

# ON THE H-SYSTEMS IN HIGHER DIMENSION

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ABSTRACT. We prove a regularity result for the weak solutions of H-systems in dimensions  $n \geq 3$ .

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## 1. INTRODUCTION

Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ . For a given bounded Borel function  $H: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  the corresponding  $H$ -system on  $\Omega$  is the elliptic system

$$(1.1) \quad -\operatorname{div}(|\nabla u|^{n-2} \nabla u) = n^{\frac{n}{2}} H(u) J_u$$

where  $u: \Omega \rightarrow \mathbb{R}^{n+1}$  is of Sobolev class  $W^{1,n}$  and  $J_u = u_{x_1} \wedge \cdots \wedge u_{x_n}$  is the *cross product* of the partial derivatives  $u_{x_j}$  of  $u$ . Recall that the cross product  $v_1 \wedge \cdots \wedge v_n$  of  $n$  vectors  $v_1, \dots, v_n$  in  $\mathbb{R}^{n+1}$  is defined as the unique vector in  $\mathbb{R}^{n+1}$  satisfying for all  $w \in \mathbb{R}^{n+1}$ ,

$$w \cdot (v_1 \wedge \cdots \wedge v_n) = \det \begin{bmatrix} w^1 & \cdots & w^{n+1} \\ v_1^1 & \cdots & v_1^{n+1} \\ \vdots & & \vdots \\ v_n^1 & \cdots & v_n^{n+1} \end{bmatrix}.$$

where we write  $v_j = (v_j^1, \dots, v_j^{n+1})$ ,  $w = (w^1, \dots, w^{n+1})$ .

**Definition 1.1.** A map  $u \in W^{1,n}(\Omega, \mathbb{R}^{n+1})$  is a weak solution to (1.1) if, for any test map  $\varphi \in C_c^\infty(\Omega, \mathbb{R}^{n+1})$ ,

$$(1.2) \quad \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot \nabla \varphi \, dx = n^{\frac{n}{2}} \int_{\Omega} H(u) \varphi \cdot J_u \, dx$$

holds.

**Remark 1.2.** By approximation it can be easily seen that equation (1.2) can be tested by any map  $\varphi \in W_0^{1,n}(\Omega, \mathbb{R}^{n+1}) \cap L^\infty(\Omega, \mathbb{R}^{n+1})$ .

It is well-known that if, in addition to (1.1),  $u$  is  $C^2$  and *conformal* meaning that for some nonnegative function  $\lambda$ ,

$$u_{x_i} \cdot u_{x_j} = \lambda \delta_{ij}$$

holds on  $\Omega$  for all  $i, j$ , then the image  $u(\Omega)$  is a surface whose mean curvature is  $H(u(x))$  at each point  $u(x)$  where  $J_u(x) \neq 0$ . For  $n = 2$  this observation is the starting point for most existence results for parametric surfaces of prescribed mean curvature, see [4, 25]. The conformality condition will play no direct role in the present paper.

The  $H$ -system (1.1) is the Euler-Lagrange system for the variational integral

$$\mathcal{E}(u) = \int_{\Omega} \left( \frac{1}{n} |\nabla u|^n + n^{\frac{n}{2}} Q_H(u) \cdot J_u \right) dx$$

where  $Q_H: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  is any vector field such that  $\operatorname{div} Q_H = H$  on  $\mathbb{R}^{n+1}$ . This has been used to prove existence of solutions by either constrained minimization (small solutions) or methods based on Mountain Pass type theorems (large solutions), see for instance [6, 10, 20] and [4], [25, Ch. III.5] for a

more comprehensive discussion. Existence has also recently been deduced by use of a heat flow for the  $H$ -system [18].

Here we shall focus entirely on the regularity of weak solutions to the  $H$ -system (1.1) in dimension  $n \geq 3$  under additional assumptions on the function  $H$  or the weak solution  $u$ . The regularity of (1.1) in the two dimensional case  $n = 2$  has been well-studied and culminated in Rivière's solution of the Heinz conjecture in [21]: under the sole assumption of boundedness of  $H$ , the weak solution defined in Definition 1.1 is continuous. Many authors contributed with important results prior to this, including [16, 26, 28, 3, 15, 2]. The higher dimensional case  $n \geq 3$  seems to be more difficult and the results are far from being conclusive. We refer to [22] for a more comprehensive discussion of the background literature. In this connection remark that (1.1) is a nonlinear degenerate elliptic system with critical growth nonlinearity  $H(u)J_u$  on the right-hand side that is merely integrable for maps  $u$  of class  $W^{1,n}$ . It is well-known that such systems in general admit very singular solutions and that a general regularity theory is only possible provided the right-hand side nonlinearity has a special structure (see for instance [1]). As mentioned, this is fully exploited for the system in two dimensions  $n = 2$  by Rivière's result, and the hope is that the structure of  $H(u)J_u$  remains strong enough to ensure regularity also for  $n \geq 3$ , at least when backed up by appropriate additional hypotheses on  $H$ . We mention the results of Mou and Yang [20] who proved that solutions of (1.1) are  $C^1$ , provided that either  $H$  is constant or  $u$  is weakly conformal. Closer to our result is that of Wang [27] who proved a higher-dimensional generalization of Heinz's result [16] under the assumption

$$(1.3) \quad \sup_{y \in \mathbb{R}^{n+1}} \left( |H(y)| + (1 + |y|)|H'(y)| \right) < \infty.$$

