Inhomogeneous Spatial Branching Processes

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Abstract

We study several spatial branching processes with inhomogeneities. Branching processes are a fundamental building block in mathematical models related to growth. Often growth happens in space: the spread of an epidemic, a selective sweep in the genetics of a population, the spread of a chemical reaction or the spread of an opinion in a social network. On the level of stochastic processes, branching Brownian motion is used to understand the interplay of effects in a spreading population, and indeed the front of branching Brownian motion exhibits wave-like properties.

We study several variants of branching Brownian motion, in all of them we move away from homogenous Euclidean space. First, we study branching Brownian motion in two-dimensional space where one direction of space improves the reproduction rate of particles. Here we show that this inhomogeneity in the branching rate leads to significant corrections to the speed of the front. Secondly, we study a discrete analogue of branching Brownian in a high dimensional random environment. Here we show that the exponential number of particles produces a law of large numbers effect and that the first order of the speed of the front is linear and non-random. Thirdly, we study branching Brownian motion in two-dimensional hyperbolic space. Here we that the limit of the empirical population measure on the boundary of the space is determined only by typical particles and we use this to compute the Hausdorff dimension of its support. The collection of these results show the emergence of interesting phenomena when inhomogeneities are added to branching Brownian motion.

Acknowledgements

I very much enjoyed my DPhil and this could have not been possible without interesting mathematical problems. For these I thank my supervisor Julien Berestycki, as well his passionate, cooperative, and relatable approach to mathematics. I also thank my second supervisor, Matthias Winkel, who especially supported me in the beginning with a well–structured style of supervision.

Mathematics can only be enjoyable if you are not stuck on problems by yourself, I thank my coauthors Julien Berestycki, Nina Gantert, Quan Shi and Michel Pain for their cooperation. I also enjoyed the probability community as a whole and many interesting conferences. I would also like to thank my former undergrad supervisor Markus Heydenreich who lead me to joining this field. I also thank my assessors Christina Goldschmidt and Bastien Mallein for carefully reading this thesis and for an interesting viva.

The probability group in the stats department was a great workplace. I would like to thank all my friends there, in particular Peter (whose relentless productivity was a shock), Rivka, Félix, Bastian, Ruairi, Matt, Serte, Gabriel, Dan, Zhixiao and Adrian. Being part of the teaching team at LMH was a stimulating experience, I especially thank the two other junior mathematicians, Sam and Ian, for sharing many free lunches.

A key part of my life in Oxford was spending nearly as much time on the river as at the desk, I thank the Wolfson college boat club for its welcoming environment. This especially applies to my former crewmates, Adam, Thomas, Dominik, Giovanni, Mads, Karim, Will, Ed, Mantas, Joe, Kirsty, Christian and Mary.

I would have not made it through nine years of studying at university without the continuing support from my parents Stefanie and Ulrich (and my brother Felix). Most importantly, I would like to thank my partner Anna (currently fiancée, to be updated soon) for her love and support. She endures my mathematical ramblings and supports me in every aspect of life.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original. Some chapters are joint work with other authors, this is detailed in the beginning of the thesis. This thesis is submitted to the University of Oxford for the degree Doctor of Philosophy, and has not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

David Geldbach
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Contents

Chapter 1

Introduction

1.1 Structure of this thesis

This thesis is an integrated thesis. This means that Chapters 2, 3 and 4 are reproductions of the preprints

[17] J. Berestycki, N. Gantert, D. Geldbach, and Q. Shi. Biased branching random walks on Bienaymé–Galton–Watson trees, 2025. *Preprint*, arXiv:2502.07363, *submitted to AIHP*,

[18] J. Berestycki, D. Geldbach, and M. Pain. Polynomial slowdown in space-inhomogeneous branching Brownian motion, 2025. *Preprint*, arXiv:2506:10623, *submitted to EJP*,

[51] D. Geldbach. Hyperbolic branching Brownian motion: the empirical limit measure, 2025. *Preprint*, arXiv:2509:06730, *submitted to CPAM*,

which differ only in typesetting from the versions submitted to the journals. I would also like to point out that *Polynomial slowdown in space-inhomogenous branching Brownian motion* is joint work with my supervisor Julien Berestycki and our co-author Michel Pain. Similarly, *Biased branching random walks on Bienaymé–Galton–Watson trees* is joint work with my supervisor Julien Berestycki and our co-authors Nina Gantert and Quan Shi. In both projects I have contributed significantly to all stages of the project, from formalising model and results, to the proofs, to polishing the presentation.

During my PhD, I have also worked on a different project,

[50] D. Geldbach. Continuum asymptotics for tree growth models, 2023. *Preprint*, arXiv:2309.04336, *submitted to ALEA*,

which is not included in this thesis for thematic reasons.

This introductory chapter discusses the background needed to understand the following chapters and to put their results into context. This means we present the basics of (spatial) branching processes in the form of branching Brownian motion and branching random walks in Section 1.2. Then we discuss how inhomogeneities can be added to these models and how this effects the methods required to understand them. We do this by looking at various existing results in the literature in Section 1.3. In Section 1.4 we present brief summaries of the projects in Chapters 2, 3 and 4.

1.2 Background on branching processes

Branching processes are a ubiquitous building block for mathematical models whenever there is growth or self similarity. This covers essentially all fields of science; biology, chemistry, epidemiology, physics, et cetera. Naturally, in parallel mathematicians are trying to understand branching processes rigorously.

1.2.1 Non–spatial branching processes

The simplest model for a probabilistic branching process is a *Bienaymé–Galton–Watson process*[1]: this is a $\mathbb{N}_0$-valued stochastic process $(Z_n, n \geq 0)$. The initial condition is $Z_0 = 1$, that is we start with one individual, and we obtain Z_{n+1} from Z_n by

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{i,n}, \qquad \xi_{i,n} \sim \mu, \ \textit{i.i.d.},$$

where $\mu = (\mu_k, k \in \mathbb{N}_0)$ is a reference probability probability measure. The interpretation here is that in every time step an individual is replaced by a random number of offspring, and everything happens independently and with the same distribution. The main result here is that there is a phase transition depending on the mean of μ. Let $m = \sum_{k\geq 1} k\mu_k$. This results goes back to Bienaymé, Galton and Watson, see for example [43, Theorems 4.3.10-12] for a proof.

Theorem 1.2.1. *Assume $\mu_1 < 1$ and $m < \infty$.*

1. *Extinction: if $m \leq 1$, then $Z_n \to 0$ almost surely.*

2. *Survival: if $m > 1$, then with positive probability we have for all n that $Z_n > 0$.*

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[1] Often also called Galton–Watson process or Bienaymé process. See [1] for a discussion on nomenclature.

This result suggests that the study of branching processes needs to differentiate between three phases: subcritical ($m < 1$), critical ($m = 1$) and supercritical ($m > 1$). In this thesis we are interested in supercritical branching processes. In this phase, the number of particles grows exponentially, which can be formalised with a small extra assumption on μ.

Theorem 1.2.2 (Kesten-Stigum 1966)**.** *Assume* $\sum_{k\geq 1} k\log(k)\mu_k < \infty$*. Then there exists a random variable* W *such that* $m^{-n}Z_n \to W$ *almost surely as* $n \to \infty$ *and* $W > 0$ *almost surely conditional on survival.*

See for example [70, Chapter 12.2] for a proof.

We also want to note that this process is endowed with a tree structure, which we may define recursively: let $\mathbf{o}$ be the root of the tree. If a vertex in generation n has $\xi_{i,n}$ children, add $\xi_{i,n}$ new vertices and connect them to their parent. We will make use of this tree structure in multiple ways later.

1.2.2 Classical spatial branching processes: BBM and BRW

In a supercritical branching process the number of particles grows exponentially. We now extend our model to endow the particles with spatial information so that the particles form a growing cloud in space. One classical way of doing this is branching Brownian motion (BBM). This is a process in continuous time and space.

A description of the process is as follows: at time 0, there is one particle located at the origin, $X_u(0) = 0$, where u is its label. This particle moves according to a Brownian motion in $\mathbb{R}$ until an exponential rate-1 clock rings. Then the particle splits into two, both offspring particles start at the position of the parent. They then (independently) perform Brownian motions and split (independently) after exponential times. See Figure 1.1 for an illustration. We denote the particles alive at time t by $\mathcal{N}(t)$ and their positions by $(X_u(t), u \in \mathcal{N}(t))$. Note that this is a simplified version of the process: we could change the branching rate from 1 to β and instead of splitting into two we could split into a random number of offspring.

One of the main objects we are interested in here is the position of the rightmost particle,

$$M_t = \max_{u\in\mathcal{N}(t)} X_u(t).$$

Bramson, Lalley and Selke [32, 33, 66] have shown the following result.

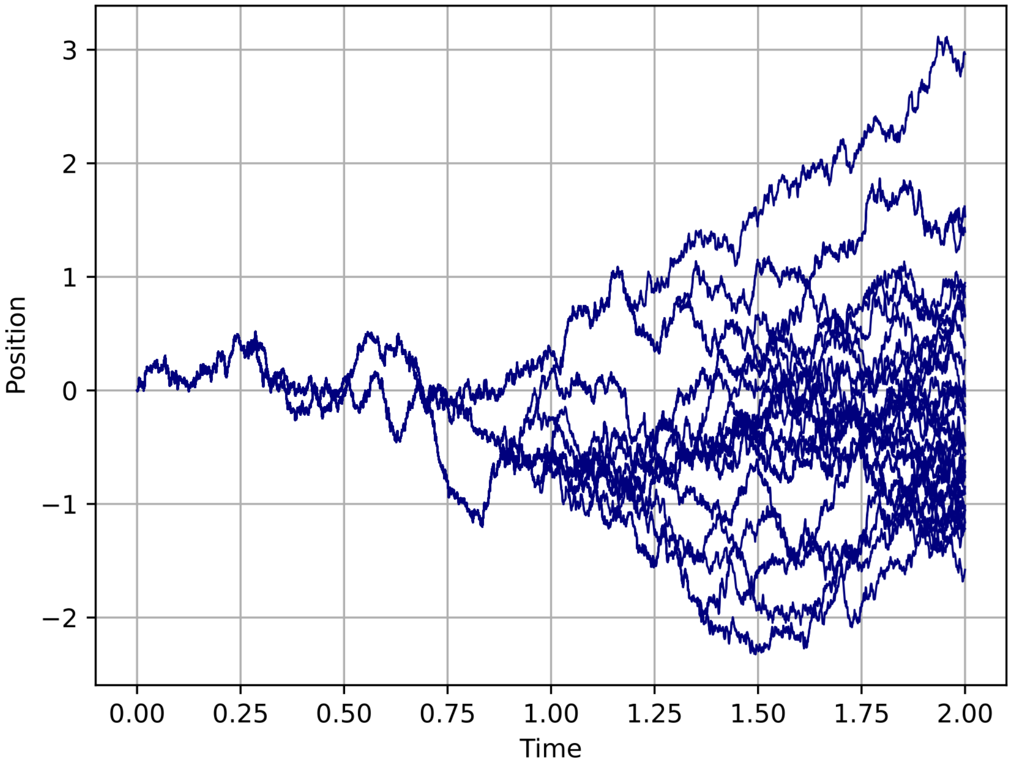

Figure 1.1: A simulation of branching Brownian motion.

Theorem 1.2.3. *Let* $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$*, then*

$$\lim_{t\to\infty} \mathbb{P}\left(M_t - m(t) \leq x\right) = \mathbb{E}\left[e^{-cD_\infty e^{-\sqrt{2}x}}\right],$$

where $c > 0$ *and* $D_\infty > 0$ *is a martingale limit.*

The martingale limit is determined by the behaviour of the so-called derivative martingale,

$$D_t = \sum_{u\in\mathcal{N}(t)} \left(\sqrt{2}t - X_u(t)\right) \exp\left(-\sqrt{2}(\sqrt{2}t - X_u(t)\right), \quad t \geq 0.$$

The idea is that it captures the behaviour of BBM at early times. The proofs of these classical results make use of the connection between BBM and the Fisher, Kolmogorov, Petrovsky, Piskunov (F-KPP) equation,

$$\begin{cases} \partial_t v = \frac{1}{2}\partial_{xx} v + \beta v(1-v), \\ v(0,x) = g(x), \end{cases}$$

where $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\beta > 0$ and $g : \mathbb{R} \to [0,1]$. McKean [75] has shown that this can be represented using branching Brownian motion,

$$v(t,x) = \mathbb{E}_x\left[\prod_{u\in\mathcal{N}t)} g(X_u(t))\right].$$

In particular, if we choose $g(x) = \mathbb{1}_{(-\infty,0]}(x)$, then $v(t,x) = \mathbb{P}(M_t \leq x)$. This also means that any result about $(X_u(t), u \in \mathcal{N}(t))$ can be translated into a result for v and vice versa.

Let us provide a quick justification for the choice of $m(t)$ in Theorem 1.2.3. By linearity

$$\mathbb{P}\left(M_t \geq m(t)\right) \leq \mathbb{E}\left[\sum_{u\in\mathcal{N}(t)} \mathbb{1}_{\{X_u(t)\geq m(t)\}}\right] = e^t\mathbb{P}(B_t \geq m(t)),$$

and using the tail of the Gaussian distribution

$$e^t\mathbb{P}(B_t \geq m(t)) = \exp\left(t - \frac{m(t)^2}{2t} - \frac{1}{2}\log(t) + O(1)\right).$$

This suggests that we should choose $m(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}}\log(t)$. Compare this to the *true* value of $m(t)$ in Theorem 1.2.3: we are missing a factor of $\frac{1}{\sqrt{2}}\log(t)$. To find this factor, we need to observe that we have a restriction on the trajectories due to the covariance structure. Let u be a particle such that $X_u(t) \approx \sqrt{2}t$. Then for any time $s \leq t$ the ancestor of u at time s will satisfy $X_u(s) \leq \sqrt{2}s + K$ with high probability where K is some large constant. If we include this as barrier in our computations, that is we look at the event $\{\forall s \leq t, \forall u \in \mathcal{N}(t) : X_u(s) \leq \sqrt{2}s\}$, then we obtain a term like

$$\mathbb{P}_0\left(\forall s \leq t : B_s \geq -1 | B_t = 0\right) \approx \frac{c}{t},$$

where c is a constant.

See Figure 1.2 for a simulation of BBM with this barrier. On the technical level this is achieved by looking at $\sqrt{2}s - B_s$ and applying Girsanov's theorem. Going back to the previous heuristic, we obtain

$$\mathbb{P}\left(M_t \geq m(t)\right) \approx \exp\left(t - \frac{m(t)^2}{2t} - \frac{1}{2}\log(t) - \log(t) + O(1)\right),$$

which now correctly suggests $m(t) = \sqrt{2}t - \frac{1}{3\sqrt{2}}\log(t)$. It is possible to make this heuristic rigorous though it is computationally intensive, see [78].

In this section we have so far considered branching Brownian motion where time and space are continuous. The discrete analogues are branching random walks (BRW).

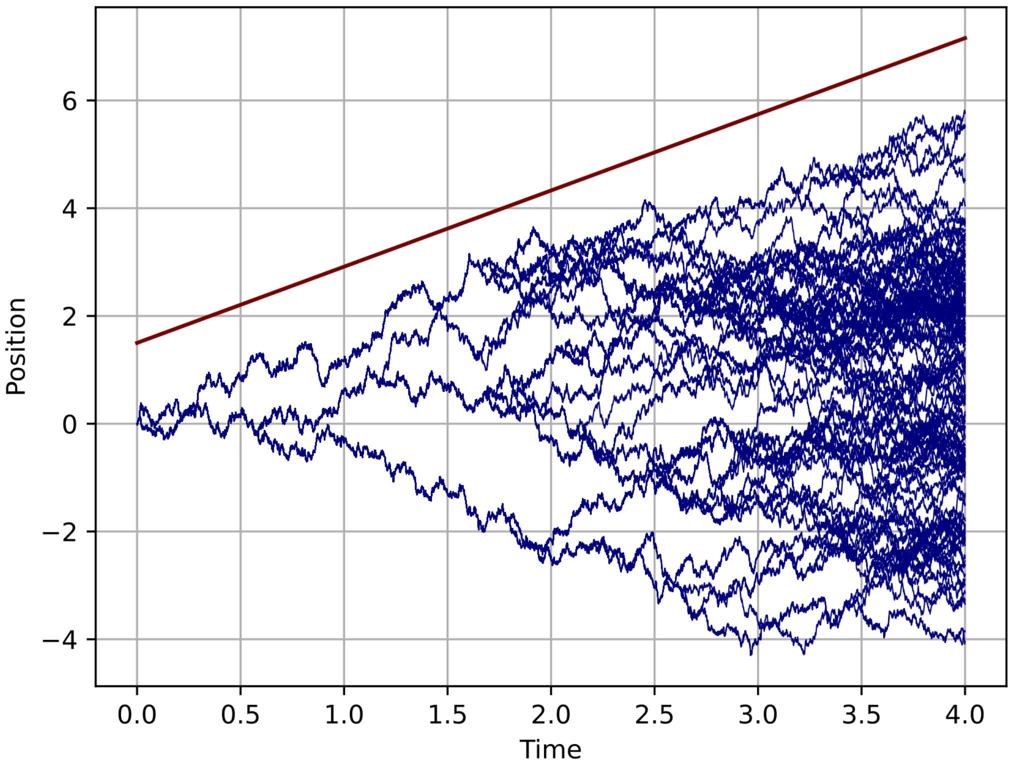

Figure 1.2: BBM with a barrier at $s \mapsto \sqrt{2}s + K$. The process will hit this barrier only finitely many times almost surely.

They are defined in discrete time: we start with one particle located at 0. In every time step, a particle located at x is replaced by offspring particles located at

$$x + \mathcal{L},$$

where $\mathcal{L}$ is a reference point process on $\mathbb{R}$ (or rather an independent copy thereof). Here we need to assume $\mathbb{E}[|\mathcal{L}|] > 1$ to be in the supercritical regime. Formulating BRW using a point process means that children of the same parent are allowed to have dependent displacements even though their offspring will be independent conditional on the position of the parent. Denote the resulting branching random walk in discrete time by $(X_u(n), u \in \mathcal{N}(n))$. Essentially any result about the asymptotics of BBM, like Theorem 1.2.3, also applies to BRWs though one has to be careful to handle phenomena arising from the discreteness of the process. For example if there are constants a, b such that $\mathcal{L}$ is supported on $a\mathbb{Z} + b$, then the BRW will always be supported on a countable set and might display periodic behaviour. Nevertheless, a slightly weaker version of Theorem 1.2.3 still holds. We refer to [74, Theorem 1.1] for an analogue of Theorem 1.2.3 for branching random walks; this was first proved by [8].

1.3 Related work - inhomogenous BBM and BRW

Although homogenous BBM and BRWs are well understood objects, they remain an active field of research. There is lot of work on related models where inhomogeneities have been added to the original models. We present some of them in this section:

1. time varying diffusivity,
2. increasing branching rates,
3. catalytic branching,
4. and random branching rates.

In each of these cases we focus on the behaviour of M_t, the maximal displacement. We also briefly state the key ideas and methods involved.

1.3.1 Time varying BBM and BRW

One simple way of adding inhomogeneity is by changing the reference process. Instead of standard Brownian motions, Maillard and Zeitouni [71] consider Brownian motions that move with a time dependent diffusivity. Let $T > 0$ and let $\sigma \in C^2([0,1],\mathbb{R})$. For time varying branching Brownian motion, particles still branch at rate 1 but move with instantaneous variance $\sigma^2(t/T)$. Define

$$m(T) = v(1)T - w(1)T^{1/3} - \sigma(1)\log(T),$$

where $v(t) = \int_0^t \sigma(s)\,\mathrm{d}s$, $w(t) = 2^{-1/3}\alpha_1 \int_0^t \sigma(s)^{1/3}|\sigma'(s)|^{2/3}\,\mathrm{d}s$, $\alpha_1 \approx 2.33$, and where $(-\alpha_1)$ is the largest zero of the Airy function of the first kind.

Theorem 1.3.1 ([71])**.** *Assume that σ is strictly decreasing, $\sigma(1) > 0$ and that $\inf_{s\in[0,1]}|\sigma'(s)| > 0$. Then the sequence $(M_t - m(t))_{t\geq 0}$ is tight.*

This result is very similar to Theorem 1.2.3. The main difference is that the correction after the linear order is polynomial rather than logarithmic. The above results improves on [47] which found the $T^{1/3}$ correction in this model. Tightness can also be improved to convergence in distribution, although this requires a less explicit choice of $m(t)$.

The proof of Theorem 1.3.1 consists of a first and second moment approach. The first moment calculation includes counting the number of particles that hit the curve γ_T where $\gamma_T(t) = Tv(t/T) - T^{1/3}w(t/T)$ for $t \in [0,T]$. See Figure 1.3 for an

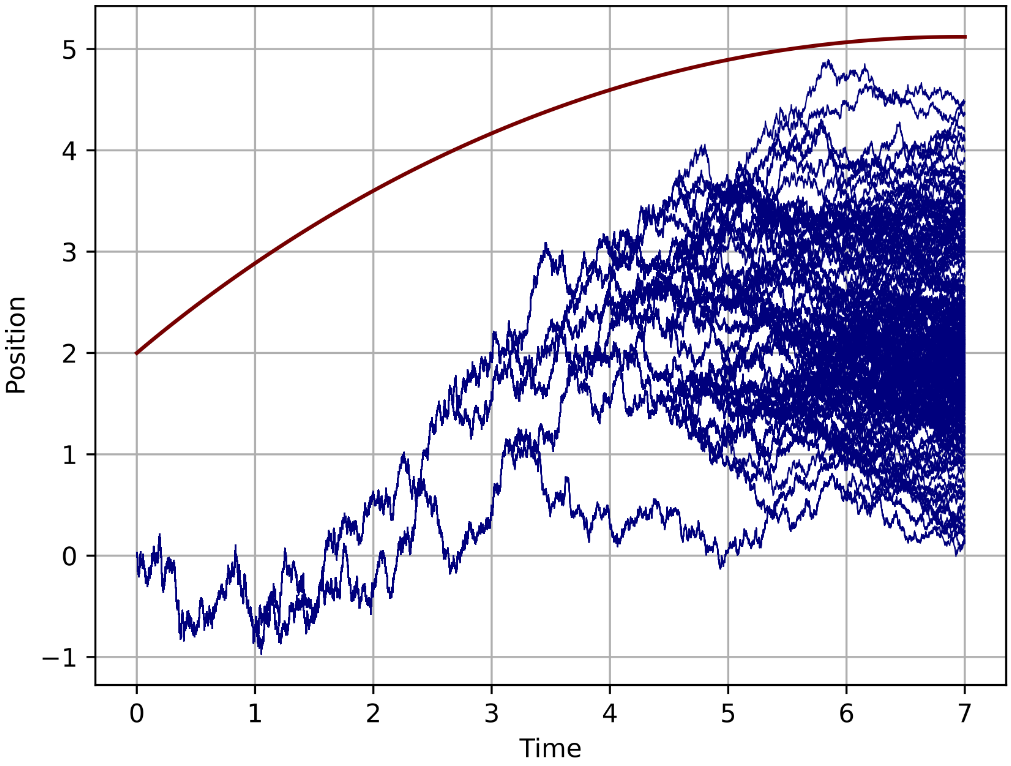

Figure 1.3: Time varying BBM with $\sigma(s) = 1 - 0.7s$ and $T = 7$ including a curved barrier $\gamma_T + K$. Observe how the diffusivity of particles decreases for larger times.

illustration. In this computation, after applying Girsanov's theorem, we encounter an expression like

$$\mathbb{E}_1 \left[\exp\left(-\frac{1}{T} \int_0^T q(t/T) B_t \, \mathrm{d}t \right) \mathbb{1}_{\{\forall t \in [0,T] : B_t \geq 0\}} \right],$$

where q is an explicit function defined in terms of σ. By rescaling the integral, this is equal to

$$\mathbb{E}_1 \left[\exp\left(-\sqrt{T} \int_0^1 q(r) B_r \, \mathrm{d}r \right) \mathbb{1}_{\{\forall t \in [0,1] : B_t \geq 0\}} \right] \\ = \mathbb{E}_1 \left[\exp\left(-\varrho \int_0^1 q(r) \sqrt{2\varrho} B_r \, \mathrm{d}r \right) \mathbb{1}_{\{\forall t \in [0,1] : B_t \geq 0\}} \right],$$

where $\varrho = 2^{-1/3} T^{1/3}$. By the Feynman–Kac formula this is connected to the PDE

$$\partial_t w = \varrho \left(\partial_{xx} w - q(t) w \right).$$

The inhomogeneity in the model is reflected in $q(t)$ not being constant. If q were constant, the PDE would be easy to understand as it is linear and falls into the framework of Sturm–Liouville theory. Thus the difficulty in the analysis of the PDE consists of quantifying the effect of a non-constant choice of q. The solutions to the PDE decay exponentially in ϱ, hence the expectation above decays like $\exp(-cT^{1/3})$ for some $c > 0$.

A similar model can be studied for branching random walks, see [72]. Analogously, it exhibits a $n^{1/3}$ correction.

1.3.2 Increasing branching rate

Instead of changing the diffusivity of the underlying Brownian motions and keeping branching homogenous, we consider non-constant branching rates. Consider branching Brownian motion where particles move as standard Brownian motions and split into two at rate $\beta|x|^p$ where $\beta > 0$ and $p \in (0, 2]$. This means that particles are *rewarded* with a higher branching rate when they deviate upwards. Consider M_t, the maximum displacement. Because the branching rate increases for the maximal particles as $M_t \to \infty$, the non-constant branching rate speeds up M_t. See Figure 1.4 for an illustration.

Theorem 1.3.2 ([54]). *The maximal displacement M_t satisfies almost surely:*

1. *If $p \in (0, 2)$:*

$$\lim_{t\to\infty} \frac{M_t}{t^{\frac{2}{2-p}}} = \left(\frac{\beta}{2}(2-p)^2\right)^{1/(2-p)}$$

2. *If $p = 2$:*

$$\lim_{t\to\infty} \frac{\log M_t}{t} = \sqrt{2\beta}$$

In the heuristics for Theorem 1.2.3 (the asymptotics for M_t for homogenous BBM) we have seen that maximal particles roughly follow the path $s \mapsto \sqrt{2}s$ up to deviations of sublinear order. Something similar is true here. However, since the environment is now inhomogeneous it is not clear what the optimal path looks like. Let $g \in C^2(\mathbb{R}_+, \mathbb{R})$ be a reference path. Combining Schilder's Theorem for large deviations of Brownian motion with the exponential growth at rate $\beta|x|^p$ yields

$$\mathbb{E}\left[\#\left\{u \in \mathcal{N}(t) \colon \forall s \leq t, X_u(s) \text{ is close to } g(s)\right\}\right] \approx \exp\left(\int_0^t \beta g(s)^p \,\mathrm{d}s - \int_0^t \frac{g'(s)^2}{2} \,\mathrm{d}s\right).$$

This motivates us to look at the martingale

$$M_g(t) = \sum_{u\in\mathcal{N}(t)} \exp\left(\int_0^t g'(t)\,\mathrm{d}X_u(s) - \int_0^t \left(\frac{g'(s)^2}{2} + \beta|X_u(t)|^p\right)\mathrm{d}s\right).$$

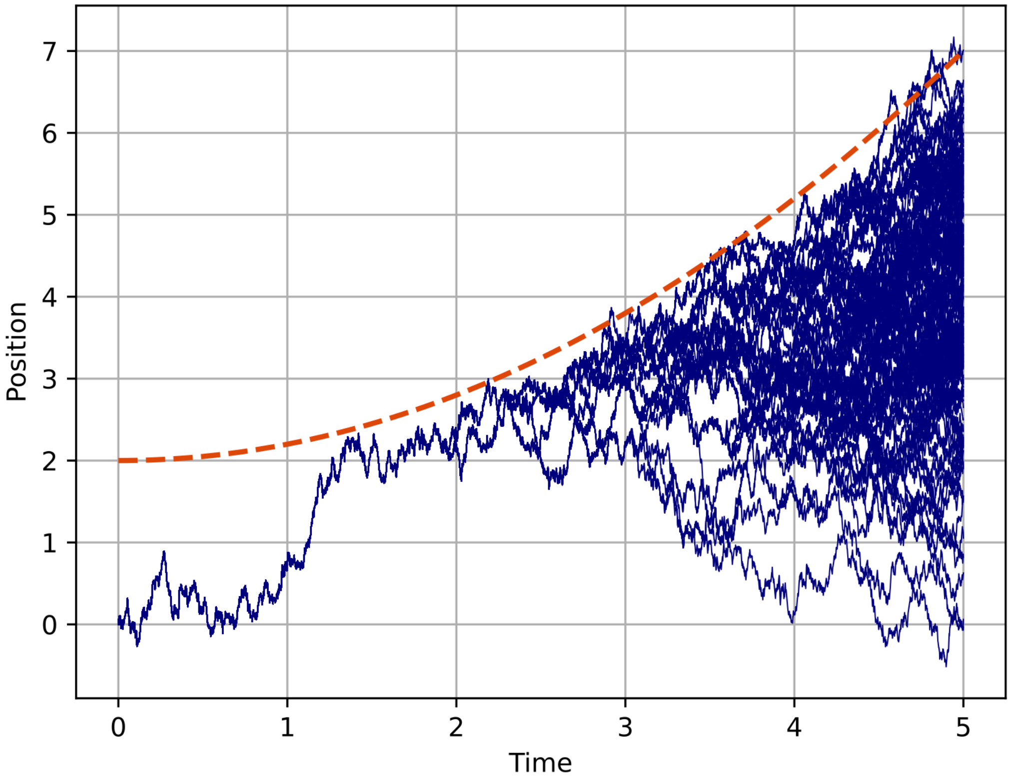

Figure 1.4: BBM with location dependent branching rate $\beta|x|^p$ for $\beta = 0.4$ and $p = 1$. The dashed line $t \mapsto 0.2t^2 + 2$ illustrates the first order of the maximal displacement.

The exponential is made up of two parts, $-\int_0^t \beta|X_u(t)|^p \,\mathrm{d}s$ compensates the growth of the process and $\int_0^t g'(t)\,\mathrm{d}X_u(s) - \int_0^t \frac{g'(s)^2}{2}\,\mathrm{d}s$ arises from Girsanov's theorem. The key idea of the proof of Theorem 1.3.2 is to determine if $M_g(t)$ has a non-trivial limit. This depends on the choice of reference path g. In the case $p \in (0, 2)$ they consider $g(s) = as^b$ for $a, b > 0$, and in the case $p = 2$ they consider $g(s) = e^{\lambda s}$ for $s > 0$. They also make use of spinal techniques: they force a particle to follow g (that is, its movement is $t \mapsto g(t) + B_t$) and look at the change of measure that is required.

The idea of looking at the process along special paths is further studied by [16] for $p \in (0, 2)$. Consider a path g that satisfies

$$\int_0^t \beta g(s)^p \,\mathrm{d}s - \int_0^t \frac{g'(s)^2}{2}\,\mathrm{d}s > 0 \quad \text{and} \quad \int_0^{t/2} \beta g(s)^p \,\mathrm{d}s - \int_0^{t/2} \frac{g'(s)^2}{2}\,\mathrm{d}s < 0.$$

This means that in the heuristics we should have an exponentially large number of particles following g at time t yet also an exponentially small number at time $t/2$. This is an indication that the almost sure behaviour of the process does not agree with the behaviour in expectation. This is indeed what the authors of [16] show: the process essentially becomes extinct along g at time $t/2$ before the large branching rate at the end of g takes effect. To show this rigorously, one needs to pick a time horizon T and rescale a reference path $g : [0,1] \to [0,\infty)$.

A similar group of authors [15] studies this in the case $p = 2$ focussing on the total population growth.

1.3.3 Catalytic branching

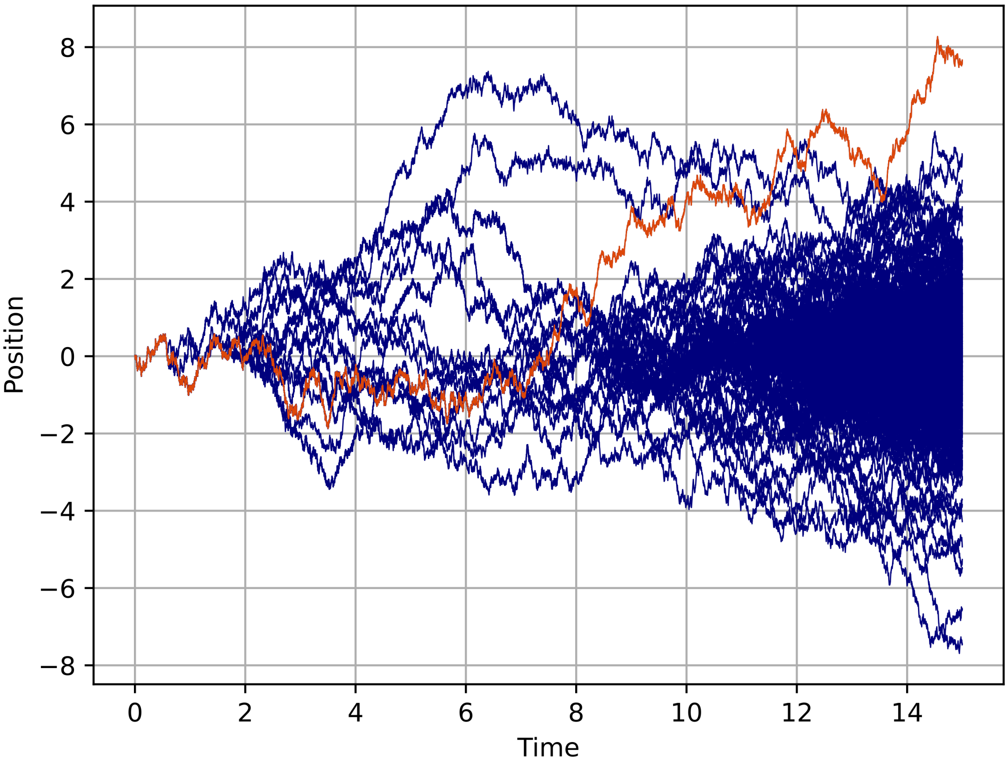

Figure 1.5: A simulation of catalytic BBM: the catalytic branching is approximated by branching at rate $\beta/(2\varepsilon)$ in the interval $[-\varepsilon, \varepsilon]$. Note that the path of the rightmost particle at the final time stays near 0 before jumping to the maximal position.

Another choice of inhomogenous branching rate on $\mathbb{R}$ is catalytic branching: particles branch at rate $\beta \, \mathrm{d}L_t$ where $\beta > 0$ and L_t is the local time at 0. Equivalently, if T is the time of the first branching event, then conditional on the path of the first particle $(X_s)_{s\geq 0}$ the distribution of T is given by $\mathbb{P}(T > t | (X_s)_{s\geq 0}) = \exp(-\beta L_t)$. This means that as soon as we move away from 0 we stop branching. Heuristically, the optimal strategy for particles to achieve the maximum should be to remain near 0 for some time and then to do a big jump. See Figure 1.5 for an illustration. The equivalent to Theorem 1.2.3 is the following.

Theorem 1.3.3 ([24, 25]). *For any $y \in \mathbb{R}$ we have*

$$\lim_{t\to\infty} \mathbb{P}\left(M_t - \frac{\beta}{2}t \le y\right) = \mathbb{E}\left[\exp\left(-G_\infty e^{-\beta y}\right)\right],$$

where G_∞ is the almost sure limit of the martingale

$$G_t = e^{-\frac{\beta^2}{2}t} \sum_{u\in\mathcal{N}(t)} \exp\left(-\beta|X_u(t)|\right),$$

and $G_\infty > 0$ almost surely.

Note that unlike in Theorem 1.2.3 there are no logarithmic corrections to the linear speed. Let L_t^u be the local time of $X_u(t)$ at 0. Consider the martingale G_t. We can write

$$G_t = \sum_{u\in X_u(t)} \exp\left(-\beta|X_u(t)| + \beta L_t^u - \frac{\beta^2}{2}t\right) \exp\left(-\beta L_t^u\right).$$

The second factor, $\exp\left(-\beta L_t^u\right)$, compensates the growth along the path of $X_u(t)$. The first factor $\exp\left(-\beta|X_u(t)| + \beta L_t^u - \frac{\beta^2}{2}t\right)$ is related to the change of measure that turns standard Brownian motion into Brownian motion with drift β towards the origin. Note that such a change of measure results in a bigger local time at 0 and hence more branching. The authors of [24] use this transform explicitly in the form of spinal techniques.

There is an interesting modification of this model studied by [26]: assume now that the branching happens at rate β everywhere and at rate $\beta_0 \, \mathrm{d}L_t$ at 0. This is a combination of catalytic BBM and homogenous BBM. The optimal strategy for a particle to be maximal will still be to remain at 0 for a while and then to do a big jump though now the ideal proportion between these two blocks depends on β and β_0. In [26] the authors show that

$$\lim_{t\to\infty} \frac{M_t}{t} = \begin{cases} \frac{\beta}{\beta_0} + \frac{1}{2}\beta_0 & \text{if} \quad \beta \le \frac{1}{2}\beta_0^2, \\ \sqrt{2\beta} & \text{if} \quad \beta \ge \frac{1}{2}\beta_0^2. \end{cases}$$

Note that $\sqrt{2\beta}$ is the speed of BBM without catalytic branching.

Catalytic branching can also be studied for branching random walks. For a branching random walk on $\mathbb{Z}$, particles branch only at 0. A result analogous to Theorem 1.3.3 is shown by [36] though it is harder to state due to the discreteness of $\mathbb{Z}$. Nevertheless, there is still a martingale limit and no logarithmic correction.

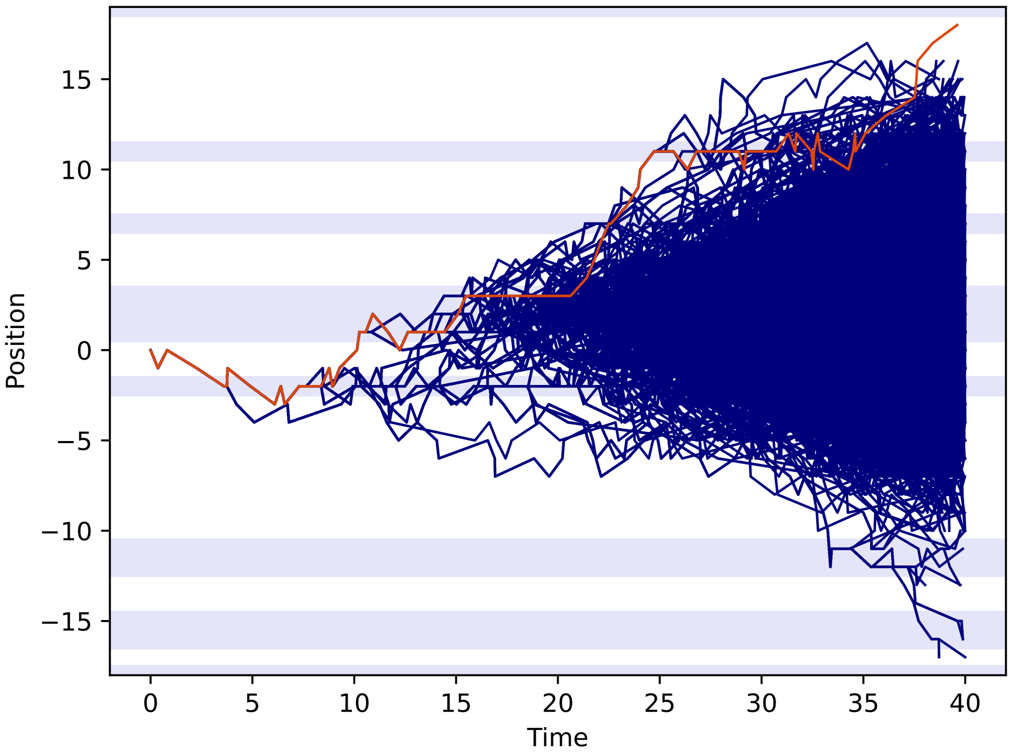

Figure 1.6: A simulation of a BRW with a random branching rates on $\mathbb{Z}$: the shaded regions correspond to integers where the branching rate is high. Observe that the path of the rightmost particle (in red) seems to spend more time in these regions.

1.3.4 Random branching rates on $\mathbb{Z}$

Instead of deterministically varying the branching rate like in the previous two sections, we can choose it to be random as well. This is best done in a discrete setting. Let $(\xi(x))_{x\in\mathbb{Z}}$ be an *i.i.d.*family of $\mathbb{R}_+$–valued random variables. Conditional on these, consider a continuous time branching random walk on $\mathbb{Z}$. Particles move at rate 1 according to a simple symmetric random walk and particles located at $x \in \mathbb{Z}$ split into two at rate $\xi(x)$. Because of the two layers of randomness, there are two probability measures: $\mathbf{P}_\xi$ considers only the randomness of the BRW, that is we condition on $(\xi(x))_{x\in\mathbb{Z}}$, whereas $\mathbb{P}$ considers $(\xi(x))_{x\in\mathbb{Z}}$ to be random as well. We call $\mathbf{P}_\xi$ the *quenched* measure and $\mathbb{P}$ the *annealed* measure, we have that $\mathbb{P}(A) = \int \mathbf{P}_\xi(A)\mathbb{P}(\mathrm{d}\xi)$ where we integrate over the distribution of ξ.

Theorem 1.3.4 ([90]). *Assume that there are* $0 < a < b < \infty$ *such that* $\mathbb{P}(\xi \in [a,b]) = 1$. *There are* $\sigma > 0, v > 0$ *such that the process*

$$\left(\frac{M_{nt} - vnt}{\sigma\sqrt{n}}, t \geq 0\right),$$

converges to standard Brownian motion as $n \to \infty$, in distribution under $\mathbb{P}$.

This is not quite analogous to Theorem 1.2.3 but similar in spirit. A key difference here is that the process is rescaled by $\sqrt{n}$. One can also show that there exists a function $m(t)$ such that $(M_t - m(t))_{t\geq 0}$ is tight, see [65], though $m(t)$ is very inexplicit. Nevertheless, already knowing that the first order of M_t is vt (which is implied by Theorem 1.3.4) is non-trivial.

Due to the random environment the proof of Theorem 1.3.4 is very technical. At its heart there is a first and second moment argument. For any $v, t > 0$ we have under the quenched measure that

$$\mathbf{E}_\xi \left[\# \left\{u \in \mathcal{N}(t) : X_u(t) = \lfloor vt \rfloor\right\}\right] = \mathbf{E}_\xi \left[\mathbb{1}_{\{Y_t = \lfloor vt \rfloor\}} \exp\left(\int_0^t \xi(Y_s)\,\mathrm{d}s\right)\right],$$

where $(Y_t)_{t\geq 0}$ is a continuous time simple random walk moving at rate 1. This suggests that one needs to understand $\int_0^t \xi(Y_s)\,\mathrm{d}s$ and the authors of [90] do this by applying a local limit theorem.

1.4 Projects in this thesis

In this section we present brief summaries of the projects founds in Chapters 2, 3 and 4. Each Chapter includes a more detailed summary of the respective project.

1.4.1 Polynomial slowdown in space-inhomogeneous branching Brownian motion

We study a spatially inhomogeneous version of BBM in $\mathbb{R}^2$, where the branching rate of a particle depends on its angular coordinate. The goal is to capture the effect of the inhomogeneity on the maximal displacement. Particles move according to 2d Brownian motions, and a particle located at a point of polar coordinates (r, θ) splits into two particles at rate $b(\theta)$ where $b : \mathbb{R} \to [0, \infty)$ is a 2π periodic, measurable function. We denote by $\mathcal{N}(t)$ the set of particles alive at time t and, for $u \in \mathcal{N}(t)$, and by $(X_u(t), Y_u(t))$ the position of particle $u \in \mathcal{N}(t)$ at time t.

We are interested in the asymptotic behaviour of

$$M_t := \max_{u\in\mathcal{N}(t)} \|(X_u(t), Y_u(t)\|_2 .$$

We need to put some assumptions on the function b, we assume that

(A1) b is continuous and reaches its maximum on $[-\pi, \pi]$ at 0 only, that is

$$\sup_{|\theta|>\delta} b(\theta) < 1, \quad \text{for all } \delta > 0.$$

(A2) there exist $\alpha \in (2/3, 2)$ and $\beta > 0$ such that $b(\theta) = 1 - \beta |\theta|^\alpha + O(\theta^2)$, as $\theta \to 0$.

To state our result, we introduce

$$\vartheta_1 := \lambda_0 \cdot \frac{2+\alpha}{2-\alpha} \cdot \frac{\beta^{2/(2+\alpha)}}{2^{2\alpha/(2+\alpha)}},$$

where $\lambda_0 > 0$ is a constant depending only on α defined as the smallest eigenvalue of a differential operator defined later, and

$$m(t) = \sqrt{2}t - \frac{\vartheta_1}{\sqrt{2}} t^{(2-\alpha)/(2+\alpha)} - \left(\frac{3}{2\sqrt{2}} - \frac{\alpha}{2\sqrt{2}(2+\alpha)} \right) \log t.$$

Theorem 1.4.1. *As $t \to \infty$, we have*

$$M_t = m(t) + O_{\mathbb{P}}(1).$$

Equivalently, the process $(M_t - m(t))_{t \geq 1}$ is tight.

This should be compared to Theorem 1.2.3 where $m_{1d}(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t)$ is the asymptotic behaviour of M_t for homogeneous 1d–branching Brownian motion. Notably, the inhomogeneous branching rate causes a polynomial correction.

The proof of Theorem 1.4.1 is based on a first and second moment argument. In particular, this means that we need to identify which trajectories lead to particles that contribute to M_t. These will be particles where

$$X_u(t) \approx \sqrt{2}t \quad \text{and} \quad |Y_u(t)| \approx t^{\alpha/(2+\alpha)}.$$

Particles like these have an angular coordinate of approximately $Y_u(t)/(\sqrt{2}t)$. Combining this with assumption (A2) on the branching rate leads us to expectations like

$$\mathbb{E}\left[\exp\left(-\beta \int_1^t \left| \frac{Y_s}{\sqrt{2}s} \right|^\alpha ds \right) \right],$$

where $(Y_s)_{s \geq 0}$ is a 1d–Brownian motion. We analyse this by using the Feynman-Kac formula and studying the resulting time–inhomogeneous partial differential equation. We provide more detailed heuristics in Section 2.1.3.

1.4.2 Biased branching random walks on Bienaymé–Galton–Watson trees

The goal of this project is to study the maximal displacement of a branching random walk in random environment. There are now three sources of randomness: the environment, the branching and the movement.

Let $p = (p_k, k = 0, 1, 2, \ldots)$ be a probability measure on $\mathbb{N}_0$ and let $\texttt{BGW}$ be the law of a Bienaymé–Galton–Watson (BGW–tree) ω with offspring distribution p. We assume that

$$p_0 = 0 \text{ and } p_1 < 1. \tag{1.1}$$

Conditional on ω, we consider the λ*–biased random walk on* ω. That is, $(X_n, n \geq 0)$ is a Markov chain which starts at the root of the tree. If $X_n = x$, then X_n moves to the parent of x with probability $\lambda/(\lambda + \deg(x))$ and to each child with probability $1/(\lambda + \deg(x))$. Here $\deg(x)$ is the number of children of x and $\lambda > 0$ is a fixed parameter. If X_n is located at the root, it moves to a uniformly chosen child of the root.

We use this random walk to construct a branching random walk: let $\mu = (\mu_k, k = 0, 1, 2, \ldots)$ be another probability measure. We start with a single particle at the root. In each step, a particle is replaced by a random number of particles, independently sampled from μ. Each offspring particle then takes a step independently according to the transition probabilities of the λ–biased random walk. This results in a branching random walk, denote it by $(X(u), u \in \mathbb{T})$ where $\mathbb{T}$ is the genealogical tree of the branching random walk. The particles alive at time n are the ones satisfying $|u| = n$.

Because of the different sources of randomness, we associate two different laws with the branching random walk: if we condition on the environment ω, we speak of the quenched law $\mathbf{P}_\omega$. If we consider the environment to be random as well, we speak of the annealed law $\mathbb{P}$. They are connected via $\mathbb{P}(\cdot) = \int \mathbf{P}_\omega(\cdot)\texttt{BGW}(d\omega)$.

We now consider the maximum displacement of the branching random walk at time n. To this end, for $x \in \omega$, let $|x|$ be the height of x, that is the distance to the root in the graph metric.

Theorem 1.4.2 (Velocity of the maximal displacement)**.** *Let* $(X(u), u \in \mathbb{T})$ *be a* λ*-biased branching random walk with reproduction law* μ*. Suppose that* $\mu_0 = 0$*,* $\mu_1 < 1$*,* $m := \sum_{k \geq 1} k\mu_k > 1$ *and* $m < \infty$*. Then for* $\texttt{BGW}$*–a.e.* ω*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \max_{|u|=n} |X(u)| = v_{\lambda,m}, \qquad \mathbf{P}_\omega - a.s.$$

where $v_{\lambda,m}$ *is a constant that only depends on* λ*,* m *and the tree offspring law* p*.*

This should be seen in relation to Theorem 1.2.3. In our theorem we show that the speed for the maximal displacement exists and is non–random. This crucially depends on the environment being *statistically transitive*, that is even though ω is not a transitive graph, different regions have the same law under `BGW` and therefore lead to similar behaviour for the branching random walk.

Our proofs depend on a careful interplay between the annealed and quenched measure. Interestingly, we need different approaches depending on the recurrence or transience of the process. Depending on the bias λ and the branching rate m, there is a regime where almost surely only finitely many particles visit the root. Conversely, outside of this regime, there are almost infinitely many particles that visit the root. We determine this regime in Theorem 3.1.2. For particle processes like this, these two regimes are called transience and strong recurrence.

1.4.3 Hyperbolic branching Brownian motion: the empirical limit measure

We study branching Brownian motion in hyperbolic space. The goal is to better understand the distribution of the bulk of the particles.

Hyperbolic space is often modelled using the hyperbolic disk: let $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ be the interior of the unit disk and consider the hyperbolic Laplacian

$$\mathcal{L}_{\mathbb{D}} = \frac{(1-(x^2+y^2))^2}{4}\left(\partial_x^2 + \partial_y^2\right).$$

We discuss hyperbolic space in more detail later. The operator $\mathcal{L}_{\mathbb{D}}$ can be used to generate hyperbolic Brownian motion, denote the corresponding stochastic process by $(X_t, Y_t)_{t\geq 0}$, started at the origin. This process behaves like regular 2d-Brownian motion except that it slows down as it approaches $\partial\mathbb{D} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$. In fact, one can show that $(X_t, Y_t)_t$ converges to a random point on $\partial\mathbb{D}$.

We use this to build a version of branching Brownian motion: we start with a single particle at the origin. Particles move independently according to hyperbolic Brownian motions, and particles split in two at rate $\beta > 0$. This results in a cloud of particles, we denote it by $((X_u(t), Y_u(t)), u \in \mathcal{N}(t))$ where $\mathcal{N}(t)$ is the set of particles alive at time t.

The goal of this project is to study the empirical measure of the process. The normalised empirical measure at time t is given by

$$\mu_t = \frac{1}{|\mathcal{N}(t)|}\sum_{u\in\mathcal{N}(t)} \delta_{(X_u(t), Y_u(t))}.$$

There is a random probability measure μ_∞ supported on $\partial\mathbb{D}$ such that almost surely μ_t converges weakly to μ_∞. This is shown with martingale arguments. We compute the Hausdorff dimension of the support of μ_∞.

Theorem 1.4.3. *For any $\beta > 0$, almost surely* $\dim \operatorname{supp} \mu_\infty = 1 \wedge (2\beta)$.

We also show more properties of μ_∞, in particular that it is almost surely non-atomic and that for $\beta > 1/2$ it admits a Lebesgue density.

This theorem has to be understood in comparison to a result by Lalley and Selke [67]. There they consider Λ, the set of all accumulation points of $\{(X_u(t), Y_u(t)), t \geq 0\}$, and show that almost surely

$$\dim \Lambda = \begin{cases} \frac{1}{2}(1 - \sqrt{1 - 8\beta}) & \beta \leq 1/8, \\ 1 & \beta > 1/8. \end{cases}$$

In particular this means that for any $\beta < 1/2$, supp μ_∞ is a proper subset of Λ. Further, $\beta = 1/8$ is also the threshold for local recurrence or transience. It is surprising that transition from transience to recurrence does not affect the behaviour of μ_∞.

The proof of Theorem 1.4.3 is based on the idea that we can approximate μ_t, the empirical measure of all particles, by the empirical measure restricted to *typical* particles. Those are particles whose trajectories match the almost sure behaviour of hyperbolic Brownian motion. That is they move at speed 1/2 in the hyperbolic metric away from the origin while exhibiting Gaussian fluctuations.

Chapter 2

Polynomial slowdown in space-inhomogeneous branching Brownian motion

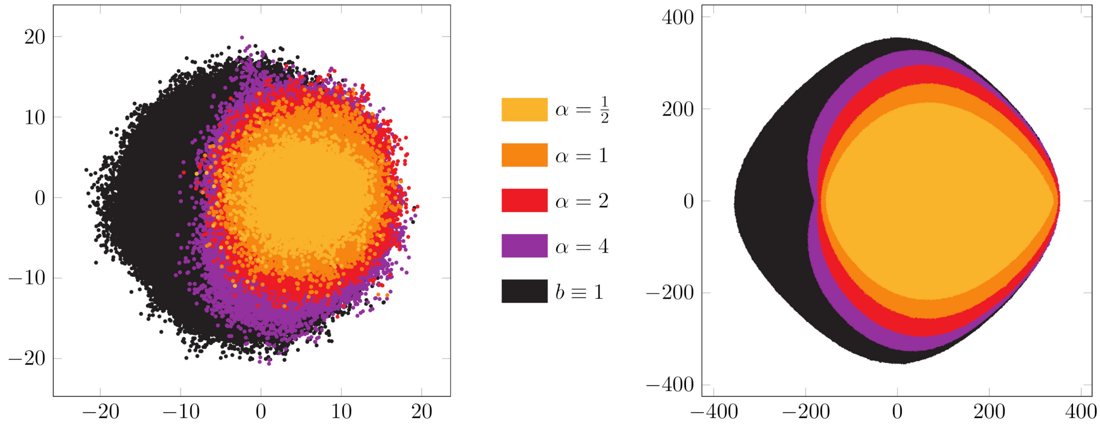

Figure 2.1: On the left, a realisation at time $t = 16$ of space–inhomogenous BBM with $\alpha \in \{1/2, 1, 2, 4\}$ and branching rate $b(\theta) = 1 - |\sin(\theta/2)|^{\alpha}$, as well as homogenous BBM in black (which can be seen as $\alpha = \infty$). The processes are coupled in such a way that every particle that exists for a smaller α also exists for a greater value of α. On the right, a realisation of a version of the model with discrete time and positions in $\mathbb{Z}^2$. In this model, each particle at time n and with angular coordinate θ has 2 children at time $n + 1$ with probability $b(\theta)$ and 1 child otherwise, and each child jumps to one of the four nearest neighbour sites, or stays at its present site with equal probability. This discretisation allows us to push the numerical simulation to larger times, here $n = 500$, but it changes the limiting shape of the process. However, the behaviour of the maximal displacement, with the appearance of a polynomial slowdown for $\alpha < 2$, should be universal.

