Model-free trading and hedging with continuous price paths

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Abstract

This thesis explores the question of model-free trading and hedging in markets where traded asset prices are continuous and where one may trade continuously in time with no transaction costs. In particular, we make no assumptions on the volatility of traded asset prices. The contributions of the thesis are as follows. First, we propose a framework of model-independent replication of financial derivatives based on solutions to systems of PDEs evaluated at market-observed inputs. This provides a model-independent extension of the paradigm of dynamic hedging to general markets with continuous prices. We then relate these replication strategies to local martingales of a certain closed form and characterise the latter for several specifications of markets. The markets we consider are: (1) a market with no traded claims, (2) a market with an underlying asset and a convex claim, (3) a market with an underlying asset and a set of co-maturing call options. The auxiliary results for the latter two markets may be of interest outside of the local martingale characterisation results. Thirdly, we propose a definition of integration with continuous paths that justifies a probability-free version of the hedging results outlined earlier. Finally, we present a number of smaller contributions related to model-free hedging and to probability-free integration with continuous paths.

Contents

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Introduction

In a landmark year for the financial derivatives industry, Black, Scholes and Merton [7, 45] revolutionised option pricing by introducing what many refer to as the paradigm of dynamic hedging. Notably, they showed that under certain assumptions on the market, the financial risk of issuing an option may be completely offset by the continuous-time limit of a hedging strategy in the underlying asset. The gains and losses incurred by the strategy match the fluctuations of the price of the option, which is guaranteed to converge to the option payoff. This methodology based on dynamic hedging eliminated the need to estimate the drift of the asset price, a task which was shown to be very difficult by Rogers [53] and Monoyios [46]. It is hence not surprising how quickly the theory was embraced in practice and has been in use thereafter.

The classical framework of dynamic hedging requires making assumptions on the volatility of the underlying asset price. Hence, even with sophisticated volatility models, classical hedging strategies are prone to model error due to volatility misspecification. With the advent of exchange-traded options, market practitioners faced with the task of pricing and hedging an over-the-counter (OTC) option have taken up the practice of calibrating an option pricing formula (such as the Black-Scholes formula) to traded option prices and using the calibrated pricing function to obtain a corresponding "delta-vega" hedging strategy. This strategy is inconsistent with the premise of the original model, but works well in practice (see Section 4.1). In fact, the study of the performance of such market practices was the starting point of this thesis. This investigation led to the results of Chapter 1, which proposes a model-independent extension of the paradigm of dynamic hedging. In particular, the chapter builds on the observation that one may obtain model-independent integral representations of pricing functions by including observed second order variations of asset prices among the inputs to the latter. A key advantage of the approach proposed in this chapter is that it allows for dynamic trading strategies in exchange traded options.

In Chapter 2, we study the connection between the results of Chapter 1 and the characterisation of local martingales which have closed form given by C^2 functions evaluated at certain functionals of asset prices. Using tools from stochastic analysis, we show that the set of trading strategies proposed in Chapter 1 in fact characterises the full set of model-independent replication strategies under certain regularity conditions.

In Chapter 3, we reinforce the model-free emphasis of the results in Chapter 1 by showing that an analogous version of the model-independent results holds in a probability-free setting of price paths which have the so-called quadratic variation property of Föllmer [22]. To this end, the chapter defines a notion of probability-free integration under which analogues of many stochastic calculus identities apply.

The final chapter of the thesis, Chapter 4, covers some topics in model-independent hedging and probability-free integration which did not fit consistently within the earlier chapters. In particular, it studies two approximate hedging strategies for fixed maturity claims, notably the delta-vega hedging strategy mentioned earlier and a super-/sub-replication strategy based on estimates of realised variance. The chapter also presents some model-independent applications of a change of numéraire identity. Finally, it proposes an alternative construction of probability-free integrals based on uniform approximations of continuous paths.

Notation

Throughout the thesis, we will consider financial markets with a riskless asset of constant value and a set of risky assets. The assets may be traded in continuous time and without transaction costs. We typically denote asset prices by A, but this may change if more suitable notation is applicable depending on the market considered.

The following summarises the basic notation used in the thesis. We will introduce further notation when the relevant theory is established.

Notation 0.0.1 (General)**.** For $x, y \in \mathbb{R}^k$, xy denotes their inner product and $x \otimes y$ denotes their outer product. For matrices $x, y \in \mathbb{R}^{k\times k}$, xy denotes their standard matrix product and $x \cdot y$ denotes their Hadamard product, notably, $x \cdot y := \sum_{i,j}^k x_{ij} y_{ij}$. For $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$, denote $(x, y) \equiv (x_1, \ldots, x_k, y_1, \ldots, y_l)$. Denote $\mathbb{R}_+ := [0, \infty)$ and denote the space of symmetric positive definite $\mathbb{R}^{k\times k}$-valued matrices by $\mathbb{S}^k_+$.

For points $x, y \in \mathbb{R}^k$, $d(x, y)$ denotes their Euclidean distance. For a point $x \in \mathbb{R}^k$ and a set $K \subset \mathbb{R}^k$, define $d(x, K) := \inf\{d(x, y) : y \in K\}$. For $x \in \mathbb{R}^k$ and $r > 0$, define the closed ball of radius r around x by $B^c_r(x) := \{y \in \mathbb{R}^k : d(x, y) \leq r\}$. For a set $K \subseteq \mathbb{R}^k$, denote $cl(K)$ and K^c to be respectively the closure and the complement of K. Denote $\partial K := cl(K) \cap cl(K^c)$ to be the boundary set of K.

For a function $f : \mathbb{R}^k \to \mathbb{R}$, denote its gradient and Hessian respectively by ∇f and $D^2 f$. For $k \geq 1$, $T > 0$ and functions $f, g : [0,T] \to \mathbb{R}^k$,

$$\int_0^t f(u)\,\mathrm{d}g(u) := \sum_{i=1}^k \int_0^t f_i(u)\,\mathrm{d}g_i(u), \quad t \in [0,T],$$

when the integrals $\int_0^t f_i(u)\,\mathrm{d}g_i(u)$ are well-defined. Similarly, for $f, g : [0,T] \to \mathbb{R}^{k\times k}$,

$$\int_0^t f(u) \cdot \mathrm{d}g(u) = \sum_{i,j=1}^k \int_0^t f_{ij}(u)\,\mathrm{d}g_{ij}(u), \quad t \in [0,T].$$

when the integrals $\int_0^t f_{ij}(u)\,\mathrm{d}g_{ij}(u)$ are well-defined.

Notation 0.0.2 (Stochastic analysis)**.** For a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$, where $\mathbb{F}_T = \{\mathcal{F}_t\}$ can be any filtration with $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s \leq t \leq T$, we denote the i-th component of an $\mathbb{F}_T$-adapted process $P : \Omega \times [0,T] \to \mathbb{R}^k$ by P^i and say that P is a (continuous) semimartingale on $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ if each of its components P^i, $i \in \{1, \ldots, k\}$ is an $\mathbb{F}_T$-adapted (continuous) semimartingale. We sometimes set $T = \infty$, in which case we use the shorthand notation $\mathbb{F} \equiv \mathbb{F}_\infty = \{\mathcal{F}_t\}_{t\geq 0}$. We use a similar notation and definition for matrix-valued semimartingales. For semimartingales $Y : \Omega \times [0,T] \to \mathbb{R}^{d_1\times d_2}$ and $P : \Omega \times [0,T] \to \mathbb{R}^{d_2}$ on $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$, define

$$\int_0^t Y_u\,\mathrm{d}P_u(\omega) := \left(\sum_{j=1}^{d_2} \int_0^t Y_u^{1,j}\,\mathrm{d}P_u^j(\omega), \ldots, \sum_{j=1}^{d_2} \int_0^t Y_u^{d_1,j}\,\mathrm{d}P_u^j(\omega) \right), \quad (\omega, t) \in \Omega \times [0,T],$$

where the $\int_0^t Y_u^{ij}\,\mathrm{d}P_u^j(\omega)$ are Itô integrals. For a semimartingale $P : \Omega \times [0,T] \to \mathbb{R}^k$ on $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$, denote sample paths of P by $P(\omega) := \{P(\omega, t)\}_{t\in[0,T]}$, $\omega \in \Omega$.

For two filtered probability spaces $(\Omega^1, \mathcal{F}^1, \mathbb{F}_T^1, \mathbb{P}^1)$ and $(\Omega^2, \mathcal{F}^2, \mathbb{F}_T^2, \mathbb{P}^2)$, define their product space by

$$(\Omega^1, \mathcal{F}^1, \mathbb{F}_T^1, \mathbb{P}^1) \otimes (\Omega^2, \mathcal{F}^2, \mathbb{F}_T^2, \mathbb{P}^2) := (\Omega^1 \times \Omega^2, \mathcal{F}^1 \otimes \mathcal{F}^2, \mathbb{F}_T^1 \otimes \mathbb{F}_T^2, \mathbb{P}^1 \otimes \mathbb{P}^2),$$

where $\mathbb{F}_T^1 \otimes \mathbb{F}_T^2 := \{\mathcal{F}_t^1 \otimes \mathcal{F}_t^2\}_{t\in[0,T]}$. Denote the canonical space of an $\mathbb{R}^k$-valued Brownian motion on $[0,T]$ by $(\mathcal{C}(\mathbb{R}^k), \mathcal{F}_T^W, \mathbb{F}_T^W, \mathbb{P}^W)$.

Denote the set of stopping times with respect to a filtration $\mathbb{F}_T$ by $\mathcal{T}(\mathbb{F}_T)$. For stopping times $\tau_1, \tau_2 \in \mathcal{T}(\mathbb{F}_T)$ on a filtered space $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$, define the notation

$$\{\tau_1 < \tau_2\} := \{\omega \in \Omega : \tau_1(\omega) < \tau_2(\omega)\} \in \mathcal{F},$$
$$(\tau_1, \tau_2) := \{(\omega, t) \in \Omega \times [0,t] : \tau_1(\omega) < t < \tau_2(\omega)\} \in \mathcal{F} \otimes \mathcal{B}([0,T]),$$

where $\mathcal{B}([0,T])$ is the Borel σ-algebra on $[0,T]$. $\{\tau_1 \le \tau_2\}$, $[\tau_1, \tau_2]$, $(\tau_1, \tau_2]$ and $[\tau_1, \tau_2)$ are defined analogously.

For an $\mathbb{R}$-valued continuous semimartingale Z, denote its local time at $a \in \mathbb{R}$ on $[0,t]$ by $L_t^a[Z]$ and its Doléans-Dade exponential $e^{Z-\frac{1}{2}\langle Z\rangle}$ by $\mathcal{E}(Z)$.

Notation 0.0.3 (Measure theory)**.** Let $\mathcal{B}(\Omega)$ denote the Borel σ-algebra on a space Ω and δ_t denote the Dirac measure at t on $([0,\infty), \mathcal{B}([0,\infty))$. For any Radon measure ν on $([0,\infty), \mathcal{B}([0,\infty))$, define the distribution function of ν by $F_\nu(t) := \nu([0,t])$ for $t \ge 0$ and note that F_ν is a function of finite variation.

We refer to functions $P : [0,\infty) \to \mathbb{R}^k$ as paths. We write $P(t)$ for P evaluated at time t and reserve the notation P_t for stochastic processes. For a path P, define its total variation by

$$TV(P;t) := \sup_{\pi} \left\{ \sum_{t_i \in \pi} |P(t_{i+1} \wedge t) - P(t_i \wedge t)| \right\}, \quad t \ge 0,$$

where the supremum is taken over all partitions π of $[0,\infty)$. We say that P is of finite variation if $TV(P;t) < \infty$ for all $t > 0$. Denote $\mathcal{C}(K)$ to be the set of K-valued continuous paths on $[0,\infty)$ for $K \subseteq \mathbb{R}^k$. Denote $\mathcal{V}(K)$ to be the subset of paths in $\mathcal{C}(K)$ which have finite variation.

A functional X which maps a path $P(\cdot)$ to another path is denoted by $X[P](\cdot)$, notably the input path is specified within square brackets, whereas the indexing by time is done with round brackets.

For a partition $\pi = \{t_0, t_1, \dots, t_m\}$, define the *mesh* of π by

$$\|\pi\| := \max_{j \in \{0,1,\dots,m-1\}} |t_{j+1} - t_j|.$$

For a sequence of partitions $\Pi = \{\pi_k\}_{j \ge 1}$ where $\pi_k = \{t_1, \dots, t_{m_k}\}$ and for a path P, introduce the shorthand notation

$$\Delta P(t_i^k) := P(t_{i+1}^k) - P(t_i^k),$$
$$\Delta P(t_i^k \wedge t) := P(t_{i+1}^k \wedge t) - P(t_i^k \wedge t).$$

For $t \ge 0$ and $k \ge 1$, define $i_k(t) := \max\{i \ge 0 : t_i^k \le t\}$. Namely, $i_k(t)$ is the index of the greatest element in π_k which is less than or equal to t.

Chapter 1

Model-independent dynamic hedging with continuous semimartingales

The work in this chapter may be seen as a model-independent extension of the dynamic hedging paradigm. In particular, the goal of the chapter is to characterise a set of payoffs one may replicate through continuous time dynamic trading in a frictionless market with continuous prices when one does not specify a model for the asset prices. This yields a general methodology which encompasses most (if not all) of the model-independent replication strategies proposed in the literature. Remarkably, the results are mainly derived using standard Itô calculus.

There are several advantages of the proposed approach, provided one can solve systems of linear parabolic PDEs. First, it yields a set of "marketable" claims which have model-independent prices under all continuous semimartingales. In Chapter 2, we will show that the replication results derived in this chapter in fact characterise the full set of model-independent strategies with wealth processes of a certain form. Second, the model-independent prices are enforced by hedging strategies with hedge ratios given in closed form. And thirdly, the proposed framework allows for considering model-independent trading strategies in traded claims whose prices are determined by the market.

1.1 Motivating examples

To motivate the results in this chapter and provide some preliminary intuition, we outline two basic examples, due to Bick [5] and Carr and Lee [10], of the kind of strategies we will be presenting in a more general setting. Within this section, we consider a market with an underlying asset S with continuous price paths and we set the time horizon to be $[0,\infty)$.

In [5], Bick showed that it is possible to replicate path-independent claims on S with maturity equal to hitting times of realised variance. In particular, the replication strategy holds under all continuous semimartingale models for S. The argument proving the result is remarkably simple.

For $q > 0$ and a payoff function f, by which we henceforth mean the function defining the value of a claim at maturity, consider the solution F to the PDE

$$\frac{\partial}{\partial x_2}F(x_1,x_2) + \frac{1}{2}x_1^2\frac{\partial^2}{\partial x_1^2}F(x_1,x_2) = 0, \quad F(x_1,q) = f(x_1). \tag{1.1.1}$$

Note that $F(x_1,x_2)$ is equal to the Black-Scholes price of the path-independent claim with payoff function f under zero interest rate and with x_1 and x_2 respectively representing the observed asset price and the realised variance to be accumulated until maturity (volatility squared multiplied by time to maturity). For $q > 0$, define τ to be the first time when $\log S$ accumulates realised variance q, notably $\tau := \inf\{t > 0 : \langle \log S\rangle_t = q\}$. Itô's formula applied to $F(S,\langle \log S\rangle)$, along with the finite variation property of $\langle \log S\rangle$ gives

$$\begin{aligned} F(S_t,\langle \log S\rangle_t) = F(S_0,0) &+ \int_0^t \frac{\partial}{\partial x_1}F(S_u,\langle \log S\rangle_u)\,\mathrm{d}S_u \\ &+ \int_0^t \frac{\partial}{\partial x_2}F(S_u,\langle \log S\rangle_u)\,\mathrm{d}\langle \log S\rangle_u \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x_1^2}F(S_u,\langle \log S\rangle_u)\,\mathrm{d}\langle S\rangle_u, \quad t\in[0,\tau]\setminus\{\infty\}. \end{aligned}$$

Now using the PDE (1.1.1) satisfied by F along with

$$\mathrm{d}\langle \log S\rangle_u = \frac{1}{S_u^2}\,\mathrm{d}\langle S\rangle_u,$$

we obtain the pathwise identity

$$F(S_t,\langle \log S\rangle_t) = F(S_0,0) + \int_0^t \frac{\partial}{\partial x_1}F(S_u,\langle \log S\rangle_u)\,\mathrm{d}S_u, \quad [0,\tau]\setminus\{\infty\}. \tag{1.1.2}$$

The interpretation of this equation is that by judicious choice of the inputs to F, and using the PDE satisfied by F, we have succeeded in mimicking the situation whereby an agent starts with initial capital $F(S_0,0)$ at time zero, uses the self-financing trading strategy which holds $H := \frac{\partial}{\partial x_1}F(S,\langle \log S\rangle)$ units of S, and at all times $t\in[0,\tau]\setminus\{\infty\}$, her wealth is equal to $F(S_t,\langle \log S\rangle_t)$. Equivalently, we trade with a delta-hedging rule generated by the hedging function F, and the resulting gain from trade, as given by the integral $\int_0^{\cdot} \frac{\partial}{\partial x_1} F(S_t, \langle \log S \rangle_t) \, \mathrm{d}S_u$, is the only term left in the pathwise integral form of the process $F(S_t, \langle \log S \rangle_{\cdot \wedge \tau}) - F(S_0, 0)$.

Now consider a claim with payoff $f(S_\tau)$ and maturity at time τ, when the quadratic variation of $\log S$ reaches q. Timer calls and puts are examples of such options which have been traded in practice (see Carr and Lee [9]). Since $\langle \log S \rangle_\tau = q$, (1.1.2) evaluated at $t = \tau$, along with the initial condition $F(x, q) = f(x)$ satisfied by F results in

$$f(S_\tau) = F(S_0, 0) + \int_0^\tau \frac{\partial}{\partial x_1} F(S_t, \langle \log S \rangle_t) \, \mathrm{d}S_t. \tag{1.1.3}$$

for $\tau < \infty$. Thus, by combining a solution F to a parabolic PDE with a suitable choice of the inputs to F, we have generated a perfect hedge for the European timer claim using a delta-hedging rule, as given by (1.1.3), under all continuous semimartingale models where $\tau < \infty$ almost surely, provided the possibility to trade the asset frictionlessly in continuous time.

The above argument is easily generalised to the case where the process $\langle \log S \rangle$ is replaced by the weighted quadratic variation for an arbitrary nonnegative weighting function w, notably

$$Q_t^w := \int_0^t w(S_u, Q_u^w) \, \mathrm{d}\langle S \rangle_u, \quad t \geq 0. \tag{1.1.4}$$

Then the hedging function F and its parabolic PDE are also suitably altered, and this leads to the result below, which is a version of Proposition 2 in [5].

Proposition 1.1.1 (Bick [5]). *For functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $w : \mathbb{R}_+^2 \to \mathbb{R}_+$ respectively representing a payoff function and a weighting function, and for $q > 0$, let F satisfy the PDE*

$$2w \frac{\partial F}{\partial x_2} + \frac{\partial^2 F}{\partial x_1^2} = 0, \quad F(x_1, q) = f(x_1). \tag{1.1.5}$$

Let Q^w be defined by (1.1.4) *and, for $q > 0$, define the stopping time $\tau \equiv \tau_q$ by $\tau_q := \inf\{t > 0 : Q_t^w = q\}$. Then,*

$$F(S_{\cdot \wedge \tau}, Q_{\cdot \wedge \tau}^w) = F(S_0, 0) + \int_0^{\cdot \wedge \tau} \frac{\partial}{\partial x_1} F(S_u, Q_u^w) \, \mathrm{d}S_u \tag{1.1.6}$$

under all models where S is a continuous semimartingale. In particular, the self-financing strategy starting with initial wealth $F(S_0, 0)$ and holding $H := \frac{\partial}{\partial x_1} F(S, Q^w)$ units of S yields replication of the claim with payoff $f(S_\tau)$ at maturity τ under all continuous semimartingale models where $\tau < \infty$ almost surely.

Similarly, Propositions 2.9 and 2.10 of Carr and Lee [10] propose model-independent replication strategies for claims on realised variance with maturity equal to the hitting time of price to a barrier or the exit time of price from an interval. We present a version of their results below.

Proposition 1.1.2 (Carr and Lee [10])**.** *For functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $w : \mathbb{R}_+^2 \to \mathbb{R}_+$ respectively representing a payoff function and a nonnegative weighting function, and for $l < S_0 < u$, let F satisfy the PDE*

$$2w\frac{\partial F}{\partial x_2} + \frac{\partial^2 F}{\partial x_1^2} = 0, \quad F(l, x_2) = F(u, x_2) = f(x_2). \tag{1.1.7}$$

Let Q^w be defined by (1.1.4) *and, for $q > 0$, define the stopping time $\tau \equiv \tau_{l,u}$ by $\tau_{l,u} := \inf\{t \geq 0 : S_t \notin (l, u)\}$. Then,*

$$F(S_{\cdot\wedge\tau}, Q^w_{\cdot\wedge\tau}) = F(S_0, 0) + \int_0^{\cdot\wedge\tau} \frac{\partial}{\partial x_1} F(S_u, Q^w_u)\,\mathrm{d}S_u \tag{1.1.8}$$

under all models where S is a continuous semimartingale. In particular, the self-financing strategy starting with initial wealth $F(S_0, 0)$ and holding $H := \frac{\partial}{\partial x_1} F(S, Q^w)$ units of S yields replication of the claim with payoff $f(Q^w_\tau)$ at maturity τ under all continuous semimartingale models where $\tau < \infty$ almost surely.

1.2 Model-independent replication

We now consider a financial market with a riskless asset and d risky assets with discounted prices $A = (A^1, \ldots, A^d)$. The time horizon is $[0, T]$. We allow for some of the asses to be traded options co-maturing at time T. When there are no traded options in the market, we may set $T = \infty$, in which case $[0, T]$ should be interpreted as $[0, \infty)$.

We begin by defining what we mean by a model for A.

Definition 1.2.1. We say that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ and a stochastic process $A : \Omega \times [0, T] \to \mathbb{R}^d$ form a *continuous semimartingale (resp. local martingale / martingale) model* $M = \{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\}$ for A if the process A is a continuous semimartingale (resp. local martingale / martingale) on $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ and its components corresponding to traded claims attain their payoff at maturity, by which we mean that the components representing traded options must satisfy their payoff conditions almost surely. Denote the family of continuous semimartingale (resp. local martingale / martingale) models for which $A_0 = a$ by $\mathcal{M} \equiv \mathcal{M}_s(a)$ (resp. $\mathcal{M}_\ell \equiv \mathcal{M}_\ell(a)$ / $\mathcal{M}_m \equiv \mathcal{M}_m(a)$).

Note that $\mathcal{M}_m \subset \mathcal{M}_\ell \subset \mathcal{M}_s$ and that, unless we specify otherwise, we will assume a common starting point $A_0 = a$ when referring to different sets of models.

Remark 1.2.2. We did not assume any no arbitrage conditions in Definition 1.2.1 since they are not required for deriving the integral representation identities and the corresponding replication results. However, one may impose the requirement of existence of an equivalent local martingale measure in the definition of continuous semimartingale models if one would only like to consider models which allow for no arbitrage in the sense of "No Free Lunch With Vanishing Risk" of Delbaen and Schachermayer [16].

Notation 1.2.3. For a non-anticipative functional X of A and a closed set B, we shall often write $X_\cdot \equiv X_\cdot[A]$ and define τ_B^X to be the first hitting time of $X_\cdot[A]$ to B. If $X_\cdot[A]$ does not attain B prior to time T, set $\tau_B^X = \infty$. Denote X^B to be the functional X stopped when hitting B (hence, $X_\cdot^B \equiv X_{\cdot\wedge\tau_B}$).

Definition 1.2.4. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two pathspaces whose paths are defined on $[0,T]$ and are continuous. A map $X : \mathcal{P}_1 \to \mathcal{P}_2$ is called a *non-anticipative functional* if $X_t[P] = X_t[P']$ for any $t \in [0,T]$ and any paths $P, P' \in \mathcal{P}_1$ which are identical on $[0,t]$.

If A is a process adapted to a filtration $\mathbb{F}_T$ and X is a non-anticipative functional (according to our definition), then $X_\cdot[A]$ is adapted to $\mathbb{F}_T$. See Cont and Fournié [12, 13] for measurability properties of non-anticipative functionals in a more general setting.

Definition 1.2.5. Define $\mathcal{X}$ to be the set of non-anticipative functionals $X = (X^1, \dots, X^n)$ such that, for each $i \in \{1, \dots, n\}$ and under all continuous semimartingale models $M \in \mathcal{M}_s$, $X_\cdot^i \equiv X_\cdot^i[A]$ has the integral form

$$X_t^i = X_0^i + \int_0^t \alpha^i(X_u)\,\mathrm{d}A_u + \int_0^t \beta^i(X_u) \cdot \mathrm{d}\langle A\rangle_u + \int_0^t \gamma^i(X_u)\,\mathrm{d}u, \quad t \in [0,T] \quad (1.2.1)$$

for a constant $X_0 = (X_0^1, \dots, X_0^n)$ and for continuous functions $\alpha^i : \mathbb{R}^n \to \mathbb{R}^d$, $\beta^i : \mathbb{R}^n \to \mathbb{R}^{d\times d}$ and $\gamma^i : \mathbb{R}^n \to \mathbb{R}$.

Remark 1.2.6. It is possible to extend the definition of the class $\mathcal{M}$ to incorporate variations in running extrema. However, we avoid doing so for two reasons. First, this would burden the notation and the formulation of the results in this chapter as well as those in Chapter 2. Second, there is a way to capture dependence on the running extrema with the current formulation of $\mathcal{M}$ via squared drawdowns and drawups, as highlighted in Section 1.3.2.

For a set of functions α^i, β^i, γ^i, $i \in \{1, \ldots, n\}$, define the following operators on smooth enough (C^2 is sufficient) functions F:

$$\mathcal{L}^{\alpha,\beta}_{ij} F := \sum_{k=1}^{n} \beta^k_{i,j} \frac{\partial}{\partial x_k} F + \frac{1}{2} \sum_{k,l=1}^{n} \alpha^k_i \alpha^l_j \frac{\partial^2}{\partial x_k \partial x_l} F, \quad i, j \in \{1, \ldots, d\} \tag{1.2.2}$$

$$\mathcal{L}^{\gamma} F := \sum_{k=1}^{n} \gamma^k \frac{\partial}{\partial x_k} F \tag{1.2.3}$$

$$\mathcal{L}^{\alpha} F := \sum_{k=1}^{n} \alpha^k \frac{\partial}{\partial x_k} F \tag{1.2.4}$$

For fixed F, note that $\mathcal{L}^{\alpha,\beta}_{i,j} F$, $i, j \in \{1, \ldots, d\}$ and $\mathcal{L}^{\gamma} F$ are functions from $\mathbb{R}^n$ to $\mathbb{R}$, whereas $\mathcal{L}^{\alpha} F$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^d$.

Definition 1.2.7. For $X \in \mathcal{X}$ with corresponding functions $\alpha^i, \beta^i, \gamma^i$, $i \in \{1, \ldots, d\}$ and for a connected set $\mathcal{D} \subseteq \mathbb{R}^n$, we define the family of functions $\mathcal{S}_X(\mathcal{D})$ by

$$\mathcal{S}_X(\mathcal{D}) := \{F \in C^2(\mathcal{D}) : \mathcal{L}^{\gamma} F = \mathcal{L}^{\alpha,\beta}_{i,j} F = 0, \quad i, j \in \{1, \ldots, d\}\}.$$

One may relax the C^2 assumption if some of the α^i are identically equal to zero.

Definition 1.2.8. Let $\mathcal{M} \subseteq \mathcal{M}_s$ be a subset of models for A and let X be a non-anticipative functional. Then, the *range of X on $\mathcal{M}$* is defined by

$$\mathcal{R}(X; \mathcal{M}) := \{x \in \mathbb{R}^n : \forall \epsilon > 0, \ \exists \{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M} \ \ s.t. \ \ \mathbb{P}(\tau^X_{B^c_\epsilon(x)} < T) > 0\},$$

where $B^c_\epsilon(x)$ is the closed ball of radius ϵ around x.

The proof of the following proposition is provided in Appendix A.

Proposition 1.2.9. *Let $\mathcal{M} \subseteq \mathcal{M}_s$ be a non-empty set of models for A and X be a non-anticipative functional. Then, for any starting point $a \in \mathbb{R}^d$ of A, the range $\mathcal{R}(X; \mathcal{M}(a))$ of X on $\mathcal{M}(a)$ is a closed and connected subset of $\mathbb{R}^n$, and $X_t[A] \in \mathcal{R}(X; \mathcal{M})$ for all $t \in [0, T)$ almost surely under all models in $\mathcal{M}(a)$.*

Henceforth, when referring to integral representation identities which hold under a set of models $M \in \mathcal{M}$, we will mean that the identities hold M-almost surely for each $M \in \mathcal{M}$. The following proposition is the main result of this section.

Proposition 1.2.10. *Consider a functional $X \in \mathcal{X}$ and a closed set $B \subseteq \mathbb{R}^n$. Denote $\mathcal{D} \equiv \mathcal{R}(X^B; \mathcal{M}_s) \setminus B$. Then, for any $F : \mathcal{D} \cup B \to \mathbb{R}$ in $\mathcal{S}_X(\mathcal{D}) \cap \mathcal{C}(\mathcal{D} \cup B)$ and under all continuous semimartingale models for $M \in \mathcal{M}_s$ for A:*

(1) $F(X^B_\cdot)$ admits the integral representation

$$F(X^B_\cdot) \equiv F(X_0) + \int_0^\cdot \mathbb{1}_{\{t<\tau_B^X\}} \mathcal{L}^\alpha F(X_t)\,\mathrm{d}A_t \tag{1.2.5}$$

(2) If $T \in (0,\infty)$, $\int_0^T F(X^B_t)\,\mathrm{d}t$ admits the integral representation

$$\int_0^T F(X^B_t)\,\mathrm{d}t = TF(X_0) + \int_0^T \mathbb{1}_{\{t<\tau_B^X\}}(T-t)\mathcal{L}^\alpha F(X_t)\,\mathrm{d}A_t. \tag{1.2.6}$$

Proof.

Proof of claim(1):

Let $M = \{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}_s$. By $X \in \mathcal{X}$, there is a set of functions α^i, β^i and γ^i, $i \in \{1\,\ldots, n\}$ such that

$$\mathrm{d}X^i_t = \alpha^i(X_t)\,\mathrm{d}A_t + \beta^i(X_t)\,\mathrm{d}\langle A\rangle_t + \gamma^i(X_t)\,\mathrm{d}t$$

for each $i \in \{1, \ldots, n\}$. Denoting $v \otimes w$ to be the outer product of vectors v and w,

$$\begin{aligned}
\mathrm{d}F(X_t) &= \sum_{k=1}^n \frac{\partial}{\partial x_k} F(X_t)\,\mathrm{d}X^k_t + \frac{1}{2}\sum_{k,l=1}^n \frac{\partial^2}{\partial x_k \partial x_l} F(X_t)\,\mathrm{d}\langle X^k, X^l\rangle_t \\
&= \sum_{k=1}^n \frac{\partial}{\partial x_k} F(X_t)\alpha^k(X_t)\,\mathrm{d}A_t + \sum_{k=1}^n \frac{\partial}{\partial x_k} F(X_t)\beta^k(X_t)\cdot\,\mathrm{d}\langle A\rangle_t \\
&\quad + \sum_{k=1}^n \frac{\partial}{\partial x_k} F(X_t)\gamma^k(X_t)\,\mathrm{d}t + \frac{1}{2}\sum_{k,l=1}^n \frac{\partial^2}{\partial x_k \partial x_l} F(X_t)(\alpha^k(X_t)\otimes\alpha^l(X_t))\cdot\,\mathrm{d}\langle A\rangle_t \\
&= \sum_{k=1}^n \frac{\partial}{\partial x_k} F(X_t)\alpha^k(X_t)\,\mathrm{d}A_t + \sum_{k=1}^n \frac{\partial}{\partial x_k} F(X_t)\gamma^k(X_t)\,\mathrm{d}t \\
&\quad + \sum_{i,j=1}^d \left(\sum_{k=1}^n \frac{\partial}{\partial x_k} F(X_t)\beta^k_{ij}(X_t) + \frac{1}{2}\sum_{k,l=1}^n \frac{\partial^2}{\partial x_k \partial x_l} F(X_t)\alpha^k_i(X_t)\alpha^l_j(X_t) \right) \mathrm{d}\langle A^i, A^j\rangle_t
\end{aligned}$$

for $t \in [0, T\wedge\tau_B^X)$. Hence, by definition of the operators in (1.2.2) – (1.2.4), we get

$$\mathrm{d}F(X_t) = \mathcal{L}^\alpha F(X_t)\,\mathrm{d}A_t + \mathcal{L}^\gamma F(X_t)\,\mathrm{d}t + \sum_{i,j=1}^d \mathcal{L}^{\alpha,\beta}_{i,j} F(X_t)\,\mathrm{d}\langle A^i, A^j\rangle_t, \quad t \in [0, T\wedge\tau_B^X).$$

By the assumption $F \in \mathcal{S}_X(\mathcal{R}(X^B; \mathcal{M}_s)\setminus B)$, notably that

$$\mathcal{L}^\gamma F = \mathcal{L}^{\alpha,\beta}_{i,j} F = 0, \quad i,j \in \{1, \ldots, d\}$$

on $\mathcal{R}(X^B; \mathcal{M}_s)\setminus B$ and by $\mathbb{P}(X_t \in \mathcal{R}(X^B; \mathcal{M}_s)\setminus B,\ t \in [0, T\wedge\tau_B^X)) = 1$ (which follows from Proposition 1.2.9 and by $X_t \notin B$ on $t \in [0, \tau_B^X)$), we conclude that

$$\mathrm{d}F(X_t) = \mathcal{L}^\alpha F(X_t)\,\mathrm{d}A_t, \quad t \in [0, T\wedge\tau_B^X).$$

Hence, by continuity of F on B,

$$F(X_t^B) = F(X_{t\wedge\tau_B^X}) = F(X_0) + \int_0^t \mathbb{1}_{\{u<\tau_B^X\}}\mathcal{L}^\alpha F(X_u)\,\mathrm{d}A_u, \quad t\in[0,T],$$

which proves (1.2.5).

Proof of claim (2):

Evaluating (1.2.5) at $t=T$ and multiplying both sides by T, we have

$$TF(X_T^B) = TF(X_0) + \int_0^T \mathbb{1}_{\{t<\tau_B^X\}}T\mathcal{L}^\alpha(X_t)\,\mathrm{d}A_t.$$

On the other hand, integration by parts (applied to $tF(X_t^B)$ at $t = T\wedge\tau_B^X$) and $\mathrm{d}F(X_t^B) = \mathcal{L}^\alpha F(X_t^B)\,\mathrm{d}A_t$ for $t\in[0,T\wedge\tau_B^X)$ give

$$\begin{aligned}(T\wedge\tau_B^X)F(X_T^B) &= \int_0^{T\wedge\tau_B^X} F(X_t^B)\,\mathrm{d}t + \int_0^{T\wedge\tau_B^X} t\,\mathrm{d}F(X_t^B)\\ &= \int_0^{T\wedge\tau_B^X} F(X_t^B)\,\mathrm{d}t + \int_0^T \mathbb{1}_{\{t<\tau_B^X\}}t\mathcal{L}^\alpha F(X_t^B)\,\mathrm{d}A_t\end{aligned}$$

By the above identities,

$$\begin{aligned}&TF(X_0) + \int_0^T \mathbb{1}_{\{t<\tau_X^B\}}(T-t)\mathcal{L}^\alpha F(X_t)\,\mathrm{d}A_t\\ &= \left(TF(X_0) + \int_0^T \mathbb{1}_{\{t<\tau_B^X\}}T\mathcal{L}^\alpha F(X_t^B)\,\mathrm{d}A_t\right) - \int_0^T \mathbb{1}_{\{t<\tau_B^X\}}t\mathcal{L}^\alpha F(X_t^B)\,\mathrm{d}A_t\\ &= TF(X_T^B) - \left((T\wedge\tau_B^X)F(X_T^B) - \int_0^{T\wedge\tau_B^X} F(X_t^B)\,\mathrm{d}t\right)\\ &= \int_0^T F(X_t^B)\,\mathrm{d}t,\end{aligned}$$

which concludes the proof.

□

For X and B as in Proposition 1.2.10, the latter yields the following corollary.

Corollary 1.2.11.

(1) $F(X_\cdot^B)$ is the model-independent wealth process of the self-financing strategy starting with capital $F(X_0)$ and holding $\mathcal{L}^\alpha F(X_t)$ units of A for $t\in[0,T\wedge\tau_B^X)$. Denoting $F=f$ on B, this strategy replicates the claim with payoff $f(X_{\tau_B^X})$ under all semimartingale models where $\tau_B^X<\infty$ almost surely.

(2) The self-financing strategy starting with capital $TF(X_0)$ and holding $(T-t)\mathcal{L}^\alpha F(X_t)$ units of A for $t\in[0,T\wedge\tau_B^X)$ model-independently replicates the cash flow $\int_0^T F(X_t^B)\,\mathrm{d}t$.

We illustrate this corollary with the examples in the following section.

1.3 Examples with single traded asset and no explicit time dependence

The goal of this section is to highlight some applications of the results in the previous section, in particular of Corollary 1.2.11. We will focus the discussion to a market with a single traded underlying asset $A \equiv S$ and time horizon $[0, \infty)$, and consider different choices of functionals $X \in \mathcal{X}$ of S. In particular, we do not consider markets with traded claims in this section (see Section 2.3 for examples with such a market). When referring to model-independent identities, we will mean that these identities apply under all continuous semimartingale models for S. When we refer to model-independent replication strategies of claims with maturity τ_B, we will mean that the strategies work under all continuous semimartingale models where the stopping time τ_B is finite almost surely.

The outline of this section is as follows. We first highlight basic examples for the case of wealth processes and corresponding claims on price and (weighted) realised variance. This includes the strategies of Bick [5] and Carr and Lee [10] outlined in Section 1.1, as well as model-independent replication of barrier versions of timer claims. We then discuss claims contingent on running extrema and justify why we did not include integrals with respect to running extrema in the integral formulation of functionals $X \in \mathcal{X}$.

1.3.1 Claims on price and quadratic variation

For the weighted realised variance Q^w defined by (1.1.4), consider $X := (S, Q^w) \in \mathcal{X}$. For a closed set $B \subset \mathbb{R}^2$ and $\mathcal{D} \equiv \mathcal{R}(X^B; \mathcal{M}_s) \setminus B$, $S_X(\mathcal{D})$ is the set of $C^{2,1}$ functions which solve the heat equation

$$2w \frac{\partial}{\partial x_2} F + \frac{\partial^2}{\partial x_1^2} F = 0 \tag{1.3.1}$$

subject to some boundary conditions. Theorem 1.2.10 and Corollary 1.2.11 yield the result in Bick [5], notably that for any function F solving (1.3.1), $F(S_{\cdot \wedge \tau_B^X}, Q^w_{\cdot \wedge \tau_B^X})$ is the wealth process of the self-financing portfolio starting with capital $F(S_0, 0)$ and holding a position $H = \frac{\partial}{\partial x_1} F(S, Q^w)$ in S on $[0, \tau_B^X)$. Different choices of the boundary set B yield different replication results. Setting $B = \{x_2 = q\}$ for $q > 0$ yields replication of timer claims, as outlined in Section 1.1. Setting $B = \{x_1 = l\} \cup \{x_1 = u\}$ for $0 \leq l < S_0 < u \leq \infty$ yields the replication strategies proposed in Carr and Lee [10].

We outline one further application with $X = (S, Q^w)$, notably replication of timer claims with barrier conditions. Let $q > 0$ be the hitting level of Q^w defining the maturity of a knock-out barrier claim with barrier levels $0 \leq l \leq S_0 \leq u \leq \infty$. Temporarily define $\tau_q := \inf\{t > 0 : Q^w_t = q\}$ to be the maturity of the claim and define $\tau_b := \inf\{t > 0 : S_t \notin (l, u)\}$ to be hitting time of S to either barrier l or u. Then, the following corollary follows directly from the integral representation identity (1.2.5).

Corollary 1.3.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be such that $f(l) = f(u) = 0$ and that a solution F to the heat equation* (1.3.1) *and boundary conditions*

$$F(x_1, q) = f(x_1), \quad x_1 \in (l, u), \tag{1.3.2}$$

$$F(l, x_2) = F(u, x_2) = 0, \quad x_2 \in [0, q], \tag{1.3.3}$$

exists. Then, the self-financing portfolio starting with capital $F(S_0, 0)$ and holding a position $H_t := \frac{\partial}{\partial x_1} F(S_t, Q^w_t)$ in S for $t \in [0, \tau_q \wedge \tau_b)$ replicates the timer claim with payoff $f(S_{\tau_q}) \mathbb{1}_{\{S_t \in (l,u),\ t \in [0,\tau_q]\}}$ under all continuous semimartingale models where $\tau_q < \infty$ almost surely.

Proof. Setting $B = \{x_1 = l\} \cup \{x_1 = u\} \cup \{x_2 = q\}$ and boundary conditions (1.3.2) – (1.3.3) in Corollary 1.2.11, we get that the self-financing strategy starting with capital $F(S_0, 0)$ and holding a position H in S until $\tau^X_B = \tau_q \wedge \tau_b$ attains a wealth at time τ^X_B equal to 0 if $\tau^X_B = \tau_b$ (in which case a barrier is hit, H is set to 0 until τ_q and we trivially replicate the knocked out barrier claim) or to $f(S_{\tau_q})$ if $\tau^X_B = \tau_q$ (the payoff of the timer claim which has not been knocked out).

□

Since knock-in barrier claims are the difference between non-barrier claims and knock-out barrier claims, their replication follows from taking the difference of the replication strategies for the latter.