His proof is based on the coarea formula and thus follows to some extent the proof of Bethuel [2] for the two dimensional case. Here we are able to prove the following result:

**Theorem 1.3.** *Let  $H : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a locally Lipschitz function satisfying the two conditions:*

$$(1.4) \quad y \mapsto H(y)y \text{ is uniformly continuous on } \mathbb{R}^{n+1}$$

*and for some exponent  $q > 1$ ,*

$$(1.5) \quad \sup_{y \in \mathbb{R}^{n+1}} \left( |H(y)| + \frac{|H'(y)|}{1 + |y|^q} \right) < \infty.$$

*Then any weak solution in the sense of Definition 1.1 is of class  $C_{\text{loc}}^{1,\alpha}$  on  $\Omega$  for some  $\alpha \in (0, 1)$ .*

We refer to Section 4 for examples of functions  $H$  that fail to satisfy the Heinz-Wang condition (1.3) but do satisfy our conditions (1.4)-(1.5). The conditions (1.4)-(1.5) serve mainly to ensure that the weak solutions are locally bounded. Indeed if we assume *a priori* that the weak solution is locally bounded, then much weaker conditions on  $H$  suffice to conclude higher regularity:

**Proposition 1.1.** *Let  $H : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be continuous. Then any locally bounded weak solution to (1.1) is locally Hölder continuous. Furthermore, if  $H$  is locally Lipschitz, then any locally bounded weak solution to (1.1) is of class  $C_{\text{loc}}^{1,\alpha}$  on  $\Omega$  for some  $\alpha \in (0, 1)$ .*

The proofs of Theorem 1.3 and Proposition 1.1 are given in Section 3.

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## 2. PRELIMINARY RESULTS

We start this section by recalling some notions from harmonic analysis that allow us to take advantage of the Jacobian structure on the right-hand side of (1.1). In the process of doing that we also fix our notation. As regards function spaces and Sobolev functions it is standard and follows [19, 29]. On

$\mathbb{R}^n, \mathbb{R}^{n+1}$  and matrix space  $\mathbb{R}^{(n+1) \times n}$  we use standard euclidean inner products and the corresponding norms. In all cases denoted by  $X \cdot Y$  and  $|X| = \sqrt{X \cdot X}$ , respectively, the meaning being clear from the context.

Recall that a function  $f \in L^1(\mathbb{R}^n)$  belongs to the *Hardy space*  $\mathcal{H}^1(\mathbb{R}^n)$  if

$$f_* := \sup_{\varepsilon > 0} |\phi_\varepsilon * f| \in L^1(\mathbb{R}^n),$$

where as usual we let  $\phi_\varepsilon(x) := \varepsilon^{-n} \phi(x/\varepsilon)$  for a fixed nonnegative function  $\phi \in C_c^\infty(B_1(0))$  with  $\int \phi(y) dy = 1$ . The definition does not depend on the choice of  $\phi$  and the expression  $\|f\|_{\mathcal{H}^1} := \|f_*\|_1$  defines a norm in  $\mathcal{H}^1(\mathbb{R}^n)$ . Hereby  $\mathcal{H}^1(\mathbb{R}^n)$  is a Banach space whose dual can be identified with the John-Nirenberg space BMO of functions of *bounded mean oscillation*. To define  $\text{BMO}(\mathbb{R}^n)$  denote for  $g \in L^1_{\text{loc}}(\mathbb{R}^n)$  and a ball  $B$  in  $\mathbb{R}^n$  the integral mean of  $g$  on  $B$  by

$$g_B := \frac{1}{\mathcal{L}^n(B)} \int_B g \, dx := \oint_B g \, dx.$$

Now  $g \in \text{BMO}(\mathbb{R}^n)$  provided  $g \in L^1_{\text{loc}}(\mathbb{R}^n)$  and

$$\|g\|_{\text{BMO}} := \sup_B \oint_B |g - g_B| \, dx$$

is finite, where the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ . The importance of BMO here is due to the elementary fact that any Sobolev function of class  $W^{1,n}(\mathbb{R}^n)$  in particular is BMO:  $\|g\|_{\text{BMO}} \leq c \|\nabla g\|_{L^n}$  for a dimensional constant  $c = c(n)$  by Poincaré's inequality.

On the other hand, Coifman, Lions, Meyer and Semmes in [9] established the crucial connection between the Jacobian determinant of a map in  $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$  and the Hardy space  $\mathcal{H}^1(\mathbb{R}^n)$  that we also record here for later reference:

**Theorem 2.1.** *If  $f \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ , then  $\det \nabla f \in \mathcal{H}^1(\mathbb{R}^n)$  and*

$$(2.1) \quad \|\det \nabla f\|_{\mathcal{H}^1} \leq C \|\nabla f\|_{L^n}^n$$

where  $C = C(n)$ .

Finally, as already mentioned BMO can be identified with the dual space of  $\mathcal{H}^1$ , and this is conveniently expressed through Fefferman's duality inequality [12]:

$$(2.2) \quad \int_{\mathbb{R}^n} f g \, dx \leq c \|f\|_{\mathcal{H}^1} \|g\|_{\text{BMO}}$$

valid for all  $f \in \mathcal{H}^1, g \in \text{BMO}$ . In general, the integral on the left-hand side of (2.2) does not converge, but there are a number of ways to give meaning to it [23].

