SOME PROBLEMS IN FUNCTIONAL ANALYSIS

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ABSTRACT

The first part of this thesis is a study of strongly continuous (or weakly) continuous one parameter semigroups and groups of bounded linear maps on Banach spaces, their perturbation and scattering, paying particular attention to operator algebras.

CHAPTER 1

We develop the spectral theory of derivations and groups of automorphisms on operator algebras, and study their relationships. The spectrum of a bounded strongly continuous one parameter group on a Banach space is characterised in terms of the spectrum of its infinitesimal generator. The relationship between Arveson's and Borchers' approach to the spectral theorey of groups of *-automorphisms on operator algebras is brought out, and we consider the problem of unitary implementation.

CHAPTER 2

Following S.C.Lin's reflexive Banach space theory [36], we consider the perturbation, similarity and scattering of strongly continuous one parameter semigroups and groups on Banach spaces, using a time dependant approach and a strong type of smoothness, with particular reference to C*-algebras and preduals of W*-algebras. Some results regarding time evolution in quasi local algebras are used to derive scattering of quasi free evolution groups in the CAR algebra by inner perturbations.

CHAPTER 3

The previous chapter leads us to study scattering of ultraweakly continuous groups of σ-weakly continuous linear maps on von Neumann algebras, by smooth perturbations of a certain weak type. We give a technique for

handling groups of *-automorphisms.

CHAPTER 4

We study time dependant perturbations and scattering of strongly continuous one parameter groups on a Banach space E. The problem is raised to the higher space $L^p(\mathbb{R};E)$ $(1 \leq p < \infty)$ or $C_0(\mathbb{R};E)$, where the perturbation is made time independant and the methods of Chapter 2 apply. We characterise certain strongly continuous groups on the higher space in terms of propogators on the lower space, and show how their scattering is related. Some examples are given.

The second part of the thesis investigates the structure of completely positive maps.

CHAPTER 5

Schwartz type inequalities for n-positive linear mappings on *-algebras are obtained. We demonstrate why the bounded completely positive linear maps for a Banach *-algebra with approximate identity, and in particular for a C*-algebra, should be regarded as higher order state spaces. A theorem of F. and M. Riesz is generalised to give a sufficient condition for the covariance of certain representations of a C*-algebra relative to a one parameter group of *-automorphisms. A completely positive analogue of Tomiyama's theorem regarding the singularity of conditional expectations on W*-algebras is obtained. We characterise the completely positive linear mappings on the CCR algebra. Operator algebra analogues of Nagy's hilbert space dilations and Stroescu's Banach space dilations are obtained. Ths infinitesimal generators of strongly continuous one parameter semi-groups and groups of linear maps with various positivity properties are studied.

CHAPTER 6

Unbounded completely positive linear maps or operator valued weights on C*-algebras are defined. We construct the Stinespring representation for an unbounded completely positive linear map α. We study the natural order structure for such maps, and following van Daele [64] for scalar valued weights, when α has dense domain, we construct a largest operator valued weight α_o majorised by α, and with the property that it is the upper envelope of continuous completely positive maps. Following Combes [7] we study the quasi equivalence, equivalence and type of the Stinespring representation associated with operator valued weights. Following van Daele [64], we study an unbounded completely positive map α which is invariant under a group of *-automorphisms and has dense domain, and construct a G-invariant projection map ϕ of the set $\mathfrak{F}$ of continuous completely positive maps dominated by α, onto the set $\mathfrak{F}_o$ of G invariant elements of $\mathfrak{F}$. This is used to derive various properties of the envelope of $\mathfrak{F}_o$. Asymptotically abelian systems and their ergodic bounded completely positive maps are studied. Observations are made regarding the possible classification of the spectral type of strongly continuous one parameter groups of *-automorphisms of a C*-algebra which possess an invariant operator valued weight.

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NOTATION

We first remark that all vector spaces are over the complex numbers, unless otherwise stated. If X and Y are Banach spaces, the following notation will be used throughout this thesis :

$\mathcal{C}(X,Y)$ = The closed linear operators form X into Y.

$\mathcal{C}_o(X,Y)$ = The closed densely defined linear operators from X into Y.

$B(X,Y)$ = The bounded linear operators from X into Y.

$C(X,Y)$ = The compact linear operators from X into Y.

$\mathcal{C}(X) = \mathcal{C}(X,X)$, $\mathcal{C}_o(X) = \mathcal{C}_o(X,X)$ etc.

$\mathcal{C}_+(X) = \{ T \in \mathcal{C}_o(X) :$ $-iT$ generates a strongly continuous semigroup e^{-itT} , $t \geq o \}$

$\mathcal{C}_-(X) = -\mathcal{C}_+(X)$.

$X \odot Y$ will denote the algebraic tensor product. Completions are written as follows :

The projective (respectively injective) tensor products are denoted by $X \check{\otimes} Y$ (respectively $X \hat{\otimes} Y$). If X,Y are hilbert spaces (respectively C*-algebras, W*-algebras), $X \otimes Y$ (respectively $X \otimes_{\alpha_o} Y$, $X \bar{\otimes} Y$) denote the hilbert space tensor product (respectively C*-tensor product [54, 1.22.8] , W*-tensor product [54, 1.22.10] .

CHAPTER 1.

This chapter is an attempt to develop some of the basic spectral theory concerning derivations and groups of automorphisms on operator algebras.

In the first section we show how certain derivations on Banach algebras and the infinitesimal generators of strongly continuous one parameter semi groups of homomorphisms are related. In §1.2, we give some spectral results for derivations. The spectral theory for automorphism groups on Banach spaces, and in particular operator algebras, has been introduced in [5,6,11,45,46] . One of our aims is to show how [5] and [6] are related. In §1.3, we recall some of the terminology and results of [5] , whilst in the fourth section we show how Borchers' approach for the group of integers can be generalised to more arbitrary locally compact abelian topological groups. In the course of §1.5 we demonstrate how the basic tools and concepts of Arveson and Borchers are related, and amongst other results, consider the unifying problem of implementation of a group of *-automorphisms on a von Neumann algebra by a unitary representation on the underlying hilbert space.

We remark that in §6.7, we will have some more comments to make regarding the spectral theory of one parameter groups of *-automorphisms on operator algebras.

§1.1

Our first objective is to show the intimate relationship between derivations and automorphisms on Banach algebras.

DEFINITION 1.1.1.

A derivation Z on an algebra A, is a linear map defined on D_Z into A, where $D_Z = D(Z) = \text{Domain of } Z$, satisfying

i) D_Z is a subalgebra of A.

ii) $Z(fg) = Zf.g + f.Zg$ for all $f,g \in D_Z$.

We remark that if Z is a derivation on an algebra A, and $D_n = D(Z^n)$, $n = 1,2,\ldots$, then the Leibnitz rule implies that D_n and $D_\infty = \bigcap_{n=1}^{\infty} D_n$ are also subalgebras.

Suppose that α_t is a strongly continuous, one parameter group on a Banach algebra A, with infinitesimal generator Z. Since Z has a dense set of analytic elements, it is clear that α_t is a group of homomorphisms iff Z is a derivation. This is well known for norm continuous groups, e.g.[54,(§4.1)]. The result still holds for strongly continuous semigroups, but we need be more careful :

PROPOSITION 1.1.2.

Let A be a Banach algebra, and $\{\alpha_t : t \geqslant 0\}$ a strongly continuous semigroup of bounded linear maps on A with infinitesimal generator Z. Then α_t is a semigroup of homomorphisms, iff Z is a derivation.

PROOF.

For the non trivial implication, we use an integral equation technique for solving Banach space valued differential equations. This method is suggested by the work of H.Tanabe [61] . Suppose Z is a derivation. Let $f,g \in D_Z$, and put $x(t) = \alpha_t(fg) - \alpha_t(f)\alpha_t(g)$ for $0 \leqslant t < \infty$. Then x is continuously differentiable, $x(t) \in D_Z$ for $0 \leqslant t < \infty$, and

$$x'(t) = Zx(t) \qquad \text{in } \mathbb{R}^+.$$

Moreover if $h \in D_Z$, then $s \to \alpha_{t-s}(h)$ is continuously differentiable in $0 \leqslant s \leqslant t$ and

$$\frac{\partial}{\partial s}\alpha_{t-s}(h) = -\alpha_{t-s}Z(h).$$

Hence, for $t \geqslant 0$,

$$x(t) - \alpha_t x(0) = \int_0^t \frac{\partial}{\partial\sigma}\alpha_{t-\sigma}x(\sigma)\,d\sigma$$

$$= -\int_0^t \alpha_{t-\sigma}Zx(\sigma)\,d\sigma + \int_0^t \alpha_{t-\sigma}\left(\frac{d}{d\sigma}x(\sigma)\right)d\sigma$$

$= 0.$

But $x(0) = 0$, hence $x(t) = 0$. By density of D_Z in A, it follows that α_t is a homomorphism for each t.

§ 1.2.

Suppose a and b are elements of an algebra A with identity. Define $\delta : A \rightarrow A$ by

$$\delta x = ax - xb \quad \text{for } x \in A.$$

Inthis section we will consider the relation between the spectrum of δ and those of a and b.

The following result is widely used, but we include a proof for completeness.

THEOREM 1.2.1.

Let A be a Banach algebra with identity. Take a,b in A, and define $\delta \in B(A)$ by $\delta x = ax - xb$, for $x \in A$.

Then $\sigma\delta \subseteq \sigma a - \sigma b$.

PROOF.

$\delta = La - Rb$, where L,R are the left and right regular representations respectively of A in B(A). Since $\sigma(a) \supseteq \sigma(La)$, and $\sigma(b) \supseteq \sigma(Rb)$, it is enough to show that if S,T are two commuting elements of a Banach algebra B with identity, that $\sigma(S-T) \subseteq \sigma(S) - \sigma(T)$.

Let M be the closed subalgebra of B generated by $(S-\lambda)^{-1}$, $(T-\mu)^{-1}$ as λ, μ range over $\rho(S)$, $\rho(T)$ respectively. Then M is commutative and contains S,T and 1 (the identity of B). If $m \in M$, then $\sigma_M(m) \supseteq \sigma_B(m)$, and moreover $\sigma_M(S) = \sigma_B(S)$, and $\sigma_M(T) = \sigma_B(T)$.

Also $\sigma_M(m) = \{\phi(m)\}$ as ϕ ranges over the non zero homomorphisms $\mathcal{S}$ on M.

Hence

$$\sigma_B(S - T) \subseteq \sigma_M(S - T) = \{\phi(S - T) : \phi \in \mathcal{S}\}$$
$$= \{\phi S - \phi T : \phi \in \mathcal{S}\} \subseteq \sigma_M(S) - \sigma_M(T) = \sigma_B(S) - \sigma_B(T).$$

This proves our assertion.

With the notation of the above theorem, if $\lambda \notin \sigma(La) - \sigma(Rb)$, we have found it possible to express the resolvent $R(\lambda,\partial)$ as a double contour integral in terms of $R(w,La)$ and $R(z,Rb)$ using a several variable functional calculus. This is similar to [48,(1.7.3)] using a single integral.

If $A = B(X)$, all the bounded linear maps on a Banach space X, then $\sigma(\partial) = \sigma(a) - \sigma(b)$ (see [51,p265]). However it is not true in general that $\sigma(\partial) = \sigma(a) - \sigma(b)$, or even if we restrictour attention to C*-algebras. For example, if A is a commutative C*-algebra, take any $a \in A$, then the zero derivation ∂ can be written

$$\partial x = ax - xa \qquad x \in A.$$

Then $\sigma(\partial) = 0$, but $\sigma(a) - \sigma(a)$ can be made larger. However we can show the following. Connes has a similar result for the spectrum of an inner *-automorphism on a factor von Neumann algebra [11,Lemme 2.3.10(a)] .

THEOREM 1.2.2.

Let M be a von Neumann algebra and a factor, with a,b normal elements in M. If $\partial \in B(M)$ is defined by

$$\partial x = ax - xb \quad \text{for } x \in M,$$

then $\sigma\partial = \sigma a - \sigma b$.

PROOF.

Take $\lambda_1 \in \sigma(a)$, $\lambda_2 \in \sigma(b)$. Then there exist spectral projections e_1, e_2 in M, such that $e_1, e_2 \neq 0$, and

$$\|(a - \lambda_1)e_1\| < \epsilon \quad , \quad \|(b - \lambda_2)e_2\| < \epsilon .$$

Since M is a factor, and e_1, e_2 are non zero, there exists an $x \in M$, such that $e_1xe_2 \neq 0$. Take $\|e_1xe_2\| = 1$. Then :

$$[\partial - (\lambda_1 - \lambda_2)](e_1xe_2) = (a - \lambda_1)e_1xe_2 - e_1xe_2(b - \lambda_2).$$

Thus

$$\|(\partial - \lambda_1 + \lambda_2)(e_1xe_2)\| \leq \|(a - \lambda_1)e_1.e_1xe_2\| + \|e_1xe_2.e_2(b - \lambda_2)\| < 2\epsilon .$$

Hence $\lambda_1 - \lambda_2 \in \sigma(\partial)$.

§1.3.

We now want a detailed spectral analysis of derivations and automorphism groups on operator algebras. In this section we explain Arveson's tools [5].

An idea of the method of approach is obtained from the following attack at the hilbert space level [40,45]. We suppose U is a strongly continuous representation of a locally compact abelian topological group G by unitary operators on a hilbert space $\mathfrak{H}$. Then we can lift U to a representation π_U of $L^1(G)$ as follows :

$$\pi_U(f)x = \int f(g)\, U_g(x)\, dg \qquad x \in \mathfrak{H}, \quad f \in L^1(G),$$

and where dg is the Haar measure on G. Then transform this representation to the transform algebra $L^1(G)^\wedge$ by

$$\pi^U(\hat{f}) = \pi_U(f) \qquad \text{for} \quad f \in L^1(G),$$

and where $\hat{f}$ is the inverse Fourier transform of f. This representation is norm continuous, $\|\pi^U \hat{f}\| \leq \|\hat{f}\|$,
hence extendable from $L^1(G)^\wedge$ to all continuous functions on $\hat{G}$, and hence to all Baire functions on $\hat{G}$. By considering characteristic functions, it is deducable that there exists a projection valued measure P_E on $\hat{G}$ such that

$$U_g = \int \langle \overline{g, \chi} \rangle \, dP_\chi \quad \forall g \in G \qquad \text{(Stone's theorem).} \quad (1.3.1)$$

We now see that

$$\pi_U(f)x = \int \hat{f}(\chi)\, dP_\chi \qquad f \in L^1(G), \quad x \in \mathfrak{H}.$$

In particular, if G = R, there is a self adjoint operator A such that $U_t = e^{iAt}$, and we conclude

(1.3.2) $\sigma(A) = \{ s \in \hat{\mathbb{R}} \ : \ \hat{f}(s)=0, \ \forall f \in L^1(\mathbb{R}) \text{ such that } \pi_U(f)=0 \}$

(1.3.3) For a closed interval $I \subseteq \hat{\mathbb{R}}$,

$P_I = \{ x \in \mathfrak{H} : \pi_U(f)x = 0, \forall f \in L^1(\mathbb{R}) \text{ such that } \hat{f} \text{ vanishes on a nhood of } I \}$.

As in [5] and [6], we will replace the reflexive hilbert space $\mathfrak{H}$ by other spaces e.g. C*- and W*-algebras, study the concepts analogous to (1.3.2) and (1.3.3), and sometimes reform the representation as in (1.3.1)

We take a complex Banach space X, and a linear subapace $X_* \subseteq X^*$, which need not be closed, such that

(1.3.4) X_* determines the norm on X, i.e. for all $x \in X$,

$$\|x\| = \sup\{|\phi(x)| : \phi \in X_*,\ \|\phi\| \leq 1\} .$$

(1.3.5) The weakly closed convex hull of every weakly compact set in X is weakly compact, where the weak topology refers to the $\sigma(X,X_*)$ topology on X.

Hypothesis 1.3.5 is a type of weak completeness although X is rarely complete in the weak topology. If $X_* = X^*$, then 1.3.5 follows from a theorem of Krein and Smulian, while if X is the dual of a Banach space, and X_* is the space of all weak*-continuous linear functionals on X, then 1.3.5 follows from Alaoglu's theorem. The former will correspond to the C*-situation, the latter to the W*-one.

If we let $B_W(X)$ be the weakly continuous operators on X, then we can define a representation.

DEFINITION. 1.3.6.

A representation of a locally compact abelian topological group G is a homomorphism α from G into the group of invertible elements of $B_W(X)$ such that $\sup_{g \in G} \|\alpha_g\| < \infty$,
and for each $x \in X$, $g \to \alpha_g(x)$ is weakly (i.e. $\sigma(X,X_*)$) continuous.

In order to lift up a representation of G to one of $L^1(G)$, we need the following proposition of Arveson.

PROPOSITION 1.3.7. [5]

Let α be a representation of G on X, and suppose (X,X_*) satisfy 1.3.4 and 1.3.5. Let μ be a complex measure on G of finite variation. Then

$$Ax = \int U_g(x)\, d\mu(g) \qquad \text{for } x \in X,$$

defines a bounded linear operator A on X. Moreover, if relative to the

X-topology on X_*, the closed convex hull of every compact set in X_* is compact, then A is weakly contiouous.

Thus 1.3.4 and 1.3.5 allow us to define the bounded representation π_α of $L^1(G)$ as follows:

$$\pi_\alpha(f)x = \int f(g)\, \alpha_g(x)\, dg \qquad \text{where } f \in L^1(G),\ x \in X.$$

DEFINITION 1.3.8.

a) The spectrum of α, written sp α is defined as the hull of the ideal

$$\ker \pi_\alpha = \{ f \in L^1(G) : \pi_\alpha(f) = 0 \}.$$

b) For each $x \in X$, the spectrum of x, written $sp_\alpha x$ is defined as the hull of the ideal $\{ f \in L^1(G) : \pi_\alpha(f)x = 0 \}$.

c) For each closed subset E contained in $\hat{G}$, we define the spectral subspace $X^\alpha(E)$ to consist of those x in X for which $sp_\alpha x \subseteq E$.

Note that in 1.3.8 a) and b), we could replace $L^1(G)$ by dense subspaces of $L^1(G)$, and still get the same spectra; e.g. if $G = \mathbb{R}^n$, we could use $\mathcal{S}(\mathbb{R}^n)$ or $C_0^\infty(\mathbb{R}^n)$.

It is well known [11] that if E is a closed subset of $\hat{G}$, that:

$sp\, \alpha = \{ \tau \in G : X^\alpha(\tau+V) \neq 0\ \forall$ compact nhds V of 0 in $\hat{G} \}$.

$X^\alpha(E) = \{ x \in X : \pi_\alpha(f)x = 0\ \forall\ f \in L^1(G)$ s.t. $\hat{f}$ vanishes on a nhd of $E \}$.

If $E = \{0\}$, then $X^\alpha\{0\} = \{x \in X : \alpha_g x = x \quad \forall g \in G \}$, the fixed point set.

So if $sp_\alpha x \subseteq 0$, then $\alpha_g x = x \quad \forall g \in G$.

§1.4.

The prime problem is to determine when a representation of a group G as *-automorphisms of a von Neumann algebra, can be implemented by a unitary representation on the underlying hilbert space. Arveson has produced the following criterion for the real line.

THEOREM 1.4.1. [5]

Let α_t be a weakly continuous one-parameter group of *-automorphisms of a von Neumann algebra M acting on a hilbert space $\mathfrak{H}$. Then the following are equivalent :

i) There exists a strongly continuous one parameter unitary group $u_t \in M$, having non negative spectrum such that

$$\alpha_t(A) = u_t A u_t^* \qquad \forall A \in M, \quad t \in R.$$

ii) $$\bigcap_{t \in \mathbb{R}} M^\alpha[t,\infty)\mathfrak{H} = \{0\} .$$

Moreover when ii) is satisfied, one may take for u, the group

$$u_t = \int_{-\infty}^{+\infty} e^{itx}\, dP(x)$$

where $P(\cdot)$ is the unique projection valued measure on M satisfying

$$[P[t,\infty)\mathfrak{H}] = \bigcap_{s<t} M^\alpha[s,\infty)\mathfrak{H} .$$

We now return to an arbitrary locally compact abelian topological group G. Let α_g be a weakly continuous group of *-automorphisms of a von Neumann algebra M, acting on a hilbert space $\mathfrak{H}$, (i.e. α is a representation of G on M,as defined in 1.3.6 , by *-automorphisms). In general, of course, an automorphism such as α_g is not necessarily inner. Borchers [6] has suggested the following classification:

DEFINITION 1.4.2 [6]

If Z denotes the center of M, and H is a closed subgroup of G, we say (M,α) belongs to the class K_H, if for all projections P in $Z \cap M^\alpha\{0\}$, that $\alpha|_{M_P}$ is inner on M_P for precisely H.

For $G = \mathbb{Z}$, the integers, we have the closed subgroups

$$H_\gamma = \{n\gamma : n \in \mathbb{Z}\} \qquad \text{for} \quad \gamma = 0,1,2,.....$$

$$H_\infty = \emptyset.$$

If α is a *-automorphism of M, then $\alpha_n = \alpha^n$ gives a representation of the integers, which Borchers decomposed as follows :

THEOREM 1.4.3. [6]

For each $\gamma = 1,2,3,\ldots ,\infty$; there exists a projection K_γ in $Z \cap M\{0\}$ such that $\left(M_{K_\gamma}, \alpha|_{M_{K_\gamma}}\right)$ belongs to class K_{H_γ} for $\gamma = 1,2,3,\ldots ,\infty$, and

$$\sum_{\gamma=1}^{\gamma=\infty} K_\gamma = 1.$$

In general, a locally compact abelian topological group G has continuously many subgroups e.g. if G is the real line $\mathbb{R}$, we have a family of closed subgroups $\{pt : t \in \mathbb{R}\}$, where $p \in \mathbb{R}^+$. We would then expect an integral decomposition. In order to avoid such complications, we will restrict ourselves at the appropriate time to M being factor. We would then hope to show that the subgroup H of G where α is inner is closed, and then perhaps implement $\alpha|_H$ by a strongly continuous unitary representation of H on $\mathfrak{H}$.

We need some more definitions :

DEFINITIONS 1.4.4.

a) For I a closed subset of $\hat{G}$, let $L^1_I = \{f \in L^1(G) : \text{support } \hat{f} \subseteq I\}$.

b) Let $M_0 = M^\alpha\{0\}$, the fixed point set, and define Z_0 to be the centre of M_0.

c) Let E be a projection in M_0, and I a closed subset of $\hat{G}$, then we denote by $\phi_I E$ the union of the ranges of

$$\{\pi_\alpha(f)xE : x \in M, f \in L^1_I\}.$$

i.e. $\phi_I E$ is the smallest projection D in M such that

$$D\pi_\alpha(f)(x)E = \pi_\alpha(f)(x)E \quad \forall x \in M, \quad f \in L^1_I.$$

The following lemma was proven by Borchers [6] for the case when the group G is the integers. The same proof extends to our general case.

LEMMA 1.4.5.

Let F be a projection in M_o, and E its central support in M_o. Then we have for all closed subsets I in $\hat{G}$:

$$\phi_I F = \phi_I E \in Z_0.$$

Again we need another definition.

DEFINITION 1.4.6. [6]

Let E_1, E_2 be projections in Z_o. Define

$$n(E_1,E_2) = \left\{ b \in \hat{G} : \exists \text{ a closed nhd } \mathcal{U} \text{ of } o \text{ in } \hat{G} \text{ s.t. } E_2 \phi_{b+\mathcal{U}} E_1 = 0 \right\} .$$

EXAMPLE 1.4.7. [6]

Suppose $G = \mathbb{Z}$, the integers and $\alpha_n x = u^n x u^{-n}$ $\forall x \in M$, where u is a unitary element in M. If E is the spectral measure for u, and I,J are closed intervals in $\hat{G} = T$, the circle group, then

$$\phi_I E(J) = E(I+J).$$

Moreover if V is a closed interval in $\hat{G} = T$, which is a neighbourhood of 0, then

$$n(E(V),E(V)) \supseteq \complement(2V).$$

So we have a family of projections E_V in M_o, satisfying

$$n(E_V,E_V) \supseteq \complement(V).$$

Then, loosely speaking, we can obtain information about an interval I in $\hat{G}$, by studying $E_I \phi_{b+\mathcal{U}} E_V$ for varying $b \in \hat{G}$, $\mathcal{U}$ closed nhd of o in $\hat{G}$, and V as before. This is our motivation. Following Borchers [6], our aim , when seeking a spectral measure, is to first find a family E_V generalising the property $n(E_V,E_V) \supseteq \complement V$, and then by considering $\bigcup_V n(E_V,E)$ for E a projection in Z_o, we shall hope to get a spectral measure.

Again, the following two theorems were proved by Borchers in [6], for the integer group $\mathbb{Z}$, and the proofs are easily adaptable to an arbitrary locally compact abelian topological group.

THEOREM 1.4.8.

Let P_o be the projections in Z_o, and consider ϕ_I, for I closed subset of $\hat{G}$, as a mapping of P_o into itself. Then it has the following properties:

a.α) $E_1 \geqslant E_2$ implies $\phi_I E_1 \geqslant \phi_I E_2$.

β) Put $E_1 \vee E_2 = E_1 + E_2 - E_1.E_2$, then we get

$$\phi_I(E_1 \vee E_2) = (\phi_I E_1) \vee (\phi_I E_2).$$

γ) Let Γ be an ordered index set and $(E_\alpha)_{\alpha\in\Gamma}$ be increasing in P_o, and let $E = \text{st}\lim\limits_\alpha E_\alpha$. Then we have

$$\phi_I E = \text{st}\lim_\alpha \phi_I E_\alpha.$$

b.α) The relation $I_1 \subseteq I_2$ implies

$$\phi_{I_1} E \leqslant \phi_{I_2} E .$$

β) Let I_1 and I_2 be two arbitrary compact subsets of $\hat{G}$, then we get

$$\phi_{I_1} \circ \phi_{I_2} E \leqslant \phi_{I_1+I_2} E.$$

c.α) If $I = \hat{G}$, then we get

$$\phi_{\hat{G}} E \in P_o \cap Z.$$

Furthermore, $\phi_{\hat{G}} E$ is the carrier of E in $P_o \cap Z$.

β) If $0 \in I^o$, $\phi_I E \geq E$.

d.α) Let $F \in P_o \cap Z$ then we get :

$$\phi_I(F.E) = F.\phi_I E .$$

β) Let $E_1, E_2 \in P_o$ be such that $E_1 \phi_I E_2 = 0$, for some closed subset I of $\hat{G}$. Then $E_2 \phi_{-I} E_1 = 0$.

THEOREM 1.4.9.

The sets $n(E_1,E_2)$ have the following properties :

a) $n(E\ ,E\ )$ is open.

b) $n(E_2,E_1) = -n(E_1,E_2)$.

c) If $E_1' \leqslant E_1$ and $E_2' \leqslant E_2$ then $n(E_1,E_2) \subseteq n(E_1',E_2')$.

d) For $F \in P_o \cap Z$, we get $n(FE_1,E_2) = n(E_1,FE_2) = n(FE_1,FE_2)$.

e) $n(E_1 \vee E_2, E_3 \vee E_4) = n(E_1,E_3) \cap n(E_1,E_4) \cap n(E_2,E_3) \cap n(E_2,E_4)$.

We shall need some deeper results regarding the sets n(,) These facts are obtained wuth the help of some work of Olesen [67].

If Ω is any subset of G, we define $R^{\alpha}(\Omega)$ to be the $\sigma(M,M_*)$ closed linear span of elements of the form $\pi_{\alpha}(f)x$ where $\hat{f}$ has support in Ω , and x runs through M.

Then [67, Proposition 2.3.4] shows that if Ω_i is any family of open subsets of G, then

$$R^{\alpha}(\cup \Omega_i) = \vee R^{\alpha}(\Omega_i) ,$$

where $\vee$ denotes the $\sigma(M,M_*)$ closed linear span of the subspaces R (Ω_i). The following characterisation of the set n(,) is an easy consequence of this.

PROPOSITION 1.4.10.

If $E_1,E_2 \in P_o$, $\mathcal{U} = n(E_1,E_2)$ is the largest open set in G for which

$$E_2R^{\alpha}(\mathcal{U})E_1 = 0.$$

This allows us to prove the following proposition, which will be useful later. It is a generalisation of [6, Lemma 3.4].

PROPOSITION 1.4.11.

Let E_1, E_2 be projections in P_o. If W is a compact neighbourhood of 0 in $\hat{G}$, let

$$n_W(E_1,E_2) = \{ \beta \in \hat{G} : \beta + W \subseteq n(E_1,E_2) \} .$$

If $E' = E_{2\ b+W}E_1 \neq 0$, where $b \notin n(E_1,E_2)$, then

$$n(E',E') \supseteq n_W(E_1,E_2) - b.$$

§1.5

In this section, we show how Arveson's and Borchers' concepts are related, and how one would proceed to construct point-fixing families, as defined in [6,(4.2)], under some general conditions. We also consider the problem of unitary implementation of representations. In the course of this work, we characterise the spectrum of a strongly continuous representation of the real line on a Banach space.

We return to an arbitrary locally compact abelian topological group G, X a Banach space, with X_* as in 1.3.4 and 1.3.5 and α a representation of G on X as defined in 1.3.6. Our first result determines the connection between Arveson's spectrum and Borchers' sets n(,).

PROPOSITION 1.5.1.

$$\mathrm{sp}\,\alpha = \complement\, n(1,1),$$

where n(1,1) denotes the set of $\lambda \in \hat{G}$, such that there exists a compact neighbourhood V of 0 in $\hat{G}$ such that $\pi_\alpha(f) = 0$, whenever $f \in L^1_{\lambda+V}$.

PROOF.

Suppose $\lambda \in \hat{G} \setminus n(1,1)$ and $\lambda \notin \mathrm{sp}\,\alpha$.

Then for some compact neighbourhood V of 0, $X^\alpha(\lambda+V) = 0$.

Take $y \in X$, and $f \in L^1(G)$ such that $\hat{f}$ has compact support in $\lambda + V$.

Then we claim $\pi_\alpha(f)y \in X^\alpha(\lambda+V)$. For let $g \in L^1(G)$, such that $\hat{g}$ vanishes on a neighbourhood of $V+\lambda$.

Then $\pi_\alpha(g)\pi_\alpha(f)y = \pi_\alpha(g*f)y$ and $(g*f)\hat{} = \hat{g}\hat{f} = 0$.

But

$$X^\alpha(\lambda+V) = \{x \in X : \pi_\alpha(g)x = 0,\ \forall g \in L^1(G) \text{ s.t. } \hat{g} \text{ vanishes on a nhd of } \lambda+V\}.$$

Hence $\pi_\alpha(f)y = 0$, i.e. $\lambda \in n(1,1)$; a contradiction.

Hence $\complement n(1,1) \subseteq \mathrm{sp}\,\alpha$.

Conversely, take $\lambda \in \mathrm{sp}\alpha$, and suppose $\lambda \in n(1,1)$.

Then for some compact nhd V of o in G, $\pi_\alpha(f) = 0$, for all $f \in L^1_{\lambda+V}$.

But there exists $f \in L^1(G)$, with support $\hat{f} \subseteq \lambda+V$, yet $\hat{f}(\lambda) \neq 0$.

Hence $\pi_\alpha(f) = 0$ and $\hat{f}(\lambda) \neq 0$. Therefore $\lambda \notin \mathrm{sp}\alpha$, a contradiction.

Thus $\mathrm{sp}\,\alpha = \complement n(1,1)$.

For completeness, we include at this point the following proposition. It has the consequence that if X is a Banach space and α_t a strongly continuous bounded representation of the real line on X, with infinitesimal generator Z, then

$$-i\sigma(Z) = \mathrm{sp}\,\alpha = \complement n(1,1) \qquad \text{(compare with 1.3.2).}$$

Olesen has recently published this for norm continuous one parameter groups of isometries [46].

PROPOSITION 1.5.2.

Let X be a Banach space, and Z the infinitesimal generator of a representation (α_t) of $\mathbb{R}$ as defined in 1.3.6 with $X_* = X^*$, (i.e. α_t is a strongly continuous one parameter bounded group on X). Let I be a compact interval in $\mathbb{R}$. Then

i) $iI \subseteq \rho(Z)$ implies that $\pi_\alpha(f) = 0$ for all $f \in L^1_I$.

ii) $\pi_\alpha(f) = 0$ for all $f \in L^1_I$ implies that $iI^o \subseteq \rho(Z)$.

PROOF.

The proof is inspired by that of [6, Lemma 5.5] .

i) Take $f \in L^1_I$. Then

$$f(t) = \int_{-\infty}^{+\infty} e^{-ita}\,\hat{f}(a)\,da \qquad \text{a.e. in } \mathbb{R}.$$

For $\delta > 0$, define

$$f^+_\delta(t) = \int_{-\infty}^{+\infty} \hat{f}(a)\,\exp(-iat-\delta t)\,da \qquad \text{for } t > 0$$

$$f^-_\delta(t) = \int_{-\infty}^{+\infty} \hat{f}(a)\,\exp(-iat+\delta t)\,da \qquad \text{for } t < 0.$$

Then $f^+_\delta(t) - f(t) = f(t)(e^{-\delta t} - 1)$ for $t > 0$,

and $f^-_\delta(t) - f(t) = f(t)(e^{+\delta t} - 1)$ for $t < 0$.

Hence by dominated convergence, there exists $\delta_0 > 0$ such that

$$\int_0^\infty |f^+_\delta(t) - f(t)|\,dt\ ,\quad \int_{-\infty}^0 |f^-_\delta(t) - f(t)|\,dt < \epsilon/2 \quad \text{for } 0 < \delta < \delta_0 .$$

Since $\|\alpha_t x\| \leq M\|x\|$ for some $M < \infty$, all $t \in R$, and

$$\pi_\alpha(f)x = \int_0^\infty f(t)\alpha_t(x)\,dt + \int_{-\infty}^0 f(t)\alpha_t(x)\,dt \quad \forall x \in X$$

we get for $0 < \delta < \delta_0$ that

$$\begin{aligned} &\|\pi_\alpha(f)x\| \\ &\leq M\|x\|\epsilon + \left\|\int_0^\infty \int \alpha_t(x)\hat{f}(a)\exp(-iat-\delta t)\,da\,dt + \int_{-\infty}^0 \int \alpha_t(x)\hat{f}(a)\exp(-iat+\delta t)\,da\,dt\right\| \\ &= M\|x\|\epsilon + \left\|\int \hat{f}(a)\int_0^\infty \exp(-iat-\delta t)\alpha_t(x)\,dt\,da + \int \hat{f}(a)\int_{-\infty}^0 \exp(-iat+\delta t)\alpha_t(x)\,dt\,da\right\| \\ &= M\|x\|\epsilon + \left\|\int_{-\infty}^\infty \hat{f}(a)\left[R(ia+\delta,Z)x - R(ia-\delta,Z)x\right]da\right\| . \end{aligned}$$

The integral tends to zero as $\delta \downarrow 0$, since $\hat{f}$ has compact support in I and $iI \subseteq \rho(Z)$. Hence $\pi_\alpha(f) = 0$.

ii) Suppose $\pi_\alpha(f) = 0 \ \forall f \in L^1_I$.

Then with the same notation, for $f \in L^1_I$, $\epsilon > 0$

$$\int_0^\infty |f^+(t) - f(t)|\,dt = \int_0^\infty |f(t)|\,|e^{-\delta t} - 1|\,dt$$

$$= \int_0^\infty (1+t^2)|f(t)|\,\frac{|e^{-\ t} - 1|}{(t^2+1)}\,dt \leq \epsilon\int_0^\infty |f(t)|^2(1+t^2)^2\,dt. \quad ,$$

for suitably small δ , independantly of f.

We now restrict ourselves to $f \in \mathscr{S}(\mathbb{R})$, such that $\hat{f}$ has support in I.

Then looking at the previous manipulations we see

$$\left\|\int_I [R(ia+\delta,Z) - R(ia-\delta,Z)]\,x\,\hat{f}(a)\,da\right\| \leq |\hat{f}|_\epsilon\|x\|\,M\ ,$$

where $|\cdot|$ is acontinuous seminorm on $\mathscr{S}(\mathbb{R})^\wedge$.

We now apply an edge of the wedge theorem [57,Theorem 2.16] and deduce that $R(\lambda,Z)$ is analytic for $\lambda \in iI^0$. Hence $iI^0 \subseteq \rho Z$.

We now retun to an arbitrary locally compact abelian topological group G, and take X to be a von Neumann algebra M, which is a factor and acts on a hilbert space $\mathfrak{H}$. X_* is the predual of M, and we consider only those representations α of G on M, which act by *-automorphisms.

DEFINITIONS 1.5.3.

a) (Borchers [6]) Define

$\hat{H}_o = \left\{ b \in \hat{G} \; : \; b+n(E_1,E_2) \subseteq n(E_1,E_2) \;\; \forall \, E_1,E_2 \text{ in } P_o \right\}$,

where as before P_o denotes the central projections in $M_o = M\{0\}$.

b) (Connes [11,Def. 2.2.1]) Define

$\Gamma(\alpha) = \bigcap_E \mathrm{sp}\left(\alpha|_{M_E}\right)$

where E ranges over the non zero projections in M_o.

We now relate these concepts.

THEOREM 1.5.4.

Suppose α is a weakly continuous representation of G by *-automorphisms on the von Neumann factor M. Then

$$\hat{H}_o = \Gamma(\alpha).$$

PROOF.

Let $\hat{H}_1 = \{b \in \hat{G} : \; b+n(E,E) \subseteq n(E,E) \;\; \forall E \in P_o\}$.

Then $\hat{H}_1 = \{b \in \hat{G} : b+\mathrm{sp}\,\alpha|_{M_E} \subseteq \mathrm{sp}\,\alpha|_{M_E} \quad \forall \, E \in P_o, \;\; E \neq 0\}$,

since $\bigcap n(E,E) = \mathrm{sp}\,\alpha|_{M_E}$ by 1.5.1.

But 1.4.8.c.ß) implies that if $E \neq 0$, then $n(E,E) \not\ni 0$.

Hence $\hat{H}_1 \subseteq \bigcap_{0 \neq E \, P_o} \mathrm{sp}\,\alpha|_{M_E} = \Gamma(\alpha)$.

Conversely, take $\lambda \in \Gamma(\alpha)$.

Now theorem 2.2.4 of [11] implies that

$$\Gamma(\alpha) + \mathrm{sp}\,\alpha = \mathrm{sp}\,\alpha \; .$$

If $0 \neq E \in P_o$, we apply this to $\left(M_E, \alpha|_{M_E}\right)$ and deduce

$$\lambda + \mathrm{sp}\,\alpha|_{M_E} \subseteq \mathrm{sp}\,\alpha|_{M_E} \; .$$

Hence $\lambda \in \hat{H}_1$, and $\Gamma(\alpha) = \hat{H}_1$.

It is clear that $\hat{H}_o \subseteq \hat{H}_1$. Suppose there exists λ in $\hat{H}_1 \setminus \hat{H}_o$. Then there exist projections E_1, E_2 in P_o such that

$$\lambda + n(E_1,E_2) \nsubseteq n(E_1,E_2).$$

So $\lambda + \mu \notin n(E_1,E_2)$ for some $\mu \in n(E_1,E_2)$.

But $n(E_1,E_2)$ is open, hence $\mu + W \subseteq n(E_1,E_2)$ for some compact nhd W of 0 in $\hat{G}$. Let $E = E_2 \, \phi_{\lambda+\mu+W} E_1$, which is non zero as $\lambda + \mu \notin n(E_1,E_2)$.

Then 1.4.11 shows that $n(E,E) \supseteq n_W(E_1,E_2) - (\lambda+\mu)$,

where $n_W(E_1,E_2) = \{ \gamma \in n(E_1,E_2) : \gamma + W \subseteq n(E_1,E_2) \}$.

Thus $-\lambda \in n(E,E)$. However 1.4.8.c.ß) implies that $0 \notin n(E,E)$, which contradicts our assumption that λ lies in $\hat{H}_1$.

Hence $\hat{H}_o = \Gamma(\alpha)$.

COROLLARY 1.5.5.

$\hat{H}_o$ is a closed subgroup of $\hat{G}$.

In order to get a spectral measure on $\hat{G}/\hat{H}_o$, we make the hypothesis that

1.5.6) $\quad sp\,\alpha / \Gamma(\alpha)$ is compact.

DEFINITION 1.5.7.

For V a compact neighbourhood of 0 in $\hat{G}$, define $\mathcal{E}_V$ to be the family of projections E in P_o such that

$$sp\,\alpha|_{M_E} \subseteq V + \Gamma(\alpha).$$

LEMMA 1.5.8.

i) $\mathcal{E}_V \neq \emptyset$, for all compact nhds V of 0 in $\hat{G}$.

ii) Given V_1, V_2 compact nhds of 0 with $V_2 \subseteq V_1$, and any $E_{V_1} \in \mathcal{E}_{V_1}$, there exists $E_{V_2} \in \mathcal{E}_{V_2}$ satisfying $E_{V_2} \leq E_{V_1}$.

PROOF.

Connes ([11, Lemma 2.3.4]) has shown that $\{ sp\,\alpha|_{M_E} + W \}$ where E runs over the non zero projections in M_o, and W over the compact nhds of 0 in $\hat{G}$, form a downward filtering family of subsets of $\hat{G}$ with intersection $\Gamma(\alpha)$.

i) Let $\beta : \hat{G} \to \hat{G}/\Gamma(\alpha)$ be the natural projection.

Then $\beta(\operatorname{sp}\alpha|_{M_E} + W)$ filter downwards to 0. Using compactness of $\operatorname{sp}\alpha/\Gamma(\alpha)$, we deduce there exists a non zero projection E in M_o, and a compact nhd W of 0 in $\hat{G}$ satisfying $\beta(\operatorname{sp}\alpha|_{M_E} + W) \subseteq \beta V$.

In particular $\operatorname{sp}\alpha|_{M_E} \subseteq V + \Gamma(\alpha)$.

Take E_V = central support of E in M_o. Then [11,Prop.2.2.2.a] says

$\operatorname{sp}\alpha|_{M_E} = \operatorname{sp}\alpha|_{M_{E_V}}$. Hence $E_V \in \mathcal{E}_V$.

ii) Connes [11,Lemma 2.3.3] has shown that

$$\Gamma\left(\alpha|_{M_{E_{V_1}}}\right) = \Gamma(\alpha).$$

Hence by using part i) on $M_{E_{V_1}}$, we get $0 \neq E'_{V_2} \leq E_{V_1}$, $E'_{V_2} \in M_o$,

and
$$\operatorname{sp}\alpha|_{M_{E'_{V_2}}} \subseteq V_2 + \Gamma\left(\alpha|_{M_{E_{V_1}}}\right) = V_2 + \Gamma(\alpha).$$

Take E_{V_2} to be the central support of E'_{V_2} in M_o.

If $\mathfrak{F}$ is a non empty family of compact neighbourhoods of 0 in $\hat{G}$, forming a base at 0, we make the following additional hypothesis on our system, for some fixed $\mathfrak{F}$.

1.5.9) For each $V \in \mathfrak{F}$, there exists E_V in $\mathcal{E}_V$, such that given V_1, V_2 in $\mathfrak{F}$, with $V_1 \subseteq V_2$, then $E_{V_1} \leq E_{V_2}$.

This condition always holds if $\mathfrak{F}$ is countable, (i.e. $\hat{G}$ first countable).

DEFINITION 1.5.10.

If $\mathfrak{F}$ is a non empty family of compact neighbourhoods of 0 in $\hat{G}$, forming a base at 0 such that 1.5.9 holds, then we define for a central projection F in M_o, the support of F to be

$$S^{\mathfrak{F}}(F) = \bigcap\left\{\bigcup n(E_V, F)\right\}.$$

THEOREM 1.5.11.

Whenever $\mathcal{F}$ is a basic family of compact nhds of 0 in $\hat{G}$, satisfying 1.5.9, $S(\cdot)$ has the following properties :

a) $S(E)$ is invariant under $\Gamma'(\alpha)$ for all E in P_o.

b) $S(E_V) \subseteq V + \Gamma'(\alpha)$ for all V in $\mathcal{F}$.

c) $S(E_1 \vee E_2) = S(E_1) \cup S(E_2)$ for all E_1, E_2 in P_o.

d) $S(\phi_{b+W} E_V) \subseteq b+V+W+\Gamma'(\alpha)$ if $b \in \hat{G}$, and $V, W \in \mathcal{F}$.

e) $S(E) = \emptyset$, $E \in P_o$ holds iff $E = 0$.

f) If $E_1 E_2 \in P_o$ and $S(E_1) \cap S(E_2) = \emptyset$, then $E_1 E_2 = 0$.

g) Let E_α be increasing in P_o and $E = \operatorname{st} \lim_\alpha E_\alpha$, then $S(E) = \text{closure} \bigcup_\alpha S(E_\alpha)$.

h) Let $K \subseteq \hat{G}$ be closed. Then there exists an unique maximal projection E^K in P_o satisfying $S(E^K) \subseteq K$, and such that for any central projection E in M_o with $(1 - E^K)E \neq 0$, it follows that $S(E)$ is not a subset of K.

PROOF.

a) is clear.

b) If $V \in \mathcal{F}$, $\complement S(E_V) = \bigcup_{W \in \mathcal{F}} n(E_W, E_V) \supseteq n(E_V, E_V)$.

c) $\complement S(E_1 \vee E_2) = \bigcup_{V \in \mathcal{F}} n(E_V, E_1 \vee E_2)$.

However $n(E_V, E_1\, E_2) = n(E_V, E_1) \cap n(E_V, E_2)$ by 1.4.9.e.

Therefore $\complement S(E_1 \vee E_2) \subseteq \left\{ \bigcup_{V \in \mathcal{F}} n(E_V, E_1) \right\} \cap \left\{ \bigcup_{V \in \mathcal{F}} n(E_V, E_2) \right\}$

$$= \complement S(E_1) \cap \complement S(E_2).$$

i.e. $S(E_1 \vee E_2) \supseteq S(E_1) \cup S(E_2)$.

If $\lambda \in \complement S(E_1) \cap \complement S(E_2)$, then there exist V, W in $\mathcal{F}$ such that

$\lambda \in n(E_V, E_1) \cap n(E_W, E_2)$.

Choose $S \in \mathcal{F}$, $S \subseteq V \wedge W$, then $\lambda \in n(E_S, E_1) \cap n(E_S, E_2) = n(E_S, E_1 \vee E_2)$.

Therefore $\lambda \in \complement S(E_1 \vee E_2)$. Hence $S(E_1 \vee E_2) = S(E_1) \cup S(E_2)$.

d) Assume $\lambda \notin b+V+W+\Gamma'(\alpha)$, which is closed as V, W are compact and $\Gamma'(\alpha)$ is closed. So $(a-U) \cap (b+V+W+\Gamma'(\alpha)) = \emptyset$ for some $U \in \mathcal{F}$.

Therefore $a-b-U-W \subseteq \complement(V+\Gamma'(\alpha)) \subseteq n(E_V, E_V)$

and $b-a+U+W \subseteq n(E_V, E_V)$.

But $E_V \phi_{-a+U} \circ \phi_{b+W} E_V \leq E_V \phi_{-a+U+b+W} E_V$, so it follows from 1.4.10 that $E_V \phi_{-a+U} \circ \phi_{b+W} E_V = 0$. Hence $-a \in n(\phi_{+b+W} E_V, E_V)$ or

$a \in n(E_V, \phi_{b+W} E_V) \subseteq \complement S(\phi_{b+W} E_V)$,

i.e. $S(\phi_{b+W} E_V) \subseteq b+W+V+\Gamma(\alpha)$.

e) If $E = 0$, then it is apparent that $SE = \emptyset$. Conversely suppose $SE = \emptyset$. Then by compactness, $\mathrm{sp}\,\alpha/\Gamma(\alpha) \subseteq \bigcup_{i=1}^{m} n(E_{V_i}, E)/\Gamma(\alpha)$ for some $V_i \in \mathcal{F}$,

Choose $W \in \mathcal{F}$ such that $W \subseteq \cap V_i$.

Then since $n(E_W, E)$ is $\Gamma(\alpha)$ invariant we have $\mathrm{sp}\,\alpha \subseteq n(E_W, E)$.

But $\mathrm{sp}\,\alpha = \complement n(1,1) \supseteq \complement n(E_W, E)$. Hence $\hat{G} = n(E_W, E)$.

The conclusion $E = 0$, follows from Proposition 1.4.10 and Theorem 1.4.8.c.α). remembering M is a factor and $E_W \neq 0$.

f) We get from c) the relation $S(E_1.E_2) \subseteq S(E_i)$ for $i=1,2$. Hence $S(E_1) \cap S(E_2) = \emptyset$ implies that $S(E_1 E_2) = \emptyset$. The conclusion follows from e).

g) From $E_\alpha \leq E$, follows $SE_\alpha \subseteq SE$. Hence closure $\bigcup_\alpha SE_\alpha \subseteq SE$.

Assume now $b \notin$ closure $\bigcup_\alpha S(E_\alpha)$. Then there exists $V \in \mathcal{F}$ such that $b+V \cap \mathrm{cl} \bigcup_\alpha S(E_\alpha) = \emptyset$. This implies $b+V \cap S(E_\alpha) = \emptyset \;\; \forall \alpha$.

Take $W \in \mathcal{F}$ such that $W+W \subseteq V$. Then by c) $S(\phi_{b+W} E_W) \subseteq b+V+\Gamma(\alpha)$.

Hence $S(\phi_{b+W} E_W) \cap S(E_\alpha) = \emptyset$. f) implies that $E_\alpha \phi_{b+W} E_W = 0, \; \forall \alpha$.

Therefore $b \in n(E_W, E)$, and $b \notin S(E)$. This gives $S(E) \subseteq$ closure $\bigcup_\alpha S(E_\alpha)$.

h)Define $\mathbb{E}_K = \{ E : SE \subseteq K \}$. Then d),g) and Zorn's lemma imply that $\mathbb{E}_K$ contains maximal element. The definition of $\mathbb{E}_K$ implies that the second statement is satisfied.

Thus under the conditions of the above theorem, we have produced a point fixing family. According to Borchers [6], is is possible to deduce unitary implementation using complex variable techniques, for the usual groups, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_n$, T, etc. If $\hat{G}/\hat{H}_0$ has an order structure, it is also possible to proceed. However, in these circumstances, it is better to consider left annihilating projections rather than bother with point fixing families ; see for instance the proofs of [5,Th.3.1], [45,Th.4.1] and [46,Th.4.2] . But for the sake of completeness, we proceed rather informally and show how under the conditions of the above theorem, we could hope to use the function E^K to implement α by a unitary representation on H, where H is the orthogonal subgroup of $\hat{H}_o$. Thus $\hat{H} = \hat{G}/\hat{H}_o$.

Suppose that S is a subset of H satisfying

1.5.12 $\quad S + S \subseteq S$

1.5.13 $\quad S \cap (-S) = \{0\}$

1.5.14 $\quad S = (S^o)^-$

Let ß be the natural projection from $\hat{G}$ onto $\hat{H}$. We assume that for $\lambda \in \hat{H}$

$$F(S+\lambda) = E^{ß^{-1}(S+\lambda)}$$

gives a projection valued measure on $\hat{H}$. Define for $h \in H$

$$u(h) = \int_{\hat{H}} \langle \chi, h\rangle dF_\chi$$

and $\quad \gamma(h)x = u(h)xu(h)^* \quad \forall\, x \in M,$

to give representations u and γ of H on $\mathfrak{H}$ and M respectively.

It is not difficult to produce a situation where

$$S\phi_{ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)} \subseteq ß^{-1}(S+\lambda+\omega) \qquad \text{for all } \lambda, \omega \in \hat{H}.$$

For example, suppose $ß^{-1}(S+\omega)$ is compact for all $\omega \in \hat{H}$.

Take $a \notin ß^{-1}(S+\lambda+\omega)$, then $(a-U) \cap ß^{-1}(S+\lambda+\omega) = \emptyset$ for some $U \in \mathcal{F}$. Then

$$a - U - ß^{-1}(S+\omega) \subseteq \complement\, ß^{-1}(S+\lambda) \subseteq \complement\, S\, E^{ß^{-1}(S+\lambda)} = \bigcup_{V\in\mathcal{F}} n(E_V, E^{ß^{-1}(S+\lambda)}).$$

Hence by compactness $a - U - ß^{-1}(S+\omega) \subseteq n(E_V, E^{ß\ (S+\lambda)})$ for some $V \in \mathcal{F}$,

i.e. $-a + U + ß^{-1}(S+\omega) \subseteq n(E^{ß^{-1}(S+\lambda)}, E_V)$.

Thus by 1.4.10 $E_V\, \phi_{-a+U+ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)} = 0$, which implies

$E_V\, \phi_{-a+U}\phi_{ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)} = 0.$ i.e. $-a \in n(\phi_{ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)}, E_V)$,

or $a \in n(E_V, \phi_{ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)})$. This means $a \notin S(\phi_{ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)})$

i.e. $S\, \phi_{ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)} \subseteq ß^{-1}(S+\omega+\lambda)$.

Hence by our maximality property 1.5.11.h

$$\phi_{ß^{-1}(S+\omega)}E^{ß^{-1}(S+\lambda)} \leq E^{ß^{-1}(S+\omega+\lambda)}.$$

It follows that $\left[1 - F(S+\lambda+\omega)\right] R^{\alpha}\left(\beta^{-1}(S+\omega)\right) F(S+\lambda) = 0,$

i.e. $A\left[F(S+\lambda)\mathfrak{h}\right] \subseteq \left[F(S+\lambda+\omega)\mathfrak{h}\right]$ for all A in $R^{\alpha}\left(\beta^{-1}(S+\omega)\right)$.

Now it is well known (e.g. [6,Cor.2]) that if $A \in M$, the following two conditions are equivalent since S is semigroup which is the closure of its interior

1.5.15. $A \in M^{\gamma}(S+\omega)$

1.5.16. $A\, F(S+\lambda)\mathfrak{h} \subseteq F(S+\lambda+\omega)\mathfrak{h}, \forall \lambda \in \hat{H}.$

Hence we deduce $R^{\alpha}\left(\beta^{-1}(S+\omega)\right) \subseteq M^{\gamma}(S+\omega) \quad \forall \omega \in \hat{H}.$ Then since S is a cone which is the closure of its interior, it is deducable from [6,Th.2.3] or [67,Prop.2.3.10] that $\alpha|_H = \gamma$ if $R^{\alpha|_H}(S+\omega) \subseteq M^{\gamma}(S+\omega)$ for all $\omega \in \hat{H}$. It follows that if

$$R^{\alpha|_H}(S+\omega) \subseteq R^{\alpha}\left(\beta^{-1}(S+\omega)\right) \qquad \forall \omega \in \hat{H}$$

(e.g. $\hat{G}=\hat{H}$) then $\alpha|_H = \gamma$.

That this would be the best possible result is seen from the following result of Connes [11,Théorème2.3.1]. This pointwise information is obtained from the result[54,Theorem4.1.19] that a *-automorphism of of M is inner when its spectrum as an element of B(M) is contained in the open right hand half plane.

THEOREM 1.5.17.

If α is a weakly continuous representation of G by * - automorphisms on a factor M, such that $\operatorname{sp}\alpha/\Gamma(\alpha)$ is compact, the orthogonal subgroup of $\Gamma(\alpha)$ is precisely the set of $g \in G$, such that there exists a unitary u (depending on g) in the centre of M_0 satisfying

$$\alpha_g x = uxu^* \qquad \text{for all } x \text{ in } M.$$

CHAPTER 2.

This chapter is concerned with perturbation and similarity of strongly continuous one parameter semigroups and groups in Banach spaces, with particular reference to C*-algebras and preduals of W*-algebras. The fundamental work on the subject, at the hilbert space level, is [29]. This was extended to reflexive Banach spaces in general by S.C.Lin in two directions; [36], a time dependant theory and [37], the stationary theory. In §2.1 we make the necessary modifications to write this time dependant theory in terms of a not necessarily reflexive, separable Banach space X. The stationary theory depends crucially on X being separable and reflexive in order to analyse the boundary values of X-valued analytic functions, (see [37,Remark2.2] and the final comments of [52,p571]), and this theory is not studied. The smoothness properties of the perturbations in §2.1 are of a strong type. Their relevance to operator algebras is studied in §2.2, whilst in §2.3 we weaken the smoothness conditions when dealing with bounded perturbations.

We also consider a particular C*-algebra, the CAR algebra of quantum theory, and derive some results regarding inner perturbations of free evolutions in §2.5. This is preceeded in §2.4 by some technicalities concerning unbounded perturbations in quasi local algebras.

2.1.

Before we extend the smoothness definitions of [29,36] to (possibly non-reflexive) separable Banach spaces, we recall that if D is any map with domain $\mathcal{D}$ in a Banach space E and codomain in another Banach space F, then the extended norm is defined by putting

$$\|Dx\| = \infty \quad \text{if } x \in X \setminus \mathcal{D} .$$

Now let X and Y be separable Banach spaces. For $1 \leq p < \infty$, we let $L^p(\mathbb{R}^+;Y)$ denote the complex Banach space of all Y-valued functions f on $\mathbb{R}^+ = [0,\infty)$ such that $f(\cdot)\|f(\cdot)\|^{p-1}$ is strongly integrable. If $1/p + 1/q = 1$, we identify $L^p(\mathbb{R}^+;Y)^*$ with the space of Y^*-valued, weakly*-measurable functions φ on $\mathbb{R}^+$ such that

$$\|\varphi\| \equiv \begin{cases} \left(\int_0^\infty \|\varphi(t)\|^q\right)^{1/q} < \infty & \text{if } q < \infty \\ \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \|\varphi(t)\| < \infty & \text{if } q = \infty . \end{cases}$$

Under this identification

$$\langle f,g\rangle = \int_0^\infty \langle f(t),\varphi(t)\rangle \, dt.$$

For further details see [15,42] and §4.6.

We will now consider some smoothness definitions c.f.[36].

DEFINITION 2.1.1.

Let $T \in \mathcal{C}_+(X)$, $A \in \mathcal{C}_0(X,Y)$ and $1 \leq p < \infty$. We say that A is $(p,+)$ smooth with respect to T, or simply $(T,p,+)$ smooth, if for each u in X, $e^{-itT}u \in \mathcal{D}(A)$ a.e. in $[0,\infty)$, and $t \mapsto Ae^{-itT}u$ gives an element of $L^p(\mathbb{R}^+;Y)$.

