



DEPARTMENT OF ECONOMICS

DISCUSSION PAPER SERIES

**ESTIMATING QUADRATIC VARIATION WHEN QUOTED PRICES
CHANGE BY A CONSTANT INCREMENT**

Jeremy Large

Number 340

August 2007

Manor Road Building, Oxford OX1 3UQ

Estimating Quadratic Variation When Quoted Prices Change by a Constant Increment

Jeremy Large *

JEREMY.LARGE@ECONOMICS.OX.AC.UK

All Souls College, University of Oxford, Oxford, OX1 4AL, U.K.

31 January 2007

first version, February 2005

Abstract

For financial assets whose best quotes almost always change by jumping by the market's price tick size (one cent, five cents, etc.), this paper proposes an estimator of Quadratic Variation which controls for microstructure effects. It measures the prevalence of *alternations*, where quotes jump back to their just-previous price. It defines a simple property called “uncorrelated alternation”, which under conditions implies that the estimator is consistent in an asymptotic limit theory, where jumps become very frequent and small. Feasible limit theory is developed, and in simulations works well.

JEL classification: C10; C22; C80

Keywords: Realized Volatility; Realized Variance; Quadratic Variation; Market Microstructure; High-Frequency Data; Pure Jump Process.

*I thank Yacine Aït-Sahalia, Julio Cacho-Díaz, Peter Hansen, Mike Ludkovski, Nour Meddahi, Per Mykland, Neil Shephard and two anonymous referees for their help and encouragement. I also thank for very helpful comments conference and seminar participants at Stanford University, at the Frontiers in Time Series Analysis Conference in Olbia, Italy, at the Princeton-Chicago Conference on the Econometrics of High Frequency Financial Data at Bonita Springs, Florida, and at the European Winter Meeting of the Econometric Society 2005, Istanbul. I am very grateful to the Bendheim Center for Finance for accommodating me at Princeton University during part of the writing of this paper, and to the US-UK Fulbright Commission for their support.

1 Introduction

There is widespread evidence of persistence in financial assets' volatility. Therefore, estimating their *ex post* volatility furthers the desirable goal of forecasting volatility. Recent research has advocated measuring for this purpose empirical Quadratic Variation (QV), or Realized Volatility, as a statistic of elapsed volatility – see for example Barndorff-Nielsen and Shephard (2002) and Andersen, Bollerslev, and Meddahi (2004). The availability of second-by-second price data has encouraged high-frequency sampling when estimating QV. However, consistent estimation is significantly complicated at the highest frequencies by market microstructure effects. This paper points out features in many markets' microstructure which can be used as structural restrictions to control for this interference. This then leads to an estimator of QV.

These features arise mainly from price discreteness. Harris (1994) points out that discreteness leads some markets to 'trade on a penny', so that their bid-ask spread is bid down to its regulatory minimum, the price tick size (a cent, five cents, etc.), practically all the time. Empirically on such a market, the best bid and ask change through sporadic jumps by the price tick size: so, they are pure jump processes of constant jump magnitude. They may also exhibit a lack of autocorrelation in reversals, herein termed "uncorrelated alternation". The paper reports both these features in quote data. It focuses on Vodafone on the London Stock Exchange (LSE), which was its busiest equity by volume in 2004, with an ancillary study of GlaxoSmithKline (GSK).

When these testable features are present, QV may be estimated either from the best bid, or from the best ask, with the statistic

$$nk^2 \frac{c}{a}, \tag{1}$$

where $n \in \mathbb{N}$ is the number of jumps in the quote, the constant $k > 0$ is the size of the price tick, and $a \leq n$ is the number of *alternations*, i.e. jumps whose direction is a reversal of the last jump. Engle and Russell (2005) calls these 'reversals'. Jumps which do not alternate are *continuations*, and number $c = (n - a)$. Under some further technical assumptions, which do *not* rule out leverage effects, the statistic in (1) is consistent for the underlying price's QV. The term nk^2 is the QV of the observed price. This is an inconsistent, and normally an upwardly biased, estimate of underlying QV because of microstructure effects. However the upwards bias implies more alternation

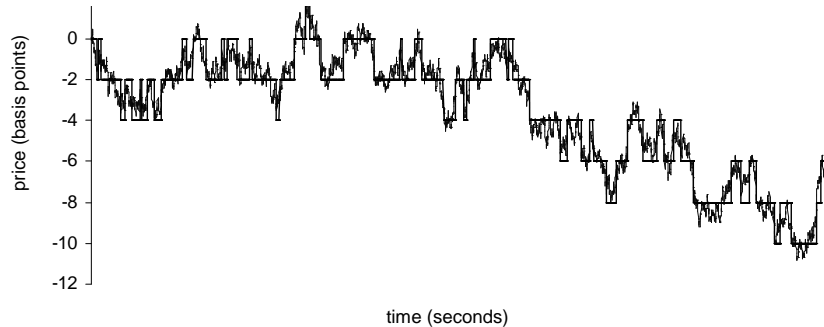


Figure 1: A simulation of this paper’s proposed model. It shows an asset’s observed price, which jumps, and its continuous underlying price, here a scaled Brownian motion. The observed price has a propensity to alternate, and its returns therefore have negative first-order autocorrelation in tick time.

than continuation, which indicates that returns in tick time have negative first-order autocorrelation (even when alternation itself is uncorrelated). In fact, multiplying by the ratio c/a compensates consistently.

Consistency is under a double asymptotic limit theory reflecting both the high-frequency and the small-scale of the market microstructure: in it the intensity of jumping grows without limit, and the squared magnitude of each jump diminishes at the same rate. Delattre and Jacod (1997) have used such an approach. This differs from the limit theory of Aït-Sahalia, Mykland and Zhang (2005), Bandi and Russell (2006b), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2006), Curci and Corsi (2005), Zhang, Mykland, and Aït-Sahalia (2005), and Zhou (1996) which present consistent estimators of QV even in cases where microstructure is not of small scale. Stochastic volatility, leverage effects and drift are introduced through a time-change, drawing on results in Monroe (1978).

Trading is in fact on a penny on important financial markets for interest rates futures, currency futures, and equities. Examples include: BNP Paribas equity (on Euronext), Vodafone equity (on the LSE), US 10-year Treasury Bond Futures (on CBOT), EURIBOR, Short Sterling and Euro-Swiss Franc Futures (all three on Euronext.liffe). However, it is not the norm, and is seldom observed for example on AMEX, Nasdaq or the NYSE, where tick sizes have fallen in recent years. To widen the range of applicable markets to include some of these cases, rounding techniques are proposed for the bid, ask or mid-quote. Using GSK data, I find empirically that the statistic can then be valid even though microstructure is more prominent.

Asset price paths, while normally nearly continuous, experience sporadic large jumps

due for example to public announcements – see for example Aït-Sahalia (2002) and Barndorff-Nielsen and Shephard (2006). These jumps are typically far greater than k . The estimator is shown to be consistent only over periods without such egregious jumps.

Transactions are less adapted than quotes to this technique. A large proportion of trades in many assets occur off-exchange (about half of volume for Vodafone), at prices which do not respect the price tick. Meanwhile, on-exchange trade price data suffers from the bid-ask bounce, an unnecessary extra disturbance compared the best bid or ask.

The paper proceeds as follows: Section 2 presents the main theorem and the asymptotic limit theory. Section 3 then outlines the central Theorem (details are left to the Appendix). Section 4 assesses the estimator and asymptotic design in a simulation study. Section 5 applies and evaluates the estimator on LSE equity data. Section 6 concludes.

2 The model and main result

This section first prepares the ground for the main result, given in Section 2.4. The probability space $\{\Omega, \mathcal{F}, P\}$ is generated by three stochastic processes on \mathbb{R}^+ : W , a standard Brownian motion, V , a finite activity pure jump process, and volatility $\sigma \geq 0$. The focus of the paper will be on X , an underlying price, and Y , an observed price (e.g. bid or ask) defined thus:

$$X = W_{[X]}, \quad Y = V_{[X]}, \quad (2)$$

where $[X]$ is a stochastic process defined by

$$[X]_t = \int_0^t \sigma_u^2 du. \quad (3)$$

So W and V are subordinated by the same process, $[X]$.