Remark 1.3.2. Provided one uses a suitable notion of solution to PDEs with non-continuous boundary conditions, the above analysis may be extended to more general payoff functions f which are not equal to zero at $x_1 = l$ and $x_1 = u$. We do not delve further into this topic since it is not specific to the model-independent aspect of the results.

The replication strategy in Corollary 1.3.1 only requires inputs S and Q^w to F. In particular, the dependence on the extrema is implicitly embedded in the stopping time defining the dynamic hedging period. This is due to the weak path-dependence of single touch barrier claims. By contrast, the following subsection considers claims which depend explicitly on the running extrema of S.

1.3.2 Dependence on running extrema

Recall that the class $\mathcal{X}$ does not contain functionals X which include the running maximum or minimum as components. Hence, Theorem 1.2.10 and Corollary 1.2.11 cannot be directly applied with such functionals. As mentioned in Remark 1.2.6, we could have extended the class $\mathcal{X}$ to incorporate dependence on running extrema and generalised Proposition 1.2.10 to this extended class, but we did not do so for the sake of clarity of notation and presentation. In this subsection, we discuss such extensions in the context of $A = S$. We then show that the squared drawdown or drawup may be included as a component of $X \in \mathcal{X}$ (as defined in Section 1.2). We conclude the subsection with an equivalence result which allows capturing dependence on running extrema by functionals in $\mathcal{X}$ as defined in Definition 1.2.5.

Notation 1.3.3. Within this subsection, denote the running maximum and minimum of S respectively by $M \equiv M^S$ and $m \equiv m^S$. Note the a slight abuse of notation due to the fact that M denotes models for assets in most of the thesis. In order to avoid notational confusion, we avoid specifying models in this subsection.

1.3.2.1 Running extrema as components of X

The following result shows that one can model-independently generate wealth processes which have closed form given by functions of X, where X includes M as a component.

Proposition 1.3.4. *Consider a closed set $B \subset \mathbb{R}^3_+$ and $\mathcal{D} \equiv \mathcal{R}(X^B; \mathcal{M}_s) \backslash B$. Suppose that $F : \mathcal{D} \cup B \to \mathbb{R}$ is a $C^{2,1,1}(\mathcal{D})$ solution to*

$$2w \frac{\partial}{\partial x_3} F + \frac{\partial^2}{\partial x_1^2} F = 0, \tag{1.3.4}$$

$$\mathbb{1}_{\{x_1 = x_2\}} \frac{\partial}{\partial x_2} F(x_1, x_2, x_3) = 0 \tag{1.3.5}$$

on $\mathcal{D}$ and is continuous on B. Then, $F(X^B_\cdot)$ admits the integral representation

$$F(X^B_\cdot) = F(X_0) + \int_0^\cdot \mathbb{1}_{\{t < \tau_B^X\}} \frac{\partial}{\partial x_1} F(X_t) \, \mathrm{d}S_t$$

under all continuous semimartingale models for S. Hence, $F(X^B_\cdot)$ is the model-independent wealth process of the self-financing strategy starting with capital $F(X_0)$

and holding $\frac{\partial}{\partial x_1}F(X_t)$ *units of* S *for* $t \in [0, \tau_B^X)$. *Denoting* $F = f$ *on* B, *this strategy replicates the claim with payoff* $f(X_{\tau_B^X})$ *and maturity* τ_B^X *under all continuous semimartingale models where* $\tau_B^X < \infty$ *almost surely. This yields replication of:*
(a) Claims on (S, M) *with maturity equal to hitting times of* Q^w, *either with or without a lower barrier condition on* S, *under all continuous semimartingale models where the hitting times are finite almost surely.*
(b) Claims on (M, Q^w) *with maturity equal to hitting times of* S *to a level* $l < S_0$ *under all continuous semimartingale models where the hitting times are finite almost surely.*

Proof. Let $F \in \mathcal{C}(B; X)$ be a solution to (1.3.4) – (1.3.5). By Itô's formula (generalised to include the running maximum),

$$\mathrm{d}F(S_t, M_t, Q_t^w) = \frac{\partial}{\partial x_1}F\,\mathrm{d}S_t + \frac{\partial}{\partial x_2}F\,\mathrm{d}M_t + \left(w\frac{\partial}{\partial x_3}F + \frac{1}{2}\frac{\partial^2}{\partial x_1^2}F\right)\mathrm{d}\langle S\rangle_t, \quad t \in [0, \tau_B^X),$$

where we used the shorthand notation $F \equiv F(S_t, M_t, Q_t^w)$. By (1.3.4) – (1.3.5), the continuity of F at B and the observation that M is carried by $\{S = M\}$, it follows that

$$F(X_t) = F(X_0) + \int_0^t \frac{\partial}{\partial x_1}F(X_u)\,\mathrm{d}u, \quad t \in [0, \tau_B^X].$$

under all continuous semimartingale models for S. Along with arguments analogous to those in Subsection 1.3.1, this implies the statements regarding model-independent replication.

□

An analogous result to Proposition 1.3.4 applies for $X = (S, m, Q^w)$.

1.3.2.2 Dependence on the squared drawdown and drawup

We now show that the square of the drawdown process $M - S$ or drawup process $S - m$ may be used as a component of X such that $X \in \mathcal{X}$. This is due to the following application of well-known results on Azéma-Yor martingales (Theorem 1 in Obłój [50]). Define $D^M := (M - S)^2$ and $D^m := (S - m)^2$.

Lemma 1.3.5. *For all continuous semimartingale models for* S,

$$D_\cdot^M = S_\cdot^2 - S_0^2 - 2\int_0^\cdot M_t\,\mathrm{d}S_t = \langle S\rangle_\cdot - 2\int_0^\cdot \sqrt{D_t^M}\,\mathrm{d}S_t, \tag{1.3.6}$$

$$D_\cdot^m = S_\cdot^2 - S_0^2 - 2\int_0^\cdot m_t\,\mathrm{d}S_t = \langle S\rangle_\cdot + 2\int_0^\cdot \sqrt{D_t^m}\,\mathrm{d}S_t, \tag{1.3.7}$$

Proof. Let g be a locally bounded function on $\mathbb{R}$ and define $G(x) := \int_0^x g(s)\,\mathrm{d}s$. Then, for $t \geq 0$,

$$g(M_t)(M_t - S_t) - g(M_0)(M_0 - S_0) = \int_0^t g(M_u)\,\mathrm{d}(M - S)_u + \int_0^t (M - S)_u\,\mathrm{d}g(M_u).$$

Since $M_0 = S_0$, $\int_0^t (M - S)_u\,\mathrm{d}g(M_u) = 0$, and $\int_0^t g(M_u)\,\mathrm{d}M_u = G(M_t) - G(S_0)$, this gives

$$\begin{aligned} & g(M_t)(M_t - S_t) = G(M_t) - G(S_0) - \int_0^t g(M_u)\,\mathrm{d}S_u \\ \Leftrightarrow\ & g(M_t)M_t - G(M_t) - S_t g(M_t) + G(S_0) + \int_0^t g(M_u)\,\mathrm{d}S_u = 0. \end{aligned}$$

The only properties we have used above are that M is carried by $\{S = M\}$, that it has finite variation and that $M_0 = S_0$. Hence, the same arguments applied to m give

$$g(m_t)m_t - G(m_t) - S_t g(m_t) + G(S_0) + \int_0^t g(m_u)\,\mathrm{d}S_u = 0.$$

For $g(x) = x$, hence $G(x) = \frac{1}{2}x^2$, this gives

$$\begin{aligned} & \frac{1}{2}M_t^2 - S_t M_t + \frac{1}{2}S_0^2 + \int_0^t M_u\,\mathrm{d}S_u = 0, \quad t \geq 0, \\ & \frac{1}{2}m_t^2 - S_t m_t + \frac{1}{2}S_0^2 + \int_0^t m_u\,\mathrm{d}S_u = 0, \quad t \geq 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & (M_t - S_t)^2 - S_t^2 + S_0^2 + 2\int_0^t M_u\,\mathrm{d}S_u = 0, \quad t \geq 0, \\ & (S_t - m_t)^2 - S_t^2 + S_0^2 + 2\int_0^t m_u\,\mathrm{d}S_u = 0, \quad t \geq 0, \end{aligned}$$

which rearrange to (1.3.6) and (1.3.7) respectively by $S_\cdot^2 = S_0^2 + 2\int_0^\cdot S_u\,\mathrm{d}S_u + \langle S\rangle_\cdot$. □

Remark 1.3.6. Let $T > 0$ and consider a market where claims with payoff $(S_T)^2$ at maturity T are traded at price C. Then, Lemma 1.3.5 implies that the claim of maturity $T > 0$ with payoff $D_T^M = (M_T - S_T)^2$ is replicated model-independently by the self-financing portfolio starting with capital $C_0 - S_0^2$ and holding a static position of one unit of C along with a dynamic position of $-2\sqrt{D^M}$ units of S. An analogous replication result holds for the claim with payoff D_T^m.

We can apply the representation of D^M and D^m in Lemma 1.3.5 along with Proposition 1.2.10 to obtain the following replication results corresponding to $X = (D^M, \langle S \rangle)$.

Corollary 1.3.7. *Consider a closed set $B \subset \mathbb{R}^2$, the set $\mathcal{D} = \mathcal{R}(X^B; \mathcal{M}_s) \setminus B$ and a function f on B such that there is a solution $F \in C^{2,1}(\mathcal{D}) \cap C(B)$ to*

$$\frac{\partial}{\partial x_2} F + \frac{\partial}{\partial x_1} F + 2x_1 \frac{\partial^2}{\partial x_1^2} F = 0. \tag{1.3.8}$$

with boundary condition $F = f$ on B. Then, the self-financing strategy starting with capital $F(0,0)$ and holding $-2\sqrt{D_t^M} \frac{\partial}{\partial x_1} F(D_t^M, \langle S \rangle_t)$ units of S for $t \in [0, \tau_B^X)$ replicates the claim with payoff $f(D^M_{\tau_B^X}, \langle S \rangle_{\tau_B^X})$ and maturity τ_B^X under all continuous semimartingale models where $\tau_B^X < \infty$ almost surely. This yields replication of:
(a) Claims on the drawdown with maturity equal to hitting times of $\langle S \rangle$, either with or without a barrier condition on the drawdown, under all continuous semimartingale models where the hitting times are finite almost surely.
(b) Claims on Q^w with maturity equal to hitting times of the drawdown under all continuous semimartingale models where the hitting times are finite almost surely.

The proof follows from Corollary 1.2.11, and arguments analogous to those in Subsection 1.3.1. A similar result holds for $X = (D^m, \langle S \rangle)$.

1.3.2.3 Equivalence via change of variables

One can generalise Corollary 1.3.7 by adding more components to X. In particular, by setting $X = (S, D^M, Q^w)$, $\mathcal{S}_X(\mathcal{D})$ is the set of $C^{2,2,1}$ functions which are solutions to

$$w \frac{\partial}{\partial x_3} F + \frac{\partial}{\partial x_2} F + \frac{1}{2} \frac{\partial^2}{\partial x_1^2} F + 2x_2 \frac{\partial^2}{\partial x_2^2} F - 2\sqrt{x_2} \frac{\partial^2}{\partial x_1 \partial x_2} F = 0, \tag{1.3.9}$$

subject to some boundary conditions. Then, one can generate portfolios with wealth process $F(X^B_\cdot)$ by starting with capital $F(X_0)$ and holding a position $H = \frac{\partial}{\partial x_1} F(X) - 2\sqrt{D^M} \frac{\partial}{\partial x_2} F(X)$ in S.

This choice of X may in fact be used to circumvent the mixed boundary PDEs in Proposition 1.3.4. Since (S, D^M, Q^w) and (S, M, Q^w) are homeomorphic, in fact diffeomorphic except on $\{D_t^M = 0\} = \{S_t = M_t\}$, one may expect the sets of self-financing strategies in S having wealth processes equal to $F^D(S_\cdot, D^M_\cdot, Q^w_\cdot)$ for solutions F^D to (1.3.9) to be equivalent to strategies with wealth processes of the form $F^M(S_\cdot, M_\cdot, Q^w_\cdot)$ for solutions F^M to (1.3.4) – (1.3.5), up to regularity conditions on F^D and F^M. For completeness, the next proposition shows this for $w = 1$. We use the following notational convention within the proposition and its proof:

- $w = 1$ when referring to (1.3.9) and (1.3.4).

- For a set K and a function g defined on it, define $g(K) := \{g(x) : x \in K\}$.

The proof of the proposition is provided in Appendix A.

Proposition 1.3.8. *Define $y(x) := (x_1, (x_2 - x_1)^2, x_3)$ and $\widetilde{y}(x) := (x_1, x_1 + \sqrt{x_2}, x_3)$. Consider a connected set $\mathcal{D} \subset \mathbb{R}^3$ with smooth boundary and a solution $F^D \in C^{2,2,1}(\mathcal{D})$ to* (1.3.9)*. Then, the function*

$$F^M := F^D \circ y \tag{1.3.10}$$

is a $C^{2,1,1}$ solution to (1.3.4) *on $\widetilde{y}(\mathcal{D})$. Furthermore,* (1.3.5) *holds on $\widetilde{y}(\mathcal{D})$ provided $\frac{\partial}{\partial y_2} F^D(y_1, y_2, y_3)$ is well-defined on $\mathcal{D} \cap \{y_2 = 0\}$.*
Conversely, consider a connected set $\widetilde{\mathcal{D}}$ with smooth boundary and a solution $\widetilde{F}^M \in C^{2,2,1}(\widetilde{\mathcal{D}})$ to (1.3.4) – (1.3.5)*. Then, the function*

$$\widetilde{F}^D := \widetilde{F}^M \circ \widetilde{y}$$

is a $C^{2,2,1}$ solution to (1.3.9) *on $\widetilde{y}(\widetilde{\mathcal{D}}) \setminus \{x_2 = 0\}$.*

Since D^M is equal to zero every time S attains its running maximum, it is important that $\widetilde{F}^D(x_1, x_2, x_3)$ be differentiable from the right in the x_2 variable on $\{x_2 = 0\} \cap y(\widetilde{\mathcal{D}})$. If one solves for $\widetilde{F}^D$ via $\widetilde{F}^M$, as in the proposition, one may check for differentiability of $\widetilde{F}^D$ after applying the change in variables, or one may refer to Lemma A.7 in Appendix A for sufficient conditions on $\widetilde{F}^M$ for differentiability of $\widetilde{F}^D$ at $\{x_2 = 0\} \cap y(\widetilde{\mathcal{D}})$.

1.3.2.4 Example: timer claims contingent on running extrema

To illustrate the approaches outlined above, we consider the following example. Suppose we would like to replicate a timer claim with payoff $f(S_\tau, M_\tau)$, where $\tau := \inf\{t > 0 : \langle S \rangle_t = q\}$ for some $q > 0$. Set $B = \{x_3 = q\} \subset \mathbb{R}^3$ and $X = (S, M, \langle S \rangle)$. Then, Proposition 1.3.4 implies that we can replicate the claim under all continuous semimartingale models where $\tau < \infty$ almost surely if, for $\mathcal{D} \equiv \mathcal{R}(X^B; \mathcal{M}_s) \setminus B = \{0 \le x_1 \le x_2\} \cap \{0 \le x_3 < q\} \subset \mathbb{R}^3$, we can solve for a solution $F \in C^{2,1,1}(\mathcal{D}) \cap C(B)$ to (1.3.4) – (1.3.5) with $w = 1$ and boundary condition $F(x_1, x_2, q) = f(x_1, x_2)$.

Proposition A.8 in Appendix A shows that, provided an implicit regularity condition on the payoff function f, the PDE has a stochastic solution given by

$$F(x_1, x_2, x_3) := \mathbb{E}\left[f\left(x_1 + W_{q-x_3}, x_2 \vee (x_1 + M^W_{q-x_3})\right)\right], \quad 0 \le x_1 \le x_2, \quad 0 \le x_3 \le q, \tag{1.3.11}$$

for a Brownian motion W (with running maximum M^W) under a probability measure $\mathbb{P}$.

By Proposition 1.3.8, a second approach would be to solve for a $C^{2,2,1}$ solution F^D to the PDE (1.3.9) with boundary condition $F^D(x_1, x_2, q) = f(x_1, x_1 + \sqrt{x_2})$ and such that $\frac{\partial}{\partial y_2} F^D(y_1, y_2, y_3)$ is well-defined on $\mathbb{R}_+ \times \{0\} \times [0, q)$, and set $F := F^D \circ \widetilde{y}$, where $\widetilde{y}$ is defined as in the proposition.

Provided with a solution F by any one of the methods above, the self-financing strategy starting with capital $F(S_0, S_0, 0)$ and holding $\frac{\partial}{\partial x_1} F(S_t, M_t, \langle S \rangle_t)$ units in S for $t \in [0, \tau)$ has wealth process equal to $F(S_{\cdot \wedge \tau}, M_{\cdot \wedge \tau}, \langle S \rangle_{\cdot \wedge \tau})$ and replicates the claim with payoff $f(S_\tau, M_\tau)$ and maturity τ under all continuous semimartingale models where $\tau < \infty$ almost surely.

1.4 Time-dependent functionals

To conclude this chapter, we turn to the question of replication of claims of fixed maturity by including time-dependent components in X. Whereas the previous examples only involved a single linear parabolic PDE, typically admitting well-defined and unique solutions for a large class of boundary conditions, the inclusion of time-dependent components in X yields a system of two PDEs and restricts the class of payoff functions (corresponding to boundary conditions) for which claims may be replicated in a model-independent manner according to Corollary 1.2.11.

In particular, this section explores the range of claims of fixed maturity $T > 0$ one may replicate via Corollary 1.2.11 by considering time as an explicit component of X. Setting $X^1 = t$ (corresponding to $\gamma^1 = 1$, $\alpha^1 = \beta^1 = 0$), and $B = \{x_1 = T\} \subset \mathbb{R}^d$, we get $\tau_B^X = T$, which by Corollary 1.2.11 yields model-independent replication of claims of the form $f(X_T)$, where f is the boundary condition on B for functions in $\mathcal{S}_X(\mathcal{R}(X; \mathcal{M}_s))$.

First, observe that unless there is $i \in \{2, \ldots, n\}$ such that $\gamma^i(x) \neq 0$, then $\mathcal{L}^\gamma F(x) = 0$ implies that $\frac{\partial}{\partial x_1} F = 0$, which would defeat the purpose of including the component $X^1 = t$ in X. The most commonly encountered quantity in finance which has an absolutely continuous variation with respect to time is the arithmetic average of the stock price defining Asian payoffs, which modulo scaling is equal to $V_\cdot \equiv \int_0^\cdot S_t \, \mathrm{d}t$.

Remark 1.4.1. As with quadratic variation, we can weight the integrand by a function of X, but this would only change the PDE formulations without providing any more insight.

Setting $X^2 = V$, we need to include S or a bijective function thereof as a component in X in order for $X \in \mathcal{X}$ to hold (since $\gamma^2(X) = S$). As a basic example, consider $X_t = (t, V_t, S_t)$, $t \in [0, T]$. Set $\mathcal{D} := \mathcal{R}(X^B; \mathcal{M}_s)$. It is easy to check that $\mathcal{D} = [0, T] \times [0, \infty) \times \mathbb{R}_+$.

Corollary 1.2.11 implies that $F(X.)$ is the model-independent wealth process of a self-financing strategy in S if $F \in \mathcal{S}_X(\mathcal{D} \setminus B)$, hence if F satisfies the system of PDEs

$$\frac{\partial^2}{\partial x_3^2} F = 0, \tag{1.4.1}$$

$$\frac{\partial}{\partial x_1} F + x_3 \frac{\partial}{\partial x_2} F = 0. \tag{1.4.2}$$

on $\mathcal{D} \setminus B = [0, T) \times [0, \infty) \times \mathbb{R}_+$.

Proposition 1.4.2. *$F \in C^2(\mathcal{D} \setminus B) \cap C(\mathcal{D})$ is a solution to* (1.4.1) – (1.4.2) *iff*

$$F(x) = c_1(x_1 x_3 - x_2) + c_2 x_3 + c_3 \tag{1.4.3}$$

for constants $c_1, c_2, c_3 \in \mathbb{R}$.

Proof. By (1.4.1), there are C^2 functions F_1 and F_2 (temporary notation) such that

$$F(x) = x_3 F_1(x_1, x_2) + F_2(x_1, x_2)$$

Applying (1.4.2) yields

$$x_3 \frac{\partial}{\partial x_1} F_1(x_1, x_3) + \frac{\partial}{\partial x_1} F_2(x_1, x_3) + x_3 \left(x_3 \frac{\partial}{\partial x_2} F_1(x_1, x_3) + \frac{\partial}{\partial x_2} F_2(x_1, x_2) \right) = 0,$$

which may be rearranged to

$$x_3^2 \frac{\partial}{\partial x_2} F_1(x_1, x_2) + x_3 \left(\frac{\partial}{\partial x_1} F_1(x_1, x_2) + \frac{\partial}{\partial x_2} F_2(x_1, x_2) \right) + \frac{\partial}{\partial x_1} F_2(x_1, x_2) = 0$$

Since the above relation applies to all $x \in [0, T) \times \mathbb{R}_+^2$, the factors of the different powers of x_3 must be zero. Hence, the above is equivalent to

$$\begin{cases} \frac{\partial}{\partial x_2} F_1(x_1, x_2) = 0 \\ \frac{\partial}{\partial x_1} F_1(x_1, x_2) + \frac{\partial}{\partial x_2} F_2(x_1, x_2) = 0 \\ \frac{\partial}{\partial x_1} F_2(x_1, x_2) = 0 \end{cases}$$

for $(x_1, x_2) \in [0, T) \times \mathbb{R}_+$. This, in turn, is equivalent to

$$\begin{cases} F_1(x_1, x_2) = c_1 x_1 + c_2 \\ F_2(x_1, x_2) = -c_1 x_2 + c_3 \end{cases}$$

for $(x_1, x_2) \in [0, T) \times \mathbb{R}_+$ and constants $c_1, c_2, c_3 \in \mathbb{R}$. This proves that $F \in \mathcal{S}_X(\mathcal{D} \setminus B)$ iff $F(x) = x_3(c_1 x_1 + c_2) + (-c_1 x_2 + c_3)$, which rearranges to (1.4.3).

□

Hence, for $X_t = (t, V_t, S_t)$, the only functions F for which Corollary 1.2.11 implies that $F(X_\cdot)$ is a model-independent wealth process of a self-financing strategy are of the form (1.4.3). In particular, this only yields linear combinations of the well-known model-independent replication strategy for the linear Asian payoff – an application of the integration by parts identity $TS_T = \int_0^T t\,\mathrm{d}S_t + \int_0^T S_t\,\mathrm{d}t$ – and of static positions in S and in bonds.

The above result illustrates the limitations implied by the system of two PDEs corresponding to functionals which depend explicitly on time or, more generally, on integrals with respect to time. To obtain non-trivial model-independent replication strategies with time-dependent functionals, we need to augment X with additional components. For instance, we can set $X_t = (t, V_t, S_t, \langle S\rangle_t)$ and $B = \{x_1 = T\} \subset \mathbb{R}^4$. Then, Corollary 1.2.11 implies that $F(X_\cdot)$ is the model-independent wealth process of a self-financing trading strategy if F solves the system of PDEs

$$\frac{\partial}{\partial x_4}F + \frac{1}{2}\frac{\partial^2}{\partial x_3^2}F = 0, \tag{1.4.4}$$

$$\frac{\partial}{\partial x_1}F + x_3\frac{\partial}{\partial x_2}F = 0. \tag{1.4.5}$$

on $\mathcal{R}(X^B; \mathcal{M}_s)\backslash B = [0, T)\times\mathbb{R}^3_+$. Provided this system of PDEs admits a solution with boundary condition $F(T, x_2, x_3, x_4) = f(x_2, x_3, x_4)$, one can replicate $f(V_T, S_T, \langle S\rangle_T)$ under all continuous semimartingale models for S.

Remark 1.4.3 (DDS). As outlined by Carr and Lee [10], there is an interpretation of the hedging strategies proposed in Bick [5] and in [10] based on a Dambis/Dubins-Schwarz time change. This interpretation, outlined below, applies to model-independent replication strategies for claims which are contingent on a single asset (or on a single self-financing portfolio, also denoted an attainable process in the language of Fukasawa [23]) and which have no explicit dependence on time.

By folklore results ([16, 25]) in the classical theory of no arbitrage, the asset price S is a local martingale under a risk-neutral measure provided that the market admits "No Free Lunch With Vanishing Risk". Fix a risk-neutral measure $\mathbb{Q}$. Hence, when $A = S$ is continuous and one-dimensional, the Dambis/Dubins-Schwarz (DDS) theorem implies that $S = W_{\langle S\rangle}$, where W is a Brownian motion under $\mathbb{Q}$. The quadratic variation based hedging strategies in Section 1.3 use the observed $\langle S\rangle$ to reduce the problem of hedging a claim for a general continuous local martingale S to hedging a claim for Brownian motion. However, this time change impacts the maturity of the claim. In particular, the maturities are specified on the clock of the underlying Brownian motion, meaning that they correspond to hitting times of $\langle S\rangle$. More generally, suppose that one of the components of X, say X^n, is $Q^w_\cdot = \int_0^\cdot w(S_t, Q^w_t)\,\mathrm{d}\langle S\rangle_t$, the boundary set B is $\{x_n = q\}$ and the payoff function is $f(x_1, \ldots, x_{n-1})$. If the claim on $f(X^1_q, \ldots, X^{n-1}_q)$ with maturity q admits a pricing function under the model

$$\mathrm{d}S_t = (w(S_u, Q^w_u))^{-1/2}\,\mathrm{d}W_t, \quad t \in [0, q),$$

where W is a Brownian motion under the risk-neutral measure, then this pricing function is in $\mathcal{S}_X(\mathcal{D})$ for a corresponding domain $\mathcal{D}$ and yields a model-independent replication strategy for the claim on $f(X^1_{\tau^X_B}, \ldots, X^{n-1}_{\tau^X_B})$ with maturity τ^X_B given by the hitting time of Q^w to q. Setting $w(x_1, x_2) = \frac{1}{x_1^2}$ yields model-independent strategies based on the Black-Scholes formula, with the time change defined by $\langle \log S\rangle$.

On the other hand, when the claim depends explicitly on S and on time, as with the examples in this section, then the above time change argument generally does not yield analogous model-independent strategies. It only does so under specific constraints on the payoff functions, which translate to constraints on the boundary conditions under which the system of PDEs corresponding to X has a solution. As will be highlighted by examples in Section 2.3, similar restrictions on payoff functions apply when A is multidimensional.

Chapter 2

Characterisation of local martingales of closed form

The main purpose of this chapter is to characterise the set of C^2 functions F such that $F(X^B_\cdot[A])$ is a local martingale for $X \in \mathcal{X}$ and for continuous local martingales A with unspecified volatility. This has direct relevance to characterising the set of wealth processes of corresponding closed form one may attain via dynamic trading in continuous time with no specification of the model except for continuity of prices and the existence of a risk-neutral (local martingale) measure.

The work has parallels with the work in Obłoj and Yor [51], where the authors characterise the set of local martingales which are functions of a continuous local martingale (with unspecified volatility) and its supremum. One of the main differences is that we have a general specification of X and B (the results we present also apply without any stopping set B, with no significant changes to the proof). In particular, throughout this chapter, X is a $\mathcal{C}_T(\mathbb{R}^n)$-valued functional in $\mathcal{X}$ with corresponding functions α^i, β^i, γ^i, $i \in \{1, \ldots, n\}$ and B is a closed subset of $\mathbb{R}^n$. Another important difference is that we consider multivariate local martingales A which may need to converge to specified payoffs if some of the components represent prices of traded claims.

Remark 2.0.4. Obłoj [50] shows that the closed form characterisation of local martingales of the form $F(S_t, \sup_{u \le t} S_u)$ (where S is a continuous local martingale on a canonical space) does not require the assumption that F be C^2 as in Obłoj and Yor [51]. Notably, the characterisation holds for all Borel functions F. The proof is based upon probabilistic arguments involving (i) martingale representation, (ii) a representation result for additive functionals of Markov processes and (iii) the property stating that functions of Brownian motion which are local martingales with respect to the canonical filtration are affine. It may be of interest to look into similar extensions of the results presented in this section, but such considerations are not within the scope of this thesis.

2.1 Characterisation in a general market setting

We begin by noting that for $\mathcal{D}_{\mathcal{M}_\ell} := \mathcal{R}(X^B; \mathcal{M}_\ell)$ and $F \in C(\mathcal{D}_{\mathcal{M}_\ell}) \cap C^2(\mathcal{D}_{\mathcal{M}_\ell} \setminus B)$, Proposition 1.2.10 implies that

$$F(X^B_\cdot) = F(X_0) + \int_0^\cdot \mathbb{1}_{\{t<\tau^X_B\}} \mathcal{L}^\alpha F(X^B_t)\, \mathrm{d}A_t$$

under all models $M \in \mathcal{M}_\ell$. Since $\mathcal{L}^\alpha F(X^B_\cdot)$ is locally bounded (by continuity) and since A is by definition a local martingale under all $M \in \mathcal{M}_\ell$, it follows that $F(X^B_\cdot)$ is a local martingale under all $M \in \mathcal{M}_\ell$.

We first show that a converse statement holds for families of models which satisfy a certain regularity property. We then consider some important cases of markets represented by A and solve for regular families of models for these markets.

We say that a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})$ contains $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ if $\exists (\Omega', \mathcal{F}', \mathbb{F}'_T, \mathbb{P}')$ such that $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}}) = (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P}) \otimes (\Omega', \mathcal{F}', \mathbb{F}'_T, \mathbb{P}')$, where we used the product space notation specified in the beginning of the thesis.

Definition 2.1.1. We say that a family of local martingale models $\mathcal{M} \subseteq \mathcal{M}_\ell$ for A is *regular* if for any model $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$, matrix $\Sigma \in \mathbb{S}^d_+$ and stopping time $\tau \in \mathcal{T}(\mathbb{F}_T)$, there is a model $\{\widetilde{A}; (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})\} \in \mathcal{M}$ and a stopping time $\widetilde{\tau} \in \mathcal{T}(\widetilde{\mathbb{F}}_T)$ such that:

- $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})$ contains $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$.
- $\widetilde{A} = A$ on $[0, \tau]$.
- $\{\tau < \widetilde{\tau} \le T\} = \{\tau < T\}$.
- $\mathrm{d}\langle \widetilde{A} \rangle_t = \Sigma\, \mathrm{d}t$ on $t \in [\tau \wedge T, \widetilde{\tau} \wedge T)$.

We will often refer to the main result below as the *characterisation theorem.*

Theorem 2.1.2. *Let $\mathcal{M}$ be a regular family of models for A and denote $\mathcal{D}_\mathcal{M} \equiv \mathcal{R}(X^B; \mathcal{M})$. Then, for any $F \in C(\mathcal{D}_\mathcal{M}) \cap C^2(\mathcal{D}_\mathcal{M} \setminus B)$, $F(X^B_\cdot)$ is a local martingale under all $M \in \mathcal{M}$ if and only if $F \in \mathcal{S}_X(\mathcal{D}_\mathcal{M} \setminus B)$ (where S_X is as defined in Definition 1.2.7).*

A subtle detail in the characterisation theorem is the role of the choice of the regular class of models $\mathcal{M}$. Let $F \in C(\mathcal{D}_{\mathcal{M}_\ell}) \cap C^2(\mathcal{D}_{\mathcal{M}_\ell} \setminus B)$. As discussed at the beginning of this section, Proposition 1.2.10 implies that $F(X^B_\cdot)$ is a local martingale under all continuous local martingale models $M \in \mathcal{M}_\ell$ if $F \in \mathcal{S}_X(\mathcal{D}_{\mathcal{M}_\ell} \setminus B)$. Is it true that the converse statement holds, notably that $F \in \mathcal{S}_X(\mathcal{D}_{\mathcal{M}_\ell} \setminus B)$ if F is a local martingale under all continuous local martingale models $M \in \mathcal{M}_\ell$? The characterisation theorem implies that $F \in \mathcal{S}_X(\mathcal{D}_{\mathcal{M}} \setminus B)$ for any regular class of models $\mathcal{M}$. Hence, F must satisfy the system of PDEs $\mathcal{L}^\gamma F = 0$ and $\mathcal{L}^{\alpha,\beta}_{i,j} F = 0$ for $i, j \in \{1, \dots, d\}$ over the domain $\mathcal{D}_{\mathcal{M}}$. Recalling that $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{M}_\ell}$ are the ranges of X^B over the classes of models $\mathcal{M}$ and $\mathcal{M}_\ell$ respectively, it is clear that $\mathcal{M} \subseteq \mathcal{M}_\ell$ implies that $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{D}_{\mathcal{M}_\ell}$. If we can find a regular class of models for A such that these ranges coincide, we can then conclude that $F \in \mathcal{S}_X(\mathcal{D}_{\mathcal{M}_\ell} \setminus B)$. This is summarised by the following corollary.

Corollary 2.1.3. *If there is a regular family $\mathcal{M}$ of models for A such that $\mathcal{R}(X^B; \mathcal{M}) = \mathcal{R}(X^B; \mathcal{M}_\ell) \equiv \mathcal{D}_{\mathcal{M}_\ell}$, then for any $F \in C(\mathcal{D}_{\mathcal{M}_\ell}) \cap C^2(\mathcal{D}_{\mathcal{M}_\ell} \setminus B)$, $F(X^B_\cdot)$ is a local martingale under all local martingale models $M \in \mathcal{M}_\ell$ iff $F \in \mathcal{S}_X(\mathcal{D}_{\mathcal{M}_\ell} \setminus B)$.*

The following lemma follows directly by checking the conditions in Definition 2.1.1.

Lemma 2.1.4. *If S is an underlying asset, then $\mathcal{M}_\ell$ is a regular class of models for a market with a single risky asset ($A \equiv S$).*

Hence, the replication strategies in Section 1.3 characterise the set of model-independent wealth processes of the form $F(X^B_\cdot)$ (for the various choices of $X \in \mathcal{X}$ considered) for a market with $A \equiv S$.

Proof of Theorem 2.1.2. Throughout the proof, we will denote $||f||_K$ to be the supremum norm of a function f on a set K.

The statement that $F(X^B_\cdot)$ is local martingale under all $M \in \mathcal{M}$ follows from Proposition 1.2.10. We hence turn to proving the converse statement.

Suppose that $F \in C(\mathcal{D}_{\mathcal{M}}) \cap C^2(\mathcal{D}_{\mathcal{M}} \setminus B)$ and that $F(X^B_\cdot)$ is a local martingale under all models in $\mathcal{M}$. Consider any such model $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$ and define

$$Y_t[A] := F(X^B_t[A]) - \int_0^t \mathbb{1}_{\{u < \tau^X_B\}} \mathcal{L}^\alpha F(X^B_u) \, \mathrm{d}A_u, \quad t \in [0, T].$$

Note that $Y_t[A] = \int_0^t \mathbb{1}_{\{u<\tau^X_B\}} \mathcal{L}^\gamma F(X^B_u) \, \mathrm{d}u + \sum_{i,j=1}^d \int_0^t \mathbb{1}_{\{u<\tau^X_B\}} \mathcal{L}^{\alpha,\beta}_{i,j} F(X^B_u) \, \mathrm{d}\langle A^i, A^j \rangle_u$ by Itô's formula. Hence, $Y_\cdot \equiv Y_\cdot[A]$ is a continuous local martingale of finite variation.

By the Burkholder-Davis-Gundy inequality, it follows that $\mathbb{P}(Y_t[A] = 0,\ t \in [0,T]) = 1$, which implies that

$$\mathbb{E}(Y_{\tau_2} - Y_{\tau_1}) = 0 \tag{2.1.1}$$

for any $\tau_1, \tau_2 \in \mathcal{T}(\mathbb{F}_T)$ such that $\tau_1 \leq \tau_2$.

We will now show that if $F \notin \mathcal{S}_X(\mathcal{D}_\mathcal{M} \setminus B)$, then there exists a model $M \in \mathcal{M}$ and stopping times $\tau_1, \tau_2 \in \mathcal{T}(\mathbb{F}_T)$ defined on it such that $\tau_1 \leq \tau_2$ and such that (2.1.1) does not hold, and thereby obtain a contradiction to $F \notin \mathcal{S}_X(\mathcal{D}_\mathcal{M} \setminus B)$. The notation introduced in the following sub-parts of the proof is temporary (it only applies within each section).

(i) We first prove that $\mathcal{L}^{\alpha,\beta}_{i,i} F = 0$ on $\mathcal{D}_\mathcal{M} \setminus B$ for all $i \in \{1, \ldots, d\}$. Suppose that for some $k \in \{1, \ldots, d\}$ and $x \in \mathcal{D}_\mathcal{M} \setminus B$,

$$\mathcal{L}^{\alpha,\beta}_{k,k} F(x) = z \neq 0.$$

Without loss of generality, assume that $z > 0$ and $x \notin \partial\mathcal{D}_\mathcal{M}$ (the proof when $z < 0$ is analogous, and if $x \in \partial\mathcal{D}_\mathcal{M}$, choose a point $x' \in \mathcal{D}_\mathcal{M} \setminus \partial\mathcal{D}_\mathcal{M}$ close to x such that $\mathcal{L}^{\alpha,\beta}_{k,k} F(x) \neq 0$).

By continuity of $\mathcal{L}^{\alpha,\beta}_{k,k} F$, $\exists \epsilon \in (0, d(x,B))$ such that

$$\inf_{y \in B^c_x(\epsilon) \cap \mathcal{D}_\mathcal{M}} \mathcal{L}^{\alpha,\beta}_{k,k} F(y) > z/2.$$

Denote $K_1 := B^c_x(\epsilon/2) \cap \mathcal{D}_\mathcal{M}$ and $K_2 := B^c_x(\epsilon) \cap \mathcal{D}_\mathcal{M}$, and note that $\partial K_1 \cap \partial K_2 = \emptyset$ due to $\epsilon < d(x,B)$. Also, define the bound b by

$$b := ||\mathcal{L}^\gamma F||_{K_2} \vee \max_{i,j \in \{1,\ldots,d\}} ||\mathcal{L}^{\alpha,\beta}_{i,j} F||_{K_2}$$

and the matrix $\Sigma \in \mathbb{S}^d_+$ by

$$\Sigma_{ij} := \begin{cases} \frac{2b(d+1)}{z} & i = j = k, \\ 1 & i = j \neq k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $x \in \mathcal{D}$, $\exists \{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$ such that $\mathbb{P}(\tau^{X^B}_{K_1} < T) > 0$ (where $\tau^{X^B}_{K_1}$ stands for the first hitting time of $X^B_\cdot$ to the set K_1). Define the stopping time

$$\tau_1 := \tau^{X^B}_{K_1} \wedge T \in \mathcal{T}(\mathbb{F}_T).$$

By $\Sigma \in \mathbb{S}^d_+$ and the regularity property of $\mathcal{M}$, there exist a model $\{\widetilde{A}; (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})\} \in \mathcal{M}$ and stopping time $\widetilde{\tau} \in \mathcal{T}(\widetilde{\mathbb{F}}_T)$ such that the properties stated in Definition 2.1.1 hold. Define the stopping time

$$\tau_2 := \widetilde{\tau} \wedge \inf\{t \geq \tau_1 : X^B_t[\widetilde{A}] \in \partial K_2\} \in \mathcal{T}(\widetilde{\mathbb{F}}_T).$$

Then,

$$
\begin{aligned}
Y_{\tau_2}[\widetilde{A}] - Y_{\tau_1}[\widetilde{A}] &= \int_{\tau_1}^{\tau_2} \mathcal{L}^\gamma F(X_t^B[\widetilde{A}])\,\mathrm{d}t + \sum_{i,j=1}^{d} \int_{\tau_1}^{\tau_2} \mathcal{L}_{i,j}^{\alpha,\beta} F(X_t^B[\widetilde{A}])\,\mathrm{d}\langle \widetilde{A}^i, \widetilde{A}^j\rangle_t \\
&= \int_{\tau_1}^{\tau_2} \left(\mathcal{L}^\gamma F(X_t^B[\widetilde{A}]) + \sum_{i,j=1}^{d} \Sigma_{ij} \mathcal{L}_{i,j}^{\alpha,\beta} F(X_t^B[\widetilde{A}]) \right) \mathrm{d}t \\
&= \int_{\tau_1}^{\tau_2} \left(\frac{2b(d+1)}{z} \mathcal{L}_{k,k}^{\alpha,\beta} F(X_t^B[\widetilde{A}]) + \mathcal{L}^\gamma F(X_t^B[\widetilde{A}]) + \sum_{i\neq k} \mathcal{L}_{i,i}^{\alpha,\beta} F(X_t^B[\widetilde{A}]) \right) \mathrm{d}t \\
&\geq \int_{\tau_1}^{\tau_2} \left(\frac{2b(d+1)}{z} \inf_{y\in K_2} \mathcal{L}_{k,k}^{\alpha,\beta} F(y) - ||\mathcal{L}^\gamma F||_{K_2} - \sum_{i\neq k} ||\mathcal{L}_{i,i}^{\alpha,\beta} F||_{K_2} \right) \mathrm{d}t \\
&\geq \int_{\tau_1}^{\tau_2} b\,\mathrm{d}t
\end{aligned}
$$

Hence,

$$
Y_{\tau_2}[\widetilde{A}] - Y_{\tau_1}[\widetilde{A}] \geq b(\tau_2 - \tau_1) \tag{2.1.2}
$$

By construction of $\widetilde{\tau}$, it follows that $\{\tau_1 < \widetilde{\tau}\} = \{\tau_1 < T\}$, whereas $\partial K_1 \cap \partial K_2 = \emptyset$ implies that $\{\tau_1 < \inf\{t \geq \tau_1 : X_t^B[\widetilde{A}] \in \partial K_2\}\} = \{\tau_1 < T\}$. Hence,

$$
\{\tau_1 < \tau_2\} = \{\tau_1 < \widetilde{\tau}\} \cap \{\tau_1 < \inf\{t \geq \tau_1 : X_t^B[\widetilde{A}] \in \partial K_2\}\} = \{\tau_1 < T\},
$$

which implies that

$$
\widetilde{\mathbb{P}}(\tau_1 < \tau_2) = \widetilde{\mathbb{P}}(\tau_1 < T) = \mathbb{P}(\tau_1 < T) > 0.
$$

By (2.1.2), this gives $\mathbb{E}^{\widetilde{\mathbb{P}}}(Y_{\tau_2}[\widetilde{A}] - Y_{\tau_1}[\widetilde{A}]) > 0$ (where $\mathbb{E}^{\widetilde{\mathbb{P}}}$ denotes the expectation under $\widetilde{\mathbb{P}}$), which contradicts (2.1.1).

