

AX-SCHANUEL FOR THE j -FUNCTION

JONATHAN PILA AND JACOB TSIMERMAN

ABSTRACT. In this paper we prove a functional transcendence statement for the j -function which is an analogue of the Ax-Schanuel theorem for the exponential function. It asserts, roughly, that atypical algebraic relations among functions and their compositions with the j -function are governed by modular relations.

To Peter Sarnak on the occasion of his 61st birthday

1. INTRODUCTION

Let $j : \mathbb{H} \rightarrow \mathbb{C}$ denote the classical modular function, or j -function, by means of which the quotient $Y(1)$ of the complex upper-half plane \mathbb{H} by the modular group is identified with \mathbb{C} . Here $Y(1)$ is the moduli space of elliptic curves over \mathbb{C} up to isomorphism. Contrary to some conventions, we shall consider \mathbb{H} as an open subset of $\mathbb{P}^1(\mathbb{C})$ rather than of \mathbb{C} . This is more natural for us since $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$. From now on, we write \mathbb{P}^1 to denote $\mathbb{P}^1(\mathbb{C})$ for brevity. By abuse of notation, we shall refer to the cartesian powers of this map as j regardless of how many variables are involved.

Let Γ denote the graph of the j function in $\mathbb{H}^n \times Y(1)^n$. For an algebraic subvariety $V \subset (\mathbb{P}^1)^n \times Y(1)^n$, a *component* of $V \cap \Gamma$ will mean a complex analytically irreducible component of $V \cap \Gamma$. The definition of a *weakly special subvariety* of $Y(1)^n$ is given in §2. A first version of our result is then the following.

Theorem 1.1 (Ax-Schanuel for $j(z)$). *Let $V \subset (\mathbb{P}^1)^n \times Y(1)^n$ be an algebraic subvariety, and let U be a component of $V \cap \Gamma$. Then*

$$\dim U = \dim V - n$$

unless the projection of U to $Y(1)^n$ is contained in a proper weakly special subvariety of $Y(1)^n$.

On dimensional grounds, a component U of the intersection of Γ with some algebraic variety V in $(\mathbb{P}^1)^n \times Y(1)^n$ has expected dimension

$$\dim U = \dim \Gamma + \dim V - \dim ((\mathbb{P}^1)^n \times Y(1)^n) = \dim V - n,$$

and the actual dimension can never be smaller than the expected dimension [14]. The theorem asserts that, for the graph Γ , any such component of larger than expected dimension is accounted for by weakly special subvarieties.

If we replace $j : \mathbb{H}^n \rightarrow \mathbb{C}^n$ by $\exp : \mathbb{C}^n \rightarrow (\mathbb{C}^\times)^n$, the cartesian power of the exponential function, and take weakly special subvarieties in $(\mathbb{C}^\times)^n$ to be cosets of algebraic subtori, then this is a formulation of “Ax-Schanuel” for the exponential function, as may be found in Ax [3]. The more usual statement of Ax’s theorem [2] is in the setting of a differential field. We will give a statement in that setting below, though in a stronger form. The connection between the two settings is afforded by the Seidenberg embedding theorem, by means of which a differential field of characteristic zero (finitely generated over \mathbb{Q} as a differential field) may be embedded in a field of meromorphic functions in several variables. We indicate in §2 how to deduce the differential version of our theorem from the complex version.

Theorem 1.1 generalises the “Ax-Lindemann” theorem for $j(z)$ established in [22] in connection with the André-Oort conjecture for $Y(1)^n$. We expect that Theorem 1.1 will be useful in connection with the broader Zilber-Pink conjecture for $Y(1)^n$: a preliminary application is given in section 7. On the Zilber-Pink conjecture generally see [6, 26, 27, 32, 33]. A special case of Theorem 1.1 (“Modular Ax-Logarithms”) is used by Habegger-Pila [11] in affirming a very special case of the Zilber-Pink conjecture for $Y(1)^n$. A more general result (but less general than 1.1) appears as one of two hypotheses in a conditional affirmation of Zilber-Pink for $Y(1)^n$ in [12]. Related functional transcendence results in connection with Zilber-Pink have also appeared in the work of Bertrand-Masser-Pillay-Zannier [17, 4] on “torsion anomalous” points.

However, we want to frame a stronger result that is a transcendence theorem not just for $j(z)$ but also for $j'(z)$ and $j''(z)$. Including further derivatives gains nothing as $j'''(z)$ is algebraic over $j(z), j'(z)$ and $j''(z)$ (see §2.4). To state a theorem encoding the derivatives geometrically we need the concept of a *jet space*.

Definition. *For a complex analytic manifold M and a positive integer k , the k 'th jet space $J_k M$ is a fiber bundle over M whose fiber over $m \in M$ consists of equivalence classes of germs of holomorphic maps from a neighborhood of $0 \in \mathbb{C}$ to M taking 0 to m , where we declare 2 such maps equivalent if they agree up to order k . For example, $J_1 M$ is the tangent bundle of M .*

A map $M \rightarrow N$ induces naturally a map $J_k M \rightarrow J_k N$. We let $J_k^0 M \subset J_k M$ denote the sub-space consisting of germs of functions with vanishing first derivative. If M is an algebraic variety, then $J_k M$ can be naturally given such a structure as well. For a complex open disc D parameterized by a holomorphic variable z , we can give local co-ordinates for $J_k D \cong D \times \mathbb{C}^k$ by letting $(z, a_1, a_2, \dots, a_k)$ denote the germ at 0 of the function

$$f(w) = z + \sum_{i=1}^k a_i w^i / i!.$$

In these co-ordinates, $J_k^0 M$ is locally defined by the equation $a_1 = 0$. An *algebraic subvariety* of $J_k \mathbb{H}^n \times J_k Y(1)^n$ will mean a complex analytically

irreducible component of the intersection with $J_k\mathbb{H}^n \times J_kY(1)^n$ with an algebraic subvariety in the ambient complex affine space. For $k \geq 0, n \geq 1$ let $J_k j : J_k\mathbb{H}^n \rightarrow J_kY(1)^n$ denote the map induced by j , and let Γ_k^n denote the graph of this map. When k, n are fixed we will just refer to the graph as Γ . Our result for j and its derivatives is then the following.

Theorem 1.2 (Ax-Schanuel with derivatives for $j(z)$). *Let $V \subset J_2\mathbb{H}^n \times J_2Y(1)^n$ be an algebraic subvariety, and let U be a component of $V \cap \Gamma$. Then*

$$\dim U = \dim V - 3n$$

unless the projection of U to $Y(1)^n$ is contained in a proper weakly special subvariety, or one of the projections of U to $J_2\mathbb{H}$ is contained inside $J_2^0\mathbb{H}$.

