Behavioral Optimal Consumption and Portfolio Selection in Continuous Time

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A thesis submitted for the degree of

MSc. by Research

Trinity 2013

Acknowledgements

I would like to thank Dr Hanqing Jin, Professor Xunyu Zhou, Professor Jian-An Yan, and Professor Jianming Xia. I am also indebted to my thesis examiners, Dr Michael Monoyios and Dr Hao Xing for their comments. In addition, I would like to thank other colleagues in the math finance group.

My thanks go out to my family, especially to my wife, for her love.

Finally, I am grateful for the financial support from the Oxford-Man Institute of Quantitative Finance and Mathematical Institute.

Abstract

This thesis mainly concerns a continuous-time behavioral consumption model under Kahneman and Tversky's cumulative prospect theory. Mathematically this is a non-concave maximization problem because of the presence of an S-shaped functional and the presence of so-called probability distortions. By using a quantile method and divide-and-conquer scheme, we solve the problem quite explicitly and the optimal consumption is in general characterized in two parts: the agent has rich consumption above the benchmark in good situations and suffers from hunger (i.e. no consumption) in bad situations. An example is given to show that judging whether the market is good or bad depends highly on the agent's benchmark. Finally we give the strategy for optimal consumption and portfolio selection to maximize behavioral utilities from both consumption and terminal wealth.

Contents

Chapter 1

Introduction

This chapter summarizes the work of the thesis. First, I give lterature review on the topic indicated by the title. Then I explain the constribution of the thesis and give its outline.

1.1 Literature Review

One problem in finance theory is utility maximization in continuous time, for example Merton [14, 15], Karatzas and Shreve [11]. Specifically speaking, an agent with an initial endowment can consume while also investing in the market. The objective of this agent is to maximize the utility of consumption over the planning horizon, or to maximize the utility of wealth at the end of time period, or to maximize the sum of the both. Conventionally people use Expected Utility Theory (EUT) to model an agent's utility while facing uncertainty. The theory, based on von Neumann and Morgenstern [19], has three key assumptions:

(i) The agent is rational and able to objectively evaluate probabilities.

(ii) The agent evaluates wealth according to final asset positions only.

(iii) The agent is uniformly risk-averse, i.e. her utility function is globally concave.

However EUT has long been challenged as these assumptions were found inconsistent with the situation in the real world. Besides there are paradoxes and puzzles that EUT fails to explain, for example Allais paradox (Allais [1]) and the equity premium puzzle (Mehra and Prescott [13]).

There are a number of alternative preference models to EUT, among which, Quiggin's rank-dependent utility theory [16], Yaari's "dual theory of choice" [21], Lopes's SP/A model [12], and Kahneman and Tversky's cumulative prospect theory (CPT;

[10, 18]) are notable. These theories among others gave birth to a new branch of economics, i.e. behavioral economics, which features the influence of the agent's psychology on making decisions in the face of uncertainty. CPT is widely regarded as the richest behavioral economic theory, which replaced the above three assumptions of EUT correspondingly with:

(i) The agent has probability distortion, which is nonlinear by amplifying a small probability and underestimating a large probability.

(ii) There is a reference point or benchmark in wealth that differentiates gains and losses.

(iii) The agent is risk-averse on gains and risk-taking on losses, i.e., her utility function (which is called value function in CPT) is concave on gains and convex on losses.

There has been burgeoning research in incorporating behavioral theories into portfolio choice selection. Most of the research is either limited to single-period models or some ingredients in (i)-(iii) are missing. For example, the single-peirod research includes Benartzi and Thaler [2], Shefrin and Statman [17], De Giorgi and Post [4], and He and Zhou [5]; while Berkelaar et al [3] consider a special case in continuous time with some ingredients of CPT but without probability distortions. The main difficulties lie in that due to the existence of probability distortions and S-shaped utility function, conventional methods for EUT like dynamic programming or convex analysis are not applicable.

For the continuous-time setting, Jin and Zhou [8, 9] first construct a general model under CPT in a complete market and solve the problem of utility maximization of lower bounded terminal wealth at the end of planning horizon. The key method in [8] includes a divide-and-conquer procedure and a quantile formulation and a technique to solve a concave Choquet minimization problem. The structure of the optimal terminal wealth is of two parts, indicating a bet on a good state of the market while accepting a fixed loss in a bad one. Zhang, Jin and Zhou [22] apply the approach in a similar model, but in which the terminal wealth has a fixed lower bound. Due to the existence of loss control (i.e. the lower bound), the structure of the solution features in three parts: the agent has gains in the good states of market, gets a moderate constant loss in the intermediate states, and suffers the maximal loss (which is the given bound for losses) in bad states.

1.2 Main Contributions

In the thesis, we extend Jin and Zhou [8] to maximizing utility of intertemporal consumption. Essentially by following the approach developed in [8], we solve the problem quite explicitly. The approach consists of several steps.

Step 1. Our problem is law invariant in that if two consumption processes (for example $c_1(\cdot), c_2(\cdot)$) have the same probability law at each time (i.e. $c_1(t) \sim c_2(t)$, $a.e.\, t \in [0, T]$), the agent would get the same utility from them. Due to this property, we use the quantile method to transform the original problem from looking for the optimal stochastic process (i.e. the consumption process) into finding an optimal two-variable function (see Lemma 2). Then we have a discussion of the well-posedness of the problem (see Theorem 1).

Step 2. In order to handle the S-shaped utility function in the transformed problem (2.5), we use the divide-and-conquer procedure, that is to decompose the problem into a positive part problem and a negative part problem, by introducing some auxiliary parameters (see Theorem 2).

The positive part problem is essentially a concave maximization problem, which we solve by using variational calculus on quantile functions (see Theorem 3).

The negative part problem involves minimizing a concave functional. We solve it by looking for the corner point in the functional space (see Theorem 4).

Step 3. After solving the positive and negative part problems, the solution to the original problem can be found by optimizing the parameters introduced in the second step (see Theorem 5 and 6).

We should mention that the quantile method is further develped by He and Zhou [6]. Like [8, 22], they only employ the method in solving different portfolio selection problems that are essentially about looking for the optimal random variables. However, in our paper, we manage to make the quantile method work for stochastic processes.

Another notable result is that the solution of our problem is surprisingly characterized by only two parts: the agent has rich consumption above the benchmark in good situations and suffers from hunger (i.e. no consumption) in bad situations (see Theorem 6). This is different from that of [22]. [As the consumption process is nonnegative (thus 0 is the lower bound), one might expect our problem is similar to that of [22].] Thus the structure of our solution is more like the one of Jin and Zhou [8], in which there is no lower bound contraint.

Then we solve a specific example with with a two-piece CRRA utility function to illustrate our result. We find that under some assumptions, judging the market is in good states or bad states highly depends on the agent's reference point or benchmark. If the agent has a lower benchmark, then she tends to believe that the market is good. Conversely, if her bechmark is higher, then she tends to believe the market is bad.

After solving the optimal consumption problem, we give the strategy on how to find the optimal consumption and portfolio selection to maximize the behavioral utilities from both consumption and terminal wealth. The strategy is to divide the initial wealth into two parts, then use one part for maximizing the consumption and the other for the terminal wealth, then the best strategy lies in the best dividing by comparing the sum of the utilities from consumption and terminal wealth.

1.3 The Outline

Chaper 2 is devoted to the main part of the thesis. Section 2.1 is devoted to the formulation of the behavioral model. We reformulate the problem by quantile method and give sufficient conditions to guarantee its well-posedness in Section 2.2. In Section 2.3 we present the divide-and-conquer scheme. In Section 2.4 and 2.5, we respectively solve a concave maximization and a concave minimization problem, which arise from the solution scheme. The main result is given in Section 2.6. An Example with a two-piece CRRA utility function is presented in Section 2.7. Section 2.8 addresses the problem of maximizing utilities from both consumption and terminal wealth in the framework of CPT. We conclude in Chapter 3. Technical preliminaries are found in the appendix.

Chapter 2

Model and Solution

2.1 The Market Model and CPT criteria

In this chapter, $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a given probability space on which is defined a standard $\mathcal{F}_t$-adapted m-dimensional Brownian Motion $W(t) \equiv (W^1(t), \ldots, W^m(t))'$ with $W(0) = 0$. Here and throughout the paper A' denotes the transpose of a matrix A. We assume that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motion and augmented by all the null sets.

We define a continuous-time financial market involving consumption following Karatzas and Shreve [11]. Fix $T > 0$ as the terminal time. In the market there are $m + 1$ assets being traded continuously. One of the assets is a bank account whose price process $S_0(t)$ is subject to the following equation:

$$dS_0(t) = r(t)S_0(t)dt,\ t \in [0, T];\ S_0(0) = s_0 > 0,$$

where the interest rate $r(\cdot)$ is an $\mathcal{F}_t$-progressively measurable, scalar-valued stochastic process with $\int_0^T |r(s)|ds < +\infty, a.s.$. The rest assets are stocks whose price processes $S_i(t), i = 1, \ldots, m$, satisfy the following stochastic differential equations (SDEs):

$$dS_i(t) = S_i(t)[\mu_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)],\ t \in [0, T];\ S_i(0) = s_i > 0,$$

where $\mu_i(\cdot)$ and $\sigma_{ij}(t)$, the appreciation and volatility rates, respectively, are scalar-valued, $\mathcal{F}_t$-progressively measurable stochastic processes with

$$\int_0^T \left[\sum_{i=1}^{m} |\mu_i(t)| + \sum_{i,j=1}^{m} \sigma_{ij}(t)^2 \right] dt < +\infty, a.s..$$

Set the excess rate of return vector process $B(t) := (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t))'$, and define the volatility matrix process $\sigma(t) := (\sigma_{ij}(t))_{m\times m}$. We impose the following basic assumptions on the market parameters:

(i) There exists $a \in \mathbb{R}$ such that $a \leq \int_0^T r(s)ds < +\infty, a.s.$;

(ii) There exists a unique $\mathbb{R}^m$-valued, uniformly bounded, $\mathcal{F}_t$-progressively measurable process $\theta(\cdot)$ such that $\sigma(t)\theta(t) = B(t), a.e.\, t \in [0,T], a.s.$.

It is well known that under these assumptions there exists a unique risk-neutral probability measure Q defined by $\frac{dQ}{dP}|_{\mathcal{F}_t} = \rho(t)S_0(t)$, where

$$\rho(t) := \exp\left\{-\int_0^t [r(s) + \frac{1}{2}|\theta(s)|^2]ds - \int_0^t \theta(s)'dW(s)\right\}$$

is pricing kernel or state density price. It is clear that $0 < \rho(t) < +\infty$, $a.s$,, and $0 < E\rho(t) < +\infty, \forall t \in [0,T]$.

A random variable ξ is said to have no atom if $P(\xi = a) = 0, \forall a \in \mathbb{R}$. We need the following assumption throughout this paper:

Assumption 1. *For each $t \in (0,T], \rho(t)$ admits no atom.*

This assumption is not essential, and is imposed to avoid undue technicality. In particular, it is satisfied when $r(\cdot)$ and $\theta(\cdot)$ are deterministic with $\int_0^t |\theta(s)|^2 ds \neq 0\, \forall t \in (0,T]$ (in which case $\rho(t)$ is a nondegenerate lognormal random variable).

A *portfolio process* $(\pi_0(\cdot), \pi(\cdot))$ consists of an $\mathcal{F}_t$-progressively measurable, real-valued process $\pi_0(\cdot)$ and an $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^m$-valued process $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_m(\cdot))'$ such that

$$\int_0^T |\sigma(t)'\pi(t)|^2 dt < +\infty, \int_0^T |B(t)'\pi(t)|dt < +\infty, a.s..$$

A *consumption process* is an $\mathcal{F}_t$-progressively measurable, nonnegative process $c(\cdot)$ satisfying $\int_0^T c(t)dt < +\infty, a.s.$. It is known that if an agent with initial endowment $x_0 \geq 0$ chooses a consumption process $c(\cdot)$, then the corresponding wealth process $x(\cdot)$ will be governed by the following SDE:

$$\begin{cases} dx(t) = [r(t)x(t) - c(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{2.1}$$

Given $x \geq 0$, we say that a consumption and portfolio process pair $(c(\cdot), \pi(\cdot))$ is *admissible* at x, and write $(c(\cdot), \pi(\cdot)) \in \mathcal{A}(x)$, if the corresponding wealth process $x(\cdot)$ satisfies $x^{x,c,\pi}(t) \geq 0, t \in [0,T]$. Note that here we omit $\pi_0(\cdot)$ since that we can recover $\pi_0(\cdot)$ from $\pi(\cdot), c(\cdot), x(\cdot)$ by the above SDE and $x(t) = \pi_0(t) + \pi(t)'\vec{1} - \int_0^t c(s)ds$, where $\vec{1} = (1, \ldots, 1)'$. The following result is from Karatzas and Shreve ([11], p. 93, Theorem 3.5).

Proposition 1. *Let $x \geq 0$ be given, let $c(\cdot)$ be a consumption process, and let ξ be a nonnegative, $\mathcal{F}_T$-measurable random variable such that $E[\int_0^T \rho(t)c(t)dt + \rho(T)\xi] = x$. Then there exists a portfolio process $\pi(\cdot)$ such that the pair $(c(\cdot), \pi(\cdot))$ is admissible at x and $\xi = x^{x,c,\pi}(T)$.*

In the conventional consumption theory, an agent's preference is modeled by the expected utility of the consumption process. In this paper, we study a consumption model featuring human behaviors by incorporating the CPT framework of Tversky and Kahneman [18]. First of all, we use a deterministic and nonnegative function $b(\cdot)$ on $[0, T]$ to denote the reference point or benchmark in CPT. Next, we are given two time-dependent utility functions $u_+(\cdot)$ and $u_-(\cdot)$, both mapping from $[0, T] \times \mathbb{R}^+$ to $\mathbb{R}^+$, that measure utilities from gains and losses respectively. There are two additional functions $w_+(\cdot)$ and $w_-(\cdot)$ from $[0, T] \times [0, 1]$ to $[0, 1]$, representing the distortions in probability for the gains and losses respectively. The technical assumptions on these functions, which will be imposed throughout this paper, are summarized as follows:

Assumption 2.

