

# Bounds for the Cubic Weyl Sum

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## 1 Introduction

In this paper we shall consider bounds for the cubic Weyl sum

$$S(\alpha, N) = \sum_{n \leq N} e(\alpha n^3),$$

where  $e(x) = \exp(2\pi i x)$  as usual. The classical bound, due essentially to Weyl [5], shows that  $S(\alpha, N) \ll_{\varepsilon} N^{3/4+\varepsilon}$  for any  $\varepsilon > 0$ , providing that there is a rational number  $a/q$  with denominator in the range  $N \leq q \leq N^2$ , for which we have  $|\alpha - a/q| \leq q^{-2}$ . It is clear that a condition on rational approximations to  $\alpha$  will be necessary, and the exact condition here is unimportant. Of greater significance is the exponent  $3/4$  in the Weyl estimate, which has never been improved on. An alternative method to bound  $S(\alpha, N)$  has been given by Vaughan [4, Theorem 3], leading to exactly the same exponent  $3/4$ . If  $\alpha$  is a real algebraic irrational then Roth's Theorem shows that the Diophantine approximation condition is met, so that

$$S(\alpha, N) \ll_{\varepsilon, \alpha} N^{3/4+\varepsilon}.$$

The goal of this paper is to show how an improvement can be made for special values of  $\alpha$ . Unfortunately our result depends on unproved hypothesis, namely the *abc*-conjecture. This states that if  $\varepsilon > 0$  is given there is a constant  $K(\varepsilon)$  such that

$$\max\{|a|, |b|, |c|\} \leq K(\varepsilon) \left( \prod_{p|abc} p \right)^{1+\varepsilon}$$

for any coprime positive integers  $a, b, c$  with  $a + b = c$ .

We shall then prove the following bound.

**Theorem 1** *Let  $\alpha \in \mathbb{R} - \mathbb{Q}$  be a quadratic irrational. Assume the truth of the *abc*-conjecture. Then*

$$S(\alpha, N) \ll_{\varepsilon, \alpha} N^{5/7+\varepsilon}$$

for any  $\varepsilon > 0$ .

Note that  $\frac{5}{7} = \frac{3}{4} - \frac{1}{28}$ .

The underlying idea is to apply the  $q$ -analogue of van der Corput's method, which requires a suitable approximation  $a/q$  to  $\alpha$ , in which  $q$  factorizes in a suitable way. Results of this type were proved in an Oxford DPhil thesis by Ringrose [3] in 1985, but not otherwise published. We therefore establish a variant of Ringrose's result here.

**Theorem 2** *Suppose that  $a$  and  $q$  are coprime integers with  $N \leq q \leq N^{3/2}$ . Suppose further that  $q = q_1 q_2 q_3$  with the factors  $q_1, q_2, q_3$  coprime in pairs and  $q_3$  square-free. Then if  $N \leq \min\{q_1 q_3, q_2 q_3\}$  we have*

$$S(\alpha, N) \ll_\varepsilon \left(1 + N^3 \left| \alpha - \frac{a}{q} \right| \right) (N^{1/2} q_1^{1/2} + N^{1/4} q^{1/4} q_2^{1/4} + N^{1/4} q^{1/4} q_3^{1/8}) q^\varepsilon,$$

for any  $\varepsilon > 0$ .

Theorem 1 is an easy consequence of Theorem 2, along with the following result on Diophantine approximation with smooth denominators.

**Theorem 3** *Let  $\alpha \in \mathbb{R}$  be a quadratic irrational, and let  $\varepsilon > 0$  be given. Then there is a constant  $C(\alpha, \varepsilon)$  such that, for any  $N \in \mathbb{N}$ , one can solve*

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{C(\alpha, \varepsilon)}{qN}, \quad (a \in \mathbb{Z}, \quad q \in \mathbb{N}, \quad q \leq N)$$

with  $q$  having no prime factors  $p > q^\varepsilon$ .

This result provides approximations almost as strong as Dirichlet's Theorem yields. However the hypothesis that  $\alpha$  is quadratic makes the proof rather simple. One can quite easily improve the statement of the theorem slightly to say that any prime power factor  $p^e$  of  $q$  has  $p^e \leq q^\varepsilon$ . However we do not need this for our application. Unfortunately we are unable to produce values of  $q$  which are "nearly square-free", and it is at this point that we must call on the *abc*-conjecture. Indeed the reader may notice that it suffices for Theorem 1 that the largest power-full divisor of  $q$  should be at most  $q^{11/21}$ , but we are unable to produce numbers  $q$  of this type.

This paper was written while the author was attending the trimestre on Diophantine Equations at the Hausdorff Institute of Mathematics in Bonn. The hospitality and financial support of the institute is gratefully acknowledged. The idea of using factorization properties of the sequence  $q_n$  to facilitate the  $q$ -analogue of van der Corput's method arose in a somewhat different context in a conversation with Jimi Truelsen. His contribution is also gratefully acknowledged.