Monroe (1978) shows that this specification of X includes all continuous semimartingales. Hence it is consistent with canonical models in work on QV estimation – see for example Barndorff-Nielsen *et al.* (2006). In particular, X may have leverage effects and drift, for W and σ may be dependent. The continuity of $[X]$ imposes continuity on X , and leads exactly to a spot volatility of σ_t . With some loss of generality, assume σ is uniformly bounded above and away from zero. For introductions to stochastic volatility, see reviews in Ghysels, Harvey, and Renault (1996) and Shephard (2005, Ch 1).

Clark (1973) argued earlier for this specification of X , so that the stochastic time-change might reflect how “[t]he number of individual effects added together to give the

price change during a day is ... random". Following in Clark (1973)'s direction, the observed price Y , while different to X , is modelled as arising from the same time-change. This will mean that the frequency, not the magnitude, of quote updates is increasing in the volatility, σ .

Definition 1 *Processes such as W and V which are subordinated by the time-change $[X]$ will be said to evolve in "business time", while X and Y evolve in "calendar time".*

See Oomen (2006) for more on this terminology. For some $T > 0$, only $\{Y_t : 0 \leq t \leq T\}$ is observed. Y has a random initial value. The quantity to be estimated is the QV of X over the period that Y is observed, namely $[X]_T$. As X is not observed, nor is $[X]_T$. Y deviates from X by a microstructure effect, the process ϵ , which is defined in calendar time by

$$\epsilon = Y - X. \quad (4)$$

Hence, $\epsilon = (V - W)_{[X]}$, and so $(V - W)$ is the microstructure effect viewed in business time. The following two conditions recur throughout the paper.

Definition 2 *The microstructure is stationary in business time if $(V - W)$ is stationary.*

Definition 3 *The microstructure has no leverage effects if (where $\perp\!\!\!\perp$ indicates independence)*

$$V|W \perp\!\!\!\perp \sigma|W. \quad (5)$$

While allowing leverage effects in X , this means that in business time the microstructure effect is conditionally independent of current volatility.

2.1 Constant observed jump magnitude

Assume that V , the observed price viewed in business time, is a pure jump process which only jumps by $\pm k$. So the true observed price, Y , is also. Then

$$Y_t = Y_0 + \int_0^t G_u dN_u, \quad (6)$$

where N is a simple counting process and G is an adapted process that only takes values $\pm k$ for some $k > 0$. This idea, that the observed price is a pure jump process which deviates from a fundamental price, is already present in *inter alia* Ball (1988), Gottlieb

and Kalay (1985), Li and Mykland (2006), Oomen (2006) and Zeng (2003).¹ The QV of Y , $[Y]_t$, is $\int_0^t G_u^2 dN_u = k^2 N_t$, a stochastic process.

Decompose the process N by $N = A + C$, where A and C are counting processes. The *alternation* process, A , counts the jumps in Y which have opposite sign to the one before, and the *continuation* process C counts jumps that continue in the same direction as the one before. Both are adapted to Y . The first jump in Y is unassigned. For all $i \in \mathbb{N}$ let t_i be the time of the i 'th jump in Y . Define the random sequence $Q = \{dA_{t_i} - dC_{t_i} : i \in \mathbb{N}\}$. So Q records $+1$ for an alternation and -1 for a continuation.

Definition 4 *Y has Uncorrelated Alternation if Q has zero first-order autocorrelation.*

2.2 Technical properties

Identification Assumption Given two events observable before any jumping time t_i , $H_1 \in \mathcal{F}_{t_i-}$ and $H_2 \in \mathcal{F}_{t_i-}$ such that $H_2 \subset H_1$,

$$\{ E(Y_{t_i}|H_1) = E(Y_{t_i}|H_2) \} \leftrightarrow \{ E(X_{t_i}|H_1) = E(X_{t_i}|H_2) \}. \quad (7)$$

Thus, if H_2 adds (no) new information to H_1 concerning the likely direction of Y 's next jump, it adds something (nothing) new about the level of X .

Definition 5 *Buy-sell symmetry* *If $(V - V_0, W)$ and $-(V - V_0, W)$ have identical distribution, then the microstructure is buy-sell symmetric.*

Even if buying and selling had identical dynamics, the behavior of a single quote, say the best bid, might differ when moving upwards when compared to the spread-widening downwards direction. But when trading is on a penny, no quote change widens the spread (other than perhaps very briefly), and buy-sell symmetry is more acceptable.

Definition 6 *Let the sequence Π be given by*

$$\Pi = \left\{ \left(\frac{[X]_{t_i} - [X]_{t_{(i-1)}}}{E([X]_{t_i} - [X]_{t_{(i-1)}})}, \frac{Q_i + 1}{2} \right) : i \in \mathbb{N} \right\}. \quad (8)$$

The left hand term here is the elapsed QV in X between the $(i - 1)$ th and i th jumps in Y , once de-averaged.

¹It explains two related effects. First, if prices are pure jump processes, then it is clear that Bipower Variation, the statistic introduced in Barndorff-Nielsen and Shephard (2004), converges to zero with finer sampling. Second, studying quotes data, Hansen and Lunde (2006) find that RV can be downwards-biased for QV. They show this implies negative covariation between efficient returns and noise. A pure jump process accounts for this mechanically, as pointed out in Bandi and Russell (2006a).

2.3 Asymptotic limit theory

A long sample leads the time series econometrician to a thought experiment where the sample is of “infinite” length. Of course, in practice the data is finite and so this provides an approximation. Similarly, high frequency market microstructure data invites the double asymptotic theory that, given an underlying price, the microstructure had evolved “infinitely” fast, with “infinitely” small jumps. For example, in Delattre and Jacod (1997), a diffusion process is observed very frequently with a very small rounding error. The current asymptotic theory is closely related, with the key difference that Y is fully and continuously observed. So – as explored in Hansen, Large, and Lunde (2006) – the full DGP, not only the extent and quality of observation, changes in the limit.

Suppose that the microstructure is stationary and has no leverage effects. So the probability measure admits the following factorization:

$$P = P_{V|W} \times P_{\sigma|W} \times P_W, \quad (9)$$

where $P_{V|W}$ and $P_{\sigma|W}$ are the conditional distributions, and P_W is the marginal. An element of the state space, $\omega \in \Omega$, may be written $\omega = (v, w, \sigma^*)$, where v , w and σ^* are the realizations in ω of V , W and σ respectively. I use a scaling parameter, $\alpha \in \mathbb{R}^+$, and let $\alpha \downarrow 0$. Define $w^{[\alpha]}$ by

$$w_t^{[\alpha]} := \frac{1}{\alpha} w_{\alpha^2 t}, \quad (10)$$

and define $v^{[\alpha]}$ similarly. So, for $\alpha < 1$, the functional $w \rightarrow w^{[\alpha]}$ slows but normalizes w so that $W^{[\alpha]}$ is also standard Brownian motion. Define a new measure $P_{V|W}^\alpha$ by:

$$P_{V|W}^\alpha(v, w) := P_{V|W}(v^{[\alpha]}, w^{[\alpha]}). \quad (11)$$

For *fixed* $v^{[\alpha]}$ in the support of P , the size of jumps in v , as well as the intervening durations, decline indefinitely as $\alpha \downarrow 0$. The asymptotic theory approximates P in (9) by $\lim_{\alpha \downarrow 0} \{P^\alpha\}$, where $P^\alpha = (P_{V|W}^\alpha \times P_{\sigma|W} \times P_W)$.

2.4 The main result

Theorem 2.1 *Consider the model $\{\Omega, \mathcal{F}, P^\alpha\}$. Suppose that*

- (A) *Y has Uncorrelated Alternation,*
- (B) *Y always jumps by a constant $\pm k$,*
- (C) *ϵ has no leverage effects, is stationary in business time, and is ergodic.*

(D) Y always jumps towards X , and

(E) The Identification Assumption and Buy-Sell Symmetry hold.