In fact, the above theorem is equivalent to the following purely differential-algebraic version. Let K be a characteristic 0 differential field with m commuting derivations D_i . Let $C = \bigcap_i \ker D_i$ be the constant field of K . In the following theorem, $F(j, j', j'', j''') = 0$ is the algebraic differential equation satisfied by $j(z)$ (see §2.4), Φ_n are the *modular polynomials*, and the *rank* of a matrix is over K .

Theorem 1.3. *Let $z_i, j_i, j'_i, j''_i, j'''_i \in K^\times, i = 1, \dots, n$, such that*

$$D_k j_i = j'_i D_k z_i, \quad D_k j'_i = j''_i D_k z_i, \quad \text{and} \quad D_k j''_i = j'''_i D_k z_i$$

for all i and k . Suppose further that $F(j_i, j'_i, j''_i, j'''_i) = 0$ for $i = 1, \dots, n$, and that $\Phi_N(j_i, j_k) \neq 0$ for all i, k, N and $j_i \notin C$ for all i . Then

$$\text{tr.deg.}_C C(z_1, j_1, j'_1, j''_1, \dots, z_n, j_n, j'_n, j''_n) \geq 3n + \text{rank}(D_k z_i)_{i,k}.$$

We show in §2 that Theorem 1.2 implies Theorem 1.3 (and *vice versa*). The rest of the paper, except section 7, is devoted to the proof of Theorem 1.2 and we adopt its hypotheses throughout. Thus $V \subset J_2\mathbb{H}^n \times J_2Y(1)^n$ is an algebraic variety, and U is an irreducible component of $V \cap \Gamma$. We assume that the projection of U to $Y(1)^n$ is not contained in a proper weakly special subvariety, and none of the projections of U to $J_2\mathbb{H}$ are contained inside $J_2^0\mathbb{H}$.

Amongst other ingredients, we rely on the theory of o-minimality. We will say that a set is *definable* if it is definable in the o-minimal structure $\mathbb{R}_{\text{an, exp}}$. For an introduction to this theory, see [8]; on $\mathbb{R}_{\text{an, exp}}$ see [9, 10]. The second author [31] gave a proof of the exponential Ax-Schanuel theorem using o-minimality.

After some preliminary results are gathered in §2, and some initial “conditioning” of V is undertaken in §3, the proof of 1.2 is carried out in §§4-6. Some applications to the Zilber-Pink conjecture are given in §7.

The main elements of the proof can be summarized as follows. We study a certain definable set Z of elements $\gamma \in \text{SL}_2(\mathbb{R})^n$ such that γV and a suitable component U_γ of $\gamma V \cap \Gamma$ have the same properties as V and U . As for analogous sets in previous applications of o-minimality to “Ax-Lindemann” theorems, we need to show that Z has “many” integer points. Here, a

key new ingredient required for this step is a volume estimate of Hwang-To [13]. This enables us, in §4, to show that U is present over “many” fundamental domains $\gamma\mathcal{F}^n$ for the action of $\mathrm{SL}_2(\mathbb{Z})^n$ on \mathbb{H}^n . Here \mathcal{F} is the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{H} . Thus Z has “many” integer points.

An application of the Counting Theorem of Pila-Wilkie [25] now shows that Z contains algebraic curves. With these we may seek to increase the dimension of V and U , or to reduce the difference $\dim V - \dim U$, which we handle by induction. Eventually, we reduce to the situation where V has a “large” stabilizer.

Studying the possible structure of the Zariski closure of the integer points of the stabilizer of V allows us to show that the image of U under projection to $Y(1)^n$ is accounted for by finitely many fundamental domains, and hence is definable. We are then in a position to apply a powerful o-minimal version of Chow’s theorem due to Peterzil-Starchenko [18, 20] to prove that the image of U in $Y(1)^n$ is algebraic. It is in this step only that our restriction to powers of the modular curve, rather than a general Shimura variety, is essential, as $\mathrm{SL}_2(\mathbb{R})^n$ has relatively few subgroups.

It is worth noting that an interesting dichotomy takes place in this last step (see §5): after quotienting by the semisimple part of the Zariski closure of the stabilizer, we are left with two cases: (i) the resulting group is finite, at which point our methods are ‘hyperbolic’ and we use the theorem of Hwang-To to get polynomial growth and arrive at a contradiction, or (ii) the resulting group might be unipotent, in which case the covering map looks like the exponential map, and our proof reduces to the proof in [31] of the ordinary Ax-Schanuel conjecture. In particular, in case (ii) we heavily rely on the ‘growth implies polynomial growth’ phenomenon for unipotent groups.

Finally, the algebraicity of the image of U in $Y(1)^n$ allows us to use monodromy properties due to Deligne-André [1, 7] to complete the proof.

Suitable “Ax-Schanuel” formulations should hold for any (mixed) Shimura variety. In particular for \mathcal{A}_g , generalising “Ax-Lindemann” [24], we expect the following.

Conjecture 1.1. *Let $\pi : \mathbb{H}_g \rightarrow \mathcal{A}_g$ be the classical uniformisation of the moduli space of principally polarised abelian varieties of dimension g . Let Γ be the cartesian power of the graph of π in $\mathbb{H}_g^n \times \mathcal{A}_g^n$. Let $V \subset \mathbb{H}_g^n \times \mathcal{A}_g^n$ be an algebraic variety and U an irreducible component of $V \cap \Gamma$. If the projection of U to \mathcal{A}_g is not contained in a proper weakly special subvariety then $\dim V = \dim U + n \frac{g(g+1)}{2}$.*

In fact, we can be more ambitious and formulate a version with derivatives for a general Shimura variety. Consider a Shimura variety S corresponding to a *semisimple* real group G and the cover $\pi : \mathbb{H} \rightarrow S$. Look at the map $J_k\pi : J_k\mathbb{H} \rightarrow J_kS$ for $k \geq 2$. Consider the graph Γ_k^S , and its Zariski closure

Z_k^S . For $S = \mathcal{A}_g$ we have $\dim Z_k^S - \dim \Gamma_k^S = \dim G$ by results of Bertrand-Zudilin [5]. This suggests the following formulation.

Conjecture 1.2. *Let $\pi : \mathbb{H} \rightarrow S$ be the uniformizing maps of a Shimura variety S , corresponding to a semisimple group G . Let $k \geq 2$ and $V \subset J_k \mathbb{H} \times J_k S$ be a Shimura variety whose projection to S is not contained in a weakly special subvariety. Moreover, assume that for a generic point $v \in V$ the projection of v to $J_1 S$ does not lie in the image of any weakly special subvariety. Let U be a positive dimensional component of $V \cap \Gamma_k^S$. Then $\dim V = \dim U + \dim G$.*

Let us explain the appearance of the assumption “the projection to $J_1 S$ does not lie in a weakly special subvariety”. Suppose \mathbb{H} splits as $\mathbb{H}_1 \times \mathbb{H}_2$, and consider a point $x \in J_1 \mathbb{H}$. This corresponds to a point (h_1, h_2) with a tangent vector (v_1, v_2) . Now $G(\mathbb{C})$ acts on $J_1 \mathbb{H}$, and if, say, $v_1 = 0$ then the action of $G(\mathbb{C})$ preserves this property. So, the variety V given by insisting that the projections to $J_1 \mathbb{H}_1$ as well as its image in the quotient have vanishing tangent vector should yield a counterexample. The hypothesis in the conjecture is designed to prevent this from happening.

