

# Homogenization of the $p$ -Laplacian with nonlinear boundary condition on critical size particles: identifying the strange term for the some non smooth and multivalued operators

J. I. Diaz, D. Gómez - Castro, A.V. Podol'skii, T.A. Shaposhnikova

Papers [2]-[10] were devoted to the study of asymptotic behavior of the solution to the boundary value problem for the  $p$ -Laplacian ( $p \in [2, n)$ ) in  $\varepsilon$ -periodically perforated domain with nonlinear Robin-type boundary condition that contains function  $\sigma(x, u)$ . It was supposed there that  $\sigma(x, u)$  is a smooth function of it's arguments, monotone by variable  $u$ . In this paper we extend the method introduced in [3], [4], [7]-[10] to deal with the problems with more general conditions on the function  $\sigma(x, u)$ . As in all papers in which the holes are critical size and the adsorption parameter has a critical power of  $\varepsilon$  ( we will precise this later) we observe a change in the nature of the nonlinearity. Our aim is to present this change in the case  $\sigma(u) = C|u|^{q-1}u$ ,  $0 < q < 1$ , which is not differentiable at 0, and the maximal monotone operator for the Heaviside function, which is a multivalued operator, and  $p \in [2, n)$ . In a further paper [12] we extend the arguments to the case of general maximal monotone graphs  $\sigma$  and  $p \in (1, n)$ .

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with a smooth boundary  $\partial\Omega$  and let  $Y = (-1/2, 1/2)^n$ . Denote by  $G_0$  the unit ball centered at the origin. For  $\delta > 0$  and  $0 < \varepsilon \ll 1$  we define sets  $\delta B = \{x \mid \delta^{-1}x \in B\}$  and  $\tilde{\Omega}_\varepsilon = \{x \in \Omega \mid \rho(x, \partial\Omega) > 2\varepsilon\}$ . Let  $a_\varepsilon = C_0\varepsilon^\alpha$ , where  $\alpha > 1$  and  $C_0$  is positive number. Define

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where  $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \tilde{\Omega}_\varepsilon \neq \emptyset\}$ ,  $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ ,  $d = \text{const} > 0$ ,  $\mathbb{Z}^n$  is the set of vectors  $z$  with integer coordinates. Define  $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$ , where  $j \in \Upsilon_\varepsilon$  and note that  $\overline{G_\varepsilon^j} \subset \overline{Y_\varepsilon^j}$  and center of the ball  $G_\varepsilon^j$  coincides

with the center of the cube  $Y_\varepsilon^j$ . Define

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma} \sigma_q(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where  $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $p \in [2, n)$ ,  $\partial_{\nu_p} u \equiv |\nabla u|^{p-2} (\nabla u, \nu)$ ,  $\nu$  is the outward unit normal vector to  $S_\varepsilon$ ,  $\gamma = \alpha(p-1)$ . We suppose that  $f \in L^{p'}(\Omega)$ ,  $p' = \frac{p}{p-1}$ .

Function  $\sigma_q(\lambda)$ ,  $q \in [0, 1]$ , is the maximal monotone continuous mapping [11], that depends on parameter  $q$

$$\sigma_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda^q, & \lambda \in (0, 1), \\ 1, & \lambda > 1. \end{cases} \quad (2)$$

We note that  $\sigma_0(\lambda)$  is the maximal monotone mapping for the Heaviside function, i.e. multivalued function

$$\sigma_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ [0, 1], & \lambda = 0, \\ 1, & \lambda > 0, \end{cases} \quad (3)$$

Boundary conditions of this type correspond to the presence of so called chemical reaction of order  $q$  on the boundary of cavities [5], [6]. The motivation to truncate the powers comes from the chemical modeling, in which concentrations impose range in  $[0, 1]$ , but it also corresponds to the case  $f \in L^\infty(\Omega)$ , for which the solution is bounded.

