda(e)$ for every $e \in \mathcal{E}$ then T commutes with $\Lambda(f)$.

(b) $\Lambda(\mathbf{1}) = I$.

(c) If $g \in \mathcal{F}_r$ then

$$\Lambda(f) + \Lambda(g) \subset \Lambda(f+g) \quad \text{and} \quad \Lambda(f)\Lambda(g) \subset \Lambda(fg). \tag{2.1}$$

Furthermore, $D(\Lambda(f)\Lambda(g)) = D(\Lambda(fg)) \cap D(\Lambda(g))$, and we have equality in the above inclusions if $g \in \mathcal{F}_b$.

(d) The map $\Lambda : \mathcal{F}_b \to \mathcal{L}(X)$ is a homomorphism of algebras.

(e) If $g \in \mathcal{F}_b$ is such that $\Lambda(g)$ is injective then $\Lambda(f) = \Lambda(g)^{-1}\Lambda(f)\Lambda(g)$.

(f) If $g \in \mathcal{F}$ is such that $fg = \mathbf{1}$ then $g \in \mathcal{F}_r$ if and only if $\Lambda(f)$ is injective, and in this case we have $\Lambda(g) = \Lambda(f)^{-1}$.

(g) Let F be a subspace of $D(\Lambda(f))$. Suppose that there exists a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\Lambda(e_n) \to I$ strongly as $n \to \infty$, and such that $R(\Lambda(e_n)) \subset F$ for all $n \in \mathbb{N}$. Then F is a core for $\Lambda(f)$.

The following result tells us that, even if the inclusions in (2.1) are strict, in certain circumstances the operator on the right-hand side of the inclusion is actually equal to the closure of the operator on the left-hand side.

Proposition 2.1.2. *Let $(\mathcal{E}, \mathcal{F}, \Lambda)$ be a proper AFC and let $f, g \in \mathcal{F}_r$. Suppose that there exist sequences $(e_n)_{n\in\mathbb{N}}$ and $(\widetilde{e}_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that*

(i) e_n *regularises* f *and* $R(\Lambda(e_n)) \subset D(\Lambda(f))$ *for every* $n \in \mathbb{N}$;

(ii) $\widetilde{e}_n$ *regularises* g *and* $R(\Lambda(\widetilde{e}_n)) \subset D(\Lambda(g))$ *for every* $n \in \mathbb{N}$;

(iii) $\Lambda(e_n)\Lambda(\widetilde{e}_n) \to I$ *strongly.*

Then $\overline{\Lambda(f) + \Lambda(g)} = \Lambda(f+g)$ *and* $\overline{\Lambda(f)\Lambda(g)} = \Lambda(fg)$.

Proof. In light of Theorem 2.1.1(c), it is enough to prove that $D(\Lambda(f) + \Lambda(g))$ is a core for $\Lambda(f+g)$, and that $D(\Lambda(f)\Lambda(g))$ is a core for $\Lambda(fg)$. Note that

$$R(\Lambda(e_n\widetilde{e}_n)) = R(\Lambda(e_n)\Lambda(\widetilde{e}_n)) \subset D(\Lambda(f)),$$

and that by symmetry we also have $R(\Lambda(e_n\widetilde{e}_n)) \subset D(\Lambda(g))$, hence Theorem 2.1.1(g) implies that $D(\Lambda(f) + \Lambda(g))$ is a core for $\Lambda(f+g)$. Moreover,

$$R(\Lambda(g)\Lambda(e_n\widetilde{e}_n)) = R(\Lambda(e_n)\Lambda(g\widetilde{e}_n)) \subset D(\Lambda(f)),$$

thus Theorem 2.1.1(g) implies that $D(\Lambda(f)\Lambda(g))$ is a core for $\Lambda(fg)$. □

This is as much as we wish to say about functional calculus at this level of abstraction. We now turn to look at more concrete examples of functional calculi which are associated to a closed operator A. In each of these examples we try to define operators $f(A)$, where f belongs to some class of functions which are holomorphic (or even meromorphic) on an open set containing $\sigma(A)$.

The precise class of functions for which $f(A)$ can be defined will of course depend heavily on A – most notably on the location of its spectrum. However, there are some similarities amongst the examples we shall consider. In each case, the primary functional calculus is constructed via contour integrals taken along a suitable contour surrounding $\sigma(A)$, and then extended by the regularisation method described above.

Since the construction of these functional calculi is essentially the same, we would expect them to have similar properties. Indeed, Haase's notion of a *meromorphic functional calculus* [22, Section 1.3] provides a model for unifying our examples. Given an open subset Ω of $\mathbb{C}$ we let $\mathcal{H}(\Omega)$ and $\mathcal{M}(\Omega)$ denote the algebras of holomorphic and meromorphic functions on Ω respectively. $\mathcal{H}^\infty(\Omega)$ will denote the space of bounded holomorphic functions on Ω. Suppose we are given an AFC $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Lambda)$ over the Banach space X, and that the following hold:

1. The function $\mathbf{z} : \Omega \to \mathbb{C}\,;\, z \mapsto z$ is regularisable with respect to $\mathcal{E}(\Omega)$. In this case the operator $A := \Lambda(\mathbf{z})$ is well-defined.

2. An operator $T \in \mathcal{L}(X)$ which commutes with A also commutes with $\Lambda(e)$ for each $e \in \mathcal{E}(\Omega)$.

The AFC is then called a **meromorphic functional calculus** (MFC) for A. We write $f(A) = \Lambda(f)$ for every $f \in \mathcal{M}(\Omega)_r$. The fundamental properties of functional calculus given in Theorem 2.1.1 can be restated for an MFC.

Theorem 2.1.3. *[22, Theorem 1.3.2] Let A be a closed operator on X and let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Lambda)$ be an MFC for A. Let $f \in \mathcal{M}(\Omega)_r$. Then the following hold:*

(a) If $T \in \mathcal{L}(X)$ commutes with A then it commutes with $f(A)$. If $f(A) \in \mathcal{L}(X)$ then $f(A)$ commutes with A.

(b) $\mathbf{1}(A) = I$ and $\mathbf{z}(A) = A$.

(c) If $g \in \mathcal{M}(\Omega)_r$ then

$$f(A) + g(A) \subset (f+g)(A) \quad \text{and} \quad f(A)g(A) \subset (fg)(A).$$

Furthermore, $D((fg)(A)) \cap D(g(A)) = D(f(A)g(A))$ and we have equality in the above inclusions if $g \in \mathcal{M}(\Omega)_b$.

(d) The mapping $(f \mapsto f(A)) : \mathcal{M}(\Omega)_b \to \mathcal{L}(X)$ is an algebra homomorphism.

(e) If $g \in \mathcal{M}(\Omega)_b$ and $g(A)$ is injective then $f(A) = g(A)^{-1}f(A)g(A)$.

(f) Let $\lambda \in \mathbb{C}$. Then

$$(\lambda - f)^{-1} \in \mathcal{M}(\Omega)_r \iff \lambda - f(A) \text{ is injective.}$$

In this case, $(\lambda - f)^{-1}(A) = (\lambda - f(A))^{-1}$. In particular, $\lambda \in \rho(f(A))$ if and only if $(\lambda - f)^{-1} \in \mathcal{M}(\Omega)_b$.

We shall feel free to make use of these properties without restating them for each particular type of functional calculus we introduce.

2.2 Examples of Meromorphic Functional Calculi

We now present some classes of operators which admit meromorphic functional calculi. In each of the following examples, the homomorphism Λ is taken to be the map $f \mapsto f(A)$, initially for f defined by the relevant primary functional calculus.

2.2.1 Sectorial Operators

An operator A is said to be **sectorial** of angle $\omega \in [0, \pi)$, written $A \in \text{Sect}(\omega)$, if

1. $\sigma(A) \subset \overline{S_\omega}$, where S_ω is as in (1.1), and

2. $\sup\{ \|\lambda R(\lambda, A)\| : \omega' \leq |\arg \lambda| \leq \pi \} < \infty$ for all $\omega' \in (\omega, \pi)$.

In this case we call

$$\omega_{sect}(A) := \inf\{ \omega \in [0, \pi) : A \in \text{Sect}(\omega) \}$$

the **spectral angle** of A. Various authors also include conditions on the kernel, domain and range of A as part of the definition of sectoriality. For our purposes it will be convenient to only consider sectorial operators satisfying

3. $D(A)$ and $R(A)$ are dense.

Note that these assumptions imply that A is injective [22, Proposition 2.1.1(d)]. If in addition A satsifies

4. $\{ \lambda R(\lambda, A) : \omega' \leq |\arg \lambda| \leq \pi \}$ is R-bounded for all $\omega' \in (\omega, \pi)$,

then A is said to be **R-sectorial** of angle ω. There is a correspondence between sectorial operators and generators of bounded holomorphic semigroups, namely, A generates a bounded holomorphic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ if and only if $-A \in \mathrm{Sect}(\frac{\pi}{2} - \theta)$ [2, Theorem 3.7.11].

We now outline the construction of the functional calculus for sectorial operators. Define a function ϕ by

$$\phi(z) = \frac{z}{(1+z)^2} \qquad (z \in S_\pi). \tag{2.2}$$

For $\theta \in (0, \pi]$ define the space $\mathcal{H}_0^\infty(S_\theta)$ by

$$\mathcal{H}_0^\infty(S_\theta) = \{ f \in \mathcal{H}^\infty(S_\theta) : \exists\, C, \varepsilon > 0 \text{ such that } |f(z)| \leq C\, |\phi(z)|^\varepsilon \quad (z \in S_\theta) \}.$$

Then if $A \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{H}_0^\infty(S_\theta)$ for some $\theta \in (\omega, \pi)$, we can define the bounded operator $f(A)$ by means of the contour integral

$$f(A) = \frac{1}{2\pi i} \int_{\partial S_{\theta'}} f(\lambda) R(\lambda, A)\, d\lambda,$$

where $\theta' \in (\omega, \theta)$. By Cauchy's Theorem this definition is independent of θ'.

The triple $(\mathcal{H}_0^\infty(S_\theta), \mathcal{M}(S_\theta), \Lambda)$ is an MFC for A (see [22, Section 2.3.2]). The function ϕ defined by (2.2) clearly lies in $\mathcal{H}_0^\infty(S_\theta)$, and $\phi(A) = A(I+A)^{-2}$ is injective, hence powers of ϕ are good candidates for regularisers in this functional calculus. Indeed $\mathcal{M}(S_\theta)_r$ contains the algebra $\mathcal{B}(S_\theta)$ defined by

$$\mathcal{B}(S_\theta) = \{ f \in \mathcal{H}(S_\theta) : \exists\, k \in \mathbb{N} \text{ such that } f\phi^k \in \mathcal{H}_0^\infty(S_\theta) \}.$$

In particular ϕ itself regularises any function in $\mathcal{H}^\infty(S_\theta)$. We say that A has a **bounded $\mathcal{H}^\infty(S_\theta)$-calculus** if $f(A) \in \mathcal{L}(X)$ for all $f \in \mathcal{H}^\infty(S_\theta)$ and if there exists a constant C such that

$$\|f(A)\| \leq C\, \|f\|_{S_\theta} \qquad (f \in \mathcal{H}^\infty(S_\theta)),$$

where $\|f\|_{S_\theta} := \sup\{\, |f(z)| : z \in S_\theta \,\}$.

For any $\alpha \in \mathbb{C}$ and $\theta \in (0, \pi)$, the function $\mathbf{z}^\alpha : (z \mapsto z^\alpha)$ lies in $\mathcal{B}(S_\theta)$ (see [22, Chapter 3]). Thus if A is injective we may define the **fractional power** A^α by

$$A^\alpha = \mathbf{z}^\alpha(A) \qquad (\alpha \in \mathbb{C}).$$

If A is invertible then the family $(A^{-\alpha})_{\operatorname{Re}\alpha>0}$ is a holomorphic semigroup of angle $\pi/2$ [2, Theorem 3.8.1]. We say that A has **bounded imaginary powers** if $A^{is} \in \mathcal{L}(X)$ for each $s \in \mathbb{R}$. In this case $(A^{is})_{s\in\mathbb{R}}$ is a C_0-group [22, Corollary 3.5.7].

Consider the sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ defined by

$$\phi_n(z) = \frac{n}{n+z} - \frac{1}{1+nz} = \frac{(n^2-1)z}{(n+z)(1+nz)} \qquad (z \in S_\pi). \tag{2.3}$$

Clearly each ϕ_n lies in $\mathcal{H}_0^\infty(S_\theta)$. Furthermore, the sequence $(\phi_n)_{n\in\mathbb{N}}$ acts as an approximate identity in the functional calculus for sectorial operators.

Proposition 2.2.1. *Let $A \in \operatorname{Sect}(\omega)$. For each $k \in \mathbb{N}$ the following hold:*

(a) $\phi_n^k(A)x \to x$ as $n \to \infty$ for each $x \in X$.

(b) $R(\phi_n^k(A)) = R(\phi^k(A)) = D(A^k) \cap R(A^k)$ for each $n \in \mathbb{N}$.

(c) Let $\theta \in (\omega, \pi)$. If $f \in \mathcal{B}(S_\theta)$ is regularised by ϕ^k then $D(A^k) \cap R(A^k)$ is a core for $f(A)$.

Proof. Statements (a) and (b) are proved in [31, Proposition 9.4]. Statement (c) now follows from these and Theorem 2.1.1(g). □

2.2.2 Invertible Sectorial Operators

Given an invertible sectorial operator, Haase has shown how to construct an extension of the usual functional calculus for sectorial operators by considering functions which only have good decay at $+\infty$ [22, Section 2.5]. This extension is consistent in the sense that, if f is regularisable in the usual functional calculus for sectorial operators, then it is regularisable in the functional calculus for invertible sectorial operators, and the resulting operator is the same in both cases. Haase also mentions that it is possible to extend this functional calculus further by considering functions which need not even be defined in a neighbourhood of the origin. We fill in the details of such a construction here.

For $\omega \in [0, \pi]$ and $r > 0$ we define $S_{\omega,r} = S_\omega \cap \{ \lambda \in \mathbb{C} : |\lambda| > r \}$. If $A \in \mathrm{Sect}(\omega)$ is invertible then there exists $r > 0$ such that $\sigma(A) \subset \overline{S_{\omega,r}}$, in which case we write $A \in \mathrm{Sect}(\omega, r)$. To construct the PFC for $A \in \mathrm{Sect}(\omega, r)$, we consider the function space

$$\mathcal{H}_0^\infty(S_{\theta,b}) = \{ f \in \mathcal{H}^\infty(S_{\theta,b}) : \exists C, \varepsilon > 0 \text{ such that } |f(z)| \le C\,|\phi(z)|^\varepsilon \quad (z \in S_{\theta,b}) \},$$

where $\theta \in (\omega, \pi)$ and $b \in (0, r)$. If $f \in \mathcal{H}_0^\infty(S_{\theta,b})$ then we define the bounded operator $f(A)$ by the contour integral

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A)\, d\lambda,$$

where $\Gamma = \partial S_{\theta',b'}$ for some $\theta' \in (\omega, \theta)$ and $b' \in (b, r)$. If we identify a function on S_θ with its restriction to $S_{\theta,b}$, we see that $\mathcal{H}_0^\infty(S_{\theta,b})$ contains the space $\mathcal{H}_0^\infty(S_\theta)$. The triple $(\mathcal{H}_0^\infty(S_{\theta,b}), \mathcal{M}(S_{\theta,b}), \Lambda)$ is an MFC for A, and is once again a consistent extension of the functional calculus for sectorial operators constructed in Section 2.2.1.

2.2.3 Strip-Type Operators

For $\omega \ge 0$ we define

$$H_\omega = \begin{cases} \{ \lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \omega \} & (\omega > 0), \\ \mathbb{R} & (\omega = 0). \end{cases}$$

Thus, for $\omega > 0$, H_ω is the horizontal strip of height ω symmetric about the real axis. An operator B is said to be a **strip-type operator** of height $\omega \ge 0$, written $B \in \mathrm{Strip}(\omega)$, if

1. $\sigma(B) \subset \overline{H_\omega}$ and

2. $\sup\{ \|R(\lambda, B)\| : |\operatorname{Im}\lambda| \geq \omega' \} < \infty$ for all $\omega' > \omega$.

In this case we call

$$\omega_{st}(B) := \inf\{ \omega \geq 0 : B \in \operatorname{Strip}(\omega) \}$$

the **spectral height** of B. Suppose in addition that, for each $\omega' > \omega$, there exists a constant $C = C(\omega') > 0$ such that

$$\|R(\lambda, B)\| \leq \frac{C}{|\operatorname{Im}\lambda| - \omega'} \qquad (|\operatorname{Im}\lambda| > \omega').$$

Then B is said to be a **strong strip-type operator** of height ω, written $B \in \operatorname{SStrip}(\omega)$, and in this case we define

$$\omega_{sst}(B) := \inf\{ \omega \geq 0 : B \in \operatorname{SStrip}(\omega) \}$$

to be the **strong spectral height** of B. It is a consequence of the Hille-Yosida Theorem (see [2, Theorem 3.3.4] for example) that if iB generates a C_0-group U then B is a strong strip-type operator such that $\omega_{sst}(B)$ is less than or equal to the group type of U.

Although we have defined the class $\operatorname{Strip}(\omega)$ for any $\omega \geq 0$, from now on we shall only be interested in strip-type operators with spectral height strictly less than π. We now outline the construction of a functional calculus for strip-type operators, following Haase's approach in [19]. For the PFC, we consider the function space

$$\mathcal{F}(H_\theta) = \{ f \in \mathcal{H}^\infty(H_\theta) : \exists C > 0 \text{ such that } |f(z)| \leq C(1 + |\operatorname{Re} z|)^{-2} \quad (z \in H_\theta) \},$$

where $\theta > 0$. If $B \in \operatorname{Strip}(\omega)$ and $f \in \mathcal{F}(H_\theta)$ for some $\theta \in (\omega, \pi)$ then we define the bounded operator $f(B)$ by

$$f(B) = \frac{1}{2\pi i} \int_{\partial H_{\theta'}} f(z) R(z, B)\, dz,$$

where $\theta' \in (\omega, \theta)$. The triple $(\mathcal{F}(H_\theta), \mathcal{M}(H_\theta), \Lambda)$ is an MFC for B [22, Lemma 4.2.2].

Fix some number $\mu > \pi$ and define a function ψ by

$$\psi(z) = \frac{1}{\mu^2 + z^2} \qquad (z \in H_\theta). \tag{2.4}$$

Then ψ lies in $\mathcal{F}(H_\theta)$ and $\psi(B) = -R(i\mu, B)R(-i\mu, B)$ is injective. If we define

$$\mathcal{G}(H_\theta) = \{\, f \in \mathcal{H}(H_\theta) : \ \exists k \in \mathbb{N} \text{ such that } f\psi^k \in \mathcal{F}(H_\theta) \ \},$$

then $\mathcal{G}(H_\theta)$ is the algebra of all functions in $\mathcal{M}(H_\theta)$ which can be regularised by some power of ψ. It contains all functions which are polynomially bounded as $|z| \to \infty$ [22, Lemma 4.2.3] and, in particular, the algebra $\mathcal{H}^\infty(H_\theta)$ of all bounded holomorphic functions on H_θ. We say that B has a **bounded $\mathcal{H}^\infty(H_\theta)$-calculus** if $f(B) \in \mathcal{L}(X)$ for every $f \in \mathcal{H}^\infty(H_\theta)$ and if there exists a constant C such that

$$\|f(B)\| \leq C \, \|f\|_{H_\theta} \qquad (f \in \mathcal{H}^\infty(H_\theta)),$$

where $\|f\|_{H_\theta} := \sup\{\, |f(z)| : z \in H_\theta \,\}$.

The final result in this section is an analogue of [22, Proposition 2.6.5] for strip-type operators. If Y is another Banach space continuously embedded in X, then the part B_Y of B in Y need not be a strip-type operator, but when it is, the functional calculus of B_Y is well-behaved with respect to that of B.

Proposition 2.2.2. *Let $B \in \mathrm{Strip}(\omega)$ and $\theta > \omega$. Suppose that the Banach space Y is continuously embedded in X. If $B_Y \in \mathrm{Strip}(\omega)$ then the following hold:*

(a) If $e \in \mathcal{F}(H_\theta)$ then Y is invariant under $e(B)$ and $e(B_Y) = e(B)_Y$.

(b) If $f \in \mathcal{M}(H_\theta)$ is regularisable in the functional calculus for B, then it is regularisable in that of B_Y and $f(B_Y) = f(B)_Y$.

Proof. For (a), let $e \in \mathcal{F}(H_\theta)$. By [22, Proposition A.2.8(d)] the space Y is invariant under the resolvents $R(\lambda, B)$ for $\lambda \notin \overline{H_\omega}$, and for such λ we have $R(\lambda, B_Y) = R(\lambda, B)_Y$. It follows that $e(B_Y) = e(B)_Y$. To prove (b), let e be a regulariser of f in the functional calculus of B. Then $e(B_Y) = e(B)_Y$ is injective, hence e also acts as a regulariser for f in the functional calculus of B_Y. Furthermore

$$\begin{aligned}(x,y) \in G(f(B)_Y) &\iff x,y \in Y,\ (ef)(B)x = e(B)y \\ &\iff x,y \in Y,\ (ef)(B_Y)x = e(B_Y)y \\ &\iff (x,y) \in G(f(B_Y))\end{aligned}$$

and this proves the result. □

2.2.4 Half-Strip-Type Operators

For $\omega \geq 0$ and $\rho \in \mathbb{R}$ we define the left- and right-hand half-strips by

$$\begin{aligned}L_{\omega,\rho} &= H_\omega \cap \{\, \lambda \in \mathbb{C} : \operatorname{Re}\lambda < \rho \,\}, \\ R_{\omega,\rho} &= H_\omega \cap \{\, \lambda \in \mathbb{C} : \operatorname{Re}\lambda > \rho \,\}.\end{aligned}$$

Let $B \in \operatorname{Strip}(\omega)$ and suppose that $\rho \in \mathbb{R}$ is such that

(i) $\sigma(B) \subset \overline{R_{\omega,\rho}}$,

(ii) $\sup\{\, \|R(\lambda, B)\| : \operatorname{Re}\lambda \leq \rho' \,\} < \infty$ for each $\rho' < \rho$.

Then we say that B is a **half-strip-type operator**, and write $B \in \operatorname{Strip}(\omega, \rho)$.

For $B \in \operatorname{Strip}(\omega, \rho)$ it is possible to construct a consistent extension of the usual functional calculus for strip-type operators, in the same way as we constructed that for invertible sectorial operators. We only need to consider functions that have good decay at $+\infty$. For the PFC we consider the function space

$$\mathcal{F}(R_{\theta,\sigma}) = \{\, f \in \mathcal{H}^\infty(R_{\theta,\sigma}) \mid \exists C > 0 \,:\, |f(z)| \leq C\,(1 + |\operatorname{Re} z|)^{-2} \quad (z \in R_{\theta,\sigma}) \,\},$$

where $\theta > 0$ and $\sigma \in \mathbb{R}$. Then if $B \in \operatorname{Strip}(\omega, \rho)$ and $f \in \mathcal{F}(R_{\theta,\sigma})$ for some $\theta > \omega$ and $\sigma < \rho$ we define the bounded operator $f(B)$ by

$$f(B) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, B)\, d\lambda,$$

where $\Gamma = \partial R_{\theta',\sigma'}$ for some $\theta' \in (\omega, \theta)$ and $\sigma' \in (\sigma, \rho)$. Identifying a function on H_θ with its restriction to $R_{\theta,\sigma}$, we find that $\mathcal{F}(R_{\theta,\sigma})$ contains the space $\mathcal{F}(H_\theta)$. The triple $(\mathcal{F}(R_{\theta,\sigma}), \mathcal{M}(R_{\theta,\sigma}), \Lambda)$ is an MFC for B.

2.3 Joint Functional Calculus

So far we have only seen examples of functional calculi associated to a single operator, but it is possible however to construct a functional calculus for two or more operators. Such a **joint functional calculus** for a pair of sectorial operators was first developed by Lancien et al. in [32], and that for n sectorial operators by Kalton and Weis in [30]. The construction follows the same abstract approach used above, namely the primary functional calculus is initially defined using contour integrals, then the regularisation method is used to extend this definition to a larger class of meromorphic functions.

We outline the construction for a pair of sectorial operators. Let $A \in \mathrm{Sect}(\omega)$ and $B \in \mathrm{Sect}(\omega')$ be resolvent commuting, and define a function Φ by

$$\Phi(w, z) = \phi(w)\,\phi(z) \qquad (w, z \in S_\pi), \tag{2.5}$$

where ϕ is as in (2.2). For the PFC we consider the function space

$$\mathcal{H}_0^\infty(S_\theta \times S_{\theta'}) := \{\, f \in \mathcal{H}^\infty(S_\theta \times S_{\theta'}) : \exists\, C, \varepsilon > 0 \text{ such that } |f| \le C\,|\Phi|^\varepsilon \,\},$$

where $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$. If $f \in \mathcal{H}_0^\infty(S_\theta \times S_{\theta'})$ then we can define the bounded operator $f(A, B)$ by the integral

$$f(A, B) = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma f(w, z) R(w, A) R(z, B)\, dw\, dz,$$

where $\Gamma = \partial S_\varphi \times \partial S_{\varphi'}$ for some $\varphi \in (\omega, \theta)$ and $\varphi' \in (\omega', \theta')$. There is a natural relationship between this PFC, and those for the individual operators A and B.

Lemma 2.3.1. *Suppose that $A \in \mathrm{Sect}(\omega)$ and $B \in \mathrm{Sect}(\omega')$ are resolvent commuting. Let $f \in \mathcal{H}_0^\infty(S_\theta)$ and $g \in \mathcal{H}_0^\infty(S_{\theta'})$ for some $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$. Then the* *function F defined by*

$$F(w,z) = f(w)\,g(z) \qquad (w \in S_\theta,\, z \in S_{\theta'})$$

lies in $\mathcal{H}_0^\infty(S_\theta \times S_{\theta'})$ and $F(A,B) = f(A)g(B) = g(B)f(A)$.

Proof. For $(w,z) \in S_\theta \times S_{\theta'}$ we have

$$|F(w,z)| \;=\; |f(w)|\,|g(z)| \;\le\; C_A|\phi(w)|^{\varepsilon_A}\,C_B|\phi(z)|^{\varepsilon_B} \;\le\; C_A\,C_B\,|\Phi(w,z)|^{\max\{\varepsilon_A,\varepsilon_B\}}$$

for some constants C_A, C_B, ε_A and ε_B, thus F certainly lies in the desired space. Furthermore, if $\varphi \in (\omega,\theta)$ and $\varphi' \in (\omega',\theta')$ then

$$\begin{aligned} F(A,B) &= \left(\frac{1}{2\pi i}\right)^2 \int_{\partial S_\varphi}\int_{\partial S_{\varphi'}} f(w)g(z)R(w,A)R(z,B)\,dz\,dw \\ &= \frac{1}{2\pi i}\int_{\partial S_\varphi} f(w)R(w,A)\left(\frac{1}{2\pi i}\int_{\partial S_{\varphi'}} g(z)R(z,B)\,dz\right)dw \\ &= \left(\frac{1}{2\pi i}\int_{\partial S_\varphi} f(w)R(w,A)\,dw\right) g(B) \\ &= f(A)g(B), \end{aligned}$$

and this is clearly equal to $g(B)f(A)$ by symmetry. □

The function Φ defined by (2.5) clearly lies in $\mathcal{H}_0^\infty(S_\theta \times S_{\theta'})$ and, by Lemma 2.3.1, $\Phi(A,B)$ is equal to the injective operator $A(I+A)^{-2}B(I+B)^{-2}$. Thus powers of Φ act as regularisers in this joint functional calculus, and $\mathcal{M}(S_\theta \times S_{\theta'})_r$ contains the algebra

$$\mathcal{B}(S_\theta \times S_{\theta'}) = \{\, f \in \mathcal{H}(S_\theta \times S_{\theta'}) \mid \exists\, k \in \mathbb{N} \,:\, f\Phi^k \in \mathcal{H}_0^\infty(S_\theta \times S_{\theta'})\,\}.$$

In analogy to the situation for a single sectorial operator, Φ regularises any member of $\mathcal{H}^\infty(S_\theta \times S_{\theta'})$, and we say that the pair (A,B) has a **bounded joint $\mathcal{H}^\infty(S_\theta \times S_{\theta'})$-calculus** if $f(A,B) \in \mathcal{L}(X)$ for each $f \in \mathcal{H}^\infty(S_\theta \times S_{\theta'})$ and if there exists a constant C such that

$$\|f(A,B)\| \le C\,\|f\|_{S_\theta \times S_{\theta'}} \qquad (f \in \mathcal{H}^\infty(S_\theta \times S_{\theta'})),$$

where $\|f\|_{S_\theta \times S_{\theta'}} := \sup\{ |f(w,z)| : (w,z) \in S_\theta \times S_{\theta'} \}$.

The next result shows that, if a function f is regularisable in the functional calculus for the single sectorial operator A, then f can easily be identified with a function which is regularisable in the joint functional calculus.

Lemma 2.3.2. *Let $A \in \mathrm{Sect}(\omega)$ and $B \in \mathrm{Sect}(\omega')$ be resolvent commuting, and let $f \in \mathcal{M}(S_\theta)_r$ for some $\theta \in (\omega, \pi)$. If we define a function F by*

$$F(w,z) = f(w) \qquad (w \in S_\theta,\ z \in S_{\theta'}),$$

for some $\theta' \in (\omega', \pi)$, then $F \in \mathcal{M}(S_\theta \times S_{\theta'})_r$ and $F(A,B) = f(A)$.

Proof. Let $e \in \mathcal{H}_0^\infty(S_\theta)$ be a regulariser for f and define a function E by

$$E(w,z) = e(w)\phi(z) \qquad (w \in S_\theta, z \in S_{\theta'}).$$

By Lemma 2.3.1, $E \in \mathcal{H}_0^\infty(S_\theta \times S_{\theta'})$ and $E(A,B) = e(A)\phi(B)$, which is injective. Furthermore, for $(w,z) \in S_\theta \times S_{\theta'}$ we have

$$|E(w,z)F(w,z)| = |e(w)f(w)|\,|\phi(z)| \le C\,|\phi(w)|^\varepsilon\,|\phi(z)| \le C\,|\Phi(w,z)|^{\max\{1,\varepsilon\}}$$

for some $C, \varepsilon > 0$, hence E is a regulariser for F. Finally, it follows from Lemma 2.3.1 that

$$\begin{aligned} F(A,B) &= E(A,B)^{-1}(EF)(A,B) \\ &= e(A)^{-1}\phi(B)^{-1}\phi(B)(ef)(A) \\ &= e(A)^{-1}(ef)(A) \\ &= f(A), \end{aligned}$$

as required. □

The triple $(\mathcal{H}_0^\infty(S_\theta \times S_{\theta'}), \mathcal{M}(S_\theta \times S_{\theta'}), \Lambda)$ is an AFC, where Λ denotes the map $(f \mapsto f(A,B)) : \mathcal{H}_0^\infty(S_\theta \times S_{\theta'}) \to \mathcal{L}(X)$. Using Theorem 2.1.1 and Lemma 2.3.2, it is easy to show that the joint functional calculus satisfies properties totally analogous to those given in Theorem 2.1.3. We do not write out the proofs in full here, rather we list these various properties for future reference.

Theorem 2.3.3. *Let $A \in \mathrm{Sect}(\omega)$ and $B \in \mathrm{Sect}(\omega')$ be resolvent commuting, and suppose that $f \in \mathcal{M}(S_\theta \times S_{\theta'})_r$ for some $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$. Then the following hold:*

(a) If $T \in \mathcal{L}(X)$ commutes with A and B then T commutes with $f(A,B)$. If $f(A,B) \in \mathcal{L}(X)$ then $f(A,B)$ commutes with A and B.

(b) If $\lambda \notin \overline{S_\theta}$, $\mu \notin \overline{S_{\theta'}}$ then the functions

$$\mathbf{1} : (w,z) \mapsto 1, \quad \mathbf{w} : (w,z) \mapsto w, \quad \mathbf{z} : (w,z) \mapsto z,$$

$$f_\lambda : (w,z) \mapsto (\lambda - w)^{-1}, \quad g_\mu : (w,z) \mapsto (\mu - z)^{-1}$$

are all regularisable, and

$$\mathbf{1}(A,B) = I, \quad \mathbf{w}(A,B) = A, \quad \mathbf{z}(A,B) = B,$$

$$f_\lambda(A,B) = R(\lambda, A), \quad g_\mu(A,B) = R(\mu, B).$$

(c) If $g \in \mathcal{M}(S_\theta \times S_{\theta'})_r$ then

$$f(A,B) + g(A,B) \subset (f+g)(A,B)$$

$$f(A,B)g(A,B) \subset (fg)(A,B).$$

Furthermore $D((FG)(A,B)) \cap D(G(A,B)) = D(F(A,B)G(A,B))$ and we have equality in the above inclusions if $g \in \mathcal{M}(S_\theta \times S_{\theta'})_b$.

(d) The map $(f \mapsto f(A,B)) : \mathcal{M}(S_\theta \times S_{\theta'})_b \to \mathcal{L}(X)$ is an algebra homomorphism.

(e) If $g \in \mathcal{M}(S_\theta \times S_{\theta'})_b$ and $g(A,B)$ is injective then

$$f(A,B) = g(A,B)^{-1} f(A,B) g(A,B).$$

(f) Let $\lambda \in \mathbb{C}$. Then

$$(\lambda - f)^{-1} \in \mathcal{M}(S_\theta \times S_{\theta'})_r \Longleftrightarrow \lambda - f(A,B) \text{ is injective.}$$

In this case we have $(\lambda - f)^{-1}(A,B) = (\lambda - f(A,B))^{-1}$. In particular, $\lambda \in \rho(f(A,B))$ if and only if $(\lambda - f)^{-1} \in \mathcal{M}(S_\theta \times S_{\theta'})_b$.

Lancien et al. [32] have constructed an approximate identity for the joint functional calculus of a pair of resolvent commuting sectorial operators. Define a sequence of functions $(\Phi_n)_{n\in\mathbb{N}}$ by

$$\Phi_n(w,z) = \phi_n(w)\phi_n(z) \qquad (w, z \in S_\pi), \tag{2.6}$$

where ϕ_n is as in (2.3).

Proposition 2.3.4. *Let $A \in \mathrm{Sect}(\omega)$ and $B \in \mathrm{Sect}(\omega')$ be resolvent commuting. For each $k \in \mathbb{N}$ the following hold:*

(a) $\Phi_n^k(A,B)x = x$ as $n \to \infty$ for all $x \in X$.

(b) $R(\Phi_n^k(A,B)) = R(\Phi^k(A,B))$ for each $n \in \mathbb{N}$.

(c) Let $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$. If $f \in \mathcal{B}(S_\theta \times S_{\theta'})$ is regularised by Φ^k then $R(\Phi^k(A,B))$ is a core for $f(A,B)$.

Proof. Statements (a) and (b) are proved in [32, Lemma 2.2]. Statement (c) then follows from Theorem 2.1.1(g). □

We shall not present explicit constructions of the joint functional calculi for the other examples in Section 2.2 (invertible sectorial, strip-type, half-strip-type). We hope that it is clear to the reader how this may be done, and that in each case the resulting functional calculus satisfies all of the properties corresponding to those listed in Theorem 2.3.3.

2.4 Operator-Valued Functional Calculus

All of the examples of functional calculi presented so far involve spaces of scalar-valued functions, but it is perfectly possible to construct functional calculi based on operator-valued functions. For a closed operator A, let $\mathcal{L}_A(X)$ denote the subalgebra of $\mathcal{L}(X)$ consisting of those bounded operators which commute with the resolvents of A. Given an open subset Ω of $\mathbb{C}$ and a space $\mathcal{E}(\Omega)$ of scalar-valued functions, we let $\mathcal{E}(\Omega; \mathcal{L}_A(X))$ denote the corresponding space of functions taking values in $\mathcal{L}_A(X)$. For example, for $A \in \mathrm{Sect}(\omega)$ and $\theta \in (\omega, \pi)$, let

$$\mathcal{H}_0^\infty(S_\theta; \mathcal{L}_A(X)) = \{ f \in \mathcal{H}^\infty(S_\theta; \mathcal{L}_A(X)) \mid \exists C, \varepsilon > 0 : \|f\| \le C\,|\phi|^\varepsilon \},$$

where ϕ is as in (2.2). If $f \in \mathcal{H}_0^\infty(S_\theta; \mathcal{L}_A(X))$ then the operator $f(A)$ can be defined by the usual method of contour integration. Setting $\Lambda(f) = f(A)$ for each $f \in \mathcal{H}_0^\infty(S_\theta; \mathcal{L}_A(X))$ we obtain the AFC $(\mathcal{H}_0^\infty(S_\theta; \mathcal{L}_A(X)), \mathcal{M}(S_\theta; \mathcal{L}_A(X)), \Lambda)$. Similar constructions can be done for each of the examples in Section 2.2.

Now let $A \in \mathrm{Sect}(\omega)$ and $B \in \mathrm{Sect}(\omega')$ be resolvent commuting. Suppose that $f \in \mathcal{H}^\infty(S_\theta \times S_{\theta'})$ for some $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$. For each $z \in S_{\theta'}$ we can define a function f_z by

$$f_z(w) = f(w, z) \qquad (w \in S_\theta).$$

Clearly $f_z \in \mathcal{H}^\infty(S_\theta)$, and we write $f(A, z)$ as a shorthand for $f_z(A)$. In the case when the function F defined by

$$F(z) = f(A, z) \qquad (z \in S_{\theta'})$$

lies in $\mathcal{H}^\infty(S_{\theta'}; \mathcal{L}_B(X))$, we can define the operator $F(B)$. In this case, taking $\varphi \in (\omega, \theta)$ and $\varphi' \in (\omega', \theta')$ we have

$$\begin{aligned} F(B) &= \phi(B)^{-1}(\phi F)(B) \\ &= \frac{\phi(B)^{-1}}{2\pi i} \int_{\partial S_{\varphi'}} \phi(z) f(A, z) R(z, B)\, dz \\ &= \frac{\Phi(A, B)^{-1}}{(2\pi i)^2} \int_{\partial S_{\varphi'}} \int_{\partial S_\varphi} \Phi(w, z) f(w, z) R(w, A) R(z, B)\, dw\, dz \\ &= f(A, B). \end{aligned}$$

Thus there is a natural relationship between the operator-valued and joint functional calculi.

