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Harder–Narasimhan filtrations of persistence modules

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Abstract

The Harder–Narasimhan (HN) type of a quiver representation is a discrete invariant parameterised by a real-valued function (called a central charge) defined on the vertices of the quiver. In this paper, we investigate the strength and limitations of HN types for several families of quiver representations which arise in the study of persistence modules. We introduce the skyscraper invariant, which amalgamates the HN types along central charges supported at single vertices, and generalise the rank invariant from multiparameter persistence modules to arbitrary quiver representations. Our four main results are as follows: (1) We show that the skyscraper invariant is strictly finer than the rank invariant and incomparable to the generalised rank invariant; (2) we characterise the set of complete central charges for zigzag (and hence, ordinary) persistence modules; (3) we extend the preceding characterisation to rectangle-decomposable multiparameter persistence modules of arbitrary dimension; and finally, (4) we show that although no single central charge is complete for interval-decomposable ladder persistence modules, a finite set of central charges is complete.

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1 | INTRODUCTION

More than 60 years ago, the notion of *semistability* was introduced by Mumford [41, 42] to construct well-behaved quotient spaces for the actions of reductive groups on algebraic varieties. Subsequently, Harder and Narasimhan described a stratification of the moduli space of finite rank vector bundles over a complex curve for the purpose of computing its cohomology groups [23]. The top stratum consists of semistable bundles, and every bundle lying in a lower stratum admits a canonical filtration of finite length whose associated graded components are semistable with strictly decreasing slopes (i.e., ratios of degree to rank). This *Harder–Narasimhan (HN) filtration* continues to play a vital role in moduli problems involving vector bundles and coherent sheaves [5, 26]; its existence and uniqueness have been established more generally for objects of certain abelian categories by Rudakov [46] to triangulated categories by Bridgeland [10] and to modular lattices by Haiden et al. [21]. The work of King [30] and Reineke [45] has extended the HN formalism to categories of quiver representations.

Our efforts here stem from the desire to use HN theory to build isomorphism invariants for multiparameter persistence modules. In general, such modules are representations of wild-type quivers, so there cannot be a complete[†] discrete invariant. Nevertheless, the quest for discriminative invariants has been a central theme within topological data analysis. The earliest work in this direction was by Carlsson and Zomorodian [13], who proposed the *rank invariant*. Subsequent efforts to study multiparameter persistence modules have involved a plethora of tools sourced from diverse subject areas — these include sheaf theory [28, 38], commutative and homological algebra [6, 24, 34, 40], lattice theory [9, 25, 39] and beyond [36].

Outline and summary of results

Fix a finite quiver Q and let $\mathbf{Rep}(Q)$ denote the category of finite-dimensional representations of Q valued in vector spaces. By a *central charge* on Q, we mean any real-valued function α defined on the set Q_0 of vertices.[‡] Consider a nontrivial representation $V \in \mathbf{Rep}(Q)$, which assigns vector spaces V_x to vertices $x \in Q_0$. The α-slope of V is the ratio

$$\mu_\alpha(V) = \frac{\sum_{x\in Q_0} \alpha(x)\cdot \dim V_x}{\sum_{x\in Q_0} \dim V_x},$$

and V is said to be α-semistable whenever the inequality $\mu_\alpha(V) \geqslant \mu_\alpha(V')$ holds for every nontrivial subrepresentation $V' \subset V$.

[†] An invariant is *complete* if two persistence modules are isomorphic whenever their invariants are equal.

[‡] This is the imaginary part of an abelian group homomorphism $K(\mathbf{Rep}(Q)) \to \mathbb{C}$; see Section 2.2.

Once we fix a central charge α on Q, every nonzero representation V admits a unique *HN filtration* [22, 45] of finite length

$$0 = \mathbf{HN}^0_\alpha(V) \subsetneq \mathbf{HN}^1_\alpha(V) \subsetneq \cdots \subsetneq \mathbf{HN}^{n-1}_\alpha(V) \subsetneq \mathbf{HN}^n_\alpha(V) = V$$

whose successive quotients $S^i := \mathbf{HN}^i_\alpha(V)/\mathbf{HN}^{i-1}_\alpha(V)$ are α-semistable and satisfy $\mu_\alpha(S^i) < \mu_\alpha(S^{i-1})$ for all i. The HN filtration thus provides a canonical method for understanding arbitrary quiver representations through the α-semistable ones. The *HN type* of V along α is the n-tuple of functions

$$\mathbf{T}[V;\alpha] = \left(\underline{\dim}_{S^1}, \underline{\dim}_{S^2} \dots, \underline{\dim}_{S^n}\right),$$

where $\underline{\dim}_{S^i} : Q_0 \to \mathbb{N}$ assigns the natural number $\dim S^i_x$ to each vertex $x \in Q_0$.

An important role in our paper is played by the *spanning* subrepresentation of V at a vertex x — this is defined as the smallest subrepresentation $\langle V_x \rangle \subset V$ containing V_x. The function $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ that sends each (x, y) to the dimension of $\langle V_x \rangle_y$ vastly generalises the rank invariant of Carlsson and Zomorodian. Consider, for each vertex x, the central charge $\delta_x : Q_0 \to \mathbb{R}$ which maps x to 1 and all other vertices to 0. The skyscraper invariant δ_V is defined on $\mathbf{Rep}(Q)$ as the collection of HN types $\mathbf{T}[V;\delta_x]$ indexed by the vertices of Q. Our first main result is presented in Section 3, and may be stated as follows.

Theorem (A). *The skyscraper invariant is finer than the rank invariant on* $\mathbf{Rep}(Q)$ *for any finite quiver* Q.

The full statement may be found in Theorem 3.5, where we provide a precise formula for recovering $\rho_V(x, y)$ in terms of $\mathbf{T}[V;\delta_x]$. We also prove in Proposition 3.9 that the skyscraper invariant is incomparable to the generalised rank invariant from [29].

We then specialise in Section 4 to the setting of zigzag persistence modules — these are finite-dimensional representations of type $\mathbb{A}$ quivers of arbitrary (but finite) length $\ell \geqslant 0$:

$$x_0 \overset{e_1}{\text{———}} x_1 \overset{e_2}{\text{———}} \cdots \overset{e_{\ell-1}}{\text{———}} x_{\ell-1} \overset{e_\ell}{\text{———}} x_\ell .$$

Here each edge e_i points either forward $x_{i-1} \to x_i$ or backward $x_{i-1} \leftarrow x_i$, and ordinary persistence modules correspond to the equioriented case where all of the e_i point forward. It is well known from Gabriel's theorem [20] that any representation V of such a quiver decomposes uniquely as a direct sum of *interval representations*, which are supported on subintervals $[a, b] \subset [0, \ell]$. The intervals which appear with nonzero multiplicity comprise the *barcode* of V — see [12]. We exploit knowledge of these indecomposables as well as some recent work of Kinser [32] to prove the following result, which is Theorem 4.3 below.

Theorem (B). *There is a classification of all complete central charges on the category of zigzag persistence modules; in the special case of ordinary persistence modules, a central charge* $\alpha : Q_0 \to \mathbb{R}$ *is complete if and only if* $\alpha(x_i) > \alpha(x_{i+1})$ *holds for all* i.

In Section 5, we study the natural d-dimensional analogues of ordinary persistence modules [35], for $d > 1$. These are certain representations of *grid quivers*, whose vertices are parameterised by integer points in the product $[0, \ell_1] \times [0, \ell_2] \times \cdots \times [0, \ell_d]$, with each $\ell_i \geqslant 1$. The $d = 2$ case is illustrated below:

$$
\begin{tikzcd}
x_{0,\ell_2} \arrow[r] & x_{1,\ell_2} \arrow[r] & \cdots \arrow[r] & x_{\ell_1,\ell_2} \\
\vdots \arrow[u] & \vdots \arrow[u] & \iddots & \vdots \arrow[u] \\
x_{0,1} \arrow[r] \arrow[u] & x_{1,1} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell_1,1} \arrow[u] \\
x_{0,0} \arrow[r] \arrow[u] & x_{1,0} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell_1,0} \arrow[u]
\end{tikzcd}
$$

The representations of interest are those for which every rectangle in sight commutes as a diagram of vector spaces. Unlike the $d = 1$ case, these quivers are of wild representation type; as mentioned above, much effort has been invested towards finding good discrete invariants for multiparameter persistence modules. A far more tractable subcategory is spanned by the *rectangle-decomposable* representations, which admit direct sum decompositions into the obvious d-dimensional analogue of interval modules. Here we establish the following result, consisting of Corollary 5.4 and Theorem 5.10.

Theorem (C). *The skyscraper invariant is strictly finer than the rank invariant on the category of multiparameter persistence modules. Moreover, for the subcategory of rectangle decomposable modules, there is a classification of complete central charges which lie outside of a hyperplane arrangement in the space of maps $Q_0 \to \mathbb{R}$.*

We further establish in Section 5.3 that no single central charge is complete on the larger category of *interval-decomposable* persistence modules [2]. Nevertheless, we show (in Proposition 5.13 below) that a finite set of central charges is complete on the subcategory of persistence modules that decompose into intervals with at most two sources.

In Section 6, we focus on the special case of 2-parameter persistence modules arising from representations of *ladder quivers* of length $\ell \geqslant 1$:

$$
\begin{tikzcd}
x_0^+ \arrow[r] \arrow[d] & x_1^+ \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{\ell-1}^+ \arrow[r] \arrow[d] & x_\ell^+ \arrow[d] \\
x_0^- \arrow[r] & x_1^- \arrow[r] & \cdots \arrow[r] & x_{\ell-1}^- \arrow[r] & x_\ell^-
\end{tikzcd}
$$

Again, every rectangle is required to commute. These *ladder persistence modules* are important because they arise from morphisms of (ordinary) persistence modules. Ladder quivers are known [11, 17] to be of finite representation type only for $\ell \leqslant 3$. Here we obtain the following results (Proposition 6.1 and Theorem 6.4).

Theorem (D). *There is no complete central charge on the category of interval-decomposable ladder persistence modules of length $\ell \geqslant 4$; however, for all ℓ there exists a finite set $A = A(\ell)$ of central charges which is complete on this category.*

The set $A(\ell)$ is explicitly described in Definition 6.8, and its cardinality grows cubically with ℓ; every constituent element of this set is expressible as an $\mathbb{R}$-linear combination of at most two skyscraper central charges.

Stability and computability

Besides high discriminative power, the most important properties that an invariant of multiparameter persistence modules should possess are computability and metric stability (with respect to the interleaving distance, see [35]). The recent work of Cheng [14] introduces a deterministic algorithm to compute HN filtrations of quiver representations along integer-valued central charges. For persistence modules, this algorithm runs in polynomial time (with respect to the number of vertices, dimensions of vector spaces and values of the central charge). We expect that various optimisations and simplifications will become available when one focuses exclusively on the skyscraper invariant, and we hope to explore these computational issues thoroughly in future work. Turning to metric stability, the first author has established in [18] the desired result for skyscraper invariants.

2 | HN TYPES OF QUIVER REPRESENTATIONS

In this section, we establish notation and recall pertinent aspects of quiver representations [33, 47], their HN filtrations and the associated HN types [22, 45].

2.1 | Quiver representations

A quiver Q consists of the following data: a set Q_0 whose elements are called vertices, a set Q_1 whose elements are called edges and a pair of functions $s, t : Q_1 \to Q_0$ called the source and target map, respectively. One often depicts each edge $e \in Q_1$ as an arrow $s(e) \longrightarrow t(e)$. A path in the quiver $Q = (s, t : Q_1 \to Q_0)$ is any finite sequence of edges $p = (e_1, \dots, e_k)$ satisfying $t(e_i) = s(e_{i+1})$ for all i, and Q is called *acyclic* if it admits no such paths with $s(e_1) = t(e_k)$. In this paper, all quivers are assumed to have only finitely many vertices and edges, and moreover, we will primarily be interested in acyclic quivers. We call Q *connected* if for every pair of distinct vertices x and y, there is a finite sequence of vertices $(x = x_0, x_1, \dots, x_k = y)$ and edges e_i satisfying $\{s(e_i), t(e_i)\} = \left\{x_i, x_{i+1}\right\}$ for all i.

Definition 2.1. By a *subquiver* of $Q = (s, t : Q_1 \to Q_0)$ we mean any quiver $Q' = (s', t' : Q_1' \to Q_0')$ with $Q_0' \subset Q_0$ and $Q_1' \subset Q_1$ such that s' and t' are the restrictions of s and t respectively. We call Q'

- *full* if Q_1' contains all edges of Q_1 between pairs of vertices of Q_0' and
- *convex* if every $e \in Q_1$ with $\{s(e), t(e)\} \subset Q_0'$ lies in Q_1'.

(Convexity holds if and only if paths of Q between vertices of Q' are also paths of Q'.)

We work over a field $\mathbb{F}$ which remains fixed throughout (and hence suppressed from the notation), so that all vector spaces encountered henceforth are defined over $\mathbb{F}$. Let us recall that a *representation* V of Q constitutes assignments of vector spaces V_x to vertices $x \in Q_0$ and linear maps $V_e : V_{s(e)} \to V_{t(e)}$ to edges $e \in Q_1$. All representations considered here are finite-dimensional in the sense that $\dim V_x < \infty$ holds for each vertex $x \in Q_0$; the *dimension vector* of any such V is the function $\underline{\dim}_V : Q_0 \to \mathbb{N}$ given by $x \mapsto \dim V_x$. The set $\mathbf{Rep}(Q)$ of all finite-dimensional representations of Q is readily upgraded to a category as follows. A morphism $\phi : V \to W$ consists of linear maps $\{\phi_x : V_x \to W_x \mid x \in Q_0\}$ which make the following diagram commute for each edge $e \in Q_1$:

$$\begin{tikzcd} V_{s(e)} \arrow[r, "\phi_{s(e)}"] \arrow[d, "V_e"'] & W_{s(e)} \arrow[d, "W_e"] \\ V_{t(e)} \arrow[r, "\phi_{t(e)}"'] & W_{t(e)} \end{tikzcd}$$

We call ϕ an (epi, iso, mono)-morphism if each ϕ_x is (sur, bi, in)-jective; V is called a *subrepresentation* of W, denoted $V \subset W$, whenever there exists a monomorphism $\phi : V \to W$. The category $\mathbf{Rep}(Q)$ is known to be abelian, with composition, (co)kernels and (co)products being defined vertex-wise [47, section 1.3].

Remark 2.2. To any quiver Q we may associate a *free category* $\mathcal{Q}$ whose objects are vertices and whose morphisms are paths in Q. With this point of view, a representation $V \in \mathbf{Rep}(Q)$ induces a functor from $\mathcal{Q}$ to the category Vect of finite-dimensional $\mathbb{F}$-vector spaces. Indeed, $\mathbf{Rep}(Q)$ is equivalent via adjunction to the functor category $\mathcal{Q} \to \mathrm{Vect}$ (for details, see [37, II.7]).

A representation of Q is called *indecomposable* if it does not admit any nontrivial direct sum decompositions in $\mathbf{Rep}(Q)$. It is known [47, Theorem 1.2] that for every finite-dimensional representation V of a finite quiver Q, there exists a unique set $\mathrm{Ind}_Q(V)$ containing isomorphism classes of indecomposables and a unique multiplicity function $d_V : \mathrm{Ind}_Q(V) \to \mathbb{Z}_{>0}$ for which there is an isomorphism

$$V \simeq \bigoplus_I I^{d_V(I)}, \tag{1}$$

with I ranging over $\mathrm{Ind}_Q(V)$. The seminal work of Gabriel [20] established that the set of indecomposables in $\mathbf{Rep}(Q)$ with a fixed dimension vector is finite if and only if the undirected graph associated to Q is a finite union of simply laced Dynkin diagrams.

Definition 2.3. Given a subquiver Q' of Q, the *identity representation* $\mathbf{I}[Q'] \in \mathbf{Rep}(Q)$ of Q' is defined for $x \in Q_0$ and $e \in Q_1$ by

$$\mathbf{I}[Q']_x = \begin{cases} \mathbb{F} & \text{if } x \in Q'_0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbf{I}[Q']_e = \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } e \in Q'_1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

When Q' is a full subquiver, the representation $\mathbf{I}[Q']$ is uniquely determined by the vertices Q'_0 of Q' and we write $\mathbf{I}[Q'_0]$ rather than $\mathbf{I}[Q']$. If Q' is moreover full and convex, then $\mathbf{I}[Q']$ is called [2, Definition 10] an *interval representation*.