2. PRELIMINARIES

2.1. Notation. We let \mathcal{F} denote the usual fundamental domain for the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{H} .

The inclusion $\mathcal{F} \subset \mathbb{H}$ gives an inclusion of jet spaces and we identify $J_k \mathcal{F}$ with its image in $J_k \mathbb{H}$. We denote $J_2 \mathcal{F} \times J_2 Y(1)$ by $\overline{\mathcal{F}}$.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on \mathbb{H} by fractional linear transformations, inducing an action on $J_2 \mathbb{H}$. We extend this action to $J_2 \mathbb{H}^n \times J_2 Y(1)^n$ by setting the action to be trivial on $J_2 Y(1)^n$. We will have use for the extension of this action to $\mathrm{SL}_2(\mathbb{C})$ acting on \mathbb{P}^1 and $J_2 \mathbb{P}^1$.

Then $\overline{\mathcal{F}}$ is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $J_2 \mathbb{H} \times J_2 Y(1)$. The reader may also check that the action of $\mathrm{SL}_2(\mathbb{Z})$ preserves the graph of $J_2 j$.

2.2. Definability. We use the basic observation, made by Peterzil and Starchenko, that the j -function restricted to \mathcal{F} is definable. Then the graph of $J_2 j$ in $J_2 \mathcal{F} \times J_2 Y(1)$ is likewise definable, along with its cartesian products.

2.3. Weakly special and Mobius subvarieties. If $g \in \mathrm{GL}_2^+(\mathbb{Q})$ then $j(z)$ and $j(gz)$ are algebraically dependent functions, related by a modular polynomial. Indeed, weakly special subvarieties, which we now define, are just those algebraic subvarieties of \mathbb{H}^n whose image under j is also algebraic. These images we refer to as weakly special subvarieties of $Y(1)^n$.

Definition. *A weakly special subvariety of $Y(1)^n$ is an irreducible (over \mathbb{C}) component of a closed algebraic subset of $Y(1)^n$ defined by an arbitrary finite collection of equations of the form $\Phi_N(x_h, x_k) = 0$ and $x_\ell = c$, a complex constant.*

We will also refer to weakly special subvarieties of \mathbb{H}^n . These are (the intersections with \mathbb{H}^n of) varieties defined by an arbitrary collection of equations of the form $z_h = gz_k$, where $k, h \in \{1, \dots, n\}$ are distinct and $g \in \mathrm{GL}_2^+(\mathbb{Q})$ or $z_\ell = c$, a complex constant.

The image under j of a weakly special subvariety of \mathbb{H}^n is a weakly special subvariety of $Y(1)^n$.

In \mathbb{H}^n we will make use of a larger class of *Mobius subvarieties*, which are defined analogously to weakly special subvarieties except that the matrices g are allowed to be any element of $\mathrm{SL}_2(\mathbb{R})$.

2.4. The differential equation satisfied by $j(z)$. The function $j(z)$ satisfies a nonlinear third order algebraic differential equation, and none of lower order [15]. Specifically (see e.g. [16]),

$$F(j, j', j'', j''') = Sj + \frac{j^2 - 1968j + 2654208}{2j^2(j - 1728)^2} (j')^2 = 0,$$

where Sf denotes the *Schwarzian derivative* $Sf = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$ and $'$ indicates differentiation with respect to z .

2.5. Proof of equivalence of Theorem 1.2 and Theorem 1.3. We introduce a further piece of notation. If f_1, \dots, f_n are complex functions of some variables w_1, \dots, w_m , we denote by $\dim(f_1, \dots, f_n)$ the (complex) dimension of the locus $\{f_1(\bar{w}), \dots, f_n(\bar{w})\}$ as \bar{w} varies over a small open disk. If f_i belong to a differential field of meromorphic functions, with derivations D_i corresponding to differentiation w.r.t. w_i , then $\dim(f_1, \dots, f_n) = \mathrm{rank}(D_k f_i)$. Similarly, the transcendence degree $\mathrm{tr.deg}_{\mathbb{C}} \mathbb{C}(f_1, \dots, f_n)$ is the dimension of the Zariski closure of the same locus, which we denote $\dim \mathrm{Zcl}(f_1, \dots, f_n)$.

Proof. We first prove that 1.2 implies 1.3. Let C' be a subfield of C finitely generated (as a field) over \mathbb{Q} , and let $K' = C'\langle z_i, j_i, j'_i, j''_i \rangle$ be the differential field generated by the indicated elements over C' . By the Seidenberg embedding theorem, K' with its derivations $D_k, k = 1, \dots, \ell$ is isomorphic to a field of meromorphic functions in variables w_1, \dots, w_ℓ with D_k acting as differentiation with respect to w_k . We identify K' with its image, in which the image of C' (whose elements vanish under all the derivations) is a subfield of \mathbb{C} . By virtue of the differential relations we have that $j_i = j(g_i z_i)$ for suitable $g_i \in \mathrm{SL}_2(\mathbb{C})$. By replacing the z_i by $g_i z_i$ (which does not change the transcendence degree of the z_i over \mathbb{C}), we may assume that $j_i = j(z_i)$. The locus of $\{z_1, \dots, z_n\}$ has dimension $r = \mathrm{rank}(D_k z_i)_{i,k}$.

Let $U \subset J_2 \mathbb{H}^n \times J_2 Y(1)^n$ denote, in local coordinates, the graph of

$$(z_1, 1, 0, j(z_1), j'(z_1), j''(z_1), \dots, z_n, 1, 0, j(z_n), j'(z_n), j''(z_n)).$$

Note that here and below j' denotes the derivative of $j : \mathbb{H} \rightarrow \mathbb{C}$ with respect to its argument, and $j'(z_i)$ denotes the composition of j' with z_i , not the derivative of $j(z_i)$ with respect to z_i .

Under the hypotheses of 1.3, the conditions of 1.2 on U are satisfied. Applying Theorem 1.2 we have that

$$\text{tr.deg.}_{\mathbb{C}} \mathbb{C}(z_1, j_1, j_1', j_1'', \dots, z_n, j_n, j_n', j_n'') \geq 3n + \text{rank}(D_k z_i)_{i,k}$$

and hence that

$$\text{tr.deg.}_{C'} C'(z_1, j_1, j_1', j_1'', \dots, z_n, j_n, j_n', j_n'') \geq 3n + \text{rank}(D_k z_i)_{i,k}.$$

Since this holds for every subfield of C finitely generated over \mathbb{Q} we get the conclusion of Theorem 1.3.