## 2 Preliminary Steps

If  $\delta = \alpha - a/q$  we have

$$\begin{aligned} S(\alpha, N) &= S(a/q, N) e(\delta N^3) - \int_0^N (2\pi i \delta) 3t^2 S(a/q, t) dt \\ &\ll (1 + N^3 |\delta|) \max_{t \leq N} |S(a/q, t)|, \end{aligned}$$

by partial summation. Moreover, if we set

$$S(a, h; q) = \sum_{n=1}^q e((an^3 + hn)/q)$$

and

$$T(h, t; q) = \sum_{n \leq t} e(-hn/q)$$

we have

$$S\left(\frac{a}{q}, t\right) = q^{-1} \sum_{-q/2 < h \leq q/2} S(a, h; q) T(h, t; q).$$

Since  $S(a, 0; q) \ll q^{2/3}$  the term  $h = 0$  contributes  $\ll Nq^{-1/3}$ , which is satisfactory.

For the remaining terms we note that

$$T(h, t; q) \ll \min(N, q/|h|), \quad \text{and} \quad \frac{\partial T(h, t; q)}{\partial h} \ll q^{-1} N \min(N, q/|h|)$$

for  $|h| \leq q/2$  and  $t \leq N$ . From now on we shall assume that

$$\left| \sum_{-q/2 < h < 0} S(a, h; q) T(h, t; q) \right| \leq \left| \sum_{0 < h \leq q/2} S(a, h; q) T(h, t; q) \right|,$$

the alternative case being treated in exactly the same way.

We proceed to define  $K = [q/N]$  and

$$\eta(r) = \max_{0 \leq L \leq K} \left| \sum_{(r-1)K < h \leq (r-1)K+L} S(a, h; q) \right|.$$

We then find by partial summation that

$$\sum_{(r-1)K \leq h \leq (r-1)K+K'} S(a, h; q) T(h, t; q) \ll \eta(r) \min\left(N, \frac{q}{(r-1)K}\right) \ll \frac{N}{r} \eta(r)$$

for any integer  $r \geq 0$  and any  $K' \leq K$ . Summing for  $r \leq q$  (which is more than adequately large) we deduce that

$$\sum_{-q/2 < h \leq q/2} S(a, h; q) T(h, t; q) \ll N \sum_{r \leq q} \frac{\eta(r)}{r}.$$

We may therefore conclude as follows.

**Lemma 1** *With the definitions above, if  $N \leq q \leq N^{3/2}$  we have*

$$S(a/q, N) \ll \frac{N}{q} \sum_{r \leq q} \frac{\eta(r)}{r}.$$

It follows as a special case of Loxton and Vaughan [2, Theorem 1] that

$$S(a, h; q) \ll_{\varepsilon} q^{1/2+\varepsilon} (q, h)^{1/4} \tag{1}$$

for any  $\varepsilon > 0$ , whence

$$\begin{aligned} \sum_{r \leq q} \frac{\eta(r)}{r} &\ll \sum_{r \leq q} \sum_{n \leq K} \frac{|S(a, (r-1)K+n; q)|}{r} \\ &\ll \sum_{m \leq qK} |S(a, m; q)| \min(1, K/m) \\ &\ll_{\varepsilon} q^{1/2+\varepsilon} \sum_{m \leq qK} (q, m)^{1/4} \min(1, K/m). \end{aligned}$$

We now use the following easy lemma, which we shall prove at the end of this section.

**Lemma 2** *Let positive integers  $v$  and  $H_1 \leq H_2$  be given. Then for any fixed  $\varepsilon > 0$  we have*

$$\sum_{H_2 - H_1 < h \leq H_2} (h, v)^\rho \ll_\varepsilon \{H_1 + \min(v, H_2)\} v^\varepsilon$$

uniformly for  $\rho \leq 1$ , and in particular

$$\sum_{1 \leq h \leq H_2} (h, v)^\rho \ll_\varepsilon H_2 v^\varepsilon.$$

From this it follows by partial summation that

$$\sum_{r \leq q} \frac{\eta(r)}{r} \ll_\varepsilon q^{1/2+\varepsilon} \sum_{m \leq qK} (q, m)^{1/4} \min(1, K/m) \ll_\varepsilon K q^{1/2+2\varepsilon}, \quad (2)$$

whence

$$S(a/q, N) \ll_\varepsilon \frac{N}{q} K q^{1/2+2\varepsilon} \ll_\varepsilon q^{1/2+2\varepsilon} \ll N^{3/4+3\varepsilon},$$

since we are assuming that  $q \leq N^{3/2}$ . We thus recover the classical exponent  $3/4$ . The argument above is equivalent to that given by Vaughan, mentioned in the introduction.