These identity representations $\mathbf{I}[Q']$ provide examples of indecomposable representations whenever Q' is connected.

2.2 | HN filtrations in abelian categories

The *Grothendieck group* of an abelian category $\mathscr{C}$ is the abelian group $K(\mathscr{C})$ freely generated by the isomorphism classes $[V]$ of objects V in $\mathscr{C}$ modulo a relation of the form $[V] = [U] + [W]$ for each short exact sequence $0 \to U \to V \to W \to 0$ in $\mathscr{C}$. Let $(\mathbb{C}, +)$ denote the abelian group of complex numbers under addition; we recall that a *stability condition*[†] [10] on $\mathscr{C}$ is any group homomorphism

$$Z : K(\mathscr{C}) \to (\mathbb{C}, +)$$

that sends nonzero objects to the open half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. The *Z-slope* of an object $V \neq 0$ is the real number

$$\mu_Z(V) := \frac{\mathrm{Im}\, Z(V)}{\mathrm{Re}\, Z(V)}.$$

We say that V is *Z-semistable* if the inequality $\mu_Z(U) \leqslant \mu_Z(V)$ holds for every nonzero subobject $U \subset V$. If this inequality is strict for all subobjects $U \notin \{0, V\}$, then V is said to be *Z-stable*.

The following result is a direct consequence of [10, Proposition 2.4]. We recall that an abelian category is said to satisfy the Noetherian (respectively, Artinian) hypothesis if every sequence $\cdots \subset a_1 \subset a_0$ of subobjects (respectively, every sequence $b_0 \twoheadrightarrow b_1 \twoheadrightarrow \cdots$ of quotient objects) eventually stabilises up to isomorphism.

Theorem 2.4. *Let $\mathscr{C}$ be any abelian category which satisfies the Noetherian and Artinian hypotheses. Fix a stability condition Z on $\mathscr{C}$. Every nonzero object V of $\mathscr{C}$ admits a unique filtration $V^\bullet$ of finite length $n \geqslant 1$:*

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^n = V, \tag{3}$$

whose successive quotients $S^i := V^i / V^{i-1}$ are Z-semistable and have strictly decreasing slopes:

$$\mu_Z(S^1) > \mu_Z(S^2) > \cdots > \mu_Z(S^n).$$

This $V^\bullet$ is called the Harder–Narasimhan (HN) filtration *of V along Z.*

Crucially, the category of finite-dimensional representations of a finite quiver satisfies the hypotheses of the above result. It follows from uniqueness that the HN filtration of a Z-semistable object V always has length one, that is, $0 \subsetneq V$.

[†] Such a homomorphism is sometimes called a *linear stability function* [32] or a *central charge* [10].

2.3 | HN types of quiver representations

Let $Q = (s, t : Q_1 \to Q_0)$ be a quiver which remains fixed throughout this subsection. The assignment $V \mapsto \underline{\dim}_V$ defines a group homomorphism from the Grothendieck group of $\mathbf{Rep}(Q)$ to the group of functions from Q_0 to $\mathbb{Z}$:

$$\underline{\dim} : K(\mathbf{Rep}(Q)) \longrightarrow \mathbb{Z}^{Q_0}$$

(For acyclic Q, this is an isomorphism [33, Theorem 1.15]). We note that any stability condition Z on $\mathbf{Rep}(Q)$ which factors through $\underline{\dim}$ amounts to a choice of two maps $\alpha : Q_0 \to \mathbb{R}$ and $\beta : Q_0 \to \mathbb{R}_{>0}$. Explicitly, for nonzero $V \in \mathbf{Rep}(Q)$, we have

$$Z(V) = \sum_{x \in Q_0} \left(\beta(x) + \sqrt{-1} \cdot \alpha(x)\right) \cdot \dim V_x,$$

and the corresponding slope is the ratio

$$\mu_Z(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \beta(x) \cdot \dim V_x}.$$

In this paper, we will only work with stability conditions Z which factor through $\underline{\dim}$. Furthermore, as in [22, 45], we further restrict attention to those stability conditions for which β is the constant map sending all vertices to 1. These are called *standard* stability conditions; and any such stability condition Z depends on a single function $\alpha : Q_0 \to \mathbb{R}$ that we will henceforth call the *central charge* of Z. In light of these simplifications, we will denote the slope of any nonzero V by

$$\mu_\alpha(V) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim V_x}{\sum_{x \in Q_0} \dim V_x}. \tag{4}$$

Similarly, the HN filtration of any nonzero $V \in \mathbf{Rep}(Q)$ along Z is indicated by $\mathbf{HN}^\bullet_\alpha(V)$. The following *seesaw lemma* is stated in [45, Lemma 2.2]; we include a proof here for completeness.

Lemma 2.5. *Let $\alpha : Q_0 \to \mathbb{R}$ be a central charge for Q. Given any three nonzero objects which fit into a short exact sequence in* $\mathbf{Rep}(Q)$*:*

$$0 \to U \to V \to W \to 0,$$

their α-slopes must satisfy one of the following chains of (in)equalities.

(1) $\mu_\alpha(U) > \mu_\alpha(V) > \mu_\alpha(W)$,
(2) $\mu_\alpha(U) = \mu_\alpha(V) = \mu_\alpha(W)$ *or*
(3) $\mu_\alpha(U) < \mu_\alpha(V) < \mu_\alpha(W)$.

In Case (2), V is α-semistable if and only if both U and W are α-semistable.

Proof. Since $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is a strictly monotone increasing function, it suffices to establish the desired inequalities for the composite $\theta := \arctan \circ \mu_\alpha$ rather than for μ_α. By definition, the (standard) stability condition Z induced by α is a group homomorphism $K(\mathbf{Rep}(Q)) \to$

$(\mathbb{C}, +)$, so we have $Z(U) + Z(W) = Z(V)$. The desired results now follow from examining the parallelogram in $\mathbb{C}$ determined by the origin, $Z(U)$ and $Z(W)$ whose fourth point must be $Z(V)$. The angle $\theta(V)$ lies between the angles $\theta(U)$ and $\theta(W)$, with equality of all three angles occurring only in the degenerate case where $Z(U)$ is an $\mathbb{R}$-multiple of $Z(W)$.

Let us now consider the case (2) where U, V and W share a common α-slope μ. If U is not α-semistable, then it admits a subrepresentation U' with $\mu_\alpha(U') > \mu$; but any such U' is automatically a subrepresentation of V which violates its α-semistability. Similarly, any quotient W' of W with $\mu_\alpha(W') < \mu$ violates the semistability of V. Thus, the semistability of V forces semistability of both U and W. Conversely, assume that U and W are α-semistable and label the maps in the short exact sequence as $\iota : U \to V$ and $\pi : V \to W$. Given any subrepresentation $V' \subset V$, we have a short exact sequence

$$0 \to \iota^{-1}(V') \to V' \to \pi(V') \to 0.$$

In the nontrivial case, $\iota^{-1}(V')$ and $\pi(V')$ are nonzero subrepresentations of U and W, respectively, so by semistability both must have α-slopes no larger than μ. By the first part of this lemma, we therefore obtain $\mu_\alpha(V') \leqslant \mu$, which confirms the α-semistability of V. □

Here is an immediate (but important) consequence of the preceding result.

Corollary 2.6. *If U and W are α-semistable objects of* $\mathbf{Rep}(Q)$ *with the same α-slope μ, then their direct sum $U \oplus W$ is also α-semistable with slope μ.*

Definition 2.7. The *HN type* of a representation $V \neq 0$ in $\mathbf{Rep}(Q)$ along a central charge $\alpha : Q_0 \to \mathbb{R}$ is denoted $\mathbf{T}[V;\alpha]$ and defined as follows. Let n be the length of the HN filtration $\mathbf{HN}^\bullet_\alpha(V)$, and let $S^i = \mathbf{HN}^i_\alpha(V)/\mathbf{HN}^{i-1}_\alpha(V)$ denote its successive quotients for $1 \leqslant i \leqslant n$. Then,

$$\mathbf{T}[V;\alpha] := \left(\underline{\dim}_{S^1}, \underline{\dim}_{S^2}, \dots, \underline{\dim}_{S^n}\right).$$

In the context of the preceding definition, it is often useful to view $\mathbf{T}[V;\alpha]$ as a function $\mathbb{R} \to \mathbb{N}^{Q_0}$ in the following manner:

$$\mathbf{T}[V;\alpha](\lambda) = \begin{cases} \underline{\dim}_{S^i} & \text{if } \lambda = \mu_\alpha(S^i) \text{ for some } i, \\ (0, 0, \dots, 0) & \text{otherwise.} \end{cases} \tag{5}$$

This function is well-defined since the successive quotients S^i have strictly decreasing slopes by Theorem 2.4. The uniqueness promised by this theorem further guarantees that $\mathbf{T}[\bullet;\alpha]$ is invariant under isomorphisms in $\mathbf{Rep}(Q)$. It is evident from Definition 2.7 that this invariant is discrete, since it only produces finite sequences of (non-negative) integer values. Moreover, this invariant is additive under direct sums; a proof of the following folklore result may be found in [19, Proposition 2.5].

Proposition 2.8. *Let V and W be two representations of a quiver Q, and let $\alpha : Q_0 \to \mathbb{R}$ be a central charge. Adopting the notation of (5), we have*

$$\mathbf{T}[V \oplus W;\alpha] = \mathbf{T}[V;\alpha] + \mathbf{T}[W;\alpha]$$

as functions $\mathbb{R} \to \mathbb{N}^{Q_0}$.

2.4 | Complete central charges

The main theme of this paper is to measure the strength of the HN type as an invariant of certain quiver representations across various choices of central charge. The definition below corresponds to the best-case scenario.

Definition 2.9. Let $Q = (s, t : Q_1 \to Q_0)$ be an acyclic quiver and $\mathscr{C}$ any subcategory of $\mathbf{Rep}(Q)$. A collection of central charges A is said to be *complete* on $\mathscr{C}$ if $\mathbf{T}[V; \alpha] = \mathbf{T}[W; \alpha]$ for all $\alpha \in A$ implies that V and W are isomorphic in $\mathscr{C}$.

If a collection of central charges A is complete on all of $\mathbf{Rep}(Q)$, then we simplify terminology by saying that A is complete for Q. For the simplest quivers, one hopes to find a single complete central charge; we will appeal to the following result frequently in our quest for such central charges.

Lemma 2.10. *If $\alpha : Q_0 \to \mathbb{R}$ is a complete central charge for the acyclic quiver Q, then every indecomposable representation in $\mathbf{Rep}(Q)$ is α-stable.*

Proof. Assume that α is a complete central charge and consider an indecomposable I in $\mathbf{Rep}(Q)$. If I is not α-semistable, then its HN filtration

$$0 \subsetneq \mathbf{HN}^1_\alpha(I) \subsetneq \mathbf{HN}^2_\alpha(I) \subsetneq \cdots \subsetneq \mathbf{HN}^n_\alpha(I) = I$$

has length $n > 1$. Abbreviating $I^i := \mathbf{HN}^i_\alpha(I)$, in particular we have $I^1 \subsetneq I$. Now consider the filtration of $I^1 \oplus (I/I^1)$ given by:

$$0 \subsetneq I^1 \subsetneq I^1 \oplus (I^2/I^1) \subsetneq \cdots \subsetneq I^1 \oplus (I^n/I^1).$$

Since the successive quotients of this filtration are identical to those of $\mathbf{T}[I; \alpha]$, it follows (from uniqueness) that this new filtration is precisely $\mathbf{HN}^\bullet_\alpha(I^1 \oplus (I/I^1))$. Moreover, since the HN type depends only on these successive quotients, we have $\mathbf{T}[I; \alpha] = \mathbf{T}[I^1 \oplus (I/I^1); \alpha]$. But since I is indecomposable, it cannot be isomorphic to $I^1 \oplus (I/I^1)$ for $I^1 \neq I$, so the completeness of α forces α-semistability of I.

Given this α-semistability, if I is not α-stable, then there exists a nonzero subrepresentation $J \subsetneq I$ with $\mu_\alpha(J) = \mu_\alpha(I)$. Using the exact sequence $0 \to J \to I \to I/J \to 0$ along with Lemma 2.5, we know that both J and I/J are α-semistable with slope $\mu_\alpha(I)$. Another appeal to the same lemma establishes that the direct sum $J \oplus (I/J)$ is also α-semistable with slope $\mu_\alpha(I)$. For dimension reasons, $\mathbf{T}[I; \alpha]$ equals $\mathbf{T}[J \oplus (I/J); \alpha]$. But once again, since I is indecomposable, it is not isomorphic to $J \oplus (I/J)$ for $J \neq I$. Thus, if I is not α-stable, then α is not a complete central charge for Q. □

Let $\alpha : Q_0 \to \mathbb{R}$ be a central charge and let $\mathscr{C}$ be a subcategory of $\mathbf{Rep}(Q)$. We stress that for a nonzero $V \in \mathscr{C}$, the HN filtration $\mathbf{HN}^\bullet_\alpha(V)$ is computed in $\mathbf{Rep}(Q)$ — in particular, we do not require $\mathscr{C}$ to be abelian. The next result describes sufficient conditions under which one can determine $\mathbf{HN}^\bullet_\alpha(V)$ by only considering objects in $\mathscr{C}$.

Lemma 2.11. *Let $\mathscr{I}$ be a family of indecomposable representations of Q and let $\mathscr{C} = \mathscr{C}(\mathscr{I})$ be the full subcategory of $\mathbf{Rep}(Q)$ spanned by those representations which decompose as a direct sum of* *elements in $\mathcal{I}$. Further assume that quotients and subrepresentations of indecomposables in $\mathcal{I}$ all lie in $\mathscr{C}$. The following hold:*

(1) *If α is complete on $\mathscr{C}$, then each indecomposable in $\mathcal{I}$ is α-stable; and*
(2) *for every nonzero $V \in \mathscr{C}$, the representations and quotients in $\mathbf{HN}^{\bullet}_{\alpha}(V)$ are objects in $\mathscr{C}$.*

Proof. Statement (1) follows from the fact that the proof of Lemma 2.10 only compares pairs of HN types of the form $(\mathbf{T}[I;\alpha], \mathbf{T}[J \bigoplus I/J;\alpha])$ with $I \in \mathcal{I}$ and $0 \subsetneq J \subsetneq I$. Indeed, the assumptions on $\mathscr{C}$ ensure that $J \bigoplus I/J$ lies in $\mathscr{C}$ and that it is not isomorphic to I in $\mathscr{C}$.

We prove Statement (2) by induction on the length ℓ of the HN filtration $V^{\bullet}$ of $V \in \mathscr{C}$ along α. When $\ell = 1$, the result is trivial as $V^{\bullet}$ is the one-step filtration $0 \subsetneq V$. Assume $\ell > 1$ and that the result holds for all lengths up to $\ell - 1$. Let $V \simeq \bigoplus_{I \in \mathcal{I}} I^{d_V(I)}$ be the decomposition of V into indecomposables. By [19, Proposition 2.5], the filtrations $\mathbf{HN}^{\bullet}_{\alpha}(I)$ have length at most ℓ and they determine $V^{\bullet}$. We can thus restrict ourselves to the case where $V \in \mathcal{I}$. By assumption, the subrepresentations $V^1, \dots, V^{\ell}$ all lie in $\mathscr{C}$, and similarly, the quotient V/V^1 is an object of $\mathscr{C}$. Finally, by the inductive hypothesis applied to V/V^1, the representations V^i and V^i/V^{i-1} are objects of $\mathscr{C}$ for all i, as desired. □

Remark 2.12. If we assume that $\mathscr{C}$ is an abelian exact subcategory of $\mathbf{Rep}(Q)$ in the setting of Lemma 2.11, then Theorem 2.4 guarantees that HN filtrations are well-defined in $\mathscr{C}$. If we further assume that the group homomorphism $F : K(\mathscr{C}) \to K(\mathbf{Rep}(Q)) \simeq \mathbb{Z}^{Q_0}$ naturally induced by the inclusion $\mathscr{C} \subset \mathbf{Rep}(Q)$ is injective, then α can be pulled back to $K(\mathscr{C})$. More precisely, the standard stability condition $Z_{\alpha} : \mathbb{Z}^{Q_0} \to \mathbb{C}$ on $\mathbf{Rep}(Q)$ associated to α induces a stability condition $F^*\alpha$ on $\mathscr{C}$ given by the composite $Z_{\alpha} \circ F$:

$$\begin{tikzcd} K(\mathscr{C}) \arrow[r, hook, "F"] \arrow[dr, dashed, "F^*\alpha"'] & \mathbb{Z}^{Q_0} \arrow[d, "Z_{\alpha}"] \\ & \mathbb{C} \end{tikzcd} .$$

By the naturality of F, for every nonzero $V \in \mathscr{C}$, we have $\mu_{F^*\alpha}(V) = \mu_{\alpha}(V)$, and the α-(semi)stability of V in $\mathbf{Rep}(Q)$ implies its $F^*\alpha$-(semi)stability in $\mathscr{C}$. By Lemma 2.11(2) and by uniqueness of HN filtrations, we have

$$\mathbf{HN}^{\bullet}_{F^*\alpha}(V) = \mathbf{HN}^{\bullet}_{\alpha}(V).$$

The above remark applies whenever $\mathscr{C}$ is the full subcategory of representations of Q subject to some commutativity relations. Indeed, in this case, $\mathscr{C}$ is stable under subrepresentations and quotients; moreover, $F : K(\mathscr{C}) \to \mathbb{Z}^{Q_0}$ is an isomorphism — see [4, III.3.5]. We will therefore treat the representations of quivers with and without relations on an equal footing throughout the rest of this paper.

