

# Symmetries of two-point sets

Ben Chad <sup>a,\*</sup>, Rolf Suabedissen <sup>b</sup>

<sup>a</sup> *St Edmund Hall, Oxford, OX1 4AR, UK*

<sup>b</sup> *Lady Margaret Hall, Oxford, OX2 6QA, UK*

Received 1 May 2007; received in revised form 27 February 2008; accepted 27 February 2008

---

## Abstract

A two-point set is a subset of the plane which meets every planar line in exactly two-points. We discuss the problem “What are the topological symmetries of a two-point set?”. Our main results assert the existence of two-point sets which are rigid and the existence of two-point sets which are invariant under the action of certain autohomeomorphism groups. We pay particular attention to the isometry group of a two-point set, and show that such groups consist only of rotations and that they may be chosen to be any subgroup of  $S^1$  having size less than  $c$ . We also construct a subgroup of  $S^1$  having size  $c$  that is contained in the isometry group of a two-point set.

© 2008 Elsevier B.V. All rights reserved.

MSC: 54G99; 54H15

Keywords: Two-point set; Rigid; Isometry group

---

## 1. Introduction

A subset of the plane is said to be a *two-point set* if and only if it meets every planar line in exactly two points and is said to be a *partial two-point set* if and only if it meets every planar line in at most two points. The existence of two-point sets was shown by Mazurkiewicz [7]. (A French translation is available in [8].)

The discussion by Mauldin [5] for Open Problems in Topology [9] gives a good indication of the problems concerning two-point sets which have subsequently been studied in mathematical literature. Of Mauldin’s three problems, two have been answered positively and one remains open. It was shown by Kulesza [3] that a two-point set must be zero-dimensional and it was shown by R. Dougherty, by Dijkstra and van Mill [2] and by Mauldin [6] that a compact zero-dimensional partial two-point set cannot always be extended to a two-point set. The remaining question is to determine if there exists a two-point set which is a Borel subset of the plane, although it was known by a proof of Larman [4] (with corrections by Baston and Bostock [1]) prior to Mauldin’s article that such a set could not be  $F_\sigma$ .

The definition of a two-point set requires only elementary terms and can be understood by someone with minimal mathematical training. So far, efforts to determine if a two-point set can be effectively described, for example as

---

\* Corresponding author.

E-mail address: [chad@maths.ox.ac.uk](mailto:chad@maths.ox.ac.uk) (B. Chad).

<sup>1</sup> The author thanks the Wingate Foundation for its financial support.

a Borel, analytic or projective set, have not returned many satisfactory results. In an attempt to better “understand” a two-point set, we undertake an initial investigation of the symmetry groups which they admit. Our motivating question is: What are the topological symmetries of a two-point set? Although it is not precise, this question immediately gives rise to ones which are.

In Theorem 2 we construct a rigid two-point set. We then proceed to show in Proposition 4 that the isometry group of any two-point set must consist of rotations only. Using a general method of construction established in Theorem 3, we proceed in Theorem 6 to show that any group of planar rotations about the origin with fewer than  $\mathfrak{c}$  many elements occurs as the isometry group of a two-point set. Using more refined techniques, Theorem 8 shows how to construct a two-point set which has  $\mathfrak{c}$  many isometries.

Unless stated otherwise, we use the notation and terminology of Willard [10] for topological concepts and we use the variables  $\alpha$ ,  $\beta$  and  $\gamma$  to range over ordinals.

The following proof is essentially due to Mazurkiewicz [7]. We have included it for later reference.

**Theorem 1.** *There exists a two-point set.*

**Proof.** Let  $\mathcal{L}$  denote the collection of all lines in the plane and let  $(L_\alpha)_{\alpha < \mathfrak{c}}$  be an enumeration of  $\mathcal{L}$ . For some  $\alpha < \mathfrak{c}$ , suppose that we have chosen a sequence  $(X_\beta)_{\beta < \alpha}$  of subsets of the plane such that:

- (1)  $|X_\beta| \leq 2$  for all  $\beta < \alpha$ ; and
- (2)  $\bigcup_{\beta < \alpha} X_\beta$  meets each member of  $\{L_\beta: \beta < \alpha\}$  in exactly two points; and
- (3)  $\bigcup_{\beta < \alpha} X_\beta$  is a partial two-point set.

Let

$$\mathcal{L}_\alpha^2 = \left\{ L \in \mathcal{L}: \left| L \cap \bigcup_{\beta < \alpha} X_\beta \right| = 2 \right\},$$

and let  $X_\alpha \subseteq L_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta$  be chosen such that  $|L_\alpha \cap \bigcup_{\beta < \alpha} X_\beta| = 2$ . The induction hypothesis has been preserved, so we now define the  $X_\alpha$  for all  $\alpha < \mathfrak{c}$  and let  $X = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$ . Then  $X$  is a two-point set.  $\square$

We remark that in this paper, the proof of every theorem of the form “There exists a two-point set such that ...” actually demonstrates the existence of  $2^{\mathfrak{c}}$  such two-point sets. The informal reasoning is simple: our processes for constructing two-point sets are such that we make a choice from  $\mathfrak{c}$  many possibilities  $\mathfrak{c}$  many times and each potential choice is only offered a limited number of times.