For example, suppose $T \in \mathcal{C}_+(X)$, and $A \in \mathcal{C}_0(X,Y)$ such that $\mathcal{D}(A^*)$ is sequentally weak*- dense in Y^* (i.e. given $\varphi \in Y^*$, there exists a sequence φ_n in $\mathcal{D}(A^*)$ such that $\varphi_n \mapsto \varphi$ $\sigma(Y^*,Y)$); with $1 \leq p < \infty$. Then A is $(T,p,+)$ smooth if for each u in X, there exists $M_u < \infty$ such that $\|Ae^{-itT}u\|$ is measurable, (where we have used the extended norm) and $\int_0^\infty \|Ae^{-itT}u\|^p \, dt \leq M_u^p$.

As in the reflexive situation [36,Lemma2.5] we can show the following using the closed graph theorem.

LEMMA 2.1.2.

Let $T \in \mathcal{C}_+(X)$ and $A \in \mathcal{C}_0(X,Y)$ with $1 \leq p < \infty$. If A is $(T,p,+)$ smooth, the mapping $u \longmapsto Ae^{-itT}u$; $0 \leq t < \infty$ defines a bounded linear operator S on X to $L^p(\mathbb{R}^+;Y)$.

If A is (T,p,+) smooth we let $|A|_{(T,p,+)} = \|S\|$, where S is the bounded linear operator defined in Lemma 2.1.2. $|A|_{(T,p,+)}$ is the infimum of all M satisfying

$$\int_0^\infty \|Ae^{-itT}u\|^p \, dt \leq M^p \|u\|^p \qquad \text{for all } u \text{ in } X.$$

Similarly we define smoothness with respect to $\hat{T}$ in $\mathcal{C}_-(X)$. If $A \in \mathcal{C}_0(X,Y)$ we say A is $(\hat{T},p,-)$ smooth (where $1 \leq p < \infty$) if it is $(-\hat{T},p,+)$ smooth, and put $|A|_{(\hat{T},p,-)} = |A|_{(-\hat{T},p+)}$.

DEFINITION 2.1.3.

Let $T \in \mathcal{C}_+(X)$, $T \in \mathcal{C}_-(X)$. If $T \subseteq \hat{T}$, we say that the pair $(T,\hat{T}) \in \mathcal{C}_\pm(X)$. If $A \in \mathcal{C}_0(X,Y)$ is (T,p,+)smooth and $(\hat{T},p,-)$ smooth with respect to a pair $(T,\hat{T})$ in $\mathcal{C}_\pm(X)$, we say A is $(T,\hat{T},p)$ smooth and set

$$|A|_{(T,\hat{T},p)} = \max\left\{ |A|_{(T,p,+)}, |A|_{(\hat{T},p-)} \right\}.$$

In the non reflexive situation, we need to consider smoothness with respect to a weakly continuous one parameter semigroup in the dual space X*.

Let $B \in \mathcal{C}_0(Y,X)$. Then Graph B has a countable dense set $\{(w_n, Bw_n) : w_n \in D_B\}$, and we can easily show (c.f. [36]) that if $f \in Y^*$ that

$$\|B^*f\| = \sup_n \frac{|\langle f, Bw_n\rangle|}{\|w_n\|} \quad : w_n \neq 0$$

It follows that $t \longmapsto \|B^*(e^{-itT})^*f\|$ is measurable on $\mathbb{R}^+$, whenever $T \in \mathcal{C}_+(X)$, $f \in X^*$.

DEFINITION 2.1.4.

Let $T \in \mathcal{C}_+(X)$, $B \in \mathcal{C}_0(Y,X)$ and $1 < p \leq \infty$. We say that B* is (p,+)* smooth with respect to T or simply (T,p,+)* smooth, if there exists a constant $M_u < \infty$ for each u in X* such that

$$\int_0^\infty \|B^*(e^{-itT})^*u\|^p \, dt \leq M_u^p \qquad \text{if } 1 < p < \infty$$

$$\operatorname*{ess\,sup}_{\mathbb{R}^+} \|B^*(e^{-itT})^*u\| \leq M_u \qquad \text{if } p = \infty.$$

It follows that of B* is (T,p,+)* smooth, than for any u in X*, $(e^{-itT})^*u \in D_{B^*}$ a.e. in $\mathbb{R}^+$.

LEMMA 2.1.5.

If B^* is $(T,p,+)^*$ smooth, the mapping

$$u \longmapsto B^*(e^{-itT})^*u \quad ; \quad 0 \leq t < \infty$$

defines a bounded linear operator R on X^* to $L^p(\mathbb{R}^+;Y)^*$.

PROOF.

Take y in Y. Since $\mathcal{D}(B)$ is dense in Y, there exist y_n in $\mathcal{D}(B)$ s.t. $y_n \longrightarrow y$. Then for u in X^*

$$\langle B^*(e^{-itT})^*u, y \rangle = \lim_{n\to\infty} \langle u, e^{-itT}By_n \rangle \quad .$$

It follows that $B^*(e^{-itT})^*u$ is weakly measurable and lies in $L^p(\mathbb{R}^+;Y)^*$. Again we use the closed graph theorem to complete the proof.

If B^* is $(T,p,+)^*$ smooth we let $|B^*|_{(T,p,+)^*} = \|R\|$, where R is the bounded operator defined in Lemma 2.1.5. $|B^*|_{(T,p,+)^*}$ is the infimum of all M satisfying

$$\int_0^\infty \|B^*(e^{-itT})^*u\|^p \, dt \leq M^p \|u\|^p \qquad \text{if } 1 < p < \infty$$

$$\operatorname{ess\,sup} \|B^*(e^{-itT})^*u\| \leq M \|u\| \qquad \text{if } p = \infty,$$

for all u in X^*.

Suppose B^* is $(T,\infty,+)^*$ smooth, and fix f in X^*. Then except on a null set N in $[0,\infty)$:

$$\|B^*(e^{-itT})^*f\| \leq M \|f\|.$$

Hence for all u in $\mathcal{D}(B)$, and all $t \notin N$

$$|\langle (e^{-itT})^*f, Bu \rangle| \leq M \|f\| \, \|u\| .$$

Hence by continuity,

$$|\langle f, Bu \rangle| \leq M \|f\| \, \|u\| \qquad \text{for all } f \in X^*, \text{ and } u \text{ in } \mathcal{D}(B).$$

It follows that B is bounded. Thus B^* is $(T,\infty,+)^*$ smooth iff $B \in B(Y,X)$, with $\sup_{\mathbb{R}^+} \|B^*(e^{-itT})^*\| < \infty$.

In which case $|B^*|_{(T,\infty,+)^*} = \sup_{t\geq 0} \|B^*(e^{-itT})^*\| = \sup_{t \geq 0} \|e^{-itT}B\|$

We similarly define smoothness with respect to $\hat{T} \in \mathcal{C}_-(X)$. If $B \in \mathcal{C}_0(Y,X)$, we say B^* is $(\hat{T},p,-)^*$ smooth if B^* is $(-\hat{T},p,+)^*$ smooth, and put $|B^*|_{(\hat{T},p,-)^*} = |B^*|_{(-\hat{T},p,+)^*}$.

If $(T,\hat{T}) \in \mathcal{C}_{\pm}(X)$ and B^* is $(T,p,+)^*$ and $(\hat{T},p,-)^*$ smooth, we say B^* is $(T,\hat{T},p)^*$ smooth and put

$$|B^*|_{(T,\hat{T},p)^*} = \max\left\{ |B^*|_{(T,p,+)^*} , |B^*|_{(\hat{T},p,-)^*} \right\} .$$

The following generalises [36, Lemma2.7]

LEMMA 2.1.7.

Let $T \in \mathcal{C}_+(X)$, and $B \in \mathcal{C}_0(Y,X)$. Assume B^* is $(T,q,+)^*$ smooth where $1 < q \leq \infty$, and put $\mathcal{Y} = L^p(\mathbb{R}^+;Y)$ where $1/p + 1/q = 1$. Then for each f in $\mathcal{Y}$, there exists an X^{**}-valued function $F(t)$ on $\mathbb{R}^+$ such that

$$\langle F(\ ),v \rangle = \int_0^t \langle f(s), B^*(e^{-i(t-s)T})^* v \rangle \, ds \qquad \text{for all } v \text{ in } X^*.$$

If the map $v \longmapsto B^*(e^{-itT})^* v$ is continuous from X^* into $\mathcal{Y}^*$ for the $\sigma(X^*,X)$ and $\sigma(\mathcal{Y}^*, \mathcal{Y})$ topologies, then $F(t) \in X$ for all t in $\mathbb{R}^+$.

DEFINITION 2.1.8.

Let $T \in \mathcal{C}_+(X)$, $A \in \mathcal{C}_0(X,Y)$, $B \in \mathcal{C}_0(Y,X)$. Assume that A is $(T,p,+)$ smooth and B^* is $(T,q,+)^*$ smooth, where $1/p + 1/q = 1$, $1 < q \leq \infty$. Moreover the map $v \longmapsto B^*(e^{-itT})^* v$ is taken to be continuous from X^* into $\mathcal{Y}^*$ for the weak*-topologies, where $\mathcal{Y} = L^p(\mathbb{R}^+;Y)$. We define an operator, symbolically denoted by $(Ae^{-i(\cdot)T}B)\theta$, on $\mathcal{Y}$ as follows.

For each f in $\mathcal{Y}$, let F^f be the function defined in Lemma 2.1.7, associated with f. If the function $F^f(\cdot)$ has the properties

i) $F^f(t) \in \mathcal{D}A$ a.e. in $\mathbb{R}^+$

ii) $AF^f(t) = g(t)$ as a function of t belongs to $\mathcal{Y}$,

then we say $f \in \mathcal{D}(Ae^{-i(\cdot)T}B)\theta$ and define

$$Ae^{-i(\cdot)T}B\theta \, f = g.$$

If $\hat{T} \in \mathcal{C}_-(X)$ and B^* is $(\hat{T},q,-)^*$ smooth, we similarly define $(Ae^{+i(\cdot)\hat{T}}B)\theta$ as $(Ae^{-i(\cdot)(-\hat{T})}B)\theta$.

Before stating our perturbation theorem, we recall that if $\omega_0 \geq 0$, and $T \in \mathcal{C}_+(X)$, then T is of type ω_o means that for all $\epsilon > o$

$$\lim_{t\to\infty} e^{-(\omega_0+\epsilon)t}\,\|e^{-itT}\| = 0 \qquad [25]$$

The following generalises [36.Th3.1] to (possibly) non reflexive spaces.

THEOREM 2.1.9.

Let X,Y be separable Banach spaces with $(T,\hat{T}) \in \mathcal{C}_{\pm}(X)$ and of type ω_o. $A \in \mathcal{C}_o(X,Y)$, $B \in \mathcal{C}_o(Y,X)$ such that A is $(T,\hat{T},p)$ smooth and B^* is $(T,\hat{T},q)^*$ smooth. Moreover $v \longmapsto B^*(e^{-itT})^*v$, $B^*(e^{+it\hat{T}})^*v$ are weak*-continuous from X^* into $\mathcal{Y}^*$ where $\mathcal{Y} = L^p(\mathbb{R}^+;Y)$ and $1 \leq p < \infty$, $1/p + 1/q = 1$. Furthermore assume that $(Ae^{-i(\cdot)T}B)\theta$ and $(Ae^{+i(\cdot)\hat{T}}B)\theta$ both belong to $B(\mathcal{Y})$, with common bound $\leq M$.

Then for any complex number κ with $|\kappa| < 1/M$, there exist $T(\kappa) \in \mathcal{C}_+(X)$, $\hat{T}(\kappa) \in \mathcal{C}_-(X)$, uniquely determined by $(T,\hat{T})$, A and B with the following properties

2.1.10) A is $(T(\kappa),\hat{T}(\kappa),p)$ smooth, B^* is $(T(\kappa),\hat{T}(\kappa),q)^*$ smooth with $v \longmapsto B^*(e^{-itT(\kappa)})^*v$, $B^*(e^{+it\hat{T}(\kappa)})^*v$ weak*-continuous from X^* into $\mathcal{Y}^*$.

Moreover

$$|A|_{(T(\kappa),\,\hat{T}(\kappa),\,p)} \leq (1-|\kappa|M)^{-1}\,|A|_{(T,\hat{T},p)}\;, \text{ and}$$

$$|B^*|_{(T(\kappa),\,\hat{T}(\kappa),\,q)^*} \leq (1-|\kappa|M)^{-1}\,|B^*|_{(T,\hat{T},q)^*}\,.$$

2.1.11) The pair $(T(\kappa), \hat{T}(\kappa)) \in \mathcal{C}_{\pm}(X)$ and are of type ω_o.

2.1.12) $T(\kappa)$ is similar to T and $\hat{T}(\kappa)$ is similar to $\hat{T}$. More specifically, there exists a non singular operator $W(\kappa)$ in $B(X)$ given by

$$\langle W(\kappa)u,\, v\rangle = \langle u,v\rangle \;-i\kappa\int_0^\infty \langle Ae^{isT}u,\; B^*(e^{-isT(\kappa)})^*v\rangle\, ds$$

for all u in X, v in X^*, such that

$$T(\kappa) = W(\kappa)\,T\,W(\kappa)^{-1} \text{ and } \hat{T}(\kappa) = W(\kappa)\,\hat{T}\,W(\kappa)^{-1}.$$

2.1.13) If $T = \hat{T}$, then $T(\kappa) = \hat{T}(\kappa)$ also, and in this case two "wave operators" $W_{\pm}(\kappa)$ exist and are given by

$$\langle W_{\pm}(\kappa)u,v\rangle = \langle u,v\rangle \;\pm i\kappa\int_0^\infty\langle Ae^{\mp isT}u,\; B^*(e^{\pm isT(\kappa)})^*v\rangle\, ds\,.$$

These operators coincide with the wave operators

$$W^{\pm}(\kappa) = \underset{t \to \pm\infty}{\text{strong limit}}\ e^{itT(\kappa)} e^{-itT}$$

and furnish the intertwining relations

$$T(\kappa) = W^{\pm}(\kappa)\, T\, W^{\pm}(\kappa)^{-1}.$$

The proof is an extension of the argument in [36].

REMARK 2.1.14

This method can also be used for more general potentials. If we consider $\mathcal{Y} = L^p(\mathbb{R}^+ \times \mathbb{N}; Y)$ in 2.1.3, 2.1.5, 2.1.7, and 2.1.8, we can treat "countably smooth" sums $\Sigma B_i A_i$.

§ 2.2

We now examine Theorem 2.1.9 in the special case when X is a hilbert space $\mathfrak{H}$. If $(T,\hat{T}) \in \mathcal{C}_{\pm}(\mathfrak{H})$, define the semigroups σ_t, $\hat{\sigma}_t$ for $t \geqslant 0$ on $B(\mathfrak{H})$ by

$$\sigma_t(x) = e^{-itT}\, x\, e^{it\hat{T}}$$
$$\hat{\sigma}_t(x) = e^{it\hat{T}}\, x\, e^{-itT} \qquad \text{for all } x \text{ in } B(\mathfrak{H}).$$

Then $\hat{\sigma}_t\, \sigma_t = 1$, and both σ_t, $\hat{\sigma}_t$ leave $C(\mathfrak{H})$ and $T(\mathfrak{H})$ invariant, for which they are strongly continuous semigroups with generating pairs in $\mathcal{C}_{\pm}C(\mathfrak{H})$ and $\mathcal{C}_{\pm}T(\mathfrak{H})$ respectively. We would hope that Theorem 2.1.9 raise Kato's and Lin's work to the algebra level, in the sense that associated with $(T,\hat{T})$-smooth operators A and B in $\mathcal{C}_o(\mathfrak{H})$ are multiplication operators on $T(\mathfrak{H})$ (or perhaps $C(\mathfrak{H})$ or $B(\mathfrak{H})$) which are smooth with respect to $(\sigma_t, \hat{\sigma}_t)$. However it is very difficult to prove this, except for the trace class operators $T(\mathfrak{H})$. There is of course no hope in $B(\mathfrak{H})$ itself because this algebra possesses an identity.

As an example of what it is possible to show, let us consider the following situation. Let $\mathfrak{H}$, $\mathfrak{H}'$ be separable hilbert spaces, and $A \in \mathcal{C}_o(\mathfrak{H}', \mathfrak{H})$. Let α be the linear operator from $B(\mathfrak{H}, \mathfrak{H}')$ into $B(\mathfrak{H})$ by defining $x \in B(\mathfrak{H}, \mathfrak{H}')$ to be in $\mathcal{D}_\alpha$ if $x\mathfrak{H} \subseteq \mathcal{D}(A)$, and putting

$\alpha x = Ax.$

Then α is closed and densely defined for the weak*-topologies.

Before we state our conclusion (Proposition 2.2.2), we recall that if X and Y are complex Banach spaces, and $D \in \mathcal{C}_o(X,Y)$ then D^* is a linear map from Y^* into X^* which is closed and densely defined for the weak*-topologies [25,65] . Similarly, we can show:

LEMMA 2.2.1.

Let C be a linear map from Y^* into X^* which is closed and densely defined for the weak*-topologies. If $x \in X$, we say $x \in \mathcal{D}(C_*)$ if there exists y in Y such that

$$\langle C\rho, x\rangle = \langle \rho, y\rangle \qquad \forall \rho \in \mathcal{D}(C),$$

and define $C_* x = y$.

Then $C_* \in \mathcal{C}_o(X,Y)$.

PROOF.

C_* is well defined and linear because of the $\sigma(Y^*,Y)$ density of $\mathcal{D}(C)$. Moreover C_* is clearly closed. Suppose $\mathcal{D}(C_*)$ is not dense in X. Then there exists $\omega_o \in X^*$ such that $\omega_o \neq 0$, but $\omega_o(x) = 0 \quad \forall x \in \mathcal{D}(C_*)$. Then $(0,\omega_o)$ does not lie in the graph of C which is weak*-closed. Hence there exists $(x,y) \in X \times Y$ such that

$$\langle C\rho, x\rangle + \langle \rho, y\rangle = 0 \qquad \forall \rho \in \mathcal{D}(C),$$

and $\langle \omega_o, x\rangle + \langle 0,y\rangle \neq 0$.

Hence $x \in \mathcal{D}(C_*)$, a contradiction. This shows $\mathcal{D}(C_*)$ to be dense in X.

PROPOSITION 2.2.2.

Let $A \in \mathcal{C}_o(\mathfrak{H},\mathfrak{H}')$ and α be the linear operator from $B(\mathfrak{H},\mathfrak{H}')$ into $B(\mathfrak{H})$ as defined previously. If $T,S \in \mathcal{C}_+(\mathfrak{H})$, where S generates a bounded semigroup, and A^* is $(-T^*,p,+)$ smooth where $1 \leqslant p < \infty$; then α_* is $(Z,p,+)$ smooth, where $Z \in \mathcal{C}_+T(\mathfrak{H})$ is defined by

$$e^{-iZt}x = e^{-itS}xe^{-itT} \qquad \text{for } t \geqslant 0, \quad x \in T(\mathfrak{H}).$$

(Here $(\cdot)^*$ refers to the hilbert space adjoint).

PROOF.

Note that $\alpha_* \in \mathcal{C}_0\left(T(\mathfrak{h}),\ T(\mathfrak{h},\mathfrak{h}')\right)$, where $T(\mathfrak{h},\mathfrak{h}')$ is the predual of $B(\mathfrak{h},\mathfrak{h}')$, and denotes the trace class operators from $\mathfrak{h}'$ into $\mathfrak{h}$ i.e. $\mathfrak{h}\check{\otimes}\bar{\mathfrak{h}}'$.

If $\xi_i,\ \eta_i \in \mathfrak{h}$ $i=1,2,\ldots$ then

$$\left\{ \int_0^\infty \left\{ \sum_{i=1}^\infty \|e^{-itS}\xi_i\|\ \|A^*(e^{-itT})^*\eta_i\| \right\}^p dt \right\}^{1/p}$$

$$\leqslant \sum_{i=1}^\infty \left\{ \int_0^\infty \left(\|e^{-isT}\xi_i\|\ \|A^*(e^{+itT^*})\ \eta_i\| \right)^p dt \right\}^{1/p}$$

$$\leqslant M_0 \sum \|\xi_i\|\ \|\eta_i\|\ |A^*| ,$$

where $M_0 < \infty$, is any constant such that $\|e^{-itS}\| \leqslant M_0$ for all $t \geqslant 0$.
The result follows if we note that $T(\mathfrak{h})$ and $T(\mathfrak{h},\mathfrak{h}')$ are projective tensor products.

Thus, we see that Theorem 2.1.9 with $p = 1$, can be used to handle strongly continuous semigroups $(e^{-iZt},\ e^{i\hat{Z}t};\ t \geqslant 0)$ on preduals M_* of von Neumann algebras. The case $1 < p < \infty$, will be studied in Chapter 3.

After seeing the importance of having an abundance of smooth operators at the lower hilbert space level, we now give a simple construction of how to produce more of them. Let $\mathfrak{h}_i$, $\mathfrak{h}_i'$ be separable hilbert spaces for $i=1,2$, with $A_i \in \mathcal{C}_0(\mathfrak{h}_i, \mathfrak{h}_i')$ for $i=1,2$. We let $A_1 \otimes A_2$ be the closure of $A_1 \odot A_2$ on $\mathfrak{D}(A_1) \odot \mathfrak{D}(A_2)$, where $\odot$ denotes the algebraic tensor product. Then $A_1 \otimes A_2 \in \mathcal{C}_0(\mathfrak{h},\mathfrak{h}')$, where $\mathfrak{h} = \mathfrak{h}_1 \otimes \mathfrak{h}_2$, $\mathfrak{h}' = \mathfrak{h}_1' \otimes \mathfrak{h}_2'$.
If $T_i \in \mathcal{C}_+(\mathfrak{h}_i)$, for $i=1,2$; define $T \in \mathcal{C}_+(\mathfrak{h})$ by

$$e^{-itT} = e^{-itT_1} \otimes e^{-itT_2} \quad \text{for } t \geqslant 0.$$

THEOREM 2.2.3.

If A_1 is $(T_1, 2, +)$ smooth and A_2 is $(T_2, \infty, +)$ smooth, then $A_1 \otimes A_2$ is $(T, 2, +)$ smooth and

$$|A_1 \otimes A_2|_{(T,2,+)} \leq |A_1|_{(T,2,+)} \; |A_2|_{(T,\infty,+)} \, .$$

PROOF.

We define a map F: $\mathfrak{H}_1 \odot^h \mathfrak{H}_2 \longrightarrow L^2(\mathbb{R}^+; \mathfrak{H}'_1) \odot^h \mathfrak{H}_2$, (where $\odot^h$ denotes the algebraic tensor product equipped with the hilbert space tensor product norm) as follows :

If $\xi_i \in \mathfrak{H}_1$, $\eta_i \in \mathfrak{H}_2$ $i = 1,2,\ldots,n$ put

$$F(\sum \xi_i \otimes \eta_i) = \sum_{i=1}^{n} A_1 \, e^{-i(\cdot)T} \, \xi_i \otimes \eta_i \, .$$

Similarly define S : $L^2(\mathbb{R}^+; \mathfrak{H}'_1) \odot \mathfrak{H}_2 \longrightarrow L^2(\mathbb{R}^+; \mathfrak{H}'_1 \otimes \mathfrak{H}'_2)$ as follows:

IF $f_i \in L^2(\mathbb{R}^+; \mathfrak{H}'_1)$, $\eta_i \in \mathfrak{H}_2$ $i = 1,2,\ldots,n$ put

$$S(\sum f_i \otimes \eta_i) = \sum f_i \otimes A_2 \, e^{-i(\cdot)T} \, \eta_i \, ,$$

which we regard as an element of $L^2(\mathbb{R}^+; \mathfrak{H}'_1 \otimes \mathfrak{H}'_2)$ in the obvious way.

Now fix ξ_i in $\mathfrak{H}_1$, η_i in $\mathfrak{H}_2$, $i = 1,2,\ldots,n$, and suppose that $\{\eta_i\}_{i=1}^{n}$ form an orthonormal set in $\mathfrak{H}_2$.

Then
$$\left\|F\left(\sum_{i=1}^{n} \xi_i \otimes \eta_i\right)\right\|^2 = \sum_{i=1}^{n} \left\|A_1 \, e^{-i(\cdot)T} \, \xi_i\right\|^2$$
$$\leq |A_1|^2_{(T,2,+)} \sum_{i=1}^{n} \|\xi_i\|^2$$
$$= |A_1|^2_{(T,2,+)} \left\|\sum_{i=1}^{n} \xi_i \otimes \eta_i\right\|^2 .$$

Now fix i. Then $A_1 e^{-itT} \xi_i \in L^2(\mathbb{R}^+; \mathfrak{H}'_1)$ and there exist $\alpha_j \in \mathfrak{H}'_1$, $h_j \in L^2(\mathbb{R}^+)$ such that

$$\left\|\sum_{j=1}^{l} \alpha_j \, h_j - A_1 \, e^{-i(\cdot)T} \, \xi_i\right\| < \epsilon/n.$$

Hence we get $g'_j \in L^2(\mathbb{R}^+)$, $\beta'_j \in \mathfrak{H}'_1$ and $\gamma'_j \in \mathfrak{H}_2$ for $j = 1,\ldots,m$ such that

$$\left\|\sum_{j=1}^{m} (g'_j \, \beta'_j) \otimes \gamma'_j - \sum_{i=1}^{m} A_1 \, e^{-i(\cdot)T} \, \xi_i \otimes \eta_i\right\| < \epsilon \, .$$

We now use the Gram Schmidt process to get $g_j \in L^2(\mathbb{R}^+)$, $\beta_j \in \mathfrak{H}'_1$, $\gamma_j \in \mathfrak{H}_2$ $j = 1,\ldots,m$, where the $\{\beta_j\}_{j=1}^{m}$ form an orthonormal set and

$$\left\|\sum_{j=1}^{m} (g_j \, \beta_j) \otimes \gamma_j - \sum_{i=1}^{n} A_1 \, e^{-i(\cdot)T} \, \xi_i \otimes \eta_i\right\| < \epsilon \, .$$

Then

$$\|S(\sum_{j=1}^{m} g_j \beta_j \otimes \gamma_j)\|^2$$

$$= \sum_{j,k} \int_0^\infty \langle g_j(s)\beta_j,\ g_k(s)\beta_k\rangle \langle A_2 e^{-isT_2}\gamma_j,\ A_2 e^{-isT_2}\gamma_k\rangle ds$$

$$= \sum_{j=1}^{m} \int_0^\infty |g_j(s)|^2 \|A_2 e^{-isT_2}\gamma_j\|^2\ ds$$

$$\leq \sum_{j=1}^{m} |A_2|(T_2,\infty,+)\ \|g_j\|^2 \|\gamma_j\|^2$$

$$= |A_2|_{(T_2,\infty,+)} \|\sum_{j=1}^{m} g_j\beta_j \otimes \gamma_j\|^2 .$$

Moreover

$$\|S(\sum_{i=1}^{n} A_1\ e^{-i(\cdot)T}\ \xi_i \otimes \eta_i\ -\ \sum_{j=1}^{m} g_j\beta_j \otimes \gamma_j\|$$

$$=\|S(\sum_{i=1}^{n} A_1\ e^{-i(\cdot)T}\ \xi_i \otimes \eta_i\ -\ \sum_{j=1}^{m} g_j'\beta_j' \otimes \gamma_j'\|\ \leq\ \epsilon\ |A_2|_{(T_2,\infty,+)}\ .$$

Hence, since ϵ was arbitrary;

$$\|S(\sum_{i=1}^{n} A_1\ e^{-i(\cdot)T}\xi_i \otimes \eta_i\|^2 \leq |A_2|^2_{(T_2,\infty,+)} \|\sum A_1\ e^{-i(\cdot)T_2}\xi_i \otimes \eta_i\|^2$$

$$\leq |A_2|^2_{(T_2,\infty,+)} |A_1|^2_{(T_2,2,+)} \|\sum_{i=1}^{n} \xi_i \otimes \eta_i\|^2$$

This means that

$$\int_0^\infty \|A_1 \otimes A_2\ e^{-itT}(\sum_{i=1}^{n} \xi_i \otimes \eta_i)\|^2 dt$$

$$\leq\ |A_1|^2_{(T_1,2,+)} |A_2|^2_{(T_2,\infty,+)} \|\sum_{i=1}^{n} \xi_i \otimes \eta_i\|^2 .$$

The result now follows easily.

We remark that it is a very difficult problem to try and replace the pairing $(2,\infty)$ in theorem 2.2.3. by another pair (p,q) where $1/p = 1/q = 1/2$; $2 < p < \infty$.

However, notice the following:

If A_i is $(T_i,q_i,+)$ smooth for i=1,2 where $1 \leq q_1 < \infty$, put $1/r = 1/q_1 + 1/q_2$.
Then $\int_0^\infty \|A_1 \otimes A_2\ e^{-iTt}\xi\|^r\ dt\ <\ \infty$
for ξ in a dense subspace of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

by looking at the proof of Proposition 2.2.2, we observe that there is an analogous statement of this weak type of smoothness at the algebra level.

We shall see in the next section that this is often sufficient information to be able to deduce scattering results at the C*-algebra level.

§2.3

In this section we will consider scattering due to a bounded time independant perturbation in a Banach space, paying particular attention to C*-algebras and the preduals of von Neumann algebras.

We take a strongly continuous group e^{iZt} acting on a Banach space X, and a perturbation $V \in B(X)$. It is seen that

$$e^{i(Z+V)t}e^{-iZt}x - e^{i(Z+V)s}e^{-iZs}x = \int_s^t e^{i(Z+V)u} V e^{-iZu}x \, du$$

for all $s < t$, and all x in X

Hence the wave operators

$$W_\pm = \underset{t \to \pm\infty}{\text{st lim}} \; e^{+i(Z+V)t} e^{-Zt;}$$

exist iff

$$\int_s^t e^{i(Z+V)u} V e^{-iZu} \, du$$

converges strongly to a bounded operator as $t \to \infty$, $s \to -\infty$.

In particular, if $e^{i(Z+V)t}$ and e^{iZt} are bounded one parameter groups (e.g. if iZ and iV are *-derivations of a C*-algebra X) then the wave operators exist if

$$(2.3.1) \qquad \int_{-\infty}^{\infty} \| V e^{iZu} x \| du < \infty \qquad \text{for all x in a dense subset } \mathcal{D} \text{ of X.}$$

This is well known at the hilbert space level, see e.g. [44,48], and the technique is well developed for such spaces. However we will be concerned with smoothness properties more general than 2.3.1 and the question of invertibility of wave operators.

Suppose V decomposes into bounded linear operators β, α in B(Y,X), B(X,Y) respectively, where Y is another Banach space. Thus $V = \beta\alpha$.

We take X and Y to be separable Banach spaces.

If $1 \leqslant q < \infty$, define

$$(2.3.2) \qquad D_q(\alpha) = \left\{ x \in X : \int_{-\infty}^{\infty} \| \alpha e^{-itZ} x \|^q \, dt < \infty \right\}$$

and

(2.3.3) $D_\infty(\alpha) = \left\{ x \in X : \sup_{t\in\mathbb{R}} \|e^{-itZ}x\| < \infty \right\}$.

From the previous section, we see that the following simultaneous situation is practical in the algebra context:

For some $1 \leq q \leq \infty$,

a) D^{α}_q is dense in X, and

b) $ß^*$ is $(Z,+Z,p)^*$ smooth where $1/p + 1/q = 1$.

i.e. if $1 \leq p < \infty$

$$\int_{-\infty}^{\infty} \|ß^*(e^{-iZt})^*\varphi\|^p \, dt \leq M^p\|\varphi\|^p \qquad \forall \varphi \in X^*, \text{ for some constant } M$$

and if $p = \infty$,

$$\sup_{t\in\mathbb{R}} \|ß^*(e^{-iZt})^*\| = \sup_{t\in\mathbb{R}} \|e^{-iZt}ß\| < \infty .$$

These conditions sill be studied in Theorems 2.3.4 and 2.3.10. The proofs employ a technique which is inspired by that of [14] and [36]. The condition

$$\int_{-\infty}^{+\infty} \|\alpha e^{-iZt} ß\| dt < \infty$$

has been particularly studied by E.B.Davies and Kato[29] in the hilbert space context.

THEOREM 2.3.4.

Let X and Y be separable Banach spaces and e^{iZt} a strongly continuous one parameter group on X. V is a bounded perturbation such that there exist $ß, \alpha$ in B(Y,X) and B(X,Y) respectvely such that $V = ß\alpha$ and

(2.3.5) $ß^*$ is $(Z,+Z,\infty)^*$ smooth

(2.3.6) $M = \int \|\alpha e^{-itZ} ß\| \, dt < \infty$.

Then the strong limits of

(2.3.7) $e^{\pm it(Z+\kappa V)} e^{\mp itZ} x$; $e^{\pm itZ} e^{\mp it(Z+\kappa V)} x$; $e^{-itZ} e^{2it(Z+\kappa V)} e^{-itZ} x$

exist as $t \to \infty$ for all x in $D_1(\alpha)$ (2.3.2) and all $\kappa \in \mathbb{C}$, $|\kappa| < 1/M$.

If moreover

(2.3.8) $ß^*$ is $(Z,+Z,1)^*$ smooth

(2.3.9) α is $(Z,+Z,\infty)$ smooth, i.e. $\sup_{t\in\mathbb{R}} \|\alpha e^{-itZ}\|$

then the limits in (2.3.7) exist as $t \to \infty$ for all x in $D_1(\alpha)^-$

and all $\kappa \in \mathbb{C}$, $|\kappa| < 1/M$.

In particular if $D_1(\alpha)$ is dense in X and (2.3.5-9) hold, then the wave operators $W_\pm = \operatorname{st\,lim}_{t\to\pm\infty} e^{+it(Z+\kappa V)} e^{-itZ}$ exist and are invertible for all $\kappa \in \mathbb{C}$, $|\kappa| < 1/M$.

PROOF.

We will consider the scattering matrix only. For any y in X, we have

$$e^{-iZt}\, e^{2i(Z+\kappa V)t}\, e^{-iZt}\, y = \sum_{n=o}^{\infty} (i\kappa)^n A_n(t)y$$

where $A_n(t) \in B(X)$ for each t, and is given by

$A_o(t) = 1$, and for $n \geq 1$,

$$A_n(t)y = \int_{-t}^{t}\int_{-t}^{s_1}\cdots\int_{-t}^{s_{n-1}} e^{-iZs_1}Ve^{iZ(s_1-s_2)}V \cdots e^{iZ(s_{n-1}-s_n)}Ve^{iZs_n}y\, ds_n \cdots ds_1 .$$

Hence if $x \in D_1(\alpha)$, and $c = \int_{-\infty}^{\infty} \|\alpha e^{iZs}x\|\, ds$, we have

$$\|A_n(t)x\| \leq$$

$$\int_{-t}^{t}\int_{-t}^{s_1}\cdots\int_{-t}^{s_{n-1}} \|e^{-iZs_1}\beta\|\, \|\alpha e^{iZ(s_1-s_2)}\beta\| \cdots \|\alpha e^{iZ(s_{n-1}-s_n)}\beta\|\, \|\alpha e^{iZs_n}x\|\, ds_n \cdots ds_1$$

$$\leq M^{n-1} c\, |\beta| \qquad \forall\, t \geq o.$$

A similar estimate shows that $\operatorname{st\,lim}_{t\to\infty} A_n(t)x$ exists $\forall\, x \in D_1(\alpha)$.

Hence for x in $D_1(\alpha)$ and $|\kappa| < 1/M$, the limit as $t \to \infty$ of

$$\sum_{n=o}^{\infty} (i\kappa)^n A_n(t)x = e^{-iZt}\, e^{2i(Z+\kappa V)t}\, e^{-iZt}x$$

exists.

Suppose also that conditions (2.3.8) and (2.3.9) hold. Then for all φ in X^*, $t \geq o$

$$\|A_n(t)^*\| \simeq$$

$$\int_{-t}^{t}\int_{-t}^{s_1}\cdots\int_{-t}^{s_{n-1}} \|(e^{iZs_n})^*\alpha^*\|\, \|\beta^*(e^{iZ(s_{n-1}-s_n)})^*\alpha^*\| \cdots \|\beta^*(e^{iZ(s_1-s_2)})^*\alpha^*\|\, \|\beta^*(e^{-iZs_1})^*\varphi\|\, ds..ds$$

$$\leq |\alpha|\, M^{n-1}\, |\beta|\, \|\varphi\| \qquad \text{for all } t \geq o,\ \varphi \in X^*.$$

Hence for fixed $|\kappa| < 1/M$

$$e^{-iZt}\, e^{2i(Z+\kappa V)t}\, e^{-iZt} = \sum_{n=o}^{\infty} A_n(t)(i\kappa)^n$$

is uniformly bounded in $\mathbb{R}^+$. This essentially proves the second half of the theorem.

For the case $(p,q) \neq (\infty,1)$, we have only been successful in showing that the wave operators and the scattering matrix exist in the weak topology.

THEOREM 2.3.10.

Let X, Y be separable Banach spaces, and e^{iZt} a strongly continuous one parameter group on X. V is a bounded perturbation such that there exist β, α in $B(Y,X)$, $B(X,Y)$ respectively such that $V = \beta\alpha$ and satisfy

(2.3.11) β^* is $(Z,+Z,p)^*$ smooth, where $\infty > p \geq 1$.

(2.3.12) $\displaystyle\int_{-\infty}^{+\infty} \|\alpha e^{-itZ}\beta\| \, dt \equiv M < \infty$.

Then the weak limits of

$$e^{\pm it(Z+\kappa V)} e^{\mp itZ} x \; ; \; e^{\pm itZ} e^{\mp it(Z+\kappa V)} x \; ; \; e^{-iZt} e^{2i(Z+\kappa V)} e^{-iZt} x$$

exist as $t \to \infty$ for all $x \in D_q(\alpha)$ (2.3.2), and all $|\kappa| < 1/M$, where $1/p + 1/q = 1$.

PROOF.

Again we only consider the scattering matrix. Then for $t \in \mathbb{R}$,

$$e^{-itZ} e^{2i(Z+\kappa V)t} e^{-iZt} = \sum_{n=0}^{\infty} (i\kappa)^n A_n(t) \quad \text{where } A_n(t) \in B(X).$$

Then for $n \geq 1$, $t \geq 0$, $x \in X$, $\varphi \in X^*$, we have by Young's and Holder's inequalities:

$$|\langle A_n(t)x, \varphi \rangle|$$
$$\leq \int_{-t}^{t}\int_{-t}^{s_1}\cdots\int_{-t}^{s_{n-1}} \|\alpha e^{iZ(s_2-s_1)}\beta\| \cdots \|\alpha e^{iZ(s_{n-1}-s_n)}\beta\| .$$
$$\|\alpha e^{iZs_n}x\| \, \|\beta^*(e^{-iZs_1})^*\varphi\| \quad ds_n \ldots\ldots ds_1$$
$$\leq M^{n-1} 2 |\beta^*| \; c_x \|\varphi\| ,$$

where

$$c_x = \begin{cases} \left\{ \displaystyle\int_{-\infty}^{\infty} \|\alpha e^{-itZ}x\|^q \, dt \right\}^{1/q} & \text{if } 1 \leq q < \infty \\ \sup_{t\in\mathbb{R}} \|\alpha e^{-itZ}x\| & \text{if } q = \infty \end{cases}$$

A similar estimate shows that $\underset{t \to \infty}{\text{weak limit}}\ A_n(t)x$ exists for all x in $D_q(\alpha)$. Hence the weak limit of

$$e^{-iZt}\, e^{2i(Z+\kappa V)t}\, e^{-iZt}x$$

exists for all x in $D_q(\alpha)$, $|\kappa| < 1/M$.

We remark that the methods of this section can also be used to analyse potentials of the form

$$\sum_{i=1}^{\infty} \beta_i\, \alpha_i \quad .$$

with analogous countably smooth conditions.

Results can also be obtained by interchanging the weak smoothness in X with the strong smoothness in X*

§2.4.

We now want to consider unbounded perturbations in the CAR algebra, and the related scattering. In order to do this we employ some of the notation and technique of time development in quasi local algebras [21, 27,49,53].

Specifically consider a C*-algebra U, with e^{Zt} a strongly continuous one parameter group of *- automorphisms of U. Also consider the finite ν-dimensional lattice $\mathbb{Z}^\nu$, and denote by $\mathcal{F}$ the family of finite subsets of $\mathbb{Z}^\nu$. Suppose that for each $\Lambda \in \mathcal{F}$, there is a subalgebra $U(\Lambda)$ of U such that

2.4.1) If $\Lambda, \Lambda_1 \in \mathcal{F}$, $\Lambda \subseteq \Lambda_1$, then $U(\Lambda) \subseteq U(\Lambda_1)$.

2.4.2) $\bigcup \{U(\Lambda) : \Lambda \in \mathcal{F}\}$ is dense in U.

2.4.3) $U(\Lambda)$ is invariant under e^{tZ} for each $\Lambda \in \mathcal{F}$, $t \in \mathbb{R}$.

We assign to each $\Lambda \in \mathcal{F}$, a potential $\Phi(\Lambda)$ in $U(\Lambda)$. We can then define, for each $\Lambda \in \mathcal{F}$, a "Hamiltonian" $H(\Lambda)$ in $U(\Lambda)$ by

2.4.4) $H(\Lambda) = \sum_{X \subset \Lambda} \Phi(X)$.

The local hamiltonian $H(\Lambda)$ gives a perturbation of Z :

2.4.5) $A \longmapsto e^{t(Z + \operatorname{ad}H(\Lambda))}(A)$; $A \in U$, $t \in \mathbb{R}$.

We will demonstrate under suitable conditions that the limit as $\Lambda \to \infty$ of (2.4.5) exists and defines a strongly continuous one parameter group of *-automorphisms of U. The infinitesimal generator of this group will be interpreted as Z + ad H , where H is taken to denote the formal sum

$$\sum_{X \in \mathcal{F}} \Phi(X) .$$

DEFINTION 2.4.6.

We let $\mathcal{B}$ denote the potentials Φ such that

2.4.7) For each $X \in \mathcal{F}$

$U \ni x \longmapsto \operatorname{ad} \Phi(X)\, x$ is a *derivation on U.

2.4.8) $\|\Phi\| = \sup_{x \in \mathbb{Z}^\nu} \left\{ \sum_{X \ni x} \|\Phi(X)\| \right\} < \infty .$

2.4.9) Φ involves only a finite number of coordinates i.e. $\Phi(X) = 0$ for all $X \in \mathcal{F}$ such that $|X| > N_\Phi$, where $N_\Phi < \infty$. (Here $|X|$ denotes the number of points in X).

2.4.10) Φ has finite range i.e. for each x in $\mathbb{Z}^\nu$, there exists $N_x < \infty$ such that $\{X \in \mathcal{F} : \Phi(X) \neq 0, \ X \ni x\}$ has cardinality $\leq N_x$.

2.4.11) If $X, \Lambda \in \mathcal{F}$ are disjoint then

$$[\Phi(X), u] = 0 \qquad \forall\, u \in U(\Lambda).$$

The following result extends [21,27,49,53] where Z = 0. Note that our translation invariance property 2.4.8 for the class of permissable potentials is weaker than what is stated in the above mentioned work.

THEOREM 2.4.12.

Suppose $\Phi \in \mathcal{B}$. Then for all $t \in \mathbb{R}$, $A \in U$,

$$\sigma_t A = \lim_{\Lambda \to \infty} e^{(Z + \operatorname{ad} H_\Lambda)t} A$$

exists, and defines a strongly continuous one parameter group of *-automorphisms of U, where the limit is taken in the sense that Λ eventually contains every finite subset of $\mathbb{Z}^\nu$.

PROOF.

Fix $\Lambda_0 \in \mathcal{F}$, and take $A \in U(\Lambda_0)$. Consider $\Lambda \in \mathcal{F}$, $\Lambda \supseteq \Lambda_0$

Then for all t in $\mathbb{R}$

$$e^{(Z+\mathrm{ad}\,H_\Lambda)t}A = e^{Zt}A + \int_0^t \left[e^{Z(t-s_1)}H_\Lambda,\ e^{Zt}A\right] ds_1$$

$$+ \int_0^t\int_0^{s_2} \left[e^{Z(t-s_2)}H_\Lambda, \left[e^{Z(t-s_1)}H_\Lambda, \left[e^{Zt}A\right]\right]\right] ds_1\, ds_2$$

$$+ \ldots\ldots\ldots$$

$$= \sum_{n=0}^{\infty} a_n(t) \qquad (2.4.13)$$

where (in terms of potentials):

$$a_n^\Lambda(t) =$$

$$\sum_{X_1,X_2,\ldots,X_n\subseteq\Lambda} \int_0^t\int_0^{s_n}\cdots\int_0^{s_2} \left[e^{Z(t-s_n)}\Phi(X_n), \left[e^{Z(t-s_{n-1})}\Phi(X_{n-1}), \left[\cdots\left[e^{Z(t-s_1)}\Phi X_1, e^{Zt}A\right]\cdots\right]\right]\right] ds_1\cdots ds_n$$

Since Φ has finite range (2.4.10), $a_n^\Lambda(t)$ becomes independant of Λ for $|\Lambda|$ sufficiently large. Hence we have

$$\lim_{\Lambda\to\infty} a_n^\Lambda(t)$$

$$= \sum_{X_1,X_2\ldots,X_n} \int_0^t\int_0^{s_n}\cdots\int_0^{s_2} \left[e^{Z(t-s_n)}\Phi(X_n), \left[\cdots\left[e^{Z(t-s_1)}\Phi X_1, e^{Zt}A\right]\cdots\right]\right] ds_1\ldots ds_n .$$

This means that the series (2.4.13) converges term by term. The following lemma is an adaptation of [49, Lemma 1].

LEMMA 2.4.14.

If $A \in U(\Lambda_0)$, $q_1\ldots q_n \in \mathbb{R}$, $\Lambda \in \mathcal{F}$

$$\left\|\left[e^{Zq_n}H(\Lambda), \left[e^{Zq_{n-1}}H(\Lambda), \left[\cdots\left[e^{Zq_1}H(\Lambda), A\right]\cdots\right]\right]\right]\right\|$$

$$\leq 2^n\|\Phi\|^n\,\|A\|\prod_{m=1}^{n}\left\{(m-1)(N_\Phi-1)+|\Lambda_0|\right\}.$$

Proof.

We have by 2.4.11 and the invariance of $U(\Lambda_1)$ under e^{Zt} for all $\Lambda_1 \in \mathcal{F}$, that

$$\left[e^{Zq_n}H(\Lambda), \left[e^{Zq_{n-1}}H(\Lambda), \left[\cdots\left[e^{Zq_1}H(\Lambda), A\right]\cdots\right]\right]\right]$$

$$= \sum_{\substack{X_1\subset\Lambda\\ X_1\cap\Lambda_0\neq\emptyset}}\ \sum_{\substack{X_2\subset\Lambda\\ X_2\cap S_1\neq\emptyset}}\cdots\sum_{\substack{X_n\subset\Lambda\\ X_n\cap S_{n-1}\neq\emptyset}} \left[e^{Zq_n}\Phi(X_n), \left[e^{Zq_{n-1}}\Phi(X_{n-1}), \cdots\left[e^{Zq_1}\Phi(X_1), A\right]\cdots\right]\right]$$

where we have introduced the notation

$S_1 = X_1 \cup \Lambda_1$, and $S_k = X_k \cup S_{k-1}$ for $1 < k < n$.

Therefore

$$\|[e^{Zq_n}H(\Lambda), [e^{Zq_{n-1}}H(\Lambda), \dots\dots [e^{Zq_1}H(\Lambda), A]]\|$$

$$\leq 2 \sum_{\substack{X_1 \subset \Lambda \\ X_1 \cap \Lambda_1 \neq \emptyset}} \dots \sum_{\substack{X_n \subset \Lambda \\ X_n \cap S_{n-1} \neq \emptyset}} \|\Phi(X_n)\| \, \|[e^{Zq_{n-1}}\Phi(X_{n-1}), [\dots [e^{Zq}\Phi(X_1), A]\dots]\|$$

But using $|S_{n-1}| \leq (n-1)(N_\Phi - 1) + |\Lambda_o|$,

we find that

$$\|[e^{Zq_n}H(\Lambda), [e^{Zq_{n-1}}H(\Lambda), \dots [e^{Zq_1}H(\Lambda), A$$

$$\leq 2\{(n-1)(N_\Phi -1) + |\Lambda_o|\}\|\Phi\| \sum_{\substack{X_1 \subset \Lambda \\ X_1 \cap \Lambda_0 \neq \emptyset}} \sum_{\substack{X_{n-1} \subset \Lambda \\ X_{n-1} \cap S_{n-2} \neq \emptyset}} \|[e^{Zq_{n-1}}\Phi(X_{n-1}) [\quad [e^{Zq_1}\Phi(X_1), A]\cdot]\|$$

The lemma follows by repeating this method of estimation.

It follows from the Lemma that

$$\|a_n^\Lambda(t)\| \leq \frac{|t|^n}{n!} 2^n \|\Phi\|^n \prod_{m=1}^{n} \{(m-1)(N_\Phi -1) + |\Lambda_o|\} .$$

Hence for $2|t|\,\|\Phi\|(N_\Phi -1) < 1$, the strong limit of

$$e^{(Z + \mathrm{ad}\, H\Lambda)t}A$$

exists as $\Lambda \to \infty$, $\Lambda \supseteq \Lambda_o$. (This convergence is uniform in $2|t|\,\|\Phi\|(N -1) \leq 1-\epsilon$ for any $\epsilon > o$). We define this limit to be $\sigma_t(A)$. But $e^{(Z + \mathrm{ad}\, H\Lambda)t}$ is a *-automorphism for all $\Lambda \in \mathcal{F}$, t in $\mathbb{R}$, and hence is isometric. It follows that

$$\|\sigma_t A\| = \|A\| \qquad \forall A \in \bigcup_{\Lambda \in \mathcal{F}} U(\Lambda).$$

Hence we can extend σ_t to the whole of U by continuity.

Take any $o < t_1 < \dfrac{1}{2\|\Phi\|(N_\Phi -1)}$.

Then the group property of $e^{(Z + \mathrm{ad}\, H\Lambda)t}$ for $|t| \leq t_1$, implies that σ_t has the same property in $|t| \leq t_1$. Hence we can extend σ_t to the whole of $\mathbb{R}$. The strong continuity of σ_t follows from the series expansion

$$\sigma_t A = \sum_{n=o}^{\infty} a_n(t) \qquad \text{for } A \in U(\Lambda_o), \text{ where } a_n(t) = \lim_{\Lambda} a_n^\Lambda(t).$$

Moreover σ_t is a *-automorphism for each t in $\mathbb{R}$.

We note that there is a natural action of $\mathbb{Z}^\nu$ on $\mathfrak{F}$. If we strengthen the translation invariance property 2.4.8, we can remove the hypothesis of finite range of our potentials.

DEFINITION 2.4.15.

We let $\mathcal{B}_0$ denote the potentials Φ such that

2.4.16) For each $X \in \mathfrak{F}$,

$U \ni x \longrightarrow \operatorname{ad}\Phi(X)\, x$ is a *-derivation on U.

2.4.17) $\|\Phi(X+a)\| = \|\Phi(X)\| \quad \forall X \in \mathfrak{F}, \quad a \in \mathbb{Z}^\nu$.

2.4.18) $\|\Phi\| \equiv \sum_{X \ni 0} \|\Phi(X)\| < \infty$

2.4.19) Φ involves only a finite number of coordinates i.e. $\Phi(X) = 0$ for all $X \in \mathfrak{F}$ such that $|X| > N_\Phi$.

2.4.20) If $X, \Lambda \in \mathfrak{F}$ are disjoint

$$[\Phi(X),u] = 0 \quad \forall u \in U(\Lambda).$$

For this class of potentials we can again take the limit as $\Lambda \to \infty$ in 2.4.5. Note that there is no need to assume that there is an action of $\mathbb{Z}^\nu$ on $\bigcup U(\Lambda)$, such that each a in $\mathbb{Z}^\nu$ takes $U(\Lambda)$ onto $U(\Lambda+a)$ for all Λ in $\mathfrak{F}$.

THEOREM 2.4.21.

Suppose $\Phi \in \mathcal{B}_0$. Then for all t in $\mathbb{R}$, A in U

$$(2.4.22) \qquad \sigma_t A = \lim_{\Lambda \to \infty} e^{(Z+\operatorname{ad} H\Lambda)t} A$$

exists and defines a strongly continuous one parameter group of *-automorphisms on U.

PROOF.

We note that under the norm $\|\cdot\|$ (2.4.18), $\mathcal{B}_0$ is a real normed space. Moreover $\mathcal{B}_0 \cap \mathcal{B}$ is dense in $\mathcal{B}_0$; if $\Phi \in \mathcal{B}_0$ define $\Phi_n \in \mathcal{B} \cap \mathcal{B}_0$ by

$$\Phi_n(X) = \begin{cases} \Phi(X) & \text{if } \operatorname{diam} X \le n \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\Phi_n - \Phi\| \to 0$ and $\|\Phi_n\| \le \|\Phi\|$.

The following is an elementary extension of [49, Lemma 2].

LEMMA 2.4.23.

If $A \in U(\Lambda_0)$, Φ_1, $\Phi_2 \in \mathcal{B}_0$, and N_Φ is such that $\Phi_1(X) = 0 = \Phi_2(X)$ if $|X| > N_\Phi$ then for all $q_1,\dots,q_n \in \mathbb{R}$,

$$\begin{aligned}\Big\| &\big[e^{Zq_n} H_{\Phi_1}(\Lambda), \big[e^{Zq_{n-1}} H_{\Phi_1}(\Lambda), \dots \big[e^{Zq_1} H_{\Phi_1}(\Lambda), A\big] \cdots\big] \\ &- \big[e^{Zq_n} H_{\Phi_2}(\Lambda), \big[e^{Zq_{n-1}} H_{\Phi_2}(\Lambda), \dots \big[e^{Zq_1} H_{\Phi_2}(\Lambda), A\big] \cdots\big]\Big\| \\ &\leq n2^n \|\Phi_1 - \Phi_2\| \|\Phi\|^{n-1} \|A\| \prod_{m=1}^{n} \left\{ (m-1)(N_\Phi - 1) + |\Lambda_0| \right\}\end{aligned}$$

Now take $A \in U(\Lambda_0)$, and $\Lambda \supset \Lambda_0 \in \mathcal{F}$. Now suppose Φ, $\Phi_1 \in \mathcal{B}_0$ and $\Phi_1(X) = 0 = \Phi(X)$ if $|X| > N_\Phi$, and also $\|\Phi_1\| < \|\Phi\|$. Then for all t in $\mathbb{R}$

$$\begin{aligned}&\| e^{t(Z + \mathrm{ad}\, H_\Phi(\Lambda))} A - e^{t(Z + \mathrm{ad}\, H_{\Phi_1}(\Lambda))} A \| \\ &\leq 2 \|A\| \|\Phi - \Phi_1\| \sum_{n=1}^{\infty} \frac{2^n |t|^n \|\Phi\|^{n-1}}{(n-1)!} \prod_{m=1}^{n} \left\{ (m-1)(N_\Phi - 1) + |\Lambda_0| \right\},\end{aligned}$$

by Lemma 2.4.23.

This estimate is uniform in Λ, and the right hand side converges for $2|t|\|\Phi\|(N_\Phi - 1) < 1$. Given $\Phi \in \mathcal{B}_0$, we take the sequence Φ_n in $\mathcal{B} \cap \mathcal{B}_0$, defined above, and express the limit in 2.4.22 as a double limit. The above estimate justifies the interchange of limits and the existence of the interchanged limit under the condition

$$2|t|\|\Phi\|(N_\Phi - 1) < 1.$$

(Here we have used Theorem 2.4.12 for the potential $\Phi_n \in \mathcal{B}$).
The remainder of the proof is similar to that in Theorem 2.4.12.

Thus if $\Phi \in \mathcal{B}_0 \cup \mathcal{B}$, $A \in U$

$$\lim_{\Lambda \to \infty} e^{(Z + \mathrm{ad}\, H\Lambda)t} e^{-Zt} A = \sigma_t e^{-Zt} A$$

where σ_t is a strongly continuous one parameter group of *-automorphisms of U, and the limit existing in the norm topology.

Now suppose that $\mathcal{D}$ is the subset of A in U such that the following limits

$$\lim_{t \to \pm\infty} e^{(Z + \mathrm{ad}\, H\Lambda)t} e^{-Zt} A$$

exist in the norm topology for all Λ in $\mathcal{F}$, with uniform convergence over Λ (for fixed A). Then both repeated limits exist on $\mathcal{D}$ (and are equal).

i.e. $\lim_{t \to \pm\infty} \lim_{\Lambda \to \infty} e^{(Z + \mathrm{ad}\, H\,)t} e^{-Zt} A$ exists for all A in $\mathcal{D}$,

i.e. $\lim_{t \to \pm\infty} \sigma_t \, e^{-Zt} A$ exists for all A in $\mathcal{D}$.

Hence if the subalgebra $\mathcal{D}$ is dense in U, the wave operators exist on U. We shall see that this approach is applicable in the CAR algebra.

§ 2.5.

We are now going to study in some detail, scattering in a particular C*-algebra, namely the algebra of the canonical anti commutation relations. We briefly recall its construction, for more details see [21].

Let E be an infinite demensional real vector space with (,) a positive definite,non degenerate symmetric form. Let $C_o(E)$ be the complex Clifford algebra i.e. the complexification of T(E)/I where T(E) is the tensorial algebra and I is the ideal generated by

$$\{ e \otimes e - (e,e)1 \; : \; e \in E \} .$$

Then we can give $C_o(E)$ a C*-norm $\| \cdot \|$, and the CAR algebra, C(E), is the completion of $C_o(E)$ with respect to this norm. C(E) is a simple C*-algebra with identity 1, and a faithful normal finite trace. Let $f \longrightarrow [f]$ be the canonical embedding of E in $C_o(E)$. Then

$$[f]\,[g] \; + \; [g]\,[f] \; = \; 2(f,g)1 \qquad \forall \; f,g \in E.$$

Whenever e^{tZ} is a strongly continuous one parameter group of orthogonal maps on E, e^{tZ} can be uniquely extended to a strongly continuous one parameter group of *-automorphisms of C(E).