2.1 Introduction

2.1.1 Model and main result

The standard, one-dimensional Branching Brownian Motion (BBM) is a system of particles on $\mathbb{R}$: at time $t = 0$, there is a single particle at the origin. This particle performs a Brownian motion up to an exponential random time with mean one when the particle is replaced by two particles at the same position. Each daughter particle then starts an independent copy of the same process, and so on. This results in an exponentially growing cloud of branching, diffusing, particles $(X_u^{1\mathrm{d}}(t), u \in \mathcal{N}^{1\mathrm{d}}(t))_{t\geq 0}$ where $X_u^{1\mathrm{d}}(t)$ is the position of particle u at time t and $\mathcal{N}^{1\mathrm{d}}(t)$ is the set of particles alive at time t. This is a well studied process with an extensive literature, including a particular focus on the study of the maximal displacement at time t,

$$M_t^{1\mathrm{d}} := \max_{u\in\mathcal{N}^{1\mathrm{d}}(t)} X_u^{1\mathrm{d}}(t).$$

Classical results by [32, 33, 66] determined the asymptotic behaviour of $M_t^{1\mathrm{d}}$, let $m^{1\mathrm{d}}(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$,

$$\lim_{t\to\infty} \mathbb{P}\left(M_t^{1\mathrm{d}} - m^{1\mathrm{d}}(t) \leq x\right) = \mathbb{E}\left[e^{-cD_\infty e^{-\sqrt{2}x}}\right], \tag{2.1}$$

where $c > 0$ and $D_\infty > 0$ is the limit of the so-called derivative martingale. Further results include for example the convergence of the extremal process [2, 5] and a precise description of its limit [39, 77, 14, 57]. More generally, the behaviour of the maximal displacement for variants of the BBM model has been a topic of intense research in the recent past and we discuss some of these results in the next subsection.

In the present work, we study a spatially inhomogeneous version of BBM in $\mathbb{R}^2$, where the branching rate of a particle depends only on its angular coordinate, with the goal of capturing the effect of the inhomogeneity on the maximal displacement. Let $b\colon \mathbb{R} \to [0,\infty)$ be a 2π-periodic function. We start with one particle at $(0,0)$ at time 0. Each particle moves according to a 2d Brownian motion and a particle located at a point of polar coordinates (r,θ) splits into two particles at rate $b(\theta)$. We denote by $\mathcal{N}(t)$ the set of particles alive at time t and, for $u \in \mathcal{N}(t)$, by $(X_u(t), Y_u(t))$ the position of particle u at time t. For $s \leq t$, we also write $(X_u(s), Y_u(s))$ for the position of the ancestor of u alive at time s. Finally, we denote by $(R_u(t), \theta_u(t))$ the polar coordinates of $(X_u(t), Y_u(t))$.

We are interested in the asymptotic behaviour of

$$M_t := \max_{u\in\mathcal{N}(t)} R_u(t)$$

in the case where the function b reaches its maximum on $[-\pi, \pi]$ at a single point, say 0. More precisely we assume

(A1) b is continuous and reaches its maximum on $[-\pi, \pi]$ at 0 only, that is

$$\sup_{|\theta|>\delta} b(\theta) < 1, \quad \text{for all } \delta > 0.$$

(A2) there exist $\alpha \in (2/3, 2)$ and $\beta > 0$ such that $b(\theta) = 1 - \beta\,|\theta|^\alpha + O(\theta^2)$, as $\theta \to 0$,

Instead of Assumption (A1), one can assume the weaker assumption

(A1') $b \leq 1$ and, if $b(\theta_n) \to 1$ for some sequence $(\theta_n)_{n\geq 0} \in (-\pi, \pi]^{\mathbb{N}}$, then $\theta_n \to 0$.

This assumption does not require continuity anymore, but ensures that b can approach the value 1 only at 0 (and other multiples of 2π). Note that by Assumption (A2) we still have continuity at 0.

To state our result, we introduce

$$\vartheta_1 := \lambda_0 \cdot \frac{2+\alpha}{2-\alpha} \cdot \frac{\beta^{2/(2+\alpha)}}{2^{2\alpha/(2+\alpha)}},$$

where $\lambda_0 > 0$ is a constant depending only on α defined as the smallest eigenvalue of a differential operator in Lemma 2.2.8, and

$$m(t) = \sqrt{2}t - \frac{\vartheta_1}{\sqrt{2}} t^{(2-\alpha)/(2+\alpha)} - \left(\frac{3}{2\sqrt{2}} - \frac{\alpha}{2\sqrt{2}(2+\alpha)} \right) \log t.$$

Theorem 2.1.1. *As $t \to \infty$, we have*

$$M_t = m(t) + O_{\mathbb{P}}(1).$$

Equivalently, the process $(M_t - m(t))_{t\geq 1}$ is tight.

Recalling that the $\theta = 0$ direction is the one with maximal branching rate, it is reasonable to expect that particles with the maximal displacement are located in that direction. More precisely, we can deduce the following result from our proof, which states that particles at a distance $m(t) + O(1)$ from the origin at time t cannot have a too large Y-coordinate, so that their displacement is essentially all due to the X-coordinate. The first part of this proposition is a direct consequence of Lemmas 2.5.3 and 2.5.4, and the second part then follows from the first part together with Theorem 2.1.1.

Proposition 2.1.2. *For any $\varepsilon > 0$ and $a \geq 0$, we have*

$$\mathbb{P}\left(\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) - a, |Y_u(t)| \geq t^{\frac{\alpha}{2+\alpha}+\varepsilon}\right) \xrightarrow[t\to\infty]{} 0.$$

In particular, this implies

$$M_t - \max_{u\in\mathcal{N}(t)} X_u(t) \xrightarrow[t\to\infty]{\mathbb{P}} 0.$$

Theorem 2.1.1 has to be understood in view of equation (2.1). Compare $m(t)$ and $m^{1\mathrm{d}}(t)$: introducing the inhomogenous branching rate results in a strong slowdown of the maximal displacement proportional to $t^{(2-\alpha)/(2+\alpha)}$. With our assumption of $\alpha \in (2/3, 2)$, the exponent takes values in $(0, 1/2)$, see Figure 2.2 for an illustration. This range of α corresponds to one of the phases of the model and we expect different behaviours outside of this range: for $\alpha \geq 2$, the polynomial correction should disappear and, for $\alpha \leq 2/3$, the fact that the exponent becomes larger than $1/2$ leads to new effects resulting in additional polynomial terms in $m(t)$. This is why we are focusing on the case $\alpha \in (2/3, 2)$ and we leave other cases as open questions which are discussed in Section 2.1.4. Another difference between Theorem 2.1.1 and (2.1) is that we obtain tightness instead of convergence in distribution. We still believe that convergence is true for our model but the random variable playing the role of D_∞ would be the limit of an approximate martingale, instead of an exact martingale; see Section 2.1.4 for more details.

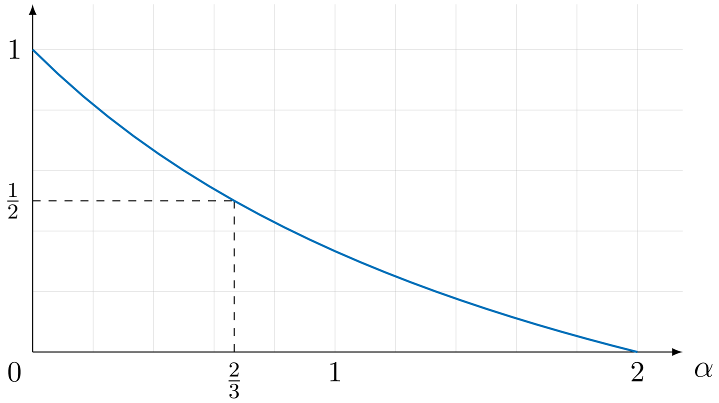

Figure 2.2: The exponent of the polynomial slowdown as function of α, $\alpha \mapsto \frac{2-\alpha}{2+\alpha}$. Note that our assumption $\alpha \in (2/3, 2)$ corresponds to the exponent taking values in $(0, 1/2)$.

2.1.2 Related work

Our result should be compared to other variants of branching Brownian motion where the maximal displacement has been studied. Firstly, BBM with homogenous branching rate, i.e. $b \equiv 1$, has been studied in dimensions higher than 1. Convergence in distribution of the maximal displacement after recentering, as well as of the extremal process, has been obtained in a series of works [73, 83, 63, 19]. In particular, in dimension 2, the proper recentering is $m^{2\mathrm{d}}(t) = \sqrt{2}t - \frac{1}{\sqrt{2}} \log t$: comparing this to (2.1), we see that changing the dimension yields a change in the logarithmic correction.

Space-inhomogeneous BBMs. Some models with a space-inhomogeneous branching rate have been studied in dimension 1. In [54], Harris and Harris considered a BBM on $\mathbb{R}$, where a particle located at x branches at rate $|x|^p$ for some $p \in (0,2]$. They proved that the maximal displacement grows like $t^{2/(2-p)}$ if $p < 2$, and exponentially fast if $p = 2$. Further results on the growth of the population along any path have been obtained in [15, 16]. In [79, 68], the authors studied a BBM with space-inhomogeneous branching and death rate and a drift, which are tuned in a specific fashion to exhibit a Gaussian travelling wave phenomenon.

Another space-inhomogenous version of branching Brownian motion is catalytic BBM. Here particles move as Brownian motions on $\mathbb{R}$ and branch at rate $\beta_0 \, \mathrm{d}L_t$ where L_t is the local time at 0 of a given particle. In [24], it has been proved that the maximum grows at speed $\beta_0/2$ a.s. and, in [25], that after centering by $(\beta_0/2)t$ the maximum converges in distribution. Note the absence of logarithmic correction. Similar results had been obtained previously in [36] in the case of the catalytic branching random walk. Finally, in [26], a BBM with branching rate $\beta + \beta_0 \, \mathrm{d}L_t$ is studied and the linear speed is obtained.

Time-inhomogeneous BBMs. Another very related model is *time-inhomogeneous* (or *speed varying*) branching Brownian motion, and its discrete time counterpart, which has been introduced by Bovier and Kurkova in [31]. In time-inhomogenous BBM, one starts with a profile $\sigma\colon [0,1] \mapsto [0,\infty)$ and a time horizon $T > 0$. Particles then branch at rate 1 and move as Brownian motions on $\mathbb{R}$ with time-inhomogeneous instantaneous variance $\sigma^2(t/T)$ at time $t \leq T$. Note that, up to a deterministic time change, the time-inhomogeneous variance can be transformed into a time-inhomogeneous branching rate. The linear speed of the maximal displacement at time T has been obtained by [31] in the discrete-time case, for general profiles σ.

Of particular interest to us here is the case where σ is strictly decreasing, because in that case the maximal displacement exhibits a polynomial correction of order $T^{1/3}$, as proved by [47] for the BBM and [72] for the branching random walk. Maillard an Zeitouni [71] proved the tightness of the maximum after centering by $m(T) = v_\sigma T - w_\sigma T^{1/3} - \frac{\sigma(1)}{\sqrt{2}} \log T$ for some explicit constants v_σ, w_σ. Hence, our space-inhomogeneous model provides another occurrence of such a polynomial slowdown for the maximal displacement. However, two differences can be noticed. First, our model is consistently defined for all time, it does not depend on the time horizon T. Second, all exponents in $(0,1)$ for the polynomial corrections can be obtained in our model, not only $1/3$: indeed, Theorem 2.1.1 includes all exponents in $(0,1/2)$ but the case $\alpha \in (0,2/3]$ should include the exponents in $[1/2,1)$ as explained in Section 2.1.4.

Other cases for the choice of σ have been studied, without any polynomial correction appearing in $m(T)$, but various possible coefficients for the logarithmic correction [46, 28, 29, 30, 3]. Similarly, we believe that, for our model in the case $\alpha \geq 2$, $m(t)$ should not include a polynomial correction but a logarithmic correction with any coefficient in $(1/\sqrt{2}, \infty)$, see Section 2.1.4 for details.

2.1.3 Heuristics and structure of the paper

Theorem 2.1.1 says that the maximal displacement at time t is found at a distance of $m(t) + O(1)$ from the origin. In this section, we discuss informally how to arrive at the right polynomial correction in $m(t)$ and describe the broad strategy of our proof.

We want to find $m(t)$ such that $\mathbb{P}(M_t \geq m(t))$ is strictly between 1 and 0. By a union bound

$$\mathbb{P}(M_t \geq m(t)) \leq \mathbb{E}[\#\{u \in \mathcal{N}(t) : R_u(t) \geq m(t)\}],$$

and so we want to find $m(t)$ so that this expectation is of order 1. Note that this union bound is not sharp enough to get the logarithmic correction in $m(t)$, but is sufficient to predict the polynomial term. By linearity we have that

$$\mathbb{E}\left[\#\{u \in \mathcal{N}(t) : R_u(t) \geq m(t)\}\right] = \mathbb{E}\left[\exp\left(\int_0^t b(\theta_s)\,\mathrm{d}s\right) \mathbb{1}_{R_t \geq m(t)}\right], \tag{2.2}$$

where $(R_t, \theta_t)_{t\geq 0}$ are the polar coordinates of a single two-dimensional Brownian motion of coordinates $(X_t, Y_t)_{t\geq 0}$ (so that X and Y are two independent standard Brownian motions).

Because b reaches its maximum at 0 only, we expect that the maximal displacement will occur in the direction $\theta = 0$. This means that, on the event $\{R_t \geq m(t)\}$, we expect that $X_t \approx R_t \approx \sqrt{2}t$ while $Y_t = o(t)$. The most efficient way for (X, Y) to achieve this is to have $X_s \approx \sqrt{2}s$ and $Y_s = o(s)$ for all $s \in [s_0, t]$ with some large but fixed s_0, meaning that the angle θ_s can be approximated by $Y_s/(\sqrt{2}s)$. Therefore, the expectation above should be close to the factorised expression

$$\mathbb{E}\left[\exp\left(\int_0^t b(\theta_s)\,\mathrm{d}s\right)\mathbb{1}_{R_t\geq m(t)}\right] \approx \mathbb{E}\left[\exp\left(\int_{s_0}^t b\left(\frac{Y_s}{\sqrt{2}s}\right)\mathrm{d}s\right)\right]\mathbb{P}\Big(X_t \geq m(t)\Big). \tag{2.3}$$

Hence, using that $b(\theta) = 1 - \beta\,|\theta|^\alpha + O(\theta^2)$ as $\theta \to 0$ by Assumption (A2) and neglecting the error term, we should choose $m(t)$ so that

$$e^t \cdot \mathbb{E}\left[\exp\left(-\beta\int_{s_0}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha \mathrm{d}s\right)\right]\cdot\mathbb{P}\Big(X_t > m(t)\Big) \approx 1. \tag{2.4}$$

Gaussian tail estimates yield $\mathbb{P}(X_t > m(t)) = \exp(-m(t)^2/2t + O(\log t))$. On the other hand, we claim that

$$\mathbb{E}\left[\exp\left(-\beta\int_{s_0}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha \mathrm{d}s\right)\right] = \exp\left(-ct^{(2-\alpha)/(2+\alpha)} + O(\log t)\right), \tag{2.5}$$

for some $c > 0$. With this, (2.4) becomes

$$\exp\left(t - ct^{(2-\alpha)/(2+\alpha)} - \frac{m(t)^2}{2t} + O(\log t)\right) \approx 1, \tag{2.6}$$

and, solving for $m(t)$, we obtain $m(t) = \sqrt{2}t - \frac{c}{\sqrt{2}}t^{(2-\alpha)/(2+\alpha)} + O(\log t)$.

Let us argue informally where (2.5) comes from. By the Feynman–Kac formula, the expectation in (2.5) is connected to solutions at time $r = 1 - \frac{s_0}{t}$ of the following PDE on $[0, 1) \times \mathbb{R}$:

$$\partial_r u = \varrho\left(\partial^2_{xx}u - (1-r)^{-\alpha}\,|x|^\alpha\,u\right), \tag{2.7}$$

where $\varrho = c(\alpha, \beta)t^{(2-\alpha)/(2+\alpha)}$. The exponential decay of these solutions is governed by the largest eigenvalue of the operator on the right-hand side of (2.7), which is proportional to ϱ and thus leads to (2.5). The exponent $(2-\alpha)/(2+\alpha)$ arises from the scaling properties of $\int_{s_0}^t |\frac{Y_s}{\sqrt{2}s}|^\alpha\,\mathrm{d}s$. Note our treatment of the time–inhomogenous PDE is inspired in large part by the approach of Maillard and Zeitouni [71].

There is another probabilistic way of seeing why $\exp(-ct^{(2-\alpha)/(2+\alpha)})$ is the right decay. The expectation in (2.5) is governed by a trade-off between the advantage for Y to stay close to zero in order to minimize the integral in the exponential, and the cost of doing so. We can expect that the optimal strategy is of the following form $A_\gamma = \{\forall s \in [s_0, t], |Y_s| \leq s^\gamma\}$ for some parameter $\gamma \in [0, 1/2]$ which we need to optimize. Note that, by the scaling property of Brownian motion, the cost of satisfying the constraint $|Y_s| \leq s^\gamma$ is roughly the same on any interval of the form $[r, r+r^{2\gamma}]$ for r large enough. Moreover, the number of such intervals needed to cover $[s_0, t]$ is proportional to $t^{1-2\gamma}$, so we deduce that $\mathbb{P}(A_\gamma) \approx \exp(-c_1 t^{1-2\gamma})$. On the other hand, on the event A_γ, we have $\int_{s_0}^t |\frac{Y_s}{\sqrt{2}s}|^\alpha \, \mathrm{d}s \approx c_2 t^{[1+\alpha(\gamma-1)]\vee 0}$. Therefore, we expect that

$$\mathbb{E}\left[\exp\left(-\beta \int_{s_0}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha \mathrm{d}s\right)\right] \approx \sup_{\gamma \in [0,1/2]} \exp\left(-c_1 t^{1-2\gamma} + c_2 t^{[1+\alpha(\gamma-1)]\vee 0}\right) \approx \exp\left(-c t^{(2-\alpha)/(2+\alpha)}\right),$$

where the supremum is achieved by $\gamma = \alpha/(2+\alpha)$. While we don't use this approach in our proofs, this heuristic tells us the scale of Y_s that should dominate the integral, namely $s^{\alpha/(2+\alpha)}$. This will be reflected in our proofs, especially in Section 2.3.

The structure of this paper follows these heuristics. In Section 2.2 we make the connection between (2.5) and the PDE (2.7) rigorous, and we show precise estimates on its solutions using Sturm–Liouville theory as well as methods inspired from [71]. Then, in Section 2.3, in order to incorporate the error term due to the approximation $b(\theta_s) \approx 1 - |\frac{Y_s}{\sqrt{2}s}|^\alpha$, we deal with more general expectations of the form

$$\mathbb{E}\left[\exp\left(-\beta \int_{s_0}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha (1 + f(Y_s, s)) \, \mathrm{d}s\right)\right], \tag{2.8}$$

where f is thought of as a small error term. The PDE approach we use in Section 2.2 does not allow us to incorporate f directly, so instead we use probabilistic methods to justify afterwards that the contribution of this error term is negligible. The main idea is to localize the trajectory of Y by showing that, on the event mainly contributing to this expectation, Y_s stays of order $s^{\alpha/(2+\alpha)}$, so that we can bound the contribution of the error term deterministically and compare with the case $f = 0$ covered in Section 2.2. We briefly state the many–to–one and many–to–two lemmas in Section 2.4, before turning to the proof of Theorem 2.1.1. In Section 2.5, we show the upper bound in Theorem 2.1.1 relying essentially on a first moment calculation. The main difficulty is to localize the trajectory of a particle u such that $R_u(t) \geq m(t)$, to be able to justify the approximations $\theta_u(s) \approx Y_u(s)/(\sqrt{2}s)$ and $R_u(t) \approx X_u(t)$ and then to add more precise barrier on the X-coordinate to determine the logarithmic corrections. In Section 2.6, we prove the lower bound in Theorem 2.1.1 by performing a first and second moment calculation on the number of particles such that $X_u(t) \geq m(t)$, while satisfying appropriate constraints on their trajectories.

2.1.4 Open questions and remarks

1. In Theorem 2.1.1 we show tightness for $(M_t - m(t))_{t\geq 1}$. Can this be extended to convergence in distribution? For this, one needs a quantity replacing the limit of the derivative martingale in (2.1). The following process should work:

$$W_t := t^{-\alpha/[2(2+\alpha)]} \exp\left(\vartheta_1 t^{(2-\alpha)/(2+\alpha)}\right) \\ \times \sum_{u\in\mathcal{N}(t)} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}(X_u(t)-\sqrt{2}t)} \varphi_0\left(\frac{\vartheta_2 Y_u(t)}{t^{\alpha/(2+\alpha)}}\right),$$

where φ_0, ϑ_1 and ϑ_2 are introduced in Proposition 2.2.1. Note that the dependence in $X_u(t)$ is similar to the definition of the derivative martingale. However, here $(W_t)_{t\geq 0}$ is not an exact martingale, but only satisfies $\mathbb{E}[W_t|\mathcal{F}_s] = W_s + o(1)$ for s and t large. We believe that $(W_t)_{t\geq 0}$ has a limit (in probability at least) which plays the role of D_∞ in the convergence in distribution of $M_t - m(t)$. To go further, we can also ask the question of the convergence of the extremal process.

2. In this work, we focus on the case $\alpha \in (2/3, 2)$. We leave the other cases as open questions and briefly discuss here what behaviour we expect for $m(t)$.

($\alpha = 2$) This is a natural case as this corresponds most naturally to the assumption that b is smooth at $\theta = 0$. Furthermore, as $\alpha \nearrow 2$ we have $\frac{2-\alpha}{2+\alpha} \to 0$, so we expect no polynomial correction. Here, one can show (for example using the methods of [49]) that

$$\mathbb{E}\left[\exp\left(-\beta \int_s^t \left(\frac{Y_r}{r}\right)^2 \mathrm{d}r\right)\right] \sim C(\beta)\left(\frac{s}{t}\right)^{\frac{1}{4}(\sqrt{1+8\beta}-1)}, \tag{2.9}$$

as $t/s \to \infty$, $C(\beta)$ can be stated explicitly. This strongly suggests that the appropriate correction for the maximum is

$$m(t) = \sqrt{2}t - \left(\frac{3}{2\sqrt{2}} + \frac{\sqrt{1+8\beta}-1}{4\sqrt{2}}\right)\log t. \tag{2.10}$$

This guess relies on the fact that trajectories of extremal particles should satisfy $Y_t = O(\sqrt{t})$ and therefore $R_t = X_t + O(1)$. The coefficient of the logarithmic correction contains two terms: the first one comes from the fact that the X-trajectory has to stay below a straight line (as for the 1d BBM), whereas the second one comes from the constraint on the Y-trajectory and (2.9). Note that this coefficient interpolates between $-3/(2\sqrt{2})$ as $\beta \to 0$ and $-\infty$ as $\beta \to \infty$.

($\alpha > 2$) In this case, the constraint on the Y-trajectory of an extremal particle is weaker: if $|Y_s|$ grows at most like s^γ with $\gamma < 1 - \frac{1}{\alpha}$, then the integral $\int_{s_0}^t |\frac{Y_s}{s}|^\alpha \, \mathrm{d}s$ does not diverge and everything behaves as if the branching rate was equal to 1. Therefore, a good proxy for our model would be a 2d BBM with branching rate 1, but where only extremal particles with an angle $\theta_t = O(t^{-1/\alpha})$ are taken into account. But heuristics coming from the study of 2d BBM [72] tell us that the behaviour of extremal particles in two disjoint cones of angle $t^{-1/2}$ is roughly independent. Altogether, this suggests that $m(t)$ should be the same as for the maximum of $t^{1/2-1/\alpha}$ independent 1d BBMs, which leads to

$$m(t) = \sqrt{2}t - \frac{1}{\sqrt{2}} \left(1 + \frac{1}{\alpha}\right) \log t.$$

Note that the coefficient of the logarithmic correction interpolates between $-3/(2\sqrt{2})$ and $-1/(2\sqrt{2})$, that is between the 1d and the 2d BBM.

($\alpha \leq \frac{2}{3}$) Results in Sections 2.2 and 2.3 concerning Brownian motion weighted by an integral are proved for any $\alpha \in (0,2)$, so in particular (2.5) still holds for $\alpha \leq 2/3$. Hence, we believe that (a weak version of) our methods should be enough to prove

$$M_t = \sqrt{2}t - \frac{\vartheta_1}{\sqrt{2}} t^{(2-\alpha)/(2+\alpha)} \left(1 + o_{\mathbb{P}}(1)\right).$$

However, getting the asymptotic expansion of M_t up to $O(1)$ seems much harder and we do not have a precise conjecture for the right choice of $m(t)$ in that case. More polynomial terms probably need to be added in the definition of $m(t)$, for several reasons listed here:

- A simple reason already appears in the simplified heuristics that led to (2.6). Indeed, solving (2.6) for $\alpha < 2/3$ requires us to add more and more polynomial terms to $m(t)$ as α decreases. However, as explained in the following points, (2.6) is not the right choice for $m(t)$ anymore.
- Because $m(s)$ is roughly the position of extremal particles at time s, an extremal particle u at time t has to satisfy $X_u(s) \leq m(s)$ for any $s \leq t$. If $\alpha \in (2/3, 2)$, the exponent $(2-\alpha)/(2+\alpha)$ is less than $1/2$ and staying below this barrier has the same cost as staying below a straight barrier with the same endpoints, that is a polynomial cost, which is responsible for the $-\frac{3}{2\sqrt{2}} \log t$ correction to $m(t)$. On the other hand, if $\alpha \in (0, 2/3)$, this exponent becomes larger than $1/2$ and staying

below this barrier has a stretched exponential cost, which results in a polynomial correction to $m(t)$.

   - Lastly, the approximation $\theta_s \approx Y_s/X_s \approx Y_s/(\sqrt{2}s)$ used in the heuristics in Section 2.1.3 is not good enough anymore. Indeed, we expect that $X_s - \sqrt{2}s$ is of order $s^{(2-\alpha)/(2+\alpha)}$, and therefore the resulting error term in the integral in (2.5) is not negligible when $\alpha \leq 2/3$: instead, it should grow polynomially fast and therefore result in another polynomial correction to $m(t)$.

3. We finally mention two models that can be seen as a limit as $\alpha \to 0$ of our model. One of them would be a 2d BBM in which a particle at (x, y) branches at rate $b(y)$, where b is continuous, $b(0) = 1$ and $b(y) \to 0$ as $|y| \to \infty$. In that case, we expect that the linear order of $m(t)$ would depend on b and would be smaller than $\sqrt{2}$, but there would be no polynomial correction, only a logarithmic one. We believe that a lot of the analysis would rely on understanding BBM in a strip $\mathbb{R} \times [-L, L]$ using the results of [55]. Another interesting model might be a catalytic 2d BBM, where a particle branches at rate $\beta + \beta_0 L_t^x$, where L_t^x is the local time of the second coordinate of the particle at 0.

2.1.5 Notation

Throughout the paper, C and c denote positive constants that can change between occurrences. They can depend on some other parameters in a way which is made clear in the statement of each result; in the proof of such a result allowed dependencies are the same as in the statement. We use standard Landau notation: for $f\colon [0,\infty) \to \mathbb{R}$ and $g\colon [0,\infty) \to (0,\infty)$, we say, as $t \to \infty$, that $f(t) = o(g(t))$ if $\lim_{t\to\infty} f(t)/g(t) = 0$, that $f(t) = O(g(t))$ if $\limsup_{t\to\infty} |f(t)|/g(t) < \infty$, and that $f(t) \sim g(t)$ if $\lim_{t\to\infty} f(t)/g(t) = 1$.

It is sometimes convenient to work under probability measures other than $\mathbb{P}$. We denote by $\mathbb{P}_{(s,x,y)}$ the probability under which our inhomogeneous BBM starts at time s from a single particle located at (x, y). Moreover, under $\mathbb{P}_{(s,x,y)}$, $(X_t, Y_t)_{t\geq 0}$ denotes a 2d Brownian motion starting at time s from (x, y). If $s = 0$, we drop the first index and simply write $\mathbb{P}_{(x,y)}$. If the event considered involves only X, then we keep only the starting point of X as index; and similarly for Y.

2.2 Brownian motion weighted by an integral via PDEs

We define the kernel G for $x, y \in \mathbb{R}$ and $0 \leq s < t$ by

$$G(s,x;t,y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right) \mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha \mathrm{d}r\right) \middle| B_t = y\right], \tag{2.11}$$

where, under $\mathbb{P}_{(s,x)}$, B is a Brownian motion starting from x at time s. Note that under $\mathbb{P}_{(s,x)}$ and given $B_t = y$, $(B_r)_{r\in[s,t]}$ is a Brownian bridge from (s,x) to (t,y). The kernel G satisfies, for any measurable function $f \colon \mathbb{R} \to \mathbb{R}_+$,

$$\int_{\mathbb{R}} f(y) G(s,x;t,y)\,\mathrm{d}y = \mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha \mathrm{d}r\right) f(B_t)\right]. \tag{2.12}$$

The goal of this section is to prove the following result.

Proposition 2.2.1. *Let $\alpha \in (0,2)$, $\kappa := 2\alpha/(2+\alpha) \in (0,1)$ and $\beta > 0$. There exist $K, C, c > 0$ such that, for any $x, y \in \mathbb{R}$, $s \geq K$ and $t \geq s + Ks^\kappa$, we have*

$$\begin{aligned}&\left|(st)^{\kappa/4} \exp\left(\vartheta_1(t^{1-\kappa} - s^{1-\kappa})\right) G(s,x;t,y) - \vartheta_2 \varphi_0\left(\frac{\vartheta_2 x}{s^{\kappa/2}}\right) \varphi_0\left(\frac{\vartheta_2 y}{t^{\kappa/2}}\right)\right| \\ &\leq C\left(s^{\kappa-1} + e^{-c(t-s)/t^\kappa}\right) \exp\left(-c\left(\frac{|x|}{s^{\kappa/2}}\right)^\alpha\right) \exp\left(-c\left(\frac{|y|}{t^{\kappa/2}}\right)^\alpha\right)\end{aligned}$$

where φ_0 is defined in Section 2.2.2 together with the constant $\lambda_0 > 0$ (and they only depend on α) and we set $\vartheta_1 := \lambda_0 \beta^{2/(2+\alpha)} 2^{-2\alpha/(2+\alpha)}/(1-\kappa)$ and $\vartheta_2 := (2\beta)^{1/(2+\alpha)}$.

By Proposition 2.2.7, φ_0 is an even positive function such that $\|\varphi_0\|_2 = 1$. See Figure 2.3 for an illustration.

By integrating these bounds in the y coordinate and bounding them uniformly in the x coordinate we obtain the following corollary.

Corollary 2.2.2. *Let $\alpha \in (0,2)$ and $\beta > 0$. There exist $K, C, c > 0$ such that the following holds for any $s \geq K$ and $t \geq s + Ks^\kappa$, the following holds.*

1. *For any $x \in \mathbb{R}$,*

$$\int_{\mathbb{R}} G(s,x;t,y)\,\mathrm{d}y \leq C(t/s)^{\kappa/4} \exp\left(\vartheta_1(s^{1-\kappa} - t^{1-\kappa})\right).$$

2. *For any $|x| \leq s^{\kappa/2}$,*

$$\int_{|y|\leq t^{\kappa/2}} G(s,x;t,y)\,\mathrm{d}y \geq c(t/s)^{\kappa/4} \exp\left(\vartheta_1(s^{1-\kappa} - t^{1-\kappa})\right).$$

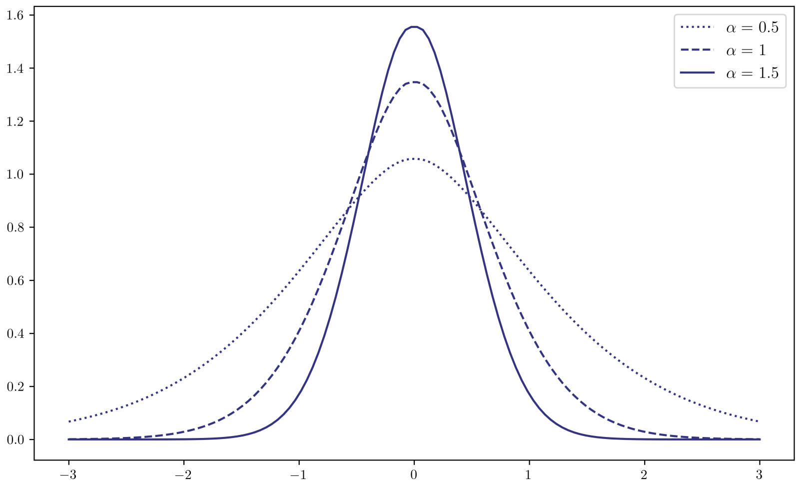

Figure 2.3: A plot of φ_0 for different values of α.

2.2.1 Connection with a PDE via Feynman–Kac formula

In this section, we first work in a slightly more general context and consider the following PDE on $[0,T]\times\mathbb{R}$, for some $T,\sigma>0$ and some function $k\colon[0,T]\times\mathbb{R}\to\mathbb{R}$,

$$\partial_t u = \frac{\sigma^2}{2}\partial^2_{xx}u + ku. \tag{2.13}$$

We write $\mathcal{C}^{1,2}(I\times\mathbb{R})$ for the space of functions u on $I\times\mathbb{R}$ such that $\partial_t u$, $\partial_x u$ and $\partial^2_{xx}u$ exists and are continuous on $I\times\mathbb{R}$. A *fundamental solution* Γ of (2.13) is a function of $(\tau,\xi;t,x)$ defined for any $0\le\tau<t\le T$ and any $\xi,x\in\mathbb{R}$ such that, for any $\tau\in[0,T)$,

- for any $\xi\in\mathbb{R}$, the function $\Gamma(\tau,\xi;\cdot,\cdot)$ is in $\mathcal{C}^{1,2}((\tau,T]\times\mathbb{R})$ and satisfies (2.13) on $(\tau,T]\times\mathbb{R}$;
- for any $f\colon\mathbb{R}\to\mathbb{R}$ continuous with compact support and any $x_0\in\mathbb{R}$,

$$\lim_{(t,x)\to(\tau^+,x_0)}\int_{\mathbb{R}}\Gamma(\tau,\xi;t,x)f(\xi)\,\mathrm{d}\xi = f(x_0). \tag{2.14}$$

The following result shows existence of a fundamental solution under appropriate conditions on the function k and expresses it in terms of a Brownian motion.

Lemma 2.2.3. *Assume $k\le 0$ and k is Hölder continuous on any compact subset of*

$[0,T]\times\mathbb{R}$. *For any* $0\le\tau<t\le T$ *and any* $\xi,x\in\mathbb{R}$, *let*

$$\begin{aligned}\Gamma(\tau,&\xi;t,x)\\ &:=\frac{1}{\sqrt{2\pi\sigma^2(t-\tau)}}\exp\left(-\frac{(\xi-x)^2}{2\sigma^2(t-\tau)}\right)\mathbb{E}_{(\tau,\frac{x}{\sigma})}\left[\exp\left(\int_\tau^t k(t+\tau-r,\sigma B_r)\,\mathrm{d}r\right)\middle|\sigma B_t=\xi\right]\\ &=\frac{1}{\sqrt{2\pi\sigma^2(t-\tau)}}\exp\left(-\frac{(\xi-x)^2}{2\sigma^2(t-\tau)}\right)\mathbb{E}_{(\tau,\frac{\xi}{\sigma})}\left[\exp\left(\int_\tau^t k(r,\sigma B_r)\,\mathrm{d}r\right)\middle|\sigma B_t=x\right].\end{aligned}$$

Then, Γ *is a fundamental solution of the PDE* (2.13).

Proof. We first mention that the fact that the two expressions for $\Gamma(\tau,\xi;t,x)$ are equal follows from the fact that if $(P_s)_{s\in[\tau,t]}$ is a Brownian bridge from $(\tau,x/\sigma)$ to $(t,\xi/\sigma)$, then $(P_{t+\tau-s})_{s\in[\tau,t]}$ is a Brownian bridge from $(\tau,\xi/\sigma)$ to $(t,x/\sigma)$.

Let $\tau\in[0,T)$ and $f\colon\mathbb{R}\to\mathbb{R}$ be continuous and bounded. Then, by [6, Theorem III] applied with $\lambda=0$ and $\alpha>0$ small enough[1], there exists a solution $u\in\mathcal{C}^0([\tau,T]\times\mathbb{R})\cap\mathcal{C}^{1,2}((\tau,T]\times\mathbb{R})$ to the PDE (2.13) with initial condition $u(\tau,x)=f(x)$ for $x\in\mathbb{R}$, and there exists $C>0$ such that

$$\forall(t,x)\in[0,T]\times\mathbb{R},\quad |u(t,x)|\le C\left(|x|^2+1\right).$$

Therefore, we can apply Feynman–Kac formula [62, Theorem 7.6] to $v(t,x)=u(T+\tau-t,x)$ to get, for any $t\in[\tau,T]$ and $x\in\mathbb{R}$,

$$\begin{aligned}u(t,x)&=\mathbb{E}_{(\tau,x/\sigma)}\left[\exp\left(\int_\tau^t k(t+\tau-r,\sigma B_r)\,\mathrm{d}r\right)f(\sigma B_t)\right]\\ &=\int_{\mathbb{R}}\Gamma(\tau,\xi;t,x)f(\xi)\,\mathrm{d}\xi,\end{aligned}\tag{2.15}$$

by conditioning on $\sigma B_t=\xi$ and using the first expression for $\Gamma(\tau,\xi;t,x)$. This, together with $u\in\mathcal{C}^0([\tau,T]\times\mathbb{R}))$ and $u(\tau,\cdot)=f$, proves that Γ satisfies (2.14).

Now, fix some $(\tau,\xi)\in[0,T)\times\mathbb{R}$. It remains to check that $\Gamma(\tau,\xi;\cdot,\cdot)$ is in $\mathcal{C}^{1,2}((\tau,T]\times\mathbb{R})$ and satisfies (2.13) on $(\tau,T]\times\mathbb{R}$. For that consider $s\in(\tau,t)$. As a consequence of Markov's property at time s together with the second expression for $\Gamma(\tau,\xi;t,x)$, we have

$$\Gamma(\tau,\xi;t,x)=\int_{\mathbb{R}}\Gamma(\tau,\xi;s,y)\Gamma(s,y;t,x)\,\mathrm{d}y,$$

But, the function $y\mapsto\Gamma(\tau,\xi;s,y)$ is bounded and continuous: this is shown using $k\le0$, continuity of k and the fact that if $(P_r)_{r\in[\tau,s]}$ is a Brownian bridge from $(\tau,\xi/\sigma)$

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[1] First note that their assumptions stated in [6, Eq. (2.1)-(2.3)] are satisfied with $\nu=k_1=\sigma^2/2$, $\lambda=0$ and any $k_2,k_3>0$. Then, existence of u is proved up to time $T_\alpha=\min(T,1/(2\beta(\alpha)))$ (see before Eq. (2.4) there) and $\beta(\alpha)$ can be made arbitrarily large by choosing α,k_2,k_3 small enough, so that we get $T_\alpha=T$.

to $(s,0)$, then $(P_r+y(r-\tau)/(s-\tau))_{r\in[\tau,s]}$ is a Brownian bridge from $(\tau,\xi/\sigma)$ to (s,y). Therefore, by (2.15) and the discussion before, $\Gamma(\tau,\xi;\cdot,\cdot)$ is in $\mathcal{C}^{1,2}((s,T]\times\mathbb{R})$ and satisfies (2.13) on $(s,T]\times\mathbb{R}$. Since this is true for any $s\in(\tau,T)$, this concludes the proof. □

Using this connection, the proof of Proposition 2.2.1 boils down to the study of the following PDE on $[0,1)\times\mathbb{R}$

$$\partial_t u = \varrho\left(\partial^2_{xx}u-(1-t)^{-\alpha}\,|x|^\alpha\,u\right), \tag{2.16}$$

where $\varrho>0$ is a parameter meant to be large. More precisely, let g denote the fundamental solution of (2.16) given by Lemma 2.2.3 (it is defined consistently on $[0,T]$ for any $T<1$). The remainder of Section 2.2 is dedicated to the proof of the following result.

Proposition 2.2.4. *Let $\alpha\in(0,2)$ and $\kappa:=2\alpha/(2+\alpha)\in(0,1)$. There exist $C,c>0$ such that, for any $\varrho\geq 40$, $x,\xi\in\mathbb{R}$, $T\in(0,1)$ and $\delta>0$ satisfying*

$$T\geq\frac{20}{\varrho},\qquad \varrho(1-T)^{1-\kappa}\geq 10,\qquad \frac{1}{\varrho(1-T)^{1-\kappa}}\leq\delta\leq\frac{1}{10}\wedge\frac{T^2\varrho}{100},$$

we have

$$\begin{aligned}\Bigg|\exp\left(\lambda_0\varrho\int_0^T\frac{\mathrm{d}s}{(1-s)^\kappa}\right)&g(0,\xi;T,x)-\varphi_0(\xi)\cdot(1-T)^{-\kappa/4}\varphi_0\left((1-T)^{-\kappa/2}x\right)\Bigg|\\
&\leq C\left(\delta+e^{-c\varrho T}\right)(1-T)^{-\kappa/4}\left(e^{-c|\xi|^{(2+\alpha)/2}}+e^{-c(\varrho\delta)^{1/2}(1+|\xi|^\alpha)}\right)\\
&\quad\times\Bigg[\exp\left(-c\left(\frac{|x|}{(1-T)^{\kappa/2}}\right)^{(2+\alpha)/2}\right)\\
&\qquad\qquad+\exp\left(-c(\varrho(1-T)^{1-\kappa}\delta)^{1/2}\left(1+\left(\frac{|x|}{(1-T)^{\kappa/2}}\right)^\alpha\right)\right)\Bigg],\end{aligned}$$

where λ_0, φ_0 are defined in Proposition 2.2.7.

Remark 2.2.5. If the constraints on ϱ and T are satisfied, then the condition on δ is non empty, that is one can check that

$$\frac{1}{\varrho(1-T)^{1-\kappa}}\leq\frac{1}{10}\wedge\frac{T^2\varrho}{100}.$$

The fact that the left-hand side is at most $1/10$ is direct. To prove that it is less than $T^2\varrho/100$, one can distinguish two cases: if $(1-T)^{1-\kappa}>1/2$, it follows from $T\geq 20/\varrho$ and, in the opposite case, one has necessarily $T\geq 1/2$ and so it follows from $\varrho(1-T)^{1-\kappa}\geq 10$ and $\varrho\geq 40$.

As a consequence of this remark, it is always allowed to take $\delta = 1/(\varrho(1-T)^{1-\kappa})$ in the previous result, if the other assumptions on ϱ and T are satisfied. This gives the following simpler bound, where we do not try to optimize the tails in ξ and x.

Corollary 2.2.6. *Let $\alpha \in (0,2)$ and $\kappa \coloneqq 2\alpha/(2+\alpha) \in (0,1)$. There exist $C, c > 0$ such that, for any $\varrho \geq 40$, $x, \xi \in \mathbb{R}$, $T \in (0,1)$ and $\delta > 0$ satisfying $T \geq 20/\varrho$ and $\varrho(1-T)^{1-\kappa} \geq 10$, we have*

$$\begin{aligned}&\left|\exp\left(\lambda_0 \varrho \int_0^T \frac{\mathrm{d}s}{(1-s)^\kappa}\right) g(0,\xi;T,x) - \varphi_0(\xi)\cdot(1-T)^{-\kappa/4}\varphi_0\left((1-T)^{-\kappa/2}x\right)\right| \\ &\leq C\left(\frac{1}{\varrho(1-T)^{1-\kappa}} + e^{-c\varrho T}\right)(1-T)^{-\kappa/4} e^{-c|\xi|^\alpha} \exp\left(-c\left(\frac{|x|}{(1-T)^{\kappa/2}}\right)^\alpha\right).\end{aligned}$$

Before diving into the proof of Proposition 2.2.4, we first explain how to deduce Proposition 2.2.1 from the corollary above.

Proof of Proposition 2.2.1. By definition of G in (2.11) and the scaling property of Brownian motion, one has, for any $0 < s < t$ and $x, y \in \mathbb{R}$,

$$\begin{aligned}&\sqrt{t}G\left(s, x\sqrt{t}; t, y\sqrt{t}\right) \\ &\quad= \frac{\sqrt{t}}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2 t}{2(t-s)}\right) \mathbb{E}_{(s/t,x)}\left[\exp\left(-\beta t^{1-\alpha/2}\int_{s/t}^1 \left|\frac{B_r}{\sqrt{2}r}\right|^\alpha \mathrm{d}r\right)\middle| B_1 = y\right] \\ &\quad= \frac{1}{\sqrt{2\pi(1-\tau)}} \exp\left(-\frac{(y-x)^2}{2(1-\tau)}\right) \mathbb{E}_{(\tau,x)}\left[\exp\left(-\varrho\int_\tau^1 \left|\frac{\sqrt{2\varrho}B_r}{r}\right|^\alpha \mathrm{d}r\right)\middle| B_1 = y\right].\end{aligned}$$

setting $\tau = s/t$ and ϱ such that $\beta t^{1-\alpha/2} 2^{-\alpha/2} = \varrho(2\varrho)^{\alpha/2}$, i.e. $\varrho = \beta^{2/(2+\alpha)} 2^{-2\alpha/(2+\alpha)} t^{1-\kappa}$. On the other hand, using the first expression for g given by Lemma 2.2.3 with $\sigma = \sqrt{2\varrho}$, we have

$$\begin{aligned}&g(0,y;1-\tau,x) \\ &= \frac{1}{\sqrt{2\pi\sigma^2(1-\tau)}} \exp\left(-\frac{(y-x)^2}{2\sigma^2(1-\tau)}\right) \mathbb{E}_{(0,x/\sigma)}\left[\exp\left(-\varrho\int_0^{1-\tau}\left|\frac{\sigma B_r}{\tau+r}\right|^\alpha \mathrm{d}r\right)\middle|\sigma B_{1-\tau} = y\right].\end{aligned}$$

Doing a time shift by τ for the Brownian motion in the last expectation, we obtain the following relation

$$G(s,x;t,y) = \sqrt{\frac{2\varrho}{t}} g\left(0, \sqrt{\frac{2\varrho}{t}}y; 1-\frac{s}{t}, \sqrt{\frac{2\varrho}{t}}x\right). \tag{2.17}$$

Then, the result follows from Corollary 2.2.6 with ϱ given above and $T = 1 - \frac{s}{t}$. □

2.2.2 Associated Sturm–Liouville operators

Fix $\alpha > 0$. An important tool in the study of the PDE (2.16) is the following family of Sturm–Liouville operators: for any $q > 0$, let $\mathcal{L}_q$ be defined by

$$(\mathcal{L}_q f)(x) \coloneqq -f''(x) + q\,|x|^\alpha\, f(x), \quad x \in \mathbb{R},$$

for any f in the domain

$$D(\mathcal{L}_q) \coloneqq \{f \in L^2(\mathbb{R}) : f, f' \text{ absolutely continuous on } \mathbb{R}, \mathcal{L}_q f \in L^2(\mathbb{R})\}.$$

These operators are densely defined on $L^2(\mathbb{R})$ [85, Lemma 9.4] and self-adjoint [85, Theorem 9.6]. In the case $q = 1$, we simply write $\mathcal{L} = \mathcal{L}_1$.

Proposition 2.2.7. *Let $\alpha > 0$.*

1. *The spectrum of $\mathcal{L}$ is discrete and consists of the eigenvalues $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ which all have multiplicity 1. Let φ_n be an eigenfunction associated to λ_n, then it has exactly n zeroes, so it can be chosen uniquely such that $\|\varphi_n\|_2 = 1$ and $\varphi_n(x) > 0$ for any x greater than its largest zero. Moreover, $(\varphi_n)_{n\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R})$.*

2. *Let $q > 0$. The previous point also holds for $\mathcal{L}_q$ with eigenvalues $(\lambda_{q,n})_{n\geq 0}$ and eigenfunctions $(\varphi_{q,n})_{n\geq 0}$ such that, for any $n \geq 0$ and $x \in \mathbb{R}$,*

$$\lambda_{q,n} = q^{2/(2+\alpha)}\lambda_n \qquad \text{and} \qquad \varphi_{q,n}(x) = q^{1/[2(2+\alpha)]}\varphi_n(q^{1/(2+\alpha)}x). \tag{2.18}$$

3. *Let $c_\alpha \coloneqq \frac{2}{\pi}\int_0^1 \sqrt{1-u^\alpha}\,\mathrm{d}u$. Then, as $n \to \infty$,*

$$\lambda_n \sim \left(\frac{n}{c_\alpha}\right)^{2\alpha/(\alpha+2)}. \tag{2.19}$$

4. *For any $n \geq 0$, the function φ_n has the same parity as n.*

5. *There exists $C > 0$ depending on α such that, for any $n \geq 0$ and $x \in \mathbb{R}$,*

$$|\varphi_n(x)| \leq C(n+1)^3\left[1 \wedge \exp\left(-\frac{1}{2+\alpha}\left(|x|^{(2+\alpha)/2} - Cn\right)\right)\right].$$

6. *There exist $C, c > 0$ depending on α such that, for any $x \in \mathbb{R}$, $|\varphi_0'(x)| \leq Ce^{-c|x|^{(2+\alpha)/2}}$.*

Proof. Part 1. The fact that the essential spectrum of $\mathcal{L}$ is empty follows from [13, Proposition 4.5.4] together with the fact that any self-adjoint realization of $-f'' + |x|^\alpha f$ in $L^2((0,\infty))$ or $L^2((-\infty,0))$ has empty essential spectrum by [13, Theorem 4.5.8]. Then, it follows from [13, Theorem 6.1.9] that the discrete spectrum is bounded from below so it can be written $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Moreover, eigenvalues have multiplicity 1 (this is more clearly stated in [88, Theorem 10.12.1.(8).(ii)]) and an eigenfunction φ_n associated with λ_n has n zeroes. We now chose φ_n uniquely as in the statement. The fact that $\lambda_0 > 0$ is obtained by contradiction: if $\lambda_0 \leq 0$, then $\varphi_0''(x) = (|x|^\alpha - \lambda_0)\varphi_0(x) \geq 0$ for any $x \in \mathbb{R}$ (using $\varphi_0 > 0$ because it has no zero), so φ_0 is convex in $\mathbb{R}$, which contradicts $\varphi_0 \in L^2$. Finally, $(\varphi_n)_{n\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R})$ by [85, Corollary 4.2.3].

Part 2. This follows from a direct verification.

Part 3. This is a consequence of [13, Theorem 6.1.12], see [13, Example 6.1.13, Part 5].

Part 4. Let $n \geq 0$. Since $\mathcal{L}$ preserves parity, the even and the odd parts of φ_n are also eigenfunctions associated to λ_n, so one of them must be zero because λ_n has multiplicity 1. So φ_n is either even or odd. If φ_n is odd, then it has an odd number of zeroes. If φ_n is even, then $\varphi_n'(0) = 0$ and therefore $\varphi_n(0) \neq 0$ (otherwise $\varphi_n = 0$), so φ_n has an even number of zeroes. But, φ_n has n zeroes by Part 1 so φ_n has the same parity as n.

Part 5. This part relies on some preliminary results established in Lemma 2.2.8 below. Let $n \geq 0$ and $x \geq 0$ (by parity the result for $x \leq 0$ follows). Let $b := \alpha \vee (2\lambda_n)^{1/\alpha}$. We consider the function ψ_n defined on $[b,\infty)$ by $\mathcal{L}\psi_n = \lambda_n\psi_n$ together with the initial conditions $\psi_n(b) = 1$ and $\psi_n'(b) = \frac{1}{2}b^{\alpha/2}$. Then, the Wronskian

$$W := \varphi_n\psi_n' - \varphi_n'\psi_n,$$

is a constant. Moreover, by Lemma 2.2.8.1, $\varphi_n > 0$ and $\varphi_n' < 0$ in $[b,\infty)$ and it follows from a similar argument that ψ_n is convex in $[b,\infty)$ and therefore $\psi_n' > 0$ and $\psi_n \geq 0$. Therefore, for any $x \geq b$,

$$|\varphi_n(x)| = \frac{W + \varphi_n'(x)\psi_n(x)}{\psi_n'(x)} \leq \frac{W}{\psi_n'(x)}. \tag{2.20}$$

We first bound W by taking its value at b: we have $W = \frac{1}{2}b^{\alpha/2}\varphi_n(b) - \varphi_n'(b)$. Moreover, using Lemma 2.2.8.1,

$$|\varphi_n'(b)| \leq \left|\varphi_n'(\lambda_n^{1/\alpha})\right| \leq \int_{y_0}^{\lambda_n^{1/\alpha}} |\varphi_n''(x)| \,\mathrm{d}x,$$

where y_0 is a zero of φ_n' in $[0, \lambda_n^{1/\alpha}]$ (one can see it exists using that either $\varphi_n'(0) = 0$ or $\varphi_n(0) = 0$ by Part 4). Then, using $\varphi_n''(x) = (|x|^\alpha - \lambda_n)\varphi_n(x)$, we get

$$|\varphi_n'(b)| \leq \int_{y_0}^{\lambda_n^{1/\alpha}} \lambda_n |\varphi_n(x)| \, \mathrm{d}x \leq C\lambda_n^{(\alpha+1)/\alpha}(n+1)^{\alpha/[2(2+\alpha)]},$$

applying Lemma 2.2.8.3. On the other hand, $b^{\alpha/2}\varphi_n(b) \leq C\lambda_n^{1/2}(n+1)^{\alpha/[2(2+\alpha)]}$ by Lemma 2.2.8.3 again. Hence, we get

$$W \leq C\lambda_n^{(\alpha+1)/\alpha}(n+1)^{\alpha/[2(2+\alpha)]} \leq C(n+1)^3, \tag{2.21}$$

using (2.19). Now, we aim at proving a lower bound for ψ_n'. For this, we introduce the following function

$$g(x) := \exp\left(\frac{1}{2+\alpha}\left(x^{(2+\alpha)/2} - b^{(2+\alpha)/2}\right)\right), \qquad x \geq b.$$

Then, $g'(x) = \frac{1}{2}x^{\alpha/2}g(x)$ and $g''(x) = (\frac{1}{4}x^\alpha + \frac{\alpha}{4}x^{(\alpha-2)/2})g(x)$. In particular, for any $x \geq b$, we have $g''(x) \leq \frac{1}{2}x^\alpha g(x) \leq (x^\alpha - \lambda_n)g(x)$, where the first inequality uses $b \geq \alpha$ and the second one $b \geq (2\lambda_n)^{1/\alpha}$. On the other hand, $g(b) = \psi_n(b)$ and $g'(b) = \psi_n'(b)$. Hence, we get $g \leq \psi_n$ and $g' \leq \psi_n'$ on $[b, \infty)$. Combining this with (2.20) and (2.21), we get, for any $x \geq b$,

$$|\varphi_n(x)| \leq C(n+1)^3 x^{-\alpha/2} \exp\left(-\frac{1}{2+\alpha}\left(x^{(2+\alpha)/2} - b^{(2+\alpha)/2}\right)\right).$$

The result follows by noting that $b^{(2+\alpha)/2} \leq Cn$ by (2.19) and using that we already now that $\|\varphi_n\|_\infty \leq C(n+1)^3$ by Lemma 2.2.8.3.