## 2. The autohomeomorphism group of a two-point set

This section, which consists of two main results, introduces techniques which we employ throughout this paper. Firstly, we will see that one particular answer to our question “What are the topological symmetries of a two-point set?” is “No more than need be!” by constructing a rigid two-point set. This seems the obvious place to start, for such a two-point set has a trivial autohomeomorphism group. Our second result is more abstract, and is intended as a tool in answering the question “Do there exist two-point sets which are invariant under non-trivial autohomeomorphisms?”. We show that given the existence of a suitable planar group action, the answer is “yes”. Subsequent sections will show the existence of such actions.

We now construct our rigid two-point set. We note that in private communication, Jan van Mill announced to have independently shown the following result.

**Theorem 2.** *There exists a rigid two-point set.*

**Proof.** We modify the construction given in the proof of Theorem 1. Let  $(f_\alpha)_{\alpha < \mathfrak{c}}$  enumerate all partial functions  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that:

- (a)  $f$  is not an identity function; and

- (b)  $\text{dom}(f) \subseteq \mathbb{R}^2$  is a  $G_\delta$  and has the property that  $\{x \in \text{dom}(f): f(x) \neq x\}$  cannot be covered by fewer than  $\mathfrak{c}$  many lines; and
- (c)  $f$  is a homeomorphism onto its image.

Let  $\mathcal{B}$  be a countable basis for the plane and let  $(U_\alpha)_{\alpha < \mathfrak{c}}$  be a sequence on  $\mathcal{B}$  which lists every member of  $\mathcal{B}$  precisely  $\mathfrak{c}$  many times. For some  $\alpha < \mathfrak{c}$ , suppose that we have chosen a sequence  $(X_\beta)_{\beta < \alpha}$  of subsets of the plane such that:

- (1)  $2 \leq |X_\beta| \leq 4$  for all  $\beta < \alpha$ ; and
- (2)  $\bigcup_{\beta < \alpha} X_\beta$  meets each member of  $\{L_\beta: \beta < \alpha\}$  in exactly two points; and
- (3)  $\bigcup_{\beta < \alpha} X_\beta$  is a partial two-point set.

Note that conditions (2) and (3) are the same as in the proof of Theorem 1. Suppose additionally that we have chosen sequences  $(x_\beta)_{\beta < \alpha}$  and  $(y_\beta)_{\beta < \alpha}$  on  $\mathbb{R}^2$  such that:

- (4)  $x_\beta, y_\beta \in X_\beta$  for each  $\beta < \alpha$ ; and
- (5)  $\bigcup_{\beta < \alpha} X_\beta \cap \{f_\beta(x_\beta): \beta < \alpha\} = \emptyset$ ; and
- (6)  $(y_\beta)_{\beta < \alpha}$  is injective and  $y_\beta \in U_\beta$  for each  $\beta < \alpha$ .

If  $L_\alpha \in \mathcal{L}_\alpha^2$  then let  $X'_\alpha = \emptyset$ . Otherwise, it can be seen that  $|L_\alpha \setminus (\bigcup \mathcal{L}_\alpha^2 \cup \{f_\beta(x_\beta): \beta < \alpha\})| = \mathfrak{c}$  and so as in the proof of Theorem 1, we let  $X'_\alpha \subseteq L_\alpha \setminus (\bigcup \mathcal{L}_\alpha^2 \cup \{f_\beta(x_\beta): \beta < \alpha\})$  be chosen such that  $|L_\alpha \cap (\bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha)| = 2$ . Letting

$$\mathcal{M}_\alpha^2 = \left\{ L \in \mathcal{L}: \left| L \cap \left( \bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha \right) \right| = 2 \right\},$$

we see that

$$\left| \{x \in \text{dom}(f_\alpha): f_\alpha(x) \neq x\} \setminus \bigcup \mathcal{M}_\alpha^2 \right| = \mathfrak{c},$$

and so we choose  $x_\alpha$  such that

$$x_\alpha \in \{x \in \text{dom}(f_\alpha): f_\alpha(x) \neq x\} \setminus \left( \bigcup \mathcal{M}_\alpha^2 \cup \{f_\beta(x_\beta): \beta < \alpha\} \cup f_\alpha^{-1} \left( \bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha \right) \right).$$

