Algebraic geometry and the Verlinde formula

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Contents

1 Introduction

This thesis will undertake to study certain projective algebraic varieties, namely the moduli spaces of stable bundles on an algebraic curve, which were first constructed and studied about thirty years ago, by David Mumford and the Tata Institute school. It will use almost exclusively algebro-geometric methods; indeed, though we work over the complex numbers for convenience, much of it will be valid over an arbitrary algebraically closed field. About the new techniques and approaches from theoretical physics which have so radically transformed the subject it will only say a little. And yet without a doubt it owes an enormous debt to physics, for it was a physicist, Erik Verlinde, working in conformal field theory, who first stated the remarkable formula on which it revolves.

A conformal field theory is a physical theory on a two-dimensional Euclidean space-time which depends only on the conformal class of the metric. Such theories arise naturally in the context of superstrings: the two-dimensional surface is thought of as a thickened Feynman diagram, or a complicated interaction between strings, and the Euclidean metric can be exchanged for a Minkowskian one through a process known as Wick rotation. If a surface is regarded as a thickening of a graph, for reasons of locality it is natural to expect that any physically significant numerical invariants of the surface can be computed as some sort of state-sum of products over the vertices of the graph, or equivalently, over the thrice-punctured spheres or "pairs of pants" which must be sewn together to produce the surface.

On the other hand, this seems quite unnatural in algebraic geometry, where it is customary to think of a smooth curve as a rigid and irreducible object, not to be cut up. So it was most surprising when in 1988 Erik Verlinde produced from conformal field theory what amounted to a formula for the Hilbert polynomials of moduli spaces of stable bundles on a curve [46]. Equally surprising, it was written not as a polynomial, but as a sum of $2g-2$th powers of trigonometric functions, where g is the genus of the curve. Since a Riemann surface of genus g is composed of $2g-2$ pairs of pants, this makes sense to a physicist, but certainly not to a mathematician. The importance of the formula for physics and mathematics was underscored a few months later, when it was used by Witten in his reinterpretation of the Jones invariants of three-manifolds in terms of Chern-Simons gauge theory [47].

However, it has not been easy to translate these impressive physical insights into rigorous mathematics, and a small army of mathematicians now seems to be struggling with them. Indeed, the author knows of at least seven independent approaches to the Verlinde formula alone. In spite of this, one of the goals of this thesis will be to contribute an eighth. The approach we shall settle on, however, has the advantage of being completely algebraic, and of producing not only the Verlinde formula itself, but also a natural generalization which, so far as the author knows, was not predicted by the physicists. It seems rather paradoxical—and may well be the result of good or bad luck—that of the two approaches explored in this thesis, the one which proves the more fruitful, both in recovering the Verlinde formula itself and in suggesting new avenues of research, is the one which has nothing at all to do with Verlinde's original techniques.

In the rank 2 case which we treat, we shall suggest two possible approaches to a proof of the Verlinde formula, although so far only one of them is complete. The first, which is discussed in part I, is formally similar to Verlinde's original work, in that it involves proving a factorization rule. When a Riemann surface is divided into two pieces, Verlinde's factorization rule decomposes its state space as a finite direct sum of tensor products of the state spaces from either side. Thus for any Riemann surface, one can inductively compute the dimension of this space, which is exactly the content of the Verlinde formula. Algebraic geometry forbids us to cut up a curve, so instead we let it degenerate to a curve with a node. This has almost the same effect as cutting, because the two sides of a curve are geometrically almost independent. (In fact, in part I we consider degenerations to an *irreducible* curve, which is more like cutting a Riemann surface along a non-separating curve.) Since Verlinde's formula, in our interpretation, concerns the moduli space of stable bundles, we need some such moduli space for a nodal curve. This has been provided by Gieseker [19]; modulo many technicalities, it is a fibre bundle with fibre a compactified algebraic group, SL_2 in our case. At the end of the proof, Verlinde's finite state-sum is revealed to be exactly the direct sum in the Peter-Weyl theorem for SL_2, but truncated after finitely many terms by the compactification.

This approach does not yet quite provide a complete proof of the formula, because of difficulties when the genus of the curve is small. Our second approach, which occupies parts II and III, is more successful. In it, we construct a new moduli space, of so-called *stable pairs,* which plays the role of the symmetric product in rank 2 Brill-Noether theory. Thus there is a natural Abel-Jacobi map to the moduli space of stable bundles, and we can pull back by this map to work on the new space. This makes explicit calculation more feasible, because there is a rational map from projective space to the new moduli space which can be factored into an explicit sequence of morphisms. In fact, it is a sequence of blow-ups and blow-downs—what in Mori theory are called flips—centred on symmetric products of the curve. Calculating the dimension of the Verlinde space can thus be reduced to calculating various Euler characteristics on the symmetric products, which in turn can be accomplished using Riemann-Roch and Macdonald's description [29] of the cohomology rings. The result of all this is not manifestly equivalent to the Verlinde formula, but this is proved in some elegant calculations by Don Zagier, which appear at the end of part III.

As a by-product of this approach, we recover a result of Bertram [5] identifying the Verlinde spaces with spaces of hypersurfaces of a certain degree in projective space, singular to a certain order on an embedded copy of the curve. Moreover, under mild hypotheses, the machine developed for the Verlinde formula can still calculate the dimension of the space for general degree and order, thereby generalizing the Verlinde formula in an unexpected direction. The trick involved is to exploit Kodaira vanishing and the behaviour of ample cones under flips to find a rational variety where the dimension we want is actually an Euler characteristic, which is then calculated inductively.

Finally, once we have proved the Verlinde formula, we proceed in part IV to an application, which is to determine the rational cohomology ring of the moduli space of stable bundles of rank 2 and fixed odd determinant, or at least to reduce the problem to a hard exercise in linear algebra. This is accomplished using the Riemann-Roch theorem, but in the reverse of the usual sense. Normally Riemann-Roch is used to translate topological information into holomorphic information; but in this case we begin with the Hilbert polynomial, and deduce the topology. It is worth mentioning, though, that several quite recent papers [12, 48] have given relatively elementary arguments to obtain the same topological information as ours. Part IV could then be used in reverse, to give an easy proof of the Verlinde formula, at least in the odd case.

A few notational habits should be mentioned: π denotes any obvious projection, such as projection on one factor, or down from a blow-up; tensor products of vector bundles are frequently indicated simply by juxtaposition; and likewise a pullback such as f^*L is often called just L. These conventions are not meant to be elliptical, but to clean up what would otherwise be some very messy formulas.

The various parts of the thesis were originally written as separate manuscripts. I have made every attempt to harmonize notation and presentation, but regret that I was unable to extirpate two minor inconsistencies. First, the notation X_i refers in part I to a nodal curve containing i lines, as explained there, but in parts II and III it refers to the ith symmetric product of a smooth curve. Second, the Verlinde vector spaces are defined at least three times! The definitions in the "prelude" and in parts I and III are all well suited to the tasks before them, but showing that they are all equivalent must be left as an exercise for the reader.

We also make the following assumptions, which are explained in the text but are repeated here for emphasis. We always assume the genus g of our curve is ≥ 2, and in part I we often assume it is ≥ 3 or 4. In the geometric invariant theory construction of §9, we assume d is large, an assumption which is justified by **(9.9)** and the discussion following it. From §11 to the end of part III we assume $d \geq 3$. However, this assumption is implicit in other inequalities—so for example our main formula **(16.8)** is valid as it stands. Finally, in part IV we assume d is odd, so that the moduli space of bundles will be smooth.

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2 Prelude: conformal field theory

We begin by discussing the Verlinde formula from a point of view very different from the one we will eventually take, namely that of conformal field theory. This has the advantage of being more or less Verlinde's original point of view and hence of clarifying how his formula was originally conjectured.

The sort of conformal field theory we shall be concerned with is known as the SU(2) Wess-Zumino-Witten model, and is discussed in detail in [15], [16], or [40]. We must first fix a positive integer k, called the *level.* Let X_g be a compact connected Riemann surface of genus g, with marked points $x_1, \ldots, x_p \in X_g$. We choose a *labelling* of the marked points, that is, we associate to each one an irreducible representation V of $SL_2(\mathbb{C}) = SU(2)_{\mathbb{C}}$ with $\dim_{\mathbb{C}} V \leq k+1$. Such representations are determined up to isomorphism by their dimension: we will write V_n for the representation of dimension $n+1$, and denote by $(X_g; x_i, n_i)$ the Riemann surface with points x_i labelled by V_{n_i}. To each such labelled Riemann surface we wish to associate a *Verlinde vector space* $Z_k(X_g; x_i, n_i)$.

Let P be a smooth principal $SL_2(\mathbb{C})$ bundle over X_g; then P is unique up to isomorphism, and the set $\mathcal{A}$ of compatible holomorphic structures on the associated rank 2 vector bundle U is an affine space modelled on $\Omega^{0,1}_{\mathbb{C}}(X_g; \text{ad } P)$. There exists a line bundle $\mathcal{L} \to \mathcal{A}$ whose fibre at any holomorphic structure is the determinant line of the associated $\overline{\partial}$-operator on U. It is easy to see that the usual action of the gauge group $\mathcal{G} = \Gamma(\text{Ad } P)$ on $\mathcal{A}$ lifts to $\mathcal{L}$, so we have an action of $\mathcal{G}$ on the tensor power $\mathcal{L}^k$. Next, at each marked point x_i, choose an identification of the fibre $(\text{Ad } P)_{x_i}$ with $SL_2(\mathbb{C})$. (Actually, Ad P always has a global trivialization in the present case, so we may as well just use a restriction of that.) Then $\mathcal{G}$ also acts on each V_{n_i} via evaluation at the point x_i. We may now define our vector space at level k as a space of $\mathcal{G}$-equivariant sections:

$$Z_k(X_g; x_i, n_i) = \Gamma_{\mathcal{G}}(\mathcal{L}^k \otimes \bigotimes_i V_{n_i}).$$

In fact, all we will really want to know about $Z_k(X_g; x_i, n_i)$ is its dimension, which we shall denote $z_k(X_g; x_i, n_i)$. Now it is well known to physicists that Z_k is an example of a *modular functor*. In particular, it ought to satisfy the axioms below, which have important implications for the computation of $z_k(X_g; x_i, n_i)$. First of all:

(2.1) *Axiom.* The value of $z_k(X_g; x_i, n_i)$ is independent of the complex structure on X_g.

Indeed, the spaces $Z_k(X_g; x_i, n_i)$ for different X_g are supposed to patch together to form a vector bundle over (decorated) Teichmüller space, and this is supposed to have a natural projectively flat connection [25, 3] enabling all the $Z_k(X_g; x_i, n_i)$ to be identified up to scalars. However, we will content ourselves with dimension, and will not pursue this.

Note that **(2.1)** implies in particular that $z_k(X_g; x_i, n_i)$ is independent of the positions of the x_i and of the trivializations of the fibres over them, because any two choices can be exchanged by a suitable smooth map. Thus we will be justified in henceforth writing simply $z_k(X_g; n_i)$ for $z_k(X_g; x_i, n_i)$.

We next have two *gluing* or *factorization axioms*, which describe the behaviour of z_k under (i) the addition of a handle and (ii) connect-sum. In light of (6.1), we don't need to worry about the choice of complex structure.

(2.2) *Axiom.*

(i) $z_k(X_{g+1}; n_i) = \sum_{n=0}^{k} z_k(X_g; n, n, n_i).$

(ii) $z_k(X_{g+g'}; n_i, n_j) = \sum_{n=0}^{k} z_k(X_g; n, n_i)\, z_k(X_{g'}; n, n_j).$

The definition of Z_k given above is well known. However, as it stands the definition involves $\mathrm{SL}_2(\mathbb{C})$ bundles over X_g, which are topologically trivial. Rank 2 bundles of any even degree would give more or less the same spaces, but we are also interested in bundles of odd degree over X_g. Hence we shall need to introduce a "twisted" version of the theory; fortunately, there is a straightforward way to do this. We consider $(X_g; x_i, n_i)$ as before. Let $\hat{P}$ be a smooth principal $\mathrm{PSL}(2,\mathbb{C})$ bundle over X_g with $w_2(\hat{P}) \neq 0$; again, $\hat{P}$ is unique up to isomorphism. The associated $\mathbb{CP}^1$ bundle lifts to a rank 2 vector bundle $\hat{U}$ of odd degree. Let $\hat{\mathcal{A}}$ denote the set of compatible holomorphic structures on $\hat{U}$ which induce a fixed holomorphic structure ω on $\Lambda^2\hat{U}$. (We didn't need to make this explicit in the untwisted case, because $\Lambda^2 U$ is canonically trivial.) As before we let $\hat{\mathcal{L}}$ be the determinant line bundle. However, we need to be careful in our definition of the gauge group; we let $\hat{\mathcal{G}} = \Gamma(\widetilde{\mathrm{Ad}}\ \hat{P})$, where $\widetilde{\mathrm{Ad}}\ \hat{P}$ is the adjoint bundle with fibre $\mathrm{SL}_2(\mathbb{C})$, *not* $\mathrm{PSL}(2,\mathbb{C})$. As in the untwisted case, we choose identifications of $(\widetilde{\mathrm{Ad}}\ \hat{P})_{x_i}$ with $\mathrm{SL}_2(\mathbb{C})$, and define

$$\hat{Z}_k(X_g; x_i, n_i) = \Gamma_{\hat{\mathcal{G}}}(\hat{\mathcal{L}}^k \otimes \bigotimes_i V_{n_i}).$$

Again, we write $\hat{z}_k(X_g; x_i, n_i) = \dim \hat{Z}_k(X_g; x_i, n_i)$. In analogy with the properties of z_k discussed above, we propose the following.

(2.3) *Axiom.* The value of $\hat{z}_k(X_g; x_i, n_i)$ is independent of the complex structure on X_g.

In particular, it is again independent of the positions of the x_i and the trivializations of the fibres, as well as the choice of the holomorphic structure ω on $\Lambda^2\hat{U}$. As in the untwisted case, we shall henceforth write simply $\hat{z}_k(X_g; n_i)$ for $\hat{z}_k(X_g; x_i, n_i)$. We also have a twisted factorization axiom:

(2.4) *Axiom.*

(i) $\hat{z}_k(X_{g+1}; n_i) = \sum_{n=0}^{k} \hat{z}_k(X_g; n, n, n_i)$.

(ii) $\hat{z}_k(X_{g+g'}; n_i, n_j) = \sum_{n=0}^{k} \hat{z}_k(X_g; n, n_i)\ z_k(X_{g'}; n, n_j)$.

Now in the untwisted case, if the genus g is at least 2, then the set $\mathcal{A}_{ss}$ of semistable holomorphic structures has complement of codimension ≥ 2 (see p. 569 of [2]), and $N = \mathcal{A}_{ss}/\mathcal{G}$ is the moduli space of semistable $\mathrm{SL}_2(\mathbb{C})$ bundles over X_g. The stabilizer $\pm I \subset \mathcal{G}$ acts as $(-1)^{n_i}$ on V_i, so $\mathcal{L}^k \otimes \bigotimes_i V_{n_i}$ descends to N if and only if $\sum n_i$ is even. In this case, $Z_k(X_g; x_i, n_i)$ can be regarded as the space of holomorphic sections of a vector bundle over N. In particular, let $\mathcal{O}(\Theta)$ denote the line bundle over N such that $\mathcal{L}$ descends to $\mathcal{O}(\Theta)$. Then if there are no marked points at all,

$$Z_k(X_g) = H^0(N; \mathcal{O}(k\Theta)).$$

The twisted case is similar: if g is at least 2, then as before the set $\hat{\mathcal{A}}_{ss}$ has complement of codimension ≥ 2, and $\hat{N} = \hat{\mathcal{A}}_{ss}/\hat{\mathcal{G}}$ is the moduli space of stable bundles of fixed odd determinant. Indeed this is in some ways even better than N, because it is smooth [42]. The section $-I \in \hat{\mathcal{G}}$ now acts nontrivially on $\hat{\mathcal{L}}$ as well as the V_{n_i}, so that $\hat{\mathcal{L}}^k \otimes \bigotimes V_{n_i}$ descends to N if and only if $k + \sum n_i$ is even. In this case, $\hat{Z}_k(X_g; n_i)$ can be identified with the space of holomorphic sections of a vector bundle over N. In particular, let $\mathcal{O}(2\Theta)$ denote the descent of $\mathcal{L}^2$ to N. Then if there are no marked points, we have for even k

$$\hat{Z}_k(X_g) = H^0(N; \mathcal{O}(k\Theta)).$$

Thus $z_k(X_g)$ and $\hat{z}_k(X_g)$ can be interpreted as spaces of sections of a line bundle over a projective variety. This is the sense in which they will be studied in subsequent sections (although we shall dispense with the hats, preferring instead to define Z_k for any degree d). Our object in the remainder of this section will hence be to give explicit formulas for $z_k(X_g)$ and $\hat{z}_k(X_g)$, and one of our main objects in the remainder of the thesis will be to prove them.

Following Verlinde [46], we define the coefficients

$$N_{m_1,m_2,m_3} = z_k(X_0; m_1, m_2, m_3),$$

which are symmetric in all three indices. Define the $(k+1) \times (k+1)$ symmetric matrices N_m by

$$(N_m)_{i,j} = N_{m,i,j}.$$

Both here and in future we index the rows and columns starting with 0 instead of 1, for convenience. We then obtain the following formula for the torus with two marked points.

(2.5) $z_k(X_1; m_1, m_2) = \operatorname{tr} N_{m_1} N_{m_2}$.

Proof.
$$\begin{aligned} z_k(X_1;m_1,m_2) &= \sum_m z_k(X_0;m,m,m_1,m_2) \\ &= \sum_{m,n} z_k(X_0;m,n,m_1)\ z_k(X_0;m,n,m_2) \\ &= \sum_{m,n} (N_{m_1})_{m,n}(N_{m_2})_{n,m} \\ &= \operatorname{tr} N_{m_1}N_{m_2}.\ \Box \end{aligned}$$

Now define another $(k+1)\times(k+1)$ symmetric matrix M by $(M)_{i,j} = \operatorname{tr}(N_iN_j)$. This enables us to generalize the previous formula to arbitrary genus.

(2.6) *For* $g \geq 1$, $z_k(X_g;m_1,m_2) = (M^g)_{m_1,m_2}$.

Proof by induction. The case $g = 1$ is **(2.5)**. Then, if $z_k(X_{g-1};m_1,m_2) = (M^{g-1})_{m_1,m_2}$, by attaching a handle we obtain

$$\begin{aligned} z_k(X_g;m_1,m_2) &= \sum_n z_k(X_{g-1};m_1,n)\ z_k(X_1;n,m_2) \\ &= \sum_n (M^{g-1})_{m_1,n}(M)_{n,m_2} \\ &= (M^g)_{m_1,m_2}.\ \Box \end{aligned}$$

(2.7)
$$z_k(X_g) = \operatorname{tr} M^{g-1}.$$

Proof. $z_k(X_g) = \sum z_k(X_{g-1};m,m) = \sum (M^{g-1})_{m,m} = \operatorname{tr} M^{g-1}$. $\Box$

Now define yet another $(k+1)\times(k+1)$ symmetric matrix J by

$$J_{m_1,m_2} = \hat{z}_k(X_0;m_1,m_2).$$

(2.8)
$$\hat{z}_k(X_g;m_1,m_2) = (M^gJ)_{m_1,m_2}.$$

Proof. $\hat{z}_k(X_g;m_1,m_2) = \sum z_k(X_g;m_1,n)\hat{z}_k(X_0;n,m_2) = (M^gJ)_{m_1,m_2}$. $\Box$

(2.9)
$$\hat{z}_k(X_g) = \operatorname{tr} M^{g-1}J.$$

Proof. $\hat{z}_k(X_g) = \sum z_k(X_{g-1};m,n)\hat{z}_k(X_0;m,n) = \sum (M^{g-1}J)_{m,m} = \operatorname{tr} M^{g-1}J$.

The dimension formula should not depend on the order in which the gluing operations are performed, so the matrices M and J ought to commute. This is indeed true; in fact, an explicit simultaneous diagonalization is given by the so-called *Verlinde conjecture.* In our SU(2) case, it can be stated as follows. Let S be the $(k+1)\times(k+1)$ matrix such that

$$S_{ij}=\sqrt{\frac{2}{k+2}}\sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right).$$

Note that S is orthogonal and symmetric, so $S=S^{-1}$.

(2.10) $SN_mS=\operatorname{diag}\left(\frac{S_{n,m}}{S_{0,m}}\right)$, *and* $SJS=\operatorname{diag}(-1)^n$.

Various proofs of the first statement appear in [9], [17], and [47]; for the second, see [44]. In any case, this implies that M is also diagonalized by S:

(2.11)
$$SMS=\operatorname{diag}\left(\frac{1}{S^2_{0,m}}\right).$$

Proof.
$$\begin{aligned}(SMS)_{i,j} &= \sum_{m,n} S_{i,m}\operatorname{tr}(N_mN_n)S_{n,j}\\ &= \sum_{m,n,p,q} S_{i,m}N_{m,p,q}N_{n,p,q}S_{n,j}\\ &= \sum_p (SN_p^2S)_{i,j}.\ \square\end{aligned}$$

Using **(2.7)**, **(2.9)**, **(2.10)**, and **(2.11)**, we then get our desired formulas.

(2.12)
$$z_k(X_g)=\left(\frac{k+2}{2}\right)^{g-1}\sum_{m=1}^{k+1}\frac{1}{(\sin\frac{m\pi}{k+2})^{2g-2}}.$$

(2.13)
$$\hat{z}_k(X_g)=\left(\frac{k+2}{2}\right)^{g-1}\sum_{m=1}^{k+1}\frac{(-1)^{m+1}}{(\sin\frac{m\pi}{k+2})^{2g-2}}.$$

Parts I, II, and III can be viewed as elaborate attempts to prove these formulas, which will, ultimately, succeed.

Before diving into the algebraic geometry, this seems a good time to prove something that from the present vantage point is quite startling—that **(2.12)** and **(2.13)** are both polynomials in k with rational coefficients. From the algebro-geometric point of view this would not be so surprising, since at least for large k they have to agree with the Hilbert polynomials of N and $\hat{N}$. Consequently, the proof below will show that they *are* the Hilbert polynomials. We follow an argument of Don Zagier in the untwisted case, but similar results were obtained by Dowker [13] well before Verlinde!

(2.14) *For fixed $g \geq 2$,*

$$z_k(X_g) = \operatorname*{Coeff}_{x^{3g-3}} \left(-\frac{k+2}{2}x\right)^{g-1} \left(\frac{x}{\sinh x}\right)^{2g-2} \frac{-(k+2)x}{\tanh(k+2)x};$$

$$\hat{z}_k(X_g) = \operatorname*{Coeff}_{x^{3g-3}} \left(-\frac{k+2}{2}x\right)^{g-1} \left(\frac{x}{\sinh x}\right)^{2g-2} \frac{(k+2)x}{\sinh(k+2)x}.$$

Proof. We first rewrite **(2.12)** in terms of roots of unity:

$$z_k(X_g) = \tfrac{1}{2}(-2k-4)^{g-1} \sum_{\substack{\zeta^{2k+4}=1 \\ \zeta\neq\pm1}} \frac{1}{(\zeta-\zeta^{-1})^{2g-2}}.$$

Substituting $\zeta^2 = \lambda$, we get

$$z_k(X_g) = (-2k-4)^{g-1} \sum_{\substack{\lambda^{k+2}=1 \\ \lambda\neq1}} \frac{\lambda^{g-1}}{(\lambda-1)^{2g-2}}.$$

Hence

$$\begin{aligned} \frac{z_k(X_g)}{(-2k-4)^{g-1}} &= \sum_{\substack{\lambda^{k+2}=1 \\ \lambda\neq1}} \operatorname*{Res}_{z=\lambda} \left[\frac{z^{g-1}}{(z-1)^{2g-2}} \frac{(k+2)}{z^{k+2}-1} \frac{dz}{z}\right] \\ &= -\operatorname*{Res}_{z=1} \left[\frac{z^{g-1}}{(z-1)^{2g-2}} \frac{(k+2)}{z^{k+2}-1} \frac{dz}{z}\right] \end{aligned}$$

for $g \geq 2$ by the residue theorem, since then the only poles of the expression in square brackets are at the $(k+2)$nd roots of unity. Since $\frac{1}{z^{k+2}-1} = \frac{1}{2}\frac{z^{k+2}+1}{z^{k+2}-1} - \frac{1}{2}$ and the residue of $\frac{z^{g-2}}{(z-1)^{2g-2}}$ at $z=1$ vanishes, this becomes

$$\frac{z_k(X_g)}{(-2k-4)^{g-1}} = -\frac{k+2}{2} \operatorname*{Res}_{z=1} \left[\frac{z^{k+2}+1}{z^{k+2}-1} \frac{z^{g-2}dz}{(z-1)^{2g-2}}\right],$$

and finally, substituting $z = e^{2x}$,

$$\begin{aligned} \frac{z_k(X_g)}{(-2k-4)^{g-1}} &= -\operatorname*{Res}_{x=0} \left[\left(\frac{1}{2\sinh x}\right)^{2g-2} \frac{k+2}{\tanh(k+2)x} dx\right] \\ &= \operatorname*{Coeff}_{x^{2g-2}} \left\{\frac{1}{2^{2g-2}} \left(\frac{x}{\sinh x}\right)^{2g-2} \frac{-(k+2)x}{\sinh(k+2)x}\right\}. \end{aligned}$$

This proves the first formula. The second proceeds similarly until we obtain

$$\frac{\hat{z}_k(X_g)}{(-2k-4)^{g-1}} = \mathrm{Res}_{z=1}\left[\frac{z^{g-1}}{(z-1)^{2g-2}}\frac{(k+2)z^{(k+2)/2}}{z^{k+2}-1}\frac{dz}{z}\right],$$

whereupon substituting $z = e^{2x}$ gives immediately

$$\begin{aligned}\frac{D(g,k)}{(-2k-4)^{g-1}} &= \mathrm{Res}_{x=0}\left[\left(\frac{1}{2\sinh x}\right)^{2g-2}\frac{k+2}{\sinh(k+2)x}dx\right]\\ &= \underset{x^{2g-2}}{\mathrm{Coeff}}\left\{\frac{1}{2^{2g-2}}\left(\frac{x}{\sinh x}\right)^{2g-2}\frac{(k+2)x}{\sinh(k+2)x}\right\}. \ \square\end{aligned}$$

Part I

The factorization axiom

3 Algebro-geometric formulation

In this part we make our first, not entirely successful attempt to prove the Verlinde formula using algebraic geometry. Unlike the later second attempt, this one is analogous to the physical approach: indeed the modular functor axioms (**2.1**) and (**2.2**) (or (**2.3**) and (**2.4**) in the twisted case) get translated into (**4.9**) and our main theorem (**3.2**), respectively. It is clear enough what the first axiom should mean in an algebro-geometric context, and it is correspondingly easy to prove. The second axiom, however, is not so easy, since of course the notions of addition of a handle and connect-sum don't make sense in algebraic geometry. So even to state it correctly we need to clarify what we mean.

We will adopt an approach which has been taken by many authors, including Tsuchiya-Ueno-Yamada [45], Bertram [5], and Narasimhan-Ramadas [32], and interpret factorization as degeneration. That is, we will let our smooth curve X degenerate to a curve with one node, and extend our definition of the Verlinde vector space to such a curve. We then prove a result (**7.1**) comparing the spaces for the nodal curve and its normalization. This is almost our factorization axiom, except that of course we really want to know about smooth curves, not nodal ones.

However, we claim that the difference is not great, since the dimension of the Verlinde vector spaces for a nodal curve is the same as for a smooth curve of the same genus. That is, the first axiom should extend to nodal curves. It is easy to prove one inequality: the usual semicontinuity results imply that the dimension can only rise on a nodal curve. This is, for example, the principle behind Bertram's partial verification of the Verlinde formula [5]. But to show that it is in fact constant is technically the most difficult part of this approach.

Indeed, it is what defeats us, since we can only prove it when the genus of the degenerating curve is at least 4. We thus find ourselves in the awkward position of having proved the inductive step for $g \geq 4$ without knowing the base case! To encompass the low-genus cases would clearly require a significant change from the present method, since it uses the moduli space N of stable bundles, which does not even exist when X has genus 0. However, this difficulty suggests its own solution: to lift upstairs to the Hilbert scheme from which N is normally constructed in geometric invariant theory, and work equivariantly. The advantage of this approach is that the Hilbert scheme will exist even when N does not. Furthermore, by removing questions of stability from the problem, this lift in some respects actually makes things easier. However, this programme has not yet been carried out, so for the time being we stick to our unreasonable hypothesis of high genus.

An outline of this part runs roughly as follows. In the remainder of this section we give an algebro-geometric definition of the Verlinde vector space $Z_k(X)$ for a smooth curve X as the space of sections of a line bundle over the moduli space of parabolic bundles, and state the factorization result we intend to prove. In §4 we show how Z_k can be interpreted in terms of ordinary stable bundles (rather than parabolic ones), so that it corresponds with the gauge theory definition at least in the case $g \geq 2$, and prove a version of the first modular functor axiom. In §§5 and 6 we review Gieseker's degeneration of the moduli space of stable bundles over a nodal curve, explain how to fix the determinant—which is not entirely straightforward—and use this to define $Z_k(X)$ for a nodal curve X. Gieseker's degeneration is isomorphic, except on a set of high codimension, to a fibre bundle S whose base is the moduli space $\tilde{N}$ of stable bundles on the normalization of X, and whose fibre is a compactification $\overline{\mathrm{SL}_2}$ of SL_2. Hence in §7 we can prove our factorization result, using a Peter-Weyl-type theorem for $\overline{\mathrm{SL}_2}$ to push down sections from S to $\tilde{N}$.

Readers familiar with the paper of Narasimhan and Ramadas [32] will have noticed that this part has quite a lot in common with it. Indeed, the main results, and much of the preparatory material, are exactly the same, except that we fix the determinant. The main novelty of our approach lies in using Gieseker's degeneration of the moduli space. Its only singularities are normal crossings, so we avoid the difficult analysis of singularities in [32]. Also, by using a slightly modified (but equivalent) definition of the Verlinde vector spaces, we avoid having to consider the generalized parabolic bundles (and sheaves) which appear in that paper.

Although it is not directly relevant to the subject matter of this part, it is a beguiling fact that all of the so-called n-birational maps described in this part are in fact finite sequences of *flips* in the sense of Mori theory. In part II we shall introduce some new moduli spaces of so-called stable pairs; as with parabolic bundles, the stability condition for pairs depends on a parameter, and again as the parameter varies, the moduli space undergoes a sequence of flips. There, however, the precise geometry of the flips will be central to our approach, and we will study them carefully. In any case, there ought to be some general reason why flips arise in moduli problems involving a parameter. This would be quite satisfying to understand properly.

