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Multiplicative dependence of k-Fibonacci numbers with the Fibonacci, Lucas, and Pell sequences

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Abstract

The k–generalized Fibonacci sequence $(F_m^{(k)})_{m\geq 2-k}$ is the linear recurrent sequence of order k whose first k terms are $0,\ldots,0,1$ and each term afterwards is the sum of the preceding k terms. The case $k=2$ corresponds to the well known Fibonacci sequence. In Gómez and Luca (Lith. Math. J. 56(4):503–517, 2016), the multiplicative independence between terms of the same k-generalized Fibonacci sequence was studied. In this paper, we find all the multiplicative dependent pairs $(F_m^{(k)}, u_n)$ where u_n is a Fibonacci, a Lucas or a Pell number.

Keywords k-generalized Fibonacci numbers · Lower bounds for nonzero linear forms in logarithms of algebraic numbers · Effective solution for exponential Diophantine equation · Diophantine approximation

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1 Introduction

Let $F := (F_n)_{n\geq 0}$ be the classical *Fibonacci sequence*, defined by $F_0 = 0$, $F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

Fibonacci numbers exhibit numerous remarkable properties and deep links to diverse mathematical areas [23]. Among their multiplicative features, two are noteworthy: Carmichael's theorem on primitive prime factors [8], asserting that every F_n with $n > 13$ has a prime divisor, namely a prime factor not dividing any smaller nonzero Fibonacci number, and the result of Bugeaud, Mignotte, and Siksek [7], showing that the only perfect powers with exponent > 1 in the sequence are 0, 1, 8, and 144. No analogue of Carmichael's theorem is known for non-binary linear recurrence sequences.

Having a primitive divisor theorem for members of a sequence of integers allows one to establish multiplicative dependence among the members of the sequence. Recall that an s–tuple of nonzero integers $(A_1, \ldots, A_s)$ is called *multiplicatively dependent* if there exist integers $x_1, \ldots, x_s$, not all zero, such that

$$A_1^{x_1} \cdots A_s^{x_s} = 1$$

and it is called *multiplicatively independent* otherwise. The study of multiplicative relations among terms of linear recurrence sequences is a classical topic in number theory, with strong connections to Diophantine equations, primitive prime factors, and the characterization of perfect powers.

Thanks to the primitive divisor theorem, there are only two nontrivial multiplicatively dependent pairs of Fibonacci numbers: $(F_1, F_2) = (1, 1)$ and $(F_3, F_6) = (2, 8)$. A general framework for studying multiplicative dependence among members of binary recurrence sequences was developed by Luca and Ziegler in [25], where it was proved that for fixed s and fixed integer exponents $x_1, \ldots, x_s$, the equation $\prod_{i=1}^{s} u_{n_i}^{x_i} = 1$ has finitely many computable nonnegative integer solutions. Later, Gómez and Luca [19] established, under certain conditions, the finiteness of multiplicatively dependent s–tuples $(u_{n_1}^{(1)}, \ldots, u_{n_s}^{(s)})$ formed from distinct binary recurrence sequences without fixing the exponents beforehand. Related works include [17] on triples of multiplicatively dependent Tribonacci numbers and [1, 2] on Padovan and Perrin sequences.

For certain types of binary linear recurrence sequences $(u_n)_{n\geq 0}$, a well-studied problem concerns the size of the largest prime factor of u_n. Denoting by $P(m)$ the largest prime factor of the positive integer m, an intriguing question is to obtain good lower bounds for $P(u_n)$. If $P(u_n)$ is "small" for "many" values of n; i.e., if numerous terms among the u_n's have only small prime factors, it is plausible that many multiplicative relations occur among such numbers. Consequently, the study of multiplicative dependence becomes relevant for estimating $P(u_n)$.

For higher-order recurrences, we consider the *k-generalized Fibonacci sequence* $F^{(k)} := (F_m^{(k)})_{m\geq -(k-2)}$, defined for $k \geq 2$ by

$$F_{-i}^{(k)} = 0 \quad (0 \leq i \leq k-2), \quad F_1^{(k)} = 1,$$

and

$$F_m^{(k)} = F_{m-1}^{(k)} + F_{m-2}^{(k)} + \cdots + F_{m-k}^{(k)} \quad (m \geq 2).$$

For $k = 2$ we recover F, and $k = 3$ corresponds to the Tribonacci sequence. The initial nonzero terms for $k = 2, 3, 4, 5, 6$ are:

$$\begin{aligned}
F^{(2)} &= \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\},\\
F^{(3)} &= \{1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, \ldots\},\\
F^{(4)} &= \{1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, \ldots\},\\
F^{(5)} &= \{1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, \ldots\},\\
F^{(6)} &= \{1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, \ldots\}.
\end{aligned}$$

Gómez and Luca [18] proved that the equation

$$\left(F_n^{(k)}\right)^x = \left(F_m^{(k)}\right)^y$$

has only trivial solutions involving powers of 2 from the initial terms of $F^{(k)}$. In [20], we examined multiplicative dependence between k–Fibonacci and k–Lucas numbers, the latter being the natural k–generalization of the Lucas sequence.

Here, we study the multiplicative dependence between k–Fibonacci numbers and some classical binary recurrence sequences. Specifically, we determine all triples of positive integers (k, m, n) and coprime positive integers (x, y) such that

$$\left(F_m^{(k)}\right)^y = u_n^x, \tag{1}$$

where u_n denotes a Fibonacci, Lucas, or Pell number, as part of an approach to studying the multiplicative dependence between sequences of different orders [18].

We establish the following theorem.

Main Theorem *The only triplets (k, m, n) such that the Diophantine equation (1) has a solution, with $k \geq 2$, $m \geq 3$ and $n \geq 2$ ($n < m$ when $k = 2$ and $\mathcal{U} = F$, and $n \geq 0$ when $\mathcal{U} = L$), for x and y coprime positive integers, are:*

- *when $\mathcal{U} = F$:*

$$\{(2, 6, 3), (3, 6, 7), (3, 9, 4), (6, 9, 5)\}$$
$$\{(t, r, a) : \ t \geq 3, \ 3 \leq r \leq t + 1, \ a = 3 \text{ or } a = 6\},$$

which correspond to the multiplicative dependent pairs $(F_m^{(k)}, F_n)$:

$$(F_6^{(2)}, F_3) = (8, 2), \quad (F_6^{(3)}, F_7) = (13, 13),$$
$$(F_9^{(3)}, F_4) = (81, 3), \quad (F_9^{(6)}, F_5) = (125, 5),$$

and

$$(F_r^{(t)}, F_3) = (2^{r-2}, 2), \quad (F_r^{(t)}, F_6) = (2^{r-2}, 8)$$

for $t \geq 3,\ 3 \leq r \leq t+1$.

- *when* $\mathcal{U} = L$*:*

$$\{(2,4,2), (2,6,0), (2,6,3), (3,9,2), (4,7,7)\}$$
$$\{(t,r,a):\ t \geq 2,\ 3 \leq r \leq t+1,\ a = 0 \text{ or } a = 3\},$$

which correspond to the multiplicative dependent pairs $(F_m^{(k)}, L_n)$*:*

$$(F_4^{(2)}, L_2) = (3,3), \quad (F_6^{(2)}, L_0) = (8,2), \quad (F_6^{(2)}, L_3) = (8,4),$$
$$(F_9^{(3)}, L_2) = (81,3), \ (F_7^{(4)}, L_7) = (29,29),$$

and

$$(F_r^{(t)}, L_0) = (2^{r-2}, 2), \quad (F_r^{(t)}, L_3) = (2^{r-2}, 4)$$

for $t \geq 2,\ 3 \leq r \leq t+1$.

- *when* $\mathcal{U} = P$*:*

$$\{(2,6,2), (2,5,3), (2,7,7), (2,12,4), (3,6,7), (4,7,5), (6,9,3)\},$$
$$\{(t,r,2):\ t \geq 2,\ 3 \leq r \leq t+1\},$$

which correspond to the multiplicative dependent pairs $(F_m^{(k)}, P_n)$*:*

$$(F_6^{(2)}, P_2) = (8,2), \quad (F_5^{(2)}, P_3) = (5,5), \quad (F_7^{(2)}, P_7) = (13,169),$$
$$(F_{12}^{(2)}, P_4) = (144,12), \ (F_6^{(3)}, P_7) = (13,169), \ (F_7^{(5)}, P_7) = (29,29),$$
$$(F_9^{(6)}, P_3) = (125,5),$$

and

$$(F_r^{(t)}, P_2) = (2^{r-2}, 2)$$

for $t \geq 2,\ 3 \leq r \leq t+1$.

The particular case of equation (1) when $\mathcal{U} = L$ and $y = 1$ was studied in the recent paper [3].

The paper is organized as follows. Section 2 outlines key properties and relations of the Fibonacci, Lucas, Pell, and $k-$Fibonacci numbers. Section 3 describes the main tools used to bound the variables, notably Matveev's theorem on lower bounds for linear forms in the logarithms of algebraic numbers, together with additional results that reduce these bounds to a computational range, enabling an exhaustive search for the pairs of interest. Finally, Sect. 4 presents the proof of our main result, organized by cases, with each step detailed alongside the necessary analytical computations.