It is easy to see that an open ball in the plane cannot be covered by fewer than  $\mathfrak{c}$  many lines. Hence we let

$$\mathcal{N}_\alpha^2 = \left\{ L \in \mathcal{L}: \left| L \cap \left( \bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha \cup \{x_\alpha\} \right) \right| = 2 \right\},$$

choose  $y_\alpha$  such that

$$y_\alpha \in U_\alpha \setminus \left( \bigcup \mathcal{N}_\alpha^2 \cup \{f_\beta(x_\beta): \beta \leq \alpha\} \right),$$

and let  $X_\alpha = X'_\alpha \cup \{x_\alpha, y_\alpha\}$ . The induction hypothesis have been preserved and so we now define the  $X_\alpha$  and  $x_\alpha$  and  $y_\alpha$  for all  $\alpha < \mathfrak{c}$  and let  $X = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$ . Then  $X$  is a two-point set which meets every non-empty open subset of the plane in  $\mathfrak{c}$  many points and is such that  $X \cap \{f_\alpha(x_\alpha): \alpha < \mathfrak{c}\} = \emptyset$ .

Suppose that  $f: X \rightarrow X$  is a non-trivial autohomeomorphism. Then  $\{x \in \text{dom}(f): f(x) \neq x\}$  is an open subset of  $X$  of size  $\mathfrak{c}$  which cannot be covered by fewer than  $\mathfrak{c}$  many lines. By an application of Lavrentieff's Theorem we can extend  $f$  to some  $f_\alpha$ , which is a contradiction.  $\square$

The following theorem gives sufficient conditions on a planar group action for there to exist a two-point set which it leaves invariant. Although it concerns an abstract group action, its hypotheses have been chosen to satisfy the properties of actions we will consider in subsequent sections.

For distinct points  $x, y \in \mathbb{R}^2$  we let  $\langle x, y \rangle$  denote the line spanned by  $x$  and  $y$ .

**Theorem 3.** *Let  $G$  be a group with identity denoted by  $e$  which acts on the plane via affine transformations and is such that  $|G| < \mathfrak{c}$ . Suppose that there exists  $E \subseteq \mathbb{R}^2$  and  $\kappa < \mathfrak{c}$  such that:*

- (a)  $|E| < \mathfrak{c}$  and  $GE = E$  and each point of  $\mathbb{R}^2 \setminus E$  has a trivial  $G$ -stabiliser group; and
- (b) the  $G$ -orbit of each  $x \in \mathbb{R}^2 \setminus E$  meets every line in at most two points; and
- (c) for each  $y \in \mathbb{R}^2 \setminus E$ , for each line  $L$  and for each  $g \in G \setminus \{e\}$ ,  $|\{x \in L: y \in \langle x, gx \rangle\}| \leq \kappa$ .

Then there exists a two-point set invariant under the action of  $G$ .

**Proof.** For each  $x \in \mathbb{R}^2 \setminus E$ , let  $O(x)$  denote the  $G$ -orbit of  $x$ . We will modify the construction given in the proof of Theorem 1. Again, suppose we are at stage  $\alpha$  of the recursion, and that we have constructed a sequence  $(X_\beta)_{\beta < \alpha}$  of subsets of the plane such that:

- (1)  $|X_\beta| \leq |G| < \mathfrak{c}$  for all  $\beta < \alpha$ ; and
- (2)  $\bigcup_{\beta < \alpha} X_\beta$  meets each member of  $\{L_\beta: \beta < \alpha\}$  in exactly two points; and
- (3)  $\bigcup_{\beta < \alpha} X_\beta$  is a partial two-point set; and
- (4)  $\bigcup_{\beta < \alpha} X_\beta \subseteq \mathbb{R}^2 \setminus E$  is  $G$ -invariant.

If  $L_\alpha \in \mathcal{L}_\alpha^2$  then as usual we set  $X_\alpha = \emptyset$ . Otherwise, let  $P = \bigcup_{\beta < \alpha} X_\beta$  be our partially constructed two-point set and note that  $\{\mathcal{L}_\alpha^0, \mathcal{L}_\alpha^1, \mathcal{L}_\alpha^2\}$  is a partition of  $\mathcal{L}$  into sets invariant under  $G$ , where for  $i = 0, 1, 2$  we let  $\mathcal{L}_\alpha^i = \{L \in \mathcal{L}: |L \cap P| = i\}$ .