3 | THE SKYSCRAPER AND RANK INVARIANTS

Here we study the HN types arising from central charges defined by delta functions on the vertices of a given quiver Q; we call this the *skyscraper invariant*. We also extend the rank invariant of [13] to the setting of arbitrary quiver representations, and show that the skyscraper invariant is finer than the rank invariant. Moreover, we prove that the skyscraper invariant is neither finer nor coarser than the generalised rank invariant of [29].

3.1 | Skyscraper invariant

Let $Q = (s, t : Q_1 \to Q_0)$ be an arbitrary (i.e. finite, but not necessarily acyclic) quiver. In the absence of specific knowledge regarding the structure of Q or its indecomposable representations, it is not immediately obvious how one might identify interesting classes of central charges for Q à la Theorem 4.2. Among the simplest nontrivial central charges which may be defined on any quiver are the ones supported on a single vertex.

Definition 3.1. The *skyscraper* central charge at a vertex $x \in Q_0$ is the map $\delta_x : Q_0 \to \mathbb{R}$ given by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

And the *skyscraper invariant* $\delta_\bullet$ on $\mathbf{Rep}(Q)$ assigns to each representation V the collection of HN types $\delta_V = \{\mathbf{T}[V; \delta_x] \mid x \in Q_0\}$ along skyscraper central charges at all of the vertices.

Definition 3.2. Let $S \subset Q_0$ be a nonempty subset of vertices. The *spanning subrepresentation* of V at S, denoted $\langle V_S \rangle$, is the intersection of all subrepresentations $W \subset V$ for which W_x is isomorphic to V_x whenever x lies in S.

We will simply write $\langle V_x \rangle$ when S is the singleton $\{x\}$. Spanning representations at singletons determine the HN filtrations along skyscraper central charges.

Proposition 3.3. *Given a vertex x of Q, let $0 = V^0 \subsetneq \cdots \subsetneq V^n = V$ be the HN filtration of V along δ_x. If j is the smallest index for which V^j_x equals V_x, then:*

(1) *either $j = n$ or $j = n - 1$ and*
(2) *for every $1 \leqslant k \leqslant j$, we have $V^k = \langle V^k_x \rangle$.*

Proof. We note that the δ_x-slope of a nonzero representation W of Q is given by

$$\mu_{\delta_x}(W) = \frac{\dim W_x}{\sum_{y \in Q_0} \dim W_y}, \tag{6}$$

which is evidently non-negative. Assuming that $V^j_x = V_x$ holds for some j in $\{0, \dots, n\}$, we have $V^k_x = V_x$ for all $k \geqslant j$, whence the successive quotients $S^k := V^k / V^{k-1}$ satisfy $S^k_x = 0$ for all $k > j$. By (6), we obtain equalities of slopes:

$$0 = \mu_{\delta_x}(S^{j+1}) = \mu_{\delta_x}(S^{j+2}) = \cdots = \mu_{\delta_x}(S^{n-1}) = \mu_{\delta_x}(S^n).$$

Since these slopes are required to strictly decrease in the HN filtration, there are only two possible options. Either $j = n - 1$, in which case only the last slope is 0; or $j = n$, in which case all slopes are nonzero. Thus, we have established assertion (1).

We now prove assertion (2) by induction on $k \in \{1, \dots, j\}$.

Base case: Since V^1 is δ_x-semistable and $\langle V^1_x \rangle$ is its subrepresentation, we must have $\mu_{\delta_x}(V^1) \geqslant \mu_{\delta_x}(\langle V^1_x \rangle)$, whence

$$\frac{\dim V^1_x}{\sum_{y \in Q_0} \dim V^1_y} \geqslant \frac{\dim V^1_x}{\sum_{y \in Q_0} \dim \langle V^1_x \rangle_y}.$$

If $\dim V^1_x = 0$, then there is nothing to check, so we assume that this dimension is nonzero. Since each $\langle V^1_x \rangle_y$ is a subspace of the corresponding V^1_y for $y \in Q_0$, we obtain $V^1 = \langle V^1_x \rangle$.

Inductive step: Assume $V^k = \langle V^k_x \rangle$ holds for some $k < j$. Now S^{k+1} is δ_x-semistable by definition of the HN filtration; and by the argument which established assertion (1), it has a strictly positive δ_x-slope. Thus, we have $\dim S^{k+1}_x > 0$, and applying the base case (to S^{k+1} instead of V^1) yields $S^{k+1} = \langle S^{k+1}_x \rangle$. Given any subrepresentation $W \subset V^{k+1}$ such that $W_x = V^{k+1}_x$, we show $W = V^{k+1}$. By the inductive hypothesis, since $(W \cap V^k)_x = V^k_x$, we have $V^k = W \cap V^k \subset W$. Similarly, since $S^{k+1} = \langle S^{k+1}_x \rangle$ and $(W/V^k)_x = S^k_x$, we have $W/V^k = S^{k+1}$. By the 5-lemma [50, Example 1.3.3] applied to the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & V^k \cap W \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, hook] & W/(V^k \cap W) \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & V^k \arrow[r] & V^{k+1} \arrow[r] & S^{k+1} \arrow[r] & 0
\end{tikzcd}$$

we obtain $W = V^{k+1}$ and thus $V^{k+1} = \langle V^{k+1}_x \rangle$, as desired. □

3.2 | Rank invariant

The following notion constitutes a substantial generalisation of the *rank invariant*, which was introduced in [13] for multiparameter persistence modules.

Definition 3.4. The *rank invariant* of a representation $V \in \mathbf{Rep}(Q)$ is the function $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ given by:

$$\rho_V(x, y) := \dim \langle V_x \rangle_y.$$

It follows from the above definition that $\rho_\bullet$ is a discrete isomorphism invariant for $\mathbf{Rep}(Q)$. The following result gives a formula for the rank invariant in terms of the skyscraper invariant $\delta_\bullet$ — in fact, we show that for any vertex x, the rank $\rho_V(x, y)$ can be recovered from the single HN type $\mathbf{T}[V; \delta_x]$.

Theorem 3.5. *Let Q be a finite quiver. The skyscraper invariant is strictly more discriminative than the rank invariant on* $\mathbf{Rep}(Q)$ *in the following sense.*

(1) *Consider* $V \in \mathbf{Rep}(Q)$ *and any vertex* x *in* Q_0. *If* $0 = V^0 \subsetneq \cdots \subsetneq V^n = V$ *is the HN filtration of* V *along* δ_x, *then for any vertex* y *of* Q *we have*

$$\rho_V(x, y) = \sum_{k=1}^{j} \dim S_y^k,$$

where $S^k := V^k / V^{k-1}$ *and* j *is the smallest index for which* V_x^j *equals* V_x.

(2) *There exist two representations* W *and* W' *of the quiver*

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd}$$

for which $\rho_W = \rho_{W'}$ *whereas* $\delta_W \neq \delta_{W'}$.

Proof. By Proposition 3.3, we have $j \in \{n, n-1\}$ and $V^j = \langle V_x \rangle$. So by Definition 3.4, the value of $\rho_V(x, y)$ equals $\dim V_y^j$. Since the $S^\bullet$ are successive quotients of the HN filtration $V^\bullet$, we have

$$\dim V_y^j = \sum_{k=1}^{j} \dim S_y^k,$$

which establishes the first assertion. Turning now to the second assertion, let us consider the representations W (left) and W' (right) depicted below:

$$\begin{tikzcd} \mathbb{F} \arrow[r] & 0 \\ \mathbb{F}^2 \arrow[u, "{[0\ 1]}"] \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \arrow[r] & 0 \\ \mathbb{F}^2 \arrow[u, "{[1\ 0]}"] \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[u] \end{tikzcd}$$

Both evidently have the same rank invariant. Let x be the $\leq$-minimal vertex of this quiver (i.e. the vertex on the bottom-left). By examining (the slopes of) subrepresentations, one readily checks that W is δ_x-semistable, so that its HN filtration is just the trivial one $0 \subsetneq W$. On the other hand, W' has a two-step HN filtration $0 \subset U \subset W'$, where U is given by

$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{F} \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Since W and W' have different HN types along δ_x, the skyscraper invariants δ_W and $\delta_{W'}$ are distinct as claimed above. □

3.3 | Generalised rank invariant

The authors of [29] introduced a generalised version of the rank invariant for multiparameter persistence modules by studying their restrictions to certain subposets. A similar approach was considered in [31] for representations of rooted tree quivers. Here we extend both constructions to the setting of general quiver representations. To this end, let Q be an arbitrary finite quiver, and denote by $\mathrm{SubQuiv}(Q)$ the set of its subquivers. Given a representation $V \in \mathbf{Rep}(Q)$ and a subquiver $Q' \in \mathrm{SubQuiv}(Q)$, the *restriction* $V_{|Q'}$ of V to Q' is the representation defined by $(V_{|Q'})_x = V_x$ for $x \in Q'_0$ and $(V_{|Q'})_e = V_e$ for $e \in Q'_0$.

As mentioned in Remark 2.2, a representation $V \in \mathbf{Rep}(Q)$ may be viewed as a functor $\mathcal{Q} \to \mathrm{Vect}$, where $\mathcal{Q}$ is the category induced by Q. One may therefore examine its categorical limit (written $\varprojlim V$) and colimit (written $\varinjlim V'$) — see [37, Chapter III and Chapter V]. Explicitly, the limit may be viewed as a subspace of the product $\prod_{x\in Q_0} V_x$:

$$\varprojlim V \simeq \left\{ v \in \prod_{x\in Q_0} V_x \,\middle|\, v_{t(e)} = V_e(v_{s(e)}) \text{ for all } e \in Q_1 \right\}. \tag{7}$$

For each vertex $x \in Q$, the desired map $\pi_x : \varprojlim V \to V_x$ is the restriction of the projection $\prod_{y\in Q_0} V_y \twoheadrightarrow V_x$. We note that limit of V is the *space of global sections* of V and may be computed efficiently using the algorithm from [48]. Conversely, the colimit of V is isomorphic to the space of its global cosections. This is the quotient of the sum $\bigoplus_{x\in Q_0} V_x$ given by

$$\varinjlim V \simeq \bigoplus_{x\in Q_0} V_x \Big/ \left\{ v - V_e(v) \,\middle|\, v \in V_{s(e)} \text{ for some } e \in Q_1 \right\}. \tag{8}$$

Given any vertex $x \in Q_0$, the desired map $\iota_x : V_x \to \varinjlim V$ now arises as the quotient of the canonical inclusion of V_x into the direct sum $\bigoplus_{x\in Q_0} V_x$.

By definition, we have the following commutative diagram for every $e \in Q_1$:

$$\begin{tikzcd}
 & \varprojlim V \arrow[dl, "\pi_{s(e)}"'] \arrow[dr, "\pi_{t(e)}"] & \\
V_{s(e)} \arrow[rr, "V_e"] \arrow[dr, "\iota_{s(e)}"'] & & V_{t(e)} \arrow[dl, "\iota_{t(e)}"] \\
 & \varprojlim V &
\end{tikzcd}$$

Thus, when Q is connected, the composite $\iota_x \circ \pi_x : \varprojlim V \to \varinjlim V$ does not depend on $x \in Q_0$ and one may unambiguously define the *limit-to-colimit* map as $\iota_x \circ \pi_x$ for any $x \in Q_0$. The limit-to-colimit map is still defined in general, as $\varprojlim V$ and $\varinjlim V$ can be obtained as the direct sums of $\varprojlim V_{|Q'}$ and $\varinjlim V_{|Q'}$ over the maximal connected subquivers Q' of Q.

Definition 3.6. The *generalised rank invariant* of $V \in \mathbf{Rep}(Q)$ is the map

$$\mathrm{GRI}_V : \mathrm{SubQuiv}(Q) \to \mathbb{N}$$

that sends each $Q' \in \mathrm{SubQuiv}(Q)$ to the rank $\mathrm{GRI}_V(Q') \in \mathbb{N}$ of the limit-to-colimit map of the restriction $V_{|Q'}$.

It follows that the assignment $V \mapsto \mathrm{GRI}_V$ constitutes an additive invariant on $\mathbf{Rep}(Q)$ (in the sense that $V \oplus W$ is sent to $\mathrm{GRI}_V + \mathrm{GRI}_W$). We note that Definition 3.6 yields an alternate characterisation of the rank invariant for multiparameter persistence modules. Explicitly, given vertices $x, y \in Q_0$, let $Q_{x,y}$ be the smallest full subquiver of Q which contains all the paths from x to y in Q. It is now readily checked that the assignment

$$\widetilde{\rho}_V(x,y) := \mathrm{GRI}_V(Q_{x,y}) \tag{9}$$

coincides with the classical rank invariant.

Remark 3.7. The restriction of GRI_V to a subset $\mathcal{S}$ of $\mathrm{SubQuiv}(Q)$ will be denoted $\mathrm{GRI}_V^{\mathcal{S}}$. In the multiparameter persistence setting which was the focus of [29], the family $\mathcal{S}$ usually consists of full and connected subquivers; see [29, Definition 3.5].

We denote by $\mathbf{Rep}_{\mathrm{id}}(Q)$ the full subcategory of representations $V \in \mathbf{Rep}(Q)$ which admit a decomposition into a direct sum of identity representations, as in (2). Adapting the proof of [29, Theorem 3.14], we arrive at the following result.

Proposition 3.8. *The generalised rank invariant* GRI_V *is complete on* $\mathbf{Rep}_{\mathrm{id}}(Q)$.

Proof. Given $Q' \in \mathrm{SubQuiv}(Q)$, we compute $\mathrm{GRI}_{\mathbf{I}[Q']}(Q)$. If $Q = Q'$, then one can check using (7) and (8) that $\varprojlim \mathbf{I}[Q'] \simeq \varinjlim \mathbf{I}[Q'] \simeq \mathbb{F}$ as vector spaces, and that the maps $(\pi_x)_{Q_0}$ and $(\iota_x)_{Q_0}$ can be chosen to be all identities, whence $\mathrm{GRI}_{\mathbf{I}[Q']}(Q) = 1$. Otherwise, if there exists a vertex $x \in Q_0 \setminus Q'_0$, then $\iota_x \circ \pi_x = 0$ and so $\mathrm{GRI}_{\mathbf{I}[Q']}(Q) = 0$. Similarly, if there is an edge $e \in Q_1 \setminus Q'_1$, then $\pi_{t(e)} = V_e \circ \pi_{s(e)} = 0$ so once again $\mathrm{GRI}_{\mathbf{I}[Q']}(Q) = 0$. Given $\widetilde{Q}, Q' \in \mathrm{SubQuiv}(Q)$, by the above (with $\widetilde{Q}$ instead of Q), we have

$$\mathrm{GRI}_{\mathbf{I}[Q']}(\widetilde{Q}) = \mathrm{GRI}_{\mathbf{I}[Q' \cap \widetilde{Q}]}(\widetilde{Q}) = \begin{cases} 1 & \text{if } Q' \supset \widetilde{Q} \\ 0 & \text{otherwise} \end{cases}.$$

Finally, let $V \in \mathbf{Rep}_{\mathrm{id}}(Q)$ and let $d_V(\mathbf{I}[Q']))_{Q' \in \mathrm{SubQuiv}(Q)}$ be the multiplicities in decomposition (1). By additivity of GRI, we have

$$\mathrm{GRI}_V(\widetilde{Q}) = \sum_{Q' \supset \widetilde{Q}} d_V(\mathbf{I}[Q']).$$

Hence, one can prove by descending induction over $Q' \in (\mathrm{SubQuiv}(Q), \subset)$ that the multiplicity $d_V(\mathbf{I}[Q'])$ can be obtained from GRI_V. $\square$

We now establish the main result of this subsection.