For the converse, parameterize U locally as $(\vec{z}, \vec{w}, \vec{r}, \vec{j}, \vec{j}_1, \vec{j}_2)$ where we set $\vec{z} = (z_1, \dots, z_n)$, $\vec{w} = (w_1, \dots, w_n)$, $\vec{r} = (r_1, \dots, r_n)$,

$$\vec{j} = (j(z_1), \dots, j(z_n)), \quad \vec{j}_1 = (j'(z_1)w_1, \dots, j'(z_n)w_n),$$

and

$$\vec{j}_2 = \left(\frac{1}{2} \cdot j''(z_1)w_1^2 + j'(z_1)r_1, \dots, \frac{1}{2} \cdot j''(z_n)w_n^2 + j'(z_n)r_n \right).$$

We must prove that

$$\dim \text{Zcl}(\vec{z}, \vec{w}, \vec{r}, \vec{j}, \vec{j}_1, \vec{j}_2) \geq 3n + \dim(\vec{z}, \vec{w}, \vec{r}).$$

As all the $w_i \neq 0$ by assumption, this is equivalent to

$$\dim \text{Zcl}(\vec{z}, \vec{w}, \vec{r}, \vec{j}, \vec{j}', \vec{j}'') \geq 3n + \dim(\vec{z}, \vec{w}, \vec{r})$$

where

$$\vec{j}' = (j'(z_1), \dots, j'(z_n)), \quad \vec{j}'' = (j''(z_1), \dots, j''(z_n)).$$

By Theorem 1.3 we get that

$$\dim \text{Zcl}(\vec{z}, \vec{j}, \vec{j}', \vec{j}'') \geq 3n + \dim(\vec{z}).$$

We now obtain the conclusion of Theorem 1.2, since

$$\dim \text{Zcl}(A, B) - \dim \text{Zcl}(A) \geq \dim(A, B) - \dim(A),$$

for any two finite sets of functions A, B . □

2.6. Deduction of 1.1 from 1.2. We have shown that 1.3 implies 1.2. Now 1.2 clearly implies the weakened conclusion (under the same hypotheses) that

$$\dim \text{Zcl}(z_1, j_1, \dots, z_n, j_n) \geq n + \dim(z_1, \dots, z_n).$$

Given the hypotheses of 1.1, if we parameterize U by (z_i, j_i) we then get the conclusion of 1.1 from the above.

3. THE FUNDAMENTAL DOMAINS THAT U GOES THROUGH

Most of the argument takes place in the setting of Theorem 1.1, without the derivatives. Accordingly we make the following definition.

Definition. *Throughout we let U_1 denote the projection of U to \mathbb{H}^n . Note that U_1 is isomorphic to U since $U \subset \Gamma$, and so U_1 is an irreducible analytic variety, and hence is pure-dimensional.*

Without loss of generality, we may assume that U_1 has non-empty intersection with the interior of \mathcal{F}^n . For an irreducible complex analytic variety $S \subset \mathbb{H}^n$ we denote by S^{Mob} the smallest Mobius subvariety containing S .

Definition. *For a complex analytic variety $X \subset \Gamma$, a positive integer d , and a Mobius subvariety $M \subset \mathbb{H}^n$, we define $G_d(X, M)$ to be the set of points $x \in X$ around which X is regular of dimension d , and such that the irreducible component X_0 of the projection of X to \mathbb{H}^n containing x satisfies $X_0^{\text{Mob}} = M$.*

Let $d_1 = \dim U_1$ and $M_1 = U_1^{\text{Mob}}$. We make the following definition:

$$Z = \{\gamma \in \text{SL}_2(\mathbb{R})^n \mid \dim G_{d_1}((\gamma^{-1} \cdot V) \cap (\Gamma \cap \overline{\mathcal{F}^n}), \gamma^{-1} M_1) = \dim U_1\}$$

so that Z is a definable set (in $\mathbb{R}_{\text{an, exp}}$). Indeed, V and $\Gamma \cap \overline{\mathcal{F}^n}$ are both definable, the conditions of being smooth and of a given dimension are definable, irreducible components are definable, and if S is complex analytic and irreducible, S^{Mob} is definable; moreover all these things are uniformly definable in parameters.

Note that, due to the invariance of Γ under $\text{SL}_2(\mathbb{Z})^n$, the integral points of Z include those $\gamma \in \text{SL}_2(\mathbb{Z})^n$ such that U_1 is present in the fundamental domain $\gamma \overline{\mathcal{F}^n}$, so that

$$\forall \gamma \in \text{SL}_2(\mathbb{Z})^n, \quad \gamma \mathcal{F}^n \cap U_1 \neq \emptyset \rightarrow \gamma \in Z(\mathbb{Z}).$$

Now $\mathbb{R}_{\text{an, exp}}$ admits analytic stratification (see [10]). So we may write $Z = \bigcup_{i=1}^m Z_i$ where each Z_i is a real-analytic cell, which is to say it is a real-analytic submanifold, and definably-analytically isomorphic to ball of some dimension. Assume Z_1 contains the identity. By replacing V by $\gamma^{-1}V$ by an appropriate $\gamma \in Z$ we assume that $\dim Z_1 \geq \dim Z_i$ for all i .

Now, define S to be the stabilizer of V . We show that by possibly modifying V and U a little (but preserving the hypotheses of 1.2 and the dimensions of both V and U) we may assume certain regularity properties of Z .

Lemma 3.1. *By replacing V by $\gamma_0^{-1} \cdot V$ for a (measure theoretically) generic set of elements $\gamma_0 \in Z_1$ we can ensure that $S(\mathbb{Z}) \subset \gamma S \gamma^{-1}$ for all $\gamma \in Z_1$. Moreover, we can ensure that, for all $k \neq 1$, for each element $t \in Z_k(\mathbb{Z})$ we have $t \in \gamma^{-1} \cdot Z_k$ for any γ in an open neighbourhood of the identity in Z_1 .*

Proof. Consider an element $t \in \text{SL}_2(\mathbb{Z})^n$, and define the set

$$Z_t := \{\gamma \in Z_1 \mid t \in \gamma^{-1} S \gamma\}.$$

Then either $Z_t = Z_1$ or Z_t has measure 0 in Z_1 . Take $Z_1^0 \subset Z_1$ to be in the complement of all Z_t for which $Z_t \neq Z_1$. Since there are only countable many integral points, Z_1^0 has full measure in Z_1 . Now we could pick $\gamma_0 \in Z_1^0$, but we have to be careful that $\gamma^{-1}M_1$ does not satisfy any equation of the form $x_k = \gamma x_j$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, otherwise we move U_1 to be inside a weakly-special variety. However, there are finitely many such equations, and so this also gives a full measure subset Z_1^{00} inside of Z_1^0 . Now pick any $\gamma_0 \in Z_1^{00}$.