Since both  $|G| < \mathfrak{c}$  and  $|P| \leq |\alpha| \cdot |G| < \mathfrak{c}$ , we see that

$$\left| \bigcup_{g \in G \setminus \{e\}} \{x \in L_\alpha \setminus E: \langle x, gx \rangle \cap P \neq \emptyset\} \right| < \mathfrak{c}.$$

Recall that  $|\mathcal{L}_\alpha^2| < \mathfrak{c}$  and  $L_\alpha \notin \mathcal{L}_\alpha^2$ . Then

$$L_\alpha \setminus \left( \bigcup_{g \in G \setminus \{e\}} \{x \in L_\alpha \setminus E: \langle x, gx \rangle \cap P \neq \emptyset\} \cup \bigcup \mathcal{L}_\alpha^2 \cup E \right)$$

is a set of size  $\mathfrak{c}$  and so we define  $X'_\alpha$  to be  $O(z)$  for one of its members  $z$ . It follows that  $P \cup X'_\alpha$  is a partial two-point set, because:

- (i)  $X'_\alpha$  meets lines in at most two points. We need not pay any attention to members of  $\mathcal{L}_\alpha^0$ .
- (ii) The restriction that  $z \notin \{x \in L_\alpha \setminus E: \langle x, gx \rangle \cap P \neq \emptyset\}$  for any  $g \in G \setminus \{e\}$  is so that  $X'_\alpha$  does not meet any member of  $\mathcal{L}_\alpha^1$  in two points. If we suppose that there exist  $y \in P$  and distinct  $g, h \in G$  such that  $y, gz$  and  $hz$  are collinear then we obtain the contradiction that  $g^{-1}y \in P$  and  $g^{-1}y \in \langle z, g^{-1}hz \rangle$ .
- (iii) The restriction that  $z \notin \bigcup \mathcal{L}_\alpha^2$  is so that we select an orbit which does not meet any member of  $\mathcal{L}_\alpha^2$ . If we suppose that there exist  $L \in \mathcal{L}_\alpha^2$  and  $g \in G$  such that  $gz \in L$  then we obtain the contradiction that  $z \in g^{-1}L \in \mathcal{L}_\alpha^2$ .

If  $P \cup X'_\alpha$  meets  $L_\alpha$  in two points then we set  $X_\alpha = X'_\alpha$  and we continue to the next stage of the recursion. Otherwise, we redefine the  $\mathcal{L}_\alpha^i$  using  $\bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha$ , repeat the above argument with  $P = \bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha$  to obtain some  $X''_\alpha$ , set  $X_\alpha = X'_\alpha \cup X''_\alpha$  and then continue to the next stage of the recursion. The resulting set two-point set  $X$  is clearly invariant under the action of  $G$ .  $\square$

### 3. The isometry group of a two-point set

We now start our search for two-point sets invariant under non-trivial autohomeomorphisms, and begin by considering the simplest class of non-trivial planar autohomeomorphisms, namely the isometries. It is here that we obtain our most pleasing results, for not only will we construct two-point sets which are invariant under certain isometries, we will construct them to have precise isometry groups.

We see immediately that it suffices to restrict our attention to rotations.

**Proposition 4.** *The isometries of a two-point set are rotations.*

**Proof.** Recall that an isometry between arbitrary subsets of the plane extends to an isometry of the plane and that the isometries of the plane are rotations, reflections, translations and glide reflections. Let  $X$  be a two-point set and let  $T$  be a non-trivial isometry of the plane. We will show that if  $T$  is not a rotation then  $X$  is not invariant under  $T$ .

Suppose that  $T$  is a reflection about a line  $L_0$ . Let  $L_1$  be a line which is both perpendicular to  $L_0$  and such that  $L_0 \cap L_1 \in X$ . Then  $|(X \cup T(X)) \cap L_1| = 3$ , giving that  $X$  is not invariant under  $T$ .

Suppose that  $T$  is a translation. All orbits of the action of  $T$  on  $\mathbb{R}^2$  are countably infinite and are contained in lines, giving that  $X$  is not invariant under  $T$ .

Suppose that  $T$  is a glide reflection. Then there exists a reflection  $A$  and some  $b \in \mathbb{R}^2$  such that for all  $x \in \mathbb{R}^2$ ,  $T(x) = Ax + b$ . For simplicity, we will assume that  $A$  is reflection about the vertical axis. Now,  $x \in X$  is easily seen to be a fixed point of  $T$  iff  $x_1 = b_1/2$  and  $b_2 = 0$ . Hence if  $T$  has a fixed point then it fixes a line point-wise, in which case the argument used for reflections shows that  $X$  is not invariant under  $T$ . Otherwise  $b_2 \neq 0$ . Let  $L_0 = \{(b_1/2, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$ . Then  $|(X \cup T(X)) \cap L_0| \geq 3$ , giving that  $X$  is not invariant under  $T$ .  $\square$

We will restrict our attention to considering two-point sets which are invariant under a group of rotations about a single point. Without loss of generality, we will take this distinguished point to be the origin. For notational purposes, we identify  $S^1$  with the group of all rotations of  $\mathbb{R}^2$  about the origin in the natural way. Thus whenever we mention subgroups of  $S^1$ , we mean groups of rotations of  $\mathbb{R}^2$  about the origin.

Lauman proved in [4] that a two-point set cannot contain an arc. Hence a two-point set cannot be invariant under  $S^1$ .