(i) *$u_+(\cdot)$ and $u_-(\cdot) : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ are jointly measurable. For each $t \in [0, T]$, $u_\pm(t, \cdot)$ are strictly increasing, concave, with $u_\pm(t, 0) = 0$. Moreover for each $t \in [0, T]$, $u_\pm(t, \cdot)$ are strictly concave. $u_+(t, \cdot)$ is twice differentiable, with the Inada conditions $\frac{\partial u_+(t,0+)}{\partial x} = +\infty$, $\frac{\partial u_+(t,+\infty)}{\partial x} = 0$. Denote $u_\pm^{-1}(t, \cdot)$ the inverse function of $u_\pm(t, \cdot)$ w.r.t the second argument. Denote $I_+(t, x)$ as the inverse function of $\frac{\partial u_+(t,x)}{\partial x}$ w.r.t x. We assume $\int_0^T u_+(t, I_+(t, z))dt < +\infty, \forall z > 0$.*

(ii) *$b(\cdot) : [0, T] \to \mathbb{R}^+$ is measurable and bounded.*

(iii) *$w_+(\cdot)$ and $w_-(\cdot) : [0, T] \times [0, 1] \to [0, 1]$, are jointly measurable, and w.r.t the second argument are differentiable and strictly increasing. $w_\pm(\cdot, 0) \equiv 0$ and $w_\pm(\cdot, 1) \equiv 1$. Denote $w'_\pm(t, x) := \frac{\partial w_\pm(t,x)}{\partial x}$.*

Now given a consumption process $c(\cdot)$, we assign it a value $V(c(\cdot))$ by

$$V(c(\cdot)) = V_+(c(\cdot)) - V_-(c(\cdot)), \tag{2.2}$$

where

$$\begin{aligned} V_+(c(\cdot)) &:= \int_0^T \int_0^\infty w_+(t, P\{u_+(t, (c(t) - b(t))^+) > y\})dydt, \\ V_-(c(\cdot)) &:= \int_0^T \int_0^\infty w_-(t, P\{u_-(t, (c(t) - b(t))^-) > y\})dydt. \end{aligned}$$

It is evident that $V(\cdot)$ is nondecreasing in the sense that $V(c_1(\cdot)) \geq V(c_2(\cdot))$ for any stochastic processes $c_1(\cdot)$ and $c_2(\cdot)$ with $c_1(t) \geq c_2(t), a.s., a.e. t \in [0, T]$.

Our problem is to find the most preferable portfolios and consumption, in terms of maximizing the value $V(c(\cdot))$, by continuously managing the portfolio and consumption. The mathematical formulation is as follows:

$$\begin{array}{l}\text{Maximize } V(c(\cdot))\\ \text{subject to } (c(\cdot),\pi(\cdot))\in\mathcal{A}(x_0).\end{array}$$

In view of Proposition 1, in order to solve the above problem, one can first to solve the following problem w.r.t the consumption process $c(\cdot)$:

$$\begin{array}{ll}\text{Maximize} & V(c(\cdot))\\ \text{subject to} & E\int_0^T \rho(t)c(t)dt = x_0, \int_0^T c(t)dt < +\infty, a.s..\\ & c(\cdot)\geq 0, \text{is}\mathcal{F}_t\text{-progressively measurable.}\end{array} \tag{2.3}$$

Set $d(t) := c(t) - b(t), a_0 := x_0 - E\int_0^T \rho(t)b(t)dt$ and

$$\begin{array}{l}J_+(d(\cdot)) := V_+(d(\cdot)+b(\cdot)),\\ J_-(d(\cdot)) := V_-(d(\cdot)+b(\cdot)),\\ J(d(\cdot)) := J_+(d(\cdot)) - J_-(d(\cdot)),\end{array}$$

then Problem (2.3) turns into

$$\begin{array}{ll}\text{Maximize} & J(d(\cdot))\\ \text{subject to} & E\int_0^T \rho(t)d(t)dt = a_0, \int_0^T d(t)dt < +\infty, a.s..\\ & d(t)\geq -b(t), \text{is } \mathcal{F}_t\text{-progressively measurable.}\end{array} \tag{2.4}$$

2.2 Quantile Transformation and well-posedness

Recall that if a random variable X has a cumulative distribution function (CDF henceafter) $F_X : (-\infty,\infty)\to[0,1]$, then F_X is nondecreasing and right-continuous. The upper quantile $G_X^+ : [0,1)\to[-\infty,\infty]$ and and lower quantile $G_X^- : (0,1]\to[-\infty,\infty]$ are defined as

$$\begin{array}{l}G_X^+(y) := \inf\{x\in\mathbb{R} : F_X(x) > y\},\ y\in[0,1)\\ G_X^-(y) := \inf\{x\in\mathbb{R} : F_X(x)\geq y\},\ y\in(0,1].\end{array}$$

It is well known that G_X^+(respectively, G_X^-) is nondecreasing and right- (left-) continuous.

Let $\mathbf{G}$ denote the set of all upper quantile functions, i.e.

$$\mathbf{G} =: \{G : [0,1)\to[-\infty,\infty],\ \text{nondecreasing, right-continous}\}.$$

Throughout this paper we denote respectively by $F(t,\cdot)$ and $F^{-1}(t,\cdot)$ the CDF and lower quantile of the pricing kernel $\rho(t)$. Denote

$$\mathbb{G} := \{g : [0,T]\times[0,1)\to[-\infty,\infty], g(t,\cdot)\in\mathbf{G}\}.$$

Lemma 1. *If $d^*(\cdot)$ solves problem (2.4), and its upper-quantile function at t is $g^*(t,\cdot)$, then $d^*(t) = g^*(t, 1-F(t,\rho(t)))$, a.s. a.e. $t \in [0,T]$.*

Proof. Denote $\bar{d}(t) := g^*(t, 1-F(t,\rho(t)))$. If the conclusion is not true, then $E\int_0^T 1_A dt > 0$, where $A := \{(t,\omega)|\bar{d}(t,\omega) \neq d^*(t,\omega)\}$. Then $\exists B \subseteq [0,T]$ with positive Lebesgue measure such that $P(A_t) > 0$ for any $t \in B$. As $E\int_0^T [\rho(t)d^*(t)]dt < +\infty$, we have $E[\rho(t)d^*(t)] < +\infty, a.e. t \in [0,T]$ by Fubini theorem.

By Theorem B.1 of Jin and Zhou [8], $E[\rho(t)\bar{d}(t)] \leq E[\rho(t)d^*(t)], a.e. t \in [0,T]$ and $E[\rho(t)\bar{d}(t)] < E[\rho(t)d^*(t)], a.e. t \in B$. Thus $E\int_0^T \rho(t)\bar{d}(t)dt < E\int_0^T \rho(t)d^*(t)dt$. Set $d_1(t) := \bar{d}(t) + \frac{a_1}{\rho(t)}$, where $a_1 := a_0 - E\int_0^T \rho(t)\bar{d}(t)dt > 0$. Then $d_1(\cdot) > \bar{d}(\cdot)$ and

$$E\int_0^T \rho(t)d_1(t)dt = a_0.$$

Thus it is feasible for problem (2.4). But $J_+(d_1(\cdot)) > J_+(\bar{d}(\cdot)) = J_+(d^*(\cdot))$, which contradicts the optimality of $d^*(\cdot)$. □

Denote $Z_t := 1 - F(t,\rho(t))$. Then $Z_t \sim U(0,1), t \in (0,1]$. Motivated by Lemma 1, we replace $d(t)$ in $J_\pm(d(\cdot)))$ with $g(t,Z_t)$, where $g(t,\cdot)$ is the upper quantile of $d(t)$. Setting $\bar{w}_+(t,x) := w_+(t,1-x)$ and by integration by parts, we can get:

$$\begin{aligned}
J_+(g(\cdot, Z_\cdot)) &= \int_0^T \int_0^\infty w_+(t, P\{u_+(t, g(t,Z_t)^+) > y\})dydt \\
&= \int_0^T \int_0^\infty w_+(P\{g(Z_t) > u_+^{-1}(y), g(Z_t) \geq 0\})dydt \\
&= \int_0^T \int_0^\infty w_+(P\{Z_t > G(u_+^{-1}(y))\})dydt \\
&= \int_0^T \int_0^\infty \bar{w}_+(G(u_+^{-1}(y)))dydt \\
&= \int_0^T \int_0^{\bar{w}_+(G(0))} u_+(g(\bar{w}_+^{-1}(x)))dxdt \\
&= \int_0^T \int_0^1 u_+(g(z))w_+'(1-z)1_{g(z)\geq 0}dzdt \\
&= E\int_0^T u_+(t, g(t,Z_t))w_+'(t, 1-Z_t)1_{g(t,Z_t)\geq 0}dt,
\end{aligned}$$

where $G(t,\cdot)$ is the CDF of $d(t)$. Here and henceforth, we sometimes omit t in any time-dependent functions for notational simplicity if no confusion occurs.

Similarly,

$$\begin{aligned} J_-(g(\cdot, Z.)) &= \int_0^T \int_0^1 u_-(-g(z))w'_-(z)1_{g(z)<0}dzdt \\ &= E\int_0^T u_-(t, -g(t,z))w'_-(t, Z_t)1_{g(t,Z_t)<0}dt. \end{aligned}$$

Moreover, note that $\rho(t) = F^{-1}(1 - Z_t)$, then Problem (2.4) turns into:

$$\begin{array}{l} \text{Maximize } v(g) := \int_0^T \int_0^1 u(t, z, g(t, z))dzdt \\ \text{subject to } g \in \hat{\mathbb{G}}, \int_0^T \int_0^1 g(t, z)F^{-1}(t, 1 - z)dzdt = a_0, \end{array} \tag{2.5}$$

where

$$u(t, z, x) := \begin{cases} u_+(t, x)w'_+(t, 1 - z), & \text{if } x \geq 0, \\ -u_-(t, -x)w'_-(t, z), & \text{if } x < 0, \end{cases}$$

and

$$\hat{\mathbb{G}} := \left\{ g \in \mathbb{G} : g(t, \cdot) \geq -b(t), \forall t \in [0, T], \int_0^T g(t, z)dt < +\infty, a.e. z \in [0, 1) \right\}.$$

The following lemma verifies the equivalence of Problems (2.4) and (2.5).

Lemma 2. *If $d^*(\cdot)$ solves Problem (2.4) and its quantile function at t is $g^*(t, \cdot)$, then $g^*(\cdot)$ solves Problem (2.5). Conversely, if Problem (2.5) admits an optimal $g^*(\cdot)$, then $g^*(\cdot, Z.)$ solves Problem (2.4).*

Proof. For the first part, by Lemma 1, $d^*(t) = g^*(t, Z_t), a.s.a.e.t \in [0, T]$. Then it is easy to see that $g^*(\cdot) \in \hat{\mathbb{G}}$. If the conclusion is not true, then there exists $g_1(\cdot)$ such that $v(g_1(\cdot)) > v(g^*(\cdot))$. Define $d_1(t) := g_1(t, Z_t)$, which is feasible for (2.4). Then $J(d_1(\cdot)) = v(g_1(\cdot)) > v(g^*(\cdot)) = J(d^*(\cdot))$, contradicting the optimality of $d^*(\cdot)$.

For the second part, set $d^*(t) := g^*(t, Z_t)$. If the conclusion does not hold, then there exists $d_1(\cdot)$ which quantile function at t is $g_1(t, \cdot)$, such that $J(d_1(\cdot)) > J(d^*(\cdot))$. As $g_1(t, Z_t) \sim d_1(t), a.e.t$, then $J(d_1(\cdot)) = J(g_1(\cdot, Z.)) = v(g_1(\cdot)) \leq v(g^*(\cdot)) = J(d^*(\cdot))$. Then we get contradiction. □

Therefore in order to solve Problem (2.4), we just need to solve (2.5). Note that in Problem (2.5), $u(\cdot)$ is an S-shaped functional on $g(\cdot)$ and the constraint on $g(\cdot)$ is linear. One might try to solve it as a whole by Lagrange method and weak duality. However by such method the solution does not necessarily have the properties in $\hat{\mathbb{G}}$, especially the monotonicity. We present the scheme of solution in the next section.

Before concluding the section, we address the issue of well-posedness for Problem (2.4) (thus also for Problem (2.5)). In general a maximization problem is ill-posed if its supremum is infinite; otherwise it is called well-posed.

Denote

$$R_{+}(t,x) := -\frac{x\partial u_{+}^{2}(t,x)/\partial x^{2}}{\partial u_{+}(t,x)/\partial x}, x > 0,$$

which is called the *Arrow-Pratt index of relative risk aversion* of the utility function $u_{+}(\cdot)$ at time t.

Another key function which will play an important role is

$$N(t,y) := -\int_{1-w_{+}^{-1}(t,1-y)}^{1} F^{-1}(t,1-z)dz. \tag{2.6}$$

It is easy to check that $N(t,\cdot)$ is continuous and strictly increasing on $[0,1)$, $\forall t \in [0,T]$. Denote $\hat{N}(t,\cdot)$ as the concave envelope of $N(t,\cdot)$.