Having disposed of these harmonic analysis tools we turn to the uniform continuity assumption (1.4) in Theorem 1.3. Define the truncation at level  $k > 0$  by  $T_k(u) := u \psi_k(|u|)$ , where

$$(2.3) \quad \psi_k(t) := \begin{cases} 1 & \text{if } t \leq k \\ \frac{k}{t} & \text{if } t > k. \end{cases}$$

**Lemma 2.2.** *If  $y \mapsto H(y)y$  is uniformly continuous and  $H$  is bounded, then there exists an increasing concave modulus of continuity  $\omega: [0, \infty) \rightarrow [0, \infty)$  such that*

$$|H(y_1)T_k(y_1) - H(y_2)T_k(y_2)| \leq \omega(|y_1 - y_2|)$$

holds for all  $y_1, y_2 \in \mathbb{R}^{n+1}$  and all  $k > 0$ .

*Proof.* Let  $\theta$  denote a modulus of continuity for  $y \mapsto H(y)y$ . We can assume that  $\theta$  is concave and increasing. We get for  $y_1, y_2 \in \mathbb{R}^{n+1}, |y_2| \leq |y_1|, k > 0$ :

$$\begin{aligned} |H(y_1)T_k(y_1) - H(y_2)T_k(y_2)| &\leq |H(y_1)y_1 - H(y_2)y_2| \psi_k(|y_1|) \\ &\quad + |H(|y_2|)y_2| |\psi_k(|y_1|) - \psi_k(|y_2|)| \\ &\leq \theta(|y_1 - y_2|) + C|y_2| |\psi_k(|y_1|) - \psi_k(|y_2|)|. \end{aligned}$$

If  $|y_1|, |y_2| \leq k$  we get  $|\psi_k(|y_1|) - \psi_k(|y_2|)| = 0$ . If  $|y_1|, |y_2| > k$ , then the last term becomes

$$C|y_2| \left| \frac{k}{|y_1|} - \frac{k}{|y_2|} \right| = C|y_2|k \frac{||y_1| - |y_2||}{|y_1||y_2|} \leq c|y_1 - y_2|.$$

If  $|y_1| > k$  and  $|y_2| \leq k$ , then the last term can be estimated as

$$C|y_2| \left| \frac{k}{|y_1|} - 1 \right| = C \frac{|y_2|}{|y_1|} |k - |y_1|| \leq C[|y_1| - k] \leq C||y_1| - |y_2|| \leq C|y_1 - y_2|.$$

Thus we have shown that we may take  $\omega(t) = \theta(t) + C|t|$ , concluding the proof.  $\square$

**Lemma 2.3.** *For  $u \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^{n+1})$  we have for any ball  $B \Subset \Omega$  and any  $k > 0$  that*

$$\int_B |T_k(u)H(u) - (T_k(u)H(u))_B| dx \leq c\omega\left(\left(\int_B |\nabla u|^n dx\right)^{\frac{1}{n}}\right),$$

where  $c = c(n)$  is a constant,  $\omega$  is determined in Lemma 2.2 and  $|\cdot|$  denotes standard euclidean norm (in  $\mathbb{R}^{n+1}$  and in  $\mathbb{R}^{(n+1) \times n}$ , respectively).

*Proof.* We simply estimate:

$$\begin{aligned} \int_B |T_k(u)H(u) - (T_k(u)H(u))_B| dx &\leq 2 \int_B |T_k(u)H(u) - T_k(u_B)H(u_B)| dx \\ &\stackrel{\text{Lemma 2.2}}{\leq} 2 \int_B \omega(|u - (u)_B|) dx \\ &\stackrel{\text{Jensen}}{\leq} 2\omega\left(\int_B |u - (u)_B| dx\right) \\ &\stackrel{\text{Poincaré-Wirtinger}}{\leq} 2\omega\left(C\left(\int_B |\nabla u|^n dx\right)^{\frac{1}{n}}\right). \end{aligned}$$

Finally since  $C \geq 1$  (without loss of generality) and  $\omega$  is concave we have for each  $t > 0$  that  $\omega(Ct)/Ct \leq \omega(t)/t$  and therefore  $\omega(Ct) \leq C\omega(t)$ , thus concluding the proof.  $\square$

The next result is well-known, but as we require it in a special form we prefer to state and derive it explicitly:

**Lemma 2.4.** *There exists a dimensional constant  $C_n$  with the following property. For each  $u \in W^{1,n}(B_r(x_0))$  there exists  $\bar{u} \in W_{\text{loc}}^{1,n}(\mathbb{R}^n)$  such that  $\bar{u} = u$  on  $B_r(x_0)$  and*

$$(2.4) \quad \int_{\mathbb{R}^n} |\nabla \bar{u}|^n dx \leq C_n \int_{B_r(x_0)} |\nabla u|^n dx.$$

Furthermore we can arrange that

$$(2.5) \quad \sup_{x \in \mathbb{R}^n \setminus B_r(x_0)} |\bar{u}(x)| \leq |u_{B_r(x_0)}| + \sup_{x \in \partial B_r(x_0)} |u(x)|$$

where the right-hand side could be infinite (making the bound vacuous).