Part 6. By Part 5, we have $|\varphi_0(x)| \leq Ce^{-c|x|^{(2+\alpha)/2}}$ and so $|\varphi_0''(x)| \leq C|x|^\alpha e^{-c|x|^{(2+\alpha)/2}}$ using $\lambda_0 > 0$. Then, for $x \geq 0$, we have $|\varphi'(x)| \leq \int_x^\infty |\varphi_0''(y)| \, \mathrm{d}y$ by Lemma 2.2.8.2 and the result follows. ☐

Lemma 2.2.8. *1. For any $n \geq 0$, φ_n is convex and decreasing in $[\lambda_n^{1/\alpha}, \infty)$;*

2. For any $n \geq 0$, φ_n and φ_n' tend to zero at infinity;

3. There exists $C = C(\alpha) > 0$ such that, for any $n \geq 0$, $\|\varphi_n\|_\infty \leq C(n+1)^{\alpha/[2(2+\alpha)]}$.

Proof. Part 1. Note that in $[\lambda_n^{1/\alpha}, \infty)$, φ_n'' and φ_n have the same sign (considering zero has both signs). Therefore, $|\varphi_n|$ is convex in $[\lambda_n^{1/\alpha}, \infty)$. By contradiction, assume φ_n has a zero x_0 in $[\lambda_n^{1/\alpha}, \infty)$. Then, $\varphi_n'(x_0) \neq 0$ (otherwise $\varphi_n = 0$). Then, convexity of $|\varphi_n|$ implies that $|\varphi_n| \to \infty$, which contradicts $\varphi_n \in L^2$. Therefore, φ_n has no zero in $[\lambda_n^{1/\alpha}, \infty)$ and hence it is positive (by our choice of φ_n) and so it is convex. If φ_n'

takes a nonnegative value at $x_1 \in [\lambda_n^{1/\alpha}, \infty)$, then $\varphi_n(x) \geq \varphi_n(x_1)$ for all $x \geq x_1$ and this contradicts $\varphi_n \in L^2$. So $\varphi_n' < 0$ and φ_n is decreasing in $[\lambda_n^{1/\alpha}, \infty)$.

Part 2. This is a consequence of Part 1, using again that $\varphi_n \in L^2$.

Part 3. The idea is to argue that, if φ_n takes a large value at some point, then it creates some mass proportional to this value to its L^2 norm, which is fixed to 1, hence providing an upper bound for this value. Since φ_n is continuous and vanishes at infinity by Part 2, there exists $y \in \mathbb{R}$ such that $|\varphi_n(y)| = \|\varphi_n\|_\infty$ and we can assume $y \geq 0$ by parity of φ_n. Then, $\varphi_n'(y) = 0$ so, for $t \geq 0$,

$$|\varphi_n(y+t)| \geq |\varphi_n(y)| - \frac{t^2}{2} \sup_{x \in [y, y+t]} |\varphi_n''(x)| .$$

Then, on the one hand, we have $y \leq \lambda_n^{1/\alpha}$ as a consequence of Part 1. On the other hand, for $x \in [0, (2\lambda_n)^{1/\alpha}]$, $|\varphi''(x)| \leq |x^\alpha - \lambda_n| \cdot |\varphi_n(x)| \leq \lambda_n \|\varphi_n\|_\infty$. Therefore, for any $t \in [0, \lambda_n^{1/\alpha}]$,

$$|\varphi_n(y+t)| \geq \|\varphi_n\|_\infty - \frac{t^2}{2} \lambda_n \|\varphi_n\|_\infty .$$

There is a constant $c \in (0,1)$ depending only on α such that $c\lambda_n^{-1/2} \leq \lambda_n^{1/\alpha}$ for any $n \geq 0$. Then, we write

$$\begin{aligned} 1 = \|\varphi_n\|_2^2 &\geq \int_0^{c\lambda_n^{-1/2}} |\varphi_n(y+t)|^2 \,\mathrm{d}t \\ &\geq \|\varphi_n\|_\infty^2 \int_0^{c\lambda_n^{-1/2}} \left(1 - \frac{t^2}{2}\lambda_n\right) \mathrm{d}t = \|\varphi_n\|_\infty^2 \lambda_n^{-1/2} \left(c - \frac{c^3}{6}\right), \end{aligned}$$

and therefore $\|\varphi_n\|_\infty \leq C\lambda_n^{1/4}$ and the result follows from (2.19). ☐

2.2.3 Proof of Proposition 2.2.4

In this section, apart from some postponed lemmas, we prove Proposition 2.2.4 concerning the fundamental solution estimate for the PDE (2.16). We follow ideas from [71, Proposition A.2], which studies the case $\alpha = 1$ and where the function $(1-t)^{-\alpha}$ is replaced by a $\mathcal{C}^1$ function on $[0,1]$ (hence, with no explosion at 1). Moreover, we aim at getting better error terms, in particular with explicit tails in terms of x and y.

Fix some horizon $T \in (0,1)$. For $q \colon [0,T] \to (0,\infty)$ a Lipschitz continuous function, we consider the more general PDE on $[0,T] \times \mathbb{R}$

$$\partial_t u = \varrho \left(\partial_{xx}^2 u - q(t) |x|^\alpha u\right), \tag{2.22}$$

for some $\alpha > 0$ fixed and $\varrho > 0$ which is a parameter meant to be large.

Fix some $\xi \in \mathbb{R}$. Recall our aim is to estimate $g(0,\xi;T,x)$. To do this, we study the PDE (2.22) with $q = q_*$ or $q = q^*$, where q_* and q^* are well-chosen functions on $[0,T]$, given by Lemma 2.2.9, which satisfy in particular $q_*(t) \leq (1-t)^{-\alpha} \leq q^*(t)$. Then, if g_q denotes the fundamental solution of the PDE (2.22) given by Lemma 2.2.3, it follows from this probabilistic representation that

$$\forall t \in (0,T], \quad \forall x \in \mathbb{R}, \quad g_{q^*}(0,\xi;t,x) \leq g(0,\xi;t,x) \leq g_{q_*}(0,\xi;t,x), \tag{2.23}$$

so it is enough to estimate g_q for $q = q_*$ or $q = q^*$.

Fix some Lipschitz continuous function $q\colon [0,T] \to (0,\infty)$. Omitting the dependence in ξ which is fixed, we define, for any $t \in (0,T]$ and $x > 0$,

$$W_t(x) \coloneqq g_q(0,\xi;t,x) \exp\left(\varrho \int_0^t \lambda_{q(s),0} \, \mathrm{d}s\right) = g_q(0,\xi;t,x) \exp\left(\lambda_0 \varrho \int_0^t q(s)^{2/(2+\alpha)} \, \mathrm{d}s\right), \tag{2.24}$$

where we used (2.18) in the second equality. Moreover, for any $t > 0$, $g_q(0,\xi;t,\cdot) \in L^2(\mathbb{R})$ (it is dominated by a Gaussian function by Lemma 2.2.3) and so $W_t \in L^2(\mathbb{R})$ and we can set, for any $n \geq 0$,

$$c_n(t) \coloneqq \langle \varphi_{q(t),n}, W_t \rangle = \langle \varphi_{q(t),n}, g_q(0,\xi;t,\cdot) \rangle \exp\left(\lambda_0 \varrho \int_0^t q(s)^{2/(2+\alpha)} \, \mathrm{d}s\right). \tag{2.25}$$

Since $(\varphi_{q(t),n})_{n \geq 0}$ is an orthonormal basis of $L^2(\mathbb{R})$ by Proposition 2.2.7.2, we get

$$\forall t \in (0,T], \quad \forall x \in \mathbb{R}, \quad W_t(x) = \sum_{n \geq 0} c_n(t) \varphi_{q(t),n}(x). \tag{2.26}$$

We also define

$$c_n(0) \coloneqq \varphi_{q(0),n}(\xi), \tag{2.27}$$

which makes c_n continuous at 0 as mentioned in the forthcoming Lemma 2.2.10.

A key property of the functions q_* and q^* is that they are constant on intervals $[0,\varepsilon_1]$ and $[T-\varepsilon_2,T]$ for some parameters $\varepsilon_1,\varepsilon_2$. On these intervals, $c_0(t)$ stays constant, while $c_n(t)$ for $n \geq 1$ decays exponentially fast (see Corollary 2.2.11). Filling the gap between times ε_1 and $T-\varepsilon_2$, we show in Lemma 2.2.13 that $c_0(T) \simeq c_0(0)$ and that $c_n(T)$ is small for $n \geq 1$. This lemma is a key tool in the proof of Proposition 2.2.4 below.

Proof of Proposition 2.2.4. We first choose the parameters $\varepsilon_1, \varepsilon_2 > 0$. The conditions we need in this proof are the following

$$\varepsilon_1, \varepsilon_2 \leq T/10 \qquad \text{and} \qquad \varepsilon_2 \leq (1-T)/10, \tag{2.28}$$

$$\varrho\varepsilon_1 \geq 1 \qquad \text{and} \qquad \varrho\varepsilon_2(1-T)^{-\kappa} \geq 1, \tag{2.29}$$

$$\varrho\varepsilon_1^2 \leq 1 \qquad \text{and} \qquad \varrho\varepsilon_2^2(1-T)^{-\kappa-1} \leq 1. \tag{2.30}$$

We set

$$\varepsilon_1 \coloneqq \left(\frac{\delta}{\varrho}\right)^{1/2} \qquad \text{and} \qquad \varepsilon_2 \coloneqq \left(\frac{\delta(1-T)^{1+\kappa}}{\varrho}\right)^{1/2} = (1-T)\left(\frac{\delta}{\varrho(1-T)^{1-\kappa}}\right)^{1/2}. \tag{2.31}$$

We now check they satisfy the conditions listed above. The first part of (2.28) follows from $\varepsilon_2 \le \varepsilon_1$ and $\delta \le (T^2\varrho)/100$, the second one from $\varrho(1-T)^{1-\kappa} \ge 10$ and $\delta \le 1/10$. The first part of (2.29) is obtained by noting that $\delta \ge 1/\varrho$ and the second one follows from $\delta \ge 1/(\varrho(1-T)^{1-\kappa})$. Finally, (2.30) only requires $\delta \le 1$ which is true. Throughout the proof, we use only the properties of ε_1 and ε_2 listed in (2.28)-(2.29)-(2.30) and express the bounds in terms of ε_1 and ε_2. Their precise choice is only used to deduce the proposition from these bounds. This highlights the separated roles of ε_1 and ε_2 and hopefully can help to understand the precise choice which is made here.

By (2.28), we can apply Lemma 2.2.9 to consider functions q_* and q^* satisfying the properties listed there. Now, note that, by (2.23), it is enough to consider $q = q_*$ or $q = q^*$ and prove the bound for

$$\left|\exp\left(\lambda_0\varrho\int_0^T \frac{\mathrm{d}s}{(1-s)^\kappa}\right) g_q(0,\xi;T,x) - \varphi_0(\xi)\varphi_{(1-T)^{-\alpha},0}(x)\right|, \tag{2.32}$$

where we rewrote $(1-T)^{-\kappa/4}\varphi_0((1-T)^{-\kappa/2}x) = \varphi_{(1-T)^{-\alpha},0}(x)$ by Proposition 2.2.7.2. By the triangle inequality, we can bound (2.32) by $T_1 + T_2 + T_3$, where

$$\begin{aligned}
T_1 &\coloneqq \left|\exp\left(\lambda_0\varrho\int_0^T \frac{\mathrm{d}s}{(1-s)^\kappa}\right) g_q(0,\xi;T,x) - W_T(x)\right|, \\
T_2 &\coloneqq \left|W_T(x) - \varphi_{q(0),0}(\xi)\varphi_{q(T),0}(x)\right|, \\
T_3 &\coloneqq \left|\varphi_{q(0),0}(\xi)\varphi_{q(T),0}(x) - \varphi_0(\xi)\varphi_{(1-T)^{-\alpha},0}(x)\right|.
\end{aligned}$$

We start with T_2. By (2.26) and recalling that $\varphi_{q(0),0}(\xi) = c_0(0)$, we have

$$\begin{aligned}
T_2 &\le |c_0(T) - c_0(0)|\,\varphi_{q(T),0}(x) + \sum_{n\ge1}\left|c_n(T)\varphi_{q(T),n}(x)\right| \\
&\le Cq(T)^{1/[2(2+\alpha)]}\left(\frac{1}{\varrho(1-T)^{1-\kappa}} + e^{-c\varrho T}\right)\left(e^{-c|\xi|^{(2+\alpha)/2}} + e^{-c\varrho\varepsilon_1(1+|\xi|^\alpha)}\right) \\
&\quad\times\left(\varphi_0\left(q(T)^{1/(2+\alpha)}x\right) + \sum_{n\ge1} e^{-c\varrho\varepsilon_2(1-T)^{-\kappa}n^\kappa}\left|\varphi_n\left(q(T)^{1/(2+\alpha)}x\right)\right|\right),
\end{aligned}$$

by Lemma 2.2.13 (using here $\varrho\varepsilon_1 \ge 1$ by (2.29)) and (2.18). Using Lemma 2.2.8.5 and that $q(T) \ge c(1-T)^{-\alpha}$, the series on the right-hand side of the last equation is at most

$$
\begin{aligned}
&C\sum_{n\geq 1} e^{-c\varrho\varepsilon_2(1-T)^{-\kappa}n^\kappa}\cdot n^3\left[1\wedge\exp\left(-c\left((1-T)^{-\alpha/2}\left|x\right|^{(2+\alpha)/2}-Cn\right)\right)\right]\\
&\quad\leq Ce^{-c\varrho\varepsilon_2(1-T)^{-\kappa}}\left(\exp\left(-c(1-T)^{-\alpha/2}\left|x\right|^{(2+\alpha)/2}\right)+\exp\left(-c\varrho\varepsilon_2(1-T)^{-\alpha}\left|x\right|^{\alpha}\right)\right),
\end{aligned}
$$

by Lemma 2.2.14 with $u=\varrho\varepsilon_2(1-T)^{-\kappa}\geq 1$ by (2.29) and $v=(1-T)^{-\alpha/2}\left|x\right|^{(2+\alpha)/2}$. Using also Lemma 2.2.8.5 to bound φ_0, we get

$$
\begin{aligned}
T_2 &\leq C(1-T)^{-\kappa/4}\left(\frac{1}{\varrho(1-T)^{1-\kappa}}+e^{-c\varrho T}\right)\left(e^{-c|\xi|^{(2+\alpha)/2}}+e^{-c\varrho\varepsilon_1(1+|\xi|^\alpha)}\right)\\
&\quad\times\left(\exp\left(-c\left(\frac{|x|}{(1-T)^{\kappa/2}}\right)^{(2+\alpha)/2}\right)+\exp\left(-c\frac{\varrho\varepsilon_2}{(1-T)^\kappa}\left(1+\left(\frac{|x|}{(1-T)^{\kappa/2}}\right)^\alpha\right)\right)\right).
\end{aligned}
$$

With our choice of ε_1 and ε_2 in (2.31) and using $\delta\geq 1/(\varrho(1-T)^{1-\kappa})$, this is smaller than the bound appearing in the statement of the proposition.

We now bound T_3. For this, we first fix some $\theta>1$ and note that, for any $q,p>0$ such that $p/q\in[\theta^{-1},\theta]$ and any $x>0$, using (2.18) and standard inequalities, with constants $C,c>0$ that depend only on θ and α,

$$
\begin{aligned}
&\left|\varphi_{p,0}(x)-\varphi_{q,0}(x)\right|\\
&\leq\left|p^{1/[2(2+\alpha)]}-q^{1/[2(2+\alpha)]}\right|\varphi_0\left(p^{1/(2+\alpha)}x\right)+q^{1/[2(2+\alpha)]}\left|\varphi_0\left(p^{1/(2+\alpha)}x\right)-\varphi_0\left(q^{1/(2+\alpha)}x\right)\right|\\
&\leq C\left(p^{1/[2(2+\alpha)]}\left|1-\frac{q}{p}\right|\varphi_0\left(p^{1/(2+\alpha)}x\right)+p^{3/[2(2+\alpha)]}\left|1-\frac{q}{p}\right|\cdot x\cdot\max_{t\geq(p\wedge q)^{1/(2+\alpha)}x}\left|\varphi_0'(t)\right|\right)\\
&\leq Cp^{1/[2(2+\alpha)]}\left|1-\frac{q}{p}\right|\exp\left(-cp^{1/2}\left|x\right|^{(2+\alpha)/2}\right),
\end{aligned}
$$

using Lemma 2.2.8.5-6 and that $(1+t^{2/(2+\alpha)})e^{-ct}\leq Ce^{-ct}$ for $t>0$ up to a modification of c. By parity, the same inequality holds for $x<0$. Therefore, we get, using again Lemma 2.2.8.5 together with Lemma 2.2.9.7,

$$
\begin{aligned}
T_3 &\leq\varphi_{q(0),0}(\xi)\left|\varphi_{q(T),0}(x)-\varphi_{(1-T)^{-\alpha},0}(x)\right|+\varphi_{(1-T)^{-\alpha},0}(x)\left|\varphi_{q(0),0}(\xi)-\varphi_0(\xi)\right|\\
&\leq C(1-T)^{-\frac{\alpha}{2(2+\alpha)}}e^{-c|\xi|^{(2+\alpha)/2}}\exp\left(-c(1-T)^{-\alpha/2}\left|x\right|^{(2+\alpha)/2}\right)\\
&\qquad\qquad\times\left(\left|1-q(0)\right|+\left|1-\frac{q(T)}{(1-T)^{-\alpha}}\right|\right)\\
&\leq C(1-T)^{-\kappa/4}e^{-c|\xi|^{(2+\alpha)/2}}\exp\left(-c\left(\frac{|x|}{(1-T)^{\kappa/2}}\right)^{(2+\alpha)/2}\right)\left(\varepsilon_1+\frac{\varepsilon_2}{1-T}\right).
\end{aligned}
$$

With our choice of ε_1 and ε_2 in (2.31), we have $\varepsilon_1\leq\varrho^{-1}+\delta$ and $\varepsilon_2/(1-T)\leq(\varrho(1-T)^{1-\kappa})^{-1}+\delta$, so, recalling $\delta\geq(\varrho(1-T)^{1-\kappa})^{-1}$, this last bound is smaller than the bound appearing in the statement of the proposition.

Finally, we bound T_1. By definition of $W_T(x)$, we have

$$T_1 = \left| \exp\left(\lambda_0 \varrho \int_0^T \left((1-s)^{-2\alpha/(2+\alpha)} - q(s)^{2/(2+\alpha)} \right) \mathrm{d}s \right) - 1 \right| \cdot W_T(x).$$

By Lemma 2.2.9.9, the quantity in the exponential above is bounded in absolute values by $10\lambda_0\varrho(\varepsilon_1^2 + \varepsilon_2^2(1-T)^{-\kappa-1})$, which is itself at most $20\lambda_0$ by (2.30). Hence, using that the exponential is Lipschitz continuous on $(-\infty, 20\lambda_0]$, we get

$$T_1 \leq C \left(\varrho\varepsilon_1^2 + \varrho\varepsilon_2^2(1-T)^{-\kappa-1} \right) W_T(x).$$

Now, note that $W_T(x) \leq \varphi_0(\xi)\varphi_{(1-T)^{-\alpha},0}(x) + T_2 + T_3$, so using again (2.18) and Lemma 2.2.8.5, we get

$$\begin{aligned} T_1 \leq C &\left(\varrho\varepsilon_1^2 + \varrho\varepsilon_2^2(1-T)^{-\kappa-1} \right) \\ &\times \left(T_2 + T_3 + (1-T)^{-\kappa/4} e^{-c|\xi|^{(2+\alpha)/2}} \exp\left(-c(1-T)^{-\alpha/2} |x|^{(2+\alpha)/2} \right) \right). \end{aligned}$$

With our choice of ε_1 and ε_2 in (2.31), we have $\varrho\varepsilon_1^2 = \varrho\varepsilon_2^2(1-T)^{-\kappa-1} = \delta$, and combining the bounds for T_1, T_2 and T_3 concludes the proof. □

2.2.4 Approximating the inhomogeneity

Lemma 2.2.9. *Let $\alpha \in (0,2)$. For any $T \in (0,1)$ and $\varepsilon_1, \varepsilon_2 \in (0, T/10)$ such that $\varepsilon_2 \leq (1-T)/10$, there exists functions $q^*, q_* \colon [0,T] \to [0,\infty)$ that satisfy the following:*

1. *q_*, q^* are non-decreasing, Lipschitz continuous on $[0,T]$, and differentiable on $[0,T]$ except at finitely many points;*

2. *q_*, q^* are constant on $[0,\varepsilon_1]$ and $[T-\varepsilon_2, T]$;*

3. *For any $t \in [2\varepsilon_1, T-2\varepsilon_2]$, $q_*(t) = (1-t)^{-\alpha} = q^*(t)$;*

4. *For any $t \in [0,T]$, $q_*(t) \leq (1-t)^{-\alpha} \leq q^*(t)$;*

5. *For any $t \in [0, 2\varepsilon_1]$,*

$$1 - 4\varepsilon_1 \leq \frac{q_*(t)}{(1-t)^{-\alpha}} \leq \frac{q^*(t)}{(1-t)^{-\alpha}} \leq 1 + 3\varepsilon_1.$$

6. *For any $t \in [T-2\varepsilon_2, T]$,*

$$1 - \frac{2\varepsilon_2}{1-T} \leq \frac{q_*(t)}{(1-t)^{-\alpha}} \leq \frac{q^*(t)}{(1-t)^{-\alpha}} \leq 1 + \frac{5\varepsilon_2}{1-T}.$$

7. *For any,* $t \in [0,T]$*,*

$$\frac{1}{2} \leq \frac{q_*(t)}{(1-t)^{-\alpha}} \leq \frac{q^*(t)}{(1-t)^{-\alpha}} \leq 2.$$

8. *For any* $t \in [0,T]$ *and* $q = q_*$ *or* q^**,*

$$\frac{q'(t)}{q(t)} \leq \frac{8}{(1-t)}.$$

9. *For any* $t \in [0,T]$ *and* $q = q_*$ *or* q^**,*

$$\int_0^T \left|(1-t)^{-2\alpha/(2+\alpha)} - q(t)^{2/(2+\alpha)}\right| \mathrm{d}t \leq 10\varepsilon_1^2 + \frac{10\varepsilon_2^2}{(1-T)^{\kappa+1}}.$$

Proof. We define q_* and q^* as follows (see Figure 2.4):

- for $t \in [0, \varepsilon_1]$, $q_*(t) = 1$ and $q^*(t) = (1-\varepsilon_1)^{-\alpha}$,
- for $t \in [\varepsilon_1, 2\varepsilon_1]$, $q^*(t) = (1-t)^{-\alpha}$ and q_* is obtained by linear interpolation between the values at the endpoints.
- for $t \in [2\varepsilon_1, T-2\varepsilon_2]$, $q_*(t) = q^*(t) = (1-t)^{-\alpha}$,
- for $t \in [T-2\varepsilon_2, T-\varepsilon_2]$, $q_*(t) = (1-t)^{-\alpha}$ and q^* is obtained by linear interpolation between the values at the endpoints.
- for $t \in [T-\varepsilon_2, T]$, $q_*(t) = (1-T+\varepsilon_2)^{-\alpha}$ and $q^*(t) = (1-T)^{-\alpha}$.

Parts 1-2-3 are clearly true. For Part 4, the only non-trivial thing to check is that $q_*(t) \leq (1-t)^{-\alpha}$ for $t \in [\varepsilon_1, 2\varepsilon_1]$. By convexity of $s \mapsto (1-s)^{-\alpha}$, it is enough to compare the left-derivatives at $2\varepsilon_1$, which amounts to show that

$$\frac{(1-2\varepsilon_1)^{-\alpha} - 1}{\varepsilon_1} \geq \alpha(1-2\varepsilon_1)^{-\alpha-1}.$$

We have $(1-2\varepsilon_1)^{-\alpha} - 1 \geq 2\alpha\varepsilon_1$, hence it is enough to show $(1-2\varepsilon_1)^{-\alpha-1} \leq 2$, which is equivalent to $\varepsilon_1 \leq \frac{1}{2}(1-2^{-1/(\alpha+1)})$. This is true by noting that $\varepsilon_1 \leq 1/10 \leq \frac{1}{2}(1-2^{-1/3}) \leq \frac{1}{2}(1-2^{-1/(\alpha+1)})$. Hence, Part 4 is proved.

For Part 5, consider $t \in [0, 2\varepsilon_1]$. Then,

$$\frac{q_*(t)}{(1-t)^{-\alpha}} \geq \frac{1}{(1-2\varepsilon_1)^{-\alpha}} = (1-2\varepsilon_1)^{\alpha} \geq \begin{cases} 1-2\alpha\varepsilon_1, & \text{if } \alpha \geq 1, \\ 1-2\varepsilon_1, & \text{if } \alpha < 1. \end{cases}$$

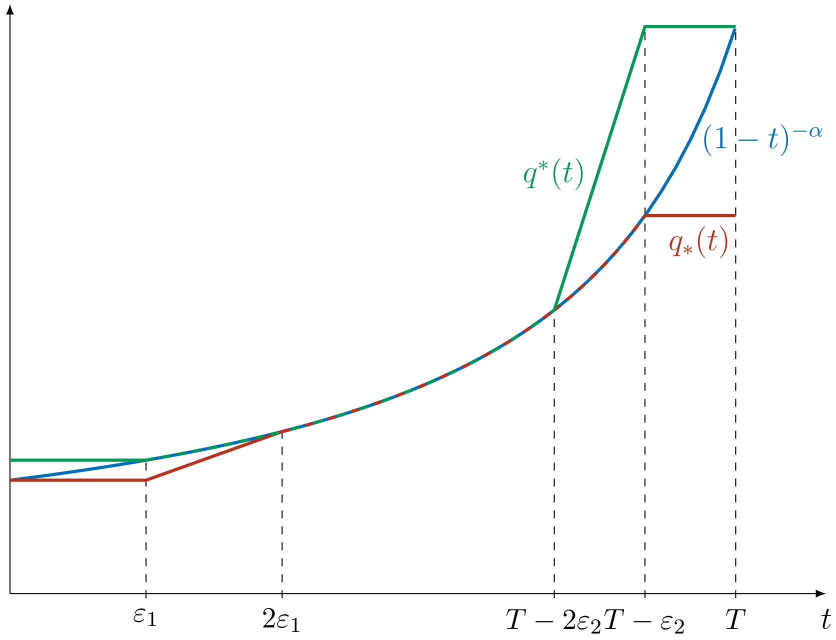

Figure 2.4: Construction of q_* and q^*.

This proves the lower bound. For the upper bound, using $(1-t)^{-\alpha} \geq 1$, we have

$$\frac{q_*(t)}{(1-t)^{-\alpha}} \leq (1-\varepsilon_1)^{-\alpha} \leq 1+\varepsilon_1\alpha(1-\varepsilon_1)^{-\alpha-1} \leq 1+2\varepsilon_1(9/10)^{-3}, \tag{2.33}$$

using $\varepsilon_1 \leq 1/10$ and $\alpha \leq 2$. This proves Part 5.

For Part 6, consider $t \in [T-2\varepsilon_2, T]$. Then,

$$\frac{q_*(t)}{(1-t)^{-\alpha}} \geq \frac{(1-T+\varepsilon_2)^{-\alpha}}{(1-T)^{-\alpha}} = \left(1+\frac{\varepsilon_2}{1-T}\right)^{-\alpha} \geq 1-\alpha\frac{\varepsilon_2}{1-T},$$

by convexity of $s \mapsto (1-s)^{-\alpha}$. On the other hand,

$$\frac{q^*(t)}{(1-t)^{-\alpha}} \leq \frac{(1-T)^{-\alpha}}{(1-T+2\varepsilon_2)^{-\alpha}} \leq \left(1+\frac{2\varepsilon_2}{1-T}\right)^2 = 1+\frac{4\varepsilon_2}{1-T}+\frac{4\varepsilon_2^2}{(1-T)^2} \leq 1+\frac{5\varepsilon_2}{1-T}, \tag{2.34}$$

using $\varepsilon_2 \leq (1-T)/10$. This proves Part 6.

Part 7 follows directly from Parts 5-6 and the assumptions on ε_1 and ε_2.

Part 8 is immediate for $t \in [0,\varepsilon_1] \cup [2\varepsilon_1, T-2\varepsilon_2] \cup [T-\varepsilon_2, T]$. For $t \in [\varepsilon_1, 2\varepsilon_1]$, it has to be checked for q_*: using $q_*(t) \geq 1$ and then $(1-2\varepsilon_1)^{-\alpha} \leq 1+4\varepsilon_1(8/10)^{-3}$ by proceeding as in (2.33), we get

$$\frac{q_*'(t)}{q_*(t)} \leq \frac{(1-2\varepsilon_1)^{-\alpha}-1}{\varepsilon_1} \leq 4\cdot(8/10)^{-3} \leq \frac{8}{(1-t)}.$$

For $t \in [T-2\varepsilon_2, T-\varepsilon_2]$, it has to be checked for q^*: using $q^*(t) \geq (1-T+2\varepsilon_2)^{-\alpha}$ and then proceeding as in (2.34), we get

$$\frac{(q^*)'(t)}{q^*(t)} \leq \frac{(1-T)^{-\alpha} - (1-T+2\varepsilon_2)^{-\alpha}}{\varepsilon_2(1-T+2\varepsilon_2)^{-\alpha}} \leq \frac{5}{1-T} \leq \frac{6}{1-t},$$

using that $1-t \leq 1-T+2\varepsilon_2 \leq \frac{6}{5}(1-T)$ by assumption on ε_2. This proves Part 8.

Finally it remains to prove Part 9. Using $|1-x^{2/(2+\alpha)}| \leq |1-x|$ for any $x > 0$, we get

$$\begin{aligned}\int_0^T &\left|(1-t)^{-2\alpha/(2+\alpha)} - q(t)^{2/(2+\alpha)}\right| \mathrm{d}t \\ &\leq \int_0^T (1-t)^{-\kappa} \left|1 - \frac{q(t)}{(1-t)^{-\alpha}}\right| \mathrm{d}t \\ &\leq \int_0^{2\varepsilon_1} (1-2\varepsilon_1)^{-\kappa} 4\varepsilon_1 \,\mathrm{d}t + \int_{T-2\varepsilon_2}^T (1-T)^{-\kappa} \frac{5\varepsilon_2}{1-T} \,\mathrm{d}t,\end{aligned}$$

using Parts 5-6. Finally, using $(1-2\varepsilon_1)^{-\kappa} \leq 5/4$ yields the result. □

2.2.5 General properties of the coefficients $c_n(t)$

In this section, we establish several results concerning the coefficients $c_n(t)$ holding for general functions q. Recall their definition in (2.25) and (2.27) and that they implicitly depend on α, ϱ and ξ. Recall also the definition of the λ_n's and φ_n's in Proposition 2.2.7.

Lemma 2.2.10. *Let $\alpha > 0$, $\varrho \geq 1$, $\xi \in \mathbb{R}$ and $T \in (0,1)$. Let $q\colon [0,T] \to (0,\infty)$ be a Lipschitz continuous function. Then, for any $n \geq 0$, c_n is continuous on $[0,T]$.*

If moreover q is differentiable at some $t \in (0,T]$, then c_n is differentiable at t and, writing $c(t) = (c_0(t), c_1(t), \ldots)$, we have $\partial_t c(t) = (-D(t) + A(t))c(t)$, where we set

$$\begin{cases} D(t) = \varrho q(t)^{2/(2+\alpha)} D, \\ D = \mathrm{Diag}((\lambda_i - \lambda_0)_{i\geq 0}), \end{cases} \qquad \begin{cases} A(t) = \frac{q'(t)}{q(t)} A, \\ A = \frac{1}{2(2+\alpha)}(\langle \varphi_j, x\varphi_i' \rangle - \langle x\varphi_j', \varphi_i \rangle)_{i,j\geq 0}. \end{cases}$$

Proof. Continuity of c_n on $(0,T]$ follows from the expression for g_q given in Lemma 2.2.3 and properties of $\varphi_{q,n}$ listed in Proposition 2.2.7. Moreover, c_n is continuous at 0 because, as $t \to 0^+$, $\varphi_{q(t),n}$ converges uniformly to $\varphi_{q(0),n}$ and $g_q(0,\xi;t,\cdot)$ converges weakly to δ_ξ.

Now, we assume q is differentiable at some $t \in (0,T]$. Recall $g_q(0,\xi;\cdot,\cdot)$ solves the PDE (2.22), hence $\partial_t g_q(0,\xi;t,\cdot) = -\varrho \mathcal{L}_{q(t)} g_q(0,\xi;t,\cdot)$. We use this together with the self–adjointness of $\mathcal{L}_{q(t)}$, we get

$$\begin{aligned}\partial_t c_n(t) &= \langle \partial_t \varphi_{q(t),n}, W_t\rangle + \varrho\langle \mathcal{L}_{q(t)}\varphi_{q(t),n}, W_t\rangle + \lambda_0 \varrho q(t)^{2/(2+\alpha)}\langle \varphi_{q(t),n}, W_t\rangle \\ &= -\varrho q(t)^{2/(2+\alpha)}(\lambda_n - \lambda_0)c_n(t) + \sum_{k\geq 0} c_k(t)\langle \varphi_{q(t),k}, \partial_t \varphi_{q(t),n}\rangle,\end{aligned}$$

where we used that $\varphi_{q(t),n}$ is an eigenfunction of $\mathcal{L}_{q(t)}$ with eigenvalue $\lambda_{q(t),n} = q(t)^{2/(2+\alpha)}\lambda_n$. Then, recalling from (2.18) that $\varphi_{q,n}(y) = q^{1/[2(2+\alpha)]}\varphi_n(q^{1/(2+\alpha)}y)$, we have

$$\langle \varphi_{q,k}, \partial_q \varphi_{q,n}\rangle = \frac{1}{(2+\alpha)q}\left(\frac{1}{2}\langle \varphi_k, \varphi_n\rangle + \langle \varphi_k, y\varphi_n'\rangle\right),$$

and therefore

$$\partial_t c_n(t) = -\varrho q(t)^{2/(2+\alpha)}(\lambda_n - \lambda_0)c_n(t) + \sum_{k\geq 0} c_k(t)\frac{1}{2+\alpha}\frac{q'(t)}{q(t)}\left(\frac{1}{2}\langle \varphi_k, \varphi_n\rangle + \langle \varphi_k, y\varphi_n'\rangle\right). \tag{2.35}$$

By integration by parts, using the decay at infinity of φ_n (see Lemma 2.2.8.5), we have

$$\langle \varphi_k, \varphi_n\rangle + 2\langle \varphi_k, x\varphi_n'\rangle = \langle \varphi_k, x\varphi_n'\rangle - \langle x\varphi_k', \varphi_n\rangle. \tag{2.36}$$

Combining this with (2.35) proves the result. ☐

A key observation is that when q is constant on some interval, then $A(t) = 0$ on this interval and the ODE satisfied by $c(t)$ can be explicitly solved as follows.

Corollary 2.2.11. *Let $0 \leq s_1 < s_2 \leq T$. Assume q is constant on $[s_1, s_2]$. Then, for any $n \geq 0$,*

$$c_n(s_2) = c_n(s_1)\exp\left(-\varrho(\lambda_n - \lambda_0)(s_2 - s_1)q(s_1)^{2/(2+\alpha)}\right).$$

In particular, $c_0(s_2) = c_0(s_1)$.

Finally, we conclude this section with some rough bounds on the coefficients $c_n(t)$.

Lemma 2.2.12. *Let $\alpha > 0$, $\varrho \geq 1$, $\xi \in \mathbb{R}$ and $T \in (0,1)$. Let $q\colon [0,T] \to (0,\infty)$ be a function Lipschitz continuous on $[0,T]$, and differentiable on $[0,T]$ except at finitely many points. Then, the following holds.*

1. *The function $t \in (0,T] \mapsto \|c(t)\|_2$ is decreasing;*
2. *Let $\bar{c}(t) := (0, c_2(t), c_3(t), \ldots)$. There exists $C = C(\alpha) > 0$ such that, for $0 < t_0 \leq t \leq T$,*

$$\begin{aligned}\|\bar{c}(t)\|_2 \leq \|\bar{c}(t_0)\|_2 &\exp\left(-\varrho(\lambda_1 - \lambda_0)\int_{t_0}^t q(s)^{2/(2+\alpha)}\,\mathrm{d}s\right) \\ &+ C\,\|c(t_0)\|_2 \int_{t_0}^t \frac{|q'(s)|}{q(s)}\exp\left(-\varrho(\lambda_1 - \lambda_0)\int_s^t q(r)^{2/(2+\alpha)}\,\mathrm{d}r\right)\mathrm{d}s.\end{aligned}$$

Proof. Part 1. We use Lemma 2.2.10, noting that D is diagonal with $D_{00} = 0$ and other entries positive and that A is anti-symmetric, to get, for any $t \in (0, T]$ where q is differentiable,

$$\partial_t \|c(t)\|_2^2 = c(t)^T (A(t)^T - D(t)^T + A(t) - D(t))c(t) = -2c(t)^T D(t)c(t) \leq 0.$$

Together with the continuity of the function $t \in [0, T] \mapsto \|c(t)\|_2^2$, this implies that it is a decreasing function and hence the same is true for $\|c(t)\|_2$.

Part 2. Using again Lemma 2.2.10 and the fact that A is antisymmetric, we have

$$\begin{aligned}
\partial_t \|\bar{c}(t)\|_2 &= \frac{1}{2\|\bar{c}(t)\|_2} \partial_t \|\bar{c}(t)\|_2^2 \\
&= -\frac{1}{\|\bar{c}(t)\|_2} \bar{c}(t)^T D(t) \bar{c}(t) - \frac{1}{\|\bar{c}(t)\|_2} c_0(t) \sum_{j=1}^{\infty} A_{0j}(t) c_j(t) \\
&\leq -\varrho q(t)^{2/(2+\alpha)} (\lambda_1 - \lambda_0) \|\bar{c}(t)\|_2 + |c_0(t)| \frac{|q'(t)|}{q(t)} \|(A_{0j})_{j\geq 1}\|_2 ,
\end{aligned}$$

using $\lambda_n \geq \lambda_1$ for the first term and the Cauchy-Schwarz inequality in ℓ^2 for the second one. By (2.36), we have $A_{0j} = \frac{1}{2(2+\alpha)}(\langle \varphi_j, \varphi_0 \rangle + 2\langle \varphi_j, x\varphi_0' \rangle)$ and, using that $(\varphi_j)_{j\geq 0}$ is an orthonormal basis, it follows that $\|(A_{0j})_{j\geq 1}\|_2 \leq \frac{1}{(2+\alpha)}(1 + 2\|x\varphi_0'\|_2)$, which is a finite constant depending only on α. Now, Grönwall's inequality yields

$$\begin{aligned}
\|\bar{c}(t)\|_2 \leq{}& \|\bar{c}(t_0)\|_2 \exp\left(-\varrho(\lambda_1 - \lambda_0) \int_{t_0}^t q(s)^{2/(2+\alpha)} \, \mathrm{d}s \right) \\
&+ C \int_{t_0}^t |c_0(s)| \frac{|q'(s)|}{q(s)} \exp\left(-\varrho(\lambda_1 - \lambda_0) \int_s^t q(r)^{2/(2+\alpha)} \, \mathrm{d}r \right) \mathrm{d}s,
\end{aligned}$$

and the result follows using that $|c_0(s)| \leq \|c(s)\|_2 \leq \|c(t_0)\|_2$ by Part 1. □

2.2.6 Precise estimates for the coefficients $c_n(t)$

In this section, we build upon the general bounds of Lemma 2.2.12 to prove more precise estimates for $c_n(T)$ in the cas $q = q_*$ or $q = q^*$. In particular, we use that these functions q are constant on $[0, \varepsilon_1]$ and $[T - \varepsilon_2, T]$ and rely on Corollary 2.2.11 on these intervals.

Lemma 2.2.13. *Let $\alpha \in (0, 2)$, $\varrho \geq 1$ and $\xi \in \mathbb{R}$. Let $T \in (0, 1)$, $\varepsilon_1, \varepsilon_2 \in (0, T/10)$ such that $\varepsilon_2 \leq (1 - T)/10$. Let q be either q_* or q^* given by Lemma 2.2.9. Assume $\varrho\varepsilon_1 \geq 1$ and $\varrho(1 - T)^{1-\kappa} \geq 1$. Then, there exist $c = c(\alpha) > 0$ and $C = C(\alpha) > 0$ such that*

$$|c_0(T) - c_0(0)| \leq \frac{C}{\varrho(1 - T)^{1-\kappa}} \left(e^{-c|\xi|^{(2+\alpha)/2}} + e^{-c\varrho\varepsilon_1(1+|\xi|^{\alpha})} \right),$$

and, for any $n \geq 1$,

$$|c_n(T)| \leq C\left(\frac{1}{\varrho(1-T)^{1-\kappa}} + e^{-c\varrho T}\right)\left(e^{-c|\xi|^{(2+\alpha)/2}} + e^{-c\varrho\varepsilon_1(1+|\xi|^\alpha)}\right)e^{-c\varrho\varepsilon_2(1-T)^{-\kappa}n^\kappa}.$$

Proof. Throughout this proof, constants c and C depend only on α.

Step 1: from 0 *to* ε_1. Since q is constant on $[0,\varepsilon_1]$ by Lemma 2.2.9.2, we have by Corollary 2.2.11, for all $n \geq 0$,

$$c_n(\varepsilon_1) = c_n(0)\exp\left(-\varrho\varepsilon_1(\lambda_n - \lambda_0)q(0)^{2/(2+\alpha)}\right). \tag{2.37}$$

Since $c_n(0) = \varphi_{q(0),n}(\xi)$, $q(0) \geq 1$ and $\lambda_n - \lambda_0 \geq cn^{2\alpha/(2+\alpha)}$ by (2.19), we get

$$\begin{aligned}\sum_{n\geq 1} c_n(\varepsilon_1)^2 &\leq \sum_{n\geq 1}\varphi_{q(0),n}(\xi)^2\exp\left(-c\varrho\varepsilon_1 n^{2\alpha/(2+\alpha)}\right)\\ &\leq C\sum_{n\geq 1} n^6\left[1\wedge\exp\left(-\frac{2}{2+\alpha}\left(|\xi|^{(2+\alpha)/2} - Cn\right)\right)\right]\exp\left(-c\varrho\varepsilon_1 n^{2\alpha/(2+\alpha)}\right),\end{aligned}$$

where we have used (2.18), Lemma 2.2.8.5 and $1 \leq q(0) \leq C$. Applying Lemma 2.2.14 (using here $\varrho\varepsilon_1 \geq 1$), we get

$$\sum_{n\geq 1} c_n(\varepsilon_1)^2 \leq Ce^{-c\varrho\varepsilon_1}\left(e^{-c|\xi|^{(2+\alpha)/2}} + e^{-c\varrho\varepsilon_1|\xi|^\alpha}\right).$$

On the other hand, $c_0(\varepsilon_1) = c_0(0) = \varphi_{q(0),0}(\xi)$. In particular, using again (2.18), Lemma 2.2.8.5 and $1 \leq q(0) \leq C$, we get

$$\|c(\varepsilon_1)\|_2 \leq C\left(e^{-c|\xi|^{(2+\alpha)/2}} + e^{-c\varrho\varepsilon_1(1+|\xi|^\alpha)}\right). \tag{2.38}$$

Step 2: from ε_1 *to* $T-\varepsilon_2$. By Lemma 2.2.10, recalling that $D_{00} = 0$, we have $\partial_t c_0(t) = \sum_{j\geq 1} A_{0j}(t)c_j(t)$ as soon as $q'(t)$ exists. Therefore,

$$|c_0(T-\varepsilon_2) - c_0(\varepsilon_1)| \leq \int_{\varepsilon_1}^{T-\varepsilon_2}\left|\sum_{j\geq 1} A_{0j}(t)c_j(t)\right|\mathrm{d}t \leq C\int_{\varepsilon_1}^{T-\varepsilon_2}\|\bar{c}(t)\|_2\frac{q'(t)}{q(t)}\,\mathrm{d}t, \tag{2.39}$$

using the Cauchy-Schwarz inequality and $\|(A_{0j})_{j\geq 1}\|_2 \leq C$ (as seen in the proof of Lemma 2.2.12). We now bound $\|\bar{c}(t)\|_2$. By Lemma 2.2.12.2, we have

$$\begin{aligned}\|\bar{c}(t)\|_2 &\leq \|\bar{c}(\varepsilon_1)\|_2\exp\left(-\varrho(\lambda_1-\lambda_0)\int_{\varepsilon_1}^t q(s)^{2/(2+\alpha)}\,\mathrm{d}s\right)\\ &\quad + C\|c(\varepsilon_1)\|_2\int_{\varepsilon_1}^t\exp\left(-\varrho(\lambda_1-\lambda_0)\int_s^t q(r)^{2/(2+\alpha)}\,\mathrm{d}r\right)\frac{q'(s)}{q(s)}\,\mathrm{d}s\\ &\leq C\|c(\varepsilon_1)\|_2\left(\exp\left(-c\varrho\int_{\varepsilon_1}^t\frac{\mathrm{d}s}{(1-s)^\kappa}\right) + \int_{1-t}^{1-\varepsilon_1}\exp\left(-c\varrho\int_{1-t}^s\frac{\mathrm{d}r}{r^\kappa}\right)\frac{\mathrm{d}s}{s}\right),\end{aligned}$$

using Lemma 2.2.9.7-8, replacing r by $1-r$ and s by $1-s$ in the second term, and recalling that $\kappa = 2\alpha/(2+\alpha)$. Then, we use that, for $s \geq 1-t$,

$$\int_{1-t}^{s} \frac{\mathrm{d}r}{r^\kappa} = \frac{1}{1-\kappa}\left(s^{1-\kappa} - (1-t)^{1-\kappa}\right) \geq \begin{cases} c(1-t)^{-\kappa}(s-(1-t)), & \text{if } s \leq 2(1-t), \\ cs^{1-\kappa}, & \text{otherwise,} \end{cases}$$

and simply bound $\int_{\varepsilon_1}^{t} \frac{\mathrm{d}s}{(1-s)^\kappa} \geq (t-\varepsilon_1)$. This yields

$$\begin{aligned} \|\bar{c}(t)\|_2 &\leq C\,\|c(\varepsilon_1)\|_2 \left(e^{-c\varrho(t-\varepsilon_1)} + \int_{1-t}^{2(1-t)} e^{-c\varrho(1-t)^{-\kappa}(s-(1-t))}\frac{\mathrm{d}s}{1-t} + \int_{2(1-t)}^{1-\varepsilon_1} e^{-c\varrho s^{1-\kappa}}\frac{\mathrm{d}s}{s}\right) \\ &\leq C\,\|c(\varepsilon_1)\|_2 \left(e^{-c\varrho(t-\varepsilon_1)} + \frac{1}{\varrho(1-t)^{1-\kappa}}\right), \end{aligned} \tag{2.40}$$

bounding by the integral from $1-t$ to ∞ for the second term and using the inequality $e^{-x} \leq \frac{1}{x}$ for $x > 0$ for the third term. Coming back to (2.39) and using again $q'(t)/q(t) \leq C(1-t)^{-1}$ by Lemma 2.2.9.8, we get

$$|c_0(T-\varepsilon_2) - c_0(\varepsilon_1)| \leq C\,\|c(\varepsilon_1)\|_2 \int_{\varepsilon_1}^{T-\varepsilon_2} \left(e^{-c\varrho(t-\varepsilon_1)} + \frac{1}{\varrho(1-t)^{1-\kappa}}\right)\frac{\mathrm{d}t}{1-t}. \tag{2.41}$$

Now, note that

$$\begin{aligned} \int_{\varepsilon_1}^{T-\varepsilon_2} e^{-c\varrho(t-\varepsilon_1)}\frac{\mathrm{d}t}{1-t} &\leq 2\int_{\varepsilon_1}^{1/2} e^{-c\varrho(t-\varepsilon_1)}\,\mathrm{d}t + \mathbb{1}_{T-\varepsilon_2>1/2}\int_{1/2}^{T-\varepsilon_2} e^{-c\varrho}\frac{\mathrm{d}t}{1-t} \\ &\leq \frac{C}{\varrho} + e^{-c\varrho}\log\frac{1}{1-T} \\ &\leq \frac{C}{\varrho}, \end{aligned}$$

using here the assumptions $(1-T)^{-1} \leq \varrho^{1/(1-\kappa)}$ and $\varrho \geq 1$. Coming back to (2.41), we get

$$|c_0(T-\varepsilon_2) - c_0(\varepsilon_1)| \leq \frac{C\,\|c(\varepsilon_1)\|_2}{\varrho(1-T)^{1-\kappa}}. \tag{2.42}$$

Step 3: from $T-\varepsilon_2$ to T. Since q is constant on $[T-\varepsilon_2, T]$, by Corollary 2.2.11, we have, for all $n \geq 0$,

$$c_n(T) = c_n(T-\varepsilon_2)\exp\left(-\varrho\varepsilon_2(\lambda_n - \lambda_0)q(T)^{2/(2+\alpha)}\right).$$

On the one hand, combining this with (2.37) and (2.42), we get

$$|c_0(T) - c_0(0)| = |c_0(T-\varepsilon_2) - c_0(\varepsilon_1)| \leq \frac{C\,\|c(\varepsilon_1)\|_2}{\varrho(1-T)^{1-\kappa}}.$$

On the other hand, for $n \geq 1$, using (2.19), this yields

$$\begin{aligned}|c_n(T)| &\leq \|\bar{c}(T-\varepsilon_2)\|_2 \exp\left(-cn^{2\alpha/(2+\alpha)}\varrho\varepsilon_2(1-T)^{-2\alpha/(2+\alpha)}\right)\\ &\leq C\,\|c(\varepsilon_1)\|_2\left(e^{-c\varrho T}+\frac{1}{\varrho(1-T)^{1-\kappa}}\right)\exp\left(-cn^{2\alpha/(2+\alpha)}\varrho\varepsilon_2(1-T)^{-2\alpha/(2+\alpha)}\right),\end{aligned}$$

using (2.40) and the fact that $\varepsilon_1 \leq T/4$. Combining the two last inequalities with (2.38) yields the result. □

2.2.7 Technical bound on a series

Lemma 2.2.14. *For any $\kappa \in (0,1)$ and $a_1, a_2, a_3, a_4 > 0$, there exists $C, c > 0$ such that, for any $u \geq 1$ and $v \geq 0$,*

$$\sum_{n\geq 1} n^{a_4}\left(1 \wedge e^{-a_1(v-a_2n)}\right)e^{-a_3un^\kappa} \leq Ce^{-cu}\left(e^{-cv}+e^{-cuv^\kappa}\right).$$

Proof. In this proof, constants C and c can depend on $\kappa, a_1, a_2, a_3, a_4$. First note that, bounding $e^{-a_3un^\kappa} \leq e^{-a_3u/2}\cdot e^{-a_3un^\kappa/2}$, we can get a factor $e^{-a_3u/2}$ in front of the series and therefore, it is enough to prove

$$\sum_{n\geq 1} n^{a_4}\left(1 \wedge e^{-a_1(v-a_2n)}\right)e^{-a_3un^\kappa} \leq C\left(e^{-cv}+e^{-cuv^\kappa}\right). \tag{2.43}$$

Letting $n_0 := \lceil v/a_2 \vee 2\rceil$, we split the series into a part $n < n_0$ and a part $n \geq n_0$.

The part $n \geq n_0$ equals

$$\sum_{n\geq n_0} n^{a_4}e^{-a_3un^\kappa} \leq \int_{n_0-1}^{\infty}(x+1)^{a_4}e^{-a_3ux^\kappa}\,\mathrm{d}x \leq Cu^{-a_4/\kappa}\int_{u(n_0-1)^\kappa}^{\infty} y^{a_4/\kappa-1}e^{-a_3y}\,\mathrm{d}y, \tag{2.44}$$

bounding $x+1 \leq 2x$ because $x \geq n_0-1 \geq 1$ and changing variables with $y = ux^\kappa$. Noting that $u^{-a_4/\kappa} \leq 1$ together with $y^{a_4/\kappa-1}e^{-a_3y} \leq Ce^{-a_3y/2}$ and $(n_0-1)^\kappa \geq (v/2a_2)^\kappa$, this shows the right-hand side of (2.44) is at most Ce^{-cuv^κ}.