All computations for this paper were carried out using `Wolfram Mathematica`. In particular, the `ParallelDo` and `ParallelTable` routines were extensively employed to handle large search intervals.

2 Linear recurrence sequences

2.1 Fibonacci, Lucas and Pell numbers

The characteristic polynomial for the Fibonacci sequence $F = (F_n)_{n\geq 0}$ is given by $p(z) = z^2 - z - 1$ and it has the real roots $\phi = (1+\sqrt{5})/2$ and $\overline{\phi} = (1-\sqrt{5})/2$ with $\phi \approx 1.69$ and $|\overline{\phi}| \approx 0.69$. The following identity is known as Binet's formula for the n^{th} Fibonacci number:

$$F_n = \frac{\phi^n - \overline{\phi}^n}{\sqrt{5}}, \quad \text{for all} \quad n \geq 0.$$

Thus, it is clear that

$$F_n = \phi^n/\sqrt{5} + \epsilon(n) \quad \text{with} \quad |\epsilon(n)| < 1/\phi^{n+1}, \quad \text{for all} \quad n \geq 1. \tag{2}$$

Besides, as a result of an induction process

$$\phi^{n-2} \leq F_n \leq \phi^{n-1} \tag{3}$$

holds for all $n \geq 1$.

Concerning the Lucas sequence $L = (L_n)_{n\geq 0}$, its characteristic polynomial coincides with that of the Fibonacci sequence, since it satisfies the binary linear recurrence $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$. Its initial values are $L_0 = 2$ and $L_1 = 1$; hence, the Binet formula for its $n-$th term is

$$L_n = \phi^n + \overline{\phi}^n, \quad \text{for all} \quad n \geq 0.$$

It is also a straightforward exercise to show that

$$L_n = \phi^n + \xi(n), \quad \text{with} \quad |\xi(n)| \leq 1/\phi^n, \quad \text{for all} \quad n \geq 1. \tag{4}$$

Similar to (3), here we have

$$\phi^{n-1} \leq L_n \leq (2\phi^2 - 1)\phi^{n-2}. \tag{5}$$

Below we present the first few Lucas numbers

$$L = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots\},$$

The Pell sequence $P = (P_n)_{n\geq 0}$ satisfies the linear recurrence $P_n = 2P_{n-1} + P_{n-2}$, with $P_0 = 0$ and $P_1 = 1$. For it, $p(z) = z^2 - 2z - 1$ is the characteristic polynomial which has the real roots $\beta = 1 + \sqrt{2}$ and $\overline{\beta} = 1 - \sqrt{2}$, with $2 < \beta < 3$ and $|\overline{\beta}| < 1$. The Binet formula for the $n-$th Pell number is

$$P_n = \frac{\beta^n - \overline{\beta}^n}{2\sqrt{2}}, \quad \text{for all} \quad n \geq 0,$$

which allows us to write

$$P_n = \beta^n/2\sqrt{2} + \zeta(n), \quad |\zeta(n)| < 1/\beta^{n+1} \quad \text{for all} \quad n \geq 1. \tag{6}$$

By an induction argument similar to the case of the Fibonacci numbers, we have

$$\beta^{n-2} \leq P_n \leq \beta^{n-1} \tag{7}$$

for all $n \geq 1$. Below we present the first non-zero Pell numbers

$$P = \{1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots\}.$$

With the intention to work in general terms, we set $\mathcal{U} = (u_n)_{n\geq 0}$ given by

$$u_n = C\gamma^n(1 + r) \quad \text{with} \quad r := e(n)/(C\gamma^n), \tag{8}$$

such that, by (2), (4) and (6), correspond to

$$\begin{aligned} C &:= 1/\sqrt{5}, \quad \gamma := \phi,\ e(n) := \epsilon(n),\ \text{when } \mathcal{U} = F,\\ C &:= 1, \qquad\ \ \gamma := \phi,\ e(n) := \xi(n),\ \text{when } \mathcal{U} = L,\\ \text{and } C &:= 1/2\sqrt{2},\ \gamma := \beta,\ e(n) := \zeta(n),\ \text{when } \mathcal{U} = P. \end{aligned} \tag{9}$$

Note that we have $|e(n)| \leq 1/\alpha^n$ in all cases. Besides, by (3), (5) and (7),

$$\gamma^{n-2} \leq u_n \leq \gamma^{n+1} \tag{10}$$

holds for all our sequences.

Since Eq. (1) has the term u_n^x, we need to study this expression. To proceed, we consider $w := xr$. Then, by the fact that $1/4 < C \leq 1$ and $\phi \leq \gamma \leq \beta$, we have $|r| = |e(n)|/C\gamma^n < 4/\phi^{2n}$, which implies that

$$|w| < 4x/\phi^{2n}. \tag{11}$$

Now, let us assume that $|w| < 0.4$. We are going to show that

$$|(1 + r)^x - 1| < 2|w| \tag{12}$$

and we proceed by cases. If $r < 0$, then

$$1 > (1+r)^x = \exp(x\log(1-|r|)) \geq \exp(-2|w|) > 1-2|w|,$$

and, if $r > 0$, then

$$1 < (1+r)^x = (1+|w|/x)^x < \exp|w| < 1+2|w|.$$

Thus, the inequality (12) holds in all cases. Thus, we have that

$$\left|u_n^x - C^x\gamma^{nx}\right| = C^x\gamma^{nx}\left|(1+r)^x - 1\right| < 2\gamma^{nx}|w|$$

therefore, by inequality (11), we have the following.

Lemma 1 *Assume* $10x < \alpha^{2n}$. *Then the estimate*

$$u_n^x = C^x\gamma^{nx} + e(n,x) \quad \textit{with} \quad |e(n,x)| < \frac{8x\gamma^{nx}}{\alpha^{2n}} \tag{13}$$

holds when $(u_n)_{n\geq 0}$ *is one of the Fibonacci, Lucas or Pell sequence.*

2.2 *k*−Fibonacci numbers

For the linear recurrence of order k associated to the k−Fibonacci sequence $F^{(k)}$, its characteristic polynomial is

$$p_k(z) = z^k - z^{k-1} - \cdots - z - 1,$$

which is irreducible in $\mathbb{Q}[z]$ with one real root outside the unit circle denoted by $\alpha_k := \alpha$. This root is real and $2(1-2^{-k}) < \alpha < 2$, see [22, 28]. Note that for $k = 2$, we have $\alpha = \phi$.

For any fixed integer $k \geq 2$, let us consider the function

$$f_k(z) = \frac{z-1}{2+(k+1)(z-2)}, \quad \text{with} \quad z > 2(1-2^{-k}).$$

Then, according to [16], if we set $f_\alpha := f_k(\alpha)$, we then have

$$1/2 < f_\alpha < 3/4 \quad \text{and} \quad |f_k(\alpha^{(i)})| < 1, \quad \text{for} \quad 2 \leq i \leq k, \tag{14}$$

where $\alpha := \alpha^{(1)}, \alpha^{(2)}, \ldots \alpha^{(k)}$ are the conjugate roots of $p_k(z)$.

Remark 1 f_α is not an algebraic integer.

By a result of Dresden and Du [13], we have, for any $k \geq 2$ and all $m \geq 1$,

$$F_m^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)})\alpha^{(i)^{m-1}} \quad \text{and} \quad F_m^{(k)} = f_\alpha\alpha^{m-1} + \varepsilon(k,m), \tag{15}$$

where in the right-hand we have $|\varepsilon(k,m)| < 1/2$. Bravo and Luca [3] showed that, for any $k \geq 2$

$$\alpha^{m-2} \leq F_m^{(k)} \leq \alpha^{m-1} \qquad \text{for all} \quad m \geq 1. \tag{16}$$

It is also well known that

$$F_m^{(k)} = 2^{m-2} \quad \text{holds for all} \quad k \geq 2 \quad \text{and} \quad 2 \leq m \leq k+1. \tag{17}$$

Additionally, Cooper and Howard [11] proved that, for any $k \geq 2$, the formula

$$F_m^{(k)} = 2^{m-2} + \sum_{j=1}^{m-1} c_{m,j} 2^{m-(k+1)j-2}, \quad \text{holds for all} \quad m \geq k+2, \tag{18}$$

where $l := \left\lfloor \frac{m+k}{k+1} \right\rfloor$ and

$$c_{m,j} = (-1)^j \left[\binom{m-jk}{j} - \binom{m-jk-2}{j-2} \right],$$

with the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of them is negative, and we use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

As a consequence of the Cooper and Howard identity (18), we have the following result (see [20]).

Lemma 2 *If $k+2 \leq r < 2^{ck}$ for some $c \in (0,1)$, then*

$$F_r^{(k)} = 2^{r-2}\,(1+\rho(k,r)) \qquad \textit{with} \qquad |\rho(k,r)| < \frac{2r}{2^k} < \frac{2}{2^{k(1-c)}}. \tag{19}$$

3 Tools

In this section, we present some classical results which allow us to explicitly find all the solutions of Diophantine exponential Eq. (1).

3.1 Baker theory results

Our main tool is a result concerning lower bounds for a nonzero linear form in logarithms of algebraic numbers. So, we have to start by establishing some related concepts.