For the second part of the lemma, the same argument works once we notice that replacing V by $\gamma_0^{-1}V$ replaces Z by $\gamma_0^{-1}Z$. \square

From now on we assume V is as in the conclusion of Lemma 3.1. Let G be the Zariski closure of $S(\mathbb{Z})$, and $N(G)$ the normalizer of G .

Proposition 3.2. *The set $Z(\mathbb{Z})$ is — and in particular the elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ such that U_1 goes through the fundamental domain $\gamma \cdot \overline{\mathcal{F}}^n$ are — contained in a finite union of right $N(G)(\mathbb{Z})$ orbits.*

Proof. By the above lemma elements of $\gamma \in Z_1(\mathbb{Z})$ satisfy $S(\mathbb{Z}) \subset \gamma^{-1}S(\mathbb{Z})\gamma$. It follows that $Z_1(\mathbb{Z}) \subset N(G)$.

Next, fix $k \neq 1$, take a point $t \in Z_k(\mathbb{Z})$. By the above lemma, $\gamma \cdot t \in Z_k$ for γ in a open neighbourhood of the identity in Z_1 . Since $\dim Z_1 \geq \dim Z_k$, this implies that if we take E to be an open neighborhood of Z_1 , then Et is open in Z_k . Now, since we saw that $S(\mathbb{Z}) \subset e^{-1}Se$ for all $e \in E$ it follows by analytic continuation that for all $z \in Z_k$ we have that

$$S(\mathbb{Z}) \subset tz^{-1}Szt^{-1}.$$

It follows that for $z \in Z_k(\mathbb{Z})$ we have $z \in N(G)(\mathbb{Z})t$. Since there are only finitely many Z_k , the proof is finished. \square

4. VOLUME ESTIMATES AND POLYNOMIAL PRESENCE

Lemma 4.1. *For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ the quantity $\mathrm{Vol}(\gamma \cdot U_1 \cap \mathcal{F}^n)$ is uniformly bounded above.*

Proof. We will prove a stronger statement. Let $d = \dim U_1$. Now, the form giving the $2d$ real-dimensional volume form is bounded above by the sum of the pullbacks of the volume forms from every $2d$ real-dimensional subspace, as follows immediately from the generalization of De Gua's theorem to arbitrary dimensions (for this oft-rediscovered result see e.g. [28]). Integrating, we get

$$\mathrm{Vol}(\gamma \cdot U_1 \cap \mathcal{F}^n) \leq \sum_{\substack{I \subset \{1, \dots, n\} \\ |I| = \dim U_1}} \deg(\pi_I)$$

where $\deg(\pi_I)$ is the degree of the projection from $\gamma \cdot U_1$ onto the I 'th coordinates of \mathbb{H}^n .

Now for $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$,

$$\gamma U_1 \cap \mathcal{F}^n \subset \pi_{\mathbb{H}^n}(\gamma \cdot V \cap \Gamma) \cap \mathcal{F}^n.$$

But now we can define this for any $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$, and since degree is a definable function taking integer values it is uniformly bounded. This completes the proof. \square

Let $p \in U_1$ be a fixed point, and consider the geodesic ball $B(p, r)$ around p of radius r for the hyperbolic metric. By a theorem of Hwang-To [13] the volume of $U_1 \cap B(p, r)$ grows at least like $c_1 c_2^r$, for constants $c_1 > 0$ and $c_2 > 1$. In view of the previous Lemma, $U_1 \cap B(p, r)$ must intersect exponentially (in r) many fundamental domains. Now, by Proposition 5.2 of [22] all of these fundamental domains are of the form $\gamma \cdot \mathcal{F}^n$, where the height of γ is at most K^r for some $K > 0$. Recall that each of these fundamental domains contributes an integer point in Z .

This is precisely the setting that the Pila-Wilkie counting theorem is useful for. By setting $T = K^r$, we see that Z is a definable subset which, for any positive real number T , has $\gg T^\delta$ integer points of height at most T , for some $\delta > 0$. Therefore, by the Counting Theorem [25], we learn that Z contains semi-algebraic curves.

Let $C_{\mathbb{R}}$ be such a curve, with C the complex algebraic curve containing $C_{\mathbb{R}}$. For each c in an open neighborhood containing $C_{\mathbb{R}}$, let U_c denote a component of $c \cdot V \cap \Gamma$ such that $\dim U_c = \dim U$ and $(\pi_{\mathbb{H}^n} U_c)^{\mathrm{Mob}} = U_1^{\mathrm{Mob}}$. Let $c_0 \in C_{\mathbb{R}}$. We now have three cases:

- (i) $c \cdot V$ is not independent of $c \in C_{\mathbb{R}}$, and $c \cdot V$ does not contain U_{c_0} for all $c \in C_{\mathbb{R}}$. Then we may take $V' = C \cdot V$ to get a new pair V', U' where U' is a component of $\cup_{c \in C} U_c$ increasing the dimension of each of U and V by one.
- (ii) $c \cdot V$ is not independent of $c \in C$ but $c \cdot V$ contains U_{c_0} for all $c \in C$. Then we replace V, U by $V' = V \cap c \cdot V, U' = U_{c_0}$ for a generic $c \in C$, which reduces the dimension of V by one but leaves the dimension of U unchanged.
- (iii) $c \cdot V$ is independent of $c \in C$. Then $C_{\mathbb{R}}$ is contained in a right coset of S .

Start of proof of Theorem 1.2. We argue by induction (upwards) on $\delta = \dim V - \dim U$. The case $\delta = 0$ is as follows. Suppose $U = V$. Then the projection $\pi_{Y(1)^n} U = \pi_{Y(1)^n} V$ is algebraic. This case can be handled with monodromy arguments, and we do this in lemma 6.1.

For a given δ we argue by induction (downwards) on $\dim U$, the case $\dim U = \dim \Gamma = 3n$ being obvious. If $C \subset Z$ is a semi-algebraic curve as above, we either reduce δ if we are in case (ii), or increase $\dim U$ maintaining the value of δ if we are in case (i), or find ourselves in case (iii).

Therefore, by induction, we may assume from now on that we are always in case (iii), and that the conclusion of Lemma 3.1 holds. We thus learn

that

(*) *All maximal algebraic sets in Z having integer points are right cosets of S .*

By the Counting Theorem of [25], the integer points in Z of height at most T can be covered by fewer than T^ϵ algebraic subsets of Z for any $\epsilon > 0$. It follows that for any T , one of the cosets gS contains at least $T^{\delta/2}$ integer points of height at most T , and thus that $S = (gS) \cdot (gS)^{-1}$ contains at least T^δ integer points of height at most T^2 , and thus G has positive dimension. The next section deals with this case.

5. GROUP THEORY: SPLITTING OF CO-ORDINATES

Throughout this entire section we assume we are in case (iii) from section 4 (so that (*) holds), and we will show that the image of U in $Y(1)^n$ is algebraic. This requires some Lie-theoretic analysis.