Hence, this proves that $\mathcal{L}_{i,i}^{\alpha,\beta} F = 0$ on $\mathcal{D}_\mathcal{M} \setminus B$ for $i \in \{1, \dots, d\}$.

(ii) We will now prove that $\mathcal{L}_{i,j}^{\alpha,\beta} = 0$ on $\mathcal{D}_\mathcal{M} \setminus B$ for $i \neq j$. As in part (i), we argue by contradiction and provide a construction of a model and stopping times τ_1 and τ_2 such that (2.1.1) does not hold. The main difference with the proof in part (i) is that the result for $i = j$ (proved in part (i)) is used in one of the steps.

Suppose that for some $k \neq l$ and $x \in \mathcal{D}_\mathcal{M} \setminus B$,

$$
\mathcal{L}_{k,l}^{\alpha,\beta} F(x) = z \neq 0.
$$

Once again, assume without loss of generality that $z > 0$ and $x \notin \partial\mathcal{D}_\mathcal{M}$. Let $\epsilon \in (0, d(x, B))$ be such that $\inf_{y\in B_x^c(\epsilon)\cap\mathcal{D}_\mathcal{M}} \mathcal{L}_{k,l}^{\alpha,\beta} F(y) > z/2$, denote $K_1 := B_x^c(\epsilon/2) \cap \mathcal{D}_\mathcal{M}$,

$K_2 := B^c_x(\epsilon) \cap \mathcal{D}_{\mathcal{M}}$ and define $b := ||\mathcal{L}^\gamma F||_{K_2} \vee \max_{i\in\{1,\dots,d\}} ||\mathcal{L}^{\alpha,\beta}_{i,j}||_{K_2}$. Define the matrix $\Sigma \in \mathbb{S}^d_+$ by

$$\Sigma_{ij} = \begin{cases} \frac{5b}{z}, & i = j \in \{k, l\}, \\ \frac{4b}{z}, & (i, j) \in \{(k, l), (l, k)\}, \\ 1, & i = j \notin \{k, l\}, \\ 0, & \text{otherwise}. \end{cases}$$

Since $x \in \mathcal{D}_{\mathcal{M}}$, $\exists\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$ such that $\mathbb{P}(\tau^{X^B}_{K_1}) > 0$. Define $\tau_1 := \tau^{X^B}_{K_1} \wedge T$, and note that, by $\Sigma \in \mathbb{S}^d_+$ and the regularity property of $\mathcal{M}$, there exists a model $\{\widetilde{A}; (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{P})\} \in \mathcal{M}$ and stopping time $\widetilde{\tau} \in \mathcal{T}(\widetilde{\mathbb{F}}_T)$ such that properties stated in Definition 2.1.1 holds. Define the $\widetilde{\mathbb{F}}_T$-stopping time $\tau_2 := \widetilde{\tau} \wedge \inf\{t \geq \tau_1 : X^B_t[\widetilde{A}] \in \partial K_2\}$. Then,

$$\begin{aligned} Y_{\tau_2}[\widetilde{A}] - Y_{\tau_1}[\widetilde{A}] &= \int_{\tau_1}^{\tau_2} \mathcal{L}^\gamma F(X^B_t[\widetilde{A}]) + \sum_{i,j=1}^d \int_{\tau_1}^{\tau_2} \mathcal{L}^{\alpha,\beta}_{i,j} F(X^B_t[\widetilde{A}])\,\mathrm{d}\langle \widetilde{A}^i, \widetilde{A}^j\rangle_t \\ &= \int_{\tau_1}^{\tau_2} \left(\mathcal{L}^\gamma F(X^B_t[\widetilde{A}]) + \sum_{i,j=1}^d \Sigma_{ij} \mathcal{L}^{\alpha,\beta}_{i,j} F(X^B_t[\widetilde{A}]) \right) \mathrm{d}t \\ &= \int_{\tau_1}^{\tau_2} \left(2\left(\frac{4b}{z}\right) \mathcal{L}^{\alpha,\beta}_{k,l} F(X^B_t[\widetilde{A}]) + \mathcal{L}^\gamma F(X^B_t[\widetilde{A}]) \right) \mathrm{d}t \\ &\geq \int_{\tau_1}^{\tau_2} \left(\frac{8b}{z} \inf_{y\in K_2} \mathcal{L}^{\alpha,\beta}_{i,j} F(y) - ||\mathcal{L}^\gamma F||_{K_2} \right) \mathrm{d}t \\ &\geq \int_{\tau_1}^{\tau_2} 3b\,\mathrm{d}t. \end{aligned}$$

where we used $\mathcal{L}^{\alpha,\beta}_{k,l} F = \mathcal{L}^{\alpha,\beta}_{l,k} F$ and $\mathcal{L}^{\alpha,\beta}_{i,i} F = 0$ on $\mathcal{D}_{\mathcal{M}} \setminus B$ for $i \in \{1, \dots, d\}$ in obtaining the third line. Hence,

$$Y_{\tau_2}[\widetilde{A}] - Y_{\tau_2}[\widetilde{A}] \geq 3b\,(\tau_2 - \tau_1). \tag{2.1.3}$$

By analogous arguments as in part (i) of the proof, $\widetilde{\mathbb{P}}(\tau_1 < \tau_2) > 0$. Hence, with (2.1.3), this again implies that $\mathbb{E}^{\widetilde{\mathbb{P}}}(Y_{\tau_2}[\widetilde{A}] - Y_{\tau_1}[\widetilde{A}]) > 0$, which contradicts (2.1.1).

This proves that $\mathcal{L}^{\alpha,\beta}_{i,j} F = 0$ on $\mathcal{D}_{\mathcal{M}} \setminus B$ for all $i \neq j$.

(iii) Parts (i) and (ii) proved that $\mathcal{L}^{\alpha,\beta}_{i,j} F = 0$ on $\mathcal{D}_{\mathcal{M}} \setminus B$ for all $i, j \in \{1, \dots, d\}$. Hence, (2.1.1) becomes

$$\mathbb{E}\left(\int_{\tau_1}^{\tau_2} \mathcal{L}^\gamma F(X^B_t[\widetilde{A}])\,\mathrm{d}t \right) = 0.$$

By a simpler version of the arguments in parts (i) and (ii), this implies that $\mathcal{L}^\gamma F = 0$ on $\mathcal{D}_{\mathcal{M}} \setminus B$.

□

2.2 Underlying asset and convex claim

In this section, we consider a market with $A = (S, C)$, where S is an underlying asset and C is a traded European path-independent claim with maturity T and payoff $g(S_T)$. We also assume that g is a non-linear convex function such that $\int_{\mathbb{R}} g(se^y)e^{-y^2/2}\,\mathrm{d}y < \infty$ and for all $s > 0$. The latter condition requires the Black-Scholes price under $r = 0$ and $\sigma = 1$ to be finite for all initial prices $S_0 = s$. Note that when we say that a convex function g is non-linear on an interval $(s_1, s_2) \subset \mathbb{R}_+$, we mean that $g'(s_1) < g'(s_2)$, where g' denotes the left derivative of g.

Definition 2.2.1. We define $V := C - g(S)$ to be the *time value* of the claim C.

Notation 2.2.2.

(1) We say that I is an $\mathbb{F}_T$-adapted open interval in $\mathbb{R}_+$ if $I = (a, b)$ for $a, b : \Omega \times [0, T] \to \mathbb{R}_+$ such that $a < b$ $\mathbb{P}$-a.e. and a, b are $\mathbb{F}_T$-adapted.

(2) For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and subsets $\xi_1, \xi_2 \in \mathcal{F}$ of Ω, $\mathbb{P}(\xi_1 \,|\, \mathcal{G}) > 0$ on ξ_2 means that for any $\xi_3 \in \mathcal{G}$ such that $\mathbb{P}(\xi_2 \cap \xi_3) > 0$, we have that $\mathbb{P}(\xi_1 \cap \xi_2 \cap \xi_3) > 0$.

(3) A process Z is a local martingale on $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ if there is a sequence of $\mathbb{F}_T$-stopping times $\tau_k \uparrow T$ such that $Z_{\cdot \wedge \tau_k}$ is a martingale with respect to $\mathbb{F}_T$.

We now define a so-called *full support* property of a stochastic process. This will be a central property in this section as well as in Section 2.4.

Definition 2.2.3. Let S be an $\mathbb{R}_+$-valued stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$. We then say that S has *full support* or S has the (FS) property if for any $\mathbb{F}_T$-adapted open interval $I \subset \mathbb{R}_+$ and for any $t \in [0, T)$, $\mathbb{P}(S_T \in I_t \,|\, \mathcal{F}_t) > 0$. Similarly, for a subset $\xi \in \mathcal{F}$ of Ω, we say that S has full support on ξ if for any $\mathbb{F}_T$-adapted open interval $I \subset \mathbb{R}_+$ and for any $t \in [0, T)$, $\mathbb{P}(S_T \in I_t \,|\, \mathcal{F}_t) > 0$ on ξ.

The requirement that $\mathbb{P}(S_T \in I_t \,|\, \mathcal{F}_t) > 0$ is an almost-sure statement on conditional probability distributions. It is equivalent to the statement $\mathbb{P}(\mathbb{P}(S_T \in I_t \,|\, \mathcal{F}_t) = 0) = 0$. In particular, any process which is absorbed at a level which it can attain prior to time T with non-zero probability does not have the full support property (since $\mathbb{P}(\mathbb{P}(S_T \in I_t \,|\, \mathcal{F}_t) = 0) > 0$ for some $t < T$).

We can now define the family of models which we will prove to be regular for $A = (S, C)$.

Definition 2.2.4. Define $\mathcal{M} \equiv \mathcal{M}_g$ to be the family of models $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}_\ell$ such that $C - \int_0^\cdot g'(S_t)\,\mathrm{d}S_t$ is a supermartingale and such that S has full support.

Note that $\mathcal{M} \subseteq \mathcal{M}_\ell$, hence S and C are local martingales (on the time interval $[0,T]$) such that $C_T = g(S_T)$ $\mathbb{P}$-a.e.. The class $\mathcal{M}$ is general enough to work with models with bubbles in the price of the traded claim, notably such that $C_t > \mathbb{E}(g(S_T)|\mathcal{F}_t)$. In particular, the class of models $\mathcal{M}$ allows for strict local martingales for the traded options even if the underlying is a true martingale. This contrasts somewhat with the context considered in Cox and Hobson [14], where the emphasis is on the asset price being a strict local martingale under the risk-neutral measure. The main result of this section is the following.

Theorem 2.2.5. *$\mathcal{M}$ defined in Definition 2.2.4 is a regular class of models for $A = (S, C)$. Furthermore, $\mathcal{M}(A_0) \neq \emptyset$ iff $C_0 > g(S_0)$.*

For $X \in \mathcal{X}$ and a closed set $B \subset \mathbb{R}^n$, denote $\mathcal{D} \equiv \mathcal{R}(X^B; \mathcal{M})$ and consider a function $F \in C(\mathcal{D}) \cap C^2(\mathcal{D} \setminus B)$. Then, Theorem 2.1.2 and Theorem 2.2.5 together imply that $F(X^B_\cdot)$ is a local martingale under all models in $\mathcal{M}$ iff $F \in \mathcal{S}_X(\mathcal{D} \setminus B)$. Whether $\mathcal{R}(X^B; \mathcal{M}) = \mathcal{R}(X^B; \mathcal{M}_\ell)$ is a topic for future research. The ranges coincide for some functionals such as $X = (S, V, \langle S \rangle)$. But the answer is yet unclear to us for other functionals such as $X = (S, V, \langle S \rangle, \langle V \rangle, \langle S, V \rangle))$. Both of these examples are considered in Section 2.3.

The result of $\mathcal{M}$ being regular (proved in the section) holds under a slightly weaker condition than S having full support, notably by considering the domain of convexity of g. This extension would be pedantic and would obfuscate the clarity of the presentation, hence we work with the class $\mathcal{M}$ in Definition 2.2.4.

We first present some results which will be helpful in proving Theorem 2.2.5. We begin with the following lemma, which we prove in Appendix B. The lemma provides an equivalent statement of the full support property based on stopping times (this will be useful in some of the upcoming proofs).

Lemma 2.2.6. *S has the full support property iff for any $\mathbb{F}_T$-adapted stochastic open interval $I \subset \mathbb{R}_+$ and any stopping time $\tau \in \mathcal{T}(\mathbb{F}_T)$, $\mathbb{P}(S_T \in I_\tau \,|\, \mathcal{F}_\tau) > 0$ on $\{\tau < T\}$.*

The following proposition provides a sufficient regularity condition on the volatility of a continuous local martingale S in order for it to have full support. In particular, it shows that processes which have a lower and upper bounded realised log-variance on $[\tau, T]$ for a stopping time τ are of full support on $\{\tau < T\}$. Hence, coupling a stochastic process at a stopping time $\tau_{coupling}$ with a local volatility process with lower and upper bounded volatility will yield a process with full support on $\{\tau_{coupling} < T\}$. The proof is based on arguments involving time-changed Brownian motion and is provided in Appendix B.

Proposition 2.2.7. *Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ with a one-dimensional Brownian motion W, a left-continuous $\mathbb{F}_T$-adapted process σ and a stopping time $\tau \in \mathcal{T}(\mathbb{F}_T)$ defined on it. Suppose that S is an $\mathbb{F}_T$-adapted process on $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ such that*

$$\mathrm{d}S_t := \sigma_t S_t \,\mathrm{d}W_t, \quad t \in [\tau \wedge T, T],$$

Then, S has full support on $\{\tau < T\}$ if there are functions $l, u : [0,T] \to \mathbb{R}$ such that $0 < l(t) \leq u(t) < \infty$ for $t \in [0,T)$, $l(T) = u(T) = 0$ and

$$\mathbb{P}\left(l(t \vee \tau) \leq \int_{t\vee\tau}^{T} \sigma_u^2 \,\mathrm{d}u \leq u(t \vee \tau) \,\middle|\, \mathcal{F}_{t\vee\tau} \right) > 0 \quad \text{on } \{\tau < T\} \tag{2.2.1}$$

for $t \geq 0$.

The following proposition is a key ingredient in the proof of Theorem 2.2.5.

Proposition 2.2.8. *$\mathcal{M}(A_0) \neq \emptyset$ iff $V_0 > 0$, and for any model $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$, $\mathbb{P}(V_t > 0,\ t \in [0,T)) = 1$.*

Proof. Suppose that $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$. The notational definitions in this proof are temporary.

(i) We begin by showing that for $a \in \mathbb{R}_+$, stopping times $\tau \in \mathcal{T}(\mathbb{F}_T)$ and $\tau_a := \inf\{t \geq \tau : S_t = a\}$, the full support property of S implies that $\mathbb{P}(\tau_a < T \,|\, \tau < T) > 0$. Let $s_1 < a < s_2$. Then,

$$\begin{aligned}
&\mathbb{P}(\tau_a < T \,|\, \tau < T) \\
&\ = \mathbb{P}(\tau_a < T, S_\tau = a \,|\, \tau < T) + \mathbb{P}(\tau_a < T, S_\tau < a \,|\, \tau < T) + \mathbb{P}(\tau_a < T, S_\tau > a \,|\, \tau < T) \\
&\ \geq \mathbb{P}(S_\tau = a \,|\, \tau < T) + \mathbb{P}(S_T > s_2, S_\tau < a \,|\, \tau < T) + \mathbb{P}(S_T < s_1, S_\tau > a \,|\, \tau < T) \\
&\ = \mathbb{P}(S_\tau = a \,|\, \tau < T) + \mathbb{P}(S_T > s_2 \,|\, S_\tau < a, \tau < T)\,\mathbb{P}(S_\tau < a \,|\, \tau < T) \\
&\qquad\qquad + \mathbb{P}(S_T < s_1, \,|\, S_\tau > a, \tau < T)\,\mathbb{P}(S_\tau > a \,|\, \tau < T)
\end{aligned}$$

Since $\mathbb{P}(S_T > s_2 \,|\, S_\tau < a, \tau < T) > 0$, $\mathbb{P}(S_T < s_1 \,|\, S_\tau > a, \tau < T) > 0$ (by $\mathbb{P}(S_T > s_2 \,|\, \tau < T) > 0$ and $\mathbb{P}(S_T < s_1 \,|\, \tau < T) > 0$ respectively) and

$$\mathbb{P}(S_\tau = a \,|\, \tau < T) + \mathbb{P}(S_\tau < a \,|\, \tau < T) + \mathbb{P}(S_\tau < a \,|\, \tau < T) = 1,$$

we conclude that $\mathbb{P}(\tau_a < T \,|\, \tau < T) > 0$ as claimed.

(ii) Next, we show that for $a \in \mathbb{R}_+$ and $\tau \in \mathcal{T}(\mathbb{F}_T)$, the full support property of S implies that $\mathbb{E}(L^a_T[S] - L^a_{\tau\wedge T}[S]) > 0$ if $\mathbb{P}(\tau < T) > 0$.

Let $b \in \mathbb{R}_+ \setminus \{a\}$ and define

$$\begin{aligned} \tau_a &:= \inf\{t \geq \tau : S_t = a\}, \\ \tau_{a,b} &:= \inf\{t \geq \tau_a : S_t = b\}. \end{aligned}$$

By monotonicity of $L^a_t[S]$ in t, Tanaka's formula and $\tau_a \wedge T \leq \tau_{a,b} \wedge T \leq T$,

$$\begin{aligned} \mathbb{E}\left(L^a_T[S] - L^a_{\tau\wedge T}[S]\right) &\geq \mathbb{E}\left(L^a_{\tau_{a,b}\wedge T}[S] - L^a_{\tau_a\wedge T}[S]\right) \\ &= \mathbb{E}\left(|S_{\tau_{a,b}\wedge T} - a| - |S_{\tau_a\wedge T} - a|\right) \\ &= \mathbb{E}\left(\left(|S_{\tau_{a,b}\wedge T} - a| - |S_{\tau_a\wedge T} - a|\right) \mathbb{1}_{\{\tau_a<T\}}\right) \\ &= \mathbb{E}\left(|S_{\tau_{a,b}\wedge T} - a| \mathbb{1}_{\{\tau_a<T\}}\right) \\ &\geq \mathbb{E}\left(|S_{\tau_{a,b}\wedge T} - a| \mathbb{1}_{\{\tau_{a,b}<T\}}\right) \\ &\geq |b-a| \mathbb{P}(\tau_{a,b} < T) \end{aligned}$$

By $\{\tau_{a,b} < T\} = \{\tau_{a,b} < T, \tau_a < T\}$ and $\{\tau_a < T\} = \{\tau_a < T, \tau < T\}$,

$$\begin{aligned} \mathbb{P}(\tau_{a,b} < T) &= \mathbb{P}(\tau_{a,b} < T \,|\, \tau_a < T)\mathbb{P}(\tau_a < T) \\ &= \mathbb{P}(\tau_{a,b} < T \,|\, \tau_a < T)\mathbb{P}(\tau_a < T \,|\, \tau < T)\mathbb{P}(\tau < T) \end{aligned}$$

Hence, by part (i) of the proof,

$$\mathbb{E}\left(L^a_T[S] - L^a_{\tau\wedge T}[S]\right) \geq |b-a| \mathbb{P}(\tau_{a,b} < T \,|\, \tau_a < T)\, \mathbb{P}(\tau_a < T \,|\, \tau < T)\, \mathbb{P}(\tau < T) > 0$$

if $\mathbb{P}(\tau < T) > 0$.

(iii) Define $\tau^V_0 := \inf\{t \geq 0 : V_t = 0\}$. Note that $\tau^V_0 \leq T$ by $V_T = 0$. We will now show that $\mathbb{P}(\tau^V_0 < T) = 0$.

Temporarily define $Z := C - \int_0^\cdot g'(S_t)\,\mathrm{d}S_t$. The Itô-Tanaka formula gives $V_t - V_0 = Z_t - Z_0 - \int_0^\infty L^a_t[S]\,\mathrm{d}g'(a)$, which along with $V_T = V_{\tau^V_0} = 0$ implies that

$$\int_0^\infty \left(L^a_T[S] - L^a_{\tau^V_0}[S]\right) \mathrm{d}g'(a) = Z_T - Z_{\tau^V_0}.$$

Fubini's theorem and the supermartingale property of Z (by the assumptions in Definition 2.2.4) then yield

$$\begin{aligned} \int_0^\infty \mathbb{E}\left(L^a_T[S] - L^a_{\tau^V_0}[S]\right) \mathrm{d}g'(a) &= \mathbb{E}\left(\int_0^\infty \left(L^a_T[S] - L^a_{\tau^V_0}[S]\right) \mathrm{d}g'(a)\right) \\ &= \mathbb{E}\left(Z_T - Z_{\tau^V_0}\right) \leq 0. \end{aligned}$$

By convexity and non-linearity of g and by part (ii) of the proof, this implies that $\mathbb{P}(\tau_0^V < T) = 0$.

(iv) The conclusion in part (iii) proves that $V_t > 0$ for all $t \in [0, T)$ almost surely under all models in $\mathcal{M}$. In particular, this implies that $\mathcal{M}(A_0) = \emptyset$ if $V_0 \leq 0$. Hence, it remains to prove the converse claim that $\mathcal{M}(A_0) \neq \emptyset$ if $V_0 > 0$. To do so, we will provide a construction of a model based on geometric Brownian motion and a strict local martingale.

Consider the canonical space $(\mathcal{C}_\infty(\mathbb{R}^2), \mathcal{F}^W, \mathbb{F}^W, \mathbb{P}^W)$ on which (W^S, W^Y) is a two-dimensional Brownian motion. Define

$$Y_t := \begin{cases} \left(\frac{T-t}{T}\right)^{1/2} \exp\{W^Y_{\log(T/(T-t))}\}, & t < T, \\ 0, & t \geq T, \end{cases}$$

and denote $(\mathcal{C}_T(\mathbb{R}^2), \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ to be the canonical space of $\{(W_t^S, Y_t)\}_{t\in[0,T]}$. Y is a geometric Brownian motion (with zero mean) when stopped at a fixed sequence of times $t_k < T$ such that $t_k \uparrow T$. Hence, it is a local martingale on $(\mathcal{C}_T(\mathbb{R}^2), \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ starting at $Y_0 = 1$. Moreover, by $\lim_{t\to\infty} e^{W_t^Y - \frac{1}{2}t} = 0$ $\mathbb{P}^W$-a.e. (apply the law of iterated logarithm, Theorem 2.9.23 of Karatzas and Shreve), it follows that $\lim_{t\to T} Y_t = 0$ $\mathbb{P}^W$-a.e..

Temporarily define the Black-Scholes price of the claim on $g(S_T)$ at time zero under zero interest rate and volatility $\sigma > 0$ by

$$F_{BS}(s, t; \sigma) := \mathbb{E}\left(g\left(s\mathcal{E}(\sigma W_t^S)\right)\right).$$

The following properties are well-known for Black-Scholes prices of convex claims, but we outline the arguments for completeness. By Fubini's theorem and the Itô-Tanaka formula, $F_{BS}(S_0, T; \sigma) = g(S_0) + \int_0^t \mathbb{E}\left(L_T^a\left[S_0 e^{\sigma W_T^1 - \frac{1}{2}\sigma^2 T}\right]\right) \mathrm{d}g'(a)$. This implies that $F_{BS}(S_0, T; \sigma)$ is increasing in σ and that $F_{BS}(S_0, T; \sigma) \downarrow g(S_0)$ as $\sigma \downarrow 0$ (by dominated convergence, using $\int_{\mathbb{R}} g(S_0 e^y) e^{-y^2/2} \,\mathrm{d}y < \infty$ and $\lim_{\sigma\to 0} L_T^a\left[S_0 e^{\sigma W_T^1 - \frac{1}{2}\sigma^2 T}\right] = 0$). Hence,

$$\sigma_0 := \sup\{\sigma > 0 : F_{BS}(S_0, T; \sigma) - g(S_0) < V_0\}$$

is well-defined and $F_{BS}(S_0, T; \sigma_0) \leq C_0$. Then, for

$$\begin{aligned} S_t &:= S_0 e^{\sigma_0 W_t^1 - \frac{1}{2}\sigma_0^2 t}, \quad t \in [0, T], \\ C_t &:= F_{BS}(S_t, T - t; \sigma_0) + (C_0 - F_{BS}(S_0, T; \sigma_0)) Y_t, \quad t \in [0, T], \end{aligned}$$

we have that:

(1) S is a geometric Brownian motion, hence has full support by Proposition 2.2.7.

(2) C is a continuous local martingale on $(\mathcal{C}_T(\mathbb{R}^3), \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ such that $C_T = g(S_T)$.

(3) For $t \in [0, T]$,

$$
\begin{aligned}
&Z_t = C_t - \int_0^t g(S_u)\,\mathrm{d}S_u \\
&= (C_0 - F_{BS}(S_0, T; \sigma_0))Y_t + (F_{BS}(S_t, T-t; \sigma_0) - g(S_t)) + \int_0^\infty L_t^a[S]\,\mathrm{d}g'(a) + g(S_0) \\
&\geq g(S_0),
\end{aligned}
$$

which implies that $Z.$ is a lower bounded local martingale, thus a supermartingale. Hence, $\{(S, C); (\mathcal{C}_T(\mathbb{R}^2), \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}(A_0)$. This proves that $\mathcal{M}(A_0) \neq \emptyset$ if $V_0 > 0$. □

Proof of Theorem 2.2.5. The claim that $\mathcal{M}(A_0) \neq \emptyset$ iff $C_0 > g(S_0)$ was shown in Proposition 2.2.8. It remains to prove that $\mathcal{M}$ is a regular class of models for $A = (S, C)$.

Consider a model $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$, a stopping time $\tau \in \mathcal{T}(\mathbb{F}_T)$ such that $\mathbb{P}(\tau < T) > 0$ and a matrix $\Sigma \in \mathbb{S}_+^2$. By similar arguments as in part (iv) of the proof of Proposition 2.2.8, consider a filtered probability space $(\Omega', \mathcal{F}', \mathbb{F}'_T, \mathbb{P}')$ on which are defined a d-dimensional Brownian motion W and an independent continuous local martingale Y on $(\mathcal{C}_T(\mathbb{R}^3), \mathcal{F}, \mathbb{F}_T, \mathbb{P})$ starting at $Y_0 = 1$ and such that $\inf\{t \geq 0 : Y_t = 0\} = T$ $\mathbb{P}'$-a.e.. Denote the Black-Scholes formula by

$$F_{BS}(s, t; \sigma) := \mathbb{E}^{\mathbb{P}'}\left(g\left(s\mathcal{E}(\sigma W_t^1)\right)\right).$$

For $c > g(s)$, define

$$\sigma_t(s, c) := \sup\{\sigma \geq 0 : F_{BS}(s, t; \sigma) \leq c\}.$$

Then, $F_{BS}(s, t; \sigma_t(s, c)) \leq c$, with strict equality holding if and only if $\sigma_t(s, c) = \infty$. Define $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}}) := (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P}) \otimes (\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$ and let $m \in \mathbb{R}^{2\times 2}$ be such that $m^T m = \Sigma$. We can now define a model $\{\widetilde{A}, (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})\} \in \mathcal{M}$ and a stopping time $\widetilde{\tau}$ satisfying the properties stated in Definition 2.1.1. In particular, define

$$
\begin{aligned}
&\widetilde{A}_t := A_{t\wedge\tau} + m(W_t - W_{t\wedge\tau}), \quad t \in [\tau, \widetilde{\tau}), \\
&\widetilde{\tau} := \inf\left\{t \geq \tau : \widetilde{C}_t - g(\widetilde{S}_t) \leq V_\tau/2\right\} \wedge \frac{T+\tau}{2},
\end{aligned}
$$

and, using the shorthand notation $\widetilde{\sigma} \equiv \sigma_{\widetilde{\tau}}(\widetilde{S}_{\widetilde{\tau}}, \widetilde{C}_{\widetilde{\tau}})$,

$$\widetilde{S}_t := \widetilde{S}_{\widetilde{\tau}} \frac{\mathcal{E}(\widetilde{\sigma} W^1_t)}{\mathcal{E}(\widetilde{\sigma} W^1_{\widetilde{\tau}})}, \quad \widetilde{C}_t := BS(\widetilde{S}_t, t; \widetilde{\sigma}) + \frac{\widetilde{C}_{\widetilde{\tau}} - BS(\widetilde{S}_{\widetilde{\tau}}, \widetilde{\tau}; \widetilde{\sigma})}{Y_{\widetilde{\tau}}} Y_t, \quad t \in [\widetilde{\tau}, T].$$

By construction, $\widetilde{A} = A$ on $[0, \tau]$, $\widetilde{S}$ is a continuous martingale and $\widetilde{C}$ is a continuous local martingale such that $\widetilde{C}_T = g(\widetilde{S}_T)$. Denote $\widetilde{Z}_\cdot = \widetilde{C}_\cdot - \int_0^\cdot g'(\widetilde{S}_t)\,\mathrm{d}\widetilde{S}_t$. Note that $\widetilde{Z}$ is a local martingale. Furthermore,

$$\begin{aligned}\widetilde{Z}_t &= \widetilde{Z}_0 + \widetilde{C}_t - \widetilde{C}_0 - \left(g(\widetilde{S}_t) - g(\widetilde{S}_0) - \int_0^\infty L^a_t[\widetilde{S}]\,\mathrm{d}g'(a) \right) \\ &= g(S_0) + \left(BS(\widetilde{S}_t, t; \widetilde{\sigma}) + \frac{\widetilde{C}_{\widetilde{\tau}} - BS(\widetilde{S}_{\widetilde{\tau}}, \widetilde{\tau}; \widetilde{\sigma})}{Y_{\widetilde{\tau}}} Y_t \right) - g(\widetilde{S}_t) + \int_0^\infty L^a_t[\widetilde{S}]\,\mathrm{d}g'(a) \\ &\geq g(S_0)\end{aligned}$$

by the convexity of g (see the standard arguments outlined in part (iv) of proof of Proposition 2.2.8). This implies that $\widetilde{Z}$ is a lower bounded lower martingale, hence a supermartingale.

To prove that $\widetilde{S}$ is of full support, we will prove that it is of full support on sets $\xi_1, \xi_2 \in \widetilde{\mathcal{F}}$ such that $\xi_1 \cup \xi_2 = \widetilde{\Omega}$. In particular, we will set $\xi_1 = \{\widetilde{\tau} < T\}$ and $\xi_2 = \{\widetilde{\tau} \geq T\}$. Then:

(1) On $\{\widetilde{\tau} < T\}$, $\mathrm{d}\widetilde{S}_t = \widetilde{\sigma}\widetilde{S}\,\mathrm{d}W^1_t$ for $t \in [\widetilde{\tau}, T]$. Hence, $\widetilde{S}$ is of full support on $\{\widetilde{\tau} < T\}$ by Proposition 2.2.7.

(2) On $\{\widetilde{\tau} \geq T\}$, $\tau \geq T$ and hence $\widetilde{S} = S$ on $[0, T]$. Since S is of full support, $\widetilde{S}$ is of full support on $\{\widetilde{\tau} \geq T\}$.

This proves that $\widetilde{S}$ is of full support (on $\widetilde{\Omega} = \{\widetilde{\tau} < T\} \cup \{\widetilde{\tau} \geq T\}$) and that $\{\widetilde{A}; (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})\} \in \mathcal{M}$. Furthermore, Proposition 2.2.8 implies that $V_\tau > 0$ on $\{\tau < T\}$. Along with the continuity of paths of $\widetilde{A}$, this yields $\{\tau < \widetilde{\tau} \leq T\} = \{\tau < T\}$. Also, note that $\mathrm{d}\langle \widetilde{A} \rangle_t = \langle mW \rangle_t = m^T m\,\mathrm{d}t$ for $t \in [\tau, \widetilde{\tau})$.

This concludes the proof that $\mathcal{M}$ is a regular class of models for A.

□

Remark 2.2.9. The above proof generalises to constructing a model $\{\widetilde{A}; (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})\} \in \mathcal{M}$ and a stopping time $\widetilde{\tau}$ under which $\mathrm{d}\langle \widetilde{A} \rangle_t = \Sigma_t\,\mathrm{d}t$ for $t \in [\tau \wedge T, \widetilde{\tau} \wedge T)$, where Σ is equal to either (i) $\Sigma^f[A]$ for a $\mathcal{C}_T(\mathbb{S}^2_+)$-valued bounded non-anticipative functional Σ^f or (ii) an $\mathbb{S}^2_+$-valued process on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})$ such that $\int_0^\cdot m_t\,\mathrm{d}W_t$ is a martingale for a process $m : \widetilde{\Omega} \times [0, T] \to \mathbb{R}^{2\times 2}$ satisfying $m^T m = \Sigma$. These constructions, though of some independent interest, are not relevant for the characterisation result, hence we do not pursue them further.

2.3 Examples with $A = (S, C)$

2.3.1 Examples with time value

Consider the same setup as in Section 2.2. Convexity of g implies that g is piecewise C^2. For clarity of presentation, assume that $g \in C^2$. By known results [9, 10, 23, 29, 48], the time value V corresponds to the model-free implied weighted variance $\mathbb{E}\left(\int_t^T \frac{1}{2} g''(S_t)\,\mathrm{d}\langle S\rangle_t \,\middle|\, \mathcal{F}_t\right)$ under any model where S is a regular enough continuous local martingale. When $g(x) = -2\log(x)$, $V = C - g(S)$ corresponds to the theoretical value of the VIX index (modulo any rolling contract features and adjustments for jumps). Since the latter is a key index in industry, it is of interest to understand what payoffs contingent upon it and upon its second variations one may replicate model-independently (see Fukasawa [23] for a discussion of the role of the cross-variation between VIX and asset (log) returns).

For the rest of this section, we will denote $Q^s \equiv \int_0^\cdot w_s(S_t)\,\mathrm{d}\langle S\rangle_t$, $Q^v \equiv \int_0^\cdot w_v(S_t)\,\mathrm{d}\langle V\rangle_t$, $L \equiv \int_0^\cdot w_l(S_t)\,\mathrm{d}\langle S, V\rangle_t$, $X \equiv (S, V, Q^s, Q^v, L)$, $X^s \equiv (S, V, Q^s)$, $X^v \equiv (S, V, Q^v)$ and $X^l \equiv (S, V, L)$, where $w_s, w_v, w_l > 0$ are continuous (weighting) functions. It is easily checked that X, X^s, X^v and X^l are functionals in $\mathcal{X}$. $\mathcal{M} = \mathcal{M}_g$ is defined as in Definition 2.2.4. Define $\mathcal{D} = \mathcal{R}(X; \mathcal{M}) \setminus \{x_2 = 0\}$, $\mathcal{D}^s = \mathcal{R}(X^s; \mathcal{M}) \setminus \{x_2 = 0\}$, $\mathcal{D}^v = \mathcal{R}(X^v; \mathcal{M}) \setminus \{x_2 = 0\}$ and $\mathcal{D}^l = \mathcal{R}(X^l; \mathcal{M}) \setminus \{x_2 = 0\}$. Note that $\mathcal{D} \subset \mathbb{R}^5$ and that $\mathcal{D}^s, \mathcal{D}^v, \mathcal{D}^l \subset \mathbb{R}^3$.

We start by simplifying the system of PDEs characterising $\mathcal{S}_X(\mathcal{D})$.

Lemma 2.3.1. *$F : \mathcal{D} \to \mathbb{R}$ is in $\mathcal{S}_X(\mathcal{D})$ iff $F \equiv F(s, v, q_s, q_v, l)$ is a solution to*

$$\begin{cases} w_s(s)\frac{\partial}{\partial q_s}F - \frac{1}{2}g''(s)\frac{\partial}{\partial v}F + \frac{1}{2}\frac{\partial^2}{\partial s^2}F = 0 \\ w_v(s)\frac{\partial}{\partial q_v}F + \frac{1}{2}\frac{\partial^2}{\partial v^2}F = 0 \\ w_l(s)\frac{\partial}{\partial l}F + \frac{\partial^2}{\partial s \partial v}F = 0 \end{cases}$$

on $\mathcal{D}$.

Proof. We start by noting that

$$\begin{aligned} \mathrm{d}V_t &= \mathrm{d}C_t - g'(S_t)\,\mathrm{d}S_t - \frac{1}{2}g''(S_t)\,\mathrm{d}\langle S\rangle_t, \\ \mathrm{d}\langle V\rangle_t &= (g'(S_t))^2\,\mathrm{d}\langle S\rangle_t + \mathrm{d}\langle C\rangle_t - 2g'(S_t)\,\mathrm{d}\langle S, C\rangle_t, \\ \mathrm{d}\langle S, V\rangle_t &= -g'(S_t)\,\mathrm{d}\langle S\rangle_t + \mathrm{d}\langle S, C\rangle_t \end{aligned}$$

For $F \equiv F(s, v, q_s, q_v, l) \in C^2(\mathbb{R}^5)$, Itô's formula gives

$$\begin{aligned}
\mathrm{d}F(X_t) &= \frac{\partial}{\partial s} F(X_t)\,\mathrm{d}S_t + \frac{\partial}{\partial v} F(X_t)\,\mathrm{d}V_t \\
&\quad + \frac{\partial}{\partial q_s} F(X_t)\,\mathrm{d}Q^s_t + \frac{\partial}{\partial q_v} F(X_t)\,\mathrm{d}Q^v_t + \frac{\partial}{\partial l} F(X_t)\,\mathrm{d}L_t \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial s^2} F(X_t)\,\mathrm{d}\langle S\rangle_t + \frac{1}{2}\frac{\partial^2}{\partial v^2} F(X_t)\,\mathrm{d}\langle V\rangle_t + \frac{\partial^2}{\partial s \partial v} F(X_t)\,\mathrm{d}\langle S, V\rangle_t \\
&= \left(\frac{\partial}{\partial s} F(X_t) - g'(S_t)\frac{\partial}{\partial v} F(X_t)\right)\,\mathrm{d}S_t + \frac{\partial}{\partial v} F(X_t)\,\mathrm{d}C_t \\
&\quad + \left(w_s(S_t)\frac{\partial}{\partial q_s} F(X_t) + \frac{1}{2}\frac{\partial^2}{\partial s^2} F(X_t) - \frac{1}{2} g''(S_t)\frac{\partial}{\partial v} F(X_t)\right)\,\mathrm{d}\langle S\rangle_t \\
&\quad + \left(w_v(S_t)\frac{\partial}{\partial q_v} F(X_t) + \frac{1}{2}\frac{\partial^2}{\partial v^2} F(X_t)\right)\,\mathrm{d}\langle V\rangle_t \\
&\quad + \left(w_l(S_t)\frac{\partial}{\partial l} F(X_t) + \frac{\partial^2}{\partial s \partial v} F(X_t)\right)\,\mathrm{d}\langle S, V\rangle_t
\end{aligned}$$

Note that the integrand term corresponding to $\langle V\rangle$ must be equal to zero in order for the integral with respect to $\langle C\rangle$ to be equal to zero (this is because $\langle S, V\rangle$ does not contain any integrals with respect to $\langle C\rangle$). Then, in order for the integral with respect to $\langle S, C\rangle$ to be zero, the integral with respect to $\langle S, V\rangle$ must be zero. These two conditions along with the condition of the integral with respect to $\langle S\rangle$ being equal to zero yield

$$\begin{cases}
w_s(S_t)\frac{\partial}{\partial q_s} F(X_t) - \frac{1}{2} g''(S_t)\frac{\partial}{\partial v} F(X_t) + \frac{1}{2}\frac{\partial^2}{\partial s^2} F(X_t) = 0 \\
w_v(S_t)\frac{\partial}{\partial q_v} F(X_t) + \frac{1}{2}\frac{\partial^2}{\partial v^2} F(X_t) = 0 \\
w_l(S_t)\frac{\partial}{\partial l} F(X_t) + \frac{\partial^2}{\partial s \partial v} F(X_t) = 0.
\end{cases}$$

Hence, the characterisation theorem implies that $F \equiv F(s, v, q_s, q_v, l)$ is in $\mathcal{S}_X(\mathcal{D})$ iff

$$\begin{cases}
w_s(s)\frac{\partial}{\partial q_s} F - \frac{1}{2} g''(s)\frac{\partial}{\partial v} F + \frac{1}{2}\frac{\partial^2}{\partial s^2} F = 0 \\
w_v(s)\frac{\partial}{\partial q_v} F + \frac{1}{2}\frac{\partial^2}{\partial v^2} F = 0 \\
w_l(s)\frac{\partial}{\partial l} F + \frac{\partial^2}{\partial s \partial v} F = 0,
\end{cases}$$

on $\mathcal{D}$, which concludes the proof.

□

By imposing further conditions on $F \in \mathcal{S}_X(\mathcal{D})$, we obtain the systems of PDEs corresponding to X^s, X^v and X^l which can be solved in closed form.