Theorem 1. *Suppose that Assumptions 1 and 2 hold, then Problem* (2.4) *is well-posed for any* $a_0 > 0$ *under the following conditions:*

(i) $\liminf_{x\to\infty} R_{+}(t,x) > 0$ *uniformly in* $t \in [0,T]$*;*

(ii) $\int_0^T I_{+}(t,\lambda\hat{N}'(t,1))dt < \infty, \forall\lambda > 0,$ *where* $\hat{N}'(t,x) := \frac{\partial}{\partial x}\hat{N}(t,x)$*.*

(iii) $\int_0^T\int_0^1 u_{+}(t,I_{+}(t,\hat{N}'(t,1-w_{+}(t,1-z)))w'_{+}(t,1-z)dzdt < \infty$*.*

Proof. See the Appendix. □

Remark 1. *The main idea behind this theorem is that in ordr to guarantee the well-posedness of Problem* (2.4)*, we consider a related problem (Problem* (A.1) *in the Appendix) which dominates* (2.4) *(see Lemma 4 in the Appendix). We solves the dominating problem explicitly using Lagrange method and calculus of variation. Roughly speaking, condition (i) is sufficient for the existence of the Lagrange multiplier. Condition (ii) guarantees that the solution for the dominating problem is feasible, i.e. it satisfies the integration condition. Finally condition (iii) makes sure the dominating problem has finite maximum.*

In view of the above theorem, we need the following assumption:

Assumption 3. *(i)-(iii) of Theorem 1 hold.*

2.3 Dividing

In the section we give the solution scheme of Problem (2.5) by constructing three sub-problems, which as a whole are equivalent to (2.5). Given $a_+ \geq a_0^+$ and $h(\cdot) : [0,T] \to [0,1]$, we consider:

(i) Positive Part Problem:

$$\begin{array}{ll} \text{Maximize} & v_+(g(\cdot)) := \int_0^T \int_0^1 u_+(t, g(t,z)) w'_+(1-z) 1_{z>h(t)} dz dt \\ \text{subject to} & g(\cdot) \in \hat{\mathbb{G}}_+, \int_0^T \int_0^1 g(z) F^{-1}(1-z) dz dt = a_+, \\ & g(\cdot, z) 1_{z \leq h(\cdot)} = 0, \forall z \in [0,1), \end{array} \tag{2.7}$$

where

$$\hat{\mathbb{G}}_+ := \{g(\cdot) : g(\cdot) \in \hat{\mathbb{G}}, g(\cdot) \geq 0\}. \tag{2.8}$$

If $h(\cdot) \not\equiv 1$, *a.e.* on $[0,1]$, then the feasible region of (2.7) is nonempty (for example, $\frac{a_+}{F^{-1}(1-z)} 1_{z \geq h(t)}$ is a feasible solution), define $v_+(a_+, h(\cdot))$ to be the supremum of (2.7). If $h(\cdot) \equiv 1$ *a.e.* on $[0,1]$, and $a_+ > 0$, the there is no feasible solution and we define $v_+(a_+, h(\cdot)) := -\infty$.

(ii) Negative Part Problem:

$$\begin{array}{ll} \text{Minimize} & v_-(g(\cdot)) := \int_0^T \int_0^1 u_-(t, g(t,z)) w'_-(t,z) 1_{z \leq h(t)} dz dt \\ \text{subject to} & g(\cdot) \in \hat{\mathbb{G}}_-, \int_0^T \int_0^1 g(z) F^{-1}(1-z) dz dt = a_+ - a_0, \\ & g(\cdot, z) 1_{z > h(\cdot)} = 0, \forall z \in [0,1), \end{array} \tag{2.9}$$

where

$$\hat{\mathbb{G}}_- := \{g : -g \in \hat{\mathbb{G}}, 0 \leq g(t,z) \leq b(t), a.e. t \in [0,T], z \in [0,1)\}.$$

If $\int_0^T \int_0^1 b(t) F^{-1}(1-z) 1_{z \leq h(t)} dz dt \geq a_+ - a_0$, then the feasible region is nonempty. In fact

$$[(a_+ - a_0) b(t) 1_{z \leq h(t)}] / \int_0^T \int_0^1 b(t) F^{-1}(1-z) dz dt$$

is a feasible solution. Otherwise there is no feasible solution. Denote $v_-(a_+, h(\cdot))$ the infimum of (2.9) when the feasible region is nonempty; otherwise set $v_-(a_+, h(\cdot)) := \infty$.

(iii)

$$\begin{array}{ll} \text{Maximize} & v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) \\ \text{subject to} & a_+ \geq a_0^+, 0 \leq h(\cdot) \leq 1; a_+ = 0 \text{ when } h(\cdot) \equiv 1; \\ & \int_0^T \int_0^1 b(t) F^{-1}(1-z) 1_{z \leq h(t)} dz dt \geq a_+ - a_0. \end{array} \tag{2.10}$$

To prove the equivalence of Problem (2.5) and the above three sub-problems, we start with the equivalence of well-posedness.

Proposition 2. *Problem* (2.5) *is ill-posed if and only if Problem* (2.10) *is ill-posed.*

Proof. Suppose (2.10) is ill-posed. For any $M > 0$ and feasbile $(a_+, h(\cdot))$ such that $v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) > M$, it is easy to see that $h(\cdot) \not\equiv 1$. If $v_+(a_+, h(\cdot)) < +\infty$, then there exists $g_+(\cdot), g_-(\cdot)$ feasible for (2.7) and (2.9) respectively, satisfying that $v_+(g_+(\cdot)) \geq v_+(a_+, h(\cdot)) - 1, v_-(g_-(\cdot)) \leq v_-(a_+, h(\cdot)) + 1$. Define $g(\cdot) := g_+(\cdot) - g_-(\cdot)$. Then $g(\cdot)$ is feasible for (2.5) and $v(g(\cdot)) = v_+(g_+(\cdot)) - v_-(g_-(\cdot)) \geq v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) - 2 > M - 2$. If $v_+(a_+, h(\cdot)) = +\infty$, choose $g_+(\cdot)$ such that $v_+(g_+(\cdot)) \geq M + v_-(a_+, h(\cdot))$.(Note that $v_-(a_+, h(\cdot)) < +\infty$ for any feasible$(a_+, h(\cdot))$.) Then $v(g(\cdot)) = v_+(g_+(\cdot)) - v_-(g_-(\cdot)) \geq M + v_-(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) - 1 \geq M - 1$. Both of the two cases show that (2.5) is ill-posed.

If (2.5) is ill-posed, then for any $M > 0$, there is a feasible $g(\cdot)$ for (2.5) such that $v(g(\cdot)) > M$. Define $h(t) := \sup\{z; g(t,z) \leq 0\}, a_+ := \int_0^T \int_0^1 g(z)^+ F^{-1}(1 - z)dzdt$. Then $(a_+, h(\cdot))$ is feasible for (2.10). $v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) \geq v_+(g(\cdot)^+) - v_-(g(\cdot)^-) = v(g(\cdot)) > M$, which implies the illposedness of (2.10). □

Now we prove the equivalence.

Theorem 2. *Given* $g^*(\cdot)$, *define* $a_+^* := \int_0^T \int_0^1 g^*(z)^+ F^{-1}(1 - z)dzdt$ *and* $h^*(t) := \sup\{z; g^*(t,z) \leq 0\}$. *Then* $g^*(\cdot)$*is optimal for Problem* (2.5) *if and only if* $(a_+^*, h^*(\cdot))$ *is feasible for Problem* (2.10) *and* $g^*(\cdot)^+, g^*(\cdot)^-$ *are respectively optimal for Problem* (2.7) *and* (2.9) *with the parameter pair* $(a_+^*, h^*(\cdot))$.

Proof. For the "if" part, we have $v(g^*(\cdot)) = v_+(a_+^*, h^*(\cdot)) - v_-(a_+^*, h^*(\cdot))$. For any feasible solution $g(\cdot)$ of (2.5), define $h(t) := \sup\{z; g(t,z) \leq 0\}$ and $a_+ := \int_0^T \int_0^1 g(z)^+ F^{-1}(1 - z)dzdt$. Then $v_+(g(\cdot)^+) \leq v_+(a_+, h(\cdot)), v_-(g(\cdot)^-) \geq v_-(a_+, h(\cdot))$. Thus we have $v(g(\cdot)) \leq v_-(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) \leq v_-(a_+^*, h^*(\cdot)) - v_-(a_+^*, h^*(\cdot)) = v(g^*(\cdot))$, implying that $g^*(\cdot)$ is optimal for (2.5).

For the "only if" part, let $g^*(\cdot)$ be optimal for (2.5). Obviously $v_+(g^*(\cdot)^+) \leq v_-(a_+^*, h^*(\cdot)), v_-(g^*(\cdot)^-) \geq v_-(a_+^*, h^*(\cdot))$. If the first holds strictly, then there exists $g_1(\cdot)$ feasible for (2.7) with $(a_+^*, h^*(\cdot))$ such that $v_+(g^*(\cdot)^+) < v_+(g_1(\cdot))$. Thus $\bar{g}(\cdot, z) := g_1(\cdot)1_{z \geq h^*(\cdot)} + g^*(\cdot)1_{z < h^*(\cdot)}$ is feasible for (2.5) and $v(\bar{g}(\cdot)) > v(g^*(\cdot))$, contradicting the optimality of $g^*(\cdot)$. Similarly we can prove that $g^*(\cdot)^-$ is optimal for (2.9). Thus we have $v_+(g^*(\cdot)^+) = v_-(a_+^*, h^*(\cdot)), v_-(g^*(\cdot)^-) = v_-(a_+^*, h^*(\cdot))$.

Next we show that $v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) \leq v_-(a_+^*, h^*(\cdot)) - v_-(a_+^*, h^*(\cdot)) \equiv v(g^*(\cdot))$ for any feasible pair $(a_+, h(\cdot))$ of (2.10). We prove it in two cases.

(i)If $h(\cdot) \equiv 1$, (hence $a_+ = 0, a_0 \leq 0$),then

$$\begin{aligned}
v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) &= -v_-(0,1) \\
&= \sup_{\int_0^T \int_0^1 g(z)F^{-1}(1-z)dzdt = -a_0, g \in \hat{\mathbb{G}}_-} -v_-(g(\cdot)) \\
&= \sup_{\int_0^T \int_0^1 g(z)F^{-1}(1-z)dzdt = a_0, g \in \hat{\mathbb{G}}, g \leq 0} v(g(\cdot)) \\
&\leq \sup_{\int_0^T \int_0^1 g(z)F^{-1}(1-z)dzdt = a_0, g \in \hat{\mathbb{G}}} v(g(\cdot)) \\
&= v(g^*(\cdot)).
\end{aligned}$$

(ii) If $h(\cdot) \not\equiv 1$, then for any feasible $(a_+, h(\cdot))$ and any $\varepsilon > 0$, there exists $g_1(\cdot), g_2(\cdot)$ feasible for (2.7) and (2.9) respectively, such that $v_+(g_1(\cdot)) > v_+(a_+, h(\cdot)) - \varepsilon, v_-(g_2(\cdot)) < v_-(a_+, h(\cdot)) + \varepsilon$. Set $g(\cdot) = g_1(\cdot) - g_2(\cdot)$. Then $g(\cdot)$ is feasible for (2.5) and $v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) < v_+(g_1(\cdot)) - v_-(g_2(\cdot)) + 2\varepsilon = v(g(\cdot)) + 2\varepsilon \leq v(g^*(\cdot)) + 2\varepsilon$.

This concludes the proof. □

2.4 Positive Part Problem

In this section we solve Problem (2.7). When $h(\cdot) \equiv 1$, (2.7) is trivial. Hece we assume that $h(\cdot) \not\equiv 1$. In such case the problem is similar to Problem (A.2) in the appendix. Obviously the supremum of Problem (2.7) is smaller than that of (A.2) with $a = a_+$. Thus the well-posedness of (A.2) implies that of (2.7).

Given $\lambda > 0$, we consider the following problem.

$$\begin{array}{l}
\text{Maximize } v_+^\lambda(g(\cdot)) := \int_0^T \int_0^1 \left[u_+(g(z))w'_+(1-z) - \lambda g(z)F^{-1}(1-z)\right] dzdt \\
\text{subject to } g(\cdot) \in \hat{\mathbb{G}}, g(\cdot, z)1_{z \leq h(\cdot)} = 0, \forall z \in [0,1).
\end{array} \tag{2.11}$$

Using the similar method discussed in the appendix (see Proposition 8 and Theorem 8), we can derive the solution

$$g_{\lambda,h}(t,z) = I_+(t, \lambda \hat{N}'(t, 1 - w_+(t, 1-z)))1_{z > h(\cdot)}.$$

Set

$$f_h(\lambda) := \int_0^T \int_0^1 g_{\lambda,h}(t,z)F^{-1}(t, 1-z)1_{z > h(\cdot)}dzdt.$$

Similar to the discussion of $f(\lambda)$ in the appendix (A.7), we have that $f_h(\cdot)$ is strictly decreasing and that if $f_h(\lambda_0) < +\infty$ for some $\lambda_0 > 0$, then $f_h(\cdot)$ is continuous on $(\lambda_0, +\infty)$. Thus the Lagrange multiplier exists for any $0 < a_1 < f_h(\lambda_0)$. As shown in the appendix that under Assumption 3 (i.e. the conditions of Theorem 1), $f(\lambda) < +\infty, \forall \lambda > 0$. Hence $f_h(\lambda) \leq f(\lambda) < +\infty, \forall \lambda > 0$, implying that $\exists \lambda^* > 0$ such that $g_{\lambda^*,h}$ is feaible for (2.7). Thus we get:

Theorem 3. *Let Assumptions 1, 2 and 3 hold. Given* $a_+ \geq a_0^+$ *and* $h(\cdot) : [0, T] \to [0, 1]$*, we have*

(i) if $a_+ = 0$*, then* $g^*(\cdot) \equiv 0$ *solves* (2.7) *with* $v_+(a_+, h(\cdot)) = 0$*.*

(ii) if $a_+ > 0, h(\cdot) \equiv 1$*, then there is no feasible solution for* (2.7) *and* $v_+(a_+, h(\cdot)) = -\infty$.