We now consider the part $n < n_0$. If $v/a_2 < 2$, then this part of sum is smaller than 1 and therefore than Ce^{-v} with $C = e^{2a_2}$. We now consider the case $v/a_2 \geq 2$. Then, this part equals

$$\sum_{n=1}^{n_0-1} n^{a_4}e^{-a_1(v-a_2n)}e^{-a_3un^\kappa} \leq (v/a_2)^{a_4+1}e^{-a_1v}\max_{x\in[1,v/a_2]} e^{a_1a_2x-a_3ux^\kappa}, \tag{2.45}$$

where we bounded $n_0 - 1 \leq v/a_2$, which holds because $v/a_2 \geq 2$ so $n_0 = \lceil v/a_2\rceil$. Now, note that the function $x \mapsto a_1a_2x - a_3ux^\kappa$ is convex on $[1, v/a_2]$, so it achieves its maximum on the boundary of the interval. This shows that the right-hand side of (2.45) is at most

$$C(v/a_2)^{a_4+1}\left(e^{-a_1 v}e^{-a_3 u}+e^{-a_3 u(v/a_2)^\kappa}\right)\leq C\left(e^{-cv}+e^{-cuv^\kappa}\right),$$

using here again that $u\geq 1$ to argue that $y^{a_4+1}e^{-a_3 uy^\kappa}\leq Ce^{-a_3 uy^\kappa/2}$ for any $y>0$. This yields (2.43) and concludes the proof ☐

2.3 Brownian motion weighted by an integral via probability

In this section, we continue studying the heat kernel of Brownian motion weighted by an integral, relying now on probabilistic arguments. Our two main goals are to generalise the estimates of the previous section to cases including error terms in the integral weight and to obtain sharper bounds on the tail of the kernel.

As in the previous section, we fix some parameters $\alpha\in(0,2)$ and $\beta>0$. For $f\colon\mathbb{R}\times(0,\infty)\to\mathbb{R}$ measurable, we define the kernel $\widetilde{G}$ for $x,y\in\mathbb{R}$ and $0\leq s<t$ by

$$\widetilde{G}(s,x;t,y)=\frac{e^{-(y-x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}}\mathbb{E}_{(s,x)}\left[\exp\left(-\beta\int_s^t\left|\frac{B_r}{\sqrt{2r}}\right|^\alpha(1+f(B_r,r))\,\mathrm{d}r\right)\middle|B_t=y\right]. \tag{2.46}$$

The contribution of f should thought as an error term. More precisely, for some $L,a,b>0$, we work under the assumption $f^-_{L,a,b}\leq f\leq f^+_{L,a,b}$, where we set

$$f^+_{L,a,b}(y,r)=\left[L\left(\left|\frac{y}{r}\right|^a+r^{-b}\right)\right]\wedge 1\quad\text{and}\quad f^-_{L,a,b}(y,r)=-\left(\left[L\left(\left|\frac{y}{r}\right|^a+r^{-b}\right)\right]\wedge\eta\right), \tag{2.47}$$

where $\eta=\eta(L,a,b,\alpha)$ is chosen to be the largest possible real number in $(0,1/2]$ such that, for any $r\geq r_0:=(2L)^{1/b}$, the function $y\in[0,\infty)\mapsto(y/r)^\alpha(1+f^-_{L,a,b}(y,r))$ is non-decreasing. To see that such an η necessarily exists, use that $Lr^{-b}\leq 1/2$ and note that the function $u\in[0,u_0]\mapsto u^\alpha(\frac{1}{2}-Lu^a)$ is non-decreasing if u_0 is chosen small enough depending on α,a,L. Finally, note that we write $f^-_{L,a,b}$ even if this function actually also depends on α through η.

2.3.1 Comparing $\widetilde{G}$ to G

Recall from Section 2.2 that, around time t, the natural time scale of G is t^κ and the natural space scale is $t^{\kappa/2}$, where $\kappa:=2\alpha/(2+\alpha)\in(0,1)$. The first result of this section establishes that, if a and b are large enough in terms of α, then $\widetilde{G}$ and G are of the same order as long as we are on a time scale longer than the natural one and on a space scale not much longer than the natural one.

Proposition 2.3.1. *Let $\alpha \in (0,2)$ and $\beta > 0$. Let $L > 0$, $a > (1-\kappa)/(1-\frac{\kappa}{2}) = 1-\alpha/2$ and $b > 1-\kappa$. Then, there exists $\varepsilon_0 = \varepsilon_0(\alpha, a, b) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there are $C, c > 0$ such that for s large enough, for any $t \geq s + s^\kappa$, any $x, y \in \mathbb{R}$ with $|x| \leq s^{(\kappa+\varepsilon)/2}$ and $|y| \leq t^{(\kappa+\varepsilon)/2}$ and any function f satisfying $f^-_{L,a,b} \leq f \leq f^+_{L,a,b}$, we have*

$$\widetilde{G}(s,x;t,y) = (1+o(1))G(s,x;t,y),$$

where the $o(1)$ holds as $s \to \infty$, uniformly in t, x, y, f (but depending on $\alpha, \beta, L, a, b, \varepsilon$).

2.3.1.1 Preliminary results

We prove here two preliminary lemmas. The first one gives some rough estimates on the natural time scale.

Lemma 2.3.2. *Let $\alpha \in (0,2)$ and $\beta > 0$. There exist $C, c > 0$ such that, for any $s \geq 1$, $t \in [s+s^\kappa, s+3s^\kappa]$ and any measurable function $f \colon \mathbb{R} \times (0,\infty) \to \mathbb{R}$, the following holds.*

1. *If $f \leq 1$, then, for any $M \geq 1$, $|x| \leq M s^{\kappa/2}$ and $|y| \leq M t^{\kappa/2}$,*
$$\widetilde{G}(s,x;t,y) \geq \frac{c e^{-CM^2}}{t^{\kappa/2}}.$$
2. *If $f \geq -1/2$, then, for any $x, y \in \mathbb{R}$ and $\eta \geq 0$,*

$$\begin{aligned}\frac{e^{-(y-x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}} \mathbb{E}_{(s,x)} \Bigg[\exp\left(-\beta \int_s^t \left| \frac{B_r}{\sqrt{2r}} \right|^\alpha (1+f(B_r,r)) \,\mathrm{d}r \right) \\ \times \mathbb{1}_{\exists r \in [s,t], |B_r| \geq r^{(\kappa+\eta)/2}} \Bigg| B_t = y \Bigg] \\ \leq C \exp\left(-c t^{\eta\alpha/2} - c \left(\frac{|x|}{s^{\kappa/2}} \right)^\alpha - c \left(\frac{|y|}{t^{\kappa/2}} \right)^\alpha \right). \qquad (2.48)\end{aligned}$$

Proof. Part 1. We use that $(y-x)^2/(t-s) \leq (2Mt^{\kappa/2})^2/s^\kappa \leq CM^2$ and that $f \leq 1$ to get

$$\begin{aligned}\widetilde{G}(s,x;t,y) &\geq \frac{e^{-CM^2}}{\sqrt{2\pi(t-s)}} \mathbb{E}_{(s,x)} \left[\exp\left(-2\beta \int_s^t \left| \frac{B_r}{\sqrt{2r}} \right|^\alpha \mathrm{d}r \right) \middle| B_t = y \right] \\ &\geq \frac{c e^{-CM^2}}{t^{\kappa/2}} \mathbb{P}_{(s,x)} \left(\forall r \in [s,t], |B_r| \leq 3Mt^{\kappa/2} \middle| B_t = y \right), \qquad (2.49)\end{aligned}$$

where we restricted ourselves to the event where the Brownian bridge is bounded by $3Mt^{\kappa/2}$ so that we can bound $\int_s^t |\frac{B_r}{r}|^\alpha \,\mathrm{d}r \leq CM^\alpha t^{\kappa+\alpha\kappa/2-\alpha} = CM^\alpha$ because $\kappa + \alpha\kappa/2 - \alpha = 0$. Now, recall the following formula (see e.g. the proof of Lemma 2 in [32]): for any $0 < s < t$ and any $x, y < K$,

$$\mathbb{P}_{(s,x)}\left(\exists r \in [s,t], B_r \geq K | B_t = y\right) = \exp\left(-\frac{2(K-x)(K-y)}{t-s}\right). \tag{2.50}$$

Using this (and its symmetric version for the event $\{\exists r \in [s,t], B_r \leq -K\}$), we get that, for s, t, x, y satisfying the assumptions of Part 1,

$$\begin{aligned}\mathbb{P}_{(s,x)}\left(\forall r \in [s,t], |B_r| \leq 3Mt^{\kappa/2} \middle| B_t = y\right) &\geq 1 - 2\exp\left(-\frac{2\cdot 2Mt^{\kappa/2}\cdot 2Mt^{\kappa/2}}{t-s}\right)\\ &\geq 1 - 2e^{-8/3} > 0,\end{aligned}$$

where we used $t - s \leq 3s^\kappa \leq 3t^\kappa$ and $M \geq 1$. Coming back to (2.49), this yields the result.

Part 2. Using $f \geq -1/2$ and $t - s \leq 3s^\kappa$, the left-hand side of (2.48) is at most

$$e^{-(y-x)^2/(6s^\kappa)}\mathbb{E}_{(s,x)}\left[\exp\left(-\frac{\beta}{2}\int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha \mathrm{d}r\right)\mathbb{1}_{\exists r\in[s,t],|B_r|\geq s^{(\kappa+\eta)/2}} \middle| B_t = y\right], \tag{2.51}$$

which we now aim at bounding. We distinguish several cases. First, assume that $|x| \vee |y| \leq s^{(\kappa+\eta)/2}/2$. Then, keeping only the event in the indicator function and then applying (2.50), we get that (2.51) is at most

$$\mathbb{P}_{(s,x)}\left(\exists r \in [s,t], |B_r| \geq s^{(\kappa+\eta)/2} \middle| B_t = y\right) \leq 2\exp\left(-\frac{s^{\kappa+\eta}}{2(t-s)}\right) \leq 2\exp\left(-\frac{s^\eta}{6}\right),$$

which is smaller than the right-hand side of (2.48) in this case (note that $s \geq t/4$ and $\alpha < 2$). We can now assume that $|x| \vee |y| > s^{(\kappa+\eta)/2}/2$. We restrict ourselves to the case where $|x| = |x| \vee |y|$ and $x > 0$, the other cases being treated similarly. On the one hand, if $y \leq x/2$, then, bounding the expectation by 1, (2.51) is at most

$$e^{-(y-x)^2/(6s^\kappa)} \leq e^{-x^2/(24s^\kappa)} \leq \exp\left(-\frac{1}{24s^\kappa}\left(\frac{(s^{(\kappa+\eta)/2}/2)^2}{3} + \frac{x^2}{3} + \frac{y^2}{3}\right)\right),$$

which is smaller than the right-hand side of (2.48). On the other hand, if $y > x/2$, then (2.51) is at most

$$\begin{aligned}&\mathbb{E}_{(s,x)}\left[\exp\left(-\frac{\beta}{2}\int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha \mathrm{d}r\right) \middle| B_t = y\right]\\ &\quad= \mathbb{E}_{(s,x)}\left[\exp\left(-\frac{\beta}{2}\int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha \mathrm{d}r\right)\mathbb{1}_{\forall r\in[s,t],B_r\geq x/4} \middle| B_t = y\right]\\ &\qquad+ \mathbb{P}_{(s,x)}\left(\exists r \in [s,t], B_r \leq x/4 | B_t = y\right)\\ &\quad\leq \exp\left(-\frac{\beta}{2}(t-s)\left|\frac{x}{4\sqrt{2s}}\right|^\alpha\right) + \exp\left(-\frac{2(x/2)(x/4)}{t-s}\right),\end{aligned} \tag{2.52}$$

using (2.50) for the second term. Using $s^\kappa \leq t-s \leq 3s^\kappa$ and $\alpha - \kappa = \alpha\kappa/2$, the right-hand side of (2.52) is at most $\exp(-cx^\alpha/s^{\alpha\kappa/2}) + \exp(-cx^2/s^\kappa)$, which is again smaller than the right-hand side of (2.48). This concludes the proof. $\square$

Before stating the next lemma, we define, for $0 < s < t$ and $x, y \geq 0$,

$$\widetilde{G}^{|\cdot|}(s,x;t,y) := \widetilde{G}(s,x;t,y) + \widetilde{G}(s,x;t,-y). \tag{2.53}$$

It can be rewritten in the following way:

$$\widetilde{G}^{|\cdot|}(s,x;t,y) = \frac{e^{-(y-x)^2/[2(t-s)]} + e^{-(y+x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}} \times \mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1 + f(B_r,r))\,\mathrm{d}r\right) \middle| |B_t| = y\right]. \tag{2.54}$$

If $f(\cdot, r)$ is even for any $r \geq s$, then $f(B_r,r)$ can be replaced by $f(|B_r|, r)$, and $\widetilde{G}^{|\cdot|}$ can be seen as the heat kernel of reflected Brownian motion weighted by an integral. Moreover under this additional assumption, for any $0 < s < t$ and $x, y \geq 0$, we have

$$\widetilde{G}^{|\cdot|}(s,x;t,y) = \widetilde{G}(s,x;t,y) + \widetilde{G}(s,-x;t,y), \tag{2.55}$$

because $\widetilde{G}(s,-x;t,y) = \widetilde{G}(s,x;t,-y)$ by symmetry of Brownian motion and of f. It is convenient to work with this new kernel in the following result for coupling reasons which appear in its proof.

Lemma 2.3.3. *Let $\alpha \in (0,2)$ and $\beta > 0$. Let $f\colon \mathbb{R} \times (0,\infty) \to \mathbb{R}$ measurable and $r_0 > 0$ be such that, for any $r \geq r_0$, the function $f(\cdot, r)$ is even and the function $y \in [0,\infty) \mapsto (y/r)^\alpha (1 + f(y,r))$ is non-decreasing. For any $r_0 \leq s < t$, $0 \leq x' \leq x$ and $0 \leq y' \leq y$,*

$$\widetilde{G}^{|\cdot|}(s,x;t,y) \leq 2 \exp\left(\frac{(x'-y')^2 - (x-y)^2}{2(t-s)}\right) \widetilde{G}^{|\cdot|}(s,x';t,y').$$

Proof. We use expression (2.54) to compare $\widetilde{G}^{|\cdot|}(s,x;t,y)$ and $\widetilde{G}^{|\cdot|}(s,x';t,y')$. For the prefactor, we bound crudely

$$\frac{e^{-(y-x)^2/[2(t-s)]} + e^{-(y+x)^2/[2(t-s)]}}{e^{-(y'-x')^2/[2(t-s)]} + e^{-(y'+x')^2/[2(t-s)]}} \leq \frac{2e^{-(y-x)^2/[2(t-s)]}}{e^{-(y'-x')^2/[2(t-s)]}}.$$

For the expectation, one can couple[2] a reflected Brownian motion bridge X^1 from (s,x) to (t,y) with a reflected Brownian motion bridge X^2 from (s,x') to (t,y') such

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[2] To do so, consider $\widetilde{X}^2$ a reflected Brownian motion bridge from (s,x') to (t,y') independent of X^1. On the event where X^1 and $\widetilde{X}^2$ do not intersect, we set $X^2 = \widetilde{X}^2$. On the complement of this event, let S be the first hitting time of X^1 and $\widetilde{X}^2$ and L the last hitting time: we set $X^2 = \widetilde{X}^2$ on $[s,t] \setminus [S,L]$ and $X^2 = X^1$ on $[S,L]$. By continuity of the paths, this ensures that $X^1 \geq X^2$. The fact that X^2 is a reflected Brownian motion bridge from (s,x') to (t,y') follows from the backward strong Markov property stated in [37, Theorem 2]. This coupling is often referred to as a Doeblin coupling.

that $X^1_r \geq X^2_r$ for any $r \in [s,t]$. [37] By the assumptions on f, we then get

$$\exp\left(-\beta \int_s^t \left|\frac{X^1_r}{\sqrt{2r}}\right|^\alpha \left(1+f(X^1_r,r)\right) \mathrm{d}r\right) \leq \exp\left(-\beta \int_s^t \left|\frac{X^2_r}{\sqrt{2r}}\right|^\alpha \left(1+f(X^2_r,r)\right) \mathrm{d}r\right).$$

Combining this gives the desired inequality. □

2.3.1.2 Localizing typical paths

The main ingredient of the proof of Proposition 2.3.1 is the following result localizing the paths contributing to $\widetilde{G}(s,y;t,z)$.

Lemma 2.3.4. *Let $\alpha \in (0,2)$ and $\beta > 0$. For any $\varepsilon \in (0, 1-\kappa]$ and $\eta > 2\varepsilon/\alpha$, there are $C, c > 0$ such that the following holds. Let $f\colon \mathbb{R} \times (0,\infty) \to [-1/2, 1]$ measurable and $r_0 > 0$ be such that, for any $r \geq r_0$, the function $f(\cdot, r)$ is even and the function $y \in [0,\infty) \mapsto (y/r)^\alpha(1+f(y,r))$ is non-decreasing. For any $s \geq r_0$, $t \geq s + s^\kappa$, $|x| \leq s^{(\kappa+\varepsilon)/2}$ and $|y| \leq t^{(\kappa+\varepsilon)/2}$,*

$$\frac{e^{-(y-x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}} \mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1+f(B_r,r))\,\mathrm{d}r\right) \mathbb{1}_{\exists r \in [s,t], |B_r| \geq r^{(\kappa+\eta)/2}} \middle| B_t = y\right]$$
$$\leq C \exp\left(-cs^{\eta\alpha/2}\right) \widetilde{G}(s,x;t,y). \tag{2.56}$$

Proof. We divide $[s,t]$ into shorter intervals by choosing $s = s_0 < s_1 < \cdots < s_n = t$ such that for any $0 \leq k \leq n-1$, $s_{k+1} \in [s_k + s_k^\kappa, s_k + 3s_k^\kappa]$. By a union bound, the left-hand side of (2.56) is at most

$$\sum_{k=0}^{n-1} \frac{e^{-(y-x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}}$$
$$\times \mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1+f(B_r,r))\,\mathrm{d}r\right) \mathbb{1}_{\exists r \in [s_k,s_{k+1}], |B_r| \geq r^{(\kappa+\eta)/2}} \middle| B_t = y\right],$$

and we now aim at proving, for any $0 \leq k \leq n-1$,

$$\frac{e^{-(y-x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}}$$
$$\times \mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1+f(B_r,r))\,\mathrm{d}r\right) \mathbb{1}_{\exists r \in [s_k,s_{k+1}], |B_r| \geq r^{(\kappa+\eta)/2}} \middle| B_t = y\right]$$
$$\leq C \exp\left(-cs_k^{\eta\alpha/2}\right) \widetilde{G}(s,x;t,y). \tag{2.57}$$

The desired result then follows by summing these inequalities and using that

$$\sum_{k=0}^{n-1} \exp\left(-cs_k^{\eta\alpha/2}\right) \leq C \exp\left(-cs^{\eta\alpha/2}\right). \tag{2.58}$$

Indeed, we have $s_{k+1} \geq s_k + s_k^\kappa \geq s_k + s^\kappa$, so $s_k \geq s + ks^\kappa$, and using this together with a sum-integral comparison proves (2.58) (up to a polynomial prefactor in s which can be absorbed in the exponential by modifying c). Hence, it remains to prove (2.57).

We start with the case $1 \leq k \leq n-2$. Applying Markov's property at times s_k and s_{k+1} and then Lemma 2.3.2.1 (with $M = 1$) after having restricted the range of integration, we get

$$\begin{aligned}\widetilde{G}(s,x;t,y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \widetilde{G}(s,x;s_k,x_k)\widetilde{G}(s_k,x_k;s_{k+1},x_{k+1})\widetilde{G}(s_{k+1},x_{k+1};t,y)\,\mathrm{d}x_k\,\mathrm{d}x_{k+1} \\ &\geq \frac{c}{s_{k+1}^{\kappa/2}}\int_{|x_{k+1}|\leq s_{k+1}^{\kappa/2}}\int_{|x_k|\leq s_k^{\kappa/2}} \widetilde{G}(s,x;s_k,x_k)\widetilde{G}(s_{k+1},x_{k+1};t,y)\,\mathrm{d}x_k\,\mathrm{d}x_{k+1} \\ &= \frac{c}{s_{k+1}^{\kappa/2}}\left(\int_0^{s_k^{\kappa/2}} \widetilde{G}^{|\cdot|}(s,x;s_k,x_k)\,\mathrm{d}x_k\right)\left(\int_0^{s_{k+1}^{\kappa/2}} \widetilde{G}^{|\cdot|}(s_{k+1},x_{k+1};t,y)\,\mathrm{d}x_{k+1}\right),\end{aligned} \tag{2.59}$$

using the definition of $\widetilde{G}^{|\cdot|}$ in (2.53) for the second integral in the last line, as well as (2.55) for the first integral. On the other hand, proceeding similarly but with Lemma 2.3.2.2, the left-hand side of (2.57) is at most

$$\begin{aligned}C\exp\left(-cs_{k+1}^{\eta\alpha/2}\right)&\left(\int_0^\infty \widetilde{G}^{|\cdot|}(s,x;s_k,x_k)\exp\left(-c\left(\frac{|x_k|}{s_k^{\kappa/2}}\right)^\alpha\right)\mathrm{d}x_k\right) \\ &\times\left(\int_0^\infty \widetilde{G}^{|\cdot|}(s_{k+1},x_{k+1};t,y)\exp\left(-c\left(\frac{|x_{k+1}|}{s_{k+1}^{\kappa/2}}\right)^\alpha\right)\mathrm{d}x_{k+1}\right).\end{aligned} \tag{2.60}$$

Then cutting $[0,\infty)$ into intervals of length $s_k^{\kappa/2}$, we get

$$\begin{aligned}\int_0^\infty \widetilde{G}^{|\cdot|}(s,x;s_k,x_k)&\exp\left(-c\left(\frac{|x_k|}{s_k^{\kappa/2}}\right)^\alpha\right)\mathrm{d}x_k \\ &\leq \sum_{\ell\geq 0} e^{-c\ell^\alpha}\int_{\ell s_k^{\kappa/2}}^{(\ell+1)s_k^{\kappa/2}} \widetilde{G}^{|\cdot|}(s,x;s_k,x_k)\,\mathrm{d}x_k \\ &\leq C\int_0^{s_k^{\kappa/2}} \exp\left(\frac{(x-x_k')^2}{2(s_k-s)}\right)\widetilde{G}^{|\cdot|}(s,x;s_k,x_k')\,\mathrm{d}x_k',\end{aligned}$$

where we used Lemma 2.3.3 to compare $\widetilde{G}^{|\cdot|}(s,x;s_k,x_k)$ and $\widetilde{G}^{|\cdot|}(s,x;s_k,x_k')$ with $x_k' = x_k - \ell s_k^{\kappa/2}$ and bounded $\exp\{(x-x_k')^2 - (x-x_k)^2/(2(s_k-s)\} \leq \exp\{(x-x_k')^2/(2(s_k-s)\}$. Using here that $|x| \leq s^{(\kappa+\varepsilon)/2}$, we have $(x-x_k')^2 \leq 4s_k^{\kappa+\varepsilon}$. Moreover, we bound $s_k - s \geq s_k - s_{k-1} \geq s_{k-1}^\kappa \geq (s_k/4)^\kappa$ and therefore we get

$$\int_0^\infty \widetilde{G}^{|\cdot|}(s,x;s_k,x_k)\exp\left(-c\left(\frac{|x_k|}{s_k^{\kappa/2}}\right)^\alpha\right)\mathrm{d}x_k \leq C\exp\left(Cs_k^\varepsilon\right)\int_0^{s_k^{\kappa/2}} \widetilde{G}^{|\cdot|}(s,x;s_k,x_k)\,\mathrm{d}x_k. \tag{2.61}$$

Proceeding similarly, the second integral in (2.60) is at most

$$\begin{aligned} C \int_0^{s_{k+1}^{\kappa/2}} \exp\left(\frac{(x_{k+1}-y)^2}{2(t-s_{k+1})}\right) \widetilde{G}^{|\cdot|}(s_{k+1},x_{k+1};t,y)\,\mathrm{d}x_{k+1} \\ \le C\exp\left(\frac{2t^{\kappa+\varepsilon}}{t-s_{k+1}}\right) \int_0^{s_{k+1}^{\kappa/2}} \widetilde{G}^{|\cdot|}(s_{k+1},x_{k+1};t,y)\,\mathrm{d}x_{k+1} \\ \le C\exp\left(Cs_{k+1}^{\varepsilon}\right) \int_0^{s_{k+1}^{\kappa/2}} \widetilde{G}^{|\cdot|}(s_{k+1},x_{k+1};t,y)\,\mathrm{d}x_{k+1}, \end{aligned} \tag{2.62}$$

where, in the first inequality, we bound $(x'_{k+1}-y)^2 \le 4t^{\kappa+\epsilon}$ and, in the second one, we note that, if $s_{k+1} \ge t/2$, then $t^{\kappa+\varepsilon}/(t-s_{k+1}) \le (2s_{k+1})^{\kappa+\varepsilon}/s_{k+1}^{\kappa} \le 2s_{k+1}^{\varepsilon}$ and, if $s_{k+1} \le t/2$, then $t^{\kappa+\varepsilon}/(t-s_{k+1}) \le 2t^{\kappa+\varepsilon-1} \le 2$ because $\varepsilon \le 1-\kappa$. Combining (2.59), (2.60), (2.61) and (2.62) proves (2.57), noting that the factors $s_{k+1}^{\kappa/2}$ and $\exp(Cs_{k+1}^{\varepsilon})$ can be absorbed into $\exp(-cs_{k+1}^{\eta\alpha/2})$ up to a modification of c (recall that $\eta > 2\varepsilon/\alpha$).

We now consider the case $k=0$. Proceeding as in (2.59) and (2.60) but applying the Markov property only at time $s_{k+1} = s_1$ yields

$$\widetilde{G}(s,x;t,y) \ge \frac{c}{s_1^{\kappa/2}} \int_0^{s_1^{\kappa/2}} \widetilde{G}^{|\cdot|}(s_1,x_1;t,y)\,\mathrm{d}x_1, \tag{2.63}$$

and shows that the left-hand side of (2.57) is at most

$$C\exp\left(-cs_1^{\eta\alpha/2}\right) \int_0^{\infty} \widetilde{G}^{|\cdot|}(s_1,x_1;t,y) \exp\left(-c\left(\frac{|x_1|}{s_1^{\kappa/2}}\right)^{\alpha}\right) \mathrm{d}x_1. \tag{2.64}$$

Then proceeding as in (2.62) proves (2.57). The case $k = n-1$ is covered similarly, but applying the Markov property only at time $s_k = s_{n-1}$. Finally note that this argument for cases $k=0$ and $k=n-1$ works only if $0 < n-1$. But in the case $n=1$, (2.56) is a direct consequence of Lemma 2.3.2.1 (with $M = t^{\varepsilon}$) and Lemma 2.3.2.2. □

2.3.1.3 Proof of Proposition 2.3.1

Proof of Proposition 2.3.1. We first note that increasing values of f results in decreasing $\widetilde{G}$. Therefore, it is enough to prove the result for $f = f^{-}_{L,a,b}$ and $f = f^{+}_{L,a,b}$. We assume we are in one of these two cases. Then, the function f satisfies the assumptions of Lemma 2.3.4: for $f = f^{+}_{L,a,b}$ this is direct with $r_0 = 1$ for example, and for $f^{-}_{L,a,b}$ this follows from our choice of η in its definition, with $r_0 = (2L)^{1/b}$).

Therefore, if $\varepsilon \leq 1-\kappa$ and $\eta > 2\varepsilon/\alpha$, we get

$$\begin{aligned}\widetilde{G}(s,x;t,y) &= (1+o(1))\frac{e^{-(y-x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}} \\ &\quad\times \mathbb{E}_{(s,x)}\left[\exp\left(-\beta\int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1+f(B_r,r))\,\mathrm{d}r\right)\mathbb{1}_{\forall r\in[s,t],|B_r|<r^{(\kappa+\eta)/2}}\middle| B_t = y\right],\end{aligned}$$

where the $o(1)$ holds as $s\to\infty$, uniformly in t,x,y,f (as required in the statement of the proposition). On the event $\{\forall r \in [s,t], |B_r| < r^{(\kappa+\eta)/2}\}$, using also $|f(y,r| \leq L(|\frac{y}{r}|^a + r^{-b})$, we can bound

$$\begin{aligned}\int_s^t \left|\frac{B_r}{r}\right|^\alpha f(B_r,r)\,\mathrm{d}r &\leq L\int_s^t r^{\alpha(\kappa+\eta)/2-\alpha}\left(r^{a(\kappa+\eta)/2-a} + r^{-b}\right)\mathrm{d}r \\ &= L\int_s^t \left(r^{-\kappa-a(1-\kappa/2)+(\alpha+a)\eta/2} + r^{-\kappa-b+\alpha\eta/2}\right)\mathrm{d}r, \qquad (2.65)\end{aligned}$$

using in particular that $\alpha\kappa/2-\alpha = -\kappa$. The two exponents on the right-hand side of (2.65) can be made smaller than -1 by choosing

$$\eta < \eta_0 \coloneqq 2\left(\frac{a(1-\frac{\kappa}{2})-(1-\kappa)}{a+\alpha} \wedge \frac{b-(1-\kappa)}{\alpha}\right). \qquad (2.66)$$

By our assumptions on a and b we have $\eta_0 > 0$, so if we choose $\varepsilon < \alpha\eta_0/2$, it is possible to choose such an η which also satisfies the previous condition $\eta > 2\varepsilon/\alpha$. Then, the right-hand side of (2.65) is a $o(1)$ in the same sense as before. Hence, we get that $\widetilde{G}(s,x;t,y)$ equals

$$\begin{aligned}&(1+o(1))\frac{e^{-(y-x)^2/[2(t-s)]}}{\sqrt{2\pi(t-s)}}\mathbb{E}_{(s,x)}\left[\exp\left(-\beta\int_s^t\left|\frac{B_r}{\sqrt{2r}}\right|^\alpha \mathrm{d}r\right)\mathbb{1}_{\forall r\in[s,t],|B_r|<r^{(\kappa+\eta)/2}}\middle| B_t = y\right] \\ &= (1+o(1))G(s,x;t,y),\end{aligned}$$

where the last equality follows from Lemma 2.3.4 but with $f=0$ (for which $\widetilde{G}=G$). This concludes the proof. □

2.3.2 Bounding the total mass of $\widetilde{G}$

We now bound the total mass of $\widetilde{G}$, which can be written as

$$\int_{\mathbb{R}} \widetilde{G}(s,x;t,y)\,\mathrm{d}y = \mathbb{E}_{(s,x)}\left[\exp\left(-\beta\int_s^t\left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1+f(B_r,r))\,\mathrm{d}r\right)\right]. \qquad (2.67)$$

This is analogous to Corollary 2.2.2 for G.

Proposition 2.3.5. *Let $\alpha \in (0,2)$ and $\beta > 0$. Let $\vartheta_1 > 0$ be defined as in Proposition 2.2.1. Let $L > 0$, $a > (1-\kappa)/(1-\frac{\kappa}{2})$ and $b > 1-\kappa$. Let $\eta > 0$. There exist $K, C, c > 0$ such that the following holds for any $s \geq K$ and $t \geq s + Ks^\kappa$ the following holds.*

1. *For any $x \in \mathbb{R}$ and any function f satisfying $f \geq f^-_{L,a,b}$,*

$$\int_{\mathbb{R}} \widetilde{G}(s,x;t,y)\,\mathrm{d}y \leq C(t/s)^{\kappa/4} \exp\left(\vartheta_1(s^{1-\kappa} - t^{1-\kappa})\right). \tag{2.68}$$

2. *For any $|x| \leq s^{\kappa/2}$ and any function f satisfying $f \leq f^+_{L,a,b}$,*

$$\mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1+f(B_r,r))\,\mathrm{d}r\right) \mathbb{1}_{|B_t| \leq t^{\kappa/2}} \mathbb{1}_{\forall r \in [s,t], |B_r| \leq r^{(\kappa+\eta)/2}}\right]$$
$$\geq c(t/s)^{\kappa/4} \exp\left(\vartheta_1(s^{1-\kappa} - t^{1-\kappa})\right). \tag{2.69}$$

Proof. Part 1. By monotonicity, it is enough to prove the result for $f = f^-_{L,a,b}$. Then, $f(\cdot, r)$ is even so we can assume $x \geq 0$ by symmetry and write $f(B_r, r) = f(|B_r|, r)$ in (2.67) so that the expectation on the right-hand side of (2.67) can be seen as depending only on a reflected Brownian motion starting at (s,x). But one can couple[3] a reflected Brownian motion X^1 starting at (s,x) with a reflected Brownian motion X^2 starting at $(s,0)$ such that $X^1_r \geq X^2_r$ for any $r \geq s$. Recalling that the function $y \in [0,\infty) \mapsto (y/r)^\alpha(1+f(y,r))$ is non-decreasing (by definition of $f^-_{L,a,b}$), this shows the expectation is maximal for $x = 0$ so we can focus on this case.

Now, we decompose

$$\int_{\mathbb{R}} \widetilde{G}(s,0;t,y)\,\mathrm{d}y = \int_{|y| \leq t^{\kappa/2}} \widetilde{G}(s,0;t,y)\,\mathrm{d}y + \int_{|y| \geq t^{\kappa/2}} \widetilde{G}(s,0;t,y)\,\mathrm{d}y \tag{2.70}$$

In the first term in (2.70), we can bound $\widetilde{G}(s,0;t,y)$ by $2G(s,0;t,y)$ by Proposition 2.3.1 and then get the desired bound by Corollary 2.2.2.1. For the second term in (2.70), applying the Markov property at time $r \in (s,t)$ such that $r + r^\kappa = t$, it is at most

$$\int_{w \in \mathbb{R}} \widetilde{G}(s,0;r,w) \left(\int_{|y| \geq t^{\kappa/2}} \widetilde{G}(r,w;t,y)\,\mathrm{d}y\right) \mathrm{d}w$$
$$\leq C \int_{w \in \mathbb{R}} \widetilde{G}(s,0;r,w) \exp\left(-c\left(\frac{|w|}{r^{\kappa/2}}\right)^\alpha\right) \mathrm{d}w, \tag{2.71}$$

---

[3] This is again a Doeblin coupling as in Footnote 2, but in the simpler context of Markov processes instead of Markov bridges. Consider $\widetilde{X}^2$ a reflected Brownian motion starting at $(s,0)$ independent of X^1, and define X^2 as $\widetilde{X}^2$ up to the first hitting time of X^1 and $\widetilde{X}^2$, and then as X^1. Here the fact that X^2 is a reflected Brownian motion starting at $(s,0)$ follows from the usual strong Markov property.

by Lemma 2.3.2.2 (with $\eta = 0$, note that the indicator is automatically satisfied because $|y| \geq t^{\kappa/2}$). Then, by definition of $\widetilde{G}^{|\cdot|}$ and cutting the integral, the right-hand side of (2.71) is at most

$$C \sum_{\ell \geq 0} e^{-c\ell^\alpha} \int_{\ell r^{\kappa/2}}^{(\ell+1)r^{\kappa/2}} \widetilde{G}^{|\cdot|}(s,0;r,w) \, \mathrm{d}w \leq C \int_0^{r^{\kappa/2}} \widetilde{G}^{|\cdot|}(s,0;r,w) \, \mathrm{d}w, \tag{2.72}$$

using that $\widetilde{G}^{|\cdot|}(s,0;r,w) \leq 2\widetilde{G}^{|\cdot|}(s,0;r,w')$ for any $0 \leq w' \leq w$ by Lemma 2.3.3. Finally, on the right-hand side of (2.71), we can bound $\widetilde{G}^{|\cdot|}(s,0;r,w)$ by $2G^{|\cdot|}(s,0;r,w)$ by Proposition 2.3.1 and then get the desired bound by Corollary 2.2.2.1 (note that $r^{1-\kappa} \geq t^{1-\kappa} - 1$). Note that we have to take the constant K larger than in Corollary 2.2.2 to ensure that we can apply Corollary 2.2.2 between times s and r.

Part 2. By monotonicity again, it is enough to prove the result for $f = f^+_{L,a,b}$. We first claim that, for any $|x| \leq s^{\kappa/2}$,

$$\int_{|y| \leq t^{\kappa/2}} \widetilde{G}(s,x;t,y) \, \mathrm{d}y \geq c(t/s)^{\kappa/4} \exp\left(\vartheta_1(s^{1-\kappa} - t^{1-\kappa})\right). \tag{2.73}$$

Indeed, we can apply Proposition 2.3.1 to write $\widetilde{G}(s,x;t,y) \geq G(s,x;t,y)/2$ on the right-hand side, and then Corollary 2.2.2.2 concludes the proof of (2.73). Then, note that the difference between the left-hand side of (2.73) and the left-hand side of (2.69) equals

$$\begin{aligned} &\mathbb{E}_{(s,x)}\left[\exp\left(-\beta \int_s^t \left|\frac{B_r}{\sqrt{2r}}\right|^\alpha (1 + f(B_r, r)) \, \mathrm{d}r\right) \mathbb{1}_{|B_t| \leq t^{\kappa/2}} \mathbb{1}_{\exists r \in [s,t], |B_r| > r^{(\kappa+\eta)/2}}\right] \\ &\leq C \exp\left(-cs^{\eta\alpha/2}\right) \int_{|y| \leq t^{\kappa/2}} \widetilde{G}(s,x;t,y) \, \mathrm{d}y \end{aligned}$$

by Lemma 2.3.4. Choosing K large enough the prefactor in front of the last integral is less than $1/2$ and we get the desired result. □

2.3.3 Bounding the tail of $\widetilde{G}$

We prove here the following bound on the tail of $\widetilde{G}(s,x;t,y)$ for y in some time-dependent window.

Proposition 2.3.6. *Let $\alpha \in (0,2)$ and $\beta > 0$. Let $L > 0$, $a > (1-\kappa)/(1-\frac{\kappa}{2})$ and $b > 1 - \kappa$. There exist $C, c > 0$ such that, for s large enough, for any $t \geq 2s$, $x \in \mathbb{R}$, $|y| \leq t$, and any function f satisfying $f \geq f^-_{L,a,b}$,*

$$\widetilde{G}(s,x;t,y) \leq \frac{C}{(st)^{\kappa/4}} \exp\left(\vartheta_1(s^{1-\kappa} - t^{1-\kappa})\right) \exp\left(-c\left(\frac{|y|}{t^{\kappa/2}}\right)^{(2+\alpha)/2}\right).$$

Note that, up to constant factors, this bound is better than the upper bound given by Proposition 2.2.1 for G: the exponent for the tail in y would be α there (because of the error term), whereas it is $(2+\alpha)/2$ here. Moreover, $(2+\alpha)/2$ is the exponent appearing in the tail of φ_0 (see Proposition 2.2.7.5) and it reappears here via a probabilistic argument.

Proof. By monotonicity, it is enough to deal with the case $f = f^-_{L,a,b}$. Then, by symmetry of $f(\cdot, r)$, we can assume w.l.o.g. that $y \geq 0$. Let $r \in [3t/4, t - t^\kappa]$ which will be chosen in terms of y later. Applying Markov's property at time r, we get

$$\widetilde{G}(s,x;t,y) = \int_{\mathbb{R}} \widetilde{G}(s,x;r,w)\widetilde{G}(r,w;t,y)\,\mathrm{d}w. \tag{2.74}$$

To bound $\widetilde{G}(r,w;t,y)$, we use $f \geq -1/2$ and then distinguish according to whether the event $E = \{\forall q \in [r,t], B_q \geq y/2\}$ holds or not: this yields

$$\begin{aligned}\widetilde{G}(r,w;t,y) &\leq \frac{e^{-(y-w)^2/[2(t-r)]}}{\sqrt{2\pi(t-r)}}\mathbb{E}_{(r,w)}\left[\exp\left(-\frac{\beta}{2}\int_r^t \left|\frac{B_q}{\sqrt{2q}}\right|^\alpha \mathrm{d}q\right)\middle| B_t = y\right] \\ &\leq \frac{e^{-(y-w)^2/[2(t-r)]}}{\sqrt{2\pi(t-r)}}\left(\exp\left(-\frac{\beta}{2}\int_r^t \left|\frac{y/2}{\sqrt{2q}}\right|^\alpha \mathrm{d}q\right) + \mathbb{P}_{(r,w)}\left(E^c | B_t = y\right)\right).\end{aligned}$$

If $w \geq 3y/4$, then $\mathbb{P}_{(r,w)}(E^c|B_t = y) \leq e^{-y^2/[4(t-r)]}$ by (2.50). On the other hand, if $w < 3y/4$ then $e^{-(y-w)^2/[2(t-r)]} \leq e^{-y^2/[8(t-r)]}$. Combining this and using $q \geq r \geq 3t/4$ yields

$$\widetilde{G}(r,w;t,y) \leq \frac{1}{\sqrt{t-r}}\left(e^{-c(t-r)y^\alpha/t^\alpha} + e^{-y^2/[8(t-r)]}\right). \tag{2.75}$$

Note that the right-hand side is optimized when r is chosen such that $t-r$ is of the same order as $y^{(2-\alpha)/2}t^{\alpha/2} = (y/t^{\kappa/2})^{(2-\alpha)/2}t^\kappa$ recalling $\kappa = 2\alpha/(2+\alpha)$. This motivates the following choice, which also fulfils the constraint $r \in [3t/4, t-t^\kappa]$ (recall $y \in [0,t]$):

$$t - r := \left(\frac{1}{4}\left(\frac{y}{t^{\kappa/2}}\right)^{(2-\alpha)/2} \vee 1\right)t^\kappa.$$

With this choice of r, using that $\sqrt{t-r} \geq t^{\kappa/2}$, (2.75) becomes

$$\widetilde{G}(r,w;t,y) \leq \frac{C}{t^{\kappa/2}}\exp\left(-c\left(\frac{y}{t^{\kappa/2}}\right)^{(2+\alpha)/2}\right), \tag{2.76}$$

where we used in the case $t - r = t^\kappa$ (or equivalently $y \leq 4^{2/(2-\alpha)}t^{\kappa/2}$) that the exponential on the right-hand side is lower bounded by a positive constant. Plugging this into (2.74) and applying Proposition 2.3.5.1 yields

$$\widetilde{G}(s,x;t,y) \leq C(r/s)^{\kappa/4}\exp\left(\vartheta_1(s^{1-\kappa} - r^{1-\kappa})\right)\cdot\frac{1}{t^{\kappa/2}}\exp\left(-c\left(\frac{y}{t^{\kappa/2}}\right)^{(2+\alpha)/2}\right). \tag{2.77}$$

Finally, using that $(1-x)^{1-\kappa} \geq 1 - Cx$ for $x \in [0,1/4]$, we get

$$t^{1-\kappa} - r^{1-\kappa} = t^{1-\kappa}\left(1 - \left(1 - \frac{t-r}{t}\right)^{1-\kappa}\right) \leq Ct^{1-\kappa}\frac{t-r}{t} = C\left(\frac{1}{4}\left(\frac{y}{t^{\kappa/2}}\right)^{(2-\alpha)/2} \vee 1\right).$$

Note that the exponent $(2-\alpha)/2$ appearing here is smaller than the exponent $(2+\alpha)/2$ appearing in the last exponential in (2.77). Therefore, when $y \geq Kt^{\kappa/2}$ with K large enough chosen in terms of the previous constants c and C, the error made when replacing $r^{1-\kappa}$ by $t^{1-\kappa}$ on the right-hand side of (2.77) can be included in the last exponential factor (up to replacing c by $c/2$). On the other hand, when $y \leq Kt^{\kappa/2}$, the error can simply be bounded by a constant factor. This gives the desired result. □

2.4 Many–to–few lemmas

Two typical tools from the study of branching processes are the many–to–one and many–to–two lemmas. They allow us to reduce the certain expectations of the branching process to expectations of just one or two particles.

Lemma 2.4.1 (many–to–one)**.** *Let f be a measurable functional of t and the path of a particle up to time t. We then have for all $x, y \in \mathbb{R}^2$ and $t > 0$ that*

$$\mathbb{E}_{(x,y)}\left[\sum_{u\in\mathcal{N}(t)} f\left((X_u(s), Y_u(s))_{s\in[0,t]}\right)\right] = \mathbb{E}_{(x,y)}\left[f\left((X_s, Y_s)_{s\in[0,t]}\right)\exp\left(\int_0^t b(X_s, Y_s)\,\mathrm{d}s\right)\right],$$

where $(X_s, Y_s)_{s\in[0,t]}$ is a Brownian motion on $\mathbb{R}^2$

For a proof see [53, Theorem 8.5], they deal with the one–dimensional case but the proof does not change. Similarly, we need a version of the many–to–two lemma for our process. Again we do not prove this as a proof is messy, a more general result is the many–to–few lemma [56, Lemma 1] which also incorporates the inhomogeneous branching. To state the many–to–two lemma, we need two particles ξ^1 and $\xi^{2,r}$ that move as $\mathbb{R}^2$–Brownian motions such that $\xi_s^{2,r} = \xi_s^1$ until time r after which they move independently.

Lemma 2.4.2 (many–to–two)**.** *Let f and g be measurable functionals of t and the path of a particle up to time t. We then have for all $x, y \in \mathbb{R}^2$ and $t > 0$ that*

$$\begin{aligned}&\mathbb{E}_{(x,y)}\left[\sum_{u,v\in\mathcal{N}(t),u\neq v} f\left((X_u(s), Y_u(s))_{s\in[0,t]}\right) g\left((X_v(s), Y_v(s))_{s\in[0,t]}\right)\right]\\ &= \int_0^t \mathbb{E}_{(x,y)}\left[f\left((\xi_s^1)_{s\in[0,t]}\right) g\left((\xi_s^{2,r})_{s\in[0,t]}\right) b(\xi_r^1)\exp\left(\int_0^t b(\xi_s^1)\,\mathrm{d}s + \int_r^t b(\xi_s^{2,r})\,\mathrm{d}s\right)\right] 2\,\mathrm{d}r.\end{aligned} \tag{2.78}$$

Proof. We derive this expression from [56, Lemma 1]. There, we have a stopping time T which is given by

$$\inf\left\{t \geq 0 : \int_0^t b(\xi_s^1)\,\mathrm{d}s > \hat{T}\right\},$$

where $\hat{T}$ is an exponential random variable of rate 2 which is independent of ξ^1 and ξ^2. Observe that conditional on $(\xi_s^1)_{s\geq 0}$ we have

$$\mathbb{P}(T > r|\xi^1) = \exp\left(-2\int_0^r b(\xi_s^1)\,\mathrm{d}s\right),$$

and therefore

$$\mathbb{P}(T \in \mathrm{d}r|\xi^1) = 2b(\xi_r^1)\exp\left(-2\int_0^r b(\xi_s^1)\,\mathrm{d}s\right)\mathrm{d}r.$$

Then [56, Lemma 1] states that the left-hand side of (2.78) equals

$$\mathbb{E}_{(x,y)}\Bigg[\mathbb{1}_{\{T\leq t\}} f\left((\xi_s^1)_{s\in[0,t]}\right) g\left((\xi_s^{2,T})_{s\in[0,t]}\right) \\ \times \exp\left(\int_0^T b(\xi_s^1)\,\mathrm{d}s + \int_0^t b(\xi_s^1)\,\mathrm{d}s + \int_0^t b(\xi_s^{2,T})\,\mathrm{d}s\right)\Bigg].$$

Conditioning on ξ^1 and ξ^2, and using the density of T yields our version of the many–to–two lemma. □

2.5 Upper bound for the maximum

The goal of this section is to show that with high probability there are no particles with a modulus much greater than $m(t)$ in the sense that

$$\limsup_{a\to\infty}\limsup_{t\to\infty}\mathbb{P}\left(M_t \geq m(t)+a\right) = 0.$$

This shows the upper half of the tightness claimed in Theorem 2.1.1.

2.5.1 An upper bound on the branching rate

We start this subsection by introducing a useful event. First note that, dominating by a one-dimensional BBM with branching rate 1, we have (see e.g. [66, Eq. (20)])

$$\max_{u\in\mathcal{N}(s)} X_u(s) - \sqrt{2}s \xrightarrow[s\to\infty]{\text{a.s.}} -\infty.$$

Therefore, setting

$$A_{s_0} := \left\{\forall s \geq s_0,\ \max_{u\in\mathcal{N}(s)} X_u(s) \leq \sqrt{2}s - 1\right\}, \tag{2.79}$$

we have

$$\mathbb{P}\left(A_{s_0}\right) \xrightarrow[s_0\to\infty]{} 1. \tag{2.80}$$

Hence, in this section, we can and will regularly restrict our study to the event A_{s_0} with s_0 large enough. It is also sometimes useful to consider the larger event

$$A_{s_0,t} := \left\{ \forall s \in [s_0,t],\, \max_{u\in\mathcal{N}(s)} X_u(s) \leq \sqrt{2}s - 1 \right\}. \tag{2.81}$$

In particular, working on these events is useful to get the following upper bound on the branching rate. Recall $(X_s, Y_s)_{s\in[0,t]}$ denotes a Brownian motion on $\mathbb{R}^2$ and $(R_s, \theta_s)_{s\in[0,t]}$ denotes a representation of its polar coordinates.

Lemma 2.5.1. *There exists $\sigma, L > 0$ such that, for any $0 \leq s_0 \leq t$, on the event $\{\forall s \in [s_0,t], X_s \leq \sqrt{2}s$ and $|\theta_s| < \sigma\}$ as well as on the event $\{\forall s \in [s_0,t], X_s \leq \sqrt{2}s$ and $|Y_s| < \sigma s\}$, we have*

$$\int_0^t b(\theta_s)\,\mathrm{d}s \leq t - \beta \int_{s_0}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha (1 + f(Y_s,s))\,\mathrm{d}s,$$

where $f = f^-_{L,2-\alpha,1} = -\left(\left[L\left(\left|\frac{y}{s}\right|^{2-\alpha} + s^{-1}\right)\right] \wedge \eta\right)$ where $\eta \in (0,1/2]$ is chosen in (2.47).

Note that the parameters $a = 2-\alpha$ and $b = 1$ appearing here in $f^-_{L,a,b}$ satisfy the assumptions $a > 1 - \frac{\alpha}{2}$ and $b > 1 - \kappa$ of the propositions of Section 2.3.

Proof. By Assumption (A2), $b(\theta) = 1 - \beta|\theta|^\alpha + O(\theta^2)$ as $\theta \to 0$, so there exist $K > 0$ and $\sigma_0 \in (0, \pi/2)$ such that, for any $|\theta| \leq \sigma_0$, $b(\theta) \leq 1 - \beta|\theta|^\alpha + K\theta^2$. Moreover, one can choose σ_0 small enough such that the function $\theta \in [0,\sigma_0] \mapsto 1 - \beta|\theta|^\alpha + K\theta^2$ is decreasing. On the other hand, by Assumption (A1) or (A1'), $\sup_{[-\pi,\pi]\setminus[-\sigma_0,\sigma_0]} b < 1$. Hence, choosing σ_0 small enough, we get

$$b(\theta) \leq \bar{b}(\theta) := \begin{cases} 1 - \beta|\theta|^\alpha + K\theta^2, & \text{if } \theta \in [-\sigma_0,\sigma_0], \\ 1 - \beta|\sigma_0|^\alpha + K\sigma_0^2, & \text{if } \theta \in [-\pi,\pi] \setminus [-\sigma_0,\sigma_0], \end{cases}$$

and the function $\bar{b}$ is non-increasing on $[0,\pi]$.

We now work on $E_1 := \{\forall s \in [s_0,t], X_s \leq \sqrt{2}s$ and $|\theta_s| < \sigma\}$ or $E_2 := \{\forall s \in [s_0,t], X_s \leq \sqrt{2}s$ and $|Y_s| < \sigma s\}$. Then, for any $s \in [s_0,t]$ such that $\theta_s \in [0,\pi/2)$, we have

$$\theta_s = \arctan\left(\frac{Y_s}{X_s}\right) \geq \arctan\left(\frac{Y_s}{\sqrt{2}s}\right) \geq \frac{Y_s}{\sqrt{2}s} - \frac{1}{3}\left(\frac{Y_s}{\sqrt{2}s}\right)^3,$$

and therefore

$$b(\theta_s) \leq \bar{b}(\theta_s) \leq \bar{b}\left(\frac{Y_s}{\sqrt{2}s} - \frac{1}{3}\left(\frac{Y_s}{\sqrt{2}s}\right)^3\right).$$

Then, one can choose σ small enough such that the last argument of $\bar{b}$ necessarily belongs to $[0, \sigma_0]$ both on E_1 and on E_2, and therefore we get, for some $L = L(K, \sigma_0, \alpha, \beta) > 0$,

$$b(\theta_s) \leq 1 - \beta\left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha \left(1 - L\left|\frac{Y_s}{s}\right|^{2-\alpha}\right), \tag{2.82}$$

and the same holds if $\theta_s \in (-\pi/2, 0]$ by the same argument. On E_1, it is enough to consider the case $\theta_s \in (-\pi/2, \pi/2)$ by choosing $\sigma < \pi/2$. On E_2, we could also have $\theta_s \in [-\pi, \pi] \backslash (-\pi/2, \pi/2)$, but then $b(\theta_s) \leq 1 - \beta\left|\sigma_0\right|^\alpha + K\sigma_0^2$ and the inequality (2.82) stays true if σ is chosen small enough. Finally, σ can be chosen small enough such that, on E_1 or on E_2, $L|Y_s/s|^{2-\alpha} \leq \eta$, where η appears in the definition of $f^-_{L,2-\alpha,1}$ in (2.47), so that (2.82) becomes

$$b(\theta_s) \leq 1 - \beta\left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha \left(1 + f^-_{L,2-\alpha,1}(Y_s, s)\right),$$

Then, the result is obtained by integrating this inequality for $s \in [s_0, t]$ and by using $b \leq 1$ for $s \in [0, s_0]$. □

2.5.2 The pseudo-derivative martingale

For $t \geq 0$, let

$$Z_t := t^{-\kappa/4} \exp\left(\vartheta_1 t^{1-\kappa}\right) \sum_{u \in \mathcal{N}(t)} (\sqrt{2}t - X_u(t))e^{\sqrt{2}(X_u(t) - \sqrt{2}t)}. \tag{2.83}$$

This quantity naturally appears in the upper bound argument when computing the conditional first moment given $\mathcal{F}_t$, so we need to bound its first moment. Note that this is not a martingale, but it is defined analogously to the derivative martingale for the standard BBM introduced by Lalley and Sellke [66]. The goal of this section is to prove the following result.