The following proposition implies that the only functions of (weighted) realised variance replicable model-independently via strategies with wealth process of the form $F(X^s)$ are linear combinations of strategies in Subsection 1.3.1 and of the well-known strategy for replicating VIX.

Proposition 2.3.2. *For $F \in C^2(\mathcal{D}^s) \cap C(\overline{\mathcal{D}^s})$, $F(S_\cdot, V_\cdot, Q^s_\cdot)$ is a local martingale under all models in $\mathcal{M}$ iff*

$$F(s, v, q_s) = c_1(v + g(s)) + c_2\left(\frac{g''(s)}{w_s(s)}q_s - g(s)\right) + F_h(s, q_s) \tag{2.3.1}$$

for a constant $c \in \mathbb{R}$ and a solution $F_h : \mathbb{R}^2_+ \to \mathbb{R}$ to the heat equation $2w_s(s)\frac{\partial}{\partial q_s}F_h(s, q_s) + \frac{\partial}{\partial s^2}F_h(s, q_s) = 0$.

Proof. Since

$$\mathcal{S}_{X^s}(\mathcal{D}^s) = \left\{\widetilde{F} \in \mathcal{S}_X(\mathcal{D}) : \frac{\partial}{\partial q_v}\widetilde{F}(s, v, q_s, q_v, l) = \frac{\partial}{\partial l}\widetilde{F}(s, v, q_s, q_v, l) = 0\right\},$$

Lemma 2.3.1 implies that $F \in \mathcal{S}_{X^s}(\mathcal{D}^s)$ iff

$$w_s(s)\frac{\partial}{\partial q_s}F(s, v, q_s) - \frac{1}{2}g''(s)\frac{\partial}{\partial v}F(s, v, q_s) + \frac{1}{2}\frac{\partial^2}{\partial s^2}F(s, v, q_s) = 0, \tag{2.3.2}$$

$$\frac{\partial^2}{\partial v^2}F(s, v, q_s) = 0, \tag{2.3.3}$$

$$\frac{\partial^2}{\partial s \partial v}F(s, v, q_s) = 0 \tag{2.3.4}$$

on $\mathcal{D}^s$. By (2.3.3) and (2.3.4), F must be of the form

$$F(s, v, q_s) = vf_3(q_s) + f_{13}(s, q_s)$$

for functions f_3 and f_{13}. Applying (2.3.2), this yields

$$w_s(s)vf_3'(q_s) + w_s(s)\frac{\partial}{\partial q_s}f_{13}(s, q_s) - \frac{1}{2}g''(s)f_3(q_s) + \frac{1}{2}\frac{\partial^2}{\partial s^2}f_{13}(s, q_s) = 0. \tag{2.3.5}$$

Differentiating the above equation with respect to v gives $f_3'(q_s) = 0$. Hence $F(s, v, q_s) = c_1 v + f_{13}(s, q_s)$ for a constant $c_1 \in \mathbb{R}$. Hence, (2.3.2) gives

$$2w_s(s)\frac{\partial}{\partial q_s}f_{13}(s, q_s) + \frac{\partial^2}{\partial s^2}f_{13}(s, q_s) = c_1 g''(s).$$

It follows that $f_{13}(s, q_s) = F_h(s, q_s) + c_2\frac{g''(s)}{w_s(s)}q_s + (c_1 - c_2)g(s)$ for a solution F_h to the heat equation $2w_s(s)\frac{\partial}{\partial q_s}F_h(s, q_s) + \frac{\partial^2}{\partial s^2}F_h(s, q_s) = 0$. This implies that $F \in \mathcal{S}_{X^s}(\mathcal{D}^s)$ iff $F(s, v, q_s) = c_1(v + g(s)) + c_2\left(\frac{g''(s)}{w_s(s)}q_s - g(s)\right) + F_h(s, q_s)$, which along with Theorem 2.1.2 and Theorem 2.2.5 concludes the proof.

□

The following proposition implies that there are no functions of Q^v_T replicable model-independently via strategies with wealth process of the form $F(X^v)$.

Proposition 2.3.3. *For $F \in C^2(\mathcal{D}^v) \cap C(\overline{\mathcal{D}^v})$, $F(S_\cdot, V_\cdot, Q^v_\cdot)$ is a local martingale under all models in $\mathcal{M}$ iff*

$$F(s, v, q_v) = c_1(g(s) + v) + c_2 s + c_3 \tag{2.3.6}$$

for constants $c_1, c_2, c_3 \in \mathbb{R}$.

Proof. Since

$$\mathcal{S}_{X^v}(\mathcal{D}^v) = \left\{ \widetilde{F} \in \mathcal{S}_X(\mathcal{D}) : \frac{\partial}{\partial q_s}\widetilde{F}(s, v, q_s, q_v, l) = \frac{\partial}{\partial l}\widetilde{F}(s, v, q_s, q_v, l) = 0 \right\},$$

Lemma 2.3.1 implies that $F \in \mathcal{S}_{X^v}(\mathcal{D}^v)$ iff

$$g''(s)\frac{\partial}{\partial v}F(s, v, q_v) - \frac{\partial^2}{\partial s^2}F(s, v, q_v) = 0, \tag{2.3.7}$$

$$w_v(s)\frac{\partial}{\partial q_v}F(s, v, q_v) + \frac{1}{2}\frac{\partial^2}{\partial v^2}F(s, v, q_v) = 0, \tag{2.3.8}$$

$$\frac{\partial^2}{\partial s \partial v}F(s, v, q_v) = 0 \tag{2.3.9}$$

on $\mathcal{D}^v$. By (2.3.9), F must be of the form

$$F(s, v, q_v) = f_{13}(s, q_v) + f_{23}(v, q_v)$$

for functions f_{13} and f_{23}. Then, (2.3.7) gives

$$\begin{aligned} g''(s)\frac{\partial}{\partial v}f_{23}(v, q_v) - \frac{\partial^2}{\partial s^2}f_{13}(s, q_v) &= 0 \\ \Rightarrow \quad \frac{\partial^2}{\partial v^2}F(s, v, q_v) = \frac{\partial^2}{\partial v^2}f_{23}(v, q_v) &= 0. \end{aligned}$$

By (2.3.8), $\frac{\partial}{\partial q_v}F(s, v, q_v) = -\frac{1}{w_v(s)}\frac{\partial^2}{\partial v^2}F(s, v, q_v) = 0$, hence F must be of the form $F(s, v, q_v) = f_1(s) + c_1 v$ for a constant $c_1 \in \mathbb{R}$ and a function f_1. Finally, by (2.3.7),

$$\frac{\partial^2}{\partial s^2}f_1(s) = c_1 g''(s) \quad \Rightarrow \quad f_1(s) = c_1 g(s) + c_2 s + c_3$$

for constants $c_2, c_3 \in \mathbb{R}$. Hence, $F \in \mathcal{S}_{X^v}(\mathcal{D}^v)$ iff $F(s, v, q_v) = c_1(g(s) + v) + c_2 s + c_3$. Along with Theorem 2.1.2 and Theorem 2.2.5, this concludes the proof.

□

Define the function

$$G(x) := \int_0^x \int_0^z z g''(z)\, \mathrm{d}z\, \mathrm{d}u, \quad x > 0. \tag{2.3.10}$$

The following proposition implies that one can replicate $\frac{1}{w_l(S_T)}L_T$ by a combination of a portfolio with wealth process $F(X_\cdot)$ and a static position in co-maturing claims on $G(S_T)$ if $\frac{1}{w_l(s)}$ is affine (this model-independent replication strategy was shown by Fukasawa [23] for $w_l(s) = \frac{1}{s}$). It also shows that there are no non-linear functions of L_T one can replicate by a portfolio of the form $F(X^l_\cdot)$.

Proposition 2.3.4. *For $F \in C^2(\mathcal{D}^l) \cap C(\overline{\mathcal{D}^l})$, $F(S_\cdot, V_\cdot, L_\cdot)$ is a local martingale under all models in $\mathcal{M}$ iff*

$$F(s, v, l) = c_1 \left(sv + \frac{1}{2}G(s) - \frac{l}{w_l(s)} \right) + c_2(v + g(s)) + c_3 s + c_4 \qquad (2.3.11)$$

for constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$, where $c_1 = 0$ if $\frac{1}{w_l(s)}$ is not affine.

Proof. Since

$$\mathcal{S}_{X^l}(\mathcal{D}^l) = \left\{ \widetilde{F} \in \mathcal{S}_X(\mathcal{D}) : \frac{\partial}{\partial q_s}\widetilde{F}(s, v, q_s, q_v, l) = \frac{\partial}{\partial q_v}\widetilde{F}(s, v, q_s, q_v, l) = 0 \right\},$$

Lemma 2.3.1 implies that $F \in \mathcal{S}_{X^l}(\mathcal{D}^l)$ iff

$$g''(s)\frac{\partial}{\partial v}F(s, v, l) - \frac{\partial^2}{\partial s^2}F(s, v, l) = 0, \qquad (2.3.12)$$

$$\frac{\partial^2}{\partial v^2}F(s, v, l) = 0, \qquad (2.3.13)$$

$$w_l(s)\frac{\partial}{\partial l}F(s, v, l) + \frac{\partial^2}{\partial s \partial v}F(s, v, l) = 0 \qquad (2.3.14)$$

on $\mathcal{D}^l$. By (2.3.13), F must be of the the form $F(s, v, l) = v f_{13a}(s, l) + f_{13b}(s, l)$ for functions f_{13a} and f_{13b}. By (2.3.14),

$$w_l(s) v \frac{\partial}{\partial l} f_{13a}(s, l) + w_l(s)\frac{\partial}{\partial l} f_{13b}(s, l) + \frac{\partial}{\partial s} f_{13a}(s, l) = 0.$$

Differentiating the above identities with respect to v gives $\frac{\partial}{\partial l} f_{13a}(s, l) = 0$. Hence, F must be of the form $F(s, v, l) = v f_1(s) + f_{13}(s, l)$ for functions $f_1 \equiv f_{13a}$ and $f_{13} \equiv f_{13b}$. By (2.3.12),

$$g''(s) f_1(s) = v f_1''(s) + \frac{\partial^2}{\partial s^2} f_{13}(s, l).$$

Differentiating the above with respect to v gives

$$f_1''(s) = 0 \quad \Rightarrow \quad f_1(s) = c_1 s + c_2$$

for constants $c_1, c_2 \in \mathbb{R}$. Hence, $F(s,v,l) = v(c_1 s + c_2) + f_{13}(s,l)$. By (2.3.12) and (2.3.14), it then follows that

$$\begin{cases} \frac{\partial^2}{\partial s^2} f_{13}(s,l) = g''(s)(c_1 s + c_2), \\ w_l(s)\frac{\partial}{\partial l} f_{13}(s,l) + c_1 = 0. \end{cases}$$

This implies that

$$f_{13}(s,l) = -\frac{1}{w_l(s)} c_1 l + c_1 G(s) + c_2 g(s) + c_3 s + c_4$$

for constants $c_2, c_3, c_4 \in \mathbb{R}$, and that $c_1 = 0$ if $\frac{1}{w_l(s)}$ is not affine. Hence,

$$F(s,v,l) = c_1 \left(sv + G(s) - \frac{l}{w_l(s)} \right) + c_2(v + g(s)) + c_3 s + c_4,$$

with $c_1 = 0$ if $\frac{1}{w(s)}$ is not affine. Along with Theorem 2.1.2 and Theorem 2.2.5, this concludes the proof.

□

2.3.2 Implied variance of log returns

Another commonly encountered quantity which converges model-independently to zero at time T, and hence may be used within the context of Corollary 1.2.11 to obtain replication results for fixed maturity claims, is the implied variance obtained by calibrating the Black-Scholes model to the observed price of a traded option C. This subsection shows that the implied variance can be used as a component in $X \in \mathcal{X}$.

Consider the solution F_{BS} to the PDE

$$\frac{\partial}{\partial x_2} F_{\text{BS}}(x) = \frac{1}{2} x_1^2 \frac{\partial^2}{\partial x_1^2} F_{\text{BS}}(x) \tag{2.3.15}$$

on $\mathbb{R}_+^2$ with boundary condition $F_{\text{BS}}(x_1, 0) = g(x_1)$. Note that for any $0 \leq t < T$, $F_{\text{BS}}(x)$ is equal to the Black-Scholes price of the claim on $g(S_T)$ at time t under the assumptions of zero interest rate, volatility $\sigma = \sqrt{x_2/(T-t)}$ and $S_t = x_1$. In particular, the variable x_2 in F_{BS} corresponds to the remaining quadratic variation of log returns of S to maturity.

Define the Black-Scholes implied variance V^{BS} implicitly by

$$F_{\text{BS}}(S, V^{\text{BS}}) = C. \tag{2.3.16}$$

In particular, the Black-Scholes implied variance V^{BS} is equal to the square of the Black-Scholes implied volatility scaled by time to maturity. By convexity of g, the discussion in Section 2.2 implies that $C_t > g(S_t)$ for $t \in [0,T)$ almost surely under all models in $\mathcal{M}_g$. Hence, by standard results regarding Black-Scholes implied volatility (see [23] and references therein), V^{BS} is well-defined, $V_t^{\mathrm{BS}} > 0$ for $t \in [0,T)$ and $V_T^{\mathrm{BS}} = 0$ almost surely under all such models.

Consider now the differential formulation of V^{BS}. By Itô's formula,

$$\begin{aligned} \mathrm{d}C_t &= \frac{\partial}{\partial x_1} F_{\mathrm{BS}}(A_t)\,\mathrm{d}S_t + \frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)\,\mathrm{d}V_t^{\mathrm{BS}} \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} F_{\mathrm{BS}}(A_t)\,\mathrm{d}\langle S\rangle_t + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} F_{\mathrm{BS}}(A_t)\,\mathrm{d}\langle V^{\mathrm{BS}}\rangle_t + \frac{\partial^2}{\partial x_1 \partial x_2} F_{\mathrm{BS}}(A_t)\,\mathrm{d}\langle S, V^{\mathrm{BS}}\rangle_t \end{aligned}$$

for $t \in [0,T)$. Recalling that $\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(x) = \frac{1}{2} x_1^2 \frac{\partial^2}{\partial x_1^2} F_{\mathrm{BS}}(x)$, this yields

$$\begin{aligned} \mathrm{d}V_t^{\mathrm{BS}} &= \frac{1}{\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)}\,\mathrm{d}C_t - \frac{\frac{\partial}{\partial x_1} F_{\mathrm{BS}}(A_t)}{\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)}\,\mathrm{d}S_t \\ &\quad - \frac{1}{S_t^2}\,\mathrm{d}\langle S\rangle_t - \frac{\frac{\partial^2}{\partial x_2^2} F_{\mathrm{BS}}(A_t)}{2\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)}\,\mathrm{d}\langle V^{\mathrm{BS}}\rangle_t - \frac{\frac{\partial^2}{\partial x_1 \partial x_2} F_{\mathrm{BS}}(A_t)}{\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)}\,\mathrm{d}\langle S, V^{\mathrm{BS}}\rangle_t, \quad t \in [0,T). \end{aligned} \tag{2.3.17}$$

Hence,

$$\begin{aligned} \mathrm{d}\langle V^{\mathrm{BS}}\rangle_t &= \left(\frac{\frac{\partial}{\partial x_1} F_{\mathrm{BS}}(A_t)}{\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)}\right)^2 \mathrm{d}\langle S\rangle_t + \frac{1}{(\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t))^2}\,\mathrm{d}\langle C\rangle_t - \frac{2\frac{\partial}{\partial x_1} F_{\mathrm{BS}}(A_t)}{(\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t))^2}\,\mathrm{d}\langle S, C\rangle_t, \\ \mathrm{d}\langle S, V^{\mathrm{BS}}\rangle_t &= -\frac{\frac{\partial}{\partial x_1} F_{\mathrm{BS}}(A_t)}{\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)}\,\mathrm{d}\langle S\rangle_t + \frac{1}{\frac{\partial}{\partial x_2} F_{\mathrm{BS}}(A_t)}\,\mathrm{d}\langle S, C\rangle_t \end{aligned}$$

for $t \in [0,T)$. This yields

$$\mathrm{d}V_t^{\mathrm{BS}} = \alpha_1(A_t)\,\mathrm{d}S_t + \alpha_2(A_t)\,\mathrm{d}C_t + \beta_{11}(A_t)\,\mathrm{d}\langle S\rangle_t + \beta_{22}(A_t)\,\mathrm{d}\langle C\rangle_t + \beta_{12}(A_t)\,\mathrm{d}\langle S, C\rangle_t,$$

for $t \in [0,T)$, where

$$
\begin{aligned}
\alpha_1(x) &= -\frac{\frac{\partial}{\partial x_1}F_{\mathrm{BS}}(x)}{\frac{\partial}{\partial x_2}F_{\mathrm{BS}}(x)},\\
\alpha_2(x) &= \frac{1}{\frac{\partial}{\partial x_2}F_{\mathrm{BS}}(x)},\\
\beta_{11}(x) &= -\frac{1}{x_1^2} - \frac{(\frac{\partial}{\partial x_1}F_{\mathrm{BS}}(x))^2\frac{\partial^2}{\partial x_2^2}F_{\mathrm{BS}}(x)}{2(\frac{\partial}{\partial x_2}F_{\mathrm{BS}}(x))^3} + \frac{\frac{\partial}{\partial x_1}F_{\mathrm{BS}}(x)\frac{\partial^2}{\partial x_1\partial x_2}F_{\mathrm{BS}}(x)}{(\frac{\partial}{\partial x_2}F_{\mathrm{BS}}(x))^2},\\
\beta_{22}(x) &= -\frac{\frac{\partial^2}{\partial x_2^2}F_{\mathrm{BS}}(x)}{2(\frac{\partial}{\partial x_2}F_{\mathrm{BS}}(x))^3},\\
\beta_{12}(x) &= \frac{\frac{\partial}{\partial x_1}F_{\mathrm{BS}}(x)\frac{\partial^2}{\partial x_2^2}F_{\mathrm{BS}}(x)}{(\frac{\partial}{\partial x_2}F_{\mathrm{BS}}(x))^3} - \frac{\frac{\partial^2}{\partial x_1\partial x_2}F_{\mathrm{BS}}(x)}{(\frac{\partial}{\partial x_2}F_{\mathrm{BS}}(x))^2}.
\end{aligned}
$$

A similar formulation of implied variance was outlined in Zhang [55]. The key point we emphasise is that by the above formulation and by the bijective map $A \leftrightarrow (S, V^{\mathrm{BS}})$ for $V > g(S)$, we may apply Corollary 1.2.11 with V^{BS} as a component of X when X also contains S (or a bijective function thereof) as a component. This yields model-independent replication strategies for various claims on the Black-Scholes implied variance and/or its second variations, provided one can solve for the system of PDEs

$$\mathcal{L}^{\alpha,\beta}_{i,j}F = \mathcal{L}^{\gamma}F = 0, \quad i,j \in \{1,\ldots,d\}$$

on a suitable domain $\mathcal{D}$ corresponding to such functionals. Note that the PDE operators will have less elegant formulations then those with time value V due to the implied variance differential (2.3.17) involving more terms compared to the simpler time value differential

$$\mathrm{d}V_t = \mathrm{d}C_t - g'(S_t)\,\mathrm{d}S_t - \frac{1}{2}g''(S_t)\,\mathrm{d}\langle S\rangle_t, \quad t \in [0,T). \tag{2.3.18}$$

Remark 2.3.5. For $g(x) = -\log(x)$, one gets $F_{\mathrm{BS}}(x_1,x_2) = -\log(x_1) + x_2$, and V^{BS} coincides with the time value $V = C + \log(S)$, hence with the model-free implied variance. This was noted by Fukasawa [23].

2.4 Underlying asset and traded calls

This section considers a market with an underlying asset S and a set of co-maturing traded calls C^i written on S (we could also have chosen a set of puts to be the set of traded options). Since the case of a single traded call option of strike $k > 0$ is covered by Section 2.2 with $g(x) = (x-k)^+$, we assume that $d \geq 3$. In particular, we consider the market $A = (S, C^2, \ldots, C^d)$, where the payoff requirements are $C^i_T = (S_T - k_i)^+$ for strikes k_i which are in increasing order $(0 = k_1 < k_2 < \ldots < k_d < \infty)$. Note that $C^1 \equiv S$ by convention.

As in Section 2.2, we will define a class of models $\mathcal{M}$, elaborate on some properties of models in this class and show that $\mathcal{M}$ is regular. By Theorem 2.1.2, the latter will imply that the characterisation result applies with this choice of class of models. Since many of the proofs of the results in this section are similar to proofs of analogous results in Section 2.2, we will provide less details in some of them. We begin by defining the class of models $\mathcal{M}$.

Definition 2.4.1. Define $\mathcal{M} \equiv \mathcal{M}_{calls}$ to be the set of models $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}_m$ such that S has full support.

As with the definition of $\mathcal{M}$ in Section 2.2, one may weaken the requirement of S having full support (by considering the range of strikes of traded options) and still obtain a class of models $\mathcal{M}$ which is regular. We do not do so for the sake of greater clarity.

Also, note that $\mathcal{M} \subseteq \mathcal{M}_m$, hence S and C are continuous martingales such that for all i, $C^i_T = (S^i_T - k_i)^+$ $\mathbb{P}$-a.e.. Hence, unlike $\mathcal{M}_g$ in Definition 2.2.4, the class $\mathcal{M} = \mathcal{M}_{calls}$ in Definition 2.4.1 only considers models under which traded option prices are equal to conditional expectations of their payoffs. In order to construct models in $\mathcal{M}$, it will therefore be sufficient to construct continuous martingales S having full support. In fact, the main proofs below will construct local volatility models with volatility bounded within a closed interval in $(0, \infty)$, which will imply that these models have full support by Proposition 2.2.7.

Definition 2.4.2. For a martingale S, a stopping time $\tau \in \mathcal{T}(\mathbb{F}_T)$ and $k : \Omega \times [0, T] \to \mathbb{R}_+$ which is $\mathbb{F}_T$-adapted, define $c_\tau(k) := \mathbb{E}((S_T - k_\tau)^+ \,|\, \mathcal{F}_{\tau \wedge T})$. Denote $c'_\tau(\cdot)$ to be the left-derivative of $c_\tau(\cdot)$.

For constants $k > 0$ and $t \in [0, T]$, $c_t(k)$ is equal to the call price with strike k at time t.

Definition 2.4.3. For $i \in \{2, \ldots, d\}$, define $V^i := C^i - (S - k_i)^+$ and $D^i := \frac{C^i - C^{i-1}}{k_i - k_{i-1}}$. For $i \in \{3, \ldots, d\}$, define $\Delta^i := D^i - D^{i-1}$. Also, define $V^{min} := \min_{i \in \{2, \ldots, d\}} V^i$ and $\Delta^{min} := \min_{i \in \{3, \ldots, d\}} \Delta^i$.

The V^i correspond to the time values of the call options C^i, whereas the D^i correspond to the left-hand slope of the linear interpolation of the traded call prices. Δ^{min} is hence positive if and only if the linear interpolation of call prices has a kink at each strike k_i ($i \in \{2, \dots, d-1\}$).

The following equivalence result shows that the (FS) property of a local martingale S is equivalent to strict convexity of the call price function it generates.

Lemma 2.4.4. *The following are equivalent statements of the (FS) property:*

1. *For any $\mathbb{F}_T$-adapted open interval I and any $t \in [0, T)$, $\mathbb{P}(S_T \in I_t \,|\, \mathcal{F}_t) > 0$.*

2. *For any $\mathbb{F}_T$-adapted open interval I and any $\tau \in \mathcal{T}(\mathbb{F}_T)$, $\mathbb{P}(S_T \in I_{\tau \wedge T} \,|\, \mathcal{F}_\tau) > 0$ on $\{\tau < T\}$.*

3. *For any $\mathbb{F}_T$-adapted $0 < a < b < \infty$ and for any $t \in [0, T)$, $c_t'(a) < c_t'(b)$.*

4. *For any $\mathbb{F}_T$-adapted $0 < a < b < \infty$ and any $\tau \in \mathcal{T}(\mathbb{F}_T)$, $c_\tau'(a) < c_\tau'(b)$ on $\{\tau < T\}$.*

Proof. (1) $\Leftrightarrow$ (2) was proved by Lemma 2.2.6. (4) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1) follows from noting that for an $\mathbb{F}_T$-adapted open interval $I = (a, b)$ and for $t \in [0, T)$,

$$\mathbb{P}(S_T \in I_t \,|\, \mathcal{F}_t) \geq \mathbb{P}(S_T \in [(a_t + b_t)/2, b_t) \,|\, \mathcal{F}_t) = c_t'(b) - c_t'((a + b)/2) > 0.$$

Similarly, (2) $\Rightarrow$ (4) follows from noting that for $\tau \in \mathcal{T}(\mathbb{F}_T)$ such that $\mathbb{P}(\tau < T) > 0$ and for $\mathbb{F}_T$-adapted $0 < a < b < \infty$,

$$c_\tau'(b) - c_\tau'(a) = \mathbb{P}(S_T \in [a_\tau, b_\tau) \,|\, \mathcal{F}_\tau) \geq \mathbb{P}(S_T \in (a_\tau, b_\tau) \,|\, \mathcal{F}_\tau) > 0 \quad \text{on } \{\tau < T\}.$$

□

The following theorem is the main result of this section.

Theorem 2.4.5. *$\mathcal{M}$ defined in Definition 2.4.1 is a regular class of models for $A = (S, C^2, \dots, C^d)$. Furthermore, $\mathcal{M}(A_0) \neq \emptyset$ iff $V_0^{min} > 0$, $\Delta_0^{min} > 0$, $D_0^2 > -1$ and $D_0^d < 0$.*

Note that Theorem 2.4.5 is the analogue of Theorem 2.2.5 for the market considered in this section. Recall that the proof of Theorem 2.2.5 relied on Proposition 2.2.8. Similarly, the proof of Theorem 2.4.5 relies on the following proposition.

Proposition 2.4.6. *$\mathcal{M}(A_0) \neq \emptyset$ iff $V_0^{min} > 0$, $\Delta_0^{min} > 0$, $D_0^2 > -1$ and $D_0^d < 0$, and for any $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$,*

$$\mathbb{P}(V_t^{min} > 0, \Delta_t^{min} > 0, D_t^2 > -1, D_t^d < 0, \ \forall t \in [0, T)) = 1.$$

Proof. Temporarily define

$$Z := V^{min} \wedge \Delta^{min} \wedge (D^2 + 1) \wedge (-D^d),$$
$$\tau_0 := \inf\{t \geq 0 : Z_t \leq 0\}.$$

In part (i) of the proof, we will first prove that $\tau_0 = T$ $\mathbb{P}$-a.e., which will imply that

$$\mathbb{P}(V_t^{min} > 0, \Delta_t^{min} > 0, D_t^2 > -1, D_t^d < 0, \ \forall t \in [0, T)) = 1,$$

and that $\mathcal{M}(A_0) = \emptyset$ if $A_0 = (S_0, C_0^1, \ldots, C_0^d)$ is such that any of the conditions $V_0^{min} > 0$, $\Delta_0^{min} > 0$, $D_0^2 > -1$ or $D_0^d < 0$ does not hold. In part (ii), we will prove that $\mathcal{M}(A_0) \neq \emptyset$ if $V_0^{min} > 0$, $\Delta_0^{min} > 0$, $D_0^2 > -1$ and $D_0^d < 0$.

(i) Temporarily introduce the notation

$$\tau^V := \inf\{t \geq 0 : V^{min} \leq 0\},$$
$$\tau^\Delta := \inf\{t \geq 0 : \Delta_t^{min} \leq 0\},$$
$$\tau_l^D := \inf\{t \geq 0 : D_t^2 \leq -1\},$$
$$\tau_r^D := \inf\{t \geq 0 : D_t^d \geq 0\},$$

and note that $\tau_0 = \tau^V \wedge \tau^\Delta \wedge \tau_l^D \wedge \tau_r^D$.

For any $i \in \{2, \ldots, d\}$ and any $\tau \in \mathcal{T}(\mathbb{F}_T)$,

$$\mathbb{E}(V_{\tau \wedge T}^i) = \mathbb{E}\left((S_T - k_i)^+ - (S_{\tau \wedge T} - k_i)^+\right) = \frac{1}{2}\mathbb{E}\left(L_T^{k_i}[S] - L_{\tau \wedge T}^{k_i}[S]\right).$$

Thus, part (ii) of the proof of Proposition 2.2.8 implies that $\mathbb{E}(V_{\tau \wedge T}^i) > 0$ if $\mathbb{P}(\tau < T) > 0$. It then follows that $\mathbb{P}(\inf\{t \geq 0 : V_t^i \leq 0\} < T) = 0$ for $i \in \{2, \ldots, d\}$. Hence, $\tau^V = T$ $\mathbb{P}$-a.e..

To conclude the proof of $\tau_0 = T$ $\mathbb{P}$-a.e., we will show that $\tau^\Delta \vee \tau_l^D \vee \tau_r^D \geq T$ $\mathbb{P}-a.e.$. For any $\tau \in \mathcal{T}(\mathbb{F}_T)$ and any $k > 0$, we have that $c_\tau'(k) = -\mathbb{P}(S_T \geq k \,|\, \mathcal{F}_{\tau \wedge T}) \in [-1, 0]$. Furthermore, by the (FS) property of S and Lemma 2.4.4, $c_\tau(\cdot)$ is a strictly convex function on $\{\tau < T\}$ for any $\tau \in \mathcal{T}(\mathbb{F}_T)$. This implies that $D_{\tau_l^D}^2 > \lim_{k \downarrow 0} c_{\tau_l^D}'(k) \geq -1$ on $\{\tau_l^D < T\}$, which contradicts $D_{\tau_l^D}^2 = -1$. Hence, $\mathbb{P}(\tau_l^D \geq T) = 1$. By similar arguments by contradiction, $c_{\tau_r^D}'(k_d) \geq 0$ and strict convexity of $c_{\tau_r^D}'(\cdot)$ on $\{\tau_r^D < T\}$

imply that $\mathbb{P}(\tau_r^D \geq T) = 1$, and strict convexity of $c'_{\tau^\Delta}(\cdot)$ on $\{\tau^\Delta < T\}$ implies that $\mathbb{P}(\tau^\Delta \geq T) = 1$. This concludes part (i) of the proof.

(ii) Suppose that A_0 is such that $V_0^{min} > 0$, $\Delta_0^{min} > 0$, $D_0^2 > -1$ and $D_0^d < 0$. To show that $\mathcal{M}(V^0) \neq \emptyset$, we will construct a local volatility process

$$S_\cdot = S_0 + \int_0^\cdot \sigma(S_t, t) S_t \, \mathrm{d}W_t$$

on the canonical space $(\mathcal{C}(\mathbb{R}), \mathcal{F}^W, \mathbb{F}_T^W, \mathbb{P}^W)$ for a Brownian motion W such that:

(1) The corresponding call price function $c_0(k)$ passes through all of the points (k_i, C_0^i).

(2) $\sigma(x) \in [l, u]$, $x > 0$ for $0 < l < du < \infty$.

Proposition 2.2.7 will then imply that S has full support. Defining the C^i to be the conditional expectations of the call payoffs under S, this will yield a model in $\mathcal{M}$.

Define $a_1 = (S_0 - C_0^2)/2$ and $a_2 := k_d - \frac{C_0^d}{D_0^d}$. By $D_0^2 \in (-1, 0)$ it follows that $a_1 \in (0, k_2 \wedge S_0)$ and that $k_2 - a_1 < k_2/2$. By $C_0^d > V_0^{min} > 0$ and $D_0^d < 0$, it follows that a_2 is well-defined and $a_2 > k_d$. In particular, a_2 is the point on the horizontal (strike) axis where the latter is intersected by the line connecting (k_{d-1}, C_0^{d-1}) and (k_d, C_0^d). By $D_0^2 > -1$, $\Delta_0^{min} > 0$, it also follows that $a_2 > S_0$.

Denote $c^{BS}(k, t; \sigma)$ to be the call price function generated by the geometric Brownian motion with zero drift and volatility σ, notably

$$c^{BS}(k, t; \sigma) := \mathbb{E}^{\mathbb{P}^W}((S_0 \mathcal{E}(\sigma W_t) - k)^+).$$

Define

$$\begin{aligned}
\sigma_1 &:= \sup\left\{\sigma > 0 : c^{BS}(a_2, T; \sigma) - \frac{\partial}{\partial k} c^{BS}(a_2, T; \sigma)(k_d - a_2) \leq \frac{1}{2} C_0^d\right\}, \\
\sigma_2 &:= \sup\left\{\sigma > 0 : \frac{\partial}{\partial k} c^{BS}(a_1, T; \sigma) \leq \frac{C_0^2 - c^{BS}(a_1, T; \sigma)}{k_2 - a_1}\right\}, \\
\sigma_0 &:= \sigma_1 \wedge \sigma_2.
\end{aligned}$$

By $\lim_{\sigma \to 0} c^{BS}(a_2, T; \sigma) = \lim_{\sigma \to 0} \frac{\partial}{\partial k} c^{BS}(a_2, T; \sigma) = 0$ and by

$$\begin{aligned}
\lim_{\sigma \to 0} \frac{C_0^2 - c^{BS}(a_1, T; \sigma)}{k_2 - a_1} &= \frac{C_0^2 - (S_0 - a_1)}{k_2 - a_1} \\
&= \frac{C_0^2 - S_0}{2(k_2 - a_1)} = \frac{C_0^2 - S_0}{k_2} = D_0^2 \\
&> -1 = \lim_{\sigma \to 0} \frac{\partial}{\partial k} c^{BS}(a_1, T; \sigma),
\end{aligned}$$

σ_0 is well-defined. Since $c^{BS}(a_2,T;\sigma_0) - \frac{\partial}{\partial k} c^{BS}(a_2,T;\sigma_0)(a_2-k_d) \le \frac{1}{2}C_0^d$, it follows that

$$\frac{\partial}{\partial k} c^{BS}(a_2,T;\sigma_0) - \frac{c^{BS}(a_2,T;\sigma_0) - C_0^d}{a_2-k_d} \ge \frac{C_0^d}{2(a_2-k_d)} > 0,$$

and that

$$\frac{c^{BS}(a_2,T;\sigma_0) - C_0^d}{a_2-k_d} - D_0^d = \frac{c^{BS}(a_2,T;\sigma_0)}{a_2-k_d} > 0.$$

By $\frac{\partial}{\partial k} c^{BS}(a_1,T;\sigma) \le \frac{C_0^2 - c^{BS}(a_1,T;\sigma)}{k_2-a_1}$, $D_0^2 > -1$, $D_0^i - D_0^{i-1} > 0$ for $i \in \{3,\dots,d\}$ and the identities derived above, we can interpolate the points

$$(a_1, c^{BS}(a_1,T;\sigma_0)), (k_2/2, c^{BS}(k,T;\sigma_0)), (k_2, C_0^2), \dots, (k_d, C_0^d), (a_2, c_0^{BS}(a_2;\sigma_0))$$

and obtain a function $c^I(k,t;\sigma_0)$ defined on $A_1 \equiv [a_1,a_2] \times [T/2,T]$ such that

- $c^I(k,t;\sigma_0) = c^{BS}(k,t;\sigma)$ for $(k,t) \in \{a_1,a_2\} \times [T/2,T]$.
- $\frac{\partial}{\partial k} c^I(k,t;\sigma_0) = \frac{\partial}{\partial k} c^{BS}(k,t;\sigma_0)$ for $(k,t) \in \{a_1,a_2\} \times [T/2,T]$.
- $c^I(k_i,T) = C_0^i$ for $i \in \{2,\dots,d\}$.
- $c^I(k,t;\sigma_0)$ is C^2 and convex in k and is C^1 and increasing in t.

Then, define $\widetilde{c}(k,t;\sigma_0)$ on $A_2 \equiv [0,\infty) \times [0,T]$ by

$$\widetilde{c}(k,t;\sigma_0) = \begin{cases} \left(\frac{t-T/2}{T/2}\right)^2 c^I(k,t;\sigma_0) + \left(1 - \left(\frac{t-T/2}{T/2}\right)^2\right) c^{BS}(k,t;\sigma_0), & (k,t) \in A_1 \\ c^{BS}(k,t;\sigma_0) & (k,t) \in A_2 \setminus A_1. \end{cases}$$

Define

$$\sigma(x,t) := \frac{1}{k}\sqrt{\frac{2\frac{\partial}{\partial t}\widetilde{c}(k,t;\sigma_0)}{\frac{\partial^2}{\partial^2 k}\widetilde{c}(k,t;\sigma_0)}}$$

By Dupire's formula [18], it follows that $\sigma(k,t) = \sigma_0$ on $A_2 \setminus A_1$. Moreover, since A_1 is compact and $\sigma(k,t)$ is continuous and positive on A_1, we get that $\sigma(k,t) \in [b_1,b_2]$ for some $0 < b_1 < b_2 < \infty$ for $(k,t) \in A_1$. Hence, $\sigma(k,t) \in [l,u]$ on $(k,t) \in A_2$, where $l = b_1 \wedge \sigma_0$ and $u = b_2 \vee \sigma_0$. Then, $S_t := S_0 \mathcal{E}(\int_0^t \sigma(S_u,u)\, S_u \,\mathrm{d}W_u)$ has full support by Proposition 2.2.7. Furthermore, the corresponding call price function $c_0(k)$ satisfies

$$c_0(k_i) = \widetilde{c}(k_i,T;\sigma_0) = c^I(k_i,T;\sigma_0) = C_0^i, \quad i \in \{2,\dots,d\}.$$

Hence, setting

$$C_t^i := \mathbb{E}^{\mathbb{P}^W}((S_T - k_i)^+ \,|\, \mathcal{F}_t), \quad t \in [0,T]$$

for $i \in \{2, \dots, d\}$, we conclude that

$$\{(S, C^2, \dots, C^d); (\mathcal{C}(\mathbb{R}), \mathcal{F}^W, \mathbb{F}_T^W, \mathbb{P}^W)\} \in \mathcal{M}(A_0).$$

and that $\mathcal{M}(A_0) \neq \emptyset$ if $V_0^{min} > 0$, $\Delta_0^{min} > 0$, $D_0^2 > -1$ and $D_0^d < 0$.

□

We can now turn to the proof of Theorem 2.4.5. The main ideas are similar to those of the proof of Theorem 2.2.5. Hence, we do not elaborate on the details as much as in the latter proof.

Proof of Theorem 2.4.5. The claim that $\mathcal{M}(A_0) \neq \emptyset$ iff $V_0^{min} > 0$, $\Delta_0^{min} > 0$, $D_0^2 > -1$ and $D_0^d < 0$ was proved in Proposition 2.4.6. It remains to prove that $\mathcal{M}$ is regular.

For a given model $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$, stopping time $\tau \in \mathcal{T}(\mathbb{F}_T)$ such that $\mathbb{P}(\tau < T) > 0$ and matrix $\Sigma \in \mathbb{S}_+^2$, we can construct a model $\{\widetilde{A}; (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}})\} \in \mathcal{M}$ and a stopping time $\widetilde{\tau}$ satisfying the conditions in Definition 2.1.1 as follows.

Define $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}_T, \widetilde{\mathbb{P}}) := (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P}) \otimes (\mathcal{C}(\mathbb{R}), \mathcal{F}^W, \mathbb{F}_T^W, \mathbb{P}^W)$ where W is a Brownian motion and let $m \in \mathbb{R}^{2\times 2}$ be such that $m^T m = \Sigma$. Denote z to be the function such that $z(A) = V^{min} \wedge \Delta^{min} \wedge (D^2 + 1) \wedge (-D^d)$. Define

$$\widetilde{A}_t := A_{t\wedge\tau} + m(W_t - W_{t\wedge\tau}), \quad t \in [\tau, \widetilde{\tau}),$$
$$\widetilde{\tau} := \inf\left\{t \geq \tau : z(\widetilde{A}_t) \leq z(A_\tau)/2\right\} \wedge \frac{T+\tau}{2},$$

and, for $t \in [\widetilde{\tau}, T]$, define

$$\widetilde{S}_t := \widetilde{S}_{\widetilde{\tau}} + \int_{\widetilde{\tau}}^t \widetilde{\sigma}(\widetilde{S}_u) S_u \, \mathrm{d}W_u, \quad \widetilde{C}_t^i := \mathbb{E}^{\widetilde{\mathbb{P}}}\left((\widetilde{S}_T - k_i)^+ \,\middle|\, \widetilde{\mathcal{F}}_t\right), \quad i \in \{2, \dots, d\},$$

where $\widetilde{\sigma}(k)$ is an $\widetilde{F}_T$-measurable function bounded in an $\widetilde{F}_\tau$-measurable interval $[l, u] \subset (0, \infty)$ such that

$$\mathbb{E}^{\widetilde{\mathbb{P}}}((\widetilde{S}_T - k_i)^+ \,|\, \widetilde{\mathcal{F}}_{\widetilde{\tau}}) = \widetilde{C}_{\widetilde{\tau}}^i, \quad i \in \{2, \dots, d\}.$$

The construction of $\widetilde{\sigma}$ can be done in a similar manner as in part (ii) of the proof of Proposition 2.2.8. Note that the obtained function $\widetilde{\sigma}$ will be measurable with respect to the sigma-algebra generated by the prices $\widetilde{A}_{\widetilde{\tau}}$, hence measurable with respect to $\widetilde{\mathbb{F}}_{\widetilde{\tau}}$.