2.5 Composition Rules

In this section we prove several results which have come to be known collectively as **composition rules**. Roughly speaking, a composition rule is an identity of the form

$$f(g(A)) = (f \circ g)(A). \tag{2.7}$$

Clearly we need to make some assumptions in order for (2.7) to make sense. Firstly, the operator A must have associated to it a functional calculus in which the operator $g(A)$ can be defined. Then $g(A)$ needs to have its own functional calculus in which f is regularisable. Finally, the composition $f \circ g$ must be well-defined, and regularisable in the functional calculus for A.

An example of a composition rule for the joint functional calculus of a pair of sectorial operators is proved in [32, Theorem 4.1]. Composition rules for the holomorphic functional calculi of sectorial and strip-type operators were proved by Haase in [19]; for further developments see [20] and [22]. Such composition rules provide a way to perform calculations within a functional calculus framework, and can be used to recover familiar results (e.g., on logarithms – compare [38, Satz 5] with [19, Lemma 3.1]), as well as to prove new ones.

At present there seems to be no such thing as *the* definitive composition rule; a result must be checked for each pair of functional calculi depending on whether A and $g(A)$ are sectorial, strip-type, half-strip-type etc. (see [22, Sections 2.4-2.5; 4.2]). The proofs in each case are very similar.

The first of our composition rules is very similar to [19, Proposition 2.3], except that we involve the joint functional calculus.

Theorem 2.5.1. *Let* $A \in \operatorname{Sect}(\omega)$ *and* $B \in \operatorname{Sect}(\omega')$ *be resolvent commuting, and suppose that the following hold:*

(i) $g \in \mathcal{B}(S_\theta \times S_{\theta'})$ *for some* $\theta \in (\omega, \pi)$ *and* $\theta' \in (\omega', \pi)$.

(ii) $g(S_\theta \times S_{\theta'}) \subset S_{\varphi'}$.

(iii) $g(A,B) \in \mathrm{Sect}(\varphi)$ *for some* $\varphi < \varphi'$.

Then $f(g(A,B)) = (f \circ g)(A,B)$ *whenever* $f \in \mathcal{B}(S_{\varphi'})$.

Proof. The assumptions ensure that the composition $f \circ g$ is well-defined and lies in $\mathcal{B}(S_\theta \times S_{\theta'})$. Choose m and n so that $f\phi^{n-1} \in \mathcal{H}_0^\infty(S_{\varphi'})$ and $(f \circ g)\Phi^m \in \mathcal{H}_0^\infty(S_\theta \times S_{\theta'})$, where ϕ is as in (2.2) and Φ is as in (2.5). Let $\Gamma = \partial S_\nu$ for some $\nu \in (\varphi, \varphi')$. Then

$$\begin{aligned} f(g(A,B)) &= \phi^{-n}(g(A,B))\,(f\phi^n)(g(A,B)) \\ &= \frac{\phi^{-n}(g(A,B))\,\Phi^{-m}(A,B)}{2\pi i} \int_\Gamma f(\lambda)\,\phi^n(\lambda)\,\Phi^m(A,B)\,R(\lambda, g(A,B))\,d\lambda. \end{aligned}$$

Let I denote the integral on the right-hand side and let $\Gamma' = \partial S_\psi \times \partial S_{\psi'}$ where $\psi \in (\omega, \theta)$ and $\psi' \in (\omega', \theta')$. Then

$$\begin{aligned} I &\stackrel{1}{=} \int_\Gamma f(\lambda)\phi^n(\lambda)\left(\frac{\Phi^m}{\lambda - g}\right)(A,B)\,d\lambda \\ &\stackrel{2}{=} \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \left(\int_\Gamma f(\lambda)\,\phi^n(\lambda)\,\Phi^m(w,z)\,\frac{R(w,A)R(z,B)}{\lambda - g(w,z)}\,d\lambda\right)(dw \times dz) \\ &\stackrel{3}{=} \frac{1}{2\pi i}\int_{\Gamma'} f(g(w,z))\,\phi^n(g(w,z))\,\Phi^m(w,z)\,R(w,A)R(z,B)\,(dw \times dz) \\ &\stackrel{4}{=} 2\pi i\left(\frac{g^n}{(1+g)^{2n}}\right)(A,B)\,(\Phi^m(f \circ g))(A,B) \\ &\stackrel{5}{=} 2\pi i\,\phi^n(g(A,B))\,(\Phi^m(f\circ g))(A,B), \end{aligned}$$

where (1) follows from Theorem 2.3.3(c),(f), the use of Fubini's Theorem in (2) is justified since

$$\frac{f(\lambda)\phi^n(\lambda)\Phi^m(w,z)}{\lambda - g(w,z)} = [f(\lambda)\phi^{n-1}(\lambda)]\Phi^m(w,z)\frac{1}{(1+\lambda)^2}\frac{\lambda}{\lambda - g(w,z)},$$

and the right-hand side is bounded on $\Gamma \times \Gamma'$. (3) is an application of Cauchy's Integral Theorem, and steps (4) and (5) follow from Theorem 2.3.3(c),(f). Therefore

$$\begin{aligned} f(g(A,B)) &= \phi^{-n}(g(A,B))\,\Phi^{-m}(A,B)\,\phi^n(g(A,B))\,(\Phi^m(f\circ g))(A,B) \\ &\stackrel{6}{=} \Phi^{-m}(A,B)\,(\Phi^m(f\circ g))(A,B) \\ &= (f \circ g)(A,B), \end{aligned}$$

where (6) follows from Theorem 2.3.3(e). □

The next result is a slight modification of [19, Proposition 4.3(1)].

Theorem 2.5.2. *Let $A \in \mathrm{Sect}(\omega, r)$ and suppose that the following hold:*

(i) $g \in \mathcal{B}(S_{\theta,b})$ for some $\theta \in (\omega, \pi)$ and $b \in (0, r)$.

(ii) $g(S_{\theta,b}) \subset R_{\varphi',\rho'}$.

(iii) $g(A) \in \mathrm{Strip}(\varphi, \rho)$ for some $\varphi < \varphi'$ and $\rho > \rho'$.

Then $f(g(A)) = (f \circ g)(A)$ whenever $f \in \mathcal{H}^\infty(R_{\varphi',\rho'})$.

Proof. The assumptions ensure that $f \circ g$ is well-defined and lies in $\mathcal{H}^\infty(S_{\theta,b})$. Hence $f\psi \in \mathcal{F}(R_{\varphi',\rho'})$ and $(f \circ g)\phi \in \mathcal{H}_0^\infty(S_{\theta,\beta})$, where ψ is as in (2.4) and ϕ is as in (2.2). Let $\Gamma = \partial R_{\sigma,\nu}$ where $\sigma \in (\varphi, \varphi')$ and $\nu \in (\rho', \rho)$. Then

$$\begin{aligned} f(g(A)) &= \psi^{-2}(g(A))\,(f\psi^2)(g(A)) \\ &= \frac{\psi^{-2}(g(A))\,\phi^{-1}(A)}{2\pi i} \int_\Gamma f(\lambda)\,\psi^2(\lambda)\,\phi(A)\,R(\lambda, g(A))\,d\lambda. \end{aligned}$$

Let I denote the integral on the right-hand side and let $\Gamma' = \partial S_{\theta',b'}$ for some $\theta' \in (\omega, \theta)$ and $b' \in (b, r)$. Then

$$\begin{aligned} I &\overset{1}{=} \int_\Gamma f(\lambda)\,\psi^2(\lambda)\left(\frac{\phi}{\lambda - g}\right)(A)\,d\lambda \\ &\overset{2}{=} \frac{1}{2\pi i}\int_{\Gamma'}\left(\int_\Gamma f(\lambda)\,\psi^2(\lambda)\,\phi(z)\,\frac{R(z,A)}{\lambda - g(z)}\,d\lambda\right)dz \\ &\overset{3}{=} \int_{\Gamma'} f(g(z))\,\psi^2(g(z))\,\phi(z)\,R(z,A)\,dz \\ &\overset{4}{=} 2\pi i\psi^2(g(A))\,(\phi(f \circ g))(A), \end{aligned}$$

where (1) follows from Theorem 2.1.3(c),(f), the use of Fubini's Theorem in (2) is justified since

$$\frac{f(\lambda)\psi^2(\lambda)\phi(z)}{\lambda - g(z)} = [f(\lambda)\psi(\lambda)]\,\phi(z)\frac{1}{(\mu + \lambda)^2(\lambda - g(z))},$$

and the right-hand side is bounded on $\Gamma \times \Gamma'$. (3) is an application of Cauchy's Integral Theorem, and step (4) follows from 2.1.3(c),(f). Therefore

$$\begin{aligned} f(g(A)) &= \psi^{-2}(g(A))\,\phi^{-1}(A)\,\psi^2(g(A))\,(\phi(f\circ g))(A) \\ &\stackrel{5}{=} \phi^{-1}(A)\,(\phi(f\circ g))(A) \\ &= (f\circ g)(A), \end{aligned}$$

where (5) follows from Theorem 2.1.3(e). □

Theorem 2.5.3. *Let $A \in \mathrm{Sect}(\omega, r)$ and $B \in \mathrm{Sect}(\omega', r')$ be resolvent commuting, and suppose that the following hold:*

(i) $g \in \mathcal{B}(S_\pi)$ is such that $g(S_{\theta,b}) \subset R_{\eta,\sigma}$ and $g(S_{\theta',b'})$, where $\theta \in (\omega, \pi)$, $\theta' \in (\omega', \pi)$, $b \in (0, r)$ and $b' \in (0, r')$.

(ii) $g(A) \in \mathrm{Strip}(\varphi, \rho)$ and $g(B) \in \mathrm{Strip}(\varphi', \rho')$ for some $\varphi < \eta$, $\varphi' < \eta'$, $\rho > \sigma$ and $\rho' > \sigma'$.

Let $f \in \mathcal{H}^\infty(R_{\eta,\sigma} \times R_{\eta',\sigma'})$ and define a function h by

$$h(w,z) = f(g(w), g(z)) \qquad (w \in S_{\theta,b},\ z \in S_{\theta',b'}).$$

Then the identity $f(g(A), g(B)) = h(A,B)$ holds.

Proof. The assumptions ensure that h is well-defined and lies in $\mathcal{H}^\infty(S_{\theta,b} \times S_{\theta',b'})$. Hence $h\Phi \in \mathcal{H}_0^\infty(S_{\theta,b} \times S_{\theta',b'})$ and $f\Psi \in \mathcal{F}(R_{\eta,\sigma} \times R_{\eta',\sigma'})$ where Φ is as in (2.5) and $\Psi(w,z) = \psi(w)\,\psi(z)$ for $w, z \in H_\pi$, where ψ as in (2.4). Let $\Gamma = \partial R_{\zeta,\tau} \times \partial R_{\zeta',\tau'}$ where $\zeta \in (\varphi, \eta)$, $\zeta' \in (\varphi', \eta')$, $\tau \in (\sigma, \rho)$ and $\tau' \in (\sigma', \rho')$. Then

$$\begin{aligned} f(g(A), g(B)) &= \Psi^{-2}(g(A), g(B))\,(f\Psi^2)(g(A), g(B)) \\ &= \left(\frac{1}{2\pi i}\right)^2 \Psi^{-2}(g(A), g(B))\,\Phi^{-1}(A,B) \\ &\quad \times \int_\Gamma (f\Psi^2)(\lambda,\xi)\,\Phi(A,B)\,R(w, g(A))\,R(z, g(B))\,(d\lambda \times d\xi) \end{aligned}$$

Let I denote the integral on the right-hand side and let $\Gamma' = \partial S_{\nu,\alpha} \times \partial S_{\nu',\alpha'}$ for some $\nu \in (\omega, \theta)$, $\nu' \in (\omega', \theta')$, $\alpha \in (b, r)$ and $\alpha' \in (b', r')$. Then

$$\begin{aligned}
I &\stackrel{1}{=} \int_{\Gamma} (f\Psi^2)(\lambda,\xi) \left(\frac{\Phi}{(\lambda - g)(\xi - g)} \right) (A,B)\,(d\lambda \times d\xi) \\
&\stackrel{2}{=} \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \int_{\Gamma} (f\Psi^2)(\lambda,\xi)\Phi(w,z) \frac{R(w,A)R(z,B)}{(\lambda - g(w))(\xi - g(z))} (d\lambda \times d\xi)(dw \times dz) \\
&\stackrel{3}{=} \int_{\Gamma'} (f\Psi^2)(g(w),g(z))\Phi(w,z)R(w,A)R(z,B)\,(dw \times dz) \\
&\stackrel{4}{=} (2\pi i)^2 (\Psi^2 \circ g)(A,B)\,(\Phi h)(A,B) \\
&\stackrel{5}{=} (2\pi i)^2 \Psi^2(g(A),g(B))(\Phi h)(A,B),
\end{aligned}$$

where we are denoting by $\Psi^2 \circ g$ the function

$$(w,z) \mapsto \Psi^2(g(w),g(z)) \qquad (w \in S_{\theta,b},\ z \in S_{\theta',b'}).$$

Here (1) follows from Theorem 2.3.3(c),(f), the use of Fubini's Theorem in (2) is justified since

$$\frac{(f\Psi)(\lambda,\xi)\Phi(w,z)}{(\lambda - g(w))(\xi - g(z))} = [(f\Psi)(\lambda,\xi)]\,\Phi(\lambda,\xi)\,\frac{1}{(\mu+\lambda)^2(\lambda - g(w))}\,\frac{1}{(\mu+\xi)^2(\xi - g(w))}$$

and the right-hand side is bounded on $\Gamma \times \Gamma'$. (3) is an application of Cauchy's Integral Theorem, and steps (4) and (5) follow from Theorem 2.3.3(c) and (f). Finally,

$$\begin{aligned}
f(g(A),g(B)) &= \Psi^{-2}(g(A),g(B))\Phi^{-1}(A,B)\Psi^{n}(g(A),g(B))(\Phi h)(A,B) \\
&\stackrel{6}{=} \Phi^{-1}(A,B)(\Phi h)(A,B) \\
&= h(A,B),
\end{aligned}$$

where (6) follows from Theorem 2.3.3(e). □

Remark 2.5.4. *We hope it is clear to the reader that a corresponding version of Theorem 2.5.3 can be stated and proved in the case when $g(A)$ and $g(B)$ are sectorial rather than strip-type.*

Chapter 3

Sums of Logarithms

The logarithm of an injective sectorial operator A was first defined by Nollau in [38]. Equivalent definitions have been given by Yoshikawa [47] and Okazawa [39], and further work has been done by Haase in [19], where in particular it was shown that the logarithm of A can be described via functional calculus methods. Indeed, if f is the function

$$f(z) = \log z \qquad (z \in S_\pi), \tag{3.1}$$

then f is regularisable in the functional calculus for A [19, Section 3], thus we can set $\log A = f(A)$. It follows from [22, Lemma 3.5.1] that this functional calculus definition of the logarithm is equivalent to earlier ones.

Nollau's Lemma [38, Satz 7] tells us that if $|\operatorname{Im}\lambda| > \pi$ then $\lambda \in \rho(\log A)$ and the identity

$$R(\lambda, \log A) = \int_0^\infty \frac{-(t+A)^{-1}}{(\lambda - \log t)^2 + \pi^2}\, dt \tag{3.2}$$

holds. Okazawa used (3.2) to prove that $\log A$ is a strong strip-type operator of height π [39, Lemma 5.1]. By considering the fractional powers of A, Haase showed that the spectral height of $\log A$ is in fact equal to the spectral angle of A [19, Proposition 3.2].

In Section 3.2 we use (3.2) to show that, if A is an injective R-sectorial operator, then $\log A$ is an *R-strong strip-type operator* of height π, in the sense that the operators $(|\operatorname{Im}\lambda| - \pi)R(\lambda, \log A)$, for $|\operatorname{Im}\lambda| > \pi$, form an R-bounded set (Proposition 3.2.1).

When A is an invertible sectorial operator, it follows from a Spectral Mapping Theorem for logarithms [21, Theorem 7.3] that the spectrum of $\log A$ is contained in some right half-strip. We adapt the proof of Nollau's Lemma to obtain a formula for the resolvent of $\log A$ outside this half-strip (Proposition 3.2.3). If in addition A is R-sectorial, we can again show that the resolvents of A satisfy an R-boundedness condition (Proposition 3.2.6).

Now let A and B be a pair of injective sectorial operators whose resolvents commute. It has been shown that the sum $A + B$, with natural domain $D(A) \cap D(B)$, is closed if A and B have bounded imaginary powers and X is a UMD space [12, 43], if A and B have a bounded joint $\mathcal{H}^\infty$-calculus [32], or if A has a bounded $\mathcal{H}^\infty$-calculus and B is R-sectorial [30].

In this chapter we consider the sum $\log A + \log B$, again with its natural domain $D(\log A) \cap D(\log B)$. In Section 3.1 we show that this sum is always closable (Proposition 3.1.2). Furthermore we show that it is possible to identify this closure in terms of the logarithm of the product AB (or more generally $\overline{AB}$), provided that AB (or $\overline{AB}$) is sectorial (Proposition 3.1.3).

In general the sum need not be closed. For example, let A be an unbounded invertible sectorial operator, and set $B = A^{-1}$. By [19, Lemma 3.1] $\log B = -\log A$ so that the sum $\log A + \log B$ is the restriction of the zero operator to $D(\log A)$. As A is unbounded, $D(\log A)$ is a proper subspace of X, hence $\log A + \log B$ is not closed. However, when A and B are invertible sectorial operators we can adapt some of the results of [30] to find sufficient conditions which guarantee that $\log A + \log B$ is a closed operator, and is in fact equal to $\log(AB)$ whenever AB is sectorial (Theorem 3.3.6). These conditions correspond to those of the Kalton-Weis Theorem for sums of sectorial operators [30, Theorem 6.3]. In Section 3.4 we present an alternative route to Theorem 3.3.6, which involves a direct application of the Kalton-Weis Theorem.

3.1 Closability of $\log A + \log B$

Let A and B be a pair of resolvent commuting sectorial operators on a Banach space X. It is known that the product AB is closable, and that if A is invertible then AB is even closed [43, Corollary 3]. There are several results which give sufficient conditions for the product AB (or more generally its closure $\overline{AB}$) to be sectorial. For example, this is the case if both operators have bounded imaginary powers and X is a UMD space [43], if the pair (A, B) has a bounded joint $\mathcal{H}^\infty$-calculus [32], or if one of the operators has a bounded $\mathcal{H}^\infty$-calculus and the other is R-sectorial [26]. The operator $\overline{AB}$ can be neatly described via the joint functional calculus for sectorial operators.

Lemma 3.1.1. *Let $A \in \mathrm{Sect}(\omega)$ and $B \in \mathrm{Sect}(\omega')$ be resolvent commuting. Choose $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$ and define the function p by*

$$p(w, z) = wz \qquad (w \in S_\theta,\ z \in S_{\theta'}). \tag{3.3}$$

Then $p(A, B) = \overline{AB}$.

Proof. Define functions p_1 and p_2 by

$$p_1(w, z) = w, \quad p_2(w, z) = z \qquad (w \in S_\theta,\ z \in S_{\theta'}).$$

By Theorem 2.3.3(b), $p_1(A, B) = A$ and $p_2(A, B) = B$. For $n \in \mathbb{N}$ we define functions e_n and $\widetilde{e}_n$ by

$$e_n(w, z) = \widetilde{e}_n(w, z) = \Phi_n(w, z)^2 \qquad (w \in S_\theta,\ z \in S_{\theta'}),$$

where Φ_n is as in (2.6). The result now follows by applying Proposition 2.1.2 to the functions p_1 and p_2. Note that Proposition 2.3.4 ensures that the conditions of Proposition 2.1.2 are satisfied. □

If A and B are also injective then we can use the properties of the joint functional calculus to prove that the sum $\log A + \log B$ is always closable.

Proposition 3.1.2. *Let* $A \in \mathrm{Sect}(\omega)$ *and* $B \in \mathrm{Sect}(\omega')$ *be resolvent commuting and injective. Then* $\log A + \log B$ *is closable.*

Proof. Choose $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$ and define functions f_1 and f_2 by

$$f_1(w, z) = \log w, \quad f_2(w, z) = \log z \qquad (w \in S_\theta,\ z \in S_{\theta'}).$$

Both f_1 and f_2 lie in $\mathcal{B}(S_\theta \times S_{\theta'})$ and by Lemma 2.3.2 we have

$$f_1(A, B) = \log A, \qquad f_2(A, B) = \log B.$$

If we now set $g = f_1 + f_2$ then it follows from Theorem 2.3.3(c) that $g(A, B)$ is a closed extension of $\log A + \log B$. □

Let f be the logarithm function as in (3.1), and let p be as in (3.3). If the angles $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$ can be chosen so that $\theta + \theta' < \pi$, then the composition $f \circ p$ is well-defined and equal to the function g given in the proof of Proposition 3.1.2. Hence

$$\log A + \log B \subset (f \circ p)(A, B).$$

We would like to use Theorem 2.5.1 to identify the operator $(f \circ p)(A, B)$. We know from Lemma 3.1.1 that $p(A, B) = \overline{AB}$, thus in order to apply Theorem 2.5.1 we require $\overline{AB}$ to be sectorial. As mentioned at the beginning of this section, there are several results giving sufficient conditions for this to be the case.

Proposition 3.1.3. *Let* $A \in \mathrm{Sect}(\omega)$ *and* $B \in \mathrm{Sect}(\omega')$ *be resolvent commuting and injective, and suppose that* $\overline{AB}$ *is sectorial. Then*

$$\overline{\log A + \log B} = \log(\overline{AB}).$$

Proof. Firstly suppose that $\omega + \omega' < \pi$, and choose $\theta \in (\omega, \pi)$ and $\theta' \in (\omega', \pi)$ such that $\theta + \theta' < \pi$. Let f_1 and f_2 be the functions defined in the proof of Proposition 3.1.2, so that $f_1(A, B) = \log A$ and $f_2(A, B) = \log B$. For $n \in \mathbb{N}$ we define the functions e_n and $\widetilde{e}_n$ by

$$e_n(w, z) = \widetilde{e}_n(w, z) = \Phi_n(w, z) \qquad (w \in S_\theta,\ z \in S_{\theta'}),$$

where Φ_n is as in (2.6). In light of Proposition 2.3.4, we may apply Proposition 2.1.2 to f_1 and f_2 to obtain

$$\overline{\log A + \log B} = (f_1 + f_2)(A, B) = (f \circ p)(A, B),$$

where f and p are as above. Since $\overline{AB}$ is sectorial, it follows from Theorem 2.5.1 that $(f \circ p)(A, B) = \log(\overline{AB})$ as required.

More generally note that $A^{1/2} \in \mathrm{Sect}(\omega/2)$ and $B^{1/2} \in \mathrm{Sect}(\omega'/2)$ by [34, Theorem 5.4.1], and certainly $(\omega + \omega')/2 < \pi$. It follows similarly that $\overline{AB}^{1/2}$ is sectorial, and by Theorem 2.5.1 and Lemma 3.1.1 we have $\overline{AB}^{1/2} = p^{1/2}(A, B)$, where p is as in (3.3). Clearly $p(w, z)^{1/2} = p(w^{1/2}, z^{1/2})$, thus by Remark 2.5.4 it follows that $\overline{AB}^{1/2} = \overline{A^{1/2}B^{1/2}}$. Hence

$$\overline{\log(A^{1/2}) + \log(B^{1/2})} = \log(\overline{AB}^{1/2}),$$

and by [19, Lemma 3.1] this completes the proof. □

We would like to know when $\log A + \log B$ is closed, rather than just closable. If one of the operators A or B is bounded and invertible then its logarithm is bounded, so by Theorem 2.3.3(c), $\log A + \log B$ actually coincides with the closed operator $g(A, B)$ in the proof of Proposition 3.1.2. Boundedness is a strong assumption however, and it will be the objective of the rest of this chapter to find conditions which guarantee closedness of $\log A + \log B$ without insisting that one or both of the operators be bounded.

3.2 Nollau's Lemma and R-boundedness

For an injective sectorial operator A, Nollau's Lemma gives us a representation of $R(\lambda, \log A)$ whenever $|\operatorname{Im} \lambda| > \pi$. It is this representation which enables us to estimate the norm of the resolvent, and hence show that $\log A$ is a strong strip-type operator [39, Lemma 5.1]. If in addition A is R-sectorial, we can use the same resolvent representation to prove the corresponding R-boundedness result.

Proposition 3.2.1. *If A is an injective R-sectorial operator then the set*

$$\{ (|\operatorname{Im}\lambda| - \pi) R(\lambda, \log A) : |\operatorname{Im}\lambda| > \pi \}$$

is R-bounded.

Proof. By [38, Satz 7] and [31, Corollary 2.14] it is enough to show that there exists a constant $C > 0$ such that

$$\int_0^\infty \frac{1}{|(\lambda - \log t)^2 + \pi^2|} \frac{dt}{t} \leq \frac{C}{|\operatorname{Im}\lambda| - \pi} \qquad (|\operatorname{Im}\lambda| > \pi),$$

and this is precisely what is shown in the proof that $\log A$ is a strong strip-type operator (see for example the proof of [22, Lemma 3.5.1]). □

Thus we can think of $\log A$ as an **R-strong strip-type operator** of height π. If A is invertible then it follows from [21, Theorem 7.3] that $\sigma(\log A)$ is contained in some right half-strip. In this case, we can adapt the proof of Nollau's Lemma to derive a formula for the resolvent of $\log A$ outside this half-strip (Proposition 3.2.3). In his original proof, Nollau approximated A with a sequence of bounded and invertible operators (a sequence which has come to be known as the *Nollau approximation*, and which we shall use later in Chapter 6). We use the same technique here, though when A is invertible we can take a slightly different sequence.

The **Yosida approximation** of the sectorial operator A is the sequence $(A_n)_{n\in\mathbb{N}}$ of operators defined by

$$A_n = A\left(I + \frac{A}{n}\right)^{-1} \qquad (n \in \mathbb{N}).$$

If A is invertible then each A_n is bounded and invertible, and the sequence $(A_n)_{n\in\mathbb{N}}$ is a **sectorial approximation** to A (see [19, Definition 2.2]). In particular this means that $R(\lambda, A_n) \to R(\lambda, A)$ in operator norm whenever $|\arg\lambda| > \omega_{sect}(A)$. When A is invertible we in fact have convergence on a larger set.

Lemma 3.2.2. *Let $A \in \mathrm{Sect}(\omega, r)$ and let $a \in (0, r)$. There exists $N \in \mathbb{N}$ such that $\{\lambda \in \mathbb{C} : |\lambda| \leq a\} \subset \rho(A_n)$ whenever $n \geq N$. Furthermore, $R(\lambda, A_n) \to R(\lambda, A)$ for $|\lambda| \leq a$.*

Proof. For $n \in \mathbb{N}$ define

$$f_n(z) = z\left(1 + \frac{z}{n}\right)^{-1} \qquad \text{and} \qquad g_n(z) = z\left(1 - \frac{z}{n}\right)^{-1}.$$

Then $f_n(A) = A_n$ and $\rho(A_n) = f_n(\rho(A))$ by the Spectral Mapping Theorem for resolvents [22, Proposition A.3.1], noting that $f_n(A) = n(I - n(nI + A)^{-1})$. Let

$$K = \{\lambda \in \mathbb{C} : |\lambda| \leq a\} \qquad \text{and} \qquad K_\varepsilon = \{\lambda \in \mathbb{C} : |\lambda| < a + \varepsilon\},$$

where $\varepsilon > 0$ is small enough that $K_\varepsilon \subset \rho(A)$. Since $g_n(z) \to z$ as $n \to \infty$, uniformly on compact subsets of $\mathbb{C}$, there exists $N \in \mathbb{N}$ such that $g_n(K) \subset K_\varepsilon$ whenever $n \geq N$. Hence for $n \geq N$ we see that

$$K = f_n(g_n(K)) \subset f_n(K_\varepsilon) \subset f_n(\rho(A)) = \rho(A_n).$$

The fact that $R(\lambda, A_n) \to R(\lambda, A)$ for $\lambda \in K$ follows from [22, Proposition A.5.3]. $\square$

Proposition 3.2.3. *Suppose that A is an invertible sectorial operator, and let $a > 0$ be such that $\{\lambda \in \mathbb{C} : |\lambda| \leq a\} \subset \rho(A)$. Let $\alpha < \log a$ and define*

$$E_\alpha = \{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| > \pi\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < \alpha\}.$$

Then $E_\alpha \subset \rho(\log A)$ and for $\lambda \in E_\alpha$ we have

$$R(\lambda, \log A) = \int_a^\infty \frac{-(t + A)^{-1}}{(\lambda - \log t)^2 + \pi^2}\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{a e^{is} R(a e^{is}, A)}{\lambda - \log a - is}\, ds.$$

Proof. The fact that $E_\alpha \subset \rho(\log A)$ follows from a Spectral Mapping Theorem for logarithms [21, Theorem 7.3]. To prove the remaining assertion we shall adapt the proof of Nollau's Lemma given in [22, Lemma 3.5.1].

For $\theta \in (\omega_{sect}(A), \pi)$ the function

$$f : z \mapsto (\lambda - \log z)^{-1} \qquad (z \in S_{\theta, a})$$

lies in $\mathcal{H}^\infty(S_{\theta,a})$ whenever $\lambda \in E_\alpha$. In the case when A is bounded we can even find $b > 0$ sufficiently large such that f lies in $\mathcal{H}^\infty(S_\theta(a,b))$, where $S_\theta(a,b) = S_\theta \cap \{ z \in \mathbb{C} : a < |z| < b \}$. If $\Gamma = \partial S_\theta(a,b)$ then

$$f(A) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda - \log z} R(z, A)\, dz$$

by the Riesz-Dunford functional calculus for bounded operators. Letting $\theta \to \pi$ and $b \to \infty$ but keeping a fixed we see that

$$f(A) = \int_a^\infty \frac{-(t+A)^{-1}}{(\lambda - \log t)^2 + \pi^2}\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ae^{is} R(ae^{is}, A)}{\lambda - \log a - is}\, ds,$$

and this is equal to $R(\lambda, \log A)$ by Theorem 2.1.3(f).

If A is unbounded then let $(A_n)_{n\in\mathbb{N}}$ be its Yosida approximation. Lemma 3.2.2 implies that $\{\lambda \in \mathbb{C} : |\lambda| \leq a\} \subset \rho(A_n)$ for all n sufficiently large. For such n we have

$$f(A_n) = R(\lambda, \log A_n) = \int_a^\infty \frac{-(t+A_n)^{-1}}{(\lambda - \log t)^2 + \pi^2}\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ae^{is} R(ae^{is}, A_n)}{\lambda - \log a - is}\, ds$$

whenever $\lambda \in E_\alpha$. The fact that $(A_n)_{n\in\mathbb{N}}$ is a sectorial approximation to A, together with Lemma 3.2.2 and the Dominated Convergence Theorem implies that $f(A_n) \to J(A)$ in $\mathcal{L}(X)$, where

$$J(A) := \int_a^\infty \frac{-(t+A)^{-1}}{(\lambda - \log t)^2 + \pi^2}\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ae^{is} R(ae^{is}, A)}{\lambda - \log a - is}\, ds.$$

Hence $f(A) = J(A)$ by [22, Proposition 2.6.9] and $f(A) = R(\lambda, \log A)$ by Theorem 2.1.3(f), completing the proof. □

If A is invertible and R-sectorial, we can use the resolvent representation obtained in Proposition 3.2.3 to show that the operators $\lambda R(\lambda, \log A)$, for λ in some left half-strip, form an R-bounded set. To simplify calculations we will initially assume that the resolvent set of A contains the unit disc.

Proposition 3.2.4. *Let A be an invertible R-sectorial operator and suppose that $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subset \rho(A)$. Then for any $\alpha < 0$ and $\beta \in (0, \pi)$ the set*

$$\{\lambda R(\lambda, \log A) : \lambda \in L_{\beta,\alpha}\}$$

is R-bounded.

Proof. It follows by [46, Proposition 2.6] that the set $\{R(e^{is}, A) : s \in [-\pi, \pi)\}$ is R-bounded. Hence by Proposition 3.2.3 and [31, Corollary 2.14] it is enough to show that

$$I_\lambda := \int_1^\infty \left| \frac{\lambda}{(\lambda - \log t)^2 + \pi^2} \right| \frac{dt}{t} \leq C \quad \text{and} \quad J_\lambda := \int_{-\pi}^{\pi} \left| \frac{\lambda}{\lambda - is} \right| ds \leq C,$$

for some positive constant C independent of $\lambda \in L_{\beta,\alpha}$. Firstly, for such λ we have

$$\begin{aligned} J_\lambda = |\lambda| \int_{-\pi}^{\pi} \frac{ds}{\sqrt{(s - \operatorname{Im}\lambda)^2 + (\operatorname{Re}\lambda)^2}} &= |\lambda| \left[\sinh^{-1}\left(\frac{s - \operatorname{Im}\lambda}{|\operatorname{Re}\lambda|} \right) \right]_{-\pi}^{\pi} \\ &\leq 2|\lambda| \sinh^{-1}\left(\frac{\pi + \beta}{|\operatorname{Re}\lambda|} \right). \end{aligned}$$

It is enough to show that there exists $\kappa < \alpha$ such that this quantity is uniformly bounded for $\lambda \in L_{\beta,\kappa}$. Indeed, choose κ such that $|\lambda| \leq 2|\operatorname{Re}\lambda|$ for $\lambda \in L_{\beta,\kappa}$, and such that $\sinh^{-1} x \leq x$ whenever $x > -\kappa$. Then, if $\lambda \in L_{\beta,\kappa}$,

$$J_\lambda \leq 2|\lambda| \sinh^{-1}\left(\frac{\pi + \beta}{|\operatorname{Re}\lambda|} \right) \leq 4|\operatorname{Re}\lambda| \frac{\pi + \beta}{|\operatorname{Re}\lambda|} = 4(\pi + \beta).$$

To show boundedness of I_λ we begin by supposing that $\lambda < \alpha$ is real, and we show that the integrals

$$R_\lambda := \int_1^\infty \left| \frac{1}{(\lambda - \log t)^2 + (\pi^2 - \beta^2)} \right| \frac{dt}{t}$$

and

$$S_\lambda := \int_1^\infty \left| \frac{\lambda}{(\lambda - \log t)^2 + (\pi^2 - \beta^2)} \right| \frac{dt}{t} = -\lambda R_\lambda$$

are uniformly bounded in λ. Indeed, we have

$$\begin{aligned} R_\lambda = \int_0^\infty \frac{1}{(s-\lambda)^2+(\pi^2-\beta^2)}\,ds &= \frac{1}{\sqrt{\pi^2-\beta^2}}\left[\tan^{-1}\left(\frac{s-\lambda}{\sqrt{\pi^2-\beta^2}}\right)\right]_0^\infty \\ &= \frac{1}{\sqrt{\pi^2-\beta^2}}\left[\frac{\pi}{2}-\tan^{-1}\left(-\frac{\lambda}{\sqrt{\pi^2-\beta^2}}\right)\right] \\ &\le \frac{\pi}{2\sqrt{\pi^2-\beta^2}} \end{aligned}$$

since $\tan^{-1}\left(-\frac{\lambda}{\sqrt{\pi^2-\beta^2}}\right) > 0$ for $\lambda < \alpha$. Similarly,

$$\begin{aligned} S_\lambda &= -\frac{\lambda}{\sqrt{\pi^2-\beta^2}}\left[\frac{\pi}{2}-\tan^{-1}\left(-\frac{\lambda}{\sqrt{\pi^2-\beta^2}}\right)\right] \\ &= -\frac{\lambda}{\sqrt{\pi^2-\beta^2}}\tan^{-1}\left(-\frac{\sqrt{\pi^2-\beta^2}}{\lambda}\right) \\ &\le 1 \end{aligned}$$

since $\tan^{-1} x \le x$ for $x > 0$.

More generally, if $\lambda \in L_{\beta,\alpha}$ then it follows by the above arguments that the integrals $R_{\operatorname{Re}\lambda}$, $S_{\operatorname{Re}\lambda}$ are uniformly bounded by $\max(1, \frac{\pi}{2}(\pi^2-\beta^2)^{-1/2})$. Now, since

$$\begin{aligned} \left|(\lambda-s)^2+\pi^2\right|^2 &= \left[(\operatorname{Re}\lambda-s)^2+\pi^2-(\operatorname{Im}\lambda)^2\right]^2+4(\operatorname{Im}\lambda)^2(\operatorname{Re}\lambda-s)^2 \\ &\ge \left[(\operatorname{Re}\lambda-s)^2+\pi^2-\beta^2\right]^2, \end{aligned}$$

we see that

$$\begin{aligned} I_\lambda^2 = |\lambda|^2\left(\int_0^\infty \frac{ds}{|(\lambda-s)^2+\pi^2|}\right)^2 &\le |\lambda|^2\left(\int_0^\infty \frac{ds}{(\operatorname{Re}\lambda-s)^2+(\pi^2-\beta^2)}\right)^2 \\ &\le S_{\operatorname{Re}\lambda}^2+\beta^2 R_{\operatorname{Re}\lambda}^2, \end{aligned}$$

completing the proof. □

Remark 3.2.5. *The same method of proof can be used to show that the set*

$$\{\, R(\lambda, \log A) : \lambda \in L_{\beta,\alpha} \,\}$$

is also R-bounded, whenever $\alpha < 0$ *and* $\beta \in (0, \pi)$. *For this it is enough to show that* $|\lambda|^{-1} I_\lambda, |\lambda|^{-1} J_\lambda \le C$ *for some constant* C *independent of* $\lambda \in L_{\beta,\alpha}$. *Indeed, since* $|\operatorname{Re}\lambda| > \alpha$ *and* $\pm \operatorname{Im}\lambda < \beta$ *we have*

$$|\lambda|^{-1} J_\lambda = \left[\sinh^{-1}\left(\frac{s - \operatorname{Im}\lambda}{|\operatorname{Re}\lambda|}\right)\right]_{-\pi}^{\pi} \le 2\sinh^{-1}\left(\frac{\pi+\beta}{|\alpha|}\right).$$

Also, since $|(\lambda - \log t)^2 + \pi^2| \ge |(\operatorname{Re}\lambda - \log t)^2 + (\pi^2 - \beta^2)|$, *it follows that* $|\lambda|^{-1} I_\lambda \le R_{\operatorname{Re}\lambda}$.