Let η be an algebraic number of degree d over $\mathbb{Q}$ with

$$f(z) := a_0 \prod_{i=1}^{d} (z - \eta_i) \in \mathbb{Z}[z]$$

as its minimal primitive polynomial over the integers. Here, $a_0 \geq 1$ and the η_i's are the conjugates of η. The logarithmic height of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta_i|, 1\}\right).$$

When $\eta = p/q$ with $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$ and $q \geq 1$, we get $h(\eta) = \log \max\{|p|, q\}$.

Many Diophantine problems can be solved by reducing them to an instance in which one can apply lower bounds for linear forms in logarithms of algebraic numbers. We will use the following theorem, which is a variation of a result of Matveev [27], proved by Bugeaud, Mignotte and Siksek [7, Theorem 9.1].

Theorem 1 *(Matveev's theorem) Let* $\mathbb{K}$ *be a number field of degree* D *over* $\mathbb{Q}$, $\tau_1, \ldots, \tau_t$ *be positive real numbers of* $\mathbb{K}$, *and* $b_1, \ldots, b_t$ *rational integers. Put*

$$\Lambda := \tau_1^{b_1} \cdots \tau_t^{b_t} - 1 \quad \textit{and} \quad B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\tau_i), |\log \tau_i|, 0.16\}$ *for* $i = 1, \ldots, t$. *If* $\Lambda \neq 0$, *then*

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

3.2 Reduction methods

Since we usually get some large upper bounds for the variables involved in our equation, we need to reduce them. The two following results require continued fractions.

Lemma 3 *Let* M *be a positive integer,* p_1/q_1, p_2/q_2, ..., *the convergents of an irrational number* τ *and* $[a_0, a_1, , \ldots]$ *its continued fraction. If*

$$a_M := \max\{a_t : 0 \leq t \leq N + 1\},$$

when N *is some positive integer such that* $q_{N+1} > M$. *Then,*

$$\left|\tau - \frac{n}{m}\right| > \frac{1}{(a_M + 2)m^2},$$

for every pair (m, n) *such that* $0 < m < M$.

Another useful result in order to perform a reduction process is a slight variation of a result of Dujella and Pethő which is Lemma 5a in [14]. Here, for a real number X, we use $||X||$ to denote $\min\{|X - n| : n \in \mathbb{Z}\}$; i.e., the distance from X to the nearest integer.

Lemma 4 *Let M and Q be positive integers such that $Q > 6M$, and A, B, τ and μ be real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := ||\mu Q|| - M||\tau Q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < A \cdot B^{-w},$$

in positive integers u, v and w with

$$u \le M \quad \text{and} \quad w \ge \frac{\log(AQ/\varepsilon)}{\log B}.$$

For practical applications, Q corresponds to the denominator of a convergent of the continued fraction of the real number τ.

The LLL–algorithm is a method from the geometry which can be used to find lower bounds on the shortest vector length in a given lattice. Let $\tau_0, \ldots, \tau_n$ be real numbers and $C > 0$. In practice, C is large. Let e_i be the i−th vector of the canonical basis for $\mathbb{R}^n$. We set $\mathbf{b}_j = e_j + \lfloor C\tau_j \rceil e_n$ for $j = 1, \ldots, n-1$ and $\mathbf{b}_n = \lfloor C\tau_n \rceil e_n$. Let $\mathcal{O}$ be the lattice generated by the $\mathbf{b}_j$'s, $\mathbf{y} = -\lfloor C\tau_0 \rceil e_n$ and $d(\mathcal{O}, \mathbf{y})$ denote the distance from $\mathbf{y}$ to the nearest element of $\mathcal{O}$ distinct from $\mathbf{y}$. The following result is Proposition 2.3.20 in Sect. 2.3.5 of [9].

Lemma 5 *Let $X_1, \ldots, X_n$ be positive integers, $Q = \sum_{i=1}^{n-1} X_i^2$, $T = (1 + \sum_{i=1}^{n} X_i)$ and assume that $d(\mathcal{O}, \mathbf{y})^2 \ge T^2 + Q$. If x_i are any integers such that $|x_i| \le X_i$, for all $i = 1, \ldots, n$, then we either have*

$$\left| \tau_0 + \sum_{i=1}^{n} x_i \tau_i \right| \ge \frac{\sqrt{d(\mathcal{O}, \mathbf{y})^2 - Q} - T}{C}, \tag{20}$$

or $x_1 = \cdots = x_{n-1} = 0$ and $x_n = -\lfloor C\tau_0 \rceil / \lfloor C\tau_n \rceil$.

Since $d(\mathcal{O}, \mathbf{y})$ is usually unknown, Corollaries 2.3.16 and 2.3.17 in [9] give us some lower bounds for this quantity which can be fed into the right–hand side of (20) to produce a lower bound.

3.3 Concerning norms and the logarithmic height

Given our need to fulfill some conditions related to the previous results, we present some algebraic concepts in this section. Let us start by setting $\mathbb{K}_1 := \mathbb{Q}(\phi, \alpha)$, $\mathbb{K}_2 := \mathbb{Q}(\beta, \alpha)$ and N_i to denote the norm of $\mathbb{K}_i$ over $\mathbb{Q}$. Then,

$$|\mathrm{N}_1(\phi)| = |\mathrm{N}_2(\beta)| = |\mathrm{N}_i(\alpha)| = 1, \quad \text{for } i = 1, 2.$$

Besides, since $\mathbb{Q}(\alpha) = \mathbb{Q}(f_\alpha)$, by [26, Lemma 2.3],

$$\mathrm{N}_i(f_\alpha) = (k-1)^2/(2^{k+1}k^k - (k+1)^{k+1}), \quad \text{for} \quad i = 1, 2. \tag{21}$$

We also will need the following result. First of all, for a prime number p and a nonzero rational number r we put $\nu_p(r)$ for the exponent of p in the factorization of r.

Lemma 6 *Let*

$$\Delta_k := \mathrm{N}_i(f_\alpha^{-1}) = \frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2}.$$

Then For all $k \geq 2$, the following hold:

(i) *[20, Formula (2.3)].*

$$\Delta_k \equiv \frac{k(k-1)^2(7k-17)}{6}2^{k-2} + k(3k-5)2^{k-2} + 2^k \pmod{(k-1)^2}.$$

(ii) *[24, Theorem 1.1 (iii)].*

$$\nu_2(\Delta_k) = \begin{cases} 0, & \text{if } k \equiv 0 \pmod 2; \\ k-1, & \text{if } k \equiv 1 \pmod 2. \end{cases}$$

As for the logarithmic height, due to its properties

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma) \\ h(\eta^s) &= |s| h(\eta) \qquad \text{for} \quad s \in \mathbb{Q}, \end{aligned}$$

we have

$$\begin{aligned} &h(\phi) < \log 2/2, \qquad h(\beta) < \log 3/2, \\ &h(\alpha) < \log 2/k, \qquad h(f_\alpha) < 2\log k, \\ &h(u_n) < n\log 3 \ \text{ and } h(F_m^{(k)}) < m\log\alpha. \end{aligned}$$

3.4 Some analytic results

In this section, we recall some analytic results.

Lemma 7 *(Lemma 2.2, [12]) Let $a, x \in \mathbb{R}$ and $0 < a < 1$. If $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a}|x| \quad \text{and} \quad |x| < \frac{a}{1-e^{-a}}|e^x - 1|.$$

The following result is from [21].

Lemma 8 *(Lemma 7, [21]) If $\ell \geq 1$, $T > (4\ell^2)^\ell$ and $T > x/(\log x)^\ell$, then*

$$x < 2^\ell T (\log T)^\ell.$$

4 The proof of the Main Theorem

4.1 Some initial considerations

Since $F_1^{(k)} = F_2^{(k)} = 1$, we assume $m \geq 3$. Recall that $L_0 = 2$. The fact that $L_3 = 4$ (a perfect power of 2), allows us to proceed with $n \geq 2$. Besides, the fact that x and y are coprime numbers, implies

$$u_n = R^y \qquad \text{and} \qquad F_m^{(k)} = R^x, \tag{22}$$

for some integer $R \geq 2$.

If we assume $y \geq 2$, then according to the left equation above, we should look for perfect powers on the sequence $\mathcal{U} = (u_n)_{n \geq 2}$, which we already know correspond to

$$F_6 = 8, \quad F_{12} = 144, \quad L_3 = 4 \quad \text{and} \quad P_7 = 169$$

(see [7, 10]). Therefore, it suffices to look for solutions of the Diophantine equation

$$F_m^{(k)} = R^x \tag{23}$$

for $R \in \{2, 12, 13\}$, with $k \geq 2$, $m \geq 3$ and $x \geq 1$.

If instead we consider $y = 1$, then we get the Diophantine equation

$$F_m^{(k)} = u_n^x, \tag{24}$$

with $k \geq 2$, $m \geq 3$ and $n \geq 2$ (with $n < m$ when $k = 2$ and $\mathcal{U} = F$, and $n \geq 0$ when $\mathcal{U} = L$). Note that this becomes either a problem of a coincidence between a member in $F^{(k)}$ and a member in $\mathcal{U}$ when $x = 1$, or a problem of finding perfect powers in $F^{(k)}$ of exponent > 1 whose base is a term from one of the binary recurrence sequences (Fibonacci, Lucas or Pell) when $x \geq 2$.