Let us, then, begin with an algebro-geometric definition of the vector spaces Z_k. Fix a *level* $k \in \mathbb{N}$ and a *degree* $d \in \mathbb{Z}$, and let X be a smooth complex projective curve of genus $g \geq 2$, and let $\Lambda \to X$ be a line bundle of degree d. Choose a finite set $\mathbf{q} = \{q_1, \dots, q_s\} \subset X$ of *marked points*, and *label* each q_j with an integer m_j such that $0 \leq m_j \leq k$. We will define a finite-dimensional *Verlinde vector space* $Z_k(X; (q_1, m_1), \dots (q_s, m_s))$. For brevity, write $\mathbf{m}$ to denote the sequence $\{m_j\}$; our Verlinde vector space will then be just $Z_k(X; (\mathbf{q}, \mathbf{m}))$.

Let M be the moduli space of semistable parabolic bundles on X of determinant Λ, with parabolic structure at the q_j, and weights $0, m_j/k$. If some $m_j = 0$, we interpret this in the usual way to mean there is no parabolic structure at q_j after all. If some $m_j = k$, this means that there is no parabolic structure at q_j, but that moreover the determinant Λ should be replaced by $\Lambda(q_j)$.

Now on the stable set M_s, there exists a universal bundle $\mathbf{E} \to M_s \times X$, at least up to a mod 2 obstruction. That is, $\mathbf{E}$ may not be a bona fide bundle, but only an element with denominator 2 in $K^0(M_s \times X) \otimes \mathbb{Q}$. In more down-to-earth terms, $\mathbf{E}$ does exist locally in M_s, but the patching of the local bundles is obstructed by a mod 2 cocycle. Since $\mathbf{E}$ exists only rationally, we can ask it to have any determinant we like, provided it restricts to $\Lambda^2 E$ on each $[E] \times X$. So let us normalize $\mathbf{E}$ so that, if

$\mathbf{E}_x \to M_s$ is the restriction of $\mathbf{E}$ to $M_s \times \{x\}$, then $\Lambda^2 \mathbf{E}_x = \mathcal{O}$ for all $x \in X$. With this requirement satisfied, $\mathbf{E}$ is determined uniquely. Hence the same can be said of the $\mathbb{Q}$-divisor classes $\mathcal{O}(\Theta) = \det^{-1} p_! \mathbf{E} \in \operatorname{Pic} M_s \otimes \mathbb{Q}$, where p is projection on M_s, and $\mathcal{O}(\mathrm{m}) = \otimes_j (\mathbf{E}_{q_j}/F_j)^{m_j} \in \operatorname{Pic} M_s \otimes \mathbb{Q}$, where F_j is the sub-line bundle defining the flag. Now every $\mathbb{Q}$-divisor class on M_s extends uniquely over M, because $M - M_s$ contains no divisors for $g \geq 3$. So $\mathcal{O}(\Theta)$ and $\mathcal{O}(\mathrm{m})$ extend over M. It is then not too hard to check that the natural ample $\mathbb{Q}$-divisor class on M given by geometric invariant theory is none other than $\mathcal{O}(k\Theta)(\mathrm{m})$. Because of the mod 2 obstruction, there is "no denominator"—that is, this is actually in $\operatorname{Pic} M$—if and only if $kd + \sum_j m_j$ is even.

(3.1) *Definition.* The SU(2) *Verlinde vector space* is

$$Z_k(X; (\mathrm{q}, \mathrm{m})) = H^0(M; \mathcal{O}(k\Theta)(\mathrm{m})),$$

with the convention that the cohomology is zero unless $\mathcal{O}(k\Theta)(\mathrm{m}) \in \operatorname{Pic} M$.

Now suppose $g \geq 4$, let $\tilde{X}$ be a curve of genus $g-1$, and let $\tilde{\Lambda} \to \tilde{X}$ be a line bundle of degree d. Choose a set $\tilde{\mathrm{q}} = \{\tilde{q}_1, \ldots \tilde{q}_s\} \subset \tilde{X}$, and label it with the same m as q. But choose also two further points $p_1, p_2 \in \tilde{X}$. Our main result will then be the following.

(3.2) $$\dim Z_k(X; (\mathrm{q}, \mathrm{m})) = \sum_{m=0}^{k} \dim Z_k(\tilde{X}; (\tilde{\mathrm{q}}, \mathrm{m}), (p_1, m), (p_2, m)).$$

The proof of **(3.2)** appears in §7. However, the first thing to be checked is that both sides depend only on g, d, k, s, and m. This is carried out in the next section.

4 Preliminaries on birational maps

This definition is the best one in general, but as we shall see, for $g \geq 3$ there are several equivalent alternatives, reflecting the fact that M has many different birational models. So it will be useful to understand to what extent cohomology of vector bundles is independent of the birational model. The key lemma, **(4.3)**, though little-known, is standard, unlike definition **(4.1)** which precedes it.

(4.1) *Definition.* For $n \in \mathbb{N}$, two varieties X, X' are said to be *n-birational* if there exist subvarieties $Y \subset X$, $Y' \subset X'$ of codimension $\geq n$ such that $X - Y \cong X' - Y'$. If $E \to X$ and $E' \to X'$ are vector bundles, the n-birational map $X \leftrightarrow X'$ is said to *lift to $E \leftrightarrow E'$* if $E|_{X-Y} \cong E'|_{X'-Y'}$.

Thus in particular 1-birational simply means birational. Note incidentally that the definition still makes sense if the bundles are only defined rationally. Actually this definition ought to refer to depth rather than dimension, but this will not matter, thanks to the following.

(4.2) *All moduli spaces of semistable or semistable parabolic bundles over a curve are Cohen-Macaulay, and indeed have only rational singularities.*

Proof. They are geometric invariant theory quotients of smooth varieties, so have rational singularities by a result of Boutot [7]; this implies that they are Cohen-Macaulay [43]. □

Our key fact is then a "cohomological version of Hartogs's theorem".

(4.3) *Let $E \to X$ and $E' \to X'$ be vector bundles over Cohen-Macaulay varieties, and suppose that there is an n-birational map $X \leftrightarrow X'$ lifting to $E \leftrightarrow E'$. Then there are natural isomorphisms*

$$H^i(X; E) \cong H^i(X'; E')$$

for $0 \leq i \leq n-2$.

An analytical proof can be found in [39]. The algebraic result follows from Theorem 3.8 of [23], using Proposition 1.4 and Corollary 1.9 there.

The crucial example for us will involve changing weights in the moduli space of parabolic bundles. Let M and M' be moduli spaces of semistable parabolic bundles of the same determinant on X with parabolic structure at the q_j, but with different nonzero weights. Given integer labels $\mathbf{m}$, we may define as before a $\mathbb{Q}$-divisor class $\mathcal{O}(k\Theta)(\mathbf{m})$ in both $\operatorname{Pic} M \otimes \mathbb{Q}$ and $\operatorname{Pic} M' \otimes \mathbb{Q}$.

(4.4) *There is a g-birational map $M \leftrightarrow M'$ lifting to $\mathcal{O}(k\Theta)(\mathrm{m}) \leftrightarrow \mathcal{O}(k\Theta)(\mathrm{m})$.*

Proof. Clearly we may suppose that all but one of the weights are the same. So if θ_i is the difference between the two weights at the ith point, we may regard θ_1 as a variable in $(0,1) \cap \mathbb{Q}$ and the other θ_i as fixed. Now any bundles which are in M but not M', or vice versa, will be semistable but not stable for some value of θ_1. There are only finitely many values where such bundles exist at all, so to show M and M' are g-birational it suffices to find finitely many families of dimension $< 2g-2+s = \dim M - g + 1$ parametrizing all the semistable but not stable parabolic bundles with fixed weights θ_i. Since we will need this result again later, we make it a separate lemma.

(4.5) *The semistable but not stable parabolic bundles with fixed determinant Λ and weights θ_i are parametrized by finitely many families of dimension $< 2g-2+s$, unless $s = 0$, $g = 2$, and d is even.*

Proof. If $s = 0$, then there are only semistable but not stable bundles if d is even; in that case they are parametrized by $\mathrm{Pic}_0 X/(\mathbb{Z}/2)^{2g}$ and 2^{2g} copies of $\mathbb{P}H^1(\mathcal{O})$. Unless $g = 2$, these have dimension $< 2g-2$. So suppose $s > 0$.

A parabolic bundle E is semistable but not stable if there is a sub-line bundle $L \subset E$ such that

$$2\deg L - \sum_{L_{q_j} \neq F_j} \theta_j + \sum_{L_{q_j} = F_j} \theta_j = d. \tag{4.6}$$

Say there are p terms in the first sum, so $s-p$ in the second. Clearly there are only finitely many possibilities for $\deg L$. Now if the underlying bundle splits as $L \oplus \Lambda L^{-1}$, then E is determined by $L \in \mathrm{Pic}\, X$ and by the p unconstrained flags, so E lies in a $(g+p)$-dimensional family of semistable parabolic bundles. Since $g \geq 3$, $g + p < 2g-2+s$, so this is small enough. If however the underlying bundle does not split, then E is determined by the extension class in $\mathbb{P}H^1(L^2\Lambda^{-1})$ and by the p unconstrained flags. As L varies over $\mathrm{Pic}\, X$, the extension classes lie in $\mathrm{Proj}(R^1\pi)P^2\Lambda^{-1}$, where P is the Poincaré line bundle over the Jacobian. This is at most $(2g-2-2\deg L + d)$-

dimensional, so E lies in a $(2g-2-2\deg L+d+p)$-dimensional family. It remains to show that this is $< 2g-2+s$, or equivalently, that $s-p+2\deg L-d>0$. But this is clear from **(4.6)**, considering that $s>0$ and all $\theta_i \in (0,1)$. □

Another example of a g-birational map, which indeed may be viewed as a limiting case of the one above, is the so-called *Hecke correspondence.* Choose $x \in X$, and let M and M' be the moduli spaces of stable bundles E of determinant Λ and $\Lambda(-x)$ respectively, with universal bundles $\mathbf{E}$ and $\mathbf{E}'$. Define $\mathcal{O}(\Theta) = \det^{-1} p_! \mathbf{E} \in \operatorname{Pic} M \otimes \mathbb{Q}$, and similarly $\mathcal{O}(\Theta) \in \operatorname{Pic} M' \otimes \mathbb{Q}$. The projectivizations $\mathbb{P}\mathbf{E}_x \to M$ and $\mathbb{P}\mathbf{E}'_x \to M'$ both exist as bona fide bundles even though either $\mathbf{E}$ or $\mathbf{E}'$ does not.

(4.7) *There is a g-birational map $\mathbb{P}\mathbf{E}_x \leftrightarrow \mathbb{P}\mathbf{E}'_x$ lifting to $\mathcal{O}(\Theta) \leftrightarrow \mathcal{O}(\Theta)(1)$, and to $\mathbf{E}_y \leftrightarrow \mathbf{E}'_y$ for any $y \in X - \{x\}$.*

Proof. The rational map $\mathbb{P}\mathbf{E}_x \to \mathbb{P}\mathbf{E}'_x$ is induced by *modifying* the bundle at x, as follows. An element of $\mathbb{P}\mathbf{E}_x$ consists of a stable bundle E and a one-dimensional subspace $F_x \subset E_x$, up to isomorphism. Regarding E_x/F_x as a skyscraper sheaf supported on x, let E' be the sheaf kernel of the obvious map:

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E_x/F_x \longrightarrow 0. \tag{4.8}$$

Then E' is a vector bundle with $\Lambda^2 E' = \Lambda^2 E(-x)$. Moreover E'_x has a canonical one-dimensional subspace F'_x, which is just the kernel of the induced map on fibres:

$$0 \longrightarrow F'_x \longrightarrow E'_x \longrightarrow E_x \longrightarrow E_x/F_x \longrightarrow 0.$$

It is not hard to see that repeating this procedure on the new pair (E', F'_x), recovers the original (E, F_x), but twisted by $\mathcal{O}(-x)$. Furthermore, it can be checked that E stable implies E' semistable. So there is a rational map $\mathbb{P}\mathbf{E}_x \to \mathbb{P}\mathbf{E}'_x$ which only fails to be an isomorphism at those pairs (E, F_x) whose modifications are semistable but not stable. By **(4.5)** all semistable but not stable bundles can be parametrized by families of dimension $< 2g-2$, so the bad set in $\mathbb{P}\mathbf{E}_x$ has dimension at most $2g-2$ (including a parameter for the flag), so codimension g. That shows that the two projective bundles are g-birational. As for the liftings, both can be seen from the "universal version" of

(4.8),

$$0 \longrightarrow F'_x \longrightarrow \mathbf{E}'_x \longrightarrow \mathbf{E}_x \longrightarrow \mathbf{E}_x/F_x \longrightarrow 0,$$

the first by applying $\det^{-1} p_!$, and the second by restricting to $M \times \{y\}$. □

Armed with these results, we can now deduce a fundamental fact about the Verlinde vector spaces: essentially axioms **(2.1)** and **(2.3)**.

(4.9) *The dimension of Z_k depends only on g and* m.

The proof uses two lemmas.

(4.10) *Let $f : X \to Y$ be a projective morphism of varieties, and let $\mathcal{F} \to X$ be a coherent sheaf flat over Y. If $H^1(X_y, \mathcal{F}_y) = 0$ for $y \in Y$, then* $\dim H^0(X_{y'}, \mathcal{F}_{y'})$ *is constant for y' in a neighbourhood of y.*

This is a special case of the theorem on cohomology and base change [24, III 12.11]. We will be interested in the case when each X_y is a moduli space and $\mathcal{F}_y$ is a vector bundle.

(4.11) *Kodaira vanishing holds for all moduli spaces of stable bundles and stable parabolic bundles on a curve.*

Proof. By **(4.2)** they have rational singularities; this implies Kodaira vanishing [43, 7.80]. □

Proof of **(4.9)**. We wish to show that dim Z_k is independent of the precise choice of X, the q_j, and Λ. There exists a flat algebraic family containing any two such choices, so for any choice θ_j of weights, there is an associated flat family of moduli spaces of parabolic bundles with weight θ_j at q_j, and a line bundle over that family whose restriction to each fibre is $\mathcal{O}(k\Theta)(\mathrm{m})$. By lemma **(4.10)** it suffices to show that for each fibre M we have $H^1(M; \mathcal{O}(k\Theta)(\mathrm{m})) = 0$. This will follow from Kodaira vanishing **(4.11)** provided that $K_M^{-1}\mathcal{O}(k\Theta)(\mathrm{m})$ is ample. Now the canonical bundles of 2-birational varieties are clearly 2-birational, and by calculating K_M for θ_j small where

M fibres over the moduli space of stable bundles except on a set of high codimension, we obtain $K_M = \mathcal{O}(-4\Theta)(-2,\ldots,-2)$. Hence $K_M^{-1}\mathcal{O}(k\Theta)(\mathbf{m}) = \mathcal{O}((k+4)\Theta)(\mathbf{m}+2)$. By **(4.4)** we are free to take whatever $\theta_j \in (0,1)\cap\mathbb{Q}$ we like—so choose $\theta_j = (m_j + 2)/(k+4)$. Then $\mathcal{O}((k+4)\Theta)(\mathbf{m}+2)$ is automatically ample, since it is the bundle given by geometric invariant theory. □

We now explain how to use these facts to transfer our spaces Z_k from moduli spaces of parabolic bundles to moduli spaces of ordinary stable bundles. So let N be the moduli space of stable bundles $E \to X$ with $\Lambda^2 E \cong \Lambda$, equipped with $\mathbf{E} \to N \times X$ and $\mathcal{O}(\Theta) \in \operatorname{Pic} N \otimes \mathbb{Q}$ as usual. On N, we use the shorthand $\mathbf{E}_x^m = S^m \mathbf{E}_x$ and $\mathbf{E}_\mathbf{q}^\mathbf{m} = \bigotimes_j \mathbf{E}_{q_j}^{m_j}$. Here then is an alternative definition of Z_k in terms of N.

(4.12) *There is a natural isomorphism*

$$Z_k(X;(\mathbf{q},\mathbf{m})) = H^0(N;\mathcal{O}(k\Theta)\otimes \mathbf{E}_\mathbf{q}^\mathbf{m}).$$

Proof. Let $\times_j \mathbb{P}\mathbf{E}_{q_j}$ be the fibred product $\mathbb{P}\mathbf{E}_{q_1} \times_N \cdots \times_N \mathbb{P}\mathbf{E}_{q_s}$ over all j such that $m_j \neq 0$, and by abuse of notation write $\mathcal{O}(\mathbf{m})$ for the line bundle $\bigotimes_j p_j^* \mathcal{O}(m_j) \to \times_j \mathbb{P}\mathbf{E}_{q_j}$, where p_j is projection on the jth factor. Then the right-hand side is clearly isomorphic to $H^0(\times_j \mathbb{P}\mathbf{E}_{q_j};\mathcal{O}(k\Theta)(\mathbf{m}))$. By repeated use of the Hecke correspondence **(4.7)** we may eliminate all those j such that $m_j = k$ by moving to a new moduli space of stable bundles with different fixed determinant. Once this is done, by **(4.3)** it suffices to show that there is a 2-birational map from $\times_j \mathbb{P}\mathbf{E}_{q_j}$ to the relevant parabolic moduli space M lifting to $\mathcal{O}(k\Theta)(\mathbf{m}) \leftrightarrow \mathcal{O}(k\Theta)(\mathbf{m})$. But $\times_j \mathbb{P}\mathbf{E}_{q_j}$ is an open set in the moduli space M' of parabolic bundles with small weights, whose complement consists precisely of those stable parabolic bundles whose underlying bundle is semistable but not stable; this is empty for d odd, and of codimension $\geq g-1$ for d even by **(4.5)**. It is easy to check that $\mathcal{O}(k\Theta)(\mathbf{m}) \to \times_j \mathbb{P}\mathbf{E}_{q_j}$ is indeed the restriction of $\mathcal{O}(k\Theta)(\mathbf{m})$ on our moduli space of parabolic bundles. So by **(4.4)**there is a 2-birational map $\times_j \mathbb{P}\mathbf{E}_{q_j} \leftrightarrow M'$ lifting to $\mathcal{O}(k\Theta)(\mathbf{m}) \leftrightarrow \mathcal{O}(k\Theta)(\mathbf{m})$. Finally **(4.3)** tells us that the discrepancy in weights does not matter. □

Hence instead of working on a moduli space M of parabolic bundles and studying sections of line bundles, we may work on a moduli space N of ordinary stable bundles and study sections of certain vector bundles. Indeed, by suitable application of the Hecke correspondence (4.7) we may even assume that the degree d is odd. So Gieseker's degeneration of the moduli space of stable bundles of odd degree is well suited for our purposes.

5 Gieseker's degenerative construction

We shall now review Gieseker's results on degeneration of the moduli space of bundles over a curve. Gieseker covers only the case when the degree d is odd, but in light of (4.7) this is all we will need. Note, however, that he does *not* fix the determinant, so we will need to do this ourselves later.

For any curve X, we say that a curve Y *resembles* X if there exists a surjection $s : Y \rightarrow X$ satisfying $s^*\omega_X = \omega_Y$ (dualizing sheaves). Thus if X is smooth, we have $Y = X$, but if X is an irreducible curve with one node, Y may be any nodal curve whose components are the normalization $\tilde{X}$ of X and a finite number of lines R_i, glued together as shown in figure 1. We write X_n for the curve having n lines. On curves of the sort described above, Gieseker defines a notion of stability for rank 2 vector bundles of odd degree, which for our purposes is best understood simply by enumerating cases. We assume that X_0 is a nodal curve of the sort described above, with normalization $\tilde{X}$. If E is a bundle on X_n, we let $E' = E \otimes \mathcal{O}_{\tilde{X}}$.

(5.1) *Definition.* A vector bundle E of rank 2 and odd degree d is *stable* if one of the following holds:

- E is defined on a smooth curve, and is stable in the usual sense;
- Type I: E is defined on a nodal curve X_0, and E' is either stable or has a line subbundle of degree $(d+1)/2$;
- Type II_1: E is defined on X_1, E' is semistable, and $E_{R_1} = \mathcal{O}(1) \oplus \mathcal{O}$;
- Type II_2: E is defined on X_1, E' is stable, and $E_{R_1} = \mathcal{O}(1) \oplus \mathcal{O}(1)$;

– Type III: E is defined on X_2, E' is stable, $E_{R_1} = \mathcal{O}(1) \oplus \mathcal{O} = E_{R_2}$, but the $\mathcal{O}(1)$ sub-line bundles of E_{R_1} and E_{R_2} are not glued together.

Now let C be a smooth affine curve and choose a point $0 \in C$. Let $\pi : \mathcal{X} \to C$ be a smooth flat family of projective curves of genus $g \geq 4$ (our hypothesis: Gieseker only needs $g \geq 2$), such that $X_z = \pi^{-1}(z)$ is smooth for $z \neq 0$, but X_0 is an irreducible curve with one node. Gieseker proves the following.

(5.2) *There is a smooth projective fine moduli space $\mathcal{W} \to C$ of stable bundles on curves resembling those in $\mathcal{X}$, such that the singularities of the zero fibre $\mathcal{W}_0$ are normal crossings.*

Sketch of proof. To construct $\mathcal{W}$, Gieseker considers flat families of curves in a certain Grassmannian $Gr_2(W)$ such that each curve in the family resembles a curve in $\mathcal{X}$. This should be thought of as a family of rank 2 vector bundles, by pulling back the tautological bundle over the Grassmannian. He restricts his attention to those families where every bundle has degree d, and is stable in the sense above. Roughly speaking, the category of such families is then represented by a smooth Hilbert scheme $\mathcal{Y} \to C$, whose fibres Y_z are all smooth except for Y_0, which has normal crossings. Gieseker shows that all of his stable bundles are *simple* in the sense that $\mathrm{Hom}(E, E) = k$. It follows that the natural action on $\mathcal{Y}$ of the automorphism group $\mathrm{PSL}(W)$ of the Grassmannian is free, so that there exists a projective geometric invariant theory quotient $\mathcal{W} \to C$. That $\mathcal{W}$ is smooth, and that $\mathcal{W}_0$ has normal crossings, follows easily from the corresponding facts for $\mathcal{Y}$. To claim that $\mathcal{W}$ is fine means that there exists a universal family $\mathcal{U} \to \mathcal{W}$ of curves resembling those in $\mathcal{X}$ and a rank 2 vector bundle $\mathbf{E} \to \mathcal{U}$ such that the fibre $\mathbf{E}_w \to \mathcal{U}_w$ over $w \in \mathcal{W}$ is in the isomorphism class specified by w. To prove this, note first that by definition $\mathcal{Y}$ parametrizes such a family of curves in $Gr_2(W)$, and that the universal bundle there is just the pullback of the tautological bundle over $Gr_2(W)$. Since stable bundles are simple, it follows by imitating standard arguments, as for example in [2], that the universal family and bundle (suitably twisted) descend from $\mathcal{Y}$ to $\mathcal{W}$ and that the bundle $\mathbf{E}$ is unique up to twisting by a $\mathbb{Q}$-divisor over $\mathcal{W}$. $\square$

In keeping with our previous policy, we will normalize $\mathbf{E}$ as follows. Any section $x : C \to \mathcal{X}$ which misses the node lifts uniquely to $x : \mathcal{W} \to \mathcal{U}$, since $\mathcal{U}$ is a family of curves resembling those in $\mathcal{X}$. Define the bundle $\mathbf{E}_x = x^*\mathbf{E} \to \mathcal{W}$; thus for $z \neq 0$ the restriction of $\mathbf{E}_x$ to $\mathcal{W}_z$ is just $\mathbf{E}_{x(z)}$ in the old sense. Then normalize $\mathbf{E}$ rationally so that $\Lambda^2\mathbf{E}_x \in \mathrm{Pic}_0(\mathcal{W})$ for all sections x missing the node.

The heart of Gieseker's paper is a construction of a birational model S for the normalization $\tilde{\mathcal{W}}_0$, and of a universal family over it, which goes more or less as follows. Continue to denote by X_0 an irreducible curve with one node, and by $\tilde{X}$ its normalization. Call p_1 and p_2 the two points in $\tilde{X}$ which map to the node.

Let $\tilde{N}$ be the moduli space of bundles of degree d on $\tilde{X}$, and let $\tilde{\mathbf{E}} \to \tilde{N} \times \tilde{X}$ be the universal bundle. Form the projective bundle $\mathbb{P}(\mathrm{Hom}(\tilde{\mathbf{E}}_{p_1}, \tilde{\mathbf{E}}_{p_2}) \oplus \mathcal{O}) \to \tilde{N}$ and define two divisors $H_2 = \{[\phi, 0] : \phi \in \mathrm{Hom}(\tilde{\mathbf{E}}_{p_1}, \tilde{\mathbf{E}}_{p_2})\}$ and $D_1 = \{[\phi, \lambda] : \ker\phi \neq 0, \lambda \in k\}$ on it. Then let S (Gieseker's S_3) be the variety obtained by blowing up this projective bundle at the zero section $\{[0, \lambda] : \lambda \in k\}$ and at the bundle of quadric surfaces $Q = H_2 \cap D_1$, and call the exceptional divisors H_1 and D_2 respectively. By abuse of notation, call the proper transforms of H_2 and D_1 again just H_2 and D_1. To sum up, the projection $\pi : S \to \tilde{N}$ is a fibration whose fibre is $\mathbb{P}^4$ blown up at a quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \subset \mathbb{P}^4$, and at one other point outside the $\mathbb{P}^3 \subset \mathbb{P}^4$. We shall henceforth call this fibre $\overline{\mathrm{GL}}$, since it is a compactification of GL_2 as a $\mathrm{GL}_2 \times \mathrm{GL}_2$-space. Schematically it looks something like figure 2.

The variety S is supposed to parametrize a family of bundles over curves resembling X_0. To cook up such a family, use the following recipe:

1. Blow up $S \times \tilde{X}$ at $H_1 \times p_2$ and $H_2 \times p_1$, pull back the universal bundle $\mathbf{E}$ to the result, and twist by the exceptional divisors E_1, E_2 to obtain $\mathbf{E}_1 = \mathbf{E}(E_1 + E_2) \to \mathcal{U}_1$;

2. Blow up $\mathcal{U}_1$ at the proper transforms of $D_1 \times p_2$ and $D_2 \times p_1$, pull back $\mathbf{E}_1$ to the result, and modify by one-dimensional quotients M_1, M_2 supported on the two exceptional divisors to obtain $\mathbf{E}_2 \to \mathcal{U}_2$ (for details, see §7 of [19]);

3. Glue the proper transforms of $S \times p_1$ and $S \times p_2$ in $\mathcal{U}_2$ together, and identify the fibres of $\mathbf{E}_2$ there, as specified by Gieseker's map ϕ_3 in (8.1.3) of [19], to obtain

$\mathbf{E}_3 \to \mathcal{U}_3$. This is a family of bundles whose types are as shown in figure 3.

By carefully examining this family, Gieseker proves the following.

(5.3) *The normalization $\tilde{\mathcal{W}}_0$ of $\mathcal{W}_0$ is $(g-2)$-birational to S. Consequently, if $\tilde{\mathcal{U}} = \mathcal{U} \times_{\mathcal{W}_0} \tilde{\mathcal{W}}_0$ and $\mathbf{E}$ is pulled back to $\tilde{\mathcal{U}}$, there is a $(g-2)$-birational map $\tilde{\mathcal{U}} \leftrightarrow \mathcal{U}_3$ lifting to $\mathbf{E} \leftrightarrow \mathbf{E}_3$.*

Indeed, Gieseker explicitly constructs the subsets $Z \subset S$, $Z' \subset \tilde{\mathcal{W}}_0$ where the birational map is not defined, and shows that actually $Z \subset D_1 \cap D_2$. The set Z' contains, among other things, those type I bundles for which E' is unstable.

We will actually need to go a bit further than this theorem, and work out the precise locations of the normal crossings in $\mathcal{W}_0$.

(5.4) *The inverse image in $\tilde{\mathcal{W}}_0$ of the singular locus of $\mathcal{W}_0$ is the proper transform of $H_1 + H_2 + D_1 + D_2$. Indeed, $\mathcal{W}_0$ is constructed from $\tilde{\mathcal{W}}_0$ by gluing the proper transform of D_1 to that of D_2 and H_1 to H_2.*

Before giving the proof, let us remark that this result reflects the existence of n different deformations of X_n to X_{n-1}, each one obtained by smoothing out a different node. Hence two smooth pieces cross each other normally at the type II bundles, and three at the type III.

Proof. Each point in $\tilde{\mathcal{W}}_0$ represents an isomorphism class of bundles over some X_n. Since $\mathcal{W}_0$ is supposed to be a moduli space of such objects, it follows that two points is $\tilde{\mathcal{W}}_0$ are identified by the map $\tilde{\mathcal{W}}_0 \to \mathcal{W}_0$ if and only if the curves and bundles they represent are isomorphic. Since by **(5.2)** $\mathcal{W}_0$ has only normal crossings, we may ignore the set Z' of codimension $\geq g-1$, and consider only the set $\tilde{\mathcal{W}}_0 - Z'$ which is isomorphic to $S - Z$. Thus we may work over S. We denote by E_s the bundle determined by $s \in S$, and by $\tilde{E}_s$ the bundle determined by $\pi(s) \in \tilde{N}$. Note that our bundles E' and $\tilde{E}$ are the reverse of Gieseker's.

Of course, points of a given type can only be glued to other points of the same type. So we will examine each type appearing in S individually, and decide whether any bundle of that type can be represented by more than one point in S.

Type I: If $E_{s_1} = E_{s_2}$ for s_1, s_2 points of type I, then certainly $\tilde{E}_{s_1} = E'_{s_1} = E'_{s_2} = E'_{s_2}$, so s_1 and s_2 are in the same fibre over $\tilde{N}$. But then if $s_1 \neq s_2$, we get a non-constant automorphism of the stable bundle $\tilde{E}_{s_1}$, which is impossible, so $s_1 = s_2$. Hence none of the points of type I are glued together.

Type II_2: From its definition as an element of S, a point $s \in H_1 - D_1$ is determined by a stable bundle $\tilde{E}_s \to \tilde{X}$ and an element of $\mathbb{P}\,\mathrm{Hom}((\tilde{E}_s)_{p_1}, (\tilde{E}_s)_{p_2})$. But we can recover these data from E_s; indeed $\tilde{E}_s = E'_s(p_2)$ and the gluing map $(\tilde{E}_s)_{p_1} \to (\tilde{E}_s)_{p_2}$ is induced by a trivialization of $E_s \otimes \mathcal{O}_{R_1}(p_2 - 2p_3)$ for any $p_3 \in R_1 - \{p_1, p_2\}$; the scalar indeterminacy corresponds to the choice of p_3. Hence no two points in $H_1 - D_1$ are glued together. A similar argument shows that no two points in $H_2 - D_2$ are glued together.

However, a point $s_1 \in H_1 - D_1$ may be glued to $s_2 \in H_2 - D_2$. Indeed, by arguing as in the previous paragraph, we can show that it is necessary and sufficient to have an isomorphism

$$E'_{s_2}(p_2 - p_1) = E'_{s_1}$$

which identifies the gluing maps. It can be checked that this actually induces an isomorphism $H_1 - D_1 \leftrightarrow H_2 - D_2$. Hence these sets are glued together in $\mathcal{W}_0$. (Of course, this means that the points of type III must also be glued together, as we will confirm shortly.)