We can now remove the assumption that the unit disc is contained in $\rho(A)$.

Proposition 3.2.6. *Let* A *be an invertible R-sectorial operator, and let* $a > 0$ *be such that* $\{\lambda \in \mathbb{C} : |\lambda| \le a\} \subset \rho(A)$. *Then for any* $\alpha < 0$ *and* $\beta \in (0, \pi)$ *the set*

$$\{\lambda R(\lambda, \log A) : \lambda \in L_{\beta,\alpha+\log a}\}$$

is R-bounded.

Proof. Choosing $\mu = a^{-1}$ we see that μA satisfies the hypotheses of Proposition 3.2.4, hence the set $\{\nu R(\nu, \log(\mu A)) : \nu \in L_{\beta,\alpha}\}$ is R-bounded. In light of the previous remark the set $\{R(\nu, \log(\mu A)) : \nu \in L_{\beta,\alpha}\}$ is also R-bounded. Since $\log(\mu A) = \log\mu + \log A$ [38, Satz 5] we have

$$(\nu - \log\mu) R(\nu - \log\mu, \log A) = \nu R(\nu, \log(\mu A)) - \log\mu R(\nu, \log(\mu A)).$$

By standard properties of R-boundedness (see [31, Fact 2.8] for example) this means that $\{(\nu - \log\mu) R(\nu - \log\mu, \log A) : \nu \in L_{\beta,\alpha}\}$ is R-bounded, i.e., $\{\lambda R(\lambda, \log A) : \lambda \in L_{\beta,\alpha-\log\mu}\}$ is R-bounded, as claimed. □

3.3 Closedness of $\log A + \log B$

In this section we return to the question of when $\log A + \log B$ is closed, rather than just closable. We shall assume that both A and B are invertible, in which case it follows from Proposition 3.2.3 and Remark 3.2.5 that $\log A$ and $\log B$ are half-strip-type operators. We begin by proving a criterion for the sum of two resolvent commuting half-strip-type operators to be closed (Proposition 3.3.1). We then prove some technical results concerning the functional calculus of invertible sectorial operators, adapted from [30, Section 4]. Using the composition rules of Section 2.5, we obtain a result on logarithms (Theorem 3.3.5). Finally we use the results of the previous section to obtain sufficient conditions for $\log A + \log B$ to be closed (Theorem 3.3.6). These conditions are of Kalton-Weis type and, with some restriction on the angles involved, ensure that AB is sectorial, so that in fact we have $\log A + \log B = \log(AB)$.

Proposition 3.3.1. *Let $A \in \mathrm{Strip}(\omega, \rho)$ and $B \in \mathrm{Strip}(\omega', \rho')$ be resolvent commuting, where $\rho + \rho' > 0$. Let $\theta > \omega$, $\theta' > \omega'$, $\eta < \rho$ and $\eta' < \rho'$ such that $\eta + \eta' > 0$. Then the function F defined by*

$$F(w, z) = w(w + z)^{-1} \qquad (w \in R_{\theta,\eta},\ z \in R_{\theta',\eta'})$$

lies in $\mathcal{H}^\infty(R_{\theta,\eta} \times R_{\theta',\eta'})$. Furthermore, if $F(A, B)$ is bounded then $A + B$ is closed.

Proof. For all $(w, z) \in R_{\theta,\eta} \times R_{\theta',\eta'}$ we have $\mathrm{Re}(w + z) > \eta + \eta' > 0$, thus F is a well-defined holomorphic function. Furthermore, for such w and z we have

$$\begin{aligned} |F(w,z)|^2 &= \frac{(\mathrm{Re}\, w)^2 + (\mathrm{Im}\, w)^2}{|w + z|^2} \\ &\leq \frac{(\mathrm{Re}\, w)^2 + \theta^2}{(\mathrm{Re}\, w + \mathrm{Re}\, z)^2} \\ &\leq \frac{(\mathrm{Re}\, w)^2}{(\mathrm{Re}\, w + \eta')^2} + \frac{\theta^2}{(\eta + \eta')^2}, \end{aligned}$$

and this quantity is uniformly bounded in w and z. Define a function G by

$$G(w, z) = w + z \qquad (w \in R_{\theta,\eta},\ z \in R_{\theta',\eta'}).$$

If $x \in D(A) \cap D(B)$ then $x \in D(G(A, B))$ and $Ax + Bx = G(A, B)x$. If $F(A, B)$ is bounded then we have

$$\|Ax\| = \|(FG)(A, B)x\| = \|F(A, B)G(A, B)x\| \leq C\|G(A, B)x\| = C\|Ax + Bx\|$$

for some constant C. A standard sequence argument (see [31, Theorem 12.13]) shows that $A+B$ is closed. □

Suppose that A belongs to the class $\mathrm{Sect}(\omega, r)$ for some $\omega \in [0,\pi)$ and $r>0$. We adapt some of the results of [30, Section 4] concerning the operator-valued functional calculus of A. We begin by proving a representation for those operators $F(A)$ which are defined by the primary functional calculus.

Proposition 3.3.2. *Let $A \in Sect(\omega, r)$ and let $F \in \mathcal{H}_0^\infty(S_{\theta,b}; \mathcal{L}_A(X))$ for some $\theta \in (\omega, \pi)$ and $b \in (0,r)$. Then for any $\sigma \in (\omega, \theta)$, $s \in (0,1)$, $\nu \in (b,r)$ and $x \in X$ we have*

$$F(A)x = \frac{1}{2\pi i}\int_1^2 (M_+(t)+M_-(t))x\,\frac{dt}{t} + \frac{1}{2\pi}\int_{-\sigma}^{\sigma} e^{it(i-s)}F(\nu e^{it})h_s^t(\nu^{-1}A)x\,dt,$$

where

$$M_\pm(t) = e^{\pm i(1-s)\sigma}\sum_{k=0}^{\infty} F(2^k\nu t e^{\pm i\sigma})h_s^{\pm\sigma}(2^{-k}t^{-1}\nu^{-1}A)$$

and $h_s^\rho(\lambda) = \lambda^s(e^{i\rho}-\lambda)^{-1}$.

Proof. By simply changing the contour of integration in the proof of [30, Proposition 4.2] we obtain

$$F(A)x = \frac{1}{2\pi i}\int_{\partial S_{\sigma,\nu}} \lambda^{-s}F(\lambda)A^sR(\lambda,A)x\,d\lambda \qquad (x\in X).$$

Since $\partial S_{\sigma,\nu}$ can be parametrised as $\partial S_{\sigma,\nu} = \{\,|t|e^{i(\operatorname{sgn} t)\sigma} : |t|\geq \nu\,\}\cup\{\,\nu e^{it} : |t|\leq\sigma\,\}$, we see that

$$\begin{aligned}
F(A)x &= \frac{e^{-i(1-s)\sigma}}{2\pi i}\int_\nu^\infty F(te^{-i\sigma})h_s^{-\sigma}(t^{-1}A)x\,\frac{dt}{t}\\
&\quad+\frac{e^{i(1-s)\sigma}}{2\pi i}\int_\nu^\infty F(te^{i\sigma})h_s^{\sigma}(t^{-1}A)x\,\frac{dt}{t}\\
&\quad+\frac{1}{2\pi}\int_{-\sigma}^{\sigma} e^{it(1-s)}F(\nu e^{it})h_s^t(\nu^{-1}A)x\,dt\\
&= \frac{1}{2\pi i}\int_1^2 (M_+(t)+M_-(t))x\,\frac{dt}{t}\\
&\quad+\frac{1}{2\pi}\int_{-\sigma}^{\sigma} e^{it(1-s)}F(\nu e^{it})h_s^t(\nu^{-1}A)x\,dt,
\end{aligned}$$

as required. □

We can now prove a sufficient condition for the operator $F(A)$ to be bounded when F is a bounded holomorphic function.

Theorem 3.3.3. *Let $A \in \mathrm{Sect}(\omega, r)$ and suppose that A admits a bounded $\mathcal{H}^\infty(S_\theta)$-calculus for some $\theta \in (\omega, \pi)$. Let $F \in \mathcal{H}^\infty(S_{\rho,b}; \mathcal{L}_A(X))$ for some $\rho \in (\theta, \pi)$ and $b \in (0, r)$, and suppose that the set $\{ F(\lambda) : \lambda \in S_{\rho,b} \}$ is U-bounded. Then $F(A)$ is a bounded operator.*

Proof. Let $F_n = \phi_n F$, where ϕ_n is as in (2.3). Then each F_n is contained in $\mathcal{H}_0^\infty(S_{\rho,b}; \mathcal{L}_A(X))$ and $F_n(A) \to F(A)$ strongly as $n \to \infty$. As in the proof of [30, Theorem 4.4], it suffices to show that $\sup_n \|F_n(A)\| < \infty$.

We can apply Proposition 3.3.2 to each F_n, for fixed $s \in (0,1)$, $\nu \in (b, r)$ and $\sigma \in (\theta, \rho)$. For $x \in X$, $x^* \in X^*$ with $\|x\|, \|x^*\| \leq 1$ and $t \in (1, 2)$ we have

$$|\langle M_\pm(t)x, x^*\rangle| \leq \sum_{k=0}^{\infty} |\langle F_n(2^k \nu t e^{\pm i\sigma}) g(2^{-k}t^{-1}\nu^{-1}A)x, g(2^{-k}t^{-1}\nu^{-1}A)^* x^*\rangle|,$$

where $g = (h_s^{\pm\sigma})^{1/2}$. Then

$$|\langle M_\pm(t)x, x^*\rangle| \leq C \sup_{\varepsilon_k = \pm 1} \sup_N \left\| \sum_{k=0}^{N} \varepsilon_k g(2^{-k}t^{-1}\nu^{-1}A) \right\|^2,$$

where C is the U-boundedness constant of the set $\{ F(\lambda) : \lambda \in S_{\rho,b} \}$. Applying [30, Lemma 4.1] to the function g we see that

$$\sup_n \left\| F_n(A) - \frac{1}{2\pi} \int_{-\sigma}^{\sigma} e^{it(i-s)} F_n(\nu e^{it}) h_s^t(\nu^{-1}A)\, dt \right\| < \infty.$$

It therefore remains to show that

$$\sup_n \left\| \int_{-\sigma}^{\sigma} e^{it(1-s)} F_n(\nu e^{it}) h_s^t(\nu^{-1}A)\, dt \right\| < \infty,$$

and for this it is sufficient to check that $\sup_n \|F_n(\nu e^{it})\| < \infty$ for each $t \in [-\sigma, \sigma]$. Now,

$$\sup_n \|F_n(\nu e^{it})\| \leq \left(\sup_n |\phi_n(\nu e^{it})| \right) \|F\|_{S_{\rho,\beta}}.$$

Clearly $\phi_n(\nu e^{it}) \to 1$ as $n \to \infty$ for each $t \in [-\sigma, \sigma]$, thus $\sup_n |\phi_n(\nu e^{it})| < \infty$ for each t, completing the proof. $\square$

We can prove a similar result for operators defined in the joint functional calculus of a pair of invertible sectorial operators.

Theorem 3.3.4. *Suppose that $A \in \mathrm{Sect}(\omega, r)$ and $B \in \mathrm{Sect}(\omega', r')$ are resolvent commuting, and that A admits a bounded $\mathcal{H}^\infty(S_\theta)$-calculus for some $\theta \in (\omega, \pi)$. Let $b \in (0, r)$, $\varphi \in (\theta, \pi)$, $b' \in (0, r')$ and $\varphi' \in (\omega', \pi)$. Suppose that $f \in \mathcal{H}^\infty(S_{\varphi,b} \times S_{\varphi',b'})$ is such that $f(w, B)$ is a bounded operator for every $w \in S_{\varphi,b}$, and that the set $\{ f(w, B) : w \in S_{\varphi,b} \}$ is U-bounded. Then $f(A, B)$ is a bounded operator.*

Proof. We define $F(w) = f(w, B)$ and note that $F \in \mathcal{H}^\infty(S_{\varphi,b}; \mathcal{L}_A(X))$. Our conditions and Theorem 3.3.3 imply that $F(A)$ is a bounded operator. Since $f(A, B) = F(A)$ this proves the theorem. $\square$

Theorem 3.3.5. *Suppose that $A \in \mathrm{Sect}(\omega, r)$ and $B \in \mathrm{Sect}(\omega', r')$ are resolvent commuting, and that A has a bounded $\mathcal{H}^\infty(S_\theta)$-calculus for some $\theta \in (\omega, \pi)$. Let $\alpha < \log r$, $\varphi \in (\theta, \pi)$, $\beta < \log r'$ and $\varphi' \in (\omega', \pi)$. Suppose that $f \in \mathcal{H}^\infty(R_{\varphi,\alpha} \times R_{\varphi',\beta})$ is such that $f(w, \log B)$ is a bounded operator for every $w \in R_{\varphi,\alpha}$, and that the set $\{ f(w, \log B) : w \in R_{\varphi,\alpha} \}$ is U-bounded. Then $f(\log A, \log B)$ is a bounded operator.*

Proof. Let $g(\lambda) = \log \lambda$ for all $\lambda \in S_\pi$. Then g maps $S_{\psi,\rho}$ into $R_{\psi,\log\rho}$ for every $\psi \in [0, \pi)$ and $\rho > 0$. Therefore, the function h defined by

$$h(w, z) = f(g(w), g(z)) \qquad (w \in S_{\varphi,e^\alpha}, z \in S_{\varphi',e^\beta})$$

is well-defined and lies in $\mathcal{H}^\infty(S_{\varphi,e^\alpha} \times S_{\varphi',e^\beta})$. Moreover, for each $w \in S_{\varphi,e^\alpha}$, the operator $h(w, B)$ is equal to $h_w(B)$, where $h_w(z) = h(w, z)$ for $z \in S_{\varphi',e^\beta}$. Also, $h_w = \mathcal{F}_w \circ g$, where $\mathcal{F}_w(z) = f(g(w), z)$, hence it follows from Theorem 2.5.2 that

$$h(w, B) = h_w(B) = (\mathcal{F}_w \circ g)(B) = \mathcal{F}_w(g(B)) = \mathcal{F}_w(\log B).$$

Hence $\{h(w, B) : w \in S_{\varphi,e^\alpha}\} \subset \{f(w, \log B) : w \in R_{\varphi,\alpha}\}$. By the assumptions of the theorem this means that $h(w, B)$ is bounded for each $w \in S_{\varphi,e^\alpha}$, and that the set $\{h(w, B) : w \in S_{\varphi,e^\alpha}\}$ is U-bounded. Theorems 3.3.4 and 2.5.3 now imply that $f(\log A, \log B)$ is bounded, as claimed. □

Theorem 3.3.6. *Let $A \in \mathrm{Sect}(\omega, r)$ and $B \in \mathrm{Sect}(\omega', r')$ be resolvent commuting. Suppose that A has a bounded $\mathcal{H}^\infty(S_\theta)$-calculus for some $\theta \in (\omega, \pi)$, and that B is R-sectorial. Suppose also that AB is sectorial. Then $\log A + \log B$ is closed and*

$$\log A + \log B = \log(AB).$$

Proof. By considering the operator αA for suitable $\alpha > 0$, we may assume that $\log r + \log r' > 0$. Once we prove closedness of $\log A + \log B$, the identity $\log A + \log B = \log(AB)$ will follow from Proposition 3.1.3. Note that by Proposition 3.2.3, $\log A \in \mathrm{Strip}(\omega, \log r)$ and $\log B \in \mathrm{Strip}(\omega', \log r')$. Thus we will aim to apply Proposition 3.3.1 in order to prove closedness of $\log A + \log B$.

By Proposition 3.2.6, the set $\{\lambda R(\lambda, \log B) : \lambda \in L_{\varphi,\alpha+\log r'}\}$ is R-bounded for every $\varphi \in (0, \pi)$ and $\alpha < 0$. In particular if $\varphi \in (\theta, \pi)$ and $\rho < \log r$ is such that $\rho + \log r' > 0$ then the set $\{w(w + \log B)^{-1} : w \in R_{\varphi,\rho}\}$ is R-bounded. The function

$$f : R_{\varphi,\rho} \times R_{\varphi',\rho'} \to \mathbb{C}, \qquad f(w, z) = w(w + z)^{-1}$$

lies in $\mathcal{H}^\infty(R_{\varphi,\rho} \times R_{\varphi',\rho'})$ for each $\varphi' \in (\omega', \pi)$ and $\rho' < \log r'$. By assumption, $f(w, \log B)$ is a bounded operator for every $w \in R_{\varphi,\rho}$ and the set $\{f(w, \log B) : w \in R_{\varphi,\rho}\}$ is R-bounded. Since R-boundedness implies U-boundedness, Theorem 3.3.5 tells us that $f(\log A, \log B)$ is bounded, and it then follows from Proposition 3.3.1 that $\log A + \log B$ is closed. □

If we make a few assumptions on the various angles involved then our conditions automatically imply that AB is sectorial.

Corollary 3.3.7. *Let $A \in \mathrm{Sect}(\omega, r)$ and $B \in \mathrm{Sect}(\omega', r')$ be resolvent commuting, where $\omega + \omega' < \pi$. Suppose that A has a bounded $\mathcal{H}^\infty(S_\theta)$-calculus for some $\theta \in (\omega, \pi)$, and that B is R-sectorial of angle $\theta' \in (\omega', \pi)$, where $\theta + \theta' < \pi$. Then*

$$\log A + \log B = \log(AB).$$

Proof. It follows from [26, Corollary 2.2] that AB is sectorial, thus Theorem 3.3.6 gives the result. □

3.4 An Alternative Proof of Theorem 3.3.6

There is an alternative route to Theorem 3.3.6, which involves a direct application of the Kalton-Weis Theorem [30, Theorem 6.3] and avoids much of the technical detail used above. If A is an invertible sectorial operator whose spectrum contains the unit disc, it can be shown that $\log A$ is also an invertible sectorial operator, of angle strictly less than $\pi/2$ (Lemma 3.4.1). It can also be shown that if A is R-sectorial or has a bounded $\mathcal{H}^\infty$-calculus, then these properties transfer to the logarithm (Lemmas 3.4.2 and 3.4.3).

Lemma 3.4.1. *If $A \in \mathrm{Sect}(\omega, r)$ for some $r > 1$ then $\log A$ is sectorial of angle less than $\pi/2$.*

Proof. We have $\log A \in \mathrm{Strip}(\omega, \log r)$, and since $\log r > 0$ it is clear that there exists $\theta < \pi/2$ such that $\sigma(\log A) \subset S_\theta$. We prove first that the required resolvent estimate holds on the negative real axis, using the representation given in Proposition 3.2.3. If $a \in (1, r)$ and $\lambda < 0$ then

$$\|R(\lambda, \log A)\| \leq C \int_a^\infty \frac{1}{|(\lambda - \log t)^2 + \pi^2|} \frac{dt}{t} + \frac{C|a|}{2\pi} \int_{-\pi}^{\pi} \frac{ds}{|\lambda - \log a - is|}, \tag{3.4}$$

since A is sectorial, and since the resolvent is bounded on compact sets. Setting $\log t = |\lambda| s$ in the first integral in (3.4) we obtain

$$\int_a^\infty \frac{1}{|(\lambda - \log t)^2 + \pi^2|} \frac{dt}{t} \leq \frac{1}{|\lambda|} \int_{\frac{\log a}{|\lambda|}}^\infty \frac{ds}{(s+1)^2} \leq \frac{1}{|\lambda|} \int_0^\infty \frac{ds}{(s+1)^2}$$

as $\log a > 0$. The second integral in (3.4) satisfies

$$\int_{-\pi}^{\pi} \frac{ds}{|\lambda - \log a - is|} = \int_{-\pi}^{\pi} \frac{ds}{\sqrt{(\lambda - \log a)^2 + s^2}} \leq \frac{2\pi}{|\lambda - \log a|}.$$

Since $\lambda < 0$ and $\log a > 0$ we see that $|\lambda - \log a| > |\lambda|$. Hence there exists a constant $C > 0$ such that

$$\|R(\lambda, \log A)\| \leq \frac{C}{|\lambda|} \qquad (\lambda < 0).$$

By [22, Proposition 2.1.1(a)] this means that there exists $\varphi > 0$ such that

$$\|R(\lambda, \log A)\| \leq \frac{C}{|\lambda|} \qquad (\pi - \varphi \leq |\arg \lambda| \leq \pi).$$

For $|\arg \lambda| \in [\theta, \pi - \varphi]$ there exists C such that $|\lambda| \leq C|\operatorname{Im} \lambda|$. Hence, since $\log A$ is a strong strip-type operator of height ω,

$$\|R(\lambda, \log A)\| \leq \frac{C}{|\operatorname{Im} \lambda| - \omega} \leq \frac{C}{|\lambda|}$$

for $|\lambda|$ large and $|\arg \lambda| \in [\theta, \pi - \varphi]$. Finally, there exists $\varepsilon > 0$ such that $\lambda \in \rho(\log A)$ whenever $|\lambda| \leq \varepsilon$. As the resolvent is bounded on compact sets, there exists C such that $\|\lambda R(\lambda, \log A)\| \leq C$ for all such λ. □

Lemma 3.4.2. *Let $r > 1$. If $A \in \operatorname{Sect}(\omega, r)$ is R-sectorial then $\log A$ is R-sectorial of angle less than $\pi/2$.*

Proof. As above, there exists $\theta < \pi/2$ such that $\sigma(\log A) \subset S_\theta$. For R-sectoriality, we first show that the set $\{\, \lambda R(\lambda, \log A) : \lambda < 0 \,\}$ is R-bounded. By [46, Proposition 2.6] the set $\{\, R(e^{is}, A) : s \in [-\pi, \pi) \,\}$ is R-bounded. Hence by Proposition 3.2.3 and [31, Corollary 2.14] it is enough to show that there exists a constant $C > 0$ such that

$$\int_a^{\infty} \frac{1}{|(\lambda - \log t)^2 + \pi^2|} \frac{dt}{t} \leq \frac{C}{|\lambda|}, \qquad \int_{-\pi}^{\pi} \frac{ds}{|\lambda - \log a - is|} \leq \frac{C}{|\lambda|}$$

for all $\lambda < 0$, where $a \in (1, r)$. But this is exactly what was shown in the proof of Lemma 3.4.1. Thus by [31, Lemma 2.21(a)], there exists $\varphi > 0$ such that

$$\{\, \lambda R(\lambda, \log A) : |\arg \lambda| \in [\pi - \varphi, \pi] \,\}$$

is R-bounded. By Proposition 3.2.1, there exists $\kappa > 0$ such that

$$\{\, \lambda R(\lambda, \log A) : |\arg \lambda| \in [\theta, \pi - \varphi], |\lambda| > \kappa \,\}$$

is R-bounded. In light of [46, Proposition 2.6] it follows that the set

$$\{\, \lambda R(\lambda, \log A) : |\arg \lambda| \in [\theta, \pi] \,\}$$

is R-bounded, completing the proof. □

Lemma 3.4.3. *Let $r > 1$. If $A \in \mathrm{Sect}(\omega, r)$ has a bounded $\mathcal{H}^\infty(S_\theta)$-calculus for some $\theta \in (\omega, \pi)$ then $\log A$ has a bounded $\mathcal{H}^\infty(S_\varphi)$-calculus for some $\varphi < \pi/2$.*

Proof. By [21, Proposition 8.3(b)] it follows that A has a bounded $\mathcal{H}^\infty(S_\theta(a))$-calculus for some $a \in (1, r)$. Hence by Theorem 2.5.1 it follows that $\log A$ has a bounded $\mathcal{H}^\infty(R_{\theta,\log a})$-calculus. Since $\log a > 0$ there exists $\varphi < \pi/2$ such that $R_{\theta,\log a} \subset S_\varphi$, thus by identifying a function in $\mathcal{H}^\infty(S_\varphi)$ with its restriction to $R_{\theta,\log a}$ we see that $\mathcal{H}^\infty(S_\varphi) \subset \mathcal{H}^\infty(R_{\theta,\log a})$, proving the result. □

Now let $A \in \mathrm{Sect}(\omega, r)$ and $B \in \mathrm{Sect}(\omega', r')$ be resolvent commuting. It is clear that $\log A$ and $\log B$ are resolvent commuting. The results of this section enable us to apply [30, Theorem 6.3] directly to the operators $\log A$ and $\log B$, thus obtaining an alternative proof of Theorem 3.3.6.

Proof. (**Alternative Proof of Theorem 3.3.6**) By considering the operators αA and βB, for appropriate scalars $\alpha, \beta > 0$, we may suppose that $r, r' > 1$. In this case Lemma 3.4.1 tells us that both $\log A$ and $\log B$ are invertible sectorial operators of angle less than $\pi/2$. Moreover, it follows from Lemma 3.4.3 that $\log A$ has a bounded $\mathcal{H}^\infty(S_\varphi)$-calculus for some $\varphi < \pi/2$, and from Lemma 3.4.2 that $\log B$ is R-sectorial of angle less than $\pi/2$. Hence [30, Theorem 6.3] implies that $\log A + \log B$ is closed. If AB is sectorial then Proposition 3.1.3 tells us that $\log A + \log B = \log(AB)$. □

Chapter 4

Sums of Logarithms in Interpolation Spaces

Let A be a sectorial operator on a Banach space X. It is possible to construct a scale of Banach spaces, known as *real interpolation spaces*, which lie between X and $D(A)$. There are several results which show that the functional calculus properties of A improve if one considers the part of A in a real interpolation space between X and $D(A)$. For example, Dore [10] has shown that every invertible sectorial operator A has a bounded $\mathcal{H}^\infty$-calculus in such an interpolation space. Haase [23] has even shown that A has an R-bounded $\mathcal{H}^\infty$-calculus; in particular this means that the part of A in the interpolation space is R-sectorial.

Recall that the conditions of Corollary 3.3.7 required A to have a bounded $\mathcal{H}^\infty$-calculus and B to be R-sectorial, where B is a second invertible sectorial operator whose resolvents commute with those of A. In light of the results cited above, we might hope that some interpolation space can be constructed on which the conditions of Corollary 3.3.7 are satisfied. The presence of two operators does complicate the matter somewhat. Obviously, forming a real interpolation space for either A or B will give a space on which one of these two conditions holds, but we would like to know whether it is possible to construct a *double interpolation space* on which both conditions are satisfied simultaneously.

In Section 4.1 we give some background to interpolation theory. We show how to construct the real interpolation spaces for a suitable pair of Banach spaces X and Y, and give a description of these spaces when Y is the domain of some sectorial operator A on X. In Section 4.2 we construct the so-called double interpolation space on which the conditions of Corollary 3.3.7 are automatically satisfied. Finally, in Section 4.3, we use the well-known Reiteration Theorem to identify this space in the special case when $A = B$.

4.1 Real Interpolation

Suppose that the Banach spaces X and Y are linearly and continuously embedded in some Hausdorff topological vector space Z. Then the pair (X, Y) is said to be an **interpolation couple**. Although in our applications we will take Y to be the domain of some linear operator on X, in general there is no need to assume that Y is contained in X, or vice versa. To any interpolation couple (X, Y) we can associate two further Banach spaces: the sum $X + Y$, with norm

$$\|a\|_{X+Y} := \inf\{\, \|x\|_X + \|y\|_Y : a = x + y,\ x \in X,\ y \in Y\} \qquad (a \in X + Y),$$

and the intersection $X \cap Y$, with norm

$$\|a\|_{X\cap Y} := \max\{\, \|a\|_X, \|a\|_Y \} \qquad (a \in X \cap Y).$$

An **intermediate space** with respect to the interpolation couple (X, Y) is a Banach space E such that the continuous embeddings $X \cap Y \subset E \subset X + Y$ hold. An intermediate space E is said to be an **interpolation space** if it satisfies the following further property:

- If $T \in \mathcal{L}(X + Y)$ is such that its restriction to X (respectively Y) is a bounded linear operator on X (respectively Y), then T restricts to a bounded linear operator on E.

The two best known examples of such spaces are the so-called *real interpolation spaces*, and the *complex interpolation spaces.* We shall not be concerned with the latter spaces here. There are several equivalent methods for constructing the scale of real interpolation spaces (see [44, Sections 1.3-1.8] for details). Here we outline the so-called **K-method** developed by Peetre, following the presentation of [44, Section 1.3] and [22, Appendix B].

If $t > 0$ then we define

$$K(t, a; X, Y) := \inf\{ \|x\|_X + t\|y\|_Y : a = x + y,\ x \in X,\ y \in Y \}$$

for $a \in X + Y$. For each $t > 0$, the map $a \mapsto K(t, a; X, Y)$ is a norm on $X + Y$, and is equivalent to the norm $\|\cdot\|_{X+Y}$ defined above [22, Lemma B.2.1(a)]. We shall sometimes write $K(t, a)$ instead of $K(t, a; X, Y)$ when it is clear what the spaces X and Y are. For $0 < \theta < 1$ and $1 \le p < \infty$ we define the **real interpolation space** $(X, Y)_{\theta,p}$ by

$$(X, Y)_{\theta,p} = \left\{ a \in X + Y : \|a\|_{(X,Y)_{\theta,p}} := \left(\int_0^\infty (t^{-\theta} K(t, a))^p \frac{dt}{t} \right)^{1/p} < \infty \right\}$$

and if $p = \infty$ we set

$$(X, Y)_{\theta,\infty} = \left\{ a \in X + Y : \|a\|_{(X,Y)_{\theta,\infty}} := \sup_{0<t<\infty} t^{-\theta} K(t, a) < \infty \right\}.$$

Remark 4.1.1. *If $p < \infty$ and either $\theta \le 0$ or $\theta \ge 1$ then the definition of $(X, Y)_{\theta,p}$ would not be meaningful, as in both cases we would have $(X, Y)_{\theta,p} = \{0\}$. Similarly, the definition of $(X, Y)_{\theta,p}$ when $p = \infty$ and either $\theta < 0$ or $\theta > 1$ would also be meaningless. However, if $p = \infty$ and either $\theta = 0$ or $\theta = 1$ then it is possible to give meaning to the space $(X, Y)_{\theta,p}$, but we do not consider such cases here.*

The following summarises some properties of the real interpolation spaces.

Proposition 4.1.2. *[44, Section 1.3] Let (X, Y) be an interpolation couple, let $0 < \theta < 1$ and $1 \le p \le \infty$. Then the following hold:*

(a) $(X,Y)_{\theta,p} = (Y,X)_{1-\theta,p}$.

(b) *There exists* $c = c(\theta,p)$ *such that* $K(t,a) \leq ct^{\theta}\|a\|_{(X,Y)_{\theta,p}}$ *for all* $t > 0$ *and* $a \in (X,Y)_{\theta,p}$.

(c) $X \cap Y \subset (X,Y)_{\theta,p} \subset X + Y$.

(d) *If* $0 < \theta < 1$ *and* $1 \leq p \leq p' \leq \infty$ *then*

$$(X,Y)_{\theta,1} \subset (X,Y)_{\theta,p} \subset (X,Y)_{\theta,p'} \subset (X,Y)_{\theta,\infty}.$$

(e) *If in addition* $Y \subset X$ *and* $0 < \theta < \theta' < 1$ *then*

$$Y \subset (X,Y)_{\theta',\infty} \subset (X,Y)_{\theta,1} \subset X.$$

(f) *If* $X = Y$ *then* $(X,Y)_{\theta,p} = X = Y$.

(g) *There exists a constant* $c = c(\theta,p)$ *such that, for all* $a \in X \cap Y$,

$$\|a\|_{(X,Y)_{\theta,p}} \leq c(\theta,p)\, \|a\|_X^{1-\theta}\, \|a\|_Y^{\theta}.$$

Being examples of abstract interpolation spaces, there is a natural relationship between the real interpolation spaces and those bounded linear operators on $X + Y$ which restrict to bounded operators on both X and Y. Furthermore, it is possible to estimate the norm of the resulting linear operator on these interpolation spaces.

Proposition 4.1.3. *[33, Theorem 1.1.6] Let* (X,Y) *be an interpolation couple and suppose that*$0 < \theta < 1$ *and* $1 \leq p \leq \infty$. *If* $T \in \mathcal{L}(X+Y)$ *is such that its restriction to* X *(respectively* Y*) is a bounded linear operator on* X *(respectively* Y*), then* T *restricts to a bounded linear operator on* $(X,Y)_{\theta,p}$ *such that*

$$\|T\|_{(X,Y)_{\theta,p}} \leq \|T\|_X^{1-\theta}\, \|T\|_Y^{\theta}.$$

In what follows we will mainly be concerned with cases where Y is the domain $D(A)$ of some sectorial operator A, endowed with the graph norm. In this case the real interpolation spaces can be characterised as follows (see [10], [44, Section 1.14]). If $1 \leq p < \infty$ then

$$(X, D(A))_{\theta,p} = \left\{ x \in X : \int_0^\infty \|t^\theta A(t+A)^{-1}x\|_X^p \frac{dt}{t} < \infty \right\} \tag{4.1}$$

and the interpolation norm $\|\cdot\|_{(X,D(A))_{\theta,p}}$ is equivalent to the norm given by

$$x \mapsto \left(\int_0^\infty \|t^\theta A(t+A)^{-1}x\|_X^p \frac{dt}{t} \right)^{1/p} \qquad (\, x \in (X, D(A))_{\theta,p}\,). \tag{4.2}$$

In the case $p = \infty$ we have

$$(X, D(A))_{\theta,\infty} = \left\{ x \in X : \sup_{t>0} \|t^\theta A(t+A)^{-1}x\|_X \leq \infty \right\} \tag{4.3}$$

and the norm $\|\cdot\|_{(X,D(A))_{\theta,\infty}}$ is equivalent to that given by

$$x \mapsto \sup_{t>0} \|t^\theta A(t+A)^{-1}x\|_X \qquad (\, x \in (X, D(A))_{\theta,\infty}\,). \tag{4.4}$$

4.2 Double Interpolation

Let A and B be resolvent commuting invertible sectorial operators. For $\theta \in (0,1)$ and $p \in [1, \infty]$ we form the real interpolation spaces

$$X_0 = (X, D(A))_{\theta,p} \qquad \text{and} \qquad X_1 = (X, D(B))_{\theta,p}$$

as in the previous section. Let B_0 denote the part of B in X_0, and let A_1 denote the part of A in X_1.

Lemma 4.2.1. *If* $\lambda \in \rho(B)$ *then* $\lambda \in \rho(B_0)$ *and* $\|R(\lambda, B_0)\|_{\mathcal{L}(X_0)} \leq \|R(\lambda, B)\|_{\mathcal{L}(X)}$.

Proof. Firstly we check that $D(A)$ is invariant under $R(\lambda, B)$. If $x \in D(A)$ then we write $x = R(\mu, A)y$ for some $\mu \in \rho(A)$ and $y \in X$. Then

$$R(\lambda, B)x = R(\lambda, B)R(\mu, A)y = R(\mu, A)R(\lambda, B)y,$$

and this clearly lies in $D(A)$. Now, if $x \in D(A)$ then

$$\begin{aligned} \|R(\lambda, B)x\|_{D(A)} &= \|R(\lambda, B)x\|_X + \|AR(\lambda, B)x\|_X \\ &= \|R(\lambda, B)x\|_X + \| - R(\lambda, B)(\mu - A)x + \mu R(\lambda, B)x\|_X \\ &\leq \|R(\lambda, B)x\|_{\mathcal{L}(X)}(\|x\|_X + \|Ax\|_X), \end{aligned}$$

hence the norm of $R(\lambda, B)$ in $\mathcal{L}(D(A))$ is less than or equal to its norm in $\mathcal{L}(X)$. Proposition 4.1.3 now tells us that the restriction of $R(\lambda, B)$ to $(X, D(A))_{\theta,p}$ is a bounded operator on $(X, D(A))_{\theta,p}$, with norm less than or equal to $\|R(\lambda, B)\|_{\mathcal{L}(X)}$. Combining this with [22, Proposition A.2.8(d)] completes the proof. □

In particular this means that B_0 is an invertible sectorial operator in X_1, and that A_1 is invertible and sectorial in X_0. Now we can form the interpolation spaces

$$(X_0, D(B_0))_{\theta,p} \qquad \text{and} \qquad (X_1, D(A_1))_{\theta,p}$$

in exactly the same way. We now show that these spaces are in fact the same, beginning with the case when $p < \infty$.

Theorem 4.2.2. *Let $\theta \in (0, 1)$ and $p \in [1, \infty)$. With notation as above, we have*

$$(X_0, D(B_0))_{\theta,p} = (X_1, D(A_1))_{\theta,p}$$

with equivalent norms.

Proof. Since A and B are sectorial, the spaces X_0 and X_1 can be described as in (4.1). Similarly, as B_0 and A_1 are sectorial in X_0 and X_1 respectively, we have

$$(X_0, D(B_0))_{\theta,p} = \left\{ x \in X_0 : \int_0^\infty \|t^\theta B_0(t + B_0)^{-1}x\|_{X_0}^p \frac{dt}{t} < \infty \right\}, \tag{4.5}$$

$$(X_1, D(A_1))_{\theta,p} = \left\{ x \in X_1 : \int_0^\infty \|t^\theta A_1(t + A_1)^{-1}x\|_{X_1}^p \frac{dt}{t} < \infty \right\}. \tag{4.6}$$

If $x \in (X_0, D(B_0))_{\theta,p}$ then it follows from (4.5) that $x \in X_0$ and

$$\int_0^\infty \|t^\theta B_0(t + B_0)^{-1}x\|_{X_0}^p \frac{dt}{t} < \infty.$$

Using the equivalent norm on X_0 given by (4.2) this means that

$$\int_0^\infty \int_0^\infty \|s^\theta A(s + A)^{-1}t^\theta B_0(t + B_0)^{-1}x\|_X^p \frac{ds}{s}\frac{dt}{t} < \infty.$$

Since B_0 is just the part of B in X_0, it follows from Fubini's Theorem that

$$\int_0^\infty \int_0^\infty \|t^\theta B(t + B)^{-1}s^\theta A(s + A)^{-1}x\|_X^p \frac{dt}{t}\frac{ds}{s} < \infty. \tag{4.7}$$

We want to show that $x \in (X_1, D(A_1))_{\theta,p}$. Firstly we show that $x \in X_1$, and for this it is enough to show that

$$\int_0^\infty \|t^\theta B(t + B)^{-1}x\|_X^p \frac{dt}{t} < \infty.$$

We know that $B(t+B)^{-1}x \in X_0$ for each $t > 0$ and, by part (c) of Proposition 4.1.2, that X_0 is continuously embedded in X. Hence there exists $C > 0$ such that

$$\|t^\theta B(t + B)^{-1}x\|_X \leq C\,\|t^\theta B(t + B)^{-1}x\|_{X_0}$$

for all $t > 0$. Since $x \in (X_0, D(B_0))_{\theta,p}$ we have

$$\int_0^\infty \|t^\theta B(t + B)^{-1}x\|_X^p \frac{dt}{t} \leq C \int_0^\infty \|t^\theta B(t + B)^{-1}x\|_{X_0}^p \frac{dt}{t} < \infty$$

so that $x \in X_1$ as required. Hence we can rewrite (4.7) as

$$\int_0^\infty \int_0^\infty \|t^\theta B(t + B)^{-1}s^\theta A_1(s + A_1)^{-1}x\|_X^p \frac{dt}{t}\frac{ds}{s} < \infty,$$

which in light of the equivalent norm on X_1 (see (4.2)) is the same as saying

$$\int_0^\infty \|s^\theta A_1(s + A_1)^{-1}x\|_{X_1}^p \frac{ds}{s} < \infty.$$

This tells us that $x \in (X_1, D(A_1))_{\theta,p}$ and hence $(X_0, D(B_0))_{\theta,p} \subset (X_1, D(A_1))_{\theta,p}$. The proof also shows that the embedding is continuous. The reverse inclusion follows by symmetry. □

The case when $p = \infty$ can be proved almost identically.