Condition on $[X]_T$, so that T is a random time. Then the following limit theory applies:

$$\lim_{\alpha \rightarrow 0} \sqrt{N_T} \left(\frac{[\hat{X}]_T}{[X]_T} - 1 \right) \sim N(0, U M U'), \quad \text{where} \quad [\hat{X}]_T = k^2 N_T \frac{C_T}{A_T} \quad (\text{or } 0 \text{ if } A_T = 0). \quad (12)$$

Here U is $\left(1, \frac{(1+R)^2}{R}\right)$; M is the long-run variance matrix of Π ; and R is the ratio $\frac{[X]_T}{E[Y]_T}$.

Proof. Section 3 provides the proof of this Theorem. ■

Figure 1 shows a process satisfying the Theorem's assumptions.

A feasible limit theory when volatility is constant Proposition 3.4 will show that R may be estimated by C_T/A_T . However, Theorem 2.1's asymptotic limit theory is still infeasible because Π is not observed. Nevertheless, if one is willing to assume that the spot variance does not change much within the day, the following approach provides a useful approximate way to characterize the limiting standard errors:

The elapsed QV in X between jumps at t_i and t_{i-1} is given by

$$[X]_{t_i} - [X]_{t_{i-1}} = \sigma^2(t_i - t_{i-1}), \quad (13)$$

and, de-averaged,

$$\frac{[X]_{t_i} - [X]_{t_{i-1}}}{E([X]_{t_i} - [X]_{t_{i-1}})} = \frac{(t_i - t_{i-1})}{T} E(N_T). \quad (14)$$

Substituting N_T for $E(N_T)$ gives an estimate $\hat{\Pi}$, on which the Newey and West (1987) method, and other long-run variance estimation techniques, can be used to estimate M .

Discussion of the result The result is semi-parametric because it does not refer to the dynamics or the intensity of N . The proposed estimator is easy to calculate. It can be viewed as arising from applying a scaling correction, C_T/A_T (which may be more or less than 1), to naive Realized Volatility at high frequency, $[Y]_T$. So it has a close statistical relationship to Realized Volatility. Many jumps are indicative of high volatility unless most of them are alternations, a possibility which the observed proportion of alternations to continuations provides a means to account for. Since it has no fixed observation frequency, the statistic does not encounter systematic biases due to intraday seasonality.

Discussion of the assumptions and theory Assumptions (A) and (B) can be tested empirically. (A) states that the likelihood that a jump is an alternation does not depend on whether the last jump was. It may be tested via a regression of Q on itself lagged. Section 5 goes on to treat cases where (B) and (A) do not hold directly using rounding techniques. Assumptions (C), (D) and (E) cannot be tested. (C) does not preclude leverage effects in X . (D) rules out transitory increases to $|\epsilon|$ through noise- or other trading. Unless the bid and ask simultaneously jump away from their underlying diffusions this would involve a change in the bid-ask spread: but the bid-ask spread is almost always constant when trading is on a penny. (E) rules out observed dynamics that do not reflect the underlying price at all.

Related asymptotic theories have conditioned on σ , equivalently on the process $[X]$. Here however, only $[X]_T$, the elapsed QV in X over the period $[0, T]$, is given.

3 Proof of Theorem 2.1

Throughout the proof, the model will be conditioned on the object of scientific interest, $[X]_T$. This implies that T is a random time. Furthermore, without loss of generality assume that $E[\epsilon_t] = 0$, so that the observed price Y has undergone a vertical shift leaving its increments unchanged.

Definition 7 Let R be the ratio

$$R = \frac{[X]_T}{E[Y]_T}. \quad (15)$$

Under Assumption (C), R is invariant to $[X]_T$.

Proposition 3.1 Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. The error just before the i 'th jump is ϵ_{t_i-} . Taking the ergodic expectation, for all i

$$E[|\epsilon_{t_i-}|] = \frac{k}{2} [R + 1]. \quad (16)$$

Proof. See Appendix A. ■

Discussion Proposition 3.1 provides a unbiased estimate of $|\epsilon_{t_i-}|$ while under Assumption (D) the direction of the jump at t_i gives the sign of ϵ_{t_i-} . Combining these, an unbiased estimate of ϵ_{t_i-} itself is available. Equally, X_{t_i} can be estimated without bias

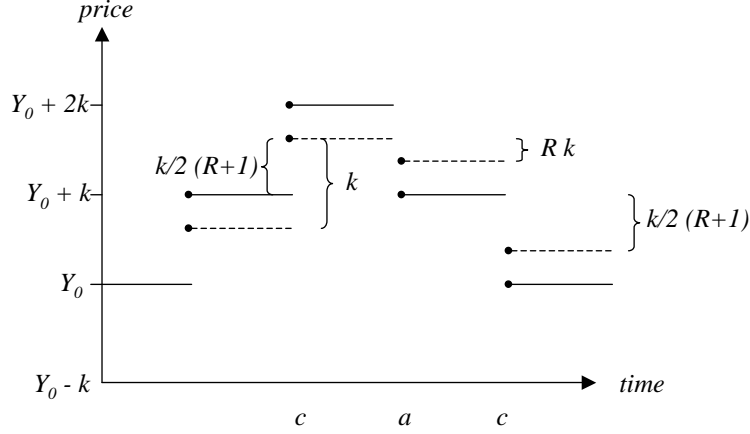


Figure 2: The solid line shows Y , while the dashed line shows Z . The letters on the time axis indicate if the jump is an alternation or a continuation. The diagram illustrates the relative contribution to the QV of Z by alternations and continuations.

by adding or subtracting $E[|\epsilon_{t_i-}|]$ to/from Y_{t_i-} , depending on the direction Y jumps at t_i . The next definition gives the name Z to this conditional estimation process, which is illustrated in Figure 2. In line with intuition, if $R \approx 0$, such as if X is almost constant, then Y jumps between $X \pm \frac{k}{2}$.

Definition 8 For each of Y 's jumping times, t_i , define Z_{t_i} by

$$Z_{t_i} = E[X_{t_i} | Y_{t_i}, G_{t_i}, R]. \quad (17)$$

(Recall that $G_{t_i} = \pm k$ is the jump in Y at t_i .) Extend the sequence $\{Z_{t_i} : i \in \mathbb{N}\}$ rightwards to a càdlàg pure jump process Z .

Note that Z is not observed because R is not observed. The evolution of Z is described in Figure 2, which also illustrates the following lemma.

Lemma 3.2 The Quadratic Variation process for Z , denoted $[Z]$, is a linear combination of the processes A and C given by $[Z] = k^2(C + R^2 A)$.

Proof. When Y jumps by continuing in the same direction as the last jump, Z jumps by k . When Y jumps by alternating in direction, Z jumps by Rk . This follows from simple calculation, and is shown in Figure 2. The QV of Z is the sum of its squared jumps. ■

Definition 9 A process S has Ideal Error if $E[[S]_T] = [X]_T$.

Proposition 3.3 *Suppose that Assumptions (B), (C), (D) and (E) of Theorem 2.1 hold. Uncorrelated Alternation then implies that Z has Ideal Error.*

Proof. See Appendix B. ■

Uncorrelated Alternation may be tested simply by regressing Q linearly on itself lagged, and testing that the regressor is significant.

Proposition 3.4 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then, conditional on $[X]_T$,*

$$E[A_T R - C_T] = 0, \quad (18)$$

and R has the Method of Moments estimator

$$\hat{R} = \frac{C_T}{A_T}. \quad (19)$$

(Define $\hat{R} = 0$ if $C_T = A_T = 0$).

Proof. See Appendix C. ■

So, recalling that $[X]_T = R E[Y]_T$, the proposed estimator of $[X]_T$ is

$$[\hat{X}]_T = \hat{R}[Y]_T. \quad (20)$$

Definition 10 *Denote by \hat{Z} the estimate of the process Z constructed by replacing R with \hat{R} in (17).²*

Straightforwardly,

$$\hat{R}[Y]_T = [\hat{Z}]_T. \quad (21)$$

The final proposition in this section provides the asymptotic limit theory for this estimator, proving its consistency.