Proposition 3.9. *The skyscraper and the generalised rank invariant are not comparable. More precisely,*

(1) *there exist two representations V and W of the quiver*

$$\begin{tikzcd}
\bullet & & \\
\bullet \arrow[u] \arrow[r] & \bullet & \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet
\end{tikzcd}$$

such that $\delta_V \neq \delta_W$ *whereas* $\mathrm{GRI}_V = \mathrm{GRI}_W$*; and*

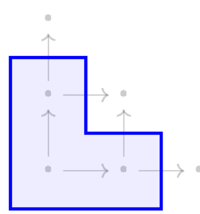
(2) *there exist two representations V' and W' of the quiver*

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \\
 & \bullet \arrow[u]
\end{tikzcd}$$

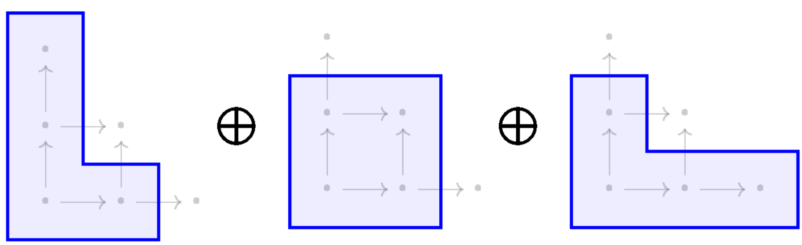
such that $\delta_{V'} = \delta_{W'}$ *whereas* $\mathrm{GRI}_{V'} \neq \mathrm{GRI}_{W'}$.

Proof. The representations V and W are given below:

$$V := \begin{tikzcd}
\mathbb{F} & & \\
\mathbb{F}^2 \arrow[u, "{[1\ 0]}"] \arrow[r, "{[0\ 1]}"] & \mathbb{F} & \\
\mathbb{F}^2 \arrow[u, equal] \arrow[r, equal] & \mathbb{F}^2 \arrow[u, "{[0\ 1]}"'] \arrow[r, "{[1\ 1]}"] & \mathbb{F}
\end{tikzcd} \quad \bigoplus$$



$W :=$



Here, following the discussion around (2), we depict a representation $\mathbf{I}[Q']$ where Q' is a full subquiver of Q by the contour (in blue) of $Q'_0 \subset Q_0$. Let $x \in Q_0$ be the bottom left vertex of Q and let $Q_{3\uparrow}$ (*resp*. $Q_{3\rightarrow}$) be the full subquiver of Q induced by its three top (*resp*. rightmost) vertices.

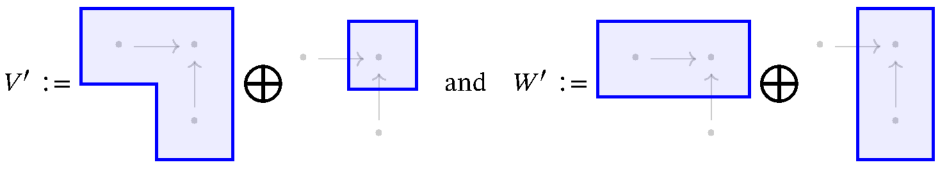
On the one hand, W is δ_x-semistable because each of its summands is δ_x-stable of slope $1/4$. On the other hand, the δ_x-slope of the second summand of V (on the right) exceeds the δ_x-slope of V; whence V is not δ_x-stable. We thus have $\mathbf{T}[V;\delta_x] \neq \mathbf{T}[W;\delta_x]$, and in particular, $\delta_V \neq \delta_W$.

We show that given $Q' \in \mathrm{SubQuiv}(Q)$, we have $\mathrm{GRI}_V(Q') = \mathrm{GRI}_W(Q')$. When $Q_{3\uparrow} \subset Q'$, the limit-to-colimit map of both $V_{|Q'}$ and $W_{|Q'}$ factors through the following commutative diagram

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[dr] & \\
\varprojlim \arrow[r] & \mathbb{F}^3 \arrow[ur, "{[1\ 0\ 0]}"] \arrow[dr, "{[0\ 1\ 0]}"'] & & \varinjlim \\
 & & \mathbb{F} \arrow[ur] &
\end{tikzcd}$$

whence the map $\mathbb{F}^3 \to \varinjlim$ is null and $\mathrm{GRI}_V(Q') = \mathrm{GRI}_W(Q') = 0$. By symmetry, we also have $\mathrm{GRI}_V(Q') = \mathrm{GRI}_W(Q') = 0$ whenever $Q_{3\rightarrow} \subset Q'$. When $Q'_0 \neq Q_0$, we claim that both $V_{Q'}$ and $W_{Q'}$ lie in $\mathbf{Rep}_{\mathrm{id}}(Q')$ and are isomorphic. Indeed, one can check this by hand for each of the six maximal proper subsets of Q_0. Finally, assume $Q'_1 \neq Q_1$ but $Q'_0 = Q_0$ and that Q' contains neither $Q_{3\uparrow}$ nor $Q_{3\rightarrow}$. Then, Q' cannot be connected and by the previous case applied to each connected component of Q', we have $V_{|Q'} \simeq W_{|Q'}$.

For the second assumption, consider the following two representations in $\mathbf{Rep}_{\mathrm{id}}(Q)$:



By Proposition 3.8, we have $\mathrm{GRI}_{V'} \neq \mathrm{GRI}_{W'}$. Since $\langle V'_x \rangle \simeq \langle W'_x \rangle$ for all $x \in Q_0$ and $\underline{\dim}_V = \underline{\dim}_W$, we have $\delta_{V'} = \delta_{W'}$. □

The remainder of this subsection will be occupied by a comparison of the four invariants introduced so far:

- δ, the skyscraper invariant from Definition 3.1,
- GRI, the generalised rank invariant from Definition 3.6,
- ρ, the version of the rank invariant from Definition 3.4, and
- $\widetilde{\rho}$, the version of the rank invariant from Equation (9).

Let us denote by $\mathscr{R}$ the set of pairs (Q, V) where Q is a finite quiver and $V \in \mathbf{Rep}(Q)$. The discriminative power of invariants on a subset $\mathscr{R}'$ of $\mathscr{R}$ defines a preorder $\preceq_{\mathscr{R}'}$ given by $I \preceq_{\mathscr{R}'} I'$ whenever I' is more discriminative than I, or equivalently if the implication

$$I'(V) = I'(W) \Rightarrow I(V) = I(W)$$

holds for all $(Q, V), (Q, W) \in \mathscr{R}'$. Let $\mathscr{R}_{\mathrm{MP}} \subset \mathscr{R}$ be the subset of pairs (Q, V) where Q is a grid quiver and V is an equalised representation of Q (see Section 5.1 for details).

Proposition 3.10. *The Hasse diagrams of the partial orders induced by $\preceq_{\mathscr{R}_{MP}}$ and $\preceq_{\mathscr{R}}$ are given below:*

$$\begin{tikzcd} \mathrm{GRI} \arrow[dr] & & \delta \arrow[dl] \\ & \widetilde{\rho} = \rho & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{GRI} \arrow[d] & \delta \arrow[d] \\ \widetilde{\rho} & \rho \end{tikzcd}$$

Multiparameter persistence setting ($\preceq_{\mathscr{R}_{MP}}$) *Quiver setting ($\preceq_{\mathscr{R}}$)*

Proof. We start with the quiver setting. We have already proved that $\delta \npreceq_{\mathscr{R}} \rho \preceq_{\mathscr{R}} \delta$ (Theorem 3.5) and that $\mathrm{GRI} \npreceq_{\mathscr{R}} \delta \npreceq_{\mathscr{R}} \mathrm{GRI}$ (Proposition 3.9). Moreover, by construction, we have $\widetilde{\rho} \preceq_{\mathscr{R}} \mathrm{GRI}$. By transitivity, showing the following three statements: (1) $\mathrm{GRI} \npreceq_{\mathscr{R}} \widetilde{\rho}$, (2) $\rho \npreceq_{\mathscr{R}} \mathrm{GRI}$ and (3) $\widetilde{\rho} \npreceq_{\mathscr{R}} \delta$ is enough to determine the whole Hasse diagram induced by $\preceq_{\mathscr{R}}$.

(1) Consider W and W' like in the proof of Theorem 3.5(2). We have $\widetilde{\rho}_W = \widetilde{\rho}_{W'}$ whereas Proposition 3.8 implies $\mathrm{GRI}_W \neq \mathrm{GRI}_{W'}$; whence $\mathrm{GRI} \npreceq_{\mathcal{R}} \widetilde{\rho}$.

(2) Consider the following representations of the quiver $Q := \begin{tikzcd} x \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & y \end{tikzcd}$

$$V := \begin{tikzcd} \mathbb{F} \arrow[r, bend left, "{[1\,0\,0]^T}"] \arrow[r, "{[0\,1\,0]^T}"] \arrow[r, bend right, "{[0\,0\,1]^T}"'] & \mathbb{F}^3 \end{tikzcd} \qquad \text{and} \qquad V' := \begin{tikzcd} \mathbb{F} \arrow[r, bend left, "{[1\,0\,0]^T}"] \arrow[r, "{[0\,1\,0]^T}"] \arrow[r, bend right, "{[1\,1\,0]^T}"'] & \mathbb{F}^3 \end{tikzcd}.$$

We have $\rho_V(x, y) = 3 \neq 2 = \rho_{V'}(x, y)$ whereas $\mathrm{GRI}_V = \mathrm{GRI}_{V'}$. Indeed, for any strict subquiver $Q' \subsetneq Q$, we have $V_{|Q'} \simeq V'_{|Q'}$ and $\dim \varprojlim V = 0 = \dim \varprojlim V'$.

(3) Consider the following two representations of the quiver $Q = \begin{tikzcd} x \arrow[r, shift left] \arrow[r, shift right] & y \end{tikzcd}$

$$V := \begin{tikzcd} \mathbb{F}^2 \arrow[r, bend left, "\mathrm{id}_{\mathbb{F}^2}"] \arrow[r, bend right, "{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}"'] & \mathbb{F}^2 \end{tikzcd} \qquad \text{and} \qquad V' := \mathbf{I}[Q]^2$$

On the one hand, $\widetilde{\rho}_V(x, y) \leqslant \dim \varprojlim V = 1$ whereas $\widetilde{\rho}_{V'}(x, y) = 2 \times \mathrm{GRI}_{\mathbf{I}[Q]}(Q) = 2$. On the other hand, both V and V' are δ_x-semistable because all the maps are injective. Hence, $\delta_V = \delta_{V'}$.

In the multiparameter persistence setting, both ρ and $\widetilde{\rho}$ both coincide with the classical rank invariant. Moreover, since positive statements of the form $I \preceq_{\mathcal{R}} I'$ still hold when restricting from $\mathcal{R}$ to $\mathcal{R}_{\mathrm{MP}}$, we have $\mathrm{GRI} \succeq_{\mathcal{R}_{\mathrm{MP}}} \rho \preceq_{\mathcal{R}_{\mathrm{MP}}} \delta$. Finally, since the example from the proof of Theorem 3.5(2) lies in $\mathcal{R}_{\mathrm{MP}}$, we have $\mathrm{GRI} \succeq_{\mathcal{R}_{\mathrm{MP}}} \rho \precneq_{\mathcal{R}_{\mathrm{MP}}} \delta$, as desired. □

Remark 3.11. Let $\mathcal{S} \subset \mathrm{SubQuiv}(Q)$ be a subset containing all the full connected convex subquivers of Q. Since the result of Proposition 3.10 only relies on examples in $\mathcal{S}$, it holds with $\mathrm{GRI}_V^{\mathcal{S}}$ instead of GRI_V. Moreover, the proof of Proposition 3.8 can be adapted to show that $\mathrm{GRI}_V^{\mathcal{S}}$ is complete on those representations which admit a decomposition into indecomposables of the form $\mathbf{I}[Q']$ with $Q' \in \mathcal{S}$.

4 | HN TYPES OF ZIGZAG PERSISTENCE MODULES

The goal of this section is to characterise complete central charges for type $\mathbb{A}_\ell$ quivers.

4.1 | Zigzag persistence modules

Fix an integer $\ell \geqslant 0$. A quiver Q is said to be of *type* $\mathbb{A}_\ell$ whenever its underlying undirected graph has the form

$$x_0 \overset{e_1}{\text{———}} x_1 \overset{e_2}{\text{———}} \cdots \overset{e_{\ell-1}}{\text{———}} x_{\ell-1} \overset{e_\ell}{\text{———}} x_\ell .$$

We describe the direction of edges via a boolean string τ of length ℓ, called the *orientation* of Q: Its ith entry τ_i indicates whether e_i points forward (1) from x_{i-1} to x_i or backward (0) from x_i to x_{i-1}. For instance, when $\ell = 3$, the sequence $\tau = 100$ implicates the following quiver:

$$x_0 \xrightarrow{e_1} x_1 \xleftarrow{e_2} x_2 \xleftarrow{e_3} x_3.$$

We say that Q is *equioriented* if every τ_i equals 1 (i.e. all edges point forward). Our goal here is to describe all complete central charges α for representations of type $\mathbb{A}_\ell$ quivers.

Representations of type $\mathbb{A}_\ell$ quivers are called *zigzag persistence modules* [12], and these specialise in the equioriented case to *ordinary persistence modules* [43]. It follows from Gabriel's theorem [20] that the indecomposable summands which appear in the decomposition (1) of a nonzero $V \in \mathbf{Rep}(Q)$ have a particularly simple form when Q is of type $\mathbb{A}_\ell$. Each such indecomposable corresponds to a subinterval $[a, b] \subset [0, \ell]$ with integral endpoints. Recalling that $\mathbb{F}$ is the ground field over which all of our vector spaces are defined, the indecomposable $\mathbf{I}_\tau[a, b] \in \mathbf{Rep}(Q)$ associated to $[a, b]$ has the form

$$0 \longleftrightarrow \cdots \longleftrightarrow 0 \longleftrightarrow \mathbb{F} \longleftrightarrow \cdots \longleftrightarrow \mathbb{F} \longleftrightarrow 0 \longleftrightarrow \cdots \longleftrightarrow 0. \tag{10}$$

Here the arrows point in accordance with the orientation τ of Q, the contiguous string of $\mathbb{F}$s spans vertex indices $\{a, a+1, \dots, b-1, b\}$, all maps with source and target $\mathbb{F}$ are identities, and all other vector spaces are trivial. These indecomposables are often called *interval* modules.

Explicitly, if Q is a type $\mathbb{A}_\ell$ quiver with orientation τ, then associated to each representation $V \in \mathbf{Rep}(Q)$ there exists a unique finite set $\mathbf{Bar}(V)$ consisting of subintervals $[a, b] \subset [0, \ell]$ with $a \leqslant b$ integers, and a unique function $\mathbf{Bar}(V) \to \mathbb{N}$ sending each $[a, b]$ to its multiplicity d_{ab}, so that there is an isomorphism

$$V \simeq \bigoplus_{[a,b]} (\mathbf{I}_\tau[a, b])^{d_{ab}}. \tag{11}$$

Here the direct sum ranges over $[a, b] \in \mathbf{Bar}(V)$. Thus, a central charge $\alpha : Q_0 \to \mathbb{R}$ is complete for Q in the sense of Definition 2.9 if and only if the multiplicity function $[a, b] \mapsto d_{ab}$ of every $V \in \mathbf{Rep}(Q)$ can be recovered from the HN type $\mathbf{T}[V; \alpha]$.

4.2 | Characterising complete central charges

The first step in our quest to describe all complete central charges of Q is a converse to Lemma 2.10. This result is mentioned (without a detailed proof) in the discussion after [45, Conjecture 7.1]. Throughout, we fix a quiver $Q = (s, t : Q_1 \to Q_0)$ of type $\mathbb{A}_\ell$ and denote its orientation by τ.