In the remainder of this section we will construct two-points sets invariant under subgroups of  $S^1$ . In particular, we show that if  $G$  is a subgroup of  $S^1$  and  $|G| < \mathfrak{c}$  then there is a two-point set whose isometry group is precisely  $G$ . Using a different technique, we construct a two-point set which is invariant under a subgroup of  $S^1$  with cardinality  $\mathfrak{c}$ .

**Lemma 5.** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a non-trivial rotation about the origin, let  $L$  be a line and let  $y \in \mathbb{R}^2 \setminus \{0\}$ . Then  $|\{x \in L : y \in \langle x, T(x) \rangle\}| \leq 3$ .

**Proof.** If  $L$  contains the origin then the slope of  $\langle x, T(x) \rangle$ , denoted by  $m\langle x, T(x) \rangle$ , is independent of  $x \in L$ , and so  $|\{x \in L : y \in \langle x, T(x) \rangle\}| = 1$ . Suppose then that  $L$  does not contain the origin. Let the angle of rotation of  $T$  be  $0 < \theta < 2\pi$  and let  $(\alpha, \beta) = (\cos \theta, \sin \theta)$ . We will demonstrate that the equation

$$m\langle x, T(x) \rangle = m\langle x, y \rangle \quad (1)$$

has at most three solutions for  $x \in L$ . Firstly, suppose that  $x_1 = y_1$  for all  $x \in L$ . Then for all  $x \in L$  we have that  $m\langle x, y \rangle = \infty$  and so (1) will hold if and only if  $T(x)_1 = x_1$ , or equivalently,

$$(\alpha - 1)x_1 - \beta x_2 = 0.$$

Since if  $\beta = 0$  then  $\alpha - 1 \neq 0$ , this is an equation having at most one solution to (1) for  $x \in L$ .

Suppose otherwise that  $x_1 \neq y_1$  for some  $x \in L$ . Then  $x_1 = y_1$  for at most a single  $x \in L$  which we assume may be a solution to our equation. If  $x \in L$  is a solution to (1) and  $x_1 \neq y_1$  then  $T(x)_1 \neq x_1$  and thus it will be sufficient to show that

$$\frac{T(x)_2 - x_2}{T(x)_1 - x_1} = \frac{y_2 - x_2}{y_1 - x_1}$$

has at most two solutions for  $x \in L$  with  $x_1 \neq y_1$ . Note that this equation simplifies to

$$-\beta x_1^2 - \beta x_2^2 + (y_1\beta - y_2(\alpha - 1))x_1 + (y_1(\alpha - 1) + y_2\beta)x_2 = 0. \quad (2)$$

Introducing a parameter  $r \in \mathbb{R}^2$ , we write  $x = ra + b$  for some linearly independent  $a, b \in \mathbb{R}^2$ . Upon substitution, (2) transforms into a polynomial equation in  $r$  of degree at most 2. We consider two cases:

*Case 1:*  $\beta \neq 0$ . The coefficient of  $r^2$  is  $-\beta a_1^2 - \beta a_2^2 \neq 0$ , and so we have a non-trivial quadratic equation with at most two solutions.

*Case 2:*  $\beta = 0$ . Then  $\alpha = -1$  and the coefficient of  $r^2$  is 0. Further, the coefficient of  $r$  and the constant term are respectively  $2y_2a_1 - 2y_1a_2$  and  $2y_2b_1 - 2y_1b_2$ , at least one of which is non-zero as  $a$  and  $b$  are linearly independent. It follows that we have a non-trivial equation with at most a single solution.  $\square$

**Theorem 6.** Let  $G \leq S^1$  be such that  $|G| < \mathfrak{c}$ . Then there exists a two-point set with isometry group isomorphic to  $G$ .

**Proof.** Let  $E = \{0\}$ , let  $\kappa = 3$  and let  $(T_\alpha)_{\alpha < \mathfrak{c}}$  enumerate  $S^1 \setminus G$ . We will modify the proof of Theorem 3. Suppose that we are at stage  $\alpha$  of the construction, and suppose that we have chosen a sequence  $(X_\beta)_{\beta < \alpha}$  of subsets of the plane such that:

- (1)  $|X_\beta| \leq |G| < \mathfrak{c}$  for all  $\beta < \alpha$ ; and
- (2)  $\bigcup_{\beta < \alpha} X_\beta$  meets each member of  $\{L_\beta: \beta < \alpha\}$  in exactly two points; and
- (3)  $\bigcup_{\beta < \alpha} X_\beta$  is a partial two-point set; and
- (4)  $\bigcup_{\beta < \alpha} X_\beta \subseteq \mathbb{R}^2 \setminus E$  is  $G$ -invariant.