Consider the Lie algebra $L(G)$. Let $L(G)_{ss}$ be a maximal semisimple subalgebra of $L(G)$. It is clear that $L(G)_{ss}$ is isomorphic to \mathfrak{sl}_2^k . Note that these \mathfrak{sl}_2 's are defined over \mathbb{Q} as the \mathbb{Q} -points are Zariski dense in G . Next set \mathfrak{b} to be the nilpotent radical of $L(G)$. Consider the projection maps $\pi_i : L(G) \rightarrow \mathfrak{sl}_2$ to the i 'th co-ordinate of \mathfrak{sl}_2^n , stemming from the inclusion $G \subset \mathrm{SL}_2^n$.

Lemma 5.1. *For each i , $\pi_i(\mathfrak{b})$ and $\pi_i(L(G)_{ss})$ are not both non-zero. In particular, \mathfrak{b} and $L(G)_{ss}$ commute.*

Proof. Since the maximal nilpotent subalgebras of \mathfrak{sl}_2 are 1-dimensional, $\pi_i(\mathfrak{b})$ is either zero, or 1-dimensional. Likewise, as semisimple lie algebras have dimension either 0 or at least 3, $\pi_i(L(G)_{ss})$ is either zero or all of \mathfrak{sl}_2 . Since \mathfrak{b} is the nilpotent radical of $L(G)$, \mathfrak{b} is an ideal of $L(G)$, and so $\pi_i(\mathfrak{b})$ is an ideal of $\pi_i(L(G))$. The claim follows, since \mathfrak{sl}_2 has no non-trivial ideals. \square

Let I_{ss} be the set of i for which $\pi_i(L(G)_{ss})$ is non-empty, I_b be the set of i for which $\pi_i(\mathfrak{b})$ is empty, and I_e be the rest of the co-ordinates. By conjugating by elements of $\mathrm{SL}_2(\mathbb{Z})$ we may (and do) assume that \mathfrak{b} is contained entirely in the upper triangular matrices. Now, we study $N(G)$. For the semi-simple part note that $\pi_{I_{ss}}(L(N(G))) = \mathbf{1}$ which implies that

$$(1) \quad \pi_{I_{ss}}(L(N(G))) = \pi_{I_{ss}}(L(G)).$$

Thus, denoting by G_{ss} the connected part of the semi-simple part of G , we see that G_{ss} is supported only on the I_{ss} co-ordinates, and $\pi_{I_{ss}}G_{ss}$ is of finite index in its own normalizer.

For the unipotent part, note that $\pi_{I_b}(L(N(G)))$ is contained in the upper triangular Borel. However, the only integral points in the Borel are unipotent, which means that $\pi_{I_b}(N(G))(\mathbb{Z})$ contains the upper triangular unipotents $M(\mathbb{Z}) \cong \mathbb{Z}^{|I_b|}$ as a subgroup of finite index. Set G_b to be the unipotent radical of G , which we know is only supported on the I_b co-ordinates.

Lemma 5.2. $G_b(\mathbb{Z}) \backslash \pi_{I_b \cup I_r}(U_1) \subset G_b(\mathbb{Z}) \backslash \mathbb{H}^{I_b \cup I_r}$ is complex analytic.

Proof. Since U_1 is $G(\mathbb{Z})$ invariant, we know that

$$G_b(\mathbb{Z}) \backslash \pi_{I_b \cup I_r}(U_1) = \text{im} \bigcup_{\gamma \in G_b \times G_{ss}(\mathbb{Z}) \backslash Z(\mathbb{Z})} \pi_{I_b \cup I_r}(U_1 \cap \gamma \cdot (\mathcal{F}^{I_{ss}} \times \mathbb{H}^{I_b \cup I_r}))$$

where both sides are considered as subsets of $G_b(\mathbb{Z}) \backslash \mathbb{H}^{I_b \cup I_r}$. By Proposition 3.2 and equation (1) the RHS is a finite union.

Now, even though $U_1 \cap \gamma \cdot (\mathcal{F}^{I_{ss}} \times \mathbb{H}^{I_b \cup I_r})$ is not necessarily definable, the projection map to $\mathbb{H}^{I_b \cup I_r}$ is ‘locally definable’, in the following sense: for any $\gamma_1 \in \text{SL}_2(\mathbb{Z})^{I_b \cup I_r}$ the fiber over $\gamma_1 \mathcal{F}^{I_b \cup I_r}$ is $U_1 \cap \gamma' \cdot \mathcal{F}^n$, which is definable, where we take γ' to be the element which acts on the I_{ss} co-ordinates as γ does, and acts on the $I_b \cup I_r$ co-ordinates as γ_1 does.

Therefore the image is closed, and so it is complex analytic by Theorem 6.1 of [19]. \square

Continuation of proof of Theorem 1.2. For the purposes of this proof, we adopt the following convention: If a group H acts on a space Y , and X is a subset of Y , we write $H \backslash X$ for the image of X in the quotient space $H \backslash Y$.

Set

$$S_b(R) := \{z \in \mathbb{H}^{I_b} \mid \text{Re}(z) \in [-R, R]^{I_b}\},$$

and let $B_r(R) \subset \mathbb{H}^{I_r}$ denote the hyperbolic ball of radius R around an arbitrary fixed point, say $\vec{i} = (i, \dots, i)$. Note that in this ball the norms of the co-ordinates are bounded above by e^{2R} while the y co-ordinates are bounded below by e^{-R} , so by [22, Prop 5.2] the heights of all fundamental domains are bounded from above by e^{mR} for some fixed m .

Consider

$$A(R) := G(\mathbb{Z}) \backslash S_b(R) \times B_r(M \ln R)$$

for $M = |I_b| + 1$. Set $U_2 = G_b(\mathbb{Z}) \backslash \pi_{I_b \cup I_r}(U_1)$. Note that since U_1 is irreducible and of pure dimension, so is U_2 . We now split into 2 cases, depending on whether U_2 intersects the boundary of $A(R)$ for arbitrarily large R :

Case A: For arbitrarily large R ,

$$U_2 \cap (\partial(G_b(\mathbb{Z}) \backslash S_b(R)) \times B_r(M \ln R)) \neq \emptyset.$$

Fix a point $u_0 \in U_2$ once and for all, and let u be a point in the intersection above which connects to u_0 . Note that \mathbb{H}^n is naturally partitioned into ‘boxes’ indexed by n -tuples of integers, where (z_1, \dots, z_n) is in box (a_1, \dots, a_n) if for all i , $[\Re(z_i) = a_i]$. Since $G_b(\mathbb{Z})$ respects this partitioning, $G_b(\mathbb{Z}) \backslash \mathbb{H}^n$ is naturally partitioned into such boxes as well. In this case, since U_2 is connected, if we fix any point $u_0 \in U_2$, one must pass at least R boxes to get from u_0 to u . Since the pre-image of a box consists of a union of translates of the standard fundamental domain, for each such box there must be a point $t_i \in Z \subset \text{SL}_2(\mathbb{Z})^n$ corresponding to it. Moreover, the height of each t_i is at most $O(R^{M+1})$ by construction, and all of these t_i are in distinct $G(\mathbb{Z})$ -orbits, since $\pi_b(t_i)$ are in distinct $G_b(\mathbb{Z})$ orbits. However, by

(*) the maximal semi-algebraic sets in Z are orbits of S , and this contradicts the theorem of Pila-Wilkie [25] that the points of height at most R^{M+1} can be covered by less than R^ϵ algebraic sets for any $\epsilon > 0$.

