

COMPATIBILITY OF t -STRUCTURES FOR QUANTUM SYMPLECTIC RESOLUTIONS

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ABSTRACT. Let W be a smooth complex variety with the action of a connected reductive group G . Adapting the stratification approach of Teleman [T] to a microlocal context, we prove a vanishing theorem for the functor of G -invariant sections—i.e., of quantum Hamiltonian reduction—for G -equivariant twisted \mathcal{D} -modules on W . As a consequence, when W is affine we establish a sufficient combinatorial condition for exactness of the global sections functors of microlocalization theory. When combined with the derived equivalence results of [McN], this gives precise criteria for “microlocalization of representation categories” in the spirit of [GS1, GS2, Ho, KR, DK, MVdB, BKu, McN, BPW].

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. Introduction. Many noncommutative algebras of intense recent interest are naturally realized via quantum Hamiltonian reduction from the ring of differential operators on a smooth complex affine variety W with the action of a complex Lie group G . Examples include spherical subalgebras of deformed preprojective algebras [Ho], of cyclotomic Cherednik algebras [Go, Ob], and more generally of wreath product symplectic reflection algebras [EGGO, Lo2]. Localizing, or rather microlocalizing, this construction, one sees that these algebras are in fact the global sections of certain of sheaves of algebras. Suitably microlocalized categories of equivariant or twisted-equivariant \mathcal{D} -modules then yield module categories for these sheaves which are also of great interest: they provide natural categorifications of representations of quantum groups (c.f. [Zh, LiA, LiB, We]).

Quantum Hamiltonian reduction depends naturally on a parameter: namely, a character c of the Lie algebra \mathfrak{g} of the group G . Under a precise, effectively computable combinatorial condition on c , we prove a vanishing theorem for the global sections functor that appears in the microlocalization theory of quantum Hamiltonian reductions.

Precise statements of our main results and their consequences appear in Section 1.2 below. However, the rough form of the main results is as follows. Suppose W is a smooth, connected complex variety (smooth, connected, separated scheme of finite type over \mathbf{C}) with an action of a connected reductive group G . Let $\chi : G \rightarrow \mathbb{G}_m$ be a group character. In the situations that interest us, the G -action on W and choice of χ determine a finite set KN of 1-parameter subgroups of a fixed maximal torus $\mathbf{T} \subset G$: these are the *Kirwan-Ness 1-parameter subgroups*. An algorithm for computing KN when W is a representation of G is explained in Section 4.5. Roughly speaking, to each β we associate a numerical shift $\text{shift}(\beta)$, defined precisely below, and a subset $I(\beta) \subseteq \mathbb{Z}_{\geq 0}$.

Rough Version of Vanishing Theorem. *Suppose that, for each $\beta \in \text{KN}$,*

$$c(\beta) \notin \text{shift}(\beta) + I(\beta) \subseteq \text{shift}(\beta) + \mathbb{Z}_{\geq 0}.$$

Then any c -twisted, G -equivariant \mathcal{D} -module with unstable singular support is in the kernel of quantum Hamiltonian reduction.

As a consequence, we establish effective, sufficient criteria for t -exactness of direct image functors in microlocalization theory à la [KR, McN, BPW], or indeed in any reasonable technical framework

for localization results in characteristic 0. Namely, if W is affine, $\mu : T^*W \rightarrow \mathfrak{g}^*$ is the moment map for the G -action, μ is flat, and the GIT quotient $\mathfrak{X} = \mu^{-1}(0) //_{\chi} G$ is smooth, there are various technical frameworks to produce a natural quantization $\mathcal{W}_{\mathfrak{X}}(c)$ of $\mathcal{O}_{\mathfrak{X}}$ depending on c and a “reasonable” category of quasicoherent $\mathcal{W}_{\mathfrak{X}}(c)$ -modules.¹ We prove:

Rough Version of Exactness Theorem. *Suppose that, for each $\beta \in \text{KN}$,*

$$c(\beta) \notin \text{shift}(\beta) + I(\beta) \subseteq \text{shift}(\beta) + \mathbb{Z}_{\geq 0}.$$

The the functor of global sections on quasicoherent $\mathcal{W}_{\mathfrak{X}}(c)$ -modules is exact.

Our results thus provide a far-reaching analogue of the exactness part of the seminal Beilinson-Bernstein localization theorem in geometric representation theory, both extending and complementing important precursors [BKu, GGS, GS1, GS2, Ho, KR, MVdB]. A crucial point is the effectiveness of the combinatorics involved, which provides us with precise guarantees for such exactness results to hold. We illustrate this effectiveness with a quick and easy derivation of an exactness assertion for the quantization of the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ yielding the spherical type A Cherednik algebra.

1.2. Precise Statement of Results. More precisely, suppose W is a smooth, connected complex algebraic variety (cf. Convention 2.1), equipped with the action of a connected complex reductive group G . A substantial menagerie of interesting examples already arises when W is a representation of G ; for example, W could be the representation space of a quiver (of a fixed dimension vector) and G the natural automorphism group. We assume that the canonical line bundle K_W of W is trivialized and is thereby G -equivariantly isomorphic to the twist of \mathcal{O}_W by a character $\gamma_G : G \rightarrow \mathbb{G}_m$ —as we explain in Section 7.2, this is not a significant restriction. Write $\rho = \frac{1}{2}d\gamma_G$ and write \mathcal{D}_W for the sheaf of differential operators on W , and $\mathcal{D}(W)$ for the algebra of global differential operators.

Let $\mathfrak{g} = \text{Lie}(G)$. Associated to the G -action there are an *infinitesimal \mathfrak{g} -action* encoded by a map $\mathfrak{g} \rightarrow \mathcal{D}(W)$, $\mathfrak{g} \ni X \mapsto \tilde{X}$, and a *canonical quantum comoment map* $\mu^{\text{can}} : \mathfrak{g} \rightarrow \mathcal{D}(W)$, $\mu^{\text{can}}(X) = \tilde{X} + \rho(X)$. Passing to the associated graded and dualizing yields a classical moment map $\mu : T^*W \rightarrow \mathfrak{g}^*$. Next, fix a character $c : \mathfrak{g} \rightarrow \mathbb{C}$, or equivalently a linear map $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \rightarrow \mathbb{C}$. Associated to c there is a category of *(canonically) c -twisted G -equivariant \mathcal{D}_W -modules*: these are \mathcal{D}_W -modules M equipped with a G -action whose derivative equals the action of \mathfrak{g} via $\mu_c^{\text{can}} := (\mu^{\text{can}} + c) : \mathfrak{g} \rightarrow \mathcal{D}(W)$. See Section 2.6 for more. The category of such modules is denoted $(\mathcal{D}, G, c) - \text{mod}$.

The *quantum Hamiltonian reduction* of \mathcal{D}_W at c is the algebra

$$U_c := H^0(\mathcal{D}_W / \mathcal{D}_W \mu_c^{\text{can}}(\mathfrak{g}))^G.$$

When W is affine this can be written $U_c = (\mathcal{D}(W) / \mathcal{D}(W) \mu_c^{\text{can}}(\mathfrak{g}))^G$. Letting $\mathcal{M}_c = \mathcal{D}_W / \mathcal{D}_W \mu_c^{\text{can}}(\mathfrak{g})$, the *quantum Hamiltonian reduction functor* is

$$(\mathcal{D}, G, c) - \text{mod} \longrightarrow U_c - \text{mod}, \quad M \mapsto \mathbb{H}(M) := \text{Hom}_{(\mathcal{D}, G, c)}(\mathcal{M}_c, M).$$

When W is affine, then, writing $M_c = \mathcal{D}(W) / \mathcal{D}(W) \mu_c^{\text{can}}(\mathfrak{g})$ and $\Gamma(\mathcal{M})$ for the global sections of \mathcal{D} -module \mathcal{M} , the functor \mathbb{H} is equivalently given by

$$\mathcal{M} \mapsto \mathbb{H}(\mathcal{M}) = \text{Hom}_{(\mathcal{D}, G, c)}(M_c, \Gamma(\mathcal{M})) \cong \Gamma(\mathcal{M})^G.$$

The main result of the paper is a characterization of a part of the kernel of this functor. This has strong implications for compatibility of t -structures in “microlocalization theory” for the algebra U_c .

To state our results, recall, following the exposition of [Kir] and terminology of [T], the notion of a *Kirwan-Ness* (or KN) stratification of the unstable locus of T^*W . The classical construction of such a stratification, as in [Kir], depends on a choice of G -equivariant line bundle \mathcal{L} on T^*W . The most interesting examples for us arise when we choose the trivial line bundle with G -action twisted

¹We note that the “quotient category” framework of [McN] works well even when \mathfrak{X} is not smooth.

by a group character $\chi : G \rightarrow \mathbb{G}_m$. As we explain in detail in Section 4 (where we also make precise our sign conventions for the unstable locus associated to χ), such a stratification of T^*W *always* exists when W is affine, and the χ -unstable locus $T^*W^{\chi-\text{uns}}$ of T^*W thus obtained agrees with that defined more concretely for affine varieties in [Kin]. For more general W , we will *assume* T^*W is equipped with a stratification satisfying properties, axiomatized in Definition 4.2, that are shared by the stratifications of [Kir]. This assumption is satisfied, in practice, in situations to which one wants to apply results like ours (besides affine W , see the discussion of $\text{Bun}_G(C)$ below).

As in [Kir, T], the KN stratification decomposes $T^*W^{\chi-\text{uns}}$ into a finite disjoint union $\coprod_{\alpha \in \text{KN}} S_\alpha$ of simpler pieces, each labelled by a pair $\alpha = (\beta, i)$ where β is a 1-parameter subgroup of a fixed maximal torus $\mathbf{T} \subseteq G$ and i labels a component $Z_{\beta, i}$ of the fixed point set $Z_\beta = (T^*W)^{\beta(\mathbb{G}_m)}$. We give an explicit, constructive recipe for computing the list of 1-parameter subgroups β in Section 4 for the KN stratification associated to a G -character when W is a vector space—in which case the Z_β are all connected. We carry out the calculation relevant to the type A spherical Cherednik algebra in Section 8. In fact we may (by Lemma 7.1) restrict attention to a subset of the strata with labels

$$(1.1) \quad \text{KN}^\circ = \{\alpha \in \text{KN} \mid S_\alpha \cap \mu^{-1}(0) \text{ contains a nonempty coisotropic subset}\}.$$

To each stratum S_α ($\alpha = (\beta, i)$) as above we associate three things. The first is the sum of all the negative β -weights on \mathfrak{g} , which we denote by $\text{wt}_{\mathbf{n}^-}(\beta)$ (since the corresponding weight spaces span a nilpotent Lie subalgebra \mathfrak{n}^-). To define the second, choose $z \in Z_{\beta, i}$. Then \mathbb{G}_m acts via β on the normal space $N_{Z_{\beta, i}/T^*W}(z)$ and we write $\text{abs-wt}_{N_{Z_{\beta, i}/T^*W}}(\beta)$ to mean the sum of absolute values of β -weights on the normal space; it does not depend on the choice of $z \in Z_{\beta, i}$. Third, \mathbb{G}_m acts on the normal space $N_{S_\alpha/T^*W}(z)$ to the stratum S_α at $z \in Z_{\beta, i}$, and we let $I_{G, T^*W}(\beta, i)$ denote the set of absolute values of weights of β on the symmetric algebra $\text{Sym}^\bullet(N_{S_\alpha/T^*W}(z))$ (on which those β -weights are in fact always negative)—it is thus again a set of non-negative integers (and similarly does not depend on $z \in Z_{\beta, i}$).

Choosing a filtration of \mathcal{D}_W or, when W is affine, $\mathcal{D}(W)$ yields a notion of the *singular support* $SS(M) \subset T^*W$ of a \mathcal{D} -module M . For a general W one only knows how to define the *operator filtration*, with functions in degree zero and vector fields in degree 1, but in special cases one knows many more: for example, if W is a G -representation, any linear \mathbb{G}_m -action on T^*W commuting with the G -action, scaling the symplectic form with positive weight, and having all weights on T^*W non-positive determines one (Section 2.4). Fix one of these filtrations.

As above, fix a choice of character $\chi : G \rightarrow \mathbb{G}_m$. Write $\mathcal{M}_c(\chi^\ell) = \mathcal{M}_{c-\ell d\chi} \otimes \chi^\ell$ and, when W is affine, $M_c(\chi^\ell) = M_{c-\ell d\chi} \otimes \chi^\ell$; as in Formula (4.2) of [McN], this is also a c -twisted G -equivariant \mathcal{D} -module. Assume further that we have a KN stratification of T^*W to which χ is adapted (see Definition 4.4) and let KN° be the subset of stratum labels as in (1.1).

Theorem 1.1 (Theorem 7.4). *Let W be a smooth complex algebraic variety. Fix a character $c : \mathfrak{g} \rightarrow \mathbb{C}$. Suppose that for every $(\beta, i) \in \text{KN}^\circ$ we have*

$$(1.2) \quad c(\beta) \notin \left(I_{G, T^*W}(\beta, i) + \text{wt}_{\mathbf{n}^-}(\beta) + \frac{1}{4} \text{abs-wt}_{N_{Z_{\beta, i}/T^*W}}(\beta) \right).$$

Then:

- (i) *If M is any object of $(\mathcal{D}, G, c) - \text{mod}$ with $SS(M) \subseteq (T^*W)^{\chi-\text{uns}}$, then*

$$\text{Hom}_{(\mathcal{D}, G, c)}(\mathcal{M}_c, M) = 0.$$

Suppose that, in addition, W is affine. Then:

- (ii) *For every $\ell \ll 0$, there is a finite-dimensional vector subspace*

$$V_\ell \subset \text{Hom}_{(\mathcal{D}, G, c)}(M_c(\chi^\ell), M_c)$$

for which the natural composite evaluation map

$$M_c(\chi^\ell) \otimes V_\ell \longrightarrow M_c(\chi^\ell) \otimes \mathrm{Hom}_{(\mathcal{D}, G, c)}(M_c(\chi^\ell), M_c) \longrightarrow M_c$$

is a split surjective homomorphism of objects of $(\mathcal{D}, G, c) - \mathrm{mod}$.

When W is a representation of G —as noted above, each Z_β is connected, and thus KN strata are labelled simply by 1-parameter subgroups β —one has a more direct combinatorial statement:

Theorem 1.2 (Theorem 7.2). *Suppose W is a representation of G . Let $\alpha_1, \dots, \alpha_d$ be the weights of the maximal torus $\mathbb{T} \subseteq G$ on a basis w_1, \dots, w_d of W . Suppose that for every $\beta \in \mathrm{KN}^\circ$ we have*

$$c(\beta) \notin \left(I_{G, T^*W}(\beta) + \mathrm{wt}_{\mathfrak{n}^-}(\beta) + \frac{1}{2} \sum_{i=1}^d |\alpha_i \bullet \beta| \right)$$

(where \bullet denotes the pairing between $\mathrm{Lie}(\mathbb{T})$ and its dual). Then conclusions (i) and (ii) of Theorem 1.1 hold.

Part (i) of Theorem 1.1 says that unstably-microsupported twisted-equivariant \mathcal{D} -modules are in the kernel of the “ G -invariant global sections,” or quantum Hamiltonian reduction, functor, except for a precise collection of values of the twist c . Note that working microlocally rather than merely locally seems essential (cf. Example 3.10) and is predicted by extensive prior work (cf. [KR]).

Part (ii) of Theorem 1.1 is a more technical-sounding assertion, but we state it explicitly here both because such statements appear in the proof of part (i) and because it provides a convenient input for “exactness of global sections” statements in microlocalization theory as in [KR] or [McN]. Condition (1.2) in Theorem 1.1 is thus the analogue of “dominant” in Beilinson-Bernstein localization. Our arguments take advantage of the flexibility of microlocal frameworks when we use DQ-modules for slices through the KN strata. In Sections 7.3 and 7.4 we make such exactness statements precise in two such frameworks, the deformation quantization approach used in [KR] and the quotient category approach of [McN]. The slogan is as follows (all undefined terms are from [KR]).

Corollary 1.3 (Theorems 7.5, 7.7). *Suppose the condition on c of Theorem 1.1 is satisfied. Suppose $\mathfrak{X} = \mu^{-1}(0) //_{\chi} G$ is a smooth Hamiltonian reduction via a GIT quotient at the character χ of G ; let $\mathcal{W}(c)$ denote the sheaf of deformation quantization algebras on \mathfrak{X} constructed by quantum Hamiltonian reduction. Then the global sections functor for good \mathbb{G}_m -equivariant $\mathcal{W}(c)$ -modules is exact, i.e., right exact for the standard t -structures.*

The same statement then follows for objects of the ind-category of good \mathbb{G}_m -equivariant $\mathcal{W}(c)$ -modules—this is the “correct” notion of quasicoherent $\mathcal{W}(c)$ -module for geometric representation theory. Theorem 1.1 similarly yields an analogue of the corollary in any other natural framework.

As an application, we quickly prove (a slightly weakened form of) the exactness part of the microlocalization of [KR] for type A spherical Cherednik algebras in Section 8; since the derived equivalence part of [KR] was handled in [McN], this completes a new approach to that problem. Similarly, calculating KN 1-parameter subgroups and applying Theorem 1.1 to the result, one expects an exactness theorem for microlocalization of spherical cyclotomic Cherednik algebras that complements the derived equivalence established in [McN], thus yielding an abelian microlocalization theory for those algebras. Progress in this direction has been achieved by Rollo Jenkins and, separately, Chunyi Li (works in preparation). A different approach to this question, relying on the theory of Procesi bundles and highest weight categories, has been carried out by Losev [Lo3].

In a different direction, one can immediately proceed from our results for varieties to similar results for algebraic stacks. We plan to return to this subject elsewhere, so for the moment we only briefly sketch it. Suppose that \mathfrak{X} is a smooth algebraic stack that is exhausted by Zariski-open substacks of the form W/G where each W is a smooth algebraic variety and G is a reductive group. Assume furthermore that $T^*\mathfrak{X}$ comes equipped with a stratification that induces a KN stratification on each

T^*W in an appropriate sense. Our theorem then implies a corresponding vanishing statement for twisted \mathcal{D} -modules on \mathfrak{X} . For example, fix a smooth projective curve C and let $\mathrm{Bun}_{\mathbb{G}}(C)$ denote the moduli stack of principal \mathbb{G} -bundles on C for a reductive group scheme \mathbb{G} over C . Let \det denote the determinant line bundle.

Corollary 1.4. *For all but countably many values of c , if $M \in D_{\mathrm{coh}}(\mathcal{D}_{\mathrm{Bun}}(\det^{\otimes c}))$ has unstable microsupport in $T^*\mathrm{Bun}_{\mathbb{G}}(C)$ then $\mathrm{Hom}(\mathcal{D}_{\mathrm{Bun}}(\det^{\otimes c}), M) = 0$.*

We will treat this and related matters in a forthcoming paper [McN3].

1.3. Methods. The main inspiration for Theorem 1.1 is the elegant proof by Teleman [T] that “quantization commutes with reduction.” Teleman’s proof uses the KN stratification to reduce to a simple analysis of weights for the β -action along S_{β} . It was understood clearly by Ian Grojnowski, Kobi Kremnizer, and possibly many others long ago that Teleman’s approach should be used to prove a result like Theorem 1.1. It was equally clear that the proof cannot reduce simply to weight-space calculations as in [T], since Theorem 1.1 depends crucially on the parameter c and nothing similar is true in the classical limit.

The new ingredient beyond [T] is provided by Kashiwara’s Equivalence, applied in a more flexible symplectic setting. Section 3 and in particular Corollary 3.8 summarize the basic tool we use to prove vanishing for \mathcal{D} -modules microsupported on a Y_{α} . We then adapt this tool to prove vanishing along a KN stratum S_{α} ($\alpha = (\beta, i)$) via a slice argument (Sections 5 and 6) to reduce from the full KN stratum to its “Morse-theoretic core” Y_{α}^{ss} , the locus that attracts to the α -component of the fixed locus Z_{α}^{ss} under the downward β -flow. Although there is a rich and beautiful theory of symplectic slices and symplectic normal forms with a long history (from [GS] to the recent achievements of [Kn, Lo1]), in the case we need—a slice for a free action of a unipotent group—it is easiest to work by hand. Alternatively, it may be possible to simplify the proof even further using the techniques of [BDMN].

The details of the symplectic geometry and its quantization are carried out in Sections 5 and 6, based on tools from Section 2 and a model case, deduced from Kashiwara’s Equivalence, in Section 3. Section 4 lays out basics of KN strata and an algorithm for computing the KN 1-parameter subgroups. Section 7 proves the main theorems, and Section 8 applies it all to type A spherical Cherednik algebras.

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2. PRELIMINARIES

In this section we lay out some preliminary conventions and facts.

Convention 2.1. “Smooth variety” will mean a smooth, separated scheme of finite type over \mathbb{C} .

Other Basic Conventions. Throughout the paper, all varieties are connected (the ground field is always \mathbb{C}). Groups G are assumed to be connected and reductive; T will always denote a torus, typically a maximal torus in an ambient reductive group G . Group actions are assumed effective.

2.1. Group Actions. Suppose a group G acts on a smooth variety E . For $f \in \mathbf{C}[E]$, $g \in G$, we let $(g \cdot f)(x) = f(g^{-1}x)$. Given a character $\chi : G \rightarrow \mathbb{G}_m$, we make the trivial line bundle $L = E \times \mathbb{A}^1$ into a G -equivariant line bundle via $g \cdot (x, z) = (g \cdot x, \chi(g)z)$. Recall that a function $f : E \rightarrow \mathbb{A}^1$ is a *relative invariant* or *semi-invariant* of weight χ if $f(g \cdot x) = \chi(g)f(x)$ for all $g \in G$ and $x \in E$. Suppose that $F : E \rightarrow L$ is a section, and write $F(x) = (x, f(x))$ for a function $f : E \rightarrow \mathbb{A}^1$. Then $g \cdot F(x) = (gx, \chi(g)f(x))$, and so F is G -equivariant if and only if f is χ -semi-invariant.