Suppose additionally that we have chosen a sequence  $(x_\beta)_{\beta < \alpha}$  on  $\mathbb{R}^2$  such that:

- (5)  $x_\beta \in X_\beta$  for all  $\beta < \alpha$ ; and
- (6)  $T_\beta(x_\beta) \notin \bigcup_{\gamma < \alpha} X_\gamma$  for all  $\beta < \alpha$ .

The sequence  $(T_\beta(x_\beta))_{\beta < \alpha}$  is a list of points which we promise never to include in our two-point set. Let  $\alpha' = \min\{\beta < \mathfrak{c}: L_\beta \notin \mathcal{L}_\alpha^2\}$ . We will choose orbits with representative points taken from  $L_{\alpha'}$  instead of  $L_\alpha$ . When selecting  $X'_\alpha = O(x_\alpha)$  for some  $x_\alpha \in L_{\alpha'}$ , we choose  $x_\alpha \in L_{\alpha'}$  such that:

- (i)  $x_\alpha \notin \bigcup_{g \in G \setminus \{e\}} \{x \in L_{\alpha'} \setminus E: \langle x, gx \rangle \cap P \neq \emptyset\} \cup \bigcup \mathcal{L}_\alpha^2$ ; and
- (ii)  $x_\alpha \notin \bigcup_{\beta < \alpha} O(T_\beta(x_\beta)) \cup T_\alpha^{-1}(P) \cup E$ ; and
- (iii)  $x_\alpha$  is not a fixed point of  $gT_\alpha$  for all  $g \in G$ .

That such a choice is possible follows from the fact that

$$\left| \bigcup_{g \in G \setminus \{e\}} \{x \in L_{\alpha'} \setminus E: \langle x, gx \rangle \cap P \neq \emptyset\} \cup \bigcup \mathcal{L}_\alpha^2 \right| < \mathfrak{c},$$

as seen in the proof of Theorem 3, the fact that

$$\left| \bigcup_{\beta < \alpha} O(T_\beta(x_\beta)) \cup T_\alpha^{-1}(P) \right| < \mathfrak{c},$$

and the fact that  $|G| < \mathfrak{c}$  and  $gT_\alpha$  is a non-trivial rotation with a unique fixed point for all  $g \in G$ .

The additional constraints are such that  $(T_\beta(x_\beta))_{\beta \leq \alpha}$  is on  $\mathbb{R}^2 \setminus (P \cup X'_\alpha)$ . To confirm this, note that clearly  $(T_\beta(x_\beta))_{\beta \leq \alpha}$  is on  $\mathbb{R}^2 \setminus P$ . If  $\beta < \alpha$  then  $x_\alpha \notin O(T_\beta(x_\beta))$  implies that  $T_\beta(x_\beta) \notin O(x_\alpha) = X'_\alpha$ . Now, suppose that  $T_\alpha(x_\alpha) \in X'_\alpha$ . Then  $gT_\alpha(x_\alpha) = x_\alpha$  for some  $g \in G$ , which cannot occur by the choice of  $x_\alpha$ .

If we need to define  $X''_\alpha = O(x)$  for some  $x \in L_{\alpha'}$  then (remembering now that the symbols  $P$  and  $\mathcal{L}_\alpha^2$  represent different sets to those represented previously in this proof) we choose  $x$  as above with the occurrence of  $<$  in condition (ii) replaced by  $\leq$ . Again, such a choice is clearly possible. We set  $X_\alpha = X'_\alpha \cup X''_\alpha$ .

The induction hypotheses have been preserved, so we now define the  $X_\alpha$  and  $x_\alpha$  for all  $\alpha < \mathfrak{c}$  and define  $X$  in the usual way. Then  $X$  is invariant under the action of  $G$ , but not under any  $T_\alpha$ , as witnessed by  $x_\alpha \in X$  and  $T_\alpha(x_\alpha) \notin X$ . If  $T$  is a rotation of  $\mathbb{R}^2$  but  $T \notin G$  and  $T \neq T_\alpha$  for any  $\alpha < \mathfrak{c}$ , then  $X$  cannot be invariant under  $T$ , as the group generated by  $G$  and  $T$  contains a non-rotation. Hence the isometry group of  $X$  is  $G$  by Theorem 5.  $\square$

We noted previously that no two-point set is invariant under  $S^1$ . Clearly the assumption that  $|G| < \mathfrak{c}$  is essential for the construction given in the previous proof. However, we can show that there exist two-point sets which are invariant under a subgroup of  $S^1$  with size  $\mathfrak{c}$ . We will achieve this by constructing the subgroup in parallel with the two-point set, instead of prescribing the group beforehand.

**Lemma 7.** Let  $X$  be a partial two-point set such that  $|X| < \mathfrak{c}$ . Then there exist  $\mathfrak{c}$  many rotations  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  about the origin such that  $\bigcup_{n \in \mathbb{Z}} T^n(X)$  is a partial two-point set.