In our first result , we consider a perturbation of the free evolution e^{tZ} by an inner derivation of the algebra C(E). A similar result was obtained by D.W.Robinson [50,Application2]. We remark that it is only necessary to show that the wave operators exist on a subset $\hat{U}$ of C(E), such that the algebra generated by $\hat{U}$ is dense in C(E).

THEOREM 2.5.1.

Let e^{Zt} be a strongly continuous one parameter group of orthogonal maps on E. Suppose $f^i_{j,k} \in E$ for $1 \leqslant K \leqslant 2i$, $i,j = 1,2,\ldots\ldots$ with $\|f^i_{j,k}\| = 1$ for all such (i,j,k); and $\alpha^i_j \in \mathbb{C}$ for (i,j) N^2, such that $\sum_{i,j} |\alpha^i_j| < \infty$

and $a = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\alpha^i_j| \left[f^i_{j,1}\right] \ldots\ldots\ldots \left[f^i_{j,2i}\right]$

gives a *-derivation $x \longmapsto \operatorname{ad} a\,(x)$ on C(E).

Also suppose that

$$\mathcal{D} = \left\{ f \in E : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{2i} |\alpha^i_j| \int_{-\infty}^{\infty} |(e^{Zt} f^i_{j,k}, f)| \, dt < \infty \right\}$$

is such that $C_o(\mathcal{D})$ is dense in C(E).

Then the wave operators

$$W_{\pm} = \operatorname*{st\,lim}_{t \longrightarrow \pm\infty} e^{t(Z + \operatorname{ad} a)} e^{-tZ}$$

exist.

PROOF.

If $(i,j) \in N^2$, and $f \in E$

$$\left[f^i_{j,1}\right] \cdots\cdots \left[f^i_{j,2i}\right] \left[f\right] - \left[f\right] \left[f^i_{j,1}\right] \cdots\cdots \left[f^i_{j,2i}\right]$$

$$= 2 \sum_{k=1}^{2i} \left[f^i_{j,1}\right] \cdots\cdots \left[f^i_{j,k-1}\right] (f, f^i_{j,k}) (-1)^k \left[f^i_{j,k+1}\right] \cdots\cdots \left[f^i_{j,2i}\right].$$

Hence

$a[f] - [f]a$

$$= 2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{2i} \alpha^i_j \left[f^i_{j,1}\right] \cdots \left[f^i_{j,k}\right] (-1)^k (f, f^i_{j,k}) \left[f^i_{j,k+1}\right] \cdots \left[f^i_{j,2i}\right].$$

Take $t > s > o$, $f \in \mathcal{D}$. Then

$$\int_s^t \| \operatorname{ad} a \; e^{-Zu} f \| \, du \leq \int_s^t \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{2i} |\alpha^i_j| \, |(e^{-Zu} f, f^i_{j,k})| \right\} du.$$

Thus by (2.3.1), the homomorphism property of $e^{t(z + \operatorname{ad} a)} e^{-tZ}$ for all t in $\mathbb{R}$, and their uniform boundedness, we see that the wave operators exist on $C_o(\mathcal{D})$, and finally on C(E).

Taking this a step further, we consider perturbations induced by formal sums, and exploit the computations of the previous section. Again let e^{Zt} be a strongly continuous one parameter group of orthogonal maps on E. Suppose $\left\{ f_i : i \in \mathbb{Z} \right\}$, is a countable in E with disjoint energy spectra i.e.

(2.5.2) $\quad (e^{Zt} f_i , f_j) = 0$ if $i \neq j$, for all t in $\mathbb{R}$.

Without much loss of generality we shall take

$$E = \text{linear span} \left\{ e^{Zt} f_i : t \in \mathbb{R}, \ i \in \mathbb{Z} \right\}.$$

In the notation of the previous section, consider $\mathcal{F}$, the family of finite subsets of $\mathbb{Z}$ (i.e. $\nu = 1$). For $\Lambda \in \mathcal{F}$, we define $U(\Lambda)$ to be the subalgebra of $C_o(E)$ generated by $\left\{ \left[e^{Zt} f_i \right] : t \in \mathbb{R}, \ i \in \Lambda \right\}$.

Clearly $U(\Lambda) : \Lambda \in \mathcal{F}$ satisfy the isotony relation 2.4.1.

Moreover $C(E) = \left\{ \bigcup U(\Lambda) : \Lambda \in \mathcal{F} \right\}^-$ and each $U(\Lambda)$ is invariant under e^{Zt}. We now define a potential Φ.

We put $\Phi(X) = 0$, unless X is even, and if $X = \{ i_1, \dots, i_{2n} \}$, put

$$\Phi(X) = \alpha_{i_1, \dots, i_{2n}} \left[f_{i_1} \right] \cdots \left[f_{i_{2n}} \right], \text{ where } \alpha_{i_1, \dots, i_{2n}} \in \mathbb{C}.$$

For convenience (and consistency with the CAR's and 2.5.2) we impose the condition that if π is any permutation of $1, \dots, 2n$, that

$$\alpha_{i_{\pi(1)}, \dots, i_{\pi(2n)}} = \operatorname{sgn}(\pi) \, \alpha_{i_1, \dots, i_{2n}} .$$

Also if $(i_1, \dots, i_{2n}) \in \mathbb{Z}^{2n}$, with not all i_r distinct, we put

$$\alpha_{i_1, \dots, i_{2n}} = 0.$$

If $i \neq j$, $\left[e^{Zt} f_i \right]$ and $\left[f_j \right]$ anticommute because of 2.5.2 and the CAR's. Hence if $X \in \mathcal{F}$, and $i \notin X$, $\left[\Phi(X), \left[e^{Zt} f_i \right] \right] = 0 \ \forall t \in \mathbb{R}$.

Thus if $\Lambda, X \in \mathcal{F}$ are disjoint

$$\left[\Phi(X), u \right] = 0 \quad \forall u \in U(\Lambda).$$

Finally under suitable conditions of the values of $\left\{ \alpha_{i_1, \dots i_{2n}} \right\}$ Φ is an element of $\mathcal{B} \cup \mathcal{B}_o$.

THEOREM 2.5.3

For all $A \in C(E)$, $t \in \mathbb{R}$,

$$\sigma_t A = \lim_{\Lambda \to \infty} e^{(Z + \text{ad } H\Lambda)t} A$$

exists in the norm topology, and defines a strongly continuous one parameter group of *-automorphisms of C(E).

If $\int_{-\infty}^{\infty} |(e^{Zs}f_i, f_i)| \, ds < \infty$ for all i in $\mathbb{Z}$, the wave operators

$$W_{\pm} = \text{st} \lim_{t \to \pm\infty} \sigma_t e^{-Zt}$$

exist on the CAR algebra C(E).

PROOF.

The first statement follows from Theorems 2.4.12 and 2.4.21.

Suppose $(e^{Zs}f_i, f_i) \in L^1(\mathbb{R})$ for each i in $\mathbb{Z}$. By the remarks at the end of §2.4, it is enough to show for fixed i, that

$$\lim_{t \to \pm\infty} e^{t(Z + \text{ad } H\Lambda)} e^{-tZ} [f_i]$$

exists uniformly over $\Lambda \in \mathcal{F}$, in the norm topology.

Now if $i \in \Lambda$,

$$\text{ad } H(\Lambda) [e^{-Zu}f_i] =$$

$$2 \sum_{n=1}^{\infty} \sum_{\{j_r \in \Lambda \,:\, r=1,\dots,2n-1\}} \alpha_{j_1,\dots,j_{2n-1},i} [f_{j_1}] \dots [f_{j_{2n-1}}] (e^{-uZ}f_i, f_i) .$$

Hence for all $t > s \in \mathbb{R}$, $\Lambda \ni i$

$$\| e^{t(Z + \text{ad } H\Lambda)} e^{-tZ} [f_i] - e^{s(Z + \text{ad } H\Lambda)} e^{-sZ} [f_i] \|$$

$$\leq \int_s^t \| \text{ad } H(\Lambda) [e^{-Zu}f_i] \| du$$

$$\leq 2 \int_s^t \sum_{n=1} \sum_{j_r} |\alpha_{j_1,\dots\dots,j_{2n-1},i}| \| [f_{j_1}] \dots\dots [f_{j_{2n-1}}] \| |(e^{-Zu}f_i, f_i)| du$$

$$\leq 2 \frac{\|\Phi\|}{\|f_i\|} \int_s^t |(e^{-Zu}f_i, f_i)| \, du$$

which tends to zero as $t, s \to \pm\infty$ independantly of Λ.

CHAPTER 3.

In this chapter we proceed with our study of smooth perturbations and scattering of one parameter groups at an operator algebra level. We have seen in § 2.2 how to handle a particular smoothness condition (p=1). We now extend this to a wider class of perturbations ($1 < p < \infty$); however, due to technical problems, we apply our method at the present time only to type I W*-factors with discrete centre.

If σ_t is a weakly continuous one parameter group of ultraweakly continuous linear maps on a von Neumann algebra M, let iT denote the infinitesimal generator of the strongly continuous one parameter group $(\sigma_t)_*$ acting on the predual M_*. For various unbounded operators β, and α, acting on M, we seek for small complex κ, a family of wekly continuous one parameter groups σ_t^κ, such that the infinitesimal generator $iT(\kappa)$ of $(\sigma_t^\kappa)_*$ can be interpreted as the formal symbol $i(T + \kappa\alpha_* \beta_*)$. Our method is based on the theory of S.C.Lin for the problem at the hilbert space level [36], using the one parameter groups σ_t themselves, rather than the resolvents $R(\lambda,T)$ as Kato does [29].

β and α can represent left multiplication operators L_B, L_A respectively, where B and A are unbounded operators affiliated with M. In general, if σ_t is a group of *-automorphisms, the perturbation σ_t^κ will not be a group of *-automorphisms for small (real) κ. By considering another perturbation of multiplication on the right, (R_{B^*}, R_{A^*}), it is sometimes possible to combine these two perturbations to give one, namely τ_t^κ, which is a weakly continuous group of *-automorphisms of M for small real κ. Moreover, the infinitesimal generator of $(\tau_t^\kappa)_*$ represents the formal symbol $i\left[T + \kappa\left((\alpha)_*\beta_* - (\bar{\alpha})_*(\bar{\beta})_*\right)\right]$.

§ 3.1.

In this section, we develop some notation and the tools for our perturbation theory, which we shall find particularly useful to handle the smoothness conditions we impose on our system.

For simplicity and clarity of notation, we will only consider groups, rather than semigroups as in [36] .

Now suppose $\mathfrak{H}$ is a separable hilbert space and $\pm T \in \mathcal{C}_+(\mathfrak{H})$. If $A \in \mathcal{L}_0(\mathfrak{H})$ is $(T,-T,q)$ smooth where $1 < q < \infty$, we have seen in §2.2 the difficulties which arise when considering smoothness in this system at an algebra level. However, if we let

$$\sigma_t(x) = e^{-itT} x e^{itT} \qquad x \in B(\mathfrak{H}),$$

then for all x in $B(\mathfrak{H})$, $\xi \in \mathfrak{H}$, $\sigma_t(x) e^{-itT}\xi \in D_A$ a.e. in $\mathbb{R}^+$,

and $$\left\{ \int_0^\infty \|A\, \sigma_t(x)\, e^{-itT}\xi\|^q \, dt \right\}^{1/q} \leq |A|\, \|x\|\, \|\xi\|$$

Our perturbation theorem 3.3.1 is based on these "weak smoothness" conditions, and a few calculations by the reader will show that we have raised [36, Thm 3.1]. to the algebra level. It is useful to keep this motivation in mind.

We are thus led to study spaces such as

$$\mathcal{Y} = B(\mathfrak{H}, L^q(\mathbb{R}^+; \mathfrak{H})) \qquad \text{where} \qquad 1 < q < \infty.$$

Then $\mathcal{Y}$ is a dual Banach space, with predual

$$\mathcal{Z} = \overline{\mathfrak{H} \otimes^\vee L^p(\mathbb{R}^+; \mathfrak{H})} \qquad \text{where} \quad 1/p + 1/q = 1.$$

Again $\mathcal{Z}$ is a dual Banach space if we take

$$\mathcal{Z}_* = \overline{\mathfrak{H} \otimes^\wedge L^q(\mathbb{R}^+; \mathfrak{H})}.$$

Let V be the space of all operators on $\mathfrak{H}$. An element of V is not necessarily closed, densely defined or even linear. Then $\mathcal{Y}$ can be identified with the space of V-valued functions f on $\mathbb{R}^+$ satisfying

3.1.1) If $\xi \in \mathfrak{H}$, then $\xi \in D(f(s))$ a.e. Note this null set may depend on ξ.

3.1.2) If $\alpha, \beta \in \mathbb{C}$, and $\xi, \eta \in \mathfrak{H}$, then

$$f(s)(\alpha\xi + \beta\eta) = \alpha f(s)\xi + \beta f(s)\eta \quad \text{a.e. (depending on } \alpha,\beta,\xi,\eta).$$

3.1.3) For each $\xi \in \mathfrak{H}$, $s \longmapsto f(s)\xi$ is measurable, and there exists a finite M independant of ξ satisfying

$$\int_0^\infty \|f(s)\xi\|^q \, ds \leq M^q \|\xi\|^q \qquad \forall \ \xi \in \mathfrak{H} \qquad (3.1.4)$$

If $f \in \mathcal{Y}$, $\|f\|$ is the infimum of all M satisfying (3.1.4).

Now let M be a von Neumann algebra acting on the separable hilbert space $\mathfrak{H}$, which is ultraweakly countably generated in the sense that there is a countable subset R of M, such that the linear span of R is $\sigma(M,M_*)$ dense in M. Define $\mathcal{Y}_M$ to be the closed subspace of $\mathcal{Y}$ consisting of these elements which commute with the action of M' on $\mathfrak{H}$ and $L^q(\mathbb{R}^+;\mathfrak{H})$. i.e. $f \in \mathcal{Y}$ is in $\mathcal{Y}_M$ iff for each ξ in $\mathfrak{H}$, and m' in M', that

$$f(s)(m'\xi) = m' f(s)\xi \quad \text{a.e. (depending on } m' \text{ and } \xi\text{)}.$$

Now $\mathcal{Y}_m$ is a $\sigma(\mathcal{Y},\mathcal{Z})$ closed subspace of $\mathcal{Y}$. Since if $f_\alpha \longrightarrow f$ $\sigma(\mathcal{Y},\mathcal{Z})$, where $f_\alpha \in \mathcal{Y}_M$, $f \in \mathcal{Y}$; then for all $\xi, \eta \in \mathfrak{H}$, $h \in L^p(\mathbb{R}^+)$, $m' \in M'$,

$$\begin{aligned}\int \langle f(s)m'\xi, \eta\rangle h(s)\, ds &= \lim_\alpha \int \langle f_\alpha(s)m'\xi, \eta\rangle h(s)\, ds \\ &= \lim_\alpha \int \langle m' f_\alpha(s)\xi, \eta\rangle h(s)\, ds \\ &= \int \langle m' f(s)\xi, \eta\rangle h(s)\, ds .\end{aligned}$$

Thus $f \in \mathcal{Y}_M$.

Hence $(\mathcal{Z}/\mathcal{Y}_M^\circ)^*$ is isometrically isomorphic to $\mathcal{Y}_M$, where $\mathcal{Y}_M^\circ$ denotes the polar of $\mathcal{Y}_M$ in $\mathcal{Z}$. Moreover, if i_M denotes the canonical injection of $\mathcal{Y}_M$ in $\mathcal{Y}$, and if $\mathcal{Z}_M$ is defined as $i_M^* \mathcal{Z}$ (the restriction of $\mathcal{Z}$ to $\mathcal{Y}_M$) , then $\mathcal{Z}_M$ is isometrically isomorphic to $\mathcal{Z}/\mathcal{Y}_M^\circ$.

Thus $\mathcal{Z}_M$ is a closed subspace of $\mathcal{Y}_M^*$, and if $z_o \in \mathcal{Z}_M$, then

3.1.5) $\quad z_o = \inf\left\{ \|x\| : x \in \mathcal{Z} \text{ s.t. } i_M^*(x) = z_o \right\}.$

Now if $z = \sum_{i=1}^\infty \xi_i \otimes \overline{h}_i$ is in $\mathcal{Z}$, where $\xi_i \in \mathfrak{H}$, $h_i \in L^p(\mathbb{R}^+;\mathfrak{H})$, for i = 1,2,..... and $\sum_i \|\xi_i\| \, \|h_i\| < \infty$, then

$$\left\{ \int \left(\sum \|\xi_i\| \, \|h_i(s)\| \right)^p ds \right\}^{1/p} \leq \sum_{i=1}^{\infty} \left\{ \int \left(\|\xi_i\| \, \|h_i(s)\| \right)^p ds \right\}^{1/p}$$

$$= \sum \|\xi_i\| \, \|h_i\| < \infty \, .$$

Hence $\sum \|\xi_i\| . \|h_i(s)\| < \infty$ a.e. in $\mathbb{R}^+$, and we can define

$$z(s) = \sum \xi_i \otimes \overline{h_i(s)}$$

a.e. in $\mathfrak{H} \check{\otimes} \bar{\mathfrak{H}} = T(\mathfrak{H})$, the trace class operators on $\mathfrak{H}$.

Note that z(s) is uniquely determined by z a.e., $s \longmapsto z(s)$ is measurable, and

3.1.6) $\left\{ \int \|z(s)\|^p \, ds \right\}^{1/p} \leq \|z\|$.

If $f \in \mathcal{Y}$ is bounded a.e. (e.g. if f is actually in $L^q(\mathbb{R}^+;B(\mathfrak{H}))$ then

$$\langle f,z \rangle = \sum_i \int \langle f(s)\xi_i \, , h_i(s) \rangle \, ds \quad = \int \langle f(s),z(s) \rangle \, ds \, .$$

Similarly, we consider $\mathcal{Z}_M$. If $x \in \mathcal{Z}_M$, then $x = i_M^*(z)$, for some z in $\mathcal{Z}$.
Then $z(s) \in T(\mathfrak{H})$ determines an element x(s) in M_* a.e. Moreover x(s) is uniquely determined by x a.e., and $x(\cdot)$ is an element of $L^p(\mathbb{R}^+;M_*)$.
It is clear that whenever $x = i_M^*(z_1)$, where $z_1 \in \mathcal{Z}$, that

$$\langle x(s),m \rangle = \langle z_1(s),m \rangle \quad \text{a.e. in } \mathbb{R}^+, \text{ for any m in M.}$$

REMARK 3.1.7.

We thus have a linear map $x \longmapsto x(\cdot)$ from $\mathcal{Z}_M$ into $L^p(\mathbb{R}^+;M_*)$. This is norm reducing by 3.1.5 and 3.1.6. ;(alternatively, we see from that there is a natural norm reducing injection from $L^p(\mathbb{R}^+;M_*)^*$ into $\mathcal{Y}_M$ Then for all x in $\mathcal{Z}_M$, $\varphi \in L^p(\mathbb{R}^+;M_*)^*$ $|\langle x,\varphi \rangle| \leq \|x\| \, \|\varphi\|$
Hence $\left\{ \int \|x(s)\|^p \, ds \right\}^{1/p} \leq \|x\|$ $\forall x$ in $\mathcal{Z}_M$).

Then if $f \in \mathcal{Y}_M$, and $f(s) \in M$ a.e. (e.g. if f is actually in $L^q(\mathbb{R}^+;M)$), then for all z in $\mathcal{Z}_M$

$$\langle f,z \rangle = \int \langle f(s),z(s) \rangle \, ds \quad .$$

If $f \in \mathcal{Y}_M$, $z \in \mathcal{Z}_M$, the notation $f(s) \in V$, $z(s) \in M_*$ will be used extensively throughout this chapter.

We shall see during the proof of Theorem 3.3.1, that technical problems lead us to restrict our attention to consider only von Neumann algebras which satisfy the following hypothesis.

HYPOTHESIS 3.1.8.

We say M satisfies this hypothesis if whenever $\varphi(s) \in M_*$ a.e. in $\mathbb{R}^+$, and x_n, n=1,2.... is a sequence in $\mathcal{Z}_M$ such that

i) $x_n(s) = \varphi(s)$ a.e. in $[0,n]$ n=1,2,.....

ii) $\|x_n\| \leq T < \infty$, for all n.

then there exists an x in $\mathcal{Z}_M$ satisfying

a) $x(s) = \varphi(s)$ a.e. in $\mathbb{R}^+$.

b) $\|x\| \leq T$.

PROPOSITION 3.1.9

Suppose $\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}_i$, where $\mathfrak{H}_i$ are separable hilbert spaces; then $M = \bigoplus_i B(\mathfrak{H}_i)$ satisfies hypothesis 3.1.8.

PROOF.

It is clear that $\mathcal{Y}_M = \bigoplus_{i=1}^{\infty} \mathcal{Y}_{B(\mathfrak{H}_i)}$ where $\|(f_i)\| = \sup \|f_i\|$, $f_i \in \mathcal{Y}_{B(\mathfrak{H}_i)}$,

and $\mathcal{Z}_M = \bigoplus_{i=1}^{\infty} \mathcal{Z}_{B(\mathfrak{H}_i)}$ where $\|(z_i)\| = \sum_i \|z_i\|$, $z_i \in \mathcal{Z}_{B(\mathfrak{H}_i)}$.

Now $\mathcal{Z}_{B(\mathfrak{H}_i)} = (X_i)^*$, where $X_i = \mathfrak{H}_i \hat{\otimes} \overline{L^q(\mathbb{R}^+; \mathfrak{H}_i)}$, i=1,2,.... .

Suppose $\varphi(s) \in M_*$ a.e., and $x_n = (x_n^i)_{i=1}^{\infty}$ is a sequence in $\mathcal{Z}_M$ such that

i) $x_n(s) = \varphi(s)$ a.e. in $[0,n]$

ii) $\|x_n\| \leq T < \infty$ all n.

Then $x_n^m \in (X_m)^*$ m,n = 1,2,...

Hence there exists a subsequence n_k and x^m in $(X_m)^*$ such that

$$x_{n_k}^m \longrightarrow x^m \quad \sigma(X_m^*, X_m) \quad \text{as } k \longrightarrow \infty .$$

Thus for each finite N

$$(x_{n_k}^m)_{m=1}^{N} \longrightarrow (x^m)_{m=1}^{N} \quad \sigma\Big(\big(\bigoplus_{m=1}^{N} X_m \big)^*, \bigoplus_{m=1}^{N} X_m \Big) \quad \text{as } k \longrightarrow \infty .$$

Thus $\sum_{m=1}^{N} \|x^m\| \leq T$ for all N, and hence

$$x = (x^m) \in \mathcal{Z}_M = \bigoplus_{m=1}^{\infty} (X_m)^*, \text{ with } \|x\| \leq T.$$

Moreover $\langle x_m, \psi \rangle = \lim_{k\to\infty} \langle x^m_{n_k}, \psi \rangle$

for all $\psi \in X_m = \mathfrak{H}_m \hat{\otimes} \overline{L^q(\mathbb{R}^+; \mathfrak{H}_m)}$.

Using property i) above, it follows easily that

$$\varphi(s) = (x_m(s)) \quad \text{a.e. in } \mathbb{R}^+,$$

i.e. $$\varphi(s) = x(s) \quad \text{a.e.}$$

§ 3.2.

Having set up our apparatus, we proceed to make our smoothness definitions.. We will use the notation of the previous section with q fixed in $(1,\infty)$. Let M be a von Neumann algebra acting on a separable hilbert space $\mathfrak{H}$, which is ultraweakly countably generated. Let σ_t be an ultraweakly continuous one parameter group of ultraweakly continuous bounded linear maps on M, (i.e. each σ_t is a $\sigma(M,M_*)$ continuous operator on M, and $t \longmapsto \langle \sigma_t(x), \varphi \rangle$ is continuous for all x in M, φ in M_*). For brevity, we will always refer to such a group as a weakly continuous group on M. We define iT to be the infinitesimalgenerator of $(\sigma_t)^*$, which is a strongly continuous group on the predual M_*. We will eventually perturb such a one parameter group. This perturbation of T will be of the form $K\alpha_*\beta_*$ where K is complex and small, and β, α are smooth operators on M to be defined. (For simplicity, we take the domain of β and the codomain of α to be in M).

In order to have smooth conditions which are practical, we introduce an auxiliary weakly continuous one parameter group Γ_t on M, which satisfies the following:

3.2.1) There exists a family $\{u_t : t \in \mathbb{R}\}$ in $B(\mathfrak{H})$, such that u_t is invertible a.e. and

i) $\Gamma_t(x) = u_t x u_t^{-1}$ a.e. $\forall x \in B(\mathfrak{H})$.

ii) $\|u_t\|, \|u_t^{-1}\| \leq M_0 < \infty$, for all t in $\mathbb{R}$.

We then let $v_t = u_t^{-1}$.

Let $A \in \mathcal{C}_0(\mathfrak{H})$ be affiliated with M.

DEFINITION 3.2.2.

We define the left multiplication operator $\alpha_A = \alpha$ on M, associated with A, as follows:

We put $D_\alpha = \{x \in M: x\mathfrak{H} \subseteq D_A\}$, and $\alpha x = Ax$ for $x \in D_\alpha$.

Then $\alpha : D_\alpha \longrightarrow M$ is a linear operator on M which is closed and densely defined for the ultraweak topology.

DEFINITION 3.2.3.

We say α is $(\sigma,\Gamma,q,+)$ smooth if

I) There exists $Y \in B(M, \mathcal{Y}_M)$

2) For all u in M, ξ in $\mathfrak{H}$ $\sigma_{-t}(u)v_t\xi \in D_A$ a.e. in $\mathbb{R}^+$ (depending on u,ξ)

such that

$$[Y(u)]\,(s)\xi = u_t A\,\sigma_{-t}(u)\,v_t\xi \quad \text{a.e. in } \mathbb{R}^+.$$

We then define $|\alpha|_{(\sigma,\Gamma,q,+)}$ to be $\|Y\|$.

$Y(u)$ is formally written as $\Gamma_t(A\sigma_{-t}(u))$.

Similarly, we define $(\sigma,\Gamma,q,-)$ smoothness with associated function $\hat{Y}$ in $B(M, \mathcal{Y}_M)$. If α is $(\sigma,\Gamma,q,\pm)$ smooth, we say α is (σ,Γ,q) smooth and set $|\alpha|_{(\sigma,\Gamma,q)} = \max\{|\alpha|_{(\sigma,\Gamma,q,+)}, |\alpha|_{(o,\Gamma,q,-)}\}$.

Let $\beta: M \longrightarrow M$ be a linear operator on M, which is closed and densely defined for the ultraweak topology. p is given by $1/p + 1/q = 1$.

DEFINITION 3.2.4.

We say β_* (as defined in Lemma 2.2.2.) is $(\sigma,\Gamma,p,+)_*$ smooth if for each $r > o$

i) There exists $Z_r \in B(M_*, \mathcal{Z}_M)$ with $\sup\limits_{r>o}\|Z_r\| < \infty$

ii) For each $\varphi \in M_*$, $e^{-itT}\varphi \in D_{\beta_*}$ a.e. in $\mathbb{R}^+$

such that

$$[Z_r(\varphi)](s) = (\Gamma_{-s})_* \beta_* e^{-i(r-s)T} \varphi \chi_{(o,r)}(s) .$$

We then define $|\beta_*|_{(\sigma,\Gamma,p,+)_*} = \sup_{r>o} \|Z_r\|$

Similarly, we define $(\sigma,\Gamma,p,-)_*$ smoothness, with associated functions $\hat{Z}_r$ in $B(M_*, \mathcal{Z}_M)$. If β_* is $(\sigma,\Gamma,p,\pm)_*$ smooth we say β_* is $(\sigma,\Gamma,p)_*$ smooth and set

$$|\beta_*|_{(\sigma,\Gamma,p)_*} = \max\left\{ |\beta_*|_{(\sigma,\Gamma,p,+)_*} , |\beta_*|_{(\sigma,\Gamma,p,-)_*} \right\} .$$

We take $Z_o = \hat{Z}_o = 0$.

Now fix f in $\mathcal{Y}_M$, $r > o$ and suppose β_* is $(\sigma,\Gamma,p)_*$ smooth. Then for all φ in M_*,

$$|\langle f, Z_r(\varphi)\rangle| \leq \|f\| |\beta_*| \|\varphi\| .$$

Moreover $\varphi \longrightarrow \langle f, Z_r(\varphi)\rangle$ is linear. Hence there exists $m^f(r)$ in $M = (M_*)^*$, such that

3.2.5) $\langle m^f(r), \varphi\rangle = \langle f, Z_r(\varphi)\rangle \qquad \forall \varphi \in M_*, \quad r \geq o, \quad f \in \mathcal{Y}_M$;

and similarly $\hat{m}^f(r)$ in M satisfying

3.2.6) $\langle \hat{m}^f(r), \varphi\rangle = \langle f, \hat{Z}_r(\varphi)\rangle \qquad \forall \varphi \in M_*, \quad r \geq o, \quad f \in \mathcal{Y}_M$.

DEFINITION 3.2.7.

If $A \in \mathcal{C}_b(\mathfrak{H})$ is affiliated with M, $\alpha = \alpha_A$, and β_* is $(\sigma,\Gamma,p)_*$ smooth, we define an operator symbolically denoted by $(\alpha\sigma_{-(\cdot)}\beta)\theta$ as follows: If $f \in \mathcal{Y}_M$, let $m^f(r)$ be the M valued function on $\mathbb{R}^+$ as defined in 3.2.5. If

i) For each $\xi \in \mathfrak{H}$, $m^f(r)v_r\xi \in D_A$ a.e. (depending on ξ)

ii) There exists $g \in \mathcal{Y}_M$, such that for each $\xi \in \mathfrak{H}$,

$$g(s)\xi = u_s A\, m^f(s)\, v_s\xi \text{ a.e. (depending on } f, \xi)$$

then we say $f \in D(\alpha\sigma_{-(\cdot)}\beta\theta)$ and put $(\alpha\sigma_{-(\cdot)}\beta\theta)f = g$.

We have the following comments to make regarding the system described in 3.2.7.

REMARKS 3.2.8.

a) We can similarly define $(\alpha\sigma_{+(\cdot)}\beta)\theta$ on $\mathcal{Y}_M$.

b) Suppose $f \in \mathcal{Y}_M$ is in $D(\alpha\sigma_{-(\cdot)}\beta\theta)$.

Take $\xi, \eta \in \mathfrak{H}$, and define $\varphi \in M_*$ by $\varphi m = \langle m\xi, \eta\rangle \ \forall m \in M$.

Then if $C = \alpha\sigma_{-(\cdot)}\beta\theta$ and m^f is as defined in (3.2.5)

$$\langle (Cf)(r)\xi, \eta\rangle = \langle u_r A\, m^f(r)\, v_r \xi, \eta\rangle \quad \text{a.e. (depending on } \xi \text{).}$$

Take $Q_n \in M$, such that $Q_n \to 1$ $\sigma(M,M_*)$ and AQ_n is bounded for each n, and if $\xi \in D_A$ then $AQ_n\xi \to A\xi$.

Then if $(\Gamma_r)_*\varphi \in D_{\alpha_*}$ a.e., we have

$$\begin{aligned}\langle (Cf)(r)\xi, \eta\rangle &= \lim_{n\to\infty} \langle u_r[AQ_n] m^f(r)\, v_r\xi, \eta\rangle \\ &= \lim_{n\to\infty} \langle AQ_n\, m^f(r), (\Gamma_r)_*\varphi\rangle \\ &= \lim_{n\to\infty} \langle Q_n\, m^f(r), \alpha_*(\Gamma_r)_*\varphi\rangle \\ &= \langle m^f(r), \alpha_*(\Gamma_r)_*\varphi\rangle \quad \text{a.e. (depending on } \varphi \in M_*)\end{aligned}$$

c) It is clear that there exists more than one sequence of projections P_n in M such that

i) $P_n \longrightarrow 1$ $\sigma(M,M_*)$

ii) P_nA is bounded for each n.

Then if $f \in \mathcal{Y}_M$, we define

$$f_n(s)\xi = \Gamma_s(P_n)\,[f(t)]\xi \quad \text{a.e.}$$

Then $f_n \in \mathcal{Y}_M$, and if $z = \sum \xi_i \otimes \bar{h}_i \in \mathcal{Z}$, where $\xi_i \in \mathfrak{H}$, $h_i \in L^p(\mathbb{R}^+; \mathfrak{H})$, and $\sum \|\xi_i\|\, \|h_i\| < \infty$, we have

$$\sum_i |\langle f_n(t)\xi_i, h_i(t)\rangle| \leq \sum_i (M_o)^2 \|f(t)\xi_i\|\, \|h_i(t)\| \quad .$$

Also as $n \longrightarrow \infty$

$$\sum_{i=1}^{\infty} \langle f_n(t)\xi_i, h_i(t)\rangle \longrightarrow \sum_{i=1}^{\infty} \langle f(t)\xi_i, h_i(t)\rangle \quad ,$$

by i) and the $\sigma(M,M_*)$ continuity of Γ_t .

It follows by dominated convergence that $f_n = \Gamma_{(\cdot)}(P_n)f \longrightarrow f$ $\sigma(\mathcal{Y}_M, \mathcal{Z}_M)$ as $n \longrightarrow \infty$.

We will use the sequences P_n , Q_n extensively in the proof of our perturbation theorem.

d) If $f \in \mathcal{Y}_M$, with $f(s)$ in M a.e., then

$$\langle f , Z_r(\varphi)\rangle = \int_0^\infty \langle f(s) , [Z_r(\varphi)](s)\rangle \, ds$$

$$= \int_0^\infty \langle f(s) , (\Gamma_{-s})_* \, \beta_* \, e^{-iT(r-s)}\varphi\rangle \, ds$$

for all $\varphi \in M_*$, $r \geqslant 0$.

§ 3.3.

We can now state our perturbation theorem (c.f. [36, Theorem 3.1.]).

THEOREM 3.3.1.

Let M be a type I von Neumann algebra with discrete centre acting on a separable hilbert space $\mathfrak{H}$, as in 3.1.9. Let σ_t be a weakly continuous one parameter group on M, with iT denoting the infinitesimal generator of $(\sigma_t)_*$, a strongly continuous group on type $\omega_0 \geqslant 0$. Let Γ_t be an auxiliary weakly continuous one parameter group on M satisfying 3.2.1. $A \in \mathcal{C}_0(\mathfrak{H})$ is affiliated with M, and gives a left multiplication operator $\alpha = \alpha_A$, as defined in 3.2.2. ß is a linear operator on M which is closed and densely defined for the ultraweak topology. Suppose that α is (σ,Γ,q) smooth with associated functions Y , $\hat{Y}$ in $B(M, \mathcal{Y}_M)$, such that $u \longmapsto Yu$, $\hat{Y}u$ are continuous for the $\sigma(M,M_*)$ and $\sigma(\mathcal{Y}_M, \mathcal{Z}_M)$ topologies. Moreover assume that if $\varphi \in M_*$, then $(\Gamma_t)_*\varphi \in D_{\alpha_*}$ a.e. in $\mathbb{R}$. Also, ß is $(\sigma,\Gamma,p)_*$ smooth with associated maps Z_t , $\hat{Z}_t$, and for each $\varphi \in M_*$, $t \longmapsto \langle f, Z_t(\varphi)\rangle$, $\langle f , \hat{Z}_t(\varphi)\rangle$ are continuous from the right on $[0,\infty)$. Moreover Graph ß is countably σ-weakly generated. Also, $\alpha\sigma_{\pm(\cdot)}\beta\theta$ belong to $B(\mathcal{Y}_M)$, and are $\sigma(\mathcal{Y}_M, \mathcal{Z}_M)$ continuous with common bound $\leqslant N$.

Then for each κ in $D = \{\lambda \in \mathbb{C} : |\lambda| < 1/N\}$, there is a weakly continuous one parameter group σ_t^κ on M, with $iT(\kappa)$ the infinitesimal generator of the strongly continuous group $(\sigma_t^\kappa)_*$, of type ω_0 , uniquely determined by (σ,α,β) with the following properties :

1) α is $(\sigma^\kappa, \Gamma, q)$ smooth, with associated functions Y^κ, $\hat{Y}^\kappa$ in $B(M, \mathcal{Y}_M)$ and

$$|\alpha|_{(\sigma^\kappa,\Gamma,q)} \leq (1-|\kappa|N)^{-1}|\alpha|_{(\sigma,\Gamma,q)} .$$

2) β_* is $(\sigma^\kappa, \Gamma, p)_*$ smooth with associated functions Z_r^κ, $\hat{Z}_r^\kappa$ in $B(M_*, \mathcal{Z}_M)$ for each $r \geq o$, with

$$|\beta_*|_{(\sigma^\kappa,\Gamma,p)_*} \leq (1-|\kappa|N)^{-1}|\beta_*|_{(\sigma,\Gamma,p)_*} .$$

3) For each κ in D, there exists Z^κ in $B(M_*, \mathcal{Z}_M)$ with

$$[Z^\kappa\varphi](s) = (\Gamma_s)_* \beta_* e^{-iT(\kappa)s}\varphi \qquad \text{a.e. in } \mathbb{R}^+, \text{ for each } \varphi \in M_*,$$

and $\|Z^\kappa\| \leq M_o^2 (1-|\kappa|N)^{-1}|\beta_*|_{(\sigma,\Gamma,p)_*}$.

Similarly for the other half line, with $\hat{Z}$ in $B(M_*, \mathcal{Z}_M)$.

4) $iT(\kappa)$ is similar to iT. More specifically, there exists for each κ in D, two non singular weakly continuous operators $W_\pm(\kappa)$ on M given by

$$\langle W_+(\kappa)m,\varphi\rangle = \langle m,\varphi\rangle + i\kappa\langle Y(u), \hat{Z}^\kappa(\varphi)\rangle$$
$$\langle W_-(\kappa)m,\varphi\rangle = \langle m,\varphi\rangle - i\kappa\langle \hat{Y}(u), Z^\kappa(\varphi)\rangle$$

for all m in M, φ in M_*, and satisfying

$$T(\kappa) = W_\pm(\kappa)_*^{-1}\, T\, W_\pm(\kappa)_* .$$

Moreover $W_\pm(\kappa) = \sigma\text{-weak}\lim\limits_{t\to\pm\infty} \sigma_t\,\sigma_{-t}$

5) $T(\kappa) \geq T + \kappa\alpha_*\beta_*$; $T \geq T(\kappa) - \kappa\alpha_*\beta_*$.

PROOF.

To construct the perturbed group, we first derive two semigroups.

Let $U_o(t) = \sigma_{-t}$ be the unperturbed group. For each fixed m in M, set

$$S_o(t)m = u_t A\sigma_{-t}(m)v_t \qquad \text{for } t \geq o.$$

By our smoothness assumptions, $S_o(\cdot)m \in \mathcal{Y}_M$, $S_o(\cdot)m = Y(m)$, and

$$\|S_o(\cdot)m\| \leq |\alpha|\|m\| \quad \forall m \in M.$$

Inductively define for $n \geq 1$,

$$S_n(\cdot)m = (\alpha\sigma_{-(\cdot)}\beta\Theta)\, S_{n-1}(\cdot)m \quad \forall m \in M.$$

Then $S_n(\cdot)m \in \mathcal{Y}_M$ and $\|S_n(\cdot)m\| \leq N^n|\alpha|\|m\|$.

To construct our perturbed semigroups, we first define operators $U_n(t)$ as follows:

For $o < t < \infty$, $n \geq 1$, consider for $m \in M, \varphi \in M_*$,

$$J_n(t,m,\varphi) = \langle S_{n-1}(\cdot)m,\ Z_t(\varphi)\rangle .$$

Then $|J_n(t,m,\varphi)| \leqslant N^{n-1}\,\|m\|\,|\alpha|\,|\beta_*|\,\|\varphi\|$

and $\varphi \longmapsto J_n(t,m,\varphi)$ is linear on M_*. Hence there exists an unique w_t in M such that $\langle w_t, \varphi\rangle = J_n(t,m,\varphi)$.

Define $U_n(t)m = w_t$. Set $U_n(o) = o$ $(n \geqslant 1)$.

Now we define

$$U(t,\kappa)m = \sum_{n=o}^{\infty} (-i\kappa)^n\, U_n(t)m \qquad \text{for all } m \text{ in } M,\ \kappa \text{ in } D.$$

We will show that $\{U(t,\kappa)\}_{o\leqslant t<\infty}$ is a weakly continuous semigroup on M.

LEMMA 3.3.2.

The family $U(t,\kappa)$ is a semigroup of operators in B(M) for each κ in D.

PROOF.

Let us consider the series $\sum_{n=1}^{\infty} (-i\kappa)^n\, U_n(t)$.

For each u in M, we have

$$\Big\|\sum_{n=1}^{\infty} (-i\kappa)^n\, U_n(t)u\Big\| \leqslant \sum_{n=1}^{\infty} \|U_n(t)u\|\,|\kappa|^n \leqslant \sum |\kappa|^n\ N^{n-1}|\alpha|\ |\beta_*|\ \|u\| .$$

Thus for any κ in D, the series converges absolutely and uniformly with respect to t, $(o \leqslant t < \infty)$.

Since $$U(t,\kappa) = U_o(t) + \sum_{n=1}^{\infty} (-i\kappa)^n\, U_n(t)$$

it follows that $U(t,\kappa)$ is in B(M) for each $t \geqslant o$.

We now observe that if $\xi \in \mathfrak{H}$ then

$$U_n(t)(u)\ v_t\xi \in D_A \quad \text{a.e. in } \mathbb{R}^+$$

and $u_t\, A\, U_n(t)(u)\ v_t\xi = S_n(t)(u)\xi$ a.e. in $\mathbb{R}^+$,

since, in the notation of 3.2.5, if we take $f = S_{n-1}(u)$, then $m^f(t) = U_n(t)u$ in $\mathbb{R}^+$. Thus if $m \in M, \varphi \in M_*$,

$$\begin{aligned}\langle U_n(t)u, \varphi\rangle &= \langle S_{n-1}(\cdot)u\ ,\ Z_t(\varphi)\rangle \\ &= \lim_{j\to\infty} \langle \Gamma_{(\cdot)}(P_j)\ S_{n-1}(\cdot)u,\ Z_t(\varphi)\rangle \\ &= \lim_{j\to\infty} \sum_i \int_o^{\infty} \langle \Gamma_t(P_j)\, [u_t A U_{n-1}(t)uv_t]\, \xi_i\ ,\ h_i(t)\rangle\ dt\ ,\end{aligned}$$

if $Z_t(\varphi) = i_M^* \left[\sum \xi_i \otimes \bar{h}_i\right]$, where $\xi_i \in \mathfrak{H}'$, $h_i \in L^p(\mathbb{R}^+;\mathfrak{H})$ and $\sum \|\xi_i\| \, \|h_i\| < \infty$.

Thus $\langle U_n(t)u,\varphi\rangle = \lim_{j\to\infty} \sum \int_0^\infty \langle u_r \, P_j \, A \, U_{n-1}(r)(u) \, v_r \, \xi_i \, , \, h_i(r)\rangle \, dr$

$$= \lim_{j\to\infty} \int_0^\infty \langle \Gamma_r\left(P_j A U_{n-1}(r)(u)\right) , |Z_t(\varphi)|(r)\rangle \, dr$$

$$= \lim_{j\to\infty} \int_0^t \langle P_j A U_{n-1}(r)u, \, \beta_* e^{-iT(t-r)}\varphi\rangle \, dr \quad .$$

Now in order to prove that $U(t,\kappa)$ is a semigroup, it is enough to show that $U_n(t+s) = \sum_{j=0}^{n} U_j(t) \, U_{n-j}(s)$ for $s,t \geq 0$.

We do this by induction on n. Suppose $n \geq 1$. Then if $u \in M$, $\varphi \in M_*$,

$$\sum_{j=0}^{n} \langle U_j(t) \, U_{n-j}(s)u, \, \varphi\rangle$$

$$= \lim_{m\to\infty} \sum_{j=1}^{n} \int_0^t \langle P_m A U_{j-1}(r) U_{n-j}(s)u, \, \beta_* e^{-iT(t-r)}\varphi\rangle \, dr + \langle U_0(t)U_n(s)u, \varphi\rangle$$

$$= \lim_{m\to\infty} \int_0^t \langle P_m A \, U_{n-1}(r+s)u, \, \beta_* \, e^{-iT(t-r)}\varphi\rangle \, dr + \langle U_0(t)U_n(s)u, \varphi\rangle$$

by our induction hypothesis

$$= \lim_{m\to\infty} \int_s^{s+t} \langle P_m A \, U_{n-1}(w)u, \, \beta_* \, e^{-iT(t+s-w)}\varphi\rangle \, dw + \langle U_0(t)U_n(s)u, \varphi\rangle$$

$$= \langle U_n(t+s)u, \varphi\rangle \quad .$$

The lemma follows.

LEMMA 3.3.3.

The semigroup $\{U(t,\kappa): t \geq 0\}$ is a weakly continuous semigroup of operators of M, of type ω_0.

PROOF.

For $n \geq 1$, $u \in M$, $\varphi \in M_*$, $t \geq 0$

$$\langle U_n(t)u,\varphi\rangle = \langle S_n(\cdot)u, \, Z_t(\varphi)\rangle \quad .$$

Now $u \longrightarrow S_0 u = Y(u)$ is $\sigma(\mathcal{Y}_M, \mathcal{Z}_M) - \sigma(M,M_*)$ continuous.

Moreover $\alpha\sigma_{-(\cdot)}\beta\theta$ is weakly continuous.

Hence $u \longrightarrow S_n u$ is weakly continuous, which shows that $u \longrightarrow \langle U_n(t)u, \varphi\rangle$ is $\sigma(M,M_*)$ continuous.

It follows that $u \longmapsto U(t,\kappa)u$ is $\sigma(M,M_*)$ continuous. Moreover, by assumption, for each fixed φ in M_*, $t \longmapsto Z_t(\varphi)$ is $\sigma(\mathcal{Z}_M, \mathcal{Y}_M)$ continuous from the right, so that $t \longrightarrow U_n(t)u$ is $\sigma(M,M_*)$ continuous from the right. Hence, being the sum of $U_o(t)u$ and $\sum_{n=1}^{\infty} (-i\kappa)^n U_n(t)u$, (which converges uniformly and absolutely with respect to t), $U(t,\kappa)u$ is $\sigma(M,M_*)$ continuous from the right. That is, $U(t,\kappa)_*$ is weakly $(\sigma(M_*,M))$ continuous from the right. Hence by [65,p233], $U(t,\kappa)_*$ is strongly continuous in t. The last claim is clear.

We denote the infinitesimal generator of $U(t,\kappa)_*$ by $-iT(\kappa)$, for κ in D, and write σ^{κ}_{-t} for $U(t,\kappa)$, $t \geqslant 0$.

REMARK 3.3.4.

We may replace T by -T, and define for $t \geqslant o$, $\hat{U}_n(t)$ by

$$\hat{U}_o(t) = \sigma_t ,$$

$$\langle \hat{U}_n(t)u, \varphi\rangle = -\langle \hat{S}_{n-1}(\cdot)u, \hat{Z}_t(\varphi)\rangle \qquad u \in M, \varphi \in M_*$$

where $\hat{S}_o(\cdot)u = \hat{Y}(u)$, and $\hat{S}_n(\cdot)u = (\alpha\sigma_{+(\cdot)}\beta\theta)\hat{S}_{n-1}(\cdot)u$.

Then if we put

$$\hat{U}(t,\kappa) = \sum_{n=o}^{\infty} (-i\kappa)^n \hat{U}_n(t)$$

we deduce the existence of a weakly continuous semigroup on M, for each κ in D. We shall denote by $i\hat{T}(\kappa)$, the infinitesimal generator of the strongly continuous semigroup $\hat{U}(t,\kappa)_*$ on M_* of type ω_o, and for $\hat{U}(t,\kappa)$ we write $\hat{\sigma}^{\kappa}_t$, $t \geq o$.

LEMMA 3.3.5.

For all κ in D, $t \geqslant o$,

$$\hat{\sigma}^{\kappa}_t \, \sigma^{\kappa}_{-t} = 1 = \sigma^{\kappa}_{-t} \, \hat{\sigma}^{\kappa}_t .$$

PROOF.

We will inductively show, for $n \geqslant 1$ that $\sum_{j=o}^{n} \hat{U}_j(t)\, U_{n-j}(t) = o$.

Take $u \in M$, $\varphi \in M_*$; now

$$\sum_{j=o}^{n} \langle \hat{U}_j(t)\ U_{n-j}(t)u, \varphi\rangle$$

$$= - \sum_{j=1}^{n} \lim_{m\to\infty} \int_o^t \langle (P_m\ A)\hat{U}_{j-1}(r)\ U_{n-j}(t)u,\ \beta_*\ e^{iT(t-r)}\varphi\rangle\, dr\ + \langle \hat{U}_o(t)U_n(t)u, \varphi\rangle$$

$$= - \sum_{l+k+s=n-1} \lim_{m\to\infty} \int_o^t \langle (P_m\ A)\hat{U}_l(r)U_k(r)U_s(t-r)u,\ \beta_* e^{+iT(t-r)}\varphi\rangle\ dr$$

$$+ \langle \hat{U}_o(t)U_n(t)u, \varphi\rangle$$

$$= - \lim_{m\to\infty} \int_o^t \langle (P_m A)\ U_{n-1}(t-r)u,\ \beta_*\ e^{iT(t-r)}\varphi\rangle\ dr \quad + \langle \hat{U}_o(t)U_n(t)u, \varphi\rangle$$

by our induction hypothesis

$= 0.$

Hence $\hat{\sigma}^\kappa_t\ \sigma^\kappa_{-t} = 1$, and similarly for the other relation.

As a corollary we see that $T(\kappa) = \hat{T}(\kappa)$ for all κ in D, and we have a one parameter weakly continuous group σ^κ_t, for $\kappa \in D$.

LEMMA 3.3.6.

α is $(\sigma^\kappa, \Gamma, q)$ smooth with $|\alpha|_{(\sigma^\kappa, \Gamma, q)} \leq (1-|\kappa|N)^{-1}|\alpha|_{(\sigma, \Gamma, q)}$.

PROOF.

If $u \in M$, $\xi \in \mathfrak{h}$, then $U_n(t)(u)\ v_t\xi \in D_A$ a.e. in $\mathbb{R}^+$ and

$$S_n(t)u\xi = u_t\ A\ U_n(t)(u)\ v_t\xi \qquad \text{a.e.}$$

Let $\quad Y_n(u) = \sum_{m=o}^{n} (-i\kappa)^m\ S_m u.$

It follows that for κ in D,

$$\sum \|(-i\kappa)^m\ S_m u\| \leq \sum |\kappa|^m\ N^m|\alpha|_{(\sigma,\Gamma,q,+)}\|u\| = (1-|\kappa|N)^{-1}|\alpha|_{(\sigma,\Gamma,q,+)}\|u\|.$$

Hence there exists Y^κ in $B(M, \mathcal{Y}_M)$ such that

$$Y^\kappa(u) = \sum_{m=o}^{\infty} (-i\kappa)^m\ S_m u \qquad u \in M.$$

Fix $u \in M$, $\xi \in \mathfrak{h}$. Then we can pick a subsequence n_k such that

$$\left[Y_{n_k}(u)\right](s)\xi \underset{n}{\longrightarrow} \left[Y^\kappa(u)\right](s)\xi \qquad \text{a.e. in } \mathbb{R}^+.$$

Note $\left[Y_n(u)\right](s) = \sum_{m=o}^{n} u_t\ A\ U_m(t)(u)\ v_t\xi \qquad$ a.e. in $\mathbb{R}^+$.

But $\sum_{m=1}^{\infty} (-i\kappa)^m\ U_m(t)u$ converges uniformly in t. It follows, since u_tA is closed, that $\sum (-i\kappa)^m\ U_m(t)\ v_t\xi \in D_A$ a.e., and

$u_t\ A\ (\sum(-i\kappa)^m\ U_m(t)(u)\ v_t\xi) = \lim\ \left[Y_{n_k}(u)\right](t)\xi \quad = \left[Y^\kappa(u)\right](t) \quad$ a.e.

In other words $\sigma^\kappa_{-t}(u)v_t\xi \in D_A$ a.e.

and $\left[Y^\kappa(u)\right](t)\xi \;=\; u_t\ A\ \sigma^\kappa_{-t}(u)\ v_t\xi$ a.e. in $\mathbb{R}^+$.

Similarly for the other half line.

REMARK 3.3.7.

We now investigate the $\sigma(\mathcal{Y}_M,\mathcal{Z}_M)$ continuous operator $C = \alpha\sigma_{-(\cdot)}\beta\Theta$ in $B(\mathcal{Y}_M)$. We denote the restriction of C^* to $\mathcal{Z}_M$ by C_* (as usual). Take v in D_β, and $[a,b]$ a compact interval in $\mathbb{R}^+$. Define f in $\mathcal{Y}_M$ by

$$f(s) = \Gamma_s(v)\,\chi_{[a,b)}(s)\ .$$

Take any ξ,η in $\mathfrak{H}$, and define $\varphi\in M_*$ by $\varphi(m) = \langle m\xi,\eta\rangle$ for $m\in M$. Then for almost all s in $\mathbb{R}^+$, depending on φ, we have

$$\begin{aligned}\langle (Cf)(s)\xi,\eta\rangle &= \int_0^s \langle f(\tau),\Gamma_{-\tau}\beta_*\ e^{-i(s-\tau)T}\alpha_*(\Gamma_s)_*\varphi\rangle\ \chi_{(a,b)}(\tau)\ d\tau\\ &= \int_0^s \langle v,\ \beta_* e^{-i(s-\tau)T}\alpha_*(\Gamma_s)_*\varphi\rangle\,\chi_{(a,b)}(\tau)\ d\tau\ .\\ &= \begin{cases} 0 & \text{if } s < a\\ \int_a^s\langle\sigma_{-(s-\tau)}\beta(v),\alpha_*(\Gamma_s)_*\varphi\rangle\ d\tau & \text{if } s\in[a,b)\\ \int_a^b\langle\sigma_{-(s-\tau)}\beta(v),\alpha_*(\Gamma_s)_*\varphi\rangle\ d\tau & \text{if } s\geq b.\end{cases}\end{aligned}$$

For $(s,t)\in(\mathbb{R})^2$, define for $h\in L^p(\mathbb{R}^+)$,

$$u(s,t) \;=\; \langle\sigma_{-(s-t)}\beta(v),\alpha_*(\Gamma_s)_*\varphi\rangle\ \ h(s),$$

which is jointly measurable. Moreover

$$\begin{aligned}\int_a^b\int_\tau^\infty |u(s,\tau)|\ ds\ d\tau &\leq \int_a^b\left(\int_\tau^\infty|\langle u_sA\ \sigma_{-(s-\tau)}(\beta(v))v_s\xi,\eta\rangle\ h(s)|ds\right)d\tau\\ &\leq \int_a^b\|\sigma_\tau\beta(v)\|\ \|\xi\|\ \ \|\eta\|\ \|h\|\ d\tau \quad < \infty\end{aligned}$$

as $[a,b]$ is compact, and for each fixed τ, we have a.e.

$$\begin{aligned}\langle u_s\ A\ \sigma_{-(s-\tau)}\ \beta(v)\ v_s\xi,\eta\rangle\,h(s) &= \lim_{n\to\infty}\ \langle u_s(AQ_n)\sigma_{-(s-\tau)}\beta(v)v_s\xi,\eta\rangle\,h(s)\\ &= \lim_{n\to\infty}\ \langle Q_n\sigma_{-(s-\tau)}\beta(v),\alpha_*(\Gamma_s)_*\varphi\rangle\,h(s)\\ &= \qquad\langle\sigma_{-(s-\tau)}\beta(v),\alpha_*(\Gamma_s)_*\varphi\rangle\,h(s).\end{aligned}$$

Hence we can apply Fubini's theorem, and deduce that if

$$g = i_M^*\left(\xi \otimes (\bar{\eta} h)\right) \in \mathcal{Z}_M, \text{ that}$$

$$\langle Cf, g\rangle = \int_a^b \langle Cf(s)\xi, \eta\rangle h(s)\, ds$$

$$= \int_a^b \left(\int_a^s \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\varphi\rangle\, d\tau\right) h(s)\, ds$$

$$+ \int_b^\infty \left(\int_a^b \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\varphi\rangle\, d\tau\right) h(s)\, ds$$

$$= \int_a^b \int_\tau^\infty \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\varphi\rangle\, h(s)\, ds\, d\tau$$

It follows that whenever $g \in \mathcal{Z}_M$, $g = i_M^*(z)$ where $z = \sum_{i=1} \xi_i \otimes \bar{k}_i$

$\xi_i \in \mathfrak{H}$, $k_i \in L^p(\mathbb{R}^+; \mathfrak{H})$ and $\sum \|\xi_i\|\, \|k_i\| < \infty$, that

$$\langle Cf, g\rangle = \int_a^b \left(\int_\tau^\infty \langle u_s A\, \sigma_{-(s-\tau)}\beta(v)\, v_s\, \xi_i, k_i(s)\rangle\, ds\right) d\tau\ .$$

On the other hand

$$\langle Cf, g\rangle = \langle f, C_* g\rangle = \int_a^b \langle \Gamma_\tau^1(v), [C_* g](\tau)\rangle\, d\tau\ .$$

Thus almost everywhere, (depending on $v \in D_\beta$)

$$\langle \Gamma_\tau^1(v), [C_* g](\tau)\rangle = \sum_i \int_\tau^\infty \langle u_s A\, \sigma_{-(s-\tau)}\left(\beta(v)\right) v_s\, \xi_i, h_i(s)\rangle\, ds\ .$$

LEMMA 3.3.8.

Fix $t > 0$, and define for $\varphi \in M_*$, $m \in \mathbb{Z}$,

$$U_{m,t}(r)_*\varphi = \begin{cases} U_m(t-r)_*\varphi & \text{if } r \in [0,t] \\ 0 & \text{if } r > t. \end{cases}$$

Then $U_{m,t}(r)_*\varphi \in D_{\beta_*}$ a.e. and

$$\Gamma_{-r}^1\, \beta_*\, U_{m,t}(r)_*\phi = \left[(C_*)^m\, Z_t(\varphi)\right](r) \text{ a.e. (depending on } \varphi \text{ and } t).$$

PROOF.

For $m = 0$, see the definition of $Z_t(\varphi)$.