Theorem 4.2.3. *Let $\theta \in (0,1)$. With notation as above, we have*

$$(X_0, D(B_0))_{\theta,\infty} = (X_1, D(A_1))_{\theta,\infty}$$

with equivalent norms.

Proof. This time the spaces X_0 and X_1 can be described as in (4.3). Similarly

$$(X_0, D(B_0))_{\theta,\infty} = \left\{ x \in X_0 : \sup_{t>0} \|t^\theta B_0(t+B_0)^{-1}x\|_{X_0} < \infty \right\}, \tag{4.8}$$

$$(X_1, D(A_1))_{\theta,\infty} = \left\{ x \in X_1 : \sup_{t>0} \|t^\theta A_1(t+A_1)^{-1}x\|_{X_1} < \infty \right\}. \tag{4.9}$$

If $x \in (X_0, D(B_0))_{\theta,\infty}$ then it follows from (4.8) that $x \in X_0$ and

$$\sup_{t>0} \|t^\theta B_0(t+B_0)^{-1}x\|_{X_0} < \infty.$$

Using the equivalent norm on X_0 given by (4.4) this means that

$$\sup_{t>0} \sup_{s>0} \|s^\theta A(s+A)^{-1} t^\theta B_0(t+B_0)^{-1}x\|_X < \infty.$$

Since B_0 is just the part of B in X_0, it follows that

$$\sup_{s>0} \sup_{t>0} \|t^\theta B(t+B)^{-1} s^\theta A(s+A)^{-1}x\|_X < \infty. \tag{4.10}$$

We want to show that $x \in (X_1, D(A_1))_{\theta,\infty}$. Firstly we show that $x \in X_1$, and for this it is enough to show that

$$\sup_{t>0} \|t^\theta B(t+B)^{-1}x\|_X < \infty.$$

We know that $B(t+B)^{-1}x \in X_0$ for each $t > 0$ and, by part (c) of Proposition 4.1.2, that X_0 is continuously embedded in X. Hence there exists $C > 0$ such that

$$\|t^\theta B(t+B)^{-1}x\|_X \leq C \, \|t^\theta B(t+B)^{-1}x\|_{X_0}$$

for all $t > 0$. Since $x \in (X_0, D(B_0))_{\theta,\infty}$ we have

$$\sup_{t>0} \|t^\theta B(t+B)^{-1}x\|_X \leq C \sup_{t>0} \|t^\theta B(t+B)^{-1}x\|_{X_0} < \infty$$

so that $x \in X_1$ as required. Hence we can rewrite (4.10) as

$$\sup_{s>0} \sup_{t>0} \|t^\theta B(t+B)^{-1} s^\theta A_1(s+A_1)^{-1}x\|_X < \infty,$$

which in light of the equivalent norm on X_1 (see (4.4)) this is the same as saying

$$\sup_{s>0} \|s^\theta A_1(s+A_1)^{-1}x\|_{X_1} < \infty.$$

This tells us that $x \in (X_1, D(A_1))_{\theta,\infty}$ and hence $(X_0, D(B_0))_{\theta,\infty} \subset (X_1, D(A_1))_{\theta,\infty}$. Again the proof also shows that the embedding is continuous, and the reverse inclusion follows by symmetry. □

We can now show that, on the space we have just constructed, the assumptions of Corollary 3.3.7 are satisfied.

Corollary 4.2.4. *Suppose that* $A \in \text{Sect}(\omega, r)$ *and* $B \in \text{Sect}(\omega', r')$ *are resolvent commuting, where* $\omega + \omega' < \pi$. *Let* $\theta \in (0,1)$ *and* $p \in [1, \infty]$. *If* Y *denotes the interpolation space constructed above, i.e.,*

$$Y = (X_0, D(B_0))_{\theta,p} = (X_1, D(A_1))_{\theta,p}.$$

then

$$\log A_Y + \log B_Y = \log(A_Y B_Y).$$

Proof. It follows from Lemma 4.2.1 that A_1 is an invertible sectorial operator of angle ω in X_1. Hence [23, Corollary 6.6] implies that the part of A_1 in Y is R-sectorial of angle φ, for each $\varphi \in (\omega, \pi)$. It is easy to check that the part of A_1 in Y is just the part of A in Y.

Similarly, [10, Theorem 3.2] implies that the part of B_0 in Y has a bounded $\mathcal{H}^\infty(S_{\varphi'})$-calculus whenever $\varphi' \in (\omega', \pi)$. Again, the part of B_0 in Y is just the part of B in Y. We can choose φ and φ' so that $\varphi + \varphi' < \pi$, hence Corollary 3.3.7 gives the result. □

4.3 The Reiteration Theorem

Now that we have constructed an interpolation space on which the conditions of Corollary 3.3.7 are automatically satisfied, we would like to be able to identify it in terms of more familiar spaces. To do this we shall use the **Reiteration Theorem** (Theorem 4.3.1), a powerful tool in interpolation theory. We shall only consider the special case when $A = B$, even though in this case the conclusions of Corollary 3.3.7 are trivial.

Let (X, Y) be an interpolation couple and suppose that E is an intermediate space with respect to (X, Y). We say that E is of **class** $\mathbf{J}(\theta)$ in (X, Y), where $\theta \in (0, 1)$, if there exists a constant $C \geq 0$ such that

$$\|x\|_E \leq C\|x\|_X^{1-\theta}\|x\|_Y^{\theta} \qquad (x \in X \cap Y).$$

We write $E \in J_\theta(X, Y)$ if this holds. By [33, Proposition 1.3.2] this is equivalent to the embedding $(X, Y)_{\theta,1} \subset E$. We say that E is of **class** $\mathbf{K}(\theta)$ in (X, Y), where $\theta \in (0, 1)$, if there exists a constant $C \geq 0$ such that

$$K(t, x; X, Y) \leq Ct^\theta\|x\|_E \qquad (x \in E,\, t > 0).$$

We write $E \in K_\theta(X, Y)$ if this holds. By Proposition 4.1.2(b) and the definition of $(X, Y)_{\theta,\infty}$, this is equivalent to the embedding $E \subset (X, Y)_{\theta,\infty}$.

There are some familiar examples of these spaces. If A is a sectorial operator on X then $D(A) \in J_{1/2}(X, D(A)) \cap K_{1/2}(X, D(A))$ by [33, Proposition 3.1.4]. Furthermore, it follows immediately from Proposition 4.1.2(d) that $(X, D(A))_{\theta,p} \in J_\theta(X, D(A)) \cap K_\theta(X, D(A))$ for each $\theta \in (0, 1)$ and $p \in [1, \infty]$.

If (X, Y) is an interpolation couple and E_1, E_2 are intermediate spaces, then (E_1, E_2) is also an interpolation couple. The Reiteration Theorem allows us to identify the interpolation spaces of (E_1, E_2) in terms of those of (X, Y).

Theorem 4.3.1. *[44, Section 1.10.2] Let (X, Y) be an interpolation couple. Suppose that $0 < \theta_1 < \theta_2 < 1$ and that $0 < \theta < 1$. Let $\sigma = (1 - \theta)\theta_1 + \theta\theta_2 \in (0, 1)$.*

(a) *If* $E_i \in K_{\theta_i}(X,Y)$ *for* $i = 1,2$ *then*

$$(E_1, E_2)_{\theta,p} \subset (X,Y)_{\sigma,p} \qquad (1 \le p \le \infty).$$

(b) *If* $E_i \in J_{\theta_i}(X,Y)$ *for* $i = 1,2$ *then*

$$(X,Y)_{\sigma,p} \subset (E_1, E_2)_{\theta,p} \qquad (1 \le p \le \infty).$$

We now introduce some notation used by Lunardi in [33]. For $0 < \theta < 1$ and $1 \le p \le \infty$ let $D_A(\theta,p) = (X, D(A))_{\theta,p}$. Furthermore, for $k \in \mathbb{N}$ define

$$D_A(\theta + k, p) = \{\, x \in D(A^k) : A^k x \in D_A(\theta,p) \,\}$$

with norm

$$\|x\|_{D_A(\theta+k,p)} = \|x\|_X + \|A^k x\|_{D_A(\theta,p)} \qquad (x \in D_A(\theta + k, p)).$$

In other words, $D_A(\theta + k, p)$ is just the domain of the part of A^k in $(X, D(A))_{\theta,p}$. Using this notation, there is a useful characterisation of the real interpolation spaces between X and $D(A^2)$.

Proposition 4.3.2. *[33, Proposition 3.1.5] Let A be a sectorial operator on the Banach space X. Then for $\theta \in (0,1)$, $\theta \ne 1/2$ and $1 \le p \le \infty$,*

$$(X, D(A^2))_{\theta,p} = D_A(2\theta, p).$$

Corollary 4.3.3. *Let A be a sectorial operator on the Banach space X. Let $\theta \in (0,1)$ and $1 \le p \le \infty$. If $A_{\theta,p}$ denotes the part of A in $(X, D(A))_{\theta,p}$ then*

(a) $(X, D(A))_{\theta,p} = (X, D(A^2))_{\frac{\theta}{2},p}$.

(b) $D(A_{\theta,p}) = (X, D(A^2))_{\frac{1}{2}(1+\theta),p}$.

Proof. With $\mu = \theta/2$, it follows from Proposition 4.3.2 that

$$(X, D(A))_{\theta,p} = D_A(2\mu, p) = (X, D(A^2))_{\mu,p} = (X, D(A^2))_{\theta/2,p},$$

which proves (a). Similarly, if we set $\mu = (1+\theta)/2$ we have

$$D(A_{\theta,p}) = D_A(\theta+1, p) = D_A(2\mu, p) = (X, D(A^2))_{\mu,p} = (X, D(A^2))_{\frac{1}{2}(1+\theta),p},$$

proving (b). □

We can now apply the Reiteration Theorem to identify the interpolation space constructed in Section 4.2.

Proposition 4.3.4. *Let A be a sectorial operator on the Banach space X. Let $0 < \alpha, \theta < 1$ and $1 \leq p, q \leq \infty$. Then*

$$((X, D(A))_{\theta,p}, D(A_{\theta,p}))_{\alpha,q} = (X, D(A^2))_{\frac{1}{2}(\theta+\alpha),q}.$$

Proof. From Corollary 4.3.3 we see that

$$(X, D(A))_{\theta,p} \in J_{\theta/2}(X, D(A^2)) \cap K_{\theta/2}(X, D(A^2))$$

and that

$$D(A_{\theta,p}) \in J_{(1+\theta)/2}(X, D(A^2)) \cap K_{(1+\theta)/2}(X, D(A^2)).$$

The result now follows from Theorem 4.3.1 □

In particular, by taking $p = q$ and $\alpha = \theta$ we obtain

Corollary 4.3.5. *Let A be a sectorial operator on the Banach space X. Let $\theta \in (0,1)$ and $p \in [1, \infty]$. Then*

$$((X, D(A))_{\theta,p}, D(A_{\theta,p}))_{\theta,p} = (X, D(A^2))_{\theta,p}.$$

Chapter 5

Exponentials of Strip-Type Operators

We now turn to look at the relationship between operator logarithms and exponentials. It follows from Nollau's Lemma that the logarithm of an injective sectorial operator A is a strong strip-type operator such that $\omega_{sst}(\log A) = \omega_{sect}(A)$ [19, Proposition 3.2, Theorem 4.1]. Moreover the identity $e^{\log A} = A$ follows from a composition rule [22, Corollary 4.2.5]. We would like to know which strong strip-type operators arise as the logarithm of some injective sectorial operator. In other words, given a strong strip-type operator B such that $\omega_{sst}(B) < \pi$, we would like to know when e^B is sectorial. This question is the so-called **inversion problem**, and is the motivation behind the remainder of this thesis.

We mentioned in Section 2.2.3 that examples of strong strip-type operators include those operators B such that iB generates a C_0-group. In Section 5.1 we survey what is known about the exponential e^B when B is such an operator. It turns out that sectoriality of e^B is equivalent to sectoriality of the *analytic generator* of the group generated by iB. We state Monniaux's Theorem, which gives sufficient conditions for the analytic generator to be sectorial, and seems to be the only substantial positive result on the inversion problem.

If B is a strip-type operator such that $-1 \in \rho(e^B)$, then $\sigma(e^B)$ lies in a sector (Lemma 5.2.1). In Section 5.2 we investigate what sort of estimate the resolvent of e^B might satisfy outside this sector. Although we may not have sectoriality, we can show that the norm of $\lambda R(\lambda, e^B)$ satisfies a logarithmic growth estimate. As greater restrictions are placed on the rate of decay of the resolvents of B itself, this estimate becomes closer to sectoriality. In Section 5.3 we prove a necessary and sufficient condition for e^B to be sectorial when B belongs to a certain subclass of strip-type operators.

In Section 5.4 we give some results on Fourier multipliers, considered firstly on L^p-spaces and then on Besov spaces, which arise as real interpolation spaces between L^p-spaces and Sobolev spaces. We make use of these results in Section 5.5, where we look at a familiar example of a strong strip-type operator – the derivative on $\mathbb{R}$. When considered on $L^1(\mathbb{R})$, the exponential of this operator is not sectorial (Proposition 5.5.1) – indeed, it has empty resolvent set. However, we show that, when considered on a suitable Besov space, the spectrum of the exponential is contained in $[0, \infty)$, and we use the results of Section 5.2 to estimate the norm of the resolvent outside $[0, \infty)$.

5.1 Monniaux's Theorem

Suppose that iB generates a C_0-group $U = (U(s))_{s\in\mathbb{R}}$ on a Banach space X. The **analytic generator** of U was introduced by Cioránescu and Zsidó in [5], and is defined as follows. Let Ω denote the strip $\{ z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \}$ and let $\mathcal{H}(\Omega) \cap C(\overline{\Omega})$ denote the space of X-valued functions which are holomorphic on Ω, with continuous extension to $\overline{\Omega}$. The analytic generator of U is the operator C given by

$$\begin{aligned} D(C) &= \{ x \in X : \exists f_x \in \mathcal{H}(\Omega) \cap C(\overline{\Omega}) \text{ such that } f_x(is) = U(s)x, \ (s \in \mathbb{R}) \} \\ Cx &= f_x(1). \end{aligned}$$

The analytic generator C is closely related to the exponential e^B.

Proposition 5.1.1. *Suppose that iB generates the C_0-group U of group type less than π and let C denote the analytic generator of U. If one of C or e^B is sectorial, then in fact both are sectorial and $C = e^B$.*

Proof. Suppose that the analytic generator C is sectorial. By [45, Corollary 4.3] C has bounded imaginary powers, therefore $i \log C$ generates a C_0-group [34, Theorem 10.1.3]. This group is in fact U by [36, Example 3.7]. Therefore $\log C = B$, so by [22, Theorem 4.2.4] we see that $C = e^B$.

On the other hand, suppose that e^B is sectorial. It follows from [22, Theorem 4.2.4] that $\log e^B = B$, hence $i \log e^B$ generates a C_0-group. The group generated by $i \log e^B$ consists of the imaginary powers of e^B [34, Theorem 10.1.4], hence $U(s) = e^{isB}$ for $s \in \mathbb{R}$. By [36, Example 3.7] the analytic generator of U is e^B. □

There are examples of C_0-groups whose analytic generator is not sectorial [45, Section 7]. In this case it is not clear whether the analytic generator coincides with e^B. Monniaux's Theorem gives sufficient conditions for the analytic generator to be sectorial.

Theorem 5.1.2. *[36, Theorem 4.3] Let U be a C_0-group on a UMD space X, with group type less than π. Then the analytic generator C is sectorial and has bounded imaginary powers.*

A slight improvement was obtained by Haase.

Theorem 5.1.3. *[19, Theorem 5.2] Let iB be the generator of a C_0-group U on a UMD space X, and suppose that $\omega_{st}(B) < \pi$. Then the analytic generator C is sectorial and has bounded imaginary powers.*

It is known that if $\rho(C)$ is non-empty then $\sigma(C)$ is contained in some sector, even if X is not a UMD space [36, Proposition 3.8(iii)]. However, it is not known whether non-emptiness of $\rho(C)$ implies that C is sectorial in general. If this was the case then by [45, Corollary 4.3] this would mean that, if the group type was less than π and

$\rho(C)$ was non-empty, then C would even have bounded imaginary powers, answering positively a conjecture of [4, p.3588].

5.2 Logarithmic Estimates

Let $B \in \text{Strip}(\omega)$ and define the function f_λ by

$$f_\lambda(z) = \frac{e^z}{(e^z - \lambda)(e^z + 1)} \qquad (z \in H_\theta), \tag{5.1}$$

where $\omega < \theta < |\arg \lambda| \leq \pi$. Then $f_\lambda \in \mathcal{F}(H_\theta)$ by [22, Lemma 4.2.3] and the identity

$$\frac{\lambda}{\lambda - e^z} = -(1+\lambda) f_\lambda(z) + \frac{1}{e^z + 1} \qquad (z \in H_\theta) \tag{5.2}$$

holds. Suppose that $-1 \in \rho(e^B)$. It follows from Theorem 2.1.3(f) that the function $z \mapsto (e^z + 1)^{-1}$ is regularisable and that $(e^z + 1)^{-1}(B) = -R(-1, e^B)$. Thus from (5.2) we obtain

Lemma 5.2.1. *Let $B \in \text{Strip}(\omega)$ and suppose that $-1 \in \rho(e^B)$. Then $\lambda \in \rho(e^B)$ whenever $|\arg \lambda| \in (\omega, \pi]$ and the operator identity*

$$\lambda R(\lambda, e^B) = -(1+\lambda) f_\lambda(B) - R(-1, e^B) \tag{5.3}$$

holds for such λ.

There is nothing unique about the point -1 here. Indeed, if $\rho(e^B)$ contains any point μ with $|\arg \mu| \in (\omega, \pi]$, then $\lambda \in \rho(e^B)$ whenever $|\arg \lambda| \in (\omega, \pi]$ [22, Lemma 4.4.2]. The identity (5.3) can be adjusted accordingly.

Although it appears to be an open question whether existence of the resolvent of e^B outside some sector is sufficient for sectoriality, we can estimate the norm of $\lambda R(\lambda, e^B)$ for $|\arg \lambda| \in (\omega, \pi]$ when $-1 \in \rho(e^B)$. In light of Lemma 5.2.1 we simply need to estimate the norm of the contour integral defining $f_\lambda(B)$.

The precise nature of this estimate depends on the rate of decay of the norms of the resolvents of B along half-lines parallel to the imaginary axis. Initially we prove a result for arbitrary strip-type operators. We show that, if $B \in \text{Strip}(\omega)$ is such that $-1 \in \rho(e^B)$, then the norm of $\lambda R(\lambda, e^B)$ grows like $\log|\lambda|$ as $|\lambda| \to \infty$ (Proposition 5.2.4). We then show that if B is a strong strip-type operator, the norm of $\lambda R(\lambda, e^B)$ actually grows like $\log\log|\lambda|$ (Proposition 5.2.9). Finally we introduce the class of *α-strong strip-type operators*, whose resolvents satisfy an estimate somewhere between that of a strip-type operator and a strong strip-type operator. For such an operator, it turns out that the norm of $\lambda R(\lambda, e^B)$ grows like a fractional power of $\log|\lambda|$ (Proposition 5.2.11).

5.2.1 Strip-Type Operators

Let $B \in \text{Strip}(\omega)$. We begin with two technical lemmas.

Lemma 5.2.2. *Let $\omega \in \mathbb{R}$. Define a function f by*

$$f(x) = \frac{|1 + xe^{i\omega}|}{1+x} \qquad (x \geq 0).$$

Then $f(x) \geq \sqrt{(1+\cos\omega)/2}$ for all $x \geq 0$.

Proof. The function f is certainly well-defined, non-negative, continuous, such that $f(0) = 1$ and $f(x) \to 1$ as $x \to \infty$. To find the minimum value of f it is enough to find the minimum value of f^2. A straightforward calculation gives

$$\frac{d(f^2)}{dx}(x) = \frac{2(1-\cos\omega)(x-1)}{(1+x)^3} \qquad (x \geq 0).$$

If $\omega = 2n\pi$ for some $n \in \mathbb{Z}$ then the derivative of f^2 is identically zero. It is easy to see that in this case f takes the constant value 1 on $[0, \infty)$. Otherwise the derivative of f^2 vanishes only when $x = 1$. The value of f at this point is $\sqrt{(1+\cos\omega)/2}$, which lies in $[0, 1)$, hence is the minimum value of f. $\square$

Lemma 5.2.3. *Let $|\omega| < \pi$ and let $\theta \in (|\omega|, \pi)$. There exists $C = C(\omega, \theta) > 0$ such that*

$$|s - \lambda e^{i\omega}| \geq C(\omega, \theta)(s + |\lambda|)$$

for all $s > 0$ and λ with $|\arg\lambda| \in [\theta, \pi]$.

Proof. Define

$$C = C(\omega, \theta) := \inf \left\{ \sqrt{\frac{1 + \cos \mu}{2}} : |\mu| \leq |\omega| + \pi - \theta \right\}.$$

Then $C > 0$ since $|\omega| < \theta < \pi$, and in fact

$$C(\omega, \theta) = \sqrt{\frac{1 + \cos(|\omega| + \pi - \theta)}{2}}.$$

Let λ be such that $|\arg \lambda| \in [\theta, \pi]$ and let $s > 0$. With $x = |\lambda|/s$ we have

$$\begin{aligned}
|s - \lambda e^{i\omega}| \geq C(s + |\lambda|) \quad &\Longleftrightarrow \quad \left|1 - \frac{\lambda}{s} e^{i\omega}\right| \geq C \left(1 + \left|\frac{\lambda}{s}\right|\right) \\
&\Longleftrightarrow \quad \left|1 + \frac{|\lambda|}{s} e^{i(\arg(-\lambda)+\omega)}\right| \geq C \left(1 + \left|\frac{\lambda}{s}\right|\right) \\
&\Longleftrightarrow \quad \left|1 + x e^{i(\arg(-\lambda)+\omega)}\right| \geq C\,(1 + x),
\end{aligned}$$

and this last inequality holds by Lemma 5.2.2, since

$$\sqrt{\frac{1 + \cos(\omega + \arg(-\lambda))}{2}} \geq C(\omega, \theta)$$

whenever $|\arg \lambda| \in [\theta, \pi]$. □

Using these results we can now estimate the norm of the resolvent of e^B when $-1 \in \rho(e^B)$.

Proposition 5.2.4. *Let $B \in \mathrm{Strip}(\omega)$ and suppose that $-1 \in \rho(e^B)$. Then for each $\theta \in (\omega, \pi)$ there exists a constant $C = C(\theta) > 0$ such that*

$$\|\lambda R(\lambda, e^B)\| \leq C\,(1 + |\log|\lambda||) \qquad (\theta \leq |\arg \lambda| \leq \pi).$$

Proof. Suppose that $\omega < \varphi < \theta < \pi$ and let λ be such that $|\arg \lambda| \geq \theta$. As mentioned earlier, we shall estimate the norm of $f_\lambda(B)$. Since B is a strip-type operator, there exists a constant $M = M(\varphi) > 0$ such that $\|R(z, B)\| \leq M$ whenever $|\mathrm{Im}\, z| \geq \varphi$. Hence

$$\begin{aligned}
\|f_\lambda(B)\| &= \left\| \frac{1}{2\pi i} \int_{\partial H_\varphi} f_\lambda(z) R(z, B)\, dz \right\| \\
&\leq \frac{M}{2\pi} \left(\int_{-\infty}^{\infty} |f_\lambda(t - i\varphi)|\, dt + \int_{-\infty}^{\infty} |f_\lambda(t + i\varphi)|\, dt \right).
\end{aligned}$$

Setting $s = e^t$ we see that

$$\int_{-\infty}^{\infty} |f_\lambda(t+i\varphi)|\, dt = \int_0^\infty \frac{ds}{|s - \lambda e^{-i\varphi}||s + e^{-i\varphi}|}.$$

Therefore, providing $|\lambda| \neq 1$, it follows from Lemma 5.2.3 that

$$\begin{aligned}
\int_{-\infty}^{\infty} |f_\lambda(t+i\varphi)|\, dt &\leq \frac{1}{C(\varphi,\theta)^2} \int_0^\infty \frac{ds}{(s+|\lambda|)(s+1)} \\
&= \frac{1}{C(\varphi,\theta)^2(1-|\lambda|)} \lim_{k\to\infty} \int_0^k \left(\frac{1}{s+|\lambda|} - \frac{1}{s+1}\right) ds \\
&= \frac{1}{C(\varphi,\theta)^2(1-|\lambda|)} \lim_{k\to\infty} \left[\log\left(\frac{s+|\lambda|}{s+1}\right)\right]_0^k \\
&= -\frac{\log|\lambda|}{C(\varphi,\theta)^2(1-|\lambda|)},
\end{aligned}$$

where $C(\varphi,\theta)^2 = (1-\cos(\theta-\varphi))/2$. We can estimate the integral along $\mathbb{R} - i\varphi$ similarly to obtain

$$\|f_\lambda(B)\| \leq \frac{M \log|\lambda|}{C(\varphi,\theta)^2 \pi(|\lambda|-1)}.$$

Hence it follows from (5.3) that

$$\|\lambda R(\lambda, e^B)\| \leq \frac{M|1+\lambda|\log|\lambda|}{C(\varphi,\theta)^2\pi(|\lambda|-1)} + \|R(-1,e^B)\| \tag{5.4}$$

for $\theta \leq |\arg\lambda| \leq \pi$ and $|\lambda| \neq -1$. Now for small $|\lambda|$, say $|\lambda| < \varepsilon_0 < 1$, we have

$$\frac{|1+\lambda|}{1-|\lambda|} \leq K_{\varepsilon_0}$$

where $K_{\varepsilon_0} = (1+\varepsilon_0)/(1-\varepsilon_0)$. On the other hand, for $|\lambda| > N_0 > 1$,

$$\frac{|1+\lambda|}{|\lambda|-1} \leq K_{N_0}$$

where $K_{N_0} = (1+\frac{1}{N_0})/(1-\frac{1}{N_0})$. Finally, the holomorphic function $\lambda \mapsto \lambda R(\lambda, e^B)$ is bounded on the compact set $\{\lambda \in \mathbb{C} : |\arg\lambda| \in [\theta,\pi], |\lambda| \in [\varepsilon_0, N_0]\}$. Hence we deduce from (5.4) that in fact there exists a constant $C = C(\theta) > 0$ such that

$$\|\lambda R(\lambda, e^B)\| \leq C\,(1 + |\log|\lambda||)$$

whenever $\theta \leq |\arg\lambda| \leq \pi$. □

5.2.2 Strong Strip-Type Operators

We have already mentioned that the class of strong strip-type operators includes logarithms of injective sectorial operators, as well as imaginary multiples of C_0-group generators. One particular example is the derivative operator $B = -i\frac{d}{dt}$ on $\mathbb{R}$. The operator iB generates a C_0-group of isometries on $L^p(\mathbb{R}; X)$ for $p \in [1, \infty)$, hence $B \in \text{SStrip}(0)$. It therefore follows from Monniaux's Theorem that e^B is sectorial when $p \in (1, \infty)$ and X is a UMD space. We look at the case $p = 1$ in Section 5.5.

As another example, let Δ_p denote the Laplacian in $L^p(\mathbb{R}; X)$, with natural maximal domain. It is shown in [22, Section 8.3] that $\Delta_p \in \text{SStrip}(0)$ for each $p \in [1, \infty)$. If $X = H$ is a Hilbert space then $i\Delta_2$ even generates a C_0-group on $L^2(\mathbb{R}; H)$, however, it was proved by Hörmander in [29] that if $p \neq 2$ then $i\Delta_p$ does not generate a C_0-group (see also [22, Proposition 8.3.8]). Haase has shown that e^{Δ_p} is a bounded sectorial operator of angle 0 on $L^p(\mathbb{R}; X)$ when $1 < p < \infty$ and X is a UMD space [22, Theorem 8.3.9]. It seems to be unknown whether e^{Δ_1} is sectorial on $L^1(\mathbb{R})$, thus Δ_1 may be an example of a strong strip-type operator which is neither the logarithm of some sectorial operator, nor a C_0-group generator.

Suppose that $B \in \text{SStrip}(\omega)$. Since $\text{SStrip}(\omega) \subset \text{Strip}(\omega)$, the discussion at the beginning of Section 5.2 is still valid, and indeed we shall prove a version of Proposition 5.2.4 for strong strip-type operators. In order to do this we will use a different contour from that in the previous section, so as to make use of the strong strip-type estimate on the resolvent of B.

Let $\omega < \theta < |\arg \lambda| \leq \pi$. The function f_λ defined in (5.1) can be extended to a function on the whole of $\mathbb{C}$, holomorphic everywhere except for simple poles at $z = (2k+1)\pi i$ and $z = (2k\pi + \arg \lambda)i$, for $k \in \mathbb{Z}$. Initially we will suppose that

$|\lambda| < 1$. Let $\varphi \in (\omega, \theta)$ and define $\Gamma_1, \dots, \Gamma_4$ as follows:

$$\begin{aligned}
\Gamma_1 &= (-\infty, \log|\lambda|] + i\varphi, \\
\Gamma_2 &= \left\{ t + i(\varphi + t - \log|\lambda|) : \log|\lambda| < t \le \frac{\log|\lambda|}{2} \right\}, \\
\Gamma_3 &= \left\{ t + i(\varphi - t) : \frac{\log|\lambda|}{2} < t \le 0 \right\}, \\
\Gamma_4 &= (0, \infty) + i\varphi.
\end{aligned}$$

Set $\Gamma_+ = \bigcup_{j=1}^4 \Gamma_j$, so that Γ_+ runs from left to right in the upper half-plane, avoiding all poles of f_λ. Let Γ_- be the reflection of Γ_+ in the real axis, and set $\Gamma = \Gamma_- \cup \Gamma_+$. Set

$$I_j = \left\| \int_{\Gamma_j} f_\lambda(z) R(z, B)\, dz \right\| \qquad (j = 1, \dots, 4).$$

We shall estimate each of $I_1, \dots, I_4$ in turn. Since Γ_1 and Γ_4 are horizontal line-segments, I_1 and I_4 can be estimated exactly as in Proposition 5.2.4, without making use of the strong strip-type estimate. Indeed, by Lemma 5.2.3,

$$\begin{aligned}
\left\| \int_{\Gamma_1} f_\lambda(z) R(z, B)\, dz \right\| &\le M \int_0^{|\lambda|} \frac{ds}{|s - \lambda e^{-i\varphi}||s + e^{-i\varphi}|} \\
&\le \frac{M}{C^2} \int_0^{|\lambda|} \frac{ds}{(s + |\lambda|)(s + 1)} \\
&= \frac{M}{C^2(1 - |\lambda|)} \left[\log\left(\frac{s + |\lambda|}{s + 1} \right) \right]_0^{|\lambda|} \\
&\le \frac{M}{C^2(1 - |\lambda|)} \log 2, \qquad (5.5)
\end{aligned}$$

where $C^2 = (1 - \cos(\theta - \varphi))/2 > 0$ and $M > 0$ depends on φ. Similarly,

$$\begin{aligned}
\left\| \int_{\Gamma_4} f_\lambda(z) R(z, B)\, dz \right\| &\le M \int_1^{\infty} \frac{ds}{|s - \lambda e^{-i\varphi}||s + e^{-i\varphi}|} \\
&\le \frac{M}{C^2} \int_1^{\infty} \frac{ds}{(s + |\lambda|)(s + 1)} \\
&= \frac{M}{C^2(1 - |\lambda|)} \lim_{k \to \infty} \left[\log\left(\frac{s + |\lambda|}{s + 1} \right) \right]_1^{k} \\
&= \frac{M}{C^2(1 - |\lambda|)} \left(-\log\left(\frac{1 + |\lambda|}{2} \right) \right) \\
&\le \frac{M}{C^2(1 - |\lambda|)} \log 2. \qquad (5.6)
\end{aligned}$$

The estimates for I_2 and I_3 require some preliminary lemmas.

Lemma 5.2.5. *Let $0 < \varphi < \theta < \pi$ and suppose that λ is such that $|\arg\lambda| \in [\theta, \pi]$ and $|\lambda| < 1$. There exists a constant $C > 0$, independent of λ, such that*

$$|se^{i(\varphi - \log|\lambda| + \log s)} - \lambda| \geq C\,(s + |\lambda|) \qquad (|\lambda| \leq s \leq |\lambda|^{1/2}).$$

Proof. Suppose initially that λ is real, i.e., $-1 < \lambda < 0$. By setting $x = s/|\lambda|$, we see that it is enough to show that there exists a constant $C > 0$ such that

$$|xe^{i(\varphi + \log x)} + 1| \geq C\,(x + 1) \qquad (x \geq 1).$$

The function

$$f : x \mapsto \frac{|xe^{i(\varphi + \log x)} + 1|}{x + 1} \qquad (x \geq 1)$$

is continuous, is strictly positive (since $\varphi < \pi$), and $f(x) \to 1$ as $x \to \infty$. Hence f has a global minimum $C > 0$. More generally, the function

$$f : x \mapsto \frac{|xe^{i(\varphi - \arg(-\lambda) + \log x)} + 1|}{x + 1} \qquad (x \geq 1)$$

has a global minimum $C > 0$ which may depend on λ, but remains uniform provided $|\arg\lambda| \in [\theta, \pi]$. □

Lemma 5.2.6. *Let $0 < \varphi < \theta < \pi$, and suppose that λ is such that $|\arg\lambda| \in [\theta, \pi]$ and $|\lambda| < \frac{1}{2}$. There exists a constant $C > 0$, independent of λ, such that*

$$|se^{i(\varphi - \log|\lambda| + \log s)} + 1| \geq C\,(s + 1) \qquad (|\lambda| \leq s \leq |\lambda|^{1/2}).$$

Proof. Since $|\lambda| < \frac{1}{2}$ we have

$$|se^{i(\varphi - \log|\lambda| + \log s)} + 1| \geq 1 - s \geq 1 - |\lambda|^{1/2} \geq 1 - \frac{1}{\sqrt{2}},$$

and as $s < 1$ this proves the result. □

We can now use the strong strip-type estimate together with Lemmas 5.2.5 and 5.2.6 to estimate I_2. Let $|\arg\lambda| \in [\theta,\pi]$ and $|\lambda| < 1/2$. With $\varepsilon = \varphi - \omega'$ for some $\omega' \in (\omega,\varphi)$, we have

$$\begin{aligned}\left\|\int_{\Gamma_2} f_\lambda(z)R(z,B)\,dz\right\| &\leq C\int_{|\lambda|}^{|\lambda|^{1/2}} \frac{ds}{(s+|\lambda|)(s+1)\,(\log(s/|\lambda|)+\varepsilon)} \\ &\leq C\int_{|\lambda|}^{|\lambda|^{1/2}} \frac{ds}{(s+|\lambda|)\,(\log(s/|\lambda|)+\varepsilon)}.\end{aligned}$$

Setting $u = s/|\lambda|$ we see that

$$\begin{aligned}\left\|\int_{\Gamma_2} f_\lambda(z)R(z,B)\,dz\right\| &\leq C\int_{1}^{|\lambda|^{-1/2}} \frac{du}{(u+1)(\log u+\varepsilon)} \\ &\leq C\int_{1}^{|\lambda|^{-1/2}} \frac{du}{u(\log u+\varepsilon)} \\ &\leq C\,[\log(\log u+\varepsilon)]_1^{|\lambda|^{-1/2}} \\ &= C\log\left(-\frac{1}{2\varepsilon}\log|\lambda|+1\right). \end{aligned} \tag{5.7}$$

Lemma 5.2.7. *Let $0 < \varphi < \theta < \pi$, and suppose that λ is such that $|\arg\lambda| \in [\theta,\pi]$ and $|\lambda| < 1/2$. There exists a constant $C > 0$, independent of λ, such that*

$$|se^{i(\varphi-\log s)} - \lambda| \geq C\,(s+|\lambda|) \qquad (|\lambda|^{1/2} \leq s \leq 1).$$

Proof. As $s > |\lambda|^{1/2} > |\lambda|$ it is sufficient to prove the existence of a constant $C > 0$ for which

$$|se^{i(\varphi-\log s)} - \lambda| \geq Cs \qquad (|\lambda|^{1/2} \leq s \leq 1).$$

Since $|\lambda| < 1/2$ we have

$$\left|e^{i(\varphi-\log s)} - \frac{\lambda}{s}\right| \geq 1 - \left|\frac{\lambda}{s}\right| \geq 1 - \frac{1}{\sqrt{2}},$$

as required. □

Lemma 5.2.8. *Let $0 < \varphi < \theta < \pi$ and suppose that λ is such that $|\arg\lambda| \in [\theta,\pi]$ and $|\lambda| < 1$. There exists a constant $C > 0$, independent of λ, such that*

$$|se^{i(\varphi-\log s)} + 1| \geq C\,(s+1) \qquad (|\lambda|^{1/2} \leq s \leq 1).$$

Proof. It is sufficient to prove the existence of $C > 0$ such that

$$|se^{i(\varphi - \log s)} + 1| \geq C \qquad (|\lambda|^{1/2} \leq s \leq 1).$$

The function

$$f : s \mapsto |se^{i(\varphi - \log s)} + 1| \qquad (0 < s \leq 1)$$

is continuous, strictly positive, and $f(s) \to 1$ as $s \to 0$. Hence it has a minimum value $C > 0$, proving the result. □

We can now use Lemmas 5.2.7 and 5.2.8 to estimate I_3. Suppose that $|\arg \lambda| \in [\theta, \pi]$ and that $|\lambda| < \frac{1}{2}$. Setting $\varepsilon = \varphi - \omega'$ where $\omega' \in (\omega, \varphi)$, we have

$$\begin{aligned} \left\| \int_{\Gamma_3} f_\lambda(z) R(z, B)\, dz \right\| &\leq C \int_{|\lambda|^{1/2}}^{1} \frac{ds}{(s + |\lambda|)(s+1)(\varepsilon - \log s)} \\ &\leq C \int_{|\lambda|^{1/2}}^{1} \frac{ds}{s(\varepsilon - \log s)} \\ &= C \left[-\log \left(\log \left(\frac{1}{s} \right) + \varepsilon \right) \right]_{|\lambda|^{1/2}}^{1} \\ &= C \log \left(-\frac{1}{2\varepsilon} \log |\lambda| + 1 \right). \end{aligned} \tag{5.8}$$

We are now in a position to prove a version of Proposition 5.2.4 for strong strip-type operators. We see that in this case the norm of the resolvent satisfies an estimate involving an iterated logarithm.