*Proof.* Considering the function  $x \mapsto u(x_0 + rx)/r$  instead of  $u$  we may assume that  $B_r(x_0) = B_1(0) =: B$ , the open unit ball in  $\mathbb{R}^n$ . For Lipschitz  $v: \bar{B} \rightarrow \mathbb{R}$  we put  $m = \sup_{\partial B} |v|$  and

$$V(x) = \begin{cases} v(x) & \text{if } x \in B \\ T_m\left(v\left(\frac{x}{|x|^2}\right)(2 - |x|)^+\right) & \text{if } x \in \mathbb{R}^n \setminus B, \end{cases}$$

where  $T_m: \mathbb{R} \rightarrow \mathbb{R}$  is the 1-dimensional version of the truncation map defined above (so  $T_m(t) := \max\{\min\{t, m\}, -m\}$ ). Then  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  is Lipschitz,  $V = v$  on  $B$ ,  $V = 0$  on  $\mathbb{R}^n \setminus 2B$  and  $|V| \leq m$  on  $\mathbb{R}^n \setminus B$ . For  $1 < |x| < 2$  with  $|v(\frac{x}{|x|^2})|(2 - |x|) < m$  we have almost everywhere

$$\nabla V(x) = (2 - |x|)\nabla v\left(\frac{x}{|x|^2}\right) \frac{I - 2\frac{x \otimes x}{|x|^2}}{|x|^2} - v\left(\frac{x}{|x|^2}\right) \frac{x}{|x|},$$

while if  $|v(\frac{x}{|x|^2})|(2 - |x|) \geq m$  we have  $\nabla V(x) = 0$  almost everywhere. Thus after a routine estimation, involving also a change of variables, we find

$$(2.6) \quad \int_{\mathbb{R}^n} |\nabla V|^n dx \leq c \int_B (|\nabla v|^n + |v|^n) dx$$

for some dimensional constant  $c = c(n)$ . It follows by approximation that the above construction and listed properties extend by continuity to functions  $v \in W^{1,n}(B)$ : there exists  $V \in W^{1,n}(\mathbb{R}^n)$  so  $V|_B = v$ ,  $V|_{\mathbb{R}^n \setminus 2B} = 0$ ,  $\sup_{\mathbb{R}^n \setminus B} |V| \leq \sup_{\partial B} |v|$  and (2.6) hold. To complete the proof let  $u \in W^{1,n}(B)$  and put  $v = u - u_B$ , where  $u_B$  is the integral mean of  $u$  over  $B$ . Then  $\bar{u} = V + u_B$  belongs to  $W_{\text{loc}}^{1,n}(\mathbb{R}^n)$  and the desired bounds follow by combination of the above and Poincaré's inequality.  $\square$

A useful feature of the determinant expression  $J_u$  on the right-hand side of (1.1) involves also the truncation maps  $T_k$  defined above and is summarized in the next

**Lemma 2.5.** *For  $\varphi \in C_c^1(\Omega)$ ,  $u \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^{n+1})$  and  $k > 0$*

$$\varphi^n T_k(u) \cdot J_u = T_k(u) \cdot J_{\varphi u}.$$

*Proof.* By definition the right-hand side equals

$$T_k(u) \cdot J_{\varphi u} = \begin{bmatrix} T_k(u)^1 & \cdots & T_k(u)^{n+1} \\ \varphi_{x_1} u^1 + \varphi u_{x_1}^1 & \cdots & \varphi_{x_1} u^{n+1} + \varphi u_{x_1}^{n+1} \\ \vdots & & \vdots \\ \varphi_{x_n} u^1 + \varphi u_{x_n}^1 & \cdots & \varphi_{x_n} u^{n+1} + \varphi u_{x_n}^{n+1} \end{bmatrix}.$$

Here the vectors  $T_k(u(x))$  and  $\varphi_{x_j}(x)u(x)$  are proportional for a.e.  $x$  and each  $1 \leq j \leq n$ , so by elementary properties of the determinant the result follows.  $\square$

### 3. PROOF OF THE THEOREM 1.3

We start by proving the local boundedness of a solution, which as mentioned in the Introduction is the main content of Theorem 1.3.

**Theorem 3.1.** *Under the assumptions of Theorem 1.3 any weak solution is locally bounded.*

*Proof.* We can split the proof in three principal steps.

**Step 1.** Let  $B_r(x_0)$  be such that  $B_{2r}(x_0) \Subset \Omega$  and

$$(3.1) \quad \int_{B_{2r}(x_0)} |\nabla u|^n dx \leq \varepsilon_0,$$

where  $\varepsilon_0 > 0$  will be chosen in the course of the proof. Without loss of generality we can assume  $x_0 = 0$  and denote  $B_r(x_0)$  and  $B_{2r}(x_0)$  simply by  $B_r$  and  $B_{2r}$ , respectively. Let us denote by  $\bar{u}$  the extension of  $u$  from  $B_{2r}$ , given in Lemma 2.4.

Let  $\varphi \in W_0^{1,n}(B_{2r}) \cap L^\infty(B_{2r})$ , extended by 0 outside  $B_{2r}$  and still denoted by  $\varphi$ ,  $\varphi \geq 0$ , such that  $\int_{B_{2r}} |\nabla u|^{n-1} |\nabla \varphi| |u| dx < \infty$  and  $\int_{B_{2r}} |\nabla \varphi|^n |u|^n dx < +\infty$ . We observe that  $T_k(u) \varphi^n \in W_0^{1,n}(\Omega, \mathbb{R}^{n+1}) \cap L^\infty(\Omega, \mathbb{R}^{n+1})$  and so we can use it as test function in the equation satisfied by  $u$  and write the integrals on the whole  $\mathbb{R}^n$  using the extensions of  $u$  and  $\varphi$  outside  $B_{2r}$ , respectively. We get