$m \geq 1$. Let $w \in D_\beta$. Then

$$\langle \Gamma_r^1(w), [C_*^m Z_t(\varphi)](r)\rangle = \langle \Gamma_r^1(w), [C_*\, C_*^{m-1} Z_t(\varphi)](r)\rangle$$

$$= \lim_{p\to\infty} \int_r^\infty \langle \Gamma_s^1[(P_p A)\sigma_{-(s-r)}\beta(w)], [C_*^{m-1} Z_t(\varphi)](s)\rangle\, ds$$

$$= \lim_{p\to\infty} \int_0^\infty \langle \Gamma_{s+r}^1[P_p A\, \sigma_{-s}\beta(w)], [C_*^{m-1} Z_t(\varphi)](s+r)\rangle\, ds.$$

Now $\lfloor C_*^{m-1} Z_t(\varphi) \rfloor (s+r) = (\Gamma_{-s-r})_* \, \beta_* \, U_{m-1,t}(s+r)_* \varphi$
$= (\Gamma_{-s-r})_* \, \beta_* \, U_{m-1,t-r}(s)_* \varphi$
$= (\Gamma_{-r})_* \, [C_*^{m-1} Z_{t-r}(\varphi)](s)$ a.e.

if $r \leqslant t$, and zero otherwise.

Hence $\langle \Gamma_r(w), [C_*^m Z_t(\varphi)](r) = \lim_{p\to\infty} \int_0^\infty \langle \Gamma_s [P_p A \; \sigma_{-s}(\beta(w))], \; [C_*^{m-1} Z_{t-r}(\varphi)](s)\rangle ds$
$= \langle S_0 \beta(w), \; C_*^{m-1} Z_{t-r}(\varphi)\rangle$
$= \langle S_{m-1}\beta(w), \; Z_{t-r}(\varphi)\rangle$
$= \langle U_m(t-r)\beta(w), \varphi \rangle$,

a.e. (depending on t, w, φ).

But Graph β is ultraweakly countably generated. Hence for all w in D_β,

$$\langle \Gamma_r(w), \; [C_*^m \; Z_t(\varphi)](r)\rangle = \langle U_m(t-r)\beta w, \varphi\rangle$$

a.e. (depending on $t > o$ and φ in M_*).

Hence $U_m(t-r)_*\varphi \in D_{\beta_*}$ a.e., and

$$(\Gamma_r)_* \, [C_*^m \; Z_t(\varphi)](r) = \beta_* \, U_{m,t}(r)_*\varphi \quad \text{a.e.}$$

LEMMA 3.3.9.

(a) β_* is $(\sigma^\kappa, \Gamma, p)_*$ smooth, with associated functions Z_r^κ , $r \geqslant o$, and

$$|\beta_*|_{(\sigma^\kappa,\Gamma,p)_*} \leqslant (1-|\kappa|N)^{-1} \, |\beta_*|_{(\sigma,\Gamma,p)_*} \quad .$$

(b) There exists Z^κ in $B(M_*, \mathcal{Z}_M)$ satisfying, for φ in M_*,

$$(Z^\kappa\varphi)(s) = (\Gamma_s)_* \, \beta_* \, e^{-iT(\kappa)s}\varphi \quad \text{a.e.}$$

and $\|Z^\kappa\| \leqslant M_o^2 \, (1-|\kappa|N)^{-1} \, |\beta_*|_{(\sigma,\Gamma,p)_*}$.

Similarly for the other half line, with associated maps $\hat{Z}_r$, $r \geqslant o$, and $\hat{Z}$ in $B(M_*, \mathcal{Z}_M)$.

PROOF.

(a) $\sum \|(-i\kappa)^m \, C_*^m \, Z_t(\varphi)\| \leqslant |\kappa|^m \, N^m \, |\beta_*| \|\varphi\|$ for each φ in M_*, $t \geqslant o$.

Hence if $\kappa \in D$, the series converges uniformly and absolutely in t to $Z_t^\kappa(\varphi)$ say. By Remark 3.1.7, we see that for each fixed $t > 0$, $\varphi \in M_*$, there is a subsequence n_k such that

$$[Z_t^\kappa(\varphi)](s) = \lim_{k\to\infty} \sum_{m=1}^{n_k} (-i\kappa)^m [C_*^m \; Z_t(\varphi)](s) \quad \text{a.e.}$$

Let W be a null set in $\mathbb{R}^+$, such that if $s \notin W$,

i) $\left[Z_t^{\kappa}(\varphi)\right](s) = \lim_{k\to\infty} \sum_{m=1}^{n_k} (-i\kappa)^m \left[C_*{}^m Z_t(\varphi)\right](s)$

ii) $\left[C_*{}^m Z_t(\varphi)\right](s) = (\Gamma_{-s})_* U_{m,t}(s)_* \qquad m = 0,1,2,\ldots..$

Take $s \notin W$, and b satisfying $\Gamma_{-s}(b) \in D_\beta$.

$$
\begin{aligned}
\text{Then } \left\langle \left[Z_t(\varphi)\right](s), b\right\rangle &= \lim_{k\to\infty} \sum_{m=1}^{n_k} \left\langle \left[C_*^m Z_t(\varphi)\right](s), b \right\rangle (-i\kappa)^m \\
&= \lim_{k\to\infty} \sum_{m=1}^{n_k} \left\langle (\Gamma_{-s})_* \beta_* U_{m,t}(s)_* \varphi, b\right\rangle (-i\kappa)^m \\
&= \lim_{k\to\infty} \sum_{m=1}^{n_k} \left\langle U_m(t-s)_* \varphi, \beta \Gamma_{-s}(b)\right\rangle (-i\kappa)^m \chi_{[0,t]}(s) \\
&= \left\langle e^{-iT(\kappa)(t-s)}\varphi, \beta\Gamma_{-s}(b)\right\rangle \chi_{[0,t]}(s)
\end{aligned}
$$

It follows that $e^{-iT(\kappa)(t-s)}\varphi \in D_\beta$ for $s \in [0,t]\cap \complement W$, and

$$\left[Z_t^{\kappa}(\varphi)\right](s) = (\Gamma_{-s})_* \beta_* e^{-iT(\kappa)(t-s)}\varphi \quad \text{a.e.}$$

b) From a) it follows that if $\varphi \subset M_*$, κ in D, that there exists z_n in $\mathcal{Z}_M$ such that

$$z_n(s) = (\Gamma_{+s})_* \beta_* e^{-iT(\kappa)s}\varphi\, \chi_{[0,n]}(s) \qquad \text{a.e.}$$

and $$\|z_n\| \leqslant M_0^2 (1 - |\kappa| N)^{-1} |\beta_*| \|\varphi\| .$$

The result follows from 3.1.9.

We can now easily deduce the following:

LEMMA 3.3.10.

The semigroups σ_{-t} and σ_{-t}^{κ} satisfy for $t \geqslant 0$ the following relations if $u \in M$, $\varphi \in M_*$, $\kappa \in D$:

$$\langle \sigma_{-t}^{\kappa}(u) - \sigma_{-t}(u), \varphi\rangle = -i\kappa \langle Y^{\kappa}(u), Z_t(\varphi)\rangle$$

$$\langle \sigma_{-t}^{\kappa}(u) - \sigma_{-t}(u), \varphi\rangle = -i\kappa \langle Y(u), Z_t^{\kappa}(\varphi)\rangle$$

with analogous results for the other half line.

LEMMA 3.3.11.

For each κ in D, $$T(\kappa) \geq T + \kappa\alpha_*\beta_*$$
$$T \geq T(\kappa) - \kappa\alpha_*\beta_*$$

PROOF.

Let $A = U|A|$ be a polar decomposition of A, and Q_n (3.2.7) to be chosen as spectral projections of A. Then take $P_n' = U Q_n U^*$.

For $z \in C$, with $\text{im } z > \omega_o$, and $\kappa \in D$, let $R(z,\kappa) = (T(\kappa) - z)^{-1}$ and $R(z) = (T - z)^{-1}$.

Then $e^{izt}(-i\kappa)\langle Y^\kappa(u), Z_t(\varphi)\rangle \in L_1(o,\infty)$, for $u \in M$, $\varphi \in M_*$; and by dominated convergence:

$$i \int_o^\infty e^{izt}(-i\kappa)\langle Y^\kappa(u), Z_t(\varphi)\rangle\, dt$$

$$= i\int \lim_{n\to\infty} (-i\kappa)\, e^{izt} \langle \Gamma_{(\cdot)}(P_n')\, Y^\kappa(u), Z_t(\varphi)\rangle\, dt$$

$$= i \int_o^\infty \lim_{n\to\infty} e^{izt}(-i\kappa)\int_o^t \langle \Gamma_s(P_n'A\,\sigma_{-s}(u)), [Z_t(\varphi)](s)\rangle\, ds\, dt$$

$$= i \lim_{n\to\infty} \int_o^\infty\int_o^t e^{izt}(-i\kappa)\langle \Gamma_s(P_n'A\,\sigma_{-s}(u)), [Z_t(\varphi)](s)\rangle\, ds\, dt$$

$$= i \lim_{n\to\infty} \int_o^\infty\int_o^\infty e^{iz(s+r)}(-i\kappa)\langle P_n'A\,\sigma_{-s}(u), \beta_* e^{-irT}\varphi\rangle\, dr\, ds,$$

where we have used Fubini's theorem, the use of which is easily justified as in Remark 3.3.7.

Now suppose $R(z)\varphi \in D_{\alpha_*\beta_*}$. Now by 3.1.7, and 3.3.9, for each ψ in M_*,

$$r \longmapsto \beta_* e^{-irT}\psi \in L^p(\mathbb{R}^+; M_*).$$

Hence for all $\psi \in M_*$; $R(z)\psi \in D_{\beta_*}$ and

$$\langle u, \beta_* R(z)\psi\rangle = +i\int_o^\infty \langle u, \beta_* e^{-irT}\psi\rangle e^{irz}\, dr \qquad \text{for all } u \text{ in } M.$$

Hence

$$i \int_o^\infty e^{izt}(-i\kappa)\langle Y^\kappa(u), Z_t(\varphi)\rangle\, dt$$

$$= \lim_{n\to\infty} \int_o^\infty (-i\kappa) e^{izs}\langle AQ_n\,\sigma_{-s}(u), \beta_* R(z)(v)\rangle\, ds$$

$$= -(\kappa)\langle u, R(z,\kappa)\alpha_*\beta_* R(z)\varphi\rangle$$

$$= \langle u, R(z,\kappa)\varphi\rangle - \langle u, R(z)\varphi\rangle \qquad \text{by taking Laplace transforms.}$$

It follows easily that $T \geq T(\kappa) - \kappa\,\alpha_*\beta_*$.

The other relation is proved similarly.

We now consider the similarity operators.

LEMMA 3.3.12.

For $u \in M$, $\varphi \in M_*$, the expressions

$$\langle W_-(\kappa)u,\varphi\rangle = \langle u,\varphi\rangle - i\kappa \langle \hat{Y}(u), Z^\kappa(\varphi)\rangle$$

$$\langle W_+(\kappa)u,\varphi\rangle = \langle u,\varphi\rangle + i\kappa \langle Y(u), \hat{Z}^\kappa(\varphi)\rangle$$

$$\langle H_-(\kappa)u,\varphi\rangle = \langle u,\varphi\rangle + i\kappa \langle \hat{Y}^\kappa(u), Z(\varphi)\rangle$$

$$\langle H_+(\kappa)u,\varphi\rangle = \langle u,\varphi\rangle - i\kappa \langle Y^\kappa(u), \hat{Z}(\varphi)\rangle \qquad (\kappa \in D)$$

define ultraweakly continuous bounded operators on M. Moreover

$$W_-(\kappa)\, H_-(\kappa) = H_-(\kappa)\, W_-(\kappa) = 1$$

$$W_+(\kappa)\, H_+(\kappa) = H_+(\kappa)\, W_+(\kappa) = 1.$$

PROOF.

The right hand sides clearly define bounded linear operators on M.
But $\hat{Y}^\kappa(u) = \sum_{n=0}^{\infty} \hat{S}_n u\, (-i\kappa)^n$.
Using the weak continuity of $\alpha\sigma_{+(\cdot)}\beta\theta$ and $u \longmapsto \hat{S}_0 u$, we see that $W_-(\kappa)$ and $H_-(\kappa)$ are ultraweakly continuous. Let us write

$$W_-(\kappa) = 1 + \kappa P, \quad \text{and} \quad H_-(\kappa) = 1 + \kappa Q.$$

Take $u \in M$, $\varphi \in M_*$, and we have

$$\begin{aligned}\langle PQu,\varphi\rangle &= \langle Qu, P_*\varphi\rangle = i\langle \hat{Y}^\kappa(u), Z(P_*\varphi)\rangle \\ &= \lim_{n\to\infty} \int_0^\infty i\langle P_n A\, \sigma_s^\kappa(u), \beta_*\, \sigma_{-s}\, P_*\varphi\rangle \, ds .\end{aligned}$$

For any w in D_β, $\varphi \in M_*$ one has

$$\begin{aligned}\langle w, \beta_*\, e^{-isT}P_*\varphi\rangle &= \langle P\, \sigma_{-s}\beta(w), \varphi\rangle \\ &= \lim_{m\to\infty} - \int_0^\infty i\langle P_m A\, \sigma_r\, \sigma_{-s}\, \beta(w), \beta_* e^{-irT(\kappa)}\varphi\rangle \, dr \\ &= -i \lim_{m\to\infty} \int_0^s \; -i \lim_{m-\infty} \int_s^\infty\end{aligned}$$

The two parts are evaluated separately as follows

$$\begin{aligned}\lim_{m\to\infty} \int_0^s &= \lim_{m\to\infty} \int_0^s \langle P_m A\, \sigma_{-(s-r)}\beta w, \beta_* e^{-irT(\kappa)}\varphi\rangle \, dr \\ &= \frac{i}{\kappa} \langle \sigma_{-s}^\kappa\, \beta w - \sigma_{-s}\beta w, \varphi\rangle \\ &= \frac{i}{\kappa} \langle w, \beta_* e^{-isT(\kappa)}\varphi\rangle - \frac{i}{\kappa}\langle w, \beta_* e^{-isT}\varphi\rangle ,\end{aligned}$$

a.e. using Lemma 3.3.10.

Also by Remark 3.3.7,

$$\lim_{m\to\infty} \int_s^\infty \langle P_m A\, \sigma_{r-s}\beta w, \beta_*\, e^{-irT(\kappa)}\varphi\rangle \, dr = \langle \Gamma_{-s}(w), \left[\hat{C}_* Z^\kappa(\varphi)\right](s)$$

a.e., depending on κ, w, φ)

Since Graph ß is ultraweakly countably generated, we conclude that

$$ß_* e^{-isT} P_* \varphi = \frac{1}{\kappa} ß_* e^{-isT(\kappa)}\varphi \quad \frac{-1}{\kappa} ß_* e^{-isT}\varphi \quad -i(\Gamma_{-s})_* \left[\hat{C}_* \, Z^{\kappa}(\varphi)\right](s) \quad .$$

Hence $\langle PQu, \varphi\rangle = \frac{i}{\kappa} \lim_{n\to\infty} \int_0^\infty \langle P_n A\, \sigma_s(u),\ ß_* e^{-isT(\kappa)}\varphi\rangle \quad ds$

$$\frac{-i}{\kappa} \lim_{n\to\infty} \int \langle P_n A\, \sigma_s(u),\ ß_* e^{-isT}\varphi\rangle \quad ds$$

$$+ \lim_{n\to\infty} \int \langle \Gamma_{-s}(P_n A\, \sigma_s(u)),\ \left[C_* \, Z^{\kappa}(\varphi)\right](s)\rangle \quad ds$$

$$= I_1 + I_2 + I_3 \text{ respectively, say.}$$

$I_3 = \langle Y^{\kappa}(u),\ \hat{C}_* \, Z^{\kappa}(\varphi)\rangle = \langle \hat{C}\, Y^{\kappa}(u),\ Z^{\kappa}(\varphi)\rangle$.

For $\xi, \eta \in \mathfrak{H}$, define $v \in M_*$ by $v(m) = \langle m\xi, \eta\rangle$, $m \in M$.

Then for almost all s, we have by 3.2.8.b, that

$$\langle \left[\hat{C}\, Y\,(u)\right](s)\,\xi, \eta\rangle = \lim_{n\to\infty} \int_0^s \langle P_n A\, \sigma_r(u),\ ß_* e^{+i(s-r)T}\alpha_*(\Gamma_{-s})_* v\rangle \ dr$$

$$= \frac{1}{i\kappa} \langle \sigma_s(u) - \sigma_s(u), \alpha_*(\Gamma_{-s})_* v\rangle$$

by Lemma 3.3.10.

Hence $\hat{C}\, Y^{\kappa}(u) = \frac{1}{i\kappa} \cdot \hat{Y}^{\kappa}(u) - \frac{1}{i\kappa} \cdot \hat{Y}\,(u)$.

It follows that

$$I_3 = \frac{1}{i\kappa} \langle \hat{Y}^{\kappa}(u),\ Z^{\kappa}(\varphi)\rangle - \frac{1}{i\kappa} \langle \hat{Y}(u),\ Z^{\kappa}(\varphi)\rangle \quad .$$

The first term of I_3 cancels I_1, and the second term of I_3 is

$-\frac{1}{\kappa} \langle Pu, \varphi\rangle$; while $I_2 = -\frac{1}{\kappa} \langle Qu, \varphi\rangle$.

Hence $\langle PQu, \varphi\rangle = -\frac{1}{\kappa} \langle Qu, \varphi\rangle \quad -\frac{1}{\kappa} \langle Pu, \varphi\rangle$.

This is equivalent to $W_-(\kappa)\, H_-(\kappa) = 1$.

The remainder can be shown in the same manner.

LEMMA 3.3.13.

With the non singular operators $W_\pm(\kappa)$ constructed above, we have that

$$W_\pm(\kappa)\, \sigma_{-r} = \sigma_{-r}\, W_\pm(\kappa) \qquad \text{for all } \kappa \text{ in } D,\ r \text{ in } \mathbb{R}.$$

PROOF.

Consider only $r \geqslant 0$; and take $u \in M$, $\varphi \in M_*$.

Then

$$\langle W_-(\kappa)\sigma_{-r}(u), \varphi\rangle = \langle\sigma_{-r}(u), \varphi\rangle - \lim_{n\to\infty}\int_0^\infty \langle P_n A\ \sigma_s\ \sigma_{-r}(u),\ \beta_* e^{-isT(\kappa)}\varphi\rangle\ ds.$$

The integral on the right hand side can be split into two terms, $\int_0^r$ and $\int_r^\infty$ where,

$$\lim_{n\to\infty} -i\kappa \int_0^r \langle P_n A\ \sigma_s\ \sigma_{-r}(u),\ \beta_* e^{-isT(\kappa)}\varphi\rangle\ ds$$

$$= \lim_{n\to\infty} -i\kappa \int_0^r \langle P_n A,\ \sigma_{-(r-s)}(u),\ \beta_* e^{-isT(\kappa)}\varphi\rangle\ ds$$

$$= \langle\sigma^\kappa_{-r}(u), \varphi\rangle - \langle\sigma_{-r}(u), \varphi\rangle \qquad \text{by Lemma 3.3.10;}$$

and $$-i\kappa \lim_{n\to\infty}\int_r^\infty \langle P_{nA}\ \sigma_{-s}\sigma_{-r}(u),\ \beta_* e^{-isT(\kappa)}\varphi\rangle\ ds$$

$$= -i\kappa \lim_{n\to\infty}\int_r^\infty \langle P_n A\ \sigma_{-s}(u),\ \beta_* e^{-isT(\kappa)}\ e^{-irT(\kappa)}\varphi\rangle\ ds\ .$$

Hence $W_-(\kappa)\sigma_{-r} = \sigma^\kappa_{-r} W_-(\kappa)$.

REMARK 3.3.14.

However, much more than the similarity relations is true. The ultraweak limit $W^\pm(\kappa)$ of $\sigma^\kappa_t\sigma_{-t}$ as $t \longrightarrow \pm\infty$ exist and coincide with $W_\pm(\kappa)$. To show this, notice that

$$\langle W_+(\kappa)\ \sigma_{-t}(u),\ e^{itT(\kappa)}\varphi\rangle = \langle\sigma^\kappa_t\sigma_{-t}u, \varphi\rangle + \lim_{n\to\infty}\int_0^\infty i\ \kappa\langle P_n A\sigma_{-(t+r)}(u), \beta_* e^{i(t+r)T(\kappa)}\varphi\rangle dr$$

Hence $\langle W_+(\kappa)(u), \varphi\rangle - \langle\sigma_t\ \sigma_{-t}(u), \varphi\rangle = i\kappa\langle Y(u).\chi_{[t,\infty)}\ ,\ Z^\kappa(\varphi)\rangle$,

which tends to zero, as $t \longrightarrow \infty$

§3.4.

As seen from our motivating example in §3.1, if the unperturbed system σ_t is a group of automorphisms, then the resulting perturbation need not be. In this section, we consider how to form groups of *-automorphisms from the perturbations σ^κ_t.

Again, M is a countably generated von Neumann algebra with discrete centre as in 3.1.8. Suppose σ_t is a weakly continuous one parameter group of *-automorphisms of M, with Γ_t an auxiliary weakly continuous group satisfying (3.2.1). $B, A \in \mathcal{C}_0(\mathfrak{H})$ are affiliated with M and give left multiplication operators β, α respectively, as defined in 3.2.2. Suppose $(\sigma, \Gamma, \alpha, \beta)$ satisfy all the hypotheses of Theorem 3.3.1.

Note that D_β is a right ideal in M, and if $x \in D_\beta$, $y \in M$ then $\beta(xy) = \beta(x).y$. This is the only new property of β that we need.

NOTATION 3.4.1.

If δ is a linear operator on M, with domain D_δ , we define $\bar{\delta}$ to be the linear map on M, with domain $(D_\delta)^*$ and $\bar{\delta}(x) = \delta(x^*)^*$ for $x \in D_\delta$.

THEOREM 3.4.2.

a) If $\kappa \in D$ $\rho_t^\kappa(x) = \sigma_t^{\bar{\kappa}}(x^*)^*$ for $x \in M$, defines a weakly continuous group on M, and if $i\mathcal{T}(\kappa)$ denotes the infinitesimal generator of the strongly continuous group $(\rho_t^\kappa)_*$, then

$$\mathcal{T}(\kappa) \supseteq T - \kappa(\bar{\alpha})_*(\bar{\beta})_*$$
$$T \supseteq \mathcal{T}(\kappa) + \kappa(\bar{\alpha})_*(\bar{\beta})_*$$

Also $\tilde{W}_\pm(\kappa)(x) = w_\pm(\bar{\kappa})(x^*)^*$ $x \in M$, defines two weakly continuous invertible operators on M, satisfying:

i) $\tilde{W}_\pm(\kappa)\sigma_t = \rho_t^\kappa \tilde{W}_\pm(\kappa)$

ii) $\tilde{W}_\pm(\kappa) = \sigma\text{-weak}\lim_{t \to \pm\infty} \rho_t^\kappa \sigma_{-t}$

b) The following hold for all x,y in M, $t \in \mathbb{R}, \kappa \in D$

iii) $\sigma_t^\kappa(xy) = \sigma_t^\kappa(x)\, \sigma_t(y)$

iv) $\rho_t^\kappa(xy) = \sigma_t(x)\, \rho_t^\kappa(y)$

v) $W_\pm(\kappa)(xy) = \left[W_\pm(\kappa)x\right] y$

vi) $\tilde{W}_\pm(\kappa)(xy) = x\left[\tilde{W}_\pm(\kappa)y\right]$

c) Suppose also, in the notation of (3.2.5), that for each $f \in \mathcal{Y}_M$, $r \longmapsto m^f(r)^*$ is strongly continuous from the left on (o, ∞).

Then for each κ in D, there is a weakly continuous one parameter group τ_t^κ on M, satisfying:

vii) $\tau_t^\kappa(xy) = \sigma_t^\kappa(x)\,\rho_t^\kappa(y)$

Also there are two σ-weakly continuous invertible bounded linear operators $\Omega_\pm(\kappa)$, for each κ in D, satisfying

viii) $\Omega_\pm(\kappa)(xy) = W_\pm(\kappa)(x)\,\tilde{W}_\pm(\kappa)(y) \qquad \forall\, x,y \in M.$

ix) $\Omega_\pm(\kappa)\tau_t^\kappa = \sigma_t\,\Omega_\pm(\kappa) \qquad \forall\, t \in \mathbb{R}.$

PROOF.

We give the proof of b) and c) only.

b) Using the notation of §3.3, we show by induction on n that

$$U_n(t)(xy) = U_n(t)(x)\,\sigma_{-t}(y), \qquad \forall x,y \in M,\ t \geq o.$$

If $\varphi \in M_*$, then

$$\langle U_n(t)(xy), \varphi\rangle = \lim_{m\to\infty}\int_o^t \langle P_m A\, U_{n-1}(r)(xy),\ \beta_* e^{-i(t-r)T}\varphi\rangle\, dr$$

$$= \lim_{m\to\infty}\int_o^t \langle P_m A\, U_{n-1}(r)(x)\,\sigma_{-r}(y),\ \beta_* e^{-i(t-r)T}\varphi\rangle\, dr$$

$$= \langle U_n(t)(x)\,\sigma_{-t}(y), \varphi\rangle \qquad ,$$

using the ideal property of D_β etc. Similarly for $t < o$.

This proves iii), and we can prove v) in a similar fashion. iv) and vi) are immediate consequences of iii) and v).

c) For each κ in D, t in $\mathbb{R}$, we can define a linear map τ_t^κ on M such that $\tau_t^\kappa(x) = \sigma_t^\kappa(x)\,\rho_t^\kappa(1) \qquad \forall\, x \in M.$

Then $\tau_t^\kappa(xy) = \sigma_t^\kappa(x)\,\rho_t^\kappa(y) \qquad \forall\, x,y \in M,\ t \in \mathbb{R},\ \kappa \in D.$

Clearly $x \longrightarrow \tau_t^\kappa(x)$ is σ-weakly continuous on M.

The additional continuity that we are now given, shows that $r \longmapsto (U_n(r)x)^*$ is strongly continuous from the left on (o,∞).

But $\tau_t^\kappa(x) = \sigma_t^\kappa(x) + \sum_{n=1}^\infty \sigma_t^\kappa(1)\,U_n(t)(x^*)^*\,(i\kappa)^n$.

Hence $\tau_t^\kappa(x)$ is $\sigma(M,M_*)$ continuous in t from the left on (o,∞). By the group property of τ_t^κ, it is continuous from the left on $\mathbb{R}$. Hence $(\tau_t^\kappa)_*$ is a strongly continuous group on $\mathbb{R}$.

The remainder is clear.

REMARKS.

3.4.3. For the motivation of the condition in 3.4.2.b, see the proof of [36, Lemma 4.2.].

3.4.4. If, as often happens in practice, it is true that

$$(W_+)_* = \operatorname*{st\,lim}_{t\to\infty} e^{-itT} e^{itT(\kappa)}$$

then $$(\Omega_+(\kappa))_* = \operatorname*{st\,lim}_{t\to\infty} e^{-itT} (\tau^{\kappa}_{-t})_*,$$

and $$(\Omega_+(\kappa)) = \sigma\text{-weak}\lim_{t\to\infty} \tau^{\kappa}_{-t}\, \sigma_t \;.$$

3.4.5. Each map τ^{κ}_t is a *-map for κ real in D, $t \in \mathbb{R}$.

3.4.6. We can regard the infinitesimal generator of $(\tau^{\kappa}_t)_*$, as representing $i\left\{T + \kappa\,(\alpha_*\beta_* - \bar{\alpha}_*\bar{\beta}_*)\right\}$. This can be formally justified as in 3.3.11.

We now consider the automorphism property.

HYPOTHESIS 3.4.7.

Suppose $v \in M_*$ has the following properties:

i) v and $v^* \in D_{\beta_*}$.

ii) There exist ξ_i, η_i, f_i, g_i in $\mathfrak{H}$ satisfying

$$\langle m, \beta_* v\rangle = \Sigma\langle m\,\xi_i, \eta_i\rangle$$

$$\langle m, \beta_* v^*\rangle = \Sigma\langle m\, f_i, g_i\rangle \qquad \forall\, m \in M,$$

where $\Sigma\|\xi_i\|\,\|\eta_i\| < \infty$, $\Sigma\|f_i\|\,\|g_i\| < \infty$.

ii) ξ_i, $f_i \in D_A$ and $\Sigma\|A\xi_i\|\,\|\eta_i\|$, $\Sigma\|Af_i\|\,\|g_i\| < \infty$.

Then we say α, β satisfy hypothesis 3.4.7, if for all such v,

$$\Sigma\langle A\xi_i, \eta_i\rangle = \Sigma\langle g_i, Af_i\rangle$$

THEOREM 3.4.8.

Under the conditions of Theorem 3.4.2 and Hypothesis 3.4.7, τ^{κ}_t is a group of automorphisms for all κ in D, t in $\mathbb{R}$.

PROOF.

It is enough to show that $\rho^{\kappa}_t(1)\,\sigma^{\kappa}_t(1) = 1$, for all t in $\mathbb{R}$, κ in D.

This is achieved by showing that

$$\sigma_{-t}\left[\sigma^{\bar{\kappa}}_t(1)^*\right] = \sigma^{\kappa}_{-t}(1) \qquad t \in \mathbb{R},\ \kappa \in D;$$

i.e. we will inductively show that

$$\sigma_{-t}\left[\hat{U}_m(t)(1)^*\right] = U_m(t)(1) \qquad \text{for } m \geqslant 0. \qquad (3.4.9)$$

Let w denote the vector state $\xi \otimes \eta$, where $\xi, \eta \in \mathfrak{H}$.

Then $\langle U_m(t)(1)\xi, \eta\rangle = \langle S_{m-1}(1),\ Z_t(w)\rangle$

$$= \lim_{n\to\infty} \int_0^t \langle (P_nA)\ U_{m-1}(s)(1),\ \beta_* e^{-iT(t-s)}w\rangle\, ds$$

and $\langle \sigma_{-t}\hat{U}_m(t)(1)\eta, \xi\rangle = \langle \hat{S}_{m-1}(1),\ \hat{Z}_t\left[(\sigma_{-t})_* w^*\right]\rangle$

$$= \langle \hat{S}_0(1),\ (C_*)^{m-1} Z_t\left[(\sigma_{-t})_* w^*\right]\rangle$$

$$= \lim_{n\to\infty} \int_0^t \langle (P_nA)\ \hat{U}_0(s)1,\ \beta_*\hat{U}_{m-1}(t-s)_*(\sigma_{-t})_* w^*\rangle\, ds.$$

Thus 3.4.9 follows by induction and our hypothesis.

Hence

$$\begin{aligned}
\rho_t^{\kappa}(1)\ \sigma_t^{\kappa}(1) &= \left[\bar{\sigma}_t^{\kappa}(1)\right]\ \sigma_t^{\kappa}(1) = \sigma_t\left[\sigma_{-t}^{\kappa}(1)\right].\sigma_t^{\kappa}(1)\\
&= \sigma_t\left\{\sigma_{-t}^{\kappa}(1)\ \sigma_{-t}\sigma_t^{\kappa}(1)\right\} = \sigma_t\left\{\sigma_{-t}^{\kappa}\left[1.\sigma_t^{\kappa}(1)\right]\right\}\\
&= \sigma_t(1) \qquad = 1.
\end{aligned}$$

The above theorem can be regarded as the algebra level analogue of [29,(4.1)].

REMARK 3.4.10.

It is possible to treat potentials of the form $\sum \beta_n \alpha_n$ using our method, if we replace $\mathcal{Y}_M(\mathbb{R}^+)$ by $\mathcal{Y}_M(\mathbb{R}^+ \times \mathbb{N})$, the natural subspace in $B(\mathfrak{H}, L^q(\mathbb{R}^+ \times \mathbb{N}; \mathfrak{H}))$.

CHAPTER 4.

We now consider time dependant perturbations and scattering of strongly continuous groups on Banach spaces. We try to solve the equation

$$\frac{i d\psi(t)}{dt} = K(t)\psi(t)$$

where $K(t) = K_o + V(t)$, and iK_o is the unperturbed infinitesimal generator of a strongly continuous group on a Banach space E, and V(t) is a time dependant perturbation. Expecting the solution to be given by a propogator U(t,s) we study the scattering operator

$$S = \lim_{t\to\infty} e^{iK_o t} U(t,-t)\, e^{iK_o t}$$

Following Howland [26], we reduce the problem to the situation handled in Chapter 2, where the perturbation V(t) is time independant. This method corresponds to a technique in classical mechanics of introducing the time and energy of the external source as conjugate variables. We consider the infinitesimal generator

$$K = -i\frac{\partial}{\partial t} + K(t)$$

on the spaces $L^p(\mathbb{R};E)$ $1 \leq p\infty$, and $C_o(\mathbb{R};E)$. Formally K satisfies

(4.0.1) $\qquad K\,M(\phi) - M(\phi)\,K = -i\,M(\dot{\phi}),$

for every $\phi \in C_c^1(\mathbb{R})$, where $(M\phi f)(t) = \phi(t)f(t)$, and $\dot{\phi} = \frac{d\phi}{dt}$.

We will characterise the infinitesimal generators on $C_o(\mathbb{R};E)$ (respectively $L^p(\mathbb{R};E)$) satisfying 4.0.1 in § 4.1 (respectively §§ 4.6-7), and show how the scattering of such operators determines the scattering of their propogators in § 4.2 (respectively § 4.8) Finally in §§4.3-5 (respectively §§ 4.9-10) we apply this theory using some amenable potentials.

Note that in practice, the choice of which space to take, namely $\{L^p(\mathbb{R};E), C_o(\mathbb{R};E)\}$, depends on what smoothness properties V(t) has. We also point out here that if E is a C*-algebra, then so is

$C_o(\mathbb{R};E)$; and if E is the separable predual of a von Neumann algebra then so is $L^1(\mathbb{R};E)$.

§ 4.1.

Let $C_o(\mathbb{R}^n)$ be the commutative C^*-algebra of all complex valued functions vanishing at infinity on Euclidean space $\mathbb{R}^n$. Let E be a Banach space, and let $C_o(\mathbb{R}^n;E)$ be the Banach space of all E-valued continuous functions vanishing at infinity on $\mathbb{R}^n$, with the supremum norm. For $f \in C_o(\mathbb{R}^n)$, $x \in E$, the mapping $\Phi: f \otimes x \longmapsto f(t)x$ of $C_o(\mathbb{R}^n) \odot E$ into $C_o(\mathbb{R}^n;E)$ can be extended to a linear map of $C_o(\mathbb{R}^n) \odot E$. Grothendieck, [23], showed that $\|\Phi(w)\| = \|w\|_\wedge$ the injective tensor product norm (for all $w \in C_o(\mathbb{R}^n) \odot E$), and so Φ can be uniquely extended to an isometric linear mapping of $C_o(\mathbb{R}^n) \otimes^\wedge E$ onto $C_o(\mathbb{R}^n;E)$. We shall identify $C_o(\mathbb{R}^n) \otimes^\wedge E$ with $C_o(\mathbb{R}^n;E)$ under this map Φ. Note that if E is a C*-algebra, $C_o(\mathbb{R}^n;E)$ becomes a C*-algebra with the product as pointwise multiplication.

For each $s \in \mathbb{R}^n$, let $A(s) \in B(E)$. Then $f \longmapsto (A(s)f(s))$; $f \in C_o(\mathbb{R}^n;E)$ defines an element of $B\big(C_o(\mathbb{R}^n;E)\big)$ iff

4.1.1) $s \longmapsto A(s)$ is strongly continuous on $\mathbb{R}^n$

4.1.2) $\sup \|A(s)\| < \infty$.

In which case $\|A\| = \sup_{s \in \mathbb{R}^n} \|A(s)\|$, and each A(s) is uniquely determined by A. We call such an A, a bounded multiplication operator on $C_o(\mathbb{R}^n;E)$.

In particular, each element ϕ of $C_o(\mathbb{R}^n)$ defines a bounded scalar multiplication operator $M(\phi)$ on $C_o(\mathbb{R}^n;E)$. If A is a bounded multiplication operator on $C_o(\mathbb{R}^n;E)$, then

$$A\, M(\phi) = M(\phi)\, A \qquad \forall\, \phi \in C_o(\mathbb{R}^n).$$

The following proposition shows that this property characterises bounded multiplicationoperators on $C_o(\mathbb{R}^n;E)$, and is analogous to the corresponding L^p-result, see 4.6 and [26].

We denote by $C_b(\mathbb{R}^n;E)$ the Banach space of all E-valued bounded continuous functions on $\mathbb{R}^n$, with the supremum norm. Each bounded multiplication operator A on $C_o(\mathbb{R}^n;E)$ has a natural extension to $C_b(\mathbb{R}^n;E)$ as follows: $(Af)(s) = A(s)f(s) \quad f \in C_b(\mathbb{R}^n;E)$.

PROPOSITION 4.1.3.

Let $A \in B\left(C_o(\mathbb{R}^n;E),\ C_b(\mathbb{R}^n:E)\right)$. Then A is a bounded multiplication operator on $C_o(\mathbb{R}^n;E)$ iff

$$A\,M(\phi) = M(\phi)\,A \qquad \forall\ \phi \in C_o(\mathbb{R}^n).$$

PROOF.

Let $A \in B\left(C_o(\mathbb{R}^n;E),\ C_b(\mathbb{R}^n;E)\right)$ such that $AM\phi = M\phi A \quad \forall\ \phi \in C_o(\mathbb{R}^n)$.

Let $x \in E$, and fix $s_o \in \mathbb{R}^n$. Take $\phi \quad C_o(\mathbb{R}^n)$ such that $\phi(s_o) = 1$.

Define $A(s_o)x = (A(\phi \otimes x))(s_o)$. We show this is well defined. Suppose $\phi_1, \phi_2 \in C_o(\mathbb{R}^n)$, $\phi_1(s_o) = 1 = \phi_2(s_o)$. Take $\psi_n \in C_o(\mathbb{R}^n)$ such that

$$\psi_n(s) = \begin{cases} 1 & \text{if } |s| \leqslant n \\ 0 & \text{if } |s| \geqslant n+1. \end{cases} \qquad \|\psi_n\| \leq 1$$

Then $\psi_n\phi_1 \otimes x,\ \psi_n\phi_2 \otimes x \longrightarrow \phi_1 \otimes x,\quad \phi_2 \otimes x$ respectively in $C_o(\mathbb{R}^n;E)$ as $n \longrightarrow \infty$. Moreover

$$A(\psi_n\phi_1 \otimes x - \psi_n\phi_2 \otimes x) = M(\phi_1)\,A(\psi_n \otimes x) - M(\phi_2)\,A(\psi_n \otimes x).$$

Hence $\quad A(\psi_n\phi_1 \otimes x)(s_o) = A(\psi_n\phi_2 \otimes x)(s_o)$.

Let $n \to \infty$, then $A(\phi_1 \otimes x)(s_o) = A(\phi_2 \otimes x)(s_o)$.

Hence A(s) is well defined on $\mathbb{R}^n$, and if $x \in E$, $\phi \in C_o(\mathbb{R}^n)$, $\phi(s_o) = 1$. we have $\|A(s_o)x\| = \|A(\phi \otimes x)(s_o)\| \leq \|A\|\,\|\phi \otimes x\| = \|A\|\,\|\phi\|\,\|x\|$.

Hence each A(s) is bounded, and clearly linear.

Let $\psi \in C_o(\mathbb{R}^n)$ be arbitrary, then

$$\begin{aligned} A(\psi \otimes x)(s_o) &= \phi(s_o)\,A(\psi \otimes x)(s_o) = A(\phi\psi \otimes x)(s_o) = \psi(s_o)\,A(\phi \otimes x)(s_o) \\ &= \psi(s_o)\,A(s_o)x \qquad = A(s_o)(\psi(s_o).x). \end{aligned}$$

Hence $(Af)(s) = A(s)f(s) \quad \forall\ f \in C_o(\mathbb{R}^n) \odot E$, and it is easily follows that this is also true for all f in $C_o(\mathbb{R}^n;E)$, and that

$$A\left(C_o(\mathbb{R}^n;E)\right) \subseteq C_o(\mathbb{R}^n;E).$$

Thus A is a bounded multiplication operator on $C_o(\mathbb{R}^n;E)$.

Multiplication operators will be used in the theory of strongly continuous propogators, which we now define.

DEFINITION 4.1.4.

A strongly continuous propogator on E is a family U(t,s)of invertible operators defined for (t,s) in $\mathbb{R}^2$ such that:

a) U(t,s) is jointly strongly continuous in (t,s).

b) The Chapman Kolmogrov equation holds i.e.

$$U(t,s)\,U(s,r) = U(t,r)\,, \qquad \forall\,(t,s,r) \in \mathbb{R}^3.$$

c) $U(t,t) = 1$, $\forall\ t \in \mathbb{R}$.

The propogator is said to be bounded if $\|U(t,s)\|$ is a bounded function on $\mathbb{R}^2$.

Given a bounded strongly continuous propogator U(,), the formula

$$(e^{-i\sigma K}f)(t) = U(t,t-\sigma)f(t-\sigma) \qquad \text{for } f \in C_o(\mathbb{R};E)$$

defines a strongly continuous bounded group on $C_o(\mathbb{R};E)$. By abuse of notation, we refer to K as the generator of the group. As a non trivial example we can consider $U(t,s) \equiv 1$ on $\mathbb{R}^2$. This gives the group of translations

$$(T_\sigma f)(t) = f(t-\sigma) \qquad \text{for } f \in C_o(\mathbb{R};E),$$

for which the generator $K = -i\dfrac{d}{dt}$.

Note that if the strongly continuous propogator U defines the strongly continuous one parameter group $e^{-i\sigma K}$ on $C_o(\mathbb{R};E)$, then

$$(e^{-i\sigma K}T_{-\sigma}f)(t) = U(t,t-\sigma)f(t) \qquad \text{for } f \in C_o(\mathbb{R};E).$$

Hence for each $\sigma \in \mathbb{R}$, $e^{-i\sigma K}T_{-\sigma}$ is a bounded multiplication operator on $C_o(\mathbb{R};E)$. Moreover if $\mathcal{U}$ is the invertible bounded multiplication operator on $C_o(\mathbb{R};E)$ defined by

$$(\mathcal{U}f)(t) = U(t,o)f(t) \qquad \text{for } f \in C_o(\mathbb{R};E),$$

then for all $\sigma \in \mathbb{R}$,

$$e^{-i\sigma K} = \mathcal{U}\,T_\sigma\,\mathcal{U}^{-1}.$$

That is, $e^{-i\sigma K}$ is similar to the group of translations T_σ, with a connecting operator $\mathcal{U}$ which is a multiplication. In order to characterise such strongly continuous groups on $C_o(\mathbb{R};E)$ we make a definition.

DEFINITION 4.1.5.

A bounded evolution group on $C_o(\mathbb{R};E)$ is a strongly continuous one parameter group $e^{-i\sigma K}$ on $C_o(\mathbb{R};E)$, such that for each σ in $\mathbb{R}$, $e^{-i\sigma K}T_{-\sigma}$ is a multiplication operator on $C_o(\mathbb{R};E)$.

By Proposition 4.1.3, a strongly continuous bounded group $e^{-i\sigma K}$ is a bounded evolution group on $C_o(\mathbb{R};E)$ iff

$$e^{-i\sigma K}T_{-\sigma}M(\phi) = M(\phi)T_{-\sigma}e^{-i\sigma K} \qquad \forall\ \phi \in C_o(\mathbb{R}),$$

or equivalently

$$e^{-i\sigma K}M(\phi)e^{i\sigma K} = T_\sigma M(\phi)T_{-\sigma} = M(\phi_\sigma) \qquad \text{where } \phi_\sigma(t) = \phi(t-\sigma).$$

We can now characterise bounded evolution groups in terms of strongly continuous propogators. This should be compared with the analogous L^p result in § 4.7 and [26].

THEOREM 4.1.6.

A strongly continuous bounded group $e^{-i\sigma K}$ on $C_o(\mathbb{R};E)$ is a bounded evolution group iff there exists an invertible bounded multiplication operator $\mathcal{U}$ on $C_o(\mathbb{R};E)$ such that

$$e^{-i\sigma K} = \mathcal{U}T_\sigma\mathcal{U}^{-1} \qquad \forall\ \sigma \in \mathbb{R}.$$

PROOF.

Let $F = C_o(\mathbb{R};E)$, and $G = C_o(\mathbb{R};F)$.

Define an operator $\mathcal{V}$ on G by

$$(\mathcal{V}f)(\sigma) = e^{-i\sigma K}T_{-\sigma}f(\sigma) \qquad \text{for } f \in C_o(\mathbb{R};F).$$

Then $\mathcal{V}$ commutes with multiplication by any continuous scalar function on $\mathbb{R}$ vanishing at infinity $\psi(\sigma)$, and also with any operator of the form $f(\sigma) \longmapsto M(\phi)f(\sigma)$, where $M(\phi)$ is multiplication on $F = C_o(\mathbb{R};E)$.

G may be also considered as $C_o(\mathbb{R}^2;E)$, by making the element $f(t,o)$ of $C_o(\mathbb{R}^2;E)$ correspond to the element $f(\sigma) = f(\cdot,\sigma)$ of $C_o(\mathbb{R};F)$. When $\mathcal{V}$ is considered as a bounded operator $C_o(\mathbb{R}^2;E)$ the above commutation properties means that $\mathcal{V}$ commutes with multiplication by $C_o(\mathbb{R}^2)$. Hence by Proposition 4.1.3., there exists an invertible operator $V(t,o)$ in $B(E)$ for each (t,σ) in $\mathbb{R}^2$ such that $(Vf)(t,\sigma) = V(t,\sigma)f(t,\sigma)$, $\forall\ f \in C_o(\mathbb{R}^2;E)$. Define $U(t,s) = V(t,t-s)$. It is easy to check using the group property of $e^{-i\sigma K}$ that $U(t,s)$ is a strongly continuous bounded propogator onE, which induces $e^{-i\sigma K}$ on $C_o(\mathbb{R};E)$.

REMARK 4.1.7.

a) If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two such operators, then $\mathcal{U}_1\mathcal{U}_2^{-1}$ is a constant multiplication. Hence $U_1(t) = U_2(t)B$, where B is invertible. This means that the propogator $\mathcal{U}$ is uniquely determined, and there is a one-one correspondence between bounded strongly continuous propogators and boundedevolution groups.

b) Since the operator K is similar to $-i\frac{d}{dt}$, we can calculate its resolvent from

$$(K-\xi)^{-1} = \mathcal{U}\left(-i\frac{d}{dt} - \xi\right)^{-1}\mathcal{U}^{-1}.$$

It is given by

$$(K-\xi)^{-1}f\ (t) = i\int_{-\infty}^{t} e^{i\xi(t-s)}\,U(t,s)\ f(s)\ ds \quad ; \quad \operatorname{im}\xi > o$$

$$= -i\int_{t}^{\infty} e^{i\xi(t-s)}\,U(t,s)\ f(s)\ ds \quad ; \quad \operatorname{im}\xi < o.$$

As in the hilbert space situation [26], we can give equivalent conditions on a bounded strongly continuous one parameter group e^{-ioK}, in order that it is a bounded evolution group. The first condition involves the unbounded operator K, and the second, the bounded resolvent $(K-\xi)^{-1}$.

THEOREM 4.1.8.

A bounded strongly continuous one parameter group $e^{-i\sigma K}$ on $C_o(\mathbb{R};E)$ is an evolution group iff one of the following equivalent conditions holds:

a) $M(\phi)D(K) \subseteq D(K)$ and $KM(\phi)f - M(\phi)Kf = -iM(\dot{\phi})f$

for every $\phi \in C_c^1(\mathbb{R})$, $f \in D(K)$ where $\dot{\phi} = \frac{d\phi}{dt}$.

b) (Let $R(\xi) = (K-\xi)^{-1}$ for $\operatorname{im}\xi \neq 0$)

For all $\phi \in C_c^1(\mathbb{R})$, $\operatorname{im}\xi \neq 0$,

$$R(\xi)M(\phi) - M(\phi)R(\xi) = i\, R(\xi)M(\dot{\phi})R(\xi).$$

§ 4.2.

We are now in a position to start studying scattering using time dependant perturbations. Again, E denotes a Banach space.

Given two bounded strongly continuous propogators $U(t,s), U_o(t,s)$, one may consider wave operators

$$W_{\pm}(s) = \operatorname*{st\,lim}_{t\to\pm\infty} U_o(s,t)U(t,s).$$

The scattering operator is formally

$$S(s) = W_+(s)\left[W_-(s)\right]^{-1}.$$

These operators for different times are related by the formulas

$$W_{\pm}(t) = U_o(t,s)\, W_{\pm}(s)\, U(s,t)$$

and
$$S(t) = U_o(t,s)\, S(s)\, U_o(s,t).$$

The usual scattering operator is associated with time $t = o$, $S = S(o)$.

Let K_o, K be the generators of the bounded evolution groups associated with U_o, U respectively on $C_o(\mathbb{R};E)$.

Then for $f \in C_o(\mathbb{R};E)$, $(\sigma,t) \in \mathbb{R}^2$,

$$(e^{i\sigma K_o} e^{-i\sigma K} f)(t) = U_o(t,t+\sigma)\, U(t+\sigma,t)\, f(t) .$$

This observation, and the following lemma yields Theorem 4.2.2.

LEMMA 4.2.1.

Let T_n be a sequence of bounded multiplication operators on $C_o(\mathbb{R};E)$ such

$$\sup \|T_n\| < \infty .$$

Then $\operatorname{st\,lim}_{n\to\infty} T_n$ exists iff $\operatorname{st\,lim}_{n\to\infty} T_n(s)$ exists for each s in $\mathbb{R}$, with uniform convergence of $T_n(s)x$ (for each fixed x in E) over compact subsets of $\mathbb{R}$. In which case $\operatorname{st\,lim}_{n\to\infty} T_n = (\operatorname{st\,lim}_{n\to\infty} T_n(s))$.

THEOREM 4.2.2.

$e^{-i\sigma K} = \mathcal{U} T_\sigma \mathcal{U}^{-1}$, $e^{-i\sigma K_0} = \mathcal{U}_0 T_\sigma \mathcal{U}_0^{-1}$ are two strongly continuous bounded evolution groups on $C_0(\mathbb{R};E)$. Then

$$\mathcal{W}_+ = \operatorname{st\,lim}_{\sigma\to\infty} e^{+i\sigma K_0} e^{-i\sigma K}$$

exists iff $W_+(s)$ exists for all $s \in \mathbb{R}$, with

$$W_+(s)f = \operatorname{st\,lim}_{\sigma\to\infty} U_0(s,\sigma)\, U(\sigma,s)f$$

converging uniformly over compact subsets of $\mathbb{R}$, for each fixed f in E. In this case $\mathcal{W}_+$, and $\mathcal{U}_0^{-1}\mathcal{W}_+\mathcal{U}$ are multiplication by $W_+(s)$ and the constant operator $W_+(0)$ respectively. If $\mathcal{S} = \mathcal{W}_+\left[\mathcal{W}_-\right]^{-1}$ exists, then $\mathcal{U}_0^{-1}\mathcal{S}\,\mathcal{U}_0$ is multiplication by the constant operator $S = W_+ W_-^{-1}$.

§ 4.3.

In § 4.3 to § 4.6 we are going to consider explicit situations where scattering of bounded evolution groups in $C_0(\mathbb{R};E)$ does occur.

Again we consider a Banach space E. Let e^{iYt} be a strongly continuous one parameter bounded group on E. Define $U(t,s) = e^{iY(s-t)}$ to be the associated bounded strongly continuous propogator on E, where $(t,s) \in \mathbb{R}^2$, and $e^{-i\sigma K}$ to be the corresponding bounded evolution group on $C_0(\mathbb{R};E)$. We let $M = \sup_{t\in\mathbb{R}} \|e^{iYt}\| < \infty$. For each s in $\mathbb{R}$, let $D(s) \in B(E)$ such that

4.3.1) $s \longmapsto D(s)$ is strongly continuous

4.3.2) $\|D(s)\| \in L^1(\mathbb{R})$, and put $C = \int_{-\infty}^{\infty} \|D(s)\|\, ds$.

LEMMA 4.3.3.

For $f \in C_o(\mathbb{R};E)$, $t \geq o$, $n \geq 1$, the Bochner integral

$$\varphi_n(f,t)(s) =$$

$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} e^{iY(s_1-t)} D(s+t-s_1) e^{iY(s_2-s_1)} D(s+t-s_2) \ldots D(s+t-s_n) e^{-iYs_n} f(s+t)\, ds_n \cdots ds_1$$

defines an element $\varphi_n(f,t)$ in $C_o(\mathbb{R};E)$ such that

$$\|\varphi_n(f,t)\| \leq M^{n+1}\, C^n\, \|f\| .$$

It follows that there exists $A_n(t)$ in $B\big(C_o(\mathbb{R};E)\big)$ such that $\varphi_n(f,t)(s) = \big(A_n(t)f\big)(s)$ and $\|A_n(t)\| \leq C^n M^{n+1}$. We can similarly consider $t \leq o$.

LEMMA 4.3.4.

$$V_t^\lambda f = e^{iKt} + \sum_{n=1}^{\infty} (i\lambda)^n A_n(t) f \qquad \text{for } f \in C_o(\mathbb{R};E) \text{ and } \lambda \in \mathbb{C} \text{ satisfying}$$

$|\lambda| < 1/CM$ defines a bounded strongly continuous group on $C_o(\mathbb{R};E)$.

We denote the infinitesimal generator of V_t^λ by $iK(\lambda)$. Also, we say $f \in C_o(\mathbb{R};E)$ is in $\mathcal{D}(D)$ if $D(s)f(s)$ lies in $C_o(\mathbb{R};E)$ and put $Df = (D(s)f(s))$. Then $D \in \mathcal{C}_o\big(C_o(\mathbb{R};E)\big)$.

LEMMA 4.3.5.

If $\lambda \in \mathbb{C}$, $|\lambda| < 1/CM$, $K \supseteq K(\lambda) - \lambda D$

$$K(\lambda) \supseteq K + \lambda D.$$

PROOF.

For $n \geq 1$, define
$$D_n(s) = \begin{cases} D(s) & |s| < n \\ D(s).(|s|-n-1) & n \leq |s| < n+1 \\ 0 & |s| \geq n+1 \end{cases}$$

and let D_n be the associated bounded multiplication operators on $C_o(\mathbb{R};E)$. Then $\int_{-\infty}^{\infty} \|D_n(s) - D(s)\|\, ds \longrightarrow 0$ as $n \to \infty$, and if $f \in \mathcal{D}(D)$, $\lim D_n f = Df$.

Moreover if $f \in C_o(\mathbb{R};E)$

$$e^{iK(\lambda)t}f = e^{iKt}f + i\lambda \lim_{n\to\infty} \int_o^t e^{iK(t-s)} D_n e^{iK(\lambda)s} f \, ds$$

where the convergence is uniform over $t \in (o,\infty)$.

However $i(K(\lambda) - z)^{-1} = iR(z,K(\lambda)) = \int_o^\infty e^{-izt} e^{iK(\lambda)t} dt$ for $\operatorname{im} z < o$.

Hence $R(z,K(\lambda))f = R(z,K) - \lim\limits_{n\to\infty} R(z,K) D_n R(z,K(\lambda))f$.

It follows easily that $K \geq K(\lambda) - \lambda D$.

The other relation can be proved similarly.

THEOREM 4.3.6.

The wave operators $\omega_\pm = \operatorname{st} \lim\limits_{t\to\pm\infty} e^{itK(\lambda)} e^{-itK}$

exist on $C_o(\mathbb{R};E)$ and are invertible, for $|\lambda| < 1/CM$. The scattering matrix $\omega_-^{-1}\omega_+ = \operatorname{st} \lim\limits_{t\to\infty} e^{-itK} e^{2itK(\lambda)} e^{-itK}$ is analytic in $|\lambda| < 1/CM$ and given by $\sum_n (i\lambda)^n S_n$ where S_n is the bounded multiplication operator on $C_o(\mathbb{R};E)$ given by

$$S_n(s)(x) = (-1)^n \int_{-\infty}^{\infty}\int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} e^{iY(s-s_1)} D(s_1) e^{iY(s_1-s_2)} \ldots D(s_n) e^{iY(s_n-s)} ds_n \ldots ds_1$$

for $x \in E$.

PROOF.

We consider only $\operatorname{st}\lim e^{-itK} e^{2itK(\lambda)} e^{-itK}$. The remainder of the proof is similar. If $f \in C_o(\mathbb{R};E)$, $t \geq o$

$$e^{-itK} e^{2itK(\lambda)} e^{-itK} f = \sum_{n=o}^{\infty} (i\lambda)^n B_n(t) f$$

where $B_n(t)$ is a bounded multiplication operator on $C_o(\mathbb{R};E)$. If $x \in E$, $s \in \mathbb{R}$, $t \geq o$

$B_n(t)(s)x =$

$$\int_{-t}^{t}\int_{-t}^{s_1} \cdots \int_{-t}^{s_{n-1}} e^{iYs} D(s-s_1) e^{iY(s_2-s_1)} D(s-s_2) e^{iY(s_3-s_2)} \ldots D(s-s_n) e^{iYs_n} x \, ds_n \ldots ds_1$$

Thus $\|B_n(t)(s)\| \leq M^{n+1} C^n$.

Moreover if $f \in C_o(\mathbb{R};E)$ $\lim\limits_{t\to\infty} \int_t^\infty \|D(s-u)\| \, \|f(s)\| \, du = 0$, uniformly over $s \in \mathbb{R}$.

In this way we see $\operatorname{st\,lim}_{t\to\infty} B_n(t)f = S_n f$, where S_n is the bounded multiplication operator given by

$S_n(s)x$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_{n-1}} e^{iYs}D(s-s_1)e^{iY(s_2-s_1)}\ldots\ldots D(s-s_n)e^{-iYs_n}x\, ds_n\ldots\, ds_1$$

$$= (-1)^n\int_{-\infty}^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{n-1}}^{\infty} e^{iY(s-s_1)}D(s_1)e^{iY(s_1-s_2)}\ldots\ldots D(s_n)e^{iY(s_n-s)}x\, ds_n\ldots ds_1.$$

It is clear that $e^{itK(\lambda)}$ is a bounded evolution group, and by Theorem 4.2.2. there is a corresponding scattering result for the propogators. Note that if E is a Banach algebra, and e^{iYt} is a group of automorphisms and D(s) is a derivation for all s, then $e^{itK(\lambda)}$ (and its propogator are also automorphism of the algebra $C_o(\mathbb{R};E)$ (E,respectively).