Proposition 5.2.9. *Let $B \in \mathrm{SStrip}(\omega)$ and suppose that $-1 \in \rho(e^B)$. Then for each $\theta \in (\omega, \pi)$ there exists a constant $C = C(\theta) > 0$ such that*

$$\|\lambda R(\lambda, e^B)\| \leq C\,(1 + \log(1 + |\log|\lambda||)) \qquad (\theta \leq |\arg \lambda| \leq \pi).$$

Proof. Let $|\arg \lambda| \in [\theta, \pi]$, let $\varphi \in (\omega, \theta)$ and set $\varepsilon = \varphi - \omega'$ for some $\omega' \in (\omega, \varphi)$. It follows from (5.5) – (5.8) that there exists a constant $C > 0$ such that

$$\left\| \int_{\Gamma_+} f_\lambda(z) R(z, B)\, dz \right\| \leq \frac{C}{1 - |\lambda|} \left(1 + \log \left(-\frac{1}{2\varepsilon} \log |\lambda| + 1 \right) \right)$$

when $|\lambda| < \frac{1}{2}$. The integral along Γ_- can be estimated similarly, hence

$$\|f_\lambda(B)\| \leq \frac{C}{1-|\lambda|}\left(1+\log\left(-\frac{1}{2\varepsilon}\log|\lambda|+1\right)\right)$$

when $|\lambda| < \frac{1}{2}$. From this we see that there exist $C > 0$ and $\delta > 0$ such that

$$\|f_\lambda(B)\| \leq \frac{C}{1-|\lambda|}\left(1+\log\left(1+|\log|\lambda||\right)\right)$$

for $|\lambda| < \delta$. Therefore it follows from (5.3) that

$$\|\lambda R(\lambda, e^B)\| \leq C\ \left(1+\log\left(1+|\log|\lambda||\right)\right) \tag{5.9}$$

for $|\lambda| < \delta$, where $C > 0$ is a constant independent of λ.

We can obtain an estimate for large $|\lambda|$ by considering the inverse of e^B. Indeed, since $\arg(1/\lambda) = -\arg\lambda$, the identity $\lambda R(\lambda, (e^B)^{-1}) = I - \lambda^{-1}R\left(\lambda^{-1}, e^B\right)$ follows from [22, Lemma A.2.1]. Now, it is clear that $-B$ is also a strong strip-type operator, and by Theorem 2.1.1(f) it follows that $e^{-B} = (e^B)^{-1}$. Therefore the norm of $\lambda R(\lambda, (e^B)^{-1})$ satisfies an estimate identical to (5.9) for $|\lambda| < \delta$. Hence, for such λ we have

$$\left\|\lambda^{-1}R\left(\lambda^{-1}, e^B\right)\right\| \leq C\ \left(1+\log\left(1+|\log|\lambda||\right)\right),$$

and setting $\mu = \lambda^{-1}$ this becomes

$$\|\mu R(\mu, e^B)\| \leq C\ \left(1+\log\left(1+|\log|\mu||\right)\right) \tag{5.10}$$

for $|\mu| > \delta^{-1}$. Clearly $\|\lambda R(\lambda, e^B)\|$ is bounded on the compact set

$$\{\,\lambda \in \mathbb{C} : |\arg\lambda| \in [\theta, \pi], |\lambda| \in [\delta, \delta^{-1}]\,\}.$$

Together with (5.9) and (5.10) this implies that

$$\|\lambda R(\lambda, e^B)\| \leq C\ \left(1+\log\left(1+|\log|\lambda||\right)\right)$$

for $|\arg\lambda| \in [\theta, \pi]$. $\square$

5.2.3 α-Strong Strip-Type Operators

We now introduce a class of operators whose spectra are contained in a horizontal strip, but whose resolvent operators satisfy an estimate somewhere between that of a strip-type operator and that of a strong strip-type operator. Namely, for $\alpha \in (0,1)$ and $\omega \in [0,\pi)$, we say that B is an **α-strong strip-type operator** of height ω, written $B \in \alpha\text{-SStrip}(\omega)$, if

1. $\sigma(B) \subset \overline{H_\omega}$ and

2. For each $\omega' \in (\omega,\pi)$ there exists $C = C(\omega') > 0$ such that

$$\|R(\lambda, B)\| \leq \frac{C}{(|\operatorname{Im}\lambda| - \omega')^\alpha} \qquad (|\operatorname{Im}\lambda| > \omega').$$

In the next result, the relationship between strip-type, strong strip-type and α-strong strip-type operators is made clear.

Proposition 5.2.10. *Let $\omega \in [0,\pi)$ and $0 < \alpha < \beta < 1$. Then*

$$\text{SStrip}(\omega) \subset \beta\text{-SStrip}(\omega) \subset \alpha\text{-SStrip}(\omega) \subset \text{Strip}(\omega).$$

Proof. Let $\omega' \in (\omega,\pi)$ and suppose that $|\operatorname{Im}\lambda| > \omega'$. We prove the three inclusions in turn.

If $B \in \text{SStrip}(\omega)$ then there exists a constant $M(\omega') > 0$ such that $\|R(\lambda,B)\| \leq M(\omega')(|\operatorname{Im}\lambda| - \omega')^{-1}$. If $|\operatorname{Im}\lambda| - \omega' \geq 1$ then $\|R(\lambda,B)\| \leq M(\omega')(|\operatorname{Im}\lambda| - \omega')^{-\beta}$. Otherwise, choose $\theta \in (\omega,\omega')$. There exists a constant $M(\theta) > 0$ such that

$$\|R(\mu, B)\| \leq \frac{M(\theta)}{|\operatorname{Im}\mu| - \theta} \qquad (|\operatorname{Im}\mu| > \theta).$$

In particular,

$$\|R(\lambda, B)\| \leq \frac{M(\theta)}{\omega' - \theta} \leq \frac{M(\theta)/(\omega' - \theta)}{(|\operatorname{Im}\lambda| - \omega')^\beta}.$$

Hence $B \in \beta\text{-SStrip}(\omega)$.

If $B \in \beta$-SStrip(ω) then there exists a constant $M(\omega') > 0$ such that $\|R(\lambda, B)\| \leq M(\omega')(|\operatorname{Im}\lambda| - \omega')^{-\beta}$. If $|\operatorname{Im}\lambda| - \omega' \geq 1$ then $\|R(\lambda, B)\| \leq M(\omega')(|\operatorname{Im}\lambda| - \omega')^{-\alpha}$. Otherwise, choose $\theta \in (\omega, \omega')$. There exists a constant $M(\theta) > 0$ such that

$$\|R(\mu, B)\| \leq \frac{M(\theta)}{(|\operatorname{Im}\mu| - \theta)^{\beta}} \qquad (|\operatorname{Im}\mu| > \theta).$$

In particular,

$$\|R(\lambda, B)\| \leq \frac{M(\theta)}{(\omega' - \theta)^{\beta}} \leq \frac{M(\theta)/(\omega' - \theta)^{\beta}}{(|\operatorname{Im}\lambda| - \omega')^{\beta}}.$$

Hence $B \in \alpha$-SStrip(ω).

If $B \in \alpha$-SStrip(ω) then there exists a constant $M(\omega') > 0$ such that $\|R(\lambda, B)\| \leq M(\omega')(|\operatorname{Im}\lambda| - \omega')^{-\alpha}$. If $|\operatorname{Im}\lambda| - \omega' \geq 1$ then $\|R(\lambda, B)\| \leq M(\omega')$. Otherwise, choose $\theta \in (\omega, \omega')$. There exists a constant $M(\theta) > 0$ such that

$$\|R(\mu, B)\| \leq \frac{M(\theta)}{(|\operatorname{Im}\mu| - \theta)^{\alpha}} \qquad (|\operatorname{Im}\mu| > \theta).$$

In particular, $\|R(\lambda, B)\| \leq M(\theta)/(\omega' - \theta)^{\alpha}$. Hence $B \in$ Strip(ω). □

If B belongs to the class α-SStrip(ω) and $-1 \in \rho(e^B)$ then it follows from Lemma 5.2.1 and Proposition 5.2.10 that $\lambda \in \rho(e^B)$ whenever $|\arg\lambda| \in (\omega, \pi]$. We can use similar methods to those in the previous sections to estimate the norm of $\lambda R(\lambda, e^B)$ in this case. As we would expect, it turns out that such an estimate is somewhere between those obtained in Propositions 5.2.4 and 5.2.9.

Proposition 5.2.11. *Let $B \in \alpha$-SStrip(ω) for some $\alpha \in (0,1)$ and suppose that $-1 \in \rho(e^B)$. Then for each $\theta \in (\omega, \pi)$ there exists a constant $C = C(\theta, \alpha) > 0$ such that*

$$\|\lambda R(\lambda, e^B)\| \leq C\left(1 + |\log|\lambda||^{1-\alpha}\right) \qquad (\theta \leq |\arg\lambda| \leq \pi).$$

Proof. Let $\theta \in (\omega, \pi)$ and suppose that $|\arg\lambda| \in [\theta, \pi]$ and $|\lambda| < 1/2$. We use the same contour Γ as defined in Section 5.2.2. Since B is a strip-type operator, the estimates obtained in (5.5) and (5.6) are still valid. We use the α-strong strip-type resolvent estimate on the remaining portions of the contour.

Let $\varphi \in (\omega, \theta)$ and set $\varepsilon = \varphi - \omega'$ for some $\omega' \in (\omega, \varphi)$. By Lemmas 5.2.5 and 5.2.6 we have

$$\begin{aligned} I_2 &\leq C \int_{|\lambda|}^{|\lambda|^{1/2}} \frac{ds}{(s+|\lambda|)(s+1)(\log(s/|\lambda|)+\varepsilon)^\alpha} \\ &\leq C \int_{|\lambda|}^{|\lambda|^{1/2}} \frac{ds}{(s+|\lambda|)(\log(s/|\lambda|)+\varepsilon)^\alpha}. \end{aligned}$$

Putting $u = s/|\lambda|$ we see that

$$\begin{aligned} I_2 &\leq C \int_{1}^{|\lambda|^{-1/2}} \frac{du}{(u+1)(\log u+\varepsilon)^\alpha} \\ &\leq C \int_{1}^{|\lambda|^{-1/2}} \frac{du}{u(\log u+\varepsilon)^\alpha} \\ &= C \left[\frac{1}{1-\alpha}(\log u + \varepsilon)^{1-\alpha}\right]_1^{|\lambda|^{-1/2}} \\ &= \frac{C}{1-\alpha}\left[\left(-\frac{1}{2}\log|\lambda| + \varepsilon\right)^{1-\alpha} - \varepsilon^{1-\alpha}\right]. \end{aligned}$$

Similarly, we can use Lemmas 5.2.7 and 5.2.8 to estimate I_3:

$$\begin{aligned} I_3 &\leq C \int_{|\lambda|^{1/2}}^{1} \frac{ds}{(s+|\lambda|)(s+1)(\log(s^{-1})+\varepsilon)^\alpha} \\ &\leq C \int_{|\lambda|^{1/2}}^{1} \frac{ds}{s(\log(s^{-1})+\varepsilon)^\alpha} \\ &= C \left[\frac{1}{\alpha-1}\left(\log\left(\frac{1}{s}\right) + \varepsilon\right)^{1-\alpha}\right]_{|\lambda|^{1/2}}^{1} \\ &= \frac{C}{1-\alpha}\left[\left(-\frac{1}{2}\log|\lambda| + \varepsilon\right)^{1-\alpha} - \varepsilon^{1-\alpha}\right]. \end{aligned}$$

Therefore,

$$\left\|\int_{\Gamma_+} f_\lambda(z) R(z,B)\,dz\right\| \leq \frac{C}{(1-|\lambda|)(1-\alpha)}\left[\left(-\frac{1}{2}\log|\lambda| + \varepsilon\right)^{1-\alpha} - \varepsilon^{1-\alpha}\right]$$

for $|\lambda| < 1/2$. We can obtain a similar estimate for Γ_-, hence

$$\|f_\lambda(B)\| \leq \frac{C}{(1-|\lambda|)(1-\alpha)}\left[1 + \left(-\frac{1}{2}\log|\lambda| + \varepsilon\right)^{1-\alpha} - \varepsilon^{1-\alpha}\right]$$

for $|\lambda| < 1/2$. It follows that there exist $C = C(\theta, \alpha) > 0$ and $\delta > 0$ such that

$$\|f_\lambda(B)\| \leq \frac{C}{1 - |\lambda|} \left(1 + |\log|\lambda||^{1-\alpha}\right) \qquad (|\lambda| < \delta).$$

As in the proof of Proposition 5.2.9 we consider the inverse of e^B to extend this estimate to large $|\lambda|$, thus

$$\|\lambda R(\lambda, e^B)\| \leq C \left(1 + |\log|\lambda||^{1-\alpha}\right)$$

whenever $\theta \leq |\arg\lambda| \leq \pi$. □

5.3 A Criterion for Sectoriality of e^B

In this section we derive a necessary and sufficient condition for e^B to be sectorial when B belongs to a certain subclass of strip-type operators whose resolvents decay to 0 (Proposition 5.3.1). This subclass includes α-strong strip-type operators for $\alpha \in (0, 1)$ and, in particular, strong strip-type operators.

Let $B \in \text{Strip}(\omega)$ and suppose that $|\arg\lambda| \in (\omega, \pi]$. We have already mentioned that the function f_λ defined by (5.1) can be thought of as a function on the whole complex plane. This function is holomorphic everywhere except for simple poles at $z = (2k+1)\pi i$ and at $z = \log|\lambda| + (2k\pi + \arg\lambda)i$, where $k \in \mathbb{Z}$. If we let $g_\lambda(z) = f_\lambda(z)R(z, B)$ then g_λ is holomorphic on $\mathbb{C} \setminus \overline{H_\theta}$ for each $\theta \in (\omega, |\arg\lambda|)$, except for simple poles at the same points. The residues of g_λ at these poles are as follows:

$$\begin{aligned} \text{Res}(g_\lambda, (2k+1)\pi i) &= -\frac{1}{1+\lambda} R((2k+1)\pi i, B) \\ \text{Res}(g_\lambda, \log|\lambda| + (2k\pi + \arg\lambda)i) &= \frac{1}{1+\lambda} R(\log|\lambda| + (2k\pi + \arg\lambda)i, B). \end{aligned}$$

If $-1 \in \rho(e^B)$ then Lemma 5.2.1 tells us that $(-\infty, 0) \subset \rho(e^B)$. Indeed, for sectoriality of e^B it is enough to look at the operators $\lambda R(\lambda, e^B)$ just for $\lambda < 0$, and in this case the contour $\Gamma_n := \partial H_{2n\pi}$ avoids all poles of f_λ. By Cauchy's Residue Theorem we obtain

$$\begin{aligned} f_\lambda(B) &= \frac{1}{2\pi i}\int_{\Gamma_n} f_\lambda(z)R(z,B)\,dz \\ &\quad - \frac{1}{1+\lambda}\sum_{k=-n}^{n-1}\left[R(\log|\lambda| + (2k+1)\pi i, B) - R((2k+1)\pi i, B)\right] \\ &= \frac{1}{2\pi i}\int_{\Gamma_n} f_\lambda(z)R(z,B)\,dz \\ &\quad + \frac{\log|\lambda|}{1+\lambda}\sum_{k=-n}^{n-1} R(\log|\lambda| + (2k+1)\pi i, B)R((2k+1)\pi i, B), \end{aligned} \tag{5.11}$$

where the last step follows from the resolvent equation. Suppose that

$$\sup_{|\operatorname{Im} z|=\theta} \|R(z,B)\| \to 0 \quad \text{as} \quad \theta \to +\infty. \tag{5.12}$$

Any B belonging to the class of strong strip-type or α-strong strip-type operators (where $\alpha \in (0,1)$) satisfies (5.12). If $\theta \in (\omega, \pi)$ then the norm of the integral in (5.11) is bounded by

$$\sup_{|\operatorname{Im} z|=2n\pi} \|R(z,B)\| \left(\int_{-\infty}^{\infty} |f_\lambda(t-2n\pi i)|\,dt + \int_{-\infty}^{\infty} |f_\lambda(t+2n\pi i)|\,dt\right).$$

Since f_λ is $2\pi i$-periodic, it follows from Lemma 5.2.3 and the proof of Proposition 5.2.4 that there exists $C > 0$ such that

$$\int_{-\infty}^{\infty} |f_\lambda(t \pm 2n\pi i)|\,dt \le \frac{C\log|\lambda|}{|\lambda|-1} \qquad (n \in \mathbb{N}).$$

Then, by (5.12), we see that $\int_{\Gamma_n} f_\lambda(z)R(z,B)\,dz \to 0$ as $n \to \infty$. Hence

$$f_\lambda(B) = \frac{\log|\lambda|}{1+\lambda}\sum_{k=-\infty}^{\infty} R(\log|\lambda| + (2k+1)\pi i, B)R((2k+1)\pi i, B), \tag{5.13}$$

where the infinite sum is interpreted as the unique limit of the Cauchy sequence

$$\left(\sum_{k=-n}^{n-1} R(\log|\lambda| + (2k+1)\pi i, B)R((2k+1)\pi i, B)\right)_{n=1}^{\infty}.$$

Hence we obtain

Proposition 5.3.1. *Let B be a strip-type operator such that (5.12) is satisfied, and such that $-1 \in \rho(e^B)$. Then e^B is sectorial if and only if there exists a constant $C > 0$ such that*

$$\left\| \sum_{k=-\infty}^{\infty} \log|\lambda|\, R(\log|\lambda| + (2k+1)\pi i, B)\, R((2k+1)\pi i, B) \right\| \leq C$$

for all $\lambda < 0$.

Proof. We know from Lemma 5.2.1 that the assumption $-1 \in \rho(e^B)$ means that $(-\infty, 0) \in \rho(e^B)$ and that the identity $\lambda R(\lambda, e^B) = -(1+\lambda) f_\lambda(B) - R(-1, e^B)$ holds for each $\lambda < 0$. From this it follows that e^B is sectorial if and only if there exists a constant C such that $\|(1+\lambda) f_\lambda(B)\| \leq C$ for all $\lambda < 0$. By (5.13) this proves the result. □

5.4 Fourier Multipliers

In this section we look at properties of Fourier multipliers on various function spaces. Specifically, given a Fourier multiplier m on some suitable function space $E(\mathbb{R}; X)$, we consider the translated functions $(t \mapsto m(t-b))$, where b and t both lie in $\mathbb{R}$. We shall show that, on L^p-spaces, the norm of the associated Fourier multiplier operator is independent of b (Lemma 5.4.1). Using interpolation theory we also derive an estimate for the norm of the corresponding Fourier multiplier operator on Besov spaces (Proposition 5.4.3).

5.4.1 L^p-Spaces

For the background material on the Fourier transform and distributions we refer to [22, Section E.4]. Let $\mathcal{S}(\mathbb{R}; X)$ denote the Schwartz space of rapidly decreasing C^∞-functions from $\mathbb{R}$ to the Banach space X, and for $f \in \mathcal{S}(\mathbb{R}; X)$ we denote its Fourier transform by $\mathcal{F}f$ or $\hat{f}$. For $m \in L^\infty(\mathbb{R}; \mathcal{L}(X))$ we consider the map

$$T_m : \mathcal{S}(\mathbb{R}; X) \to C_0(\mathbb{R}; X)\,;\; f \mapsto \mathcal{F}^{-1}(m\hat{f}). \tag{5.14}$$

We call T_m the **Fourier multiplier operator** associated to m. Furthermore, m is said to be a **bounded L^p-Fourier multiplier** ($1 \leq p < \infty$) if there is a constant C_p such that

$$\|T_m f\|_{L^p(\mathbb{R};X)} \leq C_p \|f\|_{L^p(\mathbb{R};X)} \qquad (f \in \mathcal{S}(\mathbb{R};X)).$$

In this case T_m extends uniquely to a bounded operator on $L^p(\mathbb{R};X)$, also denoted by T_m. Let $\mathcal{M}_p(\mathbb{R};X)$ denote the space of functions $m \in L^\infty(\mathbb{R};\mathcal{L}(X))$ which are bounded $L^p(\mathbb{R};X)$-Fourier multipliers, with the norm

$$\|m\|_{\mathcal{M}_p(\mathbb{R};X)} = \|T_m\|_{\mathcal{L}(L^p(\mathbb{R};X))} \qquad (m \in \mathcal{M}_p(\mathbb{R};X)).$$

For $b \in \mathbb{R}$ and a measurable function $m : \mathbb{R} \to X$, let $\tau_b m$ and $\varepsilon_b m$ denote the functions

$$(\tau_b m)(t) = m(t-b) \qquad (\varepsilon_b m)(t) = e^{ibt} m(t) \qquad (t \in \mathbb{R}).$$

Clearly both τ_b and ε_b are isometries on $L^p(\mathbb{R};X)$ for $1 \leq p < \infty$. If m (or more precisely the multiplication operator associated to m) is a bounded L^p-Fourier multiplier then we can say more about the functions $\tau_b m$.

Lemma 5.4.1. *Let $m \in \mathcal{M}_p(\mathbb{R};X)$ for some $p \in [1,\infty)$. Then $\tau_b m \in \mathcal{M}_p(\mathbb{R};X)$ for each $b \in \mathbb{R}$, and*

$$\|\tau_b m\|_{\mathcal{M}_p(\mathbb{R};X)} = \|m\|_{\mathcal{M}_p(\mathbb{R};X)} \qquad (b \in \mathbb{R}).$$

Proof. This is a straightforward consequence of the fact that m is a bounded L^p-Fourier multiplier, together with standard properties of the Fourier transform. Let $f \in \mathcal{S}(\mathbb{R};X)$. Then $\mathcal{F}^{-1}(\tau_b f) = \varepsilon_b(\mathcal{F}^{-1} f)$ and $\tau_b(\mathcal{F} f) = \mathcal{F}(\varepsilon f)$ for every $b \in \mathbb{R}$. Hence

$$\begin{aligned}
\|\mathcal{F}^{-1}(\tau_b m)\hat{f}\|_{L^p(\mathbb{R};X)} &= \|\mathcal{F}^{-1}\tau_b(m \cdot \tau_{-b}\hat{f})\|_{L^p(\mathbb{R};X)} \\
&= \|\varepsilon_b \mathcal{F}^{-1}(m \cdot \tau_{-b}\hat{f})\|_{L^p(\mathbb{R};X)} \\
&= \|\mathcal{F}^{-1}(m \cdot \mathcal{F}(\varepsilon_{-b} f))\|_{L^p(\mathbb{R};X)} \\
&\leq \|m\|_{\mathcal{M}_p(\mathbb{R};X)} \|\varepsilon_{-b} f\|_{L^p(\mathbb{R};X)} \\
&= \|m\|_{\mathcal{M}_p(\mathbb{R};X)} \|f\|_{L^p(\mathbb{R};X)}.
\end{aligned}$$

This means that $\tau_b m$ is certainly a bounded L^p-Fourier multiplier for each $b \in \mathbb{R}$, and $\|\tau_b m\|_{\mathcal{M}_p(\mathbb{R};X)} \leq \|m\|_{\mathcal{M}_p(\mathbb{R};X)}$. Moreover,

$$\|m\|_{\mathcal{M}_p(\mathbb{R};X)} = \|\tau_{-b}(\tau_b m)\|_{\mathcal{M}_p(\mathbb{R};X)} \leq \|\tau_b m\|_{\mathcal{M}_p(\mathbb{R};X)} \leq \|m\|_{\mathcal{M}_p(\mathbb{R};X)}.$$

□

Despite its rather elementary proof, we could not find this result stated explicitly in the literature. Clearly it relies on the fact that ε_b is an isometry on $L^p(\mathbb{R};X)$. The fact that τ_b is also an isometry on $L^p(\mathbb{R};X)$ can be used to prove the corresponding result for the functions $\varepsilon_b m$.

Lemma 5.4.2. *[22, Lemma E.4.1] Let $m \in \mathcal{M}_p(\mathbb{R};X)$ for some $p \in [1,\infty)$. Then $\varepsilon_b m \in \mathcal{M}_p(\mathbb{R};X)$ for each $b \in \mathbb{R}$ and*

$$\|\varepsilon_b m\|_{\mathcal{M}_p(\mathbb{R};X)} = \|m\|_{\mathcal{M}_p(\mathbb{R};X)} \qquad (b \in \mathbb{R}).$$

5.4.2 Besov Spaces

Let $n \in \mathbb{N}$ and $p \in [1,\infty)$. For $s \in (0,n)$ and $q \in [1,\infty]$ the **Besov space** $B^s_{p,q}(\mathbb{R};X)$ is the real interpolation space between $L^p(\mathbb{R};X)$ and the Sobolev space $W^{n,p}(\mathbb{R};X)$, with interpolation parameters s/n and q, i.e.,

$$B^s_{p,q}(\mathbb{R};X) = (L^p(\mathbb{R};X), W^{n,p}(\mathbb{R};X))_{\frac{s}{n},q}.$$

$B^s_{p,q}(\mathbb{R};X)$ is endowed with the interpolation norm described in Chapter 4. A function $m \in L^\infty(\mathbb{R};\mathcal{L}(X))$ is said to be a **bounded Fourier multiplier** on $B^s_{p,q}(\mathbb{R};X)$ if there exists a constant C such that

$$\|T_m f\|_{B^s_{p,q}(\mathbb{R};X)} \leq C\,\|f\|_{B^s_{p,q}(\mathbb{R};X)} \qquad (f \in \mathcal{S}(\mathbb{R};X)),$$

where T_m is as defined in (5.14). In this case T_m extends uniquely to a bounded linear operator on $B^s_{p,q}(\mathbb{R};X)$. The space of bounded Fourier multipliers on $B^s_{p,q}(\mathbb{R};X)$ shall be denoted by $\mathcal{M}^s_{p,q}(\mathbb{R};X)$, and endowed with the norm

$$\|m\|_{\mathcal{M}^s_{p,q}(\mathbb{R};X)} = \|T_m\|_{\mathcal{L}(B^s_{p,q}(\mathbb{R};X))} \qquad (m \in \mathcal{M}^s_{p,q}(\mathbb{R};X)).$$

There exist functions which are bounded Fourier multipliers on Besov spaces but not on L^p-spaces. Indeed, we shall see in Propositions 5.5.1 and 5.5.5 that the function $t \mapsto (1+e^t)^{-1}$ is a bounded Fourier multiplier on $B^s_{1,q}(\mathbb{R})$ for $s \in (0,1)$ but not on $L^1(\mathbb{R})$.

The next result shows that, if m is a bounded Fourier multiplier on some Besov space, then so is each of the functions $\tau_b m$, for $b \in \mathbb{R}$. In contrast to Lemma 5.4.1, the Fourier multiplier norms of the functions $\tau_b m$ are not uniformly bounded. This corresponds to the fact that ε_b is no longer an isometry with respect to the Besov space norm.

Proposition 5.4.3. *Let m be a bounded Fourier multiplier on $B^\theta_{p,q}(\mathbb{R};X)$ where $0 < \theta < 1$, $1 \le p < \infty$ and $1 \le q \le \infty$. Then for $b \in \mathbb{R}$, the function $\tau_b m$ is also a bounded Fourier multiplier on $B^\theta_{p,q}(\mathbb{R};X)$ and*

$$\|\tau_b m\|_{\mathcal{M}^s_{p,q}(\mathbb{R};X)} \le (1+|b|)^{2\theta} \|m\|_{\mathcal{M}^s_{p,q}(\mathbb{R};X)} \qquad (b \in \mathbb{R}).$$

Proof. We have already noted that the rotation ε_b is an isometry on $L^p(\mathbb{R};X)$ for $1 \le p < \infty$. A simple calculation shows that

$$\|\varepsilon_b f\|_{W^{1,p}(\mathbb{R};X)} \le (1+|b|) \|f\|_{W^{1,p}(\mathbb{R};X)} \qquad (b \in \mathbb{R},\ f \in W^{1,p}(\mathbb{R};X)).$$

By Proposition 4.1.3 it follows that

$$\|\varepsilon_b f\|_{B^\theta_{p,q}(\mathbb{R};X)} \le (1+|b|)^\theta \|f\|_{B^\theta_{p,q}(\mathbb{R};X)} \qquad (b \in \mathbb{R},\ f \in B^\theta_{p,q}(\mathbb{R};X)).$$

Following the proof of Lemma 5.4.1, we see that for $f \in \mathcal{S}(\mathbb{R};X)$ and $b \in \mathbb{R}$,

$$\begin{aligned}
\|\mathcal{F}^{-1}(\tau_b m)\hat{f}\|_{B^\theta_{p,q}(\mathbb{R};X)} &= \|\mathcal{F}^{-1}\tau_b(m \cdot \tau_{-b}\hat{f})\|_{B^\theta_{p,q}(\mathbb{R};X)} \\
&= \|\varepsilon_b \mathcal{F}^{-1}(m \cdot \tau_{-b}\hat{f})\|_{B^\theta_{p,q}(\mathbb{R};X)} \\
&= \|\varepsilon_b \mathcal{F}^{-1}(m\mathcal{F}(\varepsilon_{-b} f))\|_{B^\theta_{p,q}(\mathbb{R};X)} \\
&\le (1+|b|)^{2\theta} \|m\|_{\mathcal{M}^s_{p,q}(\mathbb{R};X)} \|f\|_{B^\theta_{p,q}(\mathbb{R};X)}.
\end{aligned}$$

The result now follows since $\mathcal{S}(\mathbb{R};X)$ is dense in $B^\theta_{p,q}(\mathbb{R};X)$. □

Remark 5.4.4. *As τ_b is still an isometry on Besov spaces, it follows that the functions $\varepsilon_b m$ are uniformly bounded Fourier multipliers on $B^{\theta}_{p,q}(\mathbb{R}; X)$ whenever m is.*

5.5 The Derivative on $\mathbb{R}$

We now turn to look at a specific example of a strong strip-type operator – the derivative on the real line. The functional calculus of this operator is closely linked to the theory of Fourier multipliers. When considered as an operator on $L^1(\mathbb{R})$, the derivative is a well-known counterexample to the inversion problem (Proposition 5.5.1). The situation does improve however when we consider the part of the derivative operator in a Besov space. In this case we can show that the exponential satisfies a resolvent estimate which is close to sectoriality (Proposition 5.5.6).

Let $X = L^1(\mathbb{R})$ and let $B = -i\frac{d}{dt}$ with domain $D(B) = W^{1,1}(\mathbb{R})$. It is known that iB generates a C_0-group $(U(s))_{s\in\mathbb{R}}$ of isometries [14, Example II.2.10], given by

$$(U(s)f)(t) = f(s+t) \qquad (f \in L^1(\mathbb{R}),\ s,t \in \mathbb{R}).$$

In particular, B is a strong strip-type operator of height 0.

Information on the connection between the functional calculus of B and Fourier multipliers on $L^1(\mathbb{R})$ can be found in [22, Section 8.4]. For $f \in \mathcal{F}(H_\varphi)$, where $\varphi > 0$, the operator $f(B)$ is just convolution with some function $g_f \in L^1(\mathbb{R})$ for which $\widehat{g_f} = f|_{\mathbb{R}}$. It follows from the proof of [22, Lemma 8.4.5] that if $f \in \mathcal{H}^\infty(H_\varphi)$ then $f(B)$ is bounded if and only if $f|_{\mathbb{R}}$ is a bounded $L^1(\mathbb{R})$-Fourier multiplier. In this case the norm of $f(B)$ is just the norm of the corresponding Fourier multiplier operator. As a result, we can use the theory of Fourier multipliers to show that e^B is not sectorial on $L^1(\mathbb{R})$.

Proposition 5.5.1. *[22, Example 4.4.4] Let $B = -i\frac{d}{dt}$ on $L^1(\mathbb{R})$. Then e^B is not sectorial.*

Proof. It is clear from the above discussion that e^B is sectorial if and only if the function

$$t \mapsto \frac{\lambda}{\lambda - e^t} \qquad (t \in \mathbb{R})$$

is an $L^1(\mathbb{R})$-Fourier multiplier for each $\lambda < 0$, and the corresponding Fourier multiplier norms are uniformly bounded with respect to λ. Since $\lambda(\lambda - e^t)^{-1} = (1 + e^{t - \log(-\lambda)})^{-1}$, Lemma 5.4.1 tells us that e^B is sectorial if and only if the function

$$t \mapsto \frac{1}{1 + e^t} \qquad (t \in \mathbb{R})$$

is a bounded $L^1(\mathbb{R})$-Fourier multiplier. A function f is an $L^1(\mathbb{R})$-Fourier multiplier if and only if $f = \hat{\mu}$ for some Borel measure μ on $\mathbb{R}$ (see [22, Section E.4]). However,

$$\lim_{t \to -\infty} (1 + e^t)^{-1} = 1 \neq 0 = \lim_{t \to \infty} (1 + e^t)^{-1},$$

so by [22, Proposition E.4.3], it follows that e^B is not sectorial. $\square$

Remark 5.5.2. *The same argument can be used to show that e^B has empty resolvent set – see [22, Corollary 8.4.6].*

Thus the situation on $L^1(\mathbb{R})$ is just about the worst possible. We now want to take the part of B in some real interpolation space between X and the domain of B. For $\theta \in (0, 1)$ and $q \in [1, \infty]$, the real interpolation space $(X, D(B))_{\theta,q}$ is the Besov space $B^{\theta}_{1,q}(\mathbb{R})$. Let $\widetilde{B}$ denote the part of B in $(X, D(B))_{\theta,q}$. It is clear that an operator A belongs to $\mathrm{SStrip}(\omega)$ if and only if $\omega' \pm iA \in \mathrm{Sect}(\pi/2)$ for each $\omega' \in (\omega, \pi)$. Hence it is a consequence of Lemma 4.2.1 that $\widetilde{B}$ is a strong strip-type operator of height 0 such that $i\widetilde{B}$ generates a C_0-group on $(X, D(B))_{\theta,q}$.

Again there is a link between the functional calculus of $\widetilde{B}$ and Fourier multipliers. By Proposition 2.2.2, if $f \in \mathcal{H}^\infty(H_\varphi)$ for $\varphi > 0$ then $f(\widetilde{B}) = f(B)_{B^{\theta}_{1,q}(\mathbb{R})}$. In particular this means that $f(\widetilde{B})$ is bounded if and only if $f|_{\mathbb{R}}$ is a bounded Fourier multiplier on $B^{\theta}_{1,q}(\mathbb{R})$. Hence $e^{\widetilde{B}}$ is sectorial if and only if the functions

$$t \mapsto \frac{\lambda}{\lambda - e^t} \qquad (t \in \mathbb{R}),$$

where $\lambda < 0$, are uniformly bounded Fourier multipliers on $B^{\theta}_{1,q}(\mathbb{R})$. Girardi and Weis have obtained Mihlin-type sufficient conditions for a function to be a Fourier multiplier on a Besov space.

Proposition 5.5.3. *[16, Corollary 4.11] Let $p \in [1, \infty)$, $q \in [1, \infty]$ and $s > 0$. Let $l, N \in \mathbb{N}$ and suppose that $m \in C^l(\mathbb{R}^N, \mathcal{L}(X))$. If there exists a constant A such that*

$$\sup_{t \in \mathbb{R}^N} \left\| (1 + |t|)^{|\alpha|} D^{\alpha} m(t) \right\|_{\mathcal{L}(X)} \leq A$$

for each multi-index α with $|\alpha| \leq l$, then m is a $B^s_{p,q}(\mathbb{R}^N; X)$-Fourier multiplier provided one of the following conditions hold:

(a) X is an arbitrary Banach space and $l = N + 1$.

(b) X is a uniformly convex Banach space and $l = N$.

(c) X has Fourier type λ and $l = \lfloor \frac{N}{\lambda} \rfloor + 1$.

Note that part (a), which had already been proved by Amann in [1], requires no geometrical assumptions on the underlying Banach space X. This is in contrast to the corresponding version of Mihlin's Theorem on vector-valued L^p-spaces, which only holds when $1 < p < \infty$ and X is a UMD space. Since we are considering the scalar case, we may take $N = 1$ and $X = \mathbb{C}$ in the above result. In particular X is a Hilbert space, so it has Fourier type 2 and we may take $l = 1$. Hence we obtain

Corollary 5.5.4. *Let $p \in [1, \infty)$, $q \in [1, \infty]$ and $s > 0$. Suppose that $m \in C^1(\mathbb{R})$ satisfies*

$$\sup_{t \in \mathbb{R}} |m(t)| \leq A$$

$$\sup_{t \in \mathbb{R}} |(1 + |t|) m'(t)| \leq A$$

for some constant A. Then m is a Fourier multiplier on $B^s_{p,q}(\mathbb{R})$.

We can use Corollary 5.5.4 to show that $e^{\tilde{B}}$ has a non-empty resolvent set. This is in contrast to the situation in $L^1(\mathbb{R})$, so once again we see that things improve in real interpolation spaces.