(iii) if $a_+ > 0, h(\cdot) \not\equiv 1$*, then the optimal solution for* (2.7) *is*

$$g_{\lambda^*,h}(t, z) = I_+(t, \lambda^* \hat{N}'(t, 1 - w_+(t, 1 - z)))1_{z>h(\cdot)}$$

where $\lambda^* > 0$ *uniquely solves* $\int_0^T \int_0^1 g_{\lambda^*,h}(t, z) F^{-1}(t, 1 - z) 1_{z>h(\cdot)} dz dt = a_+$*. The optimal value is*

$$v_+(a_+, h(\cdot)) = \int_0^T \int_0^1 u_+(I_+(\lambda^* \hat{N}'(t, 1 - w_+(t, 1 - z))))w'_+(1 - z)1_{z>h} dz dt. \tag{2.12}$$

Before ending the section, we prove that v_+ is strictly increasing in $h(\cdot)$ in the following sense.

Proposition 3. *Let Assumption 1, 2 and 3 hold. If* $a_+ > 0$ *and Problem* (2.7) *admits an optmial solution with* $(a_+, h(\cdot))$*, then* $v_+(a_+, \bar{h}(\cdot)) > v_+(a_+, h(\cdot))$*, where* $\bar{h}(\cdot) \leq h(\cdot)$ *and* $\int_0^T 1_{\{\bar{h}(t) < h(t)\}} dt > 0$.

Proof. Suppose by solving (2.11), we get $g_h(\cdot), g_{\bar{h}}(\cdot)$ be optimal for (2.7) respectively with parameter pairs $(a_+, h(\cdot)), (a_+, \bar{h}(\cdot))$. Then $g_h(\cdot)$ solves (2.11) with the constraint that $g(\cdot, z)1_{z \leq h(\cdot)} = 0, \forall z \in [0, 1]$ while $g_{\bar{h}}(\cdot)$ solves it with $g(\cdot, z)1_{z \leq \bar{h}(\cdot)} = 0, \forall z \in [0, 1]$.

Set $A := \{t; \bar{h}(t) < h(t)\}$. For any $y > 0$, let $\epsilon(t, y) := \inf\{x > 0 : g_h(t, h(t) + x) > y\}$ and

$$g_y(t, z) := \begin{cases} 0, & t \in A, z \in [0, \bar{h}(t)], \\ y & t \in A, z \in (\bar{h}(t), h(t) + \epsilon(t, y)] \\ g_h(t, z), & t \in A^C. \end{cases},$$

Then

$$
\begin{aligned}
& v_+^\lambda(g_y(\cdot)) - v_+^\lambda(g_h(\cdot)) \\
= & \int_A \int_{\bar{h}(t)}^{h(t)+\epsilon(t,y)} u_+(y)w_+'(1-z) - \lambda y F^{-1}(1-z)dzdt \\
& - \int_A \int_{h(t)}^{h(t)+\epsilon(t,y)} u_+(g_h(z))w_+'(1-z) - \lambda g_h(z) F^{-1}(1-z)dzdt \\
\geq & \int_A \int_{\bar{h}(t)}^{h(t)+\epsilon(t,y)} u_+(y)w_+'(1-z) - \lambda y F^{-1}(1-z)dzdt \\
& - \int_A \int_{h(t)}^{h(t)+\epsilon(t,y)} u_+(g_h(z))w_+'(1-z)dzdt \\
\geq & \int_A u_+(t,y) \int_{\bar{h}(t)}^{h(t)} w_+'(1-z)dzdt - \int_A \lambda y \int_{\bar{h}(t)}^{h(t)+\epsilon(t,y)} F^{-1}(1-z)dzdt \\
= & \; y \left[\int_A \frac{u_+(y)}{y}(w_+(1-\bar{h}(t)) - w_+(1-h(t)))dt - \lambda \int_A \int_{\bar{h}(t)}^{h(t)+\epsilon(t,y)} F^{-1}(1-z)dzdt \right].
\end{aligned}
$$

Since $\frac{u_+(t,y)}{y} \to +\infty$ as $y \to 0$, and $w_+(1-\bar{h}(t)) - w_+(1-h(t)) > 0, t \in A$, we have $\int_A \frac{u_+(y)}{y}(w_+(1-\bar{h}(t)) - w_+(1-h(t)))dt \to +\infty$ as $y \to 0$. On the other hand, as $y \to 0$, $\epsilon(t,y) \to 0$, hence

$$
\begin{aligned}
& \int_A \int_{\bar{h}(t)}^{h(t)+\epsilon(t,y)} F^{-1}(1-z)dzdt \\
\leq & \int_A (h(t)+\epsilon(t,y)-\bar{h}(t))F^{-1}(1-\bar{h}(t))dt \\
\to & \int_A (h(t)-\bar{h}(t))F^{-1}(1-\bar{h}(t))dt.
\end{aligned}
$$

Then as $y \to 0$,

$$
\int_A \frac{u_+(y)}{y}(w_+(1-\bar{h}(t)) - w_+(1-h(t)))dt - \lambda \int_A \int_{\bar{h}(t)}^{h(t)+\epsilon(t,y)} F^{-1}(1-z)dzdt \to +\infty.
$$

Fix $y > 0$ small enough such that the left side of the above is larger than 1. Then

$$
v_+^\lambda(g_y(\cdot)) - v_+^\lambda(g_h(\cdot)) \geq y > 0.
$$

Note that $g_h(\cdot), g_y(\cdot)$ both are feasible for (2.11) with the constraint that $g(\cdot,z)1_{z<\bar{h}(\cdot)} = 0, \forall z \in [0,1]$. Then

$$
v_+(g_{\bar{h}}(\cdot)) = v_+^\lambda(g_{\bar{h}}(\cdot)) \geq v_+^\lambda(g_y(\cdot)) > v_+^\lambda(g_h(\cdot)) = v_+(g_h(\cdot)),
$$

which implies that $v_+(a_+, \bar{h}(\cdot)) > v_+(a_+, h(\cdot))$. □

2.5 Negative Part Problem

As discussed in Section 2.3, we assume $\int_0^T \int_0^1 b(t)F^{-1}(1-z)1_{z\le h(t)}dzdt \ge a_+ - a_0$. If $\int_0^T \int_0^1 b(t)F^{-1}(1-z)1_{z\le h(t)}dzdt = a_+ - a_0$, then Problem (2.9) is trivial. Hence we assume $\int_0^T \int_0^1 b(t)F^{-1}(1-z)1_{z\le h(t)}dzdt > a_+ - a_0$. First in the following two propositions, we prove that the feasible solutions in the form of a 2-step function with the value of either 0 or $b(t)$ at time t are better than others.

Proposition 4. *If Problem* (2.9) *has a feasible solution in the form of* $g(t,z) = \sum_{i=1}^n a_i(t)1_{(l_{i-1}(t),l_i(t)]}(z)$ *where* $b(t) \ge a_1(\cdot) \ge ... \ge a_n(\cdot) \ge 0, 0 \equiv l_0(\cdot) \le l_1(\cdot) \le ... \le l_n(\cdot) \le h(\cdot), \int 1_{\{t;a_n(t)>0,l_n(t)<h(t)\}}dt > 0$, *then there exists* $\bar{g}(t,z) = b(t)1_{(0,l(t))}(z)$, *where* $0 \le l(\cdot) \le h(\cdot)$, *such that* $v_-(\bar{g}(\cdot)) \le v_-(g(\cdot))$.

Proof. We prove it by induction. For $n = 1$, i.e. $g(t,z) = a(t)1_{[0,l_1(t)]}(z)$, denote $A := \{t; 0 < a(t) < b(t), 0 \le l_1(t) < h(t)\}$. If the conclusion is not true, then $\int 1_A dt > 0$. Fix $l_1(\cdot)$, we consider the following problem:

$$\begin{array}{l} \text{Minimize } \bar{v}_1(\alpha(\cdot)) := \int_0^T u_-(\alpha(t))w_-(l_1(t))dt \\ \text{subject to } 0 \le \alpha(\cdot) \le b(\cdot), \int_0^T \alpha(t)\int_0^{l_1(t)} F^{-1}(1-z)dzdt = a_+ - a_0. \end{array} \tag{2.13}$$

Then $a(\cdot)$ is feasible to (2.13). Note that in (2.13), $\bar{v}_1(\alpha(\cdot))$ is concave in $\alpha(\cdot)$ while the constraint is linear in $\alpha(\cdot)$. Then we can solve it by the Lagrange method. For given $\lambda > 0$, we consider

$$\begin{array}{l} \text{Minimize } \bar{v}_1^\lambda(\alpha(\cdot)) := \int_0^T \left[u_-(\alpha(t))w_-(l_1(t))] - \lambda\alpha(t)\int_0^{l_1(t)} F^{-1}(1-z)dz\right] dt \\ \text{subject to } 0 \le \alpha(\cdot) \le b(\cdot). \end{array} \tag{2.14}$$

The solution is

$$\alpha_\lambda(t) = \begin{cases} b(t), & \text{if } u_-(b(t))w_-(l_1(t)) - \lambda b(t)\int_0^{l_1(t)} F^{-1}(1-z)dz \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

Set $\phi(\lambda) := \int_0^T \alpha_\lambda(t)\int_{l_0(t)}^{h(t)} F^{-1}(1-z)dzdt$, then it is easy to see that $\phi(\cdot)$ is continuous and increasing on $(0,+\infty)$ and $\phi(0+) = 0, \phi(+\infty) > a_+ - a_0$. Then there exists λ^* such that $\phi(\lambda^*) = a_+ - a_0$. Then by weak duality, we have

$$\begin{aligned} \inf_{(2.13)} \bar{v}_1(\alpha(\cdot)) &\ge \sup_{\lambda>0} \inf_{(2.14)} [\bar{v}_1^\lambda(\alpha(\cdot)) + \lambda(a_+ - a_0)] \\ &\ge \inf_{(2.14)} [\bar{v}_1^{\lambda^*}(\alpha(\cdot)) + \lambda^*(a_+ - a_0)] \\ &= \bar{v}_1^{\lambda^*}(\alpha_{\lambda^*}(\cdot)) + \lambda^*(a_+ - a_0) \\ &\ge \inf_{(2.13)} \bar{v}_1(\alpha(\cdot)), \end{aligned}$$

where the feasible regions of inf's are those of the coresponding equations. Then $\alpha_{\lambda^*}(\cdot)$ solves (2.13), implying that $\bar{v}_1^{\lambda^*}(\alpha_{\lambda^*}(\cdot)) \le \bar{v}_1^{\lambda^*}(a(\cdot))$. Denote

$$l(t) := \begin{cases} l_0(t), & t \in R, \\ 0, & t \notin R, \end{cases}$$

where $R := \{t; u_-(b(t))w_-(l_1(t)) - \lambda^* b(t) \int_0^{l_1(t)} F^{-1}(1-z)dz \le 0\}$. Set $\bar{g}(t,z) := b(t)1_{[0,l(t)]}(z)$. The above discussion shows that $v_-(\bar{g}(\cdot)) \le v_-(g(\cdot))$.

Suppose the conclusion holds true for the case of $n-1(n>1)$. In the case of n, suppose $g(t,z) = \sum_{i=1}^n a_i(t)1_{(l_{i-1}(t),l_i(t)]}(z)$ is a feasible solution to (2.9). If $a_n(\cdot) \equiv a_{n-1}(\cdot)$ or $l_n(\cdot) \equiv l_{n-1}(\cdot)$, the it turns to the $n-1$ case, in which the conclusion is true. Otherwise $\int 1_{A_n} dt > 0$, where $A_n := \{t; 0 < a_n(t) < a_{n-1}(t), 0 \le l_{n-1}(t) < l_n(t)\}$. We fix $l_0(\cdot), l_n(\cdot), l_j(\cdot), a_j(\cdot), j = 1, ..., n-1$, and consider:

$$\begin{aligned} &\text{Minimize } \bar{v}_n(\alpha(\cdot)) := \int_0^T u_-(\alpha(t))[w_-(l_n(t)) - w_-(l_{n-1}(t))]dt \\ &\text{subject to } 0 \le \alpha(\cdot) \le a_{n-1}(\cdot), \int_0^T \alpha(t) \int_{l_{n-1}(t)}^{l_n(t)} F^{-1}(1-z)dzdt = \bar{a}_n, \end{aligned}$$

where $\bar{a}_n = a_+ - a_0 - \int_0^T \int_0^1 g(t,z)F^{-1}(1-z)1_{\{z \le l_{n-1}(t)\}}dzdt > 0$. Then $a_n(\cdot)$ is feasible to the above problem. However, following the same procedure in the case of $n=1$, we can find

$$\bar{\alpha}(t) = \begin{cases} a_{n-1}(t), & t \in R, \\ 0, & t \notin R, \end{cases}$$

for some $R \in \mathscr{B}[0,T]$ such that $\bar{v}_n(\bar{\alpha}(\cdot)) \le \bar{v}_n(a_{n-1}(\cdot))$. Set $\bar{l}_{n-1}(t) := l_n(t)1_{t \in R} + l_{n-1}(t)1_{t \notin R}$ and $\bar{g}(t,z) := \sum_{i=1}^{n-2} a_i(t)1_{(l_{i-1}(t),l_i(t)]}(z) + \bar{\alpha}(t)1_{(l_{n-2}(t),\bar{l}_{n-1}(t)]}+$, then we get $v_-(\bar{g}(\cdot)) \le v_-(g(\cdot))$ while $\bar{g}(\cdot)$ is a $(n-1)$-step function. By induction, the conclusion holds true. □

Proposition 5. *If Problem* (2.9) *has an optimal solution, then it must be in the form* $g(t,z) = b(t)1_{[0,l(t)]}(z)$, *where* $0 \le l(\cdot) \le h(\cdot)$.