§ 4.4.

Let A be a Banach algebra and $\{e^{iKt} : t \in \mathbb{R}\}$ a one parameter strongly continuous group of automorphisms of A. Suppose a,b, $\in$ A and define $V \in B(A)$ by $Vx = ax + xb$ for $x \in A$. We suppose that there exists a family $\mathcal{R}$ of Banach algebra representations (i.e. each element π of $\mathcal{R}$ is a norm decresing algebra homomorphism from A into another algebra, which may depend on π) satisfying

4.4.1) $\quad M = \max\left\{ \sup_{\pi\in\mathcal{R}} \int_{-\infty}^{\infty} \|\pi(e^{iKt}a)\|\, dt\ ,\ \sup_{\pi\in\mathcal{R}} \int_{-\infty}^{\infty} \|\pi(e^{iKt}b)\|\, dt\right\} < \infty.$

4.4.2) $\quad$ If $x \in A$, $\lim_{t\to} \int_{t}^{\infty} \|\pi(e^{iKs}y)\|\|\pi x\|\, ds = 0 \qquad y = a,b$

uniformly over $\pi \in \mathcal{R}$; with analogous behaviour at $-\infty$.

4.4.3) $\quad \mathcal{R}$ determines the norm of A i.e. $\exists$ d > o, such that for all x in A : $\quad \|x\| = \sup_{\pi\in\mathcal{R}}\|\pi x\| \geq d\cdot\|x\|$.

PROPOSITION 4.4.4.

Under the above conditions 4.4.1 to 4.4.3, the strong limits of

$$e^{\pm(K+\lambda V)t}\, e^{\mp Kt}\ ; \qquad e^{\pm Kt}\, e^{\mp(K+\lambda V)t}$$

exist on A, for $\lambda \in \mathbb{C}$, $|\lambda| < 1/2M$, and the limits are analytic on this domain. That is, the wave operators and scattering matrix exist on A.

PROOF is similar to that of 4.3.5.

COROLLARY 4.4.5.

Let E be a Banach algebra with e^{iYt} a bounded strongly continuous one parameter group of automorphisms of E. Let e^{iKt} be the induced bounded evolution group on $C_o(\mathbb{R};E)$. Suppose $a,b \in C_o(\mathbb{R};E)$, and put $Vx = ax + xb$ for $x \in C_o(\mathbb{R};E)$. If

$$M = \max\left\{ \int_{-\infty}^{\infty} \|a(s)\|\, ds,\ \int_{-\infty}^{\infty} \|b(s)\|\, ds \right\} < \infty \quad ,$$

then the strong limits of

$$e^{\pm(K+\lambda V)t}\, e^{\mp Kt} \quad ; \quad e^{\pm Kt}\, e^{\mp(K+\lambda V)t}$$

exist on $C_o(\mathbb{R};E)$, for $|\lambda| < 1/2M$, and the limits are analytic on this domain. That is, the wave operators and scattering matrix exist on $C_o(\mathbb{R};E)$.

Thus 4.4.4 can be regarded as an abstraction of 4.3.6.

§ 4.5.

Let E be a separable Banach space and e^{iYt} a one parameter strongly continuous continuous bounded group on E. Let F be another separable Banach space and $\alpha, \beta \in B(E,F)$, $B(F,E)$ respectively, and ϕ, ψ are continuous complex valued functions on $\mathbb{R}$. Define A,B in $B\big(C_o(\mathbb{R};E), C_o(\mathbb{R};F)\big)$ $B\big(C_o\,\mathbb{R};F), C_o(\mathbb{R};E)\big)$ respectively, by

$$(Af)(s) = \phi(s)\,\alpha\, f(s) \qquad f \in C_o(\mathbb{R};E)$$

$$(Bg)(s) = \psi(s)\,\beta\, g(s) \qquad g \in C_o(\mathbb{R};F).$$

Let $V = BA \in B\big(C_o(\mathbb{R};E)\big)$, and e^{iKt} be the bounded evolution group on on $C_o(\mathbb{R};E)$ induced by e^{iYt} on E.

PROPOSITION 4.5.1.

Suppose that:

a) $C = \int_{-\infty}^{\infty} \|\alpha e^{-iYt} \beta\| \, dt < \infty$

b) $D_p(\alpha)$ is dense in E, and $\varphi \in L^q(\mathbb{R})$ for some (p,q), where $1/p + 1/q = 1$; $1 \leq p \leq \infty$.

c) β^* is $(Y,Y,1)^*$smooth.

Then the strong limits of $e^{\pm itK} e^{\mp it(K+\lambda V)}$; $e^{\pm it(K+\lambda V)} e^{\mp itK}$ exist on $C_0(\mathbb{R};E)$, for $|\lambda| < 1/M$, where $M = \|\varphi\|_\infty \|\psi\|_\infty . C$, i.e. the wave operators and scattering matrix exist on $|\lambda| < 1/M$.

The proof is an application of Theorem 2.3.4.

§ 4.6.

Let $(\Omega, \mathcal{F}, \mu)$ be a localisable measure space with a positive measure μ, and E a separable Banach space. For $1 \leqslant p < \infty$, let $L^p(\Omega;E)$ denote the complex Banach space of all E-valued functions f on Ω, such that $f(\cdot)\,\|f(\cdot)\|^{p-1}$ is strongly integrable. If $p = 1$, we can identify $L^1(\Omega;E)$ with the projective tensor product $L^1(\Omega) \check{\otimes} E$. The dual of $L^1(\Omega;E)$ is isometrically isomorphic to $L^\infty(\Omega;E^*)$, the space of all E*valued, essentially bounded, $\sigma(E^*,E)\mu$-measurable functions on Ω. If $f \in L^1(\Omega;E)$, $\varphi \in L^\infty(\Omega, E^*)$, then under this isomorphism:

$$\langle f , \varphi \rangle = \int_\Omega \langle f(\omega) , \phi(\omega) \rangle \, d\mu(\omega).$$

For details see [23].

When $1 < p < \infty$, it is much harder to identify the dual of $L^p(\Omega, E)$, although this is possible in certain circumstances; see for example [15,42]. However, since we will be concerned mainly with euclidean n-space $\mathbb{R}^n$, equipped with Lebegue measure, the following will be sufficient for our purposes [15] :

If $\varphi \in L^p(\mathbb{R}^n;E)^*$, where $1 < p < \infty$, then there exists a mapping F from $\mathbb{R}^n$ into E^* which is weakly measurable (i.e. $\langle e, F(\cdot)\rangle$ is a measurable function on $\mathbb{R}^n$ for each e in E) such that

4.6.1) $\langle f', \varphi \rangle = \int \langle F(t), f(t) \rangle \, dt$

4.6.2) and $\|\varphi\| = \left\{\int \|F(t)\|^q \, dt\right\}^{1/q}$ where $1/p + 1/q = 1$.

$F(t)$ is uniquely determined by φ almost everywhere. Conversely if F is a weakly measurable function for which the right hand side of 4.6.2 is finite, F defines via 4.6.1 a continuous linear functional ϕ on $L^p(\mathbb{R}^n;E)$ for which 4.6.2 holds. We will use the same symbol φ for an element of $L^p(\mathbb{R}^n;E)^*$ and its associated function $F(\cdot)$ defined by 4.6.1.

Let us consider the following examples:

EXAMPLE 4.6.3.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with positive measure μ, and $E = \mathfrak{H}$ is a separable hilbert space. Then $L^2(\Omega, \mathfrak{H})$ is a hilbert space, naturally isomorphic to the hilbert space tensor product $L^2(\Omega) \otimes \mathfrak{H}$.

EXAMPLE 4.6.4.

Let $(\Omega, \mathcal{F}, \mu)$ be a localisable measure space with positive measure μ . Let M be a von Neumann algebra with separable predual M_*. We take $p = 1$ and $E = M_*$. Then $L^\infty(\Omega;M)$ is a W*-algebra under pointwise multiplication with predual $L^1(\Omega;M_*)$. The mapping

$$f \otimes m \longrightarrow f(t)m \qquad f \in L^\infty(\Omega), \; m \in M$$

can be uniquely extended to a W*-isomorphism of the W*-tensor product $L^\infty(\Omega)\bar{\otimes} M$ onto $L^\infty(\Omega;M)$. For details see [54] .

Howland [26], developed a theory for handling time dependant perturbations on a separable hilbert space $\mathfrak{H}$, by considering $L^2(\mathbb{R}^n;\mathfrak{H})$, In our theory, we will treat a separable Banach space E, by studying $L^p(\mathbb{R}^n;E)$. It will reduce directly to Howland's work, by this special choice of $\mathfrak{H}$ and p. Our work will also raise Howland's theory to the von Neumann algebra level by considering Example 4.6.4.

We will first develop the theory of multiplication operators.

Let $(\Omega, \mathcal{F}, \mu)$ be a localisable measure space, and E a separable Banach, For each $\omega \in \Omega$, let $A(\omega) \in B(E)$, such that $\langle m, A(\omega)e \rangle$ is a measurable function for each $m \in E^*$, $e \in E$; i.e. $A(\omega)e$ is measurable for each e in E. Then we can define an element A of $B\left(L^p(\Omega; E)\right)$ by

4.6.5) $\quad (Af)(\omega) = A(\omega)f(\omega) \quad$ for all f in $L^p(\Omega;E)$, $\quad 1 \leq p < \infty$.

Clearly $\quad \|A\| \leq \operatorname{ess\,sup} \|A(\omega)\|$.

A special case of this is the scalar multiplication operator which we now define. For each ϕ in $L^\infty(\Omega)$, let $M(\phi)$ be the operator on $L^p(\Omega;E)$ defined by

4.6.6) $\quad (M(\phi)f)(\omega) = \phi(\omega)\, f(\omega) \quad$ a.e. on Ω, for $f \in L^p(\Omega;E)$.

Clearly if A is as defined in 4.6.5, then

$$A\, M(\phi) = M(\phi)\, A \qquad \forall\, \phi \in L^\infty(\Omega).$$

The next Theorem shows, that if $(\Omega, \mathcal{F}, \mu)$ is σ-finite, then all bounded linear operators on $L^1(\Omega;E)$ which commute with $M(\phi L(\Omega))$ can be constructed in this way.

THEOREM 4.6.7.

Let $(\Omega, \mathcal{F}, \mu)$ be a o-finite measure space, with μ a positive measure. Let E be a separable Banach space. A is a bounded linear map on $L^1(\Omega;E)$ such that $\quad A\, M(\phi) = M(\phi)\, A \quad \forall\, \phi \in L^\infty(\Omega)$.

Then for each ω in Ω, there exists $A(\omega) \in B(E)$, such that

i) $\omega \longmapsto A(\omega)e$ is measurable for each e in E.

ii) If $f \in L^1(\Omega;E)$, then $\quad (Af)(\omega) = A(\omega)f(\omega) \quad$ a.e.

$A(\omega)$ is uniquely determined up to a set of measure zero. That is if $A_1(\omega)$ lies in B(E) a.e. in , and $\quad (Af)(\omega) = A_1(\omega)\, f(\omega) \quad$ a.e. for each f in $L^1(\Omega, E)$, then $A(\omega) = A_1(\omega)$ a.e.

Moreover, $\quad \|A\| = \operatorname{ess\,sup} \|A(\omega)\|$.

PROOF.

Let $\gamma \in L^1(\Omega)$ be strictly positive almost everywhere, and ψ_n a a countable linearly independant set in E, whose linear span is dense in E. Define $\theta_n = \gamma \otimes \psi_n \in L^1(\Omega, E)$, $n = 1,2,\ldots\ldots$

i.e. $\theta_n(\omega) = \gamma(\omega)\,\psi_n$ a.e.

Take $g \in L^\infty(\Omega; E^*) = L^1(\Omega; E)^*$.

Then $\langle A^*g, \theta_n\rangle = \langle g, A\theta_n\rangle$ means that

$$\int_\Omega \langle (A^*g)(\omega), \theta_n(\omega)\rangle \, d\mu(\omega) = \int_\Omega \langle g(\omega), (A\theta_n)(\omega)\rangle \, d\mu(\omega) .$$

But $A\,M(\phi) = M(\phi)\,A$ for all ϕ in $L^\infty(\Omega)$. By considering characteristic functions of measurable subsets of Ω we see that

$$\int_\Delta \langle (A^*g)(\omega), \theta_n(\omega)\rangle \, d\mu(\omega) = \int_\Delta \langle g(\omega), (A\theta_n)(\omega)\rangle \, d\mu(\omega)$$

for all $\Delta \in \mathcal{F}$.

Hence $\langle (A^*g)(\omega), \theta_n(\omega)\rangle = \langle f(\omega), (A\theta_n)(\omega)\rangle$ a.e. in Ω ,

i.e. $\langle (A^*g)(\omega), \psi_n\rangle = \langle f(\omega), \frac{1}{\gamma(\omega)} (A\theta_n)(\omega)\rangle$ a.e.

Let E_o = linear span of $\{\psi_n : n = 1,2,\ldots\}$. For almost all ω in Ω , we will define an operator $C(\omega)$ on E, with domain E_o, putting

$$C(\omega)\psi_n = \frac{1}{\gamma(\omega)} (A\theta_n)(\omega) \quad \text{a.e.}$$

and extending linearly.

Then for all g in $L^\infty(\Omega; E^*)$, $g(\omega) \in \mathcal{D}C(\omega)^*$ a.e. (this null set may depend on g) and $(A^*g)(\omega) = C(\omega)^*\,g(\omega)$ a.e.

We will show that $C(\omega)$ is essentially bounded, and

$$\operatorname*{ess\,sup}_{\Omega} \|C(\omega)\| \leq \|A\| .$$

Let $\{\eta_n : n \in \mathbb{N}\}$ be the countable dense set in E obtained by taking linear combinations

$$\sum_{n=1}^{N} \lambda_n \psi_n$$

where λ_n lie in a countable dense subset of C.

For $n \in \mathbb{N}$, let $S_n = \{\omega \in \Omega : \|C(\omega)\eta_n\| > \|\eta_n\|\,\|A\|\}$,

and $S = \{\omega \in \Omega :$ $C(\omega)$ is either unbounded, or $\|C(\omega)\| > \|A\|\}$.

Then $S = \bigcup_{n=1}^{\infty} S_n$.

Now if $\lambda_n \in C$, $n = 1,2,\ldots\ldots,N$

$$C(\omega)(\Sigma \lambda_n \psi_n) = \frac{1}{\gamma(\omega)} \Sigma \lambda_n (A\theta_n)(\omega) = \frac{1}{\gamma(\omega)} A(\gamma \otimes (\Sigma \lambda_n \cdot \psi_n))(\omega) \quad \text{a.e.}$$

Hence each S_n is measurable. We will show that S is null. Take $\omega \in S_n$, then $\|A\| \|\eta_n\| \gamma(\omega) < \gamma(\omega) \|C(\omega)\eta_n\| = \|A(\gamma \otimes \eta_n)(\omega)\|$.

It follows that for almost all ω in Ω :

$$\|A\| (\chi_{S_n} \gamma)(\omega)\|\eta_n\| < \left\|A\left[(\chi_{S_n} \gamma) \otimes \eta_n\right](\omega)\right\|$$

We deduce

$$\begin{aligned}
\|A\| \|\chi_{S_n} \gamma \otimes \eta_n\| &= \int_\Omega (\chi_{S_n} \gamma)(\omega) \|\eta_n\| \, d\mu(\omega) \\
&\leq \int_\Omega \|A[(\chi_{S_n} \gamma) \otimes \eta_n](\omega)\| \, d\mu(\omega) \\
&= \|A[(\chi_{S_n} \gamma) \otimes \eta_n]\| \\
&\leq \|A\| \|\chi_{S_n} \gamma \otimes \eta_n\|
\end{aligned}$$

Hence $\displaystyle\int_{S_n} \left\{ \|[A(\gamma \otimes \eta_n)](\omega)\| - \|A\|\|\eta_n\| \gamma(\omega) \right\} d\mu(\omega) = 0.$

We must conclude that S_n is of measure zero, for each n. S null means that $C(\omega)$ is bounded almost everywhere, and $\|C(\omega)\| \leq \|A\|$ a.e.

Let $A(\omega)$ be the unique bounded linear extension of $C(\omega)$ to E.

Then $A(\omega)^* = (C(\omega))^*$, and if $g \in L^\infty(\Omega, E^*)$, then

$$(A^*g)(\omega) = A(\omega)^* g(\omega) \quad \text{a.e.}$$

By redefining $A(\omega)$ on a set of measure zero, it follows that $A(\cdot)$ satisfies properties i and ii asserted in the theorem. The uniqueness is clear since E is separable.

In Euclidean n-space $\mathbb{R}^n$, with Lebesgue measure, we have an a alogous result when $1 < p < \infty$. Taking $\mathfrak{h}$ to be a separable hilbert space, and p = 2, the next theorem reduces to

$$L^\infty(\mathbb{R}^n; B(\mathfrak{h})) = M(L^\infty(\mathbb{R}^n))' \quad [43] .$$

(Here $M(\phi)$, for $\phi \in L^\infty(\mathbb{R}^n)$ denotes the scalar multiplication operator on $L^2(\mathbb{R}^n; \mathfrak{h})$.)

THEOREM 4.6.8.

Let E be a separable Banach space, and $1 < p < \infty$. Let A be a bounded linear map on $L^p(\mathbb{R}^n;E)$ such that $A\,M(\phi) = M(\phi)\,A \quad \forall\ \phi \in L^\infty(\mathbb{R}^n)$.

Then for each $\omega \in \mathbb{R}^n$, there exists $A(\omega)$ in $B(E)$ such that

i) $\omega \longrightarrow A(\omega)e$ is measurable for each e in E.

ii) If $f \in L^p(\mathbb{R}^n;E)$ then $(Af)(\omega) = A(\omega)f(\omega)$ a.e.

$A(\cdot)$ is uniquely determined up to a set of measure zero, and

$$\|A\| = \operatorname*{ess\,sup}_{\mathbb{R}^n} \|A(\omega)\| .$$

PROOF.

The proof is analogous to that of Theorem 4.6.7. We use 4.6.1. and 4.6.2. to identify $L^p(\mathbb{R}^n;E)^*$.

DEFINITION 4.6.9.

If E is a separable Banach space and $1 \leq p < \infty$, a bounded multiplication operator is an element A of $B\big(L^p(\mathbb{R}^n;E)\big)$ satisfying

$$A\,M(\phi) = M(\phi)\,A \qquad \forall \phi \in L^\infty(\mathbb{R}^n).$$

For such an operator A, we define $\{A(\omega) : \omega \in \mathbb{R}^n\}$ to be a family in $B(E)$ such that $(Af)(\omega) = A(\omega)f(\omega)$ a.e. for all f in $L^p(\mathbb{R}^n;E)$.

Note that such a family exists and is uniquely determined up to a set of measure zero by Theorems 4.6.7 and 4.6.8

THEOREM 4.6.10.

Let E be a separable Banach space and $1 \leq p < \infty$, with A a bounded multiplication operator on $L^p(\mathbb{R}^n;E)$. Then $A(t)$ is equal almost everywhere to a constant operator iff

(4.6.11) $\qquad A\,T_\sigma = T_\sigma\,A \qquad \forall\ \sigma \in \mathbb{R}^n,$

where $\quad (T_\sigma f)(t) = f(t-\sigma) \quad$ for $f \in L^p(\mathbb{R}^n;E)$.

PROOF.

Let $F = L^p(\mathbb{R}^n;E)$, which is separable since $L^p(\mathbb{R}^n)$ and E are.
Define $\mathcal{J}$, $\mathcal{A}$ in $B(L^p(\mathbb{R}^n;F))$ by

$$(\mathcal{J} g)(\sigma) = T_o g(\sigma)$$

and $\quad (\mathcal{A} g)(o) = A\, g(\sigma) \qquad$ for $g \in L^p(\mathbb{R}^n;F)$.

If $AT_\sigma = T_\sigma A$ a.e. then $\mathcal{J}^{-1}\mathcal{A}\mathcal{J} = \mathcal{A}$ on $L^p(\mathbb{R}^n;F)$.
But $L^p(\mathbb{R}^n;F)$ may also be considered as $L^p(\mathbb{R}^{2n};E)$ by making the element $f(t,\sigma)$ of $L^p(\mathbb{R}^{2n};E)$ correspond to the element $f(\sigma) = f(\cdot,\sigma)$ of $L^p(\mathbb{R}^n;F)$. By considering $\mathcal{A}$ and $\mathcal{J}$ as bounded multiplication operators on $L^p(\mathbb{R}^{2n};E)$, relation 4.6.11 means that $A(t+\sigma) = A(t)$ for a.e. (σ,t) in $\mathbb{R}^{2n}$. Hence $A(t)$ is constant almost everywhere. The converse is trivial.

REMARK 4.6.12.

Let A be a bounded multiplication operator on $L^p(\mathbb{R}^n;E)$. Then if A is invertible, $\quad A^{-1}M(\phi) = M(\phi)\, A^{-1} \quad \forall\ \phi \in L^\infty(\mathbb{R}^n)$.
Hence A^{-1} is a bounded multiplication operator and clearly $A(t)$ is is invertible a.e., and $A^{-1}(t) = (A(t))^{-1}$ a.e.
Conversely, suppose $A(t)$ is invertible a.e., and for each e in E, $A(t)^{-1}e$ is measurable with $\operatorname{ess\,sup} \|A(t)^{-1}\| < \infty$. Then A is invertible and
$(A^{-1}f)(t) = A(t)^{-1}f(t)$ a.e., for f in $L^p(\mathbb{R}^n;E)$.

§ 4.7.

In this section we develop the theory of time dependant perturbations, which extends [26] to the Banach space situation, and allows us to treat weakly continuous groups on von Neumann algebras. We therefore let E be a separable Banach space, and fix p in $[1, \infty)$.

Given a bounded strongly continuous propogator $U(\cdot,\cdot)$, as defined in 4.1.4, the formula

$$(e^{-i\sigma K}f)(t) = U(t,t-\sigma)f(t-\sigma) \qquad \text{for } f \in L^p(\mathbb{R};E),$$

defines a strongly continuous bounded group on $F = L^p(\mathbb{R};E)$. By abuse of notation, we refer to K as the generator of the group. As a non trivial example we can consider $U(t,s) = 1$ on $\mathbb{R}^2$. This gives the group of translations

$$(T_\sigma f)(t) = f(t-\sigma), \qquad \text{for } f \text{ in } L^p(\mathbb{R};E)$$

for which the generator $K = -i \frac{d}{dt}$.

Note that if a strongly continuous propogator U defines the strongly continuous one parameter group $e^{-i\sigma K}$ on F, then

$$(e^{-i\sigma K}T_{-\sigma}f)(t) = U(t,t-\sigma)f(t) \quad \text{for } f \text{ in } L^p(\mathbb{R};E).$$

Hence for each σ in $\mathbb{R}$, $e^{-i\sigma K}T_{-\sigma}$ is a multiplication operator on $L^p(\mathbb{R};E)$
Moreover if $\mathcal{U}$ is the invertible bounded multiplication operator on F defined by

$$(\mathcal{U}f)(t) = U(t,o)f(t) \qquad \text{for } f \text{ in } L^p(\mathbb{R};E),$$

then for all σ in $\mathbb{R}$

$$e^{-i\sigma K} = \mathcal{U}\, T_\sigma \mathcal{U}^{-1}.$$

That is, $e^{-i\sigma K}$ is similar to the group of translations T_σ, with a connecting operator $\mathcal{U}$ which is a multiplication. In order to characterise such strongly continuous groups on F, we need some more definitions.

DEFINITION 4.7.1.

A bounded evolution group is a strongly continuous one parameter bounded group $e^{-i\sigma K}$ on $F = L^p(\mathbb{R};E)$ such that for each σ in $\mathbb{R}$, $e^{-i\sigma K}T_{-\sigma}$ is a bounded multiplication operator on F.

By Theorems 4.6.7 and 4.6.8 , a strongly continuous bounded group $e^{-i\sigma K}$ is a bounded group if and only if

$$e^{-i\sigma K}T_{-\sigma}\, M(\phi) = M(\phi)\, e^{-i\sigma K}T_{-\sigma} \qquad \forall\, \phi \in L^\infty(\mathbb{R}),$$

or equivalently

$$e^{-i\sigma K}M(\phi)e^{i\sigma K} = T_\sigma M(\phi)T_{-\sigma} = M(\phi_\sigma) \quad \text{where} \quad \phi_\sigma(t) = \phi(t-\sigma).$$

DEFINITION 4.7.2.

A bounded measurable propogator is a family of invertible operators $U(t,s)$ on E, defined for $(s,t) \in \mathbb{R}^2$, and satisfying:

a) $U(t,s) = U(s,t)^{-1}$ a.e. $(s,t) \in \mathbb{R}^2$.

b) $U(t,r)\,U(r,s) = U(t,s)$ a.e. $(t,r,s) \in \mathbb{R}^3$.

c) $\|U(t,s)\|$ is bounded on $\mathbb{R}^2$.

d) $\langle m, U(t,s)e \rangle$ is jointly measurable on $\mathbb{R}^2$ for each $m \in E^*$, $e \in E$.

As in [26], we can identify bounded measurable propogators. The proof for the hilbert space situation readily extends to show the following :

LEMMA 4.7.3.

$U(t,s)$ is a bounded measurable propogator iff it can be written in the form

$$U(t,s) = U(t)\,U(s)^{-1}$$

where for each t in $\mathbb{R}$, $U(t) \in B(E)$ satisfies :

i) $U(t)$ is invertible a.e. in $\mathbb{R}$.

ii) $\|U(t)\|$ and $\|U(t)^{-1}\|$ are bounded on $\mathbb{R}$.

iii) $U(t)e$ and $U(t)^{-1}e$ are measurable on $\mathbb{R}$, for each e in E.

We can now characterise bounded evolution groups in terms of bounded measurable propogators. This extends the hilbert space result of [26].

THEOREM 4.7.4.

A strongly continuous bounded group $e^{-i\sigma K}$ on $L^p(\mathbb{R};E)$ is a bounded evolution group iff there exists an invertible bounded multilplication operator $\mathcal{U}$ on $L^p(\mathbb{R};E)$ such that

$$e^{-i\sigma K} = \mathcal{U}\, T_\sigma \mathcal{U}^{-1} \quad \text{for all } \sigma \text{ in } \mathbb{R}.$$

PROOF.

Let $F = L^p(\mathbb{R};E)$ and $G = L^p(\mathbb{R};F)$. Define an operator $\mathcal{V}$ on G by

$$(\mathcal{V} f)(\sigma) = e^{-i\sigma K} T_{-\sigma} f(\sigma) \qquad \text{for } f \in L^p(\mathbb{R};F).$$

Then $\mathcal{V}$ commutes with multiplication by any bounded measurable function $\psi(\sigma)$, and also by 4.7.1, with any operator of the form

$f(\sigma) \longrightarrow M(\phi)f(\sigma)$ where $M(\phi)$ is multiplication on $F = L^p(\mathbb{R};E)$.

G may also be considered as $L^p(\mathbb{R}^2;E)$, and when $\mathcal{V}$ is considered as a bounded operator on $L^p(\mathbb{R}^2;E)$, the above commutation properties means that $\mathcal{V}$ commutes with multiplication by $L^\infty(\mathbb{R}^2)$. Hence by Theorems 4.6.7 and 4.6.8 and Remark 4.6.12, there exists an invertible operator $V(t,\sigma)$ in B(E) for each (t,σ) $\mathbb{R}^2$ such that

$$(\mathcal{V} f)(t,\sigma) = V(t,\sigma)\, f(t,\sigma) \quad \forall f \in L^p(\mathbb{R}^2;E).$$

Define $U(t,s) = V(t,t-s)$. Since the map $(t,s) \longrightarrow (t,t-s)$ is a one-one measure preserving transformation of $\mathbb{R}^2$ onto itself, $U(t,s)$ is well defined, invertible a.e. and $U(t,s)e$ and $U(t,s)^{-1}e$ are measurable for all e in E. Consider the following relations as multiplication operators on $L^p(\mathbb{R};F)$ and $L^p(\mathbb{R}^2;F)$ respectively :

$$(e^{-i\sigma K}T_{-\sigma})^{-1} = T_\sigma\, (e^{i\sigma K}T_\sigma)\, T_{-\sigma}$$

and $$e^{-i(\sigma+\tau)K}\, T_{-(\sigma+\tau)} = (e^{-i\sigma K}\, T_{-\sigma}) \left[T_\sigma(e^{i\tau K}\, T_{-\tau})T_{-\sigma}\right] .$$

We then express the results in terms of $L^p(\mathbb{R}^2;E)$ and $L^p(\mathbb{R}^3;E)$ respectively, as in the proof of Theorem 4.6.10, and deduce that

$$V(t-\sigma,\sigma) = V^{-1}(t,\sigma) \qquad \text{a.e. } (\sigma,t) \text{ in } \mathbb{R}^2$$

and $$V(t,\sigma+\tau) = V(t,\sigma)\, V(t-\sigma,\tau) \qquad \text{a.e. } (\sigma,t,\tau) \text{ in } \mathbb{R}^3$$

In terms of U, this means

$$U(t,s) = U^{-1}(s,t) \qquad \text{a.e. in } \mathbb{R}^2$$

and $$U(t,r)\, U(r,s) = U(t,s) \quad \text{a.e. in } \mathbb{R}^3, \text{ respectively.}$$

By Lemma 4.7.3 and Remark 4.6.12, $V(t,\sigma) = U(t)\, U^{-1}(t-\sigma)$ a.e. in $\mathbb{R}^2$, where $(\mathcal{U} f)(t) = U(t)f(t)$, $f \in L^p(\mathbb{R};E)$, defines an invertible bounded multiplication operator on F. The operation $\mathcal{V}\mathcal{J}$ of multiplication by $e^{-i\sigma K}$ on $L^p(\mathbb{R};E)$ is then given as

$$(\mathcal{V}\mathcal{J})f(t,\sigma) = U(t)U^{-1}(t-\sigma)f(t-\sigma,\sigma) \quad \text{on } L^p(\mathbb{R}^2;E).$$

But $\quad (\mathcal{U}T_\sigma\mathcal{U}^{-1}f)(t) = U(t)U^{-1}(t-\sigma)f(t-\sigma) \quad$ for $f \in L^p(\mathbb{R};E)$.

It follows that on $L^p(\mathbb{R}^2;E)$, multiplication by $e^{-i\sigma K}$ coincides with multiplication by $\mathcal{U}T_\sigma\mathcal{U}^{-1}$. Hence by Theorems 4.6.7-8,

$$e^{-i\sigma K} = \mathcal{U}T_\sigma\mathcal{U}^{-1} \quad \text{a.e. in } \mathbb{R}.$$

Strong continuity shows that this equality holds on the whole of $\mathbb{R}$. This concludes the proof of the theorem.

REMARK 4.7.5.

a) There is a one to one correspondence between bounded measurable propogators and bounded evolution groups.

b) Since the operaor K is similar to $-i\frac{d}{dt}$, we can calculate its resolvent from

$$(K-\xi)^{-1} = \mathcal{U}\left(-i\frac{d}{dt}-\xi\right)^{-1}\mathcal{U}^{-1}$$

as in 4.1.7.b and [26].

c) Suppose $e^{-i\sigma K}$ is a strongly continuousbounded one parameter group on $L^p(\mathbb{R};E)$. Then as in 4.1.8 and [26], we can give equivalent conditions on K or its resolvents in order that it generates a bounded evolution group.

§4.8.

We are again in a position to study scattering using time dependant perturbations. We will particularly note what our results mean for von Neumann algebras.

Again E denotes a separable Banach space and p is fixed in $[1,\infty)$. Given two bounded strongly continuous propogators $U(t,s)$ and $U_0(t,s)$, one may consider wave operators :

4.8.1. $$W_{\pm}(s) = \lim_{t\to\pm\infty} U_o(s,t)\, U(t,s)$$

and require that the limit exists in the strong or weak sense.

The scattering operator is formally

$$S(s) = W_+(s)\left[W_-(s)\right]^{-1}.$$

The usual scattering operator is associated with time $s = o$, $S = S(o)$

Let K, K_o be the generators of the bounded evolution groups associated with U, U_o respectively, on $L^p(\mathbb{R};E)$. Then

$$(e^{i\sigma K_o} e^{-i\sigma K} f)(t) = U_o(t,t+\sigma)\, U(t+\sigma,t)\, f(t) \qquad \text{for } f \in L^p(\mathbb{R};E).$$

Thus if $W_{\pm}(t)$ exists strongly (in the weak sense respectively) for all t in $\mathbb{R}$, then

$$\mathcal{W}_{\pm} = \lim_{\sigma\to\pm\infty} e^{i\sigma K_o} e^{-i\sigma K}$$

exists on $L^p(\mathbb{R};E)$ in the strong sense (respectively weak), and is multiplication by $W_{\pm}(t)$. The scattering operator $\mathcal{S} = \mathcal{W}_+ \mathcal{W}_-^{-1}$ is therefore multiplication by the operator $S(t)$. Hence, if $\mathcal{U}_o$ is multiplication by $U_o(t,o)$, the operator $\mathcal{U}_o \mathcal{S} \mathcal{U}_o^{-1}$ is multiplication by the constant operator $S = S(o)$.

We now consider bounded evolution groups in general. Let $e^{-i\sigma K} = \mathcal{U} T_\sigma \mathcal{U}^{-1}$ and $e^{-i\sigma K_o} = \mathcal{U}_o T_\sigma \mathcal{U}_o^{-1}$ be two bounded evolution groups on $L^p(\mathbb{R};E)$. The following theorem extends [26] to our general situation.

THEOREM 4.8.2.

$\mathcal{W}_+ = \operatorname{st}\lim_{\sigma\to\infty} e^{i\sigma K} e^{-i\sigma K}$ exists on $L^p(\mathbb{R};E)$ iff there exists $W_+ \in B(E)$ such that

(4.8.3.) $$\lim_{\sigma\to\infty} \int_0^h \| hW_+(f) - U_o^{-1}(t+\sigma)\, U(t+\sigma) f \|^p \, dt = 0,$$

for all $f \in E$, $h \geq o$.

In this case $\mathcal{U}_o^{-1} \mathcal{W}_+ \mathcal{U}$ is multiplication by the constant operator W_+.

If $\mathcal{S} = \mathcal{W}_+ \mathcal{W}_-^{-1}$ exists, then $\mathcal{U}_o^{-1} \mathcal{S}\, \mathcal{U}_o$ is multiplication by the constant operator $S = W_+(W_-)^{-1}$.

REMARK 4.8.4.

If there exists $W_+ \in B(E)$ such that 4.8.3 holds, then

$$W_+ f = \lim_{\sigma\to\infty} \frac{1}{h} \int_0^h U_0^{-1}(t+\sigma)\, U(t+\sigma)f \; dt \quad \forall f \in E,\ h > o.$$

COROLLARY 4.8.5.

If $U(t)$ and $U_o(t)$ are strongly continuous, and there is a dense set $\mathcal{D}$ in E such that $U_o^{-1}(t)U(t)x$ is uniformly continuous on $t > T(x)$, for each x in $\mathcal{D}$, then ω_+ exists strongly iff the wave operators $W_+(s)$ of 4.8.1 exists strongly.

We also give a form of [26,Thm 4] for the weak topology:

THEOREM 4.8.6.

If ω_+ exists in the weak sense, then there exists $W_+ \in B(E)$ such that

$$\langle W_+ f, \phi\rangle = \lim_{\sigma\to\infty} \frac{1}{h} \int_0^h \langle U_o(t+\sigma)^{-1} U(t+\sigma)f, \phi\rangle \; dt \quad \text{for all } f \in E,$$

$\phi \in E^*$, $h > o$. [If E is reflexive the converse also holds] .
In this case $\mathcal{U}_o^{-1}\,\omega_+\mathcal{U}$ is multiplication by the constant operator W_+.
If $\mathcal{S} = \omega_+\,\omega_-^{-1}$ exists, then $\mathcal{U}_o^{-1}\,\mathcal{S}\,\mathcal{U}_o$ is multiplication by the constant operator $W_+(W_-)^{-1}$.

COROLLARY 4.8.7.

If $U(t)$ and $U_o(t)$ are strongly continuous, and there is a dense set $\mathcal{D}$ in E such that $\langle U_o^{-1}(t)U(t)f, \phi\rangle$ is uniformly continuous on $t > T(x,\phi)$, for $x \in E$, $\phi \in E^*$, then ω_+ exists weakly iff the wave operators $W(s)$ of 4.8.1 exist.

In particular, suppose that $E = M_*$, where M is a von Neumann algebra with separable predual. Take $p = 1$, and let σ_t, σ_t^{o} be the dual(weakly continuous) one parameter groups on the von Neumann algebra $L^\infty(\mathbb{R};M)$, of e^{itK} and e^{itK_o}.

Then $\text{weak*limit}_{t \to \pm\infty}\ \sigma_t\ \sigma^{\circ}_{-t}$ exists on $L(\mathbb{R};M)$

iff $\text{weak limit}_{t \to \pm\infty}\ e^{-itK_o}e^{itK}$ exists on $L^1(\mathbb{R};M_*)$.

This is because $L^1(\mathbb{R};M_*)$ is weakly sequentually complete [1]. Using 4.8.6 and 4.8.7, the above can be related to time dependant scattering in M.

§ 4.9.

We will now show how our theory of L^p-evolution groups can be used to deduce results for time dependant perturbations from known results for time independant smooth perturbations.

Let E and F be separable Banach spaces and p_o fixed in $[1, \infty)$. Put $E_o = L^p(\mathbb{R};E)$ and $F_o = L^p(\mathbb{R};F)$. Suppose $e^{-iK_o t}$ is a bounded evolution group on E_o, and $A, B \in \mathcal{C}_o(E_o,F_o)$, $\mathcal{C}_o(F_o,E_o)$ respectively. Assume $A\,M(\phi) \supseteq M(\phi)\,A$, $B\,M(\phi) \supseteq M(\phi)\,B$ $\forall \phi \in L^\infty(\mathbb{R})$, and that there exists p,q with $1/p + 1/q = 1$, $1 < q \leq \infty$ such that

a) A is (K_o,K_o,p) smooth

b) B^* is $(K_o,K_o,q)^*$ smooth, with $u \longmapsto B^*(e^{\pm itK_o})^*u$ $(o \leq t < \infty)$ weak* continuous from E_o^* into $L^p(\mathbb{R}^+;F_o)^*$.

If $\mathcal{Y} = L^p(\mathbb{R}^+;F_o)$, we also take that $Ae^{\pm iK(\cdot)}B\theta$ are in $B(\mathcal{Y})$ with common bound $\leq M$.

Then it is easy to see that the perturbations $e^{itK(\kappa)}$ given by Theorem 2.1.9 are in fact bounded evolution groups for each $|\kappa| < 1/M$.

Let us now define for $-\infty \leq \alpha \leq \beta \leq \infty$

$$P(\alpha,\beta)\,u(t) = \chi_{(\alpha,\beta)}(t)\,u(t) \qquad u \in E_o \quad (F_o),$$

where χ is the characteristic function of (α,β). It is easily seen that $A\,P(\alpha,\beta)$ and $B\,P(\alpha,\beta)$ satisfy the same conditions as A and B.

The following Theorem is the time dependant theoretic result corresponding to the stationary one in [26, Thm 6], the proof being an obvious modification to Howland's programme.

THEOREM 4.9.1.

Suppose that E_o is weakly sequentually complete and E_o^* is separable for the weak*topology. Also assume that $U_o(t,s)$ is strongly continuous and that for every $\epsilon > o$, there exists $\delta > o$, such that

$$|A\,P(\alpha,\beta)|_{(K_o,K_o,p)} < \epsilon \qquad \text{and} \qquad |(B\,P(\alpha,\beta))^*|_{(K_o,K_o,q)^*} < \epsilon$$

whenever $|\beta-\alpha| < \delta$.

Then the propogator $U(t,s;\kappa)$ of $K(\kappa)$ is strongly continuous in t,s and analytic in $|\kappa| < 1/M$. The wave operators

$$W_+(\kappa) = \operatorname*{st\,lim}_{t\to\infty} U_o(s,t)\,U(t,s;\kappa)$$

exist, are invertible and analytic in $|\kappa| < 1/M$.

We remark that Theorem 4.9.1 can be applied to Schroedinger's equation, taking $p_o = 2$ and $E = L^2(\mathbb{R}^n)$ in a similar fashion to [26, § 4] .

These time dependant techniques can also be applied to the time independant theory of Chapter 3. However, this is unsatisfactory in that we have to apply Theorem 3.3.1 (or Theorem 3.4.7) to the larger algebra $B(L^2(\mathbb{R};\mathfrak{H}))$ instead of $L^\infty(\mathbb{R};M)$ (where M is a von Neumann algebra acting on $\mathfrak{H}$); and we therefore omit all details.

§ 4.10

Suppose we are given a bounded strongly continuous one parameter group on a separable Banach space E, and we want to consider a time dependant perturbation. The problem of which space $L^q(\mathbb{R};E)$ (or indeed $C_o(\mathbb{R};E)$) to take depends on what smoothness hypotheses hold in the space E. We give a simple example to show how q smoothness in E, and the choice of $L^q(\mathbb{R};E)$ are related.

To be specific, consider a separable Banach space E, and e^{iYt} a bounded strongly continuous one parameter group on E. Suppose $A \in \mathcal{C}_o(E,F)$ where F is another Banach space, and $\phi \in L^\infty(\mathbb{R})$. Define

$f \in E_o = L^q(\mathbb{R};E)$ where $1 \leq q < \infty$ to be in $\mathcal{D}(\alpha)$ if $\phi(s)f(s) \in \mathcal{D}(A)$ a.e. and $A\phi(s)f(s) \in L^q(\mathbb{R};F) = F_o$. In which case put $(\alpha f)(s) = A\phi(s)f(s)$. Then $\alpha \in \mathcal{C}_o(E_o,F_o)$. Let e^{iKt} be the bounded evolution group on E_o induced by e^{iYt} on E i.e. $(e^{iKt}f)(s) = e^{iYt}f(s+t) \quad f \in E_o$.

For simplicity, we take E and F to be reflexive.

PROPOSITION 4.10.1.

If A is $(Y,q,+)$ smooth on E, then α is $(K,q,+)$ smooth on $L^q(\mathbb{R};E)$.

PROOF.

Consider $f \in L^q(\mathbb{R};E)$. Then

$$(s,t) \longmapsto \|A\,\phi(s+t)\, e^{iYt}f(s+t)\| \qquad \text{(extended norm)}$$

is jointly measurable by reflexivity and [36]. Hence by Fubini:

$$\int_0^\infty \int_{-\infty}^\infty \|A\,\phi(s+t)\, e^{iYt}\, f(s+t)\|^q\, dt\, ds \leq |A|^q_{(Y,q,+)} \cdot \|\varphi\|^q_\infty \int_{-\infty}^\infty \|f(s)\|^q\, ds.$$

Hence there exists a null set N in $\mathbb{R}^+$ such that for $t \notin N$:

$$\int_{-\infty}^\infty \|A\,\varphi(s+t)\, e^{iYt}\, f(s+t)\|^q\, ds < \infty$$

i.e. $\varphi(s+t)\, e^{iYt}\, f(s+t) \in \mathcal{D}(A)$ a.e. (depending on $t \notin N$), which means $e^{iKt}f \in \mathcal{D}(\alpha)$ and

$$\int_{-\infty}^\infty \|\alpha\, e^{iKt}f\|^q\, dt \leq |A|^q_{(Y,q,+)}\, \|\varphi\|^q_\infty\, \|f\|^q$$

Similarly if $1/p + 1/q = 1$, $(Y,p,+)^*$ smoothness in E^* is amenable to $(K,p,+)^*$ smoothness in $L^q(\mathbb{R};E)$.

For $q = $ "∞" , see §4.5.

CHAPTER 5.

This chapter starts the second part of the thesis, namely the investigation of the structure of completely positive (CP) maps. The study of CP maps on C*-algebras was initiated by Stinespring [55], and consequently, some extremal problems were studied by Arveson [4].

In the first section, we produce some Schwartz ty inequalities for n-positive linear mappings on *-algebras. In § 5.2, we begin to show why CP(A; $\mathfrak{H}$) the bounded linear maps from a Banach*-algebra A with approximate identity into the bounded linear maps on a hilbert space $\mathfrak{H}$, should be regarded as a higher order state space. The pure elements of the cone CP(A; $\mathfrak{H}$) which have norm one, namely P(A; $\mathfrak{H}$), are analysed in § 5.3, whilst in § 5.4, we show how a closed two sided ideal of a C*-algebra A; gives a decomposition of P(A; $\mathfrak{H}$). If A is a C*-algebra with identity, and K is a positive operator on $\mathfrak{H}$, Arveson [4] considered CP(A; $\mathfrak{H}$,K) = $\{\omega \in$ CP(A; $\mathfrak{H}$) : $\omega(1) = K\}$. In § 5.5, we extend this definition to cover a Banach*-algebra with approximate identity and observe some properties of this convex set. In § 5.6, we generalise a theorem of F. and M. Riesz to give a sufficient condition for the covariance of certain representations of a C*-algebra relative to a one parameter group of *-automorphisms. In the seventh section, we study normal completely positive maps on W*-algebras, and in § 5.8 we discuss tensor products of CP maps on operator algebras, and derive a CP analogue of Tomiyama's theorem [63] regarding the singularity of conditional expectations on W*-algebras. In § 5.9, we characterise the CP maps on the algebra of the canonical commutation relations. Dilations of semigroups of positive maps on C*-algebras are important in the classical and quantum theories of stochastic processes. In § 5.10 we give some operator algebra analogues of Nagy's hilbert space [59,60] and Stroescu's Banach space [58] dilations. Recently, Lindblad [39] has been involved with the study of infinitesimal generators of norm continuous semigroups of CP maps. In § 5.11, we extend this by studying strongly continuous semigroups and groups on C*-algebras, which have various positivity properties.

§ 5.1.

Let M_n be the C*-algebra of all complex $n \times n$ matrices. If A is a C*-algebra, let A_n be the C*-algebra $M_n(A) = A \otimes M_n$ of all $n \times n$ matrices over A.

DEFINITION 5.1.1.

If A is a *-algebra [16, Definition 1.1.1] and B is a C*-algebra, let w be linear map from A into B.

a) w is said to be n-positive, where $n \in \mathbb{N}$, if for all $\{x_i : i = 1,2,..,n\}$ in A, $\left[w(x_i^* x_j)\right] \geq 0$ in B_n.

b) w is said to completely positive (CP) if w is n-positive for all n.

Suppose A and B are C*-algebras and w a linear map from A into B. For each n define $w_n = w \otimes 1_n$ from A_n into B_n by applying w elementwise to each matrix over A. Then w is n-positive iff w_n is positive [34]. This is so because each element of A_n^+ can be written as a sum of at most n elements of the form $\left[x_i^* x_j\right]$ where $\{x_i : i=1,\ldots,n\} \subseteq A$.

Suppose $\mathfrak{H}$ is a hilbert space and $C \in B(\mathfrak{H})^+$. Then for all y in $B(\mathfrak{H})$, $\epsilon > 0$, $y^*(C+\epsilon)^{-1}y \in B(\mathfrak{H})$ and increases monotonically as $\epsilon \downarrow 0$. Hence

$$\operatorname*{st\,lim}_{\epsilon \downarrow 0} \; y^*(C+\epsilon)^{-1}y$$

exists iff $y^*(C+\epsilon)^{-1}y$ is bounded above for all $\epsilon > 0$. We write this limit as $y^*C^{-1}y$ [35]. If V is a vector subspace of $B(\mathfrak{H})$ and $y^*C^{-1}y$ exists for all y in V, then $\operatorname{st\,lim}_{\epsilon \downarrow 0} y^*(C+\epsilon)^{-1}x$ exists for all x,y in V, and we write this as $y^*C^{-1}x$.

The following theorem is already known in the following special cases. Suppose A is a C*-algebra, and $w: A \longrightarrow B(\mathfrak{H})$ is linear. Then

a) If w is CP, $w(x^*x) \geqslant w(x^*y)\, w(y^*y)^{-1}\, w(y^*x) \quad \forall x,y \in A$ [35].

b) If A has an identity and w is 2-positive with $w(1)w(x) = w(x)\ \forall x \in A$, then $w(x^*x) \geqslant w(x^*)\, w(x) \quad \forall x \in A$ [3]

THEOREM 5.1.2.

Let A be a *-algebra, $\mathfrak{H}$ a hilbert space and $w : A \longrightarrow B(\mathfrak{H})$ an n-positive *linear map, where $n \geqslant 2$. Then if $y, x_i \in A \quad i =,2,\dots,n-1$, we have

$$\left[w(x_i^*x_j)\right] \geqslant \left[w(x_i^*y)\, w(y^*y)^{-1}\, w(y^*x_j)\right] \qquad \text{in } B(\mathfrak{H})_{n-1}$$

In particular

$$\|w(y^*y)\|\left[w(x_i^*x_j)\right] \geqslant \left[w(x_i^*y)\, w(y^*x_j)\right] \quad .$$

PROOF.

Put $x_o = y$. Then if $\xi_o, \xi_1, \dots\dots, \xi_{n-1} \in \mathfrak{H}$,

$$\sum_{i,j=o}^{n-1} \langle w(x_i^*x_j)\, \xi_j, \xi_i \rangle \geqslant o \qquad \text{since w is n-positive.}$$

i.e.
$$\sum_{i,j=1}^{n-1} \langle w(x_i^*x_j)\, \xi_j, \xi_i \rangle + \sum_{i=1}^{n-1} \langle w(x_i^*y)\, \xi_o, \xi_i \rangle$$
$$+ \sum_{i=1}^{n-1} \langle w(y^*x_i)\, \xi_i, \xi_o \rangle + \langle w(y^*y)\, \xi_o, \xi_o \rangle \geqslant 0.$$

We now fix $\xi_1, \dots\dots, \xi_{n-1}$ and put $\xi_o = -\left(w(y^*y)+\epsilon\right)^{-1} \sum_{i=1}^{n-1} w(y^*x_i)\, \xi_i$.

Then

$$\sum_{i,j=1}^{n-1} \langle w(x_i^*x_j)\, \xi_j, \xi_i \rangle -2 \sum_{i,j=1}^{n-1} \langle w(x_i^*y) \left(w(y^*y)+\epsilon\right)^{-1} w(y^*x_j)\, \xi_j, \xi_i \rangle$$
$$+ \sum_{i,j=1}^{n-1} \langle w(x_i^*y) \left(w(y^*y)+\epsilon\right)^{-1} w(y^*y) \left(w(y^*y)+\epsilon\right)^{-1} w(y^*x_j)\, \xi_j, \xi_i \rangle$$
$$\geqslant 0.$$

But $2\left(w(y^*y)+\epsilon\right)^{-1} - \left(w(y^*y)+\epsilon\right)^{-2} w(y^*y) = \left(w(y^*y)+\epsilon\right)^{-1} + \epsilon \left(w(y^*y)+\epsilon\right)^{-2}$.

Hence

$$\sum_{i,j=1}^{n-1} \langle w(x_i^* x_j)\xi_j, \ \xi_i \rangle$$
$$\geq \sum_{i,j=1}^{n-1} \langle w(x_i^* y)\left\{ \left(w(y^*y)+\epsilon\right)^{-1} + \epsilon\left(w(y^*y)+\epsilon\right)^{-2} \right\} w(y^* x_j)\ \xi_j, \ \xi_i \rangle$$
$$\geq \sum_{i,j=1}^{n-1} \langle w(x_i^* y)\ \left(w(y^*y)+\epsilon\right)^{-1} w(y^* x_j)\ \xi_j, \ \xi_i \rangle \ .$$

The remainder of the proof is clear, and is omitted.

COROLLARY 5.1.3.

If A is a Banach *-algebra [16,Definition 1.2.1] with approximate identity [16,B29] and $w: A \longrightarrow B(\mathfrak{H})$ is an n-positive bounded linear map. Then for all $x_i \in A$, $i = 1,2,\ldots,n-1$

$$\|w\| \left[w(x_i^* x_j)\right] \geq \left[w(x_i^*)\ w(x_j)\right] \quad .$$

We remark that 5.1.2 and 5.1.3 can be used to characterise CP maps.

§ 5.2.

In this section we consider a Banach *-algebra A and a hilbert space $\mathfrak{H}$, and show how we can regard $CP(A;\mathfrak{H})$, the convex cone of bounded completely positive maps from A into $B(\mathfrak{H})$, as generalised higher order state spaces. We first demonstrate how the GNS-Stinespring representation can be constructed on a Banach *-algebra A with approximate identity. This is an extension of [4,55] where A is a C*-algebra with identity.

THEOREM 5.2.1.

Let A be a Banach *-algebra with approximate identity, $\mathfrak{H}$ a hilbert space and w a map from A into $B(\mathfrak{H})$. Then $w \in CP(A;\mathfrak{H})$ iff there exists a hilbert space $\mathfrak{K}$, $V \in B(\mathfrak{H}, \mathfrak{K})$, a representation π of A in $B(\mathfrak{K})$ such that $\quad w(x) = V^* \pi(x) V \quad$ for all x in A.

PROOF.

Suppose $w \in CP(A;\mathfrak{H})$. We define a bilinear form $\langle\ ,\ \rangle$ on the algebraic tensor product $A \odot \mathfrak{H}$ as follows:

If $x_i, y_j \in A, \xi_i, \eta_j \in \mathfrak{H}$ $i = 1,\dots,m;\ j = 1,\dots,n$ put

$$\langle \sum_i x_i \otimes \xi_i , \sum_j y_j \otimes \eta_j \rangle = \sum_{i,j} \langle w(y_j^* x_i)\xi_i, \eta_j \rangle .$$

This bilinear form is positive semi definite as w is completely positive.

For each x in A, define a linear transformation $\pi_o(x)$ on $A \odot \mathfrak{H}$ by

$$\pi_o(x) : \sum x_i \otimes \xi_i \longrightarrow \sum x x_i \otimes \xi_i .$$

π_o is an algebra homomorphism for which

$$\langle \pi_o(x)u, v \rangle = \langle u, \pi_o(x^*)v \rangle \quad \forall\ u,v \in A \odot \mathfrak{H} .$$

Let $u = \sum_{i=1}^m x_i \otimes \xi_i$ be a fixed element of $A \odot \mathfrak{H}$. Then $\rho(x) = \langle \pi_o(x)u,u \rangle$ defines a positive linear functional on A. Hence if u_λ is an approximate identity for $A, x \in A$:

$$\begin{aligned}\langle \pi_o(x)u, \pi_o(x)u \rangle = \rho(x^*x) &\leq \|x^*x\| \lim_\lambda \rho(u_\lambda) \qquad \text{by } [16,2.1.5] \\ &= \|x^*x\| \lim_\lambda \sum_{j,k} \langle w(x_j^* u_\lambda x_k)\xi_k, \xi_j \rangle \\ &= \|x^*x\| \langle u,u \rangle .\end{aligned}$$

(We remark that this inequality is trivial if A is a C*-algebra).

Now let $N = \{ u \in A \odot \mathfrak{H} : \langle u,u \rangle = 0 \}$. Then N is a linear subspace of $A \odot \mathfrak{H}$, invariant under $\pi_o(x)$ for each x in A, and $\langle\ ,\ \rangle$ determines an inner product on $A \odot \mathfrak{H}/N$. Let $\mathfrak{K}$ be the hilbert space completion of $A \odot \mathfrak{H}/N$, then there is an unique representation $\pi(x)$ of A on $\mathfrak{K}$ such that $\pi(x)(u + N) = \pi_o(x)u + N \qquad x \in A,\ u \in A$.

Let $\Lambda : A \odot \mathfrak{H} \longrightarrow \mathfrak{K}$ denote the canonical projection. Then for x in A ξ in $\mathfrak{H}$, $\|\Lambda(x \otimes \xi)\|^2 = \langle w(x^*x)\xi, \xi \rangle \leq \|w\| [\|x\| \|\xi\|]^2$

Hence, with u as above, $\pi(u_\lambda)\Lambda x \otimes \xi \longrightarrow \Lambda x \otimes \xi$ as $\lambda \to \infty$.

Hence $\pi(u_\lambda) \longrightarrow 1$ strongly on $\mathfrak{K}$.

For $\xi \in \mathfrak{H}$, we will define a linear form φ_ξ on $\mathfrak{K} = [\pi(A)\mathfrak{K}]^-$. Take $a_i \in A$. $\gamma_i \in \mathfrak{H}$ $i = 1,2,\dots,m$. Then

$$\sum_i \langle \gamma_i, \Lambda a_i^* \otimes \xi \rangle = \lim_\lambda \sum_i \langle \gamma_i, \Lambda a_i^* u_\lambda \otimes \xi \rangle = \lim_\lambda \sum_i \langle \pi(a_i)\gamma_i, \Lambda u_\lambda \otimes \xi \rangle$$

If $p = \sum_{i=1}^{m} \pi(a_i)\gamma_i$, then

$$|\sum \langle \gamma_i, \Lambda a_i^* \otimes \xi \rangle| \leq \sup_\lambda \|p\| \, \|\Lambda u_\lambda \otimes \xi\| \leq \|p\| \|w\|^{1/2} \|\xi\| .$$

Hence $\varphi_\xi(p) = \sum_i \langle \gamma_i, \Lambda a_i^* \otimes \xi \rangle$

gives a bounded linear form on $\pi(A)\mathcal{K}$ which we extend by continuity to $\mathcal{K}$. Hence there exists an unique $V\xi$ in $\mathcal{K}$, such that

$$\varphi_\xi(p) = \langle p, V\xi \rangle \quad \forall \ p \in \mathcal{K}.$$

Hence $\langle \gamma, \Lambda a^* \otimes \xi \rangle = \langle \pi(a)\gamma, V\xi \rangle \quad \forall \ \xi \in \mathfrak{h}, \gamma \in \mathcal{K}, a \in A,$

which shows that $\pi(a)V\xi = \Lambda a \otimes \xi \quad \forall \ a \in A, \xi \in \mathfrak{h}$.

It is clear that $\xi \longrightarrow V\xi$ is linear. Moreover $\|V\xi\| = \|\varphi_\xi\| \leq \|w\|^{1/2} \|\xi\|$.

Hence V is bounded, and $\|V\| \leq \|w\|^{1/2}$.