Lemma 2.2. *Suppose E is a smooth variety with G -action and $\chi : G \rightarrow \mathbb{G}_m$ is a character. Then a function $f \in \mathbf{C}[E]$ is χ^q -semi-invariant if and only if f is in the χ^{-q} -isotypic component of $\mathbf{C}[E]$.*

2.2. Differential Operators. Suppose an algebraic group G acts (on the left) rationally on the smooth affine variety W . Let $\mathcal{D}(W)$ denote the algebra of differential operators on W . For $f \in \mathbf{C}[W]$, $\theta \in \mathcal{D}(W)$, and $g \in G$, we let $(g \cdot \theta)(f) = g \cdot (\theta(g^{-1} \cdot f))$. Differentiating the G -action (Section 2.1) on $\mathbf{C}[W]$ gives a Lie algebra homomorphism

$$(2.1) \quad \mathfrak{g} = \text{Lie}(G) \rightarrow \Gamma(T_W) \subset \mathcal{D}(W), \quad X \mapsto \tilde{X},$$

the *infinitesimal G - (or \mathfrak{g} -) action*.

If W is a finite-dimensional G -representation, then differentiating the homomorphism $G \rightarrow \text{Aut}(W)$ yields a Lie algebra homomorphism $\text{act} : \mathfrak{g} \rightarrow \text{End}(W) = W \otimes W^*$. Writing $\tau : W \otimes W^* \rightarrow W^* \otimes W$, $\tau(w \otimes v) = v \otimes w$, for the canonical braiding, the infinitesimal \mathfrak{g} -action on $\mathbf{C}[W] = \text{Sym}(W^*)$ is induced by

$$(2.2) \quad \text{act}^* : \mathfrak{g} \rightarrow \text{End}(W^*) = W^* \otimes W, \quad \text{act}^*(X) = -\tau(\text{act}(X)).$$

Composing with the canonical map $m : W^* \otimes W \rightarrow \mathcal{D}(W)$, we get $\tilde{X} = m(\text{act}^*(X))$.

In particular, if $G = \mathbb{G}_m$ and $W = \bigoplus W_k$ with W_k the k -weight space, then for $x \in (W_k)^* \subset \mathbf{C}[W]$ we get $\lambda \cdot x = \lambda^{-k}x$. Let $\mathfrak{t} = \text{Lie}(\mathbb{G}_m)$. Then \mathfrak{t} acts infinitesimally on W as follows. If v_1, \dots, v_n is a basis of W consisting of weight vectors and \mathbb{G}_m acts on v_i with weight $w(v_i)$, then writing $x_i = v_i^*$,

$$(2.3) \quad \mathbf{C} = \mathfrak{t} \ni \mathbf{1} \mapsto \tilde{\mathbf{1}} = \sum -w(v_i)x_i\partial_{x_i}.$$

Suppose that a reductive group G acts on a vector space W . Make a choice of isomorphism $W = \mathbf{C}^N$ under which the maximal torus $\mathbf{T} \subseteq G$ acts by diagonal matrices, and let $\mathbf{T}^{md} = \mathbb{G}_m^N$ with the canonical action on W (here the notation for \mathbf{T}^{md} is meant to convey “maximal dimensional”). We write ψ_1, \dots, ψ_N for the corresponding characters of \mathbf{T} on W and $\alpha_i = d\psi_i$. As we will do later in Section 2.4, for any subgroup K of G let $\gamma_K : K \rightarrow \mathbb{G}_m$ denote the character of the K -action on

$\bigwedge^N W^* \cong \mathbf{C}$, and let $\rho_K = \frac{1}{2}d\gamma_K$: thus, if $K = \mathbf{T}$, $\rho_{\mathbf{T}} = -\frac{1}{2} \sum_{i=1}^N \alpha_i \in \mathfrak{t}^*$. Define

$$(2.4) \quad \mu_K^{\text{can}}(X) = \tilde{X} + \rho_K(X) \in \mathcal{D}(W) \quad \text{for } X \in \mathfrak{k} = \text{Lie}(K);$$

this is the *canonical quantum comoment map* for $\mathcal{D}(W)$. In terms of (2.2),

$$(2.5) \quad \mu_K^{\text{can}}(X) = m(\text{act}^*(X)) + \frac{1}{2} \text{tr}(\text{act}^*(X)).$$

When $K = G$ we omit the subscript on μ^{can} . If $\beta : \mathbb{G}_m \rightarrow \mathbf{T} \subseteq G$ is a 1-parameter subgroup of \mathbf{T} and $e_\beta = \tilde{\mathbf{1}}$ as in (2.3) then

$$(2.6) \quad \mu^{\text{can}}(d\beta(\mathbf{1})) = e_\beta - \frac{1}{2} \sum_{i=1}^N \alpha_i \bullet \beta =: \text{eu}(\beta)$$

(cf. also (3.4)). For a Lie algebra character $c : \mathfrak{g} \rightarrow \mathbf{C}$, we write $\mu_c^{\text{can}} = \mu^{\text{can}} + c$.

More generally, suppose W is any smooth variety with trivialized canonical bundle $K_W = W \times \mathbf{C}$. Suppose $\gamma_K : K \rightarrow \mathbb{G}_m$ is a character such that $k \cdot (w, c) = (k \cdot w, \gamma_K(k)c)$ for all $k \in K$, $w \in W$, and $c \in \mathbf{C}$. As above, we define $\rho_K = \frac{1}{2}d\gamma_K$ and μ^{can} as in (2.4).

Remark 2.3. Suppose that $f : W \rightarrow V$ is any K -equivariant étale morphism of smooth varieties. Then there is a pullback morphism $f^* : \mathcal{D}(V) \rightarrow \mathcal{D}(W)$ on differential operators, and $f^* \circ \mu^{\text{can}} = \mu^{\text{can}}$.

Remark 2.4. If W is any smooth variety and \mathcal{L} is a line bundle on W , then one can define, for any $c \in \mathbf{C}$, a sheaf of twisted differential operators $\mathcal{D}_W(\mathcal{L}^{\otimes c})$. (In the case $c \in \mathbb{Z}$ this is just $\mathcal{L}^c \otimes \mathcal{D}_W \otimes \mathcal{L}^{-c}$.) All of the above also holds, suitably interpreted, for sheaves of twisted differential operators.

2.3. Deformation Quantizations. Our main theorems, at least in the setting of cotangent bundles, can be stated purely in terms of \mathcal{D} -modules and their microlocalizations. However, the nature of our proof requires us to also pass the more general setting of deformation quantizations. We briefly review what we will need of this theory in Sections 2.3 to 2.5. A reader interested only in the statements of our results could thus safely omit these subsections. An excellent general reference for deformation quantization (or DQ) algebras is [KS].

If E is a smooth affine variety with Poisson structure $\{\bullet, \bullet\}$, a DQ algebra structure is an associative, \hbar -linear product $*$ on $\mathbf{C}[E][[\hbar]]$ such that

$$(2.7) \quad f * g = fg + \frac{\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2).$$

We will write $\mathcal{O}^{\hbar}(E)$ for $(\mathbf{C}[E][[\hbar]], *)$ when $*$ is understood from context.

2.3.1. Recall, more generally, that if E is a smooth affine algebraic variety then there is (cf. [Ye1]) a canonical “Kontsevich quantization,” i.e., a bijection between formal Poisson structures $\{-, -\} = \sum_{i \geq 1} \hbar^i \{-, -\}_i$ on E modulo gauge equivalence and deformation quantizations modulo gauge equivalence. Moreover, this bijection:

- (1) preserves first-order terms, i.e., satisfies (2.7) for identified structures;
- (2) is compatible with pullback by étale morphisms;
- (3) associates to the formal Poisson structure $\hbar\{-, -\}$ on \mathbb{A}^{2n} , where $\{-, -\}$ is the Poisson bracket associated to any constant (i.e., translation-invariant) bivector field, the Moyal-Weyl product.

We elaborate on (3) in Section 2.3.5 below.

2.3.2. Suppose we equip the variety E with a \mathbb{G}_m -action for which the Poisson structure $\{-, -\}$ has weight ℓ : that is, $m_z^*\{-, -\} = z^\ell\{-, -\}$ where m_z denotes action by $z \in \mathbb{G}_m$. Then, letting $m_z(\hbar) = z^{-\ell}\hbar$, any \mathbb{G}_m -invariant formal Poisson structure defines a \mathbb{G}_m -equivariant deformation quantization (or deformation quantization “with F -structure”) as described in [KR, Section 2.3]. We say $*$ is \mathbb{G}_m -equivariant with weight ℓ .

2.3.3. Suppose the algebraic group G acts on E preserving a symplectic form ω . A (classical) moment map for the action is a G -equivariant map $\mu_G : E \rightarrow \mathfrak{g}^*$ satisfying, for every $X \in \mathfrak{g}$, $d\langle \mu_G, X \rangle = i_{\tilde{X}}\omega$. The corresponding classical comoment map is the pullback on functions, $\mu^* : \text{Sym}(\mathfrak{g}) \rightarrow \mathbf{C}[E]$.

Suppose the \mathbb{G}_m -equivariant (with weight ℓ) DQ algebra $\mathcal{O}^{\hbar}(E)$ is G -equivariant, i.e., the product $*$ is also G -equivariant. Differentiating defines a Lie algebra homomorphism $\alpha : \mathfrak{g} \rightarrow \text{End}_{\mathbf{C}[[\hbar]]}(\mathcal{O}^{\hbar}(E))$.

Definition 2.5. A quantum comoment map for the action is a G -equivariant linear map $\mu : \mathfrak{g} \rightarrow \mathcal{O}^{\hbar}(E)$ satisfying:

- (1) $[\mu(X), -] = \hbar \cdot \alpha(X)$.
- (2) For every $X \in \mathfrak{g}$, $\mu(X)$ has \mathbb{G}_m -weight ℓ .

Note that this implies that, modulo \hbar , the quantum comoment map becomes the classical one.

2.3.4. The *Liouville 1-form* θ on the cotangent bundle $E = T^*W$ is given as follows: if $\pi : T^*W \rightarrow W$ denotes projection, then for a tangent vector $X \in T_\xi(T^*W)$, $\theta(X) = \langle d\pi(X), \xi \rangle = \xi(d\pi(X))$. This yields a canonical symplectic form $\omega_{T^*W} = d\theta$. Note the lack of a sign change. If W is an affine space with coordinates x_1, \dots, x_n and dual cotangent fiber coordinates y_1, \dots, y_n then $\theta = \sum_i y_i dx_i$ and hence $\omega_{T^*W} = -\sum dx_i \wedge dy_i$. For this form, one has the Poisson bracket of coordinate functions $\{x_i, y_j\} = -\delta_{ij}$.

Lemma 2.6. *The Liouville 1-form θ on T^*W is invariant under $\text{Aut}(W)$.*

Proof. This is immediate from the definition of θ or, via an étale coordinate chart, from the explicit formula for θ in affine space. \square

Given a G -action on W and the induced G -action on T^*W , differentiating the actions gives homomorphisms $\mathfrak{g} \rightarrow \Gamma(T_{T^*W})$ and $\mathfrak{g} \rightarrow \Gamma(T_W)$; we abusively write $X \mapsto \tilde{X}$ for both. Note that for any $\xi \in T^*W$, since π is equivariant we have

$$(2.8) \quad d\pi(\tilde{X}_\xi) = \tilde{X}_{\pi(\xi)}.$$

A cotangent bundle has a *canonical classical moment map* μ , given by dualizing the linear map $\mathfrak{g} \rightarrow \Gamma(T_W) \subset \mathbf{C}[T^*W]$, $X \mapsto \tilde{X}$.

Lemma 2.7. *For $\xi \in T^*W$, $X \in \mathfrak{g}$ we have $\langle \mu(\xi), X \rangle = \theta(\tilde{X}_\xi)$.*

Proof. Immediate from the definitions of μ and θ via (2.8). \square

2.3.5. We elaborate on fact (3) from Section 2.3.1. On an affine space $E = \mathbb{A}^{2n}$ with translation-invariant symplectic form ω defining a Poisson structure $\{-, -\}$, one has the *Moyal-Weyl product*, defined by the formula:

$$(2.9) \quad f * g = m \circ e^{\frac{\hbar}{2}\{-, -\}}(f \otimes g).$$

This means we view $\{-, -\}$ as a bivector field, $\{-, -\} = \sum_{i,j} \pi_{i,j} \partial_i \wedge \partial_j$ for scalars $\pi_{i,j}$, exponentiate as a bidifferential operator with \hbar coefficients, apply the result to $f \otimes g$, and multiply in $\mathbf{C}[\mathbb{A}^{2n}]$. The Moyal-Weyl product makes $\mathbf{C}[\mathbb{A}^{2n}][\hbar]$ into an associative algebra with unit, flat over $\mathbf{C}[[\hbar]]$, whose truncation mod \hbar is $\mathbf{C}[\mathbb{A}^{2n}]$ and for which $f * g - g * f = \hbar\{f, g\} + \mathcal{O}(\hbar^2)$. The Moyal-Weyl product is compatible with localization of functions: it makes $\mathbf{C}[\mathbb{A}^{2n}][\hbar]$ the global sections of a sheaf of algebras on \mathbb{A}^{2n} , and if $E \subset \mathbb{A}^{2n}$ is an affine open subset the restriction of the Moyal product to $\mathbf{C}[E][\hbar]$ is defined by the same formula (2.9) for elements $f, g \in \mathbf{C}[E]$.

2.4. **Filtrations on \mathcal{D} .** We discuss filtrations of rings of differential operators.

First assume W is a vector space with linear G -action. Fix a linear \mathbb{G}_m -action on $E = T^*W$, commuting with the G -action, that acts with weight $\ell < 0$ on the canonical Poisson structure and with nonpositive weights on E (so nonnegative weights on $\mathbf{C}[E]$). We will call a fixed choice of such \mathbb{G}_m -action a *contracting \mathbb{G}_m -action*. A \mathbb{G}_m -stable subset for this action is called *conical*.

Such a \mathbb{G}_m -action defines a grading on the space $E^* \subseteq \mathbf{C}[E]$ of linear functions on $E = T^*W$. The Weyl algebra $\mathcal{D}(W)$ is defined by

$$\mathcal{D}(W) = T^\bullet(E^*) / \langle x \otimes y - y \otimes x - \{x, y\} \mid x, y \in E^* \rangle$$

The sign of the weight ℓ of the Poisson structure guarantees that, if we give E^* and its tensor algebra the increasing filtration by \mathbb{G}_m -weight, the algebra $\mathcal{D}(W)$ inherits a nonnegative filtration, and its associated graded comes equipped with an isomorphism to $\mathbf{C}[E]$ as graded algebras. Standard filtrations on $\mathcal{D}(W)$ are obtained this way, including the Bernstein filtration (in which $E^* \subset \mathbf{C}[E]$ has weight one) and the operator filtrations (in which one of W^* or W has weight one and the other has weight zero).

If W is an arbitrary smooth, connected variety, we equip $E = T^*W$ with the \mathbb{G}_m -action that contracts cotangent fibers and acts trivially on W , and refer to this as the contracting action. It corresponds to the filtration by order of differential operators on \mathcal{D}_W .

Given a coherent \mathcal{D}_W -module M , by taking a good filtration compatible with the chosen filtration on \mathcal{D}_W and considering the support of the associated graded module as a sheaf on T^*W one obtains its singular support $SS(M)$, a conical coisotropic subvariety of T^*W . For an arbitrary module N we let $SS(N)$ be the union of $SS(M)$ as M runs over its coherent submodules.

2.5. From \mathcal{D} to \mathbf{DQ} . In Sections 2.5.1, 2.5.2, and 2.5.3, we assume E is a symplectic vector space.

2.5.1. There is a close relationship between the Moyal product, when ω is the standard symplectic form (Section 2.3.4) on $E = \mathbb{A}^{2n}$, and the n th Weyl algebra $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$. Namely, replace the Weyl algebra by its homogenized cousin, defined by

$$\mathcal{D}_{\hbar} = \mathbf{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle[[\hbar]] / ([x_i, x_j], [y_i, y_j], [y_i, x_j] - \delta_{ij}\hbar).$$

The subalgebra \mathcal{D}_{\hbar} consists of expressions that are polynomial in \hbar . With these relations, one has the usual identification at $\hbar = 1$ with the algebra of differential operators via $y_i \leftrightarrow \frac{\partial}{\partial x_i}$.

The *symmetrization map* $\mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n] \xrightarrow{\text{Symm}} \mathcal{D}_{\hbar}$ is defined on a monomial $a_1 \cdots a_k$ in the generators x_i, y_i of the polynomial ring by

$$\text{Symm}(a_1 \cdots a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$$

(where the tensor product really means the image of that element in \mathcal{D}_{\hbar}).

Lemma 2.8. *The symmetrization map Symm , extended linearly to \hbar , intertwines the Moyal $*$ -product on $\mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n][[\hbar]]$ with the product on \mathcal{D}_{\hbar} .*

Note that

$$(2.10) \quad \text{Symm}(x_i y_i) = \frac{1}{2} y_i x_i + \frac{1}{2} x_i y_i = x_i y_i + \frac{\hbar}{2},$$

which corresponds to $x_i \frac{\partial}{\partial x_i} + \frac{\hbar}{2}$ under the usual identification of y_i with $\frac{\partial}{\partial x_i}$. More generally, for an element $X \in W^* \otimes W$, letting

$$W^* \otimes W \xrightarrow{m} \mathbf{C}[W \oplus W^*], \quad W^* \otimes W \xrightarrow{\overline{m}} \mathcal{D}(W)$$

denote the multiplication maps, we get

$$(2.11) \quad \text{Symm}(m(X)) = \frac{1}{2}(\overline{m}(X) + \overline{m}(\sigma(X))) = \overline{m}(X) + \frac{\hbar}{2} \text{tr}(X).$$

2.5.2. We continue with the assumptions of Section 2.4. Write $E^* = \oplus_{\alpha} E_{\alpha}^*$ where E_{α}^* is the \mathbb{G}_m -weight α subspace of E^* . Then the algebra

$$(2.12) \quad T^{\bullet}(\oplus_{\alpha} E_{\alpha}^* t^{\alpha})[t] / ([ut^r, vt^s] - \{u, v\}t^{r+s} \mid u \in E_r^*, v \in E_s^*)$$

is isomorphic to $\mathcal{D}_{\hbar}[\hbar^{-1/\ell}]$ (note $\ell < 0$ so we are inverting a positive, fractional, power of \hbar) via

$$ut^{\text{wt}(u)} \mapsto u \quad (\text{for } \mathbb{G}_m\text{-homogeneous } u), \quad t \mapsto \hbar^{-1/\ell}.$$

2.5.3. As in Section 2.4, fix a contracting \mathbb{G}_m -action; filter $\mathcal{D}(W)$ by \mathbb{G}_m -weight. We get a corresponding Rees algebra $\mathcal{R}(\mathcal{D}) = \oplus F_k(\mathcal{D})t^k \subset \mathcal{D}[t]$. We can map the tensor algebra $T^\bullet(\oplus_\alpha \mathbf{E}_\alpha^* t^\alpha)[t]$ to $\mathcal{R}(\mathcal{D}) \subset \mathcal{D}[t]$ in the obvious way; this map is surjective and factors, via the presentation (2.12) of the algebra $\mathcal{D}_\hbar[\hbar^{-1/\ell}]$, through an isomorphism:

$$(2.13) \quad \mathcal{R}(\mathcal{D}) \cong \mathcal{D}_\hbar[\hbar^{-1/\ell}].$$

It follows that $\mathcal{D}_\hbar[\hbar^{-1/\ell}]$ is the \hbar -completion of $\mathcal{R}(\mathcal{D})$. (2.13) is a graded isomorphism if we grade the right-hand side by \mathbb{G}_m -weight.

Convention 2.9. Henceforth, throughout the remainder of the paper:

- (1) We write \mathcal{D}_\hbar and $\mathcal{O}^\hbar(\mathbf{E})$ to mean $\mathcal{D}_\hbar[\hbar^{-1/\ell}]$ and $\mathcal{O}^\hbar(\mathbf{E})[\hbar^{-1/\ell}]$, respectively.
- (2) We use part (1) and the isomorphism (2.13) to identify $\mathcal{R}(\mathcal{D})$ with a $\mathbf{C}[\hbar]$ -subalgebra of \mathcal{D}_\hbar .

The following is immediate from Formulas (2.11) and (2.5):

Lemma 2.10. *For any character $c : \mathfrak{g} \rightarrow \mathbf{C}$, the map $\mathcal{R}(\mathcal{D}) \hookrightarrow \mathcal{D}_\hbar \xrightarrow{\text{Symm}^{-1}} \mathcal{O}^\hbar(T^*W)$ identifies the twisted canonical comoment map $(\mu^{\text{can}} + c) \in F_\ell(\mathcal{R}(\mathcal{D}))$ with a quantum comoment map for $\mathcal{O}^\hbar(T^*W)$. The latter equals the twist $\mu_c^{\text{can}} := \mu + \hbar c$ of the image in $\mathcal{O}^\hbar(T^*W)$ of the canonical classical moment map $\mu \in \mathbf{C}[T^*W]$ under the inclusion $\mathbf{C}[T^*W] \hookrightarrow \mathcal{O}^\hbar(T^*W)$.*

We can thus define a canonical quantum comoment map $\mu^{\text{can}} : \mathfrak{g} \rightarrow \mathcal{O}^\hbar(T^*W)$ as the composite of the canonical classical comoment map $\mathfrak{g} \xrightarrow{\mu} \mathbf{C}[T^*W]$ followed by the inclusion $\mathbf{C}[T^*W] \hookrightarrow \mathcal{O}^\hbar(T^*W)$; by Lemma 2.10, this agrees with the canonical comoment map to $\mathcal{D}(W)$ under the natural algebra homomorphisms.