The construction of $\widetilde{S}$ and $\widetilde{C}$ implies that $\widetilde{A}$ is a d-dimensional continuous martingale. Also, it is clear that $\{\tau < \widetilde{\tau} \leq T\} = \{\tau < T\}$ by construction. The call prices $\widetilde{C}^i$ converge to their payoffs $(\widetilde{S}_T - k_i)^+$ on $\{\tau < T\}$ since they are equal to conditional expectations of the latter on $[\widetilde{\tau}, T]$. They converge to their payoffs on $\{\tau \geq T\}$ since

$\widetilde{A} = A$ on $[0, \tau \wedge T]$ and since $C_T^i = (S_T - k_i)^+$ for $i \in \{2, \dots, d\}$. As in the proof of Theorem 2.2.5, the full support property follows by considering the sets (1) $\{\widetilde{\tau} < T\}$ and (2) $\{\widetilde{\tau} \geq T\}$ and respectively using (1) Proposition 2.2.7 with $\widetilde{\sigma}(x) \in [l, u]$ and (2) $A = \widetilde{A}$ on $[0, \tau \wedge T]$ and the full support property of A.

This concludes the proof of $\mathcal{M}$ being regular.

□

Chapter 3

Probability-free integration with applications in pathwise hedging

This chapter considers the question of replication of financial derivatives in a purely pathwise setting with no probabilistic notions involved. We will show that a probability-free version of the replication results in Section 1.2 holds under the assumption that price paths have a so-called *quadratic variation property* along a fixed sequence of partitions. Observe that the only role of probability theory and stochastic analysis in the mentioned replication results was to provide a definition of integrals as the limit of left Riemann sums and a corresponding closed form (Itô) calculus. This chapter leverages on this observation and provides a sufficiently general construction of pathwise integrals which allows for a probability-free version of Proposition 1.2.10.

The intrinsically pathwise nature of the notion of dynamic replication of financial derivatives was highlighted in early works such as Bick and Willinger [6] and Lyons [43]. The former work provides a very in-depth analysis of the pathwise notion of a self-financing portfolio in the limit of trading along a sequence of partitions Π of time whose mesh tends to zero. The limiting portfolio is called the portfolio in the limit of continuous trading along Π.

In a well-known publication, Föllmer [22] proposed a pathwise version of Itô's formula and a corresponding pathwise integral defined pointwise in time. The requirement for this construction is for the integrator path to have a so-called pathwise quadratic variation property and for the integrand to be the gradient of a C^2 function of the integrator. This theory was extended by Cont and Fournié [12], who proposed a pathwise version of the functional Itô calculus of Dupire [19] and extended the set of integrands with respect to which pointwise limits of left Riemann sums are well-defined when the integrator has the pathwise quadratic variation property. The constructions of pathwise integration in Föllmer [22] and Cont and Fournié [12] are helpful tools, but they are not general enough to accommodate probability-free versions of the stochastic integrals involved in Proposition 1.2.10 (we discuss this point further and illustrate it via an example in Section 3.1). This chapter hence develops an analytic construction of pathwise integrals which addresses the above issues and allows for a probability-free interpretation of the integral representation results in Proposition 1.2.10 and of the corresponding replication results.

We should note that we do not work with the celebrated general theory of integration with rough paths developed by Lyons for two main reasons: (1) this theory is outside the scope of this thesis, and (2) the added value of the general theory on rough paths is mainly within the context of multi-dimensional integrals, as noted in Lyons et al. [44]. When the integral takes values in $\mathbb{R}$, as in the case of wealth processes of trading strategies, we can work with basic analytic tools, as done in this chapter.

The outline of the chapter is the following. Section 3.1 outlines the probability-free integration results in Föllmer [22] and in Cont and Fournié [12]. It also explains why the theory in [12, 22] is not sufficiently general to justify a probability-free version of Proposition 1.2.10. We propose an alternative definition of pathwise integrals in Section 3.2 and present some of its basic properties. We notably show that the integral satisfies a so-called associativity property, which is a key property if one is to justify a pathwise version of Itô calculus. Section 3.3 discusses the properties of the proposed integral when the integrator has the quadratic variation property defined in Föllmer [22] and presents a probability-free version of Proposition 1.2.10 and of Corollary 1.2.11. Section 3.4 discusses the pathwise quadratic variation property in the context of typical paths of continuous semimartingales and derives a sufficient condition for almost all paths of a continuous semimartingale to have this property along a sequence of partitions without the need to subsample the sequence.

3.1 Review and motivation

Throughout the chapter we will consider a sequence $\Pi = \{\pi_k\}_{k\geq 1}$ of partitions $\pi_k = \{0 = t_0^k, t_1^k, \ldots, t_{m_k}^k\}$ of $[0, \infty)$ with $\lim_{k\to\infty} t_{m_k}^k = \infty$ and with vanishing mesh, notably $\lim_{k\to\infty} ||\pi_k|| = 0$. Note that we usually denote integrand paths by H and integrator paths by A in light of the financial interpretation of H being a hedge ratio and A being the price of a traded asset.

We begin by reviewing the probability-free construction of integrals in Föllmer [22] and in Cont and Fournié [12].

Definition 3.1.1. For continuous paths $H, A \in \mathcal{C}(\mathbb{R}^d)$ and $t > 0$, the *pointwise integral* on $[0, t]$ of H against A along Π is defined by the pointwise limit

$$\int_0^t H(u) \, \mathrm{d}^\Pi A(u) := \lim_{k \to \infty} \sum_{i=0}^{m_k - 1} H(t_i^k \wedge t) \, \Delta A(t_i^k \wedge t). \tag{3.1.1}$$

If this limit is well-defined for all $t > 0$, we say that H is pointwise integrable against A along Π.

This definition has a natural meaning in continuous-time finance as the limit in $k \to \infty$ of the wealth processes at time t of the self-financing strategies rebalanced to hold a position H in an asset with price A at times in π_k. Bick and Willinger [6] provide an excellent overview of the financial interpretation of integrals defined as a limit of left Riemann sums along a fixed sequence of partitions.

The existence of the pointwise integral $\int_0^t H(u) \, \mathrm{d}^\Pi A(u)$ is justified by Riemann-Stieltjes integration when either H or A is of finite variation. When both the integrand and integrator paths are of unbounded variation, Riemann-Stieltjes integration does not apply anymore. Recall that existence results for stochastic integrals rely on Itô isometry combined with dominated convergence applied to integrals against the stochastic quadratic variation of a continuous semimartingale (which has finite variation). A classic reference is Kunita and Watanabe [39]. For a recently published proof of the existence of stochastic quadratic variation, see Karandikar and Rao [35]. Such tools are not available in a probability-free setting. In particular, there are no dominated convergence results which apply with the pointwise integrals defined by (3.1.1). However, as shown in Föllmer [22], one can obtain a pathwise version of Itô's formula as well as prove the existence of pointwise integrals for integrands of a certain form under the assumption that the integrator has a so-called quadratic variation property along Π. This property is defined as follows.

Definition 3.1.2. We say that a continuous path $A \in \mathcal{C}(\mathbb{R}^d)$ has the *quadratic variation property* along Π (or A has quadratic variation along Π) if, for each pair $i, j \in \{1, \dots, d\}$, the sequence of discrete measures on $([0, \infty), \mathcal{B}([0, \infty)))$ defined by

$$\nu_k^{(i,j)} := \sum_{l=0}^{m_k - 1} \Delta A_i(t_l^k) \Delta A_j(t_l^k) \, \delta_{t_l^k}$$

converges vaguely (weakly on compacts) to an atomless Radon measure $\nu^{(i,j)}$. The distribution functions of $\nu^{(i,i)}$ and $\nu^{(i,j)}$ are respectively denoted by $\langle A_i \rangle \equiv \langle A_i, A_i \rangle$ and $\langle A_i, A_j \rangle$, and we denote $\langle A \rangle$ to be the $\mathbb{R}^{d \times d}$-valued path whose (i, j)-th component is $\langle A \rangle_{ij} := \langle A_i, A_j \rangle$.

Denote the set of continuous $\mathbb{R}^d$-valued paths with quadratic variation along Π by $\mathcal{Q}^\Pi(\mathbb{R}^d)$.

Remark 3.1.3. Bilinearity of the pathwise cross-variation $\langle A_i, A_j\rangle$ implies that $\mathcal{Q}^\Pi(\mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued continuous paths $A = (A_1, \dots, A_d)$ such that $A_i \in \mathcal{Q}^\Pi(\mathbb{R})$ and $(A_i + A_j) \in \mathcal{Q}^\Pi(\mathbb{R})$ for all $i, j \in \{1, \dots, d\}$, notably

$$\mathcal{Q}^\Pi(\mathbb{R}^d) = \left\{A \in \mathcal{C}(\mathbb{R}^d) : \ A_i, A_i + A_j \in \mathcal{Q}^\Pi(\mathbb{R}), \ \forall i, j \in \{1, \dots, d\}\right\}. \tag{3.1.2}$$

The definition of $\mathcal{Q}^\Pi(\mathbb{R}^d)$ in (3.1.2) is a version of the definition of quadratic variation for multivariate paths stated in [22] and in [12].

We can now present the main result in Föllmer [22]. A proof of the result for the more general case of càdlàg paths can be found in the original paper [22], in the translation in Sondermann [54] or in Davis et al. [15].

Theorem 3.1.4 (Föllmer [22])**.** *Let $A \in \mathcal{Q}^\Pi(\mathbb{R}^d)$ and $F \in C^2(\mathbb{R}^d)$. Then, $\nabla F(A)$ is pointwise integrable against A along Π and the integral satisfies the pathwise Itô formula*

$$\int_0^t \nabla F(A(u)) \,\mathrm{d}^\Pi A(u) = F(A(t)) - F(A(0)) - \frac{1}{2}\int_0^t D^2F(A(u)) \cdot \mathrm{d}\langle A\rangle(u), \quad t \geq 0.$$

Föllmer [22] showed that the pointwise integral along Π in Theorem 3.1.4 coincides almost surely with the classical stochastic integral when A are sample paths of a continuous semimartingale on a filtered probability space. The argument is based upon the observation that the limit in probability of squared differences of continuous semimartingale paths exists by classical results in stochastic analysis, which then implies almost sure convergence of pathwise quadratic variation along a subsequence of Π to the standard (stochastic) quadratic variation. We discuss conditions under which the result holds without the need to use subsequences in Section 3.4.

A key goal of this chapter is to derive a probability-free version of Proposition 1.2.10 and of Corollary 1.2.11. We first discuss why the construction in Föllmer [22] is not sufficient for this purpose. Recall that in the proof of (1.2.5), we started with the integral representation $F(X_t) = F(X_0) + \int_0^t \nabla F(X_u)\,\mathrm{d}X_u + \frac{1}{2}\int_0^t D^2F(X_u)\cdot \mathrm{d}\langle X\rangle_u$ for $t \in [0, \tau_B^X]$ (Föllmer's construction is sufficient to justify this in a probability-free context). We then used the differential form of the components of X in the above equation along with the PDE conditions on F to derive (1.2.5). An implicit assumption in these arguments is the associativity of stochastic integrals. This is the property that for $Z = \int_0^\cdot H^a_t \,\mathrm{d}A_t$ and an integrand H^b, it follows that

$$\int_0^\cdot H^b \,\mathrm{d}Z_t = \int_0^\cdot H^b H^a \,\mathrm{d}A_t.$$

The following simple construction shows that the pointwise integral in Definition 3.1.1 does not satisfy this associativity property. Consider $A(t) = t$,

$$H(t) := \begin{cases} 1 & t = i/3^k, \quad i \text{ even}, \\ -1 & t = i/3^k, \quad i \text{ odd}, \\ 0 & \text{otherwise}, \end{cases}$$

and the sequence $\Pi = \{\pi_k\}_{k \geq 1}$ of partitions $\pi_k = \{0, 1/3^k, \ldots, 1\}$. It is easily checked that

$$Z(t) \equiv \int_0^t H(u)\, \mathrm{d}^\Pi A(u) = \lim_{k\to\infty} \sum_{i=0}^{m_k - 1} \frac{1}{3^k} H(i/3^k) = 0, \quad t \geq 0,$$
$$\int_0^t (H(u))^2\, \mathrm{d}^\Pi A(u) = \lim_{k\to\infty} \sum_{i=0}^{m_k - 1} \frac{1}{3^k} \left(H(i/3^k)\right)^2 = t, \quad t \geq 0.$$

Hence, $\int_0^t H(u)\, \mathrm{d}^\Pi Z(u) \neq \int_0^t (H(u))^2\, \mathrm{d}^\Pi A(u)$, which shows that the pointwise integral along Π is not associative. This implies that many of the standard operations one is used to from stochastic calculus are not justified with pointwise integrals along Π. In particular, the pathwise equivalent of the proof of (1.2.5) does not apply with Föllmer's construction alone.

The construction in Cont and Fournié [12] provides sufficient conditions for a non-anticipative functional $Z[A]$ to allow for an integral representation involving a pointwise integral, an integral with respect to $\langle A \rangle$, and an integral with respect to time. In particular, their result proves the existence of the pointwise integral via an implicit definition involving a pathwise version of Dupire's functional Itô formula [19]. However, their construction is not sufficient either for deriving a probability-free analogue of Proposition 1.2.10. This is because one of the conditions required for their integral representation result is for Z to be left-continuous under a topology on pathspaces (see Definition 3 of [12]) which is equal to the topology of uniform convergence on compacts when paths are defined on the same time domain. This condition is too strong for many of the non-anticipative functionals $Z = F(X)$ considered in the previous chapters. For example, consider a path A of non-zero quadratic variation $\langle A \rangle$ along Π. For any $\epsilon > 0$, one can construct a path $A^{(\epsilon)}$ of finite variation, hence of zero quadratic variation along Π, such that $\sup_{t\in[0,T]} |A(t) - A^{(\epsilon)}(t)| \leq \epsilon$ (see Section 4.3 for such a construction). This implies that, for instance, $F(A, \langle A \rangle)$ is not continuous with respect to the topology used in Cont and Fournié [12] when $F \in C^2(\mathbb{R}^2)$ with $\frac{\partial}{\partial x_2} F(x) \neq 0$. Hence their integral representation does not apply to such functionals. More generally, their integral representation does not apply to functionals which are contingent on $\langle A\rangle$.

By the above discussion, we need to define an alternative construction of probability-free integrals if we are to justify a probability-free version of integral representation results such as (1.2.5).

3.2 Probability-free construction of associative integrals

This section proposes a definition of probability-free integration of an integrand path H against an integrator path A along a sequence of partitions Π which coincides with pointwise integration (hence is the pointwise limit of forward Riemann sums) and which has many of the nice properties of stochastic integrals. In particular, the integral is associative. It also allows for many of the standard operations one is used to from stochastic calculus when the integrator has the quadratic variation property along Π defined earlier.

Definition 3.2.1. For $A \in \mathcal{C}(\mathbb{R}^d)$, define the set $\mathcal{I}(A) \subseteq \mathcal{C}(\mathbb{R}^d)$ of A-*integrable* paths H as follows. A continuous path $H \in \mathcal{C}(\mathbb{R}^d)$ is in $\mathcal{I}(A)$ if there exists a continuous path $I : [0,\infty) \to \mathbb{R}^d$ starting at $I(0) = 0$ such that the sequence of measures on $([0,\infty), \mathcal{B}([0,\infty)))$ defined by

$$\nu_k^{(I)} := \sum_{i=1}^{m_k-1} \left(\Delta I(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right) \delta_{t_i^k}, \quad k \geq 1$$

converges vaguely to the trivial (zero) measure. We then say that I is the forward integral of H against A along Π and we denote it by $I^{\Pi}[H;A]$ or $I[H;A]$ when Π is fixed.

Henceforth in this chapter, we sometimes refer to measures without specifying a space. In such instances, we will mean measures on $([0,\infty), \mathcal{B}([0,\infty)))$.

Proposition 3.2.2. *The forward integral satisfies the following properties:*

1. *(Uniqueness) If $A \in \mathcal{C}(\mathbb{R}^d)$ and $H \in \mathcal{I}(A)$, then I in Definition 3.2.1 is uniquely defined and coincides with the pointwise integral in Definition 3.1.1, notably $I(t) = \int_0^t H(u)\,\mathrm{d}^{\Pi} A(u)$ for $t \geq 0$.*

2. *(Associativity) If $H^a \in \mathcal{I}(A)$ and $H^b \in \mathcal{I}(I[H^a; A])$, then $H^a H^b$ is A-integrable and*
$$I[H^a H^b; A] = I[H^b; I[H^a; A]]$$

Proof.

Uniqueness:

Consider two continuous paths $I_1, I_2 \in \mathcal{C}(\mathbb{R})$ such that the sequences of measures $\{\nu_k^1\}_{k\geq 1}$ and $\{\nu_k^2\}_{k\geq 1}$ on $([0,\infty), \mathcal{B}([0,\infty)))$ defined by

$$\nu_k^1 := \sum_{i=1}^{m_k} \left(\Delta I_1(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right) \delta_{t_i^k}, \quad k \geq 1,$$
$$\nu_k^2 := \sum_{i=1}^{m_k} \left(\Delta I_2(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right) \delta_{t_i^k}, \quad k \geq 1$$

converge to the trivial measure. Then, temporarily denoting $f := I_1 - I_2$, the sequence of measures $\{\nu_k^3\}_{k\geq 1}$ defined by $\nu_k^3 := \nu_k^1 - \nu_k^2 = \sum_{i=1}^{m_k} \Delta f(t_i^k)\delta_{t_i^k}$, $k \geq 1$ converges to the trivial measure. Since f is continuous, Lemma C.2 implies that $f(t) = f(0)$ for $t \geq 0$, hence $I_1(t) - I_2(t) = I_1(0) - I_2(0) = 0$ for $t \geq 0$. This proves uniqueness of I.

Suppose now that H is A-integrable, and denote the forward integral of H against A along Π by I. Then, the sequence of measures defined by

$$\nu_k := \sum_{i=1}^{m_k} \left(\Delta I_1(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right) \delta_{t_i^k}, \quad k \geq 1$$

converges to the trivial measure. Since I is continuous, this implies that $\lim_{n\to\infty} \nu_k([0,t]) = 0$ for $t \geq 0$, hence

$$\int_0^t H(u)\, \mathrm{d}^\Pi A(u) = \lim_{k\to\infty} \sum_{i=1}^{m_k} H(t_i^k)\Delta A(t_i^k) = I(t), \quad t \geq 0.$$

This proves that the pointwise integral is well-defined and is equal to the forward integral.

Associativity:

Let H^a be A-integrable and H^b be $I[H^a; A]$-integrable, and temporarily introduce the shorthand notation $I_1 \equiv I[H^a; A]$ and $I_2 \equiv I[H^b; I_1]$. Consider the sequences of measures on $([0,\infty),\mathcal{B}([0,\infty)))$ defined by

$$\nu_k^1 := \left(\Delta I_1(t_i^k) - H^a(t_i^k)\Delta A(t_i^k)\right)\delta_{t_i^k}, \quad k \geq 1.$$
$$\nu_k^2 := \left(\Delta I_2(t_i^k) - H^b(t_i^k)\Delta I_1(t_i^k)\right)\delta_{t_i^k}, \quad k \geq 1.$$
$$\nu_k^3 := \left(\Delta I_2(t_i^k) - H^a(t_i^k)H^b(t_i^k)\Delta A(t_i^k)\right)\delta_{t_i^k}, \quad k \geq 1.$$

Note that

$$\nu_k^3 = \nu_k^2 + \sum_{i=1}^{m_k} H^b(t_i^k)\left(\Delta I_1(t_i^k) - H^a(t_i^k)\Delta A(t_i^k)\right)\delta_{t_i^k}, \quad k \geq 1,$$

hence $\mathrm{d}\nu_k^3(t) = \mathrm{d}\nu_k^2(t) + H^b(t)\,\mathrm{d}\nu_k^1(t)$ for $k \geq 1$. Since H^b is continuous, and $\lim_{k\to\infty}\nu_k^1 = \lim_{k\to\infty}\nu_k^2 = 0$, Lemma C.1 gives $\lim_{k\to\infty}\nu_k^3 = 0$. This proves the claim that H^aH^b is A-integrable and that $I[H^aH^b;A] = I[H^b;I[H^a;A]]$.

□

Remark 3.2.3.

(1) I is a bilinear operator on (H,A).

(2) The associativity property justifies the informal expression $H^b\,\mathrm{d}^\Pi I[H^a;A] = H^aH^b\,\mathrm{d}^\Pi A$.

The following lemma shows an equivalent statement for forward integrability which is sometimes easier to work with. Its proof is in Appendix C.

Lemma 3.2.4. *For $H, A \in \mathcal{C}(\mathbb{R}^d)$, H is A-integrable iff there exists a continuous path $Z : [0,\infty) \to \mathbb{R}$ such that the sequence of measures on $([0,\infty),\mathcal{B}([0,\infty)))$ defined by*

$$\nu_k^{(Z)} := \sum_{i=1}^{m_k}\left(\Delta Z(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right)\delta_{t_i^k}, \quad k \geq 1$$

converges vaguely to an atomless Radon measure ν. Denoting the distribution function of ν by V, we then have the "probability-free Doob-Meyer decomposition" of Z with respect to A:

$$Z(t) = Z(0) + I[H;A](t) + V(t), \quad t \geq 0.$$

So far, we have only provided a definition of forward integrals and provided some properties thereof. When either the integrator A or the integrand H is of finite variation, standard Riemann-Stieltjes integration is well-defined. However, Riemann-Stieltjes integrals are defined pointwise, hence we need to justify the forward integrability of H with respect to A in the sense of Definition 3.2.1. This is done by the following lemma, which is once again proved in Appendix C.

Lemma 3.2.5. *Consider continuous paths $H, A \in \mathcal{C}(\mathbb{R}^d)$. Then $H \in \mathcal{I}(A)$ if:*

(a) A is of finite variation, in which case $I[H;A]$ is of finite variation.

(b) H is of finite variation.

In either case, the forward integral satisfies the integration by parts identity

$$H(t)A(t) = H(0)A(0) + I[H;A](t) + I[A;H](t), \quad t \geq 0.$$

3.3 Forward integration with paths of quadratic variation along Π

This section discusses properties of forward integrals when the integrator path has the quadratic variation property along Π and presents a probability-free version of Proposition 1.2.10 and of Corollary 1.2.11. We begin with the following theorem, which is a version of Theorem 3.1.4 for forward integrals. In particular, it proves that, for $F \in C^2$, the quadratic variation property assumed in Föllmer [22] is strong enough to imply that $\nabla F(A)$ is A-integrable along Π.

Proposition 3.3.1. *Let $F \in C^2(\mathbb{R}^d)$ and $A \in \mathcal{Q}^\Pi(\mathbb{R}^d)$. Then, $\nabla F(A)$ is A-integrable and*

$$F(A(t)) = F(A(0)) + I[\nabla F(A);A](t) + \frac{1}{2}\int_0^t D^2F(A(u)) \cdot \mathrm{d}\langle A\rangle(u), \quad t \geq 0.$$

Proof. By $A \in \mathcal{Q}^\Pi(\mathbb{R}^d)$, for each $i,j \in \{1,\dots,d\}$, the sequence of measures defined by

$$\nu_k^{(i,j)} := \sum_{l=0}^{m_k-1} \Delta A_i(t_l^k)\Delta A_j(t_l^k)\delta_{t_l^k}, \quad k \geq 1$$

converges to an atomless measure $\nu^{(i,j)}$ with distribution function denoted by $\langle A\rangle_{ij} = \langle A_i, A_j\rangle$. Consider the Radon measure ν defined by $\mathrm{d}\nu(t) = \frac{1}{2}\sum_{i,j=1}^d D_{ij}^2F(A(t))\,\mathrm{d}\nu^{(i,j)}(t)$. The distribution function of ν is equal to

$$F_\nu(t) = \frac{1}{2}\sum_{i,j=1}^d \int_0^t D_{ij}^2F(A(u))\,\mathrm{d}\langle A_i, A_j\rangle(u) = \frac{1}{2}\int_0^t D^2F(A(u)) \cdot \mathrm{d}\langle A\rangle(u), \quad t \geq 0.$$

Next, denote $Z = F(A)$ and consider the sequence of measures defined by

$$\nu_k^{(Z)} := \sum_{i=0}^{m_k-1} \left(\Delta Z(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right)\delta_{t_i^k}, \quad k \geq 1.$$

By exact second order Taylor expansion of the function $F(x(A(t_{l+1}^k)+(1-x)A(t_l^k)))$ around $x=0$ evaluated at $x=1$, we get

$$\begin{aligned}\Delta Z(t_l^k) - H(t_l^k)\Delta A(t_l^k) &= F(A(t_{l+1}^k)) - F(A(t_l^k)) - \nabla F(A(t_l^k))\Delta A(t_l^k) \\ &= \frac{1}{2}\sum_{i,j=1}^{d} D_{ij}^2 F(\xi_{i,j}^{l,k})\Delta A_i(t_l^k)\Delta A_j(t_l^k),\end{aligned}$$

for some $\xi_{i,j}^{l,k} \in [A(t_l^k)\wedge A(t_{l+1}^k), A(t_l^k)\vee A(t_{l+1}^k)]$. For any continuous function g of bounded support,

$$\begin{aligned}\int_0^\infty g(t)\,\mathrm{d}\nu_k^{(Z)}(t) &= \sum_{l=1}^{m_k-1} g(t_l^k)\left(\Delta Z(t_l^k) - H(t_l^k)\Delta A(t_l^k)\right) \\ &= \frac{1}{2}\sum_{l=0}^{m_k-1} g(t_l^k)\left(\sum_{i,j=1}^{d} D_{ij}^2 F(\xi_{i,j}^{l,k})\Delta A_i(t_l^k)\Delta A_j(t_l^k)\right) \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\int_0^\infty f_k^{(i,j)}(t)\,\mathrm{d}\nu_k^{(i,j)}(t),\end{aligned}$$

where $f_k^{(i,j)}(t) := g(t_{i_k(t)}^k) D_{ij}^2 F\left(\xi_{i,j}^{i_k(t),k}\right)$, $t \geq 0$, where we recall the notation $i_k(t) := \max\{i \geq 0 : t_i^k \leq t\}$. By continuity of A and g, the functions $f_k^{(i,j)}$ converge to the continuous functions $f^{(i,j)}(t) := g(t)D_{ij}^2 F(A(t))$ uniformly on compacts for each $i,j \in \{1,\dots,d\}$. By Lemma C.1 and the boundedness of support of g (hence of $f^{(i,j)}$), this implies that

$$\lim_{k\to\infty}\int_0^\infty f_k^{(i,j)}(t)\,\mathrm{d}\nu^{(i,j)}(t) = \int_0^\infty f^{(i,j)}(t)\,\mathrm{d}\nu^{(i,j)}(t).$$

Hence,

$$\begin{aligned}\lim_{k\to\infty}\int_0^\infty g(t)\,\mathrm{d}\nu_k^{(Z)}(t) &= \frac{1}{2}\lim_{k\to\infty}\sum_{i,j=1}^{d}\int_0^\infty f_k^{(i,j)}(t)\,\mathrm{d}\nu_k^{(i,j)}(t) \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\int_0^\infty f^{(i,j)}(t)\,\mathrm{d}\nu^{(i,j)}(t) \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\int_0^\infty g(t)D_{ij}^2 F(A(t))\,\mathrm{d}\nu^{(i,j)}(t) \\ &= \int_0^\infty g(t)\,\mathrm{d}\nu(t),\end{aligned}$$

which implies that $\nu_k^{(Z)}$ converges vaguely to ν. Since $Z, F_\nu \in \mathcal{C}(\mathbb{R})$, Lemma 3.2.4 then implies that $\nabla F(A)$ is A-integrable and that

$$I[\nabla F(A);A](t) = Z(t) - Z(0) - F_\nu(t) = F(A(t)) - F(A(0)) - \frac{1}{2}\int_0^t D^2F(A(u))\cdot \mathrm{d}\langle A\rangle(u)$$

for $t \geq 0$, which concludes the proof.

□

Remark 3.3.2. Similar arguments as in the above proof seem to be applicable to the non-anticipative functionals in Cont and Fournié [12], which would yield a version of Theorem 3 of the latter work with forward integrals instead of pointwise integrals. However, this would require a significant amount of additional notation, definitions and technical considerations, and would detract from the focus of the chapter. Hence, we do not pursue such extensions.

The results in the following theorem are probability-free analogues of standard results in stochastic calculus. The theorem implies that the associativity property allows for extending the set of integrands which are forward integrable against $A \in \mathcal{Q}^{\Pi}(\mathbb{R}^d)$ beyond the set of paths of the form $\nabla F(A)$ for $F \in C^2(\mathbb{R}^d)$.

Theorem 3.3.3. *Let $A \in \mathcal{Q}^{\Pi}(\mathbb{R}^d)$ and consider $X = (X_1, \ldots, X_n)$ such that for each $i \in \{1, \ldots, n\}$,*

$$X_i(t) = X_i(0) + I[H^{(i)}; A](t) + V^{(i)}(t), \quad t \geq 0,$$

for $H^{(i)} \in \mathcal{I}(A)$ and $V^{(i)} \in \mathcal{V}(\mathbb{R})$. Then:

(1) $X \in \mathcal{Q}^{\Pi}(\mathbb{R}^n)$ and

$$\langle X_i, X_j \rangle(t) = \int_0^t \left(H^{(i)}(u) \otimes H^{(j)}(u) \right) \cdot \mathrm{d}\langle A \rangle(u), \quad t \geq 0.$$

(2) For $F \in C^2(\mathbb{R}^n)$, define

$$H(t) := \sum_{i=1}^{n} H^{(i)}(t) \frac{\partial}{\partial x_i} F(X(t)), \quad t \geq 0, \tag{3.3.1}$$

$$V(t) := \sum_{i=1}^{n} \int_0^t \frac{\partial}{\partial x_i} F(X(u)) \, \mathrm{d}V^{(i)}(u) + \frac{1}{2} \int_0^t Y(u) \cdot \mathrm{d}\langle A \rangle(u), \quad t \geq 0, \tag{3.3.2}$$

$$Y_{ij}(t) := \sum_{k,l=1}^{n} H_i^k(t) H_j^l(t) \frac{\partial^2}{\partial x_k \partial x_l} F(X(t)), \quad t \geq 0, \quad i, j \in \{1, \ldots, d\}.$$

Then, $H \in \mathcal{I}(A)$ and $F(X(t)) = F(X(0)) + I[H; A](t) + V(t)$ for $t \geq 0$.

Proof. Proof of claim (1):

By assumption, the sequence of discrete measures $\{\nu_k^{(i)}\}_{k \geq 1}$ defined by

$$\nu_k^{(i)} := \sum_{l=0}^{m_k - 1} \left(\Delta X_i(t_l^k) - H^{(i)}(t_l^k) \Delta A(t_l^k) \right) \delta_{t_l^k} \tag{3.3.3}$$

converges to the Radon measure $\nu^{(i)}$ with distribution functions $V^{(i)}$. For $i, j \in \{1, \dots, n\}$ and $k \geq 1$,

$$
\begin{aligned}
&\sum_{l=0}^{m_k-1} \Delta X_i(t_l^k) \Delta X_j(t_l^k) \, \mathrm{d}\delta_{t_l^k}(t) \\
&= \sum_{l=0}^{m_k-1} \left(H^{(i)}(t_l^k) \cdot \Delta A(t_l^k)\right) \left(H^{(j)}(t_l^k) \cdot \Delta A(t_l^k)\right) \, \mathrm{d}\delta_{t_l^k}(t) \\
&\qquad\qquad + \sum_{l=0}^{m_k-1} \left(\Delta X_i(t_l^k)\right) \left(\Delta X_j(t_l^k) - H^{(j)}(t_l^k) \cdot \Delta A(t_l^k)\right) \, \mathrm{d}\delta_{t_l^k}(t) \\
&\qquad\qquad + \sum_{l=0}^{m_k-1} \left(H^{(j)}(t_l^k) \cdot \Delta A(t_l^k)\right) \left(\Delta X_i(t_l^k) - H^{(i)}(t_l^k) \cdot \Delta A(t_l^k)\right) \, \mathrm{d}\delta_{t_l^k}(t) \\
&= \sum_{l=0}^{m_k-1} \left(H^{(i)}(t_l^k) \cdot \Delta A(t_l^k)\right) \left(H^{(j)}(t_l^k) \cdot \Delta A(t_l^k)\right) \, \mathrm{d}\delta_{t_l^k}(t) \qquad (3.3.4) \\
&\qquad\qquad + \sum_{l=0}^{m_k-1} f_k^{(i)}(t) \, \mathrm{d}\nu_j^{(k)}(t) + \sum_{l=0}^{m_k-1} g_k^{(j)}(t) \, \mathrm{d}\nu_k^{(i)}(t),
\end{aligned}
$$

for $t \in [0, T]$, where $f_k^{(i)}(t) := \Delta X_i(t_{i_k(t)}^k)$, $g_k^{(j)}(t) := H^{(j)}(t_{i_k(t)}^k) \cdot \Delta A(t_{i_k(t)}^k)$. Note that $\lim_{k\to\infty} i_k(t) = t$ because of $\lim_{k\to\infty} ||\pi_k|| = 0$. We now apply Lemma C.1 to the three terms on the right-hand-side of (3.3.4). The first term can be written as

$$
\sum_{l=0}^{m_k-1} \left(H^{(i)}(t_l^k) \otimes H^{(j)}(t_l^k)\right) \cdot \left(\Delta A(t_l^k) \otimes \Delta A(t_l^k)\right) \, \mathrm{d}\delta_{t_l^k}(t),
$$

which, by Lemma C.1 and $\lim_{k\to\infty} H^{(i)}(t_{i_k(t)}^k) H^{(j)}(t_{i_k(t)}^k) = H^{(i)}(t) H^{(j)}(t)$, defines a sequence of measures converging to the atomless Radon measure with distribution function

$$
\int_0^{\cdot} \left(H^{(i)}(u) \otimes H^{(j)}(u)\right) \cdot \, \mathrm{d}\langle A \rangle(u).
$$

The second and third terms on the right-hand-side of (3.3.4) define sequences of measures converging to the trivial measure by Lemma C.1 since $\lim_{k\to\infty} f_k^{(i)} = \lim_{k\to\infty} g_k^{(j)} = 0$ and $\lim_{k\to\infty} \Delta X_i(t_l^k) = \lim_{k\to\infty} \Delta A(t_l^k) = 0$.

The above arguments imply that $\langle X_i, X_j \rangle = \lim_{k\to\infty} \sum_{l=0}^{m_k-1} (\Delta X_i(t_l^k))(\Delta X_j(t_l^k)) \delta_{t_l^k}$ is well-defined for $i, j \in \{1, \dots, n\}$ (hence $X \in \mathcal{Q}^{\Pi}(\mathbb{R}^n)$) and equal to

$$
\langle X_i, X_j \rangle(t) = \int_0^t \left(H^{(i)}(u) \otimes H^{(j)}(u)\right) \cdot \, \mathrm{d}\langle A \rangle(u), \quad t \geq 0, \quad i, j \in \{1, \dots, n\},
$$

which concludes the proof of claim (1).

Proof of claim (2):

Denote $Z \equiv F(X)$. We need to show that the sequence of measures $\{\nu_k\}_{k\geq 1}$ defined by

$$\nu_k := \sum_{j=0}^{m_k-1}\left(\Delta Z(t_j^k) - \left(\sum_{i=1}^{n} H^{(i)}(t_j^k)\frac{\partial}{\partial x_i}F(X(t_j^k))\right)\cdot \Delta A(t_j^k)\right)\delta_{t_j^k}$$

tends to the atomless Radon measure ν with distribution function $V \equiv F_\nu$ defined by (3.3.2). The first claim of the theorem showed that $X \in \mathcal{Q}^{\Pi}(\mathbb{R}^n)$. Hence, Proposition 3.3.1 implies that the sequence of measures $\{\nu_k^{(X)}\}_{k\geq 1}$ defined by

$$\nu_k^{(X)} := \sum_{j=0}^{m_k-1}\left(\Delta Z(t_j^k) - \nabla F(X(t_j^k))\cdot \Delta X(t_j^k)\right)\delta_{t_j^k}$$

converges to the atomless Radon measure ν_X with distribution function given by $F_{\nu^{(X)}}(t) = \frac{1}{2}\int_0^t D^2F(X(u))\cdot \mathrm{d}\langle X\rangle(u)$, $t \geq 0$. By assumption, for each $i \in \{1,\ldots,n\}$, the sequence of measures $\{\nu_k^{(i)}\}_{k\geq 1}$ defined by (3.3.3) converges to the atomless Radon measure ν_i with distribution function $V^{(i)}$. Hence, rearranging orders of summation yields

$$\begin{aligned}
\mathrm{d}\nu_k(t) &= \mathrm{d}\nu_k^{(X)}(t) + \sum_{j=1}^{m_k-1}\left(\sum_{i=1}^{n}\frac{\partial}{\partial x_i}F(X(t_j^k))\left(\Delta X(t_j^k) - H^{(i)}(t_j^k)\cdot \Delta A(t_j^k)\right)\right)\mathrm{d}\delta_{t_j^k}(t)\\
&= \mathrm{d}\nu_k^{(X)}(t) + \sum_{i=1}^{n}\sum_{j=0}^{m_k-1}\frac{\partial}{\partial x_i}F(X(t_j^k))\left(\Delta X(t_j^k) - H^{(i)}(t_j^k)\cdot \Delta A(t_j^k)\right)\mathrm{d}\delta_{t_j^k}(t)\\
&= \mathrm{d}\nu_k^{(X)}(t) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}F(X(t_{i_k(t)}^k))\,\mathrm{d}\nu_k^{(i)}(t),
\end{aligned}$$

By Lemma C.1 and the continuity of the paths $\frac{\partial}{\partial x_i}F(X[A])$, $i \in \{1,\ldots,n\}$ for $A \in \mathcal{Q}^{\Pi}(\mathbb{R}^d)$, we get that the sequence of measures $\{\nu_k\}_{k\geq 1}$ converges to the atomless Radon measure ν with distribution function V defined by

$$V(t) = \frac{1}{2}\int_0^t D^2F(X(u))\cdot \mathrm{d}\langle X\rangle(u) + \sum_{i=1}^{n}\int_0^t \frac{\partial}{\partial x_i}F(X(u))\,\mathrm{d}V^{(i)}(u), \quad t \geq 0.$$

This identity along with $\mathrm{d}\langle X_i, X_j\rangle(t) = \left(H^{(i)}(t)\otimes H^{(j)}(t)\right)\cdot \mathrm{d}\langle A\rangle(t)$ for $i,j \in \{1,\ldots,n\}$ (proved in claim (1)) yields (3.3.2), concluding the proof of claim (2).

□

We can now present a probability-free versions of the results in Section 1.2. We begin with the following definition.

Definition 3.3.4. Define $\mathcal{X}_p$ to be the set of non-anticipative functionals $X = (X_1, \dots, X_n)$ on $\mathcal{Q}^{\Pi}(\mathbb{R}^d)$ such that, for each $i \in \{1, \dots, n\}$, $X_i(\cdot) \equiv X_i[A](\cdot)$ has the representation

$$X_i(t) = X_i(0) + I[\alpha^i(X); A](t) + \int_0^t \beta^i(X_t) \cdot \mathrm{d}\langle A \rangle_u + \int_0^t \gamma^i(X_t)\,\mathrm{d}t, \quad t \geq 0$$

for a constant $X(0) = (X_1(0), \dots, X_n(0))$ and continuous functions $\alpha^i : \mathbb{R}^n \to \mathbb{R}^d$, $\beta^i : \mathbb{R}^n \to \mathbb{R}^{d \times d}$ and $\gamma^i : \mathbb{R}^n \to \mathbb{R}$.

The following corollary of Theorem 3.3.3 is an analogue of the results of Proposition 1.2.10 and Corollary 1.2.11, modulo stopping the functional X at a closed set B (the stopped version of the result is technically more cumbersome to write out in the probability-free setting of this chapter).

Corollary 3.3.5. *Consider a functional $X \in \mathcal{X}_p$ on $\mathcal{Q}^{\Pi}(\mathbb{R}^d)$. Then, for any $F \in C^2(\mathbb{R}^n) \cap \mathcal{S}_X(\mathbb{R}^n)$ and $A \in \mathcal{Q}^{\Pi}(\mathbb{R}^d)$:*

(1) $F(X^B[A](\cdot))$ admits the forward integral representation

$$F(X[A](\cdot)) = F(X_0) + I[\mathcal{L}^{\alpha} F(X); A](\cdot). \tag{3.3.5}$$

Hence, $F(X[A](\cdot))$ is the wealth process of the self-financing strategy starting with capital $F(X(0))$ and holding $\mathcal{L}^{\alpha} F(X[A])$ units of A.

(2) For $T > 0$, $\int_0^T F(X_t)\,\mathrm{d}t$ admits the forward integral representation

$$\int_0^T F(X[A](t))\,\mathrm{d}t = TF(X_0) + I[H; A](T). \tag{3.3.6}$$

where $H(t) = (T - t)\mathcal{L}^{\alpha} F(X[A](t))$ for $t \in [0, T]$. Hence, the self-financing strategy starting with capital $TF(X(0))$ and holding $H(t)$ units of A for $t \in [0, T]$ replicates the cash flow $\int_0^T F(X[A](t)\,\mathrm{d}t$.

3.4 Quadratic variation of continuous semimartingale paths

Since the previous section discussed properties of the pathwise integral $I(H; A)$ for paths A which have the quadratic variation property along Π, it is worth discussing the link between this pathspace and the sample paths of continuous local martingales. Consider a d-dimensional continuous semimartingale $A : \Omega \times [0, T] \to \mathbb{R}^d$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. As noted by Föllmer [22], $\mathbb{P}$-a.e. sample paths of A

have quadratic variation along a subsequence of Π. We provide sufficient conditions on the semimartingale and the partition sequence Π such that $\mathbb{P}$-a.e. sample paths have quadratic variation along Π *without subsampling* the partition sequence, notably

$$\mathbb{P}\left(A(\omega) \in \mathcal{Q}^{\Pi}(\mathbb{R}^d)\right) = 1.$$

This has been noted for the case of A being a one-dimensional Brownian motion (see Remark 1 on page 149 in Föllmer [22] and Exercise 2.9.8 in Karatzas and Shreve [36]), but we did not find a similar result for more general continuous semimartingales.