Lemma 2.5.2. *For any $s_0 > 0$, there exists $C > 0$ such that, for any $t \geq s_0$,*

$$\mathbb{E}\left[Z_t \mathbb{1}_{A_{s_0,t}}\right] \leq C.$$

Proof. By the many–to–one lemma (Lemma 2.4.1),

$$\begin{aligned}&\mathbb{E}\left[Z_t \mathbb{1}_{A_{s_0,t}}\right]\\ &\leq t^{-\kappa/4} \exp\left(\vartheta_1 t^{1-\kappa}\right) \mathbb{E}\left[\exp\left(\int_0^t b(\theta_s)\,\mathrm{d}s\right)(\sqrt{2}t - X_t)e^{\sqrt{2}X_t - 2t}\mathbb{1}_{\forall s \in [s_0,t], X_s \leq \sqrt{2}s}\right].\end{aligned}$$

Let $\sigma > 0$ be the constant given by Lemma 2.5.1. Define the events $E \coloneqq \{\forall s \in [s_0, t], |Y_s| \leq \sigma s\}$ and, for $k \geq 1$,

$$F_k \coloneqq \{\exists s \in [2^{k-1}s_0, 2^k s_0) : |Y_s| > \sigma s\} \cap \{\forall s \in [2^k s_0, t] : |Y_s| \leq \sigma s\}.$$

Let $k_t = \min\{k \geq 1 : 2^k s_0 \geq t\}$. We have $E^c \subset \bigcup_{k=1}^{k_t} F_k$, so we use the bound $1 \leq \mathbb{1}_E + \sum_{k=1}^{k_t} \mathbb{1}_{F_k}$.

Using Lemma 2.5.1, the part on the event E is at most

$$\begin{aligned} & t^{-\kappa/4} \exp\left(\vartheta_1 t^{1-\kappa}\right) \\ & \quad \times \mathbb{E}\left[\exp\left(-\beta \int_{s_0}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha (1 + f(Y_s, s))\,\mathrm{d}s\right) (\sqrt{2}t - X_t) e^{\sqrt{2}X_t - t} \mathbb{1}_{\forall s \in [s_0,t], X_s \leq \sqrt{2}s}\right] \\ & \quad \leq C s_0^{-\kappa/4} \exp\left(\vartheta_1 s_0^{1-\kappa}\right) \times \mathbb{E}\left[(\sqrt{2}s_0 - X_{s_0}) e^{\sqrt{2}X_{s_0} - s_0}\right], \end{aligned} \tag{2.84}$$

using independence between X and Y, Proposition 2.3.5.1 for the Y-contribution and the fact that $((\sqrt{2}t - X_t)e^{\sqrt{2}X_t - t}\mathbb{1}_{\forall s \in [s_0,t], X_s \leq \sqrt{2}s})_{t \geq s_0}$ is a martingale. The right-hand side of (2.84) is a constant depending on s_0, so it concludes this part.

We now bound the part on event F_k for some $1 \leq k \leq k_t$. Using Lemma 2.5.1 between times $2^k s_0$ and t, this part is at most

$$\begin{aligned} & t^{-\kappa/4} \exp\left(\vartheta_1 t^{1-\kappa}\right) \mathbb{E}\left[\exp\left(-\beta \int_{2^k s_0}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha (1 + f(Y_s, s))\,\mathrm{d}s\right) \mathbb{1}_{\exists s \in [2^{k-1}s_0, 2^k s_0) : |Y_s| > \sigma s}\right] \\ & \quad \times \mathbb{E}\left[(\sqrt{2}t - X_t) e^{\sqrt{2}X_t - t} \mathbb{1}_{\forall s \in [s_0,t], X_s \leq \sqrt{2}s}\right] \\ & \leq C(s_0)(2^k s_0)^{-\kappa/4} \exp\left(\vartheta_1 (2^k s_0)^{1-\kappa}\right) \mathbb{P}\left(\exists s \in [2^{k-1}s_0, 2^k s_0) : |Y_s| > \sigma s\right), \end{aligned} \tag{2.85}$$

proceeding as in (2.84). Then, this last probability can be bounded by

$$\begin{aligned} & \mathbb{P}\left(\max_{s \in [0, 2^k s_0]} |Y_s| > \sigma 2^{k-1} s_0\right) \\ & \qquad \leq 2 \cdot \mathbb{P}\left(\max_{s \in [0, 2^k s_0]} Y_s > \sigma 2^{k-1} s_0\right) = 2 \cdot \mathbb{P}\left(|Y_{2^k s_0}| > \sigma 2^{k-1} s_0\right), \end{aligned} \tag{2.86}$$

using that $\max_{s \in [0,T]} Y_s$ has the same distribution as $|Y_T|$ for any $T > 0$. Together with the tail bound $\mathbb{P}(|Y_T| \geq x) \leq 2e^{-x^2/(2T)}$ for $x, T > 0$, we get that the right-hand side of (2.85) is at most $C(s_0)\exp(\vartheta_1(2^k s_0)^{1-\kappa} - \frac{\sigma^2}{8} 2^k s_0)$. Summing over $k \geq 0$, this is bounded by $C(s_0)$, so it concludes the proof. □

2.5.3 Localization of the trajectory of an extremal particle

In this section, we prove several properties of the trajectory of an extremal particle, that is a particle $u \in \mathcal{N}(t)$ such that $R_u(t) \geq m(t) + O(1)$. In this first lemma, we show that such a particle typically has a small angle after time $t/2$. This allows us afterwards to restrict ourselves to angles smaller than σ given by Lemma 2.5.1, so that we can bound the branching rate.

Lemma 2.5.3. *For any $\eta > 0$, there exists $c > 0$ such that, for any $a \geq 0$, for t large enough,*

$$\mathbb{P}\left(\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) - a,\ \max_{s\in[t/2,t]} |\theta_u(s)| > \eta\right) \leq e^{-ct}.$$

Proof. Before diving into the proof, we recall some facts on the polar coordinates $(R_s, \theta_s)_{s\geq 0}$ of the planar Brownian motion. Firstly, $(R_s)_{s\geq 0}$ is a 2-dimensional Bessel process starting from $r = (x^2 + y^2)^{1/2}$ and its density at time $s > 0$ is given by (see [27, Eq. 4.1.0.6])

$$\mathbb{P}_{(x,y)}(R_s \in \mathrm{d}z) = \frac{z}{s} e^{-(r^2+z^2)/2s} I_0\left(\frac{rz}{s}\right) \mathrm{d}z, \tag{2.87}$$

where I_0 is the modified Bessel function defined by $I_0(z) = \sum_{k\geq 0}(z/2)^{2k}/(k!)^2$. Note that, for $z \geq 0$, $I_0(z) \leq (e^{z/2})^2 = e^z$, so we get the following upper bound for the density of R_s:

$$\mathbb{P}_{(x,y)}(R_s \in \mathrm{d}z) \leq \frac{z}{s} e^{-(z-r)^2/2s} \,\mathrm{d}z. \tag{2.88}$$

Concerning the angular coordinate $(\theta_s)_{s\geq 0}$, note that the representation can be chosen to be continuous, except at $s = 0$ if the Brownian motion starts from the origin. Moreover, we have the following skew-product representation: for any $s \geq 0$ ($s > 0$ if the Brownian motion starts from the origin), there exists a Brownian motion W on $\mathbb{R}$ starting from 0 at time 0 and independent of R such that

$$\forall t \geq s, \quad \theta_t = \theta_s + W_{\tau(t)}, \qquad \text{with } \tau(t) := \int_s^t R_q^{-2} \,\mathrm{d}q \tag{2.89}$$

see e.g. [61, Corollary 19.7]. We now split the proof into three steps.

Step 1. We first prove that an extremal particle needs to have a large radial coordinate on $[t/2, t]$, more precisely: for any $a \geq 0$, for t large enough,

$$\mathbb{P}\left(\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) - a,\ \min_{s\in[t/2,t]} R_u(s) < \frac{\sqrt{2}}{4}t\right) \leq e^{-ct}. \tag{2.90}$$

As in the statement of the lemma, the constant c throughout the proof does not depend on a, but the threshold involved in the "t large enough" statement can. By the many–to–one lemma (Lemma 2.4.1) with the crude bound $b(\theta_s) \leq 1$, we can bound the probability in (2.90) by

$$e^t \cdot \mathbb{P}\left(R_t \geq m(t) - a,\ \min_{s\in[t/2,t]} R_s < \frac{\sqrt{2}}{4}t\right) = e^t \cdot \mathbb{E}\left[\phi(T_0)\mathbb{1}_{T_0<t}\right], \tag{2.91}$$

using Markov property at the stopping time $T_0 \coloneqq \inf\{s \geq t/2 : R_s \leq \sqrt{2}t/4\}$ and setting $\phi(s) \coloneqq \mathbb{P}_{\sqrt{2}t/4}(R_{t-s} \geq m(t) - a)$ for $s \in [t/2, t)$, where $(R_q)_{q \geq 0}$ starts from $\sqrt{2}t/4$ at time 0 under $\mathbb{P}_{\sqrt{2}t/4}$. Using the upper bound (2.88), we get, for t large enough and uniformly in $s \in [t/2, t)$,

$$\phi(s) \leq \int_{m(t)-a}^{\infty} \frac{z}{t-s} e^{-(z-\sqrt{2}t/4)^2/2(t-s)} \, \mathrm{d}z \leq \frac{Ct}{\sqrt{t-s}} e^{-(m(t)-a-\sqrt{2}t/4)^2/(t-s)} \leq e^{-9t/8+o(t)}.$$

Coming back to (2.91), this proves (2.90).

Step 2. We now prove that the angle of an extremal particle cannot be far from 0 on the whole interval $[t/2, t]$, that is, for any $a \geq 0$, for t large enough,

$$\mathbb{P}\left(\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) - a, \min_{s \in [t/2,t]} |\theta_u(s)| > \frac{\eta}{2}\right) \leq e^{-ct}. \tag{2.92}$$

By Assumption (A1) or (A1'), we have $c(\eta) \coloneqq \sup_{[-\pi,\pi] \setminus [-\eta/2,\eta/2]} b < 1$. Therefore, using the many–to–one lemma and bounding $b(\theta_s)$ by $c(\eta)$ for $s \in [t/2, t]$ and by 1 otherwise, we get that the probability in (2.92) is at most

$$\exp\left(\frac{t}{2} + \frac{t}{2} c(\eta)\right) \cdot \mathbb{P}\left(R_t \geq m(t) - a\right) = \exp\left(\frac{t}{2}(c(\eta) - 1) + o(t)\right),$$

using that $\mathbb{P}\left(R_t \geq m(t) - a\right) = e^{-t+o(t)}$ as a consequence of (2.88) and standard Gaussian bounds. This proves (2.92).

Step 3. Finally, it is enough to prove that, for any $a \geq 0$, for t large enough,

$$\mathbb{P}\Bigg(\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) - a, \min_{s \in [t/2,t]} R_u(s) \geq \frac{\sqrt{2}}{4} t, \\ \min_{s \in [t/2,t]} |\theta_u(s)| \leq \frac{\eta}{2}, \max_{s \in [t/2,t]} |\theta_u(s)| > \eta\Bigg) \leq e^{-ct}. \tag{2.93}$$

The key idea is that this event requires an angular displacement of at least $\eta/2$ over the time interval $[t/2, t]$, which has a cost exponential in t for a 2d Brownian motion with radius of order t. We first apply the many–to–one lemma with the bound $b(\theta_s) \leq 1$ to get that the probability in (2.93) is at most

$$e^t \cdot \mathbb{P}\left(R_t \geq m(t) - a, \min_{s \in [t/2,t]} R_s \geq \frac{\sqrt{2}}{4} t, T_1 \leq t, T_2 \leq t\right), \tag{2.94}$$

with $T_1 \coloneqq \inf\{s \geq t/2 : |\theta_s| \leq \frac{\eta}{2}\}$ and $T_2 \coloneqq \inf\{s \geq t/2 : |\theta_s| \geq \eta\}$. There are two cases to distinguish, $T_1 < T_2$ and $T_2 < T_1$, but they are treated similarly, so we focus on the case $T_1 < T_2$. By Markov property at time T_1,

$$\mathbb{P}\left(R_t \geq m(t) - a, \min_{s \in [t/2,t]} R_s \geq \frac{\sqrt{2}}{4} t, T_1 < T_2 \leq t\right) \leq \mathbb{E}\left[\chi(T_1, R_{T_1}) \mathbb{1}_{T_1 < t}\right], \tag{2.95}$$

where we set, for any $s \in [t/2, t)$ and $r > 0$,

$$\chi(s,r) \coloneqq \mathbb{P}_{(s,r,\eta/2)} \left(R_t \geq m(t) - a, \min_{q \in [s,t]} R_q \geq \frac{\sqrt{2}}{4} t, T_2 \leq t \right),$$

where under $\mathbb{P}_{(s,r,\eta/2)}$ the process $(R_q, \theta_q)_{q \geq s}$ starts from $(r, \eta/2)$ at time s. Now, using the representation of the angle in (2.89), we get

$$\begin{aligned} \chi(s,r) &\leq \mathbb{P}_{(s,r,\eta/2)} \left(R_t \geq m(t) - a, \min_{q \in [s,t]} R_q \geq \frac{\sqrt{2}}{4} t, \max_{p \in [0,\tau(t)]} W_p \geq \frac{\eta}{2} \right) \\ &\leq \mathbb{P}_{(s,r,\eta/2)} \left(R_t \geq m(t) - a, \max_{p \in [0,4/t]} W_p \geq \frac{\eta}{2} \right) \\ &\leq 2 \exp\left(-\frac{\eta^2 t}{32} \right) \cdot \mathbb{P}_{(s,r,\eta/2)} \left(R_t \geq m(t) - a \right), \end{aligned}$$

using the independence of R and W and proceeding as in (2.86) to bound the probability involving W. Coming back to (2.95) and using again Markov property at time T_1, the right-hand side of (2.95) is at most

$$2 \exp\left(-\frac{\eta^2 t}{32} \right) \cdot \mathbb{P}\left(R_t \geq m(t) - a \right) = \exp\left(-\frac{\eta^2 t}{32} - t + o(t) \right) \leq e^{-t-ct}.$$

Proceeding similarly in the case $T_2 < T_1$, we get that (2.94) is bounded by e^{-ct}, which concludes the proof of (2.93) and hence of the lemma. □

In the next lemma, we prove that the final position of an extremal particle u needs to have a final y-coordinate $Y_u(t)$ of order roughly $t^{\kappa/2}$, which is the typical space scale of the Brownian motion weighted by an integral via PDEs studied in Sections 2.2 and 2.3. In particular, we deduce $Y_u(t) = o(\sqrt{t})$, which together with $R_u(t) \geq m(t)$ implies that $R_u(t) = X_u(t) + o(1)$, allowing us afterwards to replace the radial coordinate of an extremal particle by its x-coordinate. We also note that this lemma, together with the previous lemma, implies Proposition 2.1.2.

Lemma 2.5.4. *There exists $\eta > 0$, such that, for any $\varepsilon > 0$ and $a \geq 0$, we have*

$$\mathbb{P}\left(\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) - a, \max_{s \in [t/2,t]} |\theta_u(s)| \leq \eta, |Y_u(t)| \geq t^{\frac{\kappa}{2}+\varepsilon} \right) \xrightarrow[t\to\infty]{} 0.$$

Proof. We assume that $\eta \leq \sigma$, where σ is given by Lemma 2.5.1. By (2.80), it is enough to work on the event A_{s_0} for some large s_0. By a union bound over $u \in \mathcal{N}(t)$, the branching property at time $t/2$, the many–to–one lemma (Lemma 2.4.1) and Lemma 2.5.1, we get

$$\begin{aligned}
&\mathbb{P}\left(A_{s_0} \cap \left\{\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) - a,\ \max_{s\in[t/2,t]} |\theta_u(s)| \leq \eta, |Y_u(t)| \geq t^{\frac{\kappa}{2}+\varepsilon}\right\} \middle| \mathcal{F}_{t/2}\right) \\
&\leq e^{t/2}\mathbb{1}_{A_{s_0,t/2}} \sum_{u\in\mathcal{N}(t/2)} \mathbb{E}_{(t/2,X_u(t/2),Y_u(t/2))}\left[\exp\left(-\beta\int_{t/2}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha (1+f(Y_s,s))\,\mathrm{d}s\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\times\, \mathbb{1}_{R_t\geq m(t)-a, |Y_t|\in[t^{\kappa/2+\varepsilon},2\eta t]}\right],
\end{aligned} \tag{2.96}$$

also noting that, on the event in the probability, we have $|Y_u(t)| = |X_u(t)\tan\theta_u(t)| \leq 2\eta t$, by choosing η small enough such that $\tan\eta \leq \sqrt{2}\eta$. Then, for any $x \leq t/\sqrt{2} - 1$ (recall that on $A_{s_0,t/2}$ we have $X_u(t/2) \leq \sqrt{2}\cdot t/2 - 1$) and for any $y \in \mathbb{R}$, we have to bound

$$\begin{aligned}
&\mathbb{E}_{(t/2,x,y)}\left[\exp\left(-\beta\int_{t/2}^t \left|\frac{Y_s}{\sqrt{2}s}\right|^\alpha (1+f(Y_s,s))\,\mathrm{d}s\right)\mathbb{1}_{R_t\geq m(t)-a, |Y_t|\in[t^{\kappa/2+\varepsilon},2\eta t]}\right] \\
&= \int_{|z|\in[t^{\kappa/2+\varepsilon},2\eta t]} \mathbb{P}_{(t/2,x)}\left(\sqrt{X_t^2+z^2} \geq m(t)-a\right)\widetilde{G}(t/2,y;t,z)\,\mathrm{d}z,
\end{aligned} \tag{2.97}$$

where $\widetilde{G}$ is defined in (2.46). Choosing $\eta < 1/\sqrt{2}$, we have, for t large enough, for any $|z| \leq 2\eta t$, $|z| \leq m(t)-a$, and therefore $\sqrt{(m(t)-a)^2 - z^2} \geq m(t) - a - z^2/(m(t)-a)^2 \geq m(t) - a - z^2/(2t)$. Hence, for t large enough, for any $x \leq t/\sqrt{2}$ and $|z| \leq 2\eta t$, we get

$$\begin{aligned}
&\mathbb{P}_{(t/2,x)}\left(\sqrt{X_t^2+z^2} \geq m(t)-a\right) \\
&\qquad = \mathbb{P}\left(\mathcal{N}(0,t/2) \geq \sqrt{(m(t)-a)^2-z^2} - x\right) \\
&\qquad \leq \mathbb{P}\left(\mathcal{N}(0,t/2) \geq m(t) - a - \frac{z^2}{2t} - x\right) \\
&\qquad \leq \exp\left(-\frac{t}{2} - \sqrt{2}\overline{x} + \vartheta_1 t^{1-\kappa} - \frac{\overline{x}^2}{t} + \frac{z^2}{t^2}\left(\frac{t}{\sqrt{2}} + \overline{x}\right) + O(\log t)\right),
\end{aligned}$$

where we wrote $m(t) - a - x = \frac{t}{\sqrt{2}} + \overline{x} - \frac{\vartheta_1}{\sqrt{2}}t^{1-\kappa} + O(\log t)$ with $\overline{x} := t/\sqrt{2} - x \geq 1$, we used that $\kappa \in (1/2,1)$, and noted that $m(t) - a - x - z^2/(2t) \geq 0$ if η is chosen small enough in order to apply the Gaussian tail bound $\mathbb{P}(\mathcal{N}(0,v) \geq u) \leq e^{-u^2/2v}$ for $u \geq 0$. Furthermore, maximizing in $\overline{x}$, we can bound

$$\frac{z^2}{t^2}\left(\frac{t}{\sqrt{2}} + \overline{x}\right) - \frac{\overline{x}^2}{t} \leq \frac{z^2}{\sqrt{2}t} + \frac{z^4}{4t^3} \leq \frac{z^2}{t},$$

by choosing η small enough. Coming back to (2.97), we get

$$\begin{aligned}
&\mathbb{E}_{(t/2,x,y)}\left[\exp\left(-\beta\int_{t/2}^{t}\left|\frac{Y_s}{\sqrt{2}s}\right|^{\alpha}(1+f(Y_s,s))\,\mathrm{d}s\right)\mathbb{1}_{R_t\geq m(t)-a,|Y_t|\in[t^{\kappa/2+\varepsilon},2\eta t]}\right]\\
&\leq \exp\left(-\frac{t}{2}-\sqrt{2}\overline{x}+\vartheta_1 t^{1-\kappa}+O(\log t)\right)\int_{|z|\in[t^{\kappa/2+\varepsilon},2\eta t]} e^{z^2/t}\widetilde{G}(t/2,y;t,z)\,\mathrm{d}z\\
&\leq \exp\left(-\frac{t}{2}-\sqrt{2}\overline{x}+\vartheta_1 (t/2)^{1-\kappa}+O(\log t)\right)\int_{|z|\in[t^{\kappa/2+\varepsilon},2\eta t]} e^{z^2/t-c|z|^{(2+\alpha)/2}t^{-\alpha/2}}\,\mathrm{d}z,
\end{aligned}$$

using Proposition 2.3.6 and that $\kappa = 2\alpha/(2+\alpha)$. Note that, if η is chosen small enough, one has $z^2/t - c\,|z|^{(2+\alpha)/2}\,t^{-\alpha/2} \leq -\frac{1}{2}c\,|z|^{(2+\alpha)/2}\,t^{-\alpha/2}$ for $|z|\leq 2\eta t$. Thus, the last integral is a $O(\exp(-c' t^{\varepsilon(2+\alpha)/2}))$. Coming back to (2.96), we get

$$\begin{aligned}
&\mathbb{P}\left(A_{s_0}\cap\left\{\exists u\in\mathcal{N}(t): R_u(t)\geq m(t)-a, \max_{s\in[t/2,t]}|\theta_u(s)|\leq\eta, |Y_u(t)|\geq t^{\frac{\kappa}{2}+\varepsilon}\right\}\middle|\mathcal{F}_{t/2}\right)\\
&\leq \exp\left(\vartheta_1(t/2)^{1-\kappa}-c' t^{\varepsilon(2+\alpha)/2}+O(\log t)\right)\cdot\mathbb{1}_{A_{s_0,t/2}}\sum_{u\in\mathcal{N}(t/2)} e^{\sqrt{2}(X_u(t/2)-\sqrt{2}\cdot t/2)}\\
&\leq \exp\left(-c' t^{\varepsilon(2+\alpha)/2}+O(\log t)\right)\cdot\mathbb{1}_{A_{s_0,t/2}}Z_{t/2},
\end{aligned}$$

using that $1\leq(\sqrt{2}\cdot t/2 - X_u(t/2))$ on $A_{s_0,t/2}$. We conclude by taking the expectation and applying Lemma 2.5.2. $\square$

In the next lemma, we improve the straight barrier given by the event A_{s_0} to replace it by a curved barrier which includes the same polynomial correction as $m(t)$. For $s>0$, let

$$m^+(s) := \sqrt{2}s - \frac{\vartheta_1}{\sqrt{2}}s^{1-\kappa} + 10\log s. \tag{2.98}$$

Note that $m^+(s)$ is above $m(s)$, at a distance of order $\log s$. The coefficient 10 here has been arbitrarily chosen.

Lemma 2.5.5. *We have*

$$\mathbb{P}\left(\exists u\in\mathcal{N}(t): \max_{s\in[t/2,t]}(X_u(s)-m^+(s))>0\right)\xrightarrow[t\to\infty]{}0.$$

Proof. In order to discretize time afterwards, we first argue that with high probability no particle moves by more than 1 on a time interval of length t^{-2} between times $t/2$ and t: by a union bound over s and over $u\in\mathcal{N}(s)$ and the many-to-one lemma (bounding the branching rate by 1), we have

$$\begin{aligned}
&\mathbb{P}\left(\exists s\in[t/2,t+t^{-2}]\cap t^{-2}\mathbb{Z}, \exists u\in\mathcal{N}(s): \max_{r\in[s-t^{-2},s]}|X_u(r)-X_u(s)|>1\right)\\
&\leq \sum_{s\in[t/2,t+t^{-2}]\cap t^{-2}\mathbb{Z}} e^s\cdot\mathbb{P}\left(\max_{r\in[0,t^{-2}]}|B_r|>1\right)\leq t^3\cdot e^t\cdot 4e^{-t^2/2}\xrightarrow[t\to\infty]{}0,
\end{aligned}$$

using in the second inequality that $\max_{r\in[0,t^{-2}]} B_r$ is distributed as $|B_{t^{-2}}|$ and a classical Gaussian tail bound. It follows that it is sufficent to prove that

$$\mathbb{P}\left(\exists u \in \mathcal{N}(t) : \max_{s\in[t/2,t]\cap t^{-2}\mathbb{Z}} (X_u(s) - m^+(s) + 1) > 0\right) \xrightarrow[t\to\infty]{} 0. \tag{2.99}$$

Recalling that $\mathbb{P}\left(A^c_{s_0}\right) \to 0$ as $s_0 \to \infty$ by (2.80), it is enough to prove that

$$\mathbb{P}\left(A_{s_0} \cap \left\{\exists u \in \mathcal{N}(t) : \max_{s\in[t/2,t]\cap t^{-2}\mathbb{Z}} (X_u(s) - m^+(s) + 1) > 0\right\}\right) \xrightarrow[t\to\infty]{} 0, \tag{2.100}$$

for any fixed $s_0 > 0$. Using a union bound over $s \in [t/2, t] \cap t^{-2}\mathbb{Z}$, the left-hand side of (2.100) is at most

$$\begin{aligned}
&\sum_{s\in[t/2,t]\cap t^{-2}\mathbb{Z}} \mathbb{P}\left(A_{s_0} \cap \left\{\exists u \in \mathcal{N}(s) : X_u(s) > m^+(s) - 1\right\}\right) \\
&\le \sum_{s\in[t/2,t]\cap t^{-2}\mathbb{Z}} \Bigg(\mathbb{P}\left(A_{s_0} \cap \left\{\exists u \in \mathcal{N}(s) : X_u(s) > m^+(s) - 1, \max_{r\in[s/2,s]} |\theta_u(r)| \le \sigma\right\}\right) \\
&\qquad\qquad + e^{-cs}\Bigg),
\end{aligned}$$

where $\sigma > 0$ is given by Lemma 2.5.1 and c is then given by Lemma 2.5.3, noting that $X_u(s) > m^+(s) - 1$ implies $R_u(s) > m(s)$. Therefore, it is now enough to prove that

$$\mathbb{P}\left(A_{s_0} \cap \left\{\exists u \in \mathcal{N}(t) : X_u(t) > m^+(t) - 1, \max_{s\in[t/2,t]} |\theta_u(s)| \le \sigma\right\}\right) = o\left(t^{-3}\right), \tag{2.101}$$

as $t \to \infty$, for any fixed $s_0 > 0$.

We first work conditionally on $\mathcal{F}_{t/2}$. By a union bound over $u \in \mathcal{N}(t)$, the branching property at time $t/2$, the many–to–one lemma (Lemma 2.4.1) and Lemma 2.5.1, we get

$$\begin{aligned}
&\mathbb{P}\left(A_{s_0} \cap \left\{\exists u \in \mathcal{N}(t) : X_u(t) > m^+(t) - 1, \max_{s\in[t/2,t]} |\theta_u(s)| \le \sigma\right\}\middle|\mathcal{F}_{t/2}\right) \\
&\le e^{t/2}\mathbb{1}_{A_{s_0,t/2}} \sum_{u\in\mathcal{N}(t/2)} \mathbb{E}_{(t/2,X_u(t/2),Y_u(t/2))}\Bigg[\exp\left(-\beta\int_{t/2}^{t}\left|\frac{Y_s}{\sqrt{2s}}\right|^{\alpha}(1 + f(Y_s, s))\,\mathrm{d}s\right) \\
&\qquad\qquad \times \mathbb{1}_{X_t>m^+(t)-1}\Bigg] \\
&\le e^{t/2}\mathbb{1}_{A_{s_0,t/2}} \sum_{u\in\mathcal{N}(t/2)} C\exp\left(\vartheta_1\left((t/2)^{1-\kappa} - t^{1-\kappa}\right)\right)\mathbb{P}_{(t/2,X_u(t/2))}\left(X_t > m^+(t) - 1\right),
\end{aligned} \tag{2.102}$$

using the independence between X and Y and applying Proposition 2.3.5.1 to the Y part. For the X contribution, note that, under $\mathbb{P}_{(t/2,X_u(t/2))}$, X_t is Gaussian with mean $X_u(t/2)$ and variance $t/2$. Moreover, note that on event $A_{s_0,t/2}$ we have $X_u(t/2) \leq \sqrt{2} \cdot t/2$ so that $m^+(t) - X_u(t/2) \geq 0$. Hence, we can apply the Gaussian tail bound $\mathbb{P}(\mathcal{N}(0,v) \geq a) \leq e^{-a^2/2v}$ for $a \geq 0$ to get

$$\begin{aligned}&\mathbb{P}_{(t/2,X_u(t/2))}\left(X_t \geq m^+(t) - 1\right) \\ &\leq \exp\left(-t/2 - \sqrt{2}(\sqrt{2}t/2 - X_u(t/2)) + \vartheta_1 t^{1-\kappa} - 10\sqrt{2}\log t + \sqrt{2}\right).\end{aligned}$$

Therefore, the right-hand side of (2.102) is at most

$$Ct^{-10\sqrt{2}} \exp\left(\vartheta_1 (t/2)^{1-\kappa}\right) \mathbb{1}_{A_{s_0,t/2}} \sum_{u \in \mathcal{N}(t/2)} e^{-\sqrt{2}(\sqrt{2}t/2 - X_u(t/2))} \leq Ct^{-10\sqrt{2}+\kappa/4} \mathbb{1}_{A_{s_0,t/2}} Z_{t/2},$$

using that on $A_{s_0,t/2}$ we have $1 \leq (\sqrt{2}t/2 - X_u(t/2))$. Taking the expectation and applying Lemma 2.5.2 yields (2.101) and concludes the proof. ☐

We conclude this subsection by the following corollary which gathers the three previous lemmas and is the starting point of the proof of the upper bound of Theorem 2.1.1 in the next subsection.

Corollary 2.5.6. *Recall the definition of* $m^+(s)$ *in* (2.98). *For any* $\eta > 0$ *and* $a \geq 0$, *as* $t \to \infty$,

$$\begin{aligned}&\mathbb{P}\left(M_t \geq m(t) + a + 1\right) \\ &\qquad \leq \mathbb{P}\Bigg(\exists u \in \mathcal{N}(t) : X_u(t) \geq m(t) + a,\ \max_{s \in [t/2,t]} |\theta_u(s)| \leq \eta, \\ &\qquad\qquad \max_{s \in [t/2,t]} X_u(s) - m^+(s) \leq 0\Bigg) + o(1).\end{aligned}$$

Proof. Let $\eta > 0$ and $a \geq 0$. First note that if the result holds for some η, then it directly holds for any other $\eta' > \eta$, hence we can assume that η is smaller than the one given by Lemma 2.5.4. Fix $\varepsilon > 0$ such that $\kappa/2 + \varepsilon < 1/2$. By Lemmas 2.5.3 and 2.5.4, we have

$$\begin{aligned}&\mathbb{P}\left(M_t \geq m(t) + a + 1\right) \\ &= \mathbb{P}\left(\exists u \in \mathcal{N}(t) : R_u(t) \geq m(t) + a + 1,\ \max_{s \in [t/2,t]} |\theta_u(s)| \leq \eta, |Y_u(t)| < t^{\frac{\kappa}{2}+\varepsilon}\right) + o(1).\end{aligned}$$

Now consider $u \in \mathcal{N}(t)$ satisfying the properties in the last probability. We can assume $\eta < \pi/2$ so that $X_u(t) \geq 0$. Then, we have $R_u(t) = X_u(t) + O(Y_u(t)^2/X_u(t)) =$

$X_u(t) + o(1)$, because $X_u(t)$ is necessarily of order t while $Y_u(t) = o(\sqrt{t})$. Therefore, we have $X_u(t) \geq m(t) + a$ for t large enough (but deterministic). This proves

$$\mathbb{P}\left(M_t \geq m(t) + a + 1\right) \leq \mathbb{P}\left(\exists u \in \mathcal{N}(t) : X_u(t) \geq m(t) + a, \max_{s \in [t/2,t]} |\theta_u(s)| \leq \eta\right) + o(1).$$

The conclusion of the corollary then follows directly from Lemma 2.5.5. ☐

2.5.4 Proof of the upper bound

Proof of the upper bound in Theorem 2.1.1. We prove here that

$$\limsup_{a \to \infty} \limsup_{t \to \infty} \mathbb{P}\left(M_t \geq m(t) + a\right) = 0. \tag{2.103}$$

Let $\sigma > 0$ be the constant given by Lemma 2.5.1. By Corollary 2.5.6 and (2.80), it is enough to prove, for any $s_0 > 0$,

$$\limsup_{a \to \infty} \limsup_{t \to \infty} \mathbb{P}\left(E\right) = 0, \tag{2.104}$$

where we define

$$E \coloneqq A_{s_0,t/2} \cap \Bigg\{ \exists u \in \mathcal{N}(t) : X_u(t) \geq m(t) + a, \\ \max_{s \in [t/2,t]} |\theta_u(s)| \leq \sigma, \max_{s \in [t/2,t]} X_u(s) \leq m^+(s) \Bigg\},$$

which depends implicitly on t, a and s_0. We summarize shortly the proof before diving into the details. Let $0 < \delta < \kappa$ and set $t_0 = t - t^\delta$. The proof consists of three steps: we first work conditionally on $\mathcal{F}_{t_0}$, then on $\mathcal{F}_{t/2}$ and finally bound the non-conditional probability. A key idea for the first step is that the upper barrier at $m^+(s)$ is too crude to get a good bound on the end of the trajectory. Instead, we compare the x-coordinate of our BBM on time interval $[t_0, t]$ to a one-dimensional BBM with branching rate 1, for which sharp bounds are known. This comparison is sufficient because $\delta < \kappa$, so the time window t^δ is smaller than the typical time scale t^κ where the fact of having a varying branching rate plays a role. The second step deals with the time interval $[t/2, t_0]$ and consists in calculating the conditional first moment of the bound obtained in the first step, and in bounding it in terms of the pseudo-derivative martingale $Z_{t/2}$. The final step, which deals with $[0, t/2]$ is a direct application of Lemma 2.5.2.

Step 1. Fix some $a \geq 0$. We condition on $\mathcal{F}_{t_0}$ and count the number of particles at time t_0 which have a descendant that exceeds $m(t)+a$ at time t,

$$\begin{aligned}\mathbb{P}(E|\mathcal{F}_{t_0}) \leq \mathbb{1}_{A_{s_0,t/2}} \sum_{u\in\mathcal{N}(t_0)} &\mathbb{1}_{\forall s\in[t/2,t_0],|\theta_u(s)|\leq\sigma}\mathbb{1}_{\forall s\in[t/2,t_0],X_u(s)\leq m^+(s)} \\ &\times \mathbb{P}_{(t_0,X_u(t_0),Y_u(t_0))}\left(\max_{v\in\mathcal{N}(t),u\leq v} X_v(t) \geq m(t)+a\right). \end{aligned}\tag{2.105}$$

Next, we can dominate $\max_{v\in\mathcal{N}(t),u\leq v} X_v(t)$ by the maximum of a one-dimensional BBM with branching rate 1, call $M_s^{1\mathrm{d}}$ its maximum at time s,

$$\mathbb{P}_{(t_0,X_u(t_0),Y_u(t_0))}\left(\max_{v\in\mathcal{N}(t),u\leq v} X_v(t) \geq m(t)+a\right) \leq \mathbb{P}_{(0,X_u(t_0))}\left(M_{t^\delta}^{1\mathrm{d}} \geq m(t)+a\right), \tag{2.106}$$

where we used that $t-t_0=t^\delta$. Now, we use the following bound for the tail of $M_s^{1\mathrm{d}}$, which follows from [4, Corollary 10]: letting $m^{1\mathrm{d}}(s)=\sqrt{2}s-\frac{3}{2\sqrt{2}}\log s$ for $s>1$, there exists $C>0$ such that, for any $x\in\mathbb{R}$ and $s\geq 1$,

$$\mathbb{P}\left(M_s^{1\mathrm{d}} \geq m^{1\mathrm{d}}(s)+x\right) \leq C(x\vee 1)e^{-\sqrt{2}x}\exp\left(-\frac{x_+^2}{4s}\right), \tag{2.107}$$

where $x_+=\max(x,0)$. We apply this to (2.106) with $s=t^\delta$ and

$$x = m(t)-m^{1\mathrm{d}}(t^\delta)-X_u(t_0)+a = \sqrt{2}t_0-X_u(t_0)+a-\frac{\vartheta_1}{\sqrt{2}}t^{1-\kappa}-\left(\frac{3}{2}(1-\delta)-\frac{\kappa}{4}\right)\frac{\log t}{\sqrt{2}}.$$

Moreover, recall $t_0=t-t^\delta$ and $\delta\in(0,\kappa)$, so we have $t^{1-\kappa}=t_0^{1-\kappa}+o(1)$ and

$$m(t)-m^{1\mathrm{d}}(t^\delta)+a = m^+(t_0)-\left(10\sqrt{2}+\frac{3}{2}(1-\delta)-\frac{\kappa}{4}\right)\frac{\log t}{\sqrt{2}}+a+o(1) \leq m^+(t_0).$$

for t large enough depending only on a. Hence, we get

$$\begin{aligned}\mathbb{P}_{(0,X_u(t_0))}\left(M_{t^\delta}^{1\mathrm{d}} \geq m(t)+a\right) \leq C\left((m^+(t_0)-X_u(t_0))\vee 1\right)e^{-\sqrt{2}(\sqrt{2}t_0-X_u(t_0)+a)+\vartheta_1 t^{1-\kappa}} \\ \times t^{\frac{3}{2}(1-\delta)-\frac{\kappa}{4}}\exp\left(-\frac{(m(t)-m^{1\mathrm{d}}(t^\delta)-X_u(t_0)+a)_+^2}{4t^\delta}\right).\end{aligned}$$

Note that the last exponential is a non-increasing function of a, so we can replace a by 0. Going back to (2.105), this yields

$$\mathbb{P}(E|\mathcal{F}_{t_0}) \leq Ce^{-\sqrt{2}a}e^{\vartheta_1 t^{1-\kappa}}t^{\frac{3}{2}(1-\delta)-\frac{\kappa}{4}}\mathbb{1}_{A_{s_0,t/2}}\Upsilon_{t_0}, \tag{2.108}$$

where we set

$$\begin{aligned}\Upsilon_{t_0} := \sum_{u\in\mathcal{N}(t_0)} (m^+(t_0)-X_u(t_0)+1)e^{\sqrt{2}(X_u(t_0)-\sqrt{2}t_0)}\exp\left(-\frac{(m(t)-m^{1\mathrm{d}}(t^\delta)-X_u(t_0))_+^2}{4t^\delta}\right) \\ \times \mathbb{1}_{\forall s\in[t/2,t_0],|\theta_u(s)|\leq\sigma}\mathbb{1}_{\forall s\in[t/2,t_0],X_u(s)\leq m^+(s)},\end{aligned}$$

and used that $X_u(t_0) \leq m^+(t_0)$ to bound $(m^+(t_0) - X_u(t_0)) \vee 1 \leq m^+(t_0) - X_u(t_0) + 1$.

Step 2. We now aim at bounding $\mathbb{P}(E|\mathcal{F}_{t/2})$. By (2.108), it is enough to bound $\mathbb{E}[\Upsilon_{t_0}|\mathcal{F}_{t/2}]$. By the branching property at time $t/2$, the many–to–one lemma (Lemma 2.4.1) and Lemma 2.5.1 (note that $X_u(s) \leq m^+(s)$ ensures that $X_u(s) \leq \sqrt{2}s$), we get

$$\begin{aligned}
&\mathbb{E}[\Upsilon_{t_0}|\mathcal{F}_{t/2}] \\
&\quad\leq \sum_{u \in \mathcal{N}(t/2)} \mathbb{E}_{(t/2, X_u(t/2), Y_u(t/2))} \Bigg[\exp\left(\left(t_0 - \frac{t}{2} \right) - \beta \int_{t/2}^{t_0} \left| \frac{Y_s}{\sqrt{2}s} \right|^{\alpha} (1 + f(Y_s, s)) \, \mathrm{d}s \right) \\
&\qquad\qquad \times (m^+(t_0) - X_{t_0} + 1) e^{\sqrt{2}(X_{t_0} - \sqrt{2}t_0)} \exp\left(- \frac{(m(t) - m^{1\mathrm{d}}(t^\delta) - X_{t_0})_+^2}{4t^\delta} \right) \\
&\qquad\qquad\qquad \times \mathbb{1}_{\forall s \in [t/2, t_0], X_s \leq m^+(s)} \Bigg].
\end{aligned}$$

Then, using the independence between X and Y and applying Proposition 2.3.5 to the Y part, we get

$$\mathbb{E}[\Upsilon_{t_0}|\mathcal{F}_{t/2}] \leq \exp\left(t_0 - \frac{t}{2} + \vartheta_1 \left((t/2)^{1-\kappa} - t_0^{1-\kappa} \right) \right) \sum_{u \in \mathcal{N}(t/2)} \chi(\sqrt{2} \cdot t/2 - X_u(t/2)), \tag{2.109}$$

where we set, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\chi(x) \coloneqq \mathbb{E}_{(t/2, \sqrt{2} \cdot t/2 - x)} \Bigg[&(m^+(t_0) - X_{t_0} + 1) e^{\sqrt{2}(X_{t_0} - \sqrt{2}t_0)} \\
&\times \exp\left(- \frac{(m(t) - m^{1\mathrm{d}}(t^\delta) - X_{t_0})_+^2}{4t^\delta} \right) \mathbb{1}_{\forall s \in [t/2, t_0], X_s \leq m^+(s)} \Bigg].
\end{aligned}$$

By Girsanov's theorem and invariance of Brownian motion by reflection, note that, under the measure with density $e^{-\sqrt{2}(X_{t_0} - [\sqrt{2} \cdot t/2 - x]) + (t_0 - t/2)}$ w.r.t. $\mathbb{P}_{(t/2, \sqrt{2} \cdot t/2 - x)}$, the process $(\sqrt{2}s - X_s)_{s \geq t/2}$ has the same distribution as $(X_s)_{s \geq t/2}$ under $\mathbb{P}_{(t/2, x)}$. Therefore, we get

$$\begin{aligned}
\chi(x) = e^{-\sqrt{2}x - t_0 + \frac{t}{2}} \mathbb{E}_{(t/2, x)} \Bigg[&(X_{t_0} - f(t_0) + 1) \exp\left(- \frac{(m(t) - m^{1\mathrm{d}}(t^\delta) + X_{t_0} - \sqrt{2}t_0)_+^2}{4t^\delta} \right) \\
&\times \mathbb{1}_{\forall s \in [t/2, t_0], X_s \geq f(s)} \Bigg],
\end{aligned}$$

with $f(s) \coloneqq \sqrt{2}s - m^+(s) = \frac{\vartheta_1}{\sqrt{2}} s^{1-\kappa} - 10 \log s$. Note that $\chi(x) = 0$ if $x < f(t/2)$, so we consider now $x \geq f(t/2)$. Since f is concave on $[t/2, t_0]$ for t large enough, it stays above the straight line between its endpoint $(t/2, f(t/2))$ and $(t_0, f(t_0))$, that we denote by f_{line}. Therefore, for any $y \geq f(t_0)$, we have

$$\begin{aligned}\mathbb{P}_{(t/2,x)}\left(\forall s \in [t/2, t_0], X_s \geq f(s) | X_{t_0} = y\right) \\ &\leq \mathbb{P}_{(t/2,x)}\left(\forall s \in [t/2, t_0], X_s \geq f_{\text{line}}(s) | X_{t_0} = y\right) \\ &\leq \frac{2(x - f(t/2))(y - f(t_0))}{(t_0 - t/2)},\end{aligned}$$

by [32, Lemma 2]. Therefore, integrating w.r.t. X_{t_0} first and noting that the density of X_{t_0} under $\mathbb{P}_{(t/2,x)}$ is upper bounded by $1/\sqrt{2\pi(t_0 - t/2)}$, we get

$$\chi(x) \leq \frac{Cxe^{-\sqrt{2}x - t_0 + \frac{t}{2}}}{(t_0 - t/2)^{3/2}} \int_{f(t_0)}^{\infty} (y - f(t_0) + 1)^2 \exp\left(-\frac{(m(t) - m^{1\mathrm{d}}(t^\delta) + y - \sqrt{2}t_0)_+^2}{4t^\delta}\right) \mathrm{d}y.$$

Then, recalling that $t^{1-\kappa} = t_0^{1-\kappa} + o(1)$, we have $m(t) - m^{1\mathrm{d}}(t^\delta) - \sqrt{2}t_0 = -f(t_0) + O(\log t)$ and therefore, the integral in the last displayed equation is at most $Ct^{3\delta/2}$ for t large enough. On the other hand, we have $t_0 - t/2 \geq t/3$ for t large enough. Combining this, we get

$$\chi(x) \leq Cxt^{(\delta-1)3/2} e^{-\sqrt{2}x - t_0 + \frac{t}{2}}.$$

Coming back to (2.108) and (2.109) and using again that $t_0^{1-\kappa} = t^{1-\kappa} + o(1)$, we obtained that, for any $a \geq 0$ for t large enough,

$$\begin{aligned}\mathbb{P}(E|\mathcal{F}_{t/2}) &\leq Ce^{-\sqrt{2}a} e^{\vartheta_1 (t/2)^{1-\kappa}} t^{-\frac{\kappa}{4}} \mathbb{1}_{A_{s_0,t/2}} \sum_{u \in \mathcal{N}(t/2)} (\sqrt{2} \cdot t/2 - X_u(t/2)) e^{-\sqrt{2}(\sqrt{2} \cdot t/2 - X_u(t/2))} \\ &= Ce^{-\sqrt{2}a} Z_{t/2} \mathbb{1}_{A_{s_0,t/2}},\end{aligned} \tag{2.110}$$

recalling the definition of the pseudo-derivative martingale $Z_{t/2}$ in (2.83).

Step 3. Taking the expectations of (2.110) and applying Lemma 2.5.2, we finally get: for any $a \geq 0$, for t large enough,

$$\mathbb{P}(E) \leq Ce^{-\sqrt{2}a}.$$

This implies (2.104) and therefore concludes the proof. □

2.6 Lower bound for the maximum

The goal of this section is to show that with high probability there will be particles at time t with a modulus close to $m(t)$ in the sense that

$$\liminf_{a \to \infty} \liminf_{t \to \infty} \mathbb{P}\left(M_t \geq m(t) - a\right) = 1.$$

This shows the lower half of the tightness claimed in Theorem 2.1.1. The proof relies on a first and second moment calculation on the number of such particles satisfying further path conditions.

2.6.1 A lower bound on the branching rate

Before defining precisely the set of particles we consider, we present a key tool for the proof, which is a lower bound for the branching rate of particles with appropriate trajectories. Recall that Lemma 2.5.1 provided a upper bound for the branching rate.

Lemma 2.6.1. *Let $\varepsilon \in (0, 2\kappa - 1)$. There exist $\sigma, L > 0$ such that, for s_0 large enough and $t \geq s_0$, on the event $\{\forall s \in [s_0, t] : X_s \geq \frac{m(t)}{t} s - s^{(1+\varepsilon)/2}, |Y_s| \leq \sigma s\}$, we have*

$$\int_{s_0}^{t} b(\theta_s) \, \mathrm{d}s \geq (t - s_0) - \beta \int_{s_0}^{t} \left| \frac{Y_s}{\sqrt{2}s} \right|^{\alpha} (1 + f(Y_s, s)) \, \mathrm{d}s,$$

where $f = f^{+}_{L,2-\alpha,(1-\varepsilon)/2} = \left[L \left(\left| \frac{y}{s} \right|^{2-\alpha} + s^{-(1-\varepsilon)/2} \right) \right] \wedge 1$ which is defined in (2.47).

We remark that the parameters $a = 2 - \alpha$ and $b = (1 - \varepsilon)/2$ appearing in $f^{+}_{L,2-\alpha,1-\delta}$ satisfy the conditions of the propositions of Section 2.3, which are $a > 1 - \alpha/2$ and $b > 1 - \kappa$.

Proof of Lemma 2.6.1. We work on the event appearing in the statement. First note that $m(t)/t = \sqrt{2} + O(t^{-\kappa}) \geq \sqrt{2} - t^{-(1-\varepsilon)/2}$ for s_0 large enough, because $\kappa > 1/2 > (1 - \varepsilon)/2$, and therefore, we have $X_s \geq \sqrt{2}s - 2s^{(1+\varepsilon)/2}$ for any $s \in [s_0, t]$. To obtain a lower bound on the branching rate, we need an upper bound on the angle: using $\arctan(x) \leq x$ for $x \geq 0$ together with $X_s \geq \sqrt{2}s - 2s^{(1+\varepsilon)/2}$ we get

$$|\theta_s| = \arctan\left(\left| \frac{Y_s}{X_s} \right| \right) \leq \left| \frac{Y_s}{X_s} \right| \leq \left| \frac{Y_s}{\sqrt{2}s} \right| \frac{1}{1 - \sqrt{2}s^{-(1-\varepsilon)/2}} \leq \left| \frac{Y_s}{\sqrt{2}s} \right| \left(1 + 2s^{-(1-\varepsilon)/2} \right), \tag{2.111}$$

for s_0 large enough. Observe that our assumption on the trajectory also entails that $|\theta_s| \leq C\sigma$ for s_0 large enough. By Assumption (A2), $b(\theta) = 1 - \beta |\theta|^{\alpha} + O(\theta^2)$ as $\theta \to 0$, so there exist $M > 0$ and $\sigma_0 \in (0, \pi/2)$ such that, for any $|\theta| \leq \sigma_0$,

$$b(\theta) \geq 1 - \beta |\theta|^{\alpha} - M |\theta|^2 .$$

Take $\sigma \leq C^{-1}$ and combine this with (2.111), for any s_0 large enough we have

$$\begin{aligned} b(\theta_s) &\geq 1 - \beta \left| \frac{Y_s}{\sqrt{2}s} \right|^{\alpha} \left(1 + 2s^{-(1-\varepsilon)/2} \right) \left(1 + \frac{M}{\beta} \left| \frac{Y_s}{\sqrt{2}s} \right|^{2-\alpha} \left(1 + 2s^{-(1-\varepsilon)/2} \right) \right) \\ &\geq 1 - \beta \left| \frac{Y_s}{\sqrt{2}s} \right|^{\alpha} \left(1 + L \left(\left| \frac{Y_s}{s} \right|^{2-\alpha} + s^{-(1-\varepsilon)/2} \right) \right), \end{aligned}$$

for some $L > 0$. Lastly, note that $L(|Y_s/s|^{2-\alpha} + s^{-(1-\varepsilon)/2}) \leq 1$ when we choose σ small enough and s_0 large, so that this term equals $f^{+}_{L,2-\alpha,(1-\varepsilon)/2}$. The statement of the lemma then follows by integrating over s. $\square$

We can now introduce the set of particles we consider for the first and second moment argument. Recall that $\alpha \in (2/3, 2)$ implies $\kappa \in (1/2, 1)$. Let $\varepsilon \in (0, 2\kappa - 1)$ and σ be small enough so that the conclusions of Lemmas 2.5.1 and 2.6.1 are satisfied. These two parameters ε and σ are now fixed for the whole section.

For $0 \leq s_0 \leq s \leq t$, we set

$$\begin{aligned} g_t^{+}(s) &:= \frac{m(t)}{t} s - ((s - s_0) \wedge (t - s))^{(1-\varepsilon)/2}, \\ g_t^{-}(s) &:= \frac{m(t)}{t} s - s^{(1+\varepsilon)/2}, \\ \Gamma_{s_0}(t) &:= \Big\{ u \in \mathcal{N}(t) : X_u(t) \in [m(t) - 1, m(t)], \\ &\qquad\qquad \forall s \in [s_0, t] : |Y_u(s)| \leq \sigma s, X_u(s) \in [g_t^{-}(s), g_t^{+}(s)] \Big\}. \end{aligned} \tag{2.112}$$

The parameter s_0 will be chosen large enough throughout the section.

2.6.2 First moment estimate

Lemma 2.6.2. *For s_0 large enough, there exists $c > 0$ such that, for any t large enough, for any $x \in [\sqrt{2}s_0 - 2\sqrt{s_0}, \sqrt{2}s_0 - \sqrt{s_0}]$, $y \in [-s_0^{\kappa/2}, s_0^{\kappa/2}]$,*

$$\mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|\right] \geq c.$$

Proof. Throughout the proof we absorb dependencies on s_0 into the constants. There are three events that contribute to $\Gamma_{s_0}(t)$,

$$\begin{aligned} B_1 &= \{X_t \in [m(t) - 1, m(t)]\}, \qquad B_2 = \Big\{\forall s \in [s_0, t], X_s \in [g_t^{-}(s), g_t^{+}(s)]\Big\}, \\ B_3 &= \{\forall s \in [s_0, t], |Y_s| \leq \sigma s\}, \end{aligned}$$

we define them for the Brownian motions $(X_s, Y_s)_{s\geq 0}$. They arise when applying the many–to–one lemma, Lemma 2.4.1,

$$\mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|\right] = \mathbb{E}_{(s_0,x,y)}\left[\mathbb{1}_{\cap_{i=1}^3 B_i} \exp\left(\int_{s_0}^t b(\theta_s)\,\mathrm{d}s\right)\right]. \tag{2.113}$$

Because we work on the event $B_2 \cap B_3$, we can apply Lemma 2.6.1. As a consequence of this, the X and Y contributions decouple and (2.113) is at least

$$e^{t-s_0}\mathbb{P}_{(s_0,x)}\left(B_1 \cap B_2\right)\mathbb{E}_{(s_0,y)}\left[\mathbb{1}_{B_3} \exp\left(-\beta \int_{s_0}^t \left|\frac{Y_s}{\sqrt{2s}}\right|^{\alpha} (1 + f(Y_s, s))\,\mathrm{d}s\right)\right]. \tag{2.114}$$

For the Y–contribution, we have by Proposition 2.3.5.2 applied with any $\eta \in (0, 2-\kappa)$, noting that $s^{(\kappa+\eta)/2} \leq \sigma s$ for any $s \geq s_0$ by choosing s_0 large enough,

$$\mathbb{E}_{(s_0,y)}\left[\mathbb{1}_{B_3}\exp\left(-\beta\int_{s_0}^{t}\left|\frac{Y_s}{\sqrt{2s}}\right|^{\alpha}(1+f(Y_s,s))\,\mathrm{d}s\right)\right] \geq ct^{\kappa/4}\exp\left(-\vartheta_1 t^{1-\kappa}\right), \quad (2.115)$$

where the factors depending on s_0 have been absorbed in c.