2.5.4. *From \mathcal{D} to DQ for Smooth Varieties.* Finally, we assume W is an arbitrary smooth variety, equip T^*W with the scaling action of \mathbb{G}_m , and equip \mathcal{D}_W with the operator filtration. There is then a canonical choice of deformation quantization $\mathcal{O}_{T^*W}^\hbar$ of the sheaf of functions on T^*W , defined as a completion of the algebra of $K^{1/2}$ -twisted differential operators. Letting $p : T^*W \rightarrow W$ denote projection, the deformation quantization comes equipped with a homomorphism $p^{-1}\mathcal{D}_W \rightarrow \mathcal{O}_{T^*W}^\hbar$. If $W^\circ \subseteq W$ is an open set with an étale morphism $q : W^\circ \rightarrow W'$, then q determines a “wrong-way” étale morphism $dq : T^*W^\circ \rightarrow T^*W'$. The pullback $dq^{-1}\mathcal{O}_{T^*W'}^\hbar \rightarrow \mathcal{O}_{T^*W^\circ}^\hbar$ is an isomorphism of sheaves of associative $\mathbf{C}[[\hbar]]$ -algebras. The algebra $\mathcal{O}_{T^*W}^\hbar$ comes equipped with a canonical splitting of sheaves of vector spaces $\mathcal{O}_{T^*W} \rightarrow \mathcal{O}_{T^*W}^\hbar$; this splitting is compatible with the map dq induced by an étale morphism of varieties $q : W \rightarrow W'$. When $W^\circ \subset W$ is affine and $q : W^\circ \rightarrow \mathbb{A}^n$ is étale, then the isomorphism $dq^{-1}\mathcal{O}_{T^*\mathbb{A}^n}^\hbar \rightarrow \mathcal{O}_{T^*W^\circ}^\hbar$ intertwines the quantization of T^*W with the Moyal-Weyl product on $T^*\mathbb{A}^n$. See [Ye2] for more discussion. If, in addition, W has trivialized canonical bundle $K_W = W \times \mathbf{C}$ via which G acts by the character γ_G , then the analogue of Lemma 2.10 holds for $p^{-1}\mathcal{D}_W \rightarrow \mathcal{O}_{T^*W}^\hbar$. Slightly abusively, we will write \mathcal{D}_\hbar for the Rees algebra of the deformation quantization algebra $\mathcal{O}_{T^*W}^\hbar$ in the above setting.

2.6. Equivariant Modules. A weakly \mathbb{K} -equivariant \mathcal{D}_W -module is a \mathcal{D}_W -module M with a rational \mathbb{K} -action such that $g \cdot (\theta m) = (g \cdot \theta)(g \cdot m)$, for all $m \in M, g \in \mathbb{K}, \theta \in \mathcal{D}_W$. The category of such modules is denoted $(\mathcal{D}_W, \mathbb{K}) - \text{mod}$, or if W is affine, $(\mathcal{D}(W), \mathbb{K}) - \text{mod}$. One similarly defines weakly equivariant $\mathcal{O}^\hbar(\mathbf{E})$ -modules. For notational simplicity, we will assume in this subsection that W is affine, though all statements generalize appropriately to non-affine W .

Returning to a reductive G acting on W and a weakly equivariant $\mathcal{D}(W)$ -module M , let $\alpha : \mathfrak{g} \rightarrow \text{End}_{\mathbf{C}}(M)$ denote the infinitesimal action. Given a Lie algebra homomorphism $c : \mathfrak{g} \rightarrow \mathbf{C}$, let

$$\gamma_{M,c} = \alpha - (\mu^{\text{can}} + c) : \mathfrak{g} \rightarrow \text{End}_{\mathbf{C}}(M).$$

The module M is (G, c) -equivariant if $\gamma_{M,c} = 0$. The category of such modules is $(\mathcal{D}, G, c) - \text{mod}$.

We define $\Phi_c(M) = M / (\sum_{z \in \mathfrak{g}} \gamma_{M,c}(z)M)$ for weakly G -equivariant M : this yields a (G, c) -equivariant module (in either category) and Φ_c is a left adjoint to the forgetful functor. We write $M_c(\rho) = \Phi_c(\mathcal{D} \otimes \rho)$ for characters $\rho : G \rightarrow \mathbb{G}_m$, and in particular $M_c = M_c(\text{triv}) = \mathcal{D} / \mathcal{D}\mu_c^{\text{can}}(\mathfrak{g})$; this is compatible with the notation $M_c(\chi^\ell)$ of the introduction by Formula (4.2) of [McN]. The functor Φ_c of course depends on the group G , and if we want to emphasize the group we write $M_c^G(\rho)$. For a review of the basic properties of twisted equivariant and weakly equivariant \mathcal{D} -modules and these functors, see for example [McN, §4], and for a more detailed account [Ka1].

The functor $\text{Hom}_{(\mathcal{D}, G)}(M_c, -)$ is the functor of *quantum Hamiltonian reduction*. Basic properties are discussed (in notation consistent with the present paper) in [McN]. In particular, if $M \in (\mathcal{D}, G, c) - \text{mod}$, then $\text{Hom}_{(\mathcal{D}, G)}(M_c, M) = M^G$.

2.7. Induction to DQ Modules. In light of the isomorphism (2.13), we get a functor

$$\mathcal{R}(\mathcal{D}) - \text{mod} \longrightarrow \mathcal{D}_{\hbar} - \text{mod}, \quad N \mapsto \mathcal{D}_{\hbar} \otimes_{\mathcal{R}(\mathcal{D})} N =: N^{\hbar}.$$

In particular, a finitely generated \mathcal{D} -module M with a choice of good filtration yields a graded Rees module that can naturally be completed to a finitely generated \mathcal{D}_{\hbar} -module, and hence yields a module for the Moyal-Weyl algebra (or when W is not a representation, a sheaf of modules for the sheaf of DQ algebras) which we will denote by $\mathcal{R}(M)^{\hbar}$. The homomorphism $\mathcal{D}_{\hbar} \rightarrow \mathcal{D}_{\hbar}[\hbar^{-1}]$ and isomorphism (2.13) induce a composite functor

$$\mathcal{R}(\mathcal{D}) - \text{gr-mod} \longrightarrow \mathcal{D}_{\hbar}[\hbar^{-1}] - \text{mod}.$$

Recalling Convention 2.9, we also have the usual functors

$$\mathcal{D} - \text{mod} \xrightarrow{\mathcal{C}[\hbar, \hbar^{-1}] \otimes -} \mathcal{D}[\hbar, \hbar^{-1}] - \text{gr-mod} \leftarrow \mathcal{R}(\mathcal{D}) - \text{gr-mod},$$

where the second functor comes via the identification $\mathcal{R}(\mathcal{D})[\hbar^{-1}] = \mathcal{D}[\hbar, \hbar^{-1}]$.

Lemma 2.11. *If M is a finitely generated \mathcal{D} -module equipped with a choice of good filtration, the images of M and $\mathcal{R}(M)$ in $\mathcal{D}[\hbar, \hbar^{-1}] - \text{gr-mod}$ are isomorphic.*

Recall the notion of *support* of a finitely generated $\mathcal{O}_{\mathbb{E}}^{\hbar}[\hbar^{-1}]$ -module, where $\mathcal{O}_{\mathbb{E}}^{\hbar}$ is a DQ algebra. If M is a finitely generated $\mathcal{O}_{\mathbb{E}}^{\hbar}[\hbar^{-1}]$ -module, we choose a finitely generated $\mathcal{O}_{\mathbb{E}}^{\hbar}$ -submodule $M(0) \subset M$ with the property that $M(0)[\hbar^{-1}] = M$; such a submodule is called a *lattice*. Then, by definition, $\text{supp}(M) = \text{supp}(M(0)/\hbar M(0))$, where the latter means the set-theoretic support of the finitely generated $\mathcal{O}_{\mathbb{E}}$ -module $M(0)/\hbar M(0)$. By standard arguments, this notion does not depend on the choice of lattice [Ka, Proposition 2.0.5].

Proposition 2.12.

(1) *We get a commutative diagram:*

$$\begin{array}{ccc} \mathcal{R}(\mathcal{D}) - \text{gr-mod} & \longrightarrow & \mathcal{D}_{\hbar} - \text{mod} \\ \downarrow & & \downarrow \\ \mathcal{D}[\hbar, \hbar^{-1}] - \text{gr-mod} & \longrightarrow & \mathcal{D}_{\hbar}[\hbar^{-1}] - \text{mod}. \end{array}$$

- (2) *If W is a smooth variety with G -action, then all functors are compatible with the G -actions (i.e. induce a commutative diagram for weakly G -equivariant modules).*
- (3) *If M is a \mathcal{D} -module equipped with good filtration, then its (singular) support, calculated in any of the above categories, is the same.*

3. KASHIWARA EQUIVALENCE AND EQUIVARIANT MODULES

In this section we study \mathcal{D} -modules on a \mathbf{T} -representation W (where \mathbf{T} is a torus). Fix a contracting \mathbb{G}_m -action on T^*W commuting with the \mathbf{T} -action. If $\mathfrak{t} = \text{Lie } \mathbf{T}$, let \bullet denote the natural pairing $\mathfrak{t}^* \times \mathfrak{t} \rightarrow \mathbb{C}$ (and for the corresponding pairing for \mathbf{T}^{md}). Note that the derivative map embeds the characters $X(\mathbf{T})$ and 1-parameter subgroups $Y(\mathbf{T})$ into \mathfrak{t}^* and \mathfrak{t} respectively, and hence we pair them using \bullet with elements of \mathfrak{t} and \mathfrak{t}^* respectively.

3.1. Torus Weights. Suppose \mathbf{T}^{md} is a torus in $GL(W)$ of dimension $\dim(W)$ commuting with \mathbf{T} and the contracting \mathbb{G}_m -action. The action of \mathbf{T} on W may then be viewed as a homomorphism $\rho: \mathbf{T} \rightarrow \mathbf{T}^{md}$. Choose a \mathbf{T}^{md} -weight basis e_1, \dots, e_d of W (thus $d = \dim(W)$) and let x_i denote the corresponding linear functions and ∂_i the corresponding partial derivatives. Then the monomial $x^I \partial^J$ has \mathbf{T}^{md} -weight $J - I$ (in multi-index notation). In particular, $\mathcal{D}(W)^{\mathbf{T}^{md}} = \mathbb{C}[x_1 \partial_1, \dots, x_d \partial_d]$.

Lemma 3.1. *Let $\mathbf{a} = (a_1, \dots, a_d)$ be a weight of \mathbf{T}^{md} . Then the \mathbf{a} -weight subspace of $\mathcal{D}(W)$ is $\mathcal{D}(W)^{\mathbf{a}} = \psi \cdot \mathbb{C}[x_1 \partial_1, \dots, x_d \partial_d]$ where $\psi = \prod_{a_i < 0} x_i^{-a_i} \prod_{a_i \geq 0} \partial_i^{a_i}$.*

3.2. Equivariant Kashiwara Equivalence. Suppose $V \subset W$ is a \mathbf{T} -invariant subspace. Let x_1, \dots, x_k be linearly independent weight vectors in $W^* \subset \mathbb{C}[W]$ such that $V = W(x_1, \dots, x_k)$ (i.e. such that $V = \{v \mid x_i(v) = 0, i = 1, \dots, k\}$). Write $\partial_i = \partial/\partial x_i$. We let \mathbf{T} act on $\mathbb{C}[\partial_1, \dots, \partial_k]$ in the natural way (via the identification with a subring of $\text{Sym}(W)$). We can also extend the natural free, rank one $\mathbb{C}[\partial_1, \dots, \partial_k]$ -module structure to make $\mathbb{C}[\partial_1, \dots, \partial_k]$ into a $\mathbb{C}[x_1, \dots, x_k, \partial_1, \dots, \partial_k]$ -module for which $x_i \partial_j^\alpha = -\alpha \delta_{ij} \partial_j^{\alpha-1}$.

Proposition 3.2 (Kashiwara's Equivalence). *Let M be a weakly \mathbf{T} -equivariant $\mathcal{D}(W)$ -module. If M is supported on $V \subset W$, then M is \mathbf{T} -equivariantly isomorphic to $\mathbb{C}[\partial_1, \dots, \partial_k] \otimes_{\mathbb{C}} M'$ where M' is a weakly \mathbf{T} -equivariant $\mathcal{D}(V)$ -module.*

3.3. Decomposition With Respect to a 1-Parameter Subgroup. Suppose now we have a 1-parameter subgroup $\beta: \mathbb{G}_m \rightarrow \mathbf{T}$ of \mathbf{T} (and hence, if we assume the action of \mathbf{T} is effective, a 1-parameter subgroup of \mathbf{T}^{md}). Write $W_+ = W_+(\beta)$, respectively $W_0 = W_0(\beta)$, respectively $W_- = W_-(\beta)$ for the sum of positive, respectively zero, respectively negative weight subspaces of W under β . We then have an identification:

$$T^*W = (T^*W)_+ \times (T^*W)_0 \times (T^*W)_- = (W_+ \times (W_-)^*) \times T^*W_0 \times (W_- \times (W_+)^*)$$

where $W_+ \times (W_-)^*$, respectively T^*W_0 , respectively $W_- \times (W_+)^*$, is the positive, respectively zero, respectively negative weight subspace of T^*W .

Note that each of the subspaces W_\pm, W_0 (respectively $(W_\pm)^*, W_0^*$) is a direct sum of eigenlines $\mathbb{C}e_i$ (respectively $\mathbb{C}e_i^*$ where $\{e_i^*\}_{i=1}^d$ is the dual basis). Choose \mathbf{T}^{md} -weight coordinates x_1, \dots, x_k on W_- , and y_1, \dots, y_j on W_+ . The torus \mathbf{T} acts on W via a list of characters $\alpha_i, i = 1, \dots, j+k+\ell$, where $i = 1, \dots, j$ correspond to W_+ , $i = j+1, \dots, j+\ell$ correspond to W_0 , and $i = j+\ell+1, \dots, j+\ell+k$ correspond to W_- (thus we are *not* assuming that the α_i are distinct). We write

$$(3.1) \quad I(\beta) = \left\{ \sum_i n_i |\alpha_i \bullet \beta| \mid n_i \geq 0 \right\}$$

for the set of $\mathbb{Z}_{\geq 0}$ -linear combinations of the $|\alpha_i \bullet \beta|$ (cf. Remark 4.17).

3.4. Partial Fourier Transform. Suppose $W = W_1 \times W_2$ is a \mathbf{T} -invariant direct sum decomposition of W . The partial Fourier transform is an isomorphism

$$\Psi: \mathcal{D}(W) = \mathcal{D}(W_1 \times W_2) \rightarrow \mathcal{D}(W_1^* \times W_2).$$

Since $\mathcal{D}(W) = \mathcal{D}(W_1) \otimes \mathcal{D}(W_2)$ it is enough define Ψ on $\mathcal{D}(W_1)$. Taking coordinates y_1, \dots, y_j for W_1 and z_1, \dots, z_j the corresponding dual coordinates on W_1^* , the isomorphism Ψ is given by: $\Psi(\partial_{y_i}) = z_i$

and $\Psi(y_i) = -\partial_{z_i}$. Using this isomorphism we get an equivalence between $\mathcal{D}(W_1 \times W_2) - \text{mod}$ and $\mathcal{D}(W_1^* \times W_2) - \text{mod}$. Note:

$$(3.2) \quad \Psi\left(y_i \partial_{y_i} + \frac{1}{2}\right) = -\left(z_i \partial_{z_i} + \frac{1}{2}\right).$$

Remark 3.3. It would be more invariant to work with a subspace $W_1 < W$ and produce a \mathcal{D} -module on the conormal bundle $T_{W_1}^*(W) \cong W_1^* \times (W/W_1)$. This is the microlocalization functor μ_{W_1} of Kashiwara-Schapira defined for any submanifold Y of a manifold X . Since we only need the special case of subspace of a vector space, we have chosen to keep to a more hands-on approach.

3.5. Application of Kashiwara to Twisted Equivariant Modules. The infinitesimal action of $\mathbf{1} \in \text{Lie}(\mathbb{G}_m) = \mathbb{C}$ associated to the action of \mathbb{G}_m via the homomorphism β is given by the Euler operator

$$(3.3) \quad \tilde{\mathbf{1}} = e(\beta) = -\sum_{i=1}^j w_i y_i \partial_{y_i} + \sum_{i=1}^k u_i x_i \partial_{x_i} \in \mathcal{D} = \mathcal{D}(W), \quad \text{where}$$

$w_i = \alpha_i \bullet \beta$ for $i = 1, \dots, j$ and $u_i = -\alpha_{j+\ell+i} \bullet \beta$ for $i = 1, \dots, k$, and each $w_i > 0, u_i > 0$.² If $W = W_1 \oplus W_2$ as above, let $W' = W_1^* \oplus W_2$. Write μ_W^{can} and $\mu_{W'}^{\text{can}}$ for the canonical quantum comoment maps for the \mathbb{T} -actions on W and W' respectively. Then we have

$$(3.4) \quad \mu_W^{\text{can}}(d\beta(\mathbf{1})) = \text{eu}(\beta) = -\sum_{i=1}^j w_i \left(y_i \partial_{y_i} + \frac{1}{2}\right) + \sum_{i=1}^k u_i \left(x_i \partial_{x_i} + \frac{1}{2}\right) \quad \text{and}$$

$$(3.5) \quad \mu_{W'}^{\text{can}}(d\beta(\mathbf{1})) = \text{eu}'(\beta) = \sum_{i=1}^j w_i \left(z_i \partial_{z_i} + \frac{1}{2}\right) + \sum_{i=1}^k u_i \left(x_i \partial_{x_i} + \frac{1}{2}\right);$$

these are the *half-density-shifted Euler vector fields*. Under the isomorphism

$$\mathcal{D}(W) = \mathcal{D}(W_+) \otimes \mathcal{D}(W_0) \otimes \mathcal{D}(W_-) \cong \mathcal{D}((W_+)^*) \otimes \mathcal{D}(W_0) \otimes \mathcal{D}(W_-) = \mathcal{D}(W')$$

as above, we see from (3.2) that $\text{eu}(\beta)$ gets identified with $\text{eu}'(\beta)$.

Next, given a Lie algebra character $c : \mathfrak{t} \rightarrow \mathbb{C}$ (i.e. linear homomorphism) of $\mathfrak{t} = \text{Lie}(\mathbb{T})$, we write:

$$(3.6) \quad M_c^\beta = \mathcal{D}/\mathcal{D}(\text{eu}(\beta) + c \bullet \beta).$$

Under the partial Fourier transform above, our calculations show that M_c^β gets identified with $\mathcal{D}'/\mathcal{D}'(\text{eu}'(\beta) + c \bullet \beta)$ where we write $\mathcal{D}' = \mathcal{D}((W_+)^* \times W_0 \times W_-)$ to emphasize which coordinates are the base coordinates.

Lemma 3.4. *The \mathbb{T}^{md} -action on $\mathcal{D}(W)$ induces a weakly \mathbb{T}^{md} -equivariant structure on M_c^β .*

Proposition 3.5. *Define M_c^β as in (3.6) and $I(\beta)$ as in (3.1). Suppose M is a weakly \mathbb{G}_m -equivariant module and $M_c^\beta \xrightarrow{\phi} M$ is a weakly \mathbb{G}_m -equivariant (via β) \mathcal{D} -module homomorphism. Suppose that $\phi(\mathbf{1})$ is supported set-theoretically on $W_+ \times W_0 \times W_0^* \times (W_-)^*$ in the sense that, for each x_i and $\partial/\partial y_i$ (notation as in Section 3.3), there is an $N \gg 0$ such that $x_i^N \cdot \phi(\mathbf{1}) = 0$ (respectively, such that $(\partial/\partial y_i)^N \cdot \phi(\mathbf{1}) = 0$). Then $\phi = 0$ if*

$$(3.7) \quad c \bullet \beta \notin \left(I(\beta) + \frac{1}{2} \sum_{i=1}^d |\alpha_i \bullet \beta| \right).$$

²The signs are consistent since β acts with positive weights on W_+ , hence negative weights on its coordinate functions y_i , whereas β acts with negative weights on W_- , hence positive weights on its coordinate functions x_i .

Remark 3.6. Both sides of (3.7) are homogeneous of degree 1 in β . It follows that the condition on c does not depend on β , up to positive rational number multiples, and hence the statement of the proposition remains true if we allow β to be a *rational* 1-parameter subgroup of \mathbb{T} .

Proof. Given a \mathbb{G}_m -equivariant ϕ as in the statement of the proposition, let $m = \phi(1)$. Write $c = c \bullet \beta$. Then $m \in M^{\mathbb{G}_m}$ and $(\text{eu}(\beta) + c) \cdot m = 0$ in M . So it suffices to prove that any such element of M is zero. Suppose that m is such an element. Using the natural inclusion $\mathcal{D}(W_+ \times W_-) \subseteq \mathcal{D}(W_+ \times W_0 \times W_-)$, we may form $\widetilde{M} = \mathcal{D}(W_+ \times W_-) \cdot m \subseteq M$; it is a $\mathcal{D}(W_+ \times W_-)$ -submodule, and $m \in \widetilde{M}^{\mathbb{G}_m}$. Noting that $\text{eu}(\beta) \in \mathcal{D}(W_+ \times W_-)$ and replacing M by \widetilde{M} , we may assume $W_0 = \{0\}$. Now, applying a partial Fourier transform as discussed above, we get $\text{supp}(M) \subseteq \{0\} \subset (W_+)^* \times W_-$ as a $\mathcal{D}' = \mathcal{D}((W_+)^* \times W_-)$ -module.