Proposition 4.1. *A central charge $\alpha : Q_0 \to \mathbb{R}$ is complete for Q if every indecomposable $\mathbf{I}_\tau[a, b]$ in $\mathbf{Rep}(Q)$ is α-stable.*

Proof. Let $V \in \mathbf{Rep}(Q)$ have the decomposition (11); we seek to establish that the multiplicities d_{ab} which appear in this decomposition can be recovered from $\mathbf{T}[V; \alpha]$. To this end, let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the collection of all slopes contained in the set

$$\{\mu_\alpha(\mathbf{I}_\tau[a, b]) \mid [a, b] \in \mathbf{Bar}(V)\}.$$

For each i in $\{1, \dots, k\}$ we denote by $B_i \subset \mathbf{Bar}(V)$ the subset consisting of all $[a, b]$ for which $\mu_\alpha(\mathbf{I}_\tau[a, b]) \geqslant \lambda_i$. Consider the filtration $V^\bullet$ of V given by

$$V^i := \bigoplus_{[a,b]\in B_i} \mathbf{I}_\tau[a, b]^{d_{ab}}.$$

By construction, the quotient V^i/V^{i-1} is a direct sum of stable representations with α-slope equal to λ_i. Now Corollary 2.6 and uniqueness (described in Theorem 2.4) ensure that $V^\bullet$ is the HN filtration of V along α.

For each $b \in \{0, 1, \dots, \ell\}$, define $\phi_b : \{0, \dots, b\} \to \mathbb{R}$ as $\phi_b(a) := \mu_\alpha(\mathbf{I}_\tau[a, b])$. We claim that these maps are injective: Given $a' < a$, there are two cases to consider, depending on the orientation of $e_a \in Q_1$ (or equivalently, on the value of $\tau_a \in \{0, 1\}$). If $t(e_a) = x_a$, then there exists a monomorphism $\mathbf{I}_\tau[a, b] \subset \mathbf{I}_\tau[a', b]$ and the stability of the latter representation guarantees $\phi_b(a) < \phi_b(a')$. On the other hand, if $t(e_a) = x_{a-1}$, then $\mathbf{I}_\tau[a, b]$ is a quotient of $\mathbf{I}_\tau[a', b]$: We have a short exact sequence in $\mathbf{Rep}(Q)$ of the form

$$0 \longrightarrow \mathbf{I}_\tau[a', a-1] \longrightarrow \mathbf{I}_\tau[a', b] \longrightarrow \mathbf{I}_\tau[a, b] \longrightarrow 0.$$

An appeal to Lemma 2.5 along with the α-stability of $\mathbf{I}_\tau[a', b]$ gives $\phi_b(a) > \phi_b(a')$. In both cases we obtain $\phi_b(a) \neq \phi_b(a')$ for $a' < a$, whence ϕ_b is injective as claimed.

Given this injectivity, for each fixed i we may order the elements of B_i as

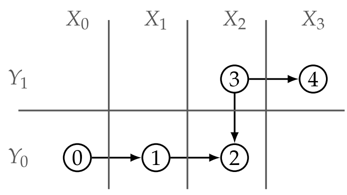
$$B_i = \{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\},$$

where $b_1 > \cdots > b_n$. Now the multiplicity $d_{a_1 b_1}$ is precisely the dimension of V^i/V^{i-1} at any vertex $x_j \in Q_0$ where $b_2 < j \leqslant b_1$. Proceeding inductively, we similarly recover the multiplicities $d_{a_k b_k}$ for all $1 < k \leqslant n$. □

The preceding result, combined with Lemma 2.10, confirms that a central charge α for Q is complete if and only if every indecomposable $\mathbf{I}_\tau[a, b] \in \mathbf{Rep}(Q)$ is α-stable. The main result of [32] is a complete characterisation of such central charges in terms of two functions: Let $\chi, \eta : \{0, 1, \dots, \ell\} \mapsto \mathbb{N}$ be defined inductively as follows. Beginning with $\chi(0) = 0$ and $\eta(0) = 0$, for each $i > 0$ we set

$$\chi(i+1) = \begin{cases} \chi(i) + 1 & \text{if } \tau_i = 1 \\ \chi(i) & \text{if } \tau_i = 0 \end{cases} \quad \text{and} \quad \eta(i+1) = \begin{cases} \eta(i) + 1 & \text{if } \tau_i = 0 \\ \eta(i) & \text{if } \tau_i = 1 \end{cases}$$

For instance, when $\tau = 1101$, the function χ takes on values (0,1,2,2,3) while the function η takes on values (0,0,0,1,1) for inputs (0,1,2,3,4):



For each $k \in \mathbb{N}$, we have the level sets $X_k := \{i \mid \chi(i) = k\}$ and $Y_k := \{i \mid \eta(i) = k\}$, both of which are subintervals of $\{0, \dots, \ell\}$ as χ and η are monotone. Writing $X_k = [a_k, b_k]$ and $Y_k = [a'_k, b'_k]$ for each k, we are able to state [32, Theorem 1.13].

Theorem 4.2. *All indecomposables $\mathbf{I}_\tau[a,b]$ in $\mathbf{Rep}(Q)$ are α-stable for a given central charge $\alpha : Q_0 \to \mathbb{R}$ if and only if the following inequalities hold:*

$$\mu_\alpha(\mathbf{I}_\tau[a_0,b_0]) > \mu_\alpha(\mathbf{I}_\tau[a_1,b_1]) > \cdots > \mu_\alpha(\mathbf{I}_\tau[a_{\chi(\ell)},b_{\chi(\ell)}]),$$

$$\mu_\alpha(\mathbf{I}_\tau[a'_0,b'_0]) < \mu_\alpha(\mathbf{I}_\tau[a'_1,b'_1]) < \cdots < \mu_\alpha(\mathbf{I}_\tau[a'_{\eta(\ell)},b'_{\eta(\ell)}]).$$

It is also shown in [32] that, for every possible orientation τ, (a) this is a *minimal* set of inequalities for characterising those central charges along which all indecomposables are stable, and (b) the set of all such central charges defines a nonempty open subset in $\mathbb{R}^{Q_0}$ which is linearly equivalent to $\mathbb{R} \times \mathbb{R}^{Q_1}_{>0}$. These results, when combined with our Proposition 4.1 and Lemma 2.10, completely describe all complete central charges for $\mathbb{A}_\ell$ quivers.

Theorem 4.3. *Given an integer $\ell \geqslant 0$, let Q be a quiver of type $\mathbb{A}_\ell$ and orientation τ. The set of complete central charges for Q is nonempty, and consists precisely of those $\alpha : Q_0 \to \mathbb{R}$ which satisfy the inequalities from Theorem 4.2.*

We note here that the set of complete central charges admits a particularly appealing description in the case where Q is equioriented.

Corollary 4.4. *For ordinary persistence modules, a central charge α is complete if and only if the inequality $\alpha(x_i) > \alpha(x_{i+1})$ holds for all $i \in \{0,1,\dots,\ell-1\}$.*

Proof. If $\tau = 11\dots1$, then the function $\chi : \{0,1,\dots,\ell\} \to \mathbb{N}$ is given by $\chi(i) = i$, whereas the function η is identically zero. We therefore seek any $\alpha : Q_0 \to \mathbb{R}$ which satisfies $\mu_\alpha(\mathbf{I}_\tau[i,i]) > \mu_\alpha(\mathbf{I}_\tau[i+1,i+1])$ for all i. By (4) and (10), this string of inequalities reduces to

$$\alpha(x_0) > \alpha(x_1) > \cdots > \alpha(x_{\ell-1}) > \alpha(x_\ell), \tag{12}$$

as desired. □

5 | HN TYPES OF MULTIPARAMETER PERSISTENCE MODULES

Finding discriminative and computable invariants for multiparameter persistence modules remains a central challenge in topological data analysis. In this setting, we prove a generalisation of Corollary 4.4 for rectangle-decomposable representations. We then establish another completeness result for a larger class of interval-decomposable representations.

5.1 | Multiparameter persistence modules as equalised representations

Let $Q = (s,t : Q_1 \to Q_0)$ be an acyclic quiver. Its source and target maps may be extended from edges to paths $\gamma = (e_1,\dots,e_k)$ by setting $s(\gamma) := s(e_1)$ and $t(\gamma) := t(e_k)$. Given a representation $V \in \mathbf{Rep}(Q)$, there is a distinguished linear map $V_\gamma : V_{s(\gamma)} \to V_{t(\gamma)}$ induced by the composite

$$V_\gamma := V_{e_k} \circ V_{e_{k-1}} \circ \cdots \circ V_{e_2} \circ V_{e_1}.$$

Definition 5.1. We say that $V \in \mathbf{Rep}(Q)$ is *equalised* if the following property holds: For any pair of vertices $x, y \in Q_0$ and any pair of paths γ, γ' with common source x and common target y, the composite maps V_γ and $V_{\gamma'}$ are identical. Let $\mathbf{Rep}_{\mathrm{eq}}(Q) \subset \mathbf{Rep}(Q)$ be the full subcategory spanned by equalised representations.

We recall from Remark 2.12 that the HN filtrations of an equalised representation coincide in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ and in $\mathbf{Rep}(Q)$.

Example 5.2. A large class of interesting equalised representations arises in the study of *cellular sheaves* [16]. Every such sheaf $\mathscr{F}$ on a regular CW complex X is a functor from the face-ordered poset of cells $(X, \prec)$ to the category $\mathbf{Vec}(\mathbb{F})$ of $\mathbb{F}$-vector spaces. The *Hasse graph* of X is the quiver $Q(X)$ whose vertices correspond bijectively to the cells of X, with a unique edge $\sigma \to \tau$ being present whenever σ is a face of τ of codimension one. Any given sheaf $\mathscr{F} : (X, \prec) \to \mathbf{Vec}(\mathbb{F})$ on X induces a representation $V(\mathscr{F})$ of $Q(X)$ as follows: Every vertex σ is assigned the vector space $\mathscr{F}(\sigma)$ and every edge $\sigma \to \tau$ is assigned the linear map $\mathscr{F}(\sigma \prec \tau)$. The fact that $\mathscr{F}$ is a functor directly implies that $V(\mathscr{F})$ is equalised.

Definition 5.3. The *flow partial order* on vertices Q_0 of the acyclic quiver Q is defined as follows: Given x and y in Q_0, we have $x \leqslant y$ if either $x = y$ or if there exists a path γ in Q with $s(\gamma) = x$ and $t(\gamma) = y$.

Given $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ and a pair of vertices $x \leqslant y$ in Q_0, we write $V_{x \leqslant y} : V_x \to V_y$ to indicate the map defined by any path γ from x to y, with the understanding that this map is the identity for $y = x$. Since V is equalised, this is well-defined and it follows that the image of $V_{x \leqslant y}$ is isomorphic to $\langle V_x \rangle_y$ (from Definition 3.2). Thus, the value of $\rho_V(x, y)$ is precisely the rank of $V_{x \leqslant y}$ when V is equalised. This is the genesis of the terminology of the rank invariant, which was introduced in [13] to study certain equalised representations of grid quivers, described below.

Let us fix a vector $L = (\ell_1, \ell_2, \dots, \ell_d)$ of $d \geqslant 2$ integers, with each $\ell_i \geqslant 1$. Here we consider the case where $Q = (s, t : Q_1 \to Q_0)$ is the d-dimensional *grid quiver* of shape L, defined as follows. Its vertices x_p are indexed by all p lying in the product

$$\Lambda(L) := \prod_{i=1}^{d} \{0, 1, \dots, \ell_i\},$$

and there exists a unique edge $x_p \to x_q$ whenever $q - p$ is a standard basis vector of $\mathbb{R}^d$. Here, for instance, is the quiver of shape $L = (\ell, 2)$ for arbitrary $\ell \geqslant 1$:

$$\begin{tikzcd}
x_{0,2} \arrow[r] & x_{1,2} \arrow[r] & \cdots \arrow[r] & x_{\ell-1,2} \arrow[r] & x_{\ell,2} \\
x_{0,1} \arrow[r] \arrow[u] & x_{1,1} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell-1,1} \arrow[r] \arrow[u] & x_{\ell,1} \arrow[u] \\
x_{0,0} \arrow[r] \arrow[u] & x_{1,0} \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{\ell-1,0} \arrow[r] \arrow[u] & x_{\ell,0} \arrow[u]
\end{tikzcd}$$

Equalised representations of d-dimensional grid quivers are also referred to as *d-parameter persistence modules* [13]. We note that every grid quiver Q contains an embedded copy of the grid quiver of shape $L = (1, 1)$, and that both the representations W and W' which appeared in the proof of Theorem 3.5 are equalised. Thus, we obtain the following consequence.

Corollary 5.4. *Given any integer $d \geqslant 2$, let Q be the grid quiver corresponding to some integer vector $L = (\ell_1, \dots, \ell_d)$ with each $\ell_i \geqslant 1$.*

(1) *The skyscraper invariant δ_V of $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ determines its rank invariant (via the formula in Theorem 3.5).*
(2) *There exist representations W and W' in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ which have identical rank invariant and satisfy $\delta_W \neq \delta_{W'}$.*

5.2 | Rectangle-decomposable representations

In general, grid quivers are of wild representation type and one cannot expect to obtain a tractable classification of all indecomposable objects in $\mathbf{Rep}_{\mathrm{eq}}(Q)$. One can, however, impose a higher dimensional analogue of (11) by passing to the subset of *rectangle-decomposable* representations, which we describe below.

As before, let $Q = (s, t : Q_1 \to Q_0)$ be the grid quiver of shape $L = (\ell_1, \ell_2, \dots, \ell_d)$ for $d \geqslant 2$ and all $\ell_i \geqslant 1$. Given any subset $P \subset \Lambda(L)$, we recall from Definition 2.3 that $\mathbf{I}[P]$ denotes the representation of Q which assigns vector spaces

$$\mathbf{I}[P]_{x_p} = \begin{cases} \mathbb{F} & \text{if } p \in P, \\ 0 & \text{otherwise;} \end{cases}$$

the linear map associated to each edge is the identity whenever both source and target spaces are $\mathbb{F}$, and it must necessarily equal zero otherwise. By a *rectangle representation*, we mean $\mathbf{I}[R]$, where $R := [a_1, b_1] \times \cdots \times [a_d, b_d]$ for some $[a_i, b_i] \subset [0, \ell_i]$ is a rectangle inside $\Lambda(L)$. We note that $\mathbf{I}[R]$ is always equalised when R is a rectangle. We write $\mathbf{Rep}_{\mathrm{rec}}(Q)$ for the full subcategory of $\mathbf{Rep}_{\mathrm{eq}}(Q)$ spanned by objects which are (isomorphic to) direct sums of rectangle representations. The rank invariant is complete when restricted to this subcategory [8, 15]; and by Corollary 5.4, so is the skyscraper invariant.

Our goal in this subsection is to prove a much sharper result — we extend Corollary 4.4 to the category of rectangle-decomposable representations of arbitrary dimension d by classifying the set of complete central charges. For this purpose, it is necessary to exclude from consideration a finite union of hyperplanes in the vector space of central charges:

$$\mathscr{H} := \bigcup_{R \neq R'} \{\alpha : Q_0 \to \mathbb{R} \mid \mu_\alpha(\mathbf{I}[R]) = \mu_\alpha(\mathbf{I}[R'])\}, \tag{13}$$

where R and R' range over distinct rectangles in $\Lambda(L)$. The following result serves to justify this exclusion.

Proposition 5.5. *If $\alpha \notin \mathscr{H}$ is a central charge for which each rectangle representation $\mathbf{I}[R] \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ is α-stable, then $\mathbf{T}[\bullet; \alpha]$ is complete on $\mathbf{Rep}_{\mathrm{rec}}(Q)$.*

Proof. Given $V \in \mathbf{Rep}_{\mathrm{rec}}(Q)$, consider its decomposition

$$V \simeq \bigoplus_R \mathbf{I}[R]^{m_R},$$

where the direct sum is indexed over all subrectangles $R \subset \Lambda(L)$, of which only finitely many have multiplicity $m_R > 0$. It suffices to recover these multiplicities from the HN type of V along α. By Proposition 2.8, for each real number $c \in \mathbb{R}$, we have

$$\mathbf{T}[V;\alpha](c) = \sum_{\mu_\alpha(R)=c} m_R \cdot \underline{\dim}_{\mathbf{I}[R]}\,.$$

Since $\alpha \notin \mathscr{H}$ by assumption, the multiplicity $m_{R'}$ of any rectangle R' can be obtained by letting $c = \mu_\alpha(R')$, so the sum simplifies to $\mathbf{T}[V;\alpha](c) = m_{R'} \cdot \underline{\dim}_{\mathbf{I}[R']}$. ☐

The following is a multiparameter generalisation of Corollary 4.4.

Theorem 5.6. *Let Q be a grid quiver of shape $L = (\ell_1, \ell_2, \dots, \ell_d)$ for any $\ell_i \geqslant 1$. A central charge $\alpha \notin \mathscr{H}$ is complete on $\mathbf{Rep}_{\mathrm{rec}}(Q)$ if and only if it satisfies the inequality $\alpha \circ s(e) > \alpha \circ t(e)$ for each edge $e \in Q_1$.*

The remainder of this subsection will be occupied by the proof.