4.2 The Diophantine equation (23)

Recall that in this case we are working under the setting $R \in \{2, 12, 13\}$, with $k \geq 2$, $m \geq 3$ and $m > x \geq 1$.

If we consider $x = 1$, then since $F_m^{(k)} \geq 13$ for $k \geq 2$ and $m \geq 7$, the sets of triplets (k, m, x), where the Diophantine Eq. (23) has a solution, are

$$\begin{aligned} &\{(t, 3, 1) : t \geq 2\}, && \text{related to } F_6 = 2^3, \\ &\{(t, 3, 1) : t \geq 2\}, && \text{related to } L_3 = 2^2, \\ \text{and } &\{(2, 7, 1), (3, 6, 1)\}, && \text{related to } P_7 = 13^2. \end{aligned}$$

However, for $x \geq 2$, if we consider $2 \leq m \leq k+1$, by (17) it is clear that $R = 2$ is the only plausible option. Thus, we get the triplets (k, m, x):

$$\begin{array}{ll} \{(t, r, r-2) : \ t \geq 3, \ 4 \leq r \leq t+1\}, & \text{related to } F_6 = 2^3, \\ \text{and } \{(t, r, r-2) : \ t \geq 2, \ 4 \leq r \leq t+1\} & \text{related to } L_3 = 2^2, \end{array}$$

all which correspond to solutions of (23).

The cases $R = 2$ and $R = 12$ have already been solved (see [3, 4]). Namely, the triplet $(k, m, x) = (2, 6, 3)$ or $\{(t, r, r-2) : \ t \geq 2, \ 3 \leq r \leq t+1\}$ when $R = 2$, and the only solution $(k, m, x) = (2, 12, 2)$ for $R = 12$.

Finally, by (16) and (23), we get

$$2^x \leq R^x = F_m^{(k)} \leq \alpha_k^{m-1},$$

but $\alpha \leq \alpha_k < 2$ for all $k \geq 2$. Thus, the previous inequalities imply $x < m$. Therefore, it remains to look for solutions of the Diophantine Eq. (23) for $R = 13$ with $k \geq 2$, $m \geq k+2$ and $m > x \geq 2$.

We proceed with R in general to show that for each positive integer R, there are only finitely many positive integer solutions (k, m, x) for (23). By the results from [4], we can assume that R is not power of two. Let us replace (15) into (23) to get

$$|f_\alpha \alpha^{m-1} - R^x| < 1/2.$$

Dividing both sides of the previous inequality by $f_{\alpha_k} \alpha_k^{m-1}$, we get

$$|\Lambda_1| := \left| f_\alpha^{-1} \alpha^{-(m-1)} R^x - 1 \right| < \frac{2}{\phi^m}, \tag{25}$$

where we have used the fact that $\phi \leq \alpha < 2$, for all $k \geq 2$.

Let us assume that $\Lambda_1 = 0$. Then $f_\alpha \alpha^{m-1} = R^x$ or $f_\alpha = R^x \alpha^{1-m}$ which implies that f_α is an algebraic integer contradicting Remark 1. Therefore $\Lambda_1 \neq 0$, and we can use Matveev's Theorem 1 ([7, Theorem 9.1]) to bound $|\Lambda_1|$ from below. In order to apply it, we set $t = 3$,

$$\begin{array}{lll} \tau_1 := f_\alpha, & \tau_2 := \alpha, & \tau_3 := R, \\ b_1 := -1, & b_2 := -(m-1), & b_3 := x \\ A_1 := 2k \log k, & A_2 := \log 2, & A_3 := k \log R, \end{array}$$

and $B = m$, where we have used $D = k$ since we are working with $\mathbb{K} = \mathbb{Q}(\alpha)$. So, we get the following lower bound for $|\Lambda_1|$:

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times k^2 (1 + \log k)(1 + \log m)(2k \log k)(\log 2)(k \log R)\right),$$

which, by (25), implies

$$\exp\left(-8 \times 10^{11} k^4 (\log k)^2 \log m \log R\right) < \frac{2}{\phi^m}.$$

Thus, after some calculations and the fact that $k+2 \le m$, we get

$$m < 1.9 \times 10^{12} k^4 (\log m)^4 (\log R).$$

Now, by Lemma 8 with $\ell := 4$ and $T := 1.9 \times 10^{12} k^4 (\log R)$, we get

$$m < 2^4 (1.9 \times 10^{12} k^4 (\log R)) \log(1.9 \times 10^{12} k^4 (\log R)).$$

Hence,

$$m < 5.8 \times 10^{14} k^5 (\log R)^2. \tag{26}$$

4.2.1 An absolute bound for *k* and *m*

We have to make some assumption on k in order to fulfill the following inequality

$$5.8 \times 10^{14} k^5 (\log R)^2 < 2^{k/2}, \tag{27}$$

with the aim to use (19) with $r = m$ and $c = 1/2$. Note that there is a positive integer k_0, which depends on R, such that, for $k > k_0$ the inequality (27) holds. Indeed, assume that (27) is not true. Then $2^{k/2} \le 5.8 \times 10^{14} k^5 \log^2 R$ and

$$\begin{aligned} k &\le 2\log(1.7 \times 10^{15} k^5 (\log R)^2)/\log 2 \\ &\le 3\left(\log(1.7 \times 10^{15}) + 5\log k + 2\log\log R\right) \\ &\le 3\log k \left(\frac{\log(1.7 \times 10^{15})}{\log 2} + 5 + 2\frac{\log\log R}{\log 2}\right) \\ &\le 3\log k(41 + 3\log\log R). \end{aligned}$$

Thus, by Lemma 8, we have $k \le 7.2(41 + 3\log\log R)(\log\log R) =: k_0$. Continuing under the assumption $k > k_0$, we have by Lemma 2

$$2^{m-2}(1 + \rho(k,m)) = R^x, \qquad \text{with} \qquad |\rho(k,m)| < \frac{2}{2^{k/2}}.$$

Rearranging the equation we get

$$|\Lambda_2| := |R^x 2^{-(m-2)} - 1| < \frac{2}{2^{k/2}}. \tag{28}$$

Since in our particular case R is not a power of 2, we have $\Lambda_2 \ne 0$. Therefore, we can apply Matveev's Theorem to bound $|\Lambda_2|$ from below. Let us set $t := 2$,

$$\begin{aligned} &\tau_1 := R, && \tau_2 := 2, \\ &b_1 := x, && b_2 := -(m-2), \\ &A_1 := \log R, && A_2 := \log 2. \end{aligned}$$

Also, we take $\mathbb{K} = \mathbb{Q}$, $D = 1$ and $B = m$. Hence, by Matveev's Theorem 1, we get the lower bound for $|\Lambda_2|$:

$$\exp\left(-1.4 \times 30^5 \times 2^{4.5}(1 + \log m)(\log R)(\log 2)\right),$$

which, in contrast with (28), implies

$$\exp\left(-1.1 \times 10^9 \log m \log R\right) < 2^{1-k/2}.$$

Thus, after some calculations, we conclude that $k < 3.5 \times 10^9 \log m \log R$. So, now we replace (26) into the previous inequality and we get

$$k < 2.8 \times 10^{11} \log k (\log R)^2,$$

which by Lemma 8 yields $k < 1.6 \times 10^{14}(\log R)^3$. To sum up we have proved the following intermediary result.

Lemma 9 *If R is a positive integer which is not a power of 2, then equation $F_m^{(k)} = R^x$ implies*

$$k < 1.6 \times 10^{14}(\log R)^3 \quad \text{and} \quad x < m < 6.1 \times 10^{85}(\log R)^{17}.$$

4.2.2 Solving the case $R = 13$

We will now deal with our initial case $R = 13$, which gives us

$$k < 2.7 \times 10^{15}.$$

In order to reduce the above bound, we set $\Gamma_2 := \log(1 + \Lambda_2)$. We take $k > 210 = k_0$ so that (27) holds. By (28), we get $|\Lambda_2| < 0.5$. Thus, Lemma 7 implies

$$|\Gamma_2| = |x \log R - (m - 2) \log 2| < 2|\Lambda_2| < \frac{4}{2^{k/2}}.$$

If we divide the above by $x \log 2$, we get

$$\left|\frac{\log R}{\log 2} - \frac{m-2}{x}\right| < \frac{6}{x2^{k/2}},$$

where $x < m < 5.5 \times 10^{86}$. Now, we proceed to set the conditions to use Lemma 3 with $\tau := \log 13/\log 2$. We take $M := 5.5 \times 10^{86}$ and after a computational search, we find $a_M = 413$. Thus,

$$\frac{1}{415x^2} < \left|\tau - \frac{m-2}{x}\right| < \frac{6}{x2^{k/2}},$$

implies $2^{k/2} < 2490x < 2490 \times 5.8 \times 10^{14} k^5 (\log 13)^2$, which implies $k \leq 202$, a contradiction. In conclusion, we have that if $R = 13$ and (k, m, x) is a solution of Diophantine Eq. (23) with $k \geq 2$, $m \geq k+2$ and $m > x \geq 1$, then

$$k \leq 210.$$

Inequality (26) and the absolute bound for k, altogether with the fact that $R = 13$, implies

$$m < 5.8 \times 10^{14} k^5 (\log R)^2 < 1.6 \times 10^{27},$$

a large absolute upper bound for m which we need to reduce to a computational range.