Type II_1: The argument is essentially the same as in the II_2 case. Again it turns out that the determining data can be recovered from E_s; for example, $\tilde{E}_s$ can be recovered from the exact sequence

$$0 \longrightarrow \tilde{E}_s(-p_2) \longrightarrow E'_s \longrightarrow \mathcal{O}_{p_2} \longrightarrow 0,$$

where $E'_s \to \mathcal{O}_{p_2}$ is the restriction of the canonical map $E_s \to \mathcal{O}_{R_1}$. So no two points in $D_1 - H_1 - D_2$, or in $D_2 - H_2 - D_1$, are glued together. However, $s_1 \in D_1 - H_1 - D_2$ will be glued to $s_2 \in D_2 - H_2 - D_1$ if there is an exact sequence of sheaves

$$0 \longrightarrow \tilde{E}_{s_1}(-p_2) \longrightarrow \tilde{E}_{s_2} \longrightarrow \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \longrightarrow 0$$

compatible with the determining data, where $\tilde{E}_{s_2} \to \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2}$ is again the restriction of the canonical map. This induces a birational map $D_1 - H_1 - D_2 \leftrightarrow D_2 - H_2 - D_1$, which is not an isomorphism on S because the modification of a stable bundle at two points need not be stable, but which goes over to an isomorphism on $\mathcal{W}_0$.

Type III: Again similar to the II_2 case. This time $\tilde{E}_s$ is recovered by

$$\tilde{E}_s = E'_s(p_2) \text{ for } s \in H_1 \cap D_1,$$
$$\tilde{E}_s = E'_s(p_1) \text{ for } s \in H_2 \cap D_2,$$
$$0 \longrightarrow \tilde{E}_s(-p_1 - p_2) \longrightarrow E'_s \longrightarrow \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \longrightarrow 0 \quad \text{for } s \in D_1 \cap D_2.$$

We conclude again that no two points in $H_1 \cap D_1$, or in $H_2 \cap D_2$, or in $D_1 \cap D_2$, are glued together. However, all three of these pieces are glued to one another as in the II_1 case; the gluing is induced by

$$\ker(E'_{s_1}, L'_{p_1} \oplus L'_{p_2})(p_1) = E'_{s_2} = \ker(E'_{s_3}, L'_{p_1} \oplus L'_{p_2})(p_2)$$

$$\text{for } s_1 \in H_1 \cap D_1, s_2 \in D_1 \cap D_2, s_3 \in H_2 \cap D_2. \ \square$$

The result above is summed up in figures 4 and 5. We ought to think of the proper transforms of the divisors D_i, H_i as shown in figure 4, so that the singular locus of $\mathcal{W}_0$ consists of two smooth divisors H and D, glued together as shown in figure 5.

6 Fixing the determinant

There is one more technicality to deal with. If we wish to study the moduli space of bundles with fixed determinant over a smooth curve, we need somehow to make sense of the notion of fixing the determinant on $\mathcal{W}$. The obvious approach will not work, since the map taking a bundle in $\mathcal{W} - \mathcal{W}_0$ to its determinant in the relevant family of Jacobians does not extend sensibly over $\mathcal{W}_0$. So we need to make a slightly more subtle definition. Let Λ be a line bundle over the family $\mathcal{X}$ of curves having degree d on each fibre. Pull it back to $\mathcal{U} \to \mathcal{W}$, and let $\mathcal{V}$ be the reduced family whose fibre over $z \in C$ consists of those points $[E] \in \mathcal{W}_z$ such that $H^0(\Lambda^2 E^* \otimes \Lambda_z) \neq 0$. By semicontinuity $\mathcal{V}$ is a closed subvariety of $\mathcal{W}$, and for $z \neq 0$, $\mathcal{V}_z$ is just the moduli space N of stable bundles with $\Lambda^2 E = \Lambda_z$. However, the situation at 0 is a little different. We write $\tilde{\Lambda} = \Lambda_0 \otimes \mathcal{O}_{\tilde{X}}$.

(6.1) *For $s \in S$, $H^0(\Lambda^2 E_s^* \otimes \Lambda_0) \neq 0$ if and only if one of the following holds:*

- $\Lambda^2 E_s = \Lambda_0$ (type I);
- $s \in D_1 \cap D_2$ *and* $\Lambda^2 \tilde{E}_s = \tilde{\Lambda}$ (type III);
- $s \in H_i$ *and* $\Lambda^2 \tilde{E}_s = \tilde{\Lambda}(p_{3-i} - p_i)$ (types II_1 and III).

Proof. For E_s of type I, clearly $H^0(\Lambda^2 E_s^* \otimes \Lambda_0) \neq 0$ if and only if $\Lambda^2 E_s = \Lambda_0$; so we need only consider types II and III. First notice that $(\Lambda^2 E_s^* \otimes \Lambda_0)_{R_i} = \mathcal{O}(-1)$ or $\mathcal{O}(-2)$, so that any section must vanish on each R_i. Hence it is necessary and sufficient to have $H^0(\tilde{X}; \Lambda^2(E_s')^* \otimes \tilde{\Lambda}(-p_1 - p_2)) \neq 0$. Now modification at p_i multiplies the determinant by $\mathcal{O}(p_i)$, so we get

$$\Lambda^2 E_s' = \begin{cases} \Lambda^2 \tilde{E}_s(-2p_{3-i}) & \text{for } s \in H_i \\ \Lambda^2 \tilde{E}_s(-p_{3-i}) & \text{for } s \in D_i - H_i - D_{3-i} \\ \Lambda^2 \tilde{E}_s(-p_1 - p_2) & \text{for } s \in D_1 \cap D_2. \end{cases}$$

Hence we find that for s a point of type II or III, $\Lambda^2(E_s')^* \otimes \tilde{\Lambda}(-p_1 - p_2)$ has a section precisely in the cases mentioned.

It is not difficult to see that in **(6.1)**, the second category is the boundary of the first. Indeed, on the fibre $\overline{\text{GL}}$, it is just the obvious completion of SL_2 by a quadric surface at infinity. We shall call this $\overline{\text{SL}_2}$; it is a double cover of $\mathbb{P}^3$ branched over the quadric surface. Hence the set $\{s \in S : \Lambda^2 E_s^* \otimes \Lambda_0 \neq 0\}$ is the disjoint union of three irreducible subvarieties:

$$P_0 = \{s \in S_3 : \Lambda^2 E_s = \Lambda_0\} \cup \{s \in D_1 \cap D_2 : \Lambda^2 \tilde{E}_s = \tilde{\Lambda}\} \text{ (types I and III)};$$
$$P_1 = \{s \in H_1 : \Lambda^2 \tilde{E}_s = \tilde{\Lambda}(p_2 - p_1)\} \text{ (types II}_1 \text{ and III)};$$
$$P_2 = \{s \in H_2 : \Lambda^2 \tilde{E}_s = \tilde{\Lambda}(p_1 - p_2)\} \text{ (types II}_1 \text{ and III)}.$$

Clearly each is a bundle over $\tilde{N}$; P_0 with fibre $\overline{\text{SL}_2}$, and P_1 and P_2 with fibre $\mathbb{P}^3$. Each P_i contains a natural bundle of quadric surfaces: to wit, $Q_0 = P_0 \cap D_1 \cap D_2$,

$Q_1 = P_1 \cap D_1$, and $Q_2 = P_2 \cap D_2$. Indeed, the gluing map of (5.4) induces an isomorphism $P_1 \leftrightarrow P_2$, and P_0 is a double cover of P_1 branched at Q_0.

(6.2) *The normalization $\tilde{\mathcal{V}}_0$ of $\mathcal{V}_0$ is $(g-1)$-birational to $P_0 \cup P_1$, and the inverse image in $\tilde{\mathcal{V}}_0$ of the singular locus of $\mathcal{V}_0$ is the proper transform of $Q_0 + Q_1$. Indeed, $\mathcal{V}_0$ is constructed from $\tilde{\mathcal{V}}_0$ by gluing the proper transform of Q_0 to Q_1.*

Proof. P_1 and P_2 are disjoint from $Z' \subset D_1 \cap D_2$, so are unchanged by the birational map $S \leftrightarrow \tilde{\mathcal{W}}_0$, and are identified by the gluing map $\tilde{\mathcal{W}}_0 \to \mathcal{W}_0$. It is straightforward from Gieseker's description of the sets Z, Z' to check that $\dim\{[E] \in Z : H^0(\Lambda^2 E^* \otimes \Lambda_0) \neq 0\} \leq 2g-2$, and likewise for Z', so the image of P_0 in the birational map is $(g-1)$-birational to P_0. Finally, note that since the transforms of $D_1 \cap D_2$ and $H_1 \cap D_1$ are identified by the map $\tilde{\mathcal{W}}_0 \to \mathcal{W}_0$, so are those of Q_0 and Q_1 by the restriction $\tilde{\mathcal{V}}_0 \to \mathcal{V}_0$.

(6.3) *$\mathcal{V} \to C$ is flat.*

Proof. Any morphism onto a smooth curve is flat if the domain is irreducible [24, III 9.7]. Clearly the inverse image in $\mathcal{V}$ of $C - \{0\}$ is irreducible, since its base and fibres are. So its closure is irreducible, hence flat over C. It suffices to show that this closure is all of $\mathcal{V}$. Its fibre over 0 is a Cartier divisor, hence equidimensional of dimension $3g-3$. So it equals $\mathcal{V}_0$ if it contains one point in $P_0 - P_1$ and one point in $P_1 - P_0$. To prove that it does, we will exhibit two curves in $\mathcal{V}$, one contained in $(\mathcal{V} - \mathcal{V}_0) \cup (P_0 - P_1)$, the other in $(\mathcal{V} - \mathcal{V}_0) \cup (P_1 - P_0)$.

The first curve is easy: we just take a line bundle $\mathbf{L} \to \mathcal{X}$ having degree $(d-1)/2$ on each fibre, choose a nonvanishing section of the coherent sheaf $R^1\pi_1 \mathbf{L}^2 \mathbf{\Lambda}^{-1}$ on a neighbourhood of $0 \in C$ and consider the corresponding extension

$$0 \to \mathbf{L} \to \mathbf{E} \to \mathbf{\Lambda} \mathbf{L}^{-1} \to 0$$

on that neighbourhood. As for the second curve, we first change base in a neighbourhood of $0 \in C$, choosing a local parameter z and pulling back by the map $z(w) = w^2$.

Since $\mathcal{X}$ is locally isomorphic to $xy - z = 0$ near the node over 0, after base change it is locally isomorphic to $xy - w^2 = 0$ and so, if we blow up the new family at the node to get $\tilde{\mathcal{X}}$, the exceptional divisor E is a -2–curve. If on a smaller neighbourhood of 0 we then choose a nonvanishing section of $R^1\pi_1\mathbf{L}^2\mathbf{\Lambda}^{-1}(E)$ whose restriction both to E and to the proper transform of X_0 is nontrivial, then as before the corresponding extension

$$0 \to \mathbf{L} \to \mathbf{E} \to \mathbf{\Lambda}\mathbf{L}^{-1}(-E) \to 0$$

does the trick. Note that $\mathbf{E}_0$ is of type II_2 because $\mathbf{\Lambda}\mathbf{L}^{-1}(-E)_E = \mathcal{O}(2)$ and the only non-split extension of $\mathcal{O}$ by $\mathcal{O}(2)$ is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

(6.4) *On S, we have $\mathcal{O}(\Theta) = \mathcal{O}(\tilde{\Theta})(2H_1 + D_1)$.*

Proof. We follow the steps in our recipe for the universal family backwards, determining how the determinant line bundle is modified at each stage. We denote by p all of the projection maps to S.

3. There is a normalization sequence on $\mathcal{U}_3$

$$0 \to \mathbf{E}_3 \to \mathbf{E}_2 \to \mathcal{O}_{S\times p_1}(\mathbf{E}_3) \to 0$$

where by abuse of notation we write $\mathbf{E}_2$ for its direct image, and where in the last term $S \times p_1$ is contained in $\mathcal{U}_3$ as the proper transform. This gives

$$\mathcal{O}(-\Theta) = \det p_!(\mathbf{E}_3) = \det p_!(\mathbf{E}_2) \otimes \Lambda^2(\mathbf{E}_3)_{p_1} = \det p_!(\mathbf{E}_2) \otimes \Lambda^2(\mathbf{E}_2)_{p_1}.$$

2. From the exact sequence

$$0 \to \mathbf{E}_1 \to \mathbf{E}_2 \to M_1 \oplus M_2 \to 0$$

we obtain

$$\det p_!(\mathbf{E}_2) = \det p_!(\mathbf{E}_1)(-D_1 - D_2)$$

and

$$(\Lambda^2\mathbf{E}_2)_{p_1} = (\Lambda^2\mathbf{E}_1)_{p_1}(D_2),$$

since $\det p_! M_1 = -D_1$, $\det p_! M_2 = -D_2$, $(M_1)_{p_1} = 0$, and $(M_2)_{p_1}$ is a one-dimensional quotient supported on D_2. Hence

$$\mathcal{O}(-\Theta) = \det p_!(\mathbf{E}_1) \otimes (\Lambda^2 \mathbf{E}_1)_{p_1}(-D_1).$$

1. From the exact sequence

$$0 \to \mathbf{E}_0 \to \mathbf{E}_1 \to \mathcal{O}_{H_1+H_2}(\mathbf{E}_0) \to 0$$

we obtain

$$\det p_!(\mathbf{E}_1) = \det p_!(\mathbf{E}_0)(-2H_1 - 2H_2) = \mathcal{O}(-\tilde{\Theta})(-2H_1 - 2H_2)$$

and

$$(\Lambda^2 \mathbf{E}_1)_{p_1} = (\Lambda^2 \tilde{E})_{p_1}(2H_2) = \mathcal{O}(2H_2),$$

according to our conventions. Putting together the last three equations gives the result.

7 Proof of the factorization rule

Finally we are ready to prove **(3.2)**. Let $\tilde{N}$ be the moduli space of rank 2 stable bundles over $\tilde{X}$ of determinant $\tilde{\Lambda}$, and let $\tilde{\mathbf{E}} \to \tilde{N} \times \tilde{X}$ and $\mathcal{O}(\tilde{\Theta}) \in \operatorname{Pic} \tilde{N} \otimes \mathbb{Q}$ be the normalized universal bundle and the determinant line bundle, both defined in the usual way. Finally, let $\mathbf{q} = \{q_1, \ldots, q_s\} \subset X_0$ be smooth points, labeled as usual by $\mathbf{m}$. As in §2, we use the shorthand $\mathbf{E}_x^m = S^m \mathbf{E}_x$ and $\mathbf{E}_\mathbf{q}^\mathbf{m} = \bigotimes_j \mathbf{E}_{q_j}^{m_j}$. The central result is then the following.

(7.1) *There are canonical isomorphisms*

(i) $H^0(\mathcal{V}_0; \mathcal{O}(k\Theta) \otimes \mathbf{E}_\mathbf{q}^\mathbf{m}) = \bigoplus_{m=0}^k H^0(\tilde{N}; \mathcal{O}(k\tilde{\Theta}) \otimes \mathbf{E}_\mathbf{q}^\mathbf{m} \otimes \mathbf{E}_{p_1}^m \otimes \mathbf{E}_{p_2}^m)$;

(ii) $H^1(\mathcal{V}_0; \mathcal{O}(k\Theta) \otimes \mathbf{E}_\mathbf{q}^\mathbf{m}) = 0$.

(We may alternatively write $\mathbf{E}_{p_2}^m = (\mathbf{E}_{p_2}^m)^*$, since all representations of SL_2 are self-dual.)

If we define Z_k for our nodal curve X_0, in analogy with **(3.1)**, to be

$$Z_k(X_0;(\mathrm{q},\mathrm{m})) = H^0(\mathcal{V}_0;\mathcal{O}(k\Theta)\otimes \mathbf{E}^{\mathrm{m}}_{\mathrm{q}}),$$

then **(7.1)**(i) can be expressed as

$$Z_k(X_0;(\mathrm{q},\mathrm{m})) = \bigoplus_{m=0}^{k} Z_k(\tilde{X};(\mathrm{q},\mathrm{m}),(p_1,m),(p_2,m)).$$

On the other hand, **(7.1)**(ii), **(4.9)**, and **(4.10)** imply that for any smooth curve X of genus g,

$$\dim Z_k(X;(\mathrm{q},\mathrm{m})) = \dim Z_k(X_0;(\mathrm{q},\mathrm{m})).$$

Thus we obtain our factorization result **(3.2)** for d odd; the corresponding result for d even then follows from the Hecke correspondence **(4.7)**. The remainder of this section is consequently devoted to the proof of **(7.1)**.

Let $\overline{\mathrm{SL}}_2$ be the closure of $\mathrm{SL}(\mathbf{E}_{p_1},\mathbf{E}_{p_2}) \subset \overline{\mathrm{GL}} = \overline{\mathrm{GL}}(\mathbf{E}_{p_1},\mathbf{E}_{p_2})$, or equivalently, the quadric threefold in $\mathbb{P}^4$ which is the zero-set of $v-1 \in S^2(\mathbf{E}^*_{p_1}\otimes\mathbf{E}_{p_2}\oplus\mathbb{C})$, where v is the image in $S^2(\mathbf{E}^*_{p_1}\otimes\mathbf{E}_{p_2})$ of the element of $\Lambda^2\mathbf{E}^*_{p_1}\otimes\Lambda^2\mathbf{E}_{p_2}$ determined by the line bundle $\Lambda_0 \to X_0$, and $1 \in S^2\mathbb{C} = \mathbb{C}$. We regard $H^0(\overline{\mathrm{SL}}_2;\mathcal{O}(k))$ as a representation of $\mathrm{SL}(\mathbf{E}_{p_1})\times\mathrm{SL}(\mathbf{E}_{p_2})$.

(7.2) *For* $k \geq 0$,

$$H^0(\overline{\mathrm{SL}}_2;\mathcal{O}(k)) = \oplus_{i=0}^{k} S^i\mathbf{E}^*_{p_1}\otimes S^i\mathbf{E}_{p_2};$$
$$H^1(\overline{\mathrm{SL}}_2;\mathcal{O}(k)) = 0.$$

Proof by induction on k. If $\mathbb{P}^3 \subset \mathbb{P}^4$ is the hyperplane at infinity, then $\overline{\mathrm{SL}}_2 \cap \mathbb{P}^3$ is the quadric surface $Q = \mathbb{P}\mathbf{E}^*_{y_1}\times\mathbb{P}\mathbf{E}_{y_2}$. Hence there is a short exact sequence

(7.3) $$0 \longrightarrow \mathcal{O}_{\overline{\mathrm{SL}}_2}((k-1)Q) \longrightarrow \mathcal{O}_{\overline{\mathrm{SL}}_2}(kQ) \longrightarrow \mathcal{O}_Q(k,k) \longrightarrow 0.$$

From the long exact sequence on $\mathbb{P}^4$ of

$$0 \longrightarrow \mathcal{O}(k-2) \longrightarrow \mathcal{O}(k) \longrightarrow \mathcal{O}_{\overline{\mathrm{SL}}_2}(k) \longrightarrow 0,$$

$H^1(\overline{\mathrm{SL}}_2;kQ) = 0$ for all $k \geq 0$, so the long exact sequence of **(7.3)** becomes

(7.4) $$0 \to H^0(\overline{\mathrm{SL}}_2;(k-1)Q) \to H^0(\overline{\mathrm{SL}}_2;kQ) \to H^0(Q;\mathcal{O}(k,k)) \to 0.$$

But $H^0(Q;\mathcal{O}(k,k)) = S^k\mathbf{E}^*_{p_1}\otimes S^k\mathbf{E}_{p_2}$ injects naturally into $H^0(\mathbb{P}^4;\mathcal{O}(k)) = S^k(\mathbf{E}^*_{p_1}\otimes \mathbf{E}_{p_2}\oplus\mathbb{C})$, and the composition of this injection with restriction to $H^0(\overline{\mathrm{SL}}_2;\mathcal{O}(k))$ splits (7.4). Hence $H^0(\overline{\mathrm{SL}}_2;\mathcal{O}(k)) = H^0(\overline{\mathrm{SL}}_2;\mathcal{O}(k-1))\oplus S^k\mathbf{E}^*_{p_1}\otimes S^k\mathbf{E}_{p_2}$, and the lemma follows by induction.

Proof of **(7.1)**. We suppose for convenience that there are no marked points q_j. The proof remains exactly the same when they are added, except that all the bundles $\mathcal{O}(k\Theta)$ need to be tensored by $\mathbf{E}^{\mathrm{m}}_{\mathbf{q}}$.

If $p:\tilde{\mathcal{V}}_0\to\mathcal{V}_0$ is the projection, then there is a normalization sequence on $\mathcal{V}_0$

$$0\longrightarrow\mathcal{O}(k\Theta)\longrightarrow p_*\mathcal{O}(k\Theta)\longrightarrow p_*\mathcal{O}(k\Theta)/\mathcal{O}(k\Theta)\longrightarrow 0, \tag{7.5}$$

where by our usual abuse of notation we write $\mathcal{O}(\Theta)$ for $p^*\mathcal{O}(\Theta)$. Since $\mathcal{V}_0$ is obtained from $\tilde{\mathcal{V}}_0$ by gluing together two smooth divisors as a normal crossing, the cohomology of $p_*\mathcal{O}(k\Theta)$ and $p_*\mathcal{O}(k\Theta)/\mathcal{O}(k\Theta)$ is easy to express in terms of $\tilde{\mathcal{V}}_0$: for example, $H^j(\mathcal{V}_0;p_*\mathcal{O}(k\Theta)) = H^j(\tilde{\mathcal{V}}_0;\mathcal{O}(k\Theta))$. Since according to **(6.2)** $\tilde{\mathcal{V}}_0$ is birational to $P_0\cup P_1$, this means that $H^j(\mathcal{V}_0;p_*\mathcal{O}(k\Theta)) = H^j(P_0;\mathcal{O}(k\Theta))\oplus H^j(P_1;\mathcal{O}(k\Theta))$ for $j=0,1$. Likewise, $H^j(\mathcal{V}_0;p_*\mathcal{O}(k\Theta)/\mathcal{O}(k\Theta)) = H^j(Q_1;\mathcal{O}(k\Theta))$ since $p_*\mathcal{O}(k\Theta)/\mathcal{O}(k\Theta) = \mathcal{O}_D(\mathcal{O}(k\Theta))$. Hence the long exact sequence of **(7.5)** becomes

$$\begin{aligned}0\longrightarrow H^0(\mathcal{V}_0;\mathcal{O}(k\Theta))\longrightarrow H^0(P_0;\mathcal{O}(k\Theta))\oplus H^0(P_1;\mathcal{O}(k\Theta))\longrightarrow H^0(Q_1;\mathcal{O}(k\Theta))\\ \longrightarrow H^1(\mathcal{V}_0;\mathcal{O}(k\Theta))\longrightarrow H^1(P_0;\mathcal{O}(k\Theta))\oplus\\ H^1(P_1;\mathcal{O}(k\Theta))\longrightarrow H^1(Q_1;\mathcal{O}(k\Theta))\longrightarrow\cdots\end{aligned}$$

But since there are isomorphisms $H^j(Q;\mathcal{O}) = H^j(\mathbb{P}^3;\mathcal{O})$ induced by restriction, there are similar isomorphisms $H^j(P_1;\mathcal{O}(k\Theta)) = H^j(Q_1;\mathcal{O}(k\Theta))$. Hence the long exact sequence above degenerates to give isomorphisms $H^j(\mathcal{V}_0;\mathcal{O}(k\Theta)) = H^j(P_0;\mathcal{O}(k\Theta))$ for $j=0,1$. But P_0 is the bundle $\overline{\mathrm{SL}}(\mathbf{E}_{p_1},\mathbf{E}_{p_2})\to\tilde{N}$, so using **(7.2)** in the Leray-Hirsch sequence we find

$$H^j(\mathcal{V}_0;\mathcal{O}(k\Theta)) = \bigoplus_{i=0}^{k} H^j(\tilde{N};\mathcal{O}(k\Theta)\otimes\mathbf{E}^i_{p_1}\otimes\mathbf{E}^i_{p_2})$$

for $j=0,1$. So now it suffices to show that the right-hand side vanishes for $j=1$. This is proved similarly to **(4.12)**, using our hypothesis that $g-1\geq 3$. Then by **(4.4)**, the birational map between moduli spaces of parabolic bundles with different weights is 3-birational, so induces isomorphisms of H^1 by (**4.3**). But as in (**4.12**), if we choose the weights appropriately, H^1 will be zero by Kodaira vanishing. □

Part II

Stable pairs and flips

8 Introduction to stable pairs

In this part we will temporarily abandon the Verlinde formula and make what appears to be a major digression. We will be studying a new moduli space, one of pairs consisting of a vector bundle $E \to X$ of fixed determinant and a section $\phi \in H^0(E) - 0$. This sounds rather a routine generalization of the well-known theory of stable bundles, and indeed, we shall start off with a faithful imitation of Gieseker's construction [18] of the ordinary moduli spaces. However, in this part and the next, we will go on to discuss some remarkable features of the moduli spaces of pairs, among which are the following.

1. Unlike bundles on curves, pairs admit many possible stability conditions. In fact, stability of a pair depends on an auxiliary parameter σ analogous to the weights of a parabolic bundle. This parameter was first detected by Bradlow-Daskalopoulos [8] in the study of vortices on Riemann surfaces, and indeed the spaces we shall construct can also be interpreted as moduli spaces of rank 2 vortices. As σ varies, we will see that the moduli space undergoes a sequence of flips in the sense of Mori theory, whose locations can be specified quite precisely.

2. For some values of σ the moduli space $M(\sigma, \Lambda)$ is the blow-up of $\mathbb{P}H^1(\Lambda^{-1})$ along X, embedded as a complete linear system. Thus we can use $M(\sigma, \Lambda)$ to study the projective embeddings of X. In particular, we will obtain a very general formula (**16.8**) for the dimension of the space of hypersurfaces of degree $m + n$ in $\mathbb{P}H^1(\Lambda^{-1})$ with a singularity at X of order $n - 1$. This formula does not depend on the precise choice of X and Λ, only on g and d, which is rather surprising.

3. For other values of σ, stability of the pair implies semistability of the bundle, so $M(\sigma, \Lambda)$ plays the role in rank 2 Brill-Noether theory of the symmetric product in the usual case, and there is an Abel-Jacobi map from $M(\sigma, \Lambda)$ to the moduli space of

semistable bundles. For large d this is generically a fibration, so we can use moduli spaces of pairs to study moduli spaces of bundles. In particular, we recover the known formulas for Poincaré polynomials [2, 22] and Picard groups [14]. And we shall attain the main goal of the thesis: to prove the rank 2 Verlinde formula **(16.9)** for both odd and even degrees.

Part II will be devoted to item 1 in the list above; Part III will be devoted to items 2 and 3. A more detailed outline is as follows. In §9 we prove some basic facts about pairs, in analogy with bundles. We then give the promised geometric invariant theory construction of the moduli space $M(\sigma, \Lambda)$ of σ-semistable pairs, and of a universal family over the stable points of $M(\sigma, \Lambda)$. The choice of σ corresponds to a choice of linearization for the group action. In §10 we discuss the deformation theory of the moduli problem. In §11 we show that the $M(\sigma, \Lambda)$ are reduced, rational, and smooth at the stable points. We then show that as σ varies, $M(\sigma, \Lambda)$ undergoes a sequence of flips whose centres are symmetric products of X. We also define the rank 2 Abel-Jacobi map mentioned above. In §12 we calculate the Poincaré polynomial of $M(\sigma, \Lambda)$, and extract from it the Harder-Narasimhan formula for the Poincaré polynomial of the moduli space of rank 2 bundles of odd degree. Part II concludes with a brief appendix, §13, explaining the relationship to Cremona transformations and Bertram's work on secant varieties.

In Part III we concentrate on studying the line bundles over $M(\sigma, \Lambda)$, and their spaces of sections. In §14 we compute the Picard group of $M(\sigma, \Lambda)$, and its ample cone. We explain how any section of a line bundle on $M(\sigma, \Lambda)$ can be interpreted as a hypersurface in projective space, singular to some order on an embedded X. We also make the connection with the Verlinde vector spaces. Finally in §§15 and 16 we use the Riemann-Roch theorem to calculate Euler characteristics of the line bundles in $M(\sigma, \Lambda)$. Combined with the information from §14, Kodaira vanishing, and some residue calculations which were carried out by Don Zagier, this gives a formula for the dimensions of the spaces of sections of line bundles on $M(\sigma, \Lambda)$, under some mild hypotheses. We conclude by extracting the Verlinde formula from this.

9 Constructing moduli spaces of σ-semistable pairs

Our main objects of study, which we refer to simply as *pairs*, will be pairs (E, ϕ) consisting of a rank 2 algebraic vector bundle E of positive degree d over our curve X, and a nonzero section $\phi \in H^0(E)$. A careful study of such pairs was made by Bradlow-Daskalopoulos [8]. They defined a stability condition for pairs and proved a Narasimhan-Seshadri-type theorem relating stable pairs to vortices on a Riemann surface. The vortex equations depend on a positive real parameter τ, and so the stability condition also depends on τ. Bradlow and Daskalopoulos went on to give a gauge-theoretic construction of the moduli space of τ-stable pairs, under certain conditions on τ and $\deg E$. We will study the moduli spaces from an algebro-geometric point of view. Hence in this section we will give a geometric invariant theory construction of the moduli space of τ-stable pairs for arbitrary τ and $\deg E$ (though for convenience we assume $\operatorname{rank} E = 2$). Aaron Bertram has informed me that he has done something similar [6], and I apologize to him for any overlap.

The Bradlow-Daskalopoulos stability condition is in general rather complicated, but in the rank 2 case it simplifies to the following. Let σ be a positive rational number. It is related to τ by $\sigma = \deg E/2 - \tau \operatorname{vol} X/4\pi$, where $\operatorname{vol} X$ is the volume of X with respect to the metric chosen in [8].

(9.1) *Definition.* The pair (E, ϕ) is *σ-semistable* if for all line bundles $L \subset E$,

$$\deg L \leq \tfrac{1}{2} \deg E - \sigma \text{ if } \phi \in H^0(L) \text{ and}$$
$$\deg L \leq \tfrac{1}{2} \deg E + \sigma \text{ if } \phi \notin H^0(L).$$

It is *σ-stable* if both inequalities are strict.

The main result of this section is then the following.

(9.2) *Let $\Lambda \to X$ be a line bundle of degree $d > 0$. There is a projective moduli space $M(\sigma, \Lambda)$ of σ-semistable pairs (E, ϕ) such that $\Lambda^2 E = \Lambda$, nonempty if and only if $\sigma \leq d/2$.*

Our construction will be modelled on that of Gieseker [18]. We begin with a few basic facts about σ-stable and semistable pairs, parallel to those for bundles. We write Λ for $\Lambda^2 E$, and d for $\deg E = \deg \Lambda$.