Proposition 3.5 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then the following limit theory applies:*

$$\lim_{\alpha \rightarrow 0} \sqrt{N_T} \left(\frac{\hat{R}[Y]_T}{[X]_T} - 1 \right) \sim N(0, U M U'). \quad (22)$$

Proof. See Appendix D. Note that Π is stationary since the microstructure is stationary in business time. ■

²Under the assumptions of Theorem 2.1, first constructing \hat{Z} provides a better estimate of X , the underlying price, than using Y directly. In this sense, \hat{Z} thus constructed is a useful “filter” for Y . The QV of \hat{Z} , $[\hat{Z}]_T$, is identical to the Alternation Estimator in the model. However, if in practice the data contains some jumps greater than the price tick size, they normally diverge. For some purposes $[\hat{Z}]_T$ may be preferable: a relevant case is when large jumps in price are expected.

	k (tick)	χ^2 (noise)	ρ	mean sampling interval, 1/m	Quantities averaged across all runs					
					'Naïve' realized variance, $RV_{1/m}$	N_T	C_T	A_T	C_T/A_T	Alternation Estimator
Rounding	0.10	-	-	10 sec	3.9	386	80	306	0.26	1.016
	0.10	-	-	1 min	2.6	153	63	91	0.70	1.073
	0.05	-	-	10 sec	2.0	766	270	497	0.54	1.042
	0.05	-	-	1 min	2.2	246	214	33	n/a	n/a
Independent noise with rounding	0.10	0.00001	-	10 sec	4.0	395	81	314	0.26	1.021
	0.10	0.00001	-	1 min	2.6	154	63	91	0.70	1.074
	0.05	0.00001	-	10 sec	2.0	783	273	510	0.54	1.050
	0.05	0.00001	-	1 min	2.2	247	214	33	n/a	n/a
	0.10	0.0001	-	10 sec	4.7	467	87	381	0.23	1.067
	0.10	0.0001	-	1 min	2.7	158	64	94	0.69	1.091
	0.05	0.0001	-	10 sec	2.4	914	305	609	0.50	1.144
	0.05	0.0001	-	1 min	2.2	251	221	30	n/a	n/a
Sluggish	0.10	-	0.08	-	1.6	156	61	96	0.64	0.997
Rounding	0.05	-	0.04	-	1.5	591	238	353	0.68	0.999
	k (tick)	χ^2 (noise)	ρ		sqrt(UMU/N_T) mean of all runs	Coverage of 1		Test of Unc. Alt.		Proportion of jumps > k
						0.95	0.99	0.05	0.01	
Rounding	0.10	-	-	10 sec	0.12	0.953	0.989	0.053	0.010	0.0%
	0.10	-	-	1 min	0.17	0.900	0.959	0.053	0.012	2.3%
	0.05	-	-	10 sec	0.08	0.908	0.970	0.059	0.014	0.6%
	0.05	-	-	1 min	n/a	0.004	0.009	0.261	0.111	24.3%
Independent noise with rounding	0.10	0.00001	-	10 sec	0.12	0.949	0.986	0.053	0.011	0.0%
	0.10	0.00001	-	1 min	0.17	0.902	0.957	0.055	0.011	2.4%
	0.05	0.00001	-	10 sec	0.08	0.893	0.962	0.066	0.015	0.8%
	0.05	0.00001	-	1 min	n/a	0.005	0.009	0.259	0.106	24.5%
	0.10	0.0001	-	10 sec	0.11	0.895	0.963	0.090	0.019	0.0%
	0.10	0.0001	-	1 min	0.17	0.881	0.950	0.052	0.011	2.8%
	0.05	0.0001	-	10 sec	0.07	0.490	0.693	0.137	0.042	2.2%
	0.05	0.0001	-	1 min	n/a	0.002	0.005	0.243	0.096	26.1%
Sluggish	0.10	-	0.08	-	0.17	0.956	0.989	0.053	0.012	0.0%
Rounding	0.05	-	0.04	-	0.09	0.953	0.992	0.051	0.010	0.0%

Table 1: Reports of simulations of the proposed estimator.

Proof of Theorem 2.1 Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. (Then Z may be constructed as in Figure 2.) If in addition Assumptions (A) and (E) hold, then Proposition 3.3 shows that Z has Ideal Error. Therefore Propositions 3.4 and 3.5 apply and the Theorem follows.

4 Simulation assessment

This Section assess the estimator and its asymptotic theory in simulations. Two widely-employed DGPs are considered, adapted so the observed price practically always jumps by k . In both, the underlying price X is specified as in Section 2. They are 1) Independent Noise with Rounding, where as in Li and Mykland (2006) X is observed with error and then rounded-down to a multiple of k ,

$$Y = k \lfloor (X + u)/k \rfloor, \quad (23)$$

with $u \perp\!\!\!\perp X$ and $u \sim NID(0, \chi^2)$; and 2) Rounding, which is a case of 1) when $\chi = 0$. These differ from this paper's DGP, and so help to evaluate the robustness of the

estimator. Both DGPs have $[Y]_T$ of unbounded expectation, so are unrealistic in this setting of full continuous observation. As was pointed out in Gottlieb and Kalay (1985), even without other noise, a rounded-off Itô process has an unbounded QV whenever it crosses a rounding threshold, for with probability 1 it crosses back and forth infinitely more times in the next instant.

This is remedied by sampling a finite $m \in \mathbb{N}$ times over the day. It is convenient to sample evenly in business time. A third DGP is simulated, where Y has finite QV and so may be sampled continuously: called here 3) Sluggish Rounding.³

Definition 11 *Sluggish Rounding. For $\rho > \frac{k}{2} > 0$, define $SR_X^{\rho,k}$ to be the process which jumps towards (possibly ‘beyond’) X by amount k whenever $|SR_X - X| \geq \rho$. The initial condition $SR_{X,0}^{\rho,k}$ is distributed such that $(SR_X - X)$ is stationary.*

The constraint $\rho > \frac{k}{2}$ is required so $SR_X^{\rho,k}$ has finite expected QV. In the case where $\rho = k$, $SR_X^{\rho,k}$ jumps to exactly the value of X whenever X reaches $SR_X^{\rho,k} \pm k$.

Proposition 4.1 *Suppose that $Y = SR_X^{\rho,k}$. Then Assumptions (A), (B), (C), (D) and (E) of Theorem 2.1 hold. By Proposition 3.1, $\rho = \frac{k}{2}[R + 1]$. Furthermore, Π is an i.i.d. sequence and*

$$UMU' = \frac{2}{3R}(1 + 4R + 2R^2). \quad (24)$$

Therefore UMU' may be estimated consistently by replacing R in (24) with $\hat{R} = C_T/A_T$.

Proof. See Appendix E. ■

The three models are calibrated various ways following the simulation study of some moving-average, kernel and time-scale based estimators in Hansen, Large, and Lunde (2006). As there, $[X]_T$ is set to 1, and T is normalized to average 6.5 hours, a typical financial market ‘trading day’. Their smaller values of k and χ^2 are excluded. All reported statistics are invariant to the distribution of σ and any leverage effects, which were therefore left unspecified. An Euler discretization broke up each day, i.e. run, into $6.5 \times 60 \times 60 \times 4$ intervals (averaging 4 per second). There were 20,000 runs.

The results are presented in Table 1. The upper panel shows the proposed estimator’s average value across all runs. The lower panel in Table 1 describes specification testing and inference. It reports rejection frequencies of first-order autocorrelation tests on Q , at

³I am grateful to Peter Hansen for suggesting this term.

5 and 1 per cent, alongside the proportion of jumps in Y exceeding k . Not surprisingly, the Sluggish Rounding models are the best specified (these simulations fixed $SR_{X,0}^{\rho,k} := X_0 = 0$). In addition, models where rounding is large ($k = 0.1$), and χ is zero or small, when sampled every 10 seconds, are quite well specified. By way of contrast, Li and Mykland (2006) finds that estimators based on models of additive Independent Noise are most effective when χ is large relative to k .