To prove Lemma 2 we merely note that

$$\begin{aligned} \sum_{H_2 - H_1 < h \leq H_2} (h, v)^\rho &\leq \sum_{H_2 - H_1 < h \leq H_2} (h, v) \\ &\leq \sum_{d|v, d \leq H_2} d \#\{H_2 - H_1 < h \leq H_2 : d|h\} \\ &\leq \sum_{d|v, d \leq H_2} d \left( \frac{H_1}{d} + 1 \right) \\ &\leq H \sum_{d|v} \{H_1 + \min(v, H_2)\} \\ &\ll_\varepsilon \{H_1 + \min(v, H_2)\} v^\varepsilon. \end{aligned}$$

### 3 The First Iteration

In order to improve on the classical bound we must demonstrate some cancellation amongst the terms  $S(a, h; q)$  in the sum  $\eta(r)$ . We begin by noting the factorization property

$$S(a, h; uv) = S(av^2, h; u) S(au^2, h; v) \quad (3)$$

for coprime  $u$  and  $v$ . We begin the van der Corput argument by writing  $\eta(r) = |\Sigma|$ , where

$$\Sigma = \sum_{h \in I} S(a, h; q)$$

for an appropriate interval  $I \subseteq ((r-1)K, rK]$ . Recall that  $q = q_1 q_2 q_3$  in Theorem 2. Thus, if we impose the condition

$$q_2 q_3 \geq N, \quad (4)$$

we will have  $K \geq q_1$ . We then set  $M = \lceil K/q_1 \rceil \geq 1$  and observe that

$$\begin{aligned} M\Sigma &= \sum_{m \leq M} \sum_{h: h+mq_1 \in I} S(a, h+mq_1; q) \\ &= \sum_{(r-2)K < h < rK} \sum_{m \leq M: h+mq_1 \in I} S(a, h+mq_1; q). \end{aligned}$$

We now apply (3) with  $u = q_1$  and  $v = q_2 q_3$ , so that

$$\begin{aligned} S(a, h+mq_1; q) &= S(a', h+mq_1; q_1) S(b, h+mq_1; q_2 q_3) \\ &= S(a', h; q_1) S(b, h+mq_1; q_2 q_3), \end{aligned}$$

where we have written  $a' = a q_2^2 q_3^2$  and  $b = a q_1^2$ . It follows that

$$M\Sigma = \sum_{(r-2)K < h < rK} S(a', h; q_1) \sum_{m \leq M: h+mq_1 \in I} S(b, h+mq_1; q_2 q_3).$$

We now apply Cauchy's inequality to produce

$$M^2 \eta(r)^2 = M^2 |\Sigma|^2 \leq \eta_1(r) \eta_2(r), \quad (5)$$

where

$$\eta_1(r) = \sum_{(r-2)K < h < rK} |S(a', h; q_1)|^2$$

and

$$\eta_2(r) = \sum_{(r-2)K < h < rK} \left| \sum_{m \leq M: h+mq_1 \in I} S(b, h+mq_1; q_2 q_3) \right|^2.$$

We may estimate  $\eta_1$  using Lemma 2. By (1) we have

$$\begin{aligned} \eta_1(r) &= \sum_{(r-2)K < h < rK} |S(a', h; q_1)|^2 \\ &\ll_{\varepsilon} q_1^{1+2\varepsilon} \sum_{(r-2)K < h < rK} (h, q_1)^{1/2} \\ &\ll_{\varepsilon} q_1^{1+3\varepsilon} (K + q_1) \\ &\ll_{\varepsilon} q_1^{1+3\varepsilon} K, \end{aligned} \quad (6)$$

since  $K \geq q_1$ , as noted above.

To handle  $\eta_2(r)$  we expand the square to produce

$$\begin{aligned} \eta_2(r) &= \sum_{m_1, m_2 \leq M} \sum_{\substack{h: h+m_1 q_1 \in I \\ h+m_2 q_1 \in I}} S(b, h+m_1 q_1; q_2 q_3) \overline{S(b, h+m_2 q_1; q_2 q_3)} \\ &= \sum_{\substack{n_1, n_2 \in I \\ q_1 | n_1 - n_2}} S(b, n_1; q_2 q_3) \overline{S(b, n_2; q_2 q_3)} N(b_1, b_2), \end{aligned}$$

where  $N(b_1, b_2)$  is the number of triples  $(h, m_1, m_2) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$  for which  $m_1, m_2 \leq M$  and  $h + m_1 q_1 = n_1$ ,  $h + m_2 q_1 = n_2$ . Then