(9.3) *For $\sigma > 0$, there exists a σ-semistable pair of determinant Λ if and only if $\sigma \leq d/2$.*

Proof. If $\sigma > d/2$, then σ-semistability implies $\deg L < 0$ if $\phi \in H^0(L)$, which is absurd. If $\sigma \leq d/2$, let $L \to X$ be a line bundle of degree $[d/2 - \sigma]$ having a nonzero section ϕ. Let E be a nonsplit extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow \Lambda L^{-1} \longrightarrow 0.$$

Then the first inequality in the definition **(9.1)** is obvious. As for the second, if $M \subset E$ and $\deg M > d/2 + \sigma$, then there is a nonzero map $M \to \Lambda L^{-1}$. Since $\deg \Lambda L^{-1} < d/2 + \sigma + 1$, this is an isomorphism, so the extension is split, which is a contradiction. □

(9.4) *Let (E, ϕ) be a pair. There is at most one σ-destabilizing bundle $L \subset E$ such that $\phi \in H^0(L)$, and at most one σ-destabilizing $M \subset E$ such that $\phi \notin H^0(M)$. If both L and M exist, then $E = L \oplus M$.*

Proof. The first statement is obvious, and the second follows from the uniqueness of ordinary destabilizing bundles, since $\deg M \geq \frac{1}{2} \deg E + \sigma > \frac{1}{2} \deg E$. If both L and M exist, then the map $M \to E \to \Lambda L^{-1}$ is nonzero since $\phi \in H^0(L)$ but $\notin H^0(M)$. But $\deg M \geq d/2 + \sigma \geq \deg \Lambda L^{-1}$, so $M = \Lambda L^{-1}$ and E is split. □

(9.5) *Let (E_1, ϕ_1) and (E_2, ϕ_2) be σ-stable pairs of degree d, and let $\psi : E_1 \to E_2$ be a map such that $\psi \phi_1 = \phi_2$. Then ψ is an isomorphism.*

Proof. The kernel of ψ is a subsheaf of a locally free sheaf on a smooth curve, so it is locally free. If $\operatorname{rank} \ker \psi = 2$, then ψ is generically zero, so $\psi = 0$ and $\psi \phi_1 \neq \phi_2$.

If $\operatorname{rank}\ker\psi = 1$, then $\ker\psi$ is a line subbundle L of E_1, since $E_1/\ker\psi$ is contained in the torsion-free sheaf E_2. Hence ψ descends to a map $\Lambda L^{-1} \to E_2$ (possibly with zeroes) such that $\phi_2 \in H^0(\Lambda L^{-1})$. Since (E_2, ϕ_2) is σ-stable, $\deg \Lambda L^{-1} < d/2 - \sigma$, so $\deg L > d/2 + \sigma$, contradicting the σ-stability of (E_1, ϕ_1). Finally, if $\operatorname{rank}\ker\psi = 0$, then $\ker\psi = 0$ and ψ is injective. Moreover, $\operatorname{coker}\psi$ is a coherent sheaf on a curve with rank and degree 0, so $\operatorname{coker}\psi = 0$ and ψ is an isomorphism. □

(9.6) *Let (E, ϕ) be a σ-stable pair. Then there are no endomorphisms of E annihilating ϕ except 0, and no endomorphisms preserving ϕ except the identity.*

Proof. Subtracting from the identity interchanges the two statements, so they are equivalent. We prove the first. Any endomorphism annihilating ϕ annihilates the subbundle L generated by ϕ, so descends to a map $E/L \to E$. But by σ-stability E/L is a line bundle of degree $\geq d/2 + \sigma$, so the image of this map, if it were nonzero, would generate a line bundle of degree $\geq d/2 + \sigma$, which would be destabilizing. □

(9.7) *Let $(\mathbf{E}, \mathbf{\Phi}), (\mathbf{E}', \mathbf{\Phi}') \to T \times X$ be two families over T parametrizing the same pairs. Then $(\mathbf{E}, \mathbf{\Phi}) = (\mathbf{E}', \mathbf{\Phi}')$.*

Proof. For any $t \in T$, the subspace of $H^0(X; \operatorname{Hom}(\mathbf{E}_t, \mathbf{E}'_t))$ consisting of homomorphisms ψ such that $\psi \mathbf{\Phi}_t = \lambda \mathbf{\Phi}'_t$ for some $\lambda \in \mathbb{C}$ is one-dimensional by **(9.6)**. This determines an invertible subsheaf of the direct image $(R^0\pi) \operatorname{Hom}(\mathbf{E}_t, \mathbf{E}'_t)$. But this subsheaf is trivialized by the section $\lambda = 1$, which produces the required isomorphism. □

The notion of a Harder-Narasimhan filtration for rank 2 pairs is quite a simple one. For (E, ϕ) stable, we of course define $\operatorname{Gr}(E, \phi) = (E, \phi)$. Otherwise, we define $\operatorname{Gr}(E, \phi)$ to be a direct sum of line bundles, one of them containing the section ϕ, as follows. If L is the destabilizing bundle and $\phi \in H^0(L)$, we define $\operatorname{Gr}(E, \phi) = (L \oplus \Lambda L^{-1}, \phi)$. If M is the destabilizing bundle and $\phi \notin H^0(M)$, we project ϕ to a nonzero section $\phi' \in H^0(\Lambda M^{-1})$ and define $\operatorname{Gr}(E, \phi) = (M \oplus \Lambda M^{-1}, \phi')$. Note that if there are destabilizing bundles of both sorts, then by **(9.4)** $E = L \oplus \Lambda L^{-1}$ and the two definitions agree.

(9.8) *There exists a degeneration of* (E,ϕ) *to* $\mathrm{Gr}\,(E,\phi)$, *but* $\mathrm{Gr}\,(E,\phi)$ *degenerates to no semistable bundle.*

Proof. The first statement is vacuous when (E,ϕ) is stable. If it is unstable, say with destabilizing bundle M, we can construct a pair $(\mathbf{E},\mathbf{\Phi})\to X\times\mathbb{C}$ such that $(\mathbf{E}_z,\mathbf{\Phi}_z)\cong(E,\phi)$ for $z\neq 0$, but $(\mathbf{E}_0,\mathbf{\Phi}_0)\cong\mathrm{Gr}\,(E,\phi)$, as follows. Pull back (E,ϕ) to $X\times\mathbb{C}$, and tensor by $\mathcal{O}(0)$ when $\phi\notin H^0(M)$. This gives a pair $(\mathbf{E}',\mathbf{\Phi}')\to X\times\mathbb{C}$ such that $\mathbf{\Phi}'$ is annihilated by the natural map $\mathbf{E}'\to\Lambda M^{-1}|_{X\times\{0\}}$. Let $\mathbf{E}$ be the kernel of this map; then $\mathbf{\Phi}'$ descends to $\mathbf{\Phi}\in H^0(\mathbf{E})$, and it is straightforward to check that $(\mathbf{E},\mathbf{\Phi})$ has the desired properties.

As for the second statement, suppose first that (E,ϕ) is stable. If C is a curve, $p\in C$, and $(\mathbf{E},\mathbf{\Phi})\to X\times C$ is a flat family of pairs such that $(\mathbf{E}_z,\mathbf{\Phi}_z)\cong(E,\phi)$ for $z\neq p$, then $\mathbf{\Phi}_p$ has the same zero-set D as ϕ, so E and $\mathbf{E}_p$ are both extensions of $L=\mathcal{O}(D)$ by $\Lambda(-D)$; indeed, $\mathbf{E}$ is a family of such extensions. The extension class varies continuously, so the extension class of $\mathbf{E}_p$ is in the same ray as that of E. If it is nonzero, $(E,\phi)\cong(\mathbf{E}_p,\mathbf{\Phi}_p)$, and if it is zero, $(\mathbf{E}_p,\mathbf{\Phi}_p)$ is destabilized by ΛL^{-1}.

Now suppose that (E,ϕ) is not stable, so that for some L, $\mathrm{Gr}\,(E,\phi)=L\oplus\Lambda L^{-1}$ and $\phi\in H^0(L)$. Then as above $\mathbf{E}_p$ is an extension of L by ΛL^{-1}, but now by continuity the extension class must be zero, so $\mathrm{Gr}\,(E,\phi)=(\mathbf{E}_p,\mathbf{\Phi}_p)$. □

(9.9) *If* (E,ϕ) *is* σ*-(semi)stable, then so is* $(E(D),\phi(D))$ *for any effective divisor* D. *Likewise, if* ϕ *vanishes on an effective divisor* D *and* (E,ϕ) *is* σ*-(semi)stable, then so is* $(E(-D),\phi(-D))$.

Proof. If $L\subset E$ is any line bundle, $\phi(D)\in H^0(L(D))$ if and only if $\phi\in H^0(L)$, and $\deg L(D)=\deg L+\deg D$. But $\frac{1}{2}\deg E(D)=\frac{1}{2}\deg E+\deg D$ also, so both inequalities are preserved by tensoring with D. The second statement is proved similarly. □

Hence if the moduli spaces $M(\sigma,\Lambda)$ exist for large enough d, then the moduli spaces for smaller d will be contained inside them as the locus of pairs (E,ϕ) such that ϕ vanishes on some effective D. So to prove our existence theorem (**9.2**) it suffices to construct $M(\sigma,\Lambda)$ for d large relative to g and σ, and *we will assume for the remainder of* §9 *that d is large in this sense.* For such a large d, we then have the following useful fact.

(9.10) *For fixed g and σ and large d, (E,ϕ) σ-semistable implies that $H^1(E)=0$ and E is globally generated.*

Proof. Suppose that $H^1(E)\neq 0$. Then $H^0(KE^*)\neq 0$, so there is an injection $0\to K^{-1}(D)\to E^*$ for some effective D. Hence there is an injection $0\to K^{-1}\Lambda(D)\to E$. Since $\deg K^{-1}\Lambda(D)\geq 2-2g+d$, the σ-semistability condition implies that $2-2g+d\leq d/2+\sigma$, so that $d\leq 4g-4+2\sigma$. So for d larger than this, $H^1(E)=0$.

Similarly, if $d>4g-2+2\sigma$, then $H^1(E(-x))=0$ for all $x\in X$, so E is globally generated. □

Since we are assuming that d is large, the above lemma implies that for (E,ϕ) σ-stable, $\dim H^0(E)=\chi(E)=d+2-2g$. Call this number χ. If we fix an isomorphism $s:\mathbb{C}^\chi\to H^0(E)$, we obtain a map $\Lambda^2\mathbb{C}^\chi\xrightarrow{s}\Lambda^2H^0(E)\xrightarrow{\wedge}H^0(\Lambda)$, which is nonzero because E is globally generated. Thus to any bundle E appearing in a σ-semistable pair, and any isomorphism s, we associate a point $T(E,s)\in\mathbb{P}\,\mathrm{Hom}(\Lambda^2\mathbb{C}^\chi,H^0(\Lambda))$. We will consider the pair $(T(E,s),s^{-1}\phi)\in\mathbb{P}\,\mathrm{Hom}\times\mathbb{P}\mathbb{C}^\chi$, where $\mathbb{P}\,\mathrm{Hom}$ is short for $\mathbb{P}\,\mathrm{Hom}(\Lambda^2\mathbb{C}^\chi,H^0(\Lambda))$. Roughly speaking, $M(\sigma,\Lambda)$ will be a geometric invariant theory quotient of the set of such pairs. The quotient is necessary to remove the dependence on the choice of s. Since two such isomorphisms are related by an element of $\mathrm{SL}(\chi)$, the group action we consider is the obvious diagonal action of $\mathrm{SL}(\chi)$ on $\mathbb{P}\,\mathrm{Hom}\times\mathbb{P}\mathbb{C}^\chi$. As usual in geometric invariant theory, we must *linearize* the action by choosing an ample line bundle and lifting the action of $\mathrm{SL}(\chi)$ to its dual. So let the ample bundle be any power of $\mathcal{O}(\chi+2\sigma,4\sigma)$, with the obvious lifting. (Of course $\chi+2\sigma$ and 4σ may not be integers, but by abuse of notation we will refrain from clearing denominators, since the choice of power does not matter.) Stable and semistable points in the sense of geometric invariant theory can then be defined using this linearization.

(9.11) *If (E,ϕ) is σ-(semi)stable, then $(T(E,s),s^{-1}\phi)$ is a (semi)stable point with respect to the linearization above.*

Proof. Suppose $T=(T(E,s),s^{-1}\phi)$ is not semistable. Then by Mumford's numerical criterion [31, 35] there exists a nontrivial 1-parameter subgroup $\lambda : \mathbb{C}^\times \to \mathrm{SL}(\chi)$ such that for any $\tilde{T}$ in the fibre of the dual of our ample bundle over T, $\lim_{t\to 0}\lambda(t)\cdot\tilde{T}=0$. We interpret this limit concretely as follows. Any 1-parameter subgroup of $\mathrm{SL}(\chi)$ can be diagonalized, so there exists a basis e_i of $\mathbb{C}^\chi$ such that $\lambda(t)\cdot e_i = t^{r_i}e_i$, where $r_i\in\mathbb{Z}$ are not all zero and satisfy $\sum_i r_i = 0$ and $r_i\le r_j$ for $i\le j$. Then $\lim_{t\to 0}\lambda(t)\cdot\tilde{T}=0$ means that any basis element $(e_i^*\wedge e_j^*\otimes v, e_k)\in \mathrm{Hom}(\Lambda^2\mathbb{C}^\chi, H^0(\Lambda))\oplus \mathbb{C}^\chi$ which is acted on with weight ≤ 0 has coefficient zero in the basis expansion of $\tilde{T}$. Because of our choice of linearization, this means that $T(E,s)(e_i,e_j)=0$ whenever

$$r_i + r_j \le \frac{2\sigma}{\chi/2+\sigma}\, r_\ell, \tag{9.12}$$

where $\ell=\max\{i : \text{coefficient of } e_i \text{ in } s^{-1}\phi \text{ is} \ne 0\}$. Let $L\subset E$ be the line bundle generated by $s(e_1)$. We distinguish between two cases, according to whether $\phi\in H^0(L)$.

First case: $\phi\in H^0(L)$. For $i\le \chi/2-\sigma+1$, note that

$$(\chi/2-\sigma)r_1 + (\chi/2+\sigma)r_i \le \sum_i r_i = 0,$$

since the left-hand side can be regarded as the integral over $[0,\chi)$ of a (two-step) step function whose value on $[j-1,j)$ is $\le r_j$. Hence for $i\le \chi/2-\sigma+1$,

$$r_1 + r_i \le \frac{2\sigma}{\chi/2+\sigma}\, r_1 \le \frac{2\sigma}{\chi/2+\sigma}\, r_\ell, .$$

so $T(E,s)(e_1,e_i)=s(e_1)\wedge s(e_i)=0$. Hence $s(e_i)$ is a section of the same line bundle as $s(e_1)$, namely L. So $\dim H^0(L) > \chi/2-\sigma$; since d is large relative to g and σ, this implies that $\deg L > d/2-\sigma$, so (E,ϕ) is not σ-semistable.

Second case: $\phi\notin H^0(L)$. For $i\le \chi/2+\sigma+1$,

$$(\chi/2+\sigma)r_1 + (\chi/2-\sigma)r_i \le 0,$$

for the same reason as above. Hence

$$r_1 + r_i \le \frac{2\sigma}{\chi/2+\sigma}\, r_i.$$

We claim that $\ell > \chi/2 + \sigma + 1$. If not, then for all $i \leq \ell$,

$$r_1 + r_i \leq \frac{2\sigma}{\chi/2 + \sigma} r_\ell,$$

so that $s(e_i)$ would be in the same line bundle as $s(e_1)$. Since ϕ is a linear combination of e_i for $i \leq \ell$, we would conclude $\phi \in H^0(L)$, a contradiction. This proves the claim.

So for $i \leq \chi/2 + \sigma + 1$, actually

$$r_1 + r_i \leq \frac{2\sigma}{\chi/2 + \sigma} r_\ell;$$

hence $s(e_i) \in H^0(L)$ as in the first case. So $\dim H^0(L) > \chi/2 + \sigma$, and again (E, ϕ) is not σ-semistable.

The proof for stability is similar: the numerical criterion now just says $\lim_{t \to 0} \lambda(t) \cdot \tilde{T} \neq \infty$, so we replace the $\leq$ in **(9.12)** by $<$. We just need to note that if $i < \chi/2 - \sigma + 1$, then

$$(\chi/2 - \sigma) r_1 + (\chi/2 + \sigma) r_i < 0$$

strictly, because either the two step functions are different just to the left of $\chi/2 - \sigma$, or the smaller one is identically $r_1 < 0$. □

(9.13) *Let (E, ϕ) be a pair, let $s : \mathbb{C}^\chi \to H^0(E)$ be a linear map, and let $v \in \mathbb{C}^\chi$ satisfy $s(v) = \phi$. Write T_s for the composition $\Lambda^2 \mathbb{C}^\chi \xrightarrow{s} \Lambda^2 H^0(E) \xrightarrow{\Delta} H^0(\Lambda)$. If (T_s, v) is semistable, then s is an isomorphism and (E, ϕ) is σ-semistable.*

Proof. First of all, note that if s is not injective, then (T_s, v) is certainly not semistable. Indeed, if $s(w) = 0$ for some w, put $e_1 = w$, $e_2 = v$, extend to a basis $\{e_i\}$ of $\mathbb{C}^\chi$, and then take the 1-parameter subgroup defined by $r_1 = -\chi + 2$, $r_2 = 0$, $r_3 = \cdots = r_\chi = 1$. Then $\ell = 2$, so

$$r_i + r_j \leq \frac{2\sigma}{\chi/2 + \sigma} r_l$$

means just $r_i + r_j \leq 0$. Hence either $i = 1$, or $j = 1$, or $i = j = 2$; in any case, clearly $T_s(e_i, e_j) = 0$.

Suppose then that s is injective and (E, ϕ) is σ-unstable. We will prove (T_s, v) is unstable. Let $L \subset E$ be the destabilizing bundle. We distinguish two cases, depending on the sign of $d - \deg L - 2g + 2$.

First case: $d - \deg L > 2g-2$. Then $H^1(\Lambda L^{-1}) = 0$, but $H^1(L) = 0$ also since $\deg L > d/2 - \sigma$ which is large relative to g. Hence from the long exact sequence of

$$\textbf{(9.14)} \qquad 0 \longrightarrow L \longrightarrow E \longrightarrow \Lambda L^{-1} \longrightarrow 0$$

we find that $H^1(E) = 0$, so $\dim H^0(E) = \chi$ and s is an isomorphism. Choose a basis $e_1, \ldots, e_p$ for $s^{-1}(H^0(L))$ and extend to a basis $e_1, \ldots, e_\chi$ for $\mathbb{C}^\chi$. Take the 1-parameter subgroup defined by $r_i = p - \chi$ for $i \leq p$, p for $i > p$. Then $r_\ell = p - \chi$ if $\phi \in H^0(L)$, p if $\phi \notin H^0(L)$. Since L is destabilizing, $p > \chi/2 - \sigma$ if $\phi \in H^0(L)$, $p > \chi/2 + \sigma$ if $\phi \notin H^0(L)$. Either way,

$$r_i + r_j \leq \frac{2\sigma}{\chi/2 + \sigma}\, r_l$$

implies $i, j \leq p$; if $\phi \in H^0(L)$, and say $i > p$, then

$$r_i + r_j - \frac{2\sigma}{\chi/2+\sigma}\, r_l \geq p + (p-\chi)(1 - \frac{2\sigma}{\chi/2+\sigma}) = p\,\frac{\chi}{\chi/2+\sigma} - \chi\,\frac{\chi/2-\sigma}{\chi/2+\sigma}$$
$$> (\chi/2 - \sigma)\frac{\chi}{\chi/2+\sigma} - \chi\,\frac{\chi/2-\sigma}{\chi/2+\sigma} = 0,$$

whereas if $\phi \notin H^0(L)$, and say $j > p$, then

$$r_i + r_j - \frac{2\sigma}{\chi/2+\sigma}\, r_l \geq p - \chi + p(1 - \frac{2\sigma}{\chi/2+\sigma}) = p\,\frac{\chi}{\chi/2+\sigma} - \chi > \chi - \chi = 0.$$

But if $i, j \leq p$, then $s(e_i), s(e_j) \in H^0(L)$, so $T_s(e_i, e_j) = 0$. Hence (T_s, v) is unstable.

Second case: $d - \deg L \leq 2g-2$. Then $\dim H^0(\Lambda L^{-1}) \leq g$, so from the long exact sequence of **(9.14)** we deduce that the codimension of $H^0(L)$ in $H^0(E)$ is $\leq g$. Hence the codimension of $s^{-1}(H^0(L))$ in $\mathbb{C}^\chi$ is $\leq g$. Choose a basis $e_1, \ldots, e_p$ for $s^{-1}(H^0(L))$ and extend to a basis $e_1, \ldots, e_\chi$ for $\mathbb{C}^\chi$. Take the 1-parameter subgroup defined by $r_i = p - \chi$ for $i \leq p$, p for $i > p$. Since $p \geq \chi - g$ and $\chi = d + 2 - 2g$ is large relative to σ and g, certainly $p > \chi/2 + \sigma$. The remainder of the proof proceeds as in the first case.

So far we have proved that if (T_s, v) is semistable, then s is injective and (E, ϕ) is σ-semistable. But then by **(9.10)**, $\dim H^0(E) = \chi$, so s is an isomorphism. □

(9.15) *Suppose (E_1, ϕ_1) and (E_2, ϕ_2) are σ-semistable, and there exist s_1, s_2 such that $(T(E_1, s_1), s_1^{-1}\phi_1) = (T(E_2, s_2), s_2^{-1}\phi_2)$. Then there is an isomorphism $(E_1, \phi_1) \cong (E_2, \phi_2)$ under which $s_1 \cong s_2$.*

Proof. By **(9.10)** each E_i is globally generated, so the components $s_i(e_j) \wedge s_i(e_k)$ of $T(E_i, s_i)$ give a map from X to the Grassmannian of $(\chi - 2)$-planes in $\mathbb{C}^\chi$ such that E_i is the pullback of the tautological rank 2 bundle, ϕ_i is the pullback of the section defined by $s_i^{-1}(\phi_i)$, and s_i is the natural map from $\mathbb{C}^\chi$ to the space of sections of the tautological bundle. So up to isomorphism we can recover (E_i, ϕ_i) and s_i from $(T(E_i, s_i), s_i^{-1}\phi_i)$. □

(9.16) *Let C be a smooth affine curve and $p \in C$. Let $(\mathbf{E}, \mathbf{\Phi})$ be a locally free family of pairs on $X \times C - \{p\}$, and suppose $\mathbf{E}$ is generated by finitely many sections s_i. Then after possibly rescaling $\mathbf{\Phi}$ by a function on $C - \{p\}$, $(\mathbf{E}, \mathbf{\Phi})$ and the s_i extend over p so that $\mathbf{E}$ is still locally free, $\mathbf{\Phi}_p \neq 0$, and the s_i generate $\mathbf{E}_p$ at the generic point.*

The reason for proving the last fact is to ensure that $T(E, s)$ is nonzero at p, so defines an element of $\mathbb{P}\,\mathrm{Hom}$.

Proof. Choose an ample line bundle L on $X \times C - \{p\}$ such that $\mathbf{E}^* \otimes L$ is globally generated. Then $\mathbf{E}$ embeds in a direct sum of copies of L, and $\oplus_j L$ can be extended over p as a sum of line bundles in such a way that the s_i extend too. Consider the subsheaf of the extended $\oplus_j L$ generated by the s_i. This is a subsheaf of a locally free sheaf, so it is torsion-free, and hence [36] has singular set S of codimension ≥ 2. Furthermore, it injects into its double dual, whose singular set has codimension ≥ 3 [36], hence is empty. Hence the double dual is a locally free extension of $\mathbf{E}$ over p, and is generated by s_i away from S. As for $\mathbf{\Phi}$, it certainly extends with a possible pole at p, so it is just necessary to multiply it by a function on C vanishing to some order at p. □

We can finally proceed to construct the geometric invariant theory quotient. Consider the Grothendieck Quot scheme [21] parametrizing flat quotients of $\mathcal{O}_X^\chi$ with degree d, let $\mathrm{Quot}(\Lambda) \subset \mathrm{Quot}$ be the locally closed subset consisting of locally free quotients E with $\Lambda^2 E = \Lambda$, and let $U \subset \mathrm{Quot}(\Lambda)$ be the open set where the quotient induces an isomorphism $s : \mathbb{C}^\chi \to H^0(E)$. Then the pair E, s specifies a point in U.

By **(9.10)**, if (E_p, ϕ) is σ-semistable for any section ϕ, then $p \in U$.

Now U is acted upon by $\mathrm{SL}(\chi)$ in the obvious way, and the map

$$T \times 1 : U \times \mathbb{P}\mathbb{C}^{\chi} \to \mathbb{P}\,\mathrm{Hom} \times \mathbb{P}\mathbb{C}^{\chi}$$

intertwines the group actions on the two sets. By **(9.11)** and **(9.13)**, the σ-semistable set $V(\sigma) \subset U \times \mathbb{P}\mathbb{C}^{\chi}$ is the inverse image of the semistable set $V'(\sigma) \subset \mathbb{P}\,\mathrm{Hom} \times \mathbb{P}\mathbb{C}^{\chi}$ with respect to the linearization $\mathcal{O}(\chi + 2\sigma, 4\sigma)$. In future, we restrict $T \times 1$ to a map $V(\sigma) \to V'(\sigma)$.

Now Gieseker proves the following.

(9.17) *Let G be a reductive group and M_1 and M_2 be two G-spaces. Suppose that $f : M_1 \to M_2$ is a finite G-morphism and that a good quotient $M_2//G$ exists. Then a good quotient $M_1//G$ exists, and the induced morphism $M_1//G \to M_2//G$ is finite.* □

So to show that $V(\sigma)$ has a good quotient it suffices to prove:

(9.18) *On $V(\sigma)$, $T \times 1$ is finite.*

Proof. By **(9.15)**, $T \times 1$ is injective. We use the valuative criterion to check that $T \times 1$ is proper. Let C be a smooth curve, $p \in C$, and let $\Psi : C - \{p\} \to V(\sigma)$ be a map such that $(T \times 1)\Psi$ extends to a map $C \to V'(\sigma)$. On $C - \{p\}$, we then have a family $(\mathbf{E}, \mathbf{\Phi})$ of pairs such that $\mathbf{E}$ is generated by the sections $s(e_1), \ldots, s(e_\chi)$. By **(9.16)**, on an open affine of C containing p, $(\mathbf{E}, \mathbf{\Phi})$ extends over p in such a way that $\mathbf{\Phi}_p \neq 0$ and the $s(e_i)$ generically generate $\mathbf{E}_p$. Thus $T(\mathbf{E}_p, s)$ is defined, and so by continuity $(T(\mathbf{E}_p, s), s^{-1}\mathbf{\Phi}_p) = ((T \times 1)\Psi)(p)$. Hence by **(9.13)** $s : \mathbb{C}^{\chi} \to H^0(\mathbf{E}_p)$ is an isomorphism and $(\mathbf{E}_p, \mathbf{\Phi}_p)$ is σ-semistable. So $(\mathbf{E}_p, s^{-1}\mathbf{\Phi}_p) \in V(\sigma)$ and Ψ extends to a map $C \to V(\sigma)$. □

Hence $V(\sigma)$ has a good projective quotient. By **(9.8)**, the closure of the orbit of (E, ϕ) contains the orbit of $\mathrm{Gr}\,(E, \phi)$, which is closed in the σ-semistable set. But the closure of any orbit in the χ-semistable set contains only one closed orbit [35, 3.14

(iii)]. Hence if two pairs are σ-semistable, then the closures of their orbits intersect if and only if they have the same Gr. This completes the proof of our main theorem **(9.2)**. □

One other pleasant fact should be mentioned: that the stable subsets of these moduli spaces are fine.

(9.19) *There exists a universal pair over the* σ*-stable set* $M_s(\sigma, \Lambda)$.

Proof. There is a universal bundle $\mathbf{E} \to \mathrm{Quot}(\Lambda) \times X$ and a surjective map $\mathcal{O}^\chi \to \mathbf{E}$. Hence there is a natural $\mathrm{SL}(\chi)$-invariant section $\mathbf{\Phi} \in H^0(\mathrm{Quot}(\Lambda) \times \mathbb{PC}^\chi \times X; \mathbf{E}(1))$, and $(\mathbf{E}(1), \mathbf{\Phi})$ is a universal pair. By **(9.6)** the only stabilizers of elements of the σ-stable subset of $V(\sigma)$ are the χth roots of unity. These act oppositely on $\mathbf{E}$ and on $\mathcal{O}(1)$, hence trivially on $\mathbf{E}(1)$, so on the σ-stable set $\mathbf{E}(1)$ is invariant under stabilizers. Hence by Kempf's descent lemma [14] $\mathbf{E}(1)$ descends to a bundle on $M_s(\sigma, \Lambda) \times X$, and the section $\mathbf{\Phi}$, being invariant, also descends. Thus we get a pair over $M_s(\sigma, \Lambda) \times X$ with the desired universal property. □

10 Their tangent spaces

We now turn to the deformation theory of our spaces. By semicontinuity σ-stability is an open condition, so the Zariski tangent spaces to our moduli spaces at the σ-stable points will just be deformation spaces. Hence we may refer to $T_{(E,\phi)}M(\sigma, \Lambda)$ simply as $T_{(E,\phi)}$.

(10.1) *If* $(E, \phi) \in M(\sigma, \Lambda)$ *is* σ*-stable, then*

(i) *(cf.* [8]*)* $T_{(E,\phi)}$ *is canonically isomorphic to* H^1 *of the complex*

$$C^0(\mathrm{End}_0\, E) \oplus \mathbb{C} \xrightarrow{p} C^1(\mathrm{End}_0\, E) \oplus C^0(E) \xrightarrow{q} C^1(E),$$

where $p(g, c) = (dg, (g + c)\phi)$ *and* $q(f, \psi) = f\phi - d\psi$*;*

(ii) H^0 *and* H^2 *of this complex vanish;*

(iii) *there is a natural exact sequence*

$$0 \longrightarrow H^0(\operatorname{End} E) \xrightarrow{\phi} H^0(E) \longrightarrow T_{(E,\phi)} \longrightarrow H^1(\operatorname{End}_0 E) \xrightarrow{\phi} H^1(E) \longrightarrow 0.$$

Proof. Let $R = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. By a well-known result [24, II Ex. 2.8] $T_{(E,\phi)}$ is the set of isomorphism classes of maps $\operatorname{Spec} R \to M(\sigma, \Lambda)$ such that $(\varepsilon) \mapsto (E, \phi)$. Since σ-stability is an open condition, $T_{(E,\phi)}$ is just the set of isomorphism classes of families $(\mathbf{E}, \mathbf{\Phi})$ of pairs on X with base $\operatorname{Spec} R$, such that $(\mathbf{E}, \mathbf{\Phi})_{(\varepsilon)} = (E, \phi)$ and $\Lambda^2 \mathbf{E}$ is the pullback of Λ. We will explain how to construct any such family.