When calculated using Proposition 4.1, the expression $\sqrt{\frac{UMU'}{N_T}}$ estimates the standard deviation of $\frac{[\hat{X}]_T}{[X]_T}$. ‘Coverage of 1’ gives the proportion of times that 1, the truth, lies within the resulting 95 and 99 per cent confidence intervals. All the aforementioned models have good such coverage. However, models with $k = 0.05$ sampled every 10 seconds, have less desirable coverage, arising perhaps from their upwards bias of 4 to 5 per cent, but are not often rejected by the proposed specification tests.

5 Empirical implementation

This part implements the proposed estimator for Vodafone stock traded on the LSE’s electronic limit order book, SETS. Vodafone was the LSE’s most heavily traded stock (in £) in 2006. The data spans a period of 7 months from August 2004 to the end of February 2005. This period comprised 147 trading days running from 8:00am to 4:30pm, except for 24 December and 31 December, when markets closed at 12:30pm. Vodafone’s best bid (ask) was revised 17,060 (17,167) times over the sampled period, on average 116 times per day.

5.1 Specification Testing

The prices at both the bid and the ask were first tested for uncorrelated alternation in a first order autoregression of the sequence Q . Over a long sample, fluctuation in the marginal propensity to alternate may introduce spurious dependence into this autoregression. For testing, the data was therefore viewed as a succession of independent trading days, over each of which parameter stability can reasonably be expected. The trading days were prepared by excising their first 15 minutes. After-effects of the opening auction are known to produce distinctive microstructure at this time. The null hypothesis tested was that Q is i.i.d., a stricter null than is needed, but simpler to test for.

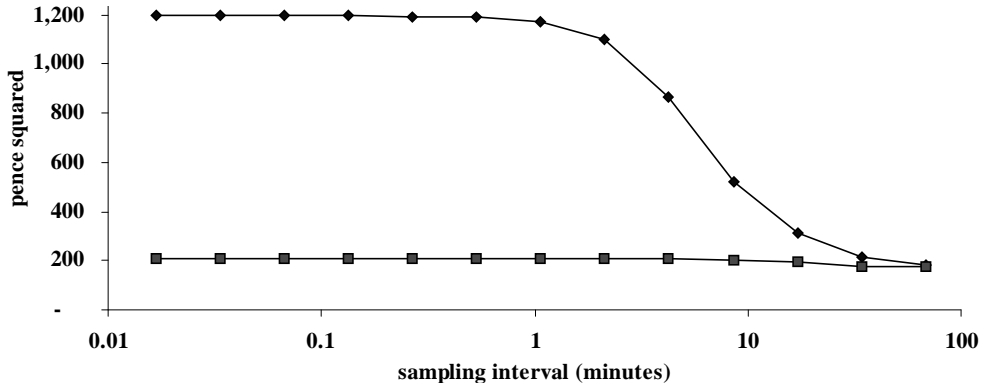


Figure 3: Volatility signature plots for Vodafone’s best bid price from August 2004 to February 2005. The diamonds and squares show $RV_{\hat{Z}}^i$, respectively for the bid and for the transformed bid, \hat{Z} , against sampling interval, $1/\zeta$.

For the best ask (bid), 14.2 (14.9) per cent of days failed an LR test for i.i.d. alternation at 5 per cent. While ideally these numbers would be close to 5 per cent, in reality a minority of days experienced episodes of abnormal market microstructure due to large price jumps, news announcements, options due dates, etc. To study this further, days were broken at 12pm into the morning and the afternoon, producing 294 periods. For the best ask (bid), 6.8 (8.5) per cent of periods now failed the LR test at 5 per cent. The test’s rejection frequencies are much improved, suggesting that an abnormal episode in the microstructure is typically brief: it does not cause both halves of a trading day to be rejected separately. As a result of its tight spread, Vodafone lacks resiliency dynamics which can induce lagged autocorrelation in quoted prices, see Degryse, de Jong, van Ravenswaaij, and Wuyts (2005) and Large (2007). This helps account for this finding of uncorrelated alternation.

Finally, only 0.5 per cent of jumps in the best bid or ask exceeded the price tick size. In conclusion, the model is found to be mis-specified mainly during infrequent brief interludes. The results of the next Section suggest that these interludes do not unduly prejudice the procedure.

5.2 Results

The estimator was calculated for each day of the sample. To study its bias, Figure 3 shows for the current data, volatility signature plots (see Andersen, Bollerslev, Diebold, and Labys 2000) of Y , and of the process \hat{Z} as given in Definition 10. Six days (among

the 14.2 per cent failing the last part's test) were excluded, since they contained large jumps in price. These were Christmas Eve, New Year's Eve, and the third Fridays in November '04, December '04, January '05 and February '05. Under the assumptions of Proposition 3.3, passing a test for i.i.d. alternation implies that Z has Ideal Error. Hence loosely, this test can be interpreted as sufficient for the hypothesis that the volatility signature plot of Z is flat. Inspection of Figure 3 suggests this may at least be so of \hat{Z} .

Definition 12 Let RV_ζ^i be the Realized Variance (or, RV) of the bid sampled at a frequency ζ on the i th observed day. This is the sum of squared changes in that price on day i between successive times in the sequence $\{0, \zeta, 2\zeta, \dots, \zeta \lfloor T/\zeta \rfloor\}$.

We will be interested in the series $\{RV_{30 \text{ min}}^i : i = 1, 2, \dots\}$, $\{RV_{15 \text{ min}}^i\}$, $\{RV_{5 \text{ min}}^i\}$, and $\{RV_{1 \text{ min}}^i\}$. In this notation the observed QV of the best bid is $\{RV_{0+}^i\}$. Figure 3 illustrates these quantities' upwards bias when viewed as estimators of underlying QV. Only at sampling intervals ≥ 30 minutes does the upwards bias due to market microstructure effects become moderate.

5.3 Forecasting assessment

To distinguish it from alternatives, the proposed estimator will now be referred to as the Alternation Estimator. On the i th day it is written Alt^i . This section uses the Vodafone best bid data (excluding the six aforementioned days) in a simple forecasting assessment of Alt^i . I follow Andersen, Bollerslev, and Diebold (2006) in turning to the HAR-RV model of Corsi (2003). Volatility is proxied both by $\{RV_{30 \text{ min}}^i\}$, and by Alt^i . The results suggests first, that the Alternation Estimator is better forecast than $\{RV_{30 \text{ min}}^i\}$. Second, forecasts of volatility proxies are improved by using either lagged values 1) of the Alternation Estimator or 2) of the biased high-frequency Realized Volatilities: $\{RV_{5 \text{ min}}^i\}$ and $\{RV_{1 \text{ min}}^i\}$. In-sample OLS regressions were used, which provide useful comparisons even in this short sample.

The dependence of $RV_{30 \text{ min}}^i$ and Alt^i on lagged variables is assessed in the models

$$\begin{aligned}
 RV_{30 \text{ min}}^i \text{ or } Alt^i = & \beta_0 + \beta_D RV_\zeta^{i-1} + \beta_W \sum_{j=1}^5 RV_\zeta^{i-j} + \beta_M \sum_{j=1}^{22} RV_\zeta^{i-j} \\
 & + \chi_D Alt^{i-1} + \chi_W \sum_{j=1}^5 Alt^{i-j} + \chi_M \sum_{j=1}^{22} Alt^{i-j} + \varepsilon_i,
 \end{aligned}$$