Let us set $\Gamma_1 := \log(1 + \Lambda_1)$. Since $m \geq k+2$, by (25) we have $|\Lambda_1| < 0.5$. Thus, by Lemma 7, we get

$$|\Gamma_1| = |(m-1)\log\alpha - x \log R + \log f_\alpha| < \frac{4}{\phi^m}.$$

After we divide by $\log R$, we get

$$|u\tau_k - v + \mu_k| < \frac{6}{\phi^m},$$

where $u := m - 1$, $\tau_k := \log\alpha / \log R$, $v := x$ and $\mu_k := \log f_\alpha / \log R$. Then, by Lemma 4, for each $k \in [2, 210]$, we look for Q_k, the denominator of a convergent of the continued fraction for τ_k, such that

$$Q_k > 3.5 \times 10^{15} k^5 (\log R)^2.$$

Furthermore, here we have $A = 6$, $B = \alpha$ and $w = m$. Thus, for $R = 13$, we get the absolute bound $m \leq 590$. In conclusion, for $R = 13$, if the triplet (k, m, x) is an integer solution of Diophantine Eq. (23), then

$$k \in [2, 210], \quad m \in [k+2, 590] \quad \text{and} \quad 2 \leq x < m.$$

A brief computational search in the above range shows that (23) with $R = 13$ has no solution for $x \geq 2$. Hence, we have established the following result.

Lemma 10 *The only solutions (k, m, x) of Diophantine Eq. (23), for $R = 13$, $k \geq 2$, $m \geq k+2$ and $m > x \geq 1$ is given by*

$$(k, m, x) = (2, 7, 1) \quad \text{or} \quad (3, 6, 1).$$

4.3 The Diophantine equation (24)

Note that, by (17), if we have $m \leq k+1$, then in this case our equation correspond to find powers of 2 on $\mathcal{U}$, which is included among the perfect powers studied in [7, 10]. In this case, we have the families of triplets (k, m, n) where (24) has a solution are given by:

$$\begin{array}{ll} \{(t,r,a):\ t\geq 2,\ 3\leq r\leq t+1,\ a=3 \text{ or } a=6\}, & \text{when } \mathcal{U}=F, \\ \{(t,r,a):\ t\geq 2,\ 3\leq r\leq t+1,\ a=0 \text{ or } a=3\} & \text{when } \mathcal{U}=L, \\ \text{and } \{(t,r,2):\ t\geq 2,\ 3\leq r\leq t+1\} & \text{when } \mathcal{U}=P. \end{array}$$

Therefore, we are going to work with $m \geq k+2$.

Concerning the coincidence problem; i.e., the case $x = 1$, we already know by work of Bravo and Luca [5], that the set of triplets (k, m, n) where we have solutions is

$$\{(3,6,7),(t,5,6):\ t\geq 4\}, \qquad \text{when} \quad \mathcal{U}=F,$$

while by work of Bravo, Gómez and Herrera [6], the set of triplets (k, m, n) is given by

$$\{(2,5,3),(4,7,5),(t,3,2):\ t\geq 2\}, \qquad \text{when} \quad \mathcal{U}=P.$$

Following an analogous argument used in the two previous research, we get the triplets (k, m, n) given by $(3, 4, 4)$, $(4, 7, 7)$ and $(2, 4, 2)$, together with

$$\{(t,3,0),(s,4,3):\ t\geq 2,\ s\geq 3\},\ \text{when}\ \mathcal{U}=L.$$

Thus, we can proceed under the assumption that $x \geq 2$.

Note that when $k = 2$ and $x \geq 2$, Eq. (24) becomes a problem of perfect powers in the Fibonacci sequence, which we already know correspond to $F_6 = 8$ and $F_{12} = 144$. Thus, we get the set of triplets (k, m, n) given by

$$\begin{array}{ll} \{(2,6,3)\}, & \text{when } \mathcal{U}=F, \\ \{(2,6,0)\}, & \text{when } \mathcal{U}=L, \\ \text{and } \{(2,6,2),(2,12,4)\} & \text{when } \mathcal{U}=P, \end{array}$$

which allow us to work with $k \geq 3$.

Finally, by (10), (16) and (24), we have

$$\gamma^{(n-2)x} \leq u_n^x = F_m^{(k)} \leq \alpha^{m-1},$$

which clearly implies $n < m$ and $x < m$. Therefore, we will look for solutions of the Diophantine Eq. (24) with $k \geq 3$, $m \geq k+2$, $m \geq n \geq 2$ and $m \geq x \geq 2$, where we have $m = \max\{k, n, m, x\}$.

Let us start by assuming that $10x < \alpha^{2n}$. Then, by (13), (15) and (24),

$$f_\alpha \alpha^{m-1} + \varepsilon(k, m) = C^x \gamma^{nx} + e(n, x).$$

Thus,

$$\left| f_\alpha \alpha^{m-1} - C^x \gamma^{nx} \right| < \frac{1}{2} + \frac{8x\gamma^{nx}}{\phi^{2n}} < \frac{8m}{\phi^{2n}} \left(1 + \gamma^{nx}\right),$$

where we have used the fact that $x < m$. Dividing the previous inequality by $C^x \gamma^{nx}$, we get

$$\left| f_\alpha \alpha^{m-1} C^{-x} \gamma^{-nx} - 1 \right| < \frac{8m}{\phi^{2n}} \left(C^{-x} \gamma^{-nx} + C^{-x}\right).$$

However, for $x \geq 2, n \geq 2, C \geq 1/4$ and $\gamma \geq \phi$, it is clear that $C^{-x}\gamma^{-nx} + C^{-x} < 15$. Therefore, we have

$$|\Lambda_3| := \left| f_\alpha \alpha^{m-1} C^{-x} \gamma^{-nx} - 1 \right| < \frac{120m}{\phi^{2n}}. \tag{29}$$

We now set the conditions required to apply Matveev's Theorem; to this end, we require the following result.

Lemma 11 *Let C and γ as in Sect. 2.1 and α and f_α as in Sect. 2.2. The only integer solutions (k, m, n, x), with $k \geq 2$, $m \geq 3$, $n \geq 2$ and $x \geq 1$, for the equation*

$$f_\alpha \alpha^{m-1} C^{-x} \gamma^{-nx} - 1 = 0 \tag{30}$$

are the 4−tuples $(2, t, t, 1)$ with $t \geq 3$ when $(C, \gamma) = (1/\sqrt{5}, \phi)$, which correspond to the trivial parametric family of solutions for Eq. (1) when $\mathcal{U} = F$, $m = n$ and $x = y = 1$.

Proof Let us start with $\mathcal{U} = L$. In this case $(C, \gamma) = (1, \phi)$, therefore Eq. (30) implies that $f_\alpha = \phi^{nx}\alpha^{1-m}$ which is not possible according to Remark 1. In the other cases, we obtain

$$f_\alpha^{-1} = \alpha^{m-1}\gamma^{-nx} D^x, \quad \text{where} \quad (D, \gamma) \in \{(\sqrt{5}, \phi), (2\sqrt{2}, \beta)\}.$$

So, we take the norm $N_{\mathbb{K}}$ in $\mathbb{K} = \mathbb{Q}(\alpha, \gamma)$ on both sides to get

$$\left(\frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} \right)^{\delta_1} = E^{x\delta_2},$$

with $E \in \{8, 5\}$, $\delta_1 \in \{1, 2\}$ and $\delta_2 \leq k$. Hence,

$$\frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} = p^a, \quad \text{for} \quad p \in \{2, 5\} \quad \text{and} \quad a \in \mathbb{Z}^+. \tag{31}$$

According to Lemma 6 (ii), the Diophantine Eq. (31) with $p = 2$ (i.e. $\mathcal{U} = P$) implies $a = k - 1$. However, we can easily prove that $\Delta_k > 2^{k-1}$ holds for all $k \geq 2$. Indeed, note that $\Delta_k > 2^{k-1}$ is equivalent to

$$2^{k+1}k^k - (k+1)^{k+1} > 2^{k-1}(k-1)^2.$$

Since $(k+1)^{k+1} = (k+1)^{k-1}(k+1)^2 > 2^{k-1}(k-1)^2$, it suffices to show that

$$2^{k+1}k^k > 2(k+1)^{k+1},$$

which is equivalent to

$$2^k > (k+1)\left(1 + \frac{1}{k}\right)^k.$$

But $(1 + 1/k)^k < e < 3$, so, the above inequality is implied by $2^k > 3(k+1)$, which is easily proved by induction to hold for all $k \geq 5$. It is a straight-forward process to check that the original inequality $\Delta_k > 2^{k-1}$ also holds for the remaining range $k \in \{2, 3, 4\}$. This proves that Eq. (30) has no solutions for all $k \geq 2$, $m \geq 3$, $n \geq 2$ and $x \geq 1$, when $\mathcal{U} = P$.