The only open set in $\operatorname{Spec} R$ containing (ε) is $\operatorname{Spec} R$ itself, so any bundle $\mathbf{E}$ over $\operatorname{Spec} R \times X$ can be trivialized on $\operatorname{Spec} R \times U_\alpha$ for some open cover $\{U_\alpha\}$ of X. Thus if $\mathbf{E}_{(\varepsilon)} = E$, the transition functions give a Čech cochain of the form $1 + \varepsilon f_{\alpha\beta}$ where $f \in C^1(\operatorname{End} E)$. In order for $\Lambda^2 \mathbf{E}$ to be isomorphic to the pullback of Λ, the transition functions of $\Lambda^2 \mathbf{E}$ must be conjugate to $1 \in C^0(\mathcal{O})$. But the transition functions are $\det(1 + \varepsilon f_{\alpha\beta}) = 1 + \varepsilon \operatorname{tr} f_{\alpha\beta}$, so we are asking that

$$(1 + \varepsilon g_\alpha)(1 + \varepsilon \operatorname{tr} f_{\alpha\beta})(1 - \varepsilon g_\beta) = 1$$

for some $g \in C^0(\mathcal{O})$, that is, $\operatorname{tr} f = -dg$. But if such a g exists, then $\tilde{f} = f + dg/2$ is trace-free, and $1 + \varepsilon \tilde{f}$ is obviously conjugate to $1 + \varepsilon f$, so determines the same bundle $\mathbf{E}$. Hence up to isomorphism we can obtain any $\mathbf{E}$ even if we consider only trace-free $f \in C^1(\operatorname{End}_0 E)$.

Now if there is a section $\mathbf{\Phi} \in H^0(\mathbf{E})$ such that $\mathbf{\Phi}_{(\varepsilon)} = .\phi$, then with respect to the local trivializations of $\mathbf{E}$ described above, $\mathbf{\Phi} = \phi + \varepsilon \psi_\alpha$ for some Čech cochain $\psi \in C^0(E)$. Of course, ψ must be compatible with the transition functions; this means that

$$(1 + \varepsilon f_{\alpha\beta})(\phi + \varepsilon \psi_\beta) = (\phi + \varepsilon \psi_\alpha),$$

that is, $f\phi = d\psi$. Hence any pair $(\mathbf{E}, \mathbf{\Phi})$ having the desired properties can be obtained from some $(f, \psi) \in C^1(\operatorname{End}_0 E) \oplus C^0(E)$ satisfying $f\phi - d\psi = 0 \in C^1(E)$.

We now need only check which (f, ψ) give us isomorphic $(\mathbf{E}, \mathbf{\Phi})$. Of course the two choices will be related by a change of trivialization on $\operatorname{Spec} R \times U_\alpha$, but we may assume that the change of trivialization is of the form $1+\varepsilon g_\alpha$ on U_α, since (E,ϕ) itself has no automorphisms (9.6). Furthermore, g must belong to $C^0(\operatorname{End}_0 E)\oplus\mathbb{C}$ in order to keep f trace-free, since the action of g is given by

$$1+\varepsilon f_{\alpha\beta}\mapsto(1+\varepsilon g_\alpha)(1+\varepsilon f_{\alpha\beta})(1-\varepsilon g_\beta),$$

that is, $f\mapsto f+dg$, and dg is trace-free if and only if $g\in C^0(\operatorname{End} E)$ is the sum of a trace-free cocycle and a constant. Similarly the action of g on ψ is

$$\phi+\varepsilon\psi_\alpha\mapsto(1+\varepsilon g_\alpha)(\phi+\varepsilon\psi_\alpha),$$

that is, $\psi\mapsto\psi+g\phi$. Hence two pairs (f,ψ) and $(\tilde f,\tilde\psi)$ determine isomorphic pairs $(\mathbf{E},\mathbf{\Phi})$ if and only if they are in the same coset of the image of the map $C^0(\operatorname{End}_0 E)\oplus\mathbb{C}\to C^1(\operatorname{End}_0 E)\oplus C^0(E)$ given by $g+c\mapsto(dg,(g+c)\phi)$. This completes the proof of (i).

As for (ii) and (iii), substituting $H^0(\operatorname{End}_0 E)\oplus\mathbb{C}=H^0(\operatorname{End} E)$ into the long exact sequence of the double complex with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & C^0(\operatorname{End}_0 E)\oplus\mathbb{C} & \longrightarrow & C^0(\operatorname{End}_0 E)\oplus\mathbb{C} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(E) & \longrightarrow & C^1(\operatorname{End}_0 E)\oplus C^0(E) & \longrightarrow & C^1(\operatorname{End}_0 E) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1(E) & \longrightarrow & C^1(E) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

gives

$$0\longrightarrow H^0\longrightarrow H^0(\operatorname{End} E)\longrightarrow H^0(E)\longrightarrow H^1\longrightarrow H^1(\operatorname{End}_0 E)\longrightarrow H^1(E)\longrightarrow H^2\longrightarrow 0,$$

where H^i is the cohomology of the complex from (i). But the map $H^0(\operatorname{End} E)\xrightarrow{\phi}H^0(E)$ is injective for (E,ϕ) σ-stable by (9.6), and the map $H^1(\operatorname{End}_0 E)\xrightarrow{\phi}H^1(E)$ is always surjective: indeed this is equivalent to the Serre dual map $H^0(KE^*)\xrightarrow{\phi}H^0(K\operatorname{End}_0 E^*)$ being injective, which is obvious since the map $KE^*\xrightarrow{\phi}K\operatorname{End}_0 E^*$ is an injection of sheaves. Hence H^0 and H^2 vanish, and we get the exact sequence in (iii). □

As a corollary, we obtain the following.

(10.2) *If* $(E,\phi) \in M(\sigma,\Lambda)$ *is* σ*-stable, then* $\dim T_{(E,\phi)} = d+g-2$.

Proof. By **(10.1)**(iii)

$$\dim T_{(E,\phi)} = \chi(E) - \chi(\mathrm{End}_0 E) - 1 = (d+2-2g) - (3-3g) - 1 = d+g-2. \ \square$$

We will see in the next section that $\dim M(\sigma,\Lambda) = d+g-2$; hence $M(\sigma,\Lambda)$ will be smooth at the stable points.

11 How they vary with σ

For obvious numerical reasons the σ-semistability condition remains the same, and implies σ-stability, for any $\sigma \in (\max(0, d/2-i-1), d/2-i)$, where i is an integer between 0 and $(d-1)/2$. Hence for σ in that interval we get a fixed projective moduli space $M(\sigma,\Lambda)$, which we will henceforth denote just M_i. In the sequel, we will concentrate on these moduli spaces M_i, ignoring the special values of σ for which there exist σ-semistable pairs which are not σ-stable.

In the extreme case $i=0$, it is then easy to construct the moduli space.

(11.1) $$M_0 = \mathbb{P}H^1(\Lambda^{-1}).$$

Proof. The first inequality in the σ-stability condition **(9.1)** says that $\phi \in H^0(L)$ implies $\deg L \leq 0$. Hence $L = \mathcal{O}$, E is an extension of $\mathcal{O}$ by Λ, and $\phi \in H^0(\mathcal{O})$ is a constant section. The second inequality says that E has no subbundles of degree $\geq d$: this is equivalent to not being split, since $M \to E \to \Lambda$ nonzero and $\deg M \geq d = \deg \Lambda$ implies $M = \Lambda$. Hence M_0 is simply the moduli space of nonsplit extensions of $\mathcal{O}$ by Λ, which is of course just $\mathbb{P}H^1(\Lambda^{-1})$. The class $id \in \mathrm{End}\, H^1(X;\Lambda^{-1}) = H^0(\mathbb{P}H^1(\Lambda^{-1}); \mathcal{O}(1)) \otimes H^1(X;\Lambda^{-1}) = H^1((\mathbb{P}H^1(\Lambda^{-1}) \times X; \Lambda^{-1}(1))$ specifies an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathbf{E} \longrightarrow \Lambda(-1) \longrightarrow 0$$

which has the required universal property. The section $\mathbf{\Phi}$ is just the constant section of $\mathcal{O}$. $\square$

We will not attempt such a direct construction of M_i for $i > 0$. Rather, we will carefully study the relationship between M_{i-1} and M_i. Of course, this will only be of interest if there exists an M_i for $i > 0$, so *we will assume for the remainder of part II, and all of part III, that* $[(d-1)/2] > 0$, *that is,* $d \geq 3$. Anyhow, the first step is to construct families parametrizing those pairs which appear in M_i but not M_{i-1}, or M_{i-1} but not M_i. To do this, we first define two vector bundles over the ith symmetric product X_i.

Let $\pi : X_i \times X \to X_i$ be the projection and let $\Delta \subset X_i \times X$ be the universal divisor. Then define $W_i^- = (R^0\pi)\mathcal{O}_\Delta\Lambda(-\Delta)$ and $W_i^+ = (R^1\pi)\Lambda^{-1}(2\Delta)$. These are locally free sheaves of rank i and $d+g-1-2i$, respectively.

(11.2) *For $i \leq (d-1)/2$, there is a family over $\mathbb{P}W_i^+$ parametrizing exactly those pairs which are represented in M_i but not M_{i-1}.*

Proof. As we pass from i to $i-1$, the first inequality in the stability condition **(9.1)** gets stronger and the second gets weaker. So we look for pairs which almost violate the first inequality. That is, E must be an extension

$$0 \longrightarrow \mathcal{O}(D) \longrightarrow E \longrightarrow \Lambda(-D) \longrightarrow 0,$$

where $\deg D = i$, and ϕ is the section of $\mathcal{O}(D)$ vanishing on D. Conversely, any such pair is stable unless it splits $E = \mathcal{O}(D) \oplus \Lambda(-D)$. Indeed, if $L \subset E$ and $\phi \notin H^0(L)$, then the map $L \to \Lambda(-D)$ is nonzero, so $\deg L \leq \deg \Lambda(-D) = d - i$, with equality only if $L = \Lambda(-D)$.

But $\mathbb{P}W_i^+$ is the base of a family parametrizing all such nonsplit pairs: indeed $\mathbf{E}$ is the tautological extension

$$0 \longrightarrow \mathcal{O}(\Delta) \longrightarrow \mathbf{E} \longrightarrow \Lambda(-\Delta)(-1) \longrightarrow 0,$$

and $\mathbf{\Phi}$ is the section of $\mathcal{O}(\Delta)$ vanishing on Δ. □

(11.3) *For $i \leq (d-1)/2$, there is a family over $\mathbb{P}W_i^-$ parametrizing exactly those pairs which are represented in M_{i-i} but not M_i.*

Proof. This time the first inequality in (9.1) gets weaker and the second gets stronger. So we look for pairs which almost violate the second inequality. That is, E is an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow \Lambda M^{-1} \longrightarrow 0$$

where $\deg M = d - i$, and $\phi \notin H^0(M)$. Hence projecting ϕ in the exact sequence, we get a nonzero $\gamma \in H^0(\Lambda M^{-1})$ vanishing on a divisor D of degree i such that $\Lambda M^{-1} = \mathcal{O}(D)$. Then at D, ϕ lifts to $M = \Lambda(-D)$, so we get an element $p(E, \phi) \in H^0(\mathcal{O}_D \Lambda(-D))$, defined up to a scalar as usual.

On the other hand, we can recover (E, ϕ) from D and p. Indeed, choose a Čech cochain $\psi \in C^0(\Lambda(-D))$ such that $\psi|_D = p$. Then $d\psi|_D = dp = 0$, so $d\psi$ vanishes on D and descends to a closed cochain $f = d\psi/\gamma \in C^1((\Lambda(-2D))$. This determines an extension

$$0 \longrightarrow \Lambda(-D) \longrightarrow E' \longrightarrow \mathcal{O}(D) \longrightarrow 0.$$

The compatibility condition for $\gamma + \psi$ to define a section $\phi' \in H^0(E')$ is $\gamma f = d\psi$, which is automatic. Thus we get a new pair (E', ϕ') satisfying $p(E', \phi') = p$.

Up to isomorphism, (E', ϕ') is independent of the choice of ψ, since adding $\xi \in C^0(\Lambda(-2D))$ to ψ is simply equivalent to acting by $\begin{pmatrix} 1 & \xi_\alpha \\ 0 & 1 \end{pmatrix}$ on the local splittings of E' with which the extension is defined. In particular, we can choose local splittings of the old E and let ψ be the projection of the old ϕ on $M = \Lambda(-D)$ with respect to these splittings. Then the construction of the previous paragraph recovers (E, ϕ), so $(E', \phi') = (E, \phi)$.

The construction above can be generalized to produce a family $(\mathbf{E}, \mathbf{\Phi}) \to \mathbb{P}W_i^- \times X$, as follows. Let $p : \mathbb{P}W_i^- \to X_i$ be the projection, and choose a cochain $\Psi \in C^0(\Lambda(-\Delta)(1))$ such that $\Psi|_{p^{-1}\Delta}$ is the tautological section. Then $d\Psi$ vanishes on $p^{-1}\Delta$, so descends to $C^1(\Lambda(-2\Delta)(1))$. This determines an extension

$$0 \longrightarrow \Lambda(-\Delta)(1) \longrightarrow \mathbf{E} \longrightarrow \mathcal{O}(\Delta) \longrightarrow 0,$$

and if $\gamma \in H^0(\mathcal{O}(\Delta))$ is the section vanishing on Δ, then $\gamma + \Psi$ defines the desired section $\mathbf{\Phi} \in H^0(\mathbf{E})$. □

By the universal properties of M_{i-1} and M_i, we thus get injections $\mathbb{P}W_i^+ \hookrightarrow M_i$ and $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$. As an example, consider the case $i = 1$. By **(11.1)**, $M_0 = \mathbb{P}H^1(\Lambda^{-1})$. Moreover, W_1^- is a line bundle and hence $\mathbb{P}W_1^- = X_1 = X$. Hence the inclusion of **(11.3)** is a map $X \hookrightarrow \mathbb{P}H^1(\Lambda^{-1})$; it can be identified explicitly as follows.

(11.4) *The inclusion* $X \hookrightarrow \mathbb{P}H^1(\Lambda^{-1})$ *is given by the complete linear system* $|K_X\Lambda|$.

Proof. There is an alternative way to see what pairs are represented in M_0 but not M_1. Any pair $(E,\phi) \in M_0$ is an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \Lambda \longrightarrow 0, \tag{11.5}$$

say with extension class $s \in H^1(\Lambda^{-1})$, and with $\phi \in H^0(\mathcal{O})$. Such a pair is the image of $x \in X$ under the injection of **(11.3)** if there is an inclusion $0 \to \Lambda(-x) \to E$ such that the composition $\gamma_x : \Lambda(-x) \to E \to \Lambda$ vanishes at x. Hence we ask for what extension classes $s \in H^1(\Lambda^{-1})$ the map $\gamma_x : \Lambda(-x) \to \Lambda$ lifts to E.

If we twist **(11.5)** by $\Lambda^{-1}(x)$ and take the long exact sequence, we get in part

$$H^0(E \otimes \Lambda^{-1}(x)) \longrightarrow H^0(\mathcal{O}(x)) \xrightarrow{s} H^1(\Lambda^{-1}(x)),$$

where the second map is the cup product with s. Thus $\gamma_x \in H^0(\mathcal{O}(x))$ lifts to $H^0(E \otimes \Lambda^{-1}(x))$ as desired if and only if $\gamma_x s = 0$. So we wish to determine the kernel of the map $\gamma_x : H^1(\Lambda^{-1}) \to H^1(\Lambda^{-1}(x))$, or Serre dually, $\gamma_x : H^0(K_X\Lambda)^* \to H^0(K_X\Lambda(-x))^*$. Since γ_x is dual to the injection $H^0(K_X\Lambda) \to H^0(K_X\Lambda(-x))$, it is surjective, so

$$\dim\ker\gamma_x = \dim H^0(K_X\Lambda(-x)) - \dim H^0(K_X\Lambda).$$

But since $\deg K_X\Lambda(-x) > 2g-2$, this is 1. Hence for each $x \in X$, there is up to a scalar a unique $s \in \mathbb{P}H^1(\Lambda^{-1})$ such that $\gamma_x s = 0$.

What is this s? Regarded as a linear functional on $H^0(K_X\Lambda)$, $s \in \ker\gamma_x$ if it annihilates all sections vanishing at x. Certainly evaluation at x does this, so this is our s generating $\ker\gamma_x$. But it is also the image of x in the map $X \hookrightarrow \mathbb{P}H^1(\Lambda^{-1})$ given by $|K_X\Lambda|$. Hence the two maps are identical. □

(11.6) *The M_i are all smooth rational integral projective varieties of dimension $d + g - 2$, and for $i > 0$, there is a birational map $M_i \leftrightarrow M_1$, which is an isomorphism except on sets of codimension ≥ 2.*

Proof. By **(11.1)**, the first statement is certainly true of M_0. For $i > 0$, suppose by induction on i that it is true of M_{i-1}. By **(11.2)** and **(11.3)** there is an isomorphism $M_{i-1} - \mathbb{P}W_i^- \leftrightarrow M_i - \mathbb{P}W_i^+$. But $\dim \mathbb{P}W_i^- = 2i - 1 < d - 1 < d + g - 2$, and $\dim \mathbb{P}W_i^+ = d + g - 2 - i < d + g - 2$, so $\dim M_i = \dim M_{i-1} = d + g - 2$ and M_i is birational to M_{i-1}, hence to M_0. Moreover by **(10.2)**, the Zariski tangent space to M_i has constant dimension $d + g - 2$, so M_i is a smooth reduced variety. The second statement is also proved by induction: we need only note that for $i > 1$, $\operatorname{codim} \mathbb{P}W_i^- / M_{i-1} = d + g - 2i - 1 \geq 2$ and $\operatorname{codim} \mathbb{P}W_i^+ / M_i = i \geq 2$. □

(11.7) *Let $(E, \phi) \in \mathbb{P}W_i^+$, let D be the zero-set of ϕ, and let γ be the map*

$$E \otimes \Lambda^{-1}(D) \to \Lambda(-D) \otimes \Lambda^{-1}(D) = \mathcal{O}.$$

Then $T_{(E,\phi)}\mathbb{P}W_i^+$ is canonically isomorphic to H^1 of the complex

$$C^0(E \otimes \Lambda^{-1}(D)) \oplus \mathbb{C} \xrightarrow{p} C^1(E \otimes \Lambda^{-1}(D)) \oplus C^0(\mathcal{O}(D)) \xrightarrow{q} C^1(\mathcal{O}(D)),$$

where $p(g, c) = (dg, (\gamma g + c)\phi)$ and $q(f, \psi) = \gamma f \phi - d\psi$; moreover, H^0 and H^2 of this complex vanish.

Proof. The proof is modelled on that of **(10.1)**. We regard $\mathbb{P}W_i^+$ as a moduli space of triples (L, E, ϕ), where L is a line bundle of degree i, E is an extension of L by ΛL^{-1}, and $\phi \in H^0(L)$, and consider the deformation theory of this moduli problem.

Let $R = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ as before. Then $T_{(L,E,\phi)}\mathbb{P}W_i^+$ is the set of isomorphism classes of families $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})$ of triples on X with base $\operatorname{Spec} R$, such that $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})_{(\varepsilon)} = (L, E, \phi)$. We will explain how to construct any such family.

Any bundle over $\operatorname{Spec} R \times X$ can be trivialized on $\operatorname{Spec} R \times U_\alpha$ for some open cover $\{U_\alpha\}$ of X. Thus if $\mathbf{L}_{(\varepsilon)} = \mathcal{O}(D)$ and $\mathbf{E}_{(\varepsilon)} = E$, then the transition functions for $\mathbf{E}$ give a Čech cochain of the form $1 + \varepsilon f_{\alpha\beta}$ where $f \in C^1(\operatorname{End} E)$. Since $\mathbf{E}$ is to be a family of extensions of L by ΛL^{-1}, it must have $\Lambda^2 \mathbf{E} = \Lambda$, so as explained in the proof of **(10.1)** we may take $f \in C^1(\operatorname{End}_0 E)$. Furthermore, the transition functions must preserve $\mathbf{L}$, so if f' is the projection of f to $C^1(\Lambda(-2D))$ in the natural exact sequence

$$0 \longrightarrow E \otimes \Lambda^{-1}(D) \longrightarrow \operatorname{End}_0 E \longrightarrow \Lambda(-2D) \longrightarrow 0,$$

then $1 + \varepsilon f'_{\alpha\beta}$ must be conjugate to 1. Hence

$$(1 - \varepsilon g_\alpha)(1 + \varepsilon f'_{\alpha\beta})(1 - \varepsilon g_\beta) = 1$$

for some $g \in C^0(\Lambda(-2D))$, that is, $f' = dg$. But if such a g exists, then for any lifting $\tilde{g}$ of g to $C^0(\operatorname{End}_0 E)$, $\tilde{f} = f - d\tilde{g}$ projects to $0 \in C^1(\Lambda(-2D))$, and $1 + \varepsilon \tilde{f}$ is obviously conjugate to $1 + \varepsilon f$, so determines the same bundle $\mathbf{E}$. Hence up to isomorphism we can obtain any $\mathbf{E}$ that is an extension of some $\mathbf{L}$ by $\Lambda \mathbf{L}^{-1}$ even if we consider only those f in the kernel of $C^1(\operatorname{End}_0 E) \to C^1(\Lambda(-2D))$, that is, in $C^1(E \otimes \Lambda^{-1}(D))$. The transition functions for $\mathbf{L}$ are then just $1 + \varepsilon \gamma f_{\alpha\beta}$.

Now if there is a section $\mathbf{\Phi} \in H^0(\mathbf{L})$ such that $\mathbf{\Phi}_{(\varepsilon)} = \phi$, then with respect to the local trivializations of $\mathbf{E}$, $\mathbf{\Phi} = \phi + \varepsilon \psi_\alpha$ for some Čech cochain $\psi \in C^0(\mathcal{O}(D))$. Of course, ψ must be compatible with the transition functions; this means that

$$(1 + \varepsilon \gamma f_{\alpha\beta})(\phi + \varepsilon \psi_\beta) = (\phi + \varepsilon \psi_\alpha),$$

that is, $\gamma f \phi = d\psi$. Hence any triple $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})$ having the desired properties can be obtained from some $(f, \psi) \in C^1(E \otimes \Lambda^{-1}(D)) \oplus C^0(\mathcal{O}(D))$ satisfying $\gamma f \phi - d\psi = 0 \in C^1(\mathcal{O}(D))$.

We now need only check which (f, ψ) give us isomorphic $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})$. This part of the argument follows that of **(10.1)** exactly, except that g ends up being in $C^1(E \otimes \Lambda^{-1}(D)) \oplus \mathbb{C}$, and acts on ψ by $\psi \mapsto \psi + \gamma g \phi$. This completes the proof of the first statement.

As for the second, taking the long exact sequence of the double complex

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & C^0(E\otimes\Lambda^{-1}(D))\oplus\mathbb{C} & \longrightarrow & C^0(E\otimes\Lambda^{-1}(D))\oplus\mathbb{C} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & C^0(\mathcal{O}(D)) & \longrightarrow & \begin{array}{c} C^1(E\otimes\Lambda^{-1}(D)) \\ \oplus C^0(\mathcal{O}(D)) \end{array} & \longrightarrow & C^1(E\otimes\Lambda^{-1}(D)) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & C^1(\mathcal{O}(D)) & \longrightarrow & C^1(\mathcal{O}(D)) & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

gives

$$0 \longrightarrow H^0 \longrightarrow H^0(E\otimes\Lambda^{-1}(D))\oplus\mathbb{C} \longrightarrow H^0(\mathcal{O}(D)) \longrightarrow H^1$$
$$\longrightarrow H^1(E\otimes\Lambda^{-1}(D)) \longrightarrow H^1(\mathcal{O}(D)) \longrightarrow H^2 \longrightarrow 0,$$

where H^i is the cohomology of the complex in the statement. But

$$H^0(E\otimes\Lambda^{-1}(D)) = H^0(\mathrm{Hom}(\Lambda(-D),E)) = 0$$

since E is a *nonsplit* extension of $\mathcal{O}(D)$ by $\Lambda(-D)$, and the map $\mathbb{C}\to H^0(\mathcal{O}(D))$ is multiplication by ϕ. Hence the map $H^0(E\otimes\Lambda^{-1}(D))\oplus\mathbb{C}\to H^0(\mathcal{O}(D))$ is injective, and $H^0 = 0$. Moreover, the map $H^1(E\otimes\Lambda^{-1}(D))\to H^1(\mathcal{O}(D))$ is surjective: indeed this is equivalent to the Serre dual map $H^0(K(-D))\to H^0(E^*\otimes K\Lambda(-D))$ being injective, which is obvious since the map $K(-D)\to K\to E^*\otimes K\Lambda(-D)$ is an injection of sheaves. Hence $H^2 = 0$. □

The following proposition is proved similarly.

(11.8) *Let $(E,\phi)\in\mathbb{P}W_i^-$, and let $D = p(E,\phi)$. Then $T_{(E,\phi)}\mathbb{P}W_i^-$ is canonically isomorphic to H^1 of the complex*

$$C^0(E(-D))\oplus\mathbb{C} \longrightarrow C^1(E(-D))\oplus C^0(E) \longrightarrow C^1(E);$$

moreover, H^0 and H^2 of this complex vanish. □

(11.9) *The injection $\mathbb{P}W_i^+ \hookrightarrow M_i$ induces an exact sequence on $\mathbb{P}W_i^+$*

$$0 \longrightarrow T\mathbb{P}W_i^+ \longrightarrow TM_i|_{\mathbb{P}W_i^+} \longrightarrow W_i^-(-1) \longrightarrow 0.$$

Proof. The complex

$$C^0(\Lambda(-2\Delta))\longrightarrow C^1(\Lambda(-2\Delta))\oplus C^0(\Lambda(-\Delta))\longrightarrow C^1(\Lambda(-\Delta))$$

with the obvious maps has $R^0\pi = 0$, $R^1\pi = W_i^-$ from the long exact sequence of the double complex

$$\begin{array}{ccccccccc}
0 \longrightarrow & C^0(\Lambda(-2\Delta)) & \longrightarrow & C^0(\Lambda(-2\Delta)) & \longrightarrow & 0 & \longrightarrow 0 \\
& \big\downarrow {\scriptstyle (1,0)} & & \big\downarrow & & \big\downarrow & \\
0 \longrightarrow & \begin{array}{c} C^0(\Lambda(-2\Delta)) \\ \oplus C^1(\Lambda(-2\Delta)) \end{array} & \longrightarrow & \begin{array}{c} C^0(\Lambda(-\Delta)) \\ \oplus C^1(\Lambda(-2\Delta)) \end{array} & \longrightarrow & C^0(\mathcal{O}_\Delta\Lambda(-\Delta)) & \longrightarrow 0 \\
& \big\downarrow {\scriptstyle (0,1)} & & \big\downarrow & & \big\downarrow & \\
0 \longrightarrow & C^1(\Lambda(-2\Delta)) & \longrightarrow & C^1(\Lambda(-\Delta)) & \longrightarrow & C^1(\mathcal{O}_\Delta\Lambda(-\Delta)) & \longrightarrow 0.
\end{array}$$

Hence the result follows from the long exact sequence of the double complex

$$\begin{array}{ccccccccc}
0 \longrightarrow & C^0(\mathbf{E}\Lambda^{-1}(\Delta))\oplus\mathbb{C} & \longrightarrow & C^0(\mathrm{End}_0\,\mathbf{E})\oplus\mathbb{C} & \longrightarrow & C^0(\Lambda(-2\Delta))(-1) & \longrightarrow 0 \\
& \big\downarrow {\scriptstyle (1,0)} & & \big\downarrow {\scriptstyle p} & & \big\downarrow & \\
0 \longrightarrow & \begin{array}{c} C^1(\mathbf{E}\Lambda^{-1}(\Delta)) \\ \oplus C^0(\mathcal{O}(\Delta)) \end{array} & \longrightarrow & \begin{array}{c} C^1(\mathrm{End}_0\,\mathbf{E}) \\ \oplus C^0(\mathbf{E}) \end{array} & \longrightarrow & \begin{array}{c} C^1(\Lambda(-2\Delta))(-1) \\ \oplus C^0(\Lambda(-\Delta))(-1) \end{array} & \longrightarrow 0 \\
& \big\downarrow {\scriptstyle (0,1)} & & \big\downarrow {\scriptstyle q} & & \big\downarrow & \\
0 \longrightarrow & C^1(\mathcal{O}(\Delta)) & \longrightarrow & C^1(\mathbf{E}) & \longrightarrow & C^1(\Lambda(-\Delta))(-1) & \longrightarrow 0,
\end{array}$$

together with **(10.1)** and **(11.7)**. □

(11.10) *The map* $\mathbb{P}W_i^+ \hookrightarrow M_i$ *is an embedding.*

Proof. By **(11.7)**, it is an injection, and by **(11.9)**, so is its derivative. □

The following proposition and corollary are proved similarly, using **(10.1)** and **(11.8)**.

(11.11) *The injection* $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$ *induces an exact sequence on* $\mathbb{P}W_i^-$

$$0\longrightarrow T\mathbb{P}W_i^- \longrightarrow TM_{i-1}|_{\mathbb{P}W_i^-} \longrightarrow W_i^+(-1)\longrightarrow 0. \ \square$$

(11.12) *The map* $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$ *is an embedding.* □

By **(11.2)** and **(11.3)** every pair in $M_i - \mathbb{P}W_i^+$ is also in $M_{i-1} - \mathbb{P}W_i^-$, and vice-versa. Hence there is a natural isomorphism $M_i - \mathbb{P}W_i^+ \to M_{i-1} - \mathbb{P}W_i^-$. Our next task is to extend this to a proper map. Let $\tilde{M}_i^+$ be the blow-up of M_i at $\mathbb{P}W_i^+$. Then by **(11.9)** the exceptional divisor is $E_i^+ = \mathbb{P}W_i^- \oplus \mathbb{P}W_i^+$, and $\mathcal{O}_{E_i^+}(E_i^+) = \mathcal{O}(-1,-1)$.

(11.13) *There is a map* $\tilde{M}_i^+ \to M_{i-1}$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} M_i - \mathbb{P}W_i^+ & \longrightarrow & \tilde{M}_i^+ & \longleftarrow & E_i^+ \\ \updownarrow & & \downarrow & & \downarrow \\ M_{i-1} - \mathbb{P}W_i^- & \longrightarrow & M_{i-1} & \longleftarrow & \mathbb{P}W_i^-. \end{array}$$

Proof. Let $(\mathbf{E}, \mathbf{\Phi}) \to \tilde{M}_i^+ \times X$ be the pullback of the universal family. We will construct a new family $(\mathbf{E}', \mathbf{\Phi}')$ of pairs all of which are in M_{i-1}.