**Proof.** For each  $\theta \in [0, 2\pi)$ , let  $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the rotation through the angle  $\theta$ . It will be sufficient to show that the collection of all  $\theta \in [0, 2\pi)$  such that  $x$ ,  $T_\theta^n(y)$  and  $T_\theta^m(z)$  are collinear for some  $n, m \in \mathbb{Z}$  and  $x, y, z \in X$  has size less than  $\mathfrak{c}$ . We proceed using polar co-ordinates. Fixing  $n, m \in \mathbb{Z}$  and  $x = (r_x, \theta_x)$ ,  $y = (r_y, \theta_y)$ ,  $z = (r_z, \theta_z) \in X$ , we will demonstrate that the equation

$$\begin{vmatrix} r_x \cos \theta_x & r_x \sin \theta_x & 1 \\ r_y \cos(\theta_y + n\theta) & r_y \sin(\theta_y + n\theta) & 1 \\ r_z \cos(\theta_z + m\theta) & r_z \sin(\theta_z + m\theta) & 1 \end{vmatrix} = 0$$

has at most finitely many solutions for  $\theta \in [0, 2\pi)$ . Supposing that the above holds, we have

$$r_y r_z \sin(\theta_z - \theta_y + (m - n)\theta) + r_x r_z \sin(\theta_x - \theta_z - m\theta) + r_x r_y \sin(\theta_y - \theta_x + n\theta) = 0.$$

Let  $z = e^{i\theta}$ . Now, for  $A, B \in \mathbb{R}$  we have that

$$\sin(A + B\theta) = \frac{e^{iA} z^B - e^{-iA} z^{-B}}{2i},$$

and so upon substitution it follows that

$$\begin{aligned} & r_y r_z (e^{i(-\theta_y + \theta_z)} z^{(m-n)} - e^{-i(-\theta_y + \theta_z)} z^{-(m-n)}) + r_x r_z (e^{i(\theta_x - \theta_z)} z^{-m} - e^{-i(\theta_x - \theta_z)} z^m) \\ & + r_x r_y (e^{i(-\theta_x + \theta_y)} z^n - e^{-i(-\theta_x + \theta_y)} z^{-n}) = 0. \end{aligned}$$

This equation is a linear combination of integer powers of  $z$ . It will be sufficient to show that it is non-trivial, for then it will be satisfied by at most finitely many  $\theta$ . Without loss of generality, we consider the following three cases:

*Case 1:  $n = m \neq 0$ .* The coefficient of  $w^n$  and  $w^{-n}$  and the constant term are respectively

$$\begin{aligned} & r_x r_y e^{i(-\theta_x + \theta_y)} - r_x r_z e^{-i(\theta_x - \theta_z)}, \\ & -r_x r_y e^{-i(-\theta_x + \theta_y)} + r_x r_z e^{i(\theta_x - \theta_z)}, \\ & r_y r_z (e^{i(-\theta_y + \theta_z)} - e^{-i(-\theta_y + \theta_z)}). \end{aligned}$$

Suppose that these are all equal to zero. Then

$$\begin{aligned} & r_y e^{i(-\theta_x + \theta_y)} - r_z e^{-i(\theta_x - \theta_z)} = 0, \\ & -r_y e^{-i(-\theta_x + \theta_y)} + r_z e^{i(\theta_x - \theta_z)} = 0, \\ & e^{i(-\theta_y + \theta_z)} - e^{-i(-\theta_y + \theta_z)} = 0. \end{aligned}$$

Multiplying the first two of these equations, we obtain

$$-r_y^2 + r_y r_z (e^{i(\theta_y - \theta_z)} + e^{-i(\theta_y - \theta_z)}) - r_z^2 = 0.$$

The third equation requires that  $e^{i(-\theta_y + \theta_z)}$  is self-conjugate, in which case either  $e^{i(-\theta_y + \theta_z)} = 1$  and  $\theta_y = \theta_z$ , or  $e^{i(-\theta_y + \theta_z)} = -1$ , and so we obtain either the contradiction that  $r_y = r_z$  and  $\theta_y = \theta_z$ , or the contradiction that  $r_y = r_z = 0$ . Hence the equation is non-trivial.

*Case 2:  $n \neq 0$  and  $m = 0$ .* The coefficient of  $w^n$  and  $w^{-n}$  and the constant term are respectively

$$\begin{aligned} & r_x r_y e^{i(\theta_y - \theta_x)} - r_y r_z e^{-i(-\theta_y + \theta_z)}, \\ & -r_x r_y e^{-i(\theta_y - \theta_x)} + r_y r_z e^{i(-\theta_y + \theta_z)}, \\ & r_x r_z (e^{i(\theta_x - \theta_z)} - e^{-i(\theta_x - \theta_z)}), \end{aligned}$$

and in a similar manner to the above, we may deduce that the equation is non-trivial.