We first establish a technical result in lattice theory. Let us recall the flow partial order on Q_0 from Definition 5.3. A subset of vertices $U \subset Q_0$ is said to be *up-closed* if $x \in U$ and $y \geqslant x$ implies $y \in U$. Given an arbitrary nonempty subset $A \subset Q_0$, we denote by A^+ the smallest up-closed subset of Q_0 containing A. The poset $(Q_0, \leqslant)$ has a structure of finite lattice with $\wedge$ and $\vee$ given by applying, respectively, min and max coordinate-wise. One can check that this lattice is distributive (the distributive law is true in each coordinate), and hence that it satisfies the following standard inequality [1, Corollary 6.1.3].

Proposition 5.7. *Let $(L, \wedge, \vee)$ be a finite distributive lattice. For any subsets $X, Y \subset L$, we have the inequality*

$$|X| \cdot |Y| \leqslant |X \wedge Y| \cdot |X \vee Y|, \text{ where:}$$

(1) $|\bullet|$ *denotes cardinality,*
(2) $X \vee Y := \{x \vee y \mid x \in X \text{ and } y \in Y\}$ *and*
(3) $X \wedge Y := \{x \wedge y \mid x \in X \text{ and } y \in Y\}$.

We will use this combinatorial inequality to establish the following result about up-closed subsets of Q_0.

Lemma 5.8. *Let $U \subset Q_0$ be an up-closed subset with complement $D := Q_0 \setminus U$. Then, for all subsets $A \subset D$, we have*

$$|A| \cdot |U| \leqslant |D| \cdot |U \cap A^+|,$$

where $|\bullet|$ denotes cardinality.

Proof. Since, $(A^+ \cap D)^+ = A^+$, it suffices to establish the desired inequality with A replaced by the larger set $A^+ \cap D$. Since $A^+ \cap D$ equals $A^+ \setminus (U \cap A^+)$, this inequality becomes

$$\left(|A^+| - |U \cap A^+|\right) \cdot |U| \leqslant (|Q_0| - |U|) \cdot |U \cap A^+|,$$

which is equivalent to

$$|A^+| \cdot |U| \leqslant |Q_0| \cdot |U \cap A^+|.$$

Since U and A^+ are up-closed, we have $U \cap A^+ = U \vee A^+$. Applying Proposition 5.7 with subsets $X = U$ and $Y = A^+$ of $L = Q_0$, we obtain

$$|A^+| \cdot |U| \leqslant |U \wedge A^+| \cdot |U \vee A^+| \leqslant |Q_0| \cdot |U \cap A^+|,$$

as desired. □

The main tool in the proof of our generalisation of Corollary 4.4 to $\mathbf{Rep}_{\mathrm{rec}}(Q)$ is the following max-flow/min-cut theorem [7, Chapter III.1].

Theorem 5.9. *Let $\Phi = (\sigma, \tau : \Phi_1 \to \Phi_0)$ be a quiver whose vertex set contains a distinguished source $s_* \notin \tau(\Phi_1)$ and target $t_* \notin \sigma(\Phi_1)$, and let $\kappa : \Phi_1 \to [0, \infty]$ be a function defined on edges. The maximum value attained by a κ-flow equals the minimum κ-capacity of a cut separating s_* from t_*.*

Recall that a κ-flow on Φ is a map $f : \Phi_1 \to [0, \infty]$ satisfying two constraints:

(1) $f(\epsilon) \leqslant \kappa(\epsilon)$ for all $\epsilon \in \Phi_1$ and
(2) $\sum_{\sigma(\epsilon)=x} f(\epsilon) = \sum_{\tau(\epsilon)=x} f(\epsilon)$ for all $x \in \Phi_0 \setminus \{s_*, t_*\}$.

The *value* of f is the sum $\nu(f) := \sum_{\sigma(\epsilon)=s_*} f(\epsilon)$; by the second constraint above, $\nu(f)$ also equals $\sum_{\tau(\epsilon)=t_*} f(\epsilon)$. On the other hand, an (s_*, t_*)-cut is any subset $E \subset \Phi_1$ whose removal disconnects s_* from t_*; the *κ-capacity* of such a cut is $c(E) := \sum_{\epsilon \in E} \kappa(\epsilon)$. We are now able to characterise complete central charges for Q which do not lie in the union of hyperplanes $\mathscr{H}$ from (13).

Theorem 5.10. *Let Q be a grid quiver of shape $L = (\ell_1, \ell_2, \ldots, \ell_d)$ for any $\ell_i \geqslant 1$. A central charge $\alpha \notin \mathscr{H}$ is complete on $\mathbf{Rep}_{\mathrm{rec}}(Q)$ if and only if it satisfies the inequality $\alpha \circ s(e) > \alpha \circ t(e)$ for each edge $e \in Q_1$.*

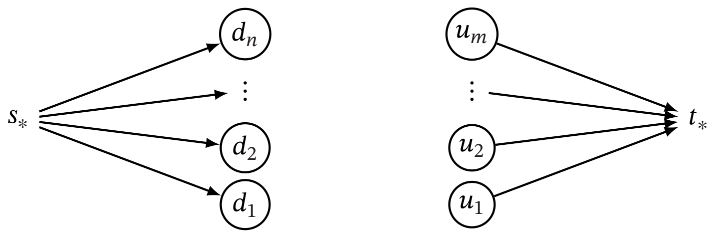
Proof. If the inequality $\alpha \circ s(e) \leqslant \alpha \circ t(e)$ holds for some edge e, then we obtain

$$\mu_\alpha(\mathbf{I}[\{t(e)\}]) \leqslant \mu_\alpha(\mathbf{I}[\{s(e), t(e)\}]).$$

We now have from Lemma 2.10 that the restriction of α to the $\mathbb{A}_2$ quiver $s(e) \to t(e)$ is not complete. It is straightforward to confirm that as a consequence α is not complete on $\mathbf{Rep}_{\mathrm{rec}}(Q)$. The remainder of the argument will be devoted to the converse implication — assuming that $\alpha \circ s(e) > \alpha \circ t(e)$ holds for all $e \in Q_1$, we will show that α is complete.

By Proposition 5.5, it is enough to prove that each rectangle representation is α-stable. By passing to the full subquiver induced by vertices lying within any such rectangle, it suffices to assume that the rectangle representation at hand is $V = \mathbf{I}[\Lambda(L)]$. Consider a subrepresentation $V' \subsetneq \mathbf{I}[\Lambda(L)]$; this assigns vector spaces of dimension at most 1 with all nontrivial maps being isomorphisms. Thus, V' has the form $\mathbf{I}[U]$ for some up-closed proper subset $U \subsetneq Q_0$. Let $\{u_1, u_2, \dots, u_m\}$ be the vertices lying in U, and similarly let $\{d_1, d_2, \dots, d_n\}$ be the vertices lying in its complement $D := Q_0 \setminus U$.

Construct a new quiver $\Phi = (\sigma, \tau : \Phi_1 \to \Phi_0)$ as follows: Its vertex set Φ_0 consists of Q_0 along with two additional vertices s_* and t_*. The edge set Φ_1 is built in two stages as follows: First we insert a unique edge from s_* to each vertex of D, and similarly a unique edge from each vertex of U to t_*, as depicted below:



In the second step, we add a unique edge $d_i \to u_j$ whenever $d_i \leqslant u_j$ holds in Q_0. Define $\kappa : \Phi_1 \to [0, \infty]$ by:

$$\kappa(\epsilon) = \begin{cases} 1/n & \text{if } \sigma(\epsilon) = s_* \\ 1/m & \text{if } \tau(\epsilon) = t_* \\ +\infty & \text{otherwise.} \end{cases}$$

Claim: Every (s_*, t_*)-cut $E \subset \Phi_1$ has κ-capacity $c(E) \geqslant 1$.

Given such a cut, let S and T denote the vertices lying in the component of s_* and t_*, respectively after the edges of E have been removed. We may safely assume that E contains no edges of the form $d_i \to u_j$ since that would immediately force $c(E) = \infty$. Therefore, writing $A := S \cap D$ and $B := T \cap U$, we know that $A^+ \cap B$ is empty because the removal of E must separate s_* from t_*. As a result, we have

$$c(E) = \frac{|D \setminus A|}{|D|} + \frac{|U \setminus B|}{|U|}.$$

Using the fact that $U \setminus B$ contains $A^+ \cap U$ followed by Lemma 5.8, we have

$$\frac{|U \setminus B|}{|U|} \geqslant \frac{|U \cap A^+|}{|U|} \geqslant \frac{|A|}{|D|} = 1 - \frac{|D \setminus A|}{|D|}.$$

Using this bound in our expression for $c(E)$ given above establishes the claim.

Returning to the main argument, we have by Theorem 5.9 that Φ admits a κ-flow $f : \Phi_1 \to [0, \infty]$ of value $\nu(f) \geqslant 1$. Select any such f and note that it must evaluate to $1/n$ on each edge $s_* \to d_i$ and to $1/m$ on each edge $u_j \to t_*$, whence its value $\nu(f)$ is exactly 1. Define $F : D \times U \to \mathbb{R}_{\geqslant 0}$ by

$$F(d_i, u_j) = \begin{cases} f(d_i \to u_j) & \text{if } d_i \leqslant u_j \text{ holds in } Q_0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha \circ s(e) > \alpha \circ t(e)$ holds for each edge $e \in Q_1$, and since F takes strictly positive values on at least some (d_i, u_j) pairs, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{m} F(d_i, u_j) \cdot \alpha(d_i) > \sum_{i=1}^{n} \sum_{j=1}^{m} F(d_i, u_j) \cdot \alpha(u_j).$$

Using the fact that f is a κ-flow, for each i we have $\sum_{j=1}^{m} F(d_i, u_j) = 1/n$, and similarly for each j we have $\sum_{i=1}^{n} F(d_i, u_j) = 1/m$. Thus, the inequality above is $\phi_\alpha(\mathbf{I}[D]) > \phi_\alpha(\mathbf{I}[U])$. Finally, the desired inequality $\phi_\alpha(V') < \phi_\alpha(V)$ follows from Lemma 2.5 applied to the short exact sequence $0 \to V' \to V \to \mathbf{I}[D] \to 0$. $\square$

5.3 | Interval-decomposable representations

The class of interval-decomposable representations (see Definition 2.3) generalises the class of rectangle-decomposable representations and plays an important role in multiparameter persistence. Let Q be a finite acyclic quiver, and let $\mathbf{Rep}_{\mathrm{int}}(Q)$ be the full subcategory of $\mathbf{Rep}(Q)$ spanned by the objects which admit a direct sum decomposition into interval representations. The subcategories of $\mathbf{Rep}(Q)$ introduced in this document are related by the following inclusions

$$\begin{tikzcd}
 & & \mathbf{Rep}_{\mathrm{id}}(Q) \arrow[dr, hook] & \\
\mathbf{Rep}_{\mathrm{rec}}(Q) \arrow[r, hook] & \mathbf{Rep}_{\mathrm{int}}(Q) \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{Rep}(Q) \\
 & & \mathbf{Rep}_{\mathrm{eq}}(Q) \arrow[ur, hook] &
\end{tikzcd}$$

where $\mathbf{Rep}_{\mathrm{id}}(Q)$ is defined in (2). The results from [2] establish that the equality

$$\mathbf{Rep}_{\mathrm{int}}(Q) = \mathbf{Rep}_{\mathrm{id}}(Q) \cap \mathbf{Rep}_{\mathrm{eq}}(Q)$$

holds when Q is a 2-dimensional grid quiver, but not in general.

Let $\mathcal{I}$ denote the set of (isomorphism classes of) indecomposables in $\mathbf{Rep}_{\mathrm{int}}(Q)$. The *support* of a representation $V \in \mathbf{Rep}(Q)$ is the subset $\mathrm{supp}(V) \subset Q_0$ consisting of all vertices x for which the vector space V_x is nontrivial (or equivalently, those vertices x where $\underline{\dim}_V(x)$ is nonzero). Note that the representations in $\mathcal{I}$ are uniquely determined by their support. We recall the flow partial order $\leq$ on Q_0 from Definition 5.3, and say that a subset $S \subset Q_0$ is convex if the full subquiver induced by S is convex. Following the proof of [3, Lemma 4.4], we show that $\mathbf{Rep}_{\mathrm{int}}(Q)$ satisfies the conditions of Lemma 2.11.

Lemma 5.11. *The subrepresentations and the quotients of an interval representation are interval-decomposable.*

Proof. Let $I \in \mathcal{I}$ and let $W \subset I$ be a nonzero subrepresentation. We prove that W and I/W both lie in $\mathbf{Rep}_{\mathrm{int}}(Q)$. For each $x \in \mathrm{supp}(I)$, let 1_x be the unit of $I_x = \mathbb{F}$. Since $\underline{\dim}_I$ takes values in $\{0, 1\}$, the vectors (1_x) constitute bases of the nonzero spaces of W and I/W. Hence, W and W/I are isomorphic to identity representations.

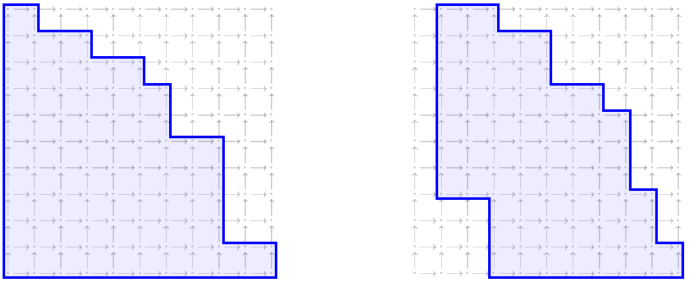
We now prove that $\mathrm{supp}(W)$ and $\mathrm{supp}(I/W)$ are convex. Let p be a path in Q from a vertex in $\mathrm{supp}(W)$ to a vertex in $\mathrm{supp}(I)$. By convexity of $\mathrm{supp}(I)$, the composite map I_p is an isomorphism; whence, W_p is injective and $t(p) \in \mathrm{supp}(W)$. So the subset $\mathrm{supp}(W) \subset \mathrm{supp}(I)$ is upward-closed with respect to $\leq$. In particular, $\mathrm{supp}(W)$ is convex in the full subquiver induced by $\mathrm{supp}(I)$ and hence in Q. Moreover, the complement $\mathrm{supp}(I/W) = \mathrm{supp}(I) \setminus \mathrm{supp}(W)$ is a downward-closed subset of $\mathrm{supp}(I)$ and is also convex in Q.

We have proven that W and I/W are isomorphic to some identity representation of the form $\mathbf{I}[P]$ where $P \subset Q$ is a full convex subquiver. Let $P^1, \ldots, P^k$ be the connected components of P, we have $\mathbf{I}[P] \simeq \bigoplus_{i=1}^k \mathbf{I}[P^i]$, and by definition each $\mathbf{I}[P^i]$ lies in $\mathscr{I}$. $\square$

Given an integer $k \geqslant 1$, we denote by $\mathscr{I}_{\leqslant k}$ the family of interval representations with at most k sources. Namely,

$$\mathscr{I}_{\leqslant k} := \{I \in \mathscr{I},\ |\min \mathrm{supp}(I)| \leqslant k\},$$

where $\min(I)$ denotes the set of minimal vertices lying in the subposet $(\mathrm{supp}(I), \leqslant)$. The supports of two indecomposables of $\mathscr{I}_{\leqslant 2}$ inside a grid of shape (10,10) are shown below in blue:



A natural question to ask is whether there is a family A of central charges which is complete on $\mathbf{Rep}_{\mathrm{int}}(Q)$. Let us first state a negative result.

Corollary 5.12. *Assume that the underlying field $\mathbb{F}$ is different from $\mathbb{Z}/2$. If Q contains a grid of shape (4,1), then no single central charge is complete on the class of $\mathscr{I}_{\leqslant 2}$-decomposable modules and hence on $\mathbf{Rep}_{\mathrm{int}}(Q)$.*

This follows from Corollary 6.2, which we will prove in the next section. The good news takes the form of the following proposition, which is the main result of this subsection.

Proposition 5.13. *There is a finite family of central charges A which is complete on $\mathscr{I}_{\leqslant 2}$-decomposable representations.*

The first step in our proof is the following lemma.