By uniqueness of families **(9.7)**, $(\mathbf{E}, \mathbf{\Phi})|_{E_i^+ \times X}$ is the pullback of the family over $\mathbb{P}W_i^+$ constructed in **(11.2)**. Thus there is a surjective sheaf map $\mathbf{E} \to \mathcal{O}_{E_i^+ \times X}\Lambda(-\Delta)(0,-1)$ annihilating $\mathbf{\Phi}$. Define $\mathbf{E}'$ to be the kernel of this map, so that

$$0 \longrightarrow \mathbf{E}' \longrightarrow \mathbf{E} \longrightarrow \mathcal{O}_{E_i^+ \times X}\Lambda(-\Delta)(0,-1) \longrightarrow 0. \tag{11.14}$$

Then $\mathbf{E}'$ is locally free, and $\mathbf{\Phi}$ descends to $\mathbf{\Phi}' \in H^0(\mathbf{E}')$. For $z \in M_i - \mathbb{P}W_i^+$, clearly $(\mathbf{E}', \mathbf{\Phi}')_z = (\mathbf{E}, \mathbf{\Phi})_z$. So to prove the proposition it suffices to show that $(\mathbf{E}', \mathbf{\Phi}')_{E_i^+}$ is the pullback of the family over $\mathbb{P}W_i^-$ constructed in **(11.3)**. The first promising thing to note is that there certainly is a surjection $\mathbf{E}' \to \mathcal{O}_{E_i^+ \times X}(\Delta) \to 0$, and $\Lambda^2 \mathbf{E}' = \Lambda^2 \mathbf{E}(-E_i^+ \times X)$, so we get an extension

$$0 \longrightarrow \Lambda(-\Delta)(1,0) \longrightarrow \mathbf{E}'_{E_i^+ \times X} \longrightarrow \mathcal{O}(\Delta) \longrightarrow 0,$$

just as in the family of **(11.3)**.

Now fix $s \in E_i^+$ over $(E, \phi) \in M_i$, and let D be the zero-set of ϕ. Let $R = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ as before, and choose a map Spec $R \to \tilde{M}_i^+$ representing an element of $T_s\tilde{M}_i^+ -$

$T_s E_i^+$. Then **(11.14)** restricts to an exact sequence

$$0 \longrightarrow \mathcal{O}_{\operatorname{Spec} R\times X}(\mathbf{E}') \longrightarrow \mathcal{O}_{\operatorname{Spec} R\times X}(\mathbf{E}) \longrightarrow \mathcal{O}_{(\varepsilon)\times X}\Lambda(-D) \longrightarrow 0.$$

On some open cover $\{U_\alpha\}$ of X, E splits as

$$E|_{U_\alpha} = \mathcal{O}(D)|_{U_\alpha} \oplus \Lambda(-D)|_{U_\alpha}, \tag{11.15}$$

and we can extend this splitting to a splitting of $\mathbf{E}|_{\operatorname{Spec} R\times U_\alpha}$. Then

$$\mathbf{E}'|_{U_\alpha} = \mathcal{O}(D)|_{U_\alpha} \oplus \Lambda(D)|_{U_\alpha} \otimes \mathcal{I}_{(\varepsilon)}. \tag{11.16}$$

The section $\mathbf{\Phi}$ is then of the form $\phi + \varepsilon\psi_\alpha$ for some $\psi \in C^0(E)$, and the transition functions are $1+\varepsilon f_{\alpha\beta}$ for some $f \in C^1(\operatorname{End}_0 E)$. The latter hence act as 1 on the second factor of **(11.16)**.

Now decompose $\psi_\beta = \psi_\beta^{\mathcal{O}(D)} + \psi_\beta^{\Lambda(-D)}$ and $f_{\alpha\beta} = f_{\alpha\beta}^{\mathcal{O}(D)} + f_{\alpha\beta}^{\Lambda(-D)}$ corresponding to the splitting on U_β. If we restrict $\mathbf{E}'$ to $(\varepsilon)\times U_\alpha$, then $\varepsilon\psi_\beta^{\mathcal{O}(D)} = 0$ and $\varepsilon f_{\alpha\beta}^{\mathcal{O}(D)} = 0$, since everything divisible by ε is now set to zero. However, $\varepsilon\psi_\beta^{\Lambda(-D)}$ and $\varepsilon f_{\alpha\beta}^{\Lambda(-D)}$ are not necessarily zero, since not everything in their images is divisible by ε *in the module* $\Lambda(-D)\otimes\mathcal{I}_{(\varepsilon)}$. Hence $\mathbf{\Phi}_{(\varepsilon)} = \phi + \varepsilon\psi_\beta^{\Lambda(-D)}$ on U_β, and $\mathbf{E}_{(\varepsilon)}$ has transition functions $\begin{pmatrix} 1 & \varepsilon f_{\alpha\beta}^{\Lambda(-D)} \\ 0 & 1 \end{pmatrix}$ with respect to the splitting **(11.16)**. In other words, the extension class of $E' = \mathbf{E}_{(\varepsilon)}$ is the projection of $f \in C^1(\operatorname{End}_0 E)$ to $C^1(\Lambda(-2D))$, and the lifting of ϕ' is the projection of $\psi \in C^1(E)$ to $C^1(\Lambda(-D))$. Hence (E',ϕ') is the bundle over the image of (E,ϕ) in $\mathbb{P}_i^-$ in the family of **(11.3)**. By uniqueness of families **(9.7)** this means that $(\mathbf{E}',\mathbf{\Phi}')|_{E_i^+\times X}$ is the pullback of the family of **(11.3)**. □

There is a result similar to **(11.13)** for the inverse map $M_{i-1} - \mathbb{P}W_i^- \to M_i - \mathbb{P}W_i^+$. Let $\tilde{M}_{i-1}^-$ be the blow-up of M_{i-1} at $\mathbb{P}W_i^-$. Hence by **(11.11)** the exceptional divisor is $E_i^- = \mathbb{P}W_i^- \oplus \mathbb{P}W_i^+$, and $\mathcal{O}_{E_i^-}(E_i^-) = \mathcal{O}(-1,-1)$. Note that there is an isomorphism $E_i^- \leftrightarrow E_i^+$.

(11.17) *There is a map $\tilde{M}_{i-1}^- \to M_i$ such that the following diagram commutes:*

$$\begin{array}{ccccc} M_{i-1} - \mathbb{P}W_i^- & \longrightarrow & \tilde{M}_{i-1}^- & \longleftarrow & E_i^- \\ \updownarrow & & \downarrow & & \downarrow \\ M_i - \mathbb{P}W_i^+ & \longrightarrow & M_i & \longleftarrow & \mathbb{P}W_i^+. \end{array}$$

Proof. Let $(\mathbf{E}, \mathbf{\Phi}) \to \tilde{M}_{i-1}^- \times X$ be the pullback of the universal family. We will construct a new family $(\mathbf{E}', \mathbf{\Phi}')$ of pairs all of which are in M_i.

By uniqueness of families **(9.7)**, $(\mathbf{E}, \mathbf{\Phi})|_{E_i^- \times X}$ is the pullback of the family over $\mathbb{P}W_i^-$ constructed in **(11.3)**. Thus there is a surjective sheaf map $\mathbf{E} \to \mathcal{O}_{E_i^- \times X}\mathcal{O}(-\Delta)$. This time, however, the map does not necessarily annihilate $\mathbf{\Phi}$. However, if we tensor by $\mathcal{O}(E_i^-)$, then the twisted map $\mathbf{E}(E_i^-) \to \mathcal{O}_{E_i^- \times X}\mathcal{O}(\Delta)(-1,-1)$ of course annihilates $\mathbf{\Phi}(E_i^-)$. If we define $\mathbf{E}'$ to be the kernel of this twisted map, so that

$$0 \longrightarrow \mathbf{E}' \longrightarrow \mathbf{E}(E_i^-) \longrightarrow \mathcal{O}_{E_i^+ \times X}(\Delta)(-1,-1) \longrightarrow 0,$$

then $\mathbf{E}'$ is locally free, and $\mathbf{\Phi}(E_i^-)$ descends to $\mathbf{\Phi}' \in H^0(\mathbf{E}')$. The remainder of the proof is analogous to that of **(11.13)**. □

At last we come to the goal of all the above work.

(11.18) *There is a natural isomorphism $\tilde{M}_i^+ \leftrightarrow \tilde{M}_{i-1}^-$ such that the following diagram commutes:*

$$\begin{array}{ccccc} M_i - \mathbb{P}W_i^+ & \longrightarrow & \tilde{M}_i^+ & \longleftarrow & E_i^+ \\ \updownarrow & & \updownarrow & & \updownarrow \\ M_{i-1} - \mathbb{P}W_i^- & \longrightarrow & \tilde{M}_{i-1}^- & \longleftarrow & E_i^-. \end{array}$$

Proof. Both $\tilde{M}_i^+$ and $\tilde{M}_{i-1}^-$ are smooth, and by **(11.13)** and **(11.17)** they both inject into $M_{i-1} \times M_i$. Indeed, both injections are embeddings, since as is easily checked they annihilate no tangent vectors, and both have the same image. This image is precisely the closure of the graph of the isomorphism $M_i - \mathbb{P}W_i^+ \leftrightarrow M_{i-1} - \mathbb{P}W_i^-$, which proves the left-hand square; for both E_i^- and E_i^+ it is the map $\mathbb{P}W_i^- \oplus \mathbb{P}W_i^+ \to \mathbb{P}W_i^- \times \mathbb{P}W_i^+$, which proves the right-hand square. □

Note. In light of this result, we will henceforth refer to $\tilde{M}_i^+ = \tilde{M}_{i-1}^-$ simply as $\tilde{M}_i$, and $E_i^+ = E_i^-$ as E_i.

Thus M_i is obtained from M_{i-1} by blowing up $\mathbb{P}W_i^-$, and then blowing down the same exceptional divisor in another direction. Such a blow-up and blow-down is an example of what is called a *flip* in Mori theory. This thesis will not use any of the deep results of Mori theory, but we will see some of its basic principles in action.

In one case the flip degenerates to an ordinary blow-up.

(11.19) *The moduli space M_1 is the blow-up of $M_0 = \mathbb{P}H^1(\Lambda^{-1})$ along X embedded via $|K_X\Lambda|$.*

Proof. Since W_1^- is a line bundle, there is nothing to blow down. □

The other extreme case is also of interest. Let $w = [(d-1)/2]$, so that M_w is the last moduli space in our sequence. Let N be the moduli space of ordinary rank 2 semistable bundles of determinant Λ.

(11.20) *There is a natural "Abel-Jacobi" map $M_w \dashrightarrow N$ with fibre $\mathbb{P}H^0(E)$ over a stable bundle E. It is surjective if $d > 2g-2$.*

Proof. If $i = w$, then $\sigma \in (0, [d/2]+1-d/2)$, so σ-stability of (E,ϕ) implies ordinary semistability of E. Thus there is a map $M_w \dashrightarrow N$. Moreover, ordinary stability of E implies σ-stability of (E,ϕ), so the fibre over a stable E is just $\mathbb{P}H^0(E)$. For $d > 2g-2$, any bundle E has a nonzero section ϕ by Riemann-Roch. Hence every stable bundle in N is certainly in the image of M_w. But M_w is complete, so its image is a complete variety containing the stable set, which must be N itself. □

We may sum up our findings in the following diagram.

$$
\begin{array}{ccccccccc}
 & \tilde{M}_2 & & \tilde{M}_3 & & \tilde{M}_4 & & \tilde{M}_w & \\
\swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow \\
M_1 & & M_2 & & M_3 & & \cdots & & M_w \\
\downarrow & & & & & & & & \downarrow \\
M_0 & & & & & & & & N
\end{array}
$$

All the arrows are birational morphisms except sometimes the one to N.

12 Their Poincaré polynomials

Before going on to our main application in part III, let us pause to see how the flips described above can be used to compute the Poincaré polynomials of our moduli spaces.

$$\textbf{(12.1)} \quad P_t(M_i) = \frac{1}{1-t^2} \operatorname*{Coeff}_{x^i} \left(\frac{t^{2d+2g-2-4i}}{xt^4-1} - \frac{t^{2i+2}}{x-t^2} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right).$$

Proof. Since $\tilde{M}_j$ is the blow-up of M_{j-1} at $\mathbb{P}W_j^-$, by the formula for the Poincaré polynomial of a blow-up [20, p. 605],

$$P_t(\tilde{M}_j) = P_t(M_{j-1}) + P_t(E_j) - P_t(\mathbb{P}W_j^-).$$

But $\tilde{M}_j$ is also the blow-up of M_j at $\mathbb{P}W_j^+$, so

$$P_t(\tilde{M}_j) = P_t(M_j) + P_t(E_j) - P_t(\mathbb{P}W_j^+)$$

as well. Hence

$$P_t(M_j) - P_t(M_{j-1}) = P_t(\mathbb{P}W_j^+) - P_t(\mathbb{P}W_j^-).$$

But the Poincaré polynomial of any projective bundle splits, so

$$\begin{aligned} P_t(\mathbb{P}W_j^+) - P_t(\mathbb{P}W_j^-) &= P_t(\mathbb{P}^{d+g-2-2j})P_t(X_j) - P_t(\mathbb{P}^{i-1})P_t(X_j) \\ &= \frac{t^{2j} - t^{2d+2g-2-4j}}{1-t^2} P_t(X_j). \end{aligned}$$

A formula for $P_t(X_j)$ was given by Macdonald [29]:

$$P_t(X_j) = \operatorname*{Coeff}_{x^j} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}. \quad .$$

Hence

$$P_t(M_j) - P_t(M_{j-1}) = \operatorname*{Coeff}_{x^j} \frac{(t^{2j} - t^{2d+2g-2-4j})(1+xt)^{2g}}{(1-t^2)(1-x)(1-xt^2)}.$$

Notice that this formula also produces $P_t(M_0)$ when $j=0$. So to sum up,

$$\begin{aligned} P_t(M_i) &= \frac{1}{1-t^2} \operatorname*{Coeff}_{x^i} \sum_{j=0}^{i} \frac{x^{i-j}(t^{2j} - t^{2d+2g-2-4j})(1+xt)^{2g}}{(1-x)(1-xt^2)} \\ &= \frac{1}{1-t^2} \operatorname*{Coeff}_{x^i} \left(\frac{x^{i+1} - t^{2i+2}}{x-t^2} + \frac{t^{2d+2g-2-4i}(1 - t^{4i-4}x^{i+1})}{xt^4-1} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right), \end{aligned}$$

which agrees with the formula stated after the terms containing x^{i+1} are removed. □

We can use this formula to recover the formula of Harder-Narasimhan [22] for the Poincaré polynomial of the moduli space N of stable bundles of rank 2, determinant Λ, and odd degree d:

$$P_t(N) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}. \tag{12.2}$$

Proof. When $d > 2g-2$ is odd and $i = w$, then by **(11.20)** there is a surjective map $M_w \to N$ with fibre $\mathbb{P}H^0(E)$ over a bundle E. If moreover $d > 4g-4$, then $H^1(E) = 0$ for all stable E (see for example the proof of **(9.10)**), so M_w is then just the $\mathbb{P}^{d-2g+1}$-bundle $\mathbb{P}(R^0\pi)\mathbf{E}$, where $\mathbf{E}$ is a universal bundle over N, and

$$P_t(N) = \frac{1-t^2}{1-t^{2d-4g+4}} P_t(M_w).$$

For simplicity we may as well assume that $d = 4g-3$. Then $w = 2g-2$ and

$$P_t(N) = \frac{1}{1-t^{4g-2}} \operatorname*{Coeff}_{x^{2g-2}} \left(\frac{t^{2g}}{xt^4 - 1} - \frac{t^{4g-2}}{x - t^2} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right).$$

The following argument, due to Don Zagier, then shows that this equals the Harder-Narasimhan formula. Let

$$F(a,b,c,t) = \operatorname*{Coeff}_{x^{2g-2}} \frac{(1+xt)^{2g}}{(1-ax)(1-bx)(1-cx)}.$$

Then

$$P_t(N) = \frac{t^{4g-4}F(1,t^2,t^{-2},t) - t^{2g}F(1,t^2,t^4,t)}{1-t^{4g-2}}.$$

On the other hand,

$$F(a,b,c,t) = \operatorname*{Res}_{x=0} \left\{ \frac{x^{1-2g}(1+xt)^{2g}dx}{(1-ax)(1-bx)(1-cx)} \right\};$$

since this has no pole at infinity, by the residue theorem

$$\begin{aligned} F(a,b,c,t) &= \left(-\operatorname*{Res}_{x=1/a} - \operatorname*{Res}_{x=1/b} - \operatorname*{Res}_{x=1/c}\right) \left\{ \frac{x^{1-2g}(1+xt)^{2g}dx}{(1-ax)(1-bx)(1-cx)} \right\} \\ &= \frac{(a+t)^{2g}}{(a-b)(a-c)} + \frac{(b+t)^{2g}}{(b-a)(b-c)} + \frac{(c+t)^{2g}}{(c-a)(c-b)}. \end{aligned}$$

After this substitution, it is a matter of high-school algebra to verify **(12.2)**. □

13 Relation with Bertram's work

In this section we explain briefly, without proving anything, how part II is related to Bertram's work on secant varieties.

In [4], Bertram considers how to resolve the rational map $\mathbb{P}H^1(\Lambda^{-1}) \dashrightarrow N$. He shows that blowing up first $X \subset \mathbb{P}H^1(\Lambda^{-1})$, then the proper transform of each of its secant varieties in turn, produces a smooth variety $\tilde{\mathbb{P}}$ having a morphism to N that agrees with the rational map away from the blow-ups. The existence of the morphism is proved by constructing a sequence of families of bundles, each obtained by an elementary transformation of the last, starting with the pullback of the tautological family on $\mathbb{P}H^1(\Lambda^{-1}) \times X$, and ending with a family of bundles that are all semistable. Bertram's families of bundles can be interpreted, after some twisting, as families of pairs in our sense, and it follows that his $\tilde{\mathbb{P}}$ dominates all of the M_σ. In other words, he performs all of our blow-ups but none of our blow-downs. In particular, our blow-up loci are birational to his, that is, our $\mathbb{P}W_i^-$ in M_{i-1} is the proper transform of the ith secant variety in $\mathbb{P}H^1(\Lambda^{-1}) = M_0$. This makes sense, since both are essentially $\mathbb{P}^{i-1}$-bundles over X_i.

However, this correspondence is a little more delicate than it seems, because the $\mathbb{P}^{i-1}$-bundles are different: ours is $\mathbb{P}W_i^- = \mathbb{P}(R^0\pi)\mathcal{O}_\Delta\Lambda(-\Delta)$, but as Bertram explains, the secant variety is the image in $\mathbb{P}H^1(\Lambda^{-1})$ of $\mathbb{P}(R^0\pi)\mathcal{O}_\Delta K\Lambda$. How is one projective bundle transformed into another? If we pull back the lower secant varieties to $\mathbb{P}(R^0\pi)\mathcal{O}_\Delta K\Lambda$ it is not hard to see that blowing them up and down induces a *Cremona transformation* on each fibre of the projective bundle. For example, consider the $\mathbb{P}^2$ fibre over $x_1 + x_2 + x_3 \in X_3$ of the 3rd secant variety. This of course meets $X \subset \mathbb{P}H^1(\Lambda^{-1})$ in the 3 points x_1, x_2, x_3, so if X is blown up, then $\mathbb{P}^2$ gets blown up at those 3 points. The proper transform of the 2nd secant variety meets this blown-up $\mathbb{P}^2$ in the proper transforms of the 3 lines between the points, so blowing it up does nothing, and blowing it down blows down the 3 lines. All in all we have blown up the vertices of a triangle in the plane, then blown down the proper transforms of the edges. This is well-known to recover $\mathbb{P}^2$ [24, V 4.2.3]; indeed it is given in coordinates by $[z_0, z_1, z_2] \mapsto [z_1z_2, z_0z_2, z_0z_1]$.

If we do the same thing to $\mathbb{P}^3$, we find ourselves blowing up the vertices of a tetrahedron, then blowing up and down—that is to say, flipping—the proper transforms of the edges, and finally blowing down the proper transforms of the faces. Notice that by the time we get to the faces, they have already undergone Cremona transformations themselves. More generally, starting with a simplex in $\mathbb{P}^n$, we may flip all of the subsimplices, starting with the vertices and working our way up. The varieties we obtain thus fit into a diagram shaped exactly like that at the end of §11. It is not so well-known that this recovers $\mathbb{P}^n$, or that it is given in coordinates by $[z_i] \mapsto [z_0 \cdots z_{i-1} z_{i+1} \cdots z_n]$, but these facts can be proved using the theory of toric varieties.

Even that is not quite the end of the story, since over divisors in X_i with multiple points the transformations are somewhat different. Over $2x_1+x_2 \in X_3$, for example, we want to blow up one reduced point and one doubled point, then blow down one reduced line and one doubled line. In coordinates, this is $[z_0, z_1, z_2] \mapsto [z_0^2, z_0 z_1, z_1 z_2]$. It is an amusing exercise to work out coordinate expressions for the Cremona tranformations over other divisors with multiple points.

Part III

Stable pairs and linear systems

14 The ample cones

We now turn to a study of the line bundles over the M_i. Indeed, our goal is a formula for the dimension of the space of sections of any line bundle over any M_i. Since M_0 is just a projective space, the first interesting case is M_1; so we first of all ask what line bundles there are on M_1.

(14.1) $\operatorname{Pic} M_1 = \mathbb{Z} \oplus \mathbb{Z}$, *generated by the hyperplane* H *and the exceptional divisor* E_1.

Proof. Obvious from **(11.19)**. □

The case of M_1 will be crucial for us, so we introduce the notation

$$\mathcal{O}_1(m,n) = \mathcal{O}((m+n)H - nE_1),$$

$$V_{m,n} = H^0(M_1; \mathcal{O}_1(m,n)).$$

Pushing down to $M_0 = \mathbb{P}H^1(\Lambda^{-1})$ then yields $V_{m,n} = H^0(M_0; \mathcal{O}(m+n) \otimes \mathcal{I}_X^n)$. That is, an element of $\mathbb{P}V_{m,n}$ is a hypersurface of degree $m+n$ with a singularity of order $n-1$ at X. The dimension of $V_{m,n}$, which we shall attempt to calculate, is thus a number canonically associated to X, Λ, m, and n.

Of course, in many cases this number is easy to compute. If $m < 0$, for example, then $V_{m,n} = 0$, since no hypersurface can have a singularity of order greater than its degree. If $n < 0$, then $V_{m,n} = H^0(M_0; \mathcal{O}(m+n) \otimes \mathcal{I}_X^n) = H^0(M_0; \mathcal{O}(m+n))$, because $\operatorname{codim} X/M_0 = d+g-3 > 1$ by our assumptions on d and g, and a section cannot have a pole on a set of codimension > 1. So in this case $\dim V_{m,n} = \binom{m+n+d+g-2}{m+n}$. However, for $m, n \geq 0$, it is quite an interesting problem to calculate $\dim V_{m,n}$. When $n = 1$, these are of course precisely the spaces whose syzygies are studied by Green and Lazarsfeld [28], but for $n > 1$ very little appears to be known.

What about M_i for $i > 1$? These give exactly the same information as M_1, for the following simple reason.

(14.2) *For $i > 0$ there is a natural isomorphism* $\operatorname{Pic} M_1 = \operatorname{Pic} M_i$. *Moreover, if by abuse of notation we denote by $\mathcal{O}_i(m,n)$ the image of $\mathcal{O}_1(m,n)$ in* $\operatorname{Pic} M_i$, *then for any m, n there is a natural isomorphism $V_{m,n} = H^0(M_i; \mathcal{O}_i(m,n))$.*

Proof. By **(11.6)**, M_1 is isomorphic to M_i except on sets of codimension ≥ 2. Hence functions, divisors, line bundles, and sections can be pulled back from one to the other and extended over the bad sets in a unique way. □

However, we will certainly not ignore the higher M_i from now on. Instead, they will be indispensable tools in the study of the cohomology of M_1, to be used as follows. A naive approach to calculating $\dim V_{m,n}$ would be to calculate $\chi(M_1; \mathcal{O}_1(m,n))$, which is easy using Riemann-Roch, and then to apply Kodaira vanishing to show that the higher cohomology all vanished. This will not work: the hypothesis of Kodaira vanishing, which is that $K_{M_1}^{-1}\mathcal{O}_1(m,n)$ must be ample, will not typically be satisfied, and the higher cohomology will not vanish. But this problem can be cured by shifting attention to some other M_i. Indeed, under some mild hypotheses on m and n, there will be some i such that $K_{M_i}^{-1}\mathcal{O}_i(m,n)$ will be ample on M_i. Hence $\dim V_{m,n} = \chi(M_i; \mathcal{O}_i(m,n))$, which will be calculated by an inductive procedure on i.

To carry out this programme, of course, we need to know the ample cone of each M_i. So our goal in this section will be to prove the following theorem, which is illustrated in figure 6.

(14.3) *For $0 < i < w$, the ample cone of M_i is bounded by $\mathcal{O}_i(1, i-1)$ and $\mathcal{O}_i(1,i)$. For $d > 2g-2$, the ample cone of M_w is bounded by $\mathcal{O}_w(1, w-1)$ and $\mathcal{O}_w(2, d-2)$; for $d \leq 2g-2$, it is bounded on one side by $\mathcal{O}_w(1, w-1)$, and contains the cone bounded on the other side by $\mathcal{O}_w(2, d-2)$.*

So as we pass from $i-1$ to i, the ample cone flips across the ray of slope $i-1$. This is exactly the behaviour which is predicted by Mori theory; indeed, flips are so named for precisely this reason.

The first thing to notice is that, since all the M_i have unique universal pairs $(\mathbf{E},\mathbf{\Phi})\rightarrow M_i \times X$, an expression such as $\det \pi_! \mathbf{E}$, or $\Lambda^2 \mathbf{E}_x$ for some $x \in X$, defines line bundles on all the M_i, which agree with one another on the open sets where the maps between different M_i are defined, and which consequently correspond under the natural isomorphism of **(14.2)**. This manner of defining a line bundle on several moduli spaces simultaneously is what is sometimes referred to as a *stack.* Since $\Lambda^2 \mathbf{E}_x$ and $\det \pi_! \mathbf{E}$ are the canonical (indeed, essentially the only) examples, we work out what they are on M_1.

(14.4) *On* M_1, $\Lambda^2 \mathbf{E}_x = \mathcal{O}_1(0,-1)$ *and* $\det \pi_! \mathbf{E} = \mathcal{O}_1(-1, g-d)$*; that is,* $\mathcal{O}_1(m,n) = \det^{-m} \pi_! \mathbf{E} \otimes (\Lambda^2 \mathbf{E}_x)^{(d-g)m-n}$.

Proof. Recall from **(11.17)** that the universal pair $(\mathbf{E}_1, \mathbf{\Phi}_1) \rightarrow M_1 \times X$ is constructed by pulling back the extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathbf{E}_0 \longrightarrow \Lambda(-1) \longrightarrow 0$$

from M_0, twisting by $\mathcal{O}(E_1^+)$, and modifying at E_1^+:

$$0 \longrightarrow \mathbf{E}_1 \longrightarrow \mathbf{E}_0(E_1^+) \longrightarrow \mathcal{O}_{E_1^+ \times X}(\Delta)(-1) \longrightarrow 0.$$

Hence $\Lambda^2(\mathbf{E}_1)_x = \Lambda^2(\mathbf{E}_0(E_1^+))_x \otimes \mathcal{O}(-E_1^+) = \Lambda^2 \mathbf{E}_0 \otimes \mathcal{O}(E_1^+) = \mathcal{O}_1(0,-1)$, and

$$\begin{aligned}
\det \pi_! \mathbf{E}_1 &= \det \pi_! \mathbf{E}_0(E_1^+) \otimes \mathcal{O}((g-2)(E_1^+)) \cdot \\
&= \det \pi_! \mathcal{O}(E_1^+) \det \pi_! \Lambda(-1)(E_1^+) \mathcal{O}_1(g-2, 2-g) \\
&= \mathcal{O}_1(1-g, g-1) \mathcal{O}_1(0, -d-1+g) \mathcal{O}_1(g-2, 2-g) \\
&= \mathcal{O}_1(-1, g-d). \ \square
\end{aligned}$$

It will also be helpful to know how $\mathcal{O}_i(m,n)$ restricts to various projective spaces inside the M_i.

(14.5) *The restriction of* $\mathcal{O}_i(m,n)$ *to*

(i) *a fibre of* $\mathbb{P}W_i^+$ *is* $\mathcal{O}(n-(i-1)m)$*;*

(ii) *a fibre of* $\mathbb{P}W_i^-$ *is* $\mathcal{O}((i-1)m-n)$*;*

(iii) $f^{-1}(E)\subset M_w$*, where* E *is a stable bundle and* f *is the Abel-Jacobi map of* **(11.20)**, *is* $\mathcal{O}(m(d-2)-2n)$.

Proof. By **(11.2)**, the bundle $\mathbf{E}$ in the universal pair restricts to an extension

$$0\longrightarrow\mathcal{O}(D)\longrightarrow\mathbf{E}\longrightarrow\Lambda(-D)(-1)\longrightarrow 0$$

on the fibre of $\mathbb{P}W_i^+$ over $D\in X_i$. Hence on this fibre $\Lambda^2\mathbf{E}_x=\mathcal{O}(-1)$ and

$$\det\pi_!\mathbf{E}=\det\pi_!\mathcal{O}(D)\det\pi_!\Lambda(-D)(-1)=\mathcal{O}(-\chi(\Lambda(-D)))=\mathcal{O}(-d+g-1+i).$$

So by **(14.4)** $\mathcal{O}_i(m,n)$ restricts to $\mathcal{O}((d-g+1-i)m-(d-g)m+n)=\mathcal{O}((1-i)m+n)$, which proves (i). Similarly by **(11.3)**, $\mathbf{E}$ restricts to an extension

$$0\longrightarrow\Lambda(-D)(1)\longrightarrow\mathbf{E}\longrightarrow\mathcal{O}(D)\longrightarrow 0$$

on the fibre of $\mathbb{P}W_i^-$ over $D\in X_i$. Hence $\Lambda^2\mathbf{E}_x=\mathcal{O}(1)$ and

$$\det\pi_!\mathbf{E}=\det\pi_!\Lambda(-D)(1)\det\pi_!\mathcal{O}(D)=\mathcal{O}(\chi(\Lambda(-D)))=\mathcal{O}(d-g+1-i).$$

So the previous situation is reversed, and $\mathcal{O}_i(m,n)$ restricts to $\mathcal{O}((i-1)m-n)$, which proves (ii). Finally, on a fibre $\mathbb{P}H^0(E)$ of the Abel-Jacobi map, the universal pair restricts to $E(1)$ with the tautological section. Hence on this fibre $\Lambda^2\mathbf{E}_x=\mathcal{O}(2)$ and $\det\pi_!\mathbf{E}=\mathcal{O}(d+2-2g)$. So by **(14.4)** $\mathcal{O}_i(m,n)$ restricts to $\mathcal{O}((2g-2-d)m+2((d-g)m-n))=\mathcal{O}(m(d-2)-2n)$, which proves (iii). □

(14.6) *On* $\tilde{M}_i$, $\mathcal{O}_i(m,n)=\mathcal{O}_{i-1}(m,n)(((i-1)m-n)E_i)$.