Now, we take $p = 5$ (i.e. $\mathcal{U} = F$). Here, we are going to show that the only solution is $k = 2$. In this case Δ_k is odd, so by Lemma 6 (ii) we have that k is even. So, we will analyse the cases $k \equiv 0 \pmod 4$ and $k \equiv 2 \pmod 4$. Besides, note that k and $k+1$ are coprime to 5 otherwise the numerator of Δ_k is coprime to 5. Let us assume first that $k \equiv 0 \pmod 4$. By Fermat's Little Theorem for the prime 5,

$$2^{k+1} \equiv 2 \pmod 5, \quad k^k \equiv 1 \pmod 5, \quad (k+1)^{k+1} \equiv k+1 \pmod 5,$$

therefore the numerator of Δ_k satisfies

$$2^{k+1}k^k - (k+1)^{k+1} \equiv 2 - (k+1) \equiv 1 - k \pmod 5.$$

So, we must have $k \equiv 1 \pmod 5$.

Next, we assume that $k \equiv 2 \pmod 4$. Then $k + 1 \equiv 3 \pmod 4$, so again by Fermat's Little Theorem for the prime 5 we have

$$\begin{aligned} 2^{k+1}k^k - (k+1)^{k+1} &\equiv (2k)^{k+1}\left(1/k - ((k+1)/(2k))^{k+1}\right) \pmod 5 \\ &\equiv (2k)^{k+1}\left(1/k - ((k+1)/(2k))^{-1}\right) \pmod 5 \\ &\equiv (2k)^{k+1}(1/k - (2k)/(k+1)) \pmod 5. \end{aligned}$$

Thus, we get that

$$1/k - (2k)/(k+1) \equiv 0 \pmod 5 \quad \text{so} \quad 2k^2 - k - 1 \equiv 0 \pmod 5,$$

which implies $(k-1)(2k+1) \equiv 0 \pmod 5$. Therefore, we have $k \equiv 1 \pmod 5$ or $k \equiv 2 \pmod 5$. Furthermore, if $k \equiv 2 \pmod 5$ then $k \equiv 2 \pmod 4$. Then, we assume $k \equiv 1 \pmod 5$. By Lemma 6 (i), we obtain

$$\Delta_k \equiv \frac{k(k-1)^2(7k-17)}{6}2^{k-2} + k(3k-5)2^{k-2} + 2^k \pmod{(k-1)^2}.$$

Since $5 \mid k-1$, we get

$$\Delta_k \equiv -2^{k-1} + 2^k \pmod{25} \equiv 2^{k-1} \pmod{25},$$

so it is clear that Δ_k is not divisible by 5 in this case.

Next, let us assume $k \equiv 2 \pmod 5$ and $k \equiv 2 \pmod 4$. Thus, $k \equiv 2 \pmod{20}$. Since $20 = \varphi(25)$, it follows by Euler's Theorem for 25 that

$$2^{k+1} \equiv 2^3 \pmod{25}, \quad k^k \equiv k^2 \pmod{25}, \quad (k+1)^{k+1} \equiv (k+1)^3 \pmod{25}.$$

Thus, the numerator of Δ_k is

$$8k^2 - (k+1)^3 \equiv -(k-1)(k^2-4k-1) \pmod{25}.$$

If we assume that $25 \mid \Delta_k$, we get that $25 \mid k^2 - 4k - 1 = (k-2)^2 - 5$, which is impossible since $25 \mid (k-2)^2$ but $25 \nmid 5$. This argument allow us to conclude that, if k is even then $\nu_5(\Delta_k) < 2$. Therefore, Eq. (31) with $p = 5$ forces to have $a = 1$ and, then, $\Delta_k = 5$, which holds only for $k = 2$. However, for $k > 2$ we have $\Delta_k > 5$ (recall that we have already shown that $\Delta_k > 2^{k-1}$ for all $k \geq 2$, and it is clear that $2^{k-1} > 5$ for $k \geq 5$. The cases $k = 3$ and 4 can be checked individually).

In summary, we have shown that $k = 2$ when $\mathcal{U} = F$. Finally, since $f_\phi = \phi/\sqrt{5}$, the Eq. (30) becomes

$$\phi^{nx-m} = \sqrt{5}^{x-1}.$$

However, α is a unit while $\sqrt{5}$ is not, so both of the exponents above must be zero. Hence, $x = 1$ and $n = m$, as we stated. □

Note that the previous result allow us to conclude that $\Lambda_3 \neq 0$ for $k \geq 3$. So, here we take $t := 4$,

$$\begin{array}{llll} \tau_1 := f_\alpha, & \tau_2 := \alpha, & \tau_3 := C, & \tau_4 := \gamma \\ b_1 := 1, & b_2 := m-1, & b_3 := -x, & b_4 := -nx \\ A_1 := 4k\log k, & A_2 := 2\log 2, & A_3 := 2k\log 5, & A_4 := 2k\log 3. \end{array}$$

Also, we have to take $\mathbb{K} := \mathbb{Q}(\gamma, \alpha)$, $D := 2k$ and $B := m^2$. Hence, by Matveev's Theorem 1, we get the following lower bound for $|\Lambda_3|$:

$$\exp\left(-2.4 \times 10^{15} \times k^5(1+\log 2k)(1+2\log m)(\log k)\right).$$

By (29), after some calculations

$$\exp\left(-2.3 \times 10^{16} \times k^5 \log^2 k \log m\right) < 120m\left(\phi^{-2n}\right),$$

which implies

$$n < 2.6 \times 10^{16} \times k^5 \log^2 k \log m. \tag{32}$$

We got the previous inequality under the assumption that $10x < \phi^{2n}$. However, if this inequality does not hold, we get $n < 6 \log m$, which clearly is a smaller bound than the one in (32).

On the other hand, if we use (15) and (24), we get

$$u_n^x = f_\alpha \alpha^{m-1} + \varepsilon(k, m).$$

Hence, rearranging, dividing by $f_{\alpha_k} \alpha_k^{m-1}$ and using the fact that $f_{\alpha_k} > 1/2$ and $\alpha < \alpha_k < 2$ for $k \geq 3$, we have

$$|\Lambda_4| := \left| u_n^x f_\alpha^{-1} \alpha^{-(m-1)} - 1 \right| < \frac{1}{\alpha^{m-1}} < \frac{2}{\phi^m}. \tag{33}$$

Let us assume that $\Lambda_4 = 0$. Then, we get

$$f_\alpha = u_n^x \alpha^{1-m},$$

which is a contradiction with Remark 1 because the right-hand is an algebraic integer. Therefore, we can use Theorem 1 to bound $|\Lambda_4|$ from below. So, let us take $t := 3$,

$$\begin{array}{lll} \tau_1 := u_n, & \tau_2 := f_\alpha, & \tau_3 := \alpha, \\ b_1 := x, & b_2 := -1, & b_3 := -(m-1), \\ A_1 := kn \log 3, & A_2 := 2k \log k, & A_3 := \log 2. \end{array}$$

Further, $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$ and $B := m$. Hence, by Theorem 1, we have the following lower bound for $|\Lambda_4|$:

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times k^2 (1 + \log k)(1 + \log m)(kn \log 3)(2k \log k)(\log 2)\right).$$

By inequality (33), we get $\exp\left(-8.8 \times 10^{11} nk^4 \log^2 k \log m\right) < 2\phi^{-m}$, so, after some calculations we get $m < 2 \times 10^{12} nk^4 \log^2 k \log m$. Now, we insert (32) in the previous inequality to get

$$m < 5.2 \times 10^{28} k^9 \log^4 k \log^2 m.$$

Finally, by Lemma 8 with $\ell := 2$, $T := 5.2 \times 10^{28} k^9 \log^4 k$ and $x := m$, we get the inequality $m < 2.7 \times 10^{34} k^9 \log^6 k$. In conclusion, we have the following intermediary result.

Lemma 12 *If the integer triplet (k, m, n) with $k \geq 3$, $m \geq k + 2$ and $n \geq 2$, is such that the Diophantine Eq. (24) holds, then*

$$m < 2.7 \times 10^{34} k^9 \log^6 k, \tag{34}$$

where $m = \max\{k, m, n, x\}$.