Proposition 3.2 now tells us that $M \cong \mathbb{C}[\partial_{x_1}, \dots, \partial_{x_k}, \partial_{z_1}, \dots, \partial_{z_j}] \otimes_{\mathbb{C}} M'$ where M' is a representation of \mathbb{G}_m . Recall that, in $\mathbb{C}[\partial_x]$, one has $x \cdot \partial_x^\ell = (-\ell) \partial_x^{\ell-1}$. Consequently, $x \partial_x \cdot \partial_x^a = (-a-1) \partial_x^a$ and thus, writing $\partial^a = \prod \partial_{x_i}^{a_i} \prod \partial_{z_i}^{a'_i}$, we get

$$\begin{aligned} \text{eu}'(\beta) \cdot \partial^a &= \sum_{i=1}^j w_i z_i \partial_{z_i} \partial^a + \sum_{i=1}^k u_i x_i \partial_{x_i} \partial^a + \frac{1}{2} \left(\sum_{i=1}^j w_i + \sum_{i=1}^k u_i \right) \partial^a \\ &= - \left[\sum_{i=1}^j w_i (a'_i + 1) + \sum_{i=1}^k u_i (a_i + 1) - \frac{1}{2} \left(\sum_{i=1}^j w_i + \sum_{i=1}^k u_i \right) \right] \partial^a. \end{aligned}$$

Thus, a nonzero \mathbb{G}_m -invariant m can only be killed by $\text{eu}(\beta) + c = \text{eu}'(\beta) + c$ if

$$\begin{aligned} 0 &= c - \left[\sum_{i=1}^j w_i \left(a'_i + \frac{1}{2} \right) + \sum_{i=1}^k u_i \left(a_i + \frac{1}{2} \right) \right], \quad \text{or} \\ c &= \sum_{i=1}^j |\alpha_i \bullet \beta| a'_i + \sum_{i=j+\ell+1}^{j+\ell+k} |\alpha_i \bullet \beta| a_i + \frac{1}{2} \sum_{i=1}^d |\alpha_i \bullet \beta|, \end{aligned}$$

where each $a_i, a'_i \geq 0$, as desired. \square

Lemma 3.7. *Suppose that M is a weakly \mathbb{T}^{md} -equivariant $\mathcal{D}(W)$ -module generated by $m \in M^{\mathbb{T}^{md}}$. Suppose that $SS(M) \subseteq W_+ \times W_0 \times W_0^* \times (W_-)^*$. Then M is also supported set-theoretically on $W_+ \times W_0 \times W_0^* \times (W_-)^*$ in the sense of Proposition 3.5.*

Proof. By the singular support hypothesis and Lemma 2.5.3(1) of [Ka], for every $\partial/\partial y_i$ and N sufficiently large, there exists a differential operator ψ_N such that ψ_N has symbol $\partial^N/\partial y_i^N$ and $\psi_N \cdot m = 0$. By a standard argument, since m is \mathbb{T}^{md} -fixed we may assume ψ_N is a \mathbb{T}^{md} -weight vector, which then is clearly of the same weight as its symbol. By Lemma 3.1, it follows that ψ_N lies in $\mathcal{D}(W)^{\mathbb{T}^{md}} \cdot \frac{\partial^N}{\partial y_i^N}$ where $\mathcal{D}(W)^{\mathbb{T}^{md}}$ is as described at Lemma 3.1. By our conditions on our

filtration of \mathcal{D} , the only element in this space with symbol $\frac{\partial^N}{\partial y_i^N}$ is $\frac{\partial^N}{\partial y_i^N}$ itself. \square

Corollary 3.8. *Assume that $\chi \circ \beta$ has positive weight. If c satisfies (3.7) then the following hold:*

- (1) *If M is a weakly \mathbb{T}^{md} -equivariant module and $\phi : M_c^\beta \rightarrow M$ is a weakly \mathbb{T}^{md} -equivariant homomorphism and M has singular support in $W_+ \times W_0 \times W_0^* \times (W_-)^*$, then $\phi = 0$.*
- (2) *For every $\ell \ll 0$, there is a finite-dimensional vector subspace*

$$V_\ell \subset \text{Hom}_{(\mathcal{D}, \beta(\mathbb{G}_m), c)}(M_c^\beta(\chi^\ell), M_c^\beta)$$

for which the natural map ψ , given as the composite (inclusion followed by evaluation)

$$(3.8) \quad M_c^\beta(\chi^\ell) \otimes V_\ell \longrightarrow M_c^\beta(\chi^\ell) \otimes \mathrm{Hom}_{(\mathcal{D}, \beta(\mathbb{G}_m), c)}(M_c^\beta(\chi^\ell), M_c^\beta) \longrightarrow M_c^\beta,$$

is a split surjective homomorphism of objects of $(\mathcal{D}, \beta(\mathbb{G}_m), c) - \mathrm{mod}$.

Proof. (1) By Lemma 3.7, the hypotheses imply that M is set-theoretically supported on $W_+ \times W_0 \times W_0^* \times (W_-)^*$ in the sense of Proposition 3.5. Thus, applying Proposition 3.5, it follows that $\phi(\mathbf{1}) = 0$, proving (1).

(2) Assume that $\ell \ll 0$. Write $M_c = M_c^\beta$, $M_c(\chi^\ell) = M_c^\beta(\chi^\ell)$, and \mathbb{G}_m for $\beta(\mathbb{G}_m)$. By adjunction,

$$(3.9) \quad \begin{aligned} \mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c(\chi^\ell), M_c) &\cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{D} \otimes \chi^\ell, M_c)^{\mathbb{G}_m} \\ &\cong \mathrm{Hom}_{\mathbb{G}_m}(\chi^\ell, M_c) \cong \mathrm{Hom}_{\mathbb{G}_m}(\mathbf{C}, M_c \otimes \chi^{-\ell}). \end{aligned}$$

The latter space is (non-canonically) isomorphic, via passing to associated graded (for the filtration on M_c induced from the surjection $\mathcal{D} \rightarrow M_c$), to $(\mathbf{C}[\mu_\beta^{-1}(0)] \otimes \chi^{-\ell})^{\mathbb{G}_m}$ where μ_β denotes the moment map for $\beta(\mathbb{G}_m)$. Applying Lemma 2.2 with $q = -\ell$, elements of $(\mathbf{C}[\mu_\beta^{-1}(0)] \otimes \chi^{-\ell})^{\mathbb{G}_m}$ are $\chi^{-\ell}$ -semi-invariants for \mathbb{G}_m . By our assumption on $\chi \circ \beta$, for $\ell \ll 0$, the common zero locus of such sections is exactly $W_+ \times W_0 \times W_0^* \times (W_-)^*$. It follows that, under the identification of (3.9), the cokernel of the evaluation map $M_c(\chi^\ell) \otimes \mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c(\chi^\ell), M_c) \rightarrow M_c$ has singular support in $W_+ \times W_0 \times W_0^* \times (W_-)^*$. We may thus choose a finite-dimensional subspace $V_\ell \subset \mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c(\chi^\ell), M_c)$ such that the cokernel of the map ψ in (3.8) also has singular support in $W_+ \times W_0 \times W_0^* \times (W_-)^*$.

It is evident from their construction that $M_c(\chi^\ell)$ and M_c are weakly \mathbb{T}^{md} -equivariant and that the evaluation map $M_c(\chi^\ell) \otimes \mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c(\chi^\ell), M_c) \rightarrow M_c$ is weakly \mathbb{T}^{md} -equivariant: note that \mathbb{G}_m acts trivially on $\mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c(\chi^\ell), M_c)$ but \mathbb{T}^{md} may not. However, since the \mathbb{T}^{md} -action is rational, the subspace V_ℓ may be chosen to be a \mathbb{T}^{md} -stable subspace, and then (3.8) is \mathbb{T}^{md} -equivariant; assume we have made such a choice. It follows from the construction that $\mathrm{coker}(\psi)$ is (\mathbb{G}_m, c) -equivariant, weakly \mathbb{T}^{md} -equivariant, and has singular support in $W_+ \times W_0 \times W_0^* \times (W_-)^*$. Applying (1) to the surjective map $M_c \rightarrow \mathrm{coker}(\psi)$ it follows that $\mathrm{coker}(\psi) = 0$ and hence ψ is surjective.

To see that ψ is split, note that since \mathbb{G}_m is reductive, ψ remains invariant after applying \mathbb{G}_m -invariants, that is we have a surjective map $\psi^{\mathbb{G}_m} : \mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c, M_c(\chi^\ell) \otimes V_\ell) \rightarrow \mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c, M_c)$ of $\mathrm{End}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c)$ -modules. An element $\nu \in \mathrm{Hom}_{(\mathcal{D}, \mathbb{G}_m, c)}(M_c, M_c(\chi^\ell) \otimes V_\ell)$ for which $\psi^{\mathbb{G}_m}(\nu) = \mathrm{Id}_{M_c}$ is a splitting of ψ . This proves (2). \square

Remark 3.9. The condition that $\chi \circ \beta$ has positive weight holds whenever β is a KN 1-parameter subgroup for a stratification to which χ is adapted.

Example 3.10. Consider $W = \mathbf{C}^{n+1}$ with coordinates x_0, \dots, x_n . Let $\beta : \mathbb{G}_m \rightarrow \mathbb{G}_m$ be given by $\beta(z) = z^{-1} \cdot \mathrm{Id}$. Let $\chi(z) = z$, so $d\chi \bullet \beta = -1$. Write

$$c = (m + (n+1)/2) \cdot d\chi, \quad \text{and} \quad \mathrm{eu}(\beta) = \sum x_i \partial_{x_i} + (n+1)/2;$$

then $M_c = \mathcal{D}(W)/\mathcal{D}(W)(\sum x_i \partial_{x_i} - m)$. This is the \mathcal{D} -module that descends to $\mathcal{D}(\mathcal{O}(m))$ on \mathbb{P}^n —indeed, note that if f is homogeneous of degree m then $(\sum x_i \partial_{x_i} - m)(f) = 0$, so $\mathrm{End}(M_c)^{\mathbb{G}_m}$ naturally acts on $H^0(\mathbb{P}^n, \mathcal{O}(m))$. For this choice of β , $W_+ = \{0\}$ and $W_+ \times (W_-)^*$ consists of the fiber $\{0\} \times W^*$ over $0 \in W$. Thus, the proposition states that for $m \notin \mathbb{Z}_{\leq -(n+1)}$, weakly equivariant twisted \mathcal{D} -modules on W supported over 0 have no invariant sections. Compare to Example 4.14 for the reason for our choice of signs.

We note that if we reverse the sign on the character χ , i.e., replace χ by $\chi'(z) = z^{-1}$, then the unstable locus flips from $T_{\{0\}}^* W$ to $T_W^* W$, i.e., the zero section (picked out by $\beta'(z) = z$). Moreover, the unstable locus in the base W becomes $W^{uns} = W$. Thus it is *not* true that the constraint on c that we identify implies that for twisted-equivariant \mathcal{D} -modules M supported on $W^{uns} = W$ have

$\text{Hom}(M_c, M) = 0$: the module $M = M_c$ itself provides a counterexample. This explains why one imposes a microlocal-in- T^*W , rather than local-in- W , support condition on M .

4. KIRWAN-NESS STRATIFICATIONS

In this section we define and study Kirwan-Ness (KN) stratifications of a G -variety, and then in the case of a G -representation give a description of the strata and an explicit description of the 1-parameter subgroups labeling them (which we will refer to as *KN 1-parameter subgroups*).

We emphasize that, for a given choice of semistable locus $X^{\text{ss}} \subset X$, a KN stratification of $X \setminus X^{\text{ss}}$ in the sense of Definition 4.2 below *a priori* may not exist. However, we explain below that such stratifications do exist in typical situations of interest. Section 4.3 explains that they exist for projective-over-affine varieties, and are induced on smooth G -stable closed subvarieties of varieties with KN stratifications; in the remainder of the section we prove that KN stratifications of T^*X exist when X is affine, and explain an algorithm to compute the KN 1-parameter subgroups. Moreover, in [McN3] we will show that the moduli stack $T^*\text{Bun}_{\mathbb{G}}(C)$ of \mathbb{G} -Higgs bundles on a smooth projective curve C , for a reductive group scheme \mathbb{G} over C , admits a KN stratification.

4.1. Preliminaries on 1-Parameter Subgroups. Let G be a connected reductive group and \mathbb{T} a maximal torus of G , with W_G the corresponding Weyl group. Let $Y(\mathbb{T}) = \text{Hom}(\mathbb{G}_m, \mathbb{T})$ be the group of 1-parameter subgroups of \mathbb{T} , and $X(\mathbb{T}) = \text{Hom}(\mathbb{T}, \mathbb{G}_m)$, the group of characters. Let $Y_{\mathbb{Q}} = Y(\mathbb{T})_{\mathbb{Q}}$ be the \mathbb{Q} -vector space $Y(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X_{\mathbb{Q}} = X(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ similarly. Fix $q: Y_{\mathbb{Q}} \rightarrow \mathbb{Q}$ a Weyl group invariant, integral, positive definite quadratic form. We will also denote by \bullet the symmetric bilinear form associated to q on $Y_{\mathbb{Q}}$. Using it, we may identify $Y_{\mathbb{Q}}$ and $X_{\mathbb{Q}}$, as we do henceforth.

Remark 4.1. The identification of $Y_{\mathbb{Q}}$ with $X_{\mathbb{Q}}$ makes our notation for the bilinear form compatible with the notation of Section 3 (cf. (3.1)) via the inclusions of $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ into \mathfrak{t}^* and \mathfrak{t} respectively.

If $M(G)$ denotes the set of rational 1-parameter subgroups of G as in [Kir, §12.4], then by W_G -invariance the form q extends to a squared-norm on all of $M(G)$. For $\beta \in M(G)$ we will write $\|\beta\|$ for the norm $\sqrt{q(\beta)}$ of β . This norm defines a natural partial order on $M(G)$, and we will write $\beta < \beta'$ if $\|\beta\| < \|\beta'\|$ for $\beta, \beta' \in M(G)$.

Given $\beta: \mathbb{G}_m \rightarrow G$ a 1-parameter subgroup, we denote by P_{β} the parabolic subgroup of G whose Lie algebra is spanned by the non-negative weight spaces for the $\beta(\mathbb{G}_m)$ -action on \mathfrak{g} . We let L_{β} be the centralizer of $\beta(\mathbb{G}_m)$, which is a Levi subgroup for P_{β} , and U_{β} the unipotent radical of P_{β} ; thus there is a canonical isomorphism $L_{\beta} \cong P_{\beta}/U_{\beta}$.

Suppose X is a G -variety (cf. Convention 2.1). If $\beta \in M(G)$ then write $Z_{\beta} = X^{\beta(\mathbb{G}_m)}$, and $Z_{\beta} = \bigsqcup_{i \in I_{\beta}} Z_{\beta,i}$ for the decomposition of Z_{β} into connected components (where I_{β} is some indexing

set). Each $Z_{\beta,i}$ is evidently an L_{β} -scheme. Define $Y_{\beta,i} = \left\{ x \in X \mid \lim_{t \rightarrow 0} \beta(t) \cdot x \in Z_{\beta,i} \right\}$. Note that the map $\text{pr}_{\beta}: Y_{\beta,i} \rightarrow Z_{\beta,i}$ given by $\text{pr}_{\beta}(x) = \lim_{t \rightarrow 0} \beta(t) \cdot x$ is P_{β} equivariant, where P_{β} acts on Y_{β} in the natural way, and on Z_{β} via $L_{\beta} = P_{\beta}/U_{\beta}$.

4.2. Axiomatic Kirwan-Ness Stratifications. Fix a finite subset $\text{KN} \subset \{(\beta, i) \mid \beta \in Y_{\mathbb{Q}}, i \in I_{\beta}\}$ where no two distinct β s are conjugate under the action of the Weyl group W_G of G . Since it consists of W_G -orbit representatives, the set KN is partially ordered by $(\beta', j) > (\beta, i)$ if $\|\beta'\| > \|\beta\|$.

Definition 4.2. A *Kirwan-Ness (KN) stratification* of X indexed by KN is a finite partition of X

$$X = X^{\text{ss}} \sqcup \bigsqcup_{\alpha \in \text{KN}} S_{\alpha},$$

into locally closed smooth pieces $S_{\alpha} = S_{\beta,i}$ (so $\alpha = (\beta, i)$), satisfying:

- (1) X^{ss} is open in X .
- (2) For each $(\beta, i) \in \text{KN}$ there is an open L_β -invariant subset $Z_{\beta,i}^{\text{ss}}$ of $Z_{\beta,i}$ such that:
 - (a) Setting $Y_{\beta,i}^{\text{ss}} = \text{pr}_\beta^{-1}(Z_{\beta,i}^{\text{ss}})$, the map pr_β defines a (necessarily P_β -equivariant) affine fibration $\text{pr}_\beta: Y_{\beta,i}^{\text{ss}} \rightarrow Z_{\beta,i}^{\text{ss}}$.
 - (b) The stratum $S_{\beta,i}$ is then given by $G \cdot Y_{\beta,i}^{\text{ss}}$, and we require $S_{\beta,i} \cong G \times_{P_\beta} Y_{\beta,i}^{\text{ss}}$ via the obvious map.
- (3) There is a refinement of $<$ (which for simplicity we also denote by $<$) such that for each $(\beta, i) \in \text{KN}$ the union $S_{\geq \beta, i} = S_{\beta, i} \bigsqcup \left(\bigsqcup_{(\beta', j) > (\beta, i)} S_{\beta', j} \right)$ is closed in X .

In typical examples of interest, the open stratum X^{ss} will be the semistable locus of X with respect to a G -equivariant line bundle. Condition (3) is a weak form of the “axiom of closure” in stratification theory, which suffices for our purposes.

Notation 4.3. For a fixed choice of partial order refining $>$ (to be understood from context), let

$$S_{\geq \beta, i} = S_{\beta, i} \bigsqcup \left(\bigsqcup_{(\beta', j) > (\beta, i)} S_{\beta', j} \right), \quad S_{> \beta, i} = S_{\geq \beta, i} \setminus S_{\beta, i}.$$

Note that it follows from the definitions that the elements of $Y_{\mathbb{Q}}$ occurring in KN can be taken to lie in a fundamental domain of the Weyl group action on $Y_{\mathbb{Q}}$.

As the next section will show, many examples of such stratifications arise from studying stability with respect to a G -equivariant line bundle \mathcal{L} , and in such cases the following condition will be automatically satisfied.

Definition 4.4. Suppose that X is a G -scheme with a fixed KN -stratification $\{S_\alpha\}_{\alpha \in \text{KN}}$ and \mathcal{L} is a G -equivariant line bundle on X . If $(\beta, i) \in \text{KN}$ then L_β acts on the fibers of $\mathcal{L}|_{Z_{\beta,i}}$ by a character $\chi_{\beta,i}$. We say that \mathcal{L} is *adapted* to the KN stratification if $\chi_{\beta,i} \circ \beta$ has positive weight.

In the next subsection we discuss situations in which a KN stratification is known to exist. However let us note first the following.

Lemma 4.5. Let X be a G -scheme with a KN stratification $X = X^{\text{ss}} \sqcup \bigsqcup_{\alpha \in \text{KN}} S_\alpha$ and let T be a smooth closed G -stable subscheme. Then the connected components of the $S_\alpha \cap T$ yield a KN stratification of T .

Proof. The only property which requires proof is the fact that the projection maps $\text{pr}_\beta: Y_{\beta,i}^{\text{ss}} \rightarrow Z_{\beta,i}^{\text{ss}}$ are affine fibrations. However this follows from [BB, Theorem 4.1]. \square

Lemma 4.6. Suppose that X is a smooth G -variety with KN stratification and that $E \subseteq X$ is a smooth G -stable locally closed subvariety that is a union of KN strata. Then the induced stratification of E is a KN stratification.

4.3. Sources of KN Stratifications. Suppose $\tilde{X} \subseteq \mathbb{C}^{n+1}$ is the affine cone of a G -stable subvariety $X \subseteq \mathbb{P}^n$ of a projective space with linear G -action, with coordinates x_0, \dots, x_n which are \mathbb{T} -weight vectors, with weights $\alpha_0, \dots, \alpha_n$. Given a rational 1-parameter subgroup $\beta: \mathbb{G}_m \rightarrow \mathbb{T}$ and $x \in X$ set

$$m(x, \beta) = \min\{\alpha_j \bullet \beta : x_j \neq 0\}$$

provided the right-hand side is non-negative, otherwise $m(x, \beta) = 0$. The Hilbert-Mumford criterion shows a point is unstable if and only if $m(x, \beta) > 0$ for some $\beta \in M(G)$. The function m should thus be thought of intuitively as measuring the rate at which β destabilizes the point x . We obtain a KN stratification of X by taking $S_\emptyset = X^{\text{ss}}$ to be the semistable locus, and then stratifying the unstable

locus by considering 1-parameter subgroups which “optimally destabilize,” in the sense that they optimize $m(x, \beta)/\|\beta\|$.

Following [Kir, §12], for $x \in X$ define

$$q_G^{-1}(x) = \inf\{q(\lambda) : \lambda \in M(G), m(x, \lambda) \geq 1\},$$

and

$$\Lambda_G(x) = \{\lambda \in M(G) : m(x, \lambda) \geq 1, q(\lambda) = q_G^{-1}(x)\}$$

(where we take $q_G^{-1}(x) = \infty$ and $\Lambda_G(x) = \emptyset$ if $m(x, \lambda) = 0$ for all $\lambda \in M(G)$, that is, when x is semistable). For $x \in X$, $\Lambda_G(x)$ is the set of its optimally destabilizing rational 1-parameter subgroups.

More precisely, define subsets \tilde{Z}_β and \tilde{Y}_β of \mathbb{P}^n by declaring that a point $(x_0 : \dots : x_n) \in \mathbb{P}^n$ lies in \tilde{Z}_β , respectively \tilde{Y}_β , if and only if, for each i such that $x_i \neq 0$, we have $\alpha_i \bullet \beta = \beta \bullet \beta$, respectively $\alpha_i \bullet \beta \geq \beta \bullet \beta$. The space \tilde{Y}_β is an affine bundle over the space \tilde{Z}_β . Let $\tilde{Y}_\beta^{\text{ss}}$ of \tilde{Y}_β be the open subset of $x \in \tilde{Y}_\beta$ for which $\beta \in \Lambda_G(x)$, and define $\tilde{Z}_\beta^{\text{ss}}$ an open subset of \tilde{Z}_β similarly. Then $\tilde{Y}_\beta^{\text{ss}}$ is the preimage under the bundle map of $\tilde{Z}_\beta^{\text{ss}}$ in Z_β ; see [Kir, §12] for more details. The *Kirwan-Ness stratum* (or KN stratum) \tilde{S}_β of \mathbb{P}^n associated to β is the G -saturation $\tilde{S}_\beta = G \cdot \tilde{Y}_\beta^{\text{ss}}$; Kirwan proves that $\tilde{S}_\beta = G \times_{P_\beta} \tilde{Y}_\beta^{\text{ss}}$. Let

$$(4.1) \quad Z_\beta = \tilde{Z}_\beta \cap X, \quad Y_\beta = \tilde{Y}_\beta \cap X, \quad \text{and write } p_\beta : \tilde{Y}_\beta \rightarrow \tilde{Z}_\beta$$

for the natural linear projection given by setting to zero those coordinates for which $\alpha_i \bullet \beta > \beta \bullet \beta$. As above, we will write $Z_\beta = \bigsqcup_{i \in I_\beta} Z_{\beta,i}$ for the decomposition of Z_β into connected components, and $Y_{\beta,i} = p_\beta^{-1}(Z_{\beta,i})$. Then setting $Z_{\beta,i}^{\text{ss}} = Z_{\beta,i} \cap \tilde{Z}_\beta^{\text{ss}}$ and $Y_{\beta,i}^{\text{ss}} = Y_{\beta,i} \cap \tilde{Y}_\beta^{\text{ss}}$, and $S_{\beta,i} = G \cdot Y_{\beta,i}^{\text{ss}}$. Note that when G is a torus, $S_{\beta,i} = Y_{\beta,i}^{\text{ss}}$. We have the following theorem, which essentially summarizes results of Kirwan.