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \bar{u}|^{n-2} \nabla \bar{u} \cdot \nabla (T_k(\bar{u})) \varphi^n dx &+ n \int_{\mathbb{R}^n} |\nabla \bar{u}|^{n-2} \nabla \bar{u} \cdot (T_k(\bar{u}) \otimes \nabla \varphi) \varphi^{n-1} dx \\ &= n^{\frac{n}{2}} \int_{\mathbb{R}^n} H(\bar{u}) \varphi^n T_k(\bar{u}) \cdot J_{\bar{u}} dx \\ &= n^{\frac{n}{2}} \int_{\mathbb{R}^n} H(\bar{u}) T_k(\bar{u}) \cdot J_{\varphi \bar{u}} dx, \end{aligned}$$

where the last equality follows by Lemma 2.5. Using (2.2), (2.1) and Lemma 2.3, we can estimate the last term as

$$\begin{aligned}
\int_{\mathbb{R}^n} H(\bar{u}) T_k(\bar{u}) \cdot J_{\varphi \bar{u}} dx &\leq C \|H(\bar{u}) T_k(\bar{u})\|_{\text{BMO}} \|\nabla(\varphi \bar{u})\|_{L^n(\mathbb{R}^n)}^n \\
&\leq C \omega \left( \int_{\mathbb{R}^n} |\nabla \bar{u}|^n \right)^{\frac{1}{n}} \|\nabla(\varphi \bar{u})\|_{L^n(\mathbb{R}^n)}^n \\
&\leq C \omega \left( \int_{B_{2r}} |\nabla u|^n dx \right)^{\frac{1}{n}} \|\nabla(\varphi u)\|_{L^n(B_{2r})}^n \\
&\leq C \omega(\varepsilon_0) \|\nabla(\varphi u)\|_{L^n(B_{2r})}^n.
\end{aligned}$$

Then, thanks to the hypotheses on  $\varphi$ , we can pass to the limit as  $k \rightarrow \infty$  to get

$$\int_{B_{2r}} |\nabla u|^n \varphi^n dx + n \int_{B_{2r}} |\nabla u|^{n-2} \nabla u \cdot (u \otimes \nabla \varphi) \varphi^{n-1} dx \leq C \omega(\varepsilon_0) \int_{B_{2r}} |\nabla(\varphi u)|^n dx.$$

**Step 2.** Now we can choose  $\varphi = T_k(|u|)^p \eta$ , where  $p \geq 1$  and  $\eta \in C_c^\infty(B_{2r})$  is a cut off function between  $B_r$  and  $B_{2r}$ , in the previous estimate. Then we find

$$\begin{aligned}
(3.2) \quad &\int_{B_{2r}} |\nabla u|^n T_k(|u|)^{pn} \eta^n dx + np \int_{\{|u| \leq k\} \cap B_{2r}} |\nabla u|^{n-2} \nabla u \cdot (u \otimes (\nabla u)) |u|^{pn-2} \eta^n dx \\
&+ n \int_{B_{2r}} |\nabla u|^{n-2} \nabla u \cdot (u \otimes \nabla \eta) T_k(|u|)^{pn} \eta^{n-1} dx \\
&\leq C \omega(\varepsilon_0) \left[ \int_{B_{2r}} \left( |\nabla u|^n T_k(|u|)^{pn} \eta^n + |\nabla \eta|^n T_k(|u|)^{pn} |u|^n \right) dx \right. \\
&\quad \left. + p^n \int_{\{|u| \leq k\} \cap B_{2r}} |u|^{(p-1)n} |u| |\nabla u|^n \eta^n dx \right] \\
&\leq C \omega(\varepsilon_0) \left[ \int_{B_{2r}} |\nabla u|^n T_k(|u|)^{pn} \eta^n dx + \int_{B_{2r}} |\nabla \eta|^n |u|^{pn+n} dx \right],
\end{aligned}$$

where now the constant  $C$  also depends on  $p$ . Observe that the second term in the left-hand side is nonnegative while the third one on the left-hand side, can be estimated by means of Young's inequality,

$$\begin{aligned}
n \int_{B_{2r}} |\nabla u|^{n-2} \nabla u \cdot (u \otimes \nabla \eta) T_k(|u|)^{pn} \eta^{n-1} dx &\geq -\frac{1}{2} \int_{B_{2r}} |\nabla u|^n T_k(|u|)^{pn} \eta^n dx \\
&\quad -c \int_{B_{2r}} |\nabla \eta|^n |u|^{pn+n} dx.
\end{aligned}$$

The first term can be absorbed in the left hand side of (3.2). Likewise with the first term in the right-hand side of (3.2), provided  $\varepsilon_0 > 0$  is suitably small. At last we gain

$$\int_{B_r} |\nabla u|^n T_k(|u|)^{pn} dx \leq C \int_{B_r} |\nabla \eta|^n |u|^{pn+n} dx,$$

so that in particular, after letting  $k$  tend to infinity,  $|\nabla u|^n |u|^{pn} \in L^1(B_r)$ .