Proof. Let $g(\cdot)$ be an optimal solution to (2.9). Denote $p(t) := \int_0^{h(t)} g(t,z)F^{-1}(t, 1-z)dz$, then $p(t) < +\infty, a.e.t \in [0,T]$. If the conclusion does not hold ture, then the set $\{(t,z); 0 < g(t,z) < b(t)\}$ must has a positive Lebesgue measure. Thus $\exists L \in \mathscr{B}[0,T], \int 1_L > 0$, such that $\forall t \in L$, $h(t) > 0$ and $\exists z_t > 0$, $0 < g(t,z_t) < b(t)$. Then $L = L_1 \cup L_2$, where $L_1 = \{t \in L; g(t,z_t) < g(t,0+)\}, L_2 = \{t \in L; g(t,z_t) = g(t,0+)\}$. We consider two cases.

Case 1:$t \in L_1$. Then for $\beta_t \in (0,1)$, define

$$g_1^\beta(t,z) := \begin{cases} \beta_t g(t,z), & t \in L_1, z > z_t, \\ \beta_t g(t,z_t) + \frac{\gamma_t g(t,0+) - \beta_t g(t,z_t)}{g(t,0+) - g(t,z_t)}(g(t,z) - g(t,z_t)), & t \in L_1, 0 < z < z_t, \\ g(t,z) & t \notin L_1, \end{cases}$$

where γ_t is uniquely determined by $\int_0^{h(t)} g_1^\beta(t,z)F^{-1}(t,1-z)dz = p(t)$. If $\gamma_t \le 1$, then it is easy to show that $g_1^\beta(t,z) \le g(t,z), z < z_t$ and hence $\int_0^{h(t)} g_1^\beta(t,z)F^{-1}(t,1-z)dz < \int_0^{h(t)} g(t,z)F^{-1}(t,1-z)dz = p(t)$, which is a contradiction. Thus $\gamma_t > 1$. A similar argument shows that $\lim_{\beta_t \uparrow 1} \gamma_t = 1$. Choose $\beta_t \in (0,1)$ appropriately such that $1 < \frac{\gamma_t g(t,0+) - \beta_t g(t,z_t)}{g(t,0+) - g(t,z_t)} < 2$. Note that by measurable section theorem we have that β_t, γ_t are measurable on t.

Case 2: $t \in L_2$. Then for $\beta_t \in (0,1)$, define

$$g_2^\beta(t,z) := \begin{cases} \beta_t g(t,z), & t \in L_2, z > z_t, \\ \gamma_t g(t,z_t), & t \in L_2, 0 < z \le z_t, \\ g(t,z), & t \notin L_2, \end{cases}$$

where γ_t is uniquely determined by $\int_0^{h(t)} g_2^\beta(t,z)F^{-1}(t,1-z)dz = p(t)$. Similar to Case 1, we can find measurable functions β_t and γ_t such that $\gamma_t > 1, \lim_{\beta_t \uparrow 1} \gamma_t = 1$. Choose β_t appropriately such that $\gamma_t g(t,z_t) \le b(t)$.

Define $g_1(t,z) := g(t,z)1_{t \in L^C} + g_1^\beta(t,z)1_{t \in L_1} + g_2^\beta(t,z)1_{t \in L_2}$ and $g_2(t,z) := 2g(t,z) - g_1(t,z)$. Then it is easy to check that $g_i(\cdot), i = 1,2$ are feasible solution to (2.9). Besides as $g(t,\cdot), g_i(\cdot), i = 1,2$ are left continuous and $g_1(t,z_t) < g(t,z_t) < g_2(t,z_t), t \in L$, there exists $\delta_t > 0$ such that $g_1(t,z) < g(t,z) < g_2(t,z), t \in L, z \in (z_t - \delta_t, z_t]$. Then by the strict concavity of $u_-(t,\cdot)$, we have $v_-(g(\cdot)) > \frac{1}{2}[v_-(g_1(\cdot)) + v_-(g_2(\cdot))]$, which implies that either $v_-(g(\cdot)) > v(g_1(\cdot))$ or $v_-(g(\cdot)) > v_-(g_2(\cdot))$. Thus we get the contradiction. ☐

In view of the preceding proposition, we need only determine the optimal $l(\cdot)$ by solving the following problem given a parameter pair $(a_+, h(\cdot))$:

$$\begin{array}{l} \text{Minimize } \bar{v}_-(l(\cdot)) := \int_0^T u_-(b(t))w_-(l(t))dt \\ \text{subject to } 0 \le l(\cdot) \le h(\cdot), \int_0^T b(t) \int_0^{l(t)} F^{-1}(1-z)dzdt = a_+ - a_0. \end{array} \tag{2.15}$$

Proposition 6. *Problem* (2.9) *and* (2.15) *have the same infimum values.*

Proof. Denote $v, \bar{v}$ seperately the infimum values of (2.9) and (2.15). Obviously, $v \le \bar{v}$. If the conclusion is not true, there exists a feasible solution $g(\cdot)$ to (2.9) such that $v(g(\cdot)) < \bar{v}$.

For each $n \geq 1, 1 \leq k \leq 2^n$, define

$$a_n(t,k) := \frac{\int_{(k-1)/2^n}^{k/2^n} g(t,z)F^{-1}(t,1-z)dz}{\int_{(k-1)/2^n}^{k/2^n} F^{-1}(t,1-z)dz}.$$

Then $g(t, \frac{k-1}{2^n}) \geq a_n(t,k) \geq g(t, \frac{k}{2^n})$, and it is easy to check that

$$g_n(t,z) := \sum_{k=1}^{2^n} a_n(t,k) 1_{(\frac{k-1}{2^n}, \frac{k}{2^n}]}(z)$$

is a feasible solution to (2.9). Since $g_n(t,z) \to g(t,z), a.s.$ and $0 \leq g_n(\cdot) \leq b(\cdot)$, we have $v_-(g_n(\cdot)) \to v_-(g(\cdot))$. Hence there exists $n > 0$, such that $v_-(g_n(\cdot)) < \bar{v}$. On the other hand, by Proposition 4, we have that there exists a two step function $\bar{g}(t,z) = b(t)1_{[0,l(t)]}(z)$, where $0 \leq l(\cdot) \leq h(\cdot)$, such that $v_-(\bar{g}(\cdot)) \leq v_-(g_n(\cdot)) < \bar{v}$. Note that $v_-(\bar{g}(\cdot)) = \bar{v}_-(l(\cdot)) \geq \bar{v}$. Thus we get the contradiction. □

The above discussions lead to:

Theorem 4. *Given $a_+ \geq a_0^+$ and $h(\cdot) : [0,T] \to [0,1]$,*

(i) if $\int_0^T \int_0^{h(t)} b(t)F^{-1}(1-z)dzdt < a_+ - a_0$, then there is no feasible solution to Problem (2.9) *and $v_-(a_+, h(\cdot)) = +\infty$;*

(ii) otherwise,

$$v_-(a_+, h(\cdot)) = \inf \int_0^T u_-(b(t))w_-(l(t))dt, \tag{2.16}$$

where the feasible region of the inf *is*

$$\{l(\cdot) : 0 \leq l(\cdot) \leq h(\cdot), \int_0^T b(t) \int_0^{l(t)} F^{-1}(1-z)dzdt = a_+ - a_0\}.$$

Moreover, Problem (2.9) *admits an optimal solution $g^*(\cdot)$ if and only if the minimization problem on the right side of* (2.16) *admits an optimal solution $l^*(\cdot)$. In such case,*

$$g^*(t,z) = b(t)1_{[0,l^*(t)]}(z).$$

Remark 2. *Note that unlike the three-piece solution in Zhang, Jin and Zhou ([22], Theorem 4.5), the structure of the solution here only has two parts. The reason is that the additional dimension of time offers us more flexibility (which can be seen in the proof of Proposition 4).*

2.6 Solution of the CPT Consumption Model

Now that we have solved both the Positive Part Problem (2.7) and the Negative Part Problem (2.9). What remains to do is to solve (2.10). By Theorem 4, Problem (2.10) can be simplified as

$$\begin{array}{ll} \text{Maximize} & v_+(a_+, h(\cdot)) - \int_0^T u_-(b(t))w_-(l(t))dt \\ \text{subject to} & a_+ \geq a_0^+, 0 \leq l(\cdot) \leq h(\cdot) \leq 1; a_+ = 0 \text{ when } h(\cdot) \equiv 1; \\ & \int_0^T b(t) \int_0^{l(t)} F^{-1}(1-z)dzdt = a_+ - a_0, \end{array} \tag{2.17}$$

where $v_+(a_+, h(\cdot))$ can be given in (2.12) if we assume Assumption 1, 2 and 3.

For any feasible $(a_+, h(\cdot), l(\cdot))$ of (2.17), as $h(\cdot) \geq l(\cdot)$, then $v_+(a_+, l(\cdot)) \geq v_+(a_+, h(\cdot))$ by Proposition 3. Thus (2.17) is equivalent to

$$\begin{array}{ll} \text{Maximize} & v_+(a_+, h(\cdot)) - \int_0^T u_-(b(t))w_-(h(t))dt \\ \text{subject to} & a_+ \geq a_0^+, 0 \leq h(\cdot) \leq 1; a_+ = 0 \text{ when } h(\cdot) \equiv 1; \\ & \int_0^T b(t) \int_0^{h(t)} F^{-1}(1-z)dzdt = a_+ - a_0. \end{array} \tag{2.18}$$

Theorem 5. *Let Assumption 1 and 2 hold.*

(i) If $g^*(\cdot)$ is optimal for Problem (2.5)*, then $a_+^* := \int_0^T \int_0^1 g^*(t,z)^+ F^{-1}(t, 1-z)dzdt$ and $h^*(t) := \sup\{z; g^*(t,z) \leq 0\}$ are optimal for Problem* (2.18)*. Moreover, $g^*(\cdot)^- = b(t)1_{[0,h^*(t)]}$.*

(ii) If $(a_+^*, h^*(\cdot))$ is optimal for Problem (2.18) *and $g_+^*(\cdot)$ is optimal for Problem* (2.7) *with parameter pair $(a_+^*, h^*(\cdot))$, then the optimal solution to Problem* (2.18) *can be represented as $g^*(t,z) = g_+^*(t,z)1_{z>h^*(t)} - b(t)1_{z\leq h^*(t)}$.*

Proof.

(i) If $g^*(\cdot)$ solves Problem (2.5), then by Theorem 2, $(a_+^*, h^*(\cdot))$ is optimal for (2.10) and $g^*(\cdot)^+, g^*(\cdot)^-$ are respectively optimal for Problem (2.7) and (2.9) with parameter pair $(a_+^*, h^*(\cdot))$. We now prove that $h^*(\cdot)$ solves Problem (2.15) with parameter pair $(a_+^*, h^*(\cdot))$, i.e.

$$v_-(a_+^*, h^*(\cdot)) = \int_0^T u_-(b(t))w_-(h^*(t))dt.$$

To this end, if it does not hold true, then by Theorem 4 there exists $0 < l(\cdot) \leq h^*(\cdot)$ such that $\int 1_{\{t; l(t) < h^*(t)\}} dt > 0$ and

$$\int_0^T u_-(b(t))w_-(l(t))dt < \int_0^T u_-(b(t))w_-(h^*(t))dt.$$

Then by Proposition 3, we have $v_+(a_+^*, l(\cdot)) > v_+(a_+^*, h^*(\cdot))$. As a result,

$$\begin{aligned} v_+(a_+^*, l(\cdot)) - v_-(a_+^*, l(\cdot)) &\geq v_+(a_+^*, l(\cdot)) - \int_0^T u_-(b(t)) w_-(l(t)) dt \\ &> v_+(a_+^*, h^*(\cdot)) - v_-(a_+^*, h^*(\cdot)), \end{aligned}$$

contradicting the optimality of $(a_+^*, h^*(\cdot))$. The other conclusions are straightforward.

(ii) As $(a_+^*, h^*(\cdot))$ is optimal for Problem (2.18), we have

$$\begin{aligned} v_+(a_+^*, h^*(\cdot)) - v_-(a_+^*, h^*(\cdot)) &\geq v_+(a_+^*, h^*(\cdot)) - \int_0^T u_-(b(t)) w_-(h^*(t)) dt \\ &= \sup_{(2.18)} \left\{ v_+(a_+, h(\cdot)) - \int_0^T u_-(b(t)) w_-(h(t)) dt \right\} \\ &= \sup_{(2.17)} \left\{ v_+(a_+, h(\cdot)) - \int_0^T u_-(b(t)) w_-(l(t)) dt \right\} \\ &= \sup_{(2.10)} \left\{ v_+(a_+, h(\cdot)) - v_-(a_+, h(\cdot)) \right\}, \end{aligned}$$

where the feasible regions of sup's are those of the coresponding equations. This shows that $(a_+^*, h^*(\cdot))$ is optimal for Problem (2.10) and the inequality above is actually an equality, leading to

$$v_-(a_+^*, h^*(\cdot)) = \int_0^T u_-(b(t)) w_-(h^*(t)) dt.$$

By Theorem (4), it indicates that $b(t)1_{z \leq h^*(t)}$ is optimal for (2.9) with parameter pair $(a_+^*, h^*(\cdot))$. The rest is from Theorem 2. □

In view of Lemma 2 and Theorem 5, we have that

Theorem 6. *If* $(a_+^*, h^*(\cdot))$ *is optimal for Problem* (2.18) *and* $g_+^*(\cdot)$ *is optimal for Problem* (2.7) *with parameter pair* $(a_+^*, h^*(\cdot))$*, then the optimal solution to Problem* (2.4) *can be represented as* $d^*(t) = g_+^*(t, 1 - F(t, \rho(t))) 1_{\rho(t) < q^*(t)} - b(t) 1_{\rho(t) \geq q^*(t)}$*, where* $q^*(t) = F^{-1}(t, 1 - h^*(t))$*. Thus the optimal solution to Problem* (2.3) *is* $c^*(t) = (g_+^*(t, 1 - F(t, \rho(t))) + b(t)) 1_{\rho(t) < q^*(t)}$.