Lemma 5.14. *Let $\mathscr{J} \subset \mathscr{I}$ be a family of interval representations and assume that for all $I \in \mathscr{J}$, there is a central charge $\alpha^I : Q_0 \to \mathbb{R}$ and a real number $\lambda^I \in \mathbb{R}$ such that I is the only α^I-semistable interval representation in $\mathscr{J}$ of slope λ^I. Then, the invariant $\mathbf{T}[V; (\alpha^I)_{I \in \mathscr{J}}]$ is complete on $\mathscr{J}$-decomposable representations.*

Proof. By Lemmas 2.11(2) and 5.11, the HN type $\mathbf{T}[J;\alpha^I](\lambda^I)$ of a nonzero interval representation $J \in \mathscr{J}$ is the dimension vector of some α^I-semistable $W \in \mathbf{Rep}_{\mathrm{int}}(Q)$. By Lemma 2.5, the indecomposable summands in the decomposition of W must have slope λ^I. Since, $\underline{\dim}_W \leqslant \underline{\dim}_J$, the dimension vector $\mathbf{T}[J;\alpha^I](\lambda^I)$ is either 0 or $\underline{\dim}_I$. Given the decomposition $V \simeq \bigoplus_{J\in\mathscr{J}} J^{d_V(J)}$, we have

$$\mathbf{T}[V;\alpha^I](\lambda^I) = \sum_{J\in\mathscr{U}^I} d_V(J)\,\underline{\dim}_I\,.$$

Since the support of every $J \in \mathscr{U}^I$ contains $\mathrm{supp}(I)$, the multiplicities $(d_V(J))_{J\in\mathscr{J}}$ can be obtained from $(\mathbf{T}[V;\alpha^J])_{J\in\mathscr{J}}$ by descending induction on $\mathscr{J}$ equipped with the partial order induced by the inclusion of supports. $\square$

We now prove the main result of this subsection by applying Lemma 5.14 with the family of interval representations $\mathscr{J} = \mathscr{I}_{\leqslant 2}$.

Proof of Proposition 5.13. Let $I \in \mathscr{I}_{\leqslant 2}$, we first prove the existence of a central charge along which I is stable. If $\mathrm{supp}(I)$ has a global minimum $x \in Q_0$, then I is δ_x-stable. If $\min \mathrm{supp}(I)$ contains two vertices $x, x' \in Q_0$, we consider a nonzero central charge $\alpha : Q_0 \to \mathbb{R}$ whose support is $\{x, x'\}$. I has two proper submodules of nonzero α-slope $\langle I_x\rangle$ and $\langle I_{x'}\rangle$. Hence, I is α-stable if and only if

$$\begin{cases} \dfrac{\alpha(x)}{|\mathrm{supp}(\langle I_x\rangle)|} & < \dfrac{\alpha(x)+\alpha(x')}{|\mathrm{supp}(I)|} \\ \dfrac{\alpha(x')}{|\mathrm{supp}(\langle I_{x'}\rangle)|} & < \dfrac{\alpha(x)+\alpha(x')}{|\mathrm{supp}(I)|} \end{cases}$$

which can be rewritten as

$$\frac{|\mathrm{supp}(I)| - |\mathrm{supp}(\langle I_{x'}\rangle)|}{|\mathrm{supp}(\langle I_{x'}\rangle)|} < \frac{\alpha(x)}{\alpha(x')} < \frac{|\mathrm{supp}(\langle I_x\rangle)|}{|\mathrm{supp}(I)| - |\mathrm{supp}(\langle I_x\rangle)|}.$$

There exist $\alpha(x), \alpha(x') \in \mathbb{R}_{>0}$ satisfying the above inequalities if and only if

$$(|\mathrm{supp}(I)| - |\mathrm{supp}(\langle I_x\rangle)|)\cdot(|\mathrm{supp}(I)| - |\mathrm{supp}(\langle I_{x'}\rangle)|) < |\mathrm{supp}(\langle I_x\rangle)|\cdot|\mathrm{supp}(\langle I_{x'}\rangle)|$$

or equivalently,

$$|\mathrm{supp}(I)| < |\mathrm{supp}(\langle I_x\rangle)| + |\mathrm{supp}(\langle I_{x'}\rangle)|.$$

This last inequality holds because $\mathrm{supp}(I) = \mathrm{supp}(\langle I_x\rangle) \cup \mathrm{supp}(\langle I_{x'}\rangle)$ and it is strict because the full subquiver induced by $\mathrm{supp}(I)$ is connected.

Let $\alpha : Q_0 \to \mathbb{R}_{\geqslant 0}$ be a central charge such that I is α-stable and let

$$m := \min_{J\subsetneq I} \mu_\alpha(I) - \mu_\alpha(J).$$

One can define a central charge α^I whose values are linearly independent over $\mathbb{Q}$ and such that for all $x \in Q_0$, we have $\alpha^I(x) \in [\alpha(x), \alpha(x)+m)$. Then for an interval representation $J \subsetneq I$, we have $\mu_{\alpha^I}(J) < \mu_\alpha(J) + m \leqslant \mu_\alpha(I) \leqslant \mu_{\alpha^I}(I)$. So I is also α^I-stable and moreover, by linear independence over $\mathbb{Q}$, each interval representation in $\mathcal{J}$ has a different α^I-slope. By Lemma 5.14, the invariant $\mathbf{T}[V;(\alpha^I)_{I\in\mathcal{J}_{\leqslant 2}}]$ is complete on $\mathcal{J}_{\leqslant 2}$-decomposable modules. □

It remains open whether the above result generalises to any family $\mathscr{J} \subset \mathcal{J}$ of interval representations. The main bottleneck for this generalisation is the existence for each $I \in \mathscr{J}$ of a central charge along which I is stable. This last problem only depends on the shapes of the supports of interval representations in $\mathscr{J}$. Moreover, the complete family A of central charges from the proof of Proposition 5.13 has the same cardinality as $\mathcal{J}_{\geqslant 2}$. It would be interesting to prove a stronger version of Proposition 5.13 with a smaller family A of central charges. Theorem 6.4 in the next section gives a partial result in this direction in the case of ladder persistence modules.

6 | HN TYPES OF INTERVAL-DECOMPOSABLE LADDER PERSISTENCE MODULES

Fix an integer $\ell \geqslant 1$. Here we will be concerned with certain equalised representations of the following *ladder* quiver $Q = (s,t : Q_1 \to Q_0)$ of length ℓ:

$$\begin{tikzcd}
x_0^+ \arrow[r, "e_1^+"] \arrow[d, "e_0"'] & x_1^+ \arrow[r, "e_2^+"] \arrow[d, "e_1"'] & \cdots \arrow[r, "e_{\ell-1}^+"] & x_{\ell-1}^+ \arrow[r, "e_\ell^+"] \arrow[d, "e_{\ell-1}"'] & x_\ell^+ \arrow[d, "e_\ell"'] \\
x_0^- \arrow[r, "e_1^-"'] & x_1^- \arrow[r, "e_2^-"'] & \cdots \arrow[r, "e_{\ell-1}^-"'] & x_{\ell-1}^- \arrow[r, "e_\ell^-"'] & x_\ell^-
\end{tikzcd}$$

Equalised representations of such quivers are sometimes called *ladder persistence modules*; these are precisely 2-parameter persistence modules of shape $L = (\ell, 1)$, but it is customary to represent them with vertical arrows pointing down instead of up. In particular, they arise when studying the morphisms of ordinary persistence modules. The authors of [17] established that $\mathbf{Rep}_{\mathrm{eq}}(Q)$ is of finite type for $\ell \leqslant 3$ and completely classified its indecomposable objects via Auslander–Reiten theory. In contrast, in this section, we focus on interval-decomposable ladder persistence modules. The last three authors introduced in [27, section 5] the nestfree condition which guarantees the interval-decomposability of a ladder persistence module. This condition was later generalised in [49].

6.1 | Interval-decomposable representations of ladders

Given $V \in \mathbf{Rep}_{\mathrm{eq}}(Q)$, we let V^+ and V^- denote its restrictions to the top and bottom rows of Q, respectively. Since both $V^\pm$ are representations of the (equioriented) $\mathbb{A}_\ell$ quiver, they admit direct sum decompositions into interval modules as described in (11). Up to isomorphism, the indecomposable objects of $\mathbf{Rep}_{\mathrm{int}}(Q)$ have one of three possible forms:

(1) Given subintervals $[a,b]$ and $[c,d]$ of $[0,\ell]$ whose endpoints satisfy $c \leqslant a \leqslant d \leqslant b$, let $\mathbf{R}^{a,b}_{c,d}$ be the representation V for which V^+ is the interval module $\mathbf{I}[a,b]$ while V^- is the interval

module $\mathbf{I}[c,d]$, and all vertical maps are 1s whenever possible and 0 otherwise:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \\
\mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

(2) Given an interval $[a,b] \subset [0,\ell]$, let $\mathbf{R}^{a,b}$ be the ladder representation V for which V^+ equals $\mathbf{I}[a,b]$ and V^- is trivial, with all vertical maps necessarily being 0:

$$\begin{tikzcd}
 & & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{F} \arrow[d] & & \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

(3) Finally, given an interval $[c,d] \subset [0,\ell]$, let $\mathbf{R}_{c,d}$ be representation V for which V^+ is trivial while V^- is $\mathbf{I}[c,d]$, so once again all vertical maps are 0:

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \\
 & & \mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & \cdots \arrow[r] & \mathbb{F} & &
\end{tikzcd}$$

It will be convenient in the sequel to unify notation by adopting the convention

$$\mathbf{R}^{a,b}_{\infty,\infty} := \mathbf{R}^{a,b} \qquad \text{and} \qquad \mathbf{R}^{\infty,\infty}_{c,d} := \mathbf{R}_{c,d}, \tag{14}$$

so that for every $V \in \mathbf{Rep}_{\mathrm{int}}(Q)$ there exist multiplicities $r^{a,b}_{c,d} \in \mathbb{N}$ satisfying:

$$V \simeq \bigoplus_{a,b,c,d} \left(\mathbf{R}^{a,b}_{c,d}\right)^{r^{a,b}_{c,d}}, \tag{15}$$

with (a,b,c,d) ranging over admissible subsets of $\{0,1,\dots,\ell,\infty\}^4$ as per (14). Our goal throughout the remainder of this section is to quantify the extent to which these multiplicities can be recovered from HN types. The first result in this direction is negative: For $\ell \geqslant 4$, there is no complete central charge $\alpha : Q_0 \to \mathbb{R}$.

Proposition 6.1. *If Q has length $\ell \geqslant 4$, then for every central charge $\alpha : Q_0 \to \mathbb{R}$ there exists at least one indecomposable in $\mathbf{Rep}_{\mathrm{int}}(Q)$ which is not α-stable.*

Proof. It suffices to consider $\ell = 4$ which embeds into all larger ladder quivers:

$$\begin{tikzcd}
x_0^+ \arrow[r] \arrow[d] & x_1^+ \arrow[r] \arrow[d] & x_2^+ \arrow[r] \arrow[d] & x_3^+ \arrow[r] \arrow[d] & x_4^+ \arrow[d] \\
x_0^- \arrow[r] & x_1^- \arrow[r] & x_2^- \arrow[r] & x_3^- \arrow[r] & x_4^-
\end{tikzcd}$$

Let $\alpha_j^\pm$ be the valued assigned by a given central charge α to the vertex $x_j^\pm$ for $0 \leqslant j \leqslant 4$. Assume, for the purposes of contradiction, that all the indecomposables in $\mathbf{Rep}_{\mathrm{int}}(Q)$ are α-stable. Consider the inequalities of slopes arising from the following inclusions of indecomposables:

$$3(\alpha_1^- + \alpha_1^+) < 2(\alpha_0^- + \alpha_1^- + \alpha_1^+) \qquad \mathbf{R}_{1,1}^{1,1} \subset \mathbf{R}_{0,1}^{1,1}$$

$$3(\alpha_0^- + \alpha_1^-) < (\alpha_0^- + \alpha_1^- + \alpha_1^+ + \alpha_2^+ + \alpha_3^+ + \alpha_4^+) \qquad \mathbf{R}_{0,1} \subset \mathbf{R}_{0,1}^{1,4}$$

$$4(\alpha_3^- + \alpha_3^+ + \alpha_4^+) < 3(\alpha_2^- + \alpha_3^- + \alpha_3^+ + \alpha_4^+) \qquad \mathbf{R}_{3,3}^{3,4} \subset \mathbf{R}_{2,3}^{3,4}$$

$$4(\alpha_2^- + \alpha_2^+ + \alpha_3^+) < 3(\alpha_1^- + \alpha_2^- + \alpha_2^+ + \alpha_3^+) \qquad \mathbf{R}_{2,2}^{2,3} \subset \mathbf{R}_{1,2}^{2,3}$$

$$4\alpha_4^+ < (\alpha_2^- + \alpha_3^- + \alpha_3^+ + \alpha_4^+) \qquad \mathbf{R}^{4,4} \subset \mathbf{R}_{2,3}^{3,4}$$

Labelling these five inequalities as $(a), (b), \dots, (e)$, respectively, we obtain

$$4(a) + 4(b) + (c) + 4(d) + (e) \text{ holds if and only if } 0 < 0,$$

which provides the desired contradiction. $\square$

When combined with Lemmas 2.11(1) and 5.11, the calculation in Proposition 6.1 yields the following consequence.

Corollary 6.2. *For $\ell \geqslant 4$, there is no central charge $\alpha : Q_0 \to \mathbb{R}$ that is complete on $\mathbf{Rep}_{\mathrm{int}}(Q)$.*

Before remedying this defect by considering a larger collection of central charges, we describe another negative result which highlights the necessity of restricting our focus to interval-decomposable representations.

Proposition 6.3. *Let $\mathbb{F}$ be any field other than $\mathbb{Z}/2$, and consider a ladder quiver Q of length $\ell \geqslant 4$. There exist nonisomorphic representations $V \ncong W$ in $\mathbf{Rep}_{\mathrm{eq}}(Q)$ such that $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$ for every central charge $\alpha : Q_0 \to \mathbb{R}$.*

Proof. For each scalar λ in $\mathbb{F}$, consider $V(\lambda) \in \mathbf{Rep}_{\mathrm{eq}}(Q)$ given by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{F} \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"'] & \mathbb{F}^2 \arrow[r, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix}1&\lambda\\1&1\end{smallmatrix}\right]}"'] & \mathbb{F}^2 \arrow[r, "{[1\ 0]}"] \arrow[d, "{[1\ \lambda]}"'] & \mathbb{F} \arrow[d] \\
\mathbb{F} \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"'] & \mathbb{F}^2 \arrow[r, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"'] & \mathbb{F}^2 \arrow[r, "{[1\ 0]}"'] & \mathbb{F} \arrow[r] & 0
\end{tikzcd}$$

(See also [11, Definition 4(2)].) Since we have assumed $\mathbb{F} \neq \mathbb{Z}/2$, there exist distinct nonzero scalars $\lambda \neq \mu$ in $\mathbb{F}$; we set $V := V(\lambda)$ and $W := V(\mu)$. Any isomorphism $\phi : V \to W$ must restrict to automorphisms $\phi^\pm : V^\pm \to W^\pm$ of the top and bottom rows. By [27, Theorem 4.4], both $\phi^\pm$ are forced to be trivial in the chosen bases, that is, represented by the identity matrix on each vertex. It is readily checked (along the middle vertical edge) that this collection of identity matrices does not constitute a morphism $V \to W$ in $\mathbf{Rep}(Q)$, whence V and W are nonisomorphic in the subcategory $\mathbf{Rep}_{\mathrm{eq}}(Q)$. On the other hand, there is an evident bijection between the subrepresentations of V and those of W that preserves dimension vectors, and hence, α-semistability for any choice of central charge $\alpha : Q_0 \to \mathbb{R}$. This yields $\mathbf{T}[V;\alpha] = \mathbf{T}[W;\alpha]$, as claimed. ☐

6.2 | A complete set of central charges

Fix a ladder quiver Q of length $\ell \geqslant 1$. Our goal in this subsection is to prove a stronger version of Proposition 5.13 in the case of ladder persistence modules. More precisely, by performing a more refined analysis, we are able to reduce the size of the collection of central charges and to make the relationship between multiplicities of interval representations and HN types more explicit.