Proposition 5.5.5. *Let $0 < \theta < 1$, $1 \leq q \leq \infty$ and let $\widetilde{B}$ denote the part of the derivative operator in the Besov space $B^{\theta}_{1,q}(\mathbb{R})$ as above. Then $-1 \in \rho(e^{\widetilde{B}})$.*

Proof. Let $m(t) = (1+e^t)^{-1}$ for $t \in \mathbb{R}$, so that $m'(t) = -e^t(1+e^t)^{-2}$. Clearly $\sup_{t\in\mathbb{R}} |m(t)| = 1$. Furthermore

$$|(1+|t|)m'(t)| = \begin{cases} (1+t)e^t(1+e^t)^{-2} & (t \geq 0), \\ (1-t)e^t(1+e^t)^{-2} & (t < 0). \end{cases}$$

Thus $|(1+|t|)m'(t)| \to 0$ as $|t| \to \infty$, and by continuity this means that $\sup_{t\in\mathbb{R}} |(1+|t|)m'(t)| < \infty$. Hence, by Corollary 5.5.4, m is a Fourier multiplier on $B^{\theta}_{1,q}(\mathbb{R})$. This means that $(I + e^{\widetilde{B}})^{-1}$ is a bounded operator on $B^{\theta}_{1,q}(\mathbb{R})$. □

Since $-1 \in \rho(e^{\widetilde{B}})$, it follows from Lemma 5.2.1 that $\sigma(e^{\widetilde{B}})$ is contained in $[0, \infty)$. Although $i\widetilde{B}$ generates a C_0-group, the space $B^{\theta}_{1,q}(\mathbb{R})$ is not UMD, and we cannot use Monniaux's Theorem to deduce sectoriality of $e^{\widetilde{B}}$. However, we can use the results of Section 5.2 to obtain a logarithmic estimate on the norm of $\lambda R(\lambda, e^{\widetilde{B}})$ for $|\arg \lambda| > 0$.

Proposition 5.5.6. *Let $B = -i\frac{d}{dt}$ on $L^1(\mathbb{R})$. Let $\theta \in (0,1)$, $q \in [1,\infty]$ and let $\widetilde{B}$ denote the part of B in the Besov space $B^{\theta}_{1,q}(\mathbb{R})$. Then for each $\omega \in (0,\pi)$ there exists $C = C(\omega) > 0$ such that*

$$\|\lambda R(\lambda, e^{\widetilde{B}})\| \leq C\,(1 + \log(1 + |\log|\lambda||)) \qquad (\omega \leq |\arg \lambda| \leq \pi).$$

Proof. We have already noted that $\widetilde{B} \in \mathrm{SStrip}(0)$, hence the result follows from Proposition 5.2.9. □

Remark 5.5.7. *(a) If we just consider $\lambda < 0$, then this estimate tells us that the norm of $\lambda R(\lambda, e^{\widetilde{B}})$ grows like* $\log\log(-\lambda)$ *for λ near $-\infty$, and like* $\log\log(-1/\lambda)$ *for λ near 0. As* $\mathrm{SStrip}(0) \subset \theta\text{-}\mathrm{SStrip}(0) \subset \mathrm{Strip}(0)$ *for each $\theta \in (0,1)$, Propositions 5.2.4 and 5.2.11 could also be applied to obtain estimates on the norm of $\lambda R(\lambda, e^{\widetilde{B}})$. However, both of these would give weaker estimates than that obtained above.*

(b) It is also possible to estimate the norm of $\lambda R(\lambda, e^{\widetilde{B}})$, *for* $\lambda < 0$, *using the Fourier multiplier theory of Section 5.4. Indeed, with* $m(t) = (1+e^t)^{-1}$ *for* $t \in \mathbb{R}$, *we have*

$$(\tau_{-\log(-\lambda)} m)(t) = \frac{\lambda}{\lambda - e^t} \qquad (t \in \mathbb{R}).$$

Hence Proposition 5.4.3 tells us that there exists a constant $C > 0$ *such that*

$$\|\lambda R(\lambda, e^{\widetilde{B}})\| \leq C\,(1 + |\log(-\lambda)|)^{2\theta} \qquad (\lambda < 0).$$

Again this estimate is weaker than that given above.

Chapter 6

Logarithms of F-Sectorial Operators

In the previous chapter we saw that the exponential of a strong strip-type operator need not be sectorial (Proposition 5.5.1). On the other hand, if B is a strong strip-type operator such that $-1 \in \rho(e^B)$, then $\sigma(e^B)$ is contained in some sector (Lemma 5.2.1), and it is possible to estimate the norm of $\lambda R(\lambda, e^B)$ for λ outside this sector (Propositions 5.2.4, 5.2.9 and 5.2.11). As greater restrictions are placed on the resolvents of B, this estimate gets closer to sectoriality.

The idea of an operator whose spectrum is contained in a sector, but which does not satisfy the exact resolvent estimate necessary for sectoriality, has been studied before. Periago and Straub [40, 41] constructed a functional calculus for an operator A whose resolvent is polynomially bounded outside a sector, and in particular considered the fractional powers A^z and exponentials e^{-tA} of such an operator. Their functional calculus is based on that developed by McIntosh for sectorial operators, and fits into the abstract framework used in this thesis. More recently, Gorodniĭ and Chaĭkovskiĭ [17] studied operators A whose resolvent grows like some function G outside a sector. They also considered fractional powers and exponentials.

In Section 6.1 we introduce the concept of an F-*sectorial operator*, where F is a positive continuous function on the positive reals. This idea is based on the work in [17]. We show how a functional calculus can be constructed for an F-sectorial operator A, and that for certain functions F it is even possible to define the logarithm of A.

Recall that, when A is injective sectorial operator, Nollau's Lemma tells us that $\lambda \in \rho(\log A)$ whenever $|\operatorname{Im} \lambda| > \pi$ and that

$$R(\lambda, \log A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2} (t + A)^{-1} \, dt \qquad (|\operatorname{Im} \lambda| > \pi). \tag{6.1}$$

It is this resolvent representation which enables us to prove that $\log A$ is a strong strip-type operator. In this chapter we investigate what can be said about the logarithm of an F-sectorial operator A.

In Section 6.2 we consider *α-log-sectorial operators*, examples of which include exponentials of $(1 - \alpha)$-strong strip-type operators. In Section 6.3 we define the class of *log log-sectorial operators*, a class which includes exponentials of strong strip-type operators. For both of these examples we show that a resolvent representation identical to (6.1) holds (Propositions 6.2.5 and 6.3.1). From such a representation we can then estimate the norm of $R(\lambda, \log A)$ for $|\operatorname{Im} \lambda| > \pi$ (Propositions 6.2.6 and 6.3.4).

In Section 6.4 we look at *log-sectorial operators*, a class which includes exponentials of strip-type operators. If we work under the assumption that $\rho(\log A)$ contains some point μ with $|\operatorname{Im} \mu| > \pi$, then we can again obtain a representation of $R(\lambda, \log A)$ (Proposition 6.4.2), from which the norm of $R(\lambda, \log A)$ can be estimated (Proposition 6.4.3).

6.1 F-Sectorial Operators

Let $\omega \in [0, \pi)$ and suppose that $F : (0, \infty) \to [1, \infty)$ is continuous. We say that an operator A is **F-sectorial** of angle ω, written $A \in F\text{-Sect}(\omega)$, if

1. $\sigma(A) \subset \overline{S_\omega}$ and

2. For each $\omega' \in (\omega, \pi)$ there exists $C = C(\omega') > 0$ such that

$$\|\lambda R(\lambda, A)\| \leq C\, F(|\lambda|) \qquad (\omega' \leq |\arg \lambda| \leq \pi).$$

Clearly if we take $F \equiv 1$ then we obtain the familiar definition of a sectorial operator.

There is no loss in assuming that F maps into $[1, \infty)$, since a Neumann series argument shows that $\liminf_{\lambda \to \infty} \|\lambda R(\lambda, A)\| > 0$, for $|\arg \lambda| > \omega$. It is also natural to only consider functions F for which $F(s) = F(1/s)$, as this means that if A is injective then A is F-sectorial if and only if A^{-1} is. In that case, since F is continuous, the definition above only depends on the behaviour of $F(s)$ for s near ∞. Later we shall consider the cases where $F(s) = (\log s)^\alpha$ for large s (where $0 < \alpha \leq 1$), and $F(s) = \log \log s$ for large s.

Let $A \in F\text{-Sect}(\omega)$ and $\theta \in (\omega, \pi)$. Although our ultimate goal is to study the logarithm of A (when it exists), we can in fact construct a functional calculus for A via the usual methods. Firstly we find a class of functions which are bounded and holomorphic on S_θ, and for which the integral $\int_\Gamma f(z) R(z, A)\, dz$ converges in $\mathcal{L}(X)$, where $\Gamma = \partial S_{\omega'}$ for some $\omega' \in (\omega, \theta)$. If we set

$$\mathcal{E}_F(S_\theta) := \left\{ f \in \mathcal{H}^\infty(S_\theta) : \int_\Gamma |f(z)| \frac{F(|z|)}{|z|}\, |dz| < \infty \right\}$$

then clearly

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz$$

is a bounded operator for each $f \in \mathcal{E}_F(S_\theta)$. Denoting by Λ the map $(f \mapsto f(A)) : \mathcal{E}_F(S_\theta) \to \mathcal{L}(X)$ we have an abstract functional calculus $(\mathcal{E}_F(S_\theta), \mathcal{M}(S_\theta), \Lambda)$.

Remark 6.1.1. *Although we have used the term abstract functional calculus, if there are constants $C, \alpha > 0$ such that*

$$|F(s)| \leq C \max\{s^\alpha, s^{-\alpha}\} \qquad (s > 0),$$

then the function $(z \mapsto z)$ is regularisable, thus we in fact have a meromorphic functional calculus.

If $\mathcal{E}_F(S_\theta)$ contains the familiar class $\mathcal{H}_0^\infty(S_\theta)$ and if A is injective, then the algebra of all regularisable functions contains the algebra $\mathcal{B}(S_\theta)$ defined in Section 2.2.1. In particular the logarithm function is regularisable and $\log A$ is well-defined. We now present an example of a function F for which this is the case.

Example 6.1.2. *Define a function F by*

$$F(s) = \begin{cases} \log s & (s > e), \\ 1 & (e^{-1} \le s \le e), \\ \log \frac{1}{s} & (0 < s < e^{-1}). \end{cases}$$

Let $A \in F\text{-Sect}(\omega)$ for some $\omega \in [0, \pi)$ and let $\theta \in (\omega, \pi)$. If $f \in \mathcal{H}_0^\infty(S_\theta)$ then by [22, Lemma 2.2.2] there exist constants $C, \varepsilon > 0$ such that

$$|f(z)| \le C \min\{|z|^\varepsilon, |z|^{-\varepsilon}\} \qquad (z \in S_\theta).$$

Let $\omega' \in (\omega, \theta)$. For $r > e$ we have

$$|f(re^{i\omega'})| \, \|R(re^{i\omega'}, A)\| \le C r^{-1-\varepsilon} \log r$$

and for $0 < r < e^{-1}$,

$$|f(re^{i\omega'})| \, \|R(re^{i\omega'}, A)\| \le C r^{-1+\varepsilon} (-\log r).$$

Identical estimates hold if we replace ω' by $-\omega'$. Hence if $\Gamma = \partial S_{\omega'}$ then

$$\begin{aligned} \left\| \int_\Gamma f(z) R(z, A)\, dz \right\| &\le \int_0^\infty |f(re^{i\omega'})| \, \|R(re^{i\omega'}, A)\| \, dr \\ &\quad + \int_0^\infty |f(re^{-i\omega'})| \, \|R(re^{-i\omega'}, A)\| \, dr \\ &< \infty \end{aligned}$$

and $f \in \mathcal{E}_F(S_\theta)$.

Haase has introduced the terms *uniform sectoriality* and *sectorial approximation* [22, Section 2.1.2]. These ideas generalise families of operators such as the Yosida approximation (defined in Section 3.2) and the Nollau approximation (defined below). Following Haase's terminology, we say that a family $(A_\iota)_{\iota\in I}$ of F-sectorial operators is **uniformly F-sectorial** of angle $\omega \in [0, \pi)$ if

$$\sup_{\iota\in I} \sup_{\lambda\notin S_{\omega'}} \left\| \frac{\lambda R(\lambda, A_\iota)}{F(|\lambda|)} \right\| < \infty$$

for all $\omega' \in (\omega, \pi)$. A uniformly F-sectorial sequence $(A_n)_{n\in\mathbb{N}}$ is called an **F-sectorial approximation** on S_ω for the operator A if $\lambda \in \rho(A)$ and $R(\lambda, A_n) \to R(\lambda, A)$ in $\mathcal{L}(X)$ for some $\lambda \notin \overline{S_\omega}$. By [22, Proposition A.5.3] it then follows that this holds for *all* $\lambda \notin \overline{S_\omega}$. Moreover, A itself belongs to F-Sect(ω). By setting $\varepsilon = 1/n$, we can also consider uniformly F-sectorial families $(A_\varepsilon)_{\varepsilon\in(0,1)}$ as F-sectorial approximations.

There is a natural relationship between sectorial approximations and the functional calculus of sectorial operators [22, Section 2.6.3]. By adapting some of the results from [22], we shall see that there is an analogous relationship between F-sectorial approximations and the functional calculus of an F-sectorial operator. The next result deals with the primary functional calculus.

Lemma 6.1.3. *Let $A \in F$-Sect(ω) and let $(A_n)_{n\in\mathbb{N}}$ be an F-sectorial approximation of A on S_ω. If $f \in \mathcal{E}_F(S_\theta)$ for some $\theta \in (\omega, \pi)$ then $f(A_n) \to f(A)$ in norm as $n \to \infty$.*

Proof. If $f \in \mathcal{E}_F(S_\theta)$ for some $\theta \in (\omega, \pi)$, then all of the operators $f(A), f(A_n)$ are well-defined and bounded. Moreover, with $\Gamma = \partial S_{\omega'}$ for some $\omega' \in (\omega, \pi)$, it follows from the Dominated Convergence Theorem that

$$\begin{aligned}
\lim_{n\to\infty} f(A_n) &= \lim_{n\to\infty} \left(\frac{1}{2\pi i} \int_\Gamma f(z) R(z, A_n)\, dz \right) \\
&= \frac{1}{2\pi i} \int_\Gamma f(z) \lim_{n\to\infty} R(z, A_n)\, dz \\
&= \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz \\
&= f(A).
\end{aligned}$$

□

The next result is a version of [22, Proposition 2.6.9]. We only consider those F-sectorial operators A for which $\log A$ can be defined. A similar result could be stated and proved in greater generality, but we shall not do this here.

Proposition 6.1.4. *Let $A \in F\text{-Sect}(\omega)$ and let $(A_n)_{n\in\mathbb{N}}$ be an F-sectorial approximation of A on S_ω. Let $\theta \in (\omega, \pi)$ and suppose that $\mathcal{H}_0^\infty(S_\theta) \subset \mathcal{E}_F(S_\theta)$. Suppose that all the operators A_n, A are injective and that $f \in \mathcal{B}(S_\theta)$. If $x_n \in D(f(A_n))$ with $x_n \to x$ and $f(A_n)x_n \to y$ as $n \to \infty$ then $x \in D(f(A))$ and $f(A)x = y$. In particular if $f(A_n) \in \mathcal{L}(X)$ with $f(A_n) \to T \in \mathcal{L}(X)$ then $f(A) = T$.*

Proof. This follows from Lemma 6.1.3 and [22, Proposition 2.6.8]. □

Let A be a sectorial operator and define

$$A_\varepsilon = (A+\varepsilon)(I+\varepsilon A)^{-1} \qquad (\varepsilon \in (0,1)), \tag{6.2}$$

so that each A_ε is a bounded invertible operator. The family $(A_\varepsilon)_{\varepsilon\in(0,1)}$ was used by Nollau in [38], and was referred to as the **Nollau approximation** by Okazawa in [39]. It is shown in [22, Proposition 2.1.3(f)] that the Nollau approximation is a sectorial approximation to A. We shall show in Proposition 6.1.6 that this same family is an F-sectorial approximation for certain F-sectorial operators.

Lemma 6.1.5. *Suppose that $\sigma(A) \subset S_\theta$. If $\varepsilon \in (0,1)$ and A_ε is as in (6.2), then $\sigma(A_\varepsilon) \subset S_\theta$ and*

$$(\lambda + A_\varepsilon)^{-1} = \frac{\varepsilon}{1+\lambda\varepsilon} I + \frac{1-\varepsilon^2}{1+\lambda\varepsilon}\left(A + \varepsilon + \frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right)^{-1} \tag{6.3}$$

for $\lambda \in S_{\pi-\theta}$.

Proof. Let $\lambda \in S_{\pi-\theta}$. The resolvent on the right-hand side of (6.3) is well-defined since

$$\left|\arg\left(\frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right)\right| = \left|\arg\left(\frac{\lambda}{\lambda+\varepsilon^{-1}}\right)\right| < |\arg\lambda|$$

and since $|\arg(\varepsilon+\mu)| < |\arg\mu|$ for all $\mu \in \mathbb{C}\setminus(-\infty, 0]$ and $\varepsilon > 0$. The identity (6.3) is then straightforward algebra. □

Proposition 6.1.6. *Suppose that F has the following properties:*

(i) The function $s \mapsto F(s)/s$ is decreasing on $(0,\infty)$.

(ii) For all $\alpha > 0$ there exists $\beta > 1$ such that $F(\alpha s) \le \beta F(s)$ for all $s > 0$.

(iii) F is non-increasing on $(0, \frac{4}{3}]$ and non-decreasing on $[\frac{3}{4}, \infty)$.

Then, if A is F-sectorial, the Nollau approximation $(A_\varepsilon)_{\varepsilon\in(0,\frac{1}{4})}$ is an F-sectorial approximation to A.

Proof. Suppose that $A \in F\text{-Sect}(\omega)$ and let $\lambda \in S_{\pi-\theta}$ for some $\theta \in (\omega, \pi)$. The fact that $(\lambda + A_\varepsilon)^{-1} \to (\lambda + A)^{-1}$ in norm as $\varepsilon \to 0$ can easily be seen from (6.3). It remains to prove that the family $(A_\varepsilon)_{\varepsilon\in(0,\frac{1}{4})}$ is uniformly F-sectorial. If $\varepsilon \in (0, \frac{1}{4})$ then by (6.3) we have

$$\begin{aligned}\|\lambda(\lambda + A_\varepsilon)^{-1}\| &\le \frac{|\lambda\varepsilon|}{|1+\lambda\varepsilon|} + \left\| \frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\left(A + \varepsilon + \frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right)^{-1} \right\| \\ &\le c_\theta + \left|\frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right| \frac{F\left(\left|\varepsilon + \frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right|\right)}{\left|\varepsilon + \frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right|},\end{aligned}$$

where

$$c_\theta := \sup\left\{ |z+1|^{-1} : z \in S_{\pi-\theta} \right\} = \begin{cases} (\sin\theta)^{-1} & (0 < \theta < \pi/2), \\ 1 & (\pi/2 \le \theta < \pi). \end{cases}$$

Let $\mu = \frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}$. From the proof of Lemma 6.1.5 we see that $\mu \in S_{\pi-\theta}$, and we also have $|\varepsilon + \mu| \ge c_\theta^{-1}|\mu|$ by the definition of c_θ. If $|\varepsilon+\mu| \ge |\mu|$ then by assumption (i),

$$\|\lambda(\lambda + A_\varepsilon)^{-1}\| \le c_\theta + |\mu|\frac{F(|\varepsilon+\mu|)}{|\varepsilon+\mu|} \le c_\theta + |\mu|\frac{F(|\mu|)}{|\mu|}.$$

If $|\mu| > |\varepsilon+\mu| \ge c_\theta^{-1}|\mu|$ then by (i) and (ii),

$$\|\lambda(\lambda + A_\varepsilon)^{-1}\| \le c_\theta + |\mu|\frac{F(|\varepsilon+\mu|)}{|\varepsilon+\mu|} \le c_\theta + \beta|\mu|\frac{F(|\mu|)}{c_\theta^{-1}|\mu|}$$

for some $\beta > 1$. In both cases we see that there exists a constant $C = C(\theta) > 0$ such that $\|\lambda(\lambda + A_\varepsilon)^{-1}\| \leq CF(|\mu|)$. Obviously for uniform F-sectoriality we need a similar estimate with $|\lambda|$ on the right-hand side. If $|\lambda| \geq 1$ then

$$\frac{3}{4} \leq \frac{1-\varepsilon^2}{1+\varepsilon} \leq \left|\frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right| \leq c_\theta|\lambda|,$$

since $|1+\lambda\varepsilon| \geq c_\theta^{-1}$. By assumptions (ii) and (iii) this means that

$$F(|\mu|) \leq F(c_\theta|\lambda|) \leq c'_\theta F(|\lambda|)$$

for some $c'_\theta > 1$. If $|\lambda| \leq 1$ then

$$\frac{3}{4}|\lambda| \leq \left|\frac{\lambda(1-\varepsilon^2)}{1+\lambda\varepsilon}\right| \leq \frac{4}{3}.$$

From (ii) and (iii) we have

$$F(|\mu|) \leq F\left(\frac{3}{4}|\lambda|\right) \leq c''_\theta F(|\lambda|)$$

for some $c''_\theta > 1$. Hence there exists a constant $C = C(\theta) > 0$ such that

$$\|\lambda(\lambda + A_\varepsilon)^{-1}\| \leq CF(|\lambda|) \qquad \left(\lambda \in S_{\pi-\theta},\ \varepsilon \in (0, \frac{1}{4})\right),$$

completing the proof. □

6.2 α-Log-Sectorial Operators

Let $\alpha \in (0,1)$. We say that A is **α-log-sectorial** of angle ω, written $A \in \alpha$-log-Sect(ω), if $A \in F$-Sect(ω) for the function F defined by

$$F(s) = \begin{cases} (\log s)^\alpha & (s > e), \\ 1 & (e^{-1} \leq s \leq e), \\ (\log \frac{1}{s})^\alpha & (0 < s < e^{-1}). \end{cases}$$

It follows from Proposition 5.2.11 that if B is a $(1-\alpha)$-strong strip-type operator such that $-1 \in \rho(e^B)$ then e^B is an α-log-sectorial operator. The following further example is taken from [17].

Example 6.2.1. *For $n \geq 3$, let A_n be the 2×2 matrix*

$$A_n = \begin{pmatrix} L_n & n \\ 0 & L_n \end{pmatrix},$$

where $L_n = n/(\log n)^\alpha$. Then $\sigma(A_n)$ consists of the single eigenvalue L_n, and the resolvent at all other points is given by

$$R(\lambda, A_n) = \frac{1}{\lambda - L_n}\left(I - \begin{pmatrix} 0 & \frac{n}{\lambda - L_n} \\ 0 & 0 \end{pmatrix}\right)^{-1} = \frac{1}{\lambda - L_n}\left(I + \begin{pmatrix} 0 & \frac{n}{\lambda - L_n} \\ 0 & 0 \end{pmatrix}\right).$$

If we define an operator A on ℓ^∞ by $A = \bigoplus_{n=3}^\infty A_n$, with maximal domain, then $\sigma(A)$ consists of the set of eigenvalues $\{ L_n : n \geq 3 \}$, and at all other points the norm of the resolvent is given by

$$\|R(\lambda, A)\| = \sup_{n \geq 3}\left(\frac{1}{|\lambda - L_n|} + \frac{n}{|\lambda - L_n|^2}\right).$$

Let $\theta \in (0, \pi)$ and define $\lambda_m = L_m e^{i\theta}$ for $m \geq 3$. Then

$$\begin{aligned} \|\lambda_m R(\lambda_m, A)\| &\geq \frac{|\lambda_m|}{|\lambda_m - L_m|} + \frac{m|\lambda_m|}{|\lambda_m - L_m|^2} \\ &= \frac{L_m}{|1 - e^{i\theta}|L_m} + \frac{mL_m}{|1 - e^{i\theta}|L_m^2} \\ &= \frac{1}{|1 - e^{i\theta}|} + \frac{(\log m)^\alpha}{|1 - e^{i\theta}|^2} \\ &\to \infty \end{aligned}$$

as $m \to \infty$. Hence A is not sectorial. However, it is shown in [17, Lemma 2 and Example 2] that A is *α-log-sectorial.*

Let $A \in \alpha\text{-log-Sect}(\omega)$. Since $(\log s)^\alpha < \log s$ for $s > e$, comparison with Example 6.1.2 tells us that $\mathcal{H}_0^\infty(S_\theta) \subset \mathcal{E}_F(S_\theta)$ for $\theta \in (\omega, \pi)$. Hence if A is injective then $\log A$ is well-defined. In order to prove a representation of the resolvent of $\log A$ similar to (6.1), we shall first show that the Nollau approximation is an F-sectorial approximation to A.

Lemma 6.2.2. *Let $c > 1$. There exists a constant $c' > 1$ such that*

$$F(cs) \leq c'F(s) \qquad (s > 0).$$

Proof. If $s \geq e$ then $cs \geq e$ and since the function $t \mapsto (\log t)^\alpha$ is concave on $[1, \infty)$ we have

$$F(cs) = (\log(cs))^\alpha \leq (\log c)^\alpha + (\log s)^\alpha \leq c'F(s)$$

taking $c' = (\log c)^\alpha + 1$. If $e^{-1} \leq s \leq e$ then

$$F(cs) \leq F(ce) \leq c'F(e) = c'F(s).$$

If $s < e^{-1} < cs$ then

$$F(cs) \leq F(ce^{-1}) \leq c'F(e^{-1}) \leq c'F(s).$$

Finally, if $s < cs < e^{-1}$ then $F(cs) \leq F(s)$. □

Corollary 6.2.3. *Let $c > 1$. Then there exists a constant $c' > 1$ such that*

$$F(s/c) \leq c'F(s) \qquad (s > 0).$$

Proof. Since $F(1/s) = F(s)$ for all $s > 0$, it follows from Lemma 6.2.2 that there exists $c' > 1$ such that $F(s/c) = F(c/s) \leq c'F(1/s) = c'F(s)$. □

Lemma 6.2.4. *The function $s \mapsto F(s)/s$ is decreasing on $(0, \infty)$.*

Proof. It is enough to consider $s \geq e$. In this case we have

$$\begin{aligned} \frac{d}{ds}\left(\frac{F(s)}{s}\right) &= -\frac{(\log s)^\alpha}{s^2} + \alpha\frac{(\log s)^{\alpha-1}}{s^2} \\ &= \frac{(\log s)^{\alpha-1}}{s^2}(\alpha - \log s) \\ &< 0 \end{aligned}$$

since $\alpha \in (0, 1)$. □

Lemma 6.2.2 and Corollary 6.2.3 imply that F satisfies assumption (ii) of Proposition 6.1.6. It follows from Lemma 6.2.4 that assumption (i) also holds. Finally, it is clear that F is decreasing on $(0, e^{-1})$ and increasing on (e, ∞), thus assumption (iii) is satisfied too. Therefore the Nollau approximation $(A_\varepsilon)_{\varepsilon\in(0,\frac{1}{4})}$ is an F-sectorial approximation to A. We now prove a version of Nollau's Lemma for α-log-sectorial operators, adapting the proof given in [22, Lemma 3.5.1].

Proposition 6.2.5. *Let $\alpha \in (0,1)$ and let $A \in \alpha$-log-Sect(ω) be injective. If $|\operatorname{Im}\lambda| > \pi$ then $\lambda \in \rho(\log A)$ and*

$$R(\lambda, \log A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\, dt. \tag{6.4}$$

Proof. Suppose initially that both A and A^{-1} lie in $\mathcal{L}(X)$. Choose $\theta \in (\omega, \pi)$, and let $a > 0$ be small enough and $b > 0$ be large enough so that $\Gamma = \partial S_\theta(a,b)$ surrounds $\sigma(A)$, where $S_\theta(a,b) = S_\theta \cap \{\, z \in \mathbb{C} : a < |z| < b \,\}$. With $\varphi \in (\theta, \pi)$, $a' \in (0,a)$ and $b' > b$, the function f defined by

$$f(z) = (\lambda - \log z)^{-1} \qquad (z \in S_\varphi(a', b'))$$

lies in $\mathcal{H}^\infty(S_\varphi(a',b'))$. By the Riesz-Dunford calculus for bounded operators we have

$$\begin{aligned} f(A) &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda - \log z} R(z, A)\, dz \\ &= \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\, dt \\ &=: J(A), \end{aligned}$$

exactly as in the proof of [22, Lemma 3.5.1], since α-log-sectoriality of A is enough to ensure that all the integrals converge. For more general $A \in \alpha$-log-Sect(ω), we know that the Nollau approximation $(A_\varepsilon)_{\varepsilon\in(0,\frac{1}{4})}$ is an α-log-sectorial approximation to A. We have already seen that $f(A_\varepsilon) = J(A_\varepsilon) \in \mathcal{L}(X)$ for each $\varepsilon \in (0, \frac{1}{4})$. Since the A_ε are uniformly α-log-sectorial, it follows from the Dominated Convergence Theorem that

$$f(A_\varepsilon) \to J(A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\, dt$$

in norm as $\varepsilon \to 0$. By Proposition 6.1.4 we have $J(A) = f(A)$ and by Theorem 2.1.3(f) it follows that $f(A) = R(\lambda, \log A)$. □

Using this result we can estimate the norm of $R(\lambda, \log A)$.

Proposition 6.2.6. *If $A \in \alpha$-log-Sect(ω) is injective then there exists a constant $C > 0$ such that*

$$\|R(\lambda, \log A)\| \le C \left(\frac{1 + |\operatorname{Re} \lambda|^\alpha}{|\operatorname{Im} \lambda| - \pi} + \frac{1}{(|\operatorname{Im} \lambda| - \pi)^{1-\alpha}} \right) \qquad (|\operatorname{Im} \lambda| > \pi).$$

Proof. We estimate the norm of the resolvent using (6.4), as in the proof of [22, Lemma 3.5.1]. If we let $a = \operatorname{Re} \lambda$ and $b = \operatorname{Im} \lambda$ then

$$\begin{aligned} \|R(\lambda, \log A)\| &\le C \int_0^\infty \frac{F(t)}{|(\lambda - \log t)^2 + \pi^2|} \frac{dt}{t} \\ &= C \int_{\mathbb{R}} \frac{F(e^{s+a})}{|(s - ib)^2 + \pi^2|} ds \\ &\le C \int_{1+a}^\infty \frac{(s-a)^\alpha}{s^2 + (b^2 - \pi^2)} ds + C \int_{-1-a}^{1-a} \frac{ds}{s^2 + (b^2 - \pi^2)} \\ &\quad + C \int_{1-a}^\infty \frac{(s+a)^\alpha}{s^2 + (b^2 - \pi^2)} ds, \end{aligned} \tag{6.5}$$

since $|(s \pm ib)^2 + \pi^2| \ge s^2 + (b^2 - \pi^2)$. Now let I_1, I_2 and I_3 denote the first, second and third integrals in (6.5). Clearly

$$I_2 \le \int_{\mathbb{R}} \frac{ds}{s^2 + (b^2 - \pi^2)} = \frac{\pi}{\sqrt{b^2 - \pi^2}}.$$

Setting $u = s(b^2 - \pi^2)^{-1/2}$ gives

$$I_3 = \frac{1}{(\sqrt{b^2 - \pi^2})^{1-\alpha}} \int_{\frac{1-a}{\sqrt{b^2 - \pi^2}}}^\infty \frac{(u + \gamma_\lambda)^\alpha}{u^2 + 1} du,$$

where $\gamma_\lambda = a(b^2 - \pi^2)^{-1/2}$. If $a < 0$ then $\gamma_\lambda < 0$ and therefore

$$I_3 \le \frac{1}{(\sqrt{b^2 - \pi^2})^{1-\alpha}} \int_0^\infty \frac{u^\alpha}{u^2 + 1} du.$$

If $0 \le a \le 1$ then since the function $s \mapsto s^\alpha$ is concave on $(0, \infty)$,

$$I_3 \le \frac{1}{(\sqrt{b^2 - \pi^2})^{1-\alpha}} \left(\int_0^\infty \frac{u^\alpha}{u^2 + 1} du + \gamma_\lambda^\alpha \int_0^\infty \frac{du}{u^2 + 1} \right).$$

If $a > 1$ then

$$\begin{aligned} I_3 &= \frac{1}{(\sqrt{b^2-\pi^2})^{1-\alpha}} \left(\int_{\frac{1-a}{\sqrt{b^2-\pi^2}}}^{0} \frac{(u+\gamma_\lambda)^\alpha}{u^2+1}\, du + \int_0^\infty \frac{(u+\gamma_\lambda)^\alpha}{u^2+1}\, du \right) \\ &\leq \frac{1}{(\sqrt{b^2-\pi^2})^{1-\alpha}} \left(\gamma_\lambda^\alpha \int_{-\infty}^{\infty} \frac{du}{u^2+1} + \int_0^\infty \frac{u^\alpha}{u^2+1}\, du \right). \end{aligned}$$

Therefore, overall we obtain

$$I_3 \leq \begin{cases} \frac{1}{(\sqrt{b^2-\pi^2})^{1-\alpha}} \int_0^\infty \frac{u^\alpha}{u^2+1}\, du & (a < 0), \\ \frac{1}{(\sqrt{b^2-\pi^2})^{1-\alpha}} \int_0^\infty \frac{u^\alpha}{u^2+1}\, du & \\ \quad + \frac{1}{(\sqrt{b^2-\pi^2})^{1-\alpha}} \frac{(\operatorname{Re}\lambda)^\alpha}{(\sqrt{b^2-\pi^2})^\alpha} \int_{-\infty}^{\infty} \frac{du}{u^2+1} & (a \geq 0). \end{cases}$$

We can estimate I_1 in exactly the same way as I_3, replacing a with $-a$. Since $\sqrt{b^2-\pi^2} \geq |b| - \pi$ it follows that

$$I_1 + I_2 + I_3 \leq \frac{\pi(1+|a|^\alpha)}{|b|-\pi} + \frac{1}{(|b|-\pi)^{1-\alpha}} \int_0^\infty \frac{u^\alpha}{u^2+1}\, du.$$

□

Thus we see that, with the real part of λ fixed, the norm of $R(\lambda, \log A)$ decays like $(|\operatorname{Im}\lambda| - \pi)^{\alpha-1}$ as $|\operatorname{Im}\lambda| \to \infty$ (compare with the definition of a $(1-\alpha)$-strong strip-type operator). On the other hand, for fixed imaginary part, the norm of $R(\lambda, \log A)$ is bounded by $|\operatorname{Re}\lambda|^\alpha$ as $|\operatorname{Re}\lambda| \to \infty$ along horizontal lines parallel to the real axis. If $A \in \alpha$-log-Sect(ω), then we can in fact extend the estimates obtained in Proposition 6.2.6 to all λ with $|\operatorname{Im}\lambda| > \omega$.

Lemma 6.2.7. *Let $A \in F$-Sect(ω) be injective, and suppose that $\mathcal{H}_0^\infty(S_\varphi) \subset \mathcal{E}_F(S_\varphi)$ for $\varphi \in (\omega, \pi)$. Then if $|\theta| < \pi - \omega$, the identity $\log(e^{i\theta}A) = i\theta + \log A$ holds.*

Proof. If $|\theta| < \pi - \omega$ then $e^{i\theta}A$ belongs to F-Sect$(\omega + |\theta|)$. Suppose that $f \in \mathcal{H}_0^\infty(S_{\omega+|\theta|+\varepsilon})$ for some $\varepsilon > 0$. By Cauchy's Theorem,

$$\begin{aligned} f(e^{i\theta}A) &= \frac{1}{2\pi i} \int_{\partial(e^{i\theta}S_{\omega+\varepsilon})} f(z) R(z, e^{i\theta}A)\, dz \\ &= \frac{1}{2\pi i} \int_{\partial S_{\omega+\varepsilon}} f(e^{i\theta}z) R(z, A)\, dz \\ &= f_\theta(A) \end{aligned}$$

where $f_\theta(z) = f(e^{i\theta}z)$ for $z \in S_{\omega+\varepsilon}$. By considering a regulariser, it follows from this that $\log(e^{i\theta}A) = g_\theta(A)$, where $g_\theta(z) = \log(e^{i\theta}z)$ for $z \in S_{\omega+\varepsilon}$. Thus the lemma follows from Theorem 2.1.3(c). □

Proposition 6.2.8. *Let $A \in \alpha$-log-Sect(ω) be injective. Then $\lambda \in \rho(\log A)$ whenever $|\operatorname{Im}\lambda| > \omega$ and, for each $\omega' \in (\omega, \pi)$, there exists a constant $C = C(\omega') > 0$ such that*

$$\|R(\lambda, \log A)\| \leq C\left(\frac{1+|\operatorname{Re}\lambda|^\alpha}{|\operatorname{Im}\lambda| - \omega'} + \frac{1}{(|\operatorname{Im}\lambda| - \omega')^{1-\alpha}}\right) \qquad (|\operatorname{Im}\lambda| > \omega').$$

Proof. Suppose that $|\theta| < \pi - \omega$. Then $e^{i\theta}A$ belongs to α-log-Sect$(\omega + |\theta|)$, hence it follows from Proposition 6.2.5 and Lemma 6.2.7 that

$$\sigma(\log A) \subset \{\lambda \in \mathbb{C} : -\pi - \theta \leq \operatorname{Im}\lambda \leq \pi - \theta\}.$$

As this holds for any θ for which $|\theta| < \pi - \omega$, we see that $\sigma(\log A) \subset \overline{H_\omega}$.

Now suppose that $|\operatorname{Im}\lambda| > \omega'$ for some $\omega' \in (\omega, \pi)$, and let $\theta = \pi - \omega'$. By Proposition 6.2.6 there exists a constant C such that

$$\|R(\mu, \log(e^{i\theta}A))\| \leq C\left(\frac{1+|\operatorname{Re}\mu|^\alpha}{|\operatorname{Im}\mu| - \pi} + \frac{1}{(|\operatorname{Im}\mu| - \pi)^{1-\alpha}}\right) \qquad (|\operatorname{Im}\mu| > \pi).$$

If $\operatorname{Im}\lambda > \omega'$ then set $\mu = \lambda + i\theta$, so that $\operatorname{Im}\mu > \pi$. Then since $R(\mu, \log(e^{i\theta}A)) = R(\lambda, \log A)$ and $|\operatorname{Im}\mu| - \pi = |\operatorname{Im}\lambda| - \omega'$, we see that

$$\|R(\lambda, \log A)\| \leq C\left(\frac{1+|\operatorname{Re}\lambda|^\alpha}{|\operatorname{Im}\lambda| - \omega'} + \frac{1}{(|\operatorname{Im}\lambda| - \omega')^{1-\alpha}}\right).$$

Similarly, if $\operatorname{Im}\lambda < -\omega'$ then set $\mu = \lambda - i\theta$ and apply Proposition 6.2.6 to the operator $e^{-i\theta}A$. □

We can compare the estimates obtained in Propositions 6.2.6 and 6.2.8 with the logarithm of the operator given in Example 6.2.1.