Proof. Certainly $\mathcal{O}_i(m,n)$ and $\mathcal{O}_{i-1}(m,n)$ are isomorphic away from E_i, so $\mathcal{O}_i(m,n)$ $=\mathcal{O}_{i-1}(m,n)(qE_i)$ for some q. But $\mathcal{O}_i(m,n)$ must be trivial on the fibres of $\mathbb{P}W_i^-$, and $\mathcal{O}_{E_i}(qE_i)=\mathcal{O}(-q,-q)$, so by **(14.5)**(ii) $q=(i-1)m-n$. □

We now pause to apply these ideas to compute the Picard group of the moduli space N of ordinary semistable bundles of determinant Λ. Before doing this, however, we need to prove the following useful lemma.

(14.7) *Let M, N be varieties with N normal, and let $f : M \to N$ be a morphism which is generically a projective bundle. Then $f_*\mathcal{O}_M = \mathcal{O}_N$.*

Proof. This is essentially Stein factorization. Let $U \subset N$ be the open set such that $f : f^{-1}(U) \to U$ is a projective bundle. Then certainly $f_*\mathcal{O}_{f^{-1}(U)} = \mathcal{O}_U$, so $N' = \mathbf{Spec}\, f_*\mathcal{O}_M$ is birational to N. By construction there is a map $f' : M \to N'$ such that $f'_*\mathcal{O}_M = \mathcal{O}_{N'}$. On the other hand, since $f_*\mathcal{O}_M$ is a coherent sheaf of $\mathcal{O}_N$-algebras, the birational morphism $N' \to N$ is finite. But a birational finite morphism to a normal variety is an isomorphism—this is essentially Zariski's main theorem; the proof in [24, III 11.4] goes through, or see [30, III.9]. Hence $N' = N$ and $f_*\mathcal{O}_M = \mathcal{O}_N$. □

The application in our case will be:

(14.8) *Let $f : M_w \to N$ be the Abel-Jacobi map of* **(11.20)**. *If $d > 2g - 2$, then $f_*\mathcal{O}_{M_w} = \mathcal{O}_N$.*

Proof. By **(11.20)** f is surjective with fibre $\mathbb{P}H^0(E)$ over a stable bundle E. If $U \subset N$ is the set of bundles E such that E is stable and $\dim H^0(E)$ is minimal, then certainly $f : f^{-1}(U) \to U$ is a projective bundle; for example it is the descent of a trivial projective bundle over the Quot scheme. Moreover, N is always normal; see for example [14]. □

Now fix a bundle $F \to X$ of rank 2 and degree $-\chi$: it might as well be $\mathcal{O} \oplus \mathcal{O}(-\chi x)$ for some $x \in X$. Let $\mathcal{O}(2\Theta)$ be the line bundle over N locally defined by $\det^{-1} \pi_! \mathbf{E} \otimes F$, where $\mathbf{E}$ is a local universal bundle. If d is even, then $\mathcal{O}(2\Theta)$ has a unique square root $\mathcal{O}(\Theta) = \det^{-1} \pi_! \mathbf{E}(-(\chi/2)x)$. Morally speaking, Θ is the *generalized theta-divisor.* A precise description of Θ (but with a different normalization for d odd) can be found in [14]. Our normalization is more consistent, because the following is true for any d:

$$f^*\mathcal{O}(2\Theta) = \mathcal{O}_w(2, d-2). \tag{14.9}$$

Proof. If $\mathbf{E} \to B \times X$ is any family of semistable bundles of determinant Λ, then the pullback of $\mathcal{O}(2\Theta)$ by the induced map $B \to N$ is precisely $\det^{-1} \pi_! \mathbf{E} \otimes F$. From the short exact sequence

$$0 \longrightarrow \mathbf{E}((i-1)x) \longrightarrow \mathbf{E}(ix) \longrightarrow \mathbf{E}(ix)_x \longrightarrow 0$$

and the isomorphism $\mathbf{E}(ix)_x = \mathbf{E}_x$,

$$\det^{-1} \pi_! \mathbf{E} \otimes F = \det^{-1} \pi_! \mathbf{E} \otimes \det^{-1} \pi_! \mathbf{E}(-\chi x) = \det^{-2} \pi_! \mathbf{E} \otimes \Lambda^2(\mathbf{E}_x)^\chi.$$

Hence if $B = M_w$,

$$\det^{-1} \pi_! \mathbf{E} \otimes F = \mathcal{O}_w(2, 2d-2g) \otimes \mathcal{O}_w(0, -\chi) = \mathcal{O}_w(2, d-2). \ \square$$

(14.10) $\operatorname{Pic} N = \mathbb{Z}$, *generated by* $\mathcal{O}(\Theta)$ *if* d *is even,* $\mathcal{O}(2\Theta)$ *if* d *is odd.*

Proof. By **(14.8)** $f_* \mathcal{O}_{M_w} = \mathcal{O}_N$, so $f^* : \operatorname{Pic} N \to \operatorname{Pic} M_w$ has a right inverse f_*, hence is injective. By **(14.9)** $f^*\mathcal{O}(\Theta)$ (in the even case) or $f^*\mathcal{O}(2\Theta)$ (in the odd case) is indivisible, so the same is true of $\mathcal{O}(\Theta)$ or $\mathcal{O}(2\Theta)$. Finally, by **(14.5)**(iii) a bundle $\mathcal{O}_w(m,n)$ can only be pulled back from N if $m(d-2) - 2n = 0$, so there are no other bundles on N besides the powers of $f^*\mathcal{O}(\Theta)$ or $f^*\mathcal{O}(2\Theta)$. $\square$

Before returning to the ample cone of M_i, let us make the connection with the Verlinde vector spaces. We first recall the definition:

(14.11) *Definition.* The *Verlinde vector spaces* are

$$Z_k(\Lambda) = H^0(N; \mathcal{O}(k\Theta)),$$

with the convention that $Z_k(\Lambda) = 0$ if d and k are both odd.

We can now prove the following result, originally due to Bertram [4].

(14.12) *For* $d > 2g-2$, *there is a natural isomorphism* $Z_k(\Lambda) = V_{k,k(d/2-1)}$.

Proof. By **(14.8)** $f_* f^* \mathcal{O}(k\Theta) = \mathcal{O}(k\Theta)$, so

$$f^* : H^0(N; \mathcal{O}(k\Theta)) \to H^0(M_w; \mathcal{O}_w(k, k(d/2-1)))$$

has inverse f_*. □

At last we return to the determination of the ample cone of M_i. It can of course be quite difficult to decide whether a given line bundle on a projective variety is ample. However, a geometric invariant theory quotient is naturally endowed with an ample bundle, which is the descent of the ample bundle used in the linearization. So we shall work out how the line bundles used in the linearizations of §9 descend to M_i. Recall that the linearization was some power of $\mathcal{O}(\chi+2\sigma,4\sigma)\to\mathbb{P}\operatorname{Hom}\times\mathbb{P}\mathbb{C}^{\chi}$, or more precisely, its pullback to $\operatorname{Quot}(\Lambda)\times\mathbb{P}\mathbb{C}^{\chi}$, which by abuse of notation we still denote $\mathcal{O}(\chi+2\sigma,4\sigma)$. By further abuse of notation we refrain from worrying about whether $\chi+2\sigma$ and 4σ are actually integers.

(14.13) *The bundle* $\mathcal{O}(\chi+2\sigma,4\sigma)\to\operatorname{Quot}(\Lambda)\times\mathbb{P}\mathbb{C}^{\chi}$ *descends to* $\mathcal{O}_i(1,d-1-2\sigma)\to M_i$.

Proof. As in §9, let $U\subset\operatorname{Quot}(\Lambda)$ be the set of quotients $\mathcal{O}^{\chi}\to E\to 0$ of determinant Λ such that the induced map $\mathbb{C}^{\chi}\to H^0(E)$ is an isomorphism. If $\mathcal{O}^{\chi}\to\mathbf{E}\to 0$ is the universal quotient over $U\times X$, then as in **(9.19)** there is a universal pair $(\mathbf{E}(1),\mathbf{\Phi})\to U\times\mathbb{P}\mathbb{C}^{\chi}\times X$ descending to the universal pair $(\mathbf{E},\mathbf{\Phi})$ on each M_i. Hence $\det\pi_!\mathbf{E}(1)\to U\times\mathbb{P}\mathbb{C}^{\chi}$ descends to $\det\pi_!\mathbf{E}=\mathcal{O}_i(-1,g-d)\to M_i$, and for any $x\in X$, $\Lambda^2\mathbf{E}(1)_x\to U\times\mathbb{P}\mathbb{C}^{\chi}$ descends to $\Lambda^2\mathbf{E}_x=\mathcal{O}_i(0,-1)\to M_i$.

By [24, III Ex. 12.6(b)] $\operatorname{Pic}(U\times\mathbb{P}\mathbb{C}^{\chi})=\operatorname{Pic}U\oplus\operatorname{Pic}\mathbb{P}\mathbb{C}^{\chi}$. So to determine a bundle on $U\times\mathbb{P}\mathbb{C}^{\chi}$, it suffices to determine it on $\{E\}\times\mathbb{P}\mathbb{C}^{\chi}$ and $U\times\{\phi\}$ for some $E\in U$, $\phi\in\mathbb{P}\mathbb{C}^{\chi}$.

On $\{E\}\times\mathbb{P}\mathbb{C}^{\chi}$, $\mathbf{E}(1)=E(1)$, so $\det\pi_!\mathbf{E}(1)=\mathcal{O}(\chi)$ and $\Lambda^2\mathbf{E}_x=\mathcal{O}(2)$. On $U\times\{\phi\}$, $\mathbf{E}(1)=\mathbf{E}$, so $\det\pi_!\mathbf{E}(1)=\det\pi_!\mathbf{E}$. But for all $E\in U$, $H^0(E)=H^0(\mathcal{O}^{\chi})$ and $H^1(E)=0$. Consequently $\det\pi_!\mathbf{E}=\mathcal{O}$. Moreover, there is a canonical map

$$\Lambda^2\mathbb{C}^{\chi}=\Lambda^2H^0(\mathcal{O}^{\chi})\longrightarrow\Lambda^2H^0(\mathbf{E})\longrightarrow H^0(\Lambda^2\mathbf{E}),$$

so the pullback of $\mathcal{O}(1)\to\mathbb{P}\operatorname{Hom}(\Lambda^2\mathbb{C}^{\chi},H^0(\Lambda))$ to U, which by abuse of notation we still write $\mathcal{O}(1)$, is precisely $(R^0\pi)\operatorname{Hom}(\Lambda,\Lambda^2\mathbf{E})$, which is clearly isomorphic to $\Lambda^2\mathbf{E}_x=\operatorname{Hom}(\Lambda,\Lambda^2\mathbf{E})_x$ since $\operatorname{Hom}(\Lambda,\Lambda^2\mathbf{E})$ is trivial on every fibre of π.

Putting it all together, we find that $\mathcal{O}(0,\chi)$ descends to $\mathcal{O}_i(-1,g-d)$ and $\mathcal{O}(1,2)$ descends to $\mathcal{O}_i(0,-1)$. The result follows after a little arithmetic. □

Proof of **(14.3)**. For any $\sigma \in (\max(0,d/2-i-1),d/2-i)$, the quotient of $U \times \mathbb{P}\mathbb{C}^{\chi}$ by the action of $\mathrm{SL}(\chi)$, linearized by $\mathcal{O}(\chi+2\sigma,4\sigma)$, gives the same quotient M_i. Hence the descent of $\mathcal{O}(\chi+2\sigma,4\sigma)$ to M_i is ample for any σ in that interval. By **(14.13)** and a little arithmetic these bundles span exactly the cones in the statement of **(14.3)**. Hence those cones are contained in the ample cones of the M_i. It remains to show that no bundles over M_i outside those cones are ample, except possibly on one side for $i=w$ and $d \leq 2g-2$.

By **(14.5)**(i), the restriction of $\mathcal{O}_i(m,n)$ to a fibre of $\mathbb{P}W_i^+$ is $\mathcal{O}(n-(i-1)m)$. So $\mathcal{O}_i(m,n)$ can only be ample over M_i if this is positive, that is, if $(i-1)m < n$. Thus one side of the ample cone of M_i is where it should be.

Likewise by **(14.5)**(ii) the restriction of $\mathcal{O}_{i-1}(m,n) \to M_{i-1}$ to a fibre of $\mathbb{P}W_i^-$ is $\mathcal{O}((i-1)m-n)$. So for $1 < i \leq w$, when the dimension of this fibre is positive, $\mathcal{O}_{i-1}(m,n)$ can only be ample over M_{i-1} if $(i-1)m > n$. Thus the other side of the ample cone of M_{i-1} is where it should be.

The only case we have not yet treated is the other side of the ample cone of M_w for $d > 2g-2$. In that case there is by **(11.20)** a surjective map $M_w \to N$ onto the moduli space of stable bundles of determinant Λ. It is not an isomorphism, since for example $\operatorname{Pic} M_w = \mathbb{Z} \oplus \mathbb{Z}$ while $\operatorname{Pic} N = \mathbb{Z}$. Hence the pullback of the ample bundle $\mathcal{O}(2\Theta) \to N$ is nef but not ample, that is, it is in the boundary of the ample cone. But by **(14.9)** this is precisely $\mathcal{O}(2,d-2)$. □

15 Computing the Euler characteristics

Now that we know the ample cones of the M_i, we can calculate $\dim V_{m,n}$ following the programme outlined at the beginning of the last section. We first need a formula for the canonical bundle of M_i:

(15.1) $$K_{M_i} = \mathcal{O}_i(-3, 4-d-g).$$

Proof. Clearly the canonical bundle is preserved by the isomorphism of **(14.2)**, so it suffices to work it out on M_1. But this is easy using **(11.19)** and the standard formulas for the canonical bundle of projective space and of a blow-up. □

(15.2) *Suppose that* $m, n \geq 0$ *and that* $m(d-2) - 2n > -d + 2g - 2$. *Let* $b = \left[\frac{n+d+g-4}{m+3}\right] + 1$. *Then* $\dim V_{m,n} = \chi(M_b; \mathcal{O}_b(m,n))$.

Figure 7 depicts what is going on. In order for $\dim V_{m,n}$ to be an Euler characteristic, Kodaira vanishing must be satisfied, which means that $\mathcal{O}(m,n)$ must lie inside some cone in the translate of the ample fan by K.

Proof. Note first that the inequality can be rewritten

$$(d/2 - 1)(m+3) > n + d + g - 4,$$

which guarantees that $b \leq [(d-1)/2]$ and hence that M_b exists.

Suppose that $\frac{n+d+g-4}{m+3}$ is not an integer. Then $b(m+3) > n + d + g - 4 > (b-1)(m+3)$, so $\mathcal{O}_b(m+3, n+d+g-4)$, which by **(15.1)** equals $K_{M_b}^{-1}\mathcal{O}_b(m,n)$, is in the ample cone of M_b by **(14.3)**. The result then follows from **(14.2)** and Kodaira vanishing.

If $\frac{n+d+g-4}{m+3}$ is an integer, then $\mathcal{O}_{b-1}(m+3, n+d+g-4)$ and $\mathcal{O}_b(m+3, n+d+g-4)$ are merely nef, so Kodaira vanishing does not apply. Instead, we move up to $\tilde{M}_b$. By **(14.7)** the 0th direct image of $\mathcal{O}_{\tilde{M}_b}$ in the projection $\tilde{M}_b \to M_b$ is $\mathcal{O}_{M_b}$, and by the theorem on cohomology and base change [24, III 12.11] the higher direct images vanish, so for all j, $H^j(\tilde{M}_b; \mathcal{O}_b(m,n)) = H^j(M_b; \mathcal{O}_b(m,n))$. By **(15.1)** and the standard formula for the canonical bundle of a blow-up, $K_{\tilde{M}_b} = \mathcal{O}_b(-3, 4-d-g)((b-1)E_b)$. Unfortunately $K_{\tilde{M}_b}^{-1}\mathcal{O}_b(m,n)$ may not be ample, so Kodaira vanishing still does not apply. Instead, we make the following two claims: first, that $H^j(\tilde{M}_b; \mathcal{O}_b(m,n)) = H^j(\tilde{M}_b; \mathcal{O}_b(m,n)((b-2)E_b))$ for all j, and second, that $\mathcal{O}_b(m+3, n+d+g-4)(-E_b)$ is ample on $\tilde{M}_b$. The desired result follows immediately from these claims, since at last Kodaira vanishing applies to $\mathcal{O}_b(m,n)((b-2)E_b)$.

To prove the first claim, note that for $0 < k < b$, $H^j(E_b; \mathcal{O}_b(m,n)(kE_b)) = 0$ for all j, since $\mathcal{O}_b(m,n)(kE_b)$ is $\mathcal{O}(-k)$ on each fibre of $\mathbb{P}^{b-1} \to E_b \to \mathbb{P}W_b^+$, so that every term in the Leray spectral sequence vanishes. Hence from the long exact sequence on $\tilde{M}_b$ of

$$0 \longrightarrow \mathcal{O}_b(m,n)((k-1)E_b) \longrightarrow \mathcal{O}_b(m,n)(kE_b) \longrightarrow \mathcal{O}_b(m,n)\mathcal{O}_{E_b}(kE_b) \longrightarrow 0,$$

we get isomorphisms $H^j(\tilde{M}_b; \mathcal{O}_b(m,n)((k-1)E_b)) = H^j(\tilde{M}_b; \mathcal{O}_b(m,n)(kE_b))$. The first claim follows by induction.

As for the second claim, note that on $\tilde{M}_b$, the line bundles $\mathcal{O}_{b-1}(1, b-2)$, $\mathcal{O}_b(1, b-1)$, and $\mathcal{O}_b(1,b)$ (or $\mathcal{O}_b(2, 2b-1)$ if $b = w$) are all nef, since they are pulled back from nef bundles on M_{b-1} or M_b. It is easy using **(14.6)**, the constraints on m and n, and a little arithmetic to check that $\mathcal{O}_b(m+3, n+d+g-4)(-E_b)$ is in the interior of the cone generated by these three bundles. □

We will have to assume in future that

$$\textbf{(15.3)} \qquad m(d-2) - 2n > -d + 2g - 2,$$

since otherwise there is no analogue of the last result and $K_{M_i}^{-1}\mathcal{O}_i(m,n)$ may not be ample for any i. However, for $d \geq 2g$, we still get a complete answer to our problem, for the following reason.

(15.4) *For $d \geq 2g$ and $m(d-2) - 2n < 0$, $V_{m,n} = 0$.*

Proof. By Riemann-Roch $\deg E \geq 2g$ implies $\dim H^0(E) \geq 2$, so for any stable bundle E, by **(11.20)** the fibre $f^{-1}(E)$ of the Abel-Jacobi map is a projective space of positive dimension. By **(14.5)**(iii), the restriction of $\mathcal{O}_w(m,n)$ to this is $\mathcal{O}(m(d-2)-2n)$, so any section of $\mathcal{O}_w(m,n)$ must vanish on $f^{-1}(E)$. Hence it must vanish on the inverse image $f^{-1}(N_s)$ of the stable subset of N. But this is open, so it must vanish everywhere. □

Let $L_i \to X_i$ be the line bundle defined by $L_i = \det^{-1} \pi_! \Lambda(-\Delta) \otimes \det^{-1} \pi_! \mathcal{O}(\Delta)$. Also put $q_i = n - (i-1)m$.

(15.5) *The restriction of $\mathcal{O}_{i-1}(m,n)$ to $\mathbb{P}W_i^-$ is $L_i^m(-q_i)$.*

Proof. Easy from **(14.4)** and the description of the universal pair over $\mathbb{P}W_i^-$ in **(11.3)**. □

Now let $U_i \to X_i$ be the vector bundle $(W_i^-) \oplus (W_i^+)^*$, and define numbers

$$N_i = \chi(X_i; L_i^m \Lambda^i W_i^- S^{q_i - i} U_i),$$

with of course the convention that this is zero when $q_i - i < 0$. On M_0, which is just projective space, make the additional convention that $\mathcal{O}_0(m,n) = \mathcal{O}(m+n)$.

(15.6) $$N_0 = \chi(M_0; \mathcal{O}_0(m,n)) = \binom{m+n+d+g-2}{m+n}.$$

Proof. Since X_0 is just a point and $W_0^- = 0$, $U_0 = (W_0^+)^*$ is just the vector space $H^1(\Lambda^{-1})^*$. Hence $S^{m+n}U_0 = H^0(M_0; \mathcal{O}_0(m,n))$ with our conventions and the result follows. □

(15.7) *Let* $0 < i \leq b$, *and suppose that* $m, n \geq 0$ *satisfy* **(15.3)**. *Then* $\chi(M_i; \mathcal{O}_i(m,n)) - \chi(M_{i-1}; \mathcal{O}_{i-1}(m,n)) = (-1)^i N_i$.

Proof. Suppose first that $q_i \leq 0$, so that $N_i = 0$. For $0 < j \leq -q_i$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{i-1}(m,n)((j-1)E_i) \longrightarrow \mathcal{O}_{i-1}(m,n)(jE_i) \longrightarrow \mathcal{O}_{i-1}(m,n) \otimes \mathcal{O}_{E_i}(jE_i) \longrightarrow 0.$$

By **(15.5)** the restriction of $\mathcal{O}_{i-1}(m,n)$ to $E_i = \mathbb{P}W_i^- \oplus \mathbb{P}W_i^+$ is $L_i^m(-q_i, 0)$, and $\mathcal{O}_{E_i}(E_i) = \mathcal{O}(-1,-1)$, so the third term of the exact sequence becomes $\mathcal{O}(-q_i - j, -j)$ and we get

$$\chi(\tilde{M}_i; \mathcal{O}_{i-1}(m,n)(jE_i)) - \chi(\tilde{M}_i; \mathcal{O}_{i-1}(m,n)((j-1)E_i)) = \chi(E_i; L_i^m(-q_i - j, -j)).$$

Summing over j and using **(14.6)** yields

$$\chi(\tilde{M}_i; \mathcal{O}_i(m,n)) - \chi(\tilde{M}_i; \mathcal{O}_{i-1}(m,n)) = \sum_{j=1}^{q_i} \chi(E_i; L_i^m(-q_i - j, -j)).$$

However, for $0 < i \leq b$ and $m, n, d, g \geq 0$, a little high-school algebra shows $-q_i < d + g - 1 - 2i$. Hence for all j in the sum above, $0 < j < d + g - 1 - 2i$, so every term in the Leray sequence of the fibration $\mathbb{P}^{d+g-2-2i} \to E_i \to \mathbb{P}W_i^-$ vanishes. Hence all terms are zero, as desired.

Now suppose $q_i > 0$. By an argument similar to the one above,

$$\chi(\tilde{M}_i; \mathcal{O}_i(m,n)) - \chi(\tilde{M}_i; \mathcal{O}_{i-1}(m,n)) = \sum_{j=0}^{q_i-1} \chi(E_i; L_i^m(-q_i+j,j)).$$

Each term of the right-hand side can be evaluated using the Leray sequence of the fibration $\mathbb{P}^{i-1} \times \mathbb{P}^{d+g-2-2i} \to E_i \to X_i$. Because $-q_i+j < 0 \leq j$, the only nonzero direct image of $L_i^m(-q_i+j,j)$ is the ith, which is just $L_i^m \Lambda^i W_i^- S^{q_i-j-i}(W_i^-) \otimes S^j(W_i^+)^*$. Here the factor of $\Lambda^i W_i^-$ comes from Serre duality, since the isomorphism $\mathcal{O}(-i) = K_{\mathbb{P}^{i-1}}$ is not canonical unless the right-hand side is tensored by such a factor. Hence

$$\chi(E_i; L_i^m(-q_i+j,j)) = (-1)^i \chi(X_i; L_i^m \Lambda^i W_i^- S^{q_i-j-i}(W_i^-) \otimes S^j(W_i^+)^*).$$

Of course the right-hand side is zero if $q_i - j - i < 0$, so our sum need only run up to $q_i - i$. The result follows because certainly

$$S^{q_i-i} U_i = \bigoplus_{j=0}^{q_i-i} S^{q_i-j}(W_i^-) \otimes S^j(W_i^+)^*. \ \square$$

(15.8) *For $i > b$, $N_i = 0$.*

Proof. It suffices to show that if $i > b$, then $q_i - i < 0$, that is, $(m+n)/(m+1) < i$. But using $m, n \geq 0$, the definition of b, and the inequality **(15.3)**, it is a matter of high-school algebra to check $(m+n)/(m+1) \leq b$. $\square$

Thus we get the simple formula

$$\textbf{(15.9)} \qquad \dim V_{m,n} = \sum_{i=0}^{\infty} (-1)^i N_i.$$

Proof. Put together **(15.2)**, **(15.6)**, **(15.7)**, and **(15.8)**. $\square$

The right-hand side clearly depends only on g, d, m, and n, not on the precise geometry of X and Λ. So even before doing the hard work of the next section, we have found that $\dim V_{m,n}$ depends only on g, d, m, and n, which is rather surprising.

16 Don Zagier to the rescue

All of the results in this section (except **(16.4)** and **(16.5)**) are due to Don Zagier and were communicated by him to the author.

In this section we will compute the N_i, using the Riemann-Roch theorem and Macdonald's description [29] of the cohomology ring of X_i. So we begin with a review of Macdonald's results. Let $e_i, \ldots, e_g, e'_1, \ldots e'_g \in H^i(X;\mathbb{Z})$ be generators such that the intersection form is $\sum_j e_j \otimes e'_j$. Define classes $\xi, \xi' \in H^1(X_i;\mathbb{Z})$ and $\eta \in H^2(X_i;\mathbb{Z})$ as the Künneth components of the divisor $\Delta \subset X_i \times X$, regarded as belonging to $H^2(X_i \times X;\mathbb{Z})$:

$$\Delta = \eta + \sum_j (\xi'_j e_j - \xi_j e'_j) + iX.$$

These generate the ring $H^*(X_i;\mathbb{Z})$. Moreover, if we put $\sigma_j = \xi_j \xi'_j$, then for any multiindex I without repeats,

(16.1) $$\langle \eta^{i-|I|}\sigma_I, X_i\rangle = 1.$$

This implies that for any two power series $A(x)$, $B(x)$,

$$\begin{aligned}\langle A(\eta)\exp(B(\eta)s), X_i\rangle &= \sum_{j=0}^{\infty}\langle A(\eta)B(\eta)^j s^j/j!, X_i\rangle \\ &= \sum_{j=0}^{g}\binom{g}{j}\operatorname*{Res}_{\eta=0}\left\{\frac{A(\eta)B(\eta)^j}{\eta^{i-j+1}}d\eta\right\} \\ &= \operatorname*{Res}_{\eta=0}\left\{\frac{A(\eta)(1+\eta B(\eta))^g}{\eta^{i+1}}d\eta\right\},\end{aligned} \qquad \textbf{(16.2)}$$

where $s = \sum_j \sigma_j$. Note that since $\sigma_j^2 = 0$, $s^k/k!$ is just the kth symmetric polynomial in the σ_j.

Since we will be doing Riemann-Roch, we need to know the Todd class of X_i; luckily this can be worked out in a useful form.

(16.3) $$\operatorname{td} X_i = \left(\frac{\eta}{1-e^{-\eta}}\right)^{i-g+1} \exp\left[s\left(\frac{1}{e^\eta - 1} - \frac{1}{\eta}\right)\right].$$

Proof. Macdonald [29] shows that the total Chern class of the tangent bundle of X_i is

$$c(X_i) = (1+\eta)^{i-2g+1}\prod_{j=1}^{g}(1+\eta-\sigma_i).$$

Let $h(x) = x/(1-e^{-x})$,so that

$$\operatorname{td} X_i = h(\eta)^{i-2g+1} \prod_{j=1}^{g} h(\eta - \sigma_j).$$

Expanding $h(\eta - \sigma_j)$ in a power series around η and using $\sigma_j^2 = 0$,

$$\begin{aligned}
\operatorname{td} X_i &= h(\eta)^{i-g+1} \prod_{j=1}^{g} \left(1 - \sigma_j \frac{h'(\eta)}{h(\eta)}\right) \\
&= h(\eta)^{i-g+1} \sum_{k=0}^{\infty} (-1)^k \frac{s^k}{k!} \left(\frac{h'(\eta)}{h(\eta)}\right)^k \\
&= h(\eta)^{i-g+1} \exp\left(-s\frac{h'(\eta)}{h(\eta)}\right),
\end{aligned}$$

which yields the desired formula. □

(16.4) *For any line bundle* $M \to X$ *and any* $k \in \mathbb{Z}$,

$$\operatorname{ch} \pi_! M(k\Delta) = ((\deg M + ki + 1 - g) - k^2 s)e^{k\eta}.$$

Proof. By Grothendieck-Riemann-Roch

$$\begin{aligned}
\operatorname{ch} \pi_! M(k\Delta) &= \pi_* \operatorname{ch} M(k\Delta) \operatorname{td} X \\
&= \pi_* \exp((\deg M + ki)X + k\Xi + k\eta)(1 + (1-g)X) \\
&= \pi_*(1 + (\deg M + ki)X)(1 + k\Xi - k^2 sX)e^{k\eta}(1 + (1-g)X) \\
&= ((\deg M + ki + 1 - g) - k^2 s)e^{k\eta},
\end{aligned}$$

where $\Xi = \sum_j (\xi_j' e_j - \xi_j e_j')$, so that $\Xi^2 = -2sX$. □

The next two lemmas give the Chern characters we shall need in our Riemann-Roch calculation.

(16.5) (i) $\operatorname{ch}(L_i) = \exp((d-2i)\eta + 2s)$;

(ii) $\operatorname{ch}(\Lambda^i W_i^-) = \exp((d-3i+1-g)\eta + 3s)$;

(iii) $\operatorname{ch}(U_i) = (d-i+1-2g)e^{-\eta} + (2g-2)e^{-2\eta} + \sum_{j=1}^{g} e^{-\eta-\sigma_i}$.