**Step 3.** The main ingredient is the construction of a suitable test function, and our argument is modelled on [3, 8]. It is well-known that, working with precise representatives, we have for the pointwise restrictions,  $u|_{\partial B_r} \in W^{1,n}(\partial B_r, \mathbb{R}^{n+1})$  hold for almost all radii  $r \in (0, \text{dist}(x_0, \partial\Omega))$ . By Morrey's embedding we then in particular have for such radii that  $u|_{\partial B_r}$  is bounded. We fix such a radius  $r$  which is simultaneously so small that also  $\int_{B_r} |\nabla u|^n dx < \varepsilon_0$ . Fix  $R_0 \geq \|u\|_{L^\infty(\partial B_r)}$ . Let  $R \geq R_0$  and  $\alpha \in (0, 1]$ . Choose a map  $\Phi \in C^1$  such that  $\Phi(s) = 0$  for  $s \leq R$ ,  $\Phi(s) = 1$  for  $s \geq (1 + \alpha)R$ ,  $\Phi' \geq 0$  and  $\Phi(s) + s\Phi'(s) \leq 2/\alpha$ . We get  $\Phi(|u|) \in W_0^{1,n}(B_r) \cap L^\infty(B_r)$  and may extend this map to all of  $\mathbb{R}^n \setminus B_r$  by 0.

Let us note that  $\Phi(|u|)$  satisfies the required conditions for the function  $\varphi$  in Step 1, therefore arguing as in the proof of Step 1, the use of  $\Phi(|u|)^{n+1}u$  to test the equation can be justified and we hereby estimate using also (2.1) and (2.2):

$$\begin{aligned} \int_{B_r} |\nabla u|^n \Phi(|u|)^{n+1} dx &+ (n+1) \int_{B_r} |\nabla u|^{n-2} \nabla u \cdot \left( u \otimes \left( \frac{u \nabla u}{|u|} \right) \right) \Phi'(|u|) \Phi(|u|)^n dx \\ &\leq C \|H(\bar{u}) \bar{u} \Phi(|u|)\|_{\text{BMO}} \|\nabla(\Phi(|u|) \bar{u})\|_{L^n(\mathbb{R}^n)}^n. \end{aligned}$$

Observe that the second term on the left-hand side is nonnegative. To estimate the BMO norm we proceed using Poincaré's inequality:

$$\|H(\bar{u}) \bar{u} \Phi(|u|)\|_{\text{BMO}} \leq C \left( \int_{B_r \cap \{|u| > R\}} |\nabla(H(u)u \Phi(|u|))|^n dx \right)^{\frac{1}{n}}.$$

Here  $H(u)u \Phi(|u|)$  is in  $W_0^{1,n}(B_r, \mathbb{R}^{n+1})$  by virtue of Step 2 since

$$\begin{aligned} |\nabla(H(u)u \Phi(|u|))|^n &\leq c |H'(u)|^n |\nabla u|^n \Phi(|u|)^n |u|^n + c |\Phi'(|u|)u|^n |\nabla u|^n |H(u)|^n + \\ &+ c |H(u)|^n \Phi(|u|)^n |\nabla u|^n \leq \frac{C}{\alpha^n} |\nabla u|^n (|u|^{n+nq} + 1) \chi_{B_r \cap \{|u| > R\}}. \end{aligned}$$

Thus we get

$$\begin{aligned} \|H(\bar{u}) \bar{u} \Phi(|u|)\|_{\text{BMO}} &\leq \frac{C}{\alpha} \left( \int_{B_r \cap \{|u| > R\}} |\nabla u|^{n-1} |\nabla u| (|u|^{n+nq} + 1) dx \right)^{\frac{1}{n}} \\ &\leq \frac{C}{\alpha} \left( \int_{B_r \cap \{|u| > R\}} |\nabla u|^n dx \right)^{\frac{n-1}{n^2}} \left( \int_{B_r} |\nabla u|^n (1 + |u|^{n^2+n^2q}) dx \right)^{\frac{1}{n^2}}. \end{aligned}$$

Finally, since

$$\|\nabla(\Phi(|u|) \bar{u})\|_{L^n(\mathbb{R}^n)}^n \leq \frac{C}{\alpha^n} \int_{B_r \cap \{|u| > R\}} |\nabla u|^n dx,$$

we obtain

$$\int_{B_r \cap \{|u| > (1+\alpha)R\}} |\nabla u|^n dx \leq \frac{C}{\alpha^{n+1}} \left( \int_{B_r \cap \{|u| > R\}} |\nabla u|^n dx \right)^{1+\frac{n-1}{n^2}}$$

where we used the fact that  $|\nabla u|^n |u|^{n^2+n^2q} \in L^1(B_r)$  by Step 2 (and we incorporated this integral into the constant). Now if we define

$$\Lambda(R) := \int_{\{|u| > R\}} |\nabla u|^n dx,$$

we obtain

$$\Lambda((1+\alpha)R) \leq \frac{C}{\alpha^{n+1}} \Lambda(R)^{1+\frac{n-1}{n^2}}.$$

Choosing  $\alpha_k = 2^{-k}$  and defining by recurrence a bounded increasing sequence  $R_{k+1} = R_k(1+2^{-k})$ , we get

$$\Lambda(R_{k+1}) \leq C A^k \Lambda(R_k)^{1+\beta},$$

where  $A = 2^{n+1}$  and  $\beta = \frac{n-1}{n^2}$ . If  $\Lambda(R_0) \leq C^{-\frac{1}{\beta}} A^{-\frac{1}{\beta^2}}$ , then it follows by induction that  $\Lambda(R_k) \leq A^{-\frac{k}{\beta}} \Lambda(R_0)$  holds for all  $k$  (see for instance [14, Lemma 7.1]). Recall that  $\int_{B_r} |\nabla u|^n dx < \varepsilon_0$ , so obviously  $\Lambda(R_0) < \varepsilon_0$  too. Hence if we choose  $\varepsilon_0$  suitably small, then we have shown that