Proposition 4.7 ([Kir], §12.16). *Suppose that $X \subseteq \mathbb{P}^n$ is a smooth, closed subvariety. Then X^{ss} together with the strata $S_{\beta,i}$ form a KN stratification of X to which the line bundle $\mathcal{O}(1)$ is adapted.*

Proof. By Lemma 4.5 it is enough to show that the $\tilde{S}_{\beta,i} = G \cdot \tilde{Y}_{\beta,i}^{\text{ss}}$ form a KN stratification of \mathbb{P}^n . Note that Kirwan’s strata are not necessarily connected; thus Proposition 12.16 of [Kir] is slightly weaker than our axiom (3). However, since the $S_{\beta,i}$ are smooth, it is easy to check that there is a refinement of the norm partial order satifying that axiom. \square

Note that in [Kir] the strata are allowed to be disconnected, and hence the labeling set (there denoted \mathbf{B}) is slightly different from the one used here. This makes no essential difference to the proofs, and one could choose instead to work with disconnected strata.

Recall that every 1-parameter subgroup is conjugate under G to a 1-parameter subgroup in \mathbf{T} . We will use the following (cf. [Kir, §12]).

- (1) If S is a KN stratum in the unstable locus, and $x \in S$, then $\Lambda_{\mathbf{T}}(x) = \Lambda_G(x) \cap Y_{\mathbb{Q}}$ consists of at most one point. Moreover there exist $x \in S$ for which $\Lambda_{\mathbf{T}}(x)$ is nonempty. Any KN stratum S contains a point x whose set of optimal 1-parameter subgroups $\Lambda_G(x)$ intersects $Y(T)_{\mathbb{Q}}$ in a unique point β .
- (2) If $Y_{\beta,i}$ denotes the KN stratum containing x with respect to the \mathbf{T} -action on X , then $S = G \cdot Y_{\beta,i}^{\text{ss}}$. Thus the W_G -orbit of the point β uniquely determines the union $\bigsqcup_{i \in I_\beta} S_{\beta,i}$ of strata.

It follows immediately that once we fix a maximal torus \mathbf{T} , the KN strata are labeled by the KN 1-parameter subgroups associated to minimal combinations of weights of the \mathbf{T} -action which lie in a single Weyl chamber. We will use this when studying KN stratifications of representations.

Minor variations of the same arguments show that KN stratifications exist in the case where X is projective over affine [T]. In fact the case where X is affine, indeed an affine space, is important in applications, so we study it in detail in the next sections. However, in general a variety need not have a KN stratification.

4.4. KN Stratification of a Representation. We are particularly interested in the case where the G -variety X is a linear representation V of G ; to do this, we need the relative version of the theory of KN strata from [T, Sections 1 and 5]. We consider the vector space V as an open subset of $\mathbb{P}(\mathbf{C} \oplus V)$ where \mathbf{C} denotes the trivial representation of G . We get an action of G on

$$\mathbb{P} = \mathbb{P}(\mathbf{C} \times V) = \text{Proj Sym}((\mathbf{C} \oplus V)^*)$$

by $g \cdot (c, v) = (c, gv)$. This action makes $V \cong \{1\} \times V \subset \mathbb{P}$ into a G -invariant open subset of \mathbb{P} consisting of semistable points. Now fix a character $\chi : G \rightarrow \mathbb{G}_m$ and consider the projective morphism $\pi : \mathbb{P} \rightarrow \mathbb{P}$ given by the identity map. Equip π with the relatively ample line bundle $M = \mathcal{O}$, but with the G -linearization twisted by χ . More precisely, this means the following. The total space of M is $\mathbb{P} \times \mathbb{A}^1$, and we let $g \in G$ act by $g \cdot ((c, v), w) = ((c, gv), \chi(g)w)$ (cf. Section 2.1).

For each $\epsilon \in \mathbb{Q}_{>0}$, we get a G -linearized \mathbb{Q} -line bundle $\mathbf{L} = M^{\otimes \epsilon} \otimes \mathcal{O}(1)$. Over $V \cong \mathbb{P}((\mathbf{C} \setminus \{0\}) \times V) \subset \mathbb{P}$, the bundle $\mathcal{O}(1)$ has a canonical nonvanishing G -invariant section, given as a linear form on $(\mathbf{C} \oplus V)^*$ by the projection $\mathbf{C} \oplus V^* \rightarrow \mathbf{C}$. In terms of this G -invariant trivialization of $\mathcal{O}(1)$, then, G acts on $\mathbf{L} = M^{\otimes \epsilon} \otimes \mathcal{O}(1)$ via a twist by the rational character χ^ϵ (see [T] for the conventions about the meaning of such statements).

On restriction of \mathbf{L}^{-1} to the open set V of \mathbb{P} , this trivialization of $\mathcal{O}(1)$ allows us to write the action on \mathbf{L}^{-1} as $g \cdot (v, w) = (gv, \chi^{-\epsilon}(g)w)$; this formula agrees with the conventions used in [Kin] for applying geometric invariant theory to a representation, and allows us to use Kirwan's set-up to directly describe the KN stratification of V for the rational character χ^ϵ .

Lemma 4.8. *The KN stratification of \mathbb{P} induced by \mathbf{L} is independent of ϵ for small $\epsilon \in \mathbb{Q}_{>0}$. The induced stratification of $V \subseteq \mathbb{P}$ is a KN stratification of V to which the line bundle \mathbf{L} is adapted.*

Proof. The first statement follows from Lemmas 1.2 and 5.1 of [T]. The second statement then follows from Lemma 4.6. Finally, the fact that the line bundle \mathbf{L} is adapted to the stratification is immediate from the construction. \square

Note that in this case, the subsets Z_β are connected, so the strata are labeled simply by rational 1-parameter subgroups.

4.5. KN 1-Parameter Subgroups for a Representation. We maintain the notation of Section 4.4. For convenience, however,

we now switch to additive notation, so $\lambda = d\chi$.

Also, to avoid a profusion of minus signs,

we assume from now on that ϵ is a small *negative* rational number,

so $\mathbf{L} = M^{\otimes(-\epsilon)} \otimes \mathcal{O}(1)$.

Let $\{e_1, e_2, \dots, e_n\}$ be a basis of V consisting of T -weight vectors, and let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the corresponding list of weights (thus we do not assume that the α_i are pairwise distinct). We extend this list by setting $\alpha_0 = 0$, so that if e_0 is the standard basis vector of $\mathbf{C} \times \{0\} \subset \mathbf{C} \oplus V$, then $\{\alpha_0, \dots, \alpha_n\} \subset X(T)_{\mathbb{Q}}$ gives a list of the weights of $\mathbf{C} \oplus V$. We will use the basis $\{e_0, e_1, \dots, e_n\}$ to give homogeneous coordinates $[t_0 : \dots : t_n]$ on $\mathbb{P}(\mathbf{C} \oplus V)$.

We will need the following standard notation from convex geometry. Let S be a nonempty subset of the \mathbb{Q} -vector space U . We write $\text{conv}(S)$ for the convex hull of S , i.e., the set

$$\text{conv}(S) = \left\{ \sum_{i=1}^m t_i s_i : m \in \mathbb{Z}_{>0}, s_i \in S, t_i \in \mathbb{Q}, 0 \leq t_i \leq 1, \sum_{i=1}^n t_i = 1 \right\}.$$

Similarly we let $\text{aff}(S)$ denote the affine hull of S , that is

$$\text{aff}(S) = \left\{ \sum_{i=1}^m t_i s_i : m \in \mathbb{Z}_{>0}, s_i \in S, t_i \in \mathbb{Q}, \sum_{i=1}^n t_i = 1 \right\}.$$

Given a subset $\emptyset \neq I \subseteq \{0, 1, \dots, n\}$, let $W_I = \text{span}\{\alpha_i : i \in I\}$ be the subspace spanned by the corresponding set of weights, and $C_I = \text{conv}\{\alpha_i : i \in I\}$, $A_I = \text{aff}\{\alpha_i : i \in I\}$. We also write $\alpha_i^\epsilon = \alpha_i + \epsilon\lambda$ (so that in particular, $\alpha_0^\epsilon = \epsilon\lambda$), and then let $A_I^\epsilon = \text{aff}\{\alpha_i^\epsilon : i \in I\}$ and $C_I^\epsilon = \text{conv}\{\alpha_i^\epsilon : i \in I\}$.

Using [Kir, §12.8], the KN 1-parameter subgroups (denoted there by \mathbf{B}) for $\mathbb{P}(\mathbf{C} \oplus V)$ are given as follows. If $I \subset \{0, 1, \dots, n\}$ then let $\beta_I = \beta_I(\epsilon)$ denote the closest point in C_I^ϵ to 0: that is, $\beta_I(\epsilon)$ minimizes the distance $d(0, C_I^\epsilon)$. If $\beta_I(\epsilon) = 0$ then there are semistable points with exactly this subset of weights.

Otherwise, the corresponding (rational) KN 1-parameter subgroup is $\beta_I/q(\beta_I)$. We write Λ_V for the set of KN 1-parameter subgroups which label KN strata intersecting $V \subset \mathbb{P}(\mathbf{C} \oplus V)$.

Note that for each nonempty subset $I \subseteq \{0, 1, \dots, n\}$, the distance $d(0, C_I^\epsilon)$ is a piecewise linear function of ϵ , as is the minimal vector $\beta_I(\epsilon)$. Since there are finitely many such subsets I , there is some $C < 0$ such that all the distance functions $d(0, C_I^\epsilon)$ and minimal vectors $\beta_I(\epsilon)$ are linear for $\epsilon \in [C, 0]$; fix one such C . Given I , fix a *minimal* subset $J \subseteq I$ with the property that $d(0, C_I^\epsilon) = d(0, C_J^\epsilon)$ for $\epsilon \in [C, 0]$. Then, since q is strictly convex, so that the closest point is unique, we must have $\beta_I(\epsilon) = \beta_J(\epsilon)$ for $\epsilon \in [C, 0]$.

For each subset I , set $p_I(\epsilon) = \text{proj}_{W_I^\perp}(\epsilon\lambda)$, the orthogonal projection to the subspace of weights perpendicular to W_I .

Proposition 4.9. *Let $x \in V \subset \mathbb{P}(\mathbf{C} \oplus V)$ have homogeneous coordinates $x = [t_0 : \dots : t_n]$ and let $I = I_x = \{i : 0 \leq i \leq n, t_i \neq 0\}$. If $J \subseteq I$ is minimal such that $d(0, C_I^\epsilon) = d(0, C_J^\epsilon)$ then $\beta_I(\epsilon) = p_J(\epsilon)$.*

Proof. Fix $J \subset I$ minimal so that $\beta_I(\epsilon) = \beta_J(\epsilon)$ for $\epsilon \in [C, 0]$. Note that C_J^ϵ is a closed subset of A_J^ϵ with nonempty interior as a subset of A_J^ϵ ; we write $\text{rel-int}(C_J^\epsilon)$ for this relative interior. We claim that

$$(4.2) \quad \beta_I(\epsilon) \in \text{rel-int}(C_J^\epsilon) \text{ for } \epsilon \in [C, 0].$$

Indeed, the boundary of C_J^ϵ in A_J^ϵ is the union of convex hulls $C_{J'}^\epsilon$, corresponding to certain proper subsets $J' \subset J$. If $\beta_I(\epsilon)$ lies in one of these boundary subsets $C_{J'}^\epsilon$ for small ϵ then we must have $\beta_{J'}(\epsilon) = \beta_I(\epsilon)$, contradicting the minimality of J .

Next we claim that $0 \in C_J$. To see this, let β' be the closest point to 0 in C_J . Then as $\beta_J(\epsilon)$ is the closest point to 0 in C_J^ϵ and $C_J^\epsilon = \epsilon\lambda + C_J$, we must have

$$(4.3) \quad \|\beta_I(\epsilon)\| \geq \|\beta'\| + \epsilon\|\lambda\|$$

(recall that ϵ is small and negative). On the other hand, since the stratum S_β intersects V , we must have $0 \in I$. Hence $\alpha_0^\epsilon = \epsilon\lambda \in C_I^\epsilon$, and as $\beta_J(\epsilon) = \beta_I(\epsilon)$ is the unique closest point, by convexity:

$$(4.4) \quad \epsilon\lambda \bullet \beta_J(\epsilon) \geq \beta_J(\epsilon) \bullet \beta_J(\epsilon) = \|\beta_J(\epsilon)\|^2.$$

For ϵ sufficiently small, the inequalities (4.3) and (4.4) can only be consistent if $\|\beta'\| = 0$, that is, if $\beta' = 0$, and hence $0 \in C_J$ as required. Note in particular since A_J is an affine subspace, this implies that A_J is in fact a subspace. It follows by (4.2) that $\beta_J(\epsilon)$ is also the closest point to 0 in $A_J^\epsilon = \epsilon\lambda + A_J$. It follows immediately that $\beta_I(\epsilon) = p_J(\epsilon)$ as required. \square

Remark 4.10.

- (1) If $\mathbb{P}(\mathbf{C} \oplus V)$ has a KN stratum with corresponding 1-parameter subgroup β of the form $\beta = p_J(\epsilon) = \epsilon \operatorname{proj}_{W_J^\perp}(\lambda)$, then S_β intersects V . Indeed, if the subset J is such that the set C_J does not contain 0, then for small enough ϵ the set C_J^ϵ clearly will not contain $p_J(\epsilon)$, contradicting the fact that $p_J(\epsilon)$ should be the closest point to 0 in C_J^ϵ ; so $0 \in C_J$. But then we may replace J by $J \cup \{0\}$ without altering C_J^ϵ , and then clearly we may find $v \in V$ with associated weights giving J , and hence $v \in S_\beta$ as required. Thus the KN strata of $\mathbb{P}(\mathbf{C} \oplus V)$ which intersect V are precisely those $\beta \in \text{KN}$ of the form $\beta = p_J(\epsilon)$ for some subset J .
- (2) Suppose $V = T^*W$ for some G -representation W . Then the collection of those $p_J(\epsilon)$ that arise as J runs over weights of T^*W is the same as that which arises as J runs over weights of W . In particular, the KN 1-parameter subgroups labelling KN strata of W and of T^*W are the same.

So far we have shown that the set of KN 1-parameter subgroups is given by a subset of the elements $\{p_J(\epsilon) : J \subset \{0, 1, \dots, n\}\}$. We now show how to refine this to a precise description of the KN 1-parameter subgroups. For this we use an elementary observation in convex geometry.

Lemma 4.11. *Let W be a finite dimensional \mathbb{Q} -vector space and let $S \subset W$ be a finite set, and $C = \operatorname{conv}(S)$ its convex hull. Then a point p lies in C if and only if, for every other point p' , there is some $s \in S$ such that $d(s, p) < d(s, p')$.*

Corollary 4.12. *Let $p_I(\epsilon)$ be as above. Then $p_I(\epsilon)$ yields a KN 1-parameter subgroup if and only if there is no $J \subset I$ with the property that $d(p_J(\epsilon), \alpha_k) < d(p_I(\epsilon), \alpha_k)$ for all $k \in I$.*

Proof. One just needs to check that $\epsilon p_I(\lambda)$ lies in C_J^ϵ . It follows from the above lemma and proposition that for all sufficiently small ϵ some $\epsilon p_I(\lambda)$ is the closest point in C_J^ϵ to 0; the result follows. \square

Example 4.13. Suppose that $V = \mathbf{C}e_1 \oplus \mathbf{C}e_2$ is two-dimensional with $\mathbb{T} = \mathbb{G}_m^2$ acting with weights $\alpha_1 = (1, 0)$ and $\alpha_2 = (1, 1)$. Then if $\lambda = (0, 1)$, and ϵ is small and negative, the modified weights on $\mathbf{C} \times V$ are $\{(0, \epsilon), (1, \epsilon), (1, 1 + \epsilon)\}$. Taking the various possible subsets of weights associated to $x = (x_0 : x_1 : x_2) \in \mathbb{P}(\mathbf{C} \times V)$ with $x_0 \neq 0$, we find that $\epsilon\lambda$ is the closest point for both subsets $\{(0, \epsilon)\}$ and $\{(0, \epsilon), (1, \epsilon)\}$ while $\frac{\epsilon}{2}(-1, 1) = \epsilon\lambda^\perp$, where the \perp is taken with respect to the subspace $\mathbf{C}(1, 1)$, is the closest point for the subsets $\{(0, \epsilon), (1, 1 + \epsilon)\}$, $\{(0, \epsilon), (1, \epsilon), (1, 1 + \epsilon)\}$.

Note however, that it is not the case that all the vectors $\epsilon\lambda^\perp$ as U runs over the span of subsets of the weights $\{\alpha_1, \dots, \alpha_k\}$ will be minimal combinations of weights: in the above case $\beta = 0$ does not occur, showing the semistable locus of this V is empty. Moreover, if we take $\lambda = (0, -1)$, then for small negative ϵ the minimal weight is always $\epsilon\lambda$ itself.

4.6. Explicit Description of KN Strata for a Representation. We next want to describe more explicitly the loci Z_β and Y_β of the KN stratification of V . As noted above, such strata are labelled by 1-parameter subgroups of the form $\beta = p_I(\epsilon)$.

Fix $I = \{i_0, i_1, \dots, i_k\} \subseteq \{0, 1, \dots, n\}$ and let $W = W_I$ and $\beta = p_I(\epsilon)$. Write $\lambda^\perp = \operatorname{proj}_{W^\perp}(\lambda)$, so $p_I(\epsilon) = \epsilon\lambda^\perp$. Then $\lambda = \lambda^\perp + \operatorname{proj}_W(\lambda)$ and $\lambda^\perp \bullet \operatorname{proj}_W(\lambda) = 0$, so $\lambda \bullet \lambda^\perp = \lambda^\perp \bullet \lambda^\perp$. It follows that

$$\alpha'_i \bullet \epsilon\lambda^\perp = (\alpha_i + \epsilon\lambda) \bullet \epsilon\lambda^\perp = \alpha_i \bullet \epsilon\lambda^\perp + \epsilon\lambda^\perp \bullet \epsilon\lambda^\perp,$$

and so for $\beta = \epsilon\lambda^\perp$, Kirwan's conditions become:

$$(4.5) \quad (x_0 : \dots : x_n) \in Z_\beta \text{ iff, for each } i \text{ such that } x_i \neq 0, \text{ we have } \alpha_i \bullet \beta = 0.$$

$$(4.6) \quad (x_0 : \dots : x_n) \in Y_\beta \text{ iff, for each } i \text{ such that } x_i \neq 0, \text{ we have } \alpha_i \bullet \beta \geq 0.$$

We introduce some notation for these subspaces. Given a rational 1-parameter subgroup β , let $V_+(\beta)$, respectively $V_0(\beta)$, respectively $V_-(\beta)$, denote the sum of the α -weight subspaces for which $\alpha \bullet \beta > 0$, respectively $\alpha \bullet \beta = 0$, respectively $\alpha \bullet \beta < 0$; this is equivalent to splitting V according

to whether the (rational) 1-parameter subgroup labelled by β acts with positive, zero, or negative weight. We observe that then:

$$(4.7) \quad Z_\beta = V_0(\beta) \quad \text{and} \quad Y_\beta = V_+(\beta) \times V_0(\beta).$$

We also find that $Z_\beta^{\text{ss}} = V_0(\beta)^{\text{ss}}$, as defined in Section 4.3 is an open subset of $V_0(\beta)$, and hence that

$$(4.8) \quad Y_\beta^{\text{ss}} = V_+(\beta) \times V_0(\beta)^{\text{ss}}.$$

Example 4.14. Let $V = T^*W = T^*\mathbf{C}^{n+1}$ with torus $\mathbf{T} = \mathbb{G}_m$ acting with weight 1 on W . Let $\chi : \mathbb{G}_m \rightarrow \mathbb{G}_m$ be the identity character (i.e. $\chi(z) = z$). Let ϵ be a small *negative* number. Then under the $\chi^{-\epsilon}$ -twisted linearization, the weights on $\mathbf{C} \times W \times W^*$ are $(\epsilon, 1 + \epsilon, \dots, 1 + \epsilon, -1 + \epsilon, \dots, -1 + \epsilon)$. By Proposition 4.9, the only relevant minimal combination of weights is $\beta = \epsilon$, a small negative multiple of the identity character, and Y_β consists of all vectors whose \mathbf{T} -weights pair non-negatively with β , i.e. whose $\mathbf{T} = \mathbb{G}_m$ -weights are nonpositive: in other words, $Y_\beta = \{0\} \times W^*$.

We now specialize to the case $V = T^*W$ where W is a representation of G of dimension d . We pick a \mathbf{T} -weight basis $\{e_1, \dots, e_d\}$ of W , so that its dual basis $\{\xi_1, \dots, \xi_d\}$ is a weight basis of W^* , and their union is a weight basis of T^*W . If β is a destabilizing 1-parameter subgroup, we then have by definition $Y_\beta = W_+(\beta) \times W_-(\beta)^* \times Z_\beta$, where $Z_\beta = T^*W_0(\beta) = W_0(\beta) \times W_0(\beta)^*$. In particular:

Lemma 4.15. *Each Y_β is coisotropic in T^*W . Furthermore,*

$$Y_\beta^{\text{ss}} = W_+(\beta) \times W_-(\beta)^* \times Z_\beta^{\text{ss}}.$$

4.7. KN Stratification of an Affine Variety. Suppose that \mathbf{E} is a smooth affine G -variety. Fix a character $\chi : G \rightarrow \mathbb{G}_m$. We may G -equivariantly embed \mathbf{E} as a closed subvariety of a representation V of G (cf. for example [Ke], Lemma 1.1). Section 4.6 describes the subsets Z_β and Y_β of V determining the KN strata of V as in (4.7) for each 1-parameter subgroup $\beta \in \text{KN}$. Thus, $Z_\beta \cap \mathbf{E} = \mathbf{E}^{\beta(\mathbb{G}_m)}$, and $Y_\beta \cap \mathbf{E} = \{x \in \mathbf{E} \mid \lim_{t \rightarrow 0} \beta(t) \cdot x \in \mathbf{E}^{\beta(\mathbb{G}_m)}\}$.