Remark 3. *It is clear from Theorem 6 that the optimal consumption is in general characterized by two parts: the agent has rich consumption above the benchmark in good states of the market and suffers from hunger (i.e. no consumption) in bad states.*

Remark 4. *From Theorem 6, we know that in order to solve the original CPT consumption problem* (2.3)*, we just need to find the optimal* $(a_+^*, h^*(\cdot))$ *for Problem* (2.18)*. Generally it is still very difficult. But we solve an example in the next section to illustrate the method.*

2.7 An Example with CRRA Utility Functions

In this section we solve a concrete example to illustrate the general results obtained in Section 2.6. We consider a model with CRRA (*constant relative risk aversion*) utility functions, which is proposed by Tversky and Kahneman [18], i.e. $u_+(x) = x^\alpha, u_-(x) = k_- x^\alpha$, where $0 < \alpha < 1$ and $k_- > 0$. We assume all the market parameters (investment opportunity set) are time-invariant: $r(\cdot) \equiv r, B(\cdot) \equiv B, \sigma(\cdot) = \sigma, \theta(\cdot) \equiv \theta$. In this case, $\rho(t) = \exp\{-(r + \|\theta\|^2/2)t - \theta' W(t)\}$, following a log-normal distribution with parameter (μ_t, σ_t^2), where $\mu_t = -(r + \|\theta\|^2/2)t, \sigma_t^2 = \|\theta\|^2 t$. Then the range of $F(t, \cdot)$ is $[0, 1)$ on $[0, +\infty)$.

When $x_0 = 0$, the optimal solution is trivially $c^*(\cdot) \equiv 0$. When $x_0 > 0$, first we solve the Positive Part Problem (2.7) with parameters $(a_+, h(\cdot))$, where $a_+ \geq a_0^+$ and $0 \leq h(\cdot) \leq 1$.

We need the following assumption.

Assumption 4.

(i) For each $t \in (0, T]$, $\frac{F^{-1}(t,z)}{w_+'(t,z)}$ *is nondecreasing in* $z \in (0, 1]$.

(ii) For any $t \in (0, T]$, $w_+'(t, 0+) > 0$ *and* $\limsup_{z \to 0} \frac{w_-(t,z)}{F^{-1}(t,1-z)} < +\infty$.

(iii) For each $t \in (0, T]$, $\frac{w_-'(t,z)}{F^{-1}(t,1-z)}$ *is nondecreasing in* $z \in [0, 1)$.

Remark 5.

(i) Generally speaking, the economic interpretation of Assumption 4 (i) is that the distortion $w_+(\cdot)$ *should not be too large in the sense that it should not increase the relative risk seeking function of the distribution by more than* 1*. In the case where there is no distortion (i.e.* $w_+(\cdot, x) = x$*), the assumption certainly holds. For more details, the readers can refer to Section 6.2 of Jin and Zhou [8].*

(ii) Based on Assumption 4 (i), one condition for Assumption 4 (iii) to hold is that

- $\frac{w_+'(t,1-z)}{w_-'(t,z)}$ *does not depend on* z, $\forall t \in [0, T]$.

This condition holds for example if the probability distortions are of Choquet integral sense. Sepcifically, given a capacity $w \circ P$ *(a non-linear probability measure), the Choquet expectation of a random variable* X *is defined as*

$$E_w^C[X] := \int_0^\infty w(P(X \geq y))dy + \int_{-\infty}^0 [w(P(X \geq y)) - 1]dy. \tag{2.19}$$

On the other hand, in CPT, the nonlinear expectation of X (taking the origin as the reference point), is defined by

$$E^{CPT}_{w_+,w_-}[X] := \int_0^\infty w_+(P(X^+ \geq y))dy + \int_0^\infty w_-(P(X^- \geq y))dy. \tag{2.20}$$

Thus if we choose $w_-(z) = 1 - w_+(1-z)$ and $w_+(z) = w(z)$, then the two distorted expectations (2.19) (2.20) *are the same, and the condtion* ($\bullet$) *holds.*

Under Assumption 4 (i), it is easy to check that N defined in (2.6) is concave itself, i.e. $N = \hat{N}$. Thus the optimal solution for (2.7) is

$$g_+^*(t,z) = \left(\frac{\alpha w_+'(t,1-z)}{\lambda^* F^{-1}(t,1-z)}\right)^{\frac{1}{1-\alpha}} 1_{z>h(t)}$$

and

$$v_+(a_+, h(\cdot)) = \int_0^T \int_0^1 \left(\frac{\alpha}{\lambda F^{-1}(t,1-z)}\right)^{\frac{\alpha}{1-\alpha}} w_+'(t,1-z)^{\frac{1}{1-\alpha}} 1_{z>h(t)} dzdt,$$

where $\lambda > 0$ is the unique real number satisfying $\int_0^T \int_0^1 g_+^*(t,z)F^{-1}(t,1-z)dzdt = a_+$. We use this equation to replace a_+ in the constraint of Problem (2.18) and set $\beta := \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}$. Note that $a_0 = x_0 - E\int_0^T \rho(t)b(t)dt = x_0 - \int_0^T\int_0^1 b(t)F^{-1}(t,1-z)dzdt$, then Problem (2.18) specializes to

$$\begin{array}{ll} \text{Maximize} & \int_0^T \int_{h(t)}^1 \beta^\alpha F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}} w_+'(1-z)^{\frac{1}{1-\alpha}} dzdt - k_- \int_0^T b(t)^\alpha w_-(h(t))dt \\ \text{subject to} & \left\{ \begin{array}{l} \int_0^T \int_{h(t)}^1 \left[\beta F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}} w_+'(1-z)^{\frac{1}{1-\alpha}} + F^{-1}(1-z)b(t)\right] dt = x_0, \\ 0 \leq h(\cdot) \leq 1, \beta > 0. \end{array}\right. \end{array} \tag{2.21}$$

Note that $w_-(t,h(t)) = 1 - \int_{h(t)}^1 w_-'(t,z)dz$, then (2.21) turns into

$$\begin{array}{ll} \text{Maximize} & \int_0^T \int_{h(t)}^1 \left[\beta^\alpha F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}} w_+'(1-z)^{\frac{1}{1-\alpha}} + k_- b(t)^\alpha w_-'(z)\right] dzdt - k_- \int_0^T b(t)^\alpha dt \\ \text{subject to} & \left\{ \begin{array}{l} \int_0^T \int_{h(t)}^1 \left[\beta F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}} w_+'(1-z)^{\frac{1}{1-\alpha}} + F^{-1}(1-z)b(t)\right] dt = x_0, \\ 0 \leq h(\cdot) \leq 1, \beta > 0. \end{array}\right. \end{array} \tag{2.22}$$

Fix $\gamma > 0$, we consider the following problem:

$$\begin{array}{l} \text{Maximize } \eta_\gamma(\beta, h(\cdot)) := \int_0^T \int_0^1 \zeta(t,z,\beta) 1_{z \geq h(t)} dt - k_- \int_0^T b(t)^\alpha dt + \frac{x_0}{\gamma} \\ \text{subject to } 0 \leq h(\cdot) \leq 1, \beta > 0, \end{array} \tag{2.23}$$

where

$$\begin{aligned}\zeta(t,z,\beta) &:= \beta^{\alpha}F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}}w_+'(1-z)^{\frac{1}{1-\alpha}}+k_-b(t)^{\alpha}w_-'(z)\\ &\quad -\frac{1}{\gamma}\left[\beta F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}}w_+'(1-z)^{\frac{1}{1-\alpha}}+F^{-1}(1-z)b(t)\right]\\ &= \left(\beta^{\alpha}-\frac{\beta}{\gamma}\right)F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}}w_+'(1-z)^{\frac{1}{1-\alpha}}\\ &\quad +k_-b(t)^{\alpha}w_-'(z)-\frac{1}{\gamma}b(t)F^{-1}(1-z).\end{aligned}$$

As $\zeta(t,z,\beta)$ is concave in β, by zero-derivative condition, we have

$$\arg\max_{\beta>0}\zeta(t,x,\beta)=(\alpha\gamma)^{\frac{1}{1-\alpha}}.$$

Then to solve (2.23) we only need to consider

$$\begin{array}{l}\text{Maximize } \int_0^T\int_0^1\zeta(t,z,(\alpha\gamma)^{\frac{1}{1-\alpha}})1_{z\geq h(t)}dzdt\\ \text{subject to } 0\leq h(\cdot)\leq 1,\end{array}\tag{2.24}$$

which is a deterministic optimal stopping problem. The optimal solution $h^*(\cdot)$ to (2.24), if it exists, must satisfy $\zeta(t,h^*(t),(\alpha\gamma)^{\frac{1}{1-\alpha}})=0$ and $\frac{\partial\zeta(t,z,(\alpha\gamma)^{\frac{1}{1-\alpha}})}{\partial z}|_{z=h^*(t)}\geq 0$ if $h^*(t)>0$; or that $h^*(t)=0$ if $\inf_{z>0}\zeta(t,z,(\alpha\gamma)^{\frac{1}{1-\alpha}})\geq 0$.

Now we explore the relation between $h^*(\cdot;\gamma):=h^*(\cdot)$ and γ. Set

$$\xi(\gamma):=\int_0^T\int_0^1\left[(\alpha\gamma)^{\frac{1}{1-\alpha}}F^{-1}(1-z)^{\frac{\alpha}{\alpha-1}}w_+'(1-z)^{\frac{1}{1-\alpha}}+F^{-1}(1-z)b(t)\right]1_{z>h^*(t)}dzdt,\tag{2.25}$$

then it is continuous and increasing in γ. Moreover note that $\lim_{z\to 1-}\zeta(t,z,(\alpha\gamma)^{\frac{1}{1-\alpha}})=+\infty$, thus $h^*(t;\gamma)<1$. Then we have

$$\lim_{\gamma\to+\infty}\xi(\gamma)=+\infty.$$

On the other hand note that

$$\zeta(t,z,(\alpha\gamma)^{\frac{1}{1-\alpha}})=\hat{\alpha}\gamma^{\frac{\alpha}{1-\alpha}}F^{-1}(1-z)\left[\left(\frac{w_+'(t,1-z)}{F^{-1}(t,1-z)}\right)^{\frac{1}{1-\alpha}}+k_-b(t)^{\alpha}\frac{w_-'(t,z)}{F^{-1}(t,1-z)}-\frac{1}{\gamma}b(t)\right],\tag{2.26}$$

where $\hat{\alpha}:=\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}>0$. By Assumption 4 (i) and (ii), it is easy to get $\lim_{\gamma\to 0}h^*(t;\gamma)=1$, which further implies that

$$\lim_{\gamma\to 0}\xi(\gamma)=0.$$

Thus there exists a γ^* such that $\xi(\gamma^*) = x_0$. Therefore $(\beta^* := (\alpha\gamma^*)^{\frac{1}{1-\alpha}}, h^*(\cdot;\gamma^*))$ is feasible to (2.23). Denote the optimal value of (2.22) and (2.23) as η and $\eta(\gamma)$ respectively, then

$$\begin{aligned}\eta &\leq \inf_{\gamma>0} \eta(\gamma)\\ &\leq \eta(\gamma^*)\\ &= \eta_{\gamma^*}(\beta^*, h(\cdot;\gamma^*))\\ &\leq \eta,\end{aligned}$$

implying that $(\beta^* := (\alpha\gamma^*)^{\frac{1}{1-\alpha}}, h^*(\cdot;\gamma^*))$ is optimal for (2.23). To sum up, we have

Theorem 7. *Under Assumption 4 (i) and (ii), if $h^*(\cdot;\gamma)$ solves Problem (2.24) for any $\gamma > 0$, then $((\alpha\gamma^*)^{\frac{1}{1-\alpha}}, h^*(\cdot;\gamma^*))$ solves Problem (2.21), where γ^* uniquely solves $\xi(\gamma^*) = x_0$. Thus we get the optimal*

$$d^*(t) = \Big(\frac{\alpha\gamma^*}{\rho(t)}\Big)^{\frac{1}{1-\alpha}} 1_{\rho(t)<q^*(t)} - b(t)1_{\rho(t)\geq q^*(t)},$$

and then the optimal consumption

$$c^*(t) = \Big[\Big(\frac{\alpha\gamma^*}{\rho(t)}\Big)^{\frac{1}{1-\alpha}} + b(t)\Big]1_{\rho(t)<q^*(t)},$$

where $q^*(t) = F^{-1}(t, 1 - h^*(t;\gamma^*))$.