Applying monotonicity tools (see, e.g. [11]) it is easy to see that problem (1) is equivalent to ask for  $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$  satisfying the integral inequality

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\psi_q(\phi) - \psi_q(u_\varepsilon)) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \quad (4)$$

for any arbitrary function  $\phi \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$  where  $\psi_q(\lambda)$  for  $q \in (0, 1]$  is the primitive of  $\sigma_q$ . For  $q \in (0, 1]$  we have that

$$\psi_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \frac{\lambda^{q+1}}{q+1}, & \lambda \in (0, 1), \\ \lambda - \frac{q}{q+1}, & \lambda > 1, \end{cases} \quad (5)$$

and if  $q = 0$  then

$$\psi_0(\lambda) = \begin{cases} 0, & \lambda \leq 0, \\ \lambda, & \lambda > 0. \end{cases} \quad (6)$$

Space  $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$  is the closure in  $W^{1,p}(\Omega_\varepsilon)$  of the set of infinitely differentiable functions in  $\overline{\Omega}_\varepsilon$ , that vanish near the boundary  $\partial\Omega$ .

It is well known that problem (1) has unique weak solution (see., e.g. [1], Theorem 8.5). The following estimation is valid

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^p + \varepsilon^{-\gamma} \|u_\varepsilon\|_{L^1(S_\varepsilon \cap \{x|u_\varepsilon > 1\})} \leq K, \quad (7)$$

where constant  $K$  here and below is independent of  $\varepsilon$ .

Let  $H_q(\lambda)$  be the function given through functional equation

$$B_0 |H_q|^{p-2} H_q \in \sigma_q(\lambda - H_q). \quad (8)$$

where  $B_0 = \text{const} > 0$ . Note that, for any prescribed  $\lambda$  equation (8) has unique solution. In the case if  $q = 0$

$$H_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda, & 0 < \lambda < B_0^{-\frac{1}{p-1}}, \\ B_0^{-\frac{1}{p-1}}, & \lambda > B_0^{-\frac{1}{p-1}}. \end{cases} \quad (9)$$

and if  $q \in (0, 1]$  then

$$H_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ (b_q)^{-1}(\lambda), & 0 < \lambda < 1 + B_0^{-\frac{1}{p-1}}, \\ B_0^{-\frac{1}{p-1}}, & \lambda > 1 + B_0^{-\frac{1}{p-1}}, \end{cases} \quad (10)$$

where  $b_q(s)$  is the strictly monotone function given for  $s \geq 0$  by

$$b_q(s) \equiv s + B_0^{\frac{1}{q}} s^{\frac{p-1}{q}} = \lambda. \quad (11)$$

Note that, in both cases,  $H_q(\lambda)$  is bounded and Lipschitz continuous.

Denote by  $\tilde{u}_\varepsilon$  a  $W^{1,p}$ -extension of function  $u_\varepsilon$  (see. [10]). Using estimate (7) we get following inequality

$$\|\tilde{u}_\varepsilon\|_{W^{1,p}(\Omega)} \leq K. \quad (13)$$

Therefore, there exists a subsequence (denoted as the original sequence), such that  $\varepsilon \rightarrow 0$ :

$$\tilde{u}_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega). \quad (14)$$

The following theorem gives a description of the limit function  $u$ .

**Theorem 1.** *let  $n \geq 3$ ,  $p \in [2, n)$ ,  $q \in [0, 1]$ ,  $\alpha = \frac{n}{n-p}$ ,  $\gamma = \alpha(p-1)$  and  $u_\varepsilon$  is the weak solution of the problem (1). Suppose that  $H_q(\lambda)$  is the function given by equation (8), in which  $B_0 = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{1-p}$ . Then the limit function  $u$ , introduced in (14), is the weak solution of the problem*

$$\begin{cases} -\Delta_p u + A(n, p) |H_q(u)|^{p-2} H_q(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (15)$$

Here  $A(n, p) = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p} \omega_n$ , with  $\omega_n$  the surface area of the unit sphere in  $\mathbb{R}^n$ .

*Proof.* a) Consider the case  $q = 0$ . Denote  $B_1 = B_0^{-\frac{1}{p-1}}$ . Note that the integral inequality in the case when  $q = 0$  has the form

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\phi^+ - u_\varepsilon^+) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \quad (16)$$

where  $\phi^+$  is the positive part of function  $\phi$ ,  $\phi = \phi^+ - \phi^-$ . From (16) we conclude

$$\varepsilon^{-\gamma} \|u_\varepsilon^+\|_{L^1(S_\varepsilon)} \leq K. \quad (17)$$

Using the monotonicity of function  $|\lambda|^{p-2}\lambda$  for  $p > 1$  we derive inequality for  $u_\varepsilon$

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\phi^+ - u_\varepsilon^+) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \quad (18)$$

that is valid for an arbitrary function  $\phi \in W_0^{1,p}(\Omega)$ .