$$\Lambda(R_\infty) \leq \lim_k \Lambda(R_k) = 0 \text{ and therefore } |u(x)| \leq R_\infty \text{ a.e. in } B_r,$$

where  $R_\infty = \lim_{k \rightarrow +\infty} R_k$ . □

*Proof of Proposition 1.1.* We will first prove a Caccioppoli inequality and fix a ball  $B_0 \Subset \Omega$  so small that

$$(3.3) \quad \int_{B_0} |\nabla u|^n dx \leq \varepsilon_0$$

holds for an  $\varepsilon_0 > 0$  that will be chosen in the course of the proof. As before, let  $B_r = B_r(x_0)$  be such that  $B_{2r} = B_{2r}(x_0) \Subset B$ . By our assumption  $\bar{R} := \|u\|_{L^\infty(B_0)}$  is finite. Denote by  $\bar{u}$  the extension of  $u$  from  $B_{2r}$  given by Lemma 2.4.

Let  $\eta$  be a cut-off function between  $B_r$  and  $B_{2r}$ , choose the function  $\eta^n(u - u_{2r})$ , where  $u_{2r} = u_{B_{2r}}$ , as test in (1.1) (see also Remark 1.2) and write the integrals in the right-hand side of the equation on the whole of  $\mathbb{R}^n$ . We gain:

$$(3.4) \quad \int_{B_{2r}} |\nabla u|^{n-2} \nabla u \cdot \nabla (\eta^n(u - u_{2r})) dx = n^{\frac{n}{2}} \int_{\mathbb{R}^n} H(\bar{u}) \eta^n(\bar{u} - u_{2r}) \cdot J_{\bar{u}} dx =: \mathcal{I}.$$

Here the left-hand side is

$$\int_{B_{2r}} |\nabla u|^{n-2} \nabla u \cdot \nabla (\eta^n(u - u_{2r})) dx = \int_{B_{2r}} |\nabla u|^n \eta^n dx + n \int_{B_{2r}} |\nabla u|^{n-2} \nabla u \cdot (\eta^{n-1}(u - u_{2r}) \otimes \nabla \eta) dx,$$

and the second term on the right-hand side can be estimated by means of Young's inequality,

$$n \int_{B_{2r}} |\nabla u|^{n-2} \nabla u \cdot \eta^{n-1}(u - u_{2r}) \otimes \nabla \eta dx \geq -\frac{1}{2} \int_{B_{2r}} |\nabla u|^n \eta^n dx - C \int_{B_{2r}} |\nabla \eta|^n |u - u_{2r}|^n dx.$$

Thus returning to (3.4) we get

$$(3.5) \quad \int_{B_{2r}} |\nabla u|^n \eta^n dx \leq C \int_{B_{2r}} |\nabla \eta|^n |u - u_{2r}|^n dx + 2\mathcal{I}.$$

Let us now estimate  $\mathcal{I}$ . We first observe

$$\mathcal{I} = n^{\frac{n}{2}} \int_{\mathbb{R}^n} H(\bar{u}) \eta^n(\bar{u} - u_{2r}) \cdot J_{\bar{u}-u_{2r}} dx = n^{\frac{n}{2}} \int_{\mathbb{R}^n} H(\bar{u})(\bar{u} - u_{2r}) \cdot J_{(\bar{u}-u_{2r})\eta} dx,$$

by Lemma 2.5. We aim to use the bounds (2.1), (2.2) and must argue that the first factor is in BMO. To that end we note that  $y \mapsto H(y)(y - u_{2r})$  is continuous, and so in particular uniformly continuous for  $|y| \leq 2\bar{R}$ . We can therefore find a modulus of continuity  $\omega = \omega_{\bar{R}}: [0, \infty) \rightarrow [0, \infty)$  (an increasing, continuous and concave functions with  $\omega(0) = 0$ ) such that

$$|H(y)(y - u_{2r}) - H(y')(y' - u_{2r})| \leq \omega(|y - y'|)$$

for all  $y, y'$  with  $|y|, |y'| \leq 2\bar{R}$ . Observe that  $\omega$  only depends on the  $L^\infty$  bound for  $u$  on the fixed ball  $B_0$  and that  $|\bar{u}| \leq 2\bar{R}$  a.e. in  $\mathbb{R}^n$  by our assumptions and Lemma 2.4. Now for any ball  $B \subset \mathbb{R}^n$  we estimate as in Lemma 2.3, using also the integral bound from Lemma 2.4 and (3.1), to get

$$\int_B |H(\bar{u})(\bar{u} - u_{2r}) - (H(\bar{u})(\bar{u} - u_{2r}))_B| dx \leq c\omega(\varepsilon_0)$$

for some dimensional constant  $c = c(n)$ . Consequently,  $\|H(\bar{u})(\bar{u} - u_{2r})\|_{\text{BMO}} \leq c\omega(\varepsilon_0)$  and we may continue with

$$\begin{aligned} \mathcal{I} &\leq C \|J_{(\bar{u}-u_{2r})\eta}\|_{\mathcal{H}^1} \|H(\bar{u})(\bar{u} - u_{2r})\|_{\text{BMO}} \\ &\leq C \int_{B_{2r}} |\nabla((u - u_{2r})\eta)|^n dx (c\omega(\varepsilon_0))^{\frac{1}{n}} \\ &\leq c\omega(\varepsilon_0)^{\frac{1}{n}} \int_{B_{2r}} (|\nabla u|^n \eta^n + |\nabla \eta|^n |u - u_{2r}|^n) dx. \end{aligned}$$