We now consider the X–contribution. By Girsanov's theorem with drift $\lambda = m(t)/t$, we have

$$e^{t-s_0}\mathbb{P}_{(s_0,x)}\left(B_1 \cap B_2\right) = e^{(1-\frac{\lambda^2}{2})(t-s_0)}\mathbb{E}_{(s_0,x-\lambda s_0)}\left[\mathbb{1}_{\hat{B}_1}\mathbb{1}_{\hat{B}_2}e^{-\lambda(X_t-X_{s_0})}\right],$$

where, recalling the definitions of g_t^+ and g_t^- in (2.112),

$$\hat{B}_1 = \{X_t \in [-1,0]\},$$
$$\hat{B}_2 = \left\{\forall s \in [s_0,t], X_s \in \left[-s^{(1+\varepsilon)/2}, -((s-s_0)\wedge(t-s))^{(1-\varepsilon)/2}\right]\right\}.$$

Noting that $1-\frac{\lambda^2}{2} = \vartheta_1 t^{-\kappa} + (\frac{3}{2}-\frac{\kappa}{4})\frac{\log t}{t} + o(\frac{1}{t})$ and bounding $e^{-\lambda(X_t-X_{s_0})} \geq c$ by using $X_t \in [-1,0]$ and our assumption on x, we get

$$e^{t-s_0}\mathbb{P}_{(s_0,x)}\left(B_1\cap B_2\right) \geq ct^{3/2-\kappa/4}e^{\vartheta_1 t^{1-\kappa}}\mathbb{P}_{(s_0,x-\lambda s_0)}\left(\hat{B}_1\cap\hat{B}_2\right). \quad (2.116)$$

Shifting time by s_0 and integrating w.r.t. the final value X_{t-s_0}, we have

$$\begin{aligned}&\mathbb{P}_{(s_0,x-\lambda s_0)}\left(\hat{B}_1\cap\hat{B}_2\right)\\ &\geq \int_{-1}^{-1/2}\mathbb{P}_{(0,x-\lambda s_0)}\left(\forall s\in[0,t-s_0], X_s\in I_s | X_{t-s_0}=y\right)\mathbb{P}_{(0,x-\lambda s_0)}\left(X_{t-s_0}\in\mathrm{d}y\right),\end{aligned} \quad (2.117)$$

where $I_s = [-(s+s_0)^{(1+\varepsilon)/2}, -(s\wedge(t-s_0-s))^{(1-\varepsilon)/2}]$. Note that we restricted ourselves to $y \in [-1,-1/2]$ to be at distance at least $1/2$ from the upper barrier at the end. Then, on the one hand, for t large enough, $\mathbb{P}_{(0,x-\lambda s_0)}(X_{t-s_0}\in\mathrm{d}y) \geq ct^{-1/2}\,\mathrm{d}y$. On the other hand, note that our lower barrier satisfies

$$-(s+s_0)^{(1+\varepsilon)/2} \leq -\frac{1}{2}s_0^{(1+\varepsilon)/2} - \frac{1}{2}s^{(1+\varepsilon)/2} \leq -\frac{1}{2}s_0^{(1+\varepsilon)/2} - \frac{1}{2}\left(s\wedge(t-s_0-s)\right)^{(1+\varepsilon)/2}$$

and by our assumption on x_0 we have $x-\lambda s_0 \in [-2\sqrt{s_0}, -\sqrt{s_0}/2]$ for s_0 large enough, so by [63, Lemma 2.8], for t large enough,

$$\mathbb{P}_{(0,x-\lambda s_0)}\left(\forall s\in[0,t-s_0], X_s\in I_s|X_{t-s_0}=y\right) \geq \frac{c}{t}(\lambda s_0-x)(-y) \geq \frac{c}{t}.$$

Combining this with (2.116) and (2.117), the total X–contribution is

$$e^{t-s_0}\mathbb{P}_{(s_0,x)}\left(B_1\cap B_2\right) \geq ct^{-\kappa/4}e^{\vartheta_1 t^{1-\kappa}}. \quad (2.118)$$

Going back to (2.114), combining (2.118) with (2.115), we get the desired result. □

2.6.3 Second moment estimate

Lemma 2.6.3. *For s_0 large enough, there exists $C > 0$ such that, for any t large enough, for any $x \in [\sqrt{2}s_0 - 2\sqrt{s_0}, \sqrt{2}s_0 - \sqrt{s_0}]$, $y \in \mathbb{R}$,*

$$\mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|^2\right] \leq C.$$

Proof. Throughout the proof, we absorb dependencies on s_0 into the constants. First, note that it is enough to prove $\mathbb{E}_{(s_0,x,y)}[|\Gamma_{s_0}(t)|\,(|\Gamma_{s_0}(t)|-1)] \leq C$. Indeed, we then have

$$\begin{aligned}\mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|^2\right] &= \mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|\right] + \mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|\,(|\Gamma_{s_0}(t)|-1)\right]\\ &\leq \mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|^2\right]^{1/2} + C,\end{aligned}$$

which yields the result because $x^2 \leq x + y$ implies $x \leq 1 + y^{1/2}$ for any $x, y \geq 0$.

To compute $\mathbb{E}_{(s_0,x,y)}[|\Gamma_{s_0}(t)|\,(|\Gamma_{s_0}(t)|-1)]$, we want to apply the many-to-two lemma. To this end, define the events for $i \in \{1,2\}$,

$$\begin{aligned}B_1^{(i)} &= \left\{\xi_{1,t}^i \in [m(t)-1, m(t)]\right\}, \qquad B_2^{(i)} = \left\{\forall s \in [s_0,t], \xi_{1,s}^i \in [g_t^-(s), g_t^+(s)]\right\},\\ B_3^{(i)} &= \left\{\forall s \in [s_0,t], |\xi_{2,s}^i| \leq \sigma s\right\},\end{aligned}$$

where $\xi_{1,t}^i$ and $\xi_{2,t}^i$ denote the X and Y–coordinate of ξ_t^i respectively. Then, the many–to–two lemma (Lemma 2.4.2) yields

$$\begin{aligned}&\mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|\,(|\Gamma_{s_0}(t)|-1)\right]\\ &= \int_{s_0}^t \mathbb{E}_{(s_0,x,y)}\left[\mathbb{1}_{\cap_{i=1}^2\cap_{j=1}^3 B_j^{(i)}} b(\xi_t^1)\exp\left(\int_{s_0}^t b(\xi_s^1)\,\mathrm{d}s + \int_r^t b(\xi_s^{2,r})\,\mathrm{d}s\right)\right] 2\,\mathrm{d}r. \qquad (2.119)\end{aligned}$$

Choosing s_0 large enough, we have $g_t^+(s) \leq \sqrt{2}s$ for any $s \in [s_0,t]$, and therefore Lemma 2.5.1 can be applied on the event $B_2^{(i)} \cap B_3^{(i)}$ to bound $b(\xi_s^i) \leq 1 - \beta|\xi_{2,s}^i/\sqrt{2}s|^\alpha(1 + f(\xi_{2,s}^i, s))$, where f satisfies the assumptions of Proposition 2.3.5.1. This then completely decouples the X and Y–coordinates of (2.119). We now bound the indicators of $B_3^{(i)}$ by 1 and use $b(\xi_t^1) \leq 1$ to bound (2.119) by

$$\begin{aligned}&\int_{s_0}^t e^{(t-s_0)}e^{(t-r)}\mathbb{P}_{(s_0,x)}\left(\bigcap_{i=1}^2\bigcap_{j=1}^2 B_j^{(i)}\right)\\ &\times \mathbb{E}_{(s_0,x,y)}\left[\exp\left(-\beta\int_{s_0}^t \left|\frac{\xi_s^1}{\sqrt{2}s}\right|(1+f(\xi_s^1,s))\,\mathrm{d}s - \beta\int_r^t\left|\frac{\xi_s^{2,r}}{\sqrt{2}s}\right|(1+f(\xi_s^{2,r},s))\,\mathrm{d}s\right)\right]\mathrm{d}r.\end{aligned}$$
(2.120)

Recalling the definition of $\widetilde{G}$ from (2.46), the expectation in (2.120) equals

$$\int_{\mathbb{R}} \widetilde{G}(s_0, y; r, \hat{y}) \left[\int_{\mathbb{R}} \widetilde{G}(r, \hat{y}; t, z) \, \mathrm{d}z \right]^2 \mathrm{d}\hat{y} \leq C t^{\kappa/2} r^{-\kappa/4} e^{\vartheta_1 r^{1-\kappa} - 2\vartheta_1 t^{1-\kappa}} \tag{2.121}$$

using Proposition 2.3.5.1 twice and incorporating the s_0 dependencies into C. We now focus on the probability in (2.120). Integrating first w.r.t. $\xi_r^{1,r} = \xi_r^{2,r}$ and weakening the barrier, we get

$$\mathbb{P}_{(s_0,x)} \left(\bigcap_{i=1}^{2} \bigcap_{j=1}^{2} B_j^{(i)} \right) \tag{2.122}$$

$$\leq \int_{-\infty}^{g_t^+(r)} \mathbb{P}_{(s_0,x)} \left(B_2^{[s_0,r]} \middle| X_r = y \right) \mathbb{P}_{(r,y)} \left(B_1 \cap B_2^{[r,t]} \right)^2 \frac{e^{-(y-x)^2/[2(r-s_0)]}}{\sqrt{2\pi(r-s_0)}} \, \mathrm{d}y, \tag{2.123}$$

where $B_1 = \{X_t \in [m(t)-1, m(t)]\}$ and $B_2^I = \{\forall s \in I, X_s \leq \lambda s\}$ with $\lambda = m(t)/t$. By [32, Lemma 2], we have

$$\mathbb{P}_{(s_0,x)} \left(B_2^{[s_0,r]} \middle| X_r = y \right) \leq \frac{2(\lambda s_0 - x)(\lambda r - y)}{r - s_0} \wedge 1 \leq \frac{C(\overline{y}+1)}{(r - s_0 + 1)},$$

writing $\overline{y} = \lambda r - y$ and using the assumption on x. On the other hand, by Girsanov's theorem, we have

$$\begin{aligned} \mathbb{P}_{(r,y)} \left(B_1 \cap B_2^{[r,t]} \right) &= \mathbb{E}_{(s_0, y - \lambda r)} \left[\mathbb{1}_{\{X_t \in [-1,0]\}} \mathbb{1}_{\{\forall s \in [r,t], X_r \leq 0\}} e^{-\lambda(X_t - X_{s_0}) - \frac{\lambda^2}{2}(t-r)} \right] \\ &\leq \frac{C(\overline{y}+1)}{(t-r+1)^{3/2}} e^{-\lambda \overline{y} - \frac{\lambda^2}{2}(t-r)}, \end{aligned}$$

where we used $e^{-\lambda(X_t - X_{s_0})} \leq e^{-\lambda \overline{y}}$ and bounded the remaining probability by first integrating w.r.t. X_t and then applying [32, Lemma 2] again. Coming back to (2.123), using also

$$e^{-(y-x)^2/[2(r-s_0)]} = e^{-(\lambda(r-s_0) - \overline{y} - x + \lambda s_0)^2/[2(r-s_0)]} \leq e^{-\frac{\lambda^2}{2}(r-s_0) + \lambda(\overline{y} + x - \lambda s_0)} \leq C e^{-\frac{\lambda^2}{2} r + \lambda \overline{y}},$$

and changing y for $\overline{y}$ in the integral, we obtain

$$\begin{aligned} &\mathbb{P}_{(s_0,x)} \left(\bigcap_{i=1}^{2} \bigcap_{j=1}^{2} B_j^{(i)} \right) \\ &\quad \leq \frac{C e^{-\lambda^2 t + \frac{\lambda^2}{2} r}}{(r - s_0 + 1)\sqrt{r - s_0}(t - r + 1)^3} \int_{((r-s_0) \wedge (t-r))^{(1-\varepsilon)/2}}^{\infty} e^{-\lambda \overline{y}} (\overline{y}+1)^3 \, \mathrm{d}\overline{y} \\ &\quad \leq \frac{C t^{3 - \frac{\kappa}{2}} e^{-2t + 2\vartheta_1 t^{-\kappa}} e^{r - \vartheta_1 r t^{-\kappa} - (\frac{3}{2} - \frac{\kappa}{4}) \frac{r}{t} \log(t)}}{(r - s_0 + 1)\sqrt{r - s_0}(t - r + 1)^3} e^{-((r-s_0) \wedge (t-r))^{(1-\varepsilon)/2}}, \end{aligned}$$

using $\frac{\lambda^2}{2} = 1 - \vartheta_1 t^{-\kappa} - (\frac{3}{2} - \frac{\kappa}{4})\frac{\log t}{t} + o(\frac{1}{t})$ for the prefactor, and $e^{-\lambda \overline{y}}(\overline{y}+1)^3 \leq Ce^{-\overline{y}}$ for the integral. Combining this with (2.121), we get that (2.120) is at most

$$Ct^3 \int_{s_0}^{t} r^{-\kappa/4} \frac{e^{\vartheta_1(r^{1-\kappa} - rt^{-\kappa}) - (\frac{3}{2} - \frac{\kappa}{4})\frac{r}{t}\log(t)}}{(r - s_0 + 1)\sqrt{r - s_0}(t - r + 1)^3} e^{-((r-s_0)\wedge(t-r))^{(1-\varepsilon)/2}} \,\mathrm{d}r. \tag{2.124}$$

Then, using that $s \geq s_0 \mapsto (\log s)/s$ is decreasing for s_0 large enough, we have $e^{\frac{\kappa}{4}\frac{r}{t}\log(t)} \leq r^{\kappa/4}$ and $e^{-\frac{3}{2}\frac{r}{t}\log(t)} = t^{-3/2}e^{\frac{3}{2}\frac{t-r}{t}\log(t)} \leq t^{-3/2}(t-r)^{3/2}$. Moreover, there exists $C(\kappa) > 0$ such that $r^{1-\kappa} - rt^{-\kappa} \leq C(r \wedge (t-r))^{1-\kappa}$ for any $r \in [s_0, t]$: this is direct for $r \leq t/2$ and, for $r \geq t/2$, one has

$$r^{1-\kappa} - rt^{-\kappa} = r^{1-\kappa}\left(1 - \left(1 - \frac{t-r}{t}\right)^{\kappa}\right) \leq C(\kappa)t^{1-\kappa}\frac{t-r}{t} \leq C(\kappa)(t-r)^{1-\kappa}.$$

It follows that $e^{\vartheta_1(r^{1-\kappa} - rt^{-\kappa}) - ((r-s_0)\wedge(t-r))^{(1-\varepsilon)/2}} \leq C$, where we recall that C can depend on s_0. Applying these facts to (2.124) shows

$$\mathbb{E}_{(s_0,x,y)}\left[|\Gamma_{s_0}(t)|\left(|\Gamma_{s_0}(t)| - 1\right)\right] \leq Ct^{3/2}\int_{s_0}^{t} \frac{1}{(r - s_0 + 1)\sqrt{r - s_0}(t - r + 1)^{3/2}} \,\mathrm{d}r \leq C,$$

which concludes the proof. □

2.6.4 Proof of the lower bound

We are now in a position to show the lower bound of Theorem 2.1.1. To be precise, we show that, for any $\delta > 0$, there exists $x \in \mathbb{R}$ such that

$$\limsup_{t\to\infty} \mathbb{P}\left(M_t \leq m(t) - x\right) \leq \delta. \tag{2.125}$$

Proof of the lower bound in Theorem 2.1.1. We use the moment bounds on $|\Gamma_K(t)|$ to show that there are particles near $m(t)$ at time t. For $s_0 \geq 0$, we consider the box

$$\mathcal{B}_{s_0} = [\sqrt{2}s_0 - 2\sqrt{s_0}, \sqrt{2}s_0 - \sqrt{s_0}] \times [-s_0^{\kappa/2}, s_0^{\kappa/2}]. \tag{2.126}$$

In the sequel, we consider s_0 large enough to satisfy Lemmas 2.6.2 and 2.6.3 as well as other properties mentioned below.

Combining Lemmas 2.6.2 and 2.6.3, as well as the Cauchy-Schwarz inequality, there exist $t_0 > s_0$ and $c_0 > 0$ such that for all $t \geq t_0$ and $(x, y) \in \mathcal{B}_{s_0}$,

$$\mathbb{P}_{(s_0,x,y)}\left(M_t \geq m(t) - 1\right) \geq \mathbb{P}_{(s_0,x,y)}\left(|\Gamma_K(t)| \geq 1\right) \geq \frac{\mathbb{E}_{(s_0,x,y)}[|\Gamma_K(t)|]^2}{\mathbb{E}_{(s_0,x,y)}[|\Gamma_K(t)|^2]} \geq c_0 > 0.$$

From this it follows that we also have, for all $t \geq t_0 - s_0$ and $(x, y) \in \mathcal{B}_{s_0}$,

$$\mathbb{P}_{(0,x,y)}\left(M_t \geq m(t) - 1\right) \geq \mathbb{P}_{(s_0,x,y)}\left(M_{t+s_0} \geq m(t + s_0) - 1\right) \geq c_0 > 0, \tag{2.127}$$

using that $m(t+s_0) \geq m(t)$, if s_0 is chosen large enough.

Let $\delta > 0$. By Lemma 2.6.4 below, for every $N \in \mathbb{N}$, there is $r > 0$ such that

$$\mathbb{P}_{(0,0,0)}\left(\#\left\{u \in \mathcal{N}(r) : (X_u(r), Y_u(r)) \in \mathcal{B}_{s_0}\right\} \geq N\right) \geq 1 - \delta/2. \tag{2.128}$$

Hence, restricting ourselves to the event in (2.128), applying the branching property at time r and then (2.127) to the BBMs starting from particles in $\mathcal{B}_{s_0}$ at time r, we get, for any $t \geq t_0 - s_0 + r$,

$$\mathbb{P}_{(0,0,0)}\left(M_t \geq m(t-r) - 1\right) \geq (1 - \varepsilon/2)\left(1 - (1-c_0)^N\right) \geq 1 - \delta,$$

for $N = N(\delta, c_0)$ large enough. And therefore we have

$$\mathbb{P}_{(0,0,0)}\left(M_t \geq m(t) - \sqrt{2}r - 1\right) \geq 1 - \delta,$$

for t large enough such that $m(t-r) \geq m(t) - \sqrt{2}r$. This proves (2.125). ☐

Lemma 2.6.4. *For s_0 large enough, recalling the definition of $\mathcal{B}_{s_0}$ in (2.126), we have $\mathbb{P}_{(0,0,0)}$-a.s.*

$$\#\left\{u \in \mathcal{N}(t) : (X_u(t), Y_u(t)) \in \mathcal{B}_{s_0}\right\} \xrightarrow[t\to\infty]{} \infty$$

Proof. As $s_0 \to \infty$, the angular coordinate of points in $\mathcal{B}_{s_0}$ is uniformly going to 0, so, using Assumption (A2), we can choose s_0 large enough so that the branching rate is at least $1/2$ on $\mathcal{B}_{s_0}$. We also introduce a family of square boxes, indexed by $L > 0$:

$$\mathcal{B}_{s_0}^L = \left(\sqrt{2}s_0 - \frac{3}{2}\sqrt{s_0} + [-L, L]\right) \times [-L, L]. \tag{2.129}$$

We also assume that s_0 is large enough such that $\mathcal{B}_{s_0}^4 \subset \mathcal{B}_{s_0}$. We now fix s_0.

We define a spine $(\xi_t)_{t\geq 0}$ with $\xi_t \in \mathcal{N}(t)$, starting with the root and then choosing one of the two children uniformly at random at each branching point. We now make the two following claims

- *Claim 1:* The spine branches infinitely many times in the box $\mathcal{B}_{s_0}^2$ almost surely.
- *Claim 2:* There exists $c > 0$ such that, for all $(x, y) \in \mathcal{B}_{s_0}^2$,

$$\mathbb{P}_{(0,x,y)}\left(\#\left\{u \in \mathcal{N}(t) : (X_u(t), Y_u(t)) \in \mathcal{B}_{s_0}^4\right\} \xrightarrow[t\to\infty]{} \infty\right) \geq c. \tag{2.130}$$

Combining these two claims together with the branching property along the spine and the second Borel–Cantelli lemma yields the result.

We now prove the claims. For Claim 1, note that the trajectory of the spine is a two-dimensional Brownian motion, which is recurrent, so a.s. the spine enters $\mathcal{B}^1_{s_0}$, then leaves $\mathcal{B}^2_{s_0}$, then enters $\mathcal{B}^1_{s_0}$, then leaves $\mathcal{B}^2_{s_0}$, and so on infinitely many times. Because the branching rate is at least $1/2$ on $\mathcal{B}^2_{s_0}$ as mentioned at the beginning of the proof, the probability that the spine branches between an entry time of $\mathcal{B}^1_{s_0}$ and the next exit time of $\mathcal{B}^2_{s_0}$ is bounded away from 0 uniformly in the entry point in $\mathcal{B}^1_{s_0}$. Therefore, Claim 1 follows from the second Borel–Cantelli lemma.

We now deal with Claim 2. Using that the branching rate is at least $1/2$ on $\mathcal{B}^4_{s_0}$, it is enough to prove the claim for a BBM with branching rate $1/2$ and where particles are killed when leaving $\mathcal{B}^4_{s_0}$. But this branching Markov process is supercritical, because the largest eigenvalue of the Laplacian with Dirichlet boundary condition on $\mathcal{B}^4_{s_0}$ is $-\frac{1}{2}(\frac{\pi^2}{4^2})$, which is larger than the opposite of the branching rate. The fact of being supercritical implies the claim as shown in [80] or [86]. More precisely, let $u_1(x, y)$ be the extinction probability when starting from $(x, y) \in \mathcal{B}^4_{s_0}$. Then, [86, Theorem 2.1] implies that u_1 is continuous on $\mathcal{B}^4_{s_0}$ and [86, Theorem 2.2] shows that $u_1 < 1$ on the interior of $\mathcal{B}^4_{s_0}$, so there exists $c > 0$ such that, for all $(x, y) \in \mathcal{B}^2_{s_0}$, $u_1(x, y) \leq 1 - c$. But [86, Lemma 2.1] proves that $1 - u_1(x, y)$ equals the probability in (2.130) $\square$

Chapter 3

Biased branching random walks on Bienaymé–Galton–Watson trees

3.1 Introduction and main results

Branching particle systems have become a staple of probability theory. On the one hand they appear naturally in random models of evolving populations, turbulence cascades, or epidemiology to name but a few of the contexts where they have proven a key tool; and on the other, in spite of their simple definition such processes have a rich and sophisticated structure while being remarkably useful to study other important mathematical objects such as reaction diffusion equations or log-correlated fields.

Unless there is extinction, a branching particle systems can be thought of as an expanding cloud of particles in some space and the first question one wants to answer is *at what linear speed does this expansion happen?* This is now very well understood in homogeneous deterministic space (e.g. for branching random walks or branching Brownian motion in $\mathbb{R}^d$). In discrete time and space, the random walk can be replaced by any irreducible Markov chain and the corresponding branching Markov chain can be described as follows. Start the system with one particle at the origin. After each time unit, particles produce offspring according to a fixed offspring law and the offspring particles choose a new location, independently of each other, according to the Markov chain, starting from the location of the mother particle. This model is also known as *tree-indexed Markov chain*. One may ask about recurrence/transience of the branching Markov chain: is the origin (or any other site in the state space of the Markov chain) visited infinitely often by some particle? Does the maximal distance of the particles to the origin grow at a linear speed? In the transient case, does the minimal distance of the particles to the origin grow at a linear speed? These questions have also been addressed in cases where the Markov chain is a random walk in random environment, see Section 3.1.4.

Here, we consider a branching random walk on Bienaymé–Galton–Watson trees where the underlying Markov chain is the λ–biased random walk on a Bienaymé–Galton–Watson tree. The motion of a single particle is a well-studied model of a random walk in random environment, although there are still fascinating open questions, see [70]. In a homogeneous environment, i.e. for a transitive Markov chain, it is well-known that the origin is visited infinitely often by some particle if and only if the product of the mean offspring number with the spectral radius of the Markov chain is strictly larger than 1. We show that the same is true in our model, see Theorem 3.1.2.

If the Markov chain is *statistically transitive* and if the distance to the origin of the Markov chain satisfies a large deviation principle, the maximal distance of the particles to the origin grows at a linear speed which should be given as follows. The linear speed v is such that the exponential growth rate of the number of particles compensates the exponential decay of the probability for a single particle to be at time n at a distance at least nv from the origin, see formula 3.4 for a precise statement. We refer to Section 3.1.4 for known examples where this heuristics has been confirmed. Indeed, this statement is true in our model: the main results of the paper, Theorem 3.1.1 and Theorem 3.1.3, characterize the linear growth rate of both the maximal and minimal distances of the particles to the origin. While the proof of the upper bound of the linear growth rate of the maximal distance to the origin follows a standard argument (relying on a union bound and the many–to–one formula), the proof of the lower bound is more complicated and involves several decoupling procedures.

The paper is organized as follows. In Section 3.1.1, we define the model and describe our assumptions. In Section 3.1.2, we state our main results, Theorems 3.1.1 and 3.1.3. We then give in Section 3.1.3 an outline of the proof of Theorem 3.1.1, which is mainly based on Proposition 3.1.4 and Proposition 3.1.5. In Section 3.1.4, we describe some related works. Turning to the proofs, we start by recalling large deviation statements for λ-biased random walk (of a single particle) on Bienaymé–Galton–Watson trees in Section 3.2.1. In Section 3.2.3 we recall a well-known zero–one law and use it to prove Proposition 3.1.5. In Section 3.2.4 we prove Proposition 3.1.4 by using a comparison with a branching process in random environment. We then show Theorem 3.1.1 and Theorem 3.1.3. Finally, we conclude with some open questions in Section 3.3.

3.1.1 The model

First, we need a distribution for the environment: Let $p := (p_k, k = 0, 1, 2, \ldots)$ be a probability measure on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Define `BGW` to be the law of a Bienaymé–

Galton–Watson (BGW–tree) ω with offspring distribution p. We will always assume that

$$p_0 = 0 \text{ and } p_1 < 1. \tag{3.1}$$

Then the Bienaymé–Galton–Watson tree is supercritical, without leaves, and

$$m_{\mathtt{BGW}} := \sum_{k \geq 1} k p_k > 1.$$

Throughout this paper, we also suppose that

$$m_{\mathtt{BGW}} < \infty.$$

The root of the tree ω is denoted by $\mathbf{o}$. For a vertex $x \in \omega$, write $|x|$ for its height, that is the (graph) distance between x and $\mathbf{o}$. Denote by x_- its parent, by κ_x the number of its children, and by $xj \in \omega$ its j-th child for $j \leq \kappa_x$. We will denote the children of the root by $1, 2, \ldots, \kappa_{\mathbf{o}}$ instead of $\mathbf{o}1, \mathbf{o}2, \ldots, \mathbf{o}\kappa_{\mathbf{o}}$. Let D_n be the set of vertices with height equal to n; we call D_n the n-th level of the tree. For every $i \leq |x|$, let $x_i \in D_i$ be its ancestor at the i-th level. Write $x \preceq y$ if x is an ancestor of y or $x = y$.

Next, we need a branching mechanism: Let $\mu := (\mu_k, k = 0, 1, 2, \ldots)$ be another probability measure. Define $\mathbb{T}$ to be another Bienaymé–Galton–Watson tree with offspring distribution μ, independent of ω. This tree will act as the genealogy of the branching random walk. To differentiate between $\mathbb{T}$ and ω in notation, we call vertices of $\mathbb{T}$ particles and typically denote them by u, v, w (as opposed to $x, y, z \in \omega$). Similarly, we denote the root of $\mathbb{T}$ by $\varnothing$, the number of children of a particle u by γ_u and we call $|u|$ the generation of u.

Thirdly, given the environment ω, we construct the λ*–biased branching random walk* (BRW) on ω in discrete time: We denote the branching random walk by $(X(u), u \in \mathbb{T})$ and its law by $\mathbf{P}_\omega$. Unless otherwise noted, the initial particle $\varnothing \in \mathbb{T}$ starts at the root $\mathbf{o} \in \omega$, i.e. $X(\varnothing) = \mathbf{o}$. Then we proceed inductively. At time $n+1$, a particle $u \in \mathbb{T}$ with $|u| = n$ is replaced independently by γ_u children with probability

$$\mathbf{P}_\omega(\gamma_u = k) = \mu_k, \qquad k \geq 0.$$

For $j \leq \gamma_u$, each child $uj \in \mathbb{T}$ is situated at a certain position in ω, independently of γ_u and the other children, according to the following rule:

$$\begin{aligned}
\mathbf{P}_\omega(X(uj) = i | X(u) = \mathbf{o}) &= \frac{1}{\kappa_{\mathbf{o}}}, \quad i = 1, 2, \ldots, \kappa_{\mathbf{o}};\\
\mathbf{P}_\omega(X(uj) = x_- | X(u) = x) &= \frac{\lambda}{\lambda + \kappa_x}, \quad x \neq \mathbf{o};\\
\mathbf{P}_\omega(X(uj) = xi | X(u) = x) &= \frac{1}{\lambda + \kappa_x}, \quad x \neq \mathbf{o},\ i = 1, 2, \ldots, \kappa_x.
\end{aligned}$$

We further assume that the BRW is supercritical with finite expectation, in the sense that

$$m := \sum_{k\geq 1} k\mu_k \in (1,\infty). \tag{3.2}$$

Under this condition, it is well-known that the BRW survives with strictly positive probability. In fact, for simplicity, throughout this paper we work under the stronger condition that

$$\mu_0 = 0 \text{ and } \mu_1 < 1, \tag{3.3}$$

so that the BRW survives almost surely. If one assumes only (3.2), similar results can be easily obtained by conditioning on the event of survival.

Lastly, when we consider the randomness of the environment together with the randomness of the process, we speak of the *annealed* law (compared to the *quenched* law $\mathbf{P}_\omega$). The annealed law is obtained by averaging $\mathbf{P}_\omega$ over the environment,

$$\mathbb{P}(\cdot) := \int \mathbf{P}_\omega(\cdot)\mathtt{BGW}(\mathrm{d}\omega).$$

We denote expectations with respect to $\mathtt{BGW}$ by $\mathbb{E}_{\mathtt{BGW}}$, e.g. $\mathbb{E}[\,\cdot\,] = \mathbb{E}_{\mathtt{BGW}}[\mathbf{E}_\omega[\,\cdot\,]]$. This is to emphasise that the remaining randomness depends only on ω but not the branching random walk.

3.1.2 Results

The main purpose of this work is to study the asymptotic behaviour of the maximal height of the branching random walk at time n, that is $\max_{|u|=n}|X(u)|$, as $n\to\infty$.

Because at generation n the branching random walk has of the order of m^n particles, a first moment argument and the so-called *many-to-one* formula suggest that the highest particle should be at a height of order $v_{\lambda,m}n$ where $v_{\lambda,m}$ corresponds to the point where the large deviations rate function of the λ-biased random walk is equal to $\log m$.

More precisely, let v_λ be the velocity of a single λ-biased random walk (see Theorem 3.2.1). Then, the velocity $v_{\lambda,m}$ is related to the rate function $I_\lambda : [v_\lambda, 1] \mapsto [0,\infty)$ of the large deviation principle for λ-biased random walks, obtained in [40] (c.f. Section 3.2.1): we have

$$v_{\lambda,m} = \sup\left\{a \in [v_\lambda, 1] : I_\lambda(a) \leq \log m\right\}. \tag{3.4}$$

In particular, if $v_{\lambda,m} < 1$, $v_{\lambda,m}$ is the unique solution to $I_\lambda(v_{\lambda,m}) = \log m$. Since $I_\lambda(v_\lambda) = 0$, we have $v_{\lambda,m} \in (v_\lambda, 1]$. Moreover, $v_{\lambda,m} = 1$ if and only if $m \geq e^{I_\lambda(1)} = (\sum_{k\geq 1}\frac{k}{k+\lambda}p_k)^{-1}$. For any fixed λ, $v_{\lambda,m}$ is strictly increasing with m in $(1, e^{I_\lambda(1)})$.

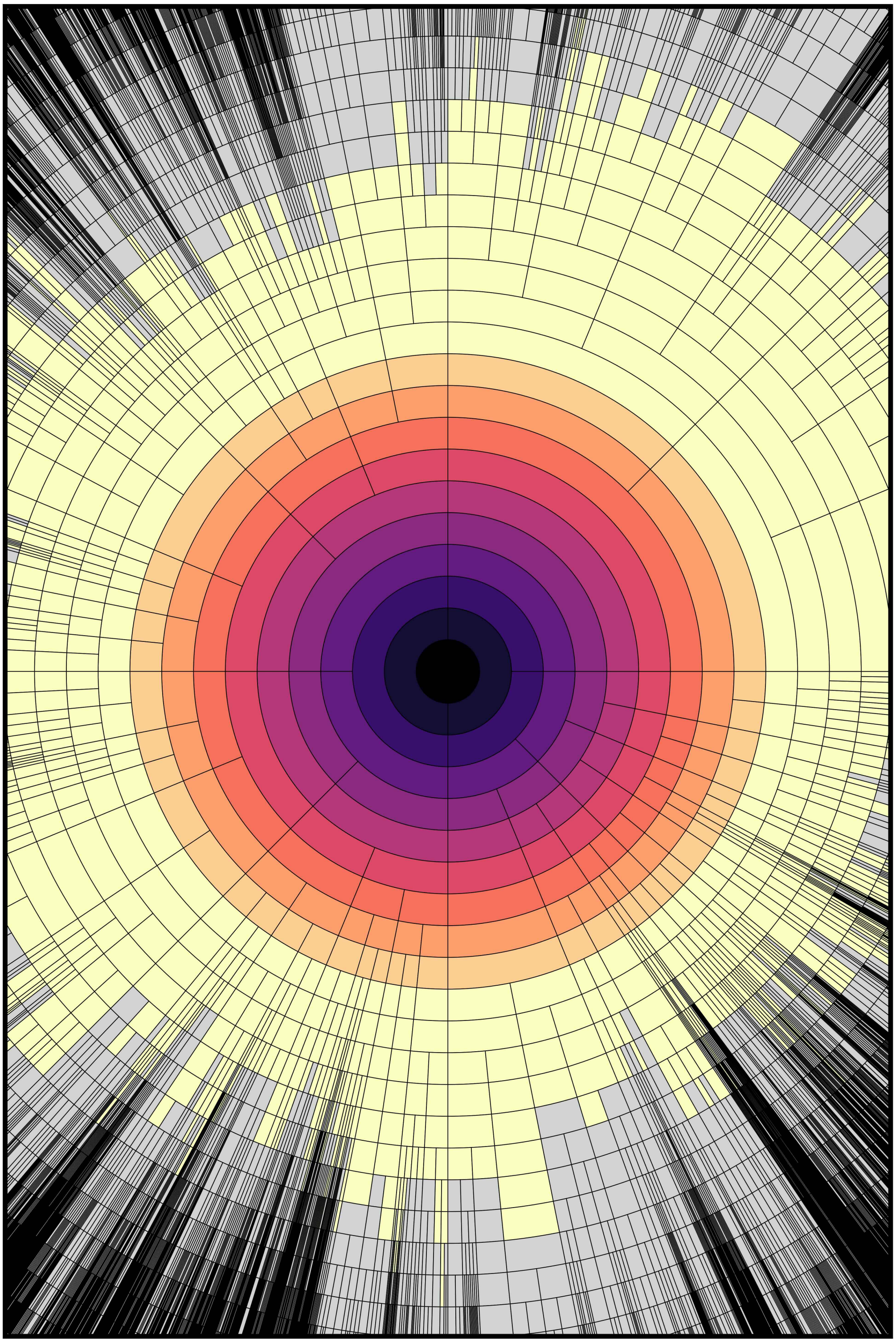

Figure 3.1: An artistic interpretation of a branching random walk on a Bienaymé–Galton–Watson tree.

Theorem 3.1.1 (Velocity of the maximal displacement)**.** *Let* $(X(u), u \in \mathbb{T})$ *be a* λ*-biased branching random walk with reproduction law* μ*. Suppose that* $\mu_0 = 0$*,* $\mu_1 < 1$*,* $m := \sum_{k \geq 1} k\mu_k > 1$ *and* $m < \infty$*. Then for* `BGW`*–a.e.* ω*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \max_{|u|=n} |X(u)| = v_{\lambda,m}, \qquad \mathbf{P}_\omega - a.s.$$

where $v_{\lambda,m}$ *is a constant that only depends on* λ*,* m *and the tree offspring law* p *and is given by* (3.4)*.*

Besides the maximal height of a branching random walk, we can also study the minimal height at time n, $\min_{|u|=n} |X(u)|$, as $n \to \infty$. The study of this quantity is related to the question of recurrence and transience of the branching random walk, or in other words its *local extinction/local survival.* Let $d_{\min} := \min\{k \colon p_k > 0\}$ be the minimum offspring number of ω and let

$$\alpha_x := \mathbf{P}_\omega \left(\forall n \in \mathbb{N}\ \exists u \in \mathbb{T} : |u| \geq n, X(u) = x | X(\varnothing) = x\right),$$

the quenched probability that starting at x, there are infinitely many particles that visit x. We show that the critical value for λ, namely $d_{\min}$, which is required for a transient phase of the branching random walk is different from the critical value of λ for recurrence/transience of the λ–biased random walk which is $m_{\mathtt{BGW}}$.

Theorem 3.1.2. *For* `BGW`*-a.e.* ω*, a BRW* $(X(u), u \in \mathbb{T})$ *is transient in the sense that* $\alpha_x = 0$ *for all* $x \in \omega$ *if and only if*

$$\lambda < d_{\min} \quad \text{and} \quad m \leq \frac{d_{\min} + \lambda}{2\sqrt{\lambda d_{\min}}}. \tag{3.5}$$

where $m = \sum_{k\geq 1} k\mu_k > 1$ *is the mean offspring of the BRW.*
Otherwise if $\lambda \geq d_{\min}$ *or* $m > \frac{d_{\min}+\lambda}{2\sqrt{\lambda d_{\min}}}$*, then the BRW is strongly recurrent, i.e.* $\alpha_x = 1$ *for all* $x \in \omega$*.*

We prove Theorem 3.1.2 in Section 3.2.2. We remark that part of this statement is the non–existence of the *weakly recurrent* phase: there is no choice of λ, m, and x such that we have $\alpha_x \in (0, 1)$. The second part of condition (3.5) can also be written as $m \leq e^{-I_\lambda(0)}$. The expression for $I_\lambda(0)$ (respectively the threshold in (3.5)) may seem complicated. We note that $I_\lambda(0) = H(\frac{1}{2}|\frac{d_{\min}}{d_{\min}+\lambda})$ where $H(s|t) = s \log \frac{s}{t} + (1-s)\frac{1-s}{1-t}$ is the relative entropy between two Bernoulli distributions with parameters $s, t \in (0, 1)$. See Figure 3.2 for an illustration of the different recurrence and transience regimes.

For the case $\lambda = 1$, it is proven in [84] that a BRW is transient in the sense that $\alpha_{\mathbf{o}} = 0$ if and only if (3.5) holds. The fact that recurrence or transience is determined only by $\operatorname{supp}\{p_k, k \geq 1\}$ and not by the actual distribution $\{p_k, k \geq 1\}$ can also be compared to [38, Theorem 1.1] where the authors study a BRW in random environment on $\mathbb{Z}^d$. There the the recurrence/transience criterion depends only on $\operatorname{supp} Q$ (in their notation).

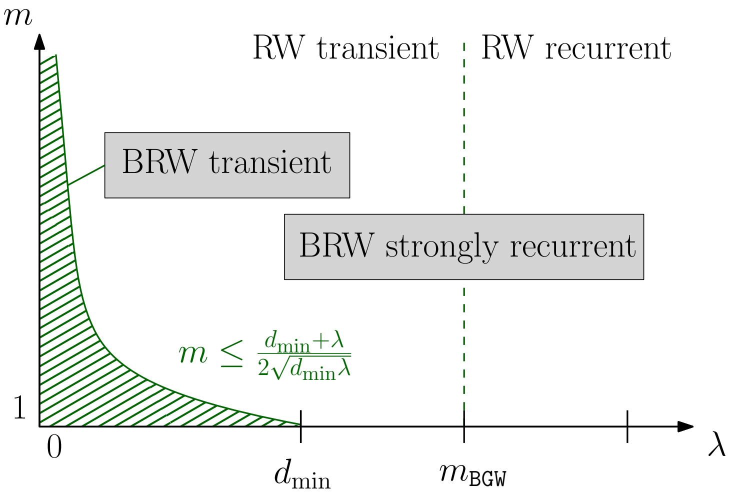

Figure 3.2: The different recurrence and transience regimes for the λ–biased random walk and λ–biased branching random walk.

Clearly, in the strongly recurrent regime we have

$$\lim_{n\to\infty} \min_{u\in\mathbb{T},|u|=n} \frac{|X(u)|}{n} = 0, \quad \mathbf{P}_\omega - a.s. \quad \text{for} \quad \texttt{BGW} - a.e.\ \omega.$$

In fact, the number of particles at the origin should grow exponentially. In the transient regime we show the following.

Theorem 3.1.3 (Velocity of the minimal displacement)**.** *Let* $(X(u), u \in \mathbb{T})$ *be a* λ*–biased branching random walk as above. Assume that we are in the transient regime, i.e. that* $m \leq \frac{d_{\min}+\lambda}{2\sqrt{\lambda d_{\min}}}$ *and* $\lambda < d_{\min}$*. We also need to assume*

$$d_{\min} \geq 2. \tag{3.6}$$

Then for `BGW`*–a.e.* ω*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \min_{|u|=n} |X(u)| = \tilde{v}_{\lambda,m}, \qquad \mathbf{P}_\omega - a.s.$$

where $\tilde{v}_{\lambda,m} \in [0, v_\lambda)$ *is a constant that only depends on* λ*,* m *and the tree offspring law* p*, see* (3.7) *below.*

The assumption (3.6) comes from the slowdown large deviations result [40, Theorem 1.2]; see Theorem 3.2.3 below. With the large deviation rate function $I_\lambda\colon [0, v_\lambda] \to [0, \infty)$ therein we have the identity

$$\tilde{v}_{\lambda,m} := \inf\left\{a \in [0, v_\lambda] : I_\lambda(a) \le \log m\right\}. \tag{3.7}$$

Let us comment on the properties of $\tilde{v}_{\lambda,m}$: because $d_{\min} \ge 2$ and $\lambda < d_{\min}$, I_λ is strictly decreasing on $[0, v_\lambda]$ and $I_\lambda(0)$ is known, see (3.11) below. This implies that $\tilde{v}_{\lambda,m} > 0$ if and only if $m < \frac{d_{\min}+\lambda}{2\sqrt{\lambda d_{\min}}}$. In the critical case when $m = \frac{d_{\min}+\lambda}{2\sqrt{\lambda d_{\min}}}$, we have $\tilde{v}_{\lambda,m} = 0$. The branching walk is still transient, which suggests that in the critical case the minimum moves at sublinear speed. Also note that $\tilde{v}_{\lambda,m} < v_{\lambda,m}$ and that $\tilde{v}_{\lambda,m}$ is strictly decreasing in m for fixed λ.

In the case of $d_{\min} = 1$, the branching random walk can only be transient (depending on m) if $\lambda < 1$. However this case is not covered by the large deviations result and therefore is out of our reach for the study of the branching random walk. Nevertheless, we believe the same results to be true.

The proof of Theorem 3.1.3 is very similar to that of Theorem 3.1.1. We will discuss the differences briefly in Section 3.2.5.

3.1.3 Outline of the proof of Theorem 3.1.1

The main difficulty in showing Theorem 3.1.1 lies in showing a lower bound, i.e. to show that there are particles that realise the speed $(v_{\lambda,m} - \varepsilon)$ for some $\varepsilon > 0$ small. Our approach is not dissimilar from the standard approach (see for example [81, Section 1.4]) and consists mainly of the following two propositions.

Proposition 3.1.4. *Under the assumptions of Theorem 3.1.1 we have for any* $a < v_{\lambda,m}$

$$\mathbb{P}\left(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \ge a\right) > 0.$$

Proposition 3.1.5. *Under the assumptions of Theorem 3.1.1 we have the following* $0 - 1$ *law for any* $a > 0$*:*

$$\mathbf{P}_\omega\left(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \ge a\right) > 0, \quad \textit{for} \quad \texttt{BGW} - \textit{a.e.}\ \omega$$
$$\implies \mathbf{P}_\omega\left(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \ge a\right) = 1, \quad \textit{for} \quad \texttt{BGW} - \textit{a.e.}\ \omega.$$

We show Propositions 3.1.5 and 3.1.4 in Sections 3.2.3 and 3.2.4 respectively.

Although Proposition 3.1.4 looks close to Theorem 3.1.1 already, there are still two more steps to do. One is to go from > 0 to $= 1$ which is provided by the $\mathbf{P}_\omega$-zero–one law, Proposition 3.1.5. The other one is to go from the annealed law $\mathbb{P}$ to the quenched law $\mathbf{P}_\omega$, for which we shall develop another zero–one law under `BGW`. The main obstacle in proving Proposition 3.1.4 is that for two particles $u, v \in \mathbb{T}$ with $|u| = |v| = n$ and $u \neq v$, the descendant populations

$$\{X(w); w \in \mathbb{T}, u \prec w\} \quad \text{and} \quad \{X(w'); w' \in \mathbb{T}, v \prec w'\},$$

while being independent under $\mathbf{P}_\omega$, are not identically distributed because they may use different parts of ω. When considered under $\mathbb{P}$, they have the same law but are not independent any more as they might depend on the same part of ω. We overcome this by careful analysis and comparing our process to an $\mathbb{N}_0$–valued branching process in random environment.

Regarding Proposition 3.1.5, it is not surprising that this holds as for fixed environment ω the quantity $\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n}$ is a tail–measurable random variable. Nevertheless, the standard tools like Kingman's subadditive ergodic theorem (as used in [81, Section 1.4]) or Kolmogorov's $0-1$ law cannot be applied. Again, the issue is that the increments, in any sense, are not identically distributed. Interestingly, our proofs are different in the strongly recurrent and the transient regimes. In the transient regime, the crucial ingredient of the proof is that ω, while not being a transitive graph in the usual sense, is *statistically transitive* in the sense that for any $x \in \omega$ the subtree $\{y \in \omega : x \preceq y\}$ has the same law as ω under $\mathbb{P}$.

Remark 3.1.6. Proposition 3.1.5 depends crucially on Bienaymé–Galton–Watson trees being statistically transitive. Consider a 3–ary and a 4–ary tree joined at a common root. While both trees are transitive, the combined tree is not. If the branching rate for the branching random walk is low enough, Proposition 3.1.5 fails (and in fact Theorem 3.1.1 as well).

3.1.4 Related work

Shape theorems for branching random walks on lattices or on $\mathbb{R}^d$ are a classical topic, going back to the work of [22, 52, 64], we also refer to [70] and to [81] for additional references. In the one-dimensional case, not only the linear growth of the maximal/minimal distance to the origin is known, but also the fluctuations, see [8]. Also, there are, for the one-dimensional case, many finer results about point process convergence of the particle configuration, seen from the maximum/minimum, we again refer to [81] for references. We remark that in the multidimensional case, there are only few results in this direction but this is a very active field of research.

The question about the linear growth of the maximal/minimal distance to the origin for branching random walks in random (spatial) environment was investigated by [41] and [89] for one-dimensional random walk in random environment. For a general model of multi-dimensional random walk in random environments, shape theorems were proven in [38]. In the discrete setup, recurrence and transience for branching Markov chains were investigated in [12],[48] and [76]. For branching random walks on Galton-Watson trees, where the underlying Markov chain is a simple random walk, the question about recurrence and transience was answered in [84]. There are several papers studying the Martin boundary of branching Markov chains, relating it to the properties of the underlying Markov chain, we refer to [20, 35, 60] among others. The authors of [42] study the range of a critical branching random walk on regular trees and are working on a similar result for random trees.

In the physics literature, there is interest in the study of F–KPP type reaction–diffusion equations in random environment [21, 34, 58]. They are linked to branching random walks by duality; in particular the front of F–KPP equation behaves similarly to the maximal displacement of the branching random walk.

For background on the λ–biased random walk we refer again to [70], to [81] and to the survey [10] on biased random walks in random environment and the references therein.

3.2 Proofs of main results

3.2.1 Preliminaries: biased random walks on a BGW tree

As mentioned in the introduction, the underlying motion to the BRW is the λ*–biased random walk* $(S_n)_{n\geq 0}$ on ω. We describe its quenched law P_ω. This is a Markov chain starting from $S_0 = \mathbf{o}$ with transition kernel

$$
\begin{aligned}
\mathrm{P}_\omega(S_{n+1} = i | S_n = \mathbf{o}) &= \frac{1}{\kappa_{\mathbf{o}}}, \quad i = 1, 2, \ldots, \kappa_{\mathbf{o}};\\
\mathrm{P}_\omega(S_{n+1} = x_- | S_n = x) &= \frac{\lambda}{\lambda + \kappa_x}, \quad x \neq \mathbf{o};\\
\mathrm{P}_\omega(S_{n+1} = xi | S_n = x) &= \frac{1}{\lambda + \kappa_x}, \quad x \neq \mathbf{o}, i = 1, 2, \ldots, \kappa_x.
\end{aligned}
$$

Let $\mathbb{P}(\cdot) := \int \mathrm{P}_\omega(\cdot)\mathtt{BGW}(\mathrm{d}\omega)$ be the annealed law.

This is one of the most well-studied models for a random walk in random environment. For an introduction to the simple random walk on random trees see [70, Chapter 17] and for a survey on biased random walks on random graphs see [10]. One key fact about the random walk is the following.

Theorem 3.2.1 (Existence of speed [69, Theorem 3.1]). *Assume that ω has no leaves, i.e. $p_0 = 0$ and that $1 < m_{\texttt{BGW}} < \infty$. Then there is $v_\lambda \in [0, \infty)$ such that*

$$\mathrm{P}_\omega\left(\lim_{n\to\infty} \frac{|S_n|}{n} = v_\lambda\right) = 1, \qquad \texttt{BGW} - a.s.$$

Moreover, $v_\lambda > 0$ if and only if $\lambda < m_{\texttt{BGW}}$.

The function $\lambda \mapsto v_\lambda$ is still not fully understood; in particular, the monotonicity is a well-known open question except for small λ, see [11].

Another result about the λ–biased random walk which is crucial to us is the existence of a large deviation principle.

Theorem 3.2.2 (Large deviations, speed-up probabilities [40, Theorem 1.1]). *Suppose that $\lambda < m_{\texttt{BGW}} < \infty$. Then, there exists a continuous, convex, strictly increasing function $I_\lambda : [v_\lambda, 1] \mapsto [0, \infty)$, with*

$$I_\lambda(v_\lambda) = 0 \quad \textit{and} \quad I_\lambda(1) = -\log \sum_{k\geq 1} \frac{k}{k+\lambda} p_k,$$

satisfying, for $b > a$, $a \in (v_\lambda, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}_\omega\Big(\frac{|S_n|}{n} \in [a, b)\Big) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{|S_n|}{n} \in [a, b)\Big) = -I_\lambda(a), \qquad \texttt{BGW} - a.s. \tag{3.8}$$

Theorem 3.2.3 (Large deviations, slow-down probabilities [40, Theorem 1.2]). *Suppose that $m_{\texttt{BGW}} < \infty$ and that*

$$\textit{either } d_{\min} \geq 2 \textit{ or } \lambda \geq 1. \tag{3.9}$$

Then, there exists a continuous, convex, decreasing function $I_\lambda : [0, v_\lambda] \mapsto [0, \infty)$, with $I_\lambda(v_\lambda) = 0$, such that for $0 \leq b < a < v_\lambda$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}_\omega\Big(\frac{|S_n|}{n} \in [b, a)\Big) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{|S_n|}{n} \in [b, a)\Big) = -I_\lambda(a), \qquad \texttt{BGW} - a.s. \tag{3.10}$$

If $\lambda \geq d_{\min}$, then $I_\lambda(a) = 0$ for $a \in [0, v_\lambda]$. If $\lambda < d_{\min}$, then I_λ is strictly decreasing on $[0, v_\lambda]$ and

$$I_\lambda(0) = \lim_{a\downarrow 0} I_\lambda(a) = \log\left(\frac{d_{\min} + \lambda}{2\sqrt{\lambda d_{\min}}}\right) \tag{3.11}$$

3.2.2 Strong recurrence and transience

The recurrence property of a BRW is closely related to the *spectral radius* of the λ-biased random walk (S_n) of a single particle on the tree ω. This is the following number, which is known to be the same for every $x, y \in \omega$:

$$\rho_\omega(\lambda) := \limsup_{n\to\infty} \mathrm{P}_\omega(S_n = y \mid S_0 = x)^{\frac{1}{n}}.$$

Proposition 3.2.4. *Suppose that* $m_{\texttt{BGW}} < \infty$*. Then for* BGW*-a.e.* ω*,*

$$\rho_\omega(\lambda) = \begin{cases} 1, & \lambda \geq d_{\min}; \\ \frac{2\sqrt{\lambda d_{\min}}}{d_{\min}+\lambda}, & \lambda \in (0, d_{\min}). \end{cases} \tag{3.12}$$

For $\lambda = 1$, this result can also be found in [84, Proposition 3.5].