Let us take a test function in inequality (18)

$$\phi(x) = v(x) - W_\varepsilon(x) H_0(v(x)),$$

where  $v \in C_0^\infty(\Omega)$ ,  $H_0(\lambda)$  is given by the formula (9), function  $W_\varepsilon(x)$  is defined as follows

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j \in \Upsilon_\varepsilon} T_\varepsilon^j, \end{cases} \quad (19)$$

Here  $T_\varepsilon^j$  is the ball of radius  $\varepsilon/4$  center of which coincides with the center  $P_\varepsilon^j$  of  $G_\varepsilon^j$ ,

$$w_\varepsilon^j(x) = \frac{|x - P_\varepsilon^j|^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}{a_\varepsilon^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}.$$

Note that

$$W_\varepsilon \rightharpoonup 0 \text{ weakly in } W^{1,p}(\Omega), \quad (20)$$

as  $\varepsilon \rightarrow 0$ . Using that

$$v = H_0(v) \iff v \in [0, B_1],$$

$$v < H_0(v) \iff v < 0,$$

$$v > H_0(v) \iff v > B_1.$$

and  $\phi^+ \Big|_{S_\varepsilon} = (v - H_0(v))^+ = v - H_0(v)$  if  $v > H_0(v)$  we get

$$\varepsilon^{-\gamma} \int_{S_\varepsilon} (\phi^+ - u_\varepsilon^+) ds \leq \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds. \quad (21)$$

Substituting introduced test function in inequality (18) and using (20) and (21), we get that the limit as  $\varepsilon \rightarrow 0$  of the left-hand side of the inequality (18) doesn't exceed the limit of

$$\begin{aligned}
& \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla (v - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \\
& - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_0|^{p-2} H_0 (v - H_0(v) - u_\varepsilon) ds - \\
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_0|^{p-2} H_0 (v - u_\varepsilon) ds
\end{aligned} \tag{22}$$

The limit of the right-hand side of inequality (18) is equal to

$$\int_{\Omega} f(v - u) dx. \tag{23}$$

Consider the integrals over  $S_\varepsilon$ , included in the expression (22):

$$\begin{aligned}
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_0|^{p-2} H_0 (v - H_0(v) - u_\varepsilon) ds + \\
& + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds = \\
& - \varepsilon^{-\gamma} \frac{B_0}{(1 - \alpha_\varepsilon)} \int_{\tilde{S}_\varepsilon} |H_0|^{p-2} H_0 (v - H_0(v) - u_\varepsilon) ds + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \\
& - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds = \\
& = - \varepsilon^{-\gamma} \frac{B_0}{(1 - \alpha_\varepsilon)} \int_{\tilde{S}_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds - \\
& - \varepsilon^{-\gamma} B_0 \int_{\tilde{S}_\varepsilon} |H_0(v(x))|^{p-2} H_0(v) (v - H_0(v) - u_\varepsilon^+) ds -
\end{aligned}$$

$$\begin{aligned}
& -\frac{B_0\varepsilon^{-\gamma}\alpha_\varepsilon}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v)(v - H_0(v) - u_\varepsilon^+) ds + \\
& -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds = \\
& = -\varepsilon^{-\gamma} \frac{B_0}{(1-\alpha_\varepsilon)} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds - \\
& -B_0\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} B_0^{-1}(v - B_1 - u_\varepsilon^+) ds + \varepsilon^{-\gamma} B_0 \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} v^{p-1}(x) u_\varepsilon^+ ds - \\
& -\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v)(v - H_0(v) - u_\varepsilon^+) ds + \\
& +\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds = \\
& = -\varepsilon^{-\gamma} \frac{B_0}{(1-\alpha_\varepsilon)} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds - \\
& -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ (1 - B_0 v^{p-1}(x)) ds - \\
& -\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v)(v - H_0(v) - u_\varepsilon^+) ds = \tag{24} \\
& = J^\varepsilon - \frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v)(v - H_0(v) - u_\varepsilon^+) ds,
\end{aligned}$$

where

$$\begin{aligned}
J^\varepsilon & \equiv -\varepsilon^{-\gamma} \frac{B_0}{(1-\alpha_\varepsilon)} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds - \\
& -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ (1 - B_0 v^{p-1}(x)) ds \leq 0, \tag{25}
\end{aligned}$$

and  $\alpha_\varepsilon \rightarrow 0$  if  $\varepsilon \rightarrow 0$ .