Case B: For sufficiently large R ,

$$U_2 \cap (\partial(G_b(\mathbb{Z}) \backslash S_b(R)) \times B_r(M \ln R)) = \emptyset.$$

This means that $\pi_{I_r}(U_2 \cap A(R)) \subset B_r(M \ln R)$ is globally analytic. Suppose $\pi_{I_r}(U_2)$ is not a point. Applying Hwang-To, we see that we get at least e^{MR} volume, which means that the volume of U_2 was at least that big, so we hit at least that many fundamental domains. At most $R^{|I_b|}$ of them could have come from the first co-ordinate so we get polynomially many (in e^R) in the second co-ordinate. But this is a contradiction since the group G is just a torus in that co-ordinate. Thus $\pi_{I_r}(U_2)$ is a point. This implies by our earlier assumption that the I_r co-ordinates are empty. Hence, since U_2 also does not intersect the boundary of $A(R)$, we must have that U_2 , and hence U_1 , has only finitely many ‘pictures’ in fundamental domains. I.e. there are finitely many fundamental domains of the form $\gamma_i \mathcal{F}^n$, $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})^n$, $i = 1, \dots, k$ such that for any such fundamental domain $\gamma \mathcal{F}^n$ there is $i \in \{1, \dots, k\}$ such that $U_1 \cap \gamma \mathcal{F}^n = \gamma \gamma_i^{-1}(U_1 \cap \gamma_i \mathcal{F}^n)$. This implies that $\pi_{Y(1)^n}(U)$ is a complex analytic and definable subset of $Y(1)^n$, an affine space. By Theorem 4.5 of Peterzil-Starchenko [20] (see alternatively Theorem 5.1 of [18]), this means that $\pi_{Y(1)^n}(U)$ is an algebraic subvariety.

6. MONODROMY AND CONCLUSION

Now we have reduced ourselves to the case that $\pi_{Y(1)^n}U$ is algebraic. The proof of Theorem 1.2 is therefore completed by the following result.

Lemma 6.1. *Theorem 1.2 holds with the additional assumption that $\pi_{Y(1)^n}U$ is algebraic.*

Proof. We can clearly assume that $V = U^{\mathrm{zar}}$. Let $\Gamma_0 \subset \mathrm{SL}_2^n(\mathbb{Z})$ denote the stabilizer of U_1 - which can also be thought of as the monodromy of $\pi_{Y(1)^n}U$. By [24, Lemma 2.4], since $\pi_{Y(1)^n}U$ is not contained in a weakly special subvariety, Γ_0 is Zariski dense in $\mathrm{SL}_2^n(\mathbb{C})$.

Now, consider a generic point $u \in \pi_{Y(1)^n}U$ and consider its fiber F_u in $\pi_{J_2 Y(1)^n}U$. Let k be the number of connected components of F_u . Then there is an action of the monodromy group $\pi_1(U, u)$ on the connected components of F_u and the kernel A of this action consists of loops in $\pi_{Y(1)^n}U$ that can be lifted to loops in $\pi_{J_2 Y(1)^n}U$. Finally, there is a natural map from $\pi_1(U, u)$ to Γ_0 given by lifting loops to paths in \mathbb{H}^n , and let Γ'_0 denote the image of A under this map. Then Γ'_0 is of index at most k in Γ_0 , and hence also Zariski dense in $\mathrm{SL}_2^n(\mathbb{C})$. Moreover, Γ'_0 consists of those elements which preserve $\pi_{J_2 \mathbb{H}^n}U$.

Therefore, this means that U is invariant under a Zariski dense subgroup of $\mathrm{SL}_2^n(\mathbb{C})$, and thus that the Zariski closure $V = U^{\mathrm{zar}}$ is invariant under all of $\mathrm{SL}_2^n(\mathbb{C})$.

Finally, observe that $J_2\mathbb{P}^1 \setminus J_2^0\mathbb{P}^1$ is acted on transitively by $\mathrm{SL}_2(\mathbb{C})$, since $\mathrm{SL}_2(\mathbb{C})$ acts transitively on \mathbb{P}^1 and the lower-triangular group acts transitively on the 2-jets at 0 which have non-vanishing first derivative. This implies that the fibers of the natural projection map $V \rightarrow J_2Y(1)^n$ are $J_2\mathbb{C}^n$, and so $3n$ dimensional, which implies the claim. \square

Remark. *This last argument is the only place where derivatives complicate the argument at all. If one were to carry out the above strategy for a general group G instead of SL_2 , it seems likely that the only place derivatives would come into the argument would be in an analogous statement that under the action of G on certain jet spaces, the generic orbit is of the same dimension as G . However, at the moment other difficulties prevent us from carrying out the proof in general, nor can we establish this statement.*

7. APPLICATION TO ZILBER-PINK

A *special subvariety* of $Y(1)^n$ is a weakly special subvariety (as defined in §2.3) in which the constant c in any defining equation of the form $x_\ell = c$ is a *singular modulus*, i.e. the j -invariant of a CM elliptic curve.

Definition. *Let $W \subset Y(1)^n$. An atypical subvariety of V is a component A of some $W \cap T$, where T is a special subvariety, such that $\dim A > \dim W + \dim T - \dim X$.*

Conjecture 7.1 (Zilber-Pink conjecture for $Y(1)^n$). *Let $W \subset Y(1)^n$. Then W contains only finitely many maximal atypical subvarieties.*

It is shown in [12] that this conjecture would follow from two statements. The first is (a less general form of) the Ax-Schanuel statement proved here. The second is a conjectural lower bound for certain Galois orbits. The purpose of this section is to show that Ax-Schanuel directly implies a partial result on the conjecture, without any assumptions on Galois orbits.

Definition. *An atypical subvariety $A \subset W$ is called strongly atypical if no coordinate is constant on A .*

Theorem 7.1. *Let $W \subset Y(1)^n$. Then W contains only finitely many maximal strongly atypical subvarieties.*

Proof. For a subvariety $A \subset Y(1)^n$ denote by $\langle A \rangle$ the smallest special subvariety containing A , and denote by $\delta(A) = \dim \langle A \rangle - \dim A$ the *defect* of A . For $W \subset Y(1)^n$, we call $A \subset W$ an *optimal subvariety* if $\delta(B) < \delta(A)$ for all strictly larger $B \supset A$ with $B \subset W$. As shown in [12], the Zilber-Pink conjecture for $Y(1)^n$ is equivalent to the assertion that, for $W \subset Y(1)^n$, W contains only finitely many optimal subvarieties.