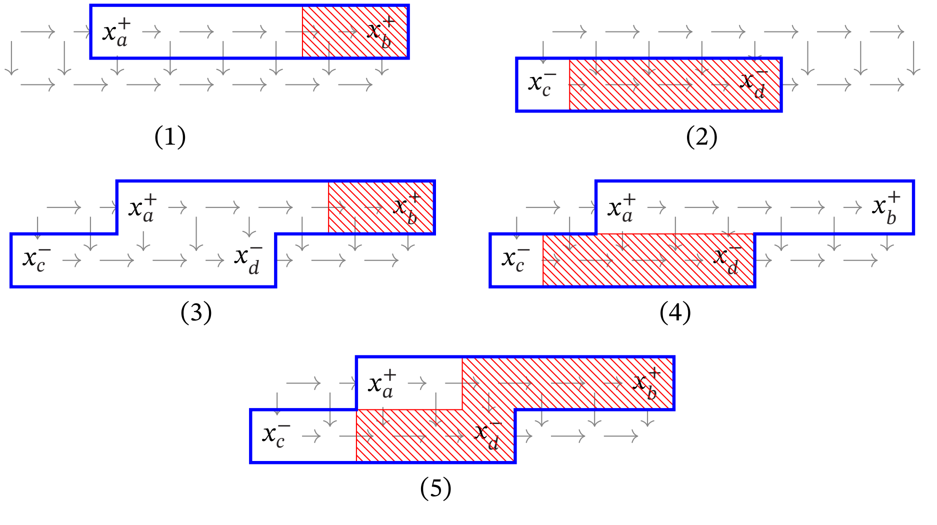
Theorem 6.4. *There exists a finite collection of central charges A whose cardinality grows cubically with ℓ and for which the HN type $\mathbf{T}[\bullet; A]$ is complete on $\mathbf{Rep}_{\text{int}}(Q)$.*

We let $\mathcal{I}$ denote the set of (isomorphism classes of) indecomposable objects which were defined at the beginning of this section. We will also adopt the convention (14) for describing these indecomposables as $\mathbf{R}^{a,b}_{c,d}$ for certain a, b, c, d in the set $[\ell]_\infty := \{0, 1, \dots, \ell, \infty\}$.

Remark 6.5. Here are all the possible strict inclusions $I' \subsetneq I$ in $\mathcal{I}$:

(1)	$\mathbf{R}^{a',b}_{\infty,\infty} \subsetneq \mathbf{R}^{a,b}_{\infty,\infty}$	if $a < a'$
(2)	$\mathbf{R}^{\infty,\infty}_{c',d} \subsetneq \mathbf{R}^{\infty,\infty}_{c,d}$	if $c < c'$
(3)	$\mathbf{R}^{a',b}_{\infty,\infty} \subsetneq \mathbf{R}^{a,b}_{c,d}$	if $d < a'$
(4)	$\mathbf{R}^{\infty,\infty}_{c',d} \subsetneq \mathbf{R}^{a,b}_{c,d}$	if $c \leqslant c'$
(5)	$\mathbf{R}^{a',b}_{c',d} \subsetneq \mathbf{R}^{a,b}_{c,d}$	if $a \leqslant a'$ and $c \leqslant c$.

We recall from Section 5.3 that the support of an indecomposable $I \in \mathcal{I}$ is the subset $\text{supp}(I) \subset Q_0$ consisting of all vertices x for which the vector space I_x is nontrivial. Since the indecomposables of $\mathbf{Rep}_{\text{int}}(Q)$ can be uniquely identified by their supports, we may illustrate these five containments $I' \subsetneq I$ from Remark 6.5 by depicting the supports of I' (shaded red) and I (outlined blue).



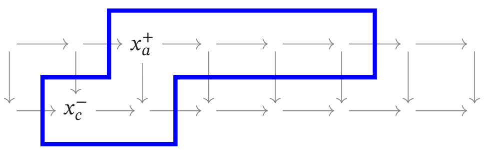
Definition 6.6. For every integer $k \geqslant 0$ and subset $S \subset Q_0$, define the set

$$\mathscr{I}_S^k := \left\{ I \in I \mid \sum_{x \in Q_0} \dim I_x = k \text{ and } \min(I) = S \right\}.$$

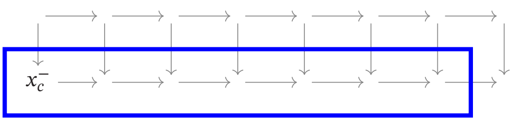
There is a partition $\mathscr{I} = \coprod_{k,S} \mathscr{I}_S^k$ where k ranges over $\mathbb{Z}_{>0}$ while S ranges over subsets of Q_0. The constituent part $\mathscr{I}_S^k$ is empty unless S has one of two possible forms:

- $\{x\}$ for any vertex $x = x_a^+$ or $x = x_c^-$,
- $\left\{x_a^+, x_c^-\right\}$ with $0 \leqslant c < a \leqslant \ell$.

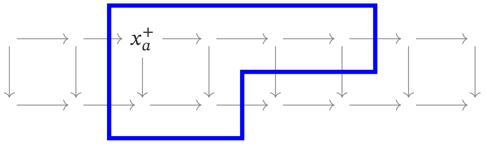
Here are the supports of certain $I \in \mathscr{I}_S^6$ for nontrivial choices of $S \subset Q_0$:

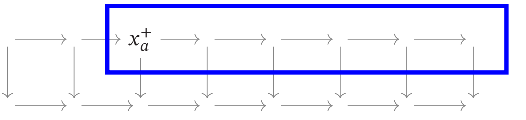


$S = \left\{x_a^+, x_c^-\right\}$



$S = \{x_c^-\}$





$S = \left\{x_a^+\right\}$

Lemma 6.7. *Given any integer $k > 0$ and subset $S \subset Q_0$ of vertices, the set of dimension vectors $\left\{\underline{\dim}_I \mid I \in \mathscr{I}_S^k\right\}$ is linearly independent in the vector space of maps $Q_0 \to \mathbb{R}$.*

Proof. It suffices to consider the nontrivial cases where $|I_S^k| > 1$, which can only occur when either $S = \{x_i^+\}$ for some i or when $S = \{x_i^+, x_j^-\}$ for some $i > j$. We claim that every indecomposable $\mathbf{R}_{c,d}^{a,b} \in \mathscr{I}_S^k$ is uniquely determined by b. To verify this claim, note first that if $S = \{x_i^+\}$ for some $i \in \{0, 1 \ldots, \ell\}$, then we must have $a = i$ and $c \in \{i, \infty\}$, with $c = \infty$ occurring if and only if $b = i + k - 1$. Otherwise, if $S = \{x_i^+, x_j^-\}$ for $i > j$, then $a = i$ and $c = j$. In both cases, d is determined by the formula

$$k = (b - a) + (d - c) + 2.$$

Thus, the map $\iota : \mathscr{I}_S^k \to [\ell]_\infty$ sending each $\mathbf{R}_{c,d}^{a,b}$ to b is injective, as claimed above. Now consider an $\mathbb{R}$-linear combination of the form

$$v := \sum_{I \in I_S^k} r_I \, \underline{\dim}_I \, .$$

A brief examination of the supports of indecomposables lying in $\mathscr{I}_S^k$ reveals that the value $v(x_j^+)$ depends only on those I which satisfy $\iota(I) \geqslant j$. Thus, the real numbers r_I may be recovered by descending induction on $\iota(I)$. $\square$

Let us fix a collection of irrational numbers $\{t_{p,q}\}$ indexed by pairs of integers $0 \leqslant q < p$ so that the following inequalities hold:

$$\frac{q}{p-q} < t_{p,q} < \frac{q+1}{p-(q+1)}. \tag{16}$$

Definition 6.8. For each integer $k > 0$ and subset $S \subset Q_0$ for which $\mathcal{I}_S^k$ is nonempty, define the central charge $\alpha_S^k : Q_0 \to \mathbb{R}$ as

$$\alpha_S^k := \begin{cases} \delta_x & \text{if } S = \{x\}. \\ \delta_{x_a^+} + t_{k,a-c} \cdot \delta_{x_c^-} & \text{if } S = \{x_a^+, x_c^-\}. \end{cases}$$

Here δ_x is the skyscraper central charge at vertex x (from Definition 3.1) while the $t_{\bullet,\bullet}$ are irrational numbers chosen to satisfy (16).

We now compute the HN type of every indecomposable $I \in \mathcal{I}$ along these central charges. The calculation below makes essential use of spanning subrepresentations $\langle I_S \rangle$, which were introduced in Definition 3.2.

Lemma 6.9. *Fix any $k > 0$ and $S \subset Q_0$ for which $\mathcal{I}_S^k$ is nonempty, and define*

$$\lambda_S^k := \begin{cases} 1/k & \text{if } |S| = 1, \\ (1 + t_{k,a-c})/k & \text{if } S = \{x_a^+, x_c^-\}. \end{cases}$$

The following assertions are equivalent for every indecomposable $I \in \mathcal{I}_{S'}^{k'}$:

(1) *The α_S^k-slope of I equals λ_S^k.*
(2) *Both $k = k'$ and $S \subset \mathrm{supp}(I)$ hold.*

If either assertion is true, then we also have that I is α_S^k-semistable if and only if $S = S'$.

Proof. We abbreviate α_S^k to α and λ_S^k to λ. Recall from Definition 6.6 that either $S = \{x\}$ or $S = \{x_a^+, x_c^-\}$ with $a > c$. In the first case, the desired equivalence follows from computing

$$\mu_\alpha(I) = \begin{cases} 1/k' & \text{if } x \in \mathrm{supp}(I) \\ 0 & \text{otherwise} \end{cases}. \tag{17}$$

In the case $S = \{x_a^+, x_c^-\}$ with $a > c$, we similarly have:

$$\mu_\alpha(I) = \begin{cases} 1/k' & \text{if } S \cap \mathrm{supp}(I) = \{x_a^+\}, \\ t_{k,a-c}/k' & \text{if } S \cap \mathrm{supp}(I) = \{x_c^-\}, \\ (1 + t_{k,a-c})/k' & \text{if } S \subset \mathrm{supp}(I), \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Since $t_{k,a-c}$ is irrational by assumption, $\mu_\alpha(I)$ determines $S \cap \mathrm{supp}(I)$, as desired.

We now assume that I is α-semistable (in addition to satisfying $\mu_\alpha(I) = \lambda$), and seek to show $S = S'$. Since $\langle I_S \rangle$ is a subrepresentation of I with $\mu_\alpha(\langle I_S \rangle) \geqslant \lambda$, we must have $I = \langle I_S \rangle$. This forces I

to lie in $\mathcal{I}_S^k$, thus ensuring $S' = S$ as desired. Conversely, if $S = S'$, then I lies in $\mathcal{I}_S^k$. By Lemma 2.5, it suffices to show that indecomposable subrepresentations $I' \subsetneq I$ have smaller α-slope than I. It is readily checked that any subrepresentation $I' \subset I$ must be equalised, and by Lemma 5.11, I' is also interval-decomposable. Therefore, it suffices to consider only those I' which have been listed in Remark 6.5. Of these, the I' with nonzero α-slope all have the form $\langle I_{S'} \rangle$ for $S' \subsetneq S$. When $|S| = 1$, there are no such I' to consider; and when $S = \{x_a^+, x_c^-\}$ for $a > c$, the only relevant I' are $\langle I_{x_a^+} \rangle$ and $\langle I_{x_c^-} \rangle$. Thus, the desired semistability of I reduces to verifying two inequalities:

$$\frac{1}{|\{x \geqslant x_a^+\} \cap \operatorname{supp}(I)|} \leqslant \frac{1 + t_{k,a-c}}{k} \geqslant \frac{t_{k,a-c}}{|\{x \geqslant x_c^-\} \cap \operatorname{supp}(I)|}.$$

Both hold by the constraints imposed on $t_{k,a-c}$ in (16). $\square$

We recall from (5) that given a central charge $\alpha : Q_0 \to \mathbb{R}$, the HN type $\mathbf{T}[V;\alpha]$ of any $V \in \mathbf{Rep}_{\mathrm{int}}(Q)$ may be viewed as a function $\mathbb{R} \to \mathbb{N}^{Q_0}$. The following result describes the values of the function associated to each central charge α_S^k from Definition 6.8 at the corresponding slope λ_S^k from Lemma 6.9.

Proposition 6.10. *Consider any (k, S) such that $\mathcal{I}_S^k$ is nonempty, and let $I \in \mathcal{I}$. Then for $\alpha := \alpha_S^k$ and $\lambda := \lambda_S^k$, we have*

$$\mathbf{T}[I;\alpha](\lambda) = \begin{cases} \underline{\dim}_{\langle I_S \rangle} & \text{if } \langle I_S \rangle \in \mathcal{I}_S^k \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Proof. Letting $I^\bullet$ be the HN filtration $\mathbf{HN}_\alpha^\bullet(I)$, we consider two possible cases:

Case 1: $I = \langle I_S \rangle$. By Lemma 6.9, we have the equivalence $I \in \mathcal{I}_S^k \Leftrightarrow \mu_\alpha(I) = \lambda$. Hence, if $\langle I_S \rangle$ is α-semistable, then $\mathbf{T}[V;\alpha]$ equals $\underline{\dim}_{\langle I_S \rangle}$ whenever $\langle I_S \rangle$ lies in $\mathcal{I}_S^k$, and is trivial otherwise. Assume now that $\langle I_S \rangle$ is α-unstable. In particular, $S \cap \operatorname{supp}(I)$ is of the form $\{x_a^+, x_c^-\}$ because otherwise $\langle I_S \rangle$ would not have any proper subrepresentations of strictly positive α-slope. In this case, we claim that $I^\bullet$ is the 3-step filtration $0 \subsetneq \langle I_s \rangle \subsetneq \langle I_S \rangle \subsetneq I$ for some $s \in S$. Indeed, $\langle I_{x_a^+} \rangle$ and $\langle I_{x_c^-} \rangle$ are the only proper subrepresentations of I of nonzero slope and they are α-semistable. Then from Definition 6.8, the slopes $\mu_\alpha(I^i/I^{i-1})$ lie in either $\mathbb{Q}$ or in $t_{k,a-c} \cdot \mathbb{Q}$, neither of which contain $\lambda = (1 + t_{k,a-c})/k$, whence $\mathbf{T}[I;\alpha](\lambda) = 0$.

Case 2: $I \neq \langle I_S \rangle$. We claim that $\mathbf{T}[I;\alpha](\lambda) = \mathbf{T}[\langle I_S \rangle;\alpha](\lambda)$. Indeed, by the argument that established Proposition 3.3, we have $I^{\ell-1} = \langle I_S \rangle$ and $\mu_\alpha(I^\ell/I^{\ell-1}) = 0 < \lambda$ where ℓ is the length of $I^\bullet$. Finally, the result follows from the first case. $\square$

To complete the proof of Theorem 6.4, recall the collection of central charges $A = \{\alpha_S^k\}$ from Definition 6.8, and consider some $V \in \mathbf{Rep}_{\mathrm{int}}(Q)$ with decomposition $V \simeq \bigoplus_{I \in \mathcal{I}} I^{r_I}$. We show that the multiplicities $(r_I)_{I \in \mathcal{I}}$ can be recovered from $\mathbf{T}[V;A]$ by descending strong induction on the partial order $\subset$ from Remark 6.5. To this end, note that for indecomposables $I \in \mathcal{I}_S^k$ and $J \in \mathcal{I}$, we have $\langle J_S \rangle = I$ if and only if I is a subrepresentation of J. By combining Propositions 2.8 and 6.10, we get

$$\mathbf{T}[V;\alpha_S^k](\lambda_S^k) = \sum_{I \in I_S^k} r_I^+ \cdot \underline{\dim}_I,$$

where $r_I^+ := \sum_J r_J$ for J ranging over representations in $\mathscr{I}$ which satisfy $J \supseteq I$. By Lemma 6.7, the integer r_I^+ can be obtained from $\mathbf{T}[V;A]$ for each $I \in \mathscr{I}_S^k$. Finally, assume by the inductive hypothesis that r_J is known for all $J \supsetneq I$ in $\mathscr{I}$. Now r_I can be recovered from $\mathbf{T}[V;A]$ as the difference $r_I = r_I^+ - \sum_{J \supsetneq I} r_J$.

Finally, it follows from Definition 6.8 that the set of complete central charges A has cardinality

$$c(\ell) := \frac{2\ell^3 + 3\ell^2 + 13\ell + 12}{6},$$

and hence grows cubically with the length ℓ of the underlying ladder quiver.

Remark 6.11. One can choose different irrational numbers $t_{p,q}$ in (16) to generate uncountably many other such complete sets of the same cardinality $c(\ell)$. It is not clear to us (for large ℓ) whether *all* complete sets of central charges for interval-decomposable representations of ladder quivers can be obtained in this fashion; nor is it clear whether $c(\ell)$ is the minimal size for such a complete set.

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