Also if $x, y \in A$, $\xi, \eta \in \mathfrak{h}$,

$$\langle w(y^*x)\xi, \eta \rangle = \langle \Lambda x \otimes \xi , \Lambda y \otimes \eta \rangle = \langle \pi(x)V\xi , \pi(y)V\eta \rangle$$
$$= \langle V^*\pi(y^*x)V\xi, \eta \rangle .$$

Hence $w(\cdot) = V^*\pi(\cdot)V$. The converse is clear.

Note that in the above theorem we can always choose $\mathcal{K} = [\pi(A)V\mathfrak{h}]^-$. The pair (π, V) is then called minimal by Arveson [5]. For a given $w \in CP(A;\mathfrak{h})$ the minimal pair is uniquely determined up to unitary equivalence, i.e. if minimal pairs (π_i, V_i) represent w , for $i = 1,2$, with π_i represenations of A on hilbert spaces $\mathcal{K}_i$, $V_i \in B(\mathfrak{h}, \mathcal{K}_i)$ $i = 1,2$ and $w(\cdot) = V_i^* \pi_i(\cdot) V_i$ $i = 1,2$, then there is a unitary map U of $\mathcal{K}_1$ onto $\mathcal{K}_2$ such that $UV_1 = V_2$ and $U\pi_1(x) = \pi_2(x)U \ \forall x \in A$. If $w \in CP(A; \mathfrak{h})$ we will alwys denote the quantities constructed in the proof of the above theorem by π_w, V_w, $\mathcal{K}_w$, Λ_w etc. Note that (π_w, V_w) is a minimal pair. We also define $K_w = V_w^* V_w$

$$= \text{st} \lim_\lambda w(u_\lambda) \in B(\mathfrak{h})^+,$$

for any approximate identity u_λ for A.

COROLLARY 5.2.2.

Let A be a Banach *-algebra with approximate identity , $\mathfrak{H}$ a hilbert space, and $w \in CP(A;\mathfrak{H})$.

Then $\|w\| = \|K_w\|$ ($= \|w(1)\|$ if A has an identity).

If A is a C*-algebra, then for all $n \geq 1$,

$$\|w_n\| = \|K_w\| \quad .$$

DEFINITION 5.2.3.

Let A be a Banach *-algebra and B C*-algebra. We define a partial ordering $\geq$ on B(A,B) by $\alpha \geq \beta$ (for α, β B(A,B)) if $\alpha - \beta \in CP(A,B)$, the completely positive bounded maps from A into B.

This partial ordering has been analysed when A is a C*-algebra with identity by Arveson [5]. He identified the following sets:

5.2.4 The extreme rays of the cone $CP(A;\mathfrak{H})$.

5.2.5 The extreme points of $[0,\varphi]$, for a fixed φ in $CP(A,\mathfrak{H})$.

5.2.6 The extreme points of the set $CP(A;\mathfrak{H},K) = \{w \in CP(A;\mathfrak{H}) : K_w = K\}$, where K is a fixed operator in $B(\mathfrak{H})^+$.

We remark that these results still hold for a Banach *-algebra with approximate identity.

We now observe what happens when an identity is adjoined to an algebra. The following reduces to [16,2.1.5] when $\mathfrak{H}$ is one dimensional.

THEOREM 5.2.7.

Let A be a Banach *-algebra with approximate identity, and $\tilde{A}$ the Banach *-algebra obtained by adjoining an identity. If $\mathfrak{H}$ is a hilbert space, take $w \in CP(A;\mathfrak{H})$. Then :

1 $w(x)^* = w(x^*) \quad \forall\ x \in A$, and

$$w(x^*y)\, w(y^*y)^{-1}\, w(y^*x) \leq w(x^*x) \quad \forall\ x,y \in A \qquad [35].$$

2 If u_λ is an approximate identity for A,

$w(u_\lambda)$, $w(u_\lambda^* u_\lambda) \longrightarrow K_w \equiv V_w^* V_w$ strongly.

3 w can be extended in an unique manner to $\tilde{w}$ in $CP(\tilde{A}; \mathfrak{H})$ such that $\tilde{w}(1) = K_w$. Every other element of $CP(\tilde{A}; \mathfrak{H})$ which majorises w, majorises $\tilde{w}$.

4 If x_i is any family in A indexed by a directed set D such that $w(x_i)$, $w(x_i^* x_i) \longrightarrow K_w$ strongly, then $\Lambda_{\tilde{w}} x_i \otimes \xi \longrightarrow \Lambda_{\tilde{w}} 1 \otimes \xi$ strongly for all $\xi \in \mathfrak{H}$. Hence $A \odot \mathfrak{H}$ is dense in $\tilde{A} \odot \mathfrak{H}$ for the prehilbertian structure defined by $\tilde{w}$. Moreover $V_w \xi = \Lambda_{\tilde{w}} 1 \otimes \xi \quad \forall \xi \in \mathfrak{H}$.

PROOF.

1. The first part is clear. The second statemant follows from 5.1.2 or [35]. However a much easier approach is to show

$$1 \geq \pi_w(y) V_w \left(V_w \pi_w(y^* y) V_w + \epsilon \right)^{-1} (\pi_w(y) V_w)^* \quad \forall\ y \in A,\ \epsilon > 0.$$

This is straightforward using a polar decomposition and the spectral theorem.

2. is clear since $\pi_w(u_\lambda) \longrightarrow 1$ strongly for any approximate identity u_λ.

3. Extend π_w to $\tilde{A}$ by defining $\tilde{\pi}_w(1) = 1$. Put $\tilde{w}(x) = V_w^* \tilde{\pi}_w(x) V_w$ for $x \in \tilde{A}$. Then $\tilde{w} \in CP(\tilde{A}; \mathfrak{H})$, extends w, moreover $(\pi_{\tilde{w}}, V_{\tilde{w}}) \simeq (\tilde{\pi}_w, V_w)$. Let $\alpha \in CP(\tilde{A}; \mathfrak{H})$ such that $\alpha|_A \geq w$. Take $x_i \in A$, $\xi_i \in \mathfrak{H}$, $i = 1,2,\ldots,n$, and let u_λ be an approximate identity for A. Then :

$$\begin{aligned}
\sum \langle \tilde{w}(x_i^* x_j)\, \xi_j, \xi_i \rangle &= \lim_\lambda \sum \langle V_w^* \tilde{\pi}_w(x_i^*) \pi_w(u_\lambda^* u_\lambda) \tilde{\pi}_w(x_j) \xi_j, \xi_i \rangle \\
&= \lim_\lambda \sum \langle w(x_i^* u_\lambda^* u_\lambda x_j)\, \xi_j, \xi_i \rangle \\
&= \lim_\lambda \sum \langle w(x_i^* u_\lambda^* u_\lambda x_j)\, \xi_j, \xi_i \rangle \\
&\leq \lim_\lambda \sum \langle \alpha(x_i^* u_\lambda^* u_\lambda x_j)\, \xi_j, \xi_i \rangle
\end{aligned}$$

But $\left[\pi_\alpha(x_i^*) \pi_\alpha(u_\lambda^* u_\lambda) \pi_\alpha(x_j)\right] \leq \| \pi_\alpha(u_\lambda^* u_\lambda) \| \,.\, \left[\pi_\alpha(x_i^*) \pi_\alpha(x_j)\right]$

It follows that $\tilde{w} \leq \alpha$.

4. The first statement is a routine calculation. Regarding $\Lambda_w A \odot \mathfrak{H}$ as a subspace of $\Lambda_{\tilde{w}} \tilde{A} \odot \mathfrak{H}$, the density is clear from 2. Moreover $V_w \xi = \Lambda_{\tilde{w}} 1 \otimes \xi$, since we already know that $V_w \xi = \text{st}\lim \Lambda_w u_\lambda \otimes \xi$.

COROLLARY 5.2.8.

If w_1 and $w_2 \in CP(A;\mathfrak{H})$, then

$$K_{w_1+w_2} = K_{w_1} + K_{w_2}$$

and
$$(w_1+w_2)^{\sim} = \tilde{w}_1 + \tilde{w}_2 .$$

Arveson [5] has an extension theorem for CP maps on self adjoint norm closed linear subspaces S of C*-algebras with identity 1 such that $S \ni 1$. For ideals in C*-algebras we have the following. The corresponding result for the usual state space ($\mathfrak{H} = C$) can be seen in [7].

THEOREM 5.2.9.

Let A be a C*-algebra, with J a closed two sided ideal of A. If $\alpha \in CP(J;\mathfrak{H})$, there exists $\beta \in CP(A;\mathfrak{H})$ such that $\beta|_J = \alpha$ and $K_\beta = K_\alpha$. β is majorised by every map in $CP(A;\mathfrak{H})$ which majorises α. If u_λ is an approximate identity for J and $x \in (A_n)^+$, then

$$\beta_n(x) = \text{st}\lim_\lambda \alpha_n\, x^{1/2}\begin{bmatrix} u_\lambda & & 0 \\ & \ddots & \\ 0 & & u_\lambda \end{bmatrix} x^{1/2}$$

If we canonically identify $_\beta$ with

$$\pi_{\beta_n}(x) = \text{st}\lim \pi_{\alpha_n}\, x^{1/2}\begin{bmatrix} u_\lambda & & 0 \\ & \ddots & \\ 0 & & u_\lambda \end{bmatrix} x^{1/2}$$

PROOF.

By [16, 2.10.4] there exists a represenation π of A on K_α extending π_α. Put $\beta(x) = V_\alpha^* \pi(x) V_\alpha$ for x in A. Since (π, V_α) is a minimal pair for β, we have $(\pi, V_\alpha) \simeq (\pi_\beta, V_\beta)$ and $K_\beta = K_\alpha$. Suppose γ is another element of $CP(A;\mathfrak{H})$ extending α. Then $\text{st}\lim \pi_\alpha(u_\lambda) = 1$, and for x in A_n^+,
$$x^{1/2}\begin{bmatrix} u_\lambda & & 0 \\ & \ddots & \\ 0 & & u_\lambda \end{bmatrix} x^{1/2} \in (J_n)^+, \text{ and } x^{1/2}\begin{bmatrix} u_\lambda & & 0 \\ & \ddots & \\ 0 & & u_\lambda \end{bmatrix} x^{1/2} \le x.$$

Hence if $\xi \in \mathfrak{H} \otimes C^n$,

$$\langle \beta_n(x)\xi,\xi\rangle = \lim_\lambda \langle \pi_n(x^{1/2})\begin{bmatrix}\pi_\gamma(u_\lambda) & & 0\\ & \ddots & \\ 0 & & \pi_\alpha(u_\lambda)\end{bmatrix}\pi_n(x^{1/2})(V_\alpha\otimes 1)\xi,\ (V_\alpha\otimes 1)\xi\rangle$$

$$= \lim_\lambda \langle \alpha_n\left(x^{1/2}\begin{bmatrix}u_\lambda & & 0\\ & \ddots & \\ 0 & & u_\lambda\end{bmatrix}x^{1/2}\right)\xi,\xi\rangle$$

$$\leq \quad \langle \gamma_n(x)\xi,\xi\rangle \ .$$

Hence $\beta \lessdot \gamma$.

Suppose also $K_\gamma = K_\alpha$.

Then $K_\gamma = K_{\gamma-\beta} + K_\beta$ implies $K_{\gamma-\beta} = 0$. Hence $\gamma-\beta = 0$.

CP maps are well behaved with respect to taking quotient C*-algebras. If A is a C*-algebra containing a closed two sided ideal J, let $\omega : A \longrightarrow A/J$ be the canonical homomorphism. If $\alpha \in CP(A;\mathfrak{H})$, for some hilbert space $\mathfrak{H}$, and $\alpha(J) = 0$, then α defines, by passing to the quotient an element β in $CP(A/J;\mathfrak{H})$. Then

$$V_\beta^* \pi_\beta(x)V_\beta \doteq \beta\omega(x) = \alpha(x).$$

Moreover $[(\pi_\beta o\omega)(A)V_\beta\mathfrak{H}]^- = [\pi_\beta(A/J)V_\beta\mathfrak{H}]^- = \mathcal{K}_\beta$. Hence $(\pi_\beta o\omega, V_\beta)$ is a minimal pair for α and $(\pi_\beta o\omega, V_\beta) \simeq (\pi_\alpha, V_\alpha)$. This leads to the following conclusions which generalises [16, 2.4.9-11] about positive linear functionals on C*-algebras.

PROPOSITION 5.2.10.

A is a C*-algebra, J is a closed two sided ideal of A, and $\mathfrak{H}$ a hilbert space. Take $\alpha \in CP(A;\mathfrak{H})$. Then :

a) $\alpha(J) = 0$ iff $\pi_\alpha(J) = 0$.

b) $\ker \pi_\alpha$ is the largest closed two sided ideal of A contained in $\ker\alpha$.

c) If also $\beta \in CP(A;\mathfrak{H})$ then the following are equivalent:

i) $\ker\pi_\alpha \subseteq \ker\pi_\beta$

ii) β is 0 on $\ker\pi_\alpha$.

§ 5.3.

It is natural now to study the pure elements in various cones of CP maps. For this, it is convenient to introduce the BW topology of Arveson [5]. Let A be a Banach space and $\mathfrak{H}$ a hilbert space. Let $B(A;\mathfrak{H})$ be the set of bounded linear maps from A into $B(\mathfrak{H})$, and $B_r(A;\mathfrak{H})$ the subset of norm at most equal to r. A net $\Phi_\alpha \in B_r(A;\mathfrak{H})$ converges to Φ in $B_r(A;\mathfrak{H})$ iff $\Phi_\alpha(x) \longrightarrow \Phi(x)$ weakly for all x in A. A convex set V is open in $B(A;\mathfrak{H})$ iff $V \cap B_r(A;\mathfrak{H})$ is open in the topology of $B_r(A;\mathfrak{H})$ for every $r > 0$. The convex sets so defined form a base for a locally convex Hausdorff topology on $B(A;\mathfrak{H})$ called the BW topology. According to [23], $B(A;\mathfrak{H})$ is isometrically isomorphic to the dual space of $A \overset{\vee}{\otimes} T(\mathfrak{H})$. This identifies the BW topology with the weak*-topology.

We first make some definitions. Let A be a Banach *-algebra and $\beta \in CP(A;\mathfrak{H})$. We say β is pure (belongs to an extreme ray) if $\alpha \in CP(A;\mathfrak{H})$ $\alpha \leqslant \beta$ implies $\alpha = \lambda\beta$ for some $\lambda \in [0,1]$. We let $P(A;\mathfrak{H})$ denote those pure elements of $CP(A;\mathfrak{H})$ which have norm one.

PROPOSITION 5.3.1.

Let A be a Banach *-algebra with approximate identity and $\mathfrak{H}$ a hilbert space. Put $B = \{ w \in CP(A;\mathfrak{H}) : \|w\| \leq 1 \}$. Then

i) B is convex and compact for the BW topology.

ii) 0 and $P(A;\mathfrak{H})$ are extreme points of B.

In the above proposition, there is a distinct difference between the case $\mathfrak{H} = \mathbb{C}$ and $\dim\mathfrak{H} = 2$. It is well known that if $\mathfrak{H} = \mathbb{C}$, then the extreme points of $B = 0 \cup P(A)$ [16,2.5.5]. However if $\dim\mathfrak{H} \geq 2$, then the extreme points of $B \neq 0 \cup P(A;\mathfrak{H})$. Consider $\varphi \in P(A)$, $\varphi \neq 0$, where A is any Banach *-algebra with approximate identity, and let P_1

be any projection in $B(\mathfrak{H})$ of rank at least two. Then $w_1(x) = \phi(x).P_1$, $x \in A$, is not extremal in $CP(A;\mathfrak{H})$ but is an extreme point of B. Another such counterexample is obtained when A has an irreducible representation π on a hilbert space $\mathfrak{H}$ of dimension at least 2. Take any projection P_2 in $B(\mathfrak{H})$, $0 \neq P_2 \neq 1$, then

$$w_2(x) = P_2\pi(x)P_2 + (1-P_2)\pi(x)(1-P_2) \qquad x \in A$$

is not extremal in $CP(A;\mathfrak{H})$ but is an extreme point of B. This means that the proof of [38, Lemma 3] is incorrect, however the conclusion remains true by [33].

Our next results describe some of the properties of pure CP maps on C*-algebras. Propositions 5.3.2-5 thus generalise [16, (2.8.5-6, 2.9.1)] concerning pure states of C*-algebras and are consequences of Kadison's transitivity theorem [28].

PROPOSITION 5.3.2.

A is a C*-algebra, $\mathfrak{H}$ a hilbert space and α a pure element of $CP(A;\mathfrak{H})$. If $N_\alpha = \left\{\sum x_i \otimes f_i \in A \odot \mathfrak{H} \;:\; \sum_{i,j=1}^{n} \langle \alpha(x_i^*x_j)f_j, f_i\rangle = 0\right\}$, then $A \odot \mathfrak{H}/N_\alpha$ with the scalar product induced by α is complete, and equal to the represenation space defined by α.

PROPOSITION 5.3.3.

Let A be a C*-algebra and $\mathfrak{H}$ a finite dimensional hilbert space. $\dot{A}$ denotes the C*-algebra obtained from A by adjoining an identity. If β_1 and β_2 are two pure elements of $CP(A;\mathfrak{H})$ then the following are equivalent:

i) π_{β_1} and π_{β_2} are equivalent, and $K_{\beta_1} = K_{\beta_2}$.

ii) There exists an unitary element u in $\dot{A}$ such that

$$\beta_2(x) = \beta_1(u^*xu) \qquad \text{for all } x \text{ in } A.$$

PROOF.

The implication ii $\Rightarrow$ i is easy, and the hypothesis that $\mathfrak{H}$ is finite dimensional is unneccesary.

i $\Rightarrow$ ii. Suppose that there exists $U: \mathcal{K}_{\beta_2} \longrightarrow \mathcal{K}_{\beta_1}$ unitary such that $\pi_{\beta_1}(x) = U\pi_{\beta_2}(x)U^* \quad \forall\, x \in A$. If $\xi \in \mathfrak{h}$, then

$$\langle V_{\beta_1}\xi , V_{\beta_1}\xi\rangle = \langle K_{\beta_1}\xi , \xi\rangle = \langle K_{\beta_2}\xi , \xi\rangle = \langle V_{\beta_2}\xi , V_{\beta_2}\xi\rangle$$
$$= \langle UV_{\beta_2}\xi , UV_{\beta_2}\xi\rangle \quad .$$

Hence there exists an unitary W in $B(\mathcal{K}_{\beta_1})$ such that

$$WV_{\beta_1}\xi = UV_{\beta_2}\xi \quad \forall \xi \in \mathfrak{h}.$$

Since π_{β_1} is irreducible and $V_{\beta_1}\mathfrak{h}$ is finite dimensional,we deduce by [16, 2.8.3] that there exists a unitary u in $\tilde{A}$ such that

$$\tilde{\pi}_{\beta_1}(u)V_{\beta_1}\xi = UV_{\beta_2}\xi \quad \forall \xi \in \mathfrak{h} \quad .$$

Hence $\beta_2(x) = \beta_1(u^*xu) \quad \forall\, x \in A$.

PROPOSITION 5.3.4.

Let A be a C*-algebra, $\mathfrak{h}$ a hilbert space and $\beta \in CP(A;\mathfrak{h})$. If

$$M = \{ x \in A : \beta(x^*x) = 0 \}$$

then $\ker\beta \supseteq M + M^*$. If β is pure and $\mathfrak{h}$ finite dimensional, then $\ker \beta = M + M^*$.

PROOF.

Since $\beta(x^*)\beta(x) \leq \|\beta\|\, \beta(x^*x) \quad \forall\, x \in A$, we have $M \subseteq \ker\beta$.

Hence $M^* \subseteq (\ker\beta)^* = \ker\beta$ and $\ker\beta \supseteq M + M^*$.

Now suppose β is pure and $\mathfrak{h}$ finite dimensional. If $b \in \ker \beta$, then for all $\xi, \eta \in \mathfrak{h}$

$$\langle \pi_\beta(b)V_\beta\xi , V_\beta\eta\rangle = \langle \beta(b)\xi , \eta\rangle = 0.$$

Hence there exists a self adjoint operator H on $\mathcal{K}_\beta$ such that

$$H\pi_\beta(b)V_\beta\xi = \pi_\beta(b)V_\beta\xi \quad \text{and} \quad HV_\beta\eta = 0 \quad \text{for all } \xi , \eta \in \mathfrak{h} .$$

Since π_β is irreducible and $\mathfrak{h}$ is finite dimensional we deduce that there exists an hermitian element a in A such that

$$\pi_\beta(a)\, \pi_\beta(b)V_\beta\xi = \pi_\beta(b)V_\beta\xi \quad \text{and} \quad \pi_\beta(a)V_\beta\eta = 0 \quad \text{for all } \xi, \eta \in \mathfrak{h}.$$

i.e. $ab - b,\ a \in M$. Then $b^* = (b - ab)^* + b^*a^* \in M^* + M$.

PROPOSITION 5.3.5.

A is a C*-algebra with identity and $\mathfrak{H}$ a hilbert space. Let α, β $P(A;\mathfrak{H})$ with $K_\alpha = K_\beta$, and $\|\alpha-\beta\| < 2$. Then π_α and π_β are equivalent.

PROOF.

Take any $f_o \in \mathfrak{H}$. If π_α and π_β are inequivalent, it follows from [16., 2.8.3] that there exists a unitary u in A such that

$$\pi_\alpha(u)\, V_\alpha f_o = .\ V_\alpha f_o$$

and $$\pi_\beta(u)\, V_\beta f_o = -\, V_\beta f_o.$$

Then $$2\|V_\alpha f_o\|^2 = \langle 2\, V_\alpha^* V_\alpha f_o, f_o\rangle = \langle V_\alpha^* V_\alpha f_o, f_o\rangle + \langle V_\beta^* V_\beta f_o, f_o\rangle$$
$$= \langle \pi_\alpha(u) V_\alpha f_o, V_\alpha f_o\rangle - \langle \pi_\beta(u) V_\beta f_o, V_\beta f_o\rangle$$
$$= \langle \left(\alpha(u) - \beta(u)\right) f_o, f_o\rangle \quad .$$

Hence $2\,\|V_\alpha f_o\|^2 \leq \|\alpha-\beta\|\,\|f_o\|^2 \qquad \forall\, f_o \in \mathfrak{H}$,

which implies $\|\alpha-\beta\| = 2$, a contradiction.

Suppose A is a Banach *-algebra with approximate identity and $\mathfrak{H}$ a hilbert space. We are now going to study the natural map from $P(A;\mathfrak{H})$ onto A, the family of equivalence classes of non zero irreducible representations of A. As when $\mathfrak{H} = \mathbf{C}$ [16, 2.5.7], the following proposition shows that $P(A;\mathfrak{H}) \longrightarrow \hat{A}$ is bijective iff every irreducible representation of A is dimension 1. For a C*-algebra A, this occurs iff A is commutative [16, 2.7.3].

PROPOSITION 5.3.6.

Suppose A is a *-algebra, π a topologically irreducible non zero representation of A, and $\mathfrak{H}$ a hilbert space. Suppose $V_i \in B(\mathfrak{H}, K_\pi)$ and β_i are the completely positive maps on A defined by (π, V_i) i = 1,2. Then $\beta_1 = \beta_2$ iff there exists $\lambda \in$ C of modulus 1 such that $V_1 = \lambda V_2$.

Now suppose A is a C*-algebra and $S \subseteq \hat{A}$. Then $\pi \in \hat{A}$ is in the closure of S, iff $\ker \pi \supseteq \bigcap\{\ker\theta : \theta \in S\}$, [16]. It is thus clear that if S is closed in A, then so is its inverse image in $P(A;\mathfrak{h})$ for any hilbert space $\mathfrak{h}$, where $P(A;\mathfrak{h})$ is given the relative BW topology. Thus $P(A;\mathfrak{h}) \longrightarrow A$ is always continuous. We have not yet been able to determine when this is also open if $\mathfrak{h}$ is infinite dimensional.

In order to look in detail at the map $P(A;\mathfrak{h}) \longrightarrow A$, when A is a C*-algebra and $\mathfrak{h}$ finite dimansional it is useful to consider an idea of Lance [34]. Let A and B be C*-algebras. Then there exists an isometric isomorphism between $(A \otimes^{\vee} B)^*$ and $B(A,B^*)$, where $\vee$ denotes the projective (Banach space) tensor product. Under this correspondence, $f \in (A \otimes^{\vee} B)^*$ goes to $T_f \in B(A,B^*)$ given by

$$T_f(a)(b) = f(a \otimes b) \qquad \forall\ a \in A,\ b \in B.$$

Lance [34] has shown that under this map, there is an isometric affine c correspondence between $(A \otimes_{\beta} B)^*_+$ and $CP(A,B^*)$, where β denotes the greatest C*-norm and $CP(A,B^*)$ the completely positive maps from A into B^*.

In particular, take $B = M_n(\mathbb{C})$, the finite dimensional matrix algebra. Then $B^* = T(\mathbb{C}^n)$, and is order isomorphic to B. Hence there is an affine correspondence between $(A_n^*)_+$ and $CP(A;\mathbb{C}^n)$, which is a homeomorphism for the weak* and BW topologies. This correspondence is not isometric; if $w \in CP(A;C^n)$, then its norm as an element iw of $(A_n^*)_+$ is $\operatorname{tr} K_w$, not $\|K_w\|$. It follows that if A has an identity, there is a natural homeomorphism from $P(A;\mathbb{C}^n)$ onto $P(A_n)$ given by

$$w \longmapsto \frac{(iw)}{\operatorname{tr} K_w} .$$

Consider the following commutative diagram :

$$\begin{array}{ccc} P(A;\mathbb{C}^n) & \longleftrightarrow & P(A_n) \\ \downarrow & & \downarrow \\ \hat{A} & \longrightarrow & \hat{A}_n \end{array}$$

The map from $\hat{A}$ onto $\hat{A}_n$ being the canonical one, namely $\pi \longrightarrow \pi_n$, which is both continuous and open. It follows that the map from $P(A;\mathbb{C}^n)$ onto A is also both continuous and open.

Using this technique, it is possible to generalise many of the results in [16,§3.4] to completely positive maps from C*-algebras into finite dimensional matrix algebras. As a further example we give the following.

THEOREM 5.3.7.

A is a C*-algebra acting on a hilbert space $\mathfrak{K}$, and $\mathfrak{H}$ is a finite dimensional hilbert space. Take $w \in CP(A;\mathfrak{H})$. Then w is a BW limit of completely positive maps of the form

$$x \longmapsto \sum_{i=1}^{N} \omega_{\xi_i}(x)\, \eta_i \otimes \bar{\eta}_i \qquad \text{where } \xi_i \in \mathfrak{K},\ \ \eta_i \in \mathfrak{H} \quad i = 1,\ldots,N.$$

PROOF.

Apply the known result, [16, § 3.4] or [22], when $\mathfrak{H} = \mathbb{C}$ to A_n and then use Lance's correspondence to deduce the result for $CP(A;\mathfrak{H})$.

§ 5.4.

Consider a C*-algebra A, and $\mathfrak{H}$ an arbitrary hilbert space. We will see how a closed two sided ideal I of A determines a decomposition of $P(A;\mathfrak{H})$ in a natural manner.

If $\omega : A \longrightarrow A/I$ is the canonical homomorphism, then

$\omega_n : A\otimes M_n \longrightarrow A\otimes M_n/I\otimes M_n = A/I\otimes M_n$ satisfies $\omega_n(A_n^{\;+}) = (A_n/I_n)^+ = (A/I)_n^{\;+}$. It follows that the map $\alpha \longrightarrow \alpha\circ\omega$ is a bijection between $CP(A/I;\mathfrak{H})$ and those elements of $CP(A;\mathfrak{H})$ which are zero on I. This bijection preserves the norm and is bicontinuous for the BW topologies.

We first decompose the whole of $CP(A;\mathfrak{H})$ relative to the ideal; the decomposition of $P(A;\mathfrak{H})$ itself follows in 5.4.2. These results generalise [16, 2.11.7-8] which are concerned with positive linear functionals and pure states.

THEOREM 5.4.1.

Let A be a C*-algebra, and I a closed two sided ideal of A. If $\mathfrak{H}$ is a hilbert space, take any $\alpha \in CP(A;\mathfrak{H})$.

i) There exists on unique decomposition $\alpha = \alpha_1 + \alpha_2$,

where $\alpha_i \in CP(A;\mathfrak{H})$ i = 1,2 , $K_{\alpha_1} = K_{\alpha_1|I}$ and $\alpha_2(I) = 0$.

ii) The pair (π_α, V_α) is identifiable with $(\pi_{\alpha_1} \oplus \pi_{\alpha_2}, V_{\alpha_1} + V_{\alpha_2})$;

π_{α_2} is zero on I, and $(\pi_{\alpha_1})|_I$ is non degenerate.

PROOF.

Let $\pi_\alpha = \pi$, and $\mathcal{K}_1$ be the essential subspace of $\pi|_I$,

i.e. $\mathcal{K}_1 = [\pi(I)\mathcal{K}_\alpha]^- = [\pi(I)\pi(A)V_\alpha\mathfrak{H}]^- = [\pi(I)V_\alpha\mathfrak{H}]^-$.

Put $\mathcal{K}_2 = \mathcal{K}_\alpha \ominus \mathcal{K}_1$, and $V_i = P_{\mathcal{K}_i} V_\alpha$ i = 1,2.

Let π_1, π_2 be the subrepresentations of π defined on $\mathcal{K}_1$, $\mathcal{K}_2$ respectively. Then $[\pi_i(A)V_i\mathfrak{H}]^- = \mathcal{K}_i$ i = 1,2.

Hence if we put $\alpha_i(x) = V_i^*\pi(x)V_i$ $x \in A$, i = 1,2; $(\pi_{\alpha_i}, V_{\alpha_i})$ is identifiable with the minimal pair (π_i, V_i) for i = 1,2.

Then $\alpha_2(I) = 0$, since $\pi_2(I) = 0$. The representations π_1 and $\pi_1|_I$ are non degenerate .Hence $K_{\alpha_1} = V_{\alpha_1}^* V_{\alpha_1} = V_1^*V_1 = \left(V_{\alpha_1|I}\right)^* V_{\alpha_1|I} = K_{\alpha_1|I}$

In particular if $K_\alpha = K_{\alpha_I}$, then

$$K_\alpha = V_\alpha^*V_\alpha = V_1^*V_1 + V_2^*V_2$$

shows that $V_2^*V_2 = 0$. Hence $\alpha_2 = 0$.

The equality $V_\alpha = V_1 + V_2$ shows that $\alpha = \alpha_1 + \alpha_2$.

We now prove the uniqueness in i). Suppose $\beta_1, \beta_2 \in CP(A;\mathfrak{H})$, such that $\alpha = \beta_1 + \beta_2$, $K_{\beta_1} = K_{\beta_1|I}$, and $\beta_2(I) = 0$. The previous argumentwith β_1 replacing α, shows that $(\pi_{\beta_1})|_I$ is non degenerate.

Hence if u_λ is an approximate identity for I, then for all x in A

$$\beta_1(x) = \text{st} \lim \beta_1(xu_\lambda).$$

Similarly $\alpha_1(x) = \text{st} \lim \alpha_1(xu_\lambda)$. But α_1 and β_1 coincide on I, as $\alpha_2(I) = 0$, and $\beta_2(I) = 0$. Hence $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$.

THEOREM 5.4.2.

A is a C*-algebra and I is a closed two sided ideal of A. $\mathfrak{h}$ is a hilbert space. Let $P_I(A;\mathfrak{h}) = \{ w \in P(A;\mathfrak{h}) : w|_I = 0 \}$,

$$P^I(A;\mathfrak{h}) = \{ w \in P(A;\mathfrak{h}) : w|_I \neq 0 \}.$$

i) For any $\alpha \in P_I(A;\mathfrak{h})$, let α' be the element of $CP(A/I;\mathfrak{h})$ obtained by going to the quotient. Then $\alpha \longmapsto \alpha'$ is a bijection from $P_I(A;\mathfrak{h})$ onto $P(A/I;\mathfrak{h})$.

ii) The map $\alpha \longmapsto \alpha|_I$ is a bijection from $P^I(A;\mathfrak{h})$ onto $P(I;\mathfrak{h})$.

PROOF.

i) See the argument of [16, 2.11.8].

ii) Take $\alpha \in P^I(A;\mathfrak{h})$. In the decomposition $\alpha = \alpha_1 + \alpha_2$ of the previous theorem, α_1, α_2 are proportional to α. α_2 is zero on I and α is non zero on I. Thus $\alpha_2 = 0$, and $\alpha = \alpha_1$. Hence $K_{\alpha_I} = K_\alpha$, which implies $\|\alpha|_I\| = 1$. By [16, 2.10.3] and Theorem 5.4.1.i, $(\pi_\alpha)|_I$ is irreducible. Hence $\alpha|_I \in P(I;\mathfrak{h})$.

If $\alpha, \alpha'' \in P^I(A;\mathfrak{h})$, and $\alpha|_I = \alpha'|_I$, there exists an isomorphism of $\mathcal{K}_\alpha$ onto $\mathcal{K}_{\alpha'}$, which transforms $V_{\alpha|_I}$ (which is the same as V_α) to $V_{\alpha'|_I}$ (which is the same as $V_{\alpha'}$), and $(\pi_\alpha)|_I$ to $(\pi_{\alpha'})|_I$, and thus π_α to $\pi_{\alpha'}$ by [16,2.10.4]. Thus $\alpha = \alpha'$, and the map $\alpha \longmapsto \alpha|_I$ of ii) is injective. Finally, if $\beta \in P(I;\mathfrak{h})$, the irreducible represenation π_β can be extended to an irreducible representation π of A [16, 2.10.4]. Since $\|K_\beta\| = 1$, $\alpha(x) = V_\beta^* \pi(x) V_\beta, x \in A$ gives an element α in $P^I(A;\mathfrak{h})$ such that $\alpha|_I = \beta$. This proves that the map $\alpha \longrightarrow \alpha_I$ of ii) is surjective.

We thus have for each hilbert space $\mathfrak{h}$, the following commutative diagram:

$$\begin{tikzcd}
P(A/I;\mathfrak{h}) \arrow[r] \arrow[d] & P(A;\mathfrak{h}) \arrow[d] & P(I;\mathfrak{h}) \arrow[l] \arrow[d] \\
(A/I)^\wedge \arrow[r] & \hat{A} & \hat{I} \arrow[l]
\end{tikzcd}$$

The vertical arrows represent surjections. On a horizontal line, two

two arrows represent injections, which map onto two complemantary subsets, whose union is the whole middle set.

§ 5.5.

Consider again a Banach *-algebra A with approximate identity, $\mathfrak{H}$ a hilbert space and K a fixed operator in $B(\mathfrak{H})_+$. We will now make some observations regarding

$$CP(A;\mathfrak{H},K) = \{ w \in CP(A;\mathfrak{H}): K_w = K \} ,$$

and in particular $P(A;\mathfrak{H},K)$ which we define to be those pure elements of the convex cone

$$\{ w \in CP(A;\mathfrak{H}): K_w = \lambda K \text{ for some } \lambda \geq 0 \}$$

which have norm $\|K\|$.

PROPOSITION 5.5.1.

A is a Banach *-algebra with approximate identity, $\mathfrak{H}$ a hilbert space and $K \in B(\mathfrak{H})_+$. Put $B = \{ w \in CP(A;\mathfrak{H}): K_w = \lambda K \text{ for some } \lambda \in [0,1] \}$.

Then :

i) B is convex, and if A has an identity, B is compact for the BW topology.

ii) The extreme points of B are O and $P(A;\mathfrak{H},K)$.

iii)If A has an identity, B is the BW-closed convex hull of o and $P(A;\mathfrak{H},K)$.

This result should be compared with [16, 2.5.5] where $\mathfrak{H} = \mathbb{C}$, and contrasted with 5.3.1 when dim $\mathfrak{H} \geq 2$. Note that if A has an identity, then, $B = \{ w \in CP(A;\mathfrak{H}): w(1) = \lambda K \text{ for some } \lambda \in [0,1] \}$.

Moreover, $CP(A;\mathfrak{H},K) = \{ w \in CP(A;\mathfrak{H}): w(1) = K \}$ is convex and BW compact. The extreme points of $CP(A;\mathfrak{H},K)$ are the extreme points of B which lie in $CP(A;\mathfrak{H},K)$, namely $P(A;\mathfrak{H},K)$. Hence $CP(A;\mathfrak{H},K)$ is the BW closed convex hull of $P(A;\mathfrak{H},K)$.

As regards taking quotient C*-algebras we can show the following, which extends [16, 2.11.8] about pure states.

PROPOSITION 5.5.2.

A is a C*-algebra, and I a closed two sided ideal of A. $\mathfrak{H}$ is a hilbert space, and $K \in B(\mathfrak{H})_+$.

Put $P_I(A;\mathfrak{H},K) = \{ w \in P(A;\mathfrak{H},K) : w|_I = 0 \}$.

If w lies in $P_I(A;\mathfrak{H},K)$ let w' be the induced map on A/I obtained by passing to the quotient. Then $w \longmapsto w'$ is a bijection from $P_I(A;\mathfrak{H},K)$ onto $P(A/I;\mathfrak{H},K)$.

§ 5.6.

We are now going to give a sufficient condition for the covariance of certain representations of C*-algebras relative to a one parameter group of *-automorphisms.

Let A be a C*-algebra and α_t a strongly continuous one parameter group of *-automorphisms of A. We will use the notation of § 1.3. Thus

$$A^{\alpha}[s,\infty)\, A^{\alpha}[t,\infty) \subseteq A^{\alpha}[s+t,\infty) \qquad \text{for all } s,t \text{ in } \mathbb{R},$$

since each α_t is multiplicative. In particular

$$A_0 = \Big(\bigcup_{\epsilon>0} A[\epsilon,\infty)\Big)^-$$

is a subalgebra of A.

If B is a Banach space, let $H^{\infty}(B)$ denote the space of all strongly measurable, essentially bounded B-valued functions h on $\mathbb{R}$ such that

$$\int f(t)\, h(t)\, dt = 0$$

for every $f \in L^1(\mathbb{R})$ such that $\hat{f}$ lives in some interval $[\epsilon,\infty)$ where $\epsilon = \epsilon_f > 0$.

The following proposition is readily deducible from [5, Prop 5.1].

PROPOSITION 5.6.1.

Let $\mu \in B(A,B)$. Then the following are equivalent:

i) $\mu \perp A_o$.

ii) $t \longrightarrow \mu\alpha_t(a)$ belongs to $H^\infty(B)$ for every a in A.

If μ satisfies i or ii above, it will be called an analytic map. Note that ii implies that there is a B-valued bounded analytic function in the upper half plane $z = x + iy$, $y > o$, which has the function $\mu\alpha_x(a)$ as its boundary value (in an appropriate sense) as $y \longrightarrow o$.

The next result generalises [5, Theorem 5.3] from covariance of cyclic represenations, with analytic linear functionals to covariance of $\mathfrak{H}$-cyclic represenations and analytic operator valued maps. As pointed out by Arveson, this can be regarded as a generalisation of the theorem of F. and M. Riesz that analytic measures are quasi invariant.

THEOREM 5.6.2.

Let $\mathfrak{H}$ be a hilbert spacem, and $w \in CP(A;\mathfrak{H})$. Let (π, V) denote the Stinespring represenation of w, and suppose there is a partial isometry W in $\pi(A)''$ such that $W^*WV = V$ and $\mu(x) = V^*W^*\pi(x)V$, $x \in A$, defines an analytic map μ. Then there is a strongly continuous one parameter unitary group U_t on the space $\mathcal{K}_w$, such that

$$\pi\alpha_t(a) = U_t\pi(a)U_t^* \qquad \forall a \in A,\ t \in \mathbb{R}.$$

PROOF.

For each t in $\mathbb{R}$, let $\mathcal{K}_t = \bigcap_{s<t}\left[\bigcup_{\xi\in\mathfrak{H}} \pi A[s,\infty)V\xi\right]$,

where if F is any subset of $\mathcal{K}_w$, $[F]$ denotes the closed subspace generated by F. If $s < t$, then $\pi A[t,\infty) \subseteq \pi A[s,\infty)$.

Hence $\mathcal{K}_t \supseteq \pi A[t,\infty)V\mathfrak{H}$. To show that $\bigcup \mathcal{K}_t$ is dense in $\mathcal{K}_w$, it suffices to show that $\bigcup_{t\in\mathbb{R}} A[t,\infty)$ is norm dense in A, since (π, V) is minimal. But $\bigcup_{t\in\mathbb{R}} A[t,\infty)$ contains every element of the form $\int f(t)\alpha_t(a)dt$ where $f \in L^1(\mathbb{R})$ is such that $\hat{f}$ has compact support and $a \in A$. So if $\lambda \in A^*$ annihilates $\bigcup_t A[t,\infty)$, then $\int f(t)\lambda\alpha_t(a)\,dt = 0$, for every such

f and a. Fixing a, it follows that $\lambda\alpha_t(a) = 0$ a.e. By continuity we see that $\lambda = 0$. This proves our density statement. Now put $\mathcal{K}_\infty = \bigcap_{t \in \mathbb{R}} \mathcal{K}_t$, and we claim $\mathcal{K}_\infty = 0$. The relation $A[s,\infty)\, A[t,\infty) \subseteq A[s+t,\infty)$ implies that $\pi A[s,\infty)[\pi A[t,\infty)V\mathfrak{h}] \subseteq [\pi A[s+t,\infty)\, V\mathfrak{h}] \quad \forall\, s,t \in \mathbb{R}$.
Hence $\pi A[s,\infty)\mathcal{K}_t \subseteq \mathcal{K}_{s+t}$. This shows that $\pi A[s,\infty)\mathcal{K}_\infty \subseteq \mathcal{K}_\infty$ for all s, and hence $\pi(A)\mathcal{K}_\infty \subseteq \mathcal{K}_\infty$. But by proposition 5.6.1, μ annihilates $A[t,\infty)$ for every $t > 0$. i.e. $\pi A[t,\infty)\, V\mathfrak{h} \perp WV\mathfrak{h}$ for $t > 0$. In particular $\mathcal{K}_t \perp WV\mathfrak{h}$ and $\mathcal{K}_\infty \perp WV\mathfrak{h}$. It follows that
$\pi(A)\mathcal{K}_\infty \perp WV\mathfrak{h}$ and $\mathcal{K}_\infty \perp \pi(A)WV\mathfrak{h}$. But $W^* \in \pi(A)''$, and $W^*WV = V$.
Thus $\mathcal{K}_\infty \perp \pi(A)V\mathfrak{h}$, and $\mathcal{K}_\infty = 0$ since (π, V) is minimal.
We can obtain a projection valued measure $P(\cdot)$ on $\mathbb{R}$ satisfying

$P[t,\infty)\mathcal{K}_W = \mathcal{K}_t$, $t \in \mathbb{R}$. If we follow the argument of [5, Thm 5.3], we can show that the unitary group $U_t = \int e^{itx}\, dP(x)$ satisfies

$$\pi\alpha_t(a) = U_t \pi(a) U_t^* \quad \forall a \in A,\ t \in \mathbb{R}.$$

The above theorem can be generalised by considering annihilating maps on the subalgebras $A_s = \left(\bigcup_{t>s} A[t,\infty)\right)^-$. This idea, for linear functionals ($\mathfrak{h} = \mathbb{C}$), is due to Oleson [67].

§ 5.7.

In this section we restrict our attention to a W*-algebra M, and in particular normal CP maps.

If $\mathfrak{h}$ is a hilbert space, it is clear that $\alpha \in CP(M;\mathfrak{h})$ is normal iff π_α is normal. Let $CP^w(M;\mathfrak{h})$ denote the normal elements in $CP(M;\mathfrak{h})$. By the Cauchy Schwartz inequality

$$\alpha(x^*)\alpha(x) \leq \alpha(x^*x)\|\alpha\|,\ x \in M$$

we see that if M is a von Neumann algebra, and $\alpha \in CP^w(M;\mathfrak{h})$, that α is ultrastrongly continuous, and its restriction to bounded sets is strongly continuous. Moreover, amongst all projections $G \in M$, such that $\alpha G = 0$, there exists an F majorising all others. Moreover $\alpha(TF) = 0 = \alpha(FT)$

$\forall\ T \in M$. The projection $1 - F$ is called the support of α , and writtten E_α.

For example, suppose M is a von Neumann algebra acting on a hilbert space $\mathcal{K}$, and $\alpha \in CP^w(M;\mathcal{K})$ can be written as

$$\alpha(x) = \sum_{i=1}^{\infty} A_i^* x A_i \quad , \quad \text{where } A_i \in B(\mathcal{H},\mathcal{K}) ,$$

and $\sum A_i^* A_i \leqslant k1$, for some $k < \infty$. Then $E = E_X^{M'}$, where $X = \bigcup_i \{A_i \mathcal{H}\}$, (see [16] for the definition of $E_X^{M'}$) . α is faithful iff X is separating for M.

Now if $\alpha \in CP^w(M;\mathcal{H})$, for a W^*-algebra M, $\ker \pi_\alpha$ is a closed two sided ideal in M. Hence there exists a central projction F in M, such that $\ker \pi_\alpha = MF$. The central support of E_α is $1 - F$.

These considerations will be dealt with further in § 6. for unbounded CP maps. The following Proposition will be useful in the next section.

PROPOSITION 5.7.1.

Let A be a C^*-algebra, $\tilde{A}$ its enveloping von Neumann algebra, and $\mathcal{H}$ a hilbert space. Every element w of $CP(A;\mathcal{H})$ can be uniquely extended to an element $\tilde{w}$ of $CP^w(\tilde{A};\mathcal{H})$. Moreover $(\pi_{\tilde{w}},V_{\tilde{w}}) \simeq (\tilde{\pi}_w,V_w)$ and $K_{\tilde{w}} = K_w$.

§ 5.8.

We will discuss here tensor products of CP maps. If A_1 and A_2 are C^*-algebras, $A_1 \otimes_{\alpha_0} A_2$ denotes the C^*-tensor product [54, 1.22.8] , and if A_1, and A_2 are W^*-algebras $A_1 \bar{\otimes} A_2$ denotes the W^*-tensor product [54, 1.22.10] . The following proposition is then well known for state spaces [16,17,54] .

PROPOSITION 5.8.1.

Let $\mathcal{H}_1, \mathcal{H}_2$ be hilbert spaces and $K_i \in B(\mathcal{H}_i)_+$ $i = 1,2$.

i) Let A_1, A_2 be C^*-algebras with $w_i \in CP(A_i;\mathcal{H}_i,K_i)$ $i = 1,2$. Then there exists an unique element of $CP(A_1 \otimes_{\alpha_0} A_2; \mathfrak{h}_1 \otimes \mathfrak{h}_2, K_1 \otimes K_2)$ written $w_1 \otimes w_2$ such that $(w_1 \otimes w_2)(a_1 \otimes a_2) = w_1(a_1) \otimes w_2(a_2)$ $\forall\, a_1 \in A_1$, $a_2 \in A_2$. Moreover $(\pi_{w_1 \otimes w_2}, V_{w_1 \otimes w_2}) \simeq (\pi_{w_1} \otimes \pi_{w_2}, V_{w_1} \otimes V_{w_2})$.

ii) Let A_1, A_2 be W*-algebras, and $w_i \in CP^{w}(A_i; \mathfrak{h}_i, K_i)$ (i.e. the normal elements of $CP(A_i; \mathfrak{h}_i, K_i)$) $i = 1,2$. Then there exists an unique element of $CP(A_1 \bar{\otimes} A_2; \mathfrak{h}_1 \otimes \mathfrak{h}_2, K_1 \otimes K_2)$ written $w_1 \otimes^{w} w_2$ such that $(w_1 \otimes^{w} w_2)(a_1 \otimes a_2) = w_1(a_1) \otimes w_2(a_2)$ $\forall\, a_i \in A_i$, $i = 1,2$. Moreover, $(\pi_{w_1 \otimes^{w} w_2}, V_{w_1 \otimes^{w} w_2}) \simeq (\pi_{w_1} \otimes^{w} \pi_{w_2}, V_{w_1} \otimes V_{w_2})$.

Now let A be a W*-algebra. Then if $(A_*)^o$ denotes the polar of A_* in $\tilde{A} = A^{**}$, we get a central projection z in A such that $\tilde{A}(1-z) = (A_*)^o$ [54, § 1.17]. If $\alpha \in CP(A; \mathfrak{h})$ for some hilbert space, $\mathfrak{h}$, let $\tilde{\alpha}$ be the extension in $CP^{w}(\tilde{A}; \mathfrak{h})$.

Put $\alpha^n(a) = \tilde{\alpha}(za)$

and $\alpha^s(a) = \tilde{\alpha}(1-z)a$ for any a in A.

Then $\alpha^n \in CP^{w}(A; \mathfrak{h})$ and $\alpha^s \in CP^s(A; \mathfrak{h})$, (singular elements of $CP(A; \mathfrak{h})$). This decomposition is unique.

Now if A_1 and A_2 are W*-algebras, and $w_i \in CP(A_i; \mathfrak{h}_i)$ $i = 1,2$ where $\mathfrak{h}_i$ are hilbert spaces, we have $w_1 \otimes w_2 \in CP(A_1 \otimes_{\alpha_0} A_2; \mathfrak{h}_1 \otimes \mathfrak{h}_2)$. We can regard $A_1 \otimes_{\alpha_0} A_2$ as a C*-subalgebra of $A_1 \bar{\otimes} A_2$, the W*-tensor product, and can therefore extend $w_1 \otimes w_2$ to an element of $CP(A_1 \bar{\otimes} A_2; \mathfrak{h}_1 \otimes \mathfrak{h}_2)$. Hence product mappings exist without σ-continuity. We can regard the following theorem as the CP analogue of [62,63] where the singularity of product conditional expectations is analysed. (A conditional expectation is CP by [56]).

THEOREM 5.8.2.

Let A_1 and A_2 be W*-algebras, $\mathfrak{h}_1$, $\mathfrak{h}_2$ hilbert spaces and $w_i \in CP(A_i; \mathfrak{h}_i)$ $i = 1,2$. Then the following are equivalent :

i) Either w_1 or w_2 is singular.

ii) Every member of

$\{\gamma \in CP(A_1 \bar{\otimes} A_2; \mathfrak{H}_1 \otimes \mathfrak{H}_2) : \gamma(a_1 \otimes a_2) = w_1(a_1) \otimes w_2(a_2) \quad \forall \quad a_i \in A_i\}$

is singular.

PROOF.

i $\Rightarrow$ ii. Take $\gamma \in CP(A_1 \bar{\otimes} A_2; \mathfrak{H}_1 \otimes \mathfrak{H}_2)$ such that

$\gamma(a_1 \otimes a_2) = w_1(a_1) \otimes w_2(a_2) \quad \forall \quad a_1, a_2 \in A_1, A_2$ respectively,

and suppose w_1 is singular. For any a_1 in A_1, $\varphi_i \in T(\mathfrak{H}_i)_+$ $i = 1,2$,

$\langle a_1 \otimes 1, \gamma^* \varphi \otimes \psi \rangle = \langle w_1(a_1) \otimes w_2(1), \varphi \otimes \psi \rangle = \langle a_1, w_1^* \varphi_1 \rangle \langle w_2(1), \varphi_2 \rangle$.

Hence the restriction of $\gamma^* \varphi_1 \otimes \varphi_2$ to $A_1 \otimes 1$ is a singular functional.

But $\gamma^* \varphi_1 \otimes \varphi_2$ is positive. Hence $\gamma^* \varphi_1 \otimes \varphi_2$ is singular by [62, Lemma 4.3] .

Thus $\gamma^* T(\mathfrak{H}_1) \odot T(\mathfrak{H}_2) = (A_1 \bar{\otimes} A_2)_*^{\perp}$. It follows that γ is singular.

ii $\Rightarrow$ i. Conversely suppose every product CP map is singular. Let

A_1, A_2 act on hilbert spaces H_1, H_2 respectively, and take CP maps

$a_i : B(H_i) \text{—} B(\mathfrak{H}^i)$ $i = 1,2$ such that $a_i|_{A_i} = w_i$, $(a_i)^n|_{A_i} = (w_i)^n$,

$i = 1,2$. Take a family of orthogonal minimal projections $\{e_i : i \in I\}$

corresponding to a base for H_2. Put $\tilde{e}_i = 1 \otimes e_i$, $e_J = \sum_{i \in J} e_i$,

and $\tilde{e}_J = 1 \otimes e_J$ for a finite subset J of I. Then

$$\begin{aligned} e_J\big(B(H_1) \bar{\otimes} B(H_2)\big) e_J &= B(H_1) \bar{\otimes} e_J B(H_2) e_J \\ = \quad B(H_1) \bar{\otimes} B(e_J H_2) &= B(H_1) \otimes_{\alpha_0} B(e_J H_2) \quad , \end{aligned}$$

the C*-tensor product, because $e_J H_2$ is finite dimensional.

Consider

$u_J' = a_1 \otimes 1_J \quad : \quad B(H_1) \otimes_{\alpha_0} B(e_J H_2) \longrightarrow B(\mathfrak{H}_1) \otimes_{\alpha_0} B(e_J H_2) \subseteq B(\mathfrak{H}_1) \bar{\otimes} B(H_2)$,

where 1_J denotes the identity map on $B(e_J H_2)$.

Define $u_J(x) = u_J'(e_J \, x \, e_J)$ for $x \in B(H_1) \bar{\otimes} B(H_2)$.

Then $u_J : B(H_1) \bar{\otimes} B(H_2) \longrightarrow B(\mathfrak{H}_1) \bar{\otimes} B(H_2)$ is CP for each finite subset

J of I, and $\|u_J\| \leq \|a_1\|$. We order $\{J : J \text{ finite} \subseteq I\}$ by inclusion.

Thus by BW compactness, there exists a CP map

$u : B(H_1) \bar{\otimes} B(H_2) \longrightarrow B(\mathfrak{H}_1) \bar{\otimes} B(H_2)$ and a subnet such that

$u_J \longrightarrow u$ in the BW topology for this subnet. If $x \in B(H_1)$, $y \in B(H_2)$,

$$u(x \otimes y) = \lim u_J(x \otimes y) = \lim u_J'(\tilde{e}_J\, x \otimes y\, \hat{e}_J) = \lim u_J'\, x \otimes e_J y e_J$$
$$= \lim a_1(x) \otimes e_J y e_J = a_1(x) \otimes y\ ,$$

where the limit is taken over the subnet for the BW topology.

Similarly, we construct v, CP , from $B(\mathfrak{H}_1) \bar{\otimes} B(H_2) \longrightarrow B(\mathfrak{H}_1) \bar{\otimes} B(\mathfrak{H}_2)$, satisfying $v(x \otimes y) = x \otimes a_2(y) \quad \forall\, x \in B(\mathfrak{H}_1),\ y \in B(H_2)$.

Put $b = vu$, which is CP from $B(H_1) \bar{\otimes} B(H_2) \longrightarrow B(\mathfrak{H}_1) \bar{\otimes} B(\mathfrak{H}_2)$ and $\gamma = b|_{A_1 \bar{\otimes} A_2}$.

Then $\gamma\, d_1 \otimes d_2 = a_1(d_1) \otimes a_2(d_2) \quad \forall\, d_i \in A_i,\ i = 1,2.$

Now $u_J' = a_1 \otimes 1_J \geqslant a_1^{\,n} \otimes 1_J$.

Take a positive alement c in $B(H_1) \bar{\otimes} B(H_2)$.

Then $u_J(c) = u_J'\ \tilde{e}_J\, c\, \hat{e}_J \geqslant a_1^{\,n} \otimes 1_J\, \hat{e}_J\, c\, \hat{e}_J = (a_1^{\,n} \otimes^{w} 1)\, \tilde{e}_J\, c\, \tilde{e}_J$

Hence since the BW limit preserves order,

$$u(c) \geqslant a_1^{\,n} \otimes^{w} 1\ (c).$$

Similarly $v \geqslant 1 \otimes^{w} a_2^{\,n}$. Thus $b \geqslant a_1^{\,n} \otimes^{w} a_2^{\,n}$, which implies

$$\gamma = b|_{A_1 \bar{\otimes} A_2} \geqslant a_1^{\,n} \otimes^{w} a_2^{\,n}|_{A_1 \bar{\otimes} A_2} = w_1^{\,n} \otimes^{w} w_2^{\,n}.$$

But γ is singular, hence $w_1^{\,n} \otimes^{w} w_2^{\,n} = 0$, which implies that either $w_1^{\,n}$ or $w_2^{\,n} = 0$.

§ 5.9.

Here we shall study completely positive maps on the C*-algebra of the canonical commutation relations. Let (H,σ) be a symplectic space, i.e. H is a real vector space with σ an $\mathbb{R}$-bilinear form on H, which is antisymmetric

$$\sigma(h,k) = -\sigma(k,h) \qquad \forall\ h,k \in H$$

and regular

$$\sigma(h,k) = 0 \qquad \forall\ k \in H \qquad \text{implies } h = o.$$

We define $\Delta(H,o)$ to be the algebra of complex valued functions on H, having finite support, with the twisted product

$$(a.b)(h) = \sum_{k \in H} a(k)\, b(h-k)\, e^{-i\sigma(k,h)} \qquad \forall\, a,b \in \Delta(H,\sigma), h \in H,$$

involution,

$$a^*(h) = \overline{a(-h)} \qquad \forall\ a \in \Delta(H,\sigma), \quad h \in H,$$

and the natural laws of addition and scalar multiplication.
A canonical basis for $\Delta(H,\sigma)$ is $\{\delta_h : h \in H\}$, where $\delta_h(k) = 0$ if $k \neq h$ and $\delta_h(h) = 1$. $\mathcal{R}(H,\sigma)$, the representations of the CCR's over (H,σ), is defined as the set of those non-degenerate represenations π of $\Delta(H,o)$, such that $t \in \mathbb{R} \longrightarrow \pi(\delta_{th})$ is weakly (or strongly continuous)for all h in H. $\Delta(\overline{H,\sigma})$ which we refer to as the C*-algebra of the canonical commutation relations, is then defined as the C*-algebra obtained by completing $\Delta(H,\sigma)$ with respect to the norm

$$\| a \| = \sup_{\pi \in \mathcal{R}(H,\sigma)} \| \pi(a) \| .$$

We also recall that a Weyl representation W of H on a hilbert space $\mathcal{R}_W$ is a map W from H into the unitary operators on $\mathcal{R}_W$ satisfying

a) $W(h)^* = W(-h) \quad \forall\ h \in H$

b) $W(h)\, W(k) = W(h+k)\, e^{-i\sigma(h,k)} \quad \forall\ h,k \in H$

c) $W(\lambda h)$ are continuous in $\lambda \in \mathbb{R}$, in the weak operator topology on $B(\mathcal{R})$ for each h in H.