If  $\varepsilon_0$  is suitable small (depending on the fixed ball  $B_0$  and the data) the first term can be absorbed in the left-hand side of (3.5). Summarising we have therefore established the following Caccioppoli inequality:

$$\int_{B_r} |\nabla u|^n dx \leq \frac{C}{r^n} \int_{B_{2r}} |u - u_{B_{2r}}|^n dx,$$



where the constant  $C = C(B_0)$  and  $B_r = B_r(x_0)$  is any ball with  $B_{2r}(x_0) \subset B_0$ . Now using Poincaré-Sobolev's inequality and Gehring's Lemma (see for instance [17]) we find in a routine manner that  $\nabla u \in L_{\text{loc}}^{n+\delta}(\Omega, \mathbb{R}^{(n+1) \times n})$  for some  $\delta > 0$ . By Morrey's embedding theorem this implies that  $u$  is  $C_{\text{loc}}^{0,\beta}(\Omega, \mathbb{R}^{n+1})$  for some  $\beta < 1$ .

It is well-known that the asserted  $C^{1,\alpha}$  regularity follows from this when  $H$  is locally Lipschitz. A streamlined approach can be obtained following [11, Lemma 5] to get that  $u$  is  $C_{\text{loc}}^{0,s}(\Omega, \mathbb{R}^{n+1})$  for each  $s < 1$ , and then finally [11, Lemma 6] to conclude that  $u$  is  $C_{\text{loc}}^{1,\alpha}(\Omega, \mathbb{R}^{n+1})$  for some  $\alpha \in (0, 1)$ .  $\square$

The proof of Theorem 1.3 follows by combination of Theorem 3.1 and Proposition 1.1.

#### 4. EXAMPLE

The condition of Heinz and Wang on the  $C^1$  function  $H: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  is

$$\sup_{y \in \mathbb{R}^{n+1}} \left( |H(y)| + |H'(y)|(1 + |y|) \right) < \infty.$$

We can rewrite this and split it into two assumptions as

$$\begin{cases} H \text{ Lipschitz} \\ y \mapsto H(y)(|y| + 1) \text{ Lipschitz continuous.} \end{cases}$$

Here we give an example of a  $C^1$  function  $H$  that does not satisfy the assumptions of Heinz and Wang but is such that our Theorem 1.3 in particular applies to it:

$$\begin{cases} H \text{ Lipschitz} \\ y \mapsto H(y)(|y| + 1) \text{ uniformly continuous.} \end{cases}$$

**Example:** Let  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be even and

$$\theta(t) := \begin{cases} 1 & 0 \leq t \leq 1 \\ t^2(t-2)^2 & 1 \leq t \leq 2 \\ 0 & t \geq 2. \end{cases}$$

Then  $\theta \in C^1$ ,  $0 \leq \theta(t) \leq \mathbf{1}_{(-2,2)}(t)$ ,  $\sup |\theta'(t)| = |\theta'(\pm t_0)| = \theta_0$ , where  $t_0 = (3 + \sqrt{3})/3 \in (1, 2)$  and  $\theta_0 = \frac{8}{9}\sqrt{3}$ . Fix an increasing function  $\omega: (0, 1] \rightarrow (0, 1]$  with  $\omega(0^+) = 0$ . Note that the intervals  $[2^j - 2^{1-j}, 2^j + 2^{1-j}]$  are pairwise disjoint and that none of them contains 0 when  $j \in \mathbb{N}$ . The function

$$H(y) := \sum_{j=1}^{\infty} \omega(2^{-j}) \frac{\theta(2^j|y| - 2^{2j})}{1 + |y|}, \quad y \in \mathbb{R}^{n+1},$$

is therefore easily seen to be  $C^1$  and  $0 \leq H(y)(1 + |y|) \leq 1$  for all  $y \in \mathbb{R}^{n+1}$ .

Now if  $\omega$  is superlinear at 0, then  $F(y) := H(y)(1 + |y|)$  cannot be Lipschitz since for  $y_\ell \in \mathbb{R}^{n+1}$  with  $|y_\ell| = 2^\ell + \frac{3+\sqrt{3}}{3}2^{-\ell}$  we have that  $|F'(y_\ell)| = \theta_0\omega(\frac{1}{2^\ell})2^\ell$  which is unbounded for  $\ell \in \mathbb{N}$ . However, it is not difficult to check that  $F$  is uniformly continuous: let  $\varepsilon > 0$ , and take  $s \in \mathbb{N}$  with  $\omega(2^{-s}) \leq \varepsilon$ . Since  $F$  is Lipschitz on  $[-2^s, 2^s]$  we can find  $\delta_1 > 0$  so

$$|F(y_1) - F(y_2)| \leq \varepsilon \quad \forall y_1, y_2 \in [-2^s, 2^s], \quad |y_1 - y_2| \leq \delta_1.$$

Next, for  $y_2 \in \mathbb{R}^{n+1}$  with  $|y_2| > 2^s$  we note that for any  $y_1 \in \mathbb{R}^{n+1}$  with  $|y_1 - y_2| < 2^{-s}$  we have that  $|F(y_1) - F(y_2)| \leq C\omega(2^{-s}) \leq C\varepsilon$ . Taking  $\delta = \min(\delta_1, 2^{-s})$  we fend off  $\varepsilon$  and conclude that  $F$  is uniformly continuous on  $\mathbb{R}^{n+1}$ .

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