Note that the existence of the limit of sums of square differences of continuous local martingale is well-known when the partitions are formed by stopping times adapted to the canonical filtration generated by the paths. For instance, see Karandikar and Rao [35] and Karandikar [33]. However, the result is less trivial when the partitions are formed by fixed times which do not depend on the integrand and integrator paths, as is the case for the sequences Π considered in this chapter.

We now present the main results in this subsection.

Proposition 3.4.1. *Let $A : \Omega \times [0,T] \to \mathbb{R}^d$ be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $\Pi = \{\pi_k\}_{k=1}^{\infty}$ be a sequence of partitions of $[0,\infty)$. Suppose that the following conditions on A and Π hold:*

1. *There exist constants $c, p > 0$ and a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which A is a local martingale and such that*

$$\mathbb{E}^{\mathbb{Q}}\left(\int_s^t \left(A_u^i - A_s^i\right)^2 \mathrm{d}\langle A^j\rangle_u\right) \leq c(t-s)^p, \quad 0 \leq s < t < \infty, \tag{3.4.1}$$

   for all $i, j \in \{1, \dots, d\}$.

2. *Π satisfies*

$$\sum_{k=1}^{\infty} \sum_{l=0}^{m_k - 1} \left|t_{l+1}^k \wedge t - t_l^k \wedge t\right|^p < \infty, \quad t > 0. \tag{3.4.2}$$

Then, $\mathbb{P}$-a.e. paths of A are in $\mathcal{Q}^{\Pi}(\mathbb{R}^d)$, notably $\mathbb{P}\left(A(\omega) \in \mathcal{Q}^{\Pi}(\mathbb{R}^d)\right) = 1$.

Corollary 3.4.2. *Under the conditions on A and Π stated in Proposition 3.4.1, $\left\{A(\omega) \in \mathcal{Q}^{\Pi}(\mathbb{R}^d)\right\} \in \mathcal{F}$ if $\mathcal{F}$ is $\mathbb{P}$-complete.*

Before proving the above proposition, we show that it implies that almost all sample paths of continuous martingales with uniformly bounded volatility have quadratic variation along $\Pi = \{\pi_k\}_{k\geq 1}$ when $\sum_{k=1}^{\infty} ||\pi_k|| < \infty$.

Corollary 3.4.3. *Consider a sequence Π of partitions of $[0,\infty)$ such that $\sum_{k=1}^{\infty} ||\pi_k|| < \infty$, and consider the martingale $A := \int_0^{\cdot} \Sigma_u \,\mathrm{d}W_u(\omega)$ for a Brownian motion $W : \Omega \times [0,T] \to \mathbb{R}^{d_W}$ and a uniformly bounded left-continuous semimartingale $\Sigma : \Omega \to \mathbb{R}^d \times \mathbb{R}^{d_W}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Then, $\mathbb{P}$-a.e. paths of A are in $\mathcal{Q}^{\Pi}(\mathbb{R}^d)$.*

Proof. Let b be a bound on Σ. By assumption, A is a continuous martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and

$$\begin{aligned}
\mathbb{E}\left(\int_s^t \left(A_u^i - A_s^i\right)^2 \mathrm{d}\langle A^j\rangle_u\right) &= \mathbb{E}\left(\int_s^t \left(A_u^i - A_s^i\right)^2 \Sigma_u^{jj}\,\mathrm{d}u\right) \\
&= \int_s^t \mathbb{E}\left(\left(A_u^i - A_s^i\right)^2 \Sigma_u^{jj}\right) \mathrm{d}u \\
&\le b \int_s^t \mathbb{E}\left(A_u^i - A_s^i\right)^2 \mathrm{d}u = b\int_s^t \mathbb{E}\left(\int_s^u \Sigma_u^{ii}\,\mathrm{d}u\right) \mathrm{d}u \\
&\le b^2 \int_s^t (u-s)\,\mathrm{d}u = \frac{b^2}{2}(t-s)^2.
\end{aligned}$$

Also, note that

$$\sum_{k=1}^{\infty}\sum_{l=0}^{m_k-1} \left|t_{l+1}^k \wedge t - t_l^k \wedge t\right|^2 \le t \sup_{l\in\{0,\dots,m_k-1\}} \sum_{k=1}^{\infty} \left|t_{l+1}^k \wedge t - t_l^k \wedge t\right| < \infty.$$

By Proposition 3.4.1, we then conclude that $\mathbb{P}$-a.e. paths of A are in $\mathcal{Q}^{\Pi}(\mathbb{R}^d)$. □

We now prove Proposition 3.4.1.

Proof of Proposition 3.4.1. For $t > 0$ and a stochastic process Z, define the shorthand notation $\Delta Z_{t_l^k \wedge t} := Z_{t_{l+1}^k \wedge t} - Z_{t_l^k \wedge t}$.

Let $i, j \in \{1, \dots, d\}$ and $t > 0$. Then,

$$\begin{aligned}
&\sum_{l=0}^{m_k-1} \Delta A^i_{t_l^k\wedge t}\,\Delta A^j_{t_l^k\wedge t} - \langle A^i, A^j\rangle_t \\
&= \sum_{l=0}^{m_k-1}\left(A^i_{t_{l+1}^k\wedge t} A^j_{t_{l+1}^k\wedge t} - A^i_{t_l^k\wedge t} A^j_{t_l^k\wedge t} - A^i_{t_l^k\wedge t}\,\Delta A^j_{t_l^k\wedge t} - A^j_{t_l^k\wedge t}\,\Delta A^i_{t_l^k\wedge t}\right) \\
&\qquad - \left(A_t^i A_t^j - A_0^i A_0^j - \int_0^t A_u^i\,\mathrm{d}A_u^j - \int_0^t A_u^j\,\mathrm{d}A_u^i\right).
\end{aligned}$$

By a telescoping argument, this implies that

$$
\begin{aligned}
&\sum_{l=0}^{m_k-1} \Delta A^i_{t^k_l\wedge t}\,\Delta A^j_{t^k_l\wedge t} - \langle A^i, A^j\rangle_t \\
&\quad = \sum_{l=0}^{m_k-1}\left(\int_{t^k_l}^{t^k_{l+1}} \left(A^i_u - A^i_{t^k_l}\right)\,\mathrm{d}A^j_u + \int_{t^k_l}^{t^k_{l+1}} \left(A^j_u - A^j_{t^k_l}\right)\,\mathrm{d}A^i_u\right).
\end{aligned}
$$

Fatou's lemma, the fact that the expected value of the product of martingales over non-overlapping periods is zero and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ then imply that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left(\sum_{k=1}^{\infty}\left(\sum_{l=0}^{m_k-1} \Delta A^i_{t^k_l\wedge t}\,\Delta A^j_{t^k_l\wedge t} - \langle A^i, A^j\rangle_t\right)^2\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=1}^{\infty}\left(\sum_{l=0}^{m_k-1}\left(\int_{t^k_l}^{t^k_{l+1}} \left(A^i_u - A^i_{t^k_l}\right)\,\mathrm{d}A^j_u + \int_{t^k_l}^{t^k_{l+1}} \left(A^j_u - A^j_{t^k_l}\right)\,\mathrm{d}A^i_u\right)\right)^2\right) \\
&\leq \sum_{k=1}^{\infty}\mathbb{E}^{\mathbb{Q}}\left(\sum_{l=0}^{m_k-1}\left(\int_{t^k_l}^{t^k_{l+1}} \left(A^i_u - A^i_{t^k_l}\right)\,\mathrm{d}A^j_u + \int_{t^k_l}^{t^k_{l+1}} \left(A^j_u - A^j_{t^k_l}\right)\,\mathrm{d}A^i_u\right)\right)^2 \\
&\leq 2\sum_{k=1}^{\infty}\sum_{l=0}^{m_k-1}\mathbb{E}^{\mathbb{Q}}\left(\int_{t^k_l}^{t^k_{l+1}} \left(A^i_u - A^i_{t^k_l}\right)\,\mathrm{d}A^j_u\right)^2 + \qquad (3.4.3) \\
&\qquad + 2\sum_{k=1}^{\infty}\sum_{l=0}^{m_k-1}\mathbb{E}^{\mathbb{Q}}\left(\int_{t^k_l}^{t^k_{l+1}} \left(A^j_u - A^j_{t^k_l}\right)\,\mathrm{d}A^i_u\right)^2.
\end{aligned}
$$

Itô isometry for local martingales, (3.4.1) and (3.4.2) yield

$$
\begin{aligned}
\sum_{k=1}^{\infty}\sum_{l=0}^{m_k-1}\mathbb{E}^{\mathbb{Q}}\left(\int_{t^k_l}^{t^k_{l+1}} \left(A^i_u - A^i_{t^k_l}\right)\,\mathrm{d}A^j_u\right)^2 &\leq \sum_{k=1}^{\infty}\sum_{l=0}^{m_k-1}\mathbb{E}^{\mathbb{Q}}\left(\int_{t^k_l}^{t^k_{l+1}} \left(A^i_u - A^i_{t^k_l}\right)^2\,\mathrm{d}\langle A^j\rangle_u\right) \\
&\leq \sum_{k=1}^{\infty}\sum_{i=0}^{m_k-1} c\left|t^k_{i+1} - t^k_i\right|^p < \infty. \qquad (3.4.4)
\end{aligned}
$$

Similarly, $\sum_{k=1}^{\infty}\sum_{l=0}^{m_k-1}\mathbb{E}^{\mathbb{Q}}\left(\int_{t^k_l}^{t^k_{l+1}}\left(A^j_u - A^j_{t^k_l}\right)\,\mathrm{d}A^i_u\right)^2 < \infty$. Hence, by (3.4.3),

$$
\mathbb{E}^{\mathbb{Q}}\left(\sum_{k=1}^{\infty}\left(\sum_{l=0}^{m_k-1} \Delta A^i_{t^k_l\wedge t}\,\Delta A^j_{t^k_l\wedge t} - \langle A^i, A^j\rangle_t\right)^2\right) < \infty,
$$

This implies that $\lim_{k\to\infty}\sum_{l=0}^{m_k-1}\Delta A^i_{t^k_l\wedge t}\,\Delta A^j_{t^k_l\wedge t} = \langle A^i, A^j\rangle_t$ $\mathbb{Q}$-a.e., where the limit is in the topology of uniform convergence. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, this implies that $\lim_{k\to\infty}\sum_{l=0}^{m_k-1}\Delta A^i_{t^k_l\wedge t}\Delta A^j_{t^k_l\wedge t} = \langle A^i, A^j\rangle_t$ $\mathbb{P}$-a.e.. Since the above holds for all $i, j \in \{1, \dots, d\}$ and since uniform convergence on compacts of distribution functions implies vague convergence of the corresponding measures, we conclude that $\mathbb{P}(A(\omega) \in \mathcal{Q}^{\Pi}(\mathbb{R}^d, T)) = 1$.

□

Chapter 4

Further model-free results for continuous prices

This chapter contains various results related to the content of the previous chapters. Section 4.1 discusses PDE-based strategies which are adaptations of strategies in Chapter 1 for hedging claims of fixed maturity and with more general payoffs than those admitting model-independent replication. Section 4.2 discusses applications of some change of numéraire identities and provides an additional tool for deriving model-independent results. In particular, there are no PDE solutions involved in this approach. The final section of the chapter, Section 4.3, proposes an alternative construction of probability-free integration than that of Föllmer [22] and of Chapter 3. In particular, the construction is based on uniform approximations of general continuous paths by continuous paths of finite variation, and it has some interesting properties relevant to trading under transaction costs.

4.1 Further PDE-based strategies for fixed maturity claims

This section presents some further applications of using a suitable function evaluated at market-observed inputs for hedging claims of fixed maturity without specifying a model for the dynamics of market prices. In particular, we study the hedging error of two adaptations of the PDE-based methodology to hedging claims with fixed maturity. The results are presented in the setting of general continuous semimartingales of Chapter 1. However, one may also cast them in the probability-free setting of Chapter 3.

The first subsection considers the common strategy of "delta-vega" hedging using a continuously calibrated Black-Scholes model. We formulate the hedging error of such a strategy and show that it only depends on the quadratic variation of the underlying asset S via a dependence on the second variations (quadratic variation and cross-variation) of the implied variance V^{BS}. In particular, we conclude that the hedging error is equal to zero (hence the strategy replicates the claim) when implied variance has no second variation.

The second subsection considers the strategy of hedging fixed maturity claims by means of estimating the realised variance of the underlying asset on $[0, T]$ and using this estimate as an input to the replication strategies in Corollary 1.2.11. Under certain conditions on the payoff function, this yields pathwise super-/sub-replication results for claims with fixed maturity T, conditional on over-/under-estimating $\langle \log S \rangle_T$. This provides an extension of results in Carr and Lee [10] to a larger set of claims.

Throughout this section, we consider a functional $X \in \mathcal{X}$ of an underlying asset S, which determines the payoff of the claim to be hedged.

4.1.1 Hedging error of delta-vega hedging with continuous calibration

A common approach to dynamic hedging implemented by practitioners is to use the Black-Scholes formula along with an implied volatility which is (in the limit) continuously calibrated to the price of a traded option, and to use this calibrated model to implement a "delta-vega" hedging strategy by trading in the underlying asset and the traded option. This strategy is inconsistent with the original stochastic framework underlying classical hedging, but the fact that it has been used in a very competitive industry for several decades indicates that there may be some robustness underlying it. The goal of this subsection is to analyse these model-inconsistent strategies.

4.1.1.1 Overview of empirical literature of delta-vega hedging

The starting point of the research in this thesis was the study of hedging strategies under model misspecification. In particular, we were initially interested in the performance of simple hedging strategies using the Black-Scholes (BS) pricing formula in a setting where the true price process of the underlying asset is not known. This question has been investigated quite heavily from an empirical perspective, where prices were either taken from the market or simulated using multi-factor models, and the performance of various hedging strategies were analysed. Although not all of the results yield the same qualitative conclusions, the consensus among much of the resulting empirical literature is that simple models which are continuously calibrated to traded option prices often perform well in terms of hedging error, and moreover that adding a second dynamic position to hedge the volatility risk, or implementing a "vega" hedge in industry parlance, typically has a better impact on hedging performance than using a more precise model for deriving the hedge ratios. We outline some of the empirical findings in more detail in the subsection.

A number of papers investigate the relative performance of local volatility models vs the BS model, with both models usually recalibrated to updated option prices quoted on the market. One of the earlier examples of this is Dumas et al. [17], where the authors compare the performance of deterministic local volatility functions against the BS model in capturing the dynamics of call option prices. Two significant conclusions are drawn in that paper. First, the simpler BS model performs better in pricing and in delta hedging the call options. Furthermore, the volatility point estimates from the recalibration are more stable (in time) for the BS model. The local volatility functions considered in the paper are polynomials fit to the data using least squares estimation, hence one may argue that the proposed conclusions are an artefact choice of parametrisation and calibration. However, the results were confirmed from a theoretical viewpoint in Hagan et al. [24], which showed that the volatility surface has the wrong smile dynamics.

The empirical work in [17] only studied call options. In practice, we are more interested in pricing and hedging path-dependent options. Recently, Fengler et al. [20] studied the relative performance of the local volatility model, the BS model, and a modification of the latter called the sticky-moneyness model in hedging reverse barrier options. Both delta and delta-vega hedging strategies were considered. The results in [20] are based on observed market data, the local volatility surface is calibrated via a numerical approximation of Dupire's theoretical formula ([18]), and all of the models are recalibrated to updated option prices. Once again, the BS model does as well, if not better, than the local volatility model. Relative to the local volatility model, the BS model performs especially well in the case of delta-vega hedging. Furthermore, the choice of delta vs delta-vega hedging strategies has a much higher impact on performance than the choice of the model used to derive the hedge ratios.

Note that Hull and Suo [31] find that the local volatility model significantly outperforms the Black-Scholes model in pricing compound options. However, the latter work is not based on market prices, but on numerical simulations. Furthermore, this substantial improvement in the pricing performance does not hold for barrier options. Most importantly, the paper does not study the relative hedging performance, which is our main focus. Other papers such as Coleman et al. [11] show that local volatility models can outperform the BS models in terms of hedging, but the obtained conclusions are specific to the context (notably the choice of option payoffs) considered.

The key point to retain is that what really matters is whether delta or delta-vega hedging is used. Similar results are observed in the study of the performance of multi-factor models compared to the BS model. For instance, Bakshi et al. [1] shows that stochastic volatility models outperform BS models in terms of pricing accuracy (both models are recalibrated to updated option prices), but that the hedging performances are similar. Again, delta-vega hedging is shown to significantly reduce the hedging error compared to delta hedging.

Remark 4.1.1. One may question the relevance of stylised models used to price and hedge derivatives. Figlewski [21] tackles this question empirically using market data. The paper proposes an ad-hoc pricing formula and shows that when the latter is recalibrated to updated option prices, it can perform as well as the BS model for pricing. The BS model is found to be marginally better for hedging. This result motivated the work in Henderson et al. [27], where the authors extend the informationally passive model proposed in the former paper to be dependent on both strike and maturity. They replicate the results in [21] with more careful considerations of static no arbitrage assumptions (the informationally passive model considered in [21] allows for static arbitrage). [27] argues that extending the passive model to vary in maturity as well as strike ties the function to the diffusion process calibrated to these prices via Dupire's local volatility formula since the latter is the unique Markovian model matching market prices (under the assumption of a deterministic interest rate). This link might be relevant for pricing but it seems questionable in the context of hedging, since the derived hedge ratios would likely be different under the passive pricing function and the diffusion model. It is important to point out that [27] does not investigate hedging errors. Its numerical results, based on the same market data used in [21], indicate that the passive models can be as accurate in pricing as the BS model, but that the Bachelier model generally performs better in pricing than both of the latter models. There are parallels between the work in [21] and [27] and the work in Chapter 1 and in this section. However, whereas the mentioned papers use a numerical approach and emphasise pricing, we use an analytical approach and focus on hedging strategies.

4.1.1.2 Hedging error of delta-vega hedging

Within this subsection, assume that $X \in \mathcal{X}$ is such that $X^1 = S$ and $\gamma^i = 0$ for $i \in \{1, \ldots, n\}$. We begin with the following observation regarding the Black-Scholes price of claims on $f(X_T)$. Consider a solution to $F_{\text{BS}}(x) \equiv F_{\text{BS}}(x_1, \ldots, x_{n+1}; f)$ to the PDE

$$\frac{1}{x_1^2} \frac{\partial}{\partial x_{n+1}} F_{\text{BS}} = \sum_{i=1}^{n} \beta^i \frac{\partial}{\partial x_i} F_{\text{BS}} + \frac{1}{2} \sum_{i,j=1}^{n} \alpha^i \alpha^j \frac{\partial^2}{\partial x_i \partial x_j} F_{\text{BS}}, \tag{4.1.1}$$

on $\mathcal{D} = \mathcal{R}(X; \mathcal{M}_s) \times \mathbb{R}_+$ and with boundary condition $F_{\text{BS}}(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n)$. Note that (4.1.1) is a generalisation of (2.3.15). In particular, for $t \in [0, T)$, $F_{\text{BS}}(x)$ is a solution to the Black-Scholes PDE for the claim with payoff $f(X_T)$ at time t with maturity T under zero interest rate and volatility $\sigma = \sqrt{x_{n+1}/(T-t)}$. Furthermore, the corresponding Black-Scholes delta hedge is equal to $H_{BS}(X, \sigma^2(T-t))$, where

$$H_{BS} := \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x_i} F_{\text{BS}} \tag{4.1.2}$$

Now, consider continuously calibrating the Black-Scholes formula using a traded claim C on $-\log(S_T)$ with maturity T. This is equivalent to setting $x_{n+1} = V^{\text{BS}}$ in $F_{\text{BS}}(x; f)$, where V^{BS} was defined implicitly by (2.3.16). Recall from Section 2.2 that for the choice of payoff function $g(x) = -\log(x)$ of the traded option (used for calibration), the Black-Scholes implied variance V^{BS} is equal to the time value V, notably $V^{\text{BS}} = V = C + \log(S)$. It is also equal to the theoretical value of the VIX index, notably the model-free implied variance. In particular, V^{BS} has differential form

$$\mathrm{d}V_t = \mathrm{d}C_t + \frac{1}{S_t} \mathrm{d}S_t - \frac{1}{2S_t^2} \mathrm{d}\langle S \rangle_t, \quad t \in [0, T]. \tag{4.1.3}$$

Under the above assumptions, the continuously calibrated Black-Scholes price at time $t \in [0, T]$ for the claim on $f(X_T)$ is equal to $F_{\text{BS}}(X_t, V_t)$. By Itô's formula, (1.2.1), (4.1.1) and (4.1.3) (where $X^1 \equiv A \equiv S$ in the context and notation of this subsection), direct computation gives

$$\begin{aligned}\mathrm{d}F_{\text{BS}}(X_t, V_t) = {}& \left(H_{\text{BS}}(X_t, V_t) + \frac{1}{S_t} \frac{\partial}{\partial x_{n+1}} F_{\text{BS}}(X_t, V_t) \right) \mathrm{d}S_t + \frac{\partial}{\partial x_{n+1}} F_{\text{BS}}(X_t, V_t) \, \mathrm{d}C_t \\ & + \left(\sum_{i=1}^{n} \alpha^i(X_t) \frac{\partial}{\partial x_i} F_{\text{BS}}(X_t, V_t) \right) \mathrm{d}\langle S, V \rangle_t + \frac{1}{2} \frac{\partial^2}{\partial x_{n+1}^2} F_{\text{BS}}(X_t, V_t) \, \mathrm{d}\langle V \rangle_t.\end{aligned}$$

In particular, all of the variation of $F_{\mathrm{BS}}(X, V)$ with respect to $\langle S\rangle$ is captured by second variations of V. Defining

$$E_t := -\int_0^t \left(\sum_{i=1}^n \alpha^i(X_u)\frac{\partial}{\partial x_i}F_{\mathrm{BS}}(X_u, V_u)\right)\mathrm{d}\langle S, V\rangle_u - \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x_{n+1}^2}F_{\mathrm{BS}}(X_u, V_u)\,\mathrm{d}\langle V\rangle_u \tag{4.1.4}$$

for $t \in [0, T]$, and noting that $F_{\mathrm{BS}}(X_T, V_T) = f(X_T)$ due to $V_T = 0$ and the boundary condition $F_{\mathrm{BS}} = f$ for $x_{n+1} = 0$, this yields

$$\begin{aligned} &f(X_T) + E_T \\ &= F_{\mathrm{BS}}(X_0, V_0) + \int_0^T \left(H_{BS}(X_t, V_t) + \frac{1}{S_t}\frac{\partial}{\partial x_{n+1}}F_{\mathrm{BS}}(X_t, V_t)\right)\mathrm{d}S_t \\ &\qquad\qquad + \int_0^T \frac{\partial}{\partial x_{n+1}}F_{\mathrm{BS}}(X_t, V_t)\,\mathrm{d}C_t. \end{aligned}$$

We summarise the above by the following proposition regarding the hedging error of the delta-vega hedging strategy based on the Black-Scholes model continuously calibrated to the model-free implied variance.

Proposition 4.1.2. *The strategy starting with capital equal to the calibrated Black-Scholes price $F_{\mathrm{BS}}(X_0, V_0)$ and holding dynamic positions $H_{BS}(X, V)+(1/S)\frac{\partial}{\partial x_{n+1}}F_{\mathrm{BS}}(X, V)$ in S and $\frac{\partial}{\partial x_{n+1}}F_{\mathrm{BS}}(X, V)$ in C generates a wealth at time T equal to $f(X_T) + E_T$ under all continuous semimartingale models for S. In particular, the hedging error of using this strategy for hedging a claim with payoff $f(X_T)$ and maturity T is E_T.*

Corollary 4.1.3. *The hedging strategy in Proposition 4.1.2 replicates the claim on $f(X_T)$ if $\langle V\rangle_T = 0$.*

The position $H_{BS}(X, V)$ in S corresponds to the standard delta hedge under the continuously calibrated Black-Scholes model, whereas the adjustment to the position in S by $(1/S)\frac{\partial}{\partial x_{n+1}}F_{\mathrm{BS}}(X, V)$ and the position $\frac{\partial}{\partial x_{n+1}}F_{\mathrm{BS}}(X, V)$ in C correspond to the vega hedge.

Remark 4.1.4. If we had chosen another calibration payoff (say a call option), the corresponding implied variance would have a differential form given by (2.3.17) instead of (4.1.3). In particular, similar results as those in this subsection would apply because the variation of V^{BS} in $\langle S\rangle$ which is not captured by second variations of V^{BS} is the same as that of V, notably it is equal to $-\frac{1}{2S_t^2}\,\mathrm{d}\langle S\rangle_t$.

Remark 4.1.5. With the formulation of the hedging error at hand, one may devise simple methods which locally offset the hedging error by adding positions in an additional set of traded claims to make the the second order sensitivities of the portfolio around $(X_0^1, \ldots, X_0^n, V_0)$ equal to zero. This may be generalised to having semi-static positions, notably by re-adjusting the positions in traded claims at stopping times defined by relationships between market prices. Although such methods may be interesting, they would require empirical considerations and would not fit consistently within the theme of this thesis.

4.1.2 Super- and sub-replication conditional on bounds on realised variance

Consider $X \in \mathcal{X}$ such that $X^1 = S$. This subsection discusses the strategy of using an initial estimate $q > 0$ for $\langle \log S \rangle_T$ along with the strategy of Corollary 1.2.11 corresponding to the replication of a timer claim with payoff $f(X_{\tau_q})$ at maturity

$$\tau_q := \inf\{t > 0 : \langle \log S \rangle_t = q\} \tag{4.1.5}$$

to hedge a claim with payoff $f(X_T)$ and maturity T. Note that estimating $\langle \log S \rangle_T$ is equivalent to estimating the average volatility on $[0, T]$, which is more sensible in practice than estimating pointwise volatilities for all $t \in [0, T]$.

Corollary 2.7 of Carr and Lee [10] and earlier work by Mykland [47] showed that for convex path-independent payoffs and under all continuous semimartingale models, such strategies are robust in the sense that over-estimation of realised variance leads to pathwise super-replication strategies, and, conversely, under-estimation leads to pathwise sub-replication strategies. This section generalises the class of claims for which this robustness property holds.

For $f \in C^2$, and $X \in \mathcal{X}$. Define $\widehat{X}$ by augmenting X with $(q - \langle \log S \rangle) \vee 0$, notably

$$\widehat{X}^i = \begin{cases} X^i, & i \in \{1, \ldots, n\}, \\ (q - \langle \log S \rangle) \vee 0 & i = n+1. \end{cases} \tag{4.1.6}$$

Note that $\widehat{X} \in \mathcal{X}$ since $X^1 = S$. For piecewise twice differentiable functions f and provided that

$$\sum_{i=1}^n \beta^i \frac{\partial}{\partial x_i} f + \frac{1}{2} \sum_{i,j=1}^n \alpha^i \alpha^j \frac{\partial^2}{\partial x_i \partial x_j} f \geq 0, \tag{4.1.7}$$

$$\sum_{i=1}^n \gamma^i \frac{\partial}{\partial x_i} f \geq 0, \tag{4.1.8}$$

on $\mathcal{R}(X;\mathcal{M}_s)$ and a condition on solutions F to $\mathcal{L}^{\alpha,\beta}F = \mathcal{L}^{\gamma}F = 0$ on $\mathcal{D} = \mathcal{R}(X;\mathcal{M}_s)\times \mathbb{R}_+$ with boundary condition $F = f$ on $B = \{x_{n+1} = 0\}$, we show that claims with payoff $f(X_T)$ and maturity T can be super-/sub-replicated provided bounds on $\langle \log S\rangle_T$.

Remark 4.1.6. The results generalise to using an estimate for weighted realised variance $Q^w_T := \int_0^T w(S_t)\,\mathrm{d}\langle S\rangle_t$ instead of an estimate for the log variance $\langle \log S\rangle_T$.

Define

$$Y_t := \int_{\tau_q}^{t\vee\tau_q}\left(\sum_{i=1}^n \beta^i(X_u)\frac{\partial}{\partial x_i}f(X_u) + \frac{1}{2}\sum_{i,j=1}^n \alpha^i(X_u)\alpha^j(X_u)\frac{\partial^2}{\partial x_i\partial x_j}f(X_u)\right)\mathrm{d}\langle S\rangle_u + \int_{\tau_q}^{t\vee\tau_q}\left(\sum_{i=1}^n \gamma^i(X_u)\frac{\partial}{\partial x_i}f(X_u)\right)\mathrm{d}u$$

for $t\in[0,T]$. In particular, $Y_t = 0$ on $\{t\le\tau_q\}$.

The following corollary generalises Corollary 2.7 of Carr and Lee [10], which corresponds to the choice of $X = S$ and convex $f\in C^2$.

Proposition 4.1.7. *Let $F\in C^2$ be a solution to $\mathcal{L}^{\alpha,\beta}F = \mathcal{L}^{\gamma}F = 0$ on $\mathcal{D} = \mathcal{R}(X;\mathcal{M}_s)\times\mathbb{R}_+$ and with boundary condition $F = f$ on B. Define*

$$H_t := \begin{cases}\sum_{i=1}^n \alpha^i(X_t)\frac{\partial}{\partial x_i}F(\widehat{X}_t), & t\in[0,\tau_q\wedge T),\\ \sum_{i=1}^n \alpha^i(X_t)\frac{\partial}{\partial x_i}f(X_t), & t\in[\tau_q\wedge T, T)\end{cases}$$

and $E^{\mathrm{RV}}(q) := F(\widehat{X}_{T\wedge\tau_q}) - f(X_{T\wedge\tau_q}) - Y_T$. Then, $F(\widehat{X}_0) + \int_0^T H_u\,\mathrm{d}S_u = f(X_T) + E^{\mathrm{RV}}(q)$ under all continuous semimartingale models for S. In particular, the hedging error of using the self-financing strategy starting with capital $F(X_0,q)$ and holding H units of S for hedging a claim with payoff $f(S_T)$ and maturity T is equal to $E^{\mathrm{RV}}(q)$.

If $F(x)\ge f(x_1,\dots,x_n)$ holds for $x\in\mathcal{D}$, then the above strategy superreplicates $f(S_T)$ on $\{\langle\log S\rangle_T\le q\}$. If f and X satisfy (4.1.7) – (4.1.8), *the above strategy subreplicates $f(S_T)$ on $\{\langle\log S\rangle_T\ge q\}$.*

Proof. Since $Y_T = 0$ on $\{\langle\log S\rangle_T\le q\}$ and by definition of H, it follows that $E^{\mathrm{RV}}(q) = F(\widehat{X}_T) - f(X_T)$ and $H = \sum_{i=1}^n\alpha^i(X)\frac{\partial}{\partial x_i}F(\widehat{X})$ on $\{\langle\log S\rangle_T\le q\}$. Hence, (1.2.5) implies that

$$F(\widehat{X}_0) + \int_0^T H_t\,\mathrm{d}S_T = F(\widehat{X}_T) = f(X_T) + E^{\mathrm{RV}}(q)$$

on $\{\langle\log S\rangle_T\le q\}$. On the other hand, the definitions of $E^{\mathrm{RV}}(q)$ and τ_q along with the boundary condition $F = f$ at B imply that $E^{\mathrm{RV}}(q) = \mathrm{Y_T}$ on $\{\langle\log S\rangle_T\ge q\}$.

Hence, by $\mathrm{d}f(X_t) = H_t \,\mathrm{d}S_t + \mathrm{d}Y_t$ for $t \geq \tau_q$ and by (1.2.5), this implies that

$$\begin{aligned} F(\widehat{X}_0) + \int_0^T H_t \,\mathrm{d}S_T &= \left(F(\widehat{X}_0) + \int_0^{\tau_q} H_t \,\mathrm{d}S_t \right) + \int_{\tau_q}^T H_t \,\mathrm{d}S_t \\ &= F(\widehat{X}_{\tau_q}) + (f(X_T) - f(X_{\tau_q})) - (Y_T - Y_{\tau_q}) \\ &= f(X_T) - Y_T \\ &= f(X_T) + E^{\mathrm{RV}}(q) \end{aligned}$$

on $\{\langle \log S \rangle_T \geq q\}$.

It remains to prove the claim regarding super- and sub-replication provided (4.1.7) – (4.1.8) and pathwise bounds on $\langle \log S \rangle_T$. By $F(\widehat{X}_0) + \int_0^T H_u \,\mathrm{d}S_u = f(X_T) + E^{\mathrm{RV}}(q)$, this is equivalent to showing that $E^{\mathrm{RV}}(q) \geq 0$ on $\{\langle \log S \rangle_T \leq q\}$ and $E^{\mathrm{RV}}(q) \leq 0$ on $\{\langle \log S \rangle_T \geq q\}$, which follow from $F(x) \geq f(x_1, \ldots, x_n)$ and from (4.1.7) – (4.1.8) respectively.

□

Well-known works of El-Karoui et al. [37], Hobson [30] and others [2, 3, 26, 28, 32] have established sufficient conditions for desirable properties of PDE solutions such as convexity in price and monotonicity in time and in volatility. In the following lemma, we provide a similar sufficiency condition tailored to the context of Proposition 4.1.7. In particular, we show via martingale arguments that under some regularity conditions, (4.1.7) – (4.1.8) imply that $F(x) \geq f(x_1, \ldots, x_n)$ for $x \in \mathcal{R}(X; \mathcal{M}_s)$.

Lemma 4.1.8. *Consider the same definitions as in Proposition 4.1.7. For $x \in \mathcal{R}(X; \mathcal{M}_s)$, consider the functional $X^{(x)}$ started at $X_0 = (x_1, \ldots, x_n)$ and having the same differential form as X. If there is a local martingale model $M^x = (\Omega^x, \mathcal{F}^x, \mathbb{F}^x, \mathbb{P}^x)$ for S under which $\mathbb{P}^x(\tau_q < \infty) = 1$ and $\int_0^\cdot \left(\sum_{i=1}^n \alpha^i(X_t^{(x)}) \frac{\partial}{\partial x_i} f(X_t^{(x)}) \right) \mathrm{d}S_t$ is a martingale, then any lower bounded solution F to $\mathcal{L}^{\alpha,\beta} F = \mathcal{L}^{\gamma} F = 0$ on $\mathcal{R}(X; \mathcal{M}_s) \times \mathbb{R}_+$ with boundary condition $F = f$ on B must satisfy $F(x) \geq f(x_1, \ldots, x_n)$.*

Proof. Let x, $\mathbb{P}^x$, f and F be as in the statement of the lemma. By (1.2.5) and the lower boundedness of F, $F(\widehat{X}_{\cdot \wedge \tau_q})$ is a lower bounded local martingale – hence a supermartingale – under M^x. On the other hand, $f(X)$ is a submartingale under M^x by (4.1.7) – (4.1.8) and the assumption that $\int_0^\cdot \left(\sum_{i=1}^n \alpha^i(X_t^{(x)}) \frac{\partial}{\partial x_i} f(X_t^{(x)}) \right) \mathrm{d}S_t$ is a martingale under M^x. Thus, $F(\widehat{X}_{\cdot \wedge \tau_q}) - f(X_{\cdot \wedge \tau_q})$ is a supermartingale under M^x which converges to 0 at $t = \tau_q$. By optional sampling and $\mathbb{P}^x(\tau_q < \infty) = 1$, it follows that $F(x) - f(x_1, \ldots, x_n) \geq \mathbb{E}^x \left(F(\widehat{X}_{\tau_q}) - f(X_{\tau_q}) \right) = 0$, which concludes the proof.

□

4.2 Applications of pathwise change of numéraire

The previous results in the thesis combined Itô calculus with solutions to systems of PDEs to derive model-free hedging results (either in a model-independent setting of continuous semimartingales or in a probability-free setting with continuous price paths having quadratic variation along a sequence of partitions). In this section, we show an alternative approach to deriving model-free results in a market where claims with fixed maturity are traded. In particular, we derive a set of pathwise identities which may be interpreted as change of numéraire identities. We present the results for general continuous semimartingales but, provided some care regarding integrability conditions, probability-free versions of the results of this chapter apply with the notion of forward integration proposed in Chapter 3.

We first derive the key identities in a general setting and then illustrates an application of these identities. In particular, we show how to link the payoff of continuous claims on price and weighted realised variance, henceforth referred to as *volatility derivatives* (which generally do not admit model-free replication, see Carr and Lee [10] for a discussion regarding minimal robust bounds), with the product of the underlying asset price and the wealth process of a trading strategy.

Remark 4.2.1. Corollary 1.2.11 implies that one can model-independently replicate claims on the underlying asset and weighted realised variance Q^w defined in (1.1.4) for payoff functions f which satisfy $w\frac{\partial}{\partial x_2}f + \frac{1}{2}\frac{\partial^2}{\partial x_1^2}f = 0$. In contrast, the volatility derivatives considered in this section have general continuous payoff functions.

4.2.1 Change of numéraire identities

For $\mathbb{R}^k$-valued continuous semimartingales H and Z, define

$$\widehat{H}_\cdot := \int_0^\cdot H_t \,\mathrm{d}Z_t - H_\cdot Z_\cdot. \tag{4.2.1}$$

We begin with the following lemma.

Lemma 4.2.2. *For any $\mathbb{R}_+$-valued continuous semimartingale N,*

$$N_t\left(\int_0^t H_u \,\mathrm{d}Z_u\right) = \int_0^t H_u \,\mathrm{d}\left(NZ\right)_u + \int_0^t \widehat{H}_u \,\mathrm{d}N_u, \quad t \geq 0. \tag{4.2.2}$$

Proof. By Itô's formula,

$$
\begin{aligned}
\mathrm{d}\left(N_t\left(\int_0^t H_u\,\mathrm{d}Z_u\right)\right) &= N_t H_t\,\mathrm{d}Z_t + \left(\int_0^t H_u\,\mathrm{d}Z_u\right)\mathrm{d}N_t + \sum_{i=1}^{k} H_t\,\mathrm{d}\langle N, Z^i\rangle_t \\
&= H_t\,\mathrm{d}(NZ)_t - (H_t Z_t)\,\mathrm{d}N_t + \left(\int_0^t H_u\,\mathrm{d}Z_u\right)\mathrm{d}N_t \\
&= H_t\,\mathrm{d}(NZ)_t + \widehat{H}_t\,\mathrm{d}N_t,
\end{aligned}
$$

which yields (4.2.2).

□

Remark 4.2.3. The identity (4.2.2) can be interpreted as a result on numéraire invariance. Suppose the Z^i, $i \in \{1, \ldots, k\}$ are prices of tradable assets. By (4.2.1), $\widehat{H}$ corresponds to the amount of bonds of the self-financing portfolio holding $H = (H^1, \ldots, H^k)$ units of $Z = (Z^1, \ldots, Z^k)$. We can denominate the gain from trade of this strategy in any positive numéraire of our choice. Letting N be such a numéraire, this implies that

$$
N_t\left(H_t Z_t + \widehat{H}_t\right) = \int_0^t H_u\,\mathrm{d}(NZ)_u + \int_0^t \widehat{H}_u\,\mathrm{d}N_u, \quad t \geq 0,
$$

which yields (4.2.2). Also, note that $\widehat{H}$ depends on H and Z, but not on N.

Even though Lemma 4.2.2 may be interpreted as a change of numéraire result in the context of a trading strategy in assets with price Z, we are rather interested in the following identities it yields within the context of traded claims of fixed maturity $T > 0$.

Corollary 4.2.4. *For an adapted process X and functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to (0, \infty)$, let C^1 and C^2 be continuous semimartingales such that $C^1_T = g_1(X_T)$, $C^2_T = g_2(X_T)$ and $C^1_t > 0$ for $t \in [0, T]$ almost surely. Define $\widehat{H}$ and $\widetilde{H}$ by*

$$
\widehat{H}_t := \int_0^t H_u\,\mathrm{d}C^1_u - H_t C^1_t, \quad t \in [0, T], \tag{4.2.3}
$$

$$
\widetilde{H}_t := \int_0^t \widehat{H}_u\,\mathrm{d}\left(C^2/C^1\right)_u - \frac{\widehat{H}_t C^2_t}{C^1_t}, \quad t \in [0, T]. \tag{4.2.4}
$$

Then, we get the following model-independent identities:

$$\frac{g_2(X_T)}{g_1(X_T)}\left(\int_0^T H_t\,\mathrm{d}C_t^1\right) = \int_0^T H_t\,\mathrm{d}C_t^2 + \int_0^T \widehat{H}_t\,\mathrm{d}\left(C^2/C^1\right)_t, \tag{4.2.5}$$

$$g_2(X_T)\left(\int_0^T H_t\,\mathrm{d}C_t^1\right) = g_1(X_T)\left(\int_0^T H_t\,\mathrm{d}C_t^2\right) + \int_0^T \widehat{H}_t\,\mathrm{d}C_t^2 + \widetilde{H}_t\,\mathrm{d}C_t^1, \tag{4.2.6}$$

$$g_2(X_T)\left(\int_0^T H_t\,\mathrm{d}C_t^1\right) = \int_0^T H_t\,\mathrm{d}\langle C^1, C^2\rangle_t + \int_0^T (H_t C_t^1 + \widehat{H}_t)\,\mathrm{d}C_t^2 \tag{4.2.7}$$

$$+ \int_0^T \left(\int_0^t H_u\,\mathrm{d}C_u^2 + \widetilde{H}_t\right)\mathrm{d}C_t^1.$$

Proof. Letting $Z = C^1$, $N = C^2/C^1$ and $t = T$ in (4.2.2) gives

$$\frac{g_2(X_T)}{g_1(X_T)}\int_0^T H_t\,\mathrm{d}C_t^1 = \frac{C_T^2}{C_T^1}\int_0^T H_t\,\mathrm{d}C_t^1 = \int_0^T H_t\,\mathrm{d}C_t^2 + \int_0^T \widehat{H}_t\,\mathrm{d}\left(C^2/C^1\right)_t,$$

which proves (4.2.5). A further application of (4.2.2) (with $Z = C^2/C^1$, $N = C^1$ and relabelling $(H, \widehat{H})$ to $(\widehat{H}, \widetilde{H})$) yields (4.2.6), which is equivalent to (4.2.7) by $C_T^1 = g_1(X_T)$ and Itô's formula applied to $C_\cdot^1\left(\int_0^\cdot H_t\,\mathrm{d}C_t^2\right)$.