There is a natural one-one correspondence between the set of all Weyl representations, $\mathcal{W}(H,\sigma)$ and $\mathcal{R}(H,\sigma)$; where $\pi \in \mathcal{R}(H,\sigma)$, gives rise to the Weyl representation

$$W(h) = \left[\pi(\delta_h)\right] \qquad h \in H.$$

For further details on these matters see [21,41].

Let $\mathfrak{H}$ be a hilbert space and let $K(\mathfrak{H})$ be the space of functions $\hat{x} : H \longrightarrow B(\mathfrak{H})$ which satisfy

i) For any $n \geqslant 1$, $h_j \in H$ $j = 1,\ldots\ldots,n$

(the matrix) $\left[\exp -\frac{1}{2}\sigma(h_j,h_k)\ \hat{x}(-h_j + h_k)\right] \geqslant 0$ in $B(\mathfrak{H})_n$

ii) If $h,k \in H$, $t \longrightarrow \hat{x}(th + k)$ is continuous from $\mathbb{R}$ into $B(\mathfrak{H})$ for the weak operator topology.

Take $\hat{x} \in K(\mathfrak{H})$, and define Δ_o to be the complex space of functions from H into $\mathfrak{h}$ with finite support. We define a bilinear form $\langle\ ,\ \rangle$ on Δ_o as follows :

For $\xi, \xi' \in \Delta_o$, put

$$\langle \xi, \xi' \rangle = \sum_{h,k \in H} \exp -\frac{1}{2}\sigma(h,k) \langle \hat{x}(-k+h)\xi(h), \xi'(k) \rangle.$$

By property i) of $K(\mathfrak{h})$, $\langle\ ,\ \rangle$ is a positive semi definite form. For $h \in H$, define a linear operator $W_{\hat{x},o}(h)$ on Δ_o by

$$W_{\hat{x},o}(h)\eta = \eta' \quad \text{for } \eta \in \Delta_o,$$

where $\eta'(k) = \exp +\frac{1}{2}\sigma(k,h)\ \eta(k-h)$ for $k \in H$, (5.9.1).

Then $\langle W_{\hat{x},o}(h)\eta_1, W_{\hat{x},o}(h)\eta_2 \rangle = \langle \eta_1, \eta_2 \rangle \quad \forall\ \eta_1, \eta_2 \in \Delta_o$.

Now let $N_{\hat{x}} = \{ \eta \in \Delta_o : \langle \eta, \eta \rangle = 0 \}$.

Then $N_{\hat{x}}$ is a linear subspace of Δ_o, invariant under $W_{\hat{x},o}(h)$ for all $h \in H$. $\langle\ ,\ \rangle$ determines an inner product on $\Delta_o/N_{\hat{x}}$, and let $\mathcal{R}_{\hat{x}}$ be the hilbert space completion of $\Delta_o/N_{\hat{x}}$. Then $W_{\hat{x},o}(h)$ induces an unitary operator $W_{\hat{x}}(h)$ on $\mathcal{R}_{\hat{x}}$, for each h in H, satisfying

$$W_{\hat{x}}(h)(\eta + N_{\hat{x}}) = \eta' + N_x,$$

where $\eta \in \Delta_o$, and η' is as defined in (5.9.1). It is straightforward to check that $W_{\hat{x}}$ is a Weyl representation. For $\xi \in \mathfrak{h}$, consider the element $\xi_o \in \Delta_o$ defined by

$$\xi_o(h) = \begin{cases} o & \text{if } h \neq o \\ \xi & h = o. \end{cases}$$

Then $\xi \longrightarrow \xi_o + N_x$ gives a bounded linear map $V_{\hat{x}}$ from $\mathfrak{h}$ into $\mathcal{R}_{\hat{x}}$ such that $\hat{x}(h) = V_{\hat{x}}^* W_{\hat{x}}(h) V_{\hat{x}} \quad \forall\ h \in H$.

Before proceeding it is convenient to introduce some definitions.

DEFINITION 5.9.2.

A Weyl representation W is said to be $\mathfrak{h}$ -cyclic if there exists $V \in B(\mathfrak{h}, \mathcal{R}_W)$ such that $V\mathfrak{h}$ is cyclic for W i.e. $[W(H)V\mathfrak{h}]^- = \mathcal{R}_W$. V is then said to be a cyclic map for W. Two $\mathfrak{h}$ -cyclic Weyl represenations W_1, W_2 with $\mathfrak{h}$-cyclic maps V_1, V_2 respectively, are said to be equivalent if there exists a unitary map $U : \mathcal{K}_{W_1} \longrightarrow \mathcal{K}_{W_2}$ such that

$$W_1(h) = U\, W_2(h)\, U^{-1} \quad \forall\, h \in H, \text{ and } V_1 = UV_2 .$$

This occurs iff $V_1^* W_1(h) V_1 = V_2^* W_2(h) V_2 \quad \forall\, h \in H$.

Analogously, we can define $\mathfrak{h}$ -cyclicity and $\mathfrak{h}$- equivalence for representations of $\Delta(H,\sigma)$, in particular for $\mathcal{R}(H,o)$.

We have thus shown that to every element $\hat{x}$ in $K(\mathfrak{h})$ there corresponds (uniquely up to $\mathfrak{h}$ -equivalence) an $\mathfrak{h}$ -cyclic Weyl representation $W_{\hat{x}}$, with $\mathfrak{h}$ -cyclic map $V_{\hat{x}}$, such that $\hat{x}(h) = V_{\hat{x}}^* W_{\hat{x}}(h) V_{\hat{x}} \quad \forall\, h \in H$. Conversely, every element of $K(\mathfrak{h})$ arises in this way.

These considerations lead us to the following characterisations of CP maps on the CCR algebra, which generalises the familiar state space results [21,41] . However note that the case $\mathfrak{h} = \mathbb{C}$, and $\mathfrak{h} \neq \mathbb{C}$, are slightly different. If $\hat{x} \in K(\mathfrak{h})$, it is clear that δ_h —— $\hat{x}(h)$ $(h \in H)$ defines a positive linear functional on $\overline{\Delta(H,\sigma)}$, and the above construction of $W_{\hat{x}}$, $V_{\hat{x}}$ etc can be bypassed by the GNS construction. However if $\mathfrak{h} \neq C$, and $\hat{x} \in K(\mathfrak{h})$, it is non trivial that $\delta_h \longrightarrow \hat{x}(h)$ defines a CP from $\overline{\Delta(H,\sigma)}$ into $B(\mathfrak{h})$, and the above construction is deemed necessary.

THEOREM 5.9.3.

The following spaces are in one-one correspondence :

i) $\mathfrak{h}$ -equivalence classes of $\mathfrak{h}$ -cyclic Weyl representations .

ii) $\mathfrak{h}$ -equivalence classes of $\mathfrak{h}$-cyclic represenations π in $\mathcal{R}(H,o)$.

iii) $K(\mathfrak{h})$.

iv) $w \in CP(\overline{\Delta(H,\sigma)},\mathfrak{h})$ satisfying $t \longrightarrow w(\delta_{th+k})$ is weakly continuous from $\mathbb{R}$ into $B(\mathfrak{h})$ for all $h,k \in H$.

§ 5.10.

We will now concern ourselves with the dilation of a family of CP maps, indexed by a group G, (from an operator algebra into itself), into a group of *-automorphisms on a larger algebra. This problem has already been considered by E.B.Davies [12] within the theory of quantum stochastic processes, and more recently by G.Lindblad for quantum dynamical semigroups. [39, Theorem 9], both working in the Schrodinger picture. For technical reasons we will omit from our treatment continuity considerations when G is a topological group.

The following theorem can be regarded as the C*-algebra analogue of E.Stroescu's Banach space dilations [58, Theorem 1].

THEOREM 5.10.1.

Let A be a C*-algebra with identity and G a group. For each g in G, w_g is a CP map of A into itself such that $w_g 1 = 1$ and $w_1 = 1$. Then there exists a C*-algebra B contatining A, and a homomorphism Φ from G into the *-automorphisms of B, and a normal conditional expectation N from B onto A satisfying :

i) $N(\Phi_g \cdot ia) = w_g(a) \quad \forall\, g \in G,\ a \in A$, where i denotes the embedding of A in B.

ii) B is the C*-algebra generated by $\{\Phi_g\, ia : g \in G,\ a \in A\}$.

iii) The identity of A is an identity for B.

PROOF.

Take A to be faithfully represented as a concrete C*-algebra on a hilbert space $\mathfrak{H}$. Then $w_g \in CP(A;\mathfrak{H},1)\ \forall g \in G$. For each g in G, let π_g, V_g, $\mathfrak{K}_g$ denote the Stinespring representation π_{w_g}, etc. Then $\mathfrak{K}_1 = \mathfrak{H}$, $V_1 = 1$, $\pi_1 = 1$, and V_g is an isometry for all g in G.

Define $\mathfrak{K} = \bigoplus_{g\in G} \mathfrak{K}_g$, and $B_g = B(\mathfrak{K})$ for all g in G, and put $B_o = \bigoplus_{g\in G} B_g$.

For each g in G, we will define a unitary operator U_g on $\mathfrak{K}$.

Put $U_1 = 1$, and for $g \in G$, $g \neq 1$,

define $$(U_g \xi)_h = \begin{cases} \xi_h & h \neq 1,g \\ V_g \xi_1 + (1 - V_g V_g^*)\xi_g & h = g \\ V_g^* \xi_g & h = 1. \end{cases}$$

Then $U_g = U_g^{-1}, \forall g \in G$.

For $g \in G$, define $\Phi_g^o : B_o \longrightarrow B_o$ by

$$(\Phi_g^o f)_h = U_h U_{hg}^{-1} f_{hg} U_{hg} U_h^{-1} \qquad \text{for } f \in B_o,\ h \in G.$$

Then Φ^o is a homomorphism from G into the *-automorphisms of B_o.

We can embedd A as a C*-subalgebra of B_o as follows: Define $i\ A \longrightarrow B_o$ by $\quad (ia)_h = \bigoplus_{g \in G} \pi_g(a) = \pi(a)$ say, $\forall\ a \in A,\ h \in G$.

The identity of A is then the identity of B_o. Now define $N^o : B_o \longrightarrow B(\mathfrak{H})$ by $N^o f = W^* f_1 W$, where W is the isometry from $\mathfrak{H}$ into $\mathfrak{K}$ defined by

$$(W\xi)_h = \begin{cases} 0 & h \neq 1 \quad \forall\ \xi \in \mathfrak{H} \\ \xi & h = 1. \end{cases}$$

Then it can be checked that $N^o \Phi_g^o(ia) = w_g(a) \quad \forall\ g \in G,\ a \in A$.

We now indicate how to prove that if $g_i \in G$, $a_i \in A$, $i = 1,2,\ldots\ldots,n$, then $N^o \Phi_{g_n}^o (a_n) \ldots\ldots \Phi_{g_1}^o (a_1) \in A$.

In order to do this, we first show by induction on n, that if $\xi \in \mathfrak{H}$ is kept fixed

$$U_{g_n} \pi(a_n) U_{g_n} U_{g_{n-1}} \pi(a_{n-1}) \ldots\ldots\ldots U_{g_2} U_{g_1} \pi(a_1) U_{g_1} W\ . \quad = \eta_h^n$$

where for each h in G, η_h^n is a finite sum of terms each of the form

$$\pi_h(x)\ V_h y\ \xi \qquad x,y \in A \text{ ,i.e.}$$

$$\eta_h^n = \sum_{i=1}^{p} \pi_h(x_i)\ V_h\ y_i\ \xi$$

where $(p, x_1, \ldots\ldots, x_p, y_1, \ldots\ldots\ldots, y_p)$ depend on n, and are independant of $\xi \in \mathfrak{H}$. It follows that

$$N^o \Phi_{g_n}^o (a_n) \ldots\ldots\ldots \Phi_{g_1}^o (a_1)\ \xi \quad = \eta_1^n.$$

But by the above induction, $\eta_1^n = x\xi$ where $x \in A$.

Hence $N^o\ \Phi_{g_n}^o (a_n) \ldots\ldots\ \Phi_{g_1}^o (a_1) \in A$.

Let B be the C*-algebra generated by $\{ \Phi_g^o(a) : g \in G,\ a \in A \}$.

Then Φ_g^o leaves B invariant and if $\Phi_g = \Phi_g^o|_B$, $g \longrightarrow \Phi_g$ is a homomorphism from G into the *-automorphisms of B. Let $N = N^o|_B$. Then N is a projection of norm one of B onto A, hence a conditional expectation. The remainder of the proof is clear.

Note that we can easily modify the proof of the above theorem to obtain similar results for W*-algebras and Σ*-algebras with families of W* and Σ* continuous CP maps respectively.

We can obtain a number of corollaries from Theorem 5.10.1. The first of which corresponds to Nagy's theorem [59,60] for unitary dilations of contraction semigroups on a hilbert space, and E.Stroescu's Banach space analogue [58, Corollary 1].

COROLLARY 5.10.2.

Let $\{w_t : t \in \mathbb{R}^+\}$ be a semigroup of CP maps from a C*-algebra A with identity into itself, such that $w_o = 1$ and $w_t 1 = 1 \ \forall\ t \geqslant o$. Then there exists a C*-algebra B containing A, a one parameter group of *-automorphisms Φ_t of B, and a normal conditional expectation N of B onto A satisfying:

i) $N\,\Phi_t(a) = w_{|t|}(a) \quad \forall\ a \in A, \quad t \in \mathbb{R}.$

ii) B is the C*-algebra generated by $\{\Phi_t(a) : t \in \mathbb{R},\ a \in A\}$.

iii) The identity of A is an identity for B.

The hilbert space and Banach space analogues of the following were obtained in [59] and [58] respectively.

COROLLARY 5.10.3.

Let w be a CP map from a C*-algebra A with identity into itself satisfying $w1 = 1$. Then there exists a C*-algebra B containing A, a normal conditional expectation N from B onto A, and a *-automorphism Φ of B satisfying :

i) $N\Phi^n(a) = w^{|n|}(a) \qquad \forall\ a \in A, \quad n \in \mathbb{Z}.$

ii) B is the C*-algebra generated by $\{\Phi^n(a) : a \in A,\ n \in \mathbb{Z}\}$.

iii) The identity of A is an identity for B.

The next corollary corresponds to Ando's result for unitary dilations of a pair of commuting contractions on a hilbert space [2]. As in the Banach space case, [58], we can take a finite number of not necessarily commuting operators.

CORLLARY 5.10.4.

Let $w_1, \ldots\ldots, w_p$ be a finite system of (not necessarily commuting) CP maps of a C*-algebra A with identity into itself satisfying $w_i 1 = 1$ for $i = 1, \ldots\ldots, p$. Then there exists a C*-algebra B containing A, a normal conditional expectation N of B onto A, a finite system of commuting *-automorphisms $\Phi_1, \ldots\ldots, \Phi_p$ of B satisfying

i) $N\, \Phi_1^{n_1} \circ \cdots\cdots \circ \Phi_p^{n_p}(a) = w_1^{|n_1|} \circ \qquad \circ w_p^{|n_p|}(a)$ for all a in A $n_1, \ldots\ldots, n_p$ in $\mathbb{Z}$.

ii) B is the C*-algebra generated by

$$\left\{ \Phi_1^{n_1} \circ \quad \cdots\cdots\cdots \circ \Phi_p^{n_p}(a) \;:\; a \in A\; n_1, \ldots\ldots, n_p \in \mathbb{Z} \right\}.$$

iii) The identity for A is also an identity for B.

Again there are W* and ξ *-analogues of Corollaries 5.10.2-5.10.4.

§ 5.11.

Several authors have studied semigroups of positive maps on C*-algebras, e.g. [12,13,30,31,32,39]. Lindblad [39] has been particularly concerned with the infinitesimal generators of norm continuous semigroups of CP maps on operator algebras. We aim to extend this work, bystudying strongly continuous one parameter semigroups and groups on C*-algebras which have various positivity properties, e.g. 2-positive or CP mappings.

In the first place we remark that if A is a C*-algebra, G a group, and $\{ w_g : g \in G \}$ a contractive group of 2-positive maps on A, then w_g is in fact a group of *-automorphisms. This follows from the following lemma.

LEMMA 5.11.1.

Let A be a C*-algebra, and $w : A \longrightarrow A$ an invertible linear map such that w and w^{-1} are 2-positive and $\|w\| = 1 = \|w^{-1}\|$. Then w is a *-automorphism.

PROOF.

This follows from 5.1.3.

We will show that it is possible to have a C*-algebra A, with identity 1, and $w : A \longrightarrow A$ an invertible linear map such that w is CP, but w^{-1} is not 2-positive.

DEFINITION 5.11.2. [39].

Let A be a C*-algebra, and $\theta \in CP(A,A)$. Define $L_\theta \in B(A,A)$ by

$$L_\theta(x) = \theta(x) - \{K_\theta, x\} \qquad x \in A.$$

LEMMA 5.11.3.

Let A be a C*-algebra, $\theta \in CP(A,A)$. Then the following conditions are equivalent :

i) L_θ is a derivation.

ii) $L_\theta = 0$.

iii) $\theta(x) = K_\theta x \quad \forall x \in A$ (and K_θ lies in the centre of A).

iv) $\pi_\theta(x)V_\theta = V_\theta x \quad \forall x \in A$.

Now let A be any C*-algebra, $\theta \in CP(A,A)$ and $\{\exp t\delta : t \geq 0\}$ a strongly continuous one parameter semigroup of *-homomorphisms of A. Then using the Lie-Trotter formula as in [13] and [39], we see that $\{\exp t(\delta + L_\theta) : t \geq 0\}$ is a strongly continuous one parameter semigroup of CP maps.

THEOREM 5.11.4.

Let A be any C*-algebra, with identity, $\theta \in CP(A,A)$, and $\exp t\delta$ a strongly continuous one parameter group of *-automorphisms of A.

Define $L = L_\theta + \delta$. Then the following conditions are equivalent:

i) $\exp tL$ is a group of *-automorphisms.

ii) $\exp tL$ is a group of CP maps.

iii) $\exp tL$ is a group of 2-positive maps.

iv) $L = \delta$.

v) $\theta(x) = K_\theta x, \ \forall x \in A.$

vi) $\pi_\theta(x)V_\theta = V_\theta x \ \forall x \in A.$

Now let $\mathfrak{h}$ be a hilbert space, and consider $A = B(\mathfrak{h})$. Take $V \in B(\mathfrak{h})$, $V \notin \mathbb{C}1$. Define $\theta(x) = V^*xV \quad x \in A$. It is then seen that $\exp tL_\theta$ is never 2-positive if $t < 0$.

It is also possible to have a group of CP maps on a C*-algebra, which is not a group of *-automorphisms. Let A be a C*-algebra, and K a non zero self adjoint element of A. Then $\alpha_t(x) = e^{Kt}xe^{Kt}$ for $t \in \mathbb{R}$, defines a norm continuous one parameter group of completely positive maps, but α_t is a *-automorphism iff $t = 0$. We shall see in Theorem 5.11.6 that this is essentially how all norm continuous one parameter groups of CP maps on C*-algebras arise.

The following Lemma generalises a part of [39, Corollary 2].

LEMMA 5.11.5.

Let A be a C*-algebra with identity and $\exp tL$ $t \geq 0$, a strongly continuous one parameter semigroup of 2-positive maps on A, such that D(L) is a subalgebra of A, and $1 \in D(L)$. Then for all x in D(L) :

$$L(x^*x) + x^*L(1)x - x^*L(x) - L(x^*)x \geq 0.$$

PROOF.

This follows from 5.1.2 if we differentiate

$$\exp tL(x^*x) \geq \exp tLx^* \cdot (\exp tL\,1 + \epsilon)^{-1} \exp tL\,x$$

and then let $\epsilon \downarrow 0$.

THEOREM 5.11.6.

Let A be a C*-algebra with identity, and $\{\exp tL : t \in \mathbb{R}\}$ a strongly continuous one parameter group of 2-positive maps on A such that D(L) is a subalgebra of A, and $1 \in D(L)$. Then if we define $D(\delta) = D(L)$, and

$$\delta x = Lx - \frac{1}{2}\{x, L(1)\}, \quad x \in D(\delta),$$

δ is a *-derivation, and $\exp tL$ is a group of CP maps on A.

Moreover, the following conditions are equivalent :

i) L is a *-derivation.

ii) $L(1) = 0$.

iii) e^{Lt} is a group of *-automorphisms.

iv) $e^{Lt}1 = 1 \quad \forall\, t \in \mathbb{R}$.

v) $\|e^{Lt}1\| = 1 \quad \forall\, t \in \mathbb{R}$.

We also have the following partial converse to 5.11.5.

THEOREM 5.11.7.

Let A be a C*-algebra, and e^{Lt} a strongly continuous one parameter semigroup of positive maps on A such that

i) D(L) is a subalgebra of A.

ii) $L(x^*x) - x^*L(x) - L(x^*)x \geq 0 \quad \forall\, x \in D(L)$.

Then $e^{Lt}(x^*x) \geq e^{Lt}(x^*)\, e^{Lt}(x) \quad \forall\, x \in A,\ t \geq 0$.

and hence e^{Lt} is a contraction semigroup.

PROOF.

The proof is similar to that of Proposition 1.1.2.

Let $f \in D(L)$ and define $x(t) = e^{Lt}f^*f - e^{Lt}f^*\, e^{Lt}f \qquad t \geq 0$.

Then x is continuously differentiable and for $t \geq 0$

$$x'(t) = Le^{Lt}f^*f - (Le^{Lt}f^*)e^{Lt}f - e^{Lt}f^*(Le^{Lt}f).$$

Moreover if $h \in D(L)$, then $s \longrightarrow e^{L(t-s)}h$ is continuously differentiable in $0 \leq s \leq t$ and

$$\frac{\partial}{\partial s}\, e^{(t-s)L}h = - e^{L(t-s)}Lh.$$

Hence $x(t) - e^{Lt}x(o)$

$$= \int_0^t \frac{\partial}{\partial\sigma} e^{(t-\sigma)L} x(\sigma)\, d\sigma$$

$$= - \int_0^t e^{(t-\sigma)L} Lx(\sigma)\, d\sigma + \int_0^t e^{(t-\sigma)L} \frac{d}{d\sigma} x(\sigma)\, d\sigma$$

$$= \int_0^t e^{(t-\sigma)L} \left[L(e^{L\sigma}f^* e^{L\sigma}f) - (Le^{L\sigma}f^*)e^{L\sigma}f - e^{L\sigma}f^*(Le^{L\sigma}f) \right] d\sigma$$

But by hypothesis, $L(e^{L\sigma}f^* e^{L\sigma}f) \geqslant L(e^{L\sigma}f^*)e^{L\sigma}f + e^{L\sigma}f^*(Le^{L\sigma}f)$

for all $f \in D(L)$, $\sigma \geqslant o$. Moreover $e^{L(t-\sigma)}$ is positive in $o \leqslant \sigma \leqslant t$.

Hence $x(t) \geqslant e^{tL}x(o) = o \quad \forall\, t \geqslant o$.

By density of $D(L)$ in A we deduce :

$$e^{Lt}(x^*x) \geqslant e^{Lt}x^*\, e^{Lt}x \quad \forall\, x \in A,\ t \geqslant o.$$

CHAPTER 6.

In this chapter we first consider unbounded completely positive linear maps on a C*-algebra A, and then those which are invariant under a group of *-automorphisms of the algebra. This generalises [7,64] for scalar valued weights. However we have left Hahn-Banach type considerations (e.g. as in [7,Lemme1.5]) for further study.

In the first section we construct the Stinespring representation for an unbounded completely positive linear map α on A, and begin an analysis of the natural order structure for such maps. In § 6.2, when has dense domain, we are concerned with the construction of a largest operator valued weight α_0 majorised by α, and with the property that it is the upper envelope of continuous CP linear maps. The third section deals with the quasi equivalence, equivalence, and type of the Stinespring representation associated with operator valued weights, whilst in § 6.4, we see how this study carries over to the enveloping von Neumann algebra. We then study in § 6.5 an unbounded completely positive linear map α with dense domain which is left invariant by a group G of *-automorphisms of A, parallel to [7] for scalar valued weights. We construct a G invariant projection map ϕ of the set $\mathcal{F}$ of continuous CP linear maps dominated by α, onto the set $\mathcal{F}_0$ of G-invariant elements of $\mathcal{F}$. This used to derive various properties of the upper envelope of $\mathcal{F}_0$. In § 6.6 we study asymptotically abelian systems (A,G) and their ergodic CP maps. We remark that a great deal of the theory of §§ 6.5-6 can be generalised to semigroups of identity preserving *-isomorphisms of C*-algebras into themselves. Finally, in § 6.7 we make some observations regarding a possible classification of the spectral type of strongly continuous one parameter groups of *-automorphisms of a C*-algebra which possesses an invariant operator valued weight.

§ 6.1

Our first problem is a suitable definition for an unbounded completely positive map on a C*-algebra, which we wish to regard as an operator valued weight.

Let A be a C*-algebra. If S is any subset of A, we denote $S \cap A^+$ by S^+. We recall that a face of A^+ is a convex hereditary subcone of A^+, where a subset S of A^+ is said to be hereditary if $o \leq x \leq y$, $y \in S$, $x \in A^+$ implies that $x \in S$. If F is a face of A^+, then lin F is a *-subalgebra $\mathcal{D}$ of A, and $\mathcal{D}^+ = F$. The set $\{x \in A: \ x^*x \in F\} = L(F)$ is a left ideal of A such that $\mathcal{D} = L(F)^* L(F)$. A subalgebra $\mathcal{D}$ of A such that $\mathcal{D} = \text{lin}\, \mathcal{D}^+$ and $\mathcal{D}^+$ is a face of A^+, is said to be a facial subalgebra of A. For details on these matters see [8,10].

It is convenient to introduce some more notation. If S is a subset of a C*-algebra A, let S_n, for each $n \geq 1$, denote the subset of $A_n = A \otimes M_n$ consisting of all $n \times n$ matrices over S; i.e.

$$S_n = \left\{ \left[a_{ij}\right] : a_{ij} \in S \quad 1 \leq i,j \leq n \right\}.$$

If α is a linear map with domain $\mathcal{D} = \mathcal{D}_\alpha$ on a C*-algebra A into another C*-algebra B, let α_n be the induced linear map with domain $\mathcal{D}_n$, by applying α elementwise to each matrix over $\mathcal{D}$. We then say that α is completely positive if

i) $\mathcal{D}_\alpha$ is a facial-subalgebra of A.

ii) Given any $n \geq 1$, $a_i \in \mathfrak{N}_\alpha \equiv L(\mathcal{D}_\alpha)$ for $i = 1,2,\ldots,n$ then $\alpha_n \left[a_i^* a_j\right] = \left[\alpha(a_i^* a_j)\right] \geq 0$ in B_n.

We denote by cp(A,B) all (posibly unbounded) completely positive linear maps from the C*-algebra A into B. If $\mathfrak{h}$ is a hilbert space, we let cp(a;$\mathfrak{h}$) denote cp(A,B($\mathfrak{h}$)).

We remark that if $\mathcal{D}$ is a facial subalgebra of a C*-algebra A, then it is an open problem whether $\mathcal{D}_n$ is a facial subalgebra of A_n for all $n \geq 2$. For a von Neumann algebra the answer is yes.

PROPOSITION 6.1.1.

If A is a von Neumann algebra, and $\mathcal{D}$ a facial subalgebra of A, then $\mathcal{D}_n$ is a facial subalgebra of A_n for all n, and moreover

$$L(\mathcal{D})_n = L(\mathcal{D}_n).$$

PROOF.

This is an application of [9, Lemme4.11] .

COROLLARY 6.1.2.

Let A be a von Neumann algebra, B a C*-algebra, and α a linear map with domain $\mathcal{D} = \mathcal{D}_\alpha \subseteq A$ into B. Then α is completely positive iff α_n maps $(\mathcal{D}_{\alpha_n})^+$ into $(B_n)^+$ for all n, i.e. α_n is positive for all n.

PROOF.

Suppose $x \in (\mathcal{D}_n)^+$. Then $x = x^{1/2}x^{1/2}$, where $x^{1/2} \in L(\mathcal{D}_n) = L(\mathcal{D})_n$. Hence x is a finite sum of at most n terms of the form $[a_i^* a_j]$ where $a_i \in L(\mathcal{D}) = \mathcal{N}_\alpha$.

REMARK 6.1.3.

We now show how operator valued maps on facial subalgebras of C*-algebras naturally arise. Consider the following definition of Haagerup :

DEFINITION 6.1.4. [24] .

Let M be a von Neumann algebra and M_* its predual. The extended positive part of M is defined as the set of functions $m : M_*^+ \longrightarrow [0, \infty)$ satisfying

1) $m\lambda\varphi = \lambda m\varphi \qquad \varphi \in M_*^+, \quad \lambda \geqslant 0.$

2) $m(\varphi + \psi) = m\varphi + m\psi \qquad \varphi, \psi \in M_*^+ .$

3) m is lower semicontinuous.

The extended positive part of M is denoted $\hat{M}_+$. Note that each x in M_+ defines an element in $\hat{M}_+$ by $\varphi \longmapsto \varphi(x)$, $\varphi \in M_*^+$. Hence we can regard M_+ as a subset of $\hat{M}_+$.

Let A be a C*-algebra, M a von Neumann algebra and ß an additive, positive homogenous map from A^+ into $\hat{M}_+$;

i.e. 1) $\beta(x + y) = \beta x + \beta y \quad \forall\ x,y \in A^+$.

2) $\beta(\lambda x) = \lambda\beta(x) \quad \forall\ x \in A^+,\ \lambda \geq 0$.

Then $\mathcal{D}^+ = \{ x \in A^+ : \beta x \in M_+\}$ is a face in A^+, and hence is the positive part of a facial subalgebra $\mathcal{D}$ in A. $\beta|_{\mathcal{D}^+}$ then has a unique linear extension to $\mathcal{D}$, which we denote by $\dot{\beta}$. We then say that ß is completely positive if $\dot{\beta}$ is completely positive.

Conversely, suppose $\mathcal{D}$ is a facial subalgebra of A, and α a linear map with domain $\mathcal{D}$ into M, satisfying $\alpha\mathcal{D}^+ \subseteq M^+$. Then $\alpha|_{\mathcal{D}^+}$ can certainly be extended in at least one way into an additive, positive homogenous map ß from A^+ into $\hat{M}_+$ as follows :

If $x \in A^+ \setminus \mathcal{D}^+$, $\varphi \in M_*^+$, $\varphi \neq 0$, define $\hat{\alpha} = \beta$ by

$$\beta(x)\varphi = \infty .$$

It is then clear that $\dot{\beta} = \kappa$.

However if $\mathfrak{h}$ is a hilbert space, and M B($\mathfrak{h}$), the cases $\mathfrak{h} = \mathbb{C}$ and $\mathfrak{h} \neq \mathbb{C}$ are strikingly different. When $\mathfrak{h} = \mathbb{C}$, the map $\varphi \longmapsto (\mathcal{D}_\varphi, \dot{\varphi})$ gives a bijection between the family of $[0,\infty]$ valued weights on A^+, and the family of pairs $(\mathcal{D}, \tau)$ where $\mathcal{D}$ is a facial subalgebra of A and τ a positive linear functional on $\mathcal{D}$. But if $\mathfrak{h} \neq \mathbb{C}$, and $\mathcal{D}$ a facial subalgebra of A, with α a positive linear map of $\mathcal{D}$ into B($\mathfrak{h}$), then by considering simple examples, $\alpha|_{\mathcal{D}^+}$ may have more than one extension to an additive positive homogenous map from A^+ into $B(\mathfrak{h})^\wedge_+$, even if α is completely positive and $\mathcal{D}$ is norm dense in A.

Having made these preliminary remarks, we now sketch our construction of the Stinespring representation. Let A be a C*-algebra, $\mathfrak{h}$ a hilbert space and $\alpha \in$ cp(A; $\mathfrak{h}$). Define $\mathcal{N}_\alpha = L(\mathcal{D}_\alpha)$. We define a bilinear form $\langle\ ,\ \rangle$ on the algebraic tensor product $\mathcal{N}_\alpha \odot \mathfrak{h}$ as follows :

If $x_i, y_j \in \mathfrak{N}_\alpha$, ξ_i, $\eta_j \in \mathfrak{H}$, $i = 1,\ldots,m$ $j = 1,\ldots,n$

put $\langle \sum x_i \otimes \xi_i , \sum y_j \otimes \eta_j \rangle = \sum_{i,j} \langle \alpha(y_j^* x_i)\, \xi_i , \eta_j \rangle$.

This bilinear form is positive semidefinite as α is completely positive.

For each x in A, define a linear transformation $\pi_o(x)$ on $\mathfrak{N}_\alpha \odot \mathfrak{H}$ by

$$\pi_o(x) : \sum x_i \otimes \xi_i \longmapsto \sum x x_i \otimes \xi_i .$$

Then π_o is an algebra homomorphism for which

$$\langle \pi_o(x)u, v \rangle = \langle u, \pi_o(x^*)v \rangle \quad \forall\ u,v \in \mathfrak{N}_\alpha \odot \mathfrak{H} .$$

Now suppose $u = \sum x_i \otimes \xi_i \in \mathfrak{N}_\alpha \odot \mathfrak{H}$, where $x_i \in \mathfrak{N}_\alpha$, $\xi_i \in \mathfrak{H}$, $i = 1,\ldots,n$, and $x \in A$. Define $y_i = \left[\|x\|^2 - x^*x\right]^{1/2} x_i \in \mathfrak{N}_\alpha$ for $i = 1,\ldots,n$. Since α is completely positive, we have

$$\sum_{i,j=1}^{n} \langle \alpha(y_j^* y_i)\, \xi_i , \xi_j \rangle \geq 0.$$

This means that $\langle \pi_o(x)u, \pi_o(x)u \rangle \leq \|x\|^2 \langle u,u \rangle$.

Let $N_\alpha = \{ u \in \mathfrak{N}_\alpha \odot \mathfrak{H} : \langle u,u \rangle = 0 \}$. Then N_α is a linear subspace of $\mathfrak{N}_\alpha \odot \mathfrak{H}$, invariant under $\pi_o(x)$ for each x in A. $\langle\ ,\ \rangle$ determines an inner product on $\mathfrak{N}_\alpha \odot \mathfrak{H}/N_\alpha$, and let $\mathcal{K}_\alpha$ be its hilbert space completion. If $\Lambda_\alpha : \mathfrak{N}_\alpha \odot \mathfrak{H} \longrightarrow \mathfrak{N}_\alpha \odot \mathfrak{H}/N_\alpha$ denotes the canonical projection, there exists an unique representation π_α of A on $\mathcal{K}_\alpha$ such that $\pi_\alpha(x)\Lambda_\alpha\, y \otimes \xi = \Lambda_\alpha\, xy \otimes \xi \quad \forall\ x \in A,\ y \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H}$.

Moreover $\langle \pi_\alpha(x)\Lambda_\alpha y \otimes \xi, \Lambda_\alpha z \otimes \eta \rangle = \langle \alpha(z^*xy)\xi , \eta \rangle \ \forall x \in A,\ y,z \in \mathfrak{N}_\alpha$, $\xi, \eta \in \mathfrak{H}$.

If $\alpha \in cp(A;\mathfrak{H})$, we will use as standard notation the objects $\mathfrak{N}_\alpha$, N_α, Λ_α, $\mathcal{K}_\alpha$, π_α constructed above.

We define a partial ordering $\geq$ on the cone $cp(A;\mathfrak{H})$, by

$\alpha \geq \beta$ $\quad \alpha, \beta \in cp(A;\mathfrak{H})$ if

1) $\mathcal{D}_\alpha \subseteq \mathcal{D}_\beta$

2) The linear map $\alpha - \beta$ with domain $\mathcal{D}_\alpha$ lies in $cp(A;\mathfrak{H})$.

With this order structure on $cp(A;\mathfrak{H})$ we can show the following theorem, which generalises [64, Lemme 2.3] for scalar valued weights, and [4] for bounded conpletely positive maps on C*-algebras. It gives one side of a Radon Nikodym story. As we shall see in § 6.5, the converse is trickier.

THEOREM 6.1.5.

Let A be a C*-algebra, $\mathfrak{H}$ a hilbert space, and $\alpha, \beta \in cp(A;\mathfrak{H})$ such that $\beta \leqslant \alpha$. The identity map on $\mathfrak{N}_\alpha$ defines on passing to the quotient a continuous linear map λ from $\Lambda_\alpha \mathfrak{N}_\alpha \odot \mathfrak{H}$ onto $\Lambda_\beta \mathfrak{N}_\alpha \odot \mathfrak{H}$, which extends to a linear continuous map λ from $\mathcal{K}_\alpha$ into $\mathcal{K}_\beta$. If $T = \lambda^*\lambda$ and $\lambda = WT^{1/2}$ is the polar decomposition of λ, then :

i) T is an operator in the commutant $\pi_\alpha(A)'$ of $\pi_\alpha(A)$ satisfying

$0 \leqslant T \leqslant 1$ and

$$\langle \beta(y^*x)\xi , \eta\rangle = \langle T\Lambda_\alpha x \otimes \xi , \Lambda_\alpha y \otimes \eta\rangle \quad \forall\, x,y \in \mathfrak{N}_\alpha , \ \xi,\eta \in \mathfrak{H}$$

ii) The partial isometry W gives an equivalence between the subrepresentation π_α^o of π_α induced by the support of T, which lies in $\pi_\alpha(A)'$, and the subrepresentation π_β^o of π_β defined by the stable subspace $[\Lambda_\beta(\mathfrak{N}_\alpha \odot \mathfrak{H})]^-$ of $\mathcal{K}_\beta$.

PROOF.

$\beta \leqslant \alpha$ implies that $\mathcal{D}_\alpha \subseteq \mathcal{D}_\beta$, $\mathfrak{N}_\alpha \subseteq \mathfrak{N}_\beta$, $N_\alpha \subseteq N_\beta$. Hence the identity map on $\mathfrak{N}_\alpha$ induces a linear map λ from $\mathfrak{N}_\alpha \odot \mathfrak{H}/N_\alpha$ onto $\mathfrak{N}_\alpha \odot \mathfrak{H}/N_\beta \subseteq \mathfrak{N}_\beta \odot \mathfrak{H}/N_\beta$. i.e. $\lambda : \Lambda_\alpha \mathfrak{N}_\alpha \odot \mathfrak{H} \longrightarrow \Lambda_\beta \mathfrak{N}_\alpha \odot \mathfrak{H}$.

Let $x_i \in \mathfrak{N}_\alpha$, $\xi_i \in \mathfrak{H}$, $i = 1,\dots,m$. Then

$$\|\lambda(\Lambda_\alpha \textstyle\sum x_i \otimes \xi_i)\|^2 = \|\Lambda_\beta(\textstyle\sum x_i \otimes \xi_i)\|^2 = \displaystyle\sum_{i,j} \langle \beta(x_i^* x_j)\xi_j, \xi_i\rangle$$

$$\leq \textstyle\sum \langle \alpha(x_i^* x_j)\xi_j, \xi_i\rangle = \|\Lambda_\alpha(\textstyle\sum x_i \otimes \xi_i)\|^2.$$

Since λ is bounded, it can be extended to a linear continuous map of $\mathcal{K}_\alpha$ into $\mathcal{K}_\beta$ which we also write as λ ,with norm $\leqslant 1$. If $T = \lambda^*\lambda$, then $0 \leqslant T \leqslant 1$. If $x,y \in \mathfrak{N}_\alpha$, $\xi,\eta \in \mathfrak{H}$ then

$$\langle T\Lambda_\alpha(x \otimes \xi), \Lambda_\alpha y \otimes \eta\rangle = \langle \lambda^*\lambda\, \Lambda_\alpha x \otimes \xi , \Lambda_\alpha y \otimes \eta\rangle$$

$$= \langle \lambda\Lambda_\alpha(x \otimes \xi), \lambda\Lambda_\alpha y \otimes \eta\rangle = \langle \Lambda_\beta x \otimes \xi , \Lambda_\beta y \otimes \eta\rangle$$

$$= \langle \beta(y^*x)\xi , \eta\rangle .$$

If also $z \in A$, then

$$\langle T\,\pi_\alpha(z)\,\Lambda_\alpha x\otimes\xi\ ,\Lambda_\alpha y\otimes\eta\rangle\ =\ \langle \beta(y^*zx)\xi\ ,\eta\rangle$$
$$=\ \langle T\Lambda_\alpha x\otimes\xi\ ,\pi_\alpha(z^*)\Lambda_\alpha y\otimes\eta\rangle\ .$$

Thus $T\in\pi_\alpha(A)'$.

If $z\in A$, $x\in\mathcal{N}_\alpha$, $\xi\in\mathfrak{H}$, $\pi_\beta(z)\Lambda_\beta x\otimes\xi = \Lambda_\beta zx\otimes\xi\in\Lambda_\beta\mathcal{N}_\alpha\odot\mathfrak{H}$.
Hence $[\Lambda_\beta\mathcal{N}_\alpha\odot\mathfrak{H}]$ is stable for π_β and defines a projection P in $\pi_\beta(A)'$, and a subrepresentation π_β^o of π_β. We now show that the partial isometry W gives an equivalence between P and the support of $T^{1/2}$, (which is the same as that of T). If $z\in A$, $x\in\mathcal{N}_\alpha$, $\xi\in\mathfrak{H}$ then

$$W\pi_\alpha^o(z)\,T^{1/2}\Lambda_\alpha x\otimes\xi = W\pi_\alpha(z)\,T^{1/2}\Lambda_\alpha x\otimes\xi = W\,T^{1/2}\pi_\alpha(z)\,\Lambda_\alpha x\otimes\xi$$
$$=\lambda\,\Lambda_\alpha zx\otimes\xi = \Lambda_\beta\,zx\otimes\xi = \pi_\beta(z)\,\Lambda_\beta\,x\otimes\xi$$
$$=\pi_\beta^o(z)\,\lambda\,\Lambda_\alpha\,x\otimes\xi = \pi_\beta^o(z)\,W\,T^{1/2}\Lambda_\alpha\,x\otimes\xi\ .$$

Since $T^{1/2}\Lambda_\alpha\mathcal{N}_\alpha\odot\mathfrak{H}$ is dense in the support of T; W gives the equivalence between π_α^o and π_β^o.

In the theory of scalar valued weights, progress has been made by studying the bounded positive linear functionals dominated by a given weight [7,64]. In order to develop our theory of operator valued weights, we find it convenient to introduce some more definitions.

DEFINITION 6.1.6.

Let A be a C*-algebra, $\mathfrak{H}$ a hilbert space and $\alpha\in cp(A;\mathfrak{H})$. We denote by $\mathcal{F}=\mathcal{F}^\alpha$, the family

$$\{w\in CP(A;\mathfrak{H})\ :\ w\leq\alpha\}\ ,$$

and by $\mathcal{K}=\mathcal{K}^\alpha$, the set of operators S in $\pi_\alpha(A)'$ such that there is a positive real number λ such that

$$\|S\Lambda_\alpha x\otimes\xi\|\ \leq\lambda\|x\|\,\|\xi\|\qquad\forall\ x\in\mathcal{N}_\alpha\ ,\ \xi\in\mathfrak{H}\ .$$

The next lemma can be proved by developing the arguments in [64, Lemma 2.3], to which it reduces if $\mathfrak{H}=\mathbb{C}$.

LEMMA 6.1.7.

A is a C*-algebra, $\mathfrak{H}$ a hilbert space and $\alpha \in \mathrm{cp}(A;\mathfrak{H})$. Then $\mathfrak{K}$ is a left ideal in $\pi_\alpha(A)'$. For any S in $\mathfrak{K}$, there exists a bounded linear map $V = V^S$ from $\mathfrak{H}$ into $[\pi_\alpha(\mathfrak{N}_\alpha^*)\mathfrak{K}_\alpha]^-$ such that $S\Lambda_\alpha x \otimes \xi = \pi_\alpha(x)V\xi$ for all x in $\mathfrak{N}_\alpha$, ξ in $\mathfrak{H}$. Moreover if $\xi \in \mathfrak{H}$ is fixed, $V\xi$ is the unique element of $[\pi_\alpha(\mathfrak{N}_\alpha^*)\mathfrak{K}_\alpha]^-$ such that $S\Lambda_\alpha x \otimes \xi = \pi_\alpha(x)V\xi$ for all x in $\mathfrak{N}_\alpha$.

Parallel to the theory for weights [64, Lemma 2.6], we can now describe the relation between $\mathfrak{F}$ and $\mathfrak{K}$.

PROPOSITION 6.1.8.

Let A be a C*-algebra, $\mathfrak{H}$ a hilbert space and $\alpha \in \mathrm{cp}(A;\mathfrak{H})$. For any w in $\mathfrak{F}$, there is a unique S in $\mathfrak{K}$ such that $0 \leq S \leq 1$, and

$$\langle w(x^*x)\xi,\xi\rangle = \|S\Lambda_\alpha x \otimes \xi\|^2 \qquad \forall\ x \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H}.$$

Conversely for any S in $\mathfrak{K}$ such that $\|S\| \leq 1$, there is an w in $\mathfrak{F}$, such that

$$\langle w(x^*x)\xi,\xi\rangle = \|S\Lambda_\alpha x \otimes \xi\|^2 \qquad \forall\ x \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H}.$$

PROOF.

Let $w \in \mathfrak{F}$. Then there is a T_w in $\pi_\alpha(A)'$ by Theorem 6.1.5 such that $0 \leq T_w \leq 1$ and

$$\langle w(x^*x)\xi,\xi\rangle = \langle T_w\Lambda_\alpha x \otimes \xi, \Lambda_\alpha x \otimes \xi\rangle \qquad \forall\ x \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H}.$$

Define $S = T_w^{1/2}$. Then

$$\|S\Lambda_\alpha x \otimes \xi\|^2 = \langle w(x^*x)\xi,\xi\rangle \leq \|w\|\,\|x\|^2\,\|\xi\|^2,$$

so that $S \in \mathfrak{K}$. The uniqueness is clear.

Conversely, let $S \in \mathfrak{K}$ such that $\|S\| \leq 1$. By Lemma 6.1.7 there exists V in $B(\mathfrak{H}, \mathfrak{K}_\alpha)$ such that $S\Lambda_\alpha x \otimes \xi = \pi_\alpha(x)V\xi \ \forall\, x \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H}$.

Define $w(z) = V^*\pi_\alpha(z)V \quad \forall\ z \in A$. If $x_i \in \mathfrak{N}_\alpha$, $\xi_i \in \mathfrak{H}$ for $i = 1,\dots,n$ then

$$\sum\langle w(x_i^*x_j)\xi_j,\xi_i\rangle = \|\sum\pi_\alpha(x_i)V\xi_i\|^2$$
$$= \|S\Lambda\sum x_i \otimes \xi_i\|^2 \leq \|\Lambda\sum x_i \otimes \xi_i\|^2 = \sum\langle\alpha(x_i^*x_j)\xi_j,\xi_i\rangle$$

i.e. $w \leq \alpha$, and $w \in \mathcal{F}$.

REMARK 6.1.9.

In the situation of Propostition 6.1.8, we note the following :

a) $\mathcal{K} = \pi_\alpha(A)'$ iff α coincides on $\mathcal{D}$ with an element of CP(A; $\mathfrak{h}$).

b) If also $\mathcal{D}_\alpha$ is norm dense in A , and $w \in \mathcal{F}$, then $\Lambda_w \mathcal{N}_\alpha \odot \mathfrak{h}$ is dense in $\mathcal{K}_w$. Thus by applying 6.1.5 to the pair (α, w), and if T is the operator defined there, π_w is a representation of A equivalent with a subrepresentation of π_α defined by the support of T.

We also observe the following, which is well known in the bounded case [4] and for weights [64, Proposition 2.5] .

PROPOSITION 6.1.10.

Let A be a C*-algebra, $\mathfrak{h}$ a hilbert space, $\alpha \in$ cp(A; $\mathfrak{h}$) such that $\mathcal{D}_\alpha$ is norm dense in A, and $\mathcal{F}$ contains a non zero element. Then π_α is irreducible iff α is the restriction to $\mathcal{D}_\alpha$ of a pure element w of CP(A; $\mathfrak{h}$). If htere exists an additive, positive homogenous map $\beta : A^+ \longrightarrow B(\mathfrak{h})^\wedge_+$ such that $\dot{\beta} = \alpha$ and $x \longmapsto \beta(x)\varphi$ is lower semicontinuous on A^+ for each φ in $T(\mathfrak{h})_+$, then $\alpha = w$.

§ 6.2.

Throughout this section, A denotes a C*-algebra, and $\mathfrak{h}$ a hilbert space. If $\alpha \in$ cp(A; $\mathfrak{h}$), we will be concerned with the construction of a largest operator valued weight α_o majorised by α , and with the property that it is the upper envelope of continuous CP maps. Therefore we will need a property of $\mathcal{F}$ called "ϵ -filtrating" by Combes [7].

DEFINITION 6.2.1.

A family $\mathcal{G}$ in CP(A; $\mathfrak{h}$) is said to be ϵ-filtering if given $\epsilon > o$,

w_1, w_2 in $\mathcal{G}$, there exists γ in $\mathcal{G}$, such that

$$(1-\epsilon)w_i \leqslant \gamma \qquad i = 1,2.$$

When $\mathfrak{H} = \mathbb{C}$, the following reduces partly to [7, Lemme 1.9] and is proved but not stated by [64].

PROPOSITION 6.2.2.

If $\alpha \in cp(A;\mathfrak{H})$ such that $\mathcal{D}_\alpha$ is dense in A, then $\mathcal{F}$ is ϵ-filtering.

PROOF.

Given $\epsilon > 0$, $w_1, w_2 \in \mathcal{F}$, there exist $S_1, S_2 \in \mathcal{K}$, such that $0 \leqslant S_i \leqslant 1$ and $\langle w_i(x^*x)\xi, \xi\rangle = \|S_i\Lambda_\alpha x \otimes \xi\|^2 \quad \forall x \in \mathcal{N}_\alpha, \xi \in \mathfrak{H}$.

Since $\mathcal{K}$ is a left ideal in $\pi_\alpha(A)'$, we can get $S \in \mathcal{K}$ by [66, Lemma 3.1] such that $(1-\epsilon)\, S_i^*S_i \leqslant S^*S \leqslant 1$ for $i = 1,2$.

Let $w \in \mathcal{F}$ be the element determined by S according to 6.1.8.

Then for all $x_j \in \mathcal{N}_\alpha$, $\xi_j \in \mathfrak{H}$, $j = 1,\dots\dots,n$

$$\sum_{j,k} (1-\epsilon)\langle w_i(x_j^*x_k)\xi_k, \xi_j\rangle = (1-\epsilon)\,\|S_i(\Lambda\sum_j x_j \otimes \xi_j)\|^2$$

$$\leqslant \|S\Lambda\sum x_j \otimes \xi_j\|^2 = \sum_{j,k}\langle w(x_j^*x_k)\xi_k, \xi_j\rangle .$$

Since $\mathcal{N}_\alpha$ is dense in A, we deduce $(1-\epsilon)w_i \leqslant w \quad i = 1,2.$

PROPOSITION 6.2.3.

A family $\mathcal{G}$ in $CP(A;\mathfrak{H})$ is ϵ-filtering. Then

$$\beta(x)(\varphi) = \sup_{w\in\mathcal{G}} \langle w(x), \varphi\rangle \qquad \varphi \in T(\mathfrak{H})_+, \; x \in A^+$$

defines a completely positive, additive, positive homogenous map ß from A^+ into $B(\mathfrak{H})^\wedge_+$. If $\alpha_0 = \dot{\beta}$ (Remark 6.1.3) then :

i) $\mathcal{D}_{\alpha_0}^{\,+} = \{x \in A^+ : \sup_{w\in\mathcal{G}} \|w(x)\| < \infty\}$

ii) $\alpha_0 \geqslant w \quad \forall\, w \in \mathcal{G}$.

iii) If $x_i \in \mathcal{N}_{\alpha_0}$, $\xi_i \in \mathfrak{H}$ $\quad i = 1,2,\dots\dots,n$ then

$$\sum_{i,j=1}^{n} \langle \alpha_0(x_i^*x_j)\xi_j, \xi_i\rangle = \sup_{\beta\in\mathcal{G}} \sum_{i,j}\langle \beta(x_i^*x_j)\,\xi_j, \xi_i\rangle$$

PROOF.

It is clear by ϵ -filtering that

$$\beta(x)(\varphi) = \sup_{w \in \mathcal{G}} \langle w(x), \varphi \rangle \qquad x \in A^+, \ \varphi \in T(\mathfrak{H})_+$$

defines an additive, positive homogenous map from A^+ into $\hat{B(\mathfrak{H})}_+$.

Moreover if $\alpha_o = \dot{\beta}$, it is trivial that

$$\mathcal{D}_{\alpha_o}^{+} \subseteq \{ x \in A^+ : \sup_{w \in \mathcal{G}} \|w(x)\| < \infty \}.$$

Now suppose $x_1 \in A^+$ and $\sup_{w \in \mathcal{G}} \|w(x_1)\| \leq c < \infty$ for some c.

Take $\varphi \in T(\mathfrak{H})$, then $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ $\varphi_i \in T(\mathfrak{H})_+$

and $\|\varphi_1 - \varphi_2\| = \|\varphi_1\| + \|\varphi_2\|$, $\|\varphi_2 - \varphi_3\| = \|\varphi_3\| + \|\varphi_4\|$ [54].

Then the linear extension of $\beta(x_1)(\varphi)$ to $T(\mathfrak{H})$ satisfies

$$|\beta(x_1)(\varphi)| \leq c \|\varphi\| \qquad \forall \ \varphi \in T(\mathfrak{H}).$$

Hence $\beta(x_1) \in B(\mathfrak{H}) = T(\mathfrak{H})^*$, and $x_1 \in \mathcal{D}_{\alpha_o}^{+}$.

In order to prove the reminder, it is enough by polarisation to show that if $y_i \in \mathcal{N}_\alpha$, $\xi_i \in \mathfrak{H}$, $i = 1,2,\ldots,m; \epsilon > 0; w_1 \in \mathcal{G}$, that there exists w in $\mathcal{G}$, satisfying

$$|\langle \alpha_o(y_i^* y_i) \xi_i, \xi_i \rangle - \langle w(y_i^* y_i) \xi_i, \xi_i \rangle| < \epsilon \qquad i = 1,\ldots,m$$

and $w \geq (1 - \epsilon) w_1$.

This follows by ϵ -filtering.

DEFINITION 6.2.4.

$\gamma \in cp(A; \mathfrak{H})$ is said to be the upper envelope of a family $\mathcal{G}$ in $CP(A; \mathfrak{H})$ if

i) If $x \in A^+$, then $x \in \mathcal{D}_\gamma$ iff $\sup_{w \in \mathcal{G}} \|w(x)\| < \infty$.

ii) For all x_i in $\mathcal{N}_\gamma$, ξ_i in $\mathfrak{H}$, $i = 1,\ldots,n$

$$\sum_{i,j=1}^{n} \langle \gamma(x_i^* x_j) \xi_j, \xi_i \rangle = \sup_{w \in \mathcal{G}} \left\{ \sum \langle w(x_i^* x_j) \xi_j, \xi_i \rangle \right\}$$

We now state our decomposition theorem. If we restrict ourselves to scalar valued weights, Theorem 6.2.5 reduces to [64, 2.9, 2.11] .

THEOREM 6.2.5.

Let $\alpha \in cp(A;\mathfrak{H})$ such that $\mathcal{D}_\alpha$ is dense in A. Then there exist α_0, α_1 in $cp(A;\mathfrak{H})$ such that

i) $\alpha = \alpha_0 + \alpha_1$

ii) $\alpha_0 \leq \alpha$ and α_0 is the upper envelope of $\mathcal{F}^\alpha$. α_0 majorises any other map $\gamma \in cp(A;\mathfrak{H})$ which is majorised by α, and such that γ is the upper envelope of a family in $CP(A;\mathfrak{H})$.

iii) $\alpha_1 \leq \alpha$ and α_1 majorises no non zero element of $CP(A;\mathfrak{H})$.

PROOF.

It is clear from Propositions 6.2.2 and 6.2.3 that α_0, the upper envelope of $\mathcal{F}$ exists, and $\alpha_0 \leq \alpha$. Let γ be any other element of $cp(A;\mathfrak{H})$ majorised by α, which is the upper envelope of a family $\mathcal{G}$ in $CP(A;\mathfrak{H})$. Let $x \in \mathcal{D}_{\alpha_0}^+$. Then $\sup_{w \in \mathcal{F}} \|w(x)\| < \infty$. However if $w \in \mathcal{G}$, $w \leq \gamma$, and $\gamma \leq \alpha$ show $w \in \mathcal{F}$. Hence $\sup_{w \in \mathcal{G}} \|w(x)\| < \infty$, i.e. $x \in \mathcal{D}_\gamma^+$. Thus $\mathcal{D}_{\alpha_0} \subseteq \mathcal{D}_\gamma$. Take $a_i \in \mathfrak{N}_{\alpha_0} \subseteq \mathfrak{N}_\gamma$, $\xi_i \in \mathfrak{H}$, $i = 1,\dots,n$. Then there exists w in $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\sum_{i,j} \langle \gamma(a_i^*a_j)\xi_j, \xi_i\rangle \leq \sum \langle w(a_i^*a_j)\xi_j, \xi_i\rangle + 1$$
$$\leq \sum \langle \alpha_0(a_i^*a_j)\xi_j, \xi_i\rangle + 1 .$$

Hence

$$\sum_{i,j} \langle \gamma(a_i^*a_j)\xi_j, \xi_i\rangle \leq \sum \langle \alpha_0(a_i^*a_j)\xi_j, \xi_i\rangle$$

and $\gamma \leq \alpha_0$.

Now define $\mathcal{D}_{\alpha_1} = \mathcal{D}$ and $\alpha_1(x) = \alpha(x) - \alpha_0(x)$ for x in $\mathcal{D}$. Then $\alpha \geq \alpha_1 \geq 0$, and $\alpha = \alpha_0 + \alpha_1$. Let $w \in CP(A;\mathfrak{H})$ such that $w \leq \alpha_1$. Then $w \leq \alpha$, hence $w \in \mathcal{F}$ so that $w \leq \alpha_0$, and $2w \leq \alpha$. Similarly $nw \leq \alpha$ $\forall n$. Hence $w = 0$, since $\mathcal{D} = \operatorname{lin} \mathcal{D}^+$ and $\mathcal{D}$ is dense in A.