$$N(b_1, b_2) = M - q^{-1}|n_1 - n_2|,$$

whence

$$\begin{aligned} \eta_2(r) &= \sum_{\substack{n_1, n_2 \in I \\ q_1 | n_1 - n_2}} (M - q^{-1}|n_1 - n_2|) S(b, n_1; q_2 q_3) \overline{S(b, n_2; q_2 q_3)} \\ &= \sum_{|m| \leq M} (M - |m|) \sum_{n, n+m q_1 \in I} S(b, n + m q_1; q_2 q_3) \overline{S(b, n; q_2 q_3)} \\ &= \sum_{|m| \leq M} (M - |m|) \sum_{n \in I(m)} S_2(b, m, n; q_2 q_3), \end{aligned}$$

where  $I(m)$  is a subinterval of  $((r-1)K, rK]$  given by

$$I(m) = \{x \in \mathbb{R} : x, x + m q_1 \in I\}$$

and  $S_2(b, m, n; q_2 q_3)$  is the exponential sum

$$S_2(b, m, n; u) = S(b, n + m q_1; u) \overline{S(b, n; u)}. \quad (7)$$

We write

$$\eta_3(r, m) = \sum_{n \in I(m)} S_2(b, m, n; q_2 q_3),$$

so that our bound becomes

$$\eta_2(r) \ll M \sum_{|m| \leq M} |\eta_3(r, m)| = M |\eta_3(r, 0)| + \sum_{1 \leq |m| \leq M} |\eta_3(r, m)|. \quad (8)$$

Notice that the only dependence of  $\eta_3(r, m)$  on  $r$  is through the interval  $I(m)$ , which our notation has suppressed.

We now combine Lemma 1 with (5), (6) and (8) to deduce that

$$\begin{aligned} S(a/q, N)^2 &\ll \left(\frac{N}{q}\right)^2 (\log q) \sum_{r \leq q} \frac{\eta(r)^2}{r} \\ &\ll_{\varepsilon} N^2 q^{-2+\varepsilon} M^{-2} q_1^{1+3\varepsilon} K \sum_{r \leq q} \frac{\eta_2(r)}{r} \\ &\ll_{\varepsilon} N^2 q^{-2+\varepsilon} M^{-2} q_1^{1+3\varepsilon} K M \sum_{r \leq q} \sum_{|m| \leq M} \frac{\eta_3(r, m)}{r} \\ &\ll_{\varepsilon} N^2 q^{-2+\varepsilon} q_1^{2+3\varepsilon} \sum_{r \leq q} \sum_{|m| \leq M} \frac{|\eta_3(r, m)|}{r}, \end{aligned}$$

whence

$$S\left(\frac{a}{q}, N\right)^2 \ll_{\varepsilon} T_1 + T_2,$$

where

$$T_1 = N^2 q^{4\varepsilon} (q_2 q_3)^{-2} \sum_{r \leq q} \frac{|\eta_3(r, 0)|}{r}$$

and

$$T_2 = N^2 q^{4\varepsilon} (q_2 q_3)^{-2} \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \frac{|\eta_3(r, m)|}{r}. \quad (9)$$

However

$$\begin{aligned} \eta_3(r, 0) &= \sum_{n \in I(m)} S_2(b, 0, n; q_2 q_3) \\ &= \sum_{n \in I(m)} |S(b, n; q_2 q_3)|^2 \\ &\ll_{\varepsilon} (q_2 q_3)^{1+2\varepsilon} \sum_{(r-1)K < n \leq rK} (q_2 q_3, n)^{1/2} \end{aligned}$$

by (1), whence

$$\begin{aligned} T_1 &\ll_{\varepsilon} N^2 q^{4\varepsilon} (q_2 q_3)^{-2} \sum_{r \leq q} r^{-1} (q_2 q_3)^{1+2\varepsilon} \sum_{(r-1)K < n \leq rK} (q_2 q_3, n)^{1/2} \\ &\ll_{\varepsilon} N^2 q^{6\varepsilon} (q_2 q_3)^{-1} \sum_{n \leq qK} \frac{K}{n} (q_2 q_3, n)^{1/2} \\ &\ll_{\varepsilon} N^2 q^{6\varepsilon} (q_2 q_3)^{-1} K q^{\varepsilon} \\ &\ll_{\varepsilon} N q^{7\varepsilon} q_1 \end{aligned}$$

using Lemma 2. This is satisfactory for Theorem 2.

We summarize the state of play as follows.

**Lemma 3** *When  $q_2 q_3 \leq N$  and  $q \leq N^{3/2}$  we have*

$$S(a/q, N)^2 \ll_{\varepsilon} N q^{\varepsilon} q_1 + T_2$$

with  $T_2$  as in (9), for any  $\varepsilon > 0$ .