Example 6.2.9. *Let A and A_n be as in Example 6.2.1, where $n \geq 3$. Using the power series expansion of* $\log(1+x)$, *we have*

$$\log A_n = \log\left[L_n\left(I + \begin{pmatrix} 0 & n/L_n \\ 0 & 0 \end{pmatrix}\right)\right] = (\log L_n)I + \begin{pmatrix} 0 & n/L_n \\ 0 & 0 \end{pmatrix}.$$

The spectrum of $\log A_n$ *consists of the single eigenvalue* $\log L_n$*, and the resolvent at all other points is given by*

$$
\begin{aligned}
R(\lambda, \log A_n) &= \frac{1}{\lambda - \log L_n}\left(I - \begin{pmatrix} 0 & \frac{n}{L_n(\lambda - \log L_n)} \\ 0 & 0 \end{pmatrix}\right)^{-1} \\
&= \frac{1}{\lambda - \log L_n}\left(I + \begin{pmatrix} 0 & \frac{(\log n)^\alpha}{\lambda - \log L_n} \\ 0 & 0 \end{pmatrix}\right).
\end{aligned}
$$

Furthermore, $\sigma(\log A)$ *consists of the set of eigenvalues* $\{\log L_n : n \geq 3\}$*, and the norm of the resolvent at all other points is given by*

$$
\|R(\lambda, \log A)\| = \sup_{n \geq 3}\left(\frac{1}{|\lambda - \log L_n|} + \frac{(\log n)^\alpha}{|\lambda - \log L_n|^2}\right).
$$

If we set $\lambda_m = \log L_m + i\beta$*, where* $m \geq 3$ *and* $\beta > 0$*, then this supremum is attained when* $n = m$*, hence*

$$
\|R(\lambda_m, \log A)\| = \frac{1}{|\beta|} + \frac{(\log m)^\alpha}{|\beta|^2} \leq \frac{1 + (\log m)^\alpha}{|\operatorname{Im}\lambda|} \sim \frac{1 + |\operatorname{Re}\lambda|^\alpha}{|\operatorname{Im}\lambda|},
$$

since $\log L_n \sim \log n$ *as* $n \to \infty$*. This shows that the estimate in Proposition 6.2.8 is sharp on horizontal lines.*

The methods used in this section break down if we consider the limiting case $\alpha = 1$, since in this case the resolvent estimate is not sufficient to ensure that the integral in (6.4) converges. We shall return to this matter in Section 6.4.

6.3 Log-Log-Sectorial Operators

We say that A is log log-**sectorial** of angle ω, written $A \in \log\log\text{-Sect}(\omega)$, if $A \in F$-Sect(ω) for the function F defined by

$$
F(s) = \begin{cases} \log\log s & (s > e^e), \\ 1 & (e^{-e} \leq s \leq e^e), \\ \log\log \frac{1}{s} & (0 < s < e^{-e}). \end{cases}
$$

Clearly $\log\log\text{-Sect}(\omega) \subset \alpha\text{-log-Sect}(\omega)$ for each $\alpha \in (0,1)$. From Proposition 5.2.9 we see that if B is a strong strip-type operator such that $-1 \in \rho(e^B)$ then e^B is log log-sectorial.

Let $A \in \log\log\text{-Sect}(\omega)$. Since $\log\log s < \log s$ for all $s > e^e$, comparison with Example 6.1.2 again shows that if A is injective then $\log A$ is well-defined. Our next aim is to prove a version of Nollau's Lemma for log log-sectorial operators. It would be possible to prove directly that F satisfies the assumptions of Proposition 6.1.6, by proving versions of Lemma 6.2.2, Corollary 6.2.3 and Lemma 6.2.4. This would tell us that the Nollau approximation was an F-sectorial approximation to A, and we could then mirror the proof of Proposition 6.2.5. However, we can use the fact that $\log\log\text{-Sect}(\omega)$ is contained in $\alpha\text{-log-Sect}(\omega)$ to simply deduce a version of Nollau's Lemma from Proposition 6.2.5.

Proposition 6.3.1. *Let $A \in \log\log\text{-Sect}(\omega)$ be injective. If $|\operatorname{Im}\lambda| > \pi$ then $\lambda \in \rho(\log A)$ and*

$$R(\lambda, \log A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\,dt. \tag{6.6}$$

Proof. Since $\log\log\text{-Sect}(\omega) \subset \alpha\text{-log-Sect}(\omega)$ for each $\alpha \in (0,1)$, this result follows from Proposition 6.2.5. □

Again we can estimate the norm of $R(\lambda, \log A)$. First we prove some technical results regarding the logarithm function.

Lemma 6.3.2. *Let $\alpha > e$ and $x \geq 1$. Then* $\log(x+\alpha) \leq 2(\log\alpha)x^{\frac{1}{2}}$.

Proof. We show that the function f defined by

$$f(x) = 2(\log\alpha)x^{1/2} - \log(x+\alpha) \qquad (x \geq 1)$$

is non-negative. We have $f(1) = \log(\alpha^2) - \log(1+\alpha) > 0$, since $\alpha > e$. Furthermore, for $x \geq 1$ we have

$$f'(x) = \frac{\log\alpha}{x^{1/2}} - \frac{1}{x+\alpha} > \frac{1}{x^{1/2}} - \frac{1}{x+e} = \frac{x+e-x^{1/2}}{x^{1/2}(x+e)} > 0,$$

which proves the result. □

Lemma 6.3.3. *Let $\delta > 0$ and chooose α so that $\alpha + \log\delta > 0$. There exists a constant $C = C(\alpha) > 0$ such that*

$$\alpha + \log s \leq C\log(1+s) \qquad (s \geq \delta).$$

Proof. By the way we have chosen α, the continuous function f defined by

$$f(s) = \frac{\alpha + \log s}{\log(1+s)} \qquad (s \geq \delta)$$

is positive for all $s \geq \delta$, and $f(s) \to 1$ as $s \to \infty$. Thus f is bounded on $[\delta, \infty)$, proving the result. □

Proposition 6.3.4. *Let $A \in \log\log\text{-Sect}(\omega)$ be injective and let $|\operatorname{Im}\lambda| > \pi$. Let $\gamma_\lambda = \operatorname{Re}\lambda/((\operatorname{Im}\lambda)^2 - \pi^2)^{1/2}$. There exists a constant C such that if $|\gamma_\lambda| \leq e$ then*

$$\|R(\lambda, \log A)\| \leq C\,\frac{\log(1 + |\operatorname{Im}\lambda|)}{|\operatorname{Im}\lambda| - \pi} \qquad (|\operatorname{Im}\lambda| > \pi).$$

Furthermore, for each $\varepsilon > 0$, there exists a constant C_ε such that if $|\gamma_\lambda| > e$ then

$$\|R(\lambda, \log A)\| \leq C_\varepsilon\,\frac{\log(1 + |\operatorname{Re}\lambda|)}{|\operatorname{Im}\lambda| - \pi} \qquad (|\operatorname{Im}\lambda| > \pi + \varepsilon).$$

Proof. Denoting $\operatorname{Re}\lambda$ by a and $\operatorname{Im}\lambda$ by b, it follows from (6.6) that

$$\begin{aligned}
\|R(\lambda, \log A)\| &\leq C\int_0^\infty \frac{F(t)}{|(\lambda - \log t)^2 + \pi^2|}\frac{dt}{t} \\
&= C\int_{\mathbb{R}} \frac{F(e^{s+a})}{|(ib - s)^2 + \pi^2|}\,ds \\
&\leq C\int_{e+a}^\infty \frac{\log(s-a)}{s^2 + (b^2 - \pi^2)}\,ds + C\int_{-e-a}^{e-a} \frac{ds}{s^2 + (b^2 - \pi^2)} \\
&\quad + C\int_{e-a}^\infty \frac{\log(s+a)}{s^2 + (b^2 - \pi^2)}\,ds, \qquad (6.7)
\end{aligned}$$

since $|(s \pm ib)^2 + \pi^2| \geq s^2 + (b^2 - \pi^2)$. Let I_1, I_2 and I_3 denote the first, second and third integrals in (6.7). Clearly

$$I_2 \leq \int_{\mathbb{R}} \frac{ds}{s^2 + (b^2 - \pi^2)} = \frac{\pi}{\sqrt{b^2 - \pi^2}}$$

whenever $|\operatorname{Im}\lambda| > \pi$.

By setting $u = s(b^2-\pi^2)^{-1/2}$ we can write $I_3 = I_{3,1} + I_{3,2}$ where

$$I_{3,1} = \frac{\log(\sqrt{b^2-\pi^2})}{\sqrt{b^2-\pi^2}} \int_{\frac{e-a}{\sqrt{b^2-\pi^2}}}^{\infty} \frac{du}{u^2+1}, \qquad I_{3,2} = \frac{1}{\sqrt{b^2-\pi^2}} \int_{\frac{e-a}{\sqrt{b^2-\pi^2}}}^{\infty} \frac{\log(u+\gamma_\lambda)}{u^2+1}\, du.$$

If $\pi < |\operatorname{Im}\lambda| \leq \sqrt{\pi^2+1}$ then $I_{3,1} \leq 0$, whereas if $|\operatorname{Im}\lambda| > \sqrt{\pi^2+1}$ we have

$$I_{3,1} \leq \frac{\log(\sqrt{b^2-\pi^2})}{\sqrt{b^2-\pi^2}} \int_{\mathbb{R}} \frac{du}{u^2+1} = \frac{\pi\log(\sqrt{b^2-\pi^2})}{\sqrt{b^2-\pi^2}}.$$

If $-\infty \leq \gamma_\lambda \leq e$ then

$$I_{3,2} \leq \frac{1}{\sqrt{b^2-\pi^2}} \int_{\frac{e-a}{\sqrt{b^2-\pi^2}}}^{\infty} \frac{\log(u+e)}{u^2+1}\, du \leq \frac{1}{\sqrt{b^2-\pi^2}} \int_{e-1}^{\infty} \frac{\log(u+e)}{u^2+1}\, du,$$

and if $\gamma_\lambda > e$ and $\frac{e-a}{\sqrt{b^2-\pi^2}} < 1$ then by Lemma 6.3.2 we have

$$\begin{aligned}
I_{3,2} &= \frac{1}{\sqrt{b^2-\pi^2}} \left[\int_{\frac{e-a}{\sqrt{b^2-\pi^2}}}^{1} \frac{\log(u+\gamma_\lambda)}{u^2+1}\, du + \int_1^{\infty} \frac{\log(u+\gamma_\lambda)}{u^2+1}\, du \right] \\
&\leq \frac{1}{\sqrt{b^2-\pi^2}} \left[\log(1+\gamma_\lambda) \int_{-\infty}^{1} \frac{du}{u^2+1} + 2(\log\gamma_\lambda) \int_1^{\infty} \frac{u^{1/2}}{u^2+1}\, du \right] \\
&\leq \frac{2(\log\gamma_\lambda)}{\sqrt{b^2-\pi^2}} \left[\int_{-\infty}^{1} \frac{du}{u^2+1} + \int_1^{\infty} \frac{u^{1/2}}{u^2+1}\, du \right].
\end{aligned}$$

Similarly if $\gamma_\lambda > e$ and $\frac{e-a}{\sqrt{b^2-\pi^2}} \geq 1$ then

$$I_{3,2} \leq \frac{1}{\sqrt{b^2-\pi^2}} \int_1^{\infty} \frac{\log(u+\gamma_\lambda)}{u^2+1}\, du \leq \frac{2(\log\gamma_\lambda)}{\sqrt{b^2-\pi^2}} \int_1^{\infty} \frac{u^{1/2}}{u^2+1}\, du.$$

The integral I_1 can be estimated in the same way as I_3, replacing a with $-a$.

Overall, if $|\gamma_\lambda| \leq e$ and $|\operatorname{Im}\lambda| > \sqrt{\pi^2+1}$ then we have

$$\|R(\lambda, \log A)\| \leq C\, \frac{1+\log(\sqrt{b^2-\pi^2})}{|b|-\pi}. \tag{6.8}$$

If $|\gamma_\lambda| \leq e$ and $\pi < |\operatorname{Im}\lambda| < \sqrt{\pi^2+1}$ then

$$\|R(\lambda, \log A)\| \leq \frac{C}{|b|-\pi}. \tag{6.9}$$

From (6.8) and (6.9) we see that there exists a constant C such that

$$\|R(\lambda, \log A)\| \leq C\, \frac{\log(1+|\operatorname{Im}\lambda|)}{|\operatorname{Im}\lambda|-\pi} \qquad (|\operatorname{Im}\lambda| > \pi,\ |\gamma_\lambda| \leq e). \tag{6.10}$$

If $|\gamma_\lambda| > e$ and $|\operatorname{Im}\lambda| > \sqrt{\pi^2+1}$ then

$$\begin{aligned}\|R(\lambda, \log A)\| &\leq C\left(\frac{\log(\sqrt{b^2-\pi^2})}{\sqrt{b^2-\pi^2}} + \frac{1}{\sqrt{b^2-\pi^2}} + \frac{\log|\gamma_\lambda|}{\sqrt{b^2-\pi^2}}\right) \\ &\leq C\left(\frac{1+\log|a|}{|b|-\pi}\right), \end{aligned} \tag{6.11}$$

since $|a| > e$ in this region of the plane. Let $\varepsilon \in (0,1)$. If $|\gamma_\lambda| > e$ and $\sqrt{\pi^2+\varepsilon^2} < |\operatorname{Im}\lambda| \leq \sqrt{\pi^2+1}$ then by Lemma 6.3.3 there exists a constant C_ε such that

$$\begin{aligned}\|R(\lambda, \log A)\| &\leq C\left(\frac{1}{\sqrt{b^2-\pi^2}} + \frac{\log|\gamma_\lambda|}{\sqrt{b^2-\pi^2}}\right) \\ &\leq C\left(\frac{1-\log\varepsilon+\log|a|}{\sqrt{b^2-\pi^2}}\right) \\ &\leq C_\varepsilon\left(\frac{1+\log(1+|a|)}{|b|-\pi}\right). \end{aligned} \tag{6.12}$$

From (6.11) and (6.12) we see that there exists a constant C_ε such that

$$\|R(\lambda, \log A)\| \leq C_\varepsilon \frac{\log(1+|\operatorname{Re}\lambda|)}{|\operatorname{Im}\lambda|-\pi} \qquad (|\operatorname{Im}\lambda| > \pi+\varepsilon,\ |\gamma_\lambda| > e). \tag{6.13}$$

(6.10) and (6.13) now give the result. □

Thus we see that, if the real part of λ is fixed, the norm of $R(\lambda, \log A)$ decays like $(\log|\operatorname{Im}\lambda|)/(|\operatorname{Im}\lambda|-\pi)$ as $|\operatorname{Im}\lambda| \to \infty$. If instead we fix the imaginary part of λ, then $R(\lambda, \log A)$ is bounded by $\log|\operatorname{Re}\lambda|$ as $|\operatorname{Re}\lambda| \to \infty$ along horizontal lines. Again, if $A \in \log\log\text{-Sect}(\omega)$ then we can estimate the resolvent whenever $|\operatorname{Im}\lambda| > \omega$.

Proposition 6.3.5. *Let $A \in \log\log$-Sect(ω). Then $\lambda \in \rho(\log A)$ when $|\operatorname{Im}\lambda| > \omega$. Furthermore, for each $\omega' \in (\omega, \pi)$, there exists a constant $C = C(\omega') > 0$ such that*

$$\|R(\lambda, \log A)\| \leq C \frac{\log(1+|\operatorname{Im}\lambda|)}{|\operatorname{Im}\lambda|-\omega'} \qquad (|\operatorname{Im}\lambda| > \omega',\ |\gamma_{\lambda'}| \leq e)$$

and

$$\|R(\lambda, \log A)\| \leq C \frac{\log(1+|\operatorname{Re}\lambda|)}{|\operatorname{Im}\lambda|-\omega'} \qquad (|\operatorname{Im}\lambda| > \omega',\ |\gamma_{\lambda'}| > e),$$

where

$$\lambda' = \begin{cases} \lambda + i(\pi-\omega') & (\operatorname{Im}\lambda > \omega'), \\ \lambda - i(\pi-\omega') & (\operatorname{Im}\lambda < -\omega'). \end{cases}$$

Proof. Since log log-Sect(ω) $\subset$ α-log-Sect(ω) for each $\alpha \in (0,1)$, it follows from Proposition 6.2.8 that $\sigma(\log A)$ is actually contained in $\overline{H_\omega}$. The resolvent estimates come from applying Proposition 6.3.4 to the operator $e^{i\theta}A$, where $\theta = \pi - \omega'$, as in the proof of Proposition 6.2.8. □

An example of a log log-sectorial operator and its logarithm can be obtained by modifying Examples 6.2.1 and 6.2.9.

Example 6.3.6. *If we set $L_n = n/\log\log n$ for $n \geq 3$, then the operator A constructed as in Example 6.2.1 is log log-sectorial but not sectorial. The spectrum of $\log A$ consists of the set of eigenvalues $\{\log L_n : n \geq 3\}$, and at all other points the norm of $R(\lambda, \log A)$ is given by*

$$\|R(\lambda, \log A)\| = \sup_{n\geq 3}\left(\frac{1}{|\lambda - \log L_n|} + \frac{\log\log n}{|\lambda - \log L_n|^2}\right).$$

Taking $\lambda_m = \log L_m + i\beta$ for $m \geq 3$ and $\beta > 0$ we have

$$\|R(\lambda_m, \log A)\| = \frac{1}{|\beta|} + \frac{\log\log m}{|\beta|^2} \leq \frac{1+\log\log m}{|\operatorname{Im}\lambda|} \sim \frac{1+\log|\operatorname{Re}\lambda|}{|\operatorname{Im}\lambda|},$$

since $\log\log L_n \sim \log\log n$ as $n \to \infty$. Again this shows that the estimate obtained in Proposition 6.3.5 is sharp on horizontal lines.

6.4 Log-Sectorial Operators

We have already noted that Propositions 6.2.5 and 6.2.6 fail to hold when $\alpha = 1$. In this section our aim is to see what *can* be said in this case. We shall see that, under the additional assumption that $\rho(\log A)$ contains a point μ with $|\operatorname{Im}\mu| > \pi$, we can indeed obtain a resolvent representation (Proposition 6.4.2), and from this we can estimate the norm of the resolvent (Proposition 6.4.3).

We say that A is **log-sectorial** of angle ω, written $A \in$ log-Sect(ω), if $A \in F$-Sect(ω) for the function F defined by

$$F(s) = \begin{cases} \log s & (s > e), \\ 1 & (e^{-1} \le s \le e), \\ \log \frac{1}{s} & (0 < s < e^{-1}). \end{cases}$$

Thus a log-sectorial operator can be thought of as the limiting case of an α-log-sectorial operator as $\alpha \to 1$. From Proposition 5.2.4 it follows that if B is a strip-type operator such that $-1 \in \rho(e^B)$ then e^B is an example of a log-sectorial operator. A further example can be obtained simply by setting $\alpha = 1$ in Example 6.2.1 (see Example 6.4.4 later on).

If $A \in$ log-Sect(ω) then it follows directly from Example 6.1.2 that $\log A$ is well-defined whenever A is injective. It is easy to see that the proof of Lemma 6.2.2 remains valid if we take $\alpha = 1$, thus F satisfies assumption (ii) of Proposition 6.1.6. It is clear that assumption (iii) is satisfied, and assumption (i) is proved in the next result:

Lemma 6.4.1. *The function $s \mapsto F(s)/s$ is decreasing on $(0, \infty)$.*

Proof. It is enough to show that $s \mapsto F(s)/s$ is decreasing on (e, ∞). For $s \in (e, \infty)$ we have

$$\frac{d}{ds}\left(\frac{F(s)}{s}\right) = -\frac{1}{s^2}\log s + \frac{1}{s^2} = \frac{1}{s^2}(1 - \log s) < 0$$

as required. □

Hence the Nollau approximation $(A_\varepsilon)_{\varepsilon \in (0, \frac{1}{4})}$ is an F-sectorial approximation to A. If we assume that $\rho(\log A)$ contains some point μ with $|\operatorname{Im} \mu| > \pi$ then it turns out that the spectrum of $\log A$ is contained in the horizontal strip of height π, and that we have a representation of the resolvent of $\log A$ outside this strip.

Proposition 6.4.2. *Let $A \in$ log-Sect(ω) be injective and suppose that there exists some point $\mu \in \rho(\log A)$ such that $|\operatorname{Im} \mu| > \pi$. Then $\lambda \in \rho(\log A)$ whenever $|\operatorname{Im} \lambda| > \pi$ and*

$$R(\lambda, \log A) = R(\mu, \log A) + \int_0^\infty \left[\frac{-(t+A)^{-1}}{(\lambda - \log t)^2 + \pi^2} - \frac{-(t+A)^{-1}}{(\mu - \log t)^2 + \pi^2} \right] dt, \quad (6.14)$$

for all such λ.

Proof. Let $|\operatorname{Im} \lambda| > \pi$, and suppose initially that $A, A^{-1} \in \mathcal{L}(X)$. Let $\theta \in (\omega, \pi)$, and choose $a > 0$ small enough and $b > 0$ large enough so that the contour $\Gamma = \partial S_\theta(a, b)$ surrounds $\sigma(A)$. With $\varphi \in (\theta, \pi)$, $a' \in (0, a)$ and $b' > b$, the function f defined by

$$f(z) = (\lambda - \log z)^{-1} \qquad (z \in S_\varphi(a', b'))$$

lies in $\mathcal{H}^\infty(S_\varphi(a', b'))$. Then by the Riesz-Dunford calculus we have

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz.$$

The operator $f(A)$ is bounded, and it follows from Theorem 2.1.3(f) that $\lambda \in \rho(\log A)$ and $R(\lambda, \log A) = f(A)$. Hence, with μ as in the statement of the Proposition,

$$R(\lambda, \log A) = R(\mu, \log A) + \frac{1}{2\pi i} \int_\Gamma \left(\frac{1}{\lambda - \log z} - \frac{1}{\mu - \log z} \right) R(z, A)\, dz.$$

Letting I denote $R(\lambda, \log A) - R(\mu, \log A)$, we have

$$\begin{aligned}
I &= \frac{1}{2\pi i} \int_a^b \left(\frac{e^{-i\theta}}{\lambda - \log t + i\theta} - \frac{e^{-i\theta}}{\mu - \log t + i\theta} \right) R(te^{-i\theta}, A)\, dt \\
&\quad + \frac{1}{2\pi} \int_{-\theta}^{\theta} \left(\frac{be^{is}}{\lambda - \log b - is} - \frac{be^{is}}{\mu - \log b - is} \right) R(be^{is}, A)\, ds \\
&\quad - \frac{1}{2\pi i} \int_a^b \left(\frac{e^{i\theta}}{\lambda - \log t - i\theta} - \frac{e^{i\theta}}{\mu - \log t - i\theta} \right) R(te^{i\theta}, A)\, dt \\
&\quad - \frac{1}{2\pi} \int_{-\theta}^{\theta} \left(\frac{ae^{is}}{\lambda - \log a - is} - \frac{ae^{is}}{\mu - \log a - is} \right) R(ae^{is}, A)\, ds \\
&\stackrel{*}{=} \int_0^\infty \left(\frac{-1}{(\lambda - \log t)^2 + \pi^2} - \frac{-1}{(\mu - \log t)^2 + \pi^2} \right) (t + A)^{-1}\, dt \\
&=: I_{\lambda,\mu}(A),
\end{aligned}$$

where we have let $\theta \to \pi$, $a \to 0$ and $b \to \infty$ in (*), just as in the proof of [22, Lemma 3.5.1]. Note that log-sectoriality *does* guarantee the convergence of this integral. More generally, we know that the Nollau approximation $(A_\varepsilon)_{\varepsilon\in(0,\frac{1}{4})}$ is a log-sectorial approximation to A. Each A_ε is bounded and invertible, thus it follows from the above that $\lambda \in \rho(\log A_\varepsilon)$ and that

$$R(\lambda, \log A_\varepsilon) = f(A_\varepsilon) = R(\mu, \log A_\varepsilon) + I_{\lambda,\mu}(A_\varepsilon).$$

It follows from the Dominated Convergence Theorem and the fact that $(A_\varepsilon)_{\varepsilon\in(0,\frac{1}{4})}$ is a log-sectorial approximation to A that $I_{\lambda,\mu}(A_\varepsilon) \to I_{\lambda,\mu}(A)$ as $\varepsilon \to 0$. Hence

$$f(A_\varepsilon) \to R(\mu, \log A) + I_{\lambda,\mu}(A),$$

which is equal to $f(A)$ by Proposition 6.1.4. From Theorem 2.1.3(f) it then follows that $\lambda \in \rho(\log A)$ and that $R(\lambda, \log A) = R(\mu, \log A) + I_{\lambda,\mu}(A)$ as claimed. □

Using this result we can estimate the norm of $R(\lambda, \log A)$ for $|\operatorname{Im}\lambda| > \pi$.

Proposition 6.4.3. *Let $A \in$ log-Sect(ω) be injective, and suppose that there exists $\mu \in \rho(\log A)$ such that $|\operatorname{Im}\mu| > \pi$. Then $\lambda \in \rho(\log A)$ whenever $|\operatorname{Im}\lambda| > \pi$ and*

$$\|R(\lambda, \log A)\| \leq C\left(\log|\lambda| + \frac{|\lambda|}{|\operatorname{Im}\lambda| - \pi}\right) \qquad (|\operatorname{Im}\lambda| > \pi,\ |\lambda| \text{ large}). \tag{6.15}$$

Proof. It follows from Proposition 6.4.2 that $\lambda \in \rho(\log A)$ whenever $|\operatorname{Im}\lambda| > \pi$. Fix $\eta \in \mathbb{R}$ with $|\eta| > \pi$ and, for $|\operatorname{Im}\lambda| > \pi$, set

$$\begin{aligned} I_{\lambda,i\eta}(A) &:= \int_0^\infty \left[\frac{-(t+A)^{-1}}{(\lambda - \log t)^2 + \pi^2} - \frac{-(t+A)^{-1}}{(i\eta - \log t)^2 + \pi^2}\right] dt \\ &= \int_0^\infty (t+A)^{-1} \frac{\eta^2 + \lambda^2 + 2(i\eta - \lambda)\log t}{[(\lambda - \log t)^2 + \pi^2]\,[(i\eta - \log t)^2 + \pi^2]}\, dt \end{aligned}$$

as in the proof of Proposition 6.4.2. Since A is log-sectorial, there exists κ such that

$\|t(t+A)^{-1}\| \leq \kappa(1+|\log t|)$ for all $t>0$. Thus, by setting $s=\log t$, we have

$$\begin{aligned}
\|I_{\lambda,i\eta}(A)\| &\leq \kappa \int_{-\infty}^{0} \frac{(1-s)\,|\eta^2+\lambda^2+2(i\eta-\lambda)s|}{|(\lambda-s)^2+\pi^2|\,|(i\eta-s)^2+\pi^2|}\,ds \\
&\quad + \kappa \int_{0}^{\infty} \frac{(1+s)\,|\eta^2+\lambda^2+2(i\eta-\lambda)s|}{|(\lambda-s)^2+\pi^2|\,|(i\eta-s)^2+\pi^2|}\,ds \\
&\leq \kappa \int_{-\infty}^{0} \frac{(1-s)\,|\eta^2+\lambda^2+2(i\eta-\lambda)s|}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds \\
&\quad + \kappa \int_{0}^{\infty} \frac{(1+s)\,|\eta^2+\lambda^2+2(i\eta-\lambda)s|}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds \\
&= \kappa \int_{-\infty}^{\infty} \frac{|\lambda-i\eta|\,(|\lambda+i\eta|+2s^2)}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds + \kappa \int_{0}^{\infty} \frac{|\lambda-i\eta|\,(2+|\lambda+i\eta|)s}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds \\
&\quad - \kappa \int_{-\infty}^{0} \frac{|\lambda-i\eta|\,(2+|\lambda+i\eta|)s}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds \qquad (6.16)
\end{aligned}$$

where $\alpha = \operatorname{Re}\lambda$, $\beta^2 = (\operatorname{Im}\lambda)^2-\pi^2$ and $\gamma^2=\eta^2-\pi^2$. We evaluate each of the integrals in (6.16). Firstly,

$$\frac{1}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]} = \frac{As+B}{(s-\alpha)^2+\beta^2} + \frac{Cs+D}{s^2+\gamma^2} \qquad (s\in\mathbb{R}),$$

where

$$\begin{aligned}
A &= -\tfrac{2\alpha}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2} & B &= \tfrac{3\alpha^2-\beta^2+\gamma^2}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2} \\
C &= -A & D &= \tfrac{\alpha^2+\beta^2-\gamma^2}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{ds}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]} &= \lim_{n\to\infty} \left\{ \int_{-n}^{n} \frac{A(s+\alpha)+B}{s^2+\beta^2}\,ds + \int_{-n}^{n} \frac{Cs+D}{s^2+\gamma^2}\,ds \right\} \\
&= \lim_{n\to\infty} \left[\frac{A\log(s^2+\beta^2)}{2} + \frac{\alpha A+B}{\beta}\tan^{-1}\left(\frac{s}{\beta}\right) \right. \\
&\quad \left. + \frac{C\log(s^2+\gamma^2)}{2} + \frac{D}{\gamma}\tan^{-1}\left(\frac{s}{\gamma}\right) \right]_{-n}^{n} \\
&= \frac{\pi(\alpha A+B)}{\beta} + \frac{\pi D}{\gamma}. \qquad (6.17)
\end{aligned}$$

Also,

$$\frac{s^2}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]} = \frac{A's+B'}{(s-\alpha)^2+\beta^2} + \frac{C's+D'}{s^2+\gamma^2} \qquad (s\in\mathbb{R}),$$

where

$$\begin{aligned}
A' &= \tfrac{2\alpha\gamma^2}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2} & B' &= \tfrac{(\alpha^2+\beta^2)(\alpha^2+\beta^2-\gamma^2)}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2} \\
C' &= -A' & D' &= -\tfrac{\gamma^2(\alpha^2+\beta^2-\gamma^2)}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2},
\end{aligned}$$

and it follows similarly that

$$\int_{-\infty}^{\infty} \frac{s^2}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds = \frac{\pi(\alpha A'+B')}{\beta} + \frac{\pi D'}{\gamma}. \tag{6.18}$$

From (6.17) and (6.18) we therefore have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{|\lambda - i\mu|\,(|\lambda+i\mu|+2s^2)}{[(s-\alpha)^2+\beta^2][s^2+\gamma^2]}\,ds \;&\leq\; \pi|\lambda-i\mu||\lambda+i\mu|\left(\frac{|\alpha A+B|}{\beta}+\frac{|D|}{\gamma}\right)\\ &\quad +2\pi|\lambda-i\mu|\left(\frac{|\alpha A'+B'|}{\beta}+\frac{|D'|}{\gamma}\right).\end{aligned}$$

Note that

$$|\alpha A+B| \leq \frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2-\gamma^2)^2} = O(|\lambda|^{-2}) \qquad (|\lambda|\to\infty),$$

and that the same holds for $|D|$ and $|D'|$. Also, since

$$|\alpha A'+B'| \leq \frac{(\alpha^2+\beta^2)(\alpha^2+\beta^2+\gamma^2)}{(\alpha^2+\beta^2-\gamma^2)^2} = O(1) \qquad (|\lambda|\to\infty),$$

it follows that

$$\int_{-\infty}^{\infty} \frac{|\lambda - i\mu|\,(|\lambda+i\mu|+2s^2)}{[(s-\alpha)^2+\beta^2][s^2+\gamma^2]}\,ds = O\left(1+\frac{|\lambda|}{\beta}\right) \qquad (|\lambda|\to\infty). \tag{6.19}$$

To estimate the remaining terms in (6.16), note that

$$\frac{s}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]} = \frac{A''s+B''}{(s-\alpha)^2+\beta^2} + \frac{C''s+D''}{s^2+\gamma^2} \qquad (s\in\mathbb{R}),$$

where

$$\begin{aligned} A'' &= \tfrac{\gamma^2-\alpha^2-\beta^2}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2} & B'' &= \tfrac{(\alpha^2+\beta^2)2\alpha}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2}\\ C'' &= -A'' & D'' &= -\tfrac{2\alpha\gamma^2}{(\alpha^2+\beta^2-\gamma^2)^2+4\alpha^2\gamma^2}. \end{aligned}$$

Therefore if we define

$$I := \int_0^{\infty} \frac{s}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds - \int_{-\infty}^{0} \frac{s}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds$$

then

$$\begin{aligned}
I &= \int_0^\infty \frac{s}{[(s-\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds + \int_0^\infty \frac{s}{[(s+\alpha)^2+\beta^2]\,[s^2+\gamma^2]}\,ds \\
&= \lim_{n\to\infty}\left\{\int_{-\alpha}^{n} \frac{A''(s+\alpha)+B''}{s^2+\beta^2}\,ds + \int_{\alpha}^{n} \frac{A''(s-\alpha)-B''}{s^2+\beta^2}\,ds - \int_0^n \frac{2A''s}{s^2+\gamma^2}\,ds\right\} \\
&= A''\lim_{n\to\infty}\log\left(\frac{n^2+\beta^2}{n^2+\gamma^2}\right) + A''\log\left(\frac{\gamma^2}{\alpha^2+\beta^2}\right) \\
&\quad + \frac{\alpha A''+B''}{\beta}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{-\alpha}{\beta}\right)\right] - \frac{\alpha A''+B''}{\beta}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right] \\
&= A''\log\left(\frac{\gamma^2}{\alpha^2+\beta^2}\right) + \frac{2(\alpha A''+B'')}{\beta}\tan^{-1}\left(\frac{\alpha}{\beta}\right). \qquad (6.20)
\end{aligned}$$

Now,

$$|A''| \le \frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2-\gamma^2)^2} = O(|\lambda|^{-2}) \qquad (|\lambda|\to\infty).$$

Also, for large $|\lambda|$ we have

$$\left|\log\left(\frac{\gamma^2}{\alpha^2+\beta^2}\right)\right| = \log\left(\frac{\alpha^2+\beta^2}{\gamma^2}\right) = O(\log|\lambda|^2) = O(\log|\lambda|).$$

Finally,

$$|\alpha A''+B''| \le \frac{|\alpha|(\alpha^2+\beta^2+\gamma^2)}{(\alpha^2+\beta^2-\gamma^2)^2} = O\left(\frac{|\operatorname{Re}\lambda|}{|\lambda|^2}\right) \qquad (|\lambda|\to\infty).$$

Hence from (6.20) we see that

$$|\lambda - i\eta|\,(2+|\lambda+i\eta|)\,I = O\left(\log|\lambda| + \frac{|\operatorname{Re}\lambda|}{\beta}\right) \qquad (|\lambda|\to\infty). \qquad (6.21)$$

The result now follows from (6.14), (6.19) and (6.21). □

The presence of the $\log|\lambda|$ term in (6.15) indicates that Proposition 6.4.3 is not obtained simply by setting $\alpha = 1$ in Proposition 6.2.6. This may be due in part to the slightly different method of proof involved. Indeed, the following example shows that the estimate obtained in Proposition 6.4.3 is not sharp.

Example 6.4.4. *If we set $\alpha = 1$ in Examples 6.2.1 and 6.2.9, we obtain a log-sectorial operator A such that $\sigma(\log A)$ consists of the set of eigenvalues $\{\log L_n : n \ge 3\}$, where $L_n = n/\log n$. The norm of the resolvent at all other points is given by*

$$\|R(\lambda, \log A)\| = \sup_{n\ge 3}\left(\frac{1}{|\lambda-\log L_n|} + \frac{\log n}{|\lambda-\log L_n|^2}\right).$$

If we set $\lambda_m = \log L_m + i\beta$, *where* $m \geq 3$ *and* $\beta > 0$, *then*

$$\|R(\lambda_m, \log A)\| = \frac{1}{|\beta|} + \frac{\log m}{|\beta|^2} \leq \frac{1 + \log m}{|\operatorname{Im}\lambda|} \sim \frac{1 + |\operatorname{Re}\lambda|}{|\operatorname{Im}\lambda|}.$$

In general, suppose that F is such that the logarithm of the F-sectorial operator A is always well-defined, and that $F(s) \to \infty$ as $s \to \infty$. Following Examples 6.2.1, 6.3.6 and 6.4.4, we can construct an F-sectorial operator A such that the resolvent of $\log A$ is unbounded on horizontal lines. This is in contrast to the classes of operators considered in Chapter 5, and is the motivation for our next chapter.

Chapter 7

Exponentials of F-Strong Strip-Type Operators

We saw in the previous chapter that it is possible to define the logarithm of certain F-sectorial operators. We showed that, under suitable conditions, the spectrum of the logarithm is contained in a horizontal strip, and that the growth or decay of the resolvent outside this strip depends on the function F. This leads us to consider the idea of an *F-strong strip-type operator*, i.e., an operator whose spectrum is contained in a horizontal strip, but which satisfies a different resolvent estimate from that of a strong strip-type operator. Just as we constructed a functional calculus for F-sectorial operators, it is possible to construct a functional calculus for F-strong strip-type operators. We outline such a construction in Section 7.1.

It is sometimes possible to define the exponential of an F-strong strip-type operator B, and in Sections 7.2 and 7.3 we look at specific examples of F-strong strip-type operators B for which this is the case. These examples are inspired by the results obtained in Sections 6.2 and 6.3. When $-1 \in \rho(e^B)$, the techniques of Chapter 5 can be used to show that $\sigma(e^B)$ is contained in a sector, and to estimate the norm of the $\lambda R(\lambda, e^B)$ for λ outside this sector. Thus these exponentials are further examples of F-sectorial operators.