□

4.2.2 Application to continuous claims on price and realised variance

As mentioned in Carr and Lee [8], pricing and hedging nonlinear contracts on realised variance such as volatility swaps can be daunting for market practitioners. To address that issue, [8] propose a dynamic hedging strategy for such contracts using the underlying asset and traded claims. This strategy works for all continuous semimartingale models where the volatility process is independent of the process driving log returns, and it is possible to relax the independence assumption up to first order in the correlation parameter. In unpublished work by the same authors, the hedging strategy in [8] is generalised to hold for time-homogeneous local volatility models time-changed by an independent volatility process, which gives more flexibility for capturing the implied volatility skewness observed in some markets.

The results in [8] imply that dynamic trading in derivatives may be useful in tightening model-independent price bounds for volatility derivatives. However, their results require asset price fluctuations to be independent of the volatility process under the risk-neutral measure, which is a strong assumption. In this subsection, we show that, modulo self-financing trading, the replication of volatility derivatives with a continuous payoff function and with maturity $T > 0$ is equivalent to replicating the product of the wealth generated by a dynamic trading strategy (in the underlying asset and in path-independent traded claims) and of the price of the underlying asset at maturity. Even if this is not implementable in practice without having to introduce a new financial product, such as a forward with notional equal to the outcome of a self-financing trading strategy, the model-independent identity may still be of interest. The result applies to continuous joint payoffs of asset price and its weighted realised variance $Q^w = \int_0^{\cdot} w(S_t)\,\mathrm{d}\langle S\rangle_t$ for a weight function w.

For the remainder of this section, we set $C^1 \equiv X \equiv S$ and $g_1(x) = x$ when referring to Corollary 4.2.4.

4.2.2.1 Exponentials of realised variance of log returns

As in Carr and Lee [8], we first consider claims on exponentials of realised variance of log returns. In particular, consider a claim on $e^{-\lambda\langle\log(S)\rangle_T}$ for $\lambda \in \mathbb{N}$.

Lemma 4.2.5. *Let $\lambda \in \mathbb{N}$ and b_λ be a solution to*

$$\frac{b_\lambda(b_\lambda - 1)}{2} = \lambda. \tag{4.2.8}$$

Then, assuming $S_T > 0$,

$$e^{-\lambda\langle\log S\rangle_T} = \frac{1}{S_T^{b_\lambda}}\left(S_0^{b_\lambda} + \int_0^T b_\lambda S_t^{b_\lambda - 1} e^{-\lambda\langle\log S\rangle_t}\,\mathrm{d}S_t\right). \tag{4.2.9}$$

Proof. By Itô's formula,

$$\begin{aligned}
\mathrm{d}\left(S_t^{b_\lambda} e^{-\frac{b_\lambda(b_\lambda-1)}{2}\langle\log S\rangle_t}\right) &= b_\lambda S_t^{b_\lambda-1} e^{-\frac{b_\lambda(b_\lambda-1)}{2}\langle\log S\rangle_t}\,\mathrm{d}S_t \\
&\quad + \frac{b_\lambda(b_\lambda-1)}{2} S_t^{b_\lambda-2} e^{-\frac{b_\lambda(b_\lambda-1)}{2}\langle\log S\rangle_t}\,\mathrm{d}\langle S\rangle_t \\
&\quad - \frac{b_\lambda(b_\lambda-1)}{2} S_t^{b_\lambda} e^{-\frac{b_\lambda(b_\lambda-1)}{2}\langle\log S\rangle_t}\,\mathrm{d}\langle\log S\rangle_t \\
&= b_\lambda S_t^{b_\lambda-1} e^{-\frac{b_\lambda(b_\lambda-1)}{2}\langle\log S\rangle_t}\,\mathrm{d}S_t.
\end{aligned}$$

With (4.2.8), this gives

$$S_T^{b_\lambda} e^{-\lambda\langle\log S\rangle_T} = S_0^{b_\lambda} + \int_0^T b_\lambda S_t^{b_\lambda-1} e^{-\lambda\langle\log S\rangle_t}\,\mathrm{d}S_t,$$

which yields (4.2.9) for $S_T > 0$. □

Remark 4.2.6. Equation (4.2.8) has two solutions for b_λ, notably $\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$. Carr and Lee [8] use this degree of freedom to mitigate their independence assumption to first order in the correlation parameter.

Theorem 4.2.7. *Let $\lambda \in \mathbb{N}$ and $b_\lambda \in \mathbb{N}$ be a solution to (4.2.8). Define $g(x) = x^{-b_\lambda}$ and suppose that claims with payoff $g(S_T)$ and maturity T are traded at price $C \equiv C^2 > 0$. For $H := b_\lambda S^{b_\lambda - 1} e^{-\lambda \langle \log S \rangle}$, define $\widehat{H}$ and $\widetilde{H}$ by (4.2.3) – (4.2.4). Then,*

$$e^{-\lambda\langle \log S\rangle_T} = S_T \left(\int_0^T H_t \, \mathrm{d}C_t \right) + \int_0^T (\widehat{H}_t + S_0^{b_\lambda}) \, \mathrm{d}C_t + \int_0^T \widetilde{H}_t \, \mathrm{d}S_t. \tag{4.2.10}$$

under all continuous semimartingale models where $S > 0$ and $C > 0$ almost surely.

Proof. Starting with (4.2.9), we get

$$\begin{aligned} e^{-\lambda\langle \log S\rangle_T} &= \frac{1}{S_T^{b_\lambda}} \left(S_0^{b_\lambda} + \int_0^T b_\lambda S_t^{b_\lambda - 1} e^{-\lambda \langle \log S \rangle_t} \, \mathrm{d}S_t \right) \\ &= S_0^{b_\lambda} g(S_T) + g(S_T) \int_0^T H_t \, \mathrm{d}S_t \\ &= S_T \left(\int_0^T H_t \, \mathrm{d}C_t \right) + \int_0^T (\widehat{H}_t + S_0^{b_\lambda}) \, \mathrm{d}C_t + \int_0^T \widetilde{H}_t \, \mathrm{d}S_t, \end{aligned}$$

where we used (4.2.6) and $C_T = g(S_T) > 0$ to get the last equality.

□

4.2.2.2 General exponentials

For a nonnegative function w, define $Q^w_\cdot := \int_0^\cdot w(S_t) \, \mathrm{d}\langle S \rangle_t$. Lemma 4.2.8 and Theorem 4.2.9 below generalise the results in Lemma 4.2.5 and Theorem 4.2.7 to the context of exponential payoffs of the form $e^{-\mu S_T - \lambda Q^w_T}$ for $\mu, \lambda \in \mathbb{N}$. Since the proofs are similar, we only provide their outlines.

Lemma 4.2.8. *Consider a function ϕ which satisfies the ODE*

$$\frac{1}{2}\phi''(x) = \lambda w(x) \phi(x). \tag{4.2.11}$$

with some initial conditions $\phi(0) > 0, \phi'(0) \geq 0$ Then, $\phi(x) > 0$ for $x > 0$, and

$$\phi(S_T) e^{-\lambda Q^w_T} = \phi(S_0) + \int_0^T \phi'(S_t) e^{-\lambda Q^w_t} \, \mathrm{d}S_t. \tag{4.2.12}$$

Proof. The proof of $\phi(x) > 0$ for $x > 0$ follows from the non-negativity of w and the assumptions $\phi(0) > 0$ and $\phi'(0) \geq 0$. The second claim follows from Itô's formula applied to $\phi(S_\cdot) e^{-\lambda Q^w_\cdot}$, along with $\mathrm{d}Q^w_t = w(S_t) \, \mathrm{d}\langle S \rangle_t$ and the ODE (4.2.11) satisfied by ϕ.

□

Note that $\phi(x) = x^{b_\lambda}$ when $w(x) = \frac{1}{x_1^2}$.

Theorem 4.2.9. *Let $\mu, \lambda \in \mathbb{N}$, w and ϕ be as in Lemma 4.2.8. Denote $g(x) := \frac{e^{-\mu x}}{\phi(x)}$ and suppose that claims with payoff $g(S_T)$ and maturity T are traded at price $C \equiv C^2 > 0$. For $H := \phi'(S)e^{-\lambda Q^w}$, define $\widehat{H}$ and $\widetilde{H}$ by (4.2.3) – (4.2.4). Then,*

$$e^{-\mu S_T - \lambda Q_T^w} = S_T \left(\int_0^T H_t \, \mathrm{d}C_t \right) + \int_0^T (\widehat{H}_t + \phi(S_0)) \, \mathrm{d}C_t + \int_0^T \widetilde{H}_t \, \mathrm{d}S_t. \tag{4.2.13}$$

under all continuous semimartingale models where $S > 0$ and $C > 0$ almost surely.

Proof. The proof is similar to that of Theorem 4.2.7, except that we use $\phi(S)$ instead of S^{b_λ} and (4.2.12) instead of (4.2.9).

□

4.2.2.3 Continuous joint payoffs of price and quadratic variation

Consider a claim with payoff $f(S_T, Q_T^w)$ where $f(x_1, x_2)$ is a continuous finite-valued function on $[0, \infty] \times [0, \infty]$. We can uniformly approximate this payoff using a finite basket of exponential claims with payoff $e^{-\mu S_T - \lambda Q_T^w}$ for $\mu, \lambda \in \mathbb{N}$ by the following method from Proposition 50 of [8]. First, define the function $\widetilde{f} : [0,1] \times [0,1] \to \mathbb{R}$ by

$$\widetilde{f}(x_1, x_2) := f(-\log(x_1), -\log(x_2)). \tag{4.2.14}$$

Since $\widetilde{f}$ is a continuous function defined on $[0,1] \times [0,1]$, it can be uniformly approximated by Bernstein polynomials, defined by

$$B_{n,m}(x_1, x_2) = \sum_{i=0}^{n} \sum_{j=0}^{m} \widetilde{f}\,(i/n, j/m) \binom{n}{i} \binom{m}{j} x_1^i (1 - x_1)^{n-i} x_2^j (1 - x_2)^{m-j}.$$

Letting

$$b_{n,m,k,l} = \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{k+l-i-j} \binom{n}{k} \binom{k}{i} \binom{m}{j} \binom{l}{j} \widetilde{f}\,(i/n, j/m)\,, \tag{4.2.15}$$

we get the representation $B_{n,m}(x_1, x_2) = \sum_{k=0}^{n} \sum_{l=0}^{m} b_{n,m,k,l}\; x_1^k x_2^l$. For $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that for any $n, m \geq N$, $\sup_{x_1, x_2 \in [0,1]} |\widetilde{f}(x_1, x_2) - B_{n,m}(x_1, x_2)| < \epsilon$, which implies that

$$|f(S_T, Q_T^w) - B_{n,m}(e^{-S_T}, e^{-Q_T^w})| = |\widetilde{f}(e^{-S_T}, e^{-Q_T^w}) - B_{n,m}(e^{-S_T}, e^{-Q_T^w})| < \epsilon.$$

Since $B_{n,m}(e^{-\mu S_T, } e^{-\lambda Q_T^w})$ is a linear combination of payoffs of the form $e^{-kS_T - lQ_T^w}$ for $k \in \{0, \dots, n\}$, $l \in \{0, \dots, m\}$, this proves that a finite basket of exponential payoffs can uniformly approximate the general continuous payoff. Each exponential payoff has the integral representation (4.2.13). Hence, this approximation yields a similar representation, up to arbitrary accuracy, for continuous payoffs defined on $[0, \infty] \times [0, \infty]$.

Remark 4.2.10. The key to the integral representation results in this subsection lies in having a function $F(x,q) = a(x)b(q)$ such that $a(x) > 0$ for $x > 0$ and $\mathrm{d}F(S_u, Q_u^w) = H_u \,\mathrm{d}S_u$ for an adapted process H. This gives $b(Q_T^w) = \frac{1}{a(S_T)}\left(a(S_0)g(Q_0^w) + \int_t^T H_u \,\mathrm{d}S_u\right)$. The identity (4.2.6) then allows incorporating the $\frac{1}{a(S_T)}$ factor into integrals with respect to prices of traded assets, with one of the integral terms being multiplied by S_T. In the above examples, b is set to the exponential functions, which form a basis for approximating more general functions by the method detailed in the subsection.

4.3 Finite variation approximations for continuous paths

This section proposes an alternative construction of probability-free integrals for continuous paths. In particular, we do not require the pathwise quadratic variation property. Instead, we define the integral via an approximating sequence of Riemann-Stieltjes integrals with integrands of finite variation. Convergence of the approximating integrals is shown to hold provided scaled truncated variation (defined by equation (4.3.1) below) of the integrator converges weakly to a measure. This limiting measure is shown to be well-defined for almost all sample paths of continuous semimartingales. Compared to the construction in Section 3.3, the advantage of this construction is that the approximating integrands minimise the variation of the integrand among the set of approximations of a given maximal distance from the original path. This is a desirable property if one considers transaction costs. Moreover, the construction does not depend on a sequence of partitions of time. It is more in line with the works of Bichteler [4] and Karandikar [34] in that the approximation is done on the vertical axis (representing the path values). On the other hand, this approach is quite limited in that it does not extend to integrals with $\mathbb{R}^d$-valued paths for $d > 1$ (i.e. to multivariate paths).

The key idea is to represent the original path A as $A^{(\epsilon)} + (A - A^{(\epsilon)})$, where $A^{(\epsilon)}$ is a path of finite variation and $D^{(\epsilon)} := A - A^{(\epsilon)}$ takes values in $[0, \epsilon]$ for some fixed $\epsilon > 0$. Our approach is motivated by the work in Łochowski [42], where the author sets $D^{(\epsilon)}$ to be the Skorokhod map of A on the interval $[-\epsilon, \epsilon]$ (see Kruk et al [38] for a discussion of Skorokhod maps on bounded intervals) and provides a pathwise construction of stochastic integrals. Note that the construction in [42] is not probability-free. We discuss this approximation and its properties in the following subsection.

4.3.1 Uniform approximation of continuous paths via Skorokhod maps

We now review the approximation method of continuous paths studied in detail by Łochowski [40, 41, 42]. For $A \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$, define the approximating path $A^{(\epsilon)} \in \mathcal{C}(\mathbb{R})$ as follows. Set $A^{(\epsilon)}(t) = \inf_{u\in[0,t]} A(u)$ up to the hitting time τ_1 of $D^{(\epsilon)}$ to ϵ, where we recall that $D^{(\epsilon)} := A - A^{(\epsilon)}$. Similarly, for $i \geq 2$, define

$$A^{(\epsilon)}(t) := \mathbb{1}_{\{D^{(\epsilon)}(\tau_i)=\epsilon\}}\left(\sup_{u\geq\tau_i} A(u) - \epsilon\right) + \mathbb{1}_{\{D^{(\epsilon)}(\tau_i)=0\}}\left(\inf_{u\geq\tau_i} A(u)\right), \quad t \in [\tau_i, \tau_{i+1}],$$
$$\tau_{i+1} := \inf\left\{t > \tau_i : D^{(\epsilon)}(t) = \epsilon\, \mathbb{1}_{\{D^{(\epsilon)}(\tau_i)=0\}}\right\}.$$

By construction, $A^{(\epsilon)}$ is at most ϵ away from A and is of finite variation. The difference $D^{(\epsilon)} = A - A^{(\epsilon)}$ is the Skorokhod map of A on the bounded interval $[0, \epsilon]$. Łochowski [42] uses the Skorokhod map on the interval $[-\epsilon, \epsilon]$, whereas we use the interval $[0, \epsilon]$ so as to reduce the notation in the integral construction we propose in this section.

Remark 4.3.1. Łochowski refers to $A^{(\epsilon)}$ as a lazy approximation of the original path since it only changes value when it has to in order to stay within a distance ϵ from A. In fact, $A^{(\epsilon)} + \epsilon/2$ has the least total variation among the uniform approximations of A of distance $\epsilon/2$, modulo the choice of the starting point of the approximation ([41]).

Since $A^{(\epsilon)}$ is of finite variation, it admits a Hahn-Jordan decomposition, notably $A^{(\epsilon)}(t) = A(0) + A_+^{(\epsilon)}(t) - A_-^{(\epsilon)}(t)$ for non-decreasing functions $A_+^{(\epsilon)}$ and $A_-^{(\epsilon)}$. Łochowski [41, 42] showed that for $t \geq 0$, $TV(A^{(\epsilon)}; t)$ is equal to the truncated variation $TV^{(\epsilon)}(t) \equiv TV_A^{(\epsilon)}(t)$ of A defined by

$$TV_A^{(\epsilon)}(t) := \sup_\pi \left\{\sum_{t_i\in\pi} ((|A(t_{i+1} \wedge t) - A(t_i \wedge t)| - \epsilon) \vee 0)\right\}, \quad t \geq 0, \tag{4.3.1}$$

where the supremum is taken over all partitions π of $[0, \infty)$. See [40] for a detailed account of truncated variation.

Consider $F \in C^2(\mathbb{R})$ and note that $\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u)$ may be defined by integration by parts since $A^{(\epsilon)}$ has finite variation, The following subsection will show that

$$\lim_{\epsilon\to 0} \int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u)$$

is well-defined provided the *scaled truncated variation* measure ν_ϵ on $([0, \infty); \mathcal{B}([0, \infty)))$ with distribution function $\epsilon TV^{(\epsilon)}$ converges to a Radon measure as $\epsilon \to 0$. This approximation sequence is similar in spirit to the approaches in Bichteler [4], Karandikar

[34] and more recently Nutz [49], except that the convergence result does not involve any probability measures and relies only on probability-free arguments (similar to the quadratic variation based constructions in Section 3.3). Furthermore, as we will show in Proposition 4.3.6, the scaled truncated variation measure of sample paths of continuous semimartingales converges almost surely to the quadratic variation $\langle A\rangle$, which provides a nice stochastic interpretation of the probability-free results.

To shorten the expressions of the results and proofs in this section, we introduce some additional notation. For $\epsilon > 0$, $A \in \mathcal{C}(\mathbb{R})$ and $t \geq 0$, denote $\rho(t)$ to be the last time up to t when $D^{(\epsilon)}$ was at either 0 or ϵ and denote $n(t)$ to be the number of times in $[0, t]$ that $D^{(\epsilon)}$ has crossed the interval $[0, \epsilon]$ in either direction (up or down). Hence,

$$\rho(t) := 0 \vee \sup\{u \leq t : D^{(\epsilon)}(u) \in \{0, \epsilon\}\}, \quad t \geq 0,$$
$$n(t) := 0 \vee \max\{i \geq 0 : \tau_i \leq t\}, \quad t \geq 0,$$

where the supremum and maximum of an empty set are equal to $-\infty$ by convention. By the continuity of A, there is only a finite number of $\tau_i \leq t$ for $t > 0$, hence $n(t) < \infty$ for finite $t > 0$. Within this section, we also denote $M \equiv M^A$ and $m \equiv m^A$ to be respectively the supremum and infimum of A since the last time $D^{(\epsilon)}$ was at 0 or ϵ, notably

$$M^A(t) := \sup_{u \in [\rho(t), t]} A(u), \quad t \geq 0,$$
$$m^A(t) := \inf_{u \in [\rho(t), t]} A(u), \quad t \geq 0.$$

With this notation, $A^{(\epsilon)}$ can be expressed in closed form by

$$A^{(\epsilon)}(t) = \mathbb{1}_{\{D^{(\epsilon)}(\rho(t))=0\}} m(t) + \mathbb{1}_{\{D^{(\epsilon)}(\rho(t))=\epsilon\}} (M(t) - \epsilon), \quad t \geq 0. \tag{4.3.2}$$

4.3.2 Integral construction by limit of Riemann-Stieltjes sums

We begin with the following result regarding the finite variation approximations $A^{(\epsilon)}$.

Proposition 4.3.2. *For $F \in C^1(\mathbb{R})$ such that F' is Lipschitz continuous and for $A \in \mathcal{C}(\mathbb{R})$ and $t \geq 0$, the Riemann-Stieltjes integral $\int_0^t F'(A^{(\epsilon)}(u)) \, \mathrm{d}A(u)$ is well-defined and equal to*

$$\begin{aligned}\int_0^t F'(A^{(\epsilon)}(u)) \, \mathrm{d}A(u) &= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) \\ &- \epsilon \sum_{i=1}^{\infty} \big(F'(A^{(\epsilon)}(\tau_{2i} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1} \wedge t)\big) + (A(t) - A^{(\epsilon)}(t)) F'(A^{(\epsilon)}(t)).\end{aligned} \tag{4.3.3}$$

Proof. By continuity of A and the finite variation property of $F'(A^{(\epsilon)})$, the Riemann-Stieltjes integral $\int_0^t A(u)\,\mathrm{d}F'(A^{(\epsilon)}(u))$ is well-defined and satisfies the integration by parts identity

$$\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) = A(t)F'(A^{(\epsilon)}(t)) - A(0)F'(A(0)) - \int_0^t A(u)\,\mathrm{d}F'(A^{(\epsilon)}(u)), \quad t \geq 0.$$

It remains to prove (4.3.3). The proof is based on first-order calculus arguments similar to the probability-free version of arguments used in proving the balayage formula (Theorem 4.2. in Revuz and Yor [52]). Let $i \geq 0$ and set $\tau_0 = 0$ by convention. By integration by parts,

$$\begin{aligned}(A(t) - m(t))F'(m(t)) &- (A(\tau_i) - m(\tau_i))F'(m(\tau_i)) \\ &= \int_{\tau_i}^t F'(m(u))\,\mathrm{d}(A - m)(u) + \int_{\tau_i}^t (A(u) - m(u))\,\mathrm{d}(F'(m(u))), \quad t \geq \tau_i.\end{aligned}$$

The second term on the left-hand-side is equal to zero by $m(\tau_i) = A(\tau_i)$, and the second term on the right-hand-side is equal to zero since m is constant on $\{m \neq A\}$. Hence, by linearity of Riemann-Stieltjes integrals with respect to the integrator and by $\int_{\tau_i}^t F'(m(u))\,\mathrm{d}m(u) = F(m(t)) - F(m(\tau_i))$,

$$(A(t) - m(t))F'(m(t)) = -(F(m(t)) - F(m(\tau_i))) + \int_{\tau_i}^t F'(m(u))\,\mathrm{d}A(u), \quad t \geq \tau_i,$$

or

$$\int_{\tau_i}^t F'(m(u))\,\mathrm{d}A(u) = F(m(t)) - F(m(\tau_i)) + (A(t) - m(t))F'(m(t)), \quad t \geq \tau_i. \tag{4.3.4}$$

Similarly,

$$\begin{aligned}&(M(t) - A(t))F'(M(t) - \epsilon) - (M(\tau_i) - A(\tau_i))F'(M(\tau_i) - \epsilon) \\ &= \int_{\tau_i}^t F'(M(u) - \epsilon)\,\mathrm{d}(M - A)(u) + \int_{\tau_i}^t (M(u) - A(u))\,\mathrm{d}(F'(M(u) - \epsilon)), \quad t \geq \tau_i,\end{aligned}$$

which gives

$$\int_{\tau_i}^t F'(M(u) - \epsilon)\,\mathrm{d}A(u) = F(M(t) - \epsilon) - F(M(\tau_i) - \epsilon) + (A(t) - M(t))F'(M(t) - \epsilon) \tag{4.3.5}$$

for $t \geq \tau_i$. If $D^{(\epsilon)}(\tau_i) = 0$, then $A^{(\epsilon)}(t) = A(\rho(t)) = m(t)$ for $t \in [\tau_i, \tau_{i+1})$, whereas if $D^{(\epsilon)}(\tau_i) = \epsilon$, then $A^{(\epsilon)}(t) = A(\rho(t)) - \epsilon = M(t) - \epsilon$ for $t \in [\tau_i, \tau_{i+1})$. Hence, (4.3.4) – (4.3.5) give

$$\int_{\tau_i}^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) = F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(\tau_i)) + (A(t) - A(\rho(t)))F'(A^{(\epsilon)}(t)) \tag{4.3.6}$$

for $t \in [\tau_i, \tau_{i+1})$. By continuity of $A^{(\epsilon)}$ and by $\lim_{t\uparrow\tau_{i+1}}(A(t) - A(\rho(t))) = (-1)^i\epsilon$, this implies

$$\lim_{t\uparrow\tau_{i+1}} \int_{\tau_i}^{t} F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) = F(A^{(\epsilon)}(\tau_{i+1})) - F(A^{(\epsilon)}(\tau_i)) + (-1)^i\epsilon F'(A^{(\epsilon)}(\tau_{i+1})). \tag{4.3.7}$$

By (4.3.6) – (4.3.7) and by

$$\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) = \sum_{i=0}^{\infty} \int_{\tau_i\wedge t}^{\tau_{i+1}\wedge t} F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u), \quad t \geq 0,$$

it follows that for $t \geq 0$,

$$\begin{aligned}
&\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) = \\
&= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \epsilon\left(\sum_{i=1}^{n(t)}(-1)^i F'(A^{(\epsilon)}(\tau_i))\right) + (A(t) - A(\rho(t)))F'(A^{(\epsilon)}(t)) \\
&= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \mathbb{1}_{\{n(t)\geq 2\}}\,\epsilon \sum_{i=1}^{\lfloor n(t)/2\rfloor} \big(F'(A^{(\epsilon)}(\tau_{2i})) - F'(A^{(\epsilon)}(\tau_{2i-1}))\big) \\
&\qquad + \mathbb{1}_{\{n(t)\ is\ odd\}}\,\epsilon\, F'(A^{(\epsilon)}(\tau_{n(t)})) + (A(t) - A(\rho(t)))F'(A^{(\epsilon)}(t)).
\end{aligned} \tag{4.3.8}$$

We conclude the proof by showing that (4.3.8) gives (4.3.3). If $D^{(\epsilon)}(\rho(t)) = 0$, then $n(t)$ is even, $A(\rho(t)) = A^{(\epsilon)}(t)$ and (4.3.8) gives

$$\begin{aligned}
&\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) = \\
&= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \mathbb{1}_{\{n(t)\geq 2\}}\,\epsilon \sum_{i=1}^{n(t)/2} \big(F'(A^{(\epsilon)}(\tau_{2i})) - F'(A^{(\epsilon)}(\tau_{2i-1}))\big) \\
&\qquad + (A(t) - A(\rho(t)))F'(A^{(\epsilon)}(t)) \\
&= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \epsilon \sum_{i=1}^{\infty} \big(F'(A^{(\epsilon)}(\tau_{2i}\wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1}\wedge t))\big) \\
&\qquad + (A(t) - A^{(\epsilon)}(t))F'(A^{(\epsilon)}(t)).
\end{aligned}$$

If $D^{(\epsilon)}(\rho(t)) = \epsilon$, then $n(t)$ is odd, $A(\rho(t)) = A^{(\epsilon)}(t) + \epsilon$, and (4.3.8) gives

$$\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) =$$

$$\begin{aligned}
&= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \mathbb{1}_{\{n(t)\geq 2\}}\,\epsilon \left(\sum_{i=1}^{(n(t)-1)/2} F'(A^{(\epsilon)}(\tau_{2i})) - F'(A^{(\epsilon)}(\tau_{2i-1})) \right) \\
&\qquad - \epsilon(F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(\tau_{n(t)}))) + (A(t) - A(\rho(t)) + \epsilon)F'(A^{(\epsilon)}(t)). \\
&= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \epsilon \left(\sum_{i=1}^{\infty} F'(A^{(\epsilon)}(\tau_{2i} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1} \wedge t)) \right) \\
&\qquad + (A(t) - A^{(\epsilon)}(t))F'(A^{(\epsilon)}(t)).
\end{aligned}$$

□

Corollary 4.3.3. *Consider $F \in C^1(\mathbb{R})$. If F' is non-decreasing, then*

$$\begin{aligned}
\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) &= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \frac{1}{2}\epsilon\, TV(F'(A^{(\epsilon)}); t) \\
&\quad - \frac{1}{2}\epsilon \left(F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(0))\right) + (A(t) - A^{(\epsilon)}(t))F'(A^{(\epsilon)}(t))
\end{aligned}$$

for $t \geq 0$. If F' is non-increasing, then

$$\begin{aligned}
\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) &= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) + \frac{1}{2}\epsilon\, TV(F'(A^{(\epsilon)}); t) \\
&\quad - \frac{1}{2}\epsilon \left(F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(0))\right) + (A(t) - A^{(\epsilon)}(t))F'(A^{(\epsilon)}(t))
\end{aligned}$$

for $t \geq 0$.

Proof. First, note that

$$\begin{aligned}
&F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(0)) \\
&= \sum_{i=0}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i+1} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i} \wedge t))\right) \\
&\qquad + \sum_{i=1}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1} \wedge t))\right).
\end{aligned}$$

By construction, $A^{(\epsilon)}$ is non-increasing on (τ_{2i}, τ_{2i+1}) for $i \geq 0$ and non-decreasing on

(τ_{2i-1}, τ_{2i}) for $i \geq 1$. If F' is non-decreasing, this implies that

$$
\begin{aligned}
&TV(F'(A^{(\epsilon)}); t) = \\
&= \sum_{i=1}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1} \wedge t))\right) \\
&\qquad\qquad - \sum_{i=0}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i+1} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i} \wedge t))\right) \\
&\Rightarrow F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(0)) + TV(F'(A^{(\epsilon)}); t) \\
&\qquad\qquad = 2\sum_{i=1}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1} \wedge t))\right).
\end{aligned}
$$

On the other hand, if F' is non-increasing, then

$$
\begin{aligned}
&TV(F'(A^{(\epsilon)}); t) \\
&= \sum_{i=0}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i+1} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i} \wedge t))\right) \\
&\qquad\qquad - \sum_{i=1}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1} \wedge t))\right) \\
&\Rightarrow F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(0)) - TV(F'(A^{(\epsilon)}); t) \\
&\qquad\qquad = 2\sum_{i=1}^{\infty} \left(F'(A^{(\epsilon)}(\tau_{2i} \wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1} \wedge t))\right).
\end{aligned}
$$

Using the above identities with (4.3.3) concludes the proof.

□

Remark 4.3.4. Since $TV(F'(A^{(\epsilon)}); t)$ defines the proportional transaction cost incurred by the trading strategy holding $F'(A^{(\epsilon)})$ units of A up to time t, the corollary could be of interest with regard to model-free applications under such transaction costs.

4.3.3 Convergence result

The following theorem is the main result of this section.

Theorem 4.3.5. *Let A be continuous and $F \in C^2(\mathbb{R})$. Then,*

$$
\begin{aligned}
&\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) \\
&= F(A^{(\epsilon)}(t)) - F(A^{(\epsilon)}(0)) - \frac{1}{2}\epsilon \int_0^t F''(A^{(\epsilon)}(u))\,\mathrm{d}TV^{(\epsilon)}(u) \qquad (4.3.9)\\
&\qquad - \frac{1}{2}\epsilon\left(F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(0))\right) + (A(t) - A^{(\epsilon)}(t))F'(A^{(\epsilon)}(t)), \quad t \geq 0.
\end{aligned}
$$

If the measure ν_ϵ on $([0,\infty),\mathcal{B}([0,\infty)))$ defined by $\mathrm{d}\nu_\epsilon(t) = \epsilon\,\mathrm{d}TV^{(\epsilon)}(t)$ converges vaguely to a Radon measure ν as $\epsilon \to 0$, then the pointwise limit

$$\lim_{\epsilon\to 0}\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u), \quad t \geq 0$$

is well-defined and equal to

$$\lim_{\epsilon\to 0}\int_0^t F'(A^{(\epsilon)}(u))\,\mathrm{d}A(u) = F(A(t)) - F(A(0)) - \frac{1}{2}\int_0^t F''(A(u))\,\mathrm{d}\nu(u). \quad (4.3.10)$$

If $\lim_{\epsilon\to 0}\epsilon TV^{(\epsilon)} = F_\nu$ in the topology of uniform convergence on compacts, then the above limit is also well-defined in the topology of uniform convergence on compacts.

Proof. Consider the Hahn-Jordan decomposition $A^{(\epsilon)} = A(0) + A_+^{(\epsilon)} - A_-^{(\epsilon)}$ of $A^{(\epsilon)}$ (A_+ and A_- are non-decreasing functions with $A_+(0) = A_-(0) = 0$) and note that $A_+^{(\epsilon)}$ and $A_-^{(\epsilon)}$ are respectively carried by $\{t \geq 0 : D^{(\epsilon)}(\rho(t)) = \epsilon\}$ and $\{t \geq 0 : D^{(\epsilon)}(\rho(t)) = 0\}$. Hence,

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left(F'(A^{(\epsilon)}(\tau_{2i}\wedge t)) - F'(A^{(\epsilon)}(\tau_{2i-1}\wedge t))\right)\\
&\qquad = \sum_{i=1}^{\infty}\int_{\tau_{2i-1}\wedge t}^{\tau_{2i}\wedge t} F''(A^{(\epsilon)}(u))\,\mathrm{d}A^{(\epsilon)}(u)\\
&\qquad = \int_0^t \mathbb{1}_{\{D(\rho(t))=\epsilon\}} F''(A^{(\epsilon)}(u))\,\mathrm{d}A^{(\epsilon)}(u)\\
&\qquad = \int_0^t F''(A^{(\epsilon)}(u))\,\mathrm{d}A_+^{(\epsilon)}(u)\\
&\qquad = \frac{1}{2}\left(\int_0^t F''(A^{(\epsilon)}(u))\,\mathrm{d}A^{(\epsilon)}(u) + \int_0^t F''(A^{(\epsilon)}(u))\,\mathrm{d}TV^{(\epsilon)}(u)\right)\\
&\qquad = \frac{1}{2}\left(F'(A^{(\epsilon)}(t)) - F'(A^{(\epsilon)}(0)) + \int_0^t F''(A^{(\epsilon)}(u))\,\mathrm{d}TV^{(\epsilon)}(u)\right),
\end{aligned}$$

where we used $\mathrm{d}A_+^{(\epsilon)} = \frac{1}{2}\,\mathrm{d}A^{(\epsilon)}(t) + \frac{1}{2}\,\mathrm{d}TV^{(\epsilon)}(t)$ for $t \geq 0$. Along with (4.3.3), this gives (4.3.9). The convergence claim follows by (4.3.9), the uniform convergence of $A^{(\epsilon)}$ to A (as $\epsilon \to 0$) and the continuity of $F''(x)$.

□

4.3.4 Semimartingale interpretation

Theorem 4.3.5 has the following interpretation within a stochastic setting where A is a continuous semimartingale.

Proposition 4.3.6. *Suppose A is a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Then, $\epsilon TV^{(\epsilon)}$ converges (under the u.c.p. topology) to the quadratic variation process $\langle A \rangle$ and $\int_0^t F'(A_u)\,\mathrm{d}A_u$ corresponds $\mathbb{P}$-a.e. to the usual stochastic integral. Furthermore, if $\{\epsilon_n\}_{n\geq 1}$ is a sequence of positive numbers such that $\sum_{i=1}^{\infty} \epsilon_n < \infty$ and if $\mathbb{E}\left(\langle A \rangle_t\right) < \infty$ for $t > 0$, then there is a version of A such that $\epsilon_n TV^{(\epsilon_n)}$ converges to $\langle A \rangle$ in the topology of uniform convergence on compacts, hence also in the weak topology.*

Proof. Note that the stochastic integral coincides with the Riemann-Stieltjes integral when the integrand is of finite variation. Also, $\lim_{\epsilon\to 0} \int_0^{\cdot} F'(A_u^{(\epsilon)})\,\mathrm{d}A_u = \int_0^{\cdot} F'(A_u)\,\mathrm{d}A_u$ in the u.c.p. topology follows from the uniform convergence of $A^{(\epsilon)}$ to A, continuity of $F'(x)$ and semimartingale convergence.

To prove that $\epsilon TV^{(\epsilon)}$ converges to $\langle A \rangle$ in the u.c.p. topology, define

$$\delta_t^{(\epsilon)} := (A_t^{(\epsilon)})^2 - A_t^2 + 2\int_0^t (A_u - A_u^{(\epsilon)})\,\mathrm{d}A_u - \epsilon(A_t^{(\epsilon)} - A_0) + 2(A_t - A_t^{(\epsilon)})A_t^{(\epsilon)}, \quad t \geq 0.$$

Semimartingale convergence and uniform convergence of $A^{(\epsilon)}$ to A imply u.c.p. convergence of $\delta^{(\epsilon)}$ to zero. By (4.3.9) applied with $F(x) = x^2$ and by $\langle A \rangle_{\cdot} = A_{\cdot}^2 - A_0^2 - \int_0^{\cdot} A_t\,\mathrm{d}A_t$, direct computation yields $\epsilon TV^{(\epsilon)} = \langle A \rangle + \delta^{(\epsilon)}$, which then implies u.c.p. convergence of $\epsilon TV^{(\epsilon)}$ to $\langle A \rangle$.

Suppose now that $\{\epsilon_n\}_{n\geq 1}$ is a sequence of positive numbers such that $\sum_{i=1}^{\infty} \epsilon_n < \infty$, which implies that $\sum_{i=1}^{\infty} \epsilon_n^2 < \infty$. Let $t > 0$. By similar arguments as in the proof of Proposition 3.4.1, Itô isometry, uniform convergence of $A^{(\epsilon)}$ to A and $\mathbb{E}\left(\langle A \rangle_t\right) < \infty$ yield $\mathbb{E}\left(\sum_{i=1}^{\infty} \sup_{u\in[0,t]} |\delta_u^{(\epsilon_n)}|\right) < \infty$. Hence, $\lim_{n\to\infty} \sup_{u\in[0,t]} \left|\epsilon_n TV_u^{(\epsilon_n)} - \langle A \rangle_u\right| = 0$ almost surely, which proves that A has a version for which $\epsilon_n TV^{(\epsilon_n)}$ converges uniformly on compacts to $\langle A \rangle$.

□

Appendix A

Auxiliary results and proofs for Chapter 1

Within this appendix, we use the shorthand notation $F_i \equiv \frac{\partial}{\partial x_i} F$ and similarly for higher order derivatives of a function F. We also sometimes write $F \equiv F(x)$ when there is no ambiguity regarding the input x to F.

Proof of Proposition 1.2.9. To simplify notation, denote $\mathcal{M} \equiv \mathcal{M}(a)$ and $\mathcal{R} \equiv \mathcal{R}(X; \mathcal{M}(a))$, and let τ_B to be the first hitting time of $X_{\cdot} \equiv X_{\cdot}[A]$ to a set B throughout the proof.

(i) Suppose $\mathcal{R}$ is not closed. Then there is a sequence $x_n \to x$ such that $x_n \in \mathcal{R}$ and $x \notin \mathcal{R}$. $x \notin \mathcal{R}$ implies that $\exists\, \epsilon > 0$ such that for any model in $\mathcal{M}$, $\tau_{B^c_\epsilon(x)} \geq T$ almost surely. Pick a point x_n such that $d(x_n, x) < \epsilon/2$. By $x_n \in \mathcal{R}$, there is a model $\{A; (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\} \in \mathcal{M}$ such that $\mathbb{P}(\tau_{B^c_{\epsilon/2}(x_n)} < T) > 0$. But $B^c_{\epsilon/2}(x_n) \subset B^c_\epsilon(x)$ implies that $\tau_{B^c_{\epsilon/2}(x_n)} \geq \tau_{B^c_\epsilon(x)}$, which yields a contradiction. This proves that $\mathcal{R}$ is closed.

(ii) We now prove an intermediate result which will be used in proving connectedness and the claim that X stays within $\mathcal{R}$ almost surely under all models in $\mathcal{M}$. We will show that for a closed set C, $C \cap \mathcal{R} = \emptyset$ implies that $\tau_C \geq T$ almost surely under all models in $\mathcal{M}$. Suppose that this is not true. Then, there is a model $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$ such that $\mathbb{P}(\tau_C < T) > 0$. By a countable partitioning of $C = \cup_{i=1}^\infty K_i$ where the K_i are compact, it follows that

$$\sum_{i=1}^{\infty} \mathbb{P}(\tau_{K_i} < T) \geq \mathbb{P}\left(\cup_{i=1}^\infty \{\tau_{K_i} < T\}\right) = \mathbb{P}\left(\tau_{\cup_{i=1}^\infty K_i} < T\right) = \mathbb{P}(\tau_C > 0) > 0.$$

Hence, there is a compact set $K \subseteq C$ such that $\mathbb{P}(\tau_K < T) > 0$. Since $K \cap \mathcal{R} = \emptyset$, for each $x \in K$ there is an $\epsilon(x) > 0$ such that $\mathbb{P}\left(\tau_{B^c_{\epsilon(x)}(x)} < T\right) = 0$. By compactness of K, there is a finite set $x_1, \dots, x_k \in K$ such that $K \subseteq \cup_{i=1}^k B^c_{\epsilon(x_i)}(x_i)$. Hence,

$$\mathbb{P}(\tau_K < T) \leq \mathbb{P}\left(\tau_{\cup_{i=1}^k B^c_{\epsilon(x_i)}(x_i)} < T\right) \leq \sum_{i=1}^k \mathbb{P}\left(\tau_{B^c_{\epsilon(x_i)}(x_i)} < T\right) = 0,$$

which yields a contradiction. Hence, $\tau_C \geq T$ almost surely under all models in $\mathcal{M}$.