§ 6.3.

We will now study quasi equivalence, equivalence and the type of representations associated with various operator valued weights. All the results of this section are operator valued versions of those in [7, 2.6 - 2.17] for the case of $[o,\infty]$ valued weights. Again A denotes a C*-algebra and $\mathfrak{H}$ a hilbert space.

DEFINITION 6.3.1.

Let $\alpha \in cp(A;\mathfrak{H})$ with domain $\mathcal{D}$, and F a family in $CP(A;\mathfrak{H})$ majorised by α . We say α belongs to the closure of F for the topology of simple convergence if any one of the following equivalent conditions are satisfied:

1) Given $n,m \geq 1$, and $a_i \in (\mathcal{D}_\alpha)_n$, $\xi_i \in \mathfrak{H}\otimes\mathbb{C}^n$ for $i = 1,\ldots m$ and $\epsilon > o$, there exists $\beta \in F$ such that

$$|\langle \alpha_n(a_i)\xi_i, \xi_i\rangle - \langle \beta_n(a_i)\xi_i, \xi_i\rangle| < \epsilon \qquad i = 1,\ldots\ldots,m.$$

2) Given $n,p \geq 1$, and $a_j^i \in \mathfrak{N}_\alpha$, $\xi_j^i \in \mathfrak{H}$ $i = 1,\ldots\ldots,p$ $j = 1,\ldots,n$ and $\epsilon > o$, there exists $\beta \in F$ such that

$$\left|\sum_{j,k=1}^{n} \langle \alpha\left((a_j^i)^* a_k^i\right)\xi_k^i , \xi_j^i\rangle - \sum_{j,k=1}^{n} \langle \beta\left((a_j^i)^* a_k^i\right)\xi_k^i , \xi_j^i\rangle\right| < \epsilon$$

for $i = 1,\ldots\ldots,p$.

3) Given $q \geq 1$, $a_i \in \mathfrak{N}_\alpha$, $\xi_i \in \mathfrak{H}$ $i = 1,\ldots.,q$, and $\epsilon > o$ there exists $\beta \in F$ such that

$$o \leq \langle \alpha(a_i^* a_i)\xi_i, \xi_i\rangle - \langle \beta(a_i^* a_i)\xi_i, \xi_i\rangle < \epsilon$$

for $i = 1,\ldots.,q$

If α is the upper envelope of an $\in$ -filtering family F in $CP(A;\mathfrak{H})$, then α belongs to the closure of F for the topology of simple convergence. In particular if $\alpha \in cp(A;\mathfrak{H})$ with $\mathcal{D}_\alpha$ dense in A, then α_o belongs to the closure of $\mathcal{F}^\alpha$ for the topology of simple convergence.

LEMMA 6.3.2.

Let $\alpha \in cp(A;\mathfrak{H})$ and $cp(A;\mathfrak{H}) \supseteq F$, a family majorised by α . If $\beta \in F$, let T_β be the corresponding element of $\pi_\alpha(A)'$ as defined in 6.1.5. Then α belongs to the closure of F for the topology of simple convergence iff 1 is in the weak closure of $\{T_\beta : \beta \in F\} \subseteq \pi_\alpha(A)'$.

PROOF.

The family $\{T_\beta\}$ satisfies $\|T_\beta\| \leq 1$ for all $\beta \in F$. Hence 1 lies in their weak closure iff given $\epsilon > 0$, $p,n \geq 1$, $\xi^i_j \in \mathfrak{H}$, $a^i_j \in \mathfrak{N}_\alpha$ $i = 1,\ldots,p$, $j = 1,\ldots,n$, there exists $\beta \in F$ such that

$$\langle (1 - T_\beta)\Lambda_\alpha \sum_{j=1}^{n} a^i_j \otimes \xi^i_j \,,\; \Lambda_\alpha \sum a^i_j \otimes \xi^i_j \rangle < \epsilon \qquad i = 1,\ldots,p$$

i.e. $$\sum_{j,k=1}^{n} \left\{ \langle \alpha\left((a^i_j)^* a^i_k\right) \xi^i_k \,,\, \xi^i_j \rangle - \langle \beta\left((a^i_j)^* a^i_k\right)\xi^i_k \,,\, \xi^i_j \rangle \right\} < \epsilon$$

for $i = 1,\ldots,p$.

PROPOSITION 6.3.3.

Let $\alpha \in cp(A;\mathfrak{H})$ and F be a family in $CP(A;\mathfrak{H})$ majorised by α. If $w \in F$, V^w denotes the corresponding element in $B(\mathfrak{H},\mathcal{K}_\alpha)$ such that $w(x) = (V^w)^* \pi_\alpha(x) V^w \quad \forall x \in A$. If α belongs to the closure of F for the topology of simple convergence, then the set $\{V^\beta \mathfrak{H} : \beta \in F\}$ is cyclic for π_α.

COROLLARY 6.3.4.

Suppose $\alpha \in cp(A;\mathfrak{H})$, A and $\mathfrak{H}$ are separable, and that there exists a countable family F in $CP(A;\mathfrak{H})$ majorised by α such that α belongs to the closure of F for the topology of simple convergence, then $\mathcal{K}_\alpha$ is separable.

PROPOSITION 6.3.5.

$\{\beta_i\}_{i\in I}$ is a family in $cp(A;\mathfrak{H})$ majorised by a single element α in $cp(A;\mathfrak{H})$ and such that α belongs to the closure of $\{\beta_i\}_{i\in I}$ for the topology of simple convergence. Moreover suppose that for each i in I, there exists a family F_i in $CP(A;\mathfrak{H})$ such that β_i majorises each element of F_i and belongs to the closure of F_i for the topology of simple convergence. If $\mathcal{D}_\alpha$ is dense in A, then the representations π_α and $\oplus_i \pi_{\beta_i}$ are quasiequivalent.

PROOF.

Suppose each $\beta_i \in CP(A;\mathfrak{h})$. Then since $\beta_i \leq \alpha$, π_{β_i} is equivalent to a subrepresentation of π_α defined by the projection P_i in $\pi_\alpha(A)'$ which is the support of the operator T_{β_i} associated with β_i (6.1.5.). Hence π_{β_i} is quasiequivalent to the subrepresenation of π_α associated to the central support Q_i of P_i in the centre of $\pi_\alpha(A)'$. Therefore $\oplus \pi_{\beta_i}$ is quasiequivalent to the subrepresentation of π_α associated with the projection $\bigvee_{i\in I} Q_i$ in the centre of $\pi_\alpha(A)'$. By 6.3.2 this is 1. In the general case put $F = \bigcup_{i\in I} F_i$. Then α belongs to the closure of F for the topology of simple convergence . Hence

$$\pi_\alpha \approx \bigoplus_{w\in F} \pi_w = \bigoplus_{i\in I}\left(\bigoplus_{w\in F_i} \pi_w\right) \approx \bigoplus_{i\in I} \pi_{w_i} .$$

COROLLARY 6.3.6.

Let α, $\beta \in cp(A;\mathfrak{h})$ such that $\beta \leq \alpha$ and $\mathcal{D}_\alpha^- = A$. Suppose that there exists a family F (respectively G) in $CP(A;\mathfrak{h})$ majorised by α (respectively β) such that α (respectively β) belongs to the closure of F (respectively G) for the topology of simple convergence, then π_β is a quasiequivalent to a subrepresenation of π_α .

COROLLARY 6.3.7.

Suppose $\alpha, \beta \in cp(A;\mathfrak{h})$ with $\mathcal{D}_\alpha \subseteq \mathcal{D}_\beta$, $\alpha = \beta|_{\mathcal{D}_\alpha}$ and $\mathcal{D}_\alpha^- = A$. If there exists a family F in $CP(A;\mathfrak{h})$, majorised by β and such that β belongs to the closure of F for the topology of simple convergence, then π_α and π_β are quasiequivalent.

DEFINITION 6.3.8.

Let $\{\alpha_i\}_{i\in I}$ be a family in $cp(A;\mathfrak{h})$. Then
$E = \left\{x \in \bigcap \mathcal{D}_{\alpha_i}^+ : \exists \lambda < \infty \text{ s.t. } \sum_J \alpha_i(x) \leq \lambda 1 \text{ for all finite subsets } J \subseteq I\right\}$
is a face in A^+, and hence is the positive part of a facial subalgebra $\mathcal{D}$

in A. If $x \in E$, $\sum_{i \in I} \alpha_i(x)$ exists as an ultraweak limit, and we define this limit to be $\alpha'(x)$. Let α be the unique linear extension of α' to $\mathcal{D}$. Then $\alpha \geq \sum_{i \in J} \alpha_i$ for all finite subsets J of I, and in particular $\alpha \in cp(A;\mathfrak{H})$. We write $\alpha = \sum_{i \in I} \alpha_i$.

(Alternatively, we could consider $(\sum \hat{\alpha}_i)^{\cdot}$).

PROPOSITION 6.3.9.

Suppose $\{\alpha_i\}_{i \in I}$ is a family in $cp(A;\mathfrak{H})$, and suppose $\alpha \in cp(A;\mathfrak{H})$ with $\alpha \leq \sum \alpha_i$. Then the representation π_α of Ais equivalent to a subrepresentation of $\bigoplus_{i \in I} \pi_{\alpha_i}$.

PROOF.

For each i in I, let $\mathcal{D}_i$, $\mathcal{N}_i$, Λ_i, $\mathcal{R}_i$, π_i be the canonical Stinespring objecects defined by α_i for each i in I. Since $\alpha_i \leq \alpha$, we have by 6.1.5 a partial isometry $W_i : \mathcal{R}_\alpha \to \mathcal{R}_i$, and an operator $T_i \in \pi_\alpha(A)'$ such that the subrepresentation induced by the support of T_i is equivalent to a subrepresentation of π_{α_i}. For $\xi \in \mathcal{R}_\alpha$ put $W\xi = (W_i(\xi)) \in \bigoplus_{i \in I} \mathcal{R}_i$.

For $x_j \in \mathcal{N}_\alpha \subseteq \mathcal{N}_i$, $\xi_j \in \mathfrak{H}$, $j = 1,2\ldots,n$ we have

$$\begin{aligned}
\|W\Lambda_\alpha \sum x_j \otimes \xi_j\|^2 &= \sum_{i \in I} \|W_i \Lambda_i \sum_{j=1}^{n} x_j \otimes \xi_j\|^2 \\
&= \sum_{i \in I} \|\Lambda_i \sum_{j=1}^{n} x_j \otimes \xi_j\|^2 \\
&= \sum_i \sum_{j,k=1}^{n} \langle \alpha_i(x_j^* x_k)\xi_k, \xi_j\rangle \\
&= \sum_{j,k} \langle \alpha(x_j^* x_k)\xi_k, \xi_j\rangle \\
&= \|\Lambda_\alpha \sum_j x_j \otimes \xi_j\|^2 .
\end{aligned}$$

Hence W is an isometry from $\mathcal{R}$ into $\oplus \mathcal{R}_i$. For each i in I, $W_i(\mathcal{R}_\alpha)$ is stable under π_i, hence $W(\mathcal{R})$ is stable under $\bigoplus_{i \in I} \pi_i$, and defines a subrepresentation ρ of $\bigoplus_i \pi_i$.

For all z in A, x in $\mathcal{N}_\alpha$, $\xi \in \mathfrak{H}$,

$$W\pi_\alpha(z)\Lambda_\alpha x \otimes \xi = W\Lambda_\alpha zx \otimes \xi = (\Lambda_i zx \otimes \xi)_{i \in I}$$

$= \quad (\pi_i(z)\Lambda_i x \otimes \xi)_{i\in I} \quad = \quad \oplus\pi_i(z) W\Lambda_\alpha x\otimes\xi \quad = \rho(z) W\Lambda_\alpha x\otimes\xi$.

Hence W gives the required equivalence between π_α and ρ.

DEFINITIONS 6.3.10.

We say $\alpha \in$ cp(A;$\mathfrak{K}$) is of type I (respectively II,III, etc) if the representation π_α is of type I (respectively II,III etc), and α is factorial if π_α is factorial.

COROLLARY 6.3.11.

Suppose $\{\alpha_i\}_{i\in I}$ is a family in cp(A; $\mathfrak{K}$), and consider $\alpha = \sum_{i\in I} \alpha_i$. If for each $i \in I$, α_i is of type I (respectively II, II_1, II_∞, III etc) then $\sum \alpha_i$ is of the same type.

COROLLARY 6.3.12.

Those elements in cp(A;$\mathfrak{K}$) of a fixed type, form a convex subcone of cp(A;$\mathfrak{K}$).

PROPOSITION 6.3.13.

Suppose $\{\alpha_i : i \in I\}$ is a family in cp(A; $\mathfrak{K}$) such that for each i in I, there exists a family F_i in CP(A; $\mathfrak{K}$) majorised by α_i , with α_i belonging to the closure of F_i for the topology of simple convergence. Suppose that $\alpha \in$ cp(A;$\mathfrak{K}$) is such that $\mathcal{D}_\alpha^- = A$ and $\alpha \leq \sum\alpha_i$. Then the representations π_α and $\bigoplus_{i\in I} \pi_{\alpha_i}$ are quasi equivalent.

PROOF.

this follows from 6.3.5 and 6.3.9.

PROPOSITION 6.3.14.

Let $\alpha \in$ cp(A;$\mathfrak{K}$) with $\mathcal{D}_\alpha^- = A$. Suppose that there exists at least one family in CP(A; $\mathfrak{K}$) whose sum is α . Then the following conditions are equivalent :

i) α is factorial.

ii) There exists a family of quasi equivalent factorial maps in $CP(A;\mathfrak{H})$ whose sum is α.

If α is also of type I, they are also equivalent to :

iii) There exists a family of pure elements of $CP(A;\mathfrak{H})$ with equivalent representations whose sum is α.

PROOF.

i $\Rightarrow$ ii. Let $\{\beta_i : i \in I\}$ be a family in $CP(A;\mathfrak{H})$ such that $\alpha = \sum \beta_i$ and $\beta_i \neq 0 \ \forall\ i$. Then by 6.1.5, for each j in I, π_{β_j} is equivalent to a subrepresentation of π_α. If π_α is factorial, π_{β_i} is factorial for all i, and quasi equivalent to π_α.

ii $\Rightarrow$ i. If $\{\beta_i : i \in I\}$ is a family of quasi equivalent factorial maps in $CP(A;\mathfrak{H})$, then $\oplus\pi_{\beta_i}$ is factorial. If $\alpha = \sum \beta_i$, α is factorial by Proposition 6.3.9.

Now suppose α is of type I.

ii $\Rightarrow$ iii. Let $\{\beta_i : i \in I\}$ be a family of quasi equivalent factorial maps in $CP(A;\mathfrak{H})$. For each i in I, π_{β_i} is factorial of type I. We can decompose π_{β_i} as a sum of irreducible representations

$$\pi_{\beta_i} = \bigoplus_{j\in I(i)} \pi_j \quad \text{and such that} \quad \mathcal{R}_{\beta_i} = \bigoplus_{j\in I(i)} H_j .$$

If we write $V_{\beta_i}\xi = (\bigoplus_{j\in I(i)} V^j\xi)\ \xi\in\mathfrak{H}$, where $V^j \in B(\mathfrak{H},H_j)$, and define $\beta^j(x) = (V^j)^*\pi_j(x)V^j \quad x \in A,\ j \in I(i)$, then

$$\pi_i(x)V_{\beta_i}\xi = \bigoplus_{j\in I(i)} \pi_j(x)\Big(\bigoplus_{j\in I(i)} V^j\xi\Big) = \bigoplus_{j\in I(i)} \pi_j(x)V^j\xi \ \forall\, x\in A.$$

Thus $\langle \beta_i(x^*x)\xi,\xi\rangle = \sum_{j\in I(i)} \langle \beta^j(x^*x)\xi,\xi\rangle \quad x \in A,\ \xi\in\mathfrak{H}$,

and $\langle \beta_i(x^*x)\xi,\xi\rangle = \sum_{i\in I}\sum_{j\in I(i)} \langle\beta^j(x^*x)\xi,\xi\rangle$.

It follows that

$$\alpha = \sum_{i\in I}\sum_{j\in I(i)} \beta^j$$

Moreover since each pair β^j, β^k are quasi equivalent and pure, they are equivalent by [16, 5.3.3].

iii $\Rightarrow$ ii is trivial.

§ 6.4.

We now wish to see how our study of the representations of operator valued weights carries over to the enveloping von Neumann algebra. It is natural to include here the following operator valued version of [7, Lemme 4.3] concerning scalar valued weights. As usual $\mathfrak{H}$ will denote a hilbert space.

PROPOSITION 6.4.1.

Let M be a von Neumann algebra, and $\alpha \in cp(M;\mathfrak{H})$. Then π_α is normal iff $z \longrightarrow \alpha(x^*zx)$ is normal for each x in $\mathcal{N}_\alpha$. In which case, if B denotes the norm closure of $\mathcal{D}_\alpha$, then every element of $CP(B;\mathfrak{H})$ majorised by α, is the restriction to B of an element in $CP^w(M;\mathfrak{H})$.

We then define $cp^w(M;\mathfrak{H})$ to be the set of α in $cp(M;\mathfrak{H})$ such that π_α is normal.

Let A be a C*-algebra. If $w \in CP(A;\mathfrak{H})$, there exists an unique CP linear extension $\hat{w}$ to $\tilde{A}$ the enveloping von Neumann algebra. The support s_w of $\hat{w}$ is called the enveloping support of w, and sc_w denotes the central support of s_w in $\tilde{A}$. $1 - sc_w$ is the largest projection in $\ker \pi_{\tilde{w}}$. If w_1, w_2 are two such maps, they give quasi equivalent representations iff $sc_{w_1} = sc_{w_2}$. We wish to extend this for unbounded completly positive maps, and in this way our results 6.4.1 - 6.4.4 generalise [7, 4.8 - 4.11] concerning scalar valued weights.

Suppose that $F \subseteq CP(A;\mathfrak{H})$ is an $\in$-filtering family, and ß given by

$$ß(x)(\varphi) = \sup_{w\in F} \langle w(x), \varphi\rangle \quad x \in A,\ \varphi \in T(\mathfrak{H})_+$$

is the associated cp additive, positive homogenous map from A^+ into $B(\mathfrak{H})^\wedge_+$. Put $\alpha = \dot{ß}$. Then $\hat{F} = \{\hat{w} : w \in F\}$ is $\in$-filtering, and define

$\tilde{\beta} : (A)^+ \longrightarrow B(\mathfrak{H})^\wedge_+$ by

$$\tilde{\beta}(x)(\varphi) = \sup_{\tilde{w} \in \hat{F}} \langle \tilde{w}(x), \varphi \rangle \qquad x \in (A)^+, \varphi \in T(\mathfrak{H})_+ .$$

We define $\hat{\alpha} = (\tilde{\beta})^\cdot$. Note that for each φ in $T(\mathfrak{H})_+$, $x \longrightarrow \tilde{\beta}(x)(\varphi)$ is ultraweakly lower semi continuous on A. Suppose M is a von Neumann algebra, and $\gamma : M^+ \longrightarrow B(\mathfrak{H})^\wedge_+$ is additive, positive homogenous and such that $x \longrightarrow \gamma(x)(\varphi)$ is u.w. ls.c. on M^+.

Then $n_\gamma = \{x \in M : \gamma(x^*x) = 0\}$ is a σ-weakly closed left ideal in M. If $\delta = \dot{\gamma}$, we let p_δ be the largest projection in n_γ. Then $1 - p_\delta = s_\delta$ is called the support of δ, and sc_δ is the central support of s_δ.

If A is a C*-algebra, and α is the upper envelope of an $\in$-filtering family F in CP(A; $\mathfrak{H}$), we let $s_\alpha = s_{\hat{\alpha}}$, and $sc_\alpha = sc_{\hat{\alpha}}$.

LEMMA 6.4.1.

If M is a von Neumann algebra, and β is the upper envelope of an $\in$-filering family G in CP $(M; \mathfrak{H})$, then $\ker \pi_\beta = \bigcap_{w \in G} \ker \pi_w$, and $1 - sc_\beta$ is the largest projection in $\ker \pi_\beta$.

PROPOSITION 6.4.2.

M is a von Neumann algebra, and $\beta \in cp(M; \mathfrak{H})$ is the upper envelope of an $\in$-filtering family G in $CP^w(M; \mathfrak{H})$. Then the normal representations π_β and $\bigcap_{w \in G} \pi_w$ are quasi equivalent.

COROLLARY 6.4.3.

A is a C*-algebra, and $\alpha \in cp(A; \mathfrak{H})$ is the upper envelope of an $\in$-filtering family F in CP(A; $\mathfrak{H}$) such that $\mathcal{D}_\alpha^- = A$. Then the representations $\tilde{\pi}_\alpha$ and $\pi_{\hat{\alpha}}$ of $\hat{A}$ are quasi equivalent.

PROOF.

$\pi_{\hat{\alpha}}$ and $\bigoplus_{w \in F} \pi_{\tilde{w}}$ are quasi equivalent by 6.4.2.

Moreover, $\hat{\pi}$ and $\bigoplus_{w \in F} \pi_{\tilde{w}}$ are quasi equivalent by 6.3.5.

COROLLARY 6.4.4.

If A is a C*-algebra, $\alpha, \beta \in cp(A; \mathfrak{H})$ such that $\mathcal{D}_\alpha^- = \mathcal{D}_\beta^- = A$, and each the upper envelopes of $\in$ -filtering families in CP(A; $\mathfrak{H}$), then π_α is quasi equivalent to a subrepresentation of π_β (respectively quasi equivalent) iff $sc_\alpha \leq sc_\beta$ (respectively $sc_\alpha = sc_\beta$).

§ 6.5.

Our investigation of the structure of operator valued weights now proceeds by studying those completely positive maps which are invariant under a group of *-automorphisms of the algebra. This work is an extension of van Daele's. If we take the hilbert space $\mathfrak{H}$ to be one dimensional, then our operator valued results 6.5.1 and 6.5.3 - 6.5.17 reduce to 2.1 - 2.4, 2.6, 2.8 - 2.10 of [64] concerning scalar valued weights.

A will always denote a C*-algebra, unless otherwise stated, and $\mathfrak{H}$ a hilbert space. We say that $\alpha \in cp(A; \mathfrak{H})$ is G-invariant, where G is a group of *-automorphisms of A, if $\mathcal{D}_\alpha$ is G-invariant and $\alpha g(a) = \alpha a$ for all a in $\mathcal{D}_\alpha'$. This is equivalent to

$$(\alpha g)^\wedge(a) = \hat{\alpha}(a) \qquad \forall\ g \in G,\ a \in A^+.$$

LEMMA 6.5.1.

Let $\alpha \in cp(A; \mathfrak{H})$ be G-invariant. Then $\mathfrak{N}_\alpha$ and N_α are G-invariant and there exists an unitary representation U_g of G in $\mathcal{K}_\alpha$ such that

i) $\quad U_g \Lambda_\alpha x \otimes \xi = \Lambda_\alpha g(x) \otimes \xi \qquad \forall\ x \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H},\ g \in G.$

ii) $\quad U_g \pi_\alpha(y) U_g^{-1} = \pi_\alpha g(y) \qquad \forall\ g \in G,\ y \in A.$

DEFINITION 6.5.2.

If $\alpha \in cp(A; \mathfrak{H})$ is G invariant, we denote by $\mathcal{F}_0 = \mathcal{F}_0^\alpha$, and $\mathcal{K}_0 = \mathcal{K}_0^\alpha$

the Ginvariant elements of $\mathcal{F} = \mathcal{F}^{\alpha}$ and $\mathcal{K} = \mathcal{K}^{\alpha}$ respectively, where $\mathcal{F}, \mathcal{K}$ are as defined in 6.1.6.

As expected from 6.1.8, there is a natural relation between $\mathcal{F}_o$ and $\mathcal{K}_o$ as the next lemmas show.

LEMMA 6.5.3.

Suppose $\alpha \in cp(A; \mathfrak{H})$ is G-invariant. Then

i) $\mathcal{K}$ is a G invariant left ideal in $\pi_\alpha(A)'$.

ii) $\mathcal{K}_o$ is a left ideal in the fixed point algebra of $\pi_\alpha(A)'$. For any S in $\mathcal{K}_o$, there is a G invariant map $V = V^S$ in $B(\mathfrak{H}, [\pi_\alpha(\mathfrak{N}^*)\mathfrak{K}]^-)$ (i.e. $U_g V = V \quad \forall\, g \in G$) such that for each fixed ξ in $\mathfrak{H}$, $V\xi$ is the unique G invariant vector in $[\pi_\alpha(\mathfrak{N}_\alpha^*)\mathfrak{K}]^-$ such that $S\Lambda_\alpha x \otimes \xi = \pi_\alpha(x)\, V\xi \quad \forall\, x \in \mathfrak{N}_\alpha$.

LEMMA 6.5.4.

Suppose $\alpha \in cp(A; \mathfrak{H})$ is G invariant. Then for any w in $\mathcal{F}_o$, there is an unique S in $\mathcal{K}_o$ such that $0 \leq S \leq 1$, and

$$\langle w(x^*x)\xi, \xi\rangle = \|S\Lambda_\alpha x \otimes \xi\|^2 \quad \forall\, x \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H} .$$

Conversely, for any S in $\mathcal{K}_o$, such that $\|S\| \leq 1$, there is a w in $\mathcal{F}_o$ such that $\langle w(x^*x)\xi, \xi\rangle = \|S\Lambda_\alpha x \otimes \xi\|^2 \quad \forall\, x \in \mathfrak{N}_\alpha,\ \xi \in \mathfrak{H}$.

COROLLARY 6.5.5.

i) Let $\alpha \in cp(A; \mathfrak{H})$ be G invariant with $\mathcal{D}_\alpha$ norm dense in A, then $\mathcal{F}_o$ is ϵ-filtering.

ii) Let A be a von Neumann algebra, $\alpha \in cp^w(A; \mathfrak{H})$ with $\mathcal{D}_\alpha$ ultraweakly dense in A Then $\mathcal{F}_o$ is in $CP^w(A; \mathfrak{H})$ and is ϵ-filtering.

PROOF.

ii) The only non trivial point is to show that $\mathcal{F}_o$ in $CP^w(A; \mathfrak{H})$ is ϵ-filtering. Suppose $\epsilon > 0$, $w_1, w_2 \in \mathcal{F}_o$ and for all a_i in $\mathfrak{N}_\alpha$, $\xi_i \in \mathfrak{H}$

$i = 1, \ldots, n$ that

$$\sum (1-\epsilon) \langle w_1(a_i^* a_j) \xi_j, \xi_i \rangle \leq \sum \langle w_2(a_i^* a_j) \xi_j, \xi_i \rangle .$$

Then $(1-\epsilon)w_1 - w_2 \leqslant 0$ follows from either 6.1.1 or [10 , 2.4] (consider $[a_i^* p\, a_j]$, where p_λ are projections in $\mathfrak{n} \cap \mathfrak{n}^*$ $p_\lambda \uparrow 1$).

This leads us naturally to :

THEOREM 6.5.6.

i) Let $\alpha \in cp(A;\mathfrak{n})$ be G invariant with $\mathcal{D}_\alpha$ norm dense in A. Then there exists a G invariant element α_0 of $cp(A;\mathfrak{n})$ which is the upper envelope of $\mathcal{F}_0$. α_0 majorises any other G invariant element of $cp(A;\mathfrak{n})$ which is majorised by α_0 and is the upper envelope of a G invariant family in $CP(A;\mathfrak{n})$.

ii) Let A be a von Neumann algebra, $\alpha \in cp^w(A;\mathfrak{n})$ with $\mathcal{D}_\alpha$ σ-weakly dense in A. Then there exists a Ginvariant map α_0 which is the upper envelope of $\mathcal{F}_0$. α_0 majorises any other G-invariant map γ in $cp(A;\mathfrak{n})$ which is majorised by α and such that γ is the upper envelope of a G-invariant family in $CP^w(A;\mathfrak{n})$.

Following van Daele's one dimensional theory [64], we will construct a unique normal G-invariant projection map ϕ of the ultraweak closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ onto the ultraweak closure $\overline{\mathcal{K}}_0$ of $\mathcal{K}_0$. It will be the case that ϕ is a projection of $\mathcal{K}$ onto $\mathcal{K}_0$, and $\mathcal{K}^*\mathcal{K}$ onto $\mathcal{K}_0^* \mathcal{K}_0$. It will then be possible to define a unique G invariant projection map ϕ' of $\mathcal{F}$ onto $\mathcal{F}_0$, which is BW continuous on bounded sets.

NOTATION 6.5.7.

Let $\alpha \in cp(A;\mathfrak{n})$ be Ginvariant. We will denote by E_0, the projection onto the fixed points in $\mathcal{R}_\alpha$.

Then $U_g . E_o = E_o . U_g = E_o \;\; \forall g \in G$. Moreover there exists a net of convex combinations $\{\sum \lambda^i(g) U_g\}_{g \in G}$ converging strongly to E_o. We can then show that if $S \in \mathfrak{K}$,

$$\sum \lambda^i(g)\, U_g\, S\, U_g^{-1}$$

converges strongly to an operator ϕS in $\mathfrak{K}_o$. The arguments of [64, Prop3.2] lead us to the following conclusions :

PROPOSITION 6.5.8.

Let $\alpha \in cp(A;\mathfrak{K})$ be G-invariant. There exists a unique normal positive G-invariant projection map ϕ of $\overline{\mathfrak{K}}$ onto $\overline{\mathfrak{K}}_o$ (the σ-weak closures). We have $\phi(S)E_o = E_o S E_o$ for any S in $\overline{\mathfrak{K}}$. In particular, $\phi(\mathfrak{K}) = \mathfrak{K}_o$, $\phi(\mathfrak{K}^*\mathfrak{K}) = \mathfrak{K}_o^* \mathfrak{K}_o$.

COROLLARY 6.5.9.

Let F, F_o be the largest projections in $\overline{\mathfrak{K}}$ and $\overline{\mathfrak{K}}_o$ respectively, (the ultraweak closures). The $\phi F = F_o$.

(6.5.10) REMARK ON COROLLARY 6.5.9.

It is clear from 6.3.2, that α belongs to the closure of $\mathfrak{F}$ (respectively $\mathfrak{F}_o$) for the topology of simple convergence iff $F = 1$ ($F_o = 1$ respectively). If α belongs to the closure of $\mathfrak{F}$ for the topology of simple convergence, it does not follow that α always belongs to the closure of $\mathfrak{F}_o$ for the topology of simple convergence. See [64, 5.2] where $F = 1$, but $F_o = 0$.

PROPOSITION 6.5.11.

Let $\alpha \in cp(A;\mathfrak{K})$ be G invariant. Then there exists a G-invariant projection map ϕ' of $\mathfrak{F}$ into $\mathfrak{F}_o$, satisfying $\phi'(\lambda f) = \lambda \phi'(f)$, and $\phi'(f+g) = \phi'(f) + \phi'(g)$ whenever $\lambda \geq 0$, and $f, g, \lambda f, f+g \in \mathfrak{F}$. If moreover $\mathcal{D}_\alpha$ is norm dense in A, then ϕ' is onto $\mathcal{F}_o$, BW continuous on bounded sets and unique.

REMARK 6.5.12.

Note that when $\mathcal{D}_\alpha$ is dense in A, the formula

$$\langle w(x^*x)\xi , \xi\rangle = \| S_w \Lambda_\alpha x \otimes \xi \|^2 \qquad x \in \mathcal{N}_\alpha , \xi \in \mathfrak{H}$$

gives a bijection $w \longmapsto S^*_w S_w$ between $\mathcal{F}$ and $\{S^*S : \|S\| \leq 1, S \in \mathcal{K}\}$ the elements of $(\mathcal{K}^*\mathcal{K})^+$ with norm ≤ 1; and similarly for $\mathcal{F}_o$ and $\mathcal{K}_o$.
Then on any bounded set in $\mathcal{F}$, $w_\alpha \longrightarrow w$ in the BW topology iff $S^*_{w_\alpha} S_{w_\alpha} \longrightarrow S^*_w S_w$ in the weak operator topology.

Our next results concern a study of the upper envelope α_o constructed in 6.5.6, and the relation of the existence of fixed points in $\mathcal{R}_\alpha$ to the existence of fixed points in $\mathcal{F}$.

THEOREM 6.5.13.

Let $\alpha \in cp(A;\mathfrak{H})$ be G-invariant, and $\mathcal{D}_\alpha$ norm dense in A. Let F_o be the largest projection in the ultraweak closure $\bar{\mathcal{K}}_o$ of $\mathcal{K}_o$, and α_o the upper envelope of $\mathcal{F}_o$. Then

$$\langle \alpha_o(x^*x)\xi , \xi\rangle = \langle F_o \Lambda_\alpha x \otimes \xi , \Lambda_\alpha x \otimes \xi\rangle \quad \text{for all } x \text{ in } \mathcal{N}_\alpha, \xi \text{ in } \mathfrak{H} .$$

COROLLARY 6.5.14.

Let $\alpha \in cp(A;\mathfrak{H})$ be G-invariant with $\mathcal{D}_\alpha$ norm dense in A, and α belonging to the closure of $\mathcal{F}$ for the topology of simple convergence.
Then $F_o = [\pi_\alpha(A) E_o \mathcal{R}_\alpha]^-$. Moreover α majorises no non zero G-invariant element of $CP(A;\mathfrak{H})$ iff $\mathcal{R}_\alpha$ has no non zero fixed points.

COROLLARY 6.5.15.

Let $\alpha \in cp(A;\mathfrak{H})$ be G-invariant with $\mathcal{D}_\alpha^- = A$. Then there is an incresing net $\{\beta_i : i \in I\}$ in $\mathcal{F}_o$ such that

$$(\alpha_o)_n(z) = \sup_i (\beta_i)_n(z) \quad \text{for all } z \text{ in } (\mathcal{D}_{\alpha_o})_n \cap (A_n)^+ .$$

COROLLARY 6.5.16.

Let $\alpha \in cp(A;\mathfrak{H})$ be G invariant with $\mathcal{D}_\alpha^- = A$, and suppose there is a family $\{\beta_i : i \in I\}$ in $\mathcal{F}$ such that $\alpha \subseteq \sum_{i\in I} \beta_i$. Then there is a family $\{\beta_i^o : i \in I\}$ in $\mathcal{F}_o$ such that

$$\alpha_o(x) = \sum_{i\in I} \beta_i^o(x) \qquad \forall\ x \in (\mathcal{D}_\alpha)^+.$$

Our final observations of this section are concerned with Radon Nikodym statements for operator valued weights. If $\alpha \in cp(A;\mathfrak{H})$, we have seen the relation between the set $\{S \in \mathcal{K}^*\mathcal{K} : 0 \leq S \leq 1\}$ and $\mathcal{F} = \{w \in CP(A;\mathfrak{H}) : w \leqslant \alpha\}$, (6.1.8). One may then wonder if for all T in $\pi_\alpha(A)'$, $0 \leq T \leq 1$, there is a β in $cp(A;\mathfrak{H})$ $\beta \leqslant \alpha$ and $\langle \beta(x^*x)\xi,\xi\rangle = \langle T\Lambda_\alpha x \otimes \xi, \Lambda_\alpha x \otimes \xi\rangle \quad \forall\ x \in \mathfrak{N}$,
This is the case if A is a von Neumann algebra. In the C*-algebra situation, the first part of the following theorem, when G is the identity automorphism, gives a partial converse to 6.1.5.

THEOREM 6.5.17.

Let $\alpha \in cp(A;\mathfrak{H})$ be G-invariantwith $\mathcal{D}_\alpha^- = A$. For any G-invariant T in $\pi_\alpha(A)'$, $0 \leq T \leq F_o$, there is a G-invariant element β of $cp(A;\mathfrak{H})$ $\beta \leqslant \alpha$, with $\langle \beta(x^*x)\xi,\xi\rangle = \langle T\Lambda_\alpha x\otimes\xi, \Lambda_\alpha x\otimes\xi\rangle \ \forall x \in \mathfrak{N}_\alpha, \xi \in \mathfrak{H}$.
For any β in $cp(A;\mathfrak{H})$ with $\beta \leqslant \alpha$, and $\beta|_{\mathcal{D}_\alpha}$ belonging to the closure of a family $\mathcal{Q}$ of G-invariant elements of $CP(A;\mathfrak{H})$ majorised by $\beta|_{\mathcal{D}_\alpha}$, there exists an operator T in $\pi_\alpha(A)'$, $0 \leq T \leq F_o$, and

$$\langle \beta(x^*x)\xi,\xi\rangle = \langle T\Lambda_\alpha x\otimes\xi, \Lambda_\alpha x\otimes\xi\rangle \quad \forall\ x \in \mathfrak{N}_\alpha, \xi \in \mathfrak{H}.$$

§ 6.6.

In the previous section, no assumptions were made regarding the automorphism group G. In this section we are concerned with the structure of G-invariant CP maps on C*-algebras, and we discuss in some detail, various cases in which the system possesses some particular properties known under the general name of asymptotic abelianness.

Let $\mathcal{Q}$ be a von Neumann algebra on a hilbert space $\mathcal{K}$, G a group and $g \longrightarrow \alpha_g$ a representation of G by *-automorphisms of $\mathcal{Q}$, which is implemented by a unitary representation U_g of G on $\mathcal{K}$. Let E_o be the projection on the G invariant elements of $\mathcal{K}$.

i.e. $E_o\mathcal{K} = \{ \gamma \in \mathcal{K} : U_g\gamma = \gamma \quad \forall g \in G \}$.

Let $\mathcal{R}$ be the von Neumann algebra generated by $\mathcal{Q}$ and U_G. Then note that $E_o \in$ strong closure $\{\text{conv } U_G\}$, and in particular $E_o \in (U_G)''$.

Throughout this section, $\mathfrak{H}$ denotes a hilbert space and $g \longrightarrow \sigma_g$ is a represenation of a group G by *-automorphisms of a C*-algebra A. If $w \in CP_o(A;\mathfrak{H})$, the G invariant part of $CP(A;\mathfrak{H})$, let U_g^w be the induced representation of G by unitary operators on $\mathcal{K}_w$ such that

i) $\pi_w\sigma_g(a) = U_g^w\,\pi_w(a)\,(U_g^w)^{-1} \qquad \forall\, g \in G,\ a \in A.$

ii) $U_g\Lambda\, a \otimes \xi = \Lambda\,\sigma_g(a) \otimes \xi \qquad \forall\, g \in G,\ a \in A,\ \xi \in \mathfrak{H}.$

The above notation will then be used with w as a suffix for the von Neumann algebra $\mathcal{Q}_w = \pi_w(A)''$, with the automorphism group

$a \longrightarrow U_g^w\, a\, (U_g^w)^{-1} \qquad a \in \mathcal{Q}_w$.

We remark at this point, that if we consider G acting elementwise on A_n (i.e. $g \longrightarrow \sigma_g \otimes 1_n$) and if $C_o(A;\mathbb{C}) \neq 0$, then A_n is never G-abelian if $n \geqslant 2$, as simple arguments using [19, Definition 0] and the non-abelianness of M_n $(n \geqslant 2)$ show.

The next proposition is a generalisation of [53, Prop 6.3.2] .

PROPOSITION 6.6.1.

If $w \in CP_o(A;\mathfrak{H})$ then

a) $\{\pi_w(A) \cup E_o^w\}' = \{\pi_w(A) \cup U_G^w\}'$

b) The mapping $\{\pi_w(A) \cup U_G^w\}' \longrightarrow E_o^w\{\pi_w(A) \cup U_G^w\}'$ is a *-isomorphism.

c) $E_o^w \{ \pi_w(A) \cup U_G^w \}' = E_o^w \{ E_o^w \pi_w(A) E_o^w \}'$

d) $E_o^w \{ E_o^w \pi_w(A) E_o^w \}''$ is the strong closure of $E_o^w \pi_w(A) E_o^w$.

We need the following lemma :

LEMMA 6.6.2. [19]

Let $\mathcal{Q}$ be a von Neumann algebra acting on a hilbert space $\mathcal{K}$, and $g \longmapsto U_g$ a representation of G by unitary operators on $\mathcal{K}$, such that $U_g \mathcal{Q} U_g^{-1} = \mathcal{Q} \;\forall g \in G$. If $E_o \mathcal{Q} E_o$ is abelian and $[\mathcal{Q} E_o] = 1$, there exists an unique normal G-invariant CP linear map M of $\mathcal{Q}$ onto $\mathcal{R} \cap \mathcal{R}'$ such that $M(A)E_o = E_o A E_o \quad \forall A \in \mathcal{Q}$.

Returning to the situation before the lemma, it follows that if A is G-abelian, then for each w in $CP_o(A; \mathfrak{H})$ there is an unique normal G-invariant CP map M of $\pi_w(A)''$ onto $\mathcal{R}_w \cap \mathcal{R}_w'$ such that $M(\pi_w(A')) E_o^w = E_o^w \pi_w(A') E_o^w$, for all A' in A . Note that if A_1 , $A_2 \in$ A, then $M(\pi_w(A_2)) \pi_w(A_1)V_w = \pi_w(A_1) E_o^w \pi_w(A_2) V_w$.

THEOREM 6.6.3.

The following conditions are equivalent, for a fixed hilbert space $\mathfrak{H}$,

1)For all w in $CP_o(A; \mathfrak{H})$, and self adjoint a in A,

[the weak closure of $\pi_w(\text{conv}\, \sigma_G(a))] \cap \pi_w(A)' \neq \emptyset$.

2)For all w in $CP_o(A; \mathfrak{H})$, self adjoint a in A, and finite sets S,T in A and $\mathfrak{H}$ respectively :

$$\inf_{a' \in \text{conv}\, \sigma_G(a)} \; \sup_{\substack{\xi, \eta \in T \\ a_1, a_2 \in S}} |\langle w(a_1[a', a_2])\xi , \eta \rangle| = 0.$$

3)A is G-abelian and for each w in $CP_o(A; \mathfrak{H})$ and self adjoint a in A, $M \pi_w(a)$ is in the weak closure of $\pi_w(\text{conv}\, \sigma_G(a))$.

PROOF.

1) $\Rightarrow$ 2). Take w in $CP_o(A; \mathfrak{H})$ and self adjoint a in A. Choose a net a_α

in conv $\sigma_G(a)$ such that $\pi_w(a_\alpha)$ converges weakly to an operator D in $(\pi_w(A))'$. Then $[\pi_w(a_\alpha), \pi_w(b_j)] \longrightarrow [D, \pi_w(b_j)] = 0$ weakly, whenever $b_1, \dots, b_n$ is a finite set in A. Therefore for a given $\epsilon > 0$, and a finite set T in $\mathfrak{H}$, there exists an α such that

$$|\langle w\, b_i[a_\alpha, b_j]\xi, \eta\rangle| = |\langle [\pi_w(a_\alpha), \pi_w(b_j)]V_w\xi, \pi_w(b_i^*)V_w\eta\rangle| < \epsilon$$

for all $\xi, \eta \in T$ $1 \le i,j \le n$.

Hence

$$\inf_{a' \in \operatorname{conv}\sigma_G(a)} \sup_{\substack{\xi,\eta \in T \\ 1 \le i,j \le n}} |\langle w(b_i[a', b_j])\xi, \eta\rangle| = 0.$$

2) $\Rightarrow$ 3). Let $w \in CP_0(A; \mathfrak{H})$, a be a self adjoint element of A, and T a finite subset of $\mathfrak{H}$. Since $E_0 \in \{\operatorname{conv} U_G\}^-$, it follows that there exists g_i, λ_i in G, $[0,1]$ respectively for $1 \le i \le n$ with $\sum \lambda_i = 1$, and satisfying

$$\| E_0^w \pi_w(a)V_w\xi - \sum_{i=1}^{n} \lambda_i U_{g_i} \pi_w(a)V_w\xi\| < \epsilon \qquad \forall\, \xi \in T,$$

i.e.

$$\| E_0^w \pi_w(a)V_w\xi - \pi_w(a')V_w\xi\| \qquad \forall\, \xi \in T,$$

where $a' = \sum\lambda_i\, \sigma_{g_i}(a) \in \operatorname{conv}\sigma_G(a)$.

Let S be a finite subset in A, then by 2) there exists a'' in $\operatorname{conv}\sigma_G(a')$ such that

$$|\langle \pi_w(a''), \pi_w(b_1)V\xi, \pi_w(b_2)V\eta\rangle| < \epsilon \qquad \forall\, b_1, b_2 \in S, \xi, \eta \in T.$$

Suppose $a'' = \sum_{i=1}^{m} \lambda_i'\ \sigma_{g_i'}(a')$ where $g_i' \in G$, $\lambda_i' \geq 0$ and $\sum\lambda_i' = 1$.

Hence

$$\begin{aligned}
&\langle M(\pi_w(a)\, \pi_w(b_1)V\xi, \pi_w(b_2)V\eta\rangle - \langle \pi_w(a'')\, \pi_w(b_1)V\xi, \pi_w(b_2)V\eta\rangle \\
&= \langle \pi_w(b_1)E_0\, \pi_w(a)V\xi, \pi_w(b_2)V\eta\rangle - \langle \pi_w(a'')\, \pi_w(b_1)V\xi, \pi_w(b_2)V\eta\rangle \\
&= \langle \pi_w(b_1)E_0\, \pi_w(a)V\xi, \pi_w(b_2)V\eta\rangle - \langle \pi_w(b_1)\, \pi_w(a'')V\xi, \pi_w(b_2)V\eta\rangle \\
&\quad + \langle \pi_w(b_1)\pi_w(a'')V\xi, \pi_w(b_2)V\eta\rangle - \langle \pi_w(a'')\, \pi_w(b_1)V\xi, \pi_w(b_2)V\eta\rangle \\
&= \sum_{i=1}^{m} \{\langle \pi_w(b_1)E_0\lambda_i'\, \pi_w(a)V\xi, \pi_w(b_2)V\eta - \lambda_i'\langle \pi_w(b_1)U_{g_i'}\pi_w(a')V\xi, \pi_w(b_2)V\eta\rangle \\
&\quad + \langle [\pi_w(b_1), \pi_w(a'')]V\xi, \pi_w(b_2)V\eta\rangle
\end{aligned}$$

Thus

$$|\langle M(\pi_w(a))\pi_w(b_1)V\xi, \pi_w(b_2)V\eta\rangle - \langle \pi_w(a'')\pi_w(b_1)V\xi, \pi_w(b_2)V\eta\rangle|$$

$$\leq \sum_{i=1}^{n} \|\pi_w(b_1)\| \quad \lambda_i' \quad \|E_o\pi_w(a)V\xi \quad - U_{g_i'}\pi_w(a')V\xi\| \quad \|\pi_w(b_2)V\eta\|$$
$$+ \quad |\langle[\pi_w(b_1), \pi_w(a'')]V\xi \;, \pi_w(b_2)V\eta\rangle|$$
$$\leq \quad \epsilon \;\|\pi_w(b_1)\| \;\; \|\pi_w(b_2)V\eta\| \qquad + \epsilon \qquad\qquad \forall\; \eta\,,\xi \;\in\; T;\; b_1\,,b_2 \in S.$$

Since $[\pi_w(A)V_w\mathfrak{H}]^- = \mathcal{K}_w$, and $\|a'\| \leq \|a\|$ whenever $a' \in \mathrm{conv}\,\sigma_G(a)$, it follows that $M(\pi_w(a))$ is in the weak closure of $\pi_w \mathrm{conv}\,\sigma_G(a)$.

3) $\Rightarrow$ 1). This is trivial, as $\mathcal{R}_w' \subseteq \pi_w(A)'$, $\forall\; w \in CP_o(A;\mathfrak{H})$.

The preceeding theorem is a generalisation of [53,Ex 6C] from the space of invariant positive linear functionals. Note that if A satisfies conditions 1-3 of 6.6.3 then G is a large group of automorphisms for A [19, Definition 2]. The truth of the converse is unclear. If (A,α) is $\mathcal{M}$ abelian [19, Definition 3], then the equivalent conditions 1-3 of Theorem 6.6.3 hold, (which can be seen using the same argument of [19] when showing that if (A,α) is $\mathcal{M}$ abelian, then G is a large group of automorphisms for A). The converse is false, using the same example in [19] which shows that (A,α) being $\mathcal{M}$ abelian is not equivalent to G being a large group of automorphisms for A.

The next result concerns the characterisation of extremal invariant (or ergodic) CP maps, and extends [19] for states.

THEOREM 6.6.4.

Let B be a C*-algebra acting on a hilbert space $\mathcal{K}$, let $\mathcal{Q}$ denote the weak closure of B. Let $g \mapsto U_g$ be a unitary representation of G on $\mathcal{K}$, such that $\sigma_g a = U_g\, a\, U_g^{-1}$ is an automorphism of B for each g in G. Suppose $V \in B(\mathfrak{H},\mathcal{K})$ such that $[BV\mathfrak{H}]^- = \mathcal{K}$ and $V\mathfrak{H} \subseteq E_o\mathcal{K}$. Then if χ_o denotes any unit vector in E_o,

a) The following are equivalent :

i) $w(x) = V^*xV \quad x \in B$, is an extremal element of $CP_o(B;\mathfrak{H})$.

ii) $\mathcal{R}'$ is the scalars.

b) If $E_o\mathcal{Q}E_o$ is abelian, the conditions in a) are equivalent also to the following :

iii) $E_o \mathcal{R}$ is one dimensional

iv) $M(a) = \langle a\chi_o, \chi_o \rangle 1 \quad \forall a \in A,$

in which case $w(a) = \langle a\chi_o, \chi_o \rangle (V^*\chi_o) \otimes (\overline{V^*\chi_o}) \quad \forall a \in B.$

c) If furthermore, $(\mathcal{A}, \alpha)$ satisfy the hypothesis of [19, Lemma 3], then the above conditions are equivalent to :

v) If β is a normal positive G-invariant map from $\mathcal{A}$ into another von Neumann algebra M, then

$$\beta(a) = \langle a\chi_o, \chi_o \rangle K, \forall a \in \mathcal{A}, \text{ and some } K \text{ in } M_+.$$

COROLLARY 6.6.5.

If $\mathcal{A}$ is a factor and $E_o \mathcal{A} E_o$ is abelian and $\mathcal{R} \cap \mathcal{R}' \subseteq \mathcal{A}$ then w is an extremal element of $CP_o(A; \mathfrak{H})$ and can be written as

$$w(a) = \langle a\chi_o, \chi_o \rangle (V^*\chi_o) \otimes (\overline{V^*\chi_o}) \quad \forall a \in B.$$

PROPOSITION 6.6.6.

Let A be G-abelian. Then if $w \in CP_o(A; \mathfrak{H})$

$$\{ \alpha \in CP_o(A; \mathfrak{H}) : \alpha \leq w \}$$

is a lattice.

In particular if A has an identity, and $\mathfrak{H}$ finite dimensional, then $\{ w \in CP_o(A; \mathfrak{H}) : \operatorname{tr} K_w = 1 \}$ forms a Choquet simplex. In this situation, an invariant CP map can be written in an unique way as an integral(in some sense) over"ergodic" CP maps.

We complete this section by returning to the general situation, to state a generalisation of a result of Arveson [4] when G is the identity automorphism.

THEOREM 6.6.7.

If $K \in B(\mathfrak{H})_+$, then $w \in CP_o(A; \mathfrak{H}) \cap CP(A; \mathfrak{H}, K)$ is an extremal element in this cone iff $[V_w \mathfrak{H}]^-$ is a faithful subspace for $\mathcal{R}'_w$.

§ 6.7.

We are now going to make some observations regarding the classification of the spectrum of a one parameter group of *-automorphisms α_t of a C*-algebra A with invariant operator valued weight w in cp(A; $\mathfrak{H}$), where $\mathfrak{H}$ is a hilbert space.

Suppose that

$$t \longmapsto \langle w(a^* \alpha_t(b)a)\xi , \xi \rangle$$

is continuous on $\mathbb{R}$ for all a in $\mathfrak{N}_w$, b in A, ξ in $\mathfrak{H}$. Then there is a strongly continuous, unitary representation U_t of $\mathbb{R}$ on $\mathcal{K}_w$ such that

1) $\pi_w \alpha_t(a) = U_t \pi_w(a) U_t^{-1}$ $\forall$ a $\in$ A, t $\in \mathbb{R}$.

2) $U_t \Lambda_w a \otimes \xi = \Lambda_w \sigma_t(a) \otimes \xi$ $\forall$ a $\in \mathfrak{N}_w$, t $\in \mathbb{R}$, $\xi \in \mathfrak{H}$.

Let $U_t = e^{iKt}$ where K is a self adjoint operator on $\mathcal{K}_w$.

THEOREM 6.7.1.

In the situation described above :

a) The following conditions are equivalent : (c.f. [11, Lemme 3.7.4])

A1 $t \longrightarrow \langle w(a^* \alpha_t(a))\xi,\xi\rangle$ is almost periodic on $\mathbb{R}$ for all a in $\mathfrak{N}_w$, $\xi \in \mathfrak{H}$.

A2 K has pure point spectrum.

b) The following conditions are equivalent :

B1 $t \longrightarrow \langle w(a^* \alpha_t(a)\xi,\xi\rangle \in L^1(\mathbb{R})^\wedge$, for all a in $\mathfrak{N}_w$, ξ in $\mathfrak{H}$.

B2 K has absolutely continuous spectrum.

c) The following two conditions are equivalent :

C1 $\int_{-\infty}^{\infty} \langle w(a^* \alpha_t(a))\xi , \xi\rangle \, h(t)\, dt = 0$

for all $\xi \in \mathfrak{H}$, a in $\mathfrak{N}_w$, and all continuous L^1 functions h whose fourier transform vanishes on the negative real axis.

C2 $K \geqslant 0$.

If moreover, α_t is strongly continuous with infinitesimal generator δ and w is bounded, then C1 and C2 are also equivalent to the following :

C3 $\left[w(a_i^* \delta a_j) \right] \geq 0$ for all $\{ a_i : 1 \leq i \leq n\}$ in $\mathcal{D}$, where $\mathcal{D}$ is a core for δ .

Note that the equivalence of C2 and C3 when $\mathfrak{H} = \mathbb{C}$ is contained in [47, Theorem 2.2] and the same argument holds here.

Now let μ be a $^{+}$Borel measure on $\mathbb{R}$, and put $\mathfrak{H}_0 = L^2(\mathbb{R},\mu)$. Define a strongly continuous unitary group e^{iHt} on $\mathfrak{H}_0$ by

$$(e^{iHt}f)(s) = e^{ist}f(s) \qquad f \in \mathfrak{H}_0 , \quad s,t \in \mathbb{R} .$$

Put $w = tr$ on $A = B(\mathfrak{H}_0)$, so that $\mathcal{D}_w = TC(\mathfrak{H}_0)$, and $\mathfrak{N}_w = HS(\mathfrak{H}_0)$.
Also define $\alpha_t(x) = e^{iHt}xe^{-iHt}$
to be the induced one parameter group of automorphisms of A. Hence we have a strongly continuous one parameter unitary group e^{iKt} on $\mathcal{K}_w = HS(\mathfrak{H}_0)$ such that

$$\pi_w(e^{iHt}xe^{-iHt}) = e^{iKt}\pi_w(x)e^{-iKt} \quad t \in \mathbb{R},\ x \in A.$$

LEMMA 6.7.2.
If $\mu * \tilde{\mu}$ is absolutely continuous with respect to Lebesgue measure, where $\tilde{\mu}(E) = \mu(-E)$, then conditions B1-B2 of 6.7.1 hold.

PROOF.
It is enough to prove that if $\xi \in \mathfrak{H}_0$, then

$$t \longmapsto \langle e^{-iHt}\xi , \xi \rangle . \langle e^{iHt}\xi , \xi \rangle \in L^1(\mathbb{R})^\wedge .$$

We do this by showing that if ν is a positive Borel measure on $\mathbb{R}$, with $\nu \ll \mu$, then $\nu * \tilde{\nu}$ is a.c. w.r.t. Lebesgue measure. Define a partial ordering on the positive Borel measures on $\mathbb{R}$, by saying $\mu_1 \leq \mu_2$ if $\mu_1(E) \leq \mu_2(E)$ $\forall$ Borel sets E. Then if $\nu_i \leq \mu_i$ $i = 1,2$ it follows that $\nu_1 * \tilde{\nu}_2 \leq \mu_1 * \tilde{\mu}_2$.

Now $\nu E = \int_E f\, d\mu$ where $f \in L^1(\mathbb{R},\mu)$.

Then $f = \sum_{n=1}^{\infty} f_n$, where $0 \leq f_n \leq 1$, and f_n are measureable.

Hence $\nu = \sum \nu_n$ where $\nu_n \leq \mu$.

Thus $\nu * \tilde{\nu} = \sum_{n,m} \nu_n * \tilde{\nu}_m$, and $\nu_n * \tilde{\nu}_m \leq \mu * \tilde{\mu}$.

Thus $\nu_n * \tilde{\nu}_m \ll \mu * \tilde{\mu}$, and it follows that $\nu * \tilde{\nu}$ is a.c. w.r.t.

Lebesgue measure.

If μ is absolutely continuous with respect to Lebegue measur, then so is $\mu * \tilde{\mu}$, but the converse is false. According to Salem [18, p 265], for each $\alpha \in (o,1)$, and $q > 2/\alpha$, there exists a positive Radon measure μ on $\mathbb{R}$, supported by a compact subset K of $[o,1]$ having Hausdorff dimension α such that $\hat{\mu}(\xi) \in L^q$. This implies that K is of Lebesgue measure zero, and μ is a singular measure. Choose $q = 4$, and $1/2 < \alpha < 1$. Then $|\hat{\mu}|^2 = (\mu * \tilde{\mu})^{\wedge} \in L^2$. Hence $\mu * \tilde{\mu}$ is absolutely continuous with repsect to Lebesgue measure, but μ itself is a singular measure.

COROLLARY 6.7.3.

There exists a hilbert space $\mathfrak{H}_o$, a self adjoint operator H on $\mathfrak{H}_o$ with singularly continuous spectrum, a faithful normal representation π of $B(\mathfrak{H}_o)$ on a hilbert space $\mathfrak{K}$, and a self adjoint operator K on $\mathfrak{K}$ with absolutely continuous spectrum such that

$$\pi(e^{iHt}xe^{-iHt}) = e^{iKt}\pi(x)e^{-iKt} \qquad \forall \; x \in B(\mathfrak{H}_o), \quad t \in \mathbb{R}.$$

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