Explained variable	Alt		30-min		Alt		30-min		Alt		30-min		Alt		30-min	
Const.	1.17	1.03	1.86	1.67	1.91	0.69	3.17	1.49	0.66	-0.78	2.55	0.51	0.23	0.13	0.88	0.69
	0.53	0.54	0.67	0.67	0.85	0.70	1.05	0.88	1.98	1.44	2.48	1.79	0.62	0.61	0.78	0.77
Day	0.35		0.10		0.23		0.10		0.32		0.29		0.23		0.33	
Alternation	0.19		0.24		0.18		0.22		0.17		0.21		0.18		0.22	
Estimator	-0.04		-0.08		-0.12		-0.21		-0.02		-0.03		0.03		0.09	
Week	0.08		0.10		0.08		0.10		0.06		0.08		0.07		0.09	
lagged terms	0.07		0.12		0.10		0.15		0.02		0.03		-0.01		-0.02	
Month	0.04		0.05		0.04		0.05		0.03		0.03		0.03		0.03	
Joint significance of lagged alternation terms	0.03		0.14		0.06		0.03		0.20		0.63		0.40		0.10	
	30-min		30-min		15-min		15-min		5-min		5-min		1-min		1-min	
Day	-0.08	0.16	-0.03	0.07	0.01	0.20	-0.06	0.07	-0.04	0.12	-0.16	-0.01	0.01	0.05	-0.07	-0.01
	0.15	0.09	0.19	0.11	0.15	0.09	0.19	0.11	0.11	0.07	0.14	0.09	0.04	0.03	0.05	0.03
Week	0.00	-0.03	0.06	0.01	0.07	0.00	0.16	0.03	-0.01	-0.02	0.02	0.01	-0.02	-0.01	-0.03	-0.01
	0.07	0.04	0.08	0.05	0.07	0.04	0.08	0.04	0.04	0.03	0.06	0.04	0.02	0.01	0.02	0.01
Month	-0.05	0.01	-0.11	0.00	-0.07	0.01	-0.12	-0.01	0.00	0.02	-0.01	0.01	0.01	0.01	0.01	0.01
	0.04	0.02	0.05	0.02	0.03	0.01	0.04	0.02	0.02	0.01	0.03	0.02	0.01	0.00	0.01	0.00
Joint significance of lagged RV terms	0.32	0.22	0.08	0.87	0.23	0.06	0.03	0.58	0.97	0.12	0.37	0.91	0.49	0.02	0.14	0.40
Joint sig. of all regressors	0.03	0.22	0.27	0.87	0.02	0.06	0.08	0.58	0.11	0.12	0.71	0.91	0.05	0.02	0.21	0.40
Specification tests																
Normal residuals	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Autocorrelated residuals	0.38	0.01	0.63	0.54	0.37	0.18	0.65	0.80	0.28	0.05	0.68	0.34	0.33	0.15	0.81	0.22
Heteroskedastic residuals	0.73	0.83	0.41	0.91	0.59	0.61	0.21	0.62	0.95	0.83	0.93	0.92	0.20	0.18	0.28	0.21

Table 2: Regressions of two proxies for Vodafone’s QV against lagged values of the Alternation Estimator and RV_{ζ}^i . Standard errors are reported below estimates. No Newey-West-type corrections were made, but specification tests were performed on the residuals. Bold type indicates significance at 5 per cent. All p-values are given in italics.

where the regressors are independent of ε_i , and the ε_i are i.d. and homoskedastic of mean zero. The frequency, ζ , is set to four values: 30, 15, 5 and 1 minutes. The coefficients β_D , β_W and β_M describe the effect of the last day’s, week’s and month’s RV respectively. The coefficients χ_D , χ_W and χ_M do likewise for Alt^i . In the construction of weekly and monthly quantities, appropriate linear adjustments were made to allow for public holidays. There were 119 daily observations. Repeats of all regressions excluded lagged values of Alt^i .

The results are reported in Table 2. Except when $\zeta = 5 \text{ min}$, Alt^i shows significant dependence on lagged terms (at 5 per cent) while $RV_{30 \text{ min}}^i$ shows none throughout. Forecasting Alt^i , when $\zeta = 30 \text{ min}$ or 15 min , lagged values of RV_{ζ} are jointly insignificant. However, adding lagged values of Alt^i gives significant regressions.

5.4 Extension to markets that do not trade on a penny

Vodafone is one of the only equities on the LSE which trades on a penny. Elsewhere, the model is mis-specified. GlaxoSmithKline (GSK) provides an example of this: over

	Per cent of half-days failing spec. test at 5 per cent	Jumps per day	Cont. / Alt.	Estimated daily QV (pence ²)
Bid				
rounded to the nearest 2 p	3.7	210	0.19	156
sluggishly rounded to 2 p	7.8	61	0.62	151
Ask				
rounded to the nearest 2 p	6.1	211	0.19	158
sluggishly rounded to 2 p	6.5	60	0.65	155
Mid-quote				
rounded to the nearest 1.5p	7.8	278	0.25	156
sluggishly rounded to 1.5p	8.2	77	0.89	155

Table 3: Results of specification testing and estimation for GSK quotes, rounded.

the same 147 days as Vodafone, the mean bid-ask spread was 1.15 pence, but the tick size was 1 pence. Jumps by more than the price tick are correspondingly more prevalent than for Vodafone, representing 4.9 (4.6) per cent of changes in the best bid (ask). To make this data applicable, initial preparation is required. I propose two techniques.

- First, the quote may be rounded down (or up) to the nearest even (or odd) multiple of the price tick.

- Second, the quote may be “sluggishly rounded”: specifically, study $SR_Y^{\rho, 2k}$ in place of the observed data, Y . It is practical to set $\rho = 2k$.

Both these may result in processes that mostly contain jumps of $2k$, making them amenable to the current method. The mid-quote, whose price increment is half the price tick size, can also be (sluggishly) rounded. A larger multiple of k than 2 can be used.

Specification testing and results For each observed day, GSK’s bid, ask and mid-quote were separately prepared using rounding and sluggish rounding. The results of specification testing and estimation are in Table 3. With the same provisos as for the Vodafone data, all the methods of preparation produce fairly well specified models. As documented in Table 3, although the six methods result in different numbers of jumps per day, and substantially differing propensities to alternate, they imply very similar estimates of underlying QV. In applications, it would be advisable to average all six.

6 Conclusion

This paper views the observed price as a pure jump process whose deviations from an underlying stochastic process are stationary in business time. Noting that on many markets the amount by which quotes jump is constant, it proposes an estimator for the underlying price's QV which scales down the quoted price's observed QV by a factor that takes into account its propensity to alternate. Under conditions, the estimator is consistent in an appropriate asymptotic theory that is confirmed in calibrated simulations. Simple rounding techniques widen the range of applicable markets. The estimator is shown to be valid for two UK equities. Analysis of its bias and use in forecasting produces favorable results. Future research could extend the results in three important ways: to account for varying jump sizes in the market microstructure, to treat large discontinuities in the underlying price, and to estimate the covariation between assets.

References

- Aït-Sahalia, Y. (2002). Telling from discrete data whether the underlying continuous-time model is a diffusion. *Journal of Finance* 57, 2075–2112.
- Aït-Sahalia, Y., P. A. Mykland, and L. Zhang (2005). How often to sample a continuous-time process in the presence of market microstructure noise. *Review of Financial Studies* 18, 351–416.
- Andersen, T. G., T. Bollerslev, and F. Diebold (2006). Roughing it up: including jump components in the measurement, modeling and forecasting of return volatility. Forthcoming, *Review of Economics and Statistics*.
- Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2000). Great realizations. *Risk* 13, 105–108.
- Andersen, T. G., T. Bollerslev, and N. Meddahi (2004). Analytic evaluation of volatility forecasts. *International Economic Review* 45, 1079–1110.
- Ball, C. A. (1988). Estimation bias induced by discrete security prices. *Journal of Finance* 43, 845–865.
- Bandi, F. M. and J. R. Russell (2006a). Comment on ‘realized variance and market microstructure noise’ by Hansen and Lunde. *Journal of Business and Economic*

Statistics 24, 167–173.