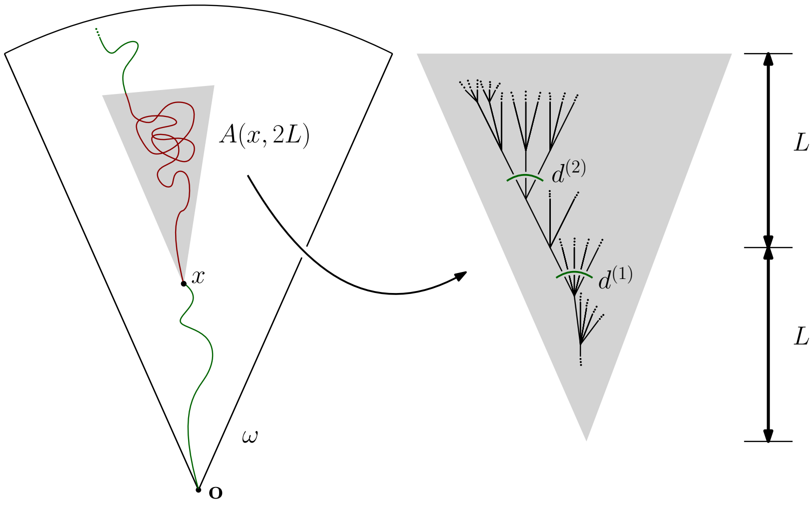

Figure 3.3: The spectral radius $\rho_\omega(\lambda)$ is determined by atypical regions of ω: the set $A(x, 2L)$ consists of L levels of a $d^{(1)}$–ary tree and L levels of a $d^{(2)}$–ary tree. The choice of $(d^{(1)}, d^{(2)})$ depends on λ: if $\lambda < d_{\min}$ we choose $d^{(1)} = d^{(2)} = d_{\min}$ and if $d_{\min} \leq \lambda < m_{\texttt{BGW}}$ we choose $d^{(1)} = d_0$ and $d^{(2)} = d_{\min}$ where $d_0 > m_{\texttt{BGW}}$. Then the set $A(x, 2L)$ acts as a trap for the random walk. Similarly, for the branching random walk, $A(x, 2L)$ acts as a seed which facilitates local survival.

Proof. If the `BGW` tree has a.s. bounded degree, then it follows from [76, Lemma 2.8] that $\rho_\omega(\lambda) = e^{-I_\lambda(0)}$, with $I_\lambda(0)$ as in Theorem 3.2.3[1], which leads to the desired statement.

---

[1] the computation of the value $I_\lambda(0)$ does not require (3.9); see [40, Remark 7.1]

In general, without the assumption of bounded degree, we provide a direct proof, using ideas from the proof of [76, Lemma 2.8]. For `BGW`-a.e. ω, we have for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}_\omega(S_n = \mathbf{o}) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}_\omega(|S_n| < n\varepsilon) = -I_\lambda(\varepsilon).$$

Letting $\varepsilon \to 0$ and by continuity of I_λ, it follows that $\rho_\omega(\lambda) \le e^{-I_\lambda(0)}$. To obtain the lower bound $\rho_\omega(\lambda) \ge e^{-I_\lambda(0)}$, we treat the two cases $\lambda < d_{\min}$ and $\lambda \ge d_{\min}$ separately.

We first consider $\lambda < d_{\min}$. Let $L \in \mathbb{N}$ and $x \in \omega$. Denote by $A(x, 2L)$ the first $2L$ levels of the fringe tree rooted at x, i.e. $A(x, 2L) = \{y \in \omega \colon x \preceq y, |y| \le |x| + 2L\}$. Call x a $(d_{\min}, 2L)$–*nice* vertex if $A(x, 2L)$ is a $(d_{\min})$–ary tree.

Such x exists for `BGW`-a.e. ω and for simplicity we write $A := A(x, 2L)$. Consider a $(d_{\min}, 2L)$–nice vertex and let P_ω^A be the law of a λ-biased random walk killed on leaving the subgraph $A \subseteq \omega$. Choose $y_0 \in A$ arbitrarily and denote the spectral radius in A by $\rho_\omega^A(\lambda) := \limsup_{n\to\infty} \mathrm{P}_\omega^A(X_n = y_0 \mid X_0 = y_0)^{\frac{1}{n}}$. Naturally we have $\mathrm{P}_\omega^A(X_n = y_0 \mid X_0 = y_0) \le \mathrm{P}_\omega(X_n = y_0 \mid X_0 = y_0)$ and therefore $\rho_\omega^A(\lambda) \le \rho_\omega(\lambda)$. Therefore it suffices to study $\rho_\omega^A(\lambda)$ in order to obtain a lower bound.

Let $y_0 \in A$ so that $|y_0| = |x| + L$. Consider a random walk (S_n) starting from $S_0 = y_0$ and denote by T the first time that the random walk reaches the boundary of A, i.e. $|S_n| = |x|$ or $|S_n| = |x| + 2L$. Because A is a $(d_{\min})$–ary tree, we can compare P_ω^A to a biased simple random walk on $\mathbb{Z}$. Let $\mathrm{P}_{\mathrm{SRW}(p)}$ be the law of a simple random walk on $\mathbb{Z}$ that moves up with probability p. Then the law of $(|S_n|, n \ge 1)$ restricted to $[|x|, |x| + 2L]$ is $\mathrm{P}_{\mathrm{SRW}(p)}$ with $p = \frac{d_{\min}}{d_{\min}+\lambda}$. For $s, t \in (0, 1)$ let $H(s|t) = s \log \frac{s}{t} + (1 - s)\frac{1-s}{1-t}$, the relative entropy between two Bernoulli distributions. We imitate the SRW–computations from [40, Section 3]. We can compare the biased SRW to the symmetric SRW by performing a change of measure. In particular we have for the large deviations of T which is now the first time of the SRW to reach the boundary of $[|x|, |x| + 2L]$ that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}_{\mathrm{SRW}(p)} (T > n) \ge \liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}_{\mathrm{SRW}(\frac{1}{2})} (T > n) - H\left(\frac{1}{2}\middle|p\right).$$

The large deviations of T for $\mathrm{P}_{\mathrm{SRW}(\frac{1}{2})}$ are well understood, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}_{\mathrm{SRW}(\frac{1}{2})} (T > n) = -\frac{\pi^2}{8L^2}.$$

Recall that $p = \frac{d_{\min}}{d_{\min}+\lambda}$ and therefore $H\left(\frac{1}{2}\middle|p\right) = I_\lambda(0)$. Thus

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}_{\mathrm{SRW}(p)} (T > n) \ge -\frac{\pi^2}{8L^2} - I_\lambda(0).$$

We return to the λ–biased random walk on ω. Let $c = \frac{1\wedge\lambda}{\lambda+d_{\min}}$ and $\varepsilon > 0$, then for an even number n with $\varepsilon n > L$, using the Markov property at time $\lceil(1-\varepsilon)n\rceil$ we have

$$\mathrm{P}_\omega^A(S_n = y_0 \mid S_0 = y_0) \geq \mathrm{P}_\omega^A(T > \lceil(1-\varepsilon)n\rceil \mid S_0 = y_0)c^{\varepsilon n}.$$

It follows that

$$\limsup_{n\to\infty} n^{-1}\log \mathrm{P}_\omega^A(S_n = y_0 \mid S_0 = y_0) \geq (1-\varepsilon)\Big(-\frac{\pi^2}{8L^2} - I_\lambda(0)\Big) + \varepsilon \log c.$$

As ε is arbitrary, this yields $\rho_\omega(\lambda) \geq \rho_\omega^A(\lambda) \geq -\frac{\pi^2}{8L^2} - I_\lambda(0)$. Letting $L \to \infty$, we conclude that $\rho_\omega(\lambda) \geq e^{-I_\lambda(0)}$.

We next turn to the case $\lambda \geq d_{\min}$. Note that when $\lambda \geq m_{\texttt{BGW}}$ the λ-biased random walk is recurrent - hence the spectral radius is 1. It remains to study the case $m_{\texttt{BGW}} > \lambda \geq d_{\min}$. The arguments are very similar as for the case $\lambda < d_{\min}$ but we consider a different subset of ω, using ideas from [40, Proof of Theorem 1.2]. Recall that $(p_k, k \geq 1)$ is the probability distribution that determines BGW. There is $d_0 \in \mathbb{N}$ such that $p_{d_0} > 0$ with $d_0 \geq m_{\texttt{BGW}} > \lambda$, fix such d_0. Next, we again consider a vertex $x \in \omega$ where $A(x, 2L)$ takes a specific form: on the first L levels $A(x, 2L)$ is a d_0–ary tree, whereas on the next L levels $A(x, 2L)$ is a $(d_{\min})$–ary tree. More precisely, for $y \succeq x$ we require $\kappa_y = d_0$ if $|y| < |x|+L$ and $\kappa_y = d_{\min}$ if $|x|+L \leq |y| \leq |x|+2L$. Such a vertex x exists BGW–almost surely, fix x and abbreviate $A = A(x, 2L)$. Starting the λ-biased random walk at any vertex $y_0 \in A$ with $|y_0| = |x|+L$, we can again compare it to a random walk on $\mathbb{Z}$ killed upon leaving $[-L, L]$. This is an asymmetric random walk on $\mathbb{Z}$ with bias $\frac{d_0}{\lambda+d_0} > \frac{1}{2}$ on $\mathbb{Z}_-$ and bias $\frac{d_{\min}}{\lambda+d_{\min}} < \frac{1}{2}$ on $\mathbb{Z}_+$. As this random walk on $\mathbb{Z}$ is recurrent (when disregarding the killing), we deduce that $\lim_{L\to\infty} \rho_\omega^A(\lambda) = 1$. This also implies that $\rho_\omega(\lambda) = 1$. □

Proof of Theorem 3.1.2. As $\lambda > 0$, the λ-biased RW is irreducible. By [76, Theorem 2.12], a BRW starting from $x \in \omega$ is recurrent (strongly or weakly), if and only if $m > 1/\rho_\omega(\lambda)$. See also [48, Theorem 3.2]. Proposition 3.2.4 provides the value of $\rho_\omega(\lambda)$. It remains to show strong recurrence when $m > 1/\rho_\omega(\lambda)$. We only give details for the case $\lambda < d_{\min}$ as only a slight modification is needed for the case $m_{\texttt{BGW}} > \lambda \geq d_{\min}$.

Fix an integer $L > 0$ (to be chosen later). As in the proof of Proposition 3.2.4, let $x \in \omega$ be a $(d_{\min}, 2L)$-nice vertex. Let $\mathbf{P}_\omega^A$ denote the law of a BRW killed upon leaving $A := A(x, 2L)$. We have shown in the proof of Proposition 3.2.4 that $\lim_{L\to\infty} \rho_\omega^A(\lambda) = \rho_\omega(\lambda)$ and therefore for all L large enough we have $m > 1/\rho_\omega^A(\lambda)$, fix such L. By the definition of the spectral radius, for all k large enough the random walk (S_n) satisfies $\mathrm{P}_\omega^A(S_k = x \mid S_0 = x) > m^{-k}$. Fix such a k and start a BRW with law $\mathbf{P}_\omega^A$ at x. The expected number of particles after time k at x is bigger than 1. Therefore, by considering the BRW at times $(k, 2k, 3k, \dots)$ and only keeping the particles located at x, this yields a supercritical branching process, which has strictly positive survival probability $c > 0$. Note that c is the same for all $(d_{\min}, 2L)$-nice vertices.

Now let us consider a BRW starting with a particle at $\mathbf{o}$. We look at the random walk $(S_n^*, n \geq 0)$ given by the ancestral lineage $(S_0^* = X(\varnothing), S_1^* = X(1), S_2^* = X(11), \dots)$. This is a λ–biased random walk. Then the trace of this random walk a.s. encounters $(d_{\min}, 2L)$–nice vertices infinitely often. Indeed, let us explore the `BGW`-tree as the random walk proceeds: let $(Z_i, i \geq 1)$ be the first vertices encountered by the random walk on level $2Li$. Note that the sets $(A(Z_i, 2L), i \geq 1)$ are disjoint. For $i \in \mathbb{N}$ let $\zeta_i = \mathbb{1}\{Z_i \text{ is a } (d_{\min}, 2L)\text{–nice vertex}\}$, the indicator that Z_i is a $(d_{\min}, 2L)$–nice vertex. Consider now $(\zeta_i, i \geq 1)$, this sequence of random variables is *i.i.d.* under $\mathbb{P}$. And furthermore, $\mathbb{P}(\zeta_1 = 1) > 0$, i.e. there is a positive probability that Z_1 is a $(d_{\min}, 2L)$–nice vertex. This implies that $\mathbb{P}$–almost surely, and hence also $\mathbf{P}_\omega$–almost surely, $\zeta_i = 1$ infinitely often.

Having established that $(S_n^*, n \geq 1)$ encounters infinitely many $(d_{\min}, 2L)$–nice vertices, let $(Z_i^*, i \geq 1)$ be a subsequence of $(Z_i, i \geq 1)$ such that for every $i \in \mathbb{N}$ Z_i^* is a $(d_{\min}, 2L)$–nice vertex. For each i, consider now a branching process with the measure $\mathbf{P}_\omega^A$ where $A = A(Z_i^*, 2L)$ starting at Z_i^* at the time when $(S_n^*, n \geq 1)$ first encounters Z_i^*. This auxiliary process is realised as a subset of particles of our BRW. As discussed above, this auxiliary branching process survives with probability $c > 0$ under $\mathbf{P}_\omega$ for every i. Also, for $i \neq j$, the auxiliary processes are independent. As survival of these processes are *i.i.d.* trials, $\mathbf{P}_\omega$–almost surely, infinitely many of them survive. In particular, there exists i such that the i–th auxiliary process survives. This means that Z_i^* is visited infinitely often and hence the branching random walk is strongly recurrent. □

3.2.3 $0-1$ laws, proof of Proposition 3.1.5

The goal of this section is to show Proposition 3.1.5. Interestingly, this requires different proofs for the strongly recurrent and transient regimes. We do this by showing various $0-1$ laws for the environment `BGW` and for the λ–biased random walk. Let us start by recalling the following classical zero-one law (c.f. [70, Proposition 5.6]). We slightly vary notation to stress that this lemma can be applied both to the law of ω and to the law of $\mathbb{T}$.

Lemma 3.2.5. *[Zero-one law for inherited properties]*
Consider a Bienaymé–Galton–Watson reproduction law satisfying (3.1) *(no leaves) and denote the corresponding probability measure on trees by* $\mathtt{BGW}$*, its expectation by* $E_{\mathtt{BGW}}$ *and the corresponding branching process by* (Z_n)*. A property* A *of trees, is called an inherited property if* $A \subseteq \{\omega : \forall j \leq \kappa_{\mathbf{o}} : \omega_j \in A\}$ *where* ω_j *denotes the subtree rooted at* j*, for* $j = 1, ..., Z_1$*. This means that if a tree satisfies the property* A *then all subtrees rooted at the children of the root also satisfy the property* A*. Then, if* A *is an inherited property, we have* $\mathtt{BGW}(A) \in \{0, 1\}$*.*

Proof. Let $\mathcal{A}$ (resp. $\mathcal{A}^{(j)}$) be the collection of trees ω (resp. ω_j) that satisfies property A. Conditional on Z_1, $(\mathcal{A}^{(j)}, j \leq Z_1)$ are independent copies of $\mathcal{A}$. Therefore

$$\alpha := \mathtt{BGW}(\mathcal{A}) \leq \mathtt{BGW}\Bigg(\bigcap_{j \leq Z_1} \mathcal{A}^{(j)}\Bigg) = \mathbb{E}_{\mathtt{BGW}}\Bigg[\prod_{j=1}^{Z_1} \mathtt{BGW}(\mathcal{A}^{(j)})\Bigg] = \mathbb{E}_{\mathtt{BGW}}[\alpha^{Z_1}],$$

hence, $\alpha \leq \varphi(\alpha)$ where $\varphi(s) := E(s^{Z_1})$. But, due to (3.1), the function φ is strictly convex with $\varphi(0) = 0$ and $\varphi(1) = 1$. We conclude that $\alpha \in \{0, 1\}$. ☐

We now turn to Proposition 3.1.5 for the strongly recurrent regime for (λ, m) as in Theorem 3.1.2. Here we have a proper $0 - 1$ law. We formulate the statements for a fixed tree ω, if the tree is chosen randomly according to $\mathtt{BGW}$ then the statement is true for $\mathtt{BGW}$–almost every ω.

Proposition 3.2.6. *Assume that* ω *is any tree such that the* λ*–biased BRW is strongly recurrent. Then for every* $a > 0$

$$\mathbf{P}_\omega\Big(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \geq a\Big) \in \{0, 1\}.$$

Proof of Proposition 3.1.5 for the strongly recurrent regime. If (λ, m) is chosen such that the λ–biased random walk is strongly recurrent for $\mathtt{BGW}$–almost every ω according to Theorem 3.1.2, then Proposition 3.1.5 follows immediately from Proposition 3.2.6. ☐

Before proving Proposition 3.2.6, we state one intermediate lemma.

Lemma 3.2.7. *Assume that* ω *is any tree such that the* λ*–biased BRW is strongly recurrent. Then for every* $a > 0$ *and* $x \in \omega$*, set*

$$q_{a,x} := \mathbf{P}_\omega\Big(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \geq a \,\Big|\, X_\varnothing = x\Big).$$

Then we have $q_{a,x} = q_{a,\mathbf{o}}$*.*

Proof. Set $T_{x,\mathbf{o}} = \inf\{n \geq 0\colon \exists u \text{ with } |u| = n, X(u) = \mathbf{o}\}$, which is a.s. finite thanks to the strong recurrence. Let v be a particle such that $|v| = T_{x,\mathbf{o}}$ and $X(v) = \mathbf{o}$. Set $X^{(v)}(u) = X(vu), vu \in \mathbb{T}$, i.e. the subpopulation descended from v. As $\max_{|u|=n} |X(u)| \geq \max_{|u|=n-T_{x,\mathbf{o}}} X(vu)$, we have

$$\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \geq \liminf_{n\to\infty} \max_{|u|=n-T_{x,\mathbf{o}}} \frac{|X^{(v)}(u)|}{n} = \liminf_{n\to\infty} \max_{|u|=n} \frac{|X^{(v)}(u)|}{n}.$$

By the branching property, $(X^{(v)})$ is a BRW started from $\mathbf{o}$. Therefore,

$$q_{a,x} \geq \mathbf{P}_\omega \left(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \geq a \,\middle|\, X(\varnothing) = \mathbf{o} \right) = q_{a,\mathbf{o}}.$$

A similar argument shows that $q_{a,\mathbf{o}} \geq q_{a,x}$. $\square$

Proof of Proposition 3.2.6. Let $A_a = \{\liminf_{n\to\infty} \max_{|u|=n} |X(u)|/n < a\}$ and $A_a^{(j)} = \{\liminf_{n\to\infty} \max_{|u|=n} |X^{(j)}(u)|/n < a\}$. We claim that A_a is an inherited property, more precisely $A_a \subseteq \bigcap_{j \leq \gamma_\varnothing} A_a^{(j)}$. This follows from the independence of the subpopulations $(X^{(j)}(u) := X(ju))$ descended from each particle $j \leq \gamma_\varnothing$ under the law $\mathbf{P}_\omega$, given $X(u), j \leq \gamma_\varnothing$. Unfortunately we cannot apply Lemma 3.2.5 as the $A_a^{(j)}$ do not have the same law as they all still depend on the whole of ω, not just their respective subtree. Nevertheless, using Lemma 3.2.7, we have

$$\begin{aligned} 1 - q_{a,\mathbf{o}} = \mathbf{P}_\omega(A_a) &\leq \mathbf{E}_\omega\left[\mathbf{P}_\omega\left(\bigcap_{j\leq\gamma_\varnothing} A_a^{(j)} \,\middle|\, X(j), j \leq \gamma_\varnothing \right) \right] \\ &= \mathbf{E}_\omega\left[\prod_{j=1}^{\gamma_\varnothing} \left(1 - q_{a,X(j)}\right) \right] = \mathbf{E}_\omega[(1 - q_{a,\mathbf{o}})^{\gamma_\varnothing}]. \end{aligned}$$

This implies that $1 - q_{a,\mathbf{o}}$ is equal to either 0 or 1, and the same applies to $q_{a,\mathbf{o}}$. $\square$

We now turn to the proof of Proposition 3.1.5 in the transient regime. The lack of strong recurrence means it is a priori difficult to compare two branching random walks started from different vertices of (the same) ω. Because of this the above methods do not apply. Instead we need to use that ω is *statistically transitive.*

To this end, we also introduce a killed version of the λ–biased random walk which is modified at the root, whose quenched law P^*_ω is given by

$$\begin{aligned} &\mathrm{P}^*_\omega(S_{n+1} = i | S_n = \mathbf{o}) = \frac{1}{\lambda + \kappa_\mathbf{o}}, \quad i = 1, \dots, \kappa_\mathbf{o}, \\ &\mathrm{P}^*_\omega(S_{n+1} = \dagger | S_n = \mathbf{o}) = \frac{\lambda}{\lambda + \kappa_\mathbf{o}}, \\ &\mathrm{P}^*_\omega(S_{n+1} = x | S_n = y) = \mathrm{P}_\omega(S_{n+1} = x | S_n = y) \text{ for } y \neq \mathbf{o}\,, \end{aligned}$$

where † is an additional cemetery point which is an absorbing state. Define the corresponding BRWs with quenched and annealed laws denoted by $\mathbf{P}^*_\omega$ and $\mathbb{P}^*$ respectively (particles absorbed at † do not branch).

Proof of Proposition 3.1.5 in the transient phase. We focus on the phase of (m, λ) in Theorem 3.1.2 where a λ–biased BRW is $\mathbf{P}_\omega$-almost surely transient for `BGW`–almost every ω.

Consider a BRW X^* of law $\mathbf{P}^*_\omega$. We can couple this killed BRW X^* with a BRW X of law $\mathbf{P}_\omega$ by adding extra randomness. More precisely, for each particle u of X situated at the root with $X(u) = \mathbf{o}$, we kill each of its children with probability $q = \frac{\lambda}{\lambda+\kappa_{\mathbf{o}}}$, independently of everything else. The resulting BRW has the same law as X^* restricted to ω without †. Let N be the total number of particles of X that are children of particles at the root $\mathbf{o}$, that is $N = \#\{uj \in \mathbb{T} \colon X(u) = \mathbf{o}, j \in \mathbb{N}\}$. Then N is $\mathbf{P}_\omega$-a.s. finite, due to the transience assumption. Set

$$B_a := A_a^c = \left\{ \liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \geq a \right\}.$$

If $\mathbf{P}_\omega\left(B_a\right) > 0$, then we have

$$\mathbf{P}^*_\omega(B_a) \geq \mathbf{P}^*_\omega\Big(B_a;\ \text{no particles are killed}\Big) = \mathbf{E}_\omega\left[\mathbb{1}_{\{B_a\}}(1-q)^N\right] > 0.$$

Now we view $\Upsilon(\omega) = \mathbf{P}_\omega\left(B_a\right)$ and $\Upsilon^*(\omega) = \mathbf{P}^*_\omega\left(B_a\right)$ as functions of the random variable ω under `BGW`. The previous considerations showed that $\Upsilon^*(\omega) > 0$, `BGW`–almost surely under the assumption that $\Upsilon(\omega) > 0$, `BGW`–almost surely.

Consider now

$$\mathcal{W}_k = \left\{u \in \mathbb{T} : |X(u)| = k, \forall v \prec u : |X(v)| < k\right\}, \tag{3.13}$$

the set of particles that first hit level k in their genealogical line of descent. Note that for $u, v \in \mathcal{W}_n$ we might have $X(u) = X(v)$ which we would like to avoid. Therefore for each $x \in \{y \in \omega : \exists u \in \mathcal{W}_k \text{ with } X(u) = y\}$ pick a representative $u \in \mathcal{W}_k$ with $X(u) = x$. Let $\widetilde{\mathcal{W}}_k \subseteq \mathcal{W}_k$ be the set of representatives. This means that for $u, v \in \widetilde{\mathcal{W}}_n$ we have $X(u) \neq X(v)$.

Fix ω. We observe that for each k

$$\begin{aligned} 1 - \Upsilon(\omega) &= \mathbf{P}_\omega\left(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} < a \right) \\ &\leq \mathbf{P}_\omega\left(\forall u \in \widetilde{\mathcal{W}}_k : \liminf_{n\to\infty} \max_{\substack{|v|=n \\ u\prec v}} \frac{|X(v)|}{n} < a \right). \end{aligned}$$

Next, observe that for $u, v \in \widetilde{\mathcal{W}}_k$, conditional on $X(u)$ and $X(v)$, the descendant populations of u and v are independent. This means that by applying the Markov property we obtain

$$\mathbf{P}_\omega\left(\forall u \in \widetilde{\mathcal{W}}_k : \liminf_{n\to\infty} \max_{\substack{|v|=n \\ u\prec v}} \frac{|X(v)|}{n} < a\right) \\ = \mathbf{E}_\omega\left[\prod_{u\in\widetilde{\mathcal{W}}_k} \mathbf{P}_{\omega,X(u)}\left(\liminf_{n\to\infty}\max_{|v|=n}\frac{|X(v)|}{n} < a\right)\right],$$

where $\mathbf{P}_{\omega,x}$ corresponds to the process started from a single particle located at x. Recall that for $x \in \omega$ we let $F_\omega(x)$ be the fringe subtree rooted at x, that is $F_\omega(x) = \{y \in \omega : y \succeq x\}$. We introduce killing again, this time we modify $\mathbf{P}_{\omega,X(u)}$ by killing all particles that cross the edge $(X(u), X(u)_-)$, the edge from $X(u)$ to its parent. Denote the corresponding probability measure by $\mathbf{P}^*_{\omega,X(u)}$. As before, the process with killing can be realised as a subset of particles of the process without killing, and thus the maximal displacement, $\max_{|v|=n}|X(v)|$ in the process without killing stochastically dominates the maximal displacement in the process with killing. Using the convention $\max\emptyset = -\infty$ we obtain

$$\mathbf{P}_{\omega,X(u)}\left(\liminf_{n\to\infty}\max_{|v|=n}\frac{|X(v)|}{n} < a\right) \le \mathbf{P}^*_{\omega,X(u)}\left(\liminf_{n\to\infty}\max_{|v|=n}\frac{|X(v)|}{n} < a\right) \\ = \mathbf{P}^*_{F_\omega(X(u))}\left(\liminf_{n\to\infty}\max_{|v|=n}\frac{|X(v)|}{n} < a\right) = 1 - \Upsilon^*(F_\omega(X(u))).$$

This yields the inequality

$$1 - \Upsilon(\omega) \le \mathbf{E}_\omega\left[\prod_{u\in\widetilde{\mathcal{W}}_k}\left(1 - \Upsilon^*\Big(F_\omega(X(u))\Big)\right)\right].$$

Going to the annealed law, i.e. integrating over ω, and rearranging this means

$$\mathbb{E}_{\texttt{BGW}}\left[\Upsilon(\omega)\right] \ge 1 - \mathbb{E}\left[\prod_{u\in\widetilde{\mathcal{W}}_k}\left(1 - \Upsilon^*\Big(F_\omega(X(u))\Big)\right)\right].$$

Note that conditional on $\widetilde{\mathcal{W}}_k$, the $(F_\omega(x), |x| = k)$ are *i.i.d.* `BGW`–trees under $\mathbb{E}$. This means that we can improve the inequality to

$$\mathbb{E}_{\texttt{BGW}}\left[\Upsilon(\omega)\right] \ge 1 - \mathbb{E}\left[\prod_{u\in\widetilde{\mathcal{W}}_k}\left(1 - \Upsilon^*\Big(F_\omega(X(u))\Big)\right)\right] = 1 - \mathbb{E}\left[\mathbb{E}_{\texttt{BGW}}\left[1 - \Upsilon^*(\omega)\right]^{|\widetilde{\mathcal{W}}_k|}\right].$$

Lastly, we claim that as $k \to \infty$ we have $|\widetilde{\mathcal{W}}_k| \to \infty$, we show this in Lemma 3.2.8 below. From this and $\Upsilon^*(\omega) > 0$, $\texttt{BGW}$–almost surely we obtain that $\mathbb{E}_{\texttt{BGW}}[\Upsilon(\omega)] \geq 1$. This implies that $\Upsilon(\omega) = 1$ for $\texttt{BGW}$–almost every ω because $\Upsilon(\omega) \leq 1$ as it is a probability. ☐

Lemma 3.2.8. *In the setting of the proof of Proposition 3.1.5 in the transient regime, we have that* $|\widetilde{\mathcal{W}}_k| \to \infty$, $\mathbb{P}$*–almost surely as* $k \to \infty$.

Proof. We consider a tagged particle like in the proof of Theorem 3.1.2: We look at the random walk $(S_n^*, n \geq 0)$ given by the ancestral lineage $(S_0^* = X(\varnothing), S_1^* = X(1), S_2^* = X(11), \ldots)$. Let 1_n denote the tagged particle in the n-th generation so that $S_n^* = X(1_n)$. This is a λ–biased random walk. Inspect the trace of S^*, $(S_n^*, n \geq 0)$. Consider the set

$$R = \Big\{x \in \omega : \exists n \in \mathbb{N} : S_n^* = x, \forall \ell \in \mathbb{N} : S_\ell^* \notin F_\omega(x) \backslash \{x\}\Big\}.$$

That is, $x \in R$ if S^* encounters x but at the same time does not explore the subtree associated with x. An event which could cause $x \in R$ is the following: S^* encounters x^*, the parent of x, moves to x, moves back to x^*, moves to a different child of x^*, and then never visits x^* again. By transience of S^* we can see that $|R| = \infty$, $\mathbb{P}$–almost surely.

For $x \in \omega$, we define an event $\texttt{split}(x)$ for the branching random walk:

1. We have $x \in R$, let τ_x be the first time S^* hits x.
2. The particle 1_{τ_x} (which corresponds to $S_{\tau_x}^*$) has at least 2 children.
3. The particle $v = 1_{\tau_x}2$ (this is the second offspring particle of 1_{τ_x} which does not correspond to $S_{\tau_x+1}^*$) moves to a child of x.
4. The lineage corresponding to v (that is particles of the form $v1_\ell$ for $\ell \geq 0$) never revisits x.

See Figure 3.4 for an illustration of this event. We use these events for a lower bound of $|\widetilde{\mathcal{W}}_k|$,

$$|\widetilde{\mathcal{W}}_k| \geq 1 + \sum_{x \in R : |x| < k} \mathbb{1}_{\{\texttt{split}(x)\}}, \tag{3.14}$$

that is, whenever $\texttt{split}(x)$ happens, $|\widetilde{\mathcal{W}}_k|$ increases by $+1$. To estimate the probabilities of $\texttt{split}(x)$, we define a sequence of σ–algebras $(\mathcal{G}_n)_{n \geq 1}$. To this end, conditional

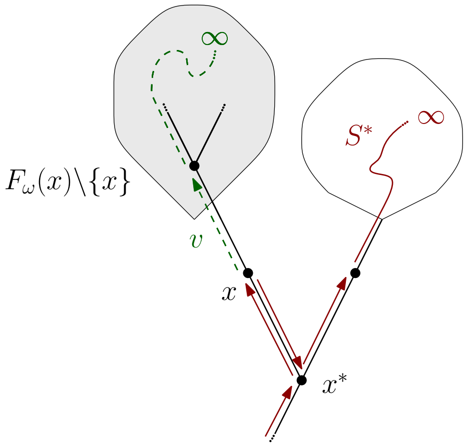

Figure 3.4: An illustration of the event $\texttt{split}(x)$: the tagged particle S^* (in solid red) visits x but not $F_\omega(x)\backslash\{x\}$ (in light grey). While S^* visits x, the BRW produces a new particle v (in dashed green) that moves to a child of x and then never revisits x.

on $(S^*_\ell)_{\ell\geq 0}$ we enumerate R_2 so that we can write $R = (x_1, x_2, \ldots)$, $\mathbb{P}$–almost surely. We set

$$\mathcal{G}_n = \sigma\left\{(S^*_\ell)_{\ell\geq 0}\,; (\kappa_{S^*_\ell})_{\ell\geq 0}; \texttt{split}(x_i) \text{ for } 1 \leq i < n\right\}.$$

Observe that $F_\omega(x_n)\backslash\{x_n\}$ is independent of $\mathcal{G}_n$ under $\mathbb{P}$ because by our choice of x_n the random walk S^* never explores $F_\omega(x_n)\backslash\{x_n\}$. Further, for $i < n$, $\texttt{split}(x_i)$ does not depend on $F_\omega(x_n)\backslash\{x_n\}$. Therefore we have

$$\begin{aligned}
&\mathbb{P}\left(\texttt{split}(x_n)|\mathcal{G}_n\right)\\
&= (1-\mu_1)\frac{\kappa_{x_n}}{\lambda+\kappa_{x_n}}\mathbb{P}(\lambda\text{–biased RW never visits } x\\
&\qquad\qquad\qquad\qquad\qquad\text{starting from a uniformly chosen child of } x|\mathcal{G}_n)\\
&\geq (1-\mu_1)\frac{\kappa_{x_n}}{\lambda+\kappa_{x_n}}\mathbb{P}^*\left(\tau_{\mathbf{o}} = \infty\right),
\end{aligned}$$

where in the second step we used that $F_\omega(x_n)\backslash\{x_n\}$ is independent of $\mathcal{G}_n$. Thus under the conditional probability the particle v sees a $\texttt{BGW}$–distributed tree which is independent of $\mathcal{G}_n$. The probability to never visit x again starting from xi is at least the probability of starting from xi, moving to a child of xi and then never visiting xi again. This probability equals $\mathbb{P}^*(\tau_{\mathbf{o}} = \infty)$ where $\tau_{\mathbf{o}}$ is the return time to the root for a λ–biased random walk. We also switch from $\mathbb{P}$ to $\mathbb{P}^*$ to ensure the correct transition probabilities at the root.

Lastly, we estimate $\frac{\kappa_{x_n}}{\lambda+\kappa_{x_n}} \geq \frac{1}{\lambda+1}$ and therefore for all $n \geq 1$ we have

$$\mathbb{P}\left(\texttt{split}(x_n)|\mathcal{G}_n\right) \geq (1-\mu_1)\frac{1}{\lambda+1}\mathbb{P}^*\left(\tau_{\mathbf{o}} = \infty\right) > 0,$$

because we are in the regime where the λ–biased random walk is transient. This estimate is uniform in n, and does not depend on $\mathcal{G}_n$. Hence

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\texttt{split}(x_n) | \mathcal{G}_n\right) \geq \sum_{n=1}^{\infty} (1-\mu_1) \frac{1}{\lambda+1} \mathbb{P}^*\left(\tau_{\mathbf{o}} = \infty\right) = \infty.$$

A conditional version of the Borel–Cantelli Lemma (see for example [61, Cor. 9.21]) tells us that therefore also $\sum_{n=1}^{\infty} \mathbb{1}_{\{\texttt{split}(x_n)\}} = \infty$, $\mathbb{P}$–almost surely. Together with (3.14), we obtain that $|\widetilde{\mathcal{W}}_k| \to \infty$ as $k \to \infty$, $\mathbb{P}$–almost surely.

□

3.2.4 Annealed probability of fast particles, proof of Proposition 3.1.4

The goal of this section is to show Proposition 3.1.4. That is we want to show that, for any $a < v_{\lambda,m}$,

$$\mathbb{P}\left(\liminf_{n\to\infty} \max_{|u|=n} \frac{|X(u)|}{n} \geq a\right) > 0.$$

The crucial idea is to look for trajectories that move from level $D_{(i-1)n}$ to level D_{in} without revisiting level $D_{(i-1)n}$ and that move quickly, that is in time less than n/a. This allows us to use an approximation with an $\mathbb{N}_0$–valued branching process in random environment. If we can show that this branching process survives with positive probability we are done.

We start by introducing some notation. For any $u \in \mathbb{T}$ and $n \in \mathbb{N}$ let

$$T_n^u := \inf\{k \leq |u| : \ |X(u_k)| = n\} \quad \text{and} \quad \mathcal{L}_n := \{u_{T_n^u}, u \in \mathbb{T}\},$$

that are the first hitting time of $D_n = \{x \in \omega : |x| = n\}$ along the ancestral lineage of u and the collection of particles that hit D_n for the first time in their lines of descent respectively. Moreover, we introduce two subclasses of $\mathcal{L}_n$: the particles that reach D_n quickly, that is in fewer than cn steps ($c \geq 1$), and the particles that reach D_n quickly without returning to the root:

$$\mathcal{L}_n^{(c)} := \{u \in \mathcal{L}_n : |u| \leq cn\}, \tag{3.15}$$

$$\mathcal{L}_n^{(c),*} := \{u \in \mathcal{L}_n \colon |u| \leq cn, X(u_k) \neq \mathbf{o}, \forall k = 1, \ldots, |u|\}. \tag{3.16}$$

Specifically, we fix $\varepsilon > 0$ and $n > 0$. Set $\mathcal{Z}_0^{(n)} = \{\varnothing\}$ and

$$\mathcal{Z}_1^{(n)} = \{u \in \mathcal{L}_n^{(1/a),*} \colon |u| > (a+\varepsilon)^{-1} n\}.$$

For $i \in \mathbb{N}$, we define recursively a family of particles

$$\mathcal{Z}_{i+1}^{(n)} = \Big\{ u \in \mathcal{L}_{(i+1)n} : u_{T^u_{in}} \in \mathcal{Z}_i^{(n)}, \\ (a+\varepsilon)^{-1} n < |u| - T^u_{in} \leq a^{-1} n, |X(u_k)| > in, T^u_{in} < k \leq |u| \Big\}.$$

These are the particles that, for $j \leq i$, move from level D_{jn} to $D_{(j+1)n}$ without revisiting D_{jn}. Further, we impose that the particles move quickly but not too quickly. The intuition behind requiring the particles not to move too quickly is that they have time to branch before reaching $D_{(i+1)n}$. We claim that particles like these exist for all i with positive probability.

Lemma 3.2.9. *There exists $n_0 > 0$, such that for every $n \geq n_0$,*

$$\mathbb{P}(\# \mathcal{Z}_i^{(n)} > 0 \text{ for all } i \geq 1) > 0.$$

Proof. We want to compare $\# \mathcal{Z}_i^{(n)}$ to an $\mathbb{N}_0$–valued branching process in random environment. We stress that $(\# \mathcal{Z}_i^{(n)}, i \geq 0)$ itself is not a branching process under the annealed law $\mathbb{P}$. The reason for this is that two particles that are counted in $\mathcal{Z}_i^{(n)}$ may or may not have been located at the same vertex in $D_{(i-1)n}$. In this case they depend differently on different regions of ω and do not have a consistent structure of independence/dependence. See Figure 3.5 for an illustration.

We compare this to a process where all particles start from the same vertex in ω at times $(in, i \geq 0)$. Informally, this increases the dependence between particles as they now all use the same environment. The increased dependence should lead to a higher probability of extinction. This also means that if this modified process survives with positive probability, so should the branching random walk.

We start by choosing n_0 large enough so that for all $n \geq n_0$ we have

$$\mathbb{E}_{\texttt{BGW}} \Big[\log \mathbf{E}_\omega \Big[\# \mathcal{Z}_1^{(n)} \Big] \Big] > 0. \tag{3.17}$$

We prove that such n_0 exists in Lemma 3.2.11 below. Fix $n \geq n_0$ and drop the superscript to write $Z_i := \# \mathcal{Z}_i^{(n)}$. We define an auxiliary branching process in random environment $(Z'_i, i \geq 0)$. Let $\vec{\omega} := (\omega_i)_{i \geq 1}$ be a family of *i.i.d.* BGW–trees with distribution `BGW`. Set $Z'_0 = 1$ and define inductively

$$Z'_{i+1} := \sum_{j=1}^{Z'_i} \xi'_{i,j},$$

where each $\xi'_{i,j}$ is an independent copy of Z_1 under the quenched law $\mathbf{P}_{\omega_i}$ using ω_i as environment. Denote the quenched law for $(Z'_i, i \geq 1)$ by $\mathbf{P}'_{\vec{\omega}}$ and the annealed law

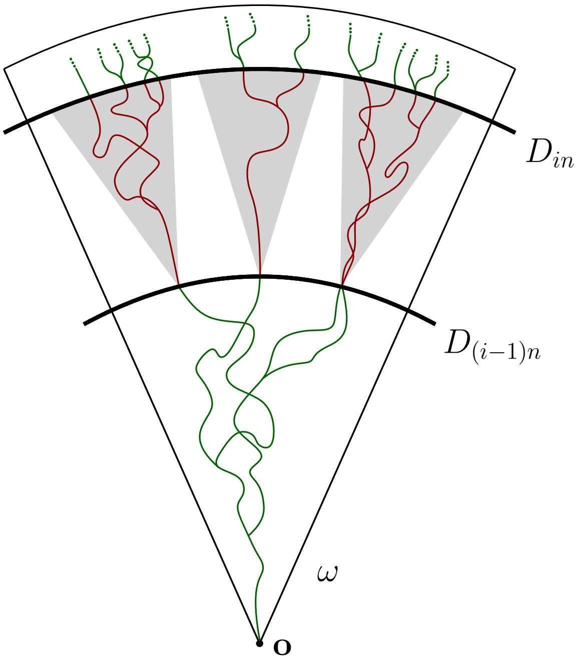

Figure 3.5: An illustration of the trajectories counted in $\mathcal{Z}_i^{(n)}$. Observe that the trajectories are allowed to backtrack but not below level $D_{(i-1)n}$ so that the trajectories in different cones are independent. Under $\mathbb{P}$ the grey cones are independent copies of ω. Each cone may have multiple particles starting in it, we compare this to a process where all particles use the same cone.

by $\mathbb{P}'$. The consequence of (3.17) is that this is a supercritical branching process in random environment, see e.g. [7, Section VI.5]. In particular, the annealed survival probability is strictly positive, i.e. there exists $c > 0$, such that

$$\mathbb{P}'(Z_i' > 0 \text{ for all } i \geq 1) \geq c.$$

We set

$$\theta_{i,\ell} := \mathbb{P}\left(Z_\ell = 0 | Z_0 = i\right) \quad \text{and} \quad \theta_{i,\ell}' := \mathbb{P}'\left(Z_\ell' = 0 | Z_0' = i\right),$$

the annealed probabilities of extinction after ℓ steps when starting with i particles. Note that for both $\mathbb{P}$ and $\mathbb{P}'$, all starting particles use the same environment. This means we have

$$\theta_{i,\ell} = \mathbb{E}_{\texttt{BGW}}\left[\mathbf{P}_\omega(Z_\ell = 0)^i\right] \quad \text{and} \quad \theta_{i,\ell}' = \mathbb{E}_{\texttt{BGW}}\left[\mathbf{P}_{\vec{\omega}}'(Z_\ell' = 0)^i\right].$$

By construction, Z_1 and Z_1' have the same distribution and therefore for all $i \geq 1$ we have $\theta_{i,1} = \theta_{i,1}'$. In particular,

$$\theta_{i,1}' \geq \theta_{i,1} \qquad \text{for all } i \geq 1.$$

We now show by induction that for all $\ell \geq 1$ we have

$$\theta'_{i,\ell} \geq \theta_{i,\ell} \qquad \text{for all } i \geq 1.$$

For any $i, j \geq 1$, Jensen's inequality leads to[2]

$$\theta_{i+j,\ell} = \mathbb{E}_{\mathtt{BGW}}\left[\mathbf{P}_\omega(Z_\ell = 0)^{i+j}\right] \geq \mathbb{E}_{\mathtt{BGW}}\left[\mathbf{P}_\omega(Z_\ell = 0)^{i}\right]\mathbb{E}_{\mathtt{BGW}}\left[\mathbf{P}_\omega(Z_\ell = 0)^{j}\right] = \theta_{i,\ell}\theta_{j,\ell}.$$

Inductively, we also have for $a_1, \dots, a_k \in \mathbb{N}^k$ with $\sum_{i=1}^k a_i = L$ that

$$\prod_{i=1}^k \theta_{a_i,j} \leq \theta_{L,j}. \tag{3.18}$$

We want to apply the branching property. To this end, for $x \in D_n$ let

$$Z_{1,x} = \#\{u \in \mathcal{Z}_1 : X(u) = x\},$$

the number of particles in $\mathcal{Z}_1$ that are located at x. Note that $\sum_{x \in D_n} Z_{1,x} = Z_1$. With this (and using the convention that the empty product is 1) we have for any $\ell \geq 1$,

$$\begin{aligned}\mathbf{P}_\omega\left(Z_{\ell+1} = 0\right) &= \mathbf{P}_\omega\left(Z_1 = 0 \text{ or } \forall u \in \mathcal{Z}_1 : \text{ the corresponding } Z_\ell^{(u)} = 0\right)\\ &= \mathbf{E}_\omega\left[\prod_{x \in D_n : Z_{1,x} > 0} \mathbf{P}_{F_\omega(x)}\left(Z_\ell = 0\right)^{Z_{1,x}}\right].\end{aligned}$$

Then by taking the annealed expectation and by applying (3.18) to $(Z_{1,x}, x \in D_n)$

$$\begin{aligned}\theta_{i,\ell+1} &= \mathbb{E}_{\mathtt{BGW}}\left[\mathbf{E}_\omega\left[\prod_{x \in D_n : Z_{1,x} > 0} \theta_{Z_{1,x},\ell}\right]^i\right]\\ &\leq \mathbb{E}_{\mathtt{BGW}}\left[\mathbf{E}_\omega\left[\mathbb{1}_{\{Z_1 = 0\}} + \theta_{Z_1,\ell}\mathbb{1}_{\{Z_1 > 0\}}\right]^i\right] = \mathbb{E}\left[(\theta_{Z_1,\ell})^i\right].\end{aligned}$$

Recall that Z_1 and Z'_1 have the same annealed distribution, let $\tilde{Z}'_1$ be a copy of Z'_1 which is independent of $(Z'_i, i \geq 1)$. Combining this with the above and the induction hypothesis yields

$$\theta_{i,\ell+1} \leq \mathbb{E}\left[(\theta_{Z_1,\ell})^i\right] \leq \mathbb{E}\left[(\theta'_{Z_1,\ell})^i\right] = \mathbb{E}'\left[(\theta'_{\tilde{Z}'_1,\ell})^i\right].$$

Lastly, we observe that the right hand side is equal to $\theta'_{i,\ell+1}$ by the Markov property for $(Z'_i, i \geq 1)$. Hence we have shown by induction

$$\theta'_{\ell,i} \geq \theta_{\ell,i} \qquad \text{for all } \ell, i \geq 1.$$

We conclude the proof using monotone convergence,

$$\mathbb{P}(Z_i > 0 \text{ for all } i \geq 1) = 1 - \lim_{\ell \to \infty} \theta_{1,\ell} \geq 1 - \lim_{\ell \to \infty} \theta'_{1,\ell} = \mathbb{P}'(Z'_i > 0 \text{ for all } i \geq 1) \geq c > 0.$$

$\square$

---

[2] The consequence of Jensen here is $\mathbb{E}[X^i]\mathbb{E}[X^j] \leq \mathbb{E}[X^{i+j}]$.

It remains to prove (3.17), see Lemma 3.2.11 below, but before doing so, let us show how one proves Proposition 3.1.4 assuming Lemma 3.2.9.

Proof of Proposition 3.1.4. We first translate Lemma 3.2.9 about hitting times into a statement in terms of the maximal displacement. We choose $\tilde{a} \in (a, v_{\lambda,m})$ to be determined later, and consider $(\mathcal{Z}_i^{(n)}, i \geq 0)$ as in Lemma 3.2.9 associated with $\tilde{a}$. Consider $u \in \mathcal{Z}_{\lfloor \tilde{a} i \rfloor}^{(n)}$, then we have

$$\frac{\lfloor \tilde{a} i \rfloor n}{\tilde{a} + \varepsilon} < |u| \leq \frac{\lfloor \tilde{a} i \rfloor n}{\tilde{a}} \leq in.$$

and $|X(u)| = \lfloor \tilde{a} i \rfloor n$. By considering the descendants of u at generation in, we observe that

$$\max_{|v|=in} |X(v)| > \lfloor \tilde{a} i \rfloor n - \left(i - \frac{\lfloor \tilde{a} i \rfloor}{\tilde{a} + \varepsilon}\right) n \geq \left(\tilde{a} - \frac{\varepsilon}{\tilde{a} + \varepsilon}\right) in - \left(1 + \frac{1}{\tilde{a} + \varepsilon}\right) n.$$

We fix $\tilde{a} \in (a, v_{\lambda,m})$ such that $\tilde{a} - \frac{\varepsilon}{\tilde{a}+\varepsilon} > a$, then for all $i \in \mathbb{N}$ large enough we have $\max_{|v|=in} |X(v)| > ain$. It follows from Lemma 3.2.9 that

$$\mathbb{P}\left(\liminf_{i\to\infty} \max_{|u|=in} \frac{|X(u)|}{in} \geq a\right) > 0.$$

This proves the statement for a subsequence of times. To complete the proof, we notice that, for $j \in [in, (i+1)n)$ with $i \geq 0$,

$$\max_{|v|=j} \frac{|X(v)|}{j} \geq \max_{|u|=(i+1)n} \frac{|X(u)| - n}{(i+1)n}.$$

Letting $i \to \infty$, we conclude that

$$\mathbb{P}\left(\liminf_{j\to\infty} \max_{|v|=j} \frac{|X(v)|}{j} \geq a\right) \geq \mathbb{P}^*\left(\liminf_{i\to\infty} \max_{|u|=in} \frac{|X(u)|}{in} \geq a\right) > 0.$$

$\square$

Remark 3.2.10. Proposition 3.1.4 still holds if $\mathbb{P}$ is replaced by $\mathbb{P}^*$.

This remark is true because the killing at the root is *built-in* into $\mathcal{Z}_i^{(n)}$ as we only consider trajectories that do not revisit level $D_{(i-1)n}$ before reaching level D_{in} - in particular they do not revisit the root before reaching level D_1.

It now remains to prove Lemma 3.2.11, see below, that we have used in the proof of Lemma 3.2.9 for Equation (3.17). Recall the definition of $\mathcal{L}_n^{(c),*}$ from (3.16). We will show that $\mathcal{L}_n^{(c),*}$ is supercritical in an appropriate sense under the annealed measure. At this point we also mention a very useful tool that connects the BRW and the random walk of a single particle, the well-known *many–to–one formula*. For fixed ω and any $f \geq 0$ that is a function of a particle trajectory we have

$$\mathbf{E}_\omega\left[\sum_{|u|=n} f\left((X(u_k), k \leq n)\right)\right] = m^n \mathrm{E}_\omega\left[f\left((S_k, k \leq n)\right)\right], \qquad n \in \mathbb{N}. \tag{3.19}$$

In our case, (3.19) follows from the linearity of expectation. See also for example [81, Theorem 1.1] for a proof for a more general model of branching random walks on $\mathbb{R}$. Analogous formulae hold for the modified λ–biased BRW as well as for the annealed laws.

Lemma 3.2.11. *Let $v_\lambda < a < v_{\lambda,m}$. Then there exists $\varepsilon > 0$ such that*

$$\liminf_{n\to\infty} \mathbb{E}_{\mathtt{BGW}}\left[\frac{a}{n}\log \mathbf{E}_\omega\Big[\#\{u \in \mathcal{L}_n^{(1/a),*} \colon |u| > (a+\varepsilon)^{-1}n\}\Big]\right] > 0. \tag{3.20}$$

In particular, there is $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ we have

$$\mathbb{E}_{\mathtt{BGW}}\left[\log \mathbf{E}_\omega\Big[\#\{u \in \mathcal{L}_n^{(1/a),*} \colon |u| > (a+\varepsilon)^{-1}n\}\Big]\right] > 0. \tag{3.21}$$

Proof. Fix any $\varepsilon > 0$. For $v \in \mathbb{T}$ with $|v| = n$ we write $\tau_{\mathbf{o}}^v > n$ if $\forall k \leq n$ we have $X(v_k) \neq \mathbf{o}$. Similarly, for the λ–biased random walk $(S_j, j \geq 0)$ we set $T_n = \inf\{j \geq 0 \colon |S_j| = n\}$ and $\tau_{\mathbf{o}} = \inf\{j > 0 \colon S_j = \mathbf{o}\}$. By the many-to-one formula (3.19),

$$\begin{aligned}\mathbf{E}_\omega&\Big[\#\{v \colon |v| = \lceil a^{-1}n\rceil, (a+\varepsilon)^{-1}n < T_n^v \leq a^{-1}n, \tau_{\mathbf{o}}^v > T_n^v\}\Big]\\ &= m^{\lceil a^{-1}n\rceil}\mathrm{P}_\omega\Big((a+\varepsilon)^{-1}n < T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big).\end{aligned}$$

On the other hand, by the branching property at the stopping line $\mathcal{L}_n = \{u_{T_n^u}, u \in \mathbb{T}\}$, we have

$$\begin{aligned}\mathbf{E}_\omega&\Big[\#\{v \colon |v| = \lceil a^{-1}n\rceil, (a+\varepsilon)^{-1}n < T_n^v \leq a^{-1}n, \tau_{\mathbf{o}}^v > T_n^v\}\Big]\\ &\leq \mathbf{E}_\omega\left[\sum_{u\in\mathcal{L}_n} m^{\lceil a^{-1}n\rceil - |u|}\mathbb{1}_{\left\{u\in\mathcal{L}_n^{(1/a),*},\, |u|>(a+\varepsilon)^{-1}n\right\}}\right].\end{aligned}$$

The right hand side can be bounded from above by

$$\leq m^{\lceil a^{-1}n\rceil - (a+\varepsilon)^{-1}n}\mathbf{E}_\omega\Big[\#\{u \in \mathcal{L}_n^{(1/a),*} \colon |u| > (a+\varepsilon)^{-1}n\}\Big].$$

Combining these considerations, taking the logarithm and rearranging leads to

$$\begin{aligned}\log \mathbf{E}_\omega&\Big[\#\{u \in \mathcal{L}_n^{(1/a),*} \colon |u| > (a+\varepsilon)^{-1}n\}\Big]\\ &\geq (a+\varepsilon)^{-1}n\log m + \log \mathrm{P}_\omega\Big((a+\varepsilon)^{-1}n < T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big).\end{aligned}$$

We integrate both sides in ω with respect to $\mathtt{BGW}$. This is the same as taking the annealed law for the branching random walk on the left-hand side and the annealed law for the non–branching random walk on the right-hand side,

$$\begin{aligned}\mathbb{E}_{\mathtt{BGW}}\Big[\log \mathbf{E}_\omega\Big[\#\{u\in\mathcal{L}_n^{(1/a),*}\colon |u|>(a+\varepsilon)^{-1}n\}\Big]\Big]\\ \geq \mathbb{E}_{\mathtt{BGW}}\Big[(a+\varepsilon)^{-1}n\log m+\log \mathrm{P}_\omega\Big((a+\varepsilon)^{-1}n<T_n\leq a^{-1}n,\tau_{\mathbf{o}}>T_n\Big)\Big].\end{aligned}\tag{3.22}$$

This means we only have to estimate the latter probability for the λ–biased random walk. By Jensen's inequality and an annealed estimate from [40, (4.7)] we have that

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{E}_{\mathtt{BGW}}\Big[\frac{1}{n}\log \mathrm{P}_\omega\Big(T_n\leq(a+\varepsilon)^{-1}n,\tau_{\mathbf{o}}>T_n\Big)\Big]\\ \leq \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(T_n\leq(a+\varepsilon)^{-1}n,\tau_{\mathbf{o}}>T_n\Big)=-\frac{I_\lambda(a+\varepsilon)}{a+\varepsilon}.\end{aligned}$$

From [40, Page 255] we know that $\frac{I_\lambda(a)}{a}$ is strictly increasing for $a>v_\lambda$. It follows that

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{E}_{\mathtt{BGW}}\Big[\frac{1}{n}\log \mathrm{P}_\omega\Big((a+\varepsilon)^{-1}n<T_n\leq a^{-1}n,\ \tau_{\mathbf{o}}>T_n\Big)\Big]\\ =\liminf_{n\to\infty}\mathbb{E}_{\mathtt{BGW}}\Big[\frac{1}{n}\log \mathrm{P}_\omega\Big(T_n\leq a^{-1}n,\tau_{\mathbf{o}}>T_n\Big)\Big].\end{aligned}$$

Going back to (3.22) we obtain

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{E}_{\mathtt{BGW}}\Big[\frac{a}{n}\log \mathbf{E}_\omega\Big[\#\{u\in\mathcal{L}_n^{(1/a),*}\colon |u|>(a+\varepsilon)^{-1}n\}\Big]\Big]\\ \geq \liminf_{n\to\infty}\mathbb{E}_{\mathtt{BGW}}\Big[\log m+\frac{a}{n}\log \mathrm{P}_\omega\Big(T_n\leq a^{-1}n,\tau_{\mathbf{o}}>T_n\Big)\Big]-\frac{\varepsilon}{a+\varepsilon}\log m.\end{aligned}\tag{3.23}$$

Due to Lemma 3.2.12 below, the right-hand side is strictly positive for ε small enough. Note that this lemma is formulated for P_ω^* instead of P_ω, i.e. here particles are killed when returning to the root. Because we consider the event $\{\tau_{\mathbf{o}}>T_n\}$ the killing will not take place and both measures agree. This completes the proof of Lemma 3.2.11. □

As we have discussed above, it remains to complete the large deviation estimate, used in (3.23), for the hitting times of the modified walk on the event that the walk does not return to the root before reaching level n. The proof requires diving deeply into [40] and the following lemma can be treated as a black box.