Proof. Since $L_i = \det^{-1} \pi_! \Lambda(-\Delta) \otimes \det^{-1} \pi_! \mathcal{O}(\Delta)$, by **(16.4)**

$$c_1(L_i) = -c_1(\pi_!\Lambda(-\Delta)) - c_1(\pi_!\mathcal{O}(\Delta)) = (d-i+1-g)\eta+s+(-i-1+g)\eta+s = (d-2i)\eta+2s,$$

which implies (i). From the exact sequence

$$0 \longrightarrow \Lambda(-2\Delta) \longrightarrow \Lambda(-\Delta) \longrightarrow \mathcal{O}_\Delta\Lambda(-\Delta) \longrightarrow 0,$$

it follows that $W_i^- = \pi_!\mathcal{O}_\Delta\Lambda(-\Delta) = \pi_!\Lambda(-\Delta) - \pi_!\Lambda(-2\Delta)$ in K-theory. Hence by **(16.4)**

$$\operatorname{ch} W_i^- = ((d-i+1-g)-s)e^{-\eta} - ((d-2i+1-g)-4s)e^{-2\eta}.$$

In particular

$$c_1(\Lambda^i W_i^-) = c_1(W_i^-) = -(d-i+1-g)\eta - s + 2(d-2i+1-g)\eta + 4s = (d-3i+1-g)\eta+3s,$$

which implies (ii). Again by **(16.4)**,

$$\operatorname{ch}(W_i^+)^* = \operatorname{ch} \pi_!\Lambda^{-1}(2\Delta) = ((d-2i+g-1)-4s)e^{-2\eta}.$$

Hence

$$\begin{aligned} \operatorname{ch} U_i &= \operatorname{ch}(W_i^-) \oplus (W_i^+)^* \\ &= ((d-i+1-g)-s)e^{-\eta} + (2g-2)e^{-2\eta} \\ &= (d-i+1-2g)e^{-\eta} + (2g-2)e^{-2\eta} + \sum_{j=1}^{g} e^{-\eta-\sigma_i}, \end{aligned}$$

which is (iii). □

(16.6) $\operatorname{ch}(L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i-i}U_i)$

$$= \operatorname*{Coeff}_{t^{q_i-i}} \left\{ e^{(m(d-2)-2n)\eta} \exp\left((2m+3)s - \frac{ts}{e^{-\eta}-t}\right) \frac{(e^{-\eta}-t)^{-d+i-1+g}}{(1-t)^{2g-2}} \right\}.$$

Proof. The Chern roots of S^kU_i are the sums of k (not necessarily distinct) Chern roots of U_i, so by **(16.5)**(iii)

$$\sum_{k=0}^{\infty} \operatorname{ch}(S^kU_i)t^k = \prod_{\substack{\text{Chern roots} \\ \alpha \text{ of } U_i}} \frac{1}{1-te^{\alpha}}$$

$$= \left(\frac{1}{1-te^{-\eta}}\right)^{d-i+1-2g}\left(\frac{1}{1-te^{-2\eta}}\right)^{2g-2}\prod_{j=1}^{g}\left(\frac{1}{1-te^{-\eta-\sigma_j}}\right)$$

$$= \frac{(1-te^{-\eta})^{-d+i-1+g}}{(1-te^{-2\eta})^{2g-2}}\exp\left(\frac{-ts}{e^{\eta}-t}\right).$$

Replacing t by $te^{2\eta}$ and taking coefficients of t^{q_i-i} yields

$$\mathrm{ch}(S^{q_i-i}U_i) = \operatorname*{Coeff}_{t^{q_i-i}}\left\{e^{-2(q_i-i)\eta}\frac{(1-te^{\eta})^{-d+i-1+g}}{\cdot(1-t)^{2g-2}}\exp\left(\frac{-ts}{e^{-\eta}-t}\right)\right\}.$$

The result then follows using **(16.5)**(i) and (ii) and the pleasing identity

$$m(d-2i)+(d-3i+1-g)-2(q_i-i) = m(d-2)-2n+(d-i+1-g). \ \square$$

We are now ready to perform our Riemann-Roch calculation:

$$\begin{aligned} N_i &= \langle \mathrm{ch}(L_i^m\Lambda^i W_i^- S^{q_i-i}U_i)\,\mathrm{td}(X_i), X_i\rangle \\ &= \operatorname*{Coeff}_{t^{q_i-i}}\Big\langle e^{(m(d-2)-2n)\eta}\exp\left((2m+3)s-\frac{ts}{e^{-\eta}-t}\right) \\ &\qquad \frac{(e^{-\eta}-t)^{-d+i-1+g}}{(1-t)^{2g-2}}\left(\frac{\eta}{1-e^{-\eta}}\right)^{i-g+1}\exp\left[s\left(\frac{1}{e^{\eta}-1}-\frac{1}{\eta}\right)\right], X_i\Big\rangle \\ &= \operatorname*{Coeff}_{t^{q_i-i}}\operatorname*{Res}_{\eta=0}\Big\{\frac{e^{((d-2)m-2n)\eta}(e^{-\eta}-t)^{-d+i-1+g}}{(1+t)^{2g-2}(1-e^{-\eta})^{i+1}} \end{aligned}$$

(16.7) $$\qquad\left(e^{-\eta}+\left(2m+3-\frac{t}{e^{-\eta}-t}\right)(1-e^{-\eta})\right)^{g}d\eta\Big\};$$

the first equality by Riemann-Roch, the second by **(16.3)** and **(16.6)**, and the third by taking

$$A(x) = \left(\frac{x}{1-e^{-x}}\right)^{i-g+1}e^{((d-2)m-2n)x}\frac{(e^{-x}-t)^{-d+i-1+g}}{(1+t)^{2g-2}}$$

and

$$B(x) = 1/(e^{x}-1)-1/x+2m+3-t/(e^{-x}-t)$$

in **(16.2)**, then amalgamating gth powers.

The term in braces is the product of $\left(\frac{e^{-\eta}-t}{1-e^{-\eta}}\right)^i$ with something independent of i, so make the substitution

$$y = \frac{e^{-\eta}-t}{1-e^{-\eta}}, \qquad e^{-\eta} = \frac{1+ty}{1+y}, \qquad 1-e^{-\eta} = \frac{(1-t)y}{1+y},$$

$$e^{-\eta} - t = \frac{1-t}{1+y}, \qquad d\eta = \frac{(1-t)dy}{(1+y)(1+ty)}.$$

Then the residue in (16.7) becomes

$$\operatorname*{Res}_{y=0}\left\{\frac{a(y)dy}{y^{i+1}}\right\} = \operatorname*{Coeff}_{y^i}\{a(y)\}$$

for

$$a(y) = \frac{(1+ty)^{q_d/2-1}(1+y)^{-q_d/2+d-2g+1}}{(1-t)^{d+g-1}}\left(1+(2m+3)(1-t)y-ty^2\right)^g.$$

Then since $q_i - i = (m+n)-(m+1)i$,

$$\begin{aligned} \dim V_{m,n} &= \sum_{i=0}^{\infty}(-1)^i N_i \\ &= \sum_{i=0}^{\infty}(-1)^i \operatorname*{Coeff}_{t^{q_i-i}} \operatorname*{Coeff}_{y^i}\{a(y)\} \\ &= \operatorname*{Coeff}_{t^{m+n}}\left(\sum_{i=0}^{\infty}(-t^{m+1})^i \operatorname*{Coeff}_{y^i}\{a(y)\}\right) \\ &= \operatorname*{Coeff}_{t^{m+n}}\{a(-t^{m+1})\}. \end{aligned}$$

Thus we obtain the following theorem. We repeat the definition of $V_{m,n}$ for convenience.

(16.8) *Let X be embedded in $\mathbb{P}H^1(\Lambda^{-1})$ via the linear system $|K_X\Lambda|$. For any $m,n \geq 0$, let $V_{m,n} = H^0(\mathbb{P}H^1(\Lambda^{-1}); \mathcal{O}(m+n)\otimes \mathcal{I}_X^n)$. Define*

$$F(t) = \frac{(1-t^{m+2})^{-h-1}(1-t^{m+1})^{-h'-1}}{(1-t)^{d+g-1}t^{m+n}}\left(1-(2m+3)(1-t)t^{m+1}-t^{2m+3}\right)^g,$$

where $h=(d-2)m-2n$ and $h' = -h-d+2g-2$. Then if $m(d-2)-2n > -d+2g-2$,

$$\dim V_{m,n} = \operatorname*{Res}_{t=0}\left\{\frac{F(t)dt}{t}\right\},$$

that is, the constant term in the Laurent expansion of $F(t)$ at $t=0$. Moreover, if $d \geq 2g$ and $m(d-2)-2n<0$, then $V_{m,n}=0$. □

This is the most explicit formula for $\dim V_{m,n}$ we will obtain in general. However, in some cases we could obtain completely explicit formulas. If $m+n$ is small, for example, we could calculate directly, since we would then be looking at the residue of a function with a pole of low order; for fixed $m+n$, we would get an explicit polynomial in g, d, m, and n. Otherwise, we can still use the residue theorem, which says that the sum of the residues at all the poles of $F(t)dt/t$ is zero. These poles are of five possible kinds: $t=0$, $t=\infty$, $t=1$, $t^{m+1}=1$ but $t\neq 1$, and $t^{m+2}=1$ but $t\neq 1$ (note that the last two cases are disjoint). But in fact $t=1$ is never a pole, since at that point $1-(2m+3)(1-t)t^{m+1}-t^{2m+3}$ has a triple zero, and hence the order of $F(t)$ is

$$(-h-1)+(-h'-1)-(d+g-1)+3g=1\geq 0.$$

Also, it is straightforward to check that $F(1/t)=-F(t)$, which implies that

$$\operatorname*{Res}_{t=\infty}\left\{\frac{F(t)dt}{t}\right\}=\operatorname*{Res}_{t=0}\left\{\frac{F(t)dt}{t}\right\}.$$

Hence

$$-2\dim V_{m,n}=\left(\sum_{\substack{\zeta^{m+1}=1\\ \zeta\neq 1}}\operatorname*{Res}_{t=\zeta}+\sum_{\substack{\zeta^{m+2}=1\\ \zeta\neq 1}}\operatorname*{Res}_{t=\zeta}\right)\left\{\frac{F(t)dt}{t}\right\}.$$

There are poles at the $(m+2)$th roots of unity if and only if $h\geq 0$, and at the $(m+1)$th roots of unity if and only if $h'\geq 0$. Thus $\dim V_{m,n}$ is a sum over the residues at the $(m+2)$th roots if $h'<0\leq h$, a sum over the residues at the $(m+1)$th roots if $h<0\leq h'$, and is 0 if $h,h'<0$. (Note that this last case agrees with **(15.4)**.) For $h\geq 0$ it is necessary to calculate the residue of a function with a pole of order $1+h$, which gets more and more difficult as h grows. However, when $h=0$, the calculation is easy, and we can prove the celebrated Verlinde formula.

(16.9)
$$\dim Z_k(\Lambda)=\left(\frac{k+2}{2}\right)^{g-1}\sum_{j=1}^{k+1}\frac{(-1)^{d(j+1)}}{(\sin\frac{j\pi}{k+2})^{2g-2}}.$$

Proof. If d and k are both odd, then on symmetry grounds the right-hand side is zero as desired. So assume d and k are not both odd. By **(14.12)** $\dim Z_k(\Lambda)=\dim V_{k,k(d/2-1)}$ for any $d>2g-2$. Then $h=0$ and $h'<0$, so

$$\begin{aligned}&-2\dim V_{k,k(d/2-1)}\\ &=\sum_{\substack{\zeta^{k+2}=1\\ \zeta\neq 1}}\operatorname*{Res}_{t=\zeta}\left(\frac{-dt/t}{t^{k+2}-1}\right)\frac{(1-\zeta^{-1})^{d-2g+1}}{(1-\zeta)^{d+g-1}\zeta^{kd/2}}\left(1-(2k+3)(\zeta^{-1}-1)-\zeta^{-1}\right)^{g}.\end{aligned}$$

But $(1-(2k+3)(\zeta^{-1}-1)-\zeta^{-1})=(2k+4)(1-\zeta^{-1})$, the residue is $-1/(k+2)$, and

$$\frac{(1-\zeta^{-1})^{d}}{(1-\zeta)^{d}\zeta^{kd/2}}=\frac{(1-\zeta^{-1})^{d}}{(1-\zeta)^{d}\zeta^{-d}\zeta^{(k+2)d/2}}=(-1)^{d}\zeta^{(k+2)d/2},$$

so

$$\begin{aligned}\dim V_{k,k(d/2-1)} &= (2k+4)^{g-1} \sum_{\substack{\zeta^{k+2}=1\\ \zeta\neq 1}} (-1)^d \zeta^{(k+2)d/2} \left(\frac{-\zeta}{(1-\zeta)^2}\right)^{g-1} \\ &= \tfrac{1}{2}(2k+4)^{g-1} \sum_{\substack{\xi^{2k+4}=1\\ \xi\neq\pm 1}} \frac{(-1)^{d+g-1}\xi^{(k+2)d}}{(\xi^{-1}-\xi)^{2g-2}},\end{aligned}$$

which is equivalent to the Verlinde formula. □

Part IV

The cohomology ring of N

17 Generators of the ring

In this final part of the thesis, we will fix the determinant Λ to have odd degree d, so that the moduli space N of rank 2 stable bundles of determinant Λ will be smooth. Our goal is to apply the Verlinde formula to study the rational cohomology of N. We have already worked out the Poincaré polynomial of N in §12, so it is the ring structure that will chiefly interest us now. Section 17 will review what was known about the cohomology ring before the advent of conformal field theory. The chief result is a theorem of Newstead giving explicit generators for the ring. Section 18 will use the Verlinde formula and the Riemann-Roch theorem to evaluate any monomial in these generators on the fundamental class of N; by Poincaré duality this is sufficient, at least in principle, to determine the ring structure of $H^*(N)$.

During this section, we shall view N in a completely different way from before, as a moduli space of representations of $\pi_1(X)$. So let us first review how N can be interpreted in this way. Regarding X as a compact Riemann surface, choose loops $e_1, e_2, \ldots, e_{2g}$ on X which generate $\pi_1(X)$ in the usual way, so that $\prod e_i e_{i+g} e_i^{-1} e_{i+g}^{-1} \sim 1$. If we cut out a small disc $D \subset X$, then $\prod e_i e_{i+g} e_i^{-1} e_{i+g}^{-1}$ is homotopic to the boundary circle of D, but is no longer contractible. Hence we lose the relation, and $\pi_1(X - D)$ is the free group on the generators e_i. Now consider the map $\mu : \mathrm{SU}(2)^{2g} \to \mathrm{SU}(2)$ defined by

$$(A_1, A_2, \ldots, A_{2g}) \longmapsto \prod_{i=1}^{g} A_i A_{i+g} A_i^{-1} A_{i+g}^{-1}$$

and in particular the subspace $S_g = \mu^{-1}(-I) \subset \mathrm{SU}(2)^{2g}$. It can be shown [26] that $-I$ is a regular value of μ, so S_g is a smooth $6g-3$-submanifold of $\mathrm{SU}(2)^{2g}$. We may regard an element $\omega \in S_g$ as a representation of $\pi_1(X - D)$ sending the boundary circle to $-I$. It must be irreducible, since if it were reducible to an abelian subgroup, it would send the boundary circle to I. Such a representation gives us a flat connection on an SU(2) bundle over $X - D$ having holonomy $-I$ around D. By passing to the associated rank 2 vector bundle and gluing in a fixed twisted unitary connection over D, we can extend our flat connection ω to a connection on a unitary vector bundle $V \to X$ of degree 1. Then the $(0,1)$ part of the associated covariant derivative is a Cauchy-Riemann operator which induces a holomorphic structure on V. Now the diagonal conjugation action of $\mathrm{SU}(2)/\pm 1 = \mathrm{SO}(3)$ on $\mathrm{SU}(2)^{2g}$ clearly preserves S_g, and by Schur's lemma the restriction of the action is free. Hence the quotient $S_g/\mathrm{SO}(3)$ is a smooth $6g-6$-manifold. But if two representations in S_g are conjugate, then the induced connections on V are isomorphic, and hence so are the holomorphic structures. Hence to each point in $S_g/\mathrm{SO}(3)$ we can associate an isomorphism class of holomorphic bundles over X. In this context, the celebrated theorem of Narasimhan and Seshadri [41] can be stated as follows.

(17.1) *All the holomorphic bundles constructed in this manner are stable, and the resulting map* $\phi : S_g/\mathrm{SO}(3) \longrightarrow N$ *is a diffeomorphism.* □

Remark. We will in future identify N with $S_g/\mathrm{SO}(3)$. Under this identification, any diffeomorphism $f : X \to X$ induces a diffeomorphism $\hat{f} : N \to N$. (To be precise, we should require that f preserves a small neighbourhood of the disc D; but this is not a serious restriction, as any diffeomorphism of X is isotopic to one of this form.)

To obtain distinguished cohomology classes in N, we construct a vector bundle $\mathcal{U}$ over $N \times X$, as follows. Let $\tilde{X}$ be the universal cover of X, and consider the twisted quotient

$$\tilde{X} \times_{\pi_1(X)} (S_g \times \mathrm{sl}(2, \mathbb{C})).$$

The action on the right-hand factor is given by $h(\rho, v) = (\rho, \mathrm{ad}\ \rho(h) \cdot v)$, where we regard $\rho \in S_g$ as determining an $\mathrm{SO}(3)$-representation of $\pi_1(X)$. This gives us a vector bundle over $S_g \times X$; to see that it descends to $N \times X$, note that the conjugation action of $\mathrm{SO}(3)$ on S_g lifts to an action on this twisted quotient, given by

$$T \cdot (s \times (\rho, v)) = (s \times (T\rho T^{-1}, \mathrm{ad}\ T \cdot v)).$$

The resulting vector bundle $\mathcal{U}$ is clearly *natural* in the sense that, for any diffeomorphism $f : X \to X$, we have $(\hat{f} \times f)^*(\mathcal{U}) \cong \mathcal{U}$. (Indeed, it is really just the universal adjoint bundle ad $\mathbf{E}$ in a representation-theoretic setting.) Because N is simply connected [33], we may write

$$c_2(\mathcal{U}) = 2\alpha\sigma - \beta + 4\psi,$$

where σ denotes the fundamental class in $H^2(X)$ and

$$\alpha \in H^2(N);\ \beta \in H^4(N);\ \psi \in H^3(N) \otimes H^1(X).$$

(The scalar factors are inserted to agree with the conventions of [34].) The Poincaré duals of our loops e_i form a basis $e^1, e^2, \dots, e^{2g}$ of $H^1(X, \mathbb{Z})$; using this, we can decompose

$$\psi = \sum_{i=1}^{2g} \psi_i e^i,$$

where $\psi_i \in H^3(N)$. We can now state the following crucial result of Newstead [34].

(17.2) *The ring $H^*(N)$ is generated by α, β, and the ψ_i.* □

Remark. Atiyah and Bott [2] later showed that the integral cohomology $H^*(N, \mathbb{Z})$ is torsion-free, so that rational multiples of the same classes generate the integral cohomology. Moreover, the ψ_i are actually integral generators of $H^3(N, \mathbb{Z})$, a fact which we will be needing later.

Since we have a set of generators, to determine the ring structure completely it suffices to find the relations. The usual commutation relations hold, so this amounts to deciding for which nonnegative integers $m_j, n_j, p_{i,j}$ we have

$$\sum_j \alpha^{m_j} \beta^{n_j} \Big(\prod_i \psi_i^{p_{i,j}}\Big) = 0. \tag{17.3}$$

But according to Poincaré duality, $\mu \in H^m(N) = 0$ if and only if $\mu\nu = 0$ for all $\nu \in H^{6g-6-m}(N)$. Hence in principle to decide when equality holds in **(17.3)** we need only evaluate the integers

$$\alpha^m \beta^n \Big(\prod_i \psi_i^{p_i}\Big)[N] \tag{17.4}$$

for those m, n, p_i such that

$$2m + 4n + 3\sum_i p_i = 6g - 6.$$

A more general discussion of these top-dimensional pairings, and of their analogues in four dimensions, can be found in [11]. Our goal will be to find an explicit formula for **(17.4)**.

The first thing to notice is the following.

(17.5) *Let* $1 \le i_0 \le g$. *If* $p_{i_0} > 1$, *or if* $p_{i_0} \ne p_{i_0+g}$, *then the pairing in* **(17.4)** *is zero.*

Proof. The first part is easy, since the commutation relations imply $\psi_{i_0}^2 = 0$. This only leaves the case $p_{i_0} + p_{i_0+g} = 1$ for the second part. Now if $f : X \to X$ is an orientation-preserving diffeomorphism, then by naturality

$$\hat{f}^*(\alpha)^m \hat{f}^*(\beta)^n (\prod_i \hat{f}^*(\psi_i)^{p_i})[N] = \alpha^m \beta^n (\prod_i \psi_i^{p_i})[N].$$

However, since $(\hat{f} \times f)^* \mathcal{U} \cong \mathcal{U}$, we have

$$\hat{f}^*(\alpha) = \alpha,\ \hat{f}^*(\beta) = \beta,\ \sum_i \hat{f}^*(\psi_i) f^*(e^i) = \sum_i \psi_i e^i.$$

Now it is easy to find a diffeomorphism of X such that $f^*(e^{i_0}) = -e^{i_0}$, $f^*(e^{i_0+g}) = -e^{i_0+g}$, but $f^*(e^i) = e^i$ for all other i. For example, a half twist of the surface in figure 8 below the loop labelled has the desired properties. If $p_{i_0} + p_{i_0+g} = 1$, we conclude

$$\alpha^m \beta^n (\prod_i \psi_i^{p_i})[N] = -\alpha^m \beta^n (\prod_i \psi_i^{p_i})[N] = 0. \ \square$$

For $1 \le i \le g$, define $\gamma_i = \psi_i \psi_{i+g}$. Then the proposition above shows that all the pairings **(17.4)** are zero except those of the form

$$\textbf{(17.6)} \qquad (\alpha^m \beta^n \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_p})[\backslash\}]$$

where $m + 2n + 3p = 3g - 3$ and $1 \le i_1 < i_2 < \cdots < i_p \le g$. Actually, the value of **(17.6)** is independent of the choice of i_j. This follows from a diffeomorphism argument similar to that in the proof of **(17.5)**, because any permutation of the handles in figure 8 can be realized by a diffeomorphism. Consequently, any pairing of the form **(17.6)** is equal to $(g-p)!/(2^p g!)$ times the even simpler expression

$$(\alpha^m \beta^n \gamma^p)[N],$$

where $\gamma = 2\sum_{i=1}^{g} \gamma_i$. The class γ has the advantage of being independent of our choice of basis $\{e^i\}$, since $\psi^2 = \gamma\sigma$.

Still using the identification $N = S_g/\mathrm{SO}(3)$, we can perform another construction that would not have made sense in the holomorphic setting. Suppose X_g and X_{g-1} have genus g and $g-1$, respectively. The map from X_g to X_{g-1} which collapses the ith handle to a point (see figure 9) is not holomorphic, but it induces an embedding $\eta_i : N_{g-1} \to N_g$ of the moduli spaces. The image of this embedding consists precisely of those conjugacy classes of representations of $\pi_1(X_g - D)$ which send the generators e_i and e_{i+g} to the identity. That is, $\eta_i(N_{g-1}) = p(\pi_i^{-1}(I) \cap \pi_{i+g}^{-1}(I))$, where π_i is the restriction to S_g of the projection of $\mathrm{SU}(2)^{2g}$ on the ith factor, and p is the quotient map $S_g \to N_g$. Since $\dim_{\mathbb{R}} N_g = 6g-6$, N_{g-1} has real codimension 6.

(17.7) *$\eta_i(N_{g-1})$ is Poincaré dual to γ_i. Consequently,*

$$\alpha^m \beta^n \gamma^p [N_g] = 2g \;\; \alpha^m \beta^n \gamma^{p-1} [N_{g-1}].$$

Proof. If we fix an element $(A_1, A_2, \ldots, A_{2g-2}) \in S_{g-1}$, then the map $\mathrm{SU}(2) \to S_g$ given by

$$T \longmapsto (A_1, \ldots, A_{i-1}, T, A_i, \ldots, A_{i+g-1}, T^{-1}, A_{i+g}, \ldots, A_{2g-2})$$

is a right inverse for π_i. There is a similar right inverse for π_{i+g}. Hence the pullbacks by π_i and π_{i+g} of the fundamental cohomology class of $\mathrm{SU}(2)$ are indivisible classes $\chi_i, \chi_{i+g} \in H^3(S_g, \mathbb{Z})$, Poincaré dual to $\pi_i^{-1}(I)$ and $\pi_{i+g}^{-1}(I)$ respectively. We claim that $\chi_i = \pm p^*(\psi_i)$ and $\chi_{i+g} = \pm p^*(\psi_{i+g})$.

To prove this, note that $\pi_i^{-1}(I)$ is a union of fibres of the $\mathrm{SU}(2)$-action on S_g, so $\chi_i = p^*(\hat{\psi}_i)$ for some $\hat{\psi}_i \in H^3(N_g, \mathbb{Z})$. This $\hat{\psi}_i$ is unique, because we can see from the Leray-Serre spectral sequence (or even the Gysin sequence) that the natural map $H^3(N_g, \mathbb{Z}) \to H^3(S_g, \mathbb{Z})$ is injective. Now for any e^j in our basis for $H^1(X_g, \mathbb{Z})$ with $j \neq i$, there exist diffeomorphisms $f : X_g \to X_g$ such that $f^*(e^i) = e^i$ but $f^*(e^j) \neq e^j$: the Dehn twists and half twists in figure 10 do the trick. Thus the only classes in $H^3(N_g, \mathbb{Z})$ which are invariant under f^* for all such f are the integer multiples of

ψ_i. But χ_i is invariant under such f^* by construction, and hence so is $\hat{\psi}_i$. Hence by the second part of (**17.2**), $\hat{\psi}_i$ is an integer multiple of ψ_i. Since both ψ_i and $\hat{\psi}_i$ are indivisible, we must have $\hat{\psi}_i = \pm\psi_i$. Likewise, $\hat{\psi}_{i+g} = \pm\psi_{i+g}$.

We shall not bother to pin down the sign, because for our purposes it is sufficient to note that the same sign holds for $\hat{\psi}_i$ and $\hat{\psi}_{i+g}$. This can be deduced from a diffeomorphism argument like the one above, using a Dehn twist that induces $e^i \mapsto e^{i+g}$ and $e^{i+g} \mapsto -e^i$ (see figure 11). Then $\chi_i\chi_{i+g} = p^*(\gamma_i)$, and so descending to N_g we obtain the first result.

As for the second, it is sufficient to show that $\eta^*(\alpha) = \alpha$, $\eta^*(\beta) = \beta$, and $\eta^*(\gamma) = \gamma$. This is straightforward from the construction of $\mathcal{U}$. □

We can use this proposition recursively to eliminate γ from the pairings. Hence it now suffices to evaluate $\alpha^m\beta^n[N]$ when $m + 2n = 3g - 3$. This will be carried out in the next section, using the Verlinde formula.

18 A formula for the pairings

We now return to a more familiar point of view, regarding N as a smooth complex projective variety with an ample bundle $\mathcal{O}(2\Theta)$. We want to use the Verlinde formula to evaluate $\alpha^m\beta^n[N]$. The natural tool for translating algebraic into cohomological information is, of course, the Riemann-Roch theorem. This tells us that

(18.1) $$\chi(N; \mathcal{O}(2k\Theta)) = (\operatorname{ch}\mathcal{O}(2k\Theta)\operatorname{td} N)[N].$$

Since the canonical bundle of N is $\mathcal{O}(-4\Theta)$ [34], which is negative, the left-hand side equals $\dim Z_k$ by Kodaira vanishing. On the other hand, we can calculate the right-hand side in terms of α and β. Newstead [34] showed that $c_1(N) = c_1\mathcal{O}(4\Theta) = 2\alpha$ and $p(N) = (1+\beta)^{2g-2}$. By restricting $\mathcal{O}(2\Theta)$ to a projective subspace of N as in [34], it is not hard to show that $c_1(\mathcal{O}(2\Theta)) = \alpha$. As for $\operatorname{td} N$, the identity

$$\frac{x}{1-e^{-x}} = \exp(\tfrac{1}{2}x)\frac{\frac{1}{2}x}{\sinh\frac{1}{2}x}$$

implies (see p. 117 of [38]) that, if y_i are the Pontrjagin roots,

$$\operatorname{td} = \exp(\tfrac{1}{2}c_1)\prod_i \frac{\frac{1}{2}\sqrt{y_i}}{\sinh\frac{1}{2}\sqrt{y_i}}.$$

Noting that $2g-2$ of the Pontrjagin roots of N are β and the rest are 0, we obtain

$$\operatorname{td} N = \exp(\alpha)\left(\frac{\frac{1}{2}\sqrt{\beta}}{\sinh\frac{1}{2}\sqrt{\beta}}\right)^{2g-2}.$$

Then by **(18.1)**

$$D(g,k) = \left(\exp\left(\frac{k+2}{2}\alpha\right)\left(\frac{\frac{1}{2}\sqrt{\beta}}{\sinh\frac{1}{2}\sqrt{\beta}}\right)^{2g-2}\right)[N] \tag{18.2}$$

for even k. For fixed g, this is a polynomial in k whose coefficients involve the pairings. On the other hand, the version of the Verlinde formula in **(2.14)** is likewise a polynomial in k. Substituting $\ell = k+2$ in **(2.14)** and **(18.2)**, and equating coefficients of ℓ^m, we get

$$-(-\tfrac{1}{2})^{g-1}\,\frac{(2^{m-g+1}-2)}{(m-g+1)!}\,B_{m-g+1}\,P_{3g-3-m} = \frac{1}{m!\,2^{3g-3}}\,P_{3g-3-m}(\alpha^m\beta^{\frac{1}{2}(3g-3-m)})[N],$$

where B_i is the ith Bernoulli number and P_i is the coefficient of x^i in the power series expansion of $(x/\sinh x)^{2g-2}$. (We use the conventions under which $B_2 = 1/6$, $B_4 = -1/30$, etc., and we interpret $B_i = 0$ if $i < 0$, because $(k+2)x/\sinh(k+2)x$ has no pole at 0.) Cancelling P_{3g-3-m} and rearranging, we obtain

$$(\alpha^m\beta^n)[N] = (-1)^{g-1}\,\frac{m!}{(m-g+1)!}\,2^{2g-2}\,(2^{m-g+1}-2)\,B_{m-g+1}. \tag{18.3}$$

Finally, we combine **(18.3)** with **(17.7)** to obtain, for $m+2n+3p = 3g-3$,

$$(\alpha^m\beta^n\gamma^p)[N] = (-1)^{p-g}\,\frac{m!}{(m+p-g+1)!}\,\frac{g!}{(g-p)!}\,2^{2g-2-p}\,(2^{m+p-g+1}-2)\,B_{m+p-g+1}.$$

This agrees with the results published for genus 2 [33] and 3 [37]. For example, the number 224 obtained by Ramanan is just $\alpha^6[\mathcal{N}_3]$. It also easily implies Newstead's second conjecture, that $\beta^g = 0$, since $n > g-1$ implies $m+p-g+1 < 0$. Indeed, we get the more general and unsuspected result that $\beta^{g-q}\gamma^q = 0$ whenever $0 \le q \le g$.

It was promised in the introduction that the determination of the ring structure would be reduced to a problem in linear algebra, and this ought to be explained. To compute the relations in any degree q, we construct a matrix whose columns are indexed by the monomials in α, β, ψ_i of total degree q and whose rows are indexed by similar monomials of complementary total degree $6g-6-q$. The matrix entry at row μ, column ν is given by $\mu\nu[N]$, as determined from the formula above and (17.5). The rank of this matrix is just the qth Betti number of N. Finding the relations at degree q is equivalent to finding a basis for the null space of this matrix. This seems an extremely difficult task, but remarkably, it has recently been carried out by Don Zagier, to give a complete set of explicit relations between Newstead's generators. Indeed, he can show that these relations are precisely equivalent to the "Mumford generators" described in [2].

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