Now we explore the relation between $b(t)$ and $q^*(t)$

When $b(t)$ increases, we consider two cases. If γ^*also increases, then by (2.25), in order to guarantee $\xi(\gamma^*) = x_0$, we must have $h^*(t)$ increases. By Theorem 6 we know that $q^*(t)$ is a decreasing function of $h^*(t)$, thus $q^*(t)$ decreases. The other case is that γ^*decreases. Note that (2.26) can be written as

$$\begin{aligned}&\zeta(t, z, (\alpha\gamma)^{\frac{1}{1-\alpha}})\\ =\ &\hat{\alpha}\gamma^{\frac{\alpha}{1-\alpha}}F^{-1}(1-z)b(t)\Big[\Big(\tfrac{w'_+(t,1-z)}{F^{-1}(t,1-z)}b(t)^{\alpha-1}\Big)^{\frac{1}{1-\alpha}} + k_-b(t)^{\alpha-1}\tfrac{w'_-(t,z)}{F^{-1}(t,1-z)} - \tfrac{1}{\gamma}\Big].\end{aligned}$$

As $\zeta(t, h^*(t), (\alpha\gamma^*)^{\frac{1}{1-\alpha}}) = 0$, then

$$\Big[\Big(\frac{w'_+(t,1-z)}{F^{-1}(t,1-z)}b(t)^{\alpha-1}\Big)^{\frac{1}{1-\alpha}} + k_-b(t)^{\alpha-1}\frac{w'_-(t,z)}{F^{-1}(t,1-z)} - \frac{1}{\gamma}\Big]|_{z=h^*(t)} = 0. \qquad (2.27)$$

When γ^*decreases, the first two terms of left hand side of (2.27) must increase. As $b(t)$ increases and $0 < \alpha < 1$, by Assumption 4 (i) and (iii), it is easy to get $h^*(t)$ increase. Thus again we get $q^*(t)$ decreases. The discussion leads to:

Proposition 7. *Let Assumption 4 (i) and (iii) hold. $q^*(t)$ is decreasing in $b(t)$.*

A special case is that $w_+(t,x) = w_-(t,x) = x$. In this situation

$$\zeta(t,z,(\alpha\gamma)^{\frac{1}{1-\alpha}}) = b(t)^\alpha\Big[\hat{\alpha}\gamma^{\frac{\alpha}{1-\alpha}}\Big(\frac{F^{-1}(t,1-z)}{b(t)^{\alpha-1}}\Big)^{\frac{\alpha}{\alpha-1}} + k_- - \frac{1}{\gamma}\Big(\frac{F^{-1}(t,1-z)}{b(t)^{\alpha-1}}\Big)\Big],$$

which is increasing in z. By solving $\zeta(t,h^*(t;\gamma),(\alpha\gamma)^{\frac{1}{1-\alpha}}) = 0$, it is easy to get $h^*(t;\gamma^*) = 1 - F(t,\alpha m_*^{\alpha-1}\gamma^* b(t)^{\alpha-1})$ and thus $q^*(t;\gamma^*) = \alpha m_*^{\alpha-1}\gamma^* b(t)^{\alpha-1}h(t)$, where m_* uniquely solves $(1-\alpha)m^\alpha + k_- - \alpha m^{\alpha-1} = 0$. Clearly $q^*(t)$ is decreasing in $b(t)$.

Remark 6. *The economic interpretation behind Proposition 7 is that when the agent lowers the reference point (i.e. $b(t)$), she tends to believe that the market is good. Conversely, if she increases the reference point, then she tends to believe the market is bad.*

2.8 Consumption and Terminal Wealth in CPT

In this section we solve the model in which the agent tries to maximize the utility of consumption over the planning horizon plus the utility of wealth at the end of the planning horizon within the framework of CPT in continuous time.

Definition 1. *Given $x \in \mathbb{R}$, we say that a consumption and portfolio process pair $(c(\cdot),\pi(\cdot))$ is* tame *at x, and write $(c(\cdot),\pi(\cdot)) \in \mathcal{T}(x)$, if the initial wealth $x(0) = x$ and the discounted wealth process $S_0(\cdot)^{-1}x(\cdot)$ is almost surely bounded from below (the bound may depend on $(c(\cdot),\pi(\cdot))$).*

Remark 7. *Normally people require $(c(\cdot),\pi(\cdot))$ to make the corresponding wealth process $x(\cdot) \geq 0$ (see for example Karatzas and Shreve [11], p.92, definition 3.2). However note that here the wealth process can be negative. One reason is that according to CPT, the agent is concerned about the magnitude of the change in wealth (or consumption) from the reference point; even if the terminal wealth is negative, the change might be positive if the reference point is set to be negative by the agent. Thus here the set of tame $(c(\cdot),\pi(\cdot))$ is larger than the one in definition 3.2 of [11].*

We define

$$\begin{aligned}\mathcal{T}_1(x) :=\{&(c(\cdot),\pi(\cdot)) \in \mathcal{T}(x) : \exists (c(\cdot),\pi_1(\cdot)) \in \mathcal{A}(x_1), (0,\pi_2(\cdot)) \in \mathcal{T}(x_2),\\ &\text{satisfying } x_1^{c,\pi_1}(\cdot) \geq 0, x_1^{c,\pi_1}(T) + x_2^{0,\pi_2}(T) = x(T)\},\end{aligned}$$

where $x^{c,\pi}(\cdot)$ is the wealth process corresponding to $(c(\cdot),\pi(\cdot))$.

Lemma 3. $\mathcal{T}(x) = \mathcal{T}_1(x), \forall x \in \mathbb{R}$.

Proof. Obviously $\mathcal{T}_1(x) \subseteq \mathcal{T}(x)$. On the other hand, for any $(c(\cdot), \pi(\cdot)) \in \mathcal{T}(x)$, suppose the corresponding wealth process is $x(\cdot)$. Set $x_1 := E\int_0^T \rho(t)c(t)dt$ and $x_2 := E[\rho(T)x(T)]$. Then $x_1 \geq 0, x_1 + x_2 = x$. By Proposition 1, we can find $\pi_1(\cdot)$ such that $(c(\cdot), \pi_1(\cdot)) \in \mathcal{A}(x_1)$ with the corresponding terminal wealth $x_1(T) = 0$. As $x(T)S_0(T)^{-1}$ is lower bounded, then we can find $(0, \pi_2(\cdot)) \in \mathcal{T}(x_2)$. It is easy to see that $x(T) = x_1(T) + x_2(T)$, where $x_1(\cdot), x_2(\cdot)$ are the wealth processes corresponding to $(c(\cdot), \pi_1(\cdot)), (0, \pi_2(\cdot))$. Thus $\mathcal{T}(x) \subseteq \mathcal{T}_1(x)$. □

We study a consumption and portfolio selection model in the framework of CPT by combining the behavior consumption model formulated in Section 2 with the behavioral terminal wealth model developed in Jin and Zhou [8]. Similar to the notations in the consumption model, we use a $\mathcal{F}_T$-measurable random variable η, satisfying $S_0(T)^{-1}\eta$ is bounded and $E[\rho(T)\eta] < +\infty$, as the reference point for the terminal wealth, use $\hat{w}_\pm(\cdot)$ as probability distortions and use two time-independent utility functions $\hat{u}_\pm(\cdot)$ to measure the gains and losses respectively. Besides we use $\iota \in R$ as the weight of terminal wealth utility to the consumption utility. The problem is formulated as follows:

$$\begin{array}{l} \text{Maximize } V(c(\cdot)) + \iota\hat{V}(x(T)) \\ \text{subject to } (c(\cdot), \pi(\cdot)) \in \mathcal{T}(x), \end{array} \tag{2.28}$$

where $V(c(\cdot))$ is from (2.2) and

$$\hat{V}(x(T)) := \hat{V}_+(x(T)) - \hat{V}_-(x(T)),$$

with

$$\begin{aligned} \hat{V}_+(x(T)) &:= \int_0^{+\infty} \hat{w}_+(P\{\hat{u}_+((x(T)-\eta)^+) > y\})dy, \\ \hat{V}_+(x(T)) &:= \int_0^{+\infty} \hat{w}_-(P\{\hat{u}_-((x(T)-\eta)^-) > y\})dy. \end{aligned}$$

In view of Lemma 3, Problem (2.28) is equivalent to

$$\begin{array}{l} \text{Maximize } V(c(\cdot)) + \iota\hat{V}(x(T)) \\ \text{subject to } (c(\cdot), \pi(\cdot)) \in \mathcal{T}_1(x), \end{array}$$

In Jin and Zhou [8], the following problem is solved:

$$\begin{array}{l} \text{Maximize } \hat{V}(x(T)) \\ \text{subject to } (0, \pi(\cdot)) \in \mathcal{T}(x), \end{array}$$

and in the previous sections we solve

$$\begin{aligned}&\text{Maximize } V(c(\cdot))\\&\text{subject to } (c(\cdot),\pi(\cdot))\in\mathcal{A}(x).\end{aligned}$$

We Set

$$\hat{v}^*(x) := \sup_{(0,\pi(\cdot))\in\mathcal{T}(x)} \hat{V}(x(T)),$$
$$v^*(x) := \sup_{(c(\cdot),\pi(\cdot))\in\mathcal{A}(x)} V(c(\cdot)).$$

In the light of Lemma 3, we have the following scheme to solve Problem (2.28).

Step 1. Find $x_1^* \geq 0, x_2^*$, satisfying $x_1^* + x_2^* = x$, such that

$$v^*(x_1^*) + \iota\hat{v}^*(x_2^*) = \sup_{x_1+x_2=x, x_1\geq 0} \{v^*(x_1) + \iota\hat{v}^*(x_2)\}.$$

Step 2. Find $(c^*(\cdot), \pi_1^*(\cdot)) \in \mathcal{A}(x_1^*)$ such that $V(c^*(\cdot)) = v^*(x_1^*)$.

Step 3. Find $(0, \pi_2^*(\cdot)) \in \mathcal{T}(x_2^*)$ such that $\hat{V}(x_2^*(T)) = \hat{v}^*(x_2^*)$, where $x_2^*(T)$ is the terminal wealth corresponding to $(0, \pi_2^*(\cdot))$.

Step 2 can be solved by Theorem 6 and Step 3 is solved by [8]. Thus the optimal solutions to Problem (2.28), if exist, must include $(c^*(\cdot), \pi_1^*(\cdot) + \pi_2^*(\cdot))$.

Remark 8. *Step 1 is similar to the classical result (see for example, Theorem 7.10 of Karatzas and Shreve [11]).*

Chapter 3

Conclusions

This thesis mainly discusses on portfolio selection and consumption with the agent's preference following cumulative prospect theory (CPT) in a continuou-time complete market driven by Brownian motion. First we consider a solely consumption model within the framework of CPT. After using quantile method and divide-and-conquer scheme, we get two sub-problems. One is essentially a concave maximization problem, which we solve by calculus of variation. Another one is essentially a concave minimization problem, which we characterize the structure of its solution, if it exists. In this way, we derive the optimal solution, which is composed of two parts, indicating that an agent of CPT preference will have rich consumption above the benchmark in good situations and suffers from hunger (i.e. no consumption) in bad situations. In an example, we further show that under some assumptions, the agent's reference poin or benchmark highly influences her judgement on whether the market is good or bad.

After that, we give the strategy on how to find the optimal consumption and portfolio selection to maximize the utilities from both consumption and terminal wealth in the framework of CPT. The strategy is very simple, i.e. first divide the initial wealth into two parts, then use one part for maximizing the consumption only and the other for the terminal wealth only, then the best strategy lies in the best dividing by comparing the sum of the utilities from consumption and terminal wealth.

Appendix A

Proof of Theorem 1

We first introduce a problem which is closely linked with Problem (2.4) as to well-posedness.

$$\begin{array}{ll} \text{Maximize} & J_+(d(\cdot)) \\ \text{subject to} & E\int_0^T \rho(t)d(t)dt = a > 0, d(\cdot) \geq 0, \int_0^T d(t)dt < +\infty, a.s., \\ & d(\cdot) \text{ is } \mathcal{F}_t\text{-progressively measurable.} \end{array} \tag{A.1}$$

Lemma 4. *If Problem* (A.1) *is well-posed with* $a = a_0 + E\int_0^T \rho(t)b(t)dt > 0$ *for some* $a_0 > -E\int_0^T \rho(t)b(t)dt$*, then Problem* (2.4) *with* a_0 *is well-posed.*

Proof. For any $d(\cdot)$ feasible for (2.4), define $\bar{d}(t) := d(t) + b(t) \geq 0$. Then $\bar{d}(\cdot)$ is feasible for (A.1) with $a = a_0 + E\int_0^T \rho(t)b(t)dt > 0$. It is easy to see that $J(d(\cdot)) \leq J_+(d(\cdot)) \leq J_+(\bar{d}(\cdot)) < +\infty$. Thus (2.4) is well-posed if (A.1) is well-posed. □

In the same way of getting (2.5), we transform (A.1) into

$$\begin{array}{l} \text{Maximize } v_+(g(\cdot)) := \int_0^T \int_0^1 u_+(g(z))w'_+(1-z)dzdt \\ \text{subject to } g(\cdot) \in \hat{\mathbb{G}}_+, \int_0^T \int_0^1 g(t,z)F^{-1}(t,1-z)dzdt = a, \end{array} \tag{A.2}$$

where $\hat{\mathbb{G}}_+$ is defined in (2.8), i.e.

$$\hat{\mathbb{G}}_+ = \left\{ g \in \mathbb{G} : g(t,\cdot) \geq 0, \forall t \in [0,T], \int_0^T g(t,z)dt < +\infty, a.e. z \in [0,1) \right\}.$$

We denote $v_+(a)$ as the supremum of Problem (A.2).

Similarly following Lemma 1 and 2, we can get:

Lemma 5. *If* $d^*(\cdot)$ *solves Problem* (A.1) *and its quantile function at* t *is* $g^*(t,\cdot)$*, then* $g^*(\cdot)$ *solves Problem* (A.2)*. Conversely, if Problem* (A.2) *admits an optimal* $g^*(\cdot)$*, then* $g^*(\cdot, Z.)$ *solves Problem* (A.1)*. Problem* (A.1) *is ill-posed if and only if Problem* (A.2) *is ill-posed.*

Since $w'_+(\cdot) > 0$, and $u_+(\cdot)$ is concave w.r.t. the second argument, $v_+(g(\cdot))$ is concave in $g(\cdot)$. On the other hand, the constrain in (A.2) is linear in $g(\cdot)$. Hence we solve it by means of two steps:

Step 1. First for a fixed Lagrange multiplier $\lambda > 0$, solve the following problem

$$\text{Maximize}_{g \in \hat{\mathbb{G}}_+} \int_0^T \int_0^1 \left[u_+(g(z))w'_+(1-z) - \lambda g(z)F^{-1}(1-z) \right] dzdt. \quad \text{(A.3)}$$

Step 2. Then determine the Lagrange multiplier λ from the following equation

$$\int_0^T \int_0^1 g_\lambda(t,z)F^{-1}(t,1-z)dzdt = a, \quad \text{(A.4)}$$

with g_λ solving (A.3).