4.3.1 An absolute upper bound for *k*

In this part, we work under the assumption that $k > 500$. Thus, we get $2.7 \times 10^{34} k^9 \log^6 k < 2^{k/2}$, which lets us use (19) with $r = m$ and $c = 1/2$. So, we have

$$F_m^{(k)} = 2^{m-2}\left(1 + \rho(k, m)\right) \quad \text{with} \quad |\rho(k, m)| < 2/2^{k/2} < 10^{-90}.$$

Besides, since $n \geq 5$, if we assume $10x < \phi^{2n}$, by (13) we have

$$u_n^x = C^x \gamma^{nx}\left(1 + \xi(n, x)\right) \quad \text{with} \quad |\xi(n, x)| < 8x/\phi^{nx} < 0.8.$$

Therefore, the Diophantine Eq. (24) implies

$$2^{m-2}\left(1 + \rho(k, m)\right) = C^x \gamma^{nx}\left(1 + \xi(n, x)\right), \tag{35}$$

which, rearranged gives

$$2^{m-2} - C^x \gamma^{nx} = C^x \gamma^{nx} \xi(n, x) - 2^{m-2} \rho(k, m).$$

Now, taking the absolute value both sides and dividing by 2^{m-2}, we get

$$\left|2^{-(m-2)} C^x \gamma^{nx} - 1\right| < \frac{8x}{\phi^{nx}}\left(\frac{C^x \gamma^{nx}}{2^{m-2}}\right) + \frac{2}{2^{k/2}}.$$

However, by (35), we have

$$\frac{C^x \gamma^{nx}}{2^{m-2}} = \frac{1 + \rho(k, m)}{1 + \xi(n, x)} < \frac{1.01}{0.2} < 5.1.$$

Then, we can conclude that

$$|\Lambda_5| := \left|2^{-(m-2)} C^x \gamma^{nx} - 1\right| < \frac{14x}{\phi^{\min\{n, k/2\}}}. \tag{36}$$

Let us assume $\Lambda_5 = 0$. Then, we get

$$2^{m-2} = C^x \gamma^{nx}.$$

However, $\mathrm{N}_i(2^{m-2}) \geq 2$ and $\mathrm{N}_i(C^x \gamma^{nx}) = \mathrm{N}_i(C^x) \leq 1$, where we have used $i = 1$ for $\mathcal{U} = F$ and $\mathcal{U} = L$ and $i = 2$ for $\mathcal{U} = P$. Thus, we conclude that $\Lambda_5 \neq 0$. So, we set the conditions to apply Matveev's Theorem to bound $|\Lambda_5|$ from below. Here we take $t := 3$,

$$\begin{array}{lll} \tau_1 := 2, & \tau_2 := C, & \tau_3 := \gamma, \\ b_1 := -(m-2), & b_2 := x, & b_3 := nx, \\ A_1 := 2\log 2, & A_2 := 2\log 5, & A_3 := 2\log 3. \end{array}$$

We also take $\mathbb{K} := \mathbb{Q}(\gamma)$, $D := 2$ and $B := m^2$. Hence, by Matveev's Theorem 1, we have the following lower bound for $|\Lambda_5|$:

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+2\log m)(2\log 2)(2\log 5)(2\log 3)\right).$$

By inequality (36), we have that

$$\exp\left(-3.9 \times 10^{13}\log m\right) < 14x\phi^{-\min\{n,k/2\}}.$$

However, by (34), we have $\log m < 130\log k$. Therefore, after some calculations, using the fact that $x < m$, we get

$$\min\{n, k/2\} < 2 \times 10^{18}\log^2 k.$$

If $\min\{n, k/2\} = k/2$, then we get

$$k < 1.1 \times 10^{22}. \tag{37}$$

If $\min\{n, k/2\} = n$, then we get

$$n < 2 \times 10^{18}\log^2 k. \tag{38}$$

However, let us consider $\alpha^{2n} \leq 10x$. Since $x < m$, by (34), we get $n < 126\log k$, which we compare with (38) and it is clear that we can keep working with (38).

We have to consider that Eq. (24) can be rewritten as

$$2^{m-2}(1+\rho(k,m)) = (u_n)^x,$$

which, by Eq. (19) with $c = 1/2$, implies

$$|\Lambda_6| := \left|u_n^x 2^{-(m-2)} - 1\right| < \frac{2}{2^{k/2}}. \tag{39}$$

If $\Lambda_6 = 0$, then we get the equation $(u_n)^x = 2^{m-2}$, that we already commented on at the beginning of this section. So, we may assume that $\Lambda_6 \neq 0$ and we apply Matveev's

Theorem 1 to bound $|\Lambda_6|$ from bellow. Let us set $t := 2$,

$$\begin{array}{ll} \tau_1 := u_n, & \tau_2 := 2, \\ b_1 := x, & b_2 := -(m-2), \\ A_1 := n \log 3, & A_2 := \log 2. \end{array}$$

We also take $\mathbb{K} = \mathbb{Q}$, $D = 1$ and $B = m$. Then, we get the following lower bound for $|\Lambda_6|$:

$$\exp\left(-1.4 \times 30^5 \times 2^{4.5}(1+\log m)(n \log 3)(\log 2)\right).$$

By (39), we have

$$\exp\left(-1.2 \times 10^{10} n \log m\right) < 2^{1-k/2}.$$

By (38) and the fact that $\log m < 130 \log k$, we get $k < 5 \times 10^{30} \log^3 k$, which implies

$$k < 3 \times 10^{36}. \tag{40}$$

In conclusion, by (34), (37) and (40), we have the following intermediate result.

Lemma 13 *Let the integer triplet (k, m, n) with $k > 500$, $m \geq k+2$ and $n \geq 2$ be such that the Diophantine Eq. (24) has a solution. Then*

$$k < 3 \times 10^{36} \quad \text{and} \quad m < 1.9 \times 10^{374},$$

where $m = \max\{k, m, n, x\}$.

4.3.2 Reducing the absolute upper bound over *k*

Let us consider $\Gamma_5 := \log(\Lambda_5 + 1)$, under the assumption that $k > 500$. So, by (36) and Lemma 7, under de assumption that $28x < \alpha^{\min\{n,k/2\}}$, we have

$$|\Gamma_5| = |(m-2)\log 2 - x \log C - nx \log \gamma| < \frac{24x}{\phi^{\min\{n,k/2\}}}. \tag{41}$$

Let us start with the case Fibonacci $(C, \gamma) = (1/\sqrt{5}, \phi)$ where we will apply the LLL algorithm. So, here we set:

$$\begin{array}{lll} \tau_1 := \log 2, & \tau_2 := \log\sqrt{5}, & \tau_3 := \log\phi, \\ x_1 := m-2, & x_2 := x, & x_3 := -nx, \end{array}$$

and let $(X_1, X_2, X_3) = (1.9 \times 10^{374}, 1.9 \times 10^{374}, 3.7 \times 10^{748})$ be upper bounds for $(x_1, x_2, |x_3|)$. Therefore, by Lemma 5, we get

$$4.4 \times 10^{-1499} < |\Gamma_5|,$$

which, together with (41), implies

$$\min\{n, k/2\} \leq 8970, \tag{42}$$

where we have used the fact that $x < m$ and that $m < 1.9 \times 10^{374}$.

On the other hand, for the Lucas and Pell cases $(C, \gamma) = (1, \phi), (1/2\sqrt{2}, \beta)$, we obtain the following expressions from (41):

$$\left|\tau_\phi - \frac{nx}{m-2}\right| < \frac{50x}{\phi^{\min\{n,k/2\}}(m-2)}, \tag{43}$$

and

$$\left|\tau_\beta - \frac{2nx}{2(m-2)+3x}\right| < \frac{10x}{\phi^{\min\{n,k/2\}}(2(m-2)+3x)}, \tag{44}$$

with $\tau_\phi := \log 2/\log \phi$ and $\tau_\beta := \log 2/\log \beta$. Then, we set the conditions to use Lemma 3 on each inequality (43), (44). By Lemma 13, we take $M_\phi := 1.9 \times 10^{374}$ as an upper bound for $m-2$ and $M_\beta := 9.5 \times 10^{374}$ as an upper bound for $2(m-2)+3x$. In the case τ_ϕ, we computationally found that $q_{722} > 2.2 \times 10^{374} > M_\alpha$ and $a_M = 1491$, then

$$\frac{1}{1493(m-2)^2} < \left|\tau_\phi - \frac{nx}{m-2}\right|,$$

which we compare with (43) and we get $\phi^{\min\{n,k/2\}} < 74650x(m-2)$, which implies

$$\min\{n, k/2\} \leq 3606, \tag{45}$$

since $x < m$ and $m < 1.9 \times 10^{374}$.

In the case of τ_β, we found that $q_{725} > 6.2 \times 10^{374}$ and $a_M = 2030$, so

$$\frac{1}{2032(2(m-2)+3x)^2} < \left|\tau_\beta - \frac{nx}{2(m-2)+3x}\right|,$$

which we compare with (44) and we get $\phi^{\min\{n,k/2\}} < 20320x(2(m-2)+3x)$, which implies

$$\min\{n, k/2\} < 3601, \tag{46}$$

since $x < m$ and $m < 1.9 \times 10^{374}$.