(iii) Suppose that $\mathcal{R}$ is not connected. Then there are open sets $O_1, O_2 \subset \mathbb{R}^n$ in the relative topology of $\mathcal{R}$ such that $O_1 \cap O_2 = \emptyset$ and $\mathcal{R} \subseteq O_1 \cup O_2$. Let H be a hyperplane separating O_1 and O_2. Without loss of generality, assume that $X_0 \in O_1$. Let $x \in O_2 \cap \mathcal{R}$. $H \cap \mathcal{R} = \emptyset$ implies that $d(x, H) > 0$. Let $0 < \epsilon < d(x, H)$. Then, $X_0 \in O_1$ and continuity of paths of X imply that $\tau_H < \tau_{B^c_\epsilon(x)}$. On the other hand, $H \cap \mathcal{R} = \emptyset$ implies that $\tau_H \geq T$ almost surely under all models in $\mathcal{M}$ by part (ii) of the proof. Hence, $\tau_{B^c_\epsilon(x)} \geq T$ almost surely under all models in $\mathcal{M}$, which contradicts $x \in \mathcal{R}$. Hence, $\mathcal{R}$ is connected.

(iv) Let $\{A; (\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P})\} \in \mathcal{M}$. For $\epsilon > 0$, define $B^c_\epsilon(\mathcal{R}) = cl\left(\cup_{x \in \mathcal{R}} B^c_\epsilon(x)\right)$ and note that $\partial B^c_\epsilon(\mathcal{R}) \cap \mathcal{R} = \emptyset$. By part (ii) of the proof, it follows that $\mathbb{P}(\tau_{B^c_\epsilon(\mathcal{R})} < T) = 0$. Hence,

$$\mathbb{P}(X_t \in \mathcal{R},\ t \in [0, T)) = 1 - \mathbb{P}\left(\cup_{n=1}^\infty \left\{\tau_{B^c_{1/n}(\mathcal{R})} < T\right\}\right) \geq 1 - \sum_{n=1}^\infty \mathbb{P}\left(\tau_{B^c_{1/n}(\mathcal{R})} < T\right) = 1,$$

which concludes the proof.

□

Proof of Proposition 1.3.8. Let F^D be a solution to (1.3.9) and F^M be defined by (1.3.10). Abusing notation by denoting $y \equiv y(x)$, direct computation gives

$$\begin{aligned}
F_1^M(x) &= F_1^D(y) - 2(x_2 - x_1)F_2^D(y) \\
F_{11}^M(x) &= F_{11}^D(y) + 2F_2^D(y) - 4(x_2 - x_1)F_{12}^D(y) + 4(x_2 - x_1)^2 F_{22}^D(y) \\
F_2^M(x) &= 2(x_2 - x_1)F_2^D(y) \\
F_3^M(x) &= F_3^D(y).
\end{aligned}$$

for $x \in \widetilde{y}(\mathcal{D})$. Hence, $F_2^M = 0$ on $\{x_1 = x_2\} \cap \widetilde{y}(D)$ provided $\frac{\partial}{\partial y_2} F^D(y_1, y_2, y_3)$ is well-defined on $\mathcal{D} \cap \{y_2 = 0\}$, and

$$\begin{aligned}
F_3^M(x) + \frac{1}{2}F_{11}^M(x) &= F_3^D(y) + \frac{1}{2}F_{11}^D(y) + F_2^D(y) - (x_2 - x_1)F_{12}^D(y) + 2(x_2 - x_1)^2 F_{22}^D(y) \\
&= F_3^D(y) + \frac{1}{2}F_{11}^D(y) + F_2^D(y) - \sqrt{y_2}F_{12}^D(y) + 2y_2 F_{22}^D(y) = 0
\end{aligned}$$

for $x \in \widetilde{y}(D)$. Conversely, let $\widetilde{F}^M$ be a solution to (1.3.4) – (1.3.5) on $\widetilde{\mathcal{D}}$. Then, denoting $\widetilde{y} = \widetilde{y}(x)$, direct computation yields

$$\begin{aligned}
\widetilde{F}^D_1(x) &= \widetilde{F}^M_1(\widetilde{y}) + \widetilde{F}^M_2(\widetilde{y}) \\
\widetilde{F}^D_{11}(x) &= \widetilde{F}^M_{11}(\widetilde{y}) + 2F^M_{12}(\widetilde{y}) + \widetilde{F}^M_{22}(\widetilde{y}) \\
\widetilde{F}^D_2(x) &= \frac{1}{2\sqrt{x_2}}\widetilde{F}^M_2(\widetilde{y}) \\
\widetilde{F}^D_{12}(x) &= \frac{1}{2\sqrt{x_2}}\left(\widetilde{F}^M_{12}(\widetilde{y}) + \widetilde{F}^M_{22}(\widetilde{y})\right) \\
\widetilde{F}^D_{22}(x) &= -\frac{1}{4(x_2)^{3/2}}\widetilde{F}^M_2(\widetilde{y}) + \frac{1}{4x_2}\widetilde{F}^M_{22}(\widetilde{y}) \\
\widetilde{F}^D_3(x) &= \widetilde{F}^M_3(\widetilde{y})
\end{aligned}$$

for $x \in y(\widetilde{\mathcal{D}}) \setminus \{x_2 = 0\}$. Hence, for such x,

$$\begin{aligned}
&\widetilde{F}^D_3(x) + \widetilde{F}^D_2(x) + \frac{1}{2}\widetilde{F}^D_{11}(x) + 2x_2\widetilde{F}^D_{22}(x) - 2\sqrt{x_2}\widetilde{F}^D_{12}(x) \\
&= \widetilde{F}^M_3(\widetilde{y}) + \frac{1}{2\sqrt{x_2}}\widetilde{F}^M_2(\widetilde{y}) + \frac{1}{2}\left(\widetilde{F}^M_{11}(\widetilde{y}) + 2\widetilde{F}^M_{12}(\widetilde{y}) + \widetilde{F}^M_{22}(\widetilde{y})\right) \\
&\qquad + 2x_2\left(-\frac{1}{4(x_2)^{3/2}}\widetilde{F}^M_2(\widetilde{y}) + \frac{1}{4x_2}\widetilde{F}^M_{22}(\widetilde{y})\right) - 2\sqrt{x_2}\left(\frac{1}{2\sqrt{x_2}}\left(\widetilde{F}^M_{12}(\widetilde{y}) + \widetilde{F}^M_{22}(\widetilde{y})\right)\right) \\
&= \widetilde{F}^M_3(\widetilde{y}) + \frac{1}{2}\widetilde{F}^M_{11}(\widetilde{y}) = 0.
\end{aligned}$$

□

We now derive sufficient conditions of second order right-differentiability of $\widetilde{F}^D$ at $x_2 = 0$.

Lemma A.7. *Consider $\widetilde{F}^M$, $\widetilde{F}^D$, y, $\widetilde{y}$ and $\widetilde{\mathcal{D}}$ as in Proposition 1.3.8. If $\widetilde{F}^M \in C^{2,5,1}(\widetilde{\mathcal{D}} \cap \{x_1 = x_2\})$ and $\frac{\partial^3}{\partial x_2^3}\widetilde{F}^M = 0$ on $\widetilde{\mathcal{D}} \cap \{x_1 = x_2\}$, then there exists a well-defined right derivative of $\widetilde{F}^D$ at $x_2 = 0$, denoted by*

$$\widetilde{F}^D_{2+}(x_1, 0, x_3) := \lim_{\epsilon \to 0} \frac{\widetilde{F}^D(x_1, \epsilon, x_3) - \widetilde{F}^D(x_1, 0, x_3)}{\epsilon},$$

and it is equal to $\lim_{x_2 \to 0} \widetilde{F}^D_2(x_1, x_2, x_3)$ for $(x_1, 0, x_3) \in y(\widetilde{\mathcal{D}})$.

In the proof below, recall that we set $w = 1$ when referring to (1.3.4) and (1.3.9).

Proof of lemma. By the definition of $\widetilde{F}^D$,

$$\widetilde{F}^D_{2+}(x_1, 0, x_3) = \lim_{\epsilon \to 0} \frac{\widetilde{F}^M(x_1, x_1 + \sqrt{\epsilon}, x_3) - \widetilde{F}^M(x_1, x_1, x_3)}{\epsilon}. \tag{A.1}$$

Since $\widetilde{F}^M \in C^{2,2,1}(\widetilde{\mathcal{D}})$ and $\widetilde{F}_2^M(x_1, x_1, x_3) = 0$ for $(x_1, 0, x_3) \in y(\widetilde{\mathcal{D}})$, a second order Taylor expansion of $\widetilde{F}^M$ around (x_1, x_1, x_3) yields

$$\widetilde{F}^M(x_1, x_1 + \sqrt{\epsilon}, x_3) - \widetilde{F}^M(x_1, x_1, x_3) = \frac{1}{2}\epsilon \widetilde{F}_{22}^M(x_1, x_1 + \xi(\epsilon), x_3)$$

for some $\xi(\epsilon) \in [0, \sqrt{\epsilon}]$. Along with (A.1) and the continuity of F_{22}^M, this gives

$$\widetilde{F}_{2+}^D(x_1, 0, x_3) = \lim_{\epsilon \to 0} \frac{1}{2} \widetilde{F}_{22}^M(x_1, x_1 + \xi(\epsilon), x_3) = \frac{1}{2} \widetilde{F}_{22}^M(x_1, x_1, x_3)$$

for $(x_1, 0, x_3) \in y(\widetilde{\mathcal{D}})$. To conclude the proof, we need to show second order continuous right-hand differentiability of $\widetilde{F}^D$ in x_2 provided the assumption that $\widetilde{F}^M$ is $C^{2,5,1}(\widetilde{\mathcal{D}} \cap \{x_1 = x_2\})$ and $\frac{\partial^3 \widetilde{F}^M}{\partial x_2^3} = 0$ on $\widetilde{\mathcal{D}} \cap \{x_1 = x_2\})$. In particular, we need to show that

$$\widetilde{F}_{22+}^D(x_1, 0, x_3) := \lim_{\epsilon \to 0} \frac{\widetilde{F}_2^D(x_1, \epsilon, x_3) - \widetilde{F}_{2+}^D(x_1, 0, x_3)}{\epsilon}$$

is well-defined and coincides with $\lim_{x_2 \to 0} \widetilde{F}_{22}^D(x_1, x_2, x_3)$ for $(x_1, 0, x_3) \in y(\widetilde{\mathcal{D}})$. First, note that

$$\widetilde{F}_{22+}^D(x_1, 0, x_3) = \lim_{\epsilon \to 0} \frac{\frac{1}{2\sqrt{\epsilon}} \widetilde{F}_2^M(x_1, x_1 + \sqrt{\epsilon}, x_3) - \frac{1}{2} \widetilde{F}_{22}^M(x_1, x_1, x_3)}{\epsilon}. \tag{A.2}$$

By $\widetilde{F}^M \in C^{2,5,1}(\widetilde{\mathcal{D}} \cap \{x_1 = x_2\})$ and $\frac{\partial^3}{\partial x_2^3} \widetilde{F}^M(x_1, x_1, x_3) = 0$, a third order Taylor expansion of $\widetilde{F}_2^M$ yields

$$\widetilde{F}_2^M(x_1, x_1 + \sqrt{\epsilon}, x_3) - \sqrt{\epsilon} \widetilde{F}_{22}^M(x_1, x_1, x_3) = \frac{1}{6} \epsilon^{3/2} \frac{\partial^4}{\partial x_2^4} \widetilde{F}^M(x_1, x_1 + \gamma(\epsilon), x_3)$$

for some $\gamma(\epsilon) \in [0, \sqrt{\epsilon}]$. Hence, (A.2) and the continuity of $\frac{\partial^4}{\partial x_2^4} \widetilde{F}^M$ at (x_1, x_1, x_3) imply

$$\widetilde{F}_{22+}^D(x_1, 0, x_3) = \lim_{\epsilon \to 0} \left(\frac{1}{12} \frac{\partial^4}{\partial x_2^4} \widetilde{F}^M(x_1, x_1 + \xi(\epsilon), x_3) \right) = \frac{1}{12} \frac{\partial^4}{\partial x_2^4} \widetilde{F}^M(x_1, x_1, x_3).$$

On the other hand, for fixed $(x_1, 0, x_3) \in y(\widetilde{\mathcal{D}})$, define $G(\,\cdot\,; x_1, x_3) : \mathbb{R}_+ \to \mathbb{R}$ by

$$G(z\,; x_1, x_3) = z \left(\widetilde{F}_{22}^M(x_1, x_1 + z, x_3) \right) - \widetilde{F}_2^M(x_1, x_1 + z, x_3)$$

Then, by $\widetilde{F}^M \in C^{2,5,1}$,

$$\begin{aligned}
G'(z\,; x_1, x_3) &= z \left(\frac{\partial^3}{\partial x_2^3} \widetilde{F}^M(x_1, x_1 + z, x_3) \right), \\
G''(z\,; x_1, x_3) &= z \left(\frac{\partial^4}{\partial x_2^4} \widetilde{F}^M(x_1, x_1 + z, x_3) \right) + \frac{\partial^3}{\partial x_2^3} \widetilde{F}^M(x_1, x_1 + z, x_3), \\
G'''(z\,; x_1, x_3) &= z \left(\frac{\partial^5}{\partial x_2^5} \widetilde{F}^M(x_1, x_1 + z, x_3) \right) + 2 \frac{\partial^4}{\partial x_2^4} \widetilde{F}^M(x_1, x_1 + z, x_3).
\end{aligned}$$

Hence,

$$\lim_{x_2\to 0} \widetilde{F}^D_{22}(x_1,x_2,x_3) = \lim_{x_2\to 0} \frac{-\widetilde{F}^M_2(x_1,x_1+\sqrt{x_2},x_3)+\sqrt{x_2}\widetilde{F}^M_{22}(x_1,x_1+\sqrt{x_2},x_3)}{4(x_2)^{3/2}}$$
$$= \lim_{x_2\to 0} \frac{G\left(\sqrt{x_2}\,;x_1,x_3\right)}{4(x_2)^{3/2}}.$$

Note that $G(0\,;x_1,x_3) = G'(0\,;x_1,x_3) = G''(0\,;x_1,x_3) = 0$ since $\frac{\partial^i \widetilde{F}^M}{\partial x_2^i}(x_1,x_1,x_3) = 0$ for $i=1$ and $i=3$. Hence, a third order Taylor expansion of G around 0 yields

$$\begin{aligned}
&G\left(\sqrt{x_2}\,;x_1,x_3\right)\\
&= \frac{1}{6}(x_2)^{3/2}\left(\sqrt{x_2}\left(\frac{\partial^5}{\partial x_2^5}\widetilde{F}^M(x_1,x_1+\rho(x_2),x_3)\right)+2\frac{\partial^4}{\partial x_2^4}\widetilde{F}^M(x_1,x_1+\rho(x_2),x_3)\right)
\end{aligned}$$

for some $\rho(x_2)\in[0,\sqrt{x_2}]$. Since $\frac{\partial^4}{\partial x_2^4}\widetilde{F}^M$ is continuous at (x_1,x_1,x_3),

$$\begin{aligned}
&\lim_{x_2\to 0} \widetilde{F}^D_{22}(x_1,x_2,x_3)\\
&= \lim_{x_2\to 0}\left(\frac{1}{24}\left(\sqrt{x_2}\left(\frac{\partial^5}{\partial x_2^5}\widetilde{F}^M(x_1,x_1+\rho(x_2),x_3)\right)+2\frac{\partial^4}{\partial x_2^4}\widetilde{F}^M(x_1,x_1+\rho(x_2),x_3)\right)\right)\\
&= \frac{1}{12}\frac{\partial^4}{\partial x_2^4}\widetilde{F}^M(x_1,x_1,x_3)\\
&= \widetilde{F}^D_{22+}(x_1,0,x_3).
\end{aligned}$$

□

The following proposition shows that there is a stochastic solution to (1.3.4) – (1.3.5) with boundary condition

$$F(x_1,x_2,q) = f(x_1,x_2), \tag{A.3}$$

provided an implicit regularity condition on f. We do not claim this to be a contribution to the literature on stochastic solutions of parabolic PDEs, but provide the result for completeness.

Proposition A.8. *Let W be a Brownian motion on the canonical space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}) \equiv (\mathcal{C}(\mathbb{R}),\mathcal{F}^W,\mathbb{F}^W,\mathbb{P}^W)$. Let $M \equiv M^W$ be the running maximum of W. Suppose that $f:\mathbb{R}^2_+\to\mathbb{R}$ is such that*

$$F(x_1,x_2,x_3) := \mathbb{E}\left(f\left(x_1+W_{q-x_3}, x_2\vee(x_1+M_{q-x_3})\right)\right), \quad 0\le x_1\le x_2, \quad 0\le x_3\le q.$$

is $C^{2,1,1}$ on $\mathcal{D}=\{x_1\le x_2\}\cap\{0\le x_3<q\}\subset\mathbb{R}^3$ and $\int_0^{\cdot} F_1(W_t,M_t,t)\,\mathrm{d}W_t$ is a martingale. Then for $0\le t\le q$,

$$\mathbb{E}\left(f(x+W_q,x+M_q)\mid\mathcal{F}_t\right) = F(W_t,M_t,t) \quad \mathbb{P}-a.e., \tag{A.4}$$

and $F(x)$ is a solution to (1.3.4) – (1.3.5) *with boundary condition* (A.3).

Proof. The claim that $F(x)$ defined by (1.3.11) satisfies (A.4) follows from the strong Markov property of Brownian motion (Obłój and Yor [51]). The boundary condition (A.3) follows from (A.4) with $t = q$. To show that $F(x)$ is a solution to (1.3.4) provided the assumed regularity conditions on F, we resort to standard martingale arguments.

Since M is continuous and of finite variation, and since F was assumed to be $C^{2,1,1}$, Itô's formula gives

$$\begin{aligned} \mathrm{d}F(W_t, M_t, t) &= F_1 \,\mathrm{d}W_t + F_2 \,\mathrm{d}M_t + F_3 \,\mathrm{d}t + \frac{1}{2}F_{11} \,\mathrm{d}t \\ &= F_1 \,\mathrm{d}W_t + F_2 \,\mathrm{d}M_t + (F_3 + F_{11}) \,\mathrm{d}t. \end{aligned} \tag{A.5}$$

Formally defining $\int_0^{\cdot} \mathbb{1}_{\{W_u=M_u\}} H_u \,\mathrm{d}W_u := \lim_{\epsilon \downarrow 0} \int_0^{\cdot} I_{\{W_u > M_u - \epsilon\}} H_u \,\mathrm{d}W_u$, this implies that for $0 \le s < t \le q$,

$$\begin{aligned} &\int_s^t \mathbb{1}_{\{W_u=M_u\}} dF(W_u, M_u, u) \\ &= \int_s^t \mathbb{1}_{\{W_u=M_u\}} \, F_1 \,\mathrm{d}W_u + \int_s^t \mathbb{1}_{\{W_u=M_u\}} F_2 \,\mathrm{d}M_u + \int_s^t \mathbb{1}_{\{W_u=M_u\}} \left(F_3 + \frac{1}{2}F_{11} \right) \mathrm{d}u. \end{aligned}$$

The set $\{u \in [s,t] : W_u = M_u\}$ has Lebesgue measure zero $\mathbb{P}$-a.e., the third term on the second line disappears. Since M is carried by the set $\{W = M\}$, it follows that

$$F_2 \,\mathrm{d}M_u = F_2 \mathbb{1}_{\{W_u=M_u\}} \,\mathrm{d}M_u = \mathbb{1}_{\{W_u=M_u\}} dF(W_u, M_u, u) - \mathbb{1}_{\{W_u=M_u\}} \, F_1(Z_u) \,\mathrm{d}W_u.$$

Since $\int_0^{\cdot} F_1(W_t, M_t, t) \,\mathrm{d}W_t$ is a martingale by assumption and $\{F(W_t, M_t, t)\}_{t \in [0,T]}$ is a martingale by (A.4),

$$\begin{aligned} 0 &\le \mathbb{E}\left(\int_s^t \mathbb{1}_{\{F_2(W_u,M_u,u)>0\}} F_2 \,\mathrm{d}M_u \right) \\ &= \mathbb{E}\Bigg(\int_s^t \mathbb{1}_{\{F_2(W_u,M_u,u)>0\} \cap \{W_u=M_u\}} dF(W_u, M_u, u) \\ &\qquad\qquad + \int_s^t \mathbb{1}_{\{F_2(W_u,M_u,u)>0\} \cap \{W_u=M_u\}} F_1 \,\mathrm{d}W_u \Bigg) = 0. \end{aligned}$$

Hence,

$$\mathbb{E}\left(\int_s^t \mathbb{1}_{\{F_2(W_u,M_u,u)>0\}} F_2 \,\mathrm{d}M_u \right) = 0. \tag{A.6}$$

By a symmetric argument,

$$\mathbb{E}\left(\int_s^t \mathbb{1}_{\{F_2(W_u,M_u,u)<0\}} F_2 \,\mathrm{d}M_u \right) = 0. \tag{A.7}$$

Combining (A.6) and (A.7) yields $\mathbb{E}\left(\int_s^t |F_2(W_u, M_u, u)| \,\mathrm{d}M_u\right) = 0$. Since M is non-decreasing, this implies $\int_s^t |F_2(W_u, M_u, u)| \,\mathrm{d}M_u = 0$ $\mathbb{P}$-a.e.. Recalling that $W = M$ on points of increase of M, we get

$$\int_s^t |F_2(M_u, M_u, u)| \,\mathrm{d}M_u = 0, \qquad \mathbb{P} - a.e. \tag{A.8}$$

By assumption, $\widehat{F}(x,t) := |F_2(x,x,t)|$ is continuous, hence absolute continuity of the Lebesgue measure with respect to the law of M_u on $[0,\infty)$ for all $u \in (0,q]$ implies that $F_2(x,x,y) = 0$ for $(x,y) \in \mathbb{R}_+ \times (0,q]$, which yields (1.3.5). Combining this with (A.5) and the martingale property of W and of $\{F(W_t, M_t, t)\}_{t\in[0,q]}$, it follows that $\int_0^t (F_3(W_u, M_u, u) + \frac{1}{2}F_{11}(W_u, M_u, u)) \,\mathrm{d}u$ is a local martingale starting at 0. Since $F_3(x) + \frac{1}{2}F_{11}(x)$ is continuous by assumption, a standard stopping time argument yields (1.3.4).

□

Appendix B

Auxiliary results and proofs for Chapter 2

Proof of Lemma 2.2.6. The 'if' statement is trivial (set $\tau = t$). We hence consider the 'only if' statement. Suppose that S has full support, let I_τ be an $\mathbb{F}_T$-measurable open interval in $\mathbb{R}_+$ and consider a stopping time $\tau \in \mathcal{T}(\mathbb{F}_T)$.

If $\mathbb{P}(\tau < T) = 0$, then there is nothing to prove. If $\mathbb{P}(\tau < T) > 0$, consider any $\xi \in \mathcal{F}_\tau \cap \{\tau < T\}$ such that $\mathbb{P}(\xi) > 0$. $\xi \subseteq \{\tau < T\}$ implies that

$$\lim_{t \to T} \mathbb{P}(\xi \cap \{\tau < t\}) = P(\xi) > 0.$$

Hence, $\exists\, t \in (0, T)$ such that $\mathbb{P}(\xi \cap \{\tau < t\}) > 0$. Note that $\xi \cap \{\tau < t\} \in \mathcal{F}_t$ since $\xi \in \mathcal{F}_\tau$. By the full support property of S, $\mathbb{P}(S_T \in I_{t\wedge\tau} \,|\, \mathcal{F}_t) > 0$. It then follows that

$$\mathbb{P}(\{S_T \in I_\tau\} \cap \xi) \geq \mathbb{P}(\{S_T \in I_\tau\} \cap \xi \cap \{\tau < t\}) = \mathbb{P}(\{S_T \in I_{t\wedge\tau}\} \cap \xi \cap \{\tau < t\}) > 0,$$

which proves that $\mathbb{P}(S_T \in I_\tau \,|\, \mathcal{F}_\tau) > 0$ on $\{\tau < T\}$ as claimed.

□

Proof of Proposition 2.2.7. Without loss of generality, assume that $S_0 = 1$. Let $t \in [0, T)$ and let I be an $\mathbb{F}_T$-adapted open interval. Assume that $\mathbb{P}(\tau < T) > 0$, else there is nothing to prove. For any $t \in [0, T)$ and $\xi \in \mathcal{F}_t$ such that $\mathbb{P}(\xi \cap \{\tau < T\}) > 0$, we need to show that $\mathbb{P}(\{S_T \in I_t\} \cap \xi \cap \{\tau < T\}) > 0$. Without loss of generality, assume that $\tau \leq T$ (else use $\tau \wedge T$ in the proof).

Consider a space $(\Omega', \mathcal{F}', \mathbb{F}'_\infty, \mathbb{P}')$ which contains $(\Omega, \mathcal{F}, \mathbb{F}_\infty := \{\mathcal{F}_{t\wedge T}\}_{t\geq 0}, \mathbb{P})$ and such that $W' : \Omega' \times [0, \infty) \to \mathbb{R}$ is a Brownian motion independent of $\mathcal{F}$ defined on it. Define

$$
\begin{aligned}
Y_\cdot &:= \int_{\cdot\wedge\tau}^{\cdot} \sigma_u \,\mathrm{d}W_u,\\
Y'_\cdot &:= Y_{\cdot\wedge t} + W'_{\langle Y\rangle_\cdot} - W'_{\langle Y\rangle_{\cdot\wedge t}},\\
S'_\cdot &:= S_{\cdot\wedge\tau}\mathcal{E}(Y').
\end{aligned}
$$

Note that $S' = S$ on $[0,t]$. By a standard time change argument (with the starting filtration being $\mathcal{F}_t$ instead of the usual trivial filtration), the laws of S_T and S'_T conditional on $\mathcal{F}_t$ coincide. Let $\xi \in \mathcal{F}_t$. Then,

$$\mathbb{P}(\{S_T \in I_t\}\cap\xi\cap\{\tau < T\}) = \mathbb{P}'(S_T \in I_t\}\cap\xi\cap\{\tau < T\}) = \mathbb{P}'(\{S'_T \in I_t\}\cap\xi\cap\{\tau < T\}).$$

Denoting $\log I + c \equiv (\log(a)+c, \log(b)+c)$ for $I = (a,b)$, define the $\mathbb{F}_T$-adapted open interval

$$I' := \log I - \log S.$$

Then, since $Y' = \log S' + \frac{1}{2}\langle Y'\rangle$ and temporarily denoting $t' \equiv t \vee \tau$ and $\langle Y\rangle_{t',T} \equiv \langle Y\rangle_T - \langle Y\rangle_{t'}$,

$$
\begin{aligned}
&\mathbb{P}(\{S_T \in I_{t'}\} \cap \xi \cap \{\tau < T\})\\
&= \mathbb{P}'\left(\{\log S'_T - \log S'_{t'} + \frac{1}{2}\langle Y\rangle_{t',T} \in \log I_{t'} - \log S'_{t'} + \frac{1}{2}\langle Y\rangle_{t',T}\} \cap \xi \cap \{\tau < T\}\right)\\
&= \mathbb{P}'\left(\{Y'_T - Y'_{t'} \in I'_{t'} + \frac{1}{2}\langle Y\rangle_{t',T}\} \cap \xi \cap \{\tau < T\}\right)\\
&= \mathbb{P}'\left(\left\{W'_{\langle Y\rangle_{t'}} + \langle Y\rangle_{t',T} - W'_{\langle Y\rangle_{t'}} \in I'_{t'} + \frac{1}{2}\langle Y\rangle_{t',T}\right\} \cap \xi \cap \{\tau < T\}\right)\\
&\geq \mathbb{P}'\left(\left\{W'_{\langle Y\rangle_{t'}+\langle Y\rangle_{t',T}} - W'_{\langle Y\rangle_{t'}} \in I'_{t'} + \frac{1}{2}\langle Y\rangle_{t',T}\right\} \cap \{l(t') \leq \langle Y\rangle_{t',T} \leq u(t')\} \cap \xi \cap \{\tau < T\}\right)\\
&= \mathbb{E}\left(\mathbb{P}'\left(W'_{\langle Y\rangle_{t'}+\langle Y\rangle_{t',T}} - W'_{\langle Y\rangle_{t'}} \in I'_{t'} + \frac{1}{2}\langle Y\rangle_{t',T} \,\middle|\, \langle Y\rangle_{t',T}, \tau < T\right) \mathbb{1}_{\{l(t')\leq\langle Y\rangle_{t',T}\leq u(t')\}\cap\xi\cap\{\tau<T\}}\right)
\end{aligned}
$$

Note that

$$\mathbb{P}'\left(W'_{\langle Y\rangle_{t'}+\langle Y\rangle_{t',T}} - W'_{\langle Y\rangle_{t'}} \in I'_{t'} + \frac{1}{2}\langle Y\rangle_{t',T} \,\middle|\, \langle Y\rangle_{t',T}, \tau < T\right) > 0$$

on $\{l(t') \leq \langle Y\rangle_{t',T} \leq u(t')\}$ by independence of W' from $\mathcal{F}$. Also,

$$\mathbb{P}'\left(\{l(t') \leq \langle Y\rangle_{t',T} \leq u(t')\} \cap \xi \cap \{\tau < T\}\right) > 0$$

by $\mathbb{P}(l(t') \leq \int_{t'}^T \sigma_u^2 \,\mathrm{d}u \leq u(t') \,|\, \mathcal{F}_{t'}) > 0$ on $\{\tau < T\}$ and by $\mathbb{P}(\xi \cap \{\tau < T\}) > 0$. Hence,

$$\mathbb{P}(\{S_T \in I_{t'}\} \cap \xi \cap \{\tau < T\}) > 0,$$

which concludes the proof of $\mathbb{P}(S_T \in I_{t'} \,|\, \mathcal{F}_{t'}) > 0$ on $\{\tau < T\}$.

□

Appendix C

Auxiliary results and proofs for Chapter 3

Many of the proofs in Chapter 3 and in this appendix rely on the following lemmas. These lemmas may be folklore in real analysis, but we provide them along with the proofs for completeness.

Lemma C.1. *Let* $\{\mu_k\}_{k\geq 1}$ *be a sequence of Radon measures on* $([0,\infty),\mathcal{B}([0,\infty)))$ *converging vaguely to a Radon measure* μ*, and let* $\{f_k\}_{k\geq 1}$ *be a sequence of bounded* $\mathbb{R}$*-valued functions on* $[0,\infty)$ *converging uniformly on compacts to a continuous and bounded function* f*. Then, the sequence of measures* $\{\nu_k\}_{k\geq 1}$ *on* $([0,\infty),\mathcal{B}([0,\infty]))$ *given by*

$$\mathrm{d}\nu_k(t) = f_k(t)\,\mathrm{d}\mu_k(t)$$

converges vaguely to the measure ν *given by*

$$\mathrm{d}\nu(t) = f(t)\,\mathrm{d}\mu(t).$$

Proof. Let g be a continuous test function of compact support in $[0,\infty)$. Then, by the triangle inequality,

$$\begin{aligned}
&\limsup_{k\to\infty}\left|\int_0^T g(u)\,\mathrm{d}\nu_k(u) - \int_0^T g(u)\,\mathrm{d}\nu(u)\right| \\
&= \limsup_{k\to\infty}\left|\int_0^T g(u)f_k(u)\,\mathrm{d}\mu_k(u) - \int_0^T g(u)f(u)\,\mathrm{d}\mu(u)\right| \\
&\leq \limsup_{k\to\infty}\left|\int_0^T g(u)\left(f_k(u)-f(u)\right)\,\mathrm{d}\nu_k(u)\right| \\
&\qquad\qquad \limsup_{k\to\infty}\left|\int_0^T g(u)f(u)\,\mathrm{d}\nu_k(u) - \int_0^T g(u)f(u)\,\mathrm{d}\nu(u)\right|
\end{aligned}$$

The function gf has compact support in $[0,\infty)$ and is bounded and continuous. By vague convergence of ν_k to ν, this yields

$$\limsup_{k\to\infty}\left|\int_0^T g(u)f(u)\,\mathrm{d}\nu_k(u) - \int_0^T g(u)f(u)\,\mathrm{d}\nu(u)\right| = 0.$$

Since the ν_k are σ-finite measures converging vaguely to ν, their total variations as well as that of their limit are uniformly bounded on the compact support K_g of g by a constant $b_g > 0$. Hence,

$$\limsup_{k\to\infty}\left|\int_0^T g(u)\left(f_k(u) - f(u)\right)\,\mathrm{d}\nu_k(u)\right| \le ||g||_{K_g} b_g \limsup_{k\to\infty} ||f_k - f||_{K_g} = 0,$$

where $||\cdot||_{K_g}$ denotes the supremum of the absolute value of a function on K_g. Hence,

$$\lim_{k\to\infty}\left|\int_0^T g(u)\,\mathrm{d}\nu_k(u) - \int_0^T g(u)\,\mathrm{d}\nu(u)\right| = 0$$

for any continuous function g of compact support, which proves vague convergence of ν_k to ν.

□

Lemma C.2. *Consider a continuous function $f \in \mathcal{C}(\mathbb{R})$ and consider the sequence of measures on $([0,\infty),\mathcal{B}([0,\infty)))$ defined by*

$$\nu_k = \sum_{i=1}^{m_k} \Delta f(t_i^k)\delta_{t_i^k}, \quad k \ge 1.$$

The following are equivalent:

1. *The sequence of measures ν_k converges to the trivial measure.*
2. *All of the measures ν_k, $k \ge 1$ are equal to the trivial measure.*
3. *f is constant on $[0,\infty)$.*

Proof. The claims (2) $\Leftrightarrow$ (3) as well as (3) $\Rightarrow$ (1) are trivial.

Suppose that (1) holds and let $t > 0$. By vague convergence of ν_k to the trivial measure, it follows that $\lim_{k\to\infty}\nu_k([0,t]) = 0$. By the definition of ν_k, this yields $\lim_{k\to\infty} f(t^k_{i^k(t)+1}) - f(0) = 0$. Continuity of f and $||\pi_k|| \to 0$ then imply that $f(t) = f(0)$, which proves the claim (1) $\Rightarrow$ (3) and concludes the proof of the lemma.

□

Proof of Lemma 3.2.4. Suppose $H \in \mathcal{I}(A)$. Then, setting $Z = I$ and ν to be the trivial measure with distribution function $F_\nu = 0$ yields the "only if" statement.

To prove the "if" statement, suppose that there is a path $Z : [0, \infty) \to \mathbb{R}$ such that the sequence of measures on $([0, \infty), \mathcal{B}([0, \infty)))$ defined by

$$\nu_k^{(Z)} := \sum_{i=0}^{m_k - 1} \left(\Delta Z(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right) \delta_{t_i^k}$$

converges vaguely to a Radon measure ν with distribution function V. To prove that $H \in \mathcal{I}(A)$ and that the integral satisfies $Z(t) = Z(0) + I[H; A](t) + V(t)$ for $t \geq 0$, it is sufficient to show that for the continuous path

$$I(t) := Z(t) - Z(0) - V(t), \quad t \geq 0,$$

the sequence of measures $([0, \infty), \mathcal{B}([0, \infty)))$ defined by

$$\nu_k^{(I)} := \left(\Delta I(t_i^k) - H(t_i^k)\Delta A(t_i^k)\right) \delta_{t_i^k}, \quad k \geq 1$$

converges to the trivial measure. Note that

$$\nu_k^{(I)} = \nu_k^{(Z)} - \sum_{i=0}^{m_k - 1} \Delta V(t_i^k)\, \delta_{t_i^k}, \quad k \geq 1.$$

Let g be a continuous test function of compact support in $[0, \infty)$. Then,

$$\lim_{k \to \infty} \int_0^\infty g(t)\, \mathrm{d}\nu_k^{(Z)}(t) = \int_0^\infty g(t)\, \mathrm{d}\nu(t)$$

by vague convergence of $\nu_k^{(Z)}$ to ν, and

$$\lim_{k \to \infty} \left(\sum_{i=0}^{m_k - 1} g(t_i^k)\Delta V(t_i^k)\right) = \int_0^\infty g(t)\, \mathrm{d}V(t) = \int_0^\infty g(t)\, \mathrm{d}\nu(t),$$

where the left-hand equality holds by Riemann-Stieltjes integration. Combining the above identities,

$$\lim_{k \to \infty} \int_0^\infty g(t)\, \mathrm{d}\nu_k^{(I)}(t) = \lim_{k \to \infty} \left(\int_0^\infty g(t)\, \mathrm{d}\nu_k^{(Z)}(t) - \sum_{i=0}^{m_k - 1} g(t_i^k)\Delta V(t_i^k)\right) = 0.$$

Hence, $\nu_k^{(I)}$ converges to the trivial measure as claimed, which concludes the proof.

□

Proof of Lemma 3.2.5. We prove each part of the claim separately.

Part (a):

Suppose that $A \in \mathcal{V}(\mathbb{R}^d)$ and $H \in \mathcal{C}(\mathbb{R}^d)$. By the finite variation property of A, we can define the Riemann-Stieltjes integral

$$I_{RS}(t) := \int_0^t H(u)\,\mathrm{d}A(u), \quad t \geq 0$$

pointwise in t. Note that $I_{RS} \in \mathcal{C}(\mathbb{R})$ and

$$TV(I_{RS}; t) \leq \left(\sup_{u \in [0,t]} |H(u)| \right) TV(A; t)$$

In particular, I_{RS} has finite variation on compacts. To show that H is A-integrable in the sense of Definition 3.2.1, consider the sequence of measures on $([0,\infty), \mathcal{B}([0,\infty)))$ defined by

$$\nu_k := \sum_{i=0}^{m_k-1} \left(\Delta I_{RS}(t_i^k) - H(t_i^k)\Delta A(t_i^k) \right) \delta_{t_i^k}, \quad k \geq 1.$$

For any continuous test function g of compact support,

$$\begin{aligned} \int_0^\infty g(t)\,\mathrm{d}\nu_k(t) &= \sum_{i=0}^{m_k-1} g(t_i^k) \int_{t_i^k}^{t_{i+1}^k} H(u)\,\mathrm{d}A(u) - \sum_{i=1}^{m_k-1} g(t_i^k) H(t_i^k) \Delta A(t_i^k) \\ &= \int_0^\infty g(t^k_{i_k(t)})(H(t) - H(t^k_{i_k(t)}))\,\mathrm{d}A(t) \end{aligned}$$

By continuity of H and by continuity and compactness of support of g, dominated convergence gives $\lim_{k\to\infty} \int_0^\infty g(t)\,\mathrm{d}\nu_k(t) = 0$, which implies that ν_k converges to the trivial measure. Hence, H is A-integrable and $I[H; A] = I_{RS}$.

Part (b):

Suppose now that $A \in \mathcal{C}(\mathbb{R}^d)$ and that $H \in \mathcal{V}(\mathbb{R}^d)$. Define $Z = HA$. Note that $Z \in \mathcal{C}(\mathbb{R})$ and that, by the first part of the proof, $I[A; H] \in \mathcal{V}(\mathbb{R})$. By Lemma 3.2.4, H is A-integrable and $I[H; A] = Z - Z(0) - I[A; H]$ if the sequence of measures on $([0,\infty), \mathcal{B}([0,\infty)))$ defined by

$$\nu_k^{(Z)} := \sum_{i=0}^{m_k-1} \left(\Delta Z(t_i^k) - H(t_i^k)\Delta A(t_i^k) \right) \delta_{t_i^k}, \quad k \geq 1$$

converges vaguely to the Radon measure ν with distribution function $I[A; H]$. Clearly, the pointwise version of this result corresponds to the standard integration by parts identity, but we justify the measure-theoretic version of the result for completeness. First, observe that simple arithmetic gives the identity

$$\begin{aligned}\Delta(HP)(t_i^k) &= H(t_{i+1}^k)A(t_{i+1}^k) - H(t_i^k)A(t_i^k)\\ &= H(t_i^k)\left(A(t_{i+1}^k) - A(t_i^k)\right) + A(t_i^k)\left(H(t_{i+1}^k) - H(t_i^k)\right)\\ &\qquad\qquad + \left(H(t_{i+1}^k) - H(t_i^k)\right)\left(A(t_{i+1}^k) - A(t_i^k)\right)\\ &= H(t_i^k)\Delta A(t_i^k) + A(t_i^k)\Delta H(t_i^k) + \Delta H(t_i^k)\Delta A(t_i^k)\end{aligned}$$

for $i \in \{0, \dots, m_k - 1\}$ and $k \geq 1$. Hence,

$$\nu_k^{(Z)} = \sum_{i=0}^{m_k-1} \left(A(t_i^k)\Delta H(t_i^k) + \Delta A(t_i^k)\Delta H(t_i^k)\right)\delta_{t_i^k}, \quad k \geq 1.$$

For any continuous test function g of compact support, Riemann-Stieltjes integration gives

$$\lim_{k\to\infty} \sum_{i=0}^{m_k-1} g(t_i^k)A(t_i^k)\Delta H(t_i^k) = \int_0^\infty g(t)A(t)\,\mathrm{d}H(t) = \int_0^\infty g(t)\,\mathrm{d}\nu(t)$$

Also, $\lim_{k\to\infty}\sum_{i=0}^{m_k-1} g(t_i^k)\Delta A(t_i^k)\Delta H(t_i^k) = 0$ by the continuity of A, finite variation of H and compactness of support of g. Hence,

$$\lim_{k\to\infty}\int_0^\infty g(t)\,\mathrm{d}\nu_k^{(Z)}(t) = \int_0^\infty g(t)\,\mathrm{d}\nu(t).$$

Hence, $\nu_k^{(Z)}$ converges vaguely to ν. By Lemma 3.2.4, this proves the claim in (b) as well as the integration by parts identity for forward integrals.

$\square$

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