This completes the first application of the van der Corput ‘‘A-process’’. We can check that nothing of significance has been lost at this stage. Thus if we use the bound (1), ignoring the highest common factor terms for simplicity, we would get a bound

$$\ll_{\varepsilon} M \sum_{|m| \leq M} \sum_{n \in I(m)} (q_2 q_3)^{1+2\varepsilon} \ll_{\varepsilon} K^3 q_1^{-2} (q_2 q_3)^{1+2\varepsilon}$$

for  $\eta_2(r)$ . Combining this with (5) and (6) would lead to  $\eta(r) \ll_{\varepsilon} K q^{(1+3\varepsilon)/2}$ , allowing us to recover the bound (2). As previously observed this in turn would lead to the classical exponent 3/4 for the original sum  $S(a/q, N)$ .

Although nothing has been lost, our manipulations have produced an advantage. We need to demonstrate that there is some cancellation in the sum  $\eta_3(r, m)$ . The range for  $n$  in this sum is of the same kind as in the previous sum  $\Sigma$ , and the exponential sum  $S_2(b, m, n; q_2 q_3)$  which occurs is more complicated than before, but crucially the modulus  $q_2 q_3$  for the exponential sum is smaller than in  $\Sigma$ , where it was  $q$ . Unfortunately the modulus is still too large, so that a second iteration of the A-process is necessary.

## 4 The Second Iteration

For the second iteration we impose the condition

$$q_1 q_3 \geq N, \quad (10)$$

whence  $q_2 \leq q/N$ . It follows that  $q_2 \leq K$ . We now set  $U = [K/q_2]$ , so that  $U \geq 1$ . For the sum (7) the product formula (3) leads to the relation

$$S_2(b, m, n; q_2 q_3) = S_2(b', m, n; q_2) S_2(c, m, n; q_3),$$

where  $b' = bq_3^2$  and  $c = bq_2^2$ . Then, by the arguments leading to (5) and (8), we find that

$$|\eta_3(r, m)|^2 \leq U^{-2} \eta_4(r, m) \eta_5(r, m), \quad (11)$$

with

$$\eta_4(r, m) = \sum_{(r-2)K < h < rK} |S_2(b', m, h; q_2)|^2$$

and

$$\eta_5(r, m) = \sum_{|u| \leq U} (U - |u|) \sum_{n \in I(m, u)} S_3(c, m, u, n; q_3).$$

Here  $I(m, u)$  is an appropriate subinterval of  $((r-1)K, rK]$ , and

$$S_3(c, m, u, n; v) = S_2(c, m, n + uq_2; v) \overline{S_2(c, m, n; v)}.$$

By (1) we will have

$$\begin{aligned} S_2(b', m, h; q_2) &\ll_{\varepsilon} q_2^{1+2\varepsilon} (h + mq_1, q_2)^{1/4} (h, q_2)^{1/4} \\ &\ll_{\varepsilon} q_2^{1+2\varepsilon} \{(h + mq_1, q_2)^{1/2} + (h, q_2)^{1/2}\}. \end{aligned}$$

It follows that

$$\begin{aligned} \eta_4(r, m) &\ll_{\varepsilon} q_2^{2+4\varepsilon} \sum_{(r-2)K < h < rK} \{(h + mq_1, q_2) + (h, q_2)\} \\ &\ll_{\varepsilon} q_2^{2+4\varepsilon} \sum_{(r-3)K < h < (r+1)K} (h, q_2) \\ &\ll_{\varepsilon} q_2^{2+5\varepsilon} (q_2 + K) \\ &\ll_{\varepsilon} q_2^{2+5\varepsilon} K, \end{aligned} \quad (12)$$

by Lemma 2. We now set

$$\eta_6(r, m, u) = \sum_{n \in I(m, u)} S_3(c, m, u, n; q_3),$$

whence

$$\eta_5(r, m) \ll U \sum_{|u| \leq U} |\eta_6(r, m, u)|. \quad (13)$$

It now follows that

$$\begin{aligned}
T_2^2 &\ll_{\varepsilon} N^4 q^{8\varepsilon} (q_2 q_3)^{-4} \left\{ M \sum_{r \leq q} r^{-1} \right\} \left\{ \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \frac{|\eta_3(r, m)|^2}{r} \right\} \\
&\ll_{\varepsilon} N^4 q^{8\varepsilon} (q_2 q_3)^{-4} \{q_2 q_3 N^{-1} q^{\varepsilon}\} \times \\
&\quad \times \left\{ U^{-1} K q_2^{2+5\varepsilon} \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \sum_{|u| \leq U} \frac{|\eta_6(r, m, u)|}{r} \right\} \\
&\ll_{\varepsilon} N^3 q^{14\varepsilon} q_3^{-3} \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \sum_{|u| \leq U} \frac{|\eta_6(r, m, u)|}{r}. \tag{14}
\end{aligned}$$