*Case 3:  $n \neq m$  and  $n \neq 0$  and  $m \neq 0$ .* Then at least one of the powers  $n, -n, m, -m, m - n$  and  $-(m - n)$  is distinct from each of the others, and each coefficient is non-zero. Hence the equation is non-trivial.  $\square$

**Theorem 8.** *There exists a two-point set which is invariant under  $\mathfrak{c}$  many rotations.*

**Proof.** As in the proofs of previous theorems, we construct the sequence  $(X_\alpha)_{\alpha < \mathfrak{c}}$  by recursion. Suppose also that at stage  $\alpha$ , we have constructed an injective sequence of rotations  $(T_\beta)_{\beta < \alpha}$  such that  $\bigcup_{\beta < \alpha} X_\beta$  is invariant under each  $T_\beta$ .

Let  $X'_\alpha$  be chosen in the way that  $X_\alpha$  would have been chosen in the proof of Theorem 3 in the case that  $G$  is the group generated by  $\{T_\beta: \beta < \alpha\}$  and  $E = \{0\}$  and  $\kappa = 3$ . Then  $\bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha$  satisfies:

- (i) a partial two-point set of cardinality less than  $\mathfrak{c}$ ; and
- (ii) is invariant under each  $T_\beta$ ; and
- (iii) meets  $L_\alpha$  in exactly two points.

By the previous lemma, there exists a rotation  $T_\alpha$  such that  $(T_\beta)_{\beta \leq \alpha}$  is injective and

$$X_\alpha := \bigcup_{n \in \mathbb{Z}} T_\alpha^n \left( \bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha \right)$$

is a partial two-point set of cardinality less than  $\mathfrak{c}$ . It can be seen that  $X_\alpha$  is invariant under each  $T_\beta$ , for if  $\beta = \alpha$  then the statement is obvious and if  $\beta < \alpha$  then the statement follows from the facts that  $T_\beta$  commutes with each  $T_\alpha^n$  and  $X_\alpha \cup X'_\alpha$  is invariant under  $T_\beta$ . The induction hypothesis has been preserved, so we now let the  $X_\alpha$  and  $T_\alpha$  be defined for all  $\alpha < \mathfrak{c}$  and let  $X = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$ . Then  $X$  is a two-point set invariant under each  $T_\alpha$ .  $\square$

**Remark 9.** We note that in the previous proof, the isometry group of  $X$  may be strictly larger than the group generated by the  $T_\alpha$ . However by fixing a well-ordering  $R$  of  $S^1$  and choosing each  $T_\alpha$  to be  $R$ -minimal in

$$A_\alpha := \left\{ T \in S^1: \bigcup_{n \in \mathbb{Z}} T^n \left( \bigcup_{\beta < \alpha} X_\beta \cup X'_\alpha \right) \text{ is a p.t.p.s. and } T \neq T_\beta \text{ for } \beta < \alpha \right\},$$

we can ensure that the intersection of the isometry group and  $S^1$  is the group generated by the  $R$ -cofinal subset  $\{T_\alpha: \alpha < \mathfrak{c}\}$  of  $S^1$ , since the sequence  $(A_\alpha)_{\alpha < \mathfrak{c}}$  is decreasing.

## References

- [1] V.J. Baston, F.A. Bostock, On a theorem of Larman, J. London Math. Soc. 5 (2) (1972) 715–718.
- [2] J.J. Dijkstra, J. van Mill, Two point set extensions—a counterexample, Proc. Amer. Math. Soc. 125 (8) (1997) 2501–2502.
- [3] J. Kulesza, A two-point set must be zero-dimensional, Proc. Amer. Math. Soc. 116 (2) (1992) 551–553.
- [4] D.G. Larman, A problem of incidence, J. London Math. Soc. 43 (1968) 407–409.
- [5] R.D. Mauldin, Problems in topology arising from analysis, Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 617–629.
- [6] R.D. Mauldin, On sets which meet each line in exactly two points, Bull. London Math. Soc. 30 (4) (1998) 397–403.
- [7] S. Mazurkiewicz, O pewnej mnogości płaskiej, która ma z każdą prostą dwa i tylko dwa punkty wspólne, C. R. Varsovie 7 (1914) 382–384.
- [8] S. Mazurkiewicz, Travaux de topologie et ses applications, Comité de rédaction: K. Borsuk, R. Engelking, B. Knaster, K. Kuratowski, J. Loś, R. Sikorski, Państwowe Wydawnictwo Naukowe-Éditions Scientifiques de Pologne, Warsaw, 1969.
- [9] J. van Mill, G.M. Reed (Eds.), Open Problems in Topology, North-Holland, Amsterdam, 1990.
- [10] S. Willard, General Topology, Addison-Wesley, Reading, MA, 1970.