In order to solve (A.3), we first fix time t, and solve

$$\text{Maximize}_{\ g \in \mathbf{G}_+} \int_0^1 \left[u_+(t,g(z))w'_+(t,1-z) - \lambda g(z)F^{-1}(t,1-z) \right] dz, \quad \text{(A.5)}$$

where

$$\mathbf{G}_+ := \{ g : [0,1) \to [0,\infty], \text{ nondecreasing and right-continous} \}.$$

Suppose $g_{\lambda;t}$ solves (A.5). Define $g_\lambda(t,z) := g_{\lambda;t}(z)$. If $g_\lambda \in \hat{\mathbb{G}}_+$, then definitely g_λ solves (A.3).

Proposition 8. *Under Assumption 1, 2 and Assumption 3 (ii), recall $N(t,\cdot)$ in* (2.6) *and its concave envelope $\hat{N}(t,\cdot)$, $\forall t \in [0,T]$. Then the optimal solution of Problem* (A.3) *is*

$$g_\lambda(t,z) := I_+(t, \lambda \hat{N}'(t, 1 - w_+(t,1-z))), \quad \text{(A.6)}$$

where $\hat{N}'(t,x) := \frac{\partial}{\partial x}\hat{N}(t,x)$.

Proof. According to section 3.2 of Xia and Zhou [20], the solution to Problem (A.5) is $g_{\lambda;t} = I_+(t, \lambda \hat{N}'(t, 1 - w_+(t,1-z)))$. Define $g_\lambda(t,z) := g_{\lambda;t}(z)$. Thus we only need to verify that $\int_0^T g_\lambda(t,z)dt < \infty$, *a.e.*$z \in [0,1)$ in order to make sure that $g_\lambda \in \hat{\mathbb{G}}_+$. Note that $\int_0^T g_\lambda(t,z)dt \leq \int_0^T I_+(t, \lambda \hat{N}'(t,1))dt < \infty$, the last inequality is guaranteed by Assumption 3 (ii). □

Now we move to Step 2, i.e. to check whether there exists some λ such that $g_\lambda(t,z)$ derived in (A.6) satisfies (A.4). Recall that

$$R_+(t,x) := -\frac{x \partial u_+^2(t,x)/\partial x^2}{\partial u_+(t,x)/\partial x}, x > 0.$$

Following the main idea of Jin, Xu and Zhou [7], we have the following two lemmas, which give conditions such that the Lagrange multiplier exists.

Lemma 6. *If* $\liminf_{x\to\infty} R_+(t,x) > 0$ *uniformly in* $t \in [0,T]$, *then*

(i) $\limsup_{x\to+\infty} \frac{\partial u_+(t,kx)/\partial x}{\partial u_+(t,x)/\partial x} < 1$*uniformly in* $t \in [0,T]$ *for any* $k > 1$.

(ii) $\limsup_{x\to 0+} \frac{I_+(t,\lambda x)}{I_+(t,x)} < +\infty$ *uniformly in* $t \in [0,T]$ *for any* $0 < \lambda < 1$.

Proof. Suppose there exists $M, K > 0$, such that $R_+(t,x) \geq K$ for any $x \geq M$. For any $x \geq M, k > 1$,

$$
\begin{aligned}
\frac{\partial u_+(t,kx)/\partial x}{\partial u_+(t,x)/\partial x} - 1 &= -\frac{\int_x^{kx} \frac{\partial^2 u_+}{\partial x^2}|_{x=y} dy}{\partial u_+/\partial x} \\
&= -\frac{\int_x^{kx} R(t,y)\frac{\partial u_+}{\partial x}|_{x=y}/y dy}{\partial u_+/\partial x} \\
&\leq -\frac{\int_x^{kx} R(t,y)\frac{\partial u_+(kx)}{\partial x}/y dy}{\partial u_+/\partial x} \\
&= -\frac{\partial u_+(kx)/\partial x}{\partial u_+/\partial x}\int_x^{kx} R(t,y)/y dy \\
&\leq -\frac{\partial u_+(kx)/\partial x}{\partial u_+/\partial x} K \int_x^{kx} \frac{1}{y} dy \\
&= -\frac{\partial u_+(kx)/\partial x}{\partial u_+/\partial x} K \ln k.
\end{aligned}
$$

Thus

$$
\frac{\partial u_+(t,kx)/\partial x}{\partial u_+(t,x)/\partial x} \leq \frac{1}{1 + K\ln k},
$$

which implies (i). Moreover (i) leads to that there exist $\delta < 1, L > +\infty$ such that for any $x > L$,

$$
\frac{\partial u_+(t,kx)/\partial x}{\partial u_+(t,x)/\partial x} \leq \delta,
$$

$$
\begin{aligned}
&\Rightarrow \quad kx \geq I_+(t, \delta\partial u_+(t,x)/\partial x), \\
&\Rightarrow \quad k \geq \tfrac{I_+(t,\delta\partial u_+(t,x)/\partial x)}{I_+(t,\partial u_+(t,x)/\partial x)}, \\
&\Rightarrow \quad \tfrac{I_+(t,\delta y)}{I_+(t,y)} \leq k, \ \forall y < I_+(t,L), \\
&\Rightarrow \quad \limsup_{x\to 0+} \tfrac{I_+(t,\delta x)}{I_+(t,x)} < +\infty.
\end{aligned}
$$

Then

$$
\begin{aligned}
\limsup_{x\to 0+} \frac{I_+(t,\delta^2 x)}{I_+(t,x)} &= \limsup_{x\to 0+} \frac{I_+(t,\delta^2 x)}{I_+(t,\delta x)} \frac{I_+(t,\delta x)}{I_+(t,x)} \\
&\leq \limsup_{x\to 0+} \frac{I_+(t,\delta^2 x)}{I_+(t,\delta x)} \limsup_{x\to 0+} \frac{I_+(t,\delta x)}{I_+(t,x)} \\
&< +\infty.
\end{aligned}
$$

In this way we can prove that $\limsup_{x\to 0+}\frac{I_+(t,\delta^n x)}{I_+(t,x)} < +\infty, \forall n \geq 1$. (ii) follows since $\limsup_{x\to 0+}\frac{I_+(\lambda x)}{I_+(x)}$ is non-increasing in λ. $\square$

Define

$$f(\lambda) := \int_0^T \int_0^1 g_\lambda(t,z)F^{-1}(t,1-z)dzdt. \tag{A.7}$$

Then it is easy to see that $f(\cdot)$ is strictly decreasing. By the monotone convergence theorem and the monotonicty of $g_\lambda(\cdot)$, we can get that if $f(\lambda_0) < +\infty$ for some $\lambda_0 > 0$, then $f(\cdot)$ is continuous on $(\lambda_0, +\infty)$. Furthermore, the Lagrange multiplier uniquely exists for any $0 < a < f(\lambda_0)$.

Lemma 7. *If* $\liminf_{x\to\infty} R_+(t,x) > 0$ *uniformly in* $t \in [0,T]$, *and* $f(1) < +\infty$, *then the Lagrange multiplier exists for any* $a > 0$.

Proof. By the monotonicity of $f(\cdot)$, we immediately get $f(\lambda) < +\infty, \forall\lambda > 1$. For $0 < \lambda \leq 1$, by Lemma 6, there exist $L > 1, \delta > 0$ such that for any $0 < x \leq \delta$, $\frac{I_+(t,x)}{I_+(t,x)} < 2L$ uniformly in $t \in [0,T]$. Define $\kappa(t,z) := \hat{N}'(t, 1 - w_+(t, 1-z))$ which is decreasing in z. Set $\kappa^{-1}(\cdot)$ as the inverse function of $\kappa(\cdot)$ w.r.t the second argument. Then

$$\begin{aligned}
& \int_0^T \int_0^1 g_\lambda(t,z)F^{-1}(t,1-z)1_{z\geq\kappa^{-1}(t,\delta)}dzdt \\
= & \int_0^T \int_0^1 I_+(t,\lambda\kappa(t,z))F^{-1}(1-z)1_{\kappa(t,z)\leq\delta}dzdt \\
= & \int_0^T \int_0^1 \frac{I_+(t,\lambda\kappa(t,z))}{I_+(t,\kappa(t,z))} I_+(t,\kappa(t,z))F^{-1}(1-z)1_{\kappa(t,z)\leq\delta}dzdt \\
\leq & \ 2L\int_0^T \int_0^1 I_+(t,\kappa(t,z))F^{-1}(1-z)1_{\kappa(t,z)\leq\delta}dzdt \\
\leq & \ 2Lf(1),
\end{aligned}$$

$$\begin{aligned}
& \int_0^T \int_0^1 g_\lambda(t,z) F^{-1}(t,1-z) 1_{z<\kappa^{-1}(t,\delta)} dz dt \\
= & \int_0^T \int_0^1 I_+(t, \lambda\kappa(t,z)) F^{-1}(1-z) 1_{\kappa(t,z)>\delta} dz dt \\
= & \frac{1}{\lambda} \int_0^T \int_0^1 I_+(t, \lambda\kappa(t,z)) \lambda\kappa(t,z) w'_+(t,1-z) 1_{\kappa(t,z)>\delta} dz dt \\
\left(u_+(x) \geq \frac{\partial u_+}{\partial x}\cdot x\right) \leq & \frac{1}{\lambda} \int_0^T \int_0^1 u_+(t, I_+(t, \lambda\kappa(t,z))) w'_+(t,1-z) 1_{\kappa(t,z)>\delta} dz dt \\
\leq & \frac{1}{\lambda} \int_0^T \int_0^1 u_+(t, I_+(t, \lambda\delta)) w'_+(t,1-z) dz dt \\
= & \frac{1}{\lambda} \int_0^T u_+(t, I_+(t, \lambda\delta)) dt.
\end{aligned}$$

Hence

$$\begin{aligned}
f(\lambda) &= \int_0^T \int_0^1 g_\lambda(t,z) F^{-1}(t,1-z) dz dt \\
&\leq 2Lf(1) + \frac{1}{\lambda} \int_0^T u_+(t, I_+(t, \lambda\delta)) dt \\
&< +\infty,
\end{aligned}$$

where the last inequality is from Assumption 2 (i).

We prove that $f(\lambda) < +\infty, \forall \lambda > 0$, which implies the conclusion. □

Now we are ready to prove the well-posedness of (A.2).

Proof. of Theorem 1

By condition (iii), we have

$$\begin{aligned}
& v_+(g_1(\cdot)) \\
= & \int_0^T \int_0^1 u_+(t, I_+(t, \hat{N}'(t, 1-w_+(t,1-z))) w'_+(t,1-z) dz dt \\
< & +\infty,
\end{aligned}$$

and

$$\begin{aligned}
a_1 &:= f(1) \\
&= \int_0^T \int_0^1 I_+(t, \hat{N}'(t, 1-w_+(t,1-z)) F^{-1}(1-z) dz dt \\
\left(u_+(x) \geq \frac{\partial u_+}{\partial x}\cdot x\right) &\leq \int_0^T \int_0^1 u_+(t, I_+(t, \hat{N}'(t, 1-w_+(t,1-z))) w'_+(t,1-z) dz dt \\
&= v_+(g_1(\cdot)) < +\infty.
\end{aligned}$$

Consequently by Lemma 7, the Lagrange multiplier exists for any $a > 0$, thus $g_1(\cdot)$ is an optimal solution for Problem (A.2) with $a = a_1$ and $v_+(a_1) = v_+(g_1(\cdot)) < +\infty$.

Denote the feasible region of (A.2) with a as

$$\Delta_a := \left\{ g(\cdot) \in \hat{\mathbb{G}}_+, \int_0^T \int_0^1 g(t,z) F^{-1}(t, 1-z) dz dt = a \right\}.$$

For any $b > a_1$, we have

$$\begin{aligned} v_+(b) &= \sup_{g \in \Delta_b} v_+(g) = \sup_{g \in \Delta_{a_1}} v_+\left(\frac{b}{a_1} g\right) \\ &\leq \sup_{g \in \Delta_{a_1}} \frac{b}{a_1} v_+(g) = \frac{b}{a_1} v_+(a_1) < +\infty, \end{aligned}$$

where the first inequality is because of the concavity of $u_+(\cdot)$ and $u_+(0) = 0$.

For any $0 < b < a_1$,

$$\begin{aligned} v_+(b) &= \sup_{g \in \Delta_b} v_+(g) = \sup_{g \in \Delta_{a_1}} v_+\left(\frac{b}{a_1} g\right) \\ &\leq \sup_{g \in \Delta_{a_1}} v_+(g) = v(a_1) < +\infty, \end{aligned}$$

where the first inequality is because of the monotonicity of $u_+(\cdot)$. □

Actually the above discussion gives the solution of Problem (A.2), which is summarized as follows.

Theorem 8. *Under Assumption 1, 2 and 3, the solution to* (A.2) *is*

$$g^*(t,z) := I_+(t, \lambda^* \hat{N}'(t, 1 - w_+(t, 1-z))),$$

where λ^* is uniquely determined by

$$\int_0^T \int_0^1 I_+(t, \lambda^* \hat{N}'(t, 1 - w_+(t, 1-z))) F^{-1}(t, 1-z) dz dt = a.$$

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