Proposition 4.16. *Let KN denote the set of KN 1-parameter subgroups for V . Let \tilde{S}_β denote the KN stratum of V corresponding to β . Let $S_\beta = \tilde{S}_\beta \cap \mathbf{E}$ denote the corresponding locally closed subset of \mathbf{E} and let $\mathbf{E}^{\text{ss}} = V^{\text{ss}} \cap \mathbf{E}$. Then $\{\mathbf{E}^{\text{ss}}\} \cup \{S_\beta \mid \beta \in \text{KN}\}$ is a KN stratification of \mathbf{E} .*

Proof. This follows immediately from the above and Lemma 4.5. \square

4.8. Parameter Shifts. Suppose V is a representation of G . Given a 1-parameter subgroup β of $\mathbf{T} \subseteq G$, we write Z_β for the β -fixed subspace of V . We let Y_β be the subspace of V spanned by β -weight vectors with non-negative β -weight. Let $K \subseteq G$ be a subgroup containing β . Suppose Y_β^{ss} is an open subset of Y_β whose intersection with Z_β is nonempty and for which the orbit space $S_\beta^\circ = K \cdot Y_\beta^{\text{ss}}$ is smooth; this will be true in the examples we need in the paper (where S_β° will be an open subset of a KN stratum for a KN 1-parameter subgroup β). We then define the set of nonnegative rationals $I_V(\beta)$ as follows. Choose a point $z \in Z_\beta \cap Y_\beta^{\text{ss}}$, which is then a fixed point of β in S_β° . Then β acts linearly on the normal space $N_{S_\beta^\circ/V}(z)$, and we let $\mathbf{w}_1, \dots, \mathbf{w}_a$ denote the β -weights on this normal space. Then we write

$$I_{K,V}(\beta) = \left\{ \sum_{i=1}^a n_i |\mathbf{w}_i| \mid n_i \geq 0 \right\}$$

for the set of nonnegative integer linear combinations of the absolute values of β -weights $|\mathbf{w}_i|$. This definition extends immediately to rational 1-parameter subgroups, i.e. formal expressions $\frac{p}{q}\beta$ where

$\frac{p}{q}$ is rational, by letting $I_{K,V}(\frac{p}{q}\beta) = \frac{p}{q} I_{K,V}(\beta)$.

Remark 4.17. Note that if $K = \mathbb{T}$ then $S_\beta^\circ = Y_\beta$ and $I_{\mathbb{T},V}(\beta)$ is just the set of nonnegative integer linear combinations of absolute values of β -weights on V .

5. EQUIVARIANT SYMPLECTIC GEOMETRY NEAR A KN STRATUM

In this section, W is a smooth, connected algebraic variety with a rational action of a connected reductive group G with maximal torus $\mathbb{T} \subseteq G$. Let $\mu : T^*W \rightarrow \mathfrak{g}^*$ denote the canonical classical moment map. We assume that T^*W is equipped with a KN stratification and a line bundle \mathcal{L} which is adapted to the KN stratification and is trivialized so that its G -linearization is given by a character $\chi : G \rightarrow \mathbb{G}_m$.

5.1. Recall from Section 4 that a KN stratum $S_{\beta,i}$ is labelled by (the Weyl group orbit of) a 1-parameter subgroup $\beta : \mathbb{G}_m \rightarrow \mathbb{T} \subseteq G$ in a fixed maximal torus \mathbb{T} of G along with a choice $Z_{\beta,i}$ of component of $Z_\beta = X^{\beta(\mathbb{G}_m)}$.

For convenience of notation we abbreviate $S_{\beta,i}$ to S_β in this section.

Let \mathfrak{n}^- be the sum of negative β -weight subspaces in \mathfrak{g} , and let $U^- = U_{P_\beta}^- \subset G$ denote the corresponding unipotent subgroup of G . This is the unipotent radical of the opposite parabolic to P_β as defined in Section 4.2. We will write

$$\mathbb{K} = U^- \rtimes \mathbb{G}_m,$$

where \mathbb{G}_m acts on U^- via β and the adjoint action of G . The group \mathbb{K} acts naturally on T^*W via G . We view U^- as a \mathbb{K} -variety where $\beta(\mathbb{G}_m)$ acts by conjugation and U^- by left translation.

Lemma 5.1. *The action map $a : U^- \times Y_\beta^{\text{ss}} \rightarrow T^*W$ is a \mathbb{K} -equivariant bijection onto an open dense subset of $S_\beta = G \cdot Y_\beta^{\text{ss}} \cong G \times_{P_\beta} Y_\beta^{\text{ss}}$. Moreover, S_β is coisotropic.*

Proof. The equivariance of a is immediate. The isomorphism $S_\beta \cong G \times_{P_\beta} Y_\beta^{\text{ss}}$ is established in Theorem 13.5 of [Kir]. That a is bijective follows from this and the fact that $U^- \cdot eP_\beta$ is the open Bruhat cell in G/P_β . Lemma 4.15 asserts that Y_β^{ss} is coisotropic. Since S_β is G -stable and $S_\beta = G \cdot Y_\beta^{\text{ss}}$, to check that S_β is coisotropic, it suffices to check that the tangent space of S_β is coisotropic at each point of Y_β^{ss} . But any linear subspace of a symplectic vector space that contains a coisotropic subspace is itself coisotropic. \square

5.2. Construction of a Slice. We will now construct a slice to the action of U^- at a point $z \in Z_\beta^{\text{ss}}$.

Suppose $z \in Z_\beta^{\text{ss}}$. The infinitesimal U^- -action induces an injective (by Lemma 5.1) map $\mathfrak{n}^- \rightarrow T_z(T^*W)$, and we get a direct sum $\mathfrak{n}^- \oplus T_z Y_\beta \subset T_z(T^*W)$. Since β acts on U^- and hence compatibly on \mathfrak{n}^- , and z is a β -fixed point, it makes sense to ask whether $\mathfrak{n}^- \rightarrow T_z(T^*W)$ is β -equivariant; it clearly is. Hence the subspace $\mathfrak{n}^- \oplus T_z Y_\beta \subset T_z(T^*W)$ is β -invariant.

Choose a β -invariant complementary subspace V , so $T_z(T^*W) = \mathfrak{n}^- \oplus T_z Y_\beta \oplus V$. If W is a G -representation, we have a canonical identification $T_z(T^*W) \cong T^*W$; hence we can identify V with a subspace of T^*W and so define

$$N := V \times Y_\beta = V \times Z_\beta \times (T^*W)_+ \subset T^*W.$$

More generally, if W is a smooth variety, let $W^\circ \subseteq W$ be a β -stable affine open subset containing z ; one exists by [Su, Corollary 3.11]. Further shrinking W° if necessary, let

$$(5.1) \quad q : W^\circ \rightarrow \mathbb{A}^n$$

be a β -equivariant étale map from a β -stable affine open subset $W^\circ \subseteq W$ with $q(z) = 0$, where \mathbb{A}^n is a linear representation of \mathbb{G}_m (such a map can be obtained by taking an étale coordinate system of β -weight functions). Since the map q is étale, it induces a “wrong way” cotangent map $dq : T^*W^\circ \rightarrow T^*\mathbb{A}^n$ (note the slightly abusive notation). Defining $Z_\beta^\dagger = (T^*\mathbb{A}^n)^{\mathbb{G}_m}$ and $Y_\beta^\dagger \subset T^*\mathbb{A}^n$ to be the \mathbb{G}_m -attracting locus of Z_β^\dagger , then $Z_\beta \cap T^*W^\circ$ is a connected component of $dq^{-1}(Z_\beta^\dagger)$ and

$Y_\beta \cap T^*W^\circ$ is a connected component of $dq^{-1}(Y_\beta^\dagger)$. The tangent map $d(dq_z) : T_z(T^*W) \rightarrow T_0(T^*\mathbb{A}^n)$ is an isomorphism. Abusively writing $V = d(dq_z)(V) \subset T_0(T^*\mathbb{A}^n) = T^*\mathbb{A}^n$, we get

$$T^*\mathbb{A}^n = T_0(T^*\mathbb{A}^n) = \mathfrak{n}^- \oplus V \oplus T_0Y_\beta^\dagger = \mathfrak{n}^- \times V \times Y_\beta^\dagger.$$

Let N denote the connected component of $dq^{-1}(V \times Y_\beta^\dagger) \subset T^*W^\circ \subseteq E$ containing $Y_\beta \cap T^*W^\circ$. Note that Y_β^\dagger is coisotropic, hence so is $V \times Y_\beta^\dagger$, hence so is N .

Write $\text{pr}_\beta : Y_\beta \rightarrow Z_\beta$ for the projection as in (4.1).

Proposition 5.2. *For any $z \in Z_\beta^{\text{ss}}$ there are an affine neighborhood $D \subset Z_\beta^{\text{ss}}$ and an open subset $U_D \subset N$ such that*

- (1) U^- acts infinitesimally transversely to U_D ;
- (2) $\text{pr}_\beta^{-1}(D) \subseteq U_D \subseteq N$; and
- (3) $(U^- \cdot U_D) \cap S_{>\beta} = \emptyset$.
- (4) The complement $N \setminus U_D$ is the hypersurface defined by a β -invariant function in $\mathbf{C}[N]$.
- (5) U_D is coisotropic.

The sets D and U_D can be chosen so that:

- (i) D and U_D are β -stable.

Furthermore, making, for each $z \in Z_\beta^{\text{ss}}$, a choice of any affine $D_z \subset Z_\beta^{\text{ss}}$ containing z and any U_{D_z} satisfying the conditions above,

- (ii) The union $\bigcup_{z \in Z_\beta^{\text{ss}}} U^- \cdot U_{D_z}$ covers $U^- \cdot Y_\beta^{\text{ss}}$.

Remark 5.3. We show in Proposition 2.4 of [McN2] that, for the standard \mathbb{G}_m -action on T^*W contracting the fibers, the subset $U_D \subset T^*W$ can be chosen to be conical. Since we do not need this in the present paper, we omit it here.

We will use the following in the proof of the proposition.

Lemma 5.4. *Suppose E is an affine variety with torus \mathbf{T} acting and $Z \subseteq E$ is a closed \mathbf{T} -stable subset. Then Z is an intersection of \mathbf{T} -stable hypersurfaces in E .*

Proof of Proposition 5.2. Note that N has complementary dimension to \mathfrak{n}^- . Pick a principal open subset $D' \subseteq Z_\beta^{\text{ss}}$ containing z that is β -invariant (this is possible since $Z_\beta \setminus Z_\beta^{\text{ss}}$ is L_β -stable, and so cut out by $\beta(\mathbb{G}_m)$ -semi-invariants) and define $D \subseteq D'$ to be the open subset consisting of points $p \in D'$ at which the composite linear map

$$(5.2) \quad \text{inf}(p) : \mathfrak{n}^- \rightarrow T_p(T^*W) \rightarrow T_p(T^*W)/T_pN$$

is an isomorphism (where the first map is the infinitesimal action at p). Since we are asking whether the linear map $\text{inf}(p)$ has vanishing top exterior power, the complement of D in D' is the zero locus of a single function $\wedge^{\text{top}} \text{inf}$, i.e. a principal affine open D of Z_β along which U^- acts infinitesimally transversely to N . Now we can define U_D to be the locus of those $p \in N$ at which (5.2) is an isomorphism; this is again principal. This establishes part (1).

Next we show that by restricting D and U_D further if necessary we can ensure (3) holds. Indeed, since $S_{>\beta}$ is G -invariant, its preimage under the restricted action map $a : U^- \times U_D \rightarrow T^*W$ above is of the form $U^- \times Z$ for some closed and β -invariant subset $Z \subseteq U_D$. Since $z \notin S_{>\beta}$, we have $z \notin Z$. Replace U_D by a β -invariant principal affine open in $U_D \setminus Z$ that contains z (this is possible by Lemma 5.4) and replace D by $U_D \cap Z_\beta^{\text{ss}}$ for this new choice of U_D . Finally, to see that (2) holds, note that if $w \in \text{pr}_\beta^{-1}(D) \setminus U_D$, then by β -invariance of U_D and closure of its complement we have

$$\text{pr}_\beta(w) = \lim_{t \rightarrow 0} \beta(t) \cdot w \in Y_\beta^{\text{ss}} \setminus U_D.$$

Hence if $\text{pr}_\beta(w) \in D$ then $w \in U_D$. Thus these new choices of D and U_D satisfy both (2) and (3), as desired.

Let $H = N \setminus U_D$, a hypersurface in N . As it is β -stable, it must be defined by a β -semi-invariant function. Note, however, that if f is any β -semi-invariant function with nonzero β -weight, then, since D consists of β -fixed points, $f(D) = 0$; hence H must be defined by a β -invariant element of $\mathbf{C}[N]$. This proves (4). It is clear from the construction that (i) and (ii) are satisfied. We have already observed that N is coisotropic, hence so is its open subset U_D , proving (5). \square

Corollary 5.5. *Keep notation as in Proposition 5.2. Then, for each weight k , the natural map of β -semi-invariants of weight k*

$$\mathbf{C}[U^- \times N]^{\beta(\mathbb{G}_m),k} \otimes \mathbf{C}[U^- \times U_D]^{\beta(\mathbb{G}_m),0} \rightarrow \mathbf{C}[U^- \times U_D]^{\beta(\mathbb{G}_m),k}$$

is surjective. In particular, the closed subset of $U^- \times U_D$ defined by the vanishing of β -semi-invariants of weight k for $k \gg 0$ is $U^- \times (Y_\beta \cap U_D) \subset U^- \times U_D$.

Proof. This is immediate from Proposition 5.2 (4). \square

5.3. Symplectic Geometry Near the Stratum. Now choose an open subset $U_D \subseteq N \subseteq T^*W$ as in Proposition 5.2. Write $\mu^{U^-} : T^*W \rightarrow (\mathfrak{n}^-)^*$ for the U^- -moment map.

Lemma 5.6. *The moment map $m := \mu^{U^-}|_{U_D}$ is regular: in particular, $m^{-1}(0)$ is a smooth subvariety of U_D . Moreover, $m(D) = \mu^{U^-}(Z_\beta) = 0$, i.e., $D \subseteq Z_\beta^{\text{ss}} \cap m^{-1}(0)$.*

Proof. If $d\mu_p^{U^-}|_{T_p U_D}$ does not have full rank (i.e., linearly independent component functions) for some $p \in U_D$, then there exists $0 \neq X \in \mathfrak{n}^-$ with $d(\mu^{U^-})^X|_{T_p U_D} \equiv 0$. The vector field \tilde{X} on T^*W generated by X thus satisfies $\omega_{T^*W}(\tilde{X}_p, -) \equiv 0$ on $T_p U_D$. Thus $\tilde{X}_p \in (T_p U_D)^\perp$ (with respect to the symplectic form). Since U_D is coisotropic, we get $\tilde{X}_p \in T_p U_D$, contradicting infinitesimal transversality of the U^- -action with respect to U_D . This proves the first statement.

To see that Z_β^{ss} lies in the zero preimage of the moment map, note that the moment map at a point $(z_1, z_2) \in Z_\beta \cong T^*(W^{\beta(\mathbb{G}_m)})$ is the fiberwise dual to the infinitesimal action map $\mathfrak{n}^- \rightarrow T_{z_1} W$. Since this map is β -equivariant and \mathfrak{n}^- has only negative β -weights, we find that it factors via $\mathfrak{n}^- \rightarrow (T_{z_1} W)_- \hookrightarrow T_{z_1} W$. Consequently, the dual map $T_{z_1}^* W \rightarrow (\mathfrak{n}^-)^*$ factors through $(T_{z_1}^* W)_-$; in particular, it kills $z_2 \in (T_{z_1}^* W)_0$, and thus $\mu^{U^-}(z_1, z_2) = 0$. \square

Consider the étale chart $dq : T^*W^\circ \rightarrow T^*\mathbb{A}^n$ and the subspace $N^\dagger := V \times Y_\beta^\dagger \subset T^*\mathbb{A}^n$. Since N^\dagger is a coisotropic linear submanifold of a symplectic vector space, it has a linear symplectic quotient

$$(5.3) \quad \Pi : N^\dagger \rightarrow \mathcal{S},$$

i.e., the quotient by the null foliation, which equals the quotient by a linear subspace of N^\dagger . It can also be realized as a symplectic vector subspace of N^\dagger . In particular, if $\omega_{\mathcal{S}}$ denotes the induced (linear) symplectic form on \mathcal{S} and ω_r denotes the pullback of $\omega_{T^*\mathbb{A}^n}$, to N^\dagger , then

$$(5.4) \quad d\theta_r = \omega_r = \Pi^* \omega_{\mathcal{S}}.$$

Next, consider the (\mathbb{K} -equivariant) action map $a : U^- \times U_D \rightarrow T^*W$. By Proposition 5.2(1), a is injective on tangent spaces at points of U_D , hence by U^- -equivariance of a it is injective on tangent spaces at all points. The pullback $a^* \omega_{T^*W} = d(a^* \theta)$ is a symplectic form on $U^- \times U_D$.

Proposition 5.7. *Let $\phi : U^- \times U_D \rightarrow T^*U^- \times \mathcal{S}$ denote the unique left U^- -equivariant map extending the map*

$$\bar{\phi} : \{e\} \times U_D \rightarrow T_e^* U^- \times \mathcal{S} = (\mathfrak{n}^-)^* \times \mathcal{S}$$

defined by $\bar{\phi}(p) = (\mu^{U^-}(p), \Pi \circ dq(p))$. Then

$$(5.5) \quad \phi^*(d\theta_{U^-} + \omega_S) = d(a^*\theta_W).$$

That is, for symplectic forms: $a^*\omega_{T^*W} = \phi^*(\omega_{T^*U^-} + \omega_S)$.

Remark 5.8. In light of Remark 5.3, the proposition could be promoted to an equality of pullbacks of Liouville 1-forms for the standard contracting \mathbb{G}_m -action on T^*W (that remark being used to guarantee that the Liouville form is well defined on \mathcal{S}). We will not use this strengthening here.

Proof. Since both sides of (5.5) are left U^- -invariant, it suffices to check the equality along $\{e\} \times U_D$. First, note that under $da : T_e U^- \times T_p U_D \rightarrow T_p T^*W$, we get $da(v, w) = \tilde{v}_p + w$. By Lemma 2.7 we get $a^*\theta(v, w) = \theta(da(v, w)) = \langle \mu^{U^-}(p), v \rangle + \theta_W(w)$.

On the other hand, let $\phi_1 : U^- \times U_D \rightarrow T^*U^-$ denote the projection of ϕ on the first factor of the target and $\phi_2 : U^- \times U_D \rightarrow \mathcal{S}$ projection on the second factor. Considering $\mu^{U^-} : T^*W \rightarrow (\mathfrak{n}^-)^*$ as a vector-valued function, for $p \in T^*W$ write

$$d\mu_p^{U^-} : T_p(T^*W) \rightarrow T_{\mu(p)}(\mathfrak{n}^-)^* \xrightarrow{\sim} (\mathfrak{n}^-)^*$$

for the differential. Under $d\phi_1 : T_e U^- \times T_p U_D \rightarrow T_{(e, \mu(p))}(T^*U^-) \cong \mathfrak{n}^- \times (\mathfrak{n}^-)^*$, we then have $d\phi_1(v, w) = \tilde{v}_{\mu(p)} + (0, d\mu_p^{U^-}(w))$. Note that

$$\pi \circ \mu^{U^-} : U_D \rightarrow (\mathfrak{n}^-)^* = T_e^* U^- \hookrightarrow T^* U^- \rightarrow U^-$$

is constant; hence $d\pi_{\mu(p)} \circ d\mu_p^{U^-} \equiv 0$. Thus

$$(5.6) \quad \theta_{U^-}(d\phi_1(v, w)) = \theta(\tilde{v}_{\mu^{U^-}(p)}) + \theta(0, d\mu_p^{U^-}(w)) = \theta(\tilde{v}_{\mu^{U^-}(p)}) = \langle \mu^{U^-}(p), v \rangle$$

by definition of θ (for the second equality) and Lemma 2.7 (for the third). Moreover,

$$(5.7) \quad \Pi^*\omega_S = d\theta_r$$

as noted above. Adding the exterior derivative of (5.6) to (5.7) and comparing to the previous paragraph gives the conclusion. \square

We have that $U_D \cap (\mu^{U^-})^{-1}(0)$ is a symplectic submanifold of $N \subset T^*W$:

Corollary 5.9. *The Hamiltonian reduction of $U^- \times U_D$ for the U^- -action is isomorphic to $m^{-1}(0) = U_D \cap (\mu^{U^-})^{-1}(0)$.*

Proof. Since ϕ is symplectic (hence étale) and U^- -equivariant, the moment map on $U^- \times U_D$ is pulled back from $T^*U^- \times \mathcal{S}$, i.e., is $\mu^{U^-} \circ \phi$ where $\mu^{U^-} : T^*U^- \rightarrow (\mathfrak{n}^-)^*$ is the moment map for the left action. Since $(\mu^{U^-})^{-1}(0) = T_{U^-}^* U^-$ (i.e., the zero section), the conclusion follows. \square

The following is an immediate consequence of the above:

Proposition 5.10. *The pullback of the symplectic form along the composite map $U_D \cap (\mu^{U^-})^{-1}(0) \rightarrow N^\dagger \rightarrow \mathcal{S}$ equals the symplectic form on $U_D \cap (\mu^{U^-})^{-1}(0) = m^{-1}(0) \subset U_D$. It follows that the maps ϕ and $m^{-1}(0) = U_D \cap (\mu^{U^-})^{-1}(0) \rightarrow \mathcal{S}$ are étale.*

6. DEFORMATION QUANTIZATION NEAR A KN STRATUM

Throughout Section 6 we assume a connected reductive G acts on the smooth variety W . We assume that W has trivialized canonical bundle and that G acts on $K_W = W \times \mathbb{C}$ via the character γ_G , yielding a canonical quantum comoment map μ^{can} . We also assume that T^*W comes equipped with a KN stratification, and that the line bundle \mathcal{L} is trivialized with its G -equivariant structure defined by a character $\chi : G \rightarrow \mathbb{G}_m$. We fix a \mathbb{G}_m -action on T^*W defining a filtration of $\mathcal{D}(W)$, if W is a representation, or the operator filtration if W is not.

6.1. A Quantum Bimodule. We fix a 1-parameter subgroup $\beta : \mathbb{G}_m \rightarrow G$ labeling a KN stratum and write $\mathbb{K} = U^- \ltimes \mathbb{G}_m$ as in Section 5.1.