When  $u = 0$  we have

$$\begin{aligned}
S_3(c, m, 0; q_3) &= |S_2(c, m, n; q_3)|^2 \\
&= |S(c, n + mq_1; q_3)|^2 |S(c, n; q_3)|^2 \\
&\ll_{\varepsilon} q_3^{2+4\varepsilon} (q_3, n + mq_1)^{1/2} (q_3, n)^{1/2}
\end{aligned}$$

by (1). The contribution to (14) from terms with  $u = 0$  is then

$$\begin{aligned}
&\ll_{\varepsilon} N^3 q^{18\varepsilon} q_3^{-1} \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \sum_{(r-1)K < n, n+mq_1 \leq rK} \frac{(q_3, n + mq_1)^{1/2} (q_3, n)^{1/2}}{r} \\
&\ll_{\varepsilon} N^3 q^{18\varepsilon} q_3^{-1} \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \sum_{(r-1)K < n, n+mq_1 \leq rK} \frac{(q_3, n + mq_1) + (q_3, n)}{n/K} \\
&\ll_{\varepsilon} N^3 q^{18\varepsilon} q_3^{-1} KM \sum_{r \leq q} \sum_{(r-1)K < n \leq rK} \frac{(q_3, n)}{n} \\
&\ll_{\varepsilon} N q^{19\varepsilon} q_1 q_2^2 q_3,
\end{aligned}$$

by Lemma 2.

We complete the second van der Corput A-process by combining this with Lemma 3 and (14) to deduce the following bound.

**Lemma 4** *When  $q_2 q_3 \geq N$ ,  $q_1 q_3 \geq N$  and  $q \leq N^{3/2}$  we have*

$$S(a/q, N)^4 \ll_{\varepsilon} q^{\varepsilon} \{N^2 q_1^2 + N q_1 q_2^2 q + N^3 q_3^{-3} \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \sum_{1 \leq |u| \leq U} \frac{|\eta_6(r, m, u)|}{r}\}$$

for any  $\varepsilon > 0$ .

The first two terms here are suitable for Theorem 2.

## 5 The van der Corput B-Process

To complete the van der Corput argument we will estimate  $\eta_6(r, m, u)$ . We have

$$\eta_6(r, m, n) = q_3^{-1} \sum_{-q_3/2 < t \leq q_3/2} S_4(c, m, u, t; q_3) \sum_{n \in I(m, u)} e(-nt/q_3),$$

where

$$S_4(c, m, u, t; v) = \sum_{n=1}^v S_3(c, m, u, n; v) e(nt/v).$$

Since  $I(m, n)$  is an interval of length at most  $K$  this leads to

$$\eta_6(r, m, u) \ll q_3^{-1} \sum_{-q_3/2 < t \leq q_3/2} \min(K, \frac{q_3}{|t|}) |S_4(c, m, u, t; q_3)|. \quad (15)$$

The sum  $S_4(c, m, u, n, t; v)$  has a multiplicative property

$$S_4(c, m, u, t; vw) = S_4(cw^2, m, u, \bar{w}t; v) S_4(cv^2, m, u, \bar{v}t; w), \quad (16)$$

where  $w\bar{w} \equiv 1 \pmod{v}$  and  $v\bar{v} \equiv 1 \pmod{w}$ . It therefore suffices to bound sums to prime-power modulus. Indeed, since we are assuming  $q_3$  to be square-free it will be enough to consider the case in which the modulus is prime. It would be good if we were able to remove the square-freeness condition, by handling  $S_4$  for prime power moduli, but this appears to be unduly complicated.

In view of the definition of  $S_3(c, m, u, n; v)$  we find that

$$\begin{aligned} S_4(c, m, u, t; v) &= \sum_{n=1}^v S^{(1)} \overline{S^{(2)}} \overline{S^{(3)}} S^{(4)} e(nt/v) \\ &= \sum_{w, x, y, z=1}^v \sum_{n=1}^v e(f(w, x, y, z, n)/v), \end{aligned}$$

where

$$\begin{aligned} S^{(1)} &= S(c, n + uq_2 + mq_1; v), & S^{(2)} &= S(c, n + uq_2; v), \\ S^{(3)} &= S(c, n + mq_1; v), & S^{(4)} &= S(c, n; v) \end{aligned}$$

and

$$\begin{aligned} f(w, x, y, z, n) &= c(w^3 - x^3 - y^3 + z^3) + w(n + uq_2 + mq_1) \\ &\quad - x(n + uq_2) - y(n + mq_1) + zn + tn. \end{aligned}$$

When we perform the summation over  $n$  this produces

$$S_4 = v \sum_{\substack{w, x, y, z \pmod{v} \\ v | w - x - y + z + t}} e \left( \frac{c(w^3 - x^3 - y^3 + z^3) + uq_2(w - x) + mq_1(w - y)}{v} \right).$$