Using that  $\varepsilon^{-\gamma} \|u_\varepsilon^+\|_{L^1(S_\varepsilon)} \leq K$ , we get that the limit of the expression (22) doesn't exceed the limit of the following expression

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx - \sum_{j \in \Upsilon_{\varepsilon} \partial T_{\varepsilon}^j} \int \partial_{\nu_p} w_{\varepsilon}^j |H_0(v)|^{p-2} H_0(v) (v - u_{\varepsilon}) ds. \quad (26)$$

Using equality proved in [3] we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sum_{j \in \Upsilon_{\varepsilon} \partial T_{\varepsilon}^j} \int \partial_{\nu_p} w_{\varepsilon}^j |H_0(v)|^{p-2} H_0(v - u_{\varepsilon}) ds = \\ = -A(n, p) \int_{\Omega} |H_0(v)|^{p-2} H_0(v) (v - u) ds, \end{aligned} \quad (27)$$

It follows from (21)-(27) that  $u$  satisfies inequality, for any  $v \in W_0^{1,p}(\Omega)$ ,

$$\begin{aligned} \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx + A(n, p) \int_{\Omega} |H_0(v)|^{p-2} H_0(v) (v - u) dx \geq \\ \geq \int_{\Omega} f(v - u) dx. \end{aligned} \quad (28)$$

Taking  $v = u + \lambda w$ , with  $w \in W_0^{1,p}(\Omega)$  arbitrary, and making  $\lambda \rightarrow 0$ , since  $H_0$  is Lipschits continuous and bounded, we get that  $u$  is a weak solution of problem (15) for  $q = 0$  in the usual sense.

b) Now we consider the case  $q \in (0, 1]$ . In this case we make similar reasoning. We set  $\phi = v - W_{\varepsilon} H_q(v)$  in inequality (18) as a test function, where  $H_q(\lambda)$  is defined by (10). Further, we only need to explain the method of the comparison of the integrals over  $S_{\varepsilon}$ , included in the obtained variational inequality. Note that in this case variational inequality has the form

$$\int_{\Omega_{\varepsilon}} |\nabla \phi|^{p-2} \nabla \phi (\nabla \phi - \nabla u_{\varepsilon}) dx + \varepsilon^{-\gamma} \int_{S_{\varepsilon}} (\psi_q(\phi) - \psi_q(u_{\varepsilon})) ds \geq \int_{\Omega_{\varepsilon}} f(\phi - u_{\varepsilon}) dx, \quad (29)$$

where

$$\varepsilon^{-\gamma} \int_{S_{\varepsilon}} (\psi_q(\phi) - \psi_q(u_{\varepsilon})) ds \leq$$



$$\begin{aligned}
&\leq \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon \in (0, 1)\}} \left( \frac{(v - H(v))^{q+1}}{q+1} - \frac{u_\varepsilon^{q+1}}{q+1} \right) ds + \\
&\quad + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon \leq 0\}} \frac{(v - H(v))^{q+1}}{q+1} ds + \\
&\quad + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon > 1\}} \left( \frac{(v - H)^{q+1}}{q+1} - u_\varepsilon + \frac{q}{q+1} \right) ds + \\
&\quad + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1+B_1\} \cap \{u_\varepsilon \in (0, 1)\}} \left( v - H - \frac{q}{q+1} - \frac{u_\varepsilon^{q+1}}{q+1} \right) ds + \\
&\quad \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1+B_1\} \cap \{u_\varepsilon \leq 0\}} \left( v - H - \frac{q}{q+1} \right) ds + \\
&\quad + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1+B_1\} \cap \{u_\varepsilon > 1\}} \left( v - H - \frac{q}{q+1} - u_\varepsilon + \frac{q}{q+1} \right) ds.
\end{aligned} \tag{30}$$