- Bandi, F. M. and J. R. Russell (2006b). Separating market microstructure noise from volatility. *Journal of Financial Economics* 79, 655–692.
- Barndorff-Nielsen, O., P. Hansen, A. Lunde, and N. Shephard (2006). Designing realized kernels to measure the ex-post variation of equity prices in the presence of noise. Unpublished Paper: Nuffield College, Oxford.
- Barndorff-Nielsen, O. and N. Shephard (2002). Econometric analysis of realised volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society B* 64, 253–280.
- Barndorff-Nielsen, O. and N. Shephard (2004). Power and bipower variation with stochastic volatility and jumps (with discussion). *Journal of Financial Econometrics* 2, 1–48.
- Barndorff-Nielsen, O. and N. Shephard (2006). Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics* 4, 1–30.
- Borodin, A. N. and P. Salminen (1996). *Handbook of Brownian Motion Facts and Formulae*. Basel: Birkhuser.
- Clark, P. K. (1973). A subordinated stochastic process model with finite variance for speculative prices. *Econometrica* 41, 135–155.
- Corsi, F. (2003). A simple long memory model of realized volatility. Manuscript, University of Southern Switzerland.
- Curci, G. and F. Corsi (2005). Discrete sine transform approach for realized volatility measurement. Unpublished Manuscript, University of Southern Switzerland.
- Degryse, H., F. de Jong, M. van Ravenswaaij, and G. Wuyts (2005). Aggressive orders and the resiliency of a limit order market. *Review of Finance* 9, 201–242.
- Delattre, S. and J. Jacod (1997). A central limit theorem for normalized functions of the increments of a diffusion process, in the presence of round-off errors. *Bernoulli* 3, 1–28.
- Engle, R. F. and J. R. Russell (2005). A discrete-state continuous-time model of financial transactions prices: the ACM-ACD model. Forthcoming, *Journal of Business*

and Economic Statistics.

- Ghysels, E., A. Harvey, and E. Renault (1996). *Stochastic Volatility*. In C. R. Rao and G. S. Maddala (Eds.), *Statistical Methods in Finance*, pp 119-191. Amsterdam: North-Holland.
- Gottlieb, G. and A. Kalay (1985). Implications of discreteness of observed stock prices. *Journal of Finance* 40, 135–153.
- Hansen, P., J. Large, and A. Lunde (2006). Moving average-based estimators of integrated variance. Forthcoming, *Econometric Reviews*.
- Hansen, P. and A. Lunde (2006). Realized variance and market microstructure noise. *Journal of Business and Economic Statistics* 24, 127–281. The 2005 Invited Address with Comments and Rejoinder.
- Harris, L. (1994). Minimum price variations, discrete bid-ask spreads, and quotation sizes. *Review of Financial Studies* 7, 149–178.
- Large, J. (2007). Measuring the resiliency of an electronic limit order book. *Journal of Financial Markets* 10, 1–25.
- Li, Y. and P. A. Mykland (2006). Are volatility estimators robust with respect to modeling assumptions. TR #564, Department of Statistics, University of Chicago.
- Monroe, I. (1978). Processes that can be embedded in Brownian motion. *The Annals of Probability* 6, 42–56.
- Newey, W. K. and K. West (1987). A simple, positive definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica* 55, 703–708.
- Oomen, R. A. C. (2006). Properties of bias-corrected realized variance under alternative sampling schemes. *Journal of Financial Econometrics* 3, 555–577.
- Shephard, N. (2005). *Stochastic Volatility: Selected Readings*. Oxford: Oxford University Press.
- Zeng, Y. (2003). A partially observed model for micromovement of asset prices with Bayes estimation via filtering. *Mathematical Finance* 13, 411–444.
- Zhang, L., P. Mykland, and Y. Aït-Sahalia (2005). A tale of two timescales: Determining integrated volatility with noisy high-frequency data. *Journal of the American Statistical Association* 100, 1394–1411.

Zhou, B. (1996). High-frequency data and volatility in foreign-exchange rates. *Journal of Business and Economic Statistics* 14, 45–52.

A Proof of Proposition 3.1

Let $u = V - W$ be the microstructure effect in business time. The proof equates the variance of u at (i.e. just after) two consecutive jumps, say the second and third, at (random) times t_2 and t_3 . Define $\bar{\lambda}$ as the average intensity of jumping in business time. Define $\{w_i\}$ and $\{v_i\}$ as the sequences of random increments in W and V :

$$w_i = W_{t_i} - W_{t_{i-1}}, \quad v_i = V_{t_i} - V_{t_{i-1}}, \quad (25)$$

so that v_3 is equal to the jump at t_3 , i.e. $\pm k$, and $\{w_i\}$ are i.i.d. of known variance: $w_i \sim N(0, t_i - t_{i-1})$. It then follows that $u_{t_i} = u_{t_{i-1}} + v_i - w_i$. So,

$$E[u_{t_3}^2] = E[(u_{t_2} + v_3 - w_3)^2] \quad (26)$$

$$= E[u_{t_2}^2 + v_3^2 + w_3^2 - 2w_3u_{t_2} + 2v_3u_{t_2} - 2v_3w_3] \quad (27)$$

$$= E[u_{t_2}^2 + v_3^2 + w_3^2 - 2w_3u_{t_2} + 2v_3(u_{t_2} - w_3)]. \quad (28)$$

But by Assumptions (B) and (D), v_3 is $-k \operatorname{sign}(u_{t_2} - w_3)$. Furthermore, $E(w_i^2)$ is $1/\bar{\lambda}$. Further, as W is a martingale, $E[w_3|u_{t_2}] = 0$, and so $E[w_3u_{t_2}] = 0$. So, (28) is

$$E[u_{t_2}^2] + k^2 + 1/\bar{\lambda} - 2kE[|u_{t_2} - w_3|]. \quad (29)$$

Moreover, $(u_{t_2} - w_3)$ is u_{t_3-} , the right limit of u before the jump at t_3 . As u is stationary, we may equate $E[u_{t_2}^2]$ and $E[u_{t_3}^2]$ to obtain the equality $E[|u_{t_3-}|] = \frac{k}{2} [\frac{1}{k^2\bar{\lambda}} + 1]$. But, conditional on $[X]_T$, $E[Y]_T = k^2\bar{\lambda}[X]_T$. As one could equally have looked at any two successive jumps, the proposition follows.

B Proof of Proposition 3.3

First suppose that Y has Ideal Error. Then $Z = Y$ has Ideal Error trivially. Now, and for the rest of the proof, assume that Y does not have Ideal Error. If Q has first lag autocorrelation of zero then it is easily checked that $E(G_{t_{i+1}}|G_{t_i}) = E(G_{t_{i+1}}|G_{t_i}, G_{t_{i-1}})$. Therefore, by the Identification Assumption, $E(\epsilon_{t_{i+1}}|G_{t_i}) = E(\epsilon_{t_{i+1}}|G_{t_i}, G_{t_{i-1}})$. But then, as no jumps occurred between t_i and t_{i-1} , $E(\epsilon_{t_i}|G_{t_i}) = E(\epsilon_{t_i}|G_{t_i}, G_{t_{i-1}})$. The Proposition now follows from Corollary B.1.

Corollary B.1 *Assume Assumptions (B), (C) and (D) of Theorem 2.1, and that Y does not have Ideal Error. Then Z has Ideal Error iff at each jump, timed t_i , $i > 1$,*

$$E(\epsilon_{t_i}|G_{t_i}) = E(\epsilon_{t_i}|G_{t_i}, G_{t_{(i-1)}}). \quad (30)$$

Proof. If Z has Ideal Error, then by Lemma B.2, for all t ,

$$E(Z_t - X_t | \text{the last two jumps in } Y \text{ went up, then down}) = 0. \quad (31)$$

So, conditional on the two jumps in Y before t going up, then down $E(Y_t - X_t) = Y_t - Z_t$. So, $E(\epsilon_t | \text{last 2 jumps in } Y \text{ went up, then down}) = E(\epsilon_t | \text{last jump in } Y \text{ went down})$. The proposition now follows by the buy-sell symmetry of the model, considering exhaustively the four cases where prior to t : the last 2 jumps in Y went up, then down; the last 2 jumps in Y went up, then up; the last 2 jumps in Y went down, then up; and the last 2 jumps in Y went down, then down. ■

So Z has Ideal Error when conditioning not only on the last jump, but also on the one before, leaves unchanged the best estimate of X_t given Y_t .

Lemma B.2 *Assume Assumptions (B), (C) and (D) of Theorem 2.1. Then for any t ,*

$$E[Z]_T - [X]_T = 2(R-1)E[Y]_T p_A E\left(\frac{Z_t - X_t}{k} \mid \text{the last two jumps in } Y \text{ went up, then down}\right), \quad (32)$$

where p_A is the probability that a jump is an alternation.