We substitute  $z = x + y - w - t$  so that the denominator becomes

$$\begin{aligned} &c(w^3 - x^3 - y^3 + (x + y - w - t)^3) + uq_2(w - x) + mq_1(w - y) \\ &= 3c(x + y)(w - x)(w - y) - 3ct(x + y - w)^2 + 3ct^2(x + y - w) - ct^3 \\ &\quad + uqw_2(w - x) + mq_1(w - y) \\ &= 3cWXY - \frac{3}{4}ct(W + X + Y)^2 + \frac{3}{2}ct^2(W + X + Y) - ct^3 \\ &\quad - uq_2X - mq_1Y, \end{aligned}$$

on writing  $W = x + y$  and  $X = x - w$ ,  $Y = y - w$ . Thus if  $v$  is a prime  $p$  which does not divide  $6t$  we find that

$$S_4 = p \sum_{W, X, Y \pmod{p}} e(g(W, X, Y)/p)$$

where  $g(W, X, Y)$  takes the shape

$$c'WXY + t'(W^2 + W^2 + Y^2 + 2XY + 2WX + 2WY) + \mu_1W + \mu_2X + \mu_3Y - ct^3$$

with  $p \nmid c't'$ . Exponential sums of this type have been treated by Bombieri and Sperber [1, Theorem 7]. Their work shows that

$$S_4(c, m, u, t; p) \ll p^{5/2}$$

for such primes. When  $p \mid t$  it is easy to see that  $S_4(c, m, u, t; p) \ll p^2(p, m, u)$ , whence in general we have  $S_4(c, m, u, t; p) \ll p^{5/2}(p, t, m, u)^{1/2}$  for all primes  $p$ . While it is convenient to call on a theorem from the literature, we remark that it is possible to evaluate  $S_4(c, m, u, t; p)$  in terms of exponential sums in one variable, for which it suffices to use Weil's theorem rather than Deligne's.

By the multiplicative property (16) we now deduce that

$$S_4(c, m, u, t; q_3) \ll_{\varepsilon} q_3^{5/2+\varepsilon} (q_3, t, m, u)^{1/2},$$

whence (15) and Lemma 2 yield

$$\begin{aligned} \eta_6(r, m, u) &\ll_{\varepsilon} q_3^{3/2+\varepsilon} \sum_{-q_3/2 < t \leq q_3/2} \min(K, q_3/|t|) (q_3, t, m, u)^{1/2} \\ &\ll_{\varepsilon} q_3^{3/2+\varepsilon} \left\{ K(q_3, m, u)^{1/2} + \sum_{1 \leq t \leq q_3/2} \frac{q_3}{|t|} (q_3, t)^{1/2} \right\} \\ &\ll_{\varepsilon} q_3^{3/2+\varepsilon} \{K(q_3, m, u)^{1/2} + q_3^{1+\varepsilon}\}, \end{aligned}$$

by partial summation. It then follows using Lemma 2 that

$$\begin{aligned} N^3 q_3^{-3} \sum_{r \leq q} \sum_{1 \leq |m| \leq M} \sum_{1 \leq |u| \leq U} \frac{|\eta_6(r, m, u)|}{r} \\ &\ll_{\varepsilon} N^3 q_3^{-3/2+\varepsilon} \{KMU q_3^{\varepsilon} + MU q_3^{1+2\varepsilon}\} \\ &\ll_{\varepsilon} q^{3\varepsilon} \{q_1^2 q_2^2 q_3^{3/2} + N q_1 q_2 q_3^{3/2}\} \\ &\ll_{\varepsilon} N q_1 q_2 q_3^{3/2} q^{3\varepsilon}, \end{aligned}$$

since

$$N = \left( \frac{N^{3/2}}{q_1 q_2} \right)^2 \frac{q_1^2 q_2^2}{N^2} \geq \left( \frac{q}{q_1 q_2} \right)^2 \frac{q_1^2 q_2^2}{N^2} = q_1 q_2 \frac{q_1 q_3}{N} \cdot \frac{q_2 q_3}{N} \geq q_1 q_2.$$

Now, when we insert this estimate into Lemma 4, we see that Theorem 2 follows, with a new value for  $\varepsilon$ .

## 6 Proof of Theorems 3 and 1

To prove Theorem 3 it clearly suffices to suppose that  $\alpha = \sqrt{d}$  for some non-square  $d \in \mathbb{N}$ . Let  $a, b \in \mathbb{N}$  be solutions of the Pell equation  $a^2 - db^2 = 1$  and define  $p_n, q_n$  by

$$p_n + q_n\sqrt{d} = \eta^n$$

where  $\eta = a + b\sqrt{d}$ . It follows that

$$q_n = (2\sqrt{d})^{-1}(\eta^n - \eta^{-n}) = (2\sqrt{d})^{-1} \prod_{k|n} \Phi_k(\eta, \eta^{-1}),$$

where

$$\Phi_k(X, Y) = \prod_{1 \leq h \leq k, (h, k)=1} (X - e^{2\pi i h/k} Y)$$

is the  $k$ -th cyclotomic polynomial. Thus  $\Phi_k(X, Y) \in \mathbb{Z}[X, Y]$  and  $\Phi_k(X, X^{-1}) = \Phi_k(X^{-1}, X)$  except for  $k = 1$ , in which case  $\Phi_1(X, Y) = X - Y$ . It follows that