We substitute the test function in (18) and we consider the remaining integrals over  $S_\varepsilon$  in the left-hand side of variational inequality (18):

$$\begin{aligned}
& - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_q|^{p-2} H_q (v - H_q(v) - u_\varepsilon) ds = \\
& = -\varepsilon^{-\gamma} \int_{S_\varepsilon} B_0 H_q^{p-1}(v) (v - H_q(v) - u_\varepsilon^+) ds - \\
& - \frac{B_0 \varepsilon^{-\gamma} \alpha_\varepsilon}{1 - \alpha_\varepsilon} \int_{S_\varepsilon} H_q^{p-1}(v) (v - H_q(v) - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \frac{B_0}{1 - \alpha_\varepsilon} \int_{S_\varepsilon} H^{p-1} u_\varepsilon^- ds. \tag{31}
\end{aligned}$$

Note that

$$\begin{aligned}
& -\varepsilon^{-\gamma} \int_{S_\varepsilon} B_0 H_q^{p-1}(v) (v - H_q(v) - u_\varepsilon^+) ds = \\
& = -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon \in (0, 1)\}} (v - H_q(v))^q (v - H_q(v) - u_\varepsilon) ds - \\
& - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon > 1\}} (v - H_q(v))^q (v - H_q(v) - u_\varepsilon) ds -
\end{aligned}$$

$$\begin{aligned}
& -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon \leq 0\}} (v - H_q(v))^{q+1} ds - \\
& -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1+B_1\} \cap \{u_\varepsilon \leq 0\}} (v - B_1) ds - \\
& -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1+B_1\} \cap \{u_\varepsilon \in (0, 1)\}} (v - B_1 - u_\varepsilon) ds - \\
& -\varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1+B_1\} \cap \{u_\varepsilon > 1\}} (v - B_1 - u_\varepsilon) ds. \tag{32}
\end{aligned}$$

Next we compare integrals over the same subsets of  $S_\varepsilon$  in the left-hand side of inequality (18). We have  $I_\varepsilon \equiv$

$$\begin{aligned}
& \varepsilon^{-\gamma} \int_{M_\varepsilon} \left\{ \frac{(v - H_q(v))^{q+1}}{q+1} - \frac{u_\varepsilon^{q+1}}{q+1} - (v - H_q(v))^{q+1} + (v - H_q(v))^q u_\varepsilon \right\} ds = \\
& = -\varepsilon^{-\gamma} \int_{M_\varepsilon} \left( \frac{q(v - H_q(v))^{q+1}}{q+1} + \frac{u_\varepsilon^{q+1}}{q+1} - u_\varepsilon (v - H_q(v))^q \right) ds. \tag{33}
\end{aligned}$$

where  $M_\varepsilon = S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon \in (0, 1)\}$ .

Using Young's inequality we get

$$u_\varepsilon (v - H_q(v))^q \leq \frac{q(v - H_q(v))^{q+1}}{q+1} + \frac{u_\varepsilon^{q+1}}{q+1}. \tag{34}$$

Therefore,

$$I_\varepsilon \leq 0. \tag{34}$$

The remaining integrals over subsets of  $S_\varepsilon$  are considered in the similar way and we establish that the sum of all integrals over the corresponding subsets of  $S_\varepsilon$  is non-positive. Therefore, the limit function  $u$  satisfy variational inequality

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx + A(n, p) \int_{\Omega} H_q^{p-1}(v) (v - u) dx \geq \int_{\Omega} f(v - u),$$

for an arbitrary function  $v \in W_0^{1,p}(\Omega)$ . Again taking  $v = u + \lambda w$ , with  $w \in W_0^{1,p}(\Omega)$  arbitrary, and making  $\lambda \rightarrow 0$  we obtain that  $u$  is a weak solution in the usual sense.

□

The research of the first two authors was partially supported as members of the Research Group MOMAT (Ref. 910480) of the UCM. The research of J.I.Díaz was partially supported by the project ref. MTM 2014-57113 of the DGISPI (Spain). The research of D.Gómez-Castro was supported by a FPU Grant from the Ministerio de Educación, Cultura y Deporte (Spain). The results of this paper were started during a visit of the last author to the UCM, on November 2015. This author wants to thank this support as well as the received hospitality from the Instituto de Matemática Interdisciplinar of the UCM.

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