Proof. See Appendix F. ■

C Proof of Proposition 3.4

Case where Y has Ideal Error Then $R = 1$. By Proposition 3.1, the expected value of $|\epsilon_t|$ just before a jump is k . Therefore, the expected value of ϵ_t conditional on Y just after an upwards jump is 0. By (E), Y then has equal probability of jumping up as down. As Q is uncorrelated, the probability that any given jump is an alternation is 0.5. Hence $E[A_T - C_T] = 0$.

Case where Y doesn't have Ideal Error This case contains an important argument. Condition on $[X]_T$. Z has Ideal Error if

$$[X]_T = E([Z]_T) = E(k^2(C_T + A_T R^2)). \quad (33)$$

Also, R is defined by $[X]_T = RE([Y]_T) = E(k^2 R(C_T + A_T))$. Subtracting and dividing by k^2 , we therefore have the moment condition,

$$E[(C_T + A_T R^2) - R(C_T + A_T)] = 0. \quad (34)$$

Or, factorizing, $(R - 1) E[(A_T R - C_T)] = 0$. Since Y does not have Ideal Error, $R \neq 1$. Divide through by $(R - 1)$ to obtain $E[A_T R - C_T] = 0$.

D Proof of Proposition 3.5

Condition on $[X]_T$, and define a business time, S , by $S = [X]_T$. Let $\{s_1, s_2, s_3 \dots\}$ be the business times of the observed jumps in Y , i.e. the times of the jumps in V . Then Π reduces to $\{ (\bar{\lambda}(s_i - s_{(i-1)}), \frac{1+Q_i}{2}) : i \in \mathbb{N} \}$. The RH fraction is 1 when Y alternates, and 0 when Y continues. We take the limit as $\alpha \rightarrow 0$ of $P_{V|W}(v^{[\alpha]}, w^{[\alpha]})$. Consider $(V^{[\alpha]}, W^{[\alpha]})$. This pair's distribution is unchanged as $\alpha \downarrow 0$, but $V^{[\alpha]}$ is observed for a longer time, until time S/α^2 . For given α , N_T is the number of jumps in $V^{[\alpha]}$ before time S/α^2 . As $\alpha \downarrow 0$, $N_T \rightarrow \infty$ with probability 1. As S/α^2 is the sum of the durations between the observed jumps (in business time), plus the time after the last jump in the sample, by a standard CLT,

$$\lim_{\alpha \rightarrow 0} \sqrt{N_T} \left(\left(\frac{\frac{S}{\alpha^2} \bar{\lambda}}{\frac{A_T}{N_T}} \right) - \binom{1}{p_A} \right) \sim N(0, M), \quad (35)$$

where p_A is the probability that a jump is an alternation. Let $f : (x, y) \rightarrow (1 - y)/xy$. Then f has positive derivative in $\mathbb{R}^+ \times \mathbb{R}^+$, so by the Delta Method,

$$\lim_{\alpha \rightarrow 0} \sqrt{N_T} \left(\frac{N_T C_T \alpha^2}{S \bar{\lambda} A_T} - \frac{(1 - p_A)}{p_A} \right) \sim N(0, df' M df), \quad (36)$$

where df is evaluated at $(1, p_A)'$. By Proposition 3.4 $\frac{(1-p_A)}{p_A} = R$, so $df|_{(1, p_A)'} = -RU'$ (after algebra) and

$$\lim_{\alpha \rightarrow 0} \sqrt{N_T} \left(\frac{N_T (\alpha k)^2 C_T}{S k^2 \bar{\lambda} R A_T} - 1 \right) \sim N(0, U M U'). \quad (37)$$

Finally, recall: $N_T (\alpha k)^2 = [Y]_T$; $k^2 \bar{\lambda} R = 1$; $S = [X]_T$, and substitute into (37).

E Proof of Proposition 4.1

Condition on $[X]_T$. In business time, the duration between jumps is the time taken for a standard Brownian motion to exit the interval $(-k, Rk)$. Hence Q and Π are

i.i.d. The probability that the level Rk is reached before the level $-k$ is known to be $\frac{k}{k+Rk}$, or $1/(1+R)$. This also equals p_A . The expected time to the first exit is Rk^2 , which is therefore also $E[s_i - s_{(i-1)}]$. The following formulae are derived from results in Borodin and Salminen (1996): The variance of the time between jumps is $\frac{1}{3}Rk^4(1+R^2)$. So $\text{var}[\bar{\lambda}(s_i - s_{(i-1)})]$, is $\frac{1+R^2}{3R}$. And $E[s_i - s_{(i-1)}|Q_i = 1] = \frac{2Rk^2(2R+1)}{3(R+1)}$. Therefore $\text{cov}[Q_i, \bar{\lambda}(s_i - s_{(i-1)})] = -\frac{1-R}{3(1+R)}$. Also $\text{var}[Q_i] = \frac{R}{(1+R)^2}$. These give all entries of M . UMU' follows.

F Proof of Lemma B.2

Let $S = [X]_T$ be known. Let \tilde{Z} be $Z_{[X]^{-1}}$, i.e. \tilde{Z} is Z as it evolves in business time. Define $\eta_s = \tilde{Z}_s - W_s$. So, η is the error in Z , as it evolves in business time. As $V - W$ is stationary, η is too. Let it follows the differential equation $d\eta_s = H_s dN'_s - dW_s$, so that N' is the driving counting process of V . Say that the adapted intensity process of this counting process is λ . H is a process which takes value $\pm k$, and $\pm Rk$, depending on whether V is alternating or continuing, up or down. Then, $E((\eta_s + d\eta_s)^2) = E(\eta_s^2)$. Therefore, $E(\eta_s^2 + 2\eta_s d\eta_s + d\eta_s^2) = E(\eta_s^2)$. And so, $-2E(\eta_s d\eta_s) = E(d\eta_s^2)$. Or,

$$-2E(\eta_s(H_s dN'_s - dW_s)) = E((H_s dN'_s - dW_s)^2). \quad (38)$$

So $-2E(\eta_s H_s \lambda_s) ds = E(H_s^2 \lambda_s) ds + ds$. Multiplying by $-S/ds$ and adding on a constant,

$$2SE((\eta_s + H_s)H_s \lambda_s) = SE(H_s^2 \lambda_s) - S. \quad (39)$$

But the left hand side of this is $2S\bar{\lambda}E(\eta_s H_s | \text{jump at } t)$, while the right hand side is $E[Z]_T - [X]_T$. Putting this together, given the buy-sell symmetry of η ,

$$E[Z]_T - [X]_T = 2S\bar{\lambda}E(\eta_s H_s | Y \text{ jumped up at } t). \quad (40)$$

But, $E[Y]_T = S\bar{\lambda}k^2$, so $E[Z]_T - [X]_T = \frac{2}{k^2}E[Y]_T E(\eta_s H_s | Y \text{ jumped up at } t)$. So, distinguishing alternation from continuation in order to extract $|H|$ from this,

$$\begin{aligned} & \frac{2}{k^2}E[Y]_T \{p_A Rk E(\eta_s | Y \text{ alternated up at } s) + (1 - p_A)k E(\eta_s | Y \text{ continued up at } s)\} \\ &= \frac{2}{k}E[Y]_T E(\eta_s | Y \text{ jumped up at } s) + \frac{2}{k}E[Y]_T p_A (R - 1) E(\eta_s | Y \text{ alternated up at } s) \\ &= 0 + \frac{2}{k}E[Y]_T p_A (R - 1) E(\eta_s | Y \text{ alternated up at } s). \end{aligned}$$

Therefore, $E[Z]_T - [X]_T = -\frac{2}{k}E[Y]_T (1 - R)p_A E(\eta_s | Y \text{ alternated up at } s)$. Or,

$$E[Z]_T - [X]_T = \frac{2}{k}E[Y]_T (1 - R)p_A E(\eta_s | Y \text{ alternated down at } s). \quad (41)$$