Now let $W \subset Y(1)^n$. We consider the uniformization $\pi : \mathbb{H}^n \rightarrow Y(1)^n$ and the definable set $Z = \pi^{-1}(W) \cap \mathcal{F}^n$. An atypical intersection $W \cap T$ gives rise to an atypical intersection of Z with a weakly special subvariety of \mathbb{H}^n . We consider the definable family of Mobius subvarieties. Among

these we can define the subset of M which intersect optimally (in the sense $\dim M - \dim(M \cap Z)$ can't be maintained for any strictly larger M) and with $M \cap Z$ having no constant coordinate (which is preserved under taking larger M). By Ax-Schanuel in the form of 5.12 of [12], such M are weakly special. As they have no constant coordinates they are in fact special, defined only by $\mathrm{GL}_2^+(\mathbb{Q})$ conditions, and the definable subset of these must be a finite set. \square

We remark that the corresponding statement for the André-Oort conjecture (“A subvariety $W \subset Y(1)^n$ contains only finitely many strongly special subvarieties”) follows in a similar way from the Ax-Lindemann theorem for $Y(1)^n$ in [22], though this was not highlighted there.

Acknowledgements. We thank Vivek Shende for pointing us to the concept of a jet space. We also thank Gareth Jones for a correction. JP thanks the EPSRC for supporting his research under grant EP/J019232/1, and JT thanks NSERC for the support of his research through a discovery grant.

REFERENCES

- [1] Y. André, Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part, *Compositio Math.* **82** (1992), no. 1, 1–24.
- [2] J. Ax, On Schanuel’s conjectures, *Annals* **93** (1971), 252–268.
- [3] J. Ax, Some topics in differential algebraic geometry I: Analytic subgroups of algebraic groups, *Amer. J. Math.* **94** (1972), 1195–1204.
- [4] D. Bertrand, D. Masser, A. Pillay, and U. Zannier, Relative Manin-Mumford for semi-abelian surfaces, *Proc. Edinburgh Math. Soc.*, to appear.
- [5] D. Bertrand and W. Zudilin, On the transcendence degree of the differential field generated by Siegel modular functions, *J. reine angew. Math.* **554** (2003), 47–68.
- [6] E. Bombieri, D. Masser, U. Zannier, Anomalous subvarieties - structure theorems and applications, *IMRN* (2007), 1–33.
- [7] P. Deligne, Théorie de Hodge. II, *Publications Mathématiques de l’IHÉS* **40** (1971), 5–57.
- [8] L. van den Dries, *Tame Topology and O-minimal Structures*, London Mathematical Society Lecture Note Series **248**, CUP, Cambridge, 1998.
- [9] L. van den Dries, A. Macintyre, and D. Marker, The elementary theory of restricted analytic fields with exponentiation, *Ann. of Math. (2)* **140** (1994), 183–205.
- [10] L. van den Dries and C. Miller, On the real exponential field with restricted analytic functions, *Israel J. Math.* **85** (1994), no. 1-3, 19–56.
- [11] P. Habegger and J. Pila, Some unlikely intersections beyond André-Oort, *Compositio Math.* **148** (2012), 1–27.
- [12] P. Habegger and J. Pila, O-minimality and certain atypical intersections, preprint 2014.
- [13] J.-M. Hwang and W.-K. To, Volumes of complex analytic subvarieties of Hermitian symmetric spaces, *Amer. J. of Math.*, **124**, No. 6, Dec. 2002, pp. 1221–1246.
- [14] S. Lojasiewicz, *Introduction to complex analytic geometry*, Birkhäuser, Basel, 1991, III.4.6.
- [15] K. Mahler, On algebraic differential equations satisfied by automorphic functions, *J. Austral. Math. Soc.* **10** (1969), 445–450.

- [16] D. Masser, Heights, transcendence, and linear Independence on commutative group varieties, *Lecture Notes in Mathematics* **1819**, Amoroso and Zannier, eds, 1-51, Springer-Verlag, Berlin, 2003.
- [17] D. Masser and U. Zannier, Torsion anomalous points and families of elliptic curves, *Amer. J. Math.* **132** (2010), 1677–1691.
- [18] Y. Peterzil and S. Starchenko, Complex analysis in a non-standard setting, *Model Theory with Applications to Algebra and Analysis*, Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, editors, LMS Lecture Note Series, CUP, 2008.
- [19] Y. Peterzil and S. Starchenko, Complex analytic geometry and analytic-geometric categories, *Crelle* **626** (2009), 39–74.
- [20] Y. Peterzil and S. Starchenko, Tame complex analysis and o-minimality, *Proceedings of the ICM, Hyderabad 2010*.
- [21] Y. Peterzil and S. Starchenko, Definability of restricted theta functions and families of abelian varieties, *Duke. Math. J.* **162** (2013), 731–765.
- [22] J. Pila, O-minimality and the André-Oort conjecture for \mathbb{C}^n , *Annals* **173** (2011), 1779–1840.
- [23] J. Pila Modular Ax-Lindemann-Weierstrass with derivatives, *Notre Dame J. Formal Logic* **54** (2013), 553–565.
- [24] J. Pila and J. Tsimerman, Ax-Lindemann for \mathcal{A}_g , *Annals* **179** (2014), 659–681.
- [25] J. Pila and A. J. Wilkie, The rational points of a definable set, *Duke Math. J.* **133** (2006), 591–616.
- [26] R. Pink, A combination of the conjectures of Mordell-Lang and André-Oort, *Geometric methods in algebra and number theory*, F. Bogomolov, Y. Tschinkel, editors, pp 251–282, Prog. Math. **253**, Birkhauser, Boston MA, 2005.
- [27] R. Pink, A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang, 2005 preprint, available from the author’s webpage.
- [28] G. J. Porter, k -volume in \mathbb{R}^n and the Generalised Pythagorean Theorem, *Amer. Math. Monthly* **103** (1996), 252–256.
- [29] A. Seidenberg, Abstract differential algebra and the analytic case, *Proc. A. M. S.* **9** (1958), 159–164.
- [30] A. Seidenberg, Abstract differential algebra and the analytic case. II, *Proc. A. M. S.* **23** (1969), 689–691.
- [31] J. Tsimerman, Ax-Schanuel and o-minimality, to appear in *O-minimality and Diophantine geometry*, A. J. Wilkie and G. O. Jones, editors, pp216–221, London Mathematical Society Lecture Note Series **421**, CUP, 2015.
- [32] U. Zannier, *Some Problems of Unlikely Intersections in Arithmetic and Geometry*, with appendices by D. Masser, *Annals of Mathematics Studies* **181**, Princeton University Press, 2012.
- [33] B. Zilber, Exponential sums equations and the Schanuel conjecture, *J. London Math. Soc. (2)* **65** (2002), 27–44.

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, UK

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, CANADA