In conclusion, by (42), (45) and (46), we get

$$\min\{n, k/2\} \leq 8970.$$

If $\min\{n, k/2\} = k/2$, then, by (34), we get

$$k \leq 17940 \quad \text{and} \quad m < 4.6 \times 10^{78}. \tag{47}$$

If $\min\{n, k/2\} = n$, then we get $n \leq 8970$. So, let us take $\Gamma_6 := \log(\Lambda_6 + 1)$. Since $k > 500$, by (7) and (39), we have

$$|\Gamma_6| = |x \log u_n - (m-2)\log 2| < \frac{4}{2^{k/2}}.$$

Thus, after we divide by $(m-2)\log u_n$, we get

$$\left| \tau_u - \frac{x}{m-2} \right| < \frac{6}{2^{k/2}(m-2)}, \tag{48}$$

where $\tau_u := \log 2/\log u_n$. We have to apply Lemma 3 for each $n \in [n_U, 8970]$ with $M_U := 1.9 \times 10^{374}$. Note that n_U is taken such that u_n is not a power of 2, which is easy to determine since we know where to find their perfect powers. In addition, we consider those $n \in [2, n_U)$ where u_n is not a power of two:

$$\begin{array}{lll} n_U = 7, \text{ and } n \in \{4, 5\}, & \text{when } \mathcal{U} = F, \\ n_U = 4, \text{ and } n \in \{2\}, & \text{when } \mathcal{U} = L, \\ n_U = 3, & \text{when } \mathcal{U} = P. \end{array}$$

Our computations show that

$$\max\{a_{M_U} : \mathcal{U} = F,\ \mathcal{U} = L,\ \mathcal{U} = P\} \leq 5.7 \times 10^7.$$

Thus, we get $2^{k/2} < 6(5.7 \times 10^7)(m-2)$, which implies $k \leq 2543$, where we used the fact that $m < 1.9 \times 10^{374}$. Comparing this new absolute bound for k with the one given in (47), it is clear that we can conclude that (47) holds in all cases.

Finally, we use the new bounds on k and m to implement a couple of times more an analogous reduction process like the one we have already done. So, with Γ_5 we get $\min\{n, k/2\} \leq 1750$, and we proceed as before with Γ_6 to conclude that $k \leq 3500$, a bound we get under the assumption that $k > 500$. Therefore, we have the following result.

Lemma 14 *Let the integer triplet (k, m, n) with $k > 500$, $m \geq k+2$ and $n \geq 2$ be such that the Diophantine Eq. (24) has a solution. Then*

$$k \leq 3500 \quad \textit{and} \quad m < 6.3 \times 10^{71},$$

where $m = \max\{k, m, n, x\}$.

4.3.3 Reducing the absolute upper bound on *m*

To start, we have to reduce the absolute upper bound on n which is the same for m since $n < m$. In order to do this, we have to consider $\Gamma_3 := \log(\Lambda_3 + 1)$. So, under the assumption that $240m < \phi^{2n}$, by Lemma 7, we get

$$|\Gamma_3| = |\log f_\alpha + (m-1)\log\alpha - x\log C - nx\log\gamma| < \frac{240m}{\phi^{2n}}. \tag{49}$$

Let us start with the Fibonacci case; i.e., $(C, \gamma) = (1/\sqrt{5}, \phi)$. Wel apply LLL–algorithm. Let us set:

$$\begin{aligned}
&\tau_1 = \log f_\alpha, \ \tau_2 := \log\alpha, \quad \tau_3 := \log\sqrt{5}, \ \tau_4 := \log\phi,\\
&x_1 := 1, \qquad x_2 := m-1, \ x_3 := x, \qquad x_4 := -nx,
\end{aligned}$$

and let $(X_1, X_2, X_3, X_4) = (1, 6.3\times10^{71}, 6.3\times10^{71}, 4\times10^{143})$ be a vector of upper bounds for $(x_1, x_2, x_3, |x_4|)$. By Lemma 5, we get $1.5 \times 10^{-1515} < |\Gamma_3|$ for each $k \in [3, 3500]$, which, together with (49) implies

$$n \le 3805, \tag{50}$$

where we have used the fact that $m < 6.3 \times 10^{71}$.

In the Pell case; i.e., $(C, \gamma) = (1/2\sqrt{2}, \beta)$, we set:

$$\begin{aligned}
&\tau_1 = \log f_\alpha, \ \tau_2 := \log\alpha, \quad \tau_3 := \log 2\sqrt{2}, \ \tau_4 := \log\beta,\\
&x_1 := 1, \qquad x_2 := m-1, \ x_3 := x, \qquad x_4 := -nx,
\end{aligned}$$

and $(X_1, X_2, X_3, X_4) = (1, 6.3 \times 10^{71}, 6.3 \times 10^{71}, 4 \times 10^{143})$ be a vector of upper bounds for $(x_1, x_2, x_3, |x_4|)$, respectively. In this case, we get $3.7 \times 10^{-1624} < |\Gamma_3|$ for each $k \in [3, 3500]$, which, together with (49), implies

$$n \le 4070. \tag{51}$$

Finally, in the Lucas case; i.e., $(C, \gamma) = (1, \phi)$, for each $k \in [3, 3500]$ we use the Dujella and Pethő result, Lemma 4. So, we divide by $\log\phi$ inequality (49), and we set:

$$\begin{aligned}
&u := m-1, \qquad v := nx, \qquad w := n,\\
&\tau := \log\alpha/\log\phi, \ \ \mu := \log f_\alpha/\log\phi,\\
&A := 240m/\log\phi, \ B := \phi^2.
\end{aligned}$$

We take $M := (6.3 \times 10^{71})^2$ as an bound for xn, which holds given that $x < m$ together to Lemma 14. In this case, we get $0.0010603 < \min_k \varepsilon$, and, by Lemma 4, we get

$$n \le 362. \tag{52}$$

Note that, if $240m < \alpha^{2n}$ does not hold, then we get $n \leq 177$, so, if we put this this together with (50), (52), and (51), we can conclude that $n \leq 4070$.

Next, we proceed to reduce the absolute upper bound on m. Here, we have to consider $\Gamma_4 := \log(\Lambda_4 + 1)$. So, if we assume that $4 < \alpha^m$, then, by Lemma 7,

$$|\Gamma_4| = |x \log u_n - \log f_\alpha - (m-1)\log\alpha| < \frac{4}{\phi^m}. \tag{53}$$

As before, for each $k \in [3, 3500]$ and $n \in [2, 4070]$ we use the Lemma 4. Therefore, we divide the previous inequality by $\log \alpha_k$ and we set:

$$\begin{array}{lll} u := x, & v := m-1, & w := m, \\ \tau := \log u_n / \log\alpha, & \mu := -\log f_\alpha / \log\alpha, & \\ \alpha := 4/\log\phi, & B := \phi, & \end{array}$$

and we use $M := 6.3 \times 10^{71}$ which is our bound for $m-1$ given by Lemma 14. In this case, we get

$$3.6 \times 10^{-494} < \min\{\varepsilon : k \in [3, 3500],\ n \in [2, 4070],\ \mathcal{U} = F,\ \mathcal{U} = P,\ \mathcal{U} = L\},$$

and, by Lemma 4, we get

$$m \leq 5050.$$

Now that we have reduced the upper bound for m, we can proceed to do a final reduction cycle on inequalities (41), (43–44), (48), (49) and (53), with the corresponding selection of M, X_i for $i = 1, 2, 3, 4$ and C. In order to do so, we have to assume $k > 500$. By (41) and (43–44), we obtain $\min\{n, k/2\} \leq 110$, so it is clear that $\min\{n, k/2\} = n$ and then $n \leq 110$. Now, by (48), we have that $\max\{a_{M_U}\} \leq 700$ and $k \leq 50$, a contradiction with our assumption over k. Therefore, we conclude that $k \leq 500$. Now, by (49), we get $n \leq 730$ and finally by (53) we get $m \leq 750$. To summarize, we have the following result

Lemma 15 *Let the integer triplet* (k, m, n) *with* $k \geq 3$, $m \geq k+2$ *and* $n \geq 2$ *be a solution of the Diophantine equation (24). Then*

$$k \leq 500, \quad n \leq 730 \quad \textit{and} \quad m \leq 750,$$

where $m = \max\{k, m, n, x\}$.

4.3.4 The computational search for solutions

Thanks to Lemma 15, we are now ready to do a direct computational search for solutions of (24). Our strategy is to compare the last 30 digits on each side. For each solution obtained in this comparison, we find (k, m, n) satisfying (24).

To start, for each $k \in [3, 500]$ we create the list

$$A^{(k)} := \{F_m^{(k)} \pmod{10^{30}} : m \in [k+2, 750]\},$$

where we used the *Wolfram Mathematica* function `Mod[m,n]` which gives the remainder on division of m by n. On the other side, since $x < m$, we used the function `PowerMod[a,b,m]`, which gives $a^b \pmod{m}$, to create the list

$$B := \{u_n^x \pmod{10^{30}} : n \in [2, 730],\ x \in [2, 750]\},$$

for $\mathcal{U} = F, \mathcal{U} = L$ and $\mathcal{U} = P$. So, we look for $I^{(k)} = A^{(k)} \cap B$ and we get

$$\begin{array}{ll} I^{(3)} = \{81\},\quad I^{(6)} = \{125\}, & \text{when } \mathcal{U} = F,\\ I^{(3)} = \{81\}, & \text{when } \mathcal{U} = L,\\ \text{and}\quad I^{(6)} = \{125\}, & \text{when } \mathcal{U} = P, \end{array}$$

which correspond to the set of triplets (k, m, n):

$$\begin{array}{ll} \{(3, 9, 4), (6, 9, 5)\}, & \text{when } \mathcal{U} = F,\\ \{(3, 9, 2)\}, & \text{when } \mathcal{U} = L,\\ \text{and}\quad \{(6, 9, 3)\}, & \text{when } \mathcal{U} = P. \end{array}$$

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