As in Section 5, for convenience of notation we abbreviate $S_{\beta,i}$ to S_β in this section.

Passing to the subgroup \mathbb{K} and choosing a slice U_D near a point $z \in Z_\beta^{\text{ss}}$ as in Section 5, we write

$$E := U^- \times U_D.$$

An equivariant deformation quantization $(\mathbf{C}[E][\hbar], *)$ is provided by the Moyal-Weyl product $*$. Henceforth we write $\mathcal{O}^\hbar(E)$ for this deformation quantization (with the Moyal-Weyl product); similarly, we write $\mathcal{O}^\hbar(\mathcal{S})$ for $\mathbf{C}[\mathcal{S}][\hbar]$ with its Moyal-Weyl product. Fix the canonical quantum comoment map μ^{U^-} as defined at the end of Section 2.5.1, and abusively use the same notation for the classical comoment map associated to the quantum map at $\hbar = 0$. Let

$$M_{U^-} = \mathcal{O}^\hbar(E) / \mathcal{O}^\hbar(E) \mu^{U^-}(\mathfrak{n}^-);$$

this is a left $\mathcal{O}^\hbar(E)$ -module. Define a linear map $\text{act} : \mathbf{C}[\mathcal{S}] \rightarrow \text{End}_{\mathcal{O}^\hbar(E)}(M_{U^-})$ by $\text{act}(f)(m) = m * (\pi_{\mathcal{S}} \circ \phi)^*(f)$, where ϕ is the map from Proposition 5.7 and $\pi_{\mathcal{S}} : T^*U^- \times \mathcal{S} \rightarrow \mathcal{S}$ is projection.

Proposition 6.1.

- (1) *The map act is well-defined: for each $f \in \mathbf{C}[\mathcal{S}]$, $\text{act}(f)$ is an $\mathcal{O}^\hbar(E)$ -module endomorphism of M_{U^-} .*
- (2) *The \hbar -linear extension of the map act defines an algebra homomorphism*

$$\mathcal{O}^\hbar(\mathcal{S}) \longrightarrow \text{End}_{\mathcal{O}^\hbar(E)}(M_{U^-}).$$

Proof. Elements of $(\pi_{\mathcal{S}} \circ \phi)^* \mathbf{C}[\mathcal{S}]$ are U^- -invariant, hence normalize $\mu^{U^-}(\mathfrak{n}^-)$ by part (1) of Definition 2.5; the map act is thus well-defined. By Proposition 5.7, pullback by $\pi_{\mathcal{S}} \circ \phi$ intertwines symplectic forms, hence Poisson brackets, hence Moyal-Weyl quantizations. \square

6.2. Comparison of Canonical Comoment Maps. Keeping the notation of Section 6.1, compatibility of quantization with étale maps implies:

Lemma 6.2. *The map $a^{-1} : \mathcal{O}^\hbar(T^*W^\circ) \rightarrow \mathcal{O}^\hbar(E)$ intertwines the Moyal-Weyl products.*

Lemma 6.2 allows us to consider the composite homomorphism

$$\mathfrak{g} \xrightarrow{\mu^{\text{can}}} \mathcal{O}^\hbar(T^*W^\circ) \xrightarrow{a^{-1}} \mathcal{O}^\hbar(E).$$

Proposition 6.3. *Let $\text{eu}_{T^*W}(\beta)$, $\text{eu}_{\mathcal{S}}(\beta)$ denote the (canonically-shifted) Euler operators for the β -action on T^*W and \mathcal{S} , respectively: that is, the images of the canonical generator of $\text{Lie}(\mathbb{G}_m)$ under the canonical quantum comoment maps associated to the β -actions (cf. Section 2.2). Then, letting $1 \in M_{U^-}$ denote the canonical generator, we have*

$$(6.1) \quad \text{eu}_{T^*W}(\beta) * 1 = 1 * \text{eu}_{\mathcal{S}}(\beta) - \frac{\hbar}{2} \text{wt}_{\mathfrak{n}^-}(\beta),$$

where $\text{wt}_{\mathfrak{n}^-}(\beta)$ denotes the weight of β on $\bigwedge^{\text{top}} \mathfrak{n}^-$.

We have maps

$$T^*U^- \times \mathcal{S} \leftarrow U^- \times U_D \rightarrow T^*W$$

which are étale, symplectic, and equivariant for the \mathbb{G}_m -actions via β . It follows that the canonically shifted Euler operators for the β -action are identified via pullbacks. Hence it suffices to prove the desired equality of actions on the bimodule $\mathcal{O}^\hbar(T^*U^- \times \mathcal{S}) / \mathcal{O}^\hbar(T^*U^- \times \mathcal{S}) \mu(\mathfrak{n}^-)$ (since this pulls back to M_{U^-}). Using the exponential map to identify \mathfrak{n}^- with U^- equivariantly for β , one calculates in coordinates. We explain this in more detail in Section 6.3 below.

6.3. Proof of Proposition 6.3. Since $U^- \times U_D \xrightarrow{a} T^*W^\circ \xrightarrow{dq} T^*\mathbb{A}^n$ and $\phi : U^- \times U_D \rightarrow T^*U^- \times \mathcal{S}$ are étale and \mathbb{G}_m -equivariant (for the β -actions), we obtain classical moment maps

$$\mu_{\beta, U^- \times U_D} = a^* \mu_{\beta, T^*W} = a^* dq^* \mu_{\beta, T^*\mathbb{A}^n}, \quad \phi^* \mu_{\beta, T^*U^- \times \mathcal{S}}$$

on $U^- \times U_D$ by pullback. Proposition 5.7 and Lemma 2.7 now guarantee that they agree: $\mu_{\beta, U^- \times U_D} = \phi^* \mu_{\beta, T^*U^- \times \mathcal{S}}$. It now follows from Lemma 2.10 that the image of $\text{eu}_{T^*U^- \times \mathcal{S}}(\beta)$ in $\mathcal{O}^h(\mathbf{E})$ equals the image of $\text{eu}_{T^*W}(\beta)$. Thus, as explained above, it suffices to prove the equality of actions of Proposition 6.3—with $\text{eu}_{T^*W}(\beta)$ on the left-hand side of (6.1) replaced by $\text{eu}_{T^*U^- \times \mathcal{S}}(\beta)$ —on

$$\mathcal{O}^h(T^*U^- \times \mathcal{S}) / \mathcal{O}^h(T^*U^- \times \mathcal{S}) \mu(\mathfrak{n}^-).$$

Note that $\text{eu}_{T^*U^- \times \mathcal{S}}(\beta) = \text{eu}_{T^*U^-}(\beta) + \text{eu}_{\mathcal{S}}(\beta)$.

The exponential map identifies U^- as a \mathbb{G}_m -variety (for the β -action) with \mathfrak{n}^- . Choose a basis e_1, \dots, e_s for \mathfrak{n}^- consisting of β -weight vectors with corresponding coordinate functions e_i^* on U^- . We may then write $\mu_{\beta, T^*U^-} = -\sum_i \text{wt}(e_i) e_i^* e_i$, where e_i is viewed as a function on $(\mathfrak{n}^-)^*$ and hence on T^*U^- via projection and $\text{wt}(e_i)$ denotes the β -weight on e_i . As in Section 2.5.1,

$$(6.2) \quad e_i^* * e_i = e_i^* \cdot e_i - \frac{\hbar}{2}.$$

To complete the proof, it therefore suffices to compute:

$$\begin{aligned} \text{Symm}^{-1}(\text{eu}_{T^*U^- \times \mathcal{S}}(\beta)) * 1 &= \text{Symm}^{-1}(\text{eu}_{\mathcal{S}}(\beta) + \text{eu}_{T^*U^-}(\beta)) * 1 \\ &= 1 * \left[\text{Symm}^{-1}(\text{eu}_{\mathcal{S}}(\beta)) - \sum_i \text{wt}(e_i) \left(e_i^* * e_i + \frac{\hbar}{2} \right) \right] \quad \text{from (6.2)} \\ &= 1 * \text{Symm}^{-1}(\text{eu}_{\mathcal{S}}(\beta)) - \frac{\hbar}{2} \text{wt}_{\mathfrak{n}^-}(\beta), \end{aligned}$$

as claimed. \square

6.4. A Split Surjection of DQ Modules. We maintain the notation of Sections 6.1 and 6.2, we write $\mathbf{A}_{\mathcal{S}}$ for the Weyl algebra associated to the linear symplectic space \mathcal{S} . Suppose that \mathbf{c} satisfies condition (3.7) for $\mathbf{A}_{\mathcal{S}}$, and let $M_{\mathbf{c}}^{\beta}(\chi^{\ell}) \otimes V_{\ell} \rightarrow M_{\mathbf{c}}^{\beta}$ denote the split surjection of $\mathbf{A}_{\mathcal{S}}$ -modules from (3.8). Note that by [McN, page 10], $M_{\mathbf{c}}^{\beta}(\chi^{\ell}) \cong M_{\mathbf{c}-\ell d\chi} \otimes \chi^{\ell}$.

Recall that $\mathbb{K} = U^- \ltimes \beta(\mathbb{G}_m)$. Let $\mathbf{k} = \text{Lie}(\mathbb{K})$. By Section 2.5 and Proposition 6.1, we have a homomorphism $\mathbf{A}_{\mathcal{S}} \rightarrow \text{End}(M_{U^-})[\hbar^{-1}]$; tensoring up yields a split surjection of \mathbb{K} -equivariant $\mathcal{O}^h(U^- \times U_D)$ -modules

$$(6.3) \quad M_{U^-}[\hbar^{-1}] \otimes_{\mathbf{A}_{\mathcal{S}}} M_{\mathbf{c}}^{\beta}(\chi^{\ell}) \otimes V_{\ell} \rightarrow M_{U^-}[\hbar^{-1}] \otimes_{\mathbf{A}_{\mathcal{S}}} M_{\mathbf{c}}^{\beta}.$$

Proposition 6.4. *Suppose that \mathbf{c} satisfies the condition (3.7) for $\mathbf{A}_{\mathcal{S}}$. Then the split surjection (6.3) induces a split surjection of \mathbb{K} -equivariant DQ modules*

$$(6.4) \quad (\mathcal{O}^h(\mathbf{E}) / \mathcal{O}^h(\mathbf{E}) \mu_{\mathbf{c}-\ell d\chi}^{\text{can}}(\mathbf{k}))[\hbar^{-1}] \otimes \chi^{\ell} \otimes V_{\ell} \longrightarrow (\mathcal{O}^h(\mathbf{E}) / \mathcal{O}^h(\mathbf{E}) \mu_{\mathbf{c}}^{\text{can}}(\mathbf{k}))[\hbar^{-1}],$$

where $c := \mathbf{c} + \frac{1}{2} \text{wt}_{\mathfrak{n}^-}(\beta)$.

Proof. This is immediate from Lemma 2.10 and Proposition 6.3. \square

6.5. Vanishing for \mathcal{D} -Modules Along a KN Stratum. We now want to study (G, c) -equivariant \mathcal{D} -modules microlocally near a KN stratum using the tools described in the previous section. In particular, we want to establish a vanishing statement that we can use in inductive arguments.

Thus, suppose M is a (G, c) -equivariant \mathcal{D}_W -module and that, with respect to the filtration that we are fixing, we have $SS(M) \subseteq S_{\geq \beta, i}$ (a union of KN strata). The main aim of Section 6.5 is to prove that, if an appropriate condition on c is satisfied, then any G -invariant section of M lies in a submodule of M with singular support in $S_{> \beta, i}$. We do this by first reducing M to an equivariant DQ module on an open set of \mathcal{S} and then using Corollary 3.8 for the residual β -action.

Hence, let $M_c^G = \mathcal{D}/\mathcal{D}\mu_c^{\text{can}}(\mathfrak{g})$ (notation as in Section 2.4), and suppose we have a (weakly) G -equivariant map $\phi : M_c^G \rightarrow M$.

First, suppose that W is a G -representation. As before, we write the list of \mathbf{T} -weights on the d -dimensional vector space W as $\{\alpha_1, \dots, \alpha_d\}$. Recall the definition of $I_{G, T^*W}(\beta)$ from Section 4.8. We will prove:

Theorem 6.5. *Suppose $SS(M) \subseteq S_{\geq \beta, i}$ and that c satisfies:*

$$(6.5) \quad c(\beta) \notin \left(I_{G, T^*W}(\beta) + \text{wt}_{\mathbf{n}^-}(\beta) + \frac{1}{2} \sum_{i=1}^d |\alpha_i \bullet \beta| \right).$$

Then $m = \phi(\mathbf{1})$ lies in a G -invariant submodule M' of M with singular support $SS(M') \subseteq S_{> \beta, i}$.

We begin by fixing some notation and observing a formula. We use the conventions of Section 5. Thus, with $\beta : \mathbb{G}_m \rightarrow G$ a fixed KN 1-parameter subgroup, $U^- = U^-(\beta)$ denotes the unipotent subgroup associated to the Lie subalgebra $\mathfrak{n}^- = \mathfrak{n}^-(\beta) \subset \mathfrak{g}$ on which β has negative weights. Our construction of the slice U_D to the infinitesimal \mathfrak{n}^- -action near a point of Z_β and of the reduced space \mathcal{S} implies that, as \mathbb{G}_m -representations via β , we have

$$(6.6) \quad T^*W \cong \mathcal{S} \oplus T^*\mathfrak{n}^-.$$

Note that this implies the following formula. For a \mathbb{G}_m -representation \mathbf{V} , let $\text{wt}(\mathbf{V})$ denote the list of weights *with multiplicities included*: so if the weight 3 appears four times, then the list $\text{wt}(\mathbf{V})$ will include the number 3 four times. Recall that we write $\text{wt}_{\mathbf{n}^-}(\beta) = -\sum_{w_k \in \text{wt}(\mathfrak{n}^-)} |w_k|$ for the sum of β -weights on \mathfrak{n}^- (which, by definition of \mathfrak{n}^- , are all negative). It follows from the direct sum decomposition (6.6) that the following holds for β -weights:

$$(6.7) \quad 2 \sum_{i=1}^d |\alpha_i \bullet \beta| = \sum_{w_j \in \text{wt}(T^*W)} |w_j| = \left(\sum_{w_j \in \text{wt}(\mathcal{S})} |w_j| \right) - 2 \text{wt}_{\mathbf{n}^-}(\beta).$$

Thus, the condition (6.5) becomes

$$c(\beta) \notin I_{G, T^*W}(\beta) + \frac{1}{4} \left(\sum_{w_j \in \text{wt}(\mathcal{S})} |w_j| \right) + \frac{1}{2} \text{wt}_{\mathbf{n}^-}(\beta).$$

The splitting (6.6) implies that $I_{G, T^*W}(\beta) = I_{\mathbb{G}_m, \mathcal{S}}(\beta)$. Thus, (6.5) is equivalently written as

$$(6.8) \quad c(\beta) - \frac{1}{2} \text{wt}_{\mathbf{n}^-}(\beta) \notin I_{\mathbb{G}_m, \mathcal{S}}(\beta) + \frac{1}{4} \left(\sum_{w_j \in \text{wt}(\mathcal{S})} |w_j| \right).$$

Consequently, Theorem 6.5 is a special case of the following.

Let W be a smooth variety with trivialized canonical bundle and canonical quantum comoment map μ^{can} . Suppose that T^*W is equipped with a KN stratification.

Theorem 6.6. *Suppose $SS(M) \subseteq S_{\geq \beta, i}$ and that c satisfies (6.8). If $\phi : \mathcal{M}_c \rightarrow M$ is a G -equivariant homomorphism, then $m = \phi(\mathbf{1})$ lies in a G -invariant submodule M' of M with singular support $SS(M') \subseteq S_{> \beta, i}$.*

Proof of Theorem 6.6. Throughout the proof we label strata S_β rather than $S_{\beta, i}$, i.e., we suppress the mention of connected components of Z_β .

We may replace M by $\phi(\mathcal{M}_c)$ and hence assume that M is cyclic, generated by the G -invariant vector $m = \phi(\mathbf{1})$. We give M the induced filtration from \mathcal{M}_c . We get $\mathcal{R}(\mathcal{M}_c) \twoheadrightarrow \mathcal{R}(M)$ and

$$(6.9) \quad \mathcal{R}(\mathcal{M}_c)^h \twoheadrightarrow \mathcal{R}(M)^h,$$

surjections of Rees modules and DQ modules. Keep the notation of Section 6.4.

It clearly suffices to show that there is an open cover of $T^*W \setminus S_{> \beta}$ by affines U_α such that the map (6.9) (equivalently, its target) vanishes on restriction to each U_α , or in other words that $\mathcal{R}(M)^h|_{U_\alpha} = 0$. In fact, however, it suffices to do something weaker. By Proposition 2.12, $\mathcal{R}(M)^h$ is supported on $S_{\geq \beta}$. Since $\mathcal{R}(M)^h$ is weakly G -equivariant, $U^- \cdot Y_\beta^{\text{ss}}$ is open in S_β , and $S_\beta = G \cdot (U^- \cdot Y_\beta^{\text{ss}})$ (where, as in Section 5, U^- is the negative unipotent subgroup associated to β), it suffices to show that each restriction $\mathcal{R}(M)^h|_{U_\alpha}$ is zero for some collection $\{U_\alpha\}$ of open sets in $T^*W \setminus S_{> \beta}$ whose union contains $U^- \cdot Y_\beta^{\text{ss}}$. By Proposition 5.2(ii), the images $a(U^- \times U_D)$ of the affine varieties $U^- \times U_D$ constructed in Section 5 form such a collection of U_α . Now, applying a^{-1} to (6.9), inverting \hbar , and “forgetting” to $\mathbb{K} = U^- \ltimes \beta(\mathbb{G}_m)$ gives a \mathbb{K} -equivariant surjection

$$(6.10) \quad \alpha_m : (\mathcal{O}_E^h / \mathcal{O}_E^h \mu_c^{\text{can}}(\mathbf{k}))[\hbar^{-1}] \rightarrow a^{-1} \mathcal{R}(M)^h[\hbar^{-1}],$$

where $E = U^- \times U_D$ and α_m denotes the evaluation on the cyclic vector $m \in M$.

By the above discussion it suffices to show that for each $z \in Z_\beta^{\text{ss}}$ and the corresponding map $a : U^- \times U_D \rightarrow T^*W$ constructed in Proposition 5.2, the map (6.10) vanishes. In fact, it is better to replace $a^{-1} \mathcal{R}(M)^h[\hbar^{-1}]$ by a slightly smaller module. Namely, note that, since S_β is smooth and a is étale, $a^{-1}(S_\beta)$ is a finite disjoint union $a^{-1}(S_\beta) = \coprod C_i$ of closed, U^- -invariant subvarieties C_i , one connected component C_0 of which is $U^- \times (Y_\beta^{\text{ss}} \cap U_D) \subseteq U^- \times U_D$. The decomposition of $a^{-1}(S_\beta)$ into connected components determines a direct sum decomposition $a^{-1} \mathcal{R}(M)^h[\hbar^{-1}] = \oplus M_i := \oplus a^{-1} \mathcal{R}(M)^h[\hbar^{-1}]|_{C_i}$. Because a is étale, the section m will be zero in a neighborhood of $z \in Z_\beta^{\text{ss}}$ if and only if its preimage $a^{-1}(m)$ is zero when projected to some direct summand $a^{-1} \mathcal{R}(M)^h[\hbar^{-1}]|_{C_i}$ for which $a^{-1}(z) \cap C_i \neq \emptyset$. Thus, in order to prove that the map (6.10) vanishes in a neighborhood of $a^{-1}(z)$, it suffices to prove that the composite of (6.10) with the projection to the direct summand $M' = a^{-1} \mathcal{R}(M)^h[\hbar^{-1}]|_{C_0}$ vanishes. Let m' be the summand of m generating the module M' .

Thus, we have a \mathbb{K} -equivariant surjective map

$$(\mathcal{O}^h(E) / \mathcal{O}^h(E) \mu_c^{\text{can}}(\mathbf{k}))[\hbar^{-1}] \xrightarrow{- \cdot m'} M',$$

with $\text{supp}(M') \subset U^- \times Y_\beta^{\text{ss}}$. Let $M'(0) = \mathcal{O}^h(E) \cdot m' \subset M'$; this is a lattice in M' .

Assume the condition (6.8) is satisfied. Then, writing $c = c(\beta) - \frac{1}{2} \text{wt}_{\mathbf{n}^-}(\beta)$, Proposition 6.4 gives a split surjection as in (6.4).

Claim 6.7. If $M' \neq \{0\}$, then $M'(0)/\hbar M'(0)$ has nonzero $(\chi \circ \beta)^\ell$ -isotypic component for all $\ell \ll 0$:

$$\text{Hom}_{\mathbb{G}_m}((\chi \circ \beta)^\ell, M'(0)/\hbar M'(0)) = [M'(0)/\hbar M'(0) \otimes (\chi \circ \beta)^{-\ell}]^{\mathbb{G}_m} \neq 0.$$

Proof of Claim. If $M' \neq \{0\}$, then the split surjection of (6.4) shows that

$$\text{Hom}((\mathcal{O}^h(E) / \mathcal{O}^h(E) \mu_{c-\ell d\chi}^{\text{can}}(\mathbf{k}))[\hbar^{-1}] \otimes \chi^\ell, M')^{\mathbb{K}} \neq 0$$

for $\ell \ll 0$. In particular, $\text{Hom}_{\mathbb{G}_m}((\chi \circ \beta)^\ell, M') \neq 0$. If $n \in M'$ is a nonzero $(\chi \circ \beta)^\ell$ -isotypic vector, then there is some a for which $\hbar^a \cdot n \in M'(0) \setminus \hbar M'(0)$, and the image of $\hbar^a \cdot n$ in $M'(0)/\hbar M'(0)$ is thus a nonzero $(\chi \circ \beta)^\ell$ -isotypic vector. \square

Claim 6.8. $\text{Hom}_{\mathbb{G}_m}((\chi \circ \beta)^\ell, M'(0)/\hbar M'(0)) = 0$ for $\ell \ll 0$.