In Section 7.2 we define the class of *log-strong strip-type operators*, which includes logarithms of $\log$ log-sectorial operators. We show that, when $-1 \in \rho(e^B)$, the norm of $\lambda R(\lambda, e^B)$ grows like the square of $\log\log|\lambda|$ (Proposition 7.2.2). In Section 7.3 we look at λ^α*-strong strip-type operators*, examples of which include logarithms of α-log-sectorial operators. Again we obtain an estimate on $\lambda R(\lambda, e^B)$ under the assumption $-1 \in \rho(e^B)$ (Proposition 7.3.1).

Finally, in Section 7.4, we summarise the relationships between the various classes of operators considered in Chapters 5 – 7, and discuss the impact of our results on the inversion problem.

7.1 F-Strong Strip-Type Operators

In this section we shall define the concept of an F*-strong strip-type operator*, and describe how a functional calculus can be constructed for such an operator. In the case when the exponential of the F-strong strip-type operator B can be defined, we explain how the techniques of Chapter 5 can be used to estimate the norm of the resolvent of e^B.

Let $\omega \in [0, \pi)$ and suppose that $F : (0, \infty) \to [\delta, \infty)$ is continuous, where $\delta > 0$. We say that an operator B is an **F-strong strip-type operator** of height ω, written $B \in F\text{-SStrip}(\omega)$, if

1. $\sigma(B) \subset \overline{H_\omega}$ and

2. For each $\omega' \in (\omega, \pi)$ there exists $C = C(\omega') > 0$ such that

$$\|R(\lambda, B)\| \le C \frac{F(|\lambda|)}{|\operatorname{Im}\lambda| - \omega'} \qquad (|\operatorname{Im}\lambda| > \omega').$$

From this definition we see that $B \in F\text{-SStrip}(\omega)$ if and only if $-B \in F\text{-SStrip}(\omega)$. Clearly, taking $F \equiv 1$ gives us the usual definition of a strong strip-type operator.

A functional calculus for $B \in F$-SStrip(ω) can be constructed in the usual way. To construct the primary functional calculus we consider the set

$$\mathcal{E}_F(H_\theta) := \left\{ f \in \mathcal{H}^\infty(H_\theta) : \int_{\partial H_\varphi} |f(z)|\, F(|z|)\, |dz| < \infty \right\}, \tag{7.1}$$

where $\omega < \varphi < \theta < \pi$. For each $f \in \mathcal{E}_F(H_\theta)$ we can define the bounded operator

$$f(B) := \frac{1}{2\pi i} \int_{\partial H_\varphi} f(z) R(z, B)\, dz.$$

The properties of this functional calculus, including the set of regularisable functions, will clearly depend heavily on the function F. In particular, the choice of F will affect whether the exponential e^B can be defined. By Theorem 2.1.1(f) it follows that if e^B can be defined then it will be injective, with $(e^B)^{-1} = e^{-B}$.

Given an F-strong strip-type operator B for which the exponential e^B can be defined, we want to know what properties e^B has. We have seen that, when B is a strong strip-type operator such that $-1 \in \rho(e^B)$, the spectrum of e^B is contained in a sector, and the resolvent outside this sector satisfies a logarithmic estimate (Proposition 5.2.9). We use the same methods as in Section 5.2 to study the exponentials of F-strong strip-type operators.

Let $B \in F$-SStrip(ω) and define the function f_λ by

$$f_\lambda(z) = \frac{e^z}{(e^z - \lambda)(e^z + 1)} \qquad (z \in H_\theta), \tag{7.2}$$

where $\omega < \theta < |\arg \lambda| \leq \pi$. Then $f_\lambda \in \mathcal{H}^\infty(H_\theta)$ and the functional identity

$$\frac{\lambda}{\lambda - e^z} = -(1+\lambda) f_\lambda(z) + \frac{1}{e^z + 1} \qquad (z \in H_\theta) \tag{7.3}$$

holds. Suppose that $f_\lambda \in \mathcal{E}_F(H_\theta)$, that e^B is well-defined and that $-1 \in \rho(e^B)$. It follows from Theorem 2.1.1(f) that the function $z \mapsto (e^z+1)^{-1}$ is regularisable and that $(e^z + 1)^{-1}(B) = -R(-1, e^B)$. Hence, under these assumptions, it follows from (7.3) that $\lambda \in \rho(e^B)$ whenever $|\arg \lambda| \in (\omega, \pi]$, and that the operator identity

$$\lambda R(\lambda, e^B) = -(1+\lambda) f_\lambda(B) - R(-1, e^B) \tag{7.4}$$

holds for such λ. Thus, as was the case for strong strip-type operators. estimating the norm of $f_\lambda(B)$ will give us an estimate on the norm of $\lambda R(\lambda, e^B)$.

Remark 7.1.1. *There is nothing unique about the point* -1 *here. If we assume that there is some point* $\mu \in \rho(e^B)$ *with* $|\arg\mu| \in (\omega, \pi]$, *then it will follow that* $\lambda \in \rho(e^B)$ *whenever* $|\arg\lambda| \in (\omega, \pi]$.

As mentioned in Section 5.2, the function f_λ can in fact be thought of as a function on the whole of $\mathbb{C}$, with simple poles at $z = (2k+1)\pi i$ and at $z = \log|\lambda| + (2k\pi + \arg\lambda)i$ for $k \in \mathbb{Z}$. We shall estimate the norm of the contour integral defining $f_\lambda(B)$ for a suitably chosen contour Γ. Initially we suppose that $|\lambda| < 1$. Let $\varphi \in (\omega, \theta)$ and define $\Gamma_1, \dots, \Gamma_4$ as follows:

$$\begin{aligned}
\Gamma_1 &= (-\infty, \log|\lambda|] + i\varphi \\
\Gamma_2 &= \left\{ t + i(\varphi - \log|\lambda| + t) : \log|\lambda| < t \le \frac{\log|\lambda|}{2} \right\} \\
\Gamma_3 &= \left\{ t + i(\varphi - t) : \frac{\log|\lambda|}{2} < t \le 0 \right\} \\
\Gamma_4 &= (0, \infty) + i\varphi.
\end{aligned}$$

Take $\Gamma_+ = \bigcup_{j=1}^4 \Gamma_j$ and let Γ_- denote the reflection of Γ_+ in the real axis. Finally, set $\Gamma = \Gamma_+ \cup \Gamma_-$. Using this contour Γ, we can obtain an estimate on the norm of $\lambda R(\lambda, e^B)$, at first for small $|\lambda|$. This estimate can then be extended to large $|\lambda|$ by considering the inverse of e^B, namely the exponential of $-B$.

7.2 Log-Strong Strip-Type Operators

Let $\omega \in [0, \pi)$ and suppose that the operator B satisfies

1. $\sigma(B) \subset \overline{H_\omega}$ and

2. For each $\omega' \in (\omega, \pi)$ there exists $C = C(\omega') > 0$ such that

$$\|R(\lambda, B)\| \le C \frac{\log(1 + |\lambda|)}{|\operatorname{Im}\lambda| - \omega'} \qquad (|\operatorname{Im}\lambda| > \omega'). \tag{7.5}$$

Then we say that B is a **log-strong strip-type operator** of height ω, and write $B \in \text{log-SStrip}(\omega)$. It follows from Proposition 6.3.4 that the logarithm of an injective log log-sectorial operator is an example of a log-strong strip-type operator.

If $B \in \text{log-SStrip}(\omega)$ then we have $\|R(\lambda, B)\| = O(\log|\operatorname{Re}\lambda|)$ as $|\operatorname{Re}\lambda| \to \infty$ on horizontal lines. Therefore, if $\theta \in (\omega, \pi)$ and $F(s) = \log(1+s)$ for $s > 0$, the set $\mathcal{E}_F(H_\theta)$ defined by (7.1) contains the algebra

$$\mathcal{F}(H_\theta) = \{ f \in \mathcal{H}^\infty(H_\theta) : f(z) = O(|\operatorname{Re} z|^{-\alpha}) \text{ as } |\operatorname{Re} z| \to \infty \text{ for some } \alpha > 1 \}$$

just as in the case of strong strip-type operators. The function f_λ defined by (7.2) belongs to $\mathcal{F}(H_\theta)$ whenever $|\arg\lambda| \in (\theta, \pi]$. This means that $f(B)$ is well-defined whenever $f \in \mathcal{H}(H_\theta)$ is such that $f(z) = O(e^{\alpha|\operatorname{Re} z|})$ as $|\operatorname{Re} z| \to \infty$ for some $\alpha \in (0, \pi/\omega)$ (see [22, Lemma 4.2.3]). In particular, the exponential e^B of B is well-defined.

Let $\omega < \theta < |\arg\lambda| \leq \pi$, and suppose initially that $|\lambda| < e^{-2}$. Let $\varphi \in (\omega, \theta)$ and, with the contour Γ defined as above, set

$$I_j = \left\| \int_{\Gamma_j} f_\lambda(z) R(z, B)\, dz \right\| \qquad (j = 1, \ldots, 4).$$

We shall estimate each of $I_1, \ldots, I_4$ in turn. Note that, on each of $\Gamma_1, \ldots, \Gamma_4$, we can modify slightly the estimate given in (7.5). Indeed, there exists a constant C such that $|z| \leq C|\operatorname{Re} z|$ for all $z \in \Gamma_1 \cup \Gamma_2$. Hence we actually have the estimates

$$\|R(z, B)\| \leq C \frac{\log|\operatorname{Re} z|}{|\operatorname{Im} z| - \omega'} \qquad (z \in \Gamma_1)$$

and

$$\|R(z, B)\| \leq C \frac{\log(1 + |\operatorname{Re} z|)}{|\operatorname{Im} z| - \omega'} \qquad (z \in \Gamma_2),$$

where $\omega' \in (\omega, \varphi)$. Furthermore, there exists C such that $1 + |z| \leq C(1 + |\operatorname{Re} z|)$ for all $z \in \Gamma_3 \cup \Gamma_4$. Hence the estimate

$$\|R(z, B)\| \leq C \frac{1 + \log(1 + |\operatorname{Re} z|)}{|\operatorname{Im} z| - \omega'} \qquad (z \in \Gamma_3 \cup \Gamma_4)$$

also holds. The following lemma will be used to estimate I_1.

Lemma 7.2.1. *If* $x, y \geq 2$ *then* $\log(x+y) \leq \log x + \log y$.

Proof. If $x, y \geq 2$ then clearly $x + y \leq xy$, hence by monotonicity of the logarithm function we have $\log(x+y) \leq \log(xy) = \log x + \log y$. □

On Γ_1 we have the estimate

$$\|R(t+i\varphi, B)\| \leq C \frac{\log|t|}{\varphi - \omega'} \qquad (-\infty < t < \log|\lambda|).$$

If we set $\varepsilon = \varphi - \omega'$ then it follows from Lemma 5.2.3 that

$$I_1 \leq \frac{C}{\varepsilon} \int_0^{|\lambda|} \frac{\log\log(1/s)}{(s+1)(s+|\lambda|)}\, ds \leq \frac{C}{\varepsilon} \int_0^{|\lambda|} \frac{\log\log(1/s)}{s+|\lambda|}\, ds.$$

Putting $u = s/|\lambda|$ we see that

$$\begin{aligned} I_1 &\leq \frac{C}{\varepsilon} \int_0^1 \frac{\log\log\left(\frac{1}{|\lambda|u}\right)}{u+1}\, du \\ &\leq \frac{C}{\varepsilon} \left[\int_0^{e^{-2}} \frac{\log\log(1/|\lambda|) + \log\log(1/u)}{u+1}\, du + \int_{e^{-2}}^1 \frac{\log\log\left(\frac{1}{|\lambda|u}\right)}{u+1}\, du, \right] \end{aligned}$$

where we have applied Lemma 7.2.1 to the integral between 0 and e^{-2}. A second application of Lemma 7.2.1 gives

$$\begin{aligned} \int_{e^{-2}}^1 \frac{\log\log\left(\frac{1}{|\lambda|u}\right)}{u+1}\, du &\leq \int_{e^{-2}}^1 \frac{\log\log(e^2/|\lambda|)}{u+1}\, du \\ &\leq \log 2 \int_{e^{-2}}^1 \frac{du}{u+1} + \left(\log\log\frac{1}{|\lambda|}\right) \int_{e^{-2}}^1 \frac{du}{u+1}. \end{aligned}$$

Therefore

$$I_1 \leq \frac{C}{\varepsilon}\left(1 + \log\log\frac{1}{|\lambda|}\right). \tag{7.6}$$

On Γ_2 we have

$$\|R(t + i(\varphi - \log|\lambda| + t), B)\| \leq C \frac{\log(1+|t|)}{(\varphi - \log|\lambda| + t) - \omega'} \qquad \left(\log|\lambda| \leq t \leq \frac{\log|\lambda|}{2}\right).$$

Using Lemmas 5.2.5 and 5.2.6 we see that

$$\begin{aligned} I_2 &\leq C\int_{|\lambda|}^{|\lambda|^{1/2}} \frac{\log(1+\log(1/s))}{(s+|\lambda|)(s+1)(\log(s/|\lambda|)+\varepsilon)}\,ds \\ &\leq C\log\left(1+\log\frac{1}{|\lambda|}\right)\int_{|\lambda|}^{|\lambda|^{1/2}} \frac{ds}{(s+|\lambda|)(\log(s/|\lambda|)+\varepsilon)}. \end{aligned}$$

Setting $u = s/|\lambda|$ gives

$$\begin{aligned} I_2 &\leq C\,\log\left(1+\log\frac{1}{|\lambda|}\right)\int_1^{|\lambda|^{-1/2}} \frac{du}{u(\log u+\varepsilon)} \\ &= C\,\log\left(1+\log\frac{1}{|\lambda|}\right)\left[\log(\log u+\varepsilon)\right]_1^{|\lambda|^{-1/2}} \\ &= C\,\log\left(1+\log\frac{1}{|\lambda|}\right)\left(\log(\log|\lambda|^{-1/2}+\varepsilon)-\log\varepsilon\right). \end{aligned}$$

Note that, since $|\lambda| < e^{-2}$, there exists a constant C such that

$$\log(\log|\lambda|^{-1/2}+\varepsilon) \leq \log(\log|\lambda|^{-1}+\varepsilon) \leq C\log\log|\lambda|^{-1}.$$

Hence

$$I_2 \leq C\left(\log\log\frac{1}{|\lambda|}\right)^2 + C\left(\log\frac{1}{\varepsilon}\right)\log\log\frac{1}{|\lambda|}. \tag{7.7}$$

On Γ_3 we have

$$\|R(t+i(\varphi-t),B)\| \leq C\frac{1+\log(|t|+1)}{(\varphi-t)-\omega'} \qquad \left(\frac{\log|\lambda|}{2} \leq t \leq 0\right).$$

Using Lemmas 5.2.7 and 5.2.8 we obtain

$$\begin{aligned} I_3 &\leq C\int_{|\lambda|^{1/2}}^{1} \frac{1+\log(\log(1/s)+1)}{(s+|\lambda|)(s+1)(\log(1/s)+\varepsilon)}\,ds \\ &\leq C\left[1+\log\left(\log\left(\frac{1}{|\lambda|^{1/2}}\right)+1\right)\right]\int_{|\lambda|^{1/2}}^{1}\frac{ds}{s(\log(1/s)+\varepsilon)} \\ &= C\left[1+\log\left(\log\left(\frac{1}{|\lambda|^{1/2}}\right)+1\right)\right]\left[-\log(\log(1/s)+\varepsilon)\right]_{|\lambda|^{1/2}}^{1} \\ &= C\left[1+\log\left(\log\left(\frac{1}{|\lambda|^{1/2}}\right)+1\right)\right]\left(\log(\log(|\lambda|^{-1/2})+\varepsilon)-\log\varepsilon\right). \end{aligned}$$

Again, since $|\lambda| < e^{-2}$ we actually have

$$I_3 \leq C\left(\log\log\frac{1}{|\lambda|}\right)^2 + C\left(\log\frac{1}{\varepsilon}\right)\log\log\frac{1}{|\lambda|}. \tag{7.8}$$

Finally, on Γ_4 we have the estimate

$$\|R(t+i\varphi, B)\| \leq C\,\frac{1+\log(t+1)}{\varphi-\omega'} \qquad (t>0),$$

hence it follows from Lemma 5.2.3 that

$$I_4 \leq \frac{C}{\varepsilon}\int_1^\infty \frac{1+\log(\log s+1)}{(s+|\lambda|)(s+1)}\,ds \leq \frac{C}{\varepsilon}\int_1^\infty \frac{1+\log(\log s+1)}{s(s+1)}\,ds. \tag{7.9}$$

From (7.6) – (7.9) we see that there exists a constant $C = C(\varphi) > 0$ such that

$$\left\|\int_{\Gamma_+} f_\lambda(z)R(z,B)\,dz\right\| \leq C\left(1+\log\log\frac{1}{|\lambda|}+\left(\log\log\frac{1}{|\lambda|}\right)^2\right)$$

for $|\lambda| < e^{-2}$. As the third term on the right-hand side is dominant for small $|\lambda|$, it follows that actually

$$\left\|\int_{\Gamma_+} f_\lambda(z)R(z,B)\,dz\right\| \leq C\left(\log\log\left(\frac{1}{|\lambda|}\right)\right)^2 \tag{7.10}$$

for all such λ. We are now able to prove the main result of this section.

Proposition 7.2.2. *Let $B \in$ log-SStrip(ω) and suppose that $-1 \in \rho(e^B)$. Then for each $\theta \in (\omega, \pi)$, there exists a constant $C = C(\theta) > 0$ such that*

$$\|\lambda R(\lambda, e^B)\| \leq C(\log\log|\lambda|)^2 \qquad (|\arg\lambda| \geq \theta,\ |\lambda| > e^2) \tag{7.11}$$

and

$$\|\lambda R(\lambda, e^B)\| \leq C\left(\log\log\frac{1}{|\lambda|}\right)^2 \qquad (|\arg\lambda| \geq \theta,\ |\lambda| < e^{-2}) \tag{7.12}$$

Proof. Let $\theta \in (\omega, \pi)$ and suppose that $|\arg\lambda| \geq \theta$. By symmetry an identical estimate to (7.10) holds on Γ_-. Hence (7.12) now follows from (7.4). To prove (7.11), note that the identity $\lambda R(\lambda, (e^B)^{-1}) = I - \lambda^{-1}R(\lambda^{-1}, e^B)$ follows from [22, Lemma A.2.1]. Since $-B$ also belongs to log-SStrip(ω) and $e^{-B} = (e^B)^{-1}$, it follows that the norm of $\lambda R(\lambda, (e^B)^{-1})$ satisfies an estimate identical to (7.12) for $|\lambda| < e^{-2}$. Hence, for such λ we have

$$\|\lambda^{-1} R\left(\lambda^{-1}, e^B\right)\| \leq C \left(\log\log \frac{1}{|\lambda|}\right)^2 .$$

Setting $\mu = \lambda^{-1}$ this becomes $\|\mu R(\mu, e^B)\| \leq C\,(\log\log|\mu|)^2$ for $|\mu| > e^2$, completing the proof. $\square$

Hence, if $B \in$ log-SStrip(ω) is such that $\rho(e^B)$ contains some point μ with $|\arg\mu| \in (\omega, \pi]$, the exponential e^B belongs to the class F-Sect(ω) for the function F defined by

$$F(s) = \begin{cases} (\log\log s)^2 & (s > e^e), \\ 1 & (e^{-e} \leq s \leq e^e), \\ (\log\log \frac{1}{s})^2 & (0 < s < e^{-e}), \end{cases}$$

and e^B could be referred to as a $(\log\log)^2$**-sectorial operator**.

7.3 λ^α-Strong Strip-Type Operators

Let $\omega \in [0, \pi)$, $\alpha \in (0,1)$ and suppose that A is an injective α-log-sectorial operator of height ω. It follows from Proposition 6.2.8 that, for each $\omega' \in (\omega, \pi)$, there exists a constant $C = C(\omega') > 0$ such that the estimate

$$\|R(\lambda, \log A)\| \leq C \left(\frac{1 + |\operatorname{Re}\lambda|^\alpha}{|\operatorname{Im}\lambda| - \omega'} + \frac{1}{(|\operatorname{Im}\lambda| - \omega')^{1-\alpha}}\right) \qquad (|\operatorname{Im}\lambda| > \omega')$$

holds. In fact then $R(\lambda, \log A)$ satisfies

$$\begin{aligned} \|R(\lambda, \log A)\| &\leq C \left(\frac{1 + |\operatorname{Re}\lambda|^\alpha + (|\operatorname{Im}\lambda| - \omega')^\alpha}{|\operatorname{Im}\lambda| - \omega'}\right) \\ &\leq C \left(\frac{|\operatorname{Re}\lambda|^\alpha + (1 + |\operatorname{Im}\lambda|^\alpha)}{|\operatorname{Im}\lambda| - \omega'}\right) \\ &\leq C \left(\frac{|\operatorname{Re}\lambda|^\alpha + 2|\operatorname{Im}\lambda|^\alpha}{|\operatorname{Im}\lambda| - \omega'}\right) \\ &\leq 2\,C \frac{|\lambda|^\alpha}{|\operatorname{Im}\lambda| - \omega'}. \end{aligned} \qquad (7.13)$$

In light of (7.13) we say that B is a **λ^α-strong strip-type operator** of height ω, and write $B \in \lambda^\alpha$-SStrip(ω), if

1. $\sigma(B) \subset \overline{H_\omega}$ and

2. For each $\omega' \in (\omega, \pi)$ there exists $C = C(\omega') > 0$ such that

$$\|R(\lambda, B)\| \leq C \frac{|\lambda|^\alpha}{|\operatorname{Im}\lambda| - \omega'} \qquad (|\operatorname{Im}\lambda| > \omega'). \tag{7.14}$$

Clearly if $A \in \alpha$-log-Sect(ω) is injective then $\log A \in \lambda^\alpha$-SStrip$(\omega)$.

Although the definition of a λ^α-strong strip-type operator also makes sense when $\alpha \geq 1$, it is unclear whether the logarithm of an injective log-sectorial operator is always a λ-strong strip-type operator (compare the estimate obtained in Proposition 6.4.3 with Example 6.4.4). However, it follows from [28, Theorem 3.5] that if iB generates an **α-times integrated group**, where $\alpha > 0$, then B is a λ^α-strong strip-type operator. A particular example is the Laplacian Δ_p on $L^p(\mathbb{R}^n)$, where $1 \leq p < \infty$ and $n \geq 2$ (the case $n = 1$ has already been considered in Section 5.2.2). Hieber showed in [27] that $i\Delta_p$ generates an α-times integrated group on $L^p(\mathbb{R}^n)$ whenever $\alpha > n|\frac{1}{2} - \frac{1}{p}|$, and that $\Delta_p \in \lambda^\alpha$-SStrip$(0)$ for such α.

From now on we shall restrict ourselves to the case $\alpha \in (0, 1]$. If $B \in \lambda^\alpha$-SStrip(ω) then we have $\|R(\lambda, B)\| = O(|\operatorname{Re}\lambda|^\alpha)$ as $|\operatorname{Re}\lambda| \to \infty$ on horizontal lines. Therefore, if $\theta \in (\omega, \pi)$ and $F(s) = s^\alpha$ for $s > 0$, the set $\mathcal{E}_F(H_\theta)$ defined by (7.1) contains

$$\mathcal{F}'(H_\theta) := \{\, f \in \mathcal{H}^\infty(H_\theta) : f(z) = O(|\operatorname{Re} z|^{-\beta}) \text{ as } |\operatorname{Re} z| \to \infty \text{ for some } \beta > 2 \,\}.$$

If $|\arg\lambda| \in (\theta, \pi]$ then $\mathcal{F}'(H_\theta)$ contains the function f_λ, since for large positive $\operatorname{Re} z$ we have

$$|f_\lambda(z)|\,\|R(z, B)\| \leq C \frac{|\operatorname{Re} z|^\alpha e^{\operatorname{Re} z}}{(e^{\operatorname{Re} z} - |\lambda|)(e^{\operatorname{Re} z} - 1)} = O(|\operatorname{Re} z|^\alpha e^{-\operatorname{Re} z}) \qquad (\operatorname{Re} z \to +\infty)$$

and for large negative $\operatorname{Re} z$, say $\operatorname{Re} z < \log\varepsilon$ for some $\varepsilon > 0$, we have

$$|f_\lambda(z)|\,\|R(z, B)\| \leq C \frac{|\operatorname{Re} z|^\alpha e^{\operatorname{Re} z}}{(|\lambda| - \varepsilon)(1 - \varepsilon)} = O(|\operatorname{Re} z|^\alpha e^{\operatorname{Re} z}) \qquad (\operatorname{Re} z \to -\infty).$$

It follows once again that the exponential e^B is well-defined.

Let $\omega < \theta < |\arg \lambda| \le \pi$ and suppose initially that $|\lambda| < e^{-2}$. Let $\varphi \in (\omega, \theta)$ and take Γ to be the same contour as before. Again we estimate each of $I_1, \ldots, I_4$ in turn. As before, the estimate given in (7.14) can be modified slightly on each of $\Gamma_1, \ldots, \Gamma_4$. Indeed, as mentioned earlier there exists C such that $|z| \le C|\operatorname{Re} z|$ for all $z \in \Gamma_1 \cup \Gamma_2$, hence

$$\|R(z, B)\| \le C \frac{|\operatorname{Re} z|^\alpha}{|\operatorname{Im} \lambda| - \omega'} \qquad (z \in \Gamma_1 \cup \Gamma_2),$$

where $\omega' \in (\omega, \varphi)$. Furthermore, there exists C such that $|z| \le C(1 + |\operatorname{Re} z|)$ for all $z \in \Gamma_3 \cup \Gamma_4$, hence

$$\|R(z, B)\| \le C \frac{1 + |\operatorname{Re} z|^\alpha}{|\operatorname{Im} \lambda| - \omega'} \qquad (z \in \Gamma_3 \cup \Gamma_4).$$

On Γ_1 we therefore have the estimate

$$\|R(t + i\varphi, B)\| \le C \frac{|t|^\alpha}{\varphi - \omega'} \qquad (-\infty < t < \log |\lambda|).$$

If we set $\varepsilon = \varphi - \omega'$ then from Lemma 5.2.3 we obtain

$$I_1 \le \frac{C}{\varepsilon} \int_0^{|\lambda|} \frac{(\log(1/s))^\alpha}{(s + |\lambda|)(s+1)}\, ds \le \frac{C}{\varepsilon} \int_0^{|\lambda|} \frac{(\log(1/s))^\alpha}{(s + |\lambda|)}\, ds.$$

Setting $u = s/|\lambda|$ we see that

$$\begin{aligned} I_1 &\le \frac{C}{\varepsilon} \int_0^1 \frac{\left(\log \frac{1}{|\lambda| u}\right)^\alpha}{u+1}\, du \\ &\le \frac{C}{\varepsilon} \left[\left(\log \frac{1}{|\lambda|}\right)^\alpha \int_0^1 \frac{du}{u+1} + \int_0^1 \frac{(\log(1/u))^\alpha}{u+1}\, du \right], \end{aligned}$$

by the fact that the function $s \mapsto s^\alpha$ is concave on $(0, \infty)$. Hence

$$I_1 \le C \left(1 + \left(\log \frac{1}{|\lambda|}\right)^\alpha\right). \tag{7.15}$$

On Γ_2 we have the estimate

$$\|R(t + i(\varphi - \log|\lambda| + t), B)\| \le C \frac{|t|^\alpha}{(\varphi - \log|\lambda| + t) - \omega'} \qquad \left(\log|\lambda| \le t \le \frac{\log|\lambda|}{2}\right).$$

From Lemmas 5.2.5 and 5.2.6 we have

$$
\begin{aligned}
I_2 &\le C\int_{|\lambda|}^{|\lambda|^{1/2}} \frac{(\log(1/s))^\alpha}{(s+|\lambda|)(s+1)(\log(s/|\lambda|)+\varepsilon)}\,ds \\
&\le C\left(\log\frac{1}{|\lambda|}\right)^\alpha \int_{|\lambda|}^{|\lambda|^{1/2}} \frac{ds}{(s+|\lambda|)(\log(s/|\lambda|)+\varepsilon)},
\end{aligned}
$$

and setting $u = s/|\lambda|$ again gives

$$
\begin{aligned}
I_2 &\le C\left(\log\frac{1}{|\lambda|}\right)^\alpha \int_1^{|\lambda|^{-1/2}} \frac{du}{(u+1)(\log u+\varepsilon)} \\
&\le C\left(\log\frac{1}{|\lambda|}\right)^\alpha \int_1^{|\lambda|^{-1/2}} \frac{du}{u(\log u+\varepsilon)} \\
&= C\left(\log\frac{1}{|\lambda|}\right)^\alpha [\log(\log u+\varepsilon)]_1^{|\lambda|^{-1/2}} \\
&= C\left(\log\frac{1}{|\lambda|}\right)^\alpha \left(\log(\log|\lambda|^{-1/2}+\varepsilon) - \log\varepsilon\right).
\end{aligned}
$$

Hence, since $|\lambda| < e^{-2}$,

$$
I_2 \le C\left(\log\frac{1}{|\lambda|}\right)^\alpha \log\log\frac{1}{|\lambda|} + C\left(\log\frac{1}{\varepsilon}\right)\left(\log\frac{1}{|\lambda|}\right)^\alpha. \tag{7.16}
$$

On Γ_3 we have

$$
\|R(t+i(\varphi-t),B)\| \le C\,\frac{1+|t|^\alpha}{(\varphi-t)-\omega'} \qquad \left(\frac{\log|\lambda|}{2} \le t \le 0\right).
$$

From Lemmas 5.2.7 and 5.2.8 it follows that

$$
\begin{aligned}
I_3 &\le C\int_{|\lambda|^{1/2}}^{1} \frac{1+(\log(1/s))^\alpha}{(s+|\lambda|)(s+1)(\log(1/s)+\varepsilon)}\,ds \\
&\le C\left(1+\left(\log\frac{1}{|\lambda|^{1/2}}\right)^\alpha\right) \int_{|\lambda|^{1/2}}^{1} \frac{ds}{s(\log(1/s)+\varepsilon)} \\
&= C\left(1+\left(\log\frac{1}{|\lambda|^{1/2}}\right)^\alpha\right) [-\log(\log(1/s)+\varepsilon)]_{|\lambda|^{1/2}}^{1} \\
&= C\left(1+\left(\log\frac{1}{|\lambda|^{1/2}}\right)^\alpha\right) \left(\log(\log|\lambda|^{-1/2}+\varepsilon) - \log\varepsilon\right).
\end{aligned}
$$

Thus

$$
I_3 \le C\left(\log\frac{1}{|\lambda|}\right)^\alpha \log\log\frac{1}{|\lambda|} + C\left(\log\frac{1}{\varepsilon}\right)\left(\log\frac{1}{|\lambda|}\right)^\alpha. \tag{7.17}
$$

Finally, on Γ_4 we have

$$\|R(t+i\varphi,B)\| \le C\,\frac{1+t^\alpha}{\varphi-\omega'} \qquad (0<t<\infty),$$

hence

$$I_4 \le \frac{C}{\varepsilon}\int_1^\infty \frac{1+(\log s)^\alpha}{(s+1)(s+|\lambda|)}\,ds \le \frac{C}{\varepsilon}\int_1^\infty \frac{1+(\log s)^\alpha}{s(s+1)}\,ds. \tag{7.18}$$

From (7.15) – (7.18) we see that there exists $C=C(\varphi)>0$ such that

$$\left\|\int_{\Gamma_+} f_\lambda(z)R(z,B)\,dz\right\| \le C\left(1+\left(\log\frac{1}{|\lambda|}\right)^\alpha+\left(\log\frac{1}{|\lambda|}\right)^\alpha \log\log\frac{1}{|\lambda|}\right)$$

for $|\lambda|<e^{-2}$, and since the third term on the right-hand side is dominant we actually have

$$\left\|\int_{\Gamma_+} f_\lambda(z)R(z,B)\,dz\right\| \le C\left(\log\frac{1}{|\lambda|}\right)^\alpha \log\log\frac{1}{|\lambda|} \tag{7.19}$$

for such λ. We now come to an analogue of Proposition 7.2.2.

Proposition 7.3.1. *Let $B\in\lambda^\alpha$-SStrip(ω) and suppose that $-1\in\rho(e^B)$. Then for each $\theta\in(\omega,\pi)$, there exists $C=C(\theta)>0$ such that*

$$\|\lambda R(\lambda,e^B)\| \le C\,(\log|\lambda|)^\alpha \log\log|\lambda| \qquad (|\arg\lambda|\ge\theta,\ |\lambda|>e^2) \tag{7.20}$$

and

$$\|\lambda R(\lambda,e^B)\| \le C\left(\log\frac{1}{|\lambda|}\right)^\alpha \log\log\frac{1}{|\lambda|} \qquad (|\arg\lambda|\ge\theta,\ |\lambda|<e^{-2}). \tag{7.21}$$

Proof. Let $\theta\in(\omega,\pi)$ and suppose that $|\arg\lambda|\ge\theta$. By symmetry an identical estimate to (7.19) holds on Γ_-. Hence (7.21) now follows from (7.4), and we can deduce (7.20) in exactly the same way as in the proof of Proposition 7.2.2. □

Hence, if $B\in\log$-SStrip(ω) is such that $\rho(e^B)$ contains some point μ with $|\arg\mu|\in(\omega,\pi]$, the exponential e^B belongs to the class F-Sect(ω) for the function F defined by

$$F(s)=\begin{cases}(\log s)^\alpha\log\log s & (s>e^e),\\ e^\alpha & (e^{-e}\le s\le e^e),\\ (\log\frac{1}{s})^\alpha\log\log\frac{1}{s} & (0<s<e^{-e}),\end{cases}$$

and e^B could be referred to as a $(\log^\alpha\times\log\log)$-**sectorial operator**.

7.4 Connection with the Inversion Problem

In this final section we present a summary of how our results allow us to pass through the various classes of operators considered in Chapters 5 – 7. If we begin with an injective sectorial operator A, then by repeatedly taking logarithms and exponentials, we eventually arrive at the class of $(\log\log)^2$-sectorial operators as follows:

$$\begin{aligned}\mathrm{Sect}(\omega) &\xrightarrow[A\to\log A]{[19,\text{ Proposition 3.2}]} \mathrm{SStrip}(\omega) \xrightarrow[B\to e^B]{\text{Prop}^{\underline{\text{n}}}\ 5.2.9} \log\log\text{-}\mathrm{Sect}(\omega)\\ &\xrightarrow[A\to\log A]{\text{Prop}^{\underline{\text{n}}}\ 6.3.5} \log\text{-}\mathrm{SStrip}(\omega) \xrightarrow[B\to e^B]{\text{Prop}^{\underline{\text{n}}}\ 7.2.2} (\log\log)^2\text{-}\mathrm{Sect}(\omega).\end{aligned}$$

Recall that for this chain of results to be valid we assume that -1 is contained in the resolvent sets of all exponentials.

Moving to the fractional case, let $\alpha \in (0,1)$. Proposition 5.2.11 tells us that if B is an α-strong strip-type operator such that $-1 \in \rho(e^B)$, then e^B is $(1-\alpha)$-log-sectorial. On the other hand, suppose that we begin with an injective $(1-\alpha)$-log-sectorial operator A. The estimates obtained in Proposition 6.2.6 tell us that, for $\operatorname{Re}\lambda$ fixed, the norm of $R(\lambda, \log A)$ decays like $|\operatorname{Im}\lambda|^{-\alpha}$ as $|\operatorname{Im}\lambda| \to \infty$, as is also the case for α-strong strip-type operators. However, we also obtain new information on the growth of the resolvent of $\log A$ along horizontal lines. Indeed, we have already observed that $\log A$ is an example of a $\lambda^{1-\alpha}$-strong strip-type operator. If B is now a $\lambda^{1-\alpha}$-strong strip-type operator such that $-1 \in \rho(e^B)$ then Proposition 7.3.1 tells us that e^B is a $(\log^{1-\alpha} \times \log\log)$-sectorial operator. Thus we have the following sequence:

$$\begin{aligned}\alpha\text{-}\mathrm{SStrip}(\omega) &\xrightarrow[B\to e^B]{\text{Prop}^{\underline{\text{n}}}\ 5.2.11} (1-\alpha)\text{-log-}\mathrm{Sect}(\omega)\\ &\xrightarrow[A\to\log A]{\text{Prop}^{\underline{\text{n}}}\ 6.2.8} \lambda^{1-\alpha}\text{-}\mathrm{SStrip}(\omega) \xrightarrow[B\to e^B]{\text{Prop}^{\underline{\text{n}}}\ 7.3.1} (\log^{1-\alpha}\times\log\log)\text{-}\mathrm{Sect}(\omega).\end{aligned}$$

The estimates obtained in Proposition 7.3.1 involve the product of an iterated logarithm with a fractional power of the logarithm. For any $\beta \in (0,1)$, there exists a constant C such that $\log\log|\lambda| \leq C(\log|\lambda|)^\beta$ for large $|\lambda|$. Hence if $B \in \lambda^{1-\alpha}$-SStrip(ω) with $-1 \in \rho(e^B)$ then e^B is in fact a $(1-\alpha+\beta)$-log-sectorial operator, where β is chosen so that $1-\alpha+\beta \in (0,1)$. From this we see that there is a 1-1 correspondence between the class

$$\bigcup_{\alpha\in(0,1)} \{\, A \in \alpha\text{-log-Sect}(\omega) \;:\; A \text{ is injective} \,\}$$

and

$$\bigcup_{\alpha\in(0,1)} \{\, B \in \lambda^{\alpha}\text{-SStrip}(\omega) : -1 \in \rho(e^B) \,\}.$$

Although the class of λ^{α}-strong strip-type operators has been defined for $\alpha = 1$, this case is not quite so satisfying. Beginning with a strip-type operator B, we have the following chain of results:

$$\text{Strip}(\omega) \xrightarrow[B\to e^B]{\text{Prop}^{\underline{\text{n}}}\ 5.2.4} \text{log-Sect}(\omega) \xrightarrow[A\to \log A]{\text{Prop}^{\underline{\text{n}}}\ 6.4.3} \|R(\lambda, \log A)\| = O\left(\log|\lambda| + \frac{|\lambda|}{|\operatorname{Im}\lambda|}\right).$$

It is unclear whether there is any correspondence between λ-strong strip-type operators B such that $-1 \in \rho(e^B)$, and injective log-sectorial operators A. Our method of proof leads to an extra $\log|\lambda|$ term in the resolvent estimates for $\log A$ shown in Proposition 6.4.3.

It also seems that the question of whether the exponential of a strong strip-type operator is sectorial, if it has non-empty resolvent set, remains open.

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