$$q_n = b \prod_{k|n, k \geq 2} r_k$$

for integers  $r_k = |\Phi_k(\eta, \eta^{-1})|$ . Moreover

$$r_k \leq (\eta + \eta^{-1})^{\phi(k)} = (2a)^{\phi(k)} \leq (2a)^{\phi(n)}.$$

Now fix an integer  $m$  such that

$$\frac{\phi(m)}{m} \leq \varepsilon \frac{\log \eta}{2 \log(2a)}.$$

Then if  $m|n$  we have  $\phi(n)/n \leq \phi(m)/m$ , whence

$$r_k \leq (2a)^{\phi(n)} \leq \{(2a)^n\}^{\phi(m)/m} \leq \{(2a)^n\}^{\varepsilon(\log \eta)/(2 \log(2a))} = (\eta^{n/2})^\varepsilon.$$

However

$$q_n = (2\sqrt{d})^{-1}(\eta^n - \eta^{-n}) \geq (4\sqrt{d})^{-1}\eta^n \geq \eta^{n/2}$$

for large enough  $n$ , whence  $r_k \leq q_n^\varepsilon$ . It follows that every prime factor of  $q_n$  is at most  $q_n^\varepsilon$ , if  $m|n$ .

Finally we observe that  $q_{n+1} \ll q_n$  with an implied constant depending on the choice of  $a, b$  and  $d$ , so that there is a value of  $n$  which is a multiple of  $m$  and for which  $N \geq q_n \gg N$ . Since

$$|p_n - q_n\sqrt{d}| = \frac{1}{p_n + q_n\sqrt{d}} \leq \frac{1}{q_n}$$

the theorem then follows.

To deduce Theorem 1 we write  $\alpha = (f + g\sqrt{d})/c$  and approximate  $\sqrt{d}$  as above, with

$$\left| \sqrt{d} - \frac{u}{v} \right| \leq \frac{1}{vV}, \quad V \geq v \gg V,$$

where we choose  $V = \lceil N^{3/2}/c \rceil$ . Let  $v_0$  be the product of all prime powers  $p^e \parallel q$  for which  $e \geq 2$ . Then the product of the prime divisors of  $v_0$  can be at most  $v_0^{1/2}$ . It follows that

$$\prod_{p|u^2v^2d} p \leq u \frac{v}{v_0} v_0^{1/2} d.$$

Since  $1 + v^2d = u^2$  the *abc*-conjecture would imply that  $u \ll_{\varepsilon, \alpha} (uvv_0^{-1/2})^{1+\varepsilon}$ , whence  $v_0 \ll_{\varepsilon, \alpha} v^{2\varepsilon}$ .

We now write  $a_1 = fv + gu$  and  $q_1 = cv$ , and set  $a = a_1/(a_1, q_1)$  and  $q = q_1/(a_1, q_1)$ , so that  $a$  and  $q$  are coprime, with

$$q \leq q_1 = cv \leq cV \leq N^{3/2}$$

and

$$q \gg_{\alpha} q_1 \gg_{\alpha} v \gg_{\varepsilon, \alpha} V \gg_{\alpha} N^{3/2}.$$

Then

$$\left| \alpha - \frac{a}{q} \right| \ll_{\alpha} \left| \sqrt{d} - \frac{u}{v} \right| \leq \frac{1}{vV} \ll_{\varepsilon, \alpha} \frac{1}{qN^{3/2}}.$$

Moreover every prime factor of  $q$  is  $O_{\varepsilon, \alpha}(q^{\varepsilon})$ , and if  $q_0$  is the product of all  $p^e \parallel q$  with  $e \geq 2$ , then  $q_0 \ll_{\varepsilon, \alpha} q^{2\varepsilon}$ .

We proceed to build up coprime square-free divisors  $q_2, q_3$  of  $q/q_0$ , one prime factor at a time, to produce products in the ranges

$$q^{5/21} \leq q_2 \ll_{\varepsilon, \alpha} q^{5/21+\varepsilon}, \quad q^{10/21} \leq q_3 \ll_{\varepsilon, \alpha} q^{10/21+\varepsilon}.$$

We will then have  $q = q_1 q_2 q_3$  with

$$q^{2/7-2\varepsilon} \ll_{\varepsilon, \alpha} q_1 \leq q^{2/7}.$$

One may then verify that the hypotheses of Theorem 2 are satisfied, and that

$$S(\alpha, N) \ll_{\varepsilon, \alpha} q^{10/21+2\varepsilon} \ll_{\varepsilon, \alpha} N^{5/7+3\varepsilon}.$$

This suffices for Theorem 1, on re-defining  $\varepsilon$ .

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