Proof of Claim. The quotient $M'(0)/\hbar M'(0)$ is a finitely generated $\mathbf{C}[E]$ -module set-theoretically supported on $U^- \times (Y_\beta^{\text{ss}} \cap U_D)$. By Corollary 5.5, this support is cut out by the $(\chi \circ \beta)^q$ -semi-invariants in $\mathbf{C}[E]$ for $q \gg 0$. Thus, for $q \gg 0$, every $(\chi \circ \beta)^q$ -semi-invariant in $\mathbf{C}[E]$ kills $M'(0)/\hbar M'(0)$. It follows by (2.2) that for $q \gg 0$, every $(\chi \circ \beta)^{-q}$ -isotypic vector $f \in \mathbf{C}[E]$ kills $M'(0)/\hbar M'(0)$. But now $M'(0)/\hbar M'(0)$ is generated by a $\beta(\mathbb{G}_m)$ -invariant vector, the image of $\mathbf{1}$. Thus, if $n \in M'(0)/\hbar M'(0)$, $n = f \cdot \mathbf{1}$ for some $f \in \mathbf{C}[E]$; and if n is $(\chi \circ \beta)^\ell$ -isotypic, we may choose a $(\chi \circ \beta)^\ell$ -isotypic $f \in \mathbf{C}[E]$ in this expression (using reductivity of \mathbb{G}_m). Now for $q = -\ell \gg 0$, we use that every such f kills $M'(0)/\hbar M'(0)$ to conclude that $n = f\mathbf{1} = 0$. This proves the claim. \square

Claims 6.7 and 6.8 imply that $M' = 0$; by the discussion above, this suffices. \square

7. t -EXACTNESS FOR QUANTUM DIRECT IMAGES AND MICROLOCALIZATION

In this section we first prove the main vanishing statement for quantum Hamiltonian reduction and the resulting existence of a certain split surjective homomorphism, treating first the case in which the variety W is a G -representation (Theorem 7.2, Section 7.1) and then its extension to more general smooth W (Theorem 7.4, Section 7.2).

At the time of writing, there are several different technical frameworks available for quantum geometry. Although one essentially knows that all of these are equivalent, there is not yet a systematic treatment of such equivalences between all frameworks available in the literature. Hence, in Sections 7.3 and 7.4 we briefly explain how Theorem 7.2 implies t -exactness results in two such frameworks, the \mathcal{W} -algebras of Kashiwara, Schapira, et al. and the categorical framework of [McN]. Extensions of our results to other frameworks are equally straightforward.

It is convenient to use the following, where we assume the same hypotheses as in Section 6.

Lemma 7.1. *Suppose that M is a (G, c) -equivariant \mathcal{D} -module with $SS(M) \subseteq S_{\geq \beta, i}$. If $SS(M) \cap S_{\beta, i} \neq \emptyset$, then $S_{\beta, i} \cap \mu^{-1}(0)$ contains a nonempty coisotropic subvariety.*

Proof. Since M is (G, c) -equivariant, $SS(M) \subseteq \mu^{-1}(0)$. Also, $SS(M)$ is coisotropic; in particular, each irreducible component of $SS(M)$ is coisotropic. Since $S_{\beta, i}$ is open in $S_{\geq \beta, i}$, it follows from the hypothesis of the lemma that $(S_\beta \cap \mu^{-1}(0)) \cap SS(M) = S_\beta \cap SS(M)$ is empty or coisotropic. \square

Recall from the introduction the subset KN° of KN stratum labels:

$$\text{KN}^\circ = \{(\beta, i) \in \text{KN} \mid S_{\beta, i} \cap \mu^{-1}(0) \text{ contains a nonempty coisotropic subset}\}.$$

7.1. Vanishing and Split Surjections for Representations. Suppose first that W is a representation of G . Choose any total ordering refining $>$ on KN 1-parameter subgroups.

An ascending induction on β (meaning a descending induction on strata!) yields:

Theorem 7.2. *Let W be a representation of G . Assume that, for every KN 1-parameter subgroup $\beta \in \text{KN}^\circ$ for the G -action on T^*W , c satisfies*

$$(7.1) \quad c(\beta) \notin \left(I_{G, T^*W}(\beta) + \text{wt}_{\mathbf{n}^-}(\beta) + \frac{1}{2} \sum_{i=1}^d |\alpha_i \bullet \beta| \right).$$

Then:

- (1) *If $M \in (\mathcal{D}, G, c) - \text{mod}$ with $SS(M) \subseteq (T^*W)^{\text{uns}}$, then $\text{Hom}_{(\mathcal{D}, G, c)}(M_c, M) = 0$.*

(2) For every $\ell \ll 0$, there is a finite-dimensional vector subspace

$$V_\ell \subset \mathrm{Hom}_{(\mathcal{D}, G, c)}(M_c(\chi^\ell), M_c)$$

for which the natural composite evaluation map

$$(7.2) \quad M_c(\chi^\ell) \otimes V_\ell \longrightarrow M_c(\chi^\ell) \otimes \mathrm{Hom}_{(\mathcal{D}, G, c)}(M_c(\chi^\ell), M_c) \longrightarrow M_c$$

is a split surjective homomorphism of objects of $(\mathcal{D}, G, c) - \mathrm{mod}$.

Proof. (1) follows by induction on β using Theorem 6.5. For (2), the argument is similar to that of part (2) of Corollary 3.8: Given $\ell \ll 0$, the cokernel of the evaluation map $M_c(\chi^\ell) \otimes \mathrm{Hom}_{(\mathcal{D}, G, c)}(M_c(\chi^\ell), M_c) \longrightarrow M_c$ is a (\mathcal{D}, G, c) -module with χ -unstable support. Choose any finite-dimensional subspace V_ℓ as in the statement of the theorem such that the cokernel of the composite map (7.2) still has χ -unstable support. Part (1) of the theorem then yields a surjection

$$\mathrm{Hom}(M_c, M_c(\chi^\ell) \otimes V_\ell) \twoheadrightarrow \mathrm{Hom}(M_c, M_c),$$

and any element in $\mathrm{Hom}(M_c, M_c(\chi^\ell) \otimes V_\ell)$ in the preimage of $\mathrm{Id} \in \mathrm{Hom}(M_c, M_c)$ provides a splitting as claimed in part (2). \square

7.2. Vanishing for More General Varieties. Now suppose that W is a smooth, connected G -variety. We fix the operator filtration on \mathcal{D}_W .³ We assume:

- (i) the canonical bundle K_W is trivial and is G -equivariantly isomorphic to \mathcal{O}_W twisted by a character $\gamma_G : G \rightarrow \mathbb{G}_m$.

We note that this assumption is harmless. Indeed, replace W by the principal \mathbb{G}_m -bundle $\widetilde{W} \xrightarrow{\pi} W$ whose points correspond to nonzero vectors in the canonical bundle K_W . The G -action on W lifts to a G -action on \widetilde{W} , commuting with the \mathbb{G}_m -action scaling the fibers. As usual, the long exact sequence of cotangent bundles for the fibration $\widetilde{W} \rightarrow W$ induces a $G \times \mathbb{G}_m$ -equivariant isomorphism

$$(7.3) \quad K_{\widetilde{W}} \cong \pi^* K_W \otimes K_{\widetilde{W}/W}.$$

The pullback $\pi^* K_W$ has a nonvanishing universal section σ_1 which is G -invariant; under the \mathbb{G}_m -action, σ_1 is rescaled by a character. Furthermore, $K_{\widetilde{W}/W}$ has a nonvanishing $G \times \mathbb{G}_m$ -invariant section σ_2 , dual to the infinitesimal generator of the \mathbb{G}_m -action on \widetilde{W} (it is \mathbb{G}_m -invariant since \mathbb{G}_m is commutative, and it is G -invariant since the G -action commutes with the \mathbb{G}_m -action). Under the isomorphism (7.3), the tensor product $\sigma_1 \otimes \sigma_2$ defines a nonvanishing section of $K_{\widetilde{W}}$ that, by the preceding discussion, is scaled by a character of $G \times \mathbb{G}_m$. Thus \widetilde{W} satisfies condition (i) for the $G \times \mathbb{G}_m$ -action. Now any quantum Hamiltonian reduction of \mathcal{D}_W by G comes from a quantum Hamiltonian reduction of $\mathcal{D}_{\widetilde{W}}$ under $G \times \mathbb{G}_m$.

We define the ρ -shift and canonical quantum comoment map as in Section 2.2; we use these to define (G, c) -equivariant \mathcal{D} -modules. We assume also that:

- (ii) T^*W comes with a KN stratification, and
- (iii) the G -equivariant line bundle \mathcal{L} defined by twisting \mathcal{O}_{T^*W} by a character $\chi : G \rightarrow \mathbb{G}_m$ is adapted to the KN stratification in the sense of Definition 4.4.

Remark 7.3. By Proposition 4.16, if W is affine and the polarization \mathcal{L} is trivial and determined by a character χ , then T^*W possesses a KN stratification to which \mathcal{L} is adapted.

For each $\alpha = (\beta, i) \in \mathrm{KN}$ we define $I_{G, T^*W}(\beta, i)$ and $\mathrm{abs}\text{-}\mathrm{wt}_{N_{Z_{\beta, i}/T^*W}}(\beta)$ as in the introduction.

³We remark, however, that the statements and proofs work more generally; for example, if W is affine and equipped with a contracting \mathbb{G}_m -action commuting with G , we can give $\mathcal{D}(W)$ a *Kazhdan-type filtration* as in Section 4 of [GG].

Theorem 7.4. *Let W be a smooth connected variety satisfying (i), (ii), and (iii) above. Suppose that for every $(\beta, i) \in \text{KN}^\circ$, we have*

$$(7.4) \quad c(\beta) \notin \left(I_{G, T^*W}(\beta, i) + \text{wt}_{\mathfrak{n}^-}(\beta) + \frac{1}{4} \text{abs-wt}_{N_{Z_{\beta, i}/T^*W}}(\beta) \right).$$

Then:

- (1) *If M is any object of $(\mathcal{D}_W, G, c) - \text{mod}$ with $SS(M) \subseteq (T^*W)^{uns}$, then*

$$\text{Hom}_{(\mathcal{D}, G, c)}(\mathcal{M}_c, M) = 0.$$

Suppose in addition that W is an affine variety. Then:

- (2) *For every $\ell \ll 0$, there is a finite-dimensional vector subspace*

$$V_\ell \subset \text{Hom}_{(\mathcal{D}, G, c)}(M_c(\chi^\ell), M_c)$$

for which the natural composite evaluation map

$$M_c(\chi^\ell) \otimes V_\ell \longrightarrow M_c(\chi^\ell) \otimes \text{Hom}_{(\mathcal{D}, G, c)}(M_c(\chi^\ell), M_c) \longrightarrow M_c$$

is a split surjective homomorphism of objects of $(\mathcal{D}, G, c) - \text{mod}$.

Proof. Repeat the proof of 7.2 using Theorem 6.6 in place of Theorem 6.5. \square

7.3. Application to Microlocalization for \mathcal{W} -Algebras. The split surjection (7.2) provides a versatile tool: applying any additive functor to it, we obtain a split surjection of the resulting objects. Thus, for example, inducing to modules over a \mathbb{G}_m -equivariant formal deformation quantization as in [KS, Chapter 6] we also obtain split surjections of \mathbb{G}_m -equivariant DQ-modules; further applying symplectic reduction as in [KR, §2.5] then yields split surjections of quantized line bundles on the symplectic quotient corresponding to the characters χ^ℓ .

More precisely, starting from the canonical \mathcal{W} -algebra on T^*W for a smooth affine variety W with action of a connected reductive G for which the classical moment map μ is flat, suppose one gets as the GIT quotient at the character χ a smooth symplectic variety $\mathfrak{X} = \mu^{-1}(0) //_\chi G$. Let $\mathcal{W}_{\mathfrak{X}}(c)$ be the \mathcal{W} -algebra on \mathfrak{X} obtained by quantum Hamiltonian reduction using the quantum comoment map μ_c^{can} , as in [KR, §2.5]. It is standard in GIT that the sequence of line bundles on \mathfrak{X} associated to the characters χ^ℓ is ample. Hence:

Theorem 7.5. *Suppose the hypothesis on c of Theorem 7.2 is satisfied. Then for every good $\mathcal{W}_{\mathfrak{X}}(c)$ -module M , we have $H^i(M) = 0$ for $i \neq 0$. In particular, the global section functor is an exact functor of good $\mathcal{W}_{\mathfrak{X}}(c)$ -modules.*

Proof. Inducing the split surjection of Theorem 7.2 to $\mathcal{W}_{\mathfrak{X}}(c)$ -modules implies that condition (2.5) of Theorem 2.9 of [KR] is satisfied. \square

Remark 7.6. The argument of [KR] using the split surjection is in essence extremely general, and can be readily adapted to essentially any other reasonable framework for sheaves of quantum algebras deforming a symplectic variety.

7.4. Application to Microlocalization for Localized Categories. Assume that the classical moment map μ is flat. We write

$$\mathcal{E}_{\mathfrak{X}}(c) - \text{mod} \stackrel{\text{def}}{=} (\mathcal{D}(W), G, c) - \text{mod} / (\mathcal{D}(W), G, c) - \text{mod}^{uns}$$

for the quotient of the category of (G, c) -equivariant \mathcal{D} -modules by the full subcategory of modules with unstable singular support; we let $D(\mathcal{E}_{\mathfrak{X}}(c))$ denote its unbounded derived category. In [McN] we define a functor $D(\mathcal{E}_{\mathfrak{X}}(c)) \xrightarrow{\text{Rf}_*} D(U_c)$ from the microlocal derived category to the derived category of left modules for the algebra $U_c = M_c^G$. Note that the microlocal derived category depends on the choice of group character $\chi : G \rightarrow \mathbb{G}_m$. Corollary 7.2 yields:

Theorem 7.7. *Assume that for all $(\beta, i) \in \text{KN}^\circ$, c satisfies (7.1). Then the functor $\mathbb{R}f_*$ is t -exact.*

As an alternative to the use of Theorem 7.2(2), we give a proof based on Theorem 7.2(1).

Proof. Recall the following from [McN]. In the notation of [McN], $\pi_c : (\mathcal{D}, G, c) - \text{mod} \rightarrow \mathcal{E}_{\mathcal{X}}(c) - \text{mod}$ denotes the projection on the quotient category and $\Gamma_c : \mathcal{E}_{\mathcal{X}}(c) - \text{mod} \rightarrow (\mathcal{D}, G, c) - \text{mod}$ is the right adjoint to the projection π_c . Then, for any object M of $\mathcal{E}_{\mathcal{X}}(c) - \text{mod}$, we have

$$f_*(M) = \text{Hom}_{\mathcal{D}}(M_c, \Gamma_c(M))^G,$$

as in Lemma 5.1 of [McN]. Now suppose $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is exact in $\mathcal{E}_{\mathcal{X}}(c) - \text{mod}$. Then $0 \rightarrow \Gamma_c(M_1) \rightarrow \Gamma_c(M_2) \rightarrow \Gamma_c(M_3) \rightarrow 0$ is exact in $(\mathcal{D}, G, c) - \text{mod}$, and, furthermore, $SS[\text{coker}(\Gamma_c(M_2) \rightarrow \Gamma_c(M_3))] \subseteq T^*W^{uns}$ (this last property is standard: it is immediate, for example, from Theorem 5.8 of [McN]). Theorem 7.2 thus implies that $\text{Hom}_{\mathcal{D}}(M_c, \text{coker}(\Gamma_c(M_2) \rightarrow \Gamma_c(M_3)))^G = 0$. Since $\text{Hom}_{\mathcal{D}}(M_c, -)^G$ is an exact functor of (G, c) -equivariant \mathcal{D} -modules (cf. Lemma 3.4 of [McN]), it follows that $0 \rightarrow f_*M_1 \rightarrow f_*M_2 \rightarrow f_*M_3 \rightarrow 0$ is exact in $U_c - \text{mod}$. The theorem follows. \square

Theorem 1.1 of [McN] states that the left adjoint $\mathbb{L}f^*$ of $\mathbb{R}f_*$ is cohomologically bounded if and only if $\mathbb{R}f_*$ is an equivalence of derived categories. In particular, combining Theorem 1.1 of [McN] and Theorem 7.7 above, we find:

Corollary 7.8. *Suppose that*

- (1) $\mathbb{L}f^*$ *is cohomologically bounded.*
- (2) *The Lie algebra character c satisfies the hypothesis of Theorem 7.2.*

Then f^, f_* form mutually quasi-inverse equivalences of abelian categories.*

8. EXAMPLE: TYPE A SPHERICAL RATIONAL CHEREDNIK ALGEBRA

In this section we derive a slightly weaker form of the exactness part of [KR].

Fix $n \geq 1$. Let $W = \mathfrak{gl}_n \times \mathbf{C}^n$ with $G = GL_n$ acting in the usual way; thus, as usual, identify $T^*W \cong \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbf{C}^n \times (\mathbf{C}^n)^*$ with $g \cdot (X, Y, i, j) = (gXg^{-1}, gYg^{-1}, g \cdot i, j \cdot g^{-1})$. Let \mathbb{T} denote the maximal torus of diagonal matrices in G , with Lie algebra \mathfrak{t} and standard rational inner product. If e_i denotes the i th standard basis vector in $\mathbf{C}^n = \mathfrak{t} \cong \mathfrak{t}^*$, the weights of \mathbb{T} on W are $e_i - e_j$, $1 \leq i, j \leq n$, and e_i , $1 \leq i \leq n$. Fix the determinant character $\det : G \rightarrow \mathbb{G}_m$. Then $\lambda = d\det|_{\mathbb{T}} = \sum_{i=1}^n e_i$. Thus $\lambda \bullet (e_i - e_j) = 0$ for all i, j and $\lambda \bullet e_i = 1$ for all i .

Choose a subset $\alpha = \{\alpha_i\}$ of the weights of \mathbb{T} on W to produce a KN 1-parameter subgroup (by Remark 4.10, we may choose subsets of weights of W or of T^*W ; the 1-parameter subgroups that arise will be the same). Via the action of the Weyl group S_n , we may assume that the subset of the weights e_1, \dots, e_n that appear in α is exactly e_{k+1}, \dots, e_n for some $0 \leq k \leq n-1$, or is empty. Suppose that the subset of such weights appearing is nonempty, and let $e_i - e_j$ be an additional weight in α . If $i, j \geq k+1$ then $e_i - e_j$ lies in the span of e_{k+1}, \dots, e_n . If $i, j \leq k$ then $e_i - e_j$ is orthogonal to the span of e_{k+1}, \dots, e_n . If exactly one of i, j —say, i —lies in $1, \dots, k$ then the span of $e_i - e_j, e_{k+1}, \dots, e_n$ equals the span of e_i, e_{k+1}, \dots, e_n . Thus, we may assume, when computing the span of the elements of α , that α consists of the weights e_{k+1}, \dots, e_n together with some subset of the weights $e_i - e_j$ with $1 \leq i, j \leq k$. The projection of λ on the orthocomplement to the span of weights is thus either 0 (if $k = 0$) or $\sum_{i=1}^k e_i$. The first of these corresponds to the trivial 1-parameter subgroup and may be discarded. Write β_k for the 1-parameter subgroup corresponding to $\sum_{i=1}^k e_i$.

The stratification of T^*W given by these 1-parameter subgroups is as follows: For each $k \in \{0, 1, \dots, n\}$ let

$$S_k = \left\{ (X, Y, i, j) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbf{C}^n \times (\mathbf{C}^n)^* \mid \dim(\mathbf{C}\langle X, Y \rangle \cdot i) = n - k \right\}.$$

Then S_0 is the semistable locus, while for $k > 0$ the set S_k is the KN stratum associated to β_k .

We now compute the terms in Formula (7.1) for β_k . Since $\beta_k \bullet \alpha_i$ is ± 1 or 0 for every i , we find that $I(\beta_k) = \mathbb{Z}_{\geq 0}$. The shift that appears in Formula (7.1) is

$$\frac{1}{2} \sum_i |\alpha_i \bullet \beta_k| + \sum_{\gamma_i \in \text{wt}(\mathfrak{n}_{\beta_k}^-)} \gamma_i \bullet \beta.$$

Note that $|(e_i - e_j) \bullet \beta_k| = 1$ if and only if exactly one of i, j lies in $\{1, 2, \dots, k\}$ and is zero otherwise. We thus get for the shift

$$\begin{aligned} & \frac{1}{2} \sum_{i \in \{1, \dots, k\}, j \in \{k+1, \dots, n\}} 2|(e_i - e_j) \bullet \beta_k| + \frac{1}{2} \sum_{i=1}^n e_i \bullet \beta + \sum_{\gamma_i \in \text{wt}(\mathfrak{n}_{\beta_k}^-)} \gamma_i \bullet \beta \\ &= \frac{1}{2} (2(n-k)k) + \frac{1}{2} k + -(n-k)k = \frac{k}{2}. \end{aligned}$$

Consider the character $c \sum_{i=1}^n e_i$ on \mathfrak{gl}_n . Note that, for the space W above, $\rho = \frac{-1}{2} \sum_{i=1}^n e_i$ (since \mathfrak{gl}_n is reductive, its weights sum to zero). Write $c' \sum_{i=1}^n e_i = c \sum_{i=1}^n e_i - \rho$. Theorem 7.7 says that t -exactness holds provided that for $1 \leq k \leq n$,

$$\left(c' \sum_{i=1}^n e_i \right) \bullet \beta_k \notin I(\beta_k) + \frac{k}{2} = \mathbb{Z}_{\geq 0} + \frac{k}{2}.$$

Since $\sum_{i=1}^n e_i \bullet \beta_k = k$, this becomes $c'k \notin \mathbb{Z}_{\geq 0} + \frac{k}{2}$ or $c' \notin \frac{1}{k} \mathbb{Z}_{\geq 0} + \frac{1}{2}$. Since $c = c' - \frac{1}{2}$, we conclude that t -exactness holds provided $c \notin \bigcup_{k=1}^n \frac{1}{k} \mathbb{Z}_{\geq 0}$. Under the conventions of [GGS], our c corresponds to their $-c$; hence in the notation of [GGS] we have shown that t -exactness holds provided c is not a rational number of the form $\frac{a}{b}$ for $a \leq 0$, $1 \leq b \leq n$. Now by [GGS], Theorem 2.8, we conclude:

Corollary 8.1. *Exactness holds for microlocalization of the type A spherical Cherednik algebra eH_c provided*

$$c \notin \left\{ -1 + \frac{a}{b} \mid b \in \{1, 2, \dots, n\}, a \in \mathbb{Z}_{\leq 0} \right\}.$$

Remark 8.2. As noted above, this is slightly weaker than the main result of [KR], where they show exactness also holds when $b = 1$ in the above set.

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