Lemma 3.2.12. *Let* $(S_j, j \geq 0)$ *be the* λ*–biased random walk starting from* $\mathbf{o}$*. Set* $T_n = \inf\{j \geq 0\colon |S_j| = n\}$ *and* $\tau_{\mathbf{o}} = \inf\{j > 0\colon S_j = \mathbf{o}\}$*. Let* $a \in (v_\lambda, 1]$*. Then we have*

$$\liminf_{n\to\infty} \mathbb{E}_{\mathtt{BGW}}\Big[\frac{a}{n}\log \mathrm{P}^*_\omega\Big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big)\Big] \geq -I_\lambda(a).$$

In particular, if $v_\lambda < a < v_{\lambda,m}$*, with* $v_{\lambda,m}$ *given by* (3.4)*, then* $I_\lambda(a) < \log m$ *and*

$$\liminf_{n\to\infty} \mathbb{E}_{\mathtt{BGW}}\Big[\log m + \frac{a}{n}\log \mathrm{P}^*_\omega\Big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big)\Big] > 0. \tag{3.24}$$

Proof. For any $\varepsilon > 0$, $\delta \in (0,1)$ and any large $\Delta \in \mathbb{N}$, it follows from [40, Page 253] that there exists $n_0 > 0$, such that for every $n \geq n_0$, which can be written as $n = n_1 + N\Delta$ with $n_1 \in [n_0, n_0 + \Delta)$, the following inequality holds:

$$\begin{aligned}\mathtt{BGW}\Big(\log \mathrm{P}^*_\omega\Big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big) \geq \log \mathrm{P}^*_\omega\Big(T_{n_1} \leq a^{-1}n_1, \tau_{\mathbf{o}} > T_{n_1}\Big) + N\log(\delta e_\Delta)\Big)&\\ \geq 1 - \varepsilon,&\end{aligned} \tag{3.25}$$

where $e_\Delta := \mathbb{P}^*(T_\Delta \leq a^{-1}\Delta, \tau_{\mathbf{o}} > T_\Delta)$ is the annealed probability. Note that $e_\Delta \geq \frac{1}{1+\lambda}\mathbb{P}(T_\Delta \leq a^{-1}\Delta, \tau_{\mathbf{o}} > T_\Delta)$.

The inequality (3.25) does not appear in this form in [40], so let us justify it here. Note that, in [40] a slightly different version of the λ–biased random walk is used, which has a positive probability to stay at the root. Denote its quenched law by $\mathrm{P}_{\omega,\mathbf{o}}$ which is given by

$$\begin{aligned}\mathrm{P}_{\omega,\mathbf{o}}(S_{n+1} = i | S_n = \mathbf{o}) &= \frac{1}{\lambda + \kappa_{\mathbf{o}}}, \quad i = 1, \ldots, \kappa_{\mathbf{o}};\\ \mathrm{P}_{\omega,\mathbf{o}}(S_{n+1} = \mathbf{o} | S_n = \mathbf{o}) &= \frac{\lambda}{\lambda + \kappa_{\mathbf{o}}};\\ \mathrm{P}_{\omega,\mathbf{o}}(S_{n+1} = \mathbf{o} | S_n = y) &= \mathrm{P}_\omega(S_{n+1} = \mathbf{o} | S_n = y) \text{ for } y \neq \mathbf{o}\,.\end{aligned}$$

A random walk of law P^*_ω differs from $\mathrm{P}_{\omega,\mathbf{o}}$ only in its behaviour at the root: under P^*_ω the random walker located at $\mathbf{o}$ is killed with probability $\frac{\lambda}{\lambda+\kappa_{\mathbf{o}}}$ in the next step, whereas under $\mathrm{P}_{\omega,\mathbf{o}}$ it remains at $\mathbf{o}$ with probability $\frac{\lambda}{\lambda+\kappa_{\mathbf{o}}}$. In particular, when we restrict to not revisiting the root, their laws are the same and therefore we have

$$\mathrm{P}_{\omega,\mathbf{o}}\Big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big) = \mathrm{P}^*_\omega\Big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big).$$

This enables us to translate the arguments in [40] into estimates about P^*_ω. Specifically, using the notation of [40], we have by [40, Equation (4.2) and Page 253] that

$$\mathrm{P}^*_\omega\Big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\Big) \geq \mathrm{P}^*_\omega\Big(T_{n_1} \leq a^{-1}n_1, \tau_{\mathbf{o}} > T_{n_1}\Big)e_\Delta^N \prod_{i=0}^{N-1} \mathsf{Z}_{n_1+i\Delta}(\omega),$$

and, for every $k \geq 1$,

$$\sum_{j=1}^{\infty} \mathtt{BGW}(\mathtt{Z}_j \leq \delta, B_k) < \infty,$$

where $(\mathtt{Z}_j, j \geq 1)$ and $(B_k, k \geq 1)$ are certain measurable functions and sets, respectively, defined on the space of trees. On the other hand, the uncertainty estimate [40, Proposition 4.1] yields that there exists k_0 large enough such that $\mathtt{BGW}(B_{k_0}) > 1-\varepsilon/2$. Taking n_0 large enough such that

$$\sum_{j \geq n_0} \mathtt{BGW}(\mathtt{Z}_j \leq \delta, B_{k_0}) < \varepsilon/2,$$

then we have for every $n > n_0$, written as $n = n_1 + N\Delta$ with $n_1 \in [n_0, n_0 + \Delta)$,

$$\mathtt{BGW}\Big(\log \mathrm{P}^*_\omega\big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\big) \geq \log \mathrm{P}^*_\omega\big(T_{n_1} \leq a^{-1}n_1, \tau_{\mathbf{o}} > T_{n_1}\big) + N\log(\delta e_\Delta)\Big)$$
$$\geq \mathtt{BGW}\Big(\bigcap_{j \geq n_0} \{\mathtt{Z}_j > \delta\}\Big) \geq 1 - \mathtt{BGW}(B_{k_0}^c) - \sum_{j \geq n_0} \mathtt{BGW}(\mathtt{Z}_j \leq \delta, B_{k_0}) \geq 1 - \varepsilon.$$

This completes the proof of (3.25).

Combining (3.25) and the trivial bound

$$\mathrm{P}^*_\omega\big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\big) \geq \mathrm{P}^*_\omega\big(T_n = n\big) \geq (1+\lambda)^{-n},$$

we deduce that

$$\begin{aligned} &\mathbb{E}_{\mathtt{BGW}}\Big[\frac{a}{n}\log \mathrm{P}^*_\omega\big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\big)\Big] \\ &\qquad \geq \Big(- \frac{an_1}{N\Delta + n_1}\log(1+\lambda) - \frac{aN}{N\Delta + n_1}\log(\delta e_\Delta)\Big) - a\log(1+\lambda)\varepsilon. \end{aligned}$$

Letting $N \to \infty$, since δ and ε were arbitrary, we derive that

$$\liminf_{n\to\infty} \mathbb{E}_{\mathtt{BGW}}\Big[\frac{a}{n}\log \mathrm{P}^*_\omega\big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\big)\Big] \geq a\Delta^{-1}\log e_\Delta.$$

As $\lim_{\Delta\to\infty} \Delta^{-1}\log e_\Delta = -a^{-1}I_\lambda(a)$ (see [40, Equations (4.7) and (4.10)]), we have that

$$\liminf_{n\to\infty} \mathbb{E}_{\mathtt{BGW}}\Big[\frac{a}{n}\log \mathrm{P}^*_\omega\big(T_n \leq a^{-1}n, \tau_{\mathbf{o}} > T_n\big)\Big] \geq -I_\lambda(a),$$

proving the claim. □

3.2.5 Proof of Theorem 3.1.1

With Propositions 3.1.4 and 3.1.5 in hand, we turn to the proof of Theorem 3.1.1. That is we show under our assumptions that for `BGW`–almost every ω we have

$$\lim_{n\to\infty}\frac{1}{n}\max_{|u|=n}|X_u| = v_{\lambda,m}, \qquad \mathbf{P}_\omega - a.s.$$

Proof of Theorem 3.1.1. We show upper and lower bounds separately. The upper bound follows almost immediately from Theorem 3.2.2. We first assume $v_{\lambda,m} < 1$ and let $a > v_{\lambda,m}$. By the many–to–one formula (3.19) we have

$$\mathbf{P}_\omega\left(\max_{|u|=n}\frac{|X(u)|}{n} \geq a\right) \leq \mathbf{E}_\omega\left[\sum_{|u|=n}\mathbb{1}_{\{|X(u)|\geq an\}}\right] = m^n \mathrm{P}_\omega\left(|S_n| \geq an\right).$$

Since $a > v_{\lambda,m}$, we have $I_\lambda(a) > \log m$ and it follows from Theorem 3.2.2 that, for `BGW`-a.e. ω,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{P}_\omega\left(\max_{|u|=n}\frac{|X(u)|}{n} \geq a\right) < 0.$$

The Borel-Cantelli lemma implies that

$$\limsup_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n} \leq a, \qquad \mathbf{P}_\omega - a.s.,$$

for `BGW`–almost every ω. Because $a > v_{\lambda,m}$ was arbitrary, we get that

$$\limsup_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n} \leq v_{\lambda,m}, \qquad \mathbf{P}_\omega - a.s., \tag{3.26}$$

for `BGW`–almost every ω. When $v_{\lambda,m} = 1$, this upper bound also holds automatically.

Most of the work for the lower bound has been done in Propositions 3.1.4 and 3.1.5. We consider $\mathbf{P}^*_\omega$ again as introduced in Section 3.2.3 For $a > 0$ set

$$G_a = \left\{\omega\colon \mathbf{P}^*_\omega\left(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n} \geq a\right) = 0\right\}.$$

Let $1, 2, \ldots, \kappa_{\mathbf{o}}$ be the children of the root in ω, and $\omega^i = F_\omega(i)$ be the subtree rooted at $i = 1, 2, \ldots, \kappa_{\mathbf{o}}$. Then we claim that

$$\{\omega \in G_a\} \subseteq \bigcap_{i=1}^{\kappa_{\mathbf{o}}}\{\omega^i \in G_a\}. \tag{3.27}$$

To see that, let us consider a BRW under law $\mathbf{P}^*_\omega$, starting from one particle at the root. With positive probability, we have one particle in the first generation, say u, located at the vertex i. If $\omega^i \notin G_a$, i.e. $\mathbf{P}^*_{\omega^i}(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n} \geq a) > 0$, then $\mathbf{P}^*_\omega(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n}\geq a)>0$, i.e. $\omega\notin G_a$. This shows (3.27) and therefore G_a is an inherited property in the sense of Lemma 3.2.5. This yields that $\mathtt{BGW}(G_a)\in\{0,1\}$.

Let $\omega\in G_a^c$, i.e. $\mathbf{P}^*_\omega(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n}\geq a)>0$. Recall that by a coupling argument $0<\mathbf{P}^*_\omega(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n}\geq a)\leq\mathbf{P}_\omega(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n}\geq a)$.

We now conclude the proof by using Propositions 3.1.4 and 3.1.5. Let $a<v_{\lambda,m}$. By Proposition 3.1.4 we have

$$0<\mathbb{P}\left(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n}\geq a\right)\leq\mathtt{BGW}(G_a^c),$$

which implies $\mathtt{BGW}(G_a^c)=1$. This means that

$$\mathbf{P}_\omega\left(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n}\geq a\right)>0,\quad\text{for}\quad\mathtt{BGW}-a.e.\ \omega.$$

This allows us to apply Proposition 3.1.5 which implies

$$\mathbf{P}_\omega\left(\liminf_{n\to\infty}\max_{|u|=n}\frac{|X(u)|}{n}\geq a\right)=1,\quad\text{for}\quad\mathtt{BGW}-a.e.\ \omega.$$

Together with (3.26), this shows Theorem 3.1.1. ☐

3.2.6 Proof of Theorem 3.1.3

In this section we consider the minimal distance to the root in the transient regime and prove Theorem 3.1.3. Our proof does not provide full details, as the approach closely follows that of Theorem 3.1.1 for the maximal displacement.

Switching from maximum to minimum, the trivial direction – corresponding to (3.26) – now serves as the lower bound:

$$\mathbf{P}_\omega\left(\liminf_{n\to\infty}\min_{|u|=n}\frac{|X(u)|}{n}\geq\tilde{v}_{\lambda,m}\right)=1,\qquad\text{for }\mathtt{BGW}-a.e.\ \omega. \tag{3.28}$$

Indeed, letting $\varepsilon>0$, by the many–to–one formula (3.19) we have

$$\begin{aligned}\mathbf{P}_\omega\left(\min_{|u|=n}\frac{|X(u)|}{n}\leq\tilde{v}_{\lambda,m}-\varepsilon\right)&\leq\mathbf{E}_\omega\left[\sum_{|u|=n}\mathbb{1}_{\{|X(u)|\geq(\tilde{v}_{\lambda,m}-\varepsilon)n\}}\right]\\&=m^n\mathrm{P}_\omega\left(|S_n|\leq\tilde{v}_{\lambda,m}-\varepsilon\right).\end{aligned}$$

By our choice of $\tilde{v}_{\lambda,m}$, we have $I_\lambda(\tilde{v}_{\lambda,m}-\varepsilon)>\log m$ and it follows from Theorem 3.2.2 that, for $\mathtt{BGW}$-a.e. ω,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{P}_\omega\left(\min_{|u|=n}\frac{|X(u)|}{n}\leq\tilde{v}_{\lambda,m}-\varepsilon\right)<0.$$

The Borel-Cantelli lemma implies that

$$\liminf_{n\to\infty}\min_{|u|=n}\frac{|X(u)|}{n}\geq \tilde{v}_{\lambda,m}-\varepsilon,\qquad \mathbf{P}_\omega - a.s.,$$

As ε was arbitrary, this yields (3.28).

The upper bound for the minimum, that is, to show the existence of slow particles with small linear velocity close to $\tilde{v}_{\lambda,m}$, is similar to the lower bound for the maximum. For this we need two statements analogous to Propositions 3.1.4 and 3.1.5. For Proposition 3.1.5, only the transient case is relevant and the arguments requires very little modification when considering the minimal displacement instead of the maximal one. The statement corresponding to Proposition 3.1.4 is as follows.

Proposition 3.2.13. *Under the assumption of Theorem 3.1.3 we have for any $\tilde{v}_{\lambda,m} < a < v_\lambda$,*

$$\mathbb{P}\left(\liminf_{n\to\infty}\min_{|u|=n}\frac{|X(u)|}{n}\leq a\right)>0.$$

We comment on the proof in Section 3.2.7. Combining Proposition 3.2.13 and the zero–one law completes the proof of Theorem 3.1.3.

3.2.7 Annealed probability of slow particles, proof of Proposition 3.2.13

We now prove Proposition 3.2.13. Similarly to Lemma 3.2.12, the first ingredient we need is a large deviation estimate for the λ–biased random walk that we show along the lines of [40].

Lemma 3.2.14. *Let $(S_j, j\geq 0)$ be a random walk starting from $\mathbf{o}$ with law $\mathrm{P}_{\omega,\mathbf{o}}$. Set $T_n=\inf\{j\geq 0\colon |S_j|=n\}$ and $\tau_{\mathbf{o}}=\inf\{j\geq 0\colon S_j=\mathbf{o}\}$. Let $a\in[0,v_\lambda)$. Then*

$$\liminf_{n\to\infty}\mathbb{E}_{\texttt{BGW}}\left[\frac{a}{n}\log \mathrm{P}^*_\omega\Big(\tau_{\mathbf{o}}>T_n\geq a^{-1}n\Big)\right]\geq -I_\lambda(a).$$

In particular, for $a\in(\tilde{v}_{\lambda,m},v_\lambda)$ with $\tilde{v}_{\lambda,m}$ as in (3.7), such that $I_\lambda(a)<\log m$,

$$\liminf_{n\to\infty}\mathbb{E}_{\texttt{BGW}}\left[\log m+\frac{a}{n}\log \mathrm{P}^*_\omega\Big(\tau_{\mathbf{o}}>T_n\geq a^{-1}n\Big)\right]>0. \tag{3.29}$$

Proof. For any $\varepsilon>0$, $\delta\in(0,1)$ and $\Delta\in\mathbb{N}$, it follows from [40, Page 267] that there exists $n_0>0$, such that for every $n\geq n_0$, which can be written as $n=n_1+N\Delta$ with $n_1\in[n_0,n_0+\Delta)$, the following inequality holds:

$$\texttt{BGW}\bigg(\log \mathrm{P}^*_\omega\Big(\tau_{\mathbf{o}}>T_n\geq a^{-1}n\Big)\geq \log \mathrm{P}^*_\omega\Big(\tau_{\mathbf{o}}>T_{n_1}\geq a^{-1}n_1\Big)+N\log(\delta e_\Delta)\bigg)\geq 1-\varepsilon, \tag{3.30}$$

where $e_\Delta := \mathrm{P}^*(\tau_{\mathbf{o}} > T_\Delta \geq a^{-1}\Delta)$ is the annealed probability. Indeed, this is again a consequence of the uncertainty estimate [40, Proposition 5.1], which holds under the assumption that $d_{\min} \geq 2$ and $\lambda < d_{\min}$.

Moreover, for any $k \in \mathbb{N}$, by considering the event that a random walk moves forward for the first two steps and then oscillates between the two levels D_2 and D_1 for $\lceil a^{-1}k \rceil - 2$ steps, we have a lower bound

$$\mathrm{P}^*_\omega\Big(\tau_{\mathbf{o}} > T_k \geq a^{-1}k\Big) \geq \left(\frac{1}{\lambda+1}\right)^{\lceil a^{-1}k \rceil} \left(\frac{1}{\lambda+K_\omega}\right)^{\lceil a^{-1}k \rceil},$$

where K_ω is the maximal outer degree of vertices in D_1 of the tree ω. Combining this bound and (3.30), we deduce that (cf. proof of Lemma 3.2.12)

$$\liminf_{n\to\infty} \mathbb{E}_{\mathtt{BGW}}\left[\frac{a}{n}\log \mathrm{P}^*_\omega\Big(\tau_{\mathbf{o}} > T_n \geq a^{-1}n\Big)\right] \geq a\Delta^{-1}\log e_\Delta.$$

As $\lim_{\Delta\to\infty}\Delta^{-1}\log e_\Delta = -a^{-1}I_\lambda(a)$ (see [40, Equation (5.3)]), we have that

$$\liminf_{n\to\infty} \mathbb{E}_{\mathtt{BGW}}\left[\frac{a}{n}\log \mathrm{P}^*_\omega\Big(\tau_{\mathbf{o}} > T_n \geq a^{-1}n\Big)\right] \geq -I_\lambda(a).$$

□

We turn to the BRW under $\mathbf{P}^*_\omega$. As in the maximum case, we consider an embedded discrete-time branching process. More precisely, instead of $\mathcal{L}_n^{(c)}$ and its variations, see (3.15), we now consider particles that take very long to reach level n without return to the root:

$$\widehat{\mathcal{L}}_n^{(1/a),*} = \Big\{u \in \mathcal{L}_n : |u| \geq a^{-1}n, X(u_k) \neq \mathbf{o}, \forall k = 1, \dots, |u|\Big\}.$$

We regard this set as the first generation of the branching process. For the second generation, we select from the descendants of the first generation those particles that reach level D_{2n} slow enough without returning to D_n. We continue inductively.

More precisely, the branching process is defined as follows. For any particle $u \in \mathbb{T}$ and $i, n \in \mathbb{N}$ let $T^u_{in} := \inf\{k \geq 1 : |X(u_k)| = in\}$ be the hitting time of level D_{in} for the ancestral lineage of u. Set $\widehat{\mathcal{Z}}_0^{(n)} = \{\varnothing\}$. Fix $\varepsilon > 0$ and define recursively the set of particles, for $i \geq 1$,

$$\begin{aligned}\widehat{\mathcal{Z}}_{i+1}^{(n)} = \Big\{&u \in \mathcal{L}_{(i+1)n} : u_{T^u_{in}} \in \widehat{\mathcal{Z}}_i^{(n)},\\ &(a+\varepsilon)^{-1}n > |u| - T^u_{in} \geq a^{-1}n, |X(u_k)| > in,\, T^u_{in} < k \leq |u|\Big\}.\end{aligned}$$

For the first generation, we have $\widehat{\mathcal{Z}}_1^{(n)} = \{u \in \widehat{\mathcal{L}}_n^{(1/a),*} \colon |u| < (a+\varepsilon)^{-1}n\}$. Using Lemma 3.2.14 and the many-to-one formula, we deduce the following analogue of Lemma 3.2.11 by similar arguments.

Lemma 3.2.15. *Let $a < v_{\lambda,m}$. Then there exists $\varepsilon > 0$ and $n' \in \mathbb{N}$ such that for all $n \geq n'$*

$$\mathbb{E}_{\texttt{BGW}}\left[\log \mathbf{E}^*_\omega\left[\#\{u \in \widehat{\mathcal{L}}^{(1/a),*}_n \colon |u| < (a+\varepsilon)^{-1}n\}\right]\right] > 0.$$

This lemma essentially states that $(\#\widehat{\mathcal{Z}}^{(n)}_i, i \geq 1)$ is supercritical for each n large enough under the annealed measure. Analogously to Lemma 3.2.9 we can now prove that the branching process $(\widehat{\mathcal{Z}}^{(n)}_i, i \geq 0)$ survives with positive probability. That is, for every $n \geq n'$ we have

$$\mathbb{P}^*(\#\widehat{\mathcal{Z}}^{(n)}_i > 0 \text{ for all } i \geq 1) > 0.$$

The remaining part of the proof is essentially the same as for the maximal displacement part, and we omit the details.

3.3 Open questions

We conclude with some open questions regarding our model and similar models. We look forward to addressing some of these in future work.

1. Our result describes the leading linear term of the growth of the maximal displacement. What can be said about the second term, i.e. the fluctuations of $\max_{|u|=n}|X(u)| - n \cdot v_{\lambda,m}$? This is a active direction of research for branching Brownian motion and branching random walks in the multi-dimensional case. We refer to [8] for the “classical" result for one-dimensional branching random walks.

2. To what extent can this result be generalised to other branching Markov chains? More precisely, given a Markov chain with large deviation rate function I, what assumptions have to be made so that the maximal displacement of the corresponding branching Markov chain (with mean offspring m) has a linear speed with velocity given by $\sup\{a : I(a) \leq \log m\}$?

3. For a branching Markov chain which is statistically transitive, i.e. the underlying Markov chain is a random walk in a homogeneous random environment, is it true in general that there is no weak recurrence phase?

4. In this work we have mostly treated the large deviation principle and the rate function I_λ as a black box. What can be said about the environment in the neighbourhood of the particles that achieve the maximum? What can be said

about their ancestral path to the root? This can be compared to [9], where it is proved that the environment seen from the λ–biased random walk is absolutely continuous with respect to `BGW`, but not identical to `BGW` itself.

5. In our model, in the critical case when $m = \frac{d_{\min}+\lambda}{2\sqrt{\lambda d_{\min}}}$, the minimal distance to the root exhibits zero velocity, despite the biased branching random walk being transient. What is the correct (sublinear) rate of escape in this scenario?

Chapter 4

Hyperbolic branching Brownian motion: the empirical limit measure

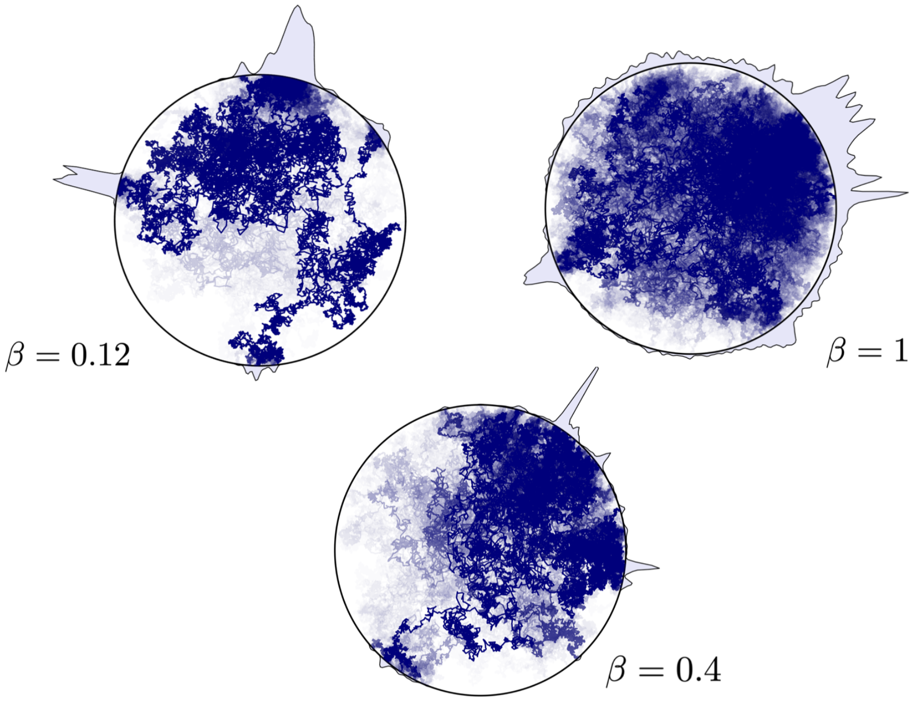

Figure 4.1: Three simulations of hyperbolic BBM and the limit of its empirical distribution μ_∞ on the boundary. The branching rates are $\beta \in \{0.12, 0.4, 1\}$ from left to right. The Hausdorff dimensions of the support of μ_∞ are $\{0.24, 0.8, 1\}$ whereas the Hausdorff dimensions of the set of accumulation points on the boundary are $\{0.4, 1, 1\}$. Observe that in the middle picture where $\beta = 0.4$ there are a lot of paths accumulating on the boundary that do not contribute significantly to μ_∞.

4.1 Introduction

Given a transient stochastic process, one can often define a natural extension of the state space so that the process converges almost surely to a point on the boundary of this space. We can think of the distribution of this limit as an exit distribution of the process. These exit distributions form a well studied topic in their own right, but it becomes even richer when combined with branching processes. The idea here is that we have multiple, possibly infinitely many, correlated particles that all escape towards the boundary. This induces different random subsets of the boundary, notably the set Λ of accumulation points on the boundary, and the set $\mathcal{S} := \operatorname{supp} \mu_\infty$ which is the support of the limit of the empirical measure μ_∞. Loosely speaking, Λ is determined by all particles including rare exceptional particles, while $\mathcal{S}$ is determined only by the bulk of the particles. We always have that $\mathcal{S} \subseteq \Lambda$, and it is natural to ask if $\mathcal{S}$ can be a proper subset of Λ and, if yes, how we can quantify the difference. This is the aim of this paper in the case of branching Brownian motion in hyperbolic space.

Branching Brownian motion (BBM) on $\mathbb{R}$ is an interacting particle system. Particles move as independent Brownian motions and split in two at a given rate β. Here the exceptionally fast particles, that is the maximal displacement at time t, is a major object of interest [2, 32]. On the other hand, the number of particles near the origin always grows exponentially for any $\beta > 0$. If the underlying space is hyperbolic, the behaviour of BBM is markedly different. Let $\beta_c = 1/8$, then for any $\beta < \beta_c$ the process eventually vacates any compact set almost surely. On the other hand, for any $\beta > \beta_c$, the number of particles near the origin grows exponentially as in $\mathbb{R}$. The same is true for a discrete version of this model, a branching random walk on a homogeneous tree. We give a precise definition of branching Brownian motion in hyperbolic space in the next section, but also refer to the recent survey by Woess [87] and the references therein for background on hyperbolic BBM.

The limit set Λ of hyperbolic BBM was first studied by Lalley and Selke [67] who showed that Λ is a fractal–like random set and compute its Hausdorff dimension. (See for example [45] for some background on fractals and Hausdorff dimension.) Others have studied similar sets of accumulation points of branching random walks on the boundary on discrete hyperbolic spaces, see for example [44, 59, 82]. Much less is known about μ_∞ and its support. Even the existence of μ_∞ has only been shown recently [35, 60]. In fact, we believe that this paper is the first work to show quantitative properties of μ_∞.

4.1.1 The model

Hyperbolic space is usually modelled with the Poincaré disk model $\mathbb{D}$ or the upper half plane model $\mathbb{H}$. We use them interchangeably. They are Riemannian manifolds with metrics given by

$$\frac{2\sqrt{dx^2+dy^2}}{1-x^2-y^2} \text{ for } (x,y)\in\mathbb{D}=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\},$$

for the disk model and for the upper half plane model by

$$\frac{\sqrt{dx^2+dy^2}}{y^2} \text{ for } (x,y)\in\mathbb{H}=\{(x,y)\in\mathbb{R}\times\mathbb{R}_+\}.$$

The two models are isometric, an isometry $f:\mathbb{D}\to\mathbb{H}$ is given by $f(z)=i\frac{1+z}{1-z}$ where we identify $\mathbb{D}$ and $\mathbb{H}$ with subsets of $\mathbb{C}$ by $z=x+iy$. Note that origin $0\in\mathbb{D}$ corresponds to $i\in\mathbb{H}$. Both $\mathbb{D}$ and $\mathbb{H}$ are endowed with natural boundaries $\partial\mathbb{D}$ and $\partial\mathbb{H}$ given by $\partial\mathbb{D}=\{z\in\mathbb{C}:|z|=1\}$ and $\partial\mathbb{H}=\{z\in\mathbb{C}:\Im(z)=0\}$. The hyperbolic Laplacian is given by

$$\mathcal{L}_{\mathbb{D}}=\frac{(1-|z|^2)^2}{4}\left(\partial_x^2+\partial_y^2\right), \qquad \text{respectively} \qquad \mathcal{L}_{\mathbb{H}}=y^2\left(\partial_x^2+\partial_y^2\right).$$

From this we define hyperbolic Brownian motion to be the stochastic process with generator $\frac{1}{2}\mathcal{L}_{\mathbb{D}}$ (respectively $\frac{1}{2}\mathcal{L}_{\mathbb{H}}$). In the upper half plane model we could also do this by solving the pair of stochastic differential equations

$$dX_t=Y_t dW_t, \quad dY_t=Y_t dB_t,$$

where $(W_t)_t$ and $(B_t)_t$ are independent Brownian motions. This process is then canonically started from $(X_0,Y_0)=(0,1)$. Observe that $(Y_t)_t$ is a geometric Brownian motion, hence we can solve the SDE explicitly in the second coordinate,

$$Y_t=\exp\left(-\frac{t}{2}+B_t\right).$$

This also tells us that X_t is Gaussian with mean 0 and variance $\int_0^t \exp\left(-s+2B_s\right)ds$, conditional on $(B_s)_{s\geq 0}$. From this we can see that hyperbolic Brownian motion converges to a random point $(X_\infty,0)$ on the boundary $\partial\mathbb{H}$ where X_∞ is Gaussian with (random) variance $\int_0^\infty \exp\left(-s+2B_s\right)ds$.

Having defined hyperbolic Brownian motion, we define hyperbolic branching Brownian motion (BBM) to be the following particle process on $\mathbb{D}$: At time 0, we start with one particle at the origin. Particles move as independent hyperbolic Brownian motions. At rate β, each particle independently branches in two; both offspring particles branch and move independently. This results in a cloud of particles, we denote those positions by $((X_u(t), Y_u(t)), u \in \mathcal{N}(t))$, where $\mathcal{N}(t)$ is the set of particles alive at time t. By abuse of notation, we also denote the (isometric) process on $\mathbb{H}$ by $((X_u(t), Y_u(t)), u \in \mathcal{N}(t))$. Here the process is started from a single particle at $(0, 1)$.

We can relate certain expectations for hyperbolic BBM to expectations of hyperbolic Brownian motion by the many–to–one formula,

$$\mathbb{E}_{(x,y)}\left[\sum_{u \in \mathcal{N}(t)} f((X_u(s), Y_u(s))_{0 \le s \le t})\right] = e^{\beta t}\mathbb{E}_{(x,y)}\left[f((X_s, Y_s)_{0 \le s \le t})\right], \tag{4.1}$$

for any $(x, y) \in \mathbb{D}$ and measurable non–negative f. This follows from linearity due to the independence of movement and branching.

4.1.2 Results

We are interested in the long term behaviour of the cloud $((X_u(t), Y_u(t)), u \in \mathcal{N}(t))$, especially related to the boundary. We define the normalised empirical measure at time t to be

$$\mu_t = \frac{1}{|\mathcal{N}(t)|} \sum_{u \in \mathcal{N}(t)} \delta_{(X_u(t), Y_u(t))}.$$

One can show that there is a measure μ_∞, supported on the boundary, such that μ_t converges weakly to μ_∞ with probability one, see [87]. This follows a simple argument: let $h : \mathbb{D} \to \mathbb{R}$ be a non-negative, bounded function which is harmonic with respect to hyperbolic Brownian motion. Then $(e^{-\beta t}|\mathcal{N}(t)|)\langle h, \mu_t\rangle$ is a martingale for hyperbolic BBM and hence converges almost surely. To obtain weak convergence, one then only needs to check that the space of harmonic functions is sufficiently rich. One can also see (essentially from the many–to–one formula (4.1)) that for any measurable set $A \subseteq \mathbb{D} \cup \partial\mathbb{D}$ we have

$$\mathbb{E}\left[\mu_\infty(A)\right] = \mathbb{P}\left(\lim_{t \to \infty}(X_t, Y_t) \in A\right),$$

from which it follows that μ_∞ is supported on the boundary almost surely. The goal of this paper is to better understand μ_∞. See Figure 4.1 for a simulation of hyperbolic BBM and μ_∞. One object that is slightly easier to understand is

$$\Lambda = \{\text{accumulation points of } ((X_u(t), Y_u(t)), u \in \mathcal{N}(t))_{t \ge 0} \text{ in } \partial\mathbb{D}\}.$$

Lalley and Selke [67] have analysed this set and shown that its Hausdorff dimension is almost surely given by

$$\dim \Lambda = \begin{cases} \frac{1}{2}(1-\sqrt{1-8\beta}) & \text{for} \quad 0 < \beta \leq 1/8, \\ 1 & \text{for} \quad \beta > 1/8. \end{cases}$$

Note the discontinuity at $\beta = 1/8$. Further, they have shown that for $\beta > 1/8$ we actually have $\Lambda = \partial\mathbb{D}$ almost surely. The threshold $1/8$ isunsurprisingly also the threshold for recurrence/transience, and at $\beta = 1/8$ the process is transient. In his survey article about hyperbolic BBM [87], Woess asks several questions about the relationship between Λ and μ_∞, in particular if the support of μ_∞ is a proper subset of Λ. We answer these questions.

Theorem 4.1.1. *The Hausdorff dimension of the support of μ_∞ is almost surely given by*

$$\dim \operatorname{supp} \, \mu_\infty = \begin{cases} 2\beta & \textit{for} \quad 0 < \beta < 1/2, \\ 1 & \textit{for} \quad \beta \geq 1/2. \end{cases}$$

Consequently, supp μ_∞ *is a proper subset of* Λ *for* $\beta < 1/2$.

Note that this quantity is continuous in β and that the threshold $\beta = 1/8$ does not appear here. This is quite surprising given that $1/8$ is the threshold for local survival. Also note that $\lim_{\beta \to 0} \frac{\dim \Lambda}{\dim \operatorname{supp} \, \mu} = 1$. See Figure 4.2 for a plot of $\dim \operatorname{supp} \mu_\infty$ compared to $\dim \Lambda$. We also give some more quantitative statements about the nature of μ_∞. Call

$$\theta \mapsto \mu_\infty([0,\theta]), \quad \theta \in [0, 2\pi],$$

the (random) cumulative distribution function of μ_∞ where $[0,\theta]$ denotes the arc segment of $\partial\mathbb{D}$ with angles between 0 and θ.

Theorem 4.1.2. *Almost surely the following statements hold:*

(i) For any $\beta > 0$, μ_∞ is purely non-atomic.

(ii) The (random) cumulative distribution function of μ_∞ is γ–Hölder–continuous for every exponent $\gamma < (1/2) \wedge (\beta/3)$.

(iii) For $\beta > 1/2$, μ_∞ has a density with respect the the Lebesgue measure on $\partial\mathbb{D}$.

In the case $\beta = 1/2$ we believe that μ_∞ should not admit a Lebesgue density almost surely though we do not prove this. The bound on the Hölder exponent γ in 4.1.2 (ii) is not sharp, we believe it should hold for any $\gamma < 1 \wedge (2\beta)$.

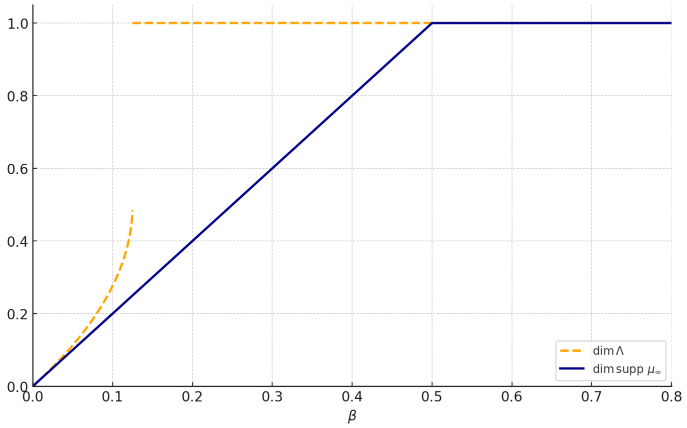

Figure 4.2: A plot of $\dim \operatorname{supp} \mu_\infty$ and $\dim \Lambda$ as functions of β.

Theorems 4.1.1 and 4.1.2 also partially answer some of the questions posed by Candellero and Hutchcroft [35, Problems 4.2, 4.3]. In particular, they ask about the behaviour of μ_∞ for branching random walks in hyperbolic space. While they pose their questions about branching random walks in discrete time and discrete space, this should not change the overall behaviour.

The main idea behind the proofs of Theorems 4.1.1 and 4.1.2 is that μ_∞ is determined by typical particles. In this context, these are particles for which $Y_u(t) \approx e^{-t/2}$. The structure of the paper follows this idea. In Section 4.2 we rigorously define what it means for a particle to be typical and we compute the Hausdorff dimension of the accumulation set of typical particles. In Section 4.3 we show that indeed μ_∞ is determined by typical particles, we prove the upper bound of Theorem 4.1.1 and a sketch the lower bound. In Section 4.4 we show Theorem 4.4 by computing the expected moments of $\mu(I)$ for intervals I. As a corollary, we obtain the lower bound for Theorem 4.1.1. Lastly we discuss some open questions in Section 4.5, in particular we discuss what should happen if you add a repulsive or attractive drift towards the origin, and we pose a conjecture regarding branching random walks on hyperbolic groups.

4.2 Typical particles

We work in $\mathbb{H}$. We start by looking at *typical* particles and their accumulation set on the boundary. Define the set of typical particles to be

$$\mathcal{T}(K) = \left\{ u \in \mathcal{N}(\infty) : \forall t \geq K : \log Y_u(s) + t/2 \in -[t^{2/3}, t^{2/3}] \right\}, \tag{4.2}$$

where $K > 0$ is a parameter. We also consider the typical particles at time t,

$$\mathcal{T}_t(K) = \{u \in \mathcal{N}(t) : \exists v \in \mathcal{T} \text{ such that } u \preceq v\}.$$

Note that $\mathcal{T}_t(K)$ is not measurable with respect to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the BBM. Let $u \in \mathcal{T}(K)$. Then we necessarily have that $Y_u(t) \to 0$ as $t \to \infty$ and $X_u(t)$ converges almost surely on $\partial\mathbb{H}$. Let

$$\Upsilon(K) = \{X_u(\infty), u \in \mathcal{T}(K)\} \tag{4.3}$$

be the accumulation set of typical particles on the boundary. The goal of this section is to determine the Hausdorff dimension of this set.

Proposition 4.2.1. *For any $\beta > 0$ and any $K > 0$, $\dim \Upsilon(K) = 2\beta \wedge 1$ almost surely on the event that $\Upsilon(K)$ in non-empty.*

We show this proposition in two steps: Lemma 4.2.3 for the upper bound and Lemma 4.2.4 for the lower bound.

Lemma 4.2.2. *For any $\beta < 1/2$ and any $K > 0$, there is $C < \infty$ such that $\mathbb{E}[\operatorname{diam} \Upsilon(K)] \leq C$.*

Proof. Let $M_t = \sup_{u\in\mathcal{T}_t(K)} X_u(t)$ be the maximal displacement of a typical particle in the x–direction at time t. Here we use the convention $\sup \emptyset = 0$ as there is nothing to show in the case when $\Upsilon(K) = \emptyset$. It suffices to show that $\mathbb{E}[\limsup_{t\to\infty} M_t] < \infty$.

Let $t \geq K$ and choose $n \in \mathbb{N}_0$ such that $t \in K + [n, n+1)$. By a telescoping sum

$$M_t = M_K + \left(\sum_{j=0}^{n-1}(M_{K+j+1} - M_{K+j})\right) + (M_t - M_{K+n}).$$

For each summand, we bound the difference between the maxima by the maximal positive increment at an intermediate time. This means that

$$\begin{aligned}(M_{K+j+1} - M_{K+j}) &\leq \sup_{u\in\mathcal{T}_{K+j+1}(K)} \sup_{s\in[0,1]} (X_u(K+j+s) - X_u(K+j)) \\ &\leq \sum_{u\in\mathcal{T}_{K+j+1}(K)} \sup_{s\in[0,1]} (X_u(K+j+s) - X_u(K+j)),\end{aligned}$$

where also used a union bound. This estimate is also true for the last summand,

$$\begin{aligned}(M_t - M_{K+n}) &\leq \sum_{u\in\mathcal{T}_t(K)} \sup_{s\in[0,t-\lfloor t\rfloor]} (X_u(\lfloor t\rfloor + s) - X_u(\lfloor t\rfloor)) \\ &\leq \sum_{u\in\mathcal{T}_{K+n+1}(K)} \sup_{s\in[0,1]} (X_u(\lfloor t\rfloor + s) - X_u(\lfloor t\rfloor)).\end{aligned}$$

We therefore have that

$$\begin{aligned}
M_t &\leq M_K + \sum_{j=0}^{n} \left(\sum_{u \in \mathcal{T}_{K+j+1}(K)} \sup_{s\in[0,1]} \left(X_u(K+j+s) - X_u(K+j)\right) \right) \\
&\leq M_K + \sum_{j=0}^{\infty} \left(\sum_{u \in \mathcal{T}_{K+j+1}(K)} \sup_{s\in[0,1]} \left(X_u(K+j+s) - X_u(K+j)\right) \right), \qquad (4.4)
\end{aligned}$$

note that all summands in the infinite series are non-negative and that this bound is uniform in $t \geq K$.

We estimate the expected value of these summands. For fixed u and j, the process $(X_u(K+j+s) - X_u(K+j))_{s\in[0,1]}$ is a Brownian motion with diffusivity $Y_u(K+j+s)$. Hence, by conditioning on $(Y_u(K+j+s))_{s\in[0,1]}$ and by the reflection principle we have the distributional equality

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,1]} \left(X_u(K+j+s) - X_u(K+j)\right) \middle| (Y_u(K+j+s))_{s\in[0,1]} \right] \\
&\qquad \overset{d}{=} \mathbb{E}\left[\left| X_u(K+j+1) - X_u(K+j) \right| \middle| (Y_u(K+j+s))_{s\in[0,1]} \right].
\end{aligned}$$

Now $(X_u(K+j+1) - X_u(K+j))$ is Gaussian with mean zero and with variance

$$\int_{K+j}^{K+j+1} Y_u(s)^2 ds \leq c \exp\left(-j(1+o_j(1)\right),$$

where we used that u is a typical particle and where $c = c(K)$ is a constant. Therefore

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,1]} \left(X_u(K+j+s) - X_u(K+j)\right) \right] &= \mathbb{E}\left[\left| X_u(K+j+1) - X_u(K+j) \right| \right] \\
&\leq \sqrt{\frac{2c}{\pi}} \exp\left(-\frac{j}{2}(1+o_j(1)) \right).
\end{aligned}$$

This means we also have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{u \in \mathcal{T}_{K+j+1}(K)} \sup_{s\in[0,1]} \left(X_u(K+j+s) - X_u(K+j)\right) \right] \\
&\qquad \leq \sqrt{\frac{2c}{\pi}} \mathbb{E}\left[|\mathcal{T}_{K+j+1}(K)| \right] \exp\left(-\frac{j}{2}(1+o_j(1)) \right) \\
&\qquad \leq \sqrt{\frac{2c}{\pi}} \exp\left(\beta(K+j+1)\right) \exp\left(-\frac{j}{2}(1+o_j(1)) \right), \qquad (4.5)
\end{aligned}$$

where we also used the bound

$$\mathbb{E}\left[|\mathcal{T}_{K+j+1}(K)| \right] \leq \mathbb{E}\left[|\mathcal{N}(K+j+1)| \right] = \exp\left(\beta(K+j+1)\right).$$

We apply this to (4.4) to see that uniformly in t

$$\mathbb{E}[M_t] \leq \mathbb{E}[M_K] + \sqrt{\frac{2c}{\pi}} \sum_{j=0}^{\infty} \exp\left(\beta(K+j+1)\right) \exp\left(-\frac{j}{2}(1+o_j(1))\right),$$

this sum converges because we assumed that $\beta < 1/2$. Further we have by the many–to–one formula (4.1) that

$$\mathbb{E}[M_K] \leq \mathbb{E}\left[\sum_{u \in \mathcal{N}(K)} |X_u(K)|\right] = e^{\beta K}\mathbb{E}[|X_t|] < \infty.$$

Looking back at (4.4), the monotone convergence theorem now implies that the right-hand side is almost surely finite. Because this is true for any $t \geq K$, this implies that $\mathbb{E}[\limsup_{t\to\infty} M_t] < \infty$ by Fatou's Lemma. □

Remark 4.2.1. The same proof strategy can also show that $\mathbb{E}[(\text{diam } \Upsilon(K))^k] < \infty$ for any $k \in \mathbb{N}$.

Clearly $\dim \Upsilon(K) \leq 1$. Therefore we need to show an upper for $\dim \Upsilon(K)$ only in the case $\beta < 1/2$.

Lemma 4.2.3. *For $\beta < 1/2$ and any $K > 0$, $\dim \Upsilon(K) \leq 2\beta$ almost surely.*

Proof. We follow a similar idea to [67, Proposition 11]. For a particle $u \in \mathcal{T}_t(K)$, consider

$$\Upsilon_u^t(K) = \left\{X_v(\infty) : v \in \mathcal{T}(K) \text{ with } u \preceq v\right\},$$

that is, the set of limits in $\partial\mathbb{H}$ of all typical descendents of u. Naturally, we have for any t that

$$\Upsilon(K) = \bigcup_{u \in \mathcal{T}_t(K)} \Upsilon_u^t(K). \tag{4.6}$$

For $u \neq v \in \mathcal{T}_t(K)$, $\Upsilon_u^t(K)$ and $\Upsilon_v^t(K)$ are independent conditional on $(X_u(t), Y_u(t))$ and $(X_v(t), Y_v(t))$. Let $I_u^t \subset \partial\mathbb{H}$ be the smallest closed interval that contains $\Upsilon_u^t(K)$. By isometries of $\mathbb{H}$, $\Upsilon_u^t(K)$ is contained in an independent copy of $\Upsilon(K)$ scaled by $Y_u(t)$, provided that $t \geq K$. In particular, we have by Lemma 4.2.2 that for any $0 < \eta \leq 1$

$$\mathbb{E}\Big[|I_u^t|^\eta \Big| Y_u(t)\Big] \leq Y_u(t)^\eta \mathbb{E}\left[(\text{diam } \Upsilon(K))^\eta\right] \leq (C+1)Y_u(t)^\eta. \tag{4.7}$$

Let us return to (4.6). This decomposition implies that $\{I_u^t\}_{u \in \mathcal{T}_t(K)}$ is an interval cover for $\Upsilon(K)$. Let $\varepsilon \geq 0$ such that $2\beta + \varepsilon < 1$. We apply (4.7) to get

$$\begin{aligned}\mathbb{E}\left[\sum_{t \in \mathcal{T}_t(K)} |I_u^t|^{2\beta+\varepsilon}\right] &\leq (C+1)\mathbb{E}\left[\sum_{t \in \mathcal{T}_t(K)} Y_u(t)^{2\beta+\varepsilon}\right] \\ &\leq (C+1)\mathbb{E}\left[|\mathcal{T}_t(K)|\right] \exp\left(-(2\beta+\varepsilon)(t/2)(1+o_t(1)\right),\end{aligned}$$

where we also used that $Y_u(t) = \exp(-(t/2)(1 + o_t(1))$ for typical particles. Further we can bound $\mathbb{E}\left[|\mathcal{T}_t(K)|\right] \leq \mathbb{E}[|\mathcal{N}(t)|] = \exp(\beta t)$, therefore

$$\begin{aligned}\mathbb{E}\left[\sum_{t\in\mathcal{T}_t(K)} |I_u^t|^{2\beta+\varepsilon}\right] &\leq (C+1)\exp\left(\beta t - \frac{2\beta+\varepsilon}{2}t(1+o_t(1))\right)\\ &= (C+1)\exp\left(-\frac{\varepsilon}{2}t(1+o_t(1))\right). \qquad (4.8)\end{aligned}$$

Using this for $\varepsilon > 0$ shows that

$$\sup_{u\in\mathcal{T}_t(K)} |I_u^t| \xrightarrow{t\to\infty} 0,$$

almost surely. Lastly, by Fatou's Lemma and applying (4.8) with $\varepsilon = 0$, we estimate for 2β–dimensional Hausdorff measure $\mathcal{H}^{2\beta}$ of $\Upsilon(K)$,

$$\mathbb{E}\left[\mathcal{H}^{2\beta}(\Upsilon(K))\right] \leq \mathbb{E}\left[\liminf_{t\to\infty}\sum_{t\in\mathcal{T}_t(K)} |I_u^t|^{2\beta+\varepsilon}\right] \leq \liminf_{t\to\infty}\mathbb{E}\left[\sum_{t\in\mathcal{T}_t(K)} |I_u^t|^{2\beta+\varepsilon}\right] < \infty.$$

In particular, this implies that $\mathcal{H}^{2\beta}(\Upsilon(K)) < \infty$ almost surely. ☐

Lemma 4.2.4. *For $\beta > 0$ and any $K > 0$, $\dim\Upsilon(K) \geq 2\beta \wedge 1$ almost surely on the event that $\Upsilon(K)$ is non-empty.*

A common tool to show lower bounds for Hausdorff dimensions is Frostman's Lemma, a corollary of which we state below as Lemma 4.2.5. See [45, Theorem 4.13] for a reference.

Lemma 4.2.5. *Let A be a compact subset of Euclidean space. Assume that there exists a probability measure ν on A such that*

$$\iint_{A\times A} |x-y|^{-\eta}\nu(dx)\nu(dy) < \infty,$$

where $\eta > 0$. Then the Hausdorff dimension of A is at least η.

Proof of Lemma 4.2.4. Throughout the proof, we work on the event that $\mathcal{T}(K)$ is non-empty. To use Frostman's Lemma, we need to define a probability distribution on $\Upsilon(K)$. We do this by defining a sequence $(U_n)_{n\in\mathbb{N}_0}$ of random variables such that $U_n \in \mathcal{T}_{nK}(K)$.

1. Let $U_0 = u$, where $u \in \mathcal{T}_0(K)$ is the unique initial particle.
2. Given U_{n-1}, let U_n be a uniform choice from $\{u \in \mathcal{T}_{nK}(t) : U_{n-1} \preceq u\}$.

Let $U = \lim_{n\to\infty} U_n$ be the natural limit in $\mathcal{T}(K)$ and let ν be the distribution of $X_U(\infty)$.

Now let $(U'_n)_n$ be a copy of $(U_n)_n$, independent conditional on $\mathcal{T}(K)$. Let $\tau = \inf\{n : U_n \neq U'_n\}$, the first time that U_n and U'_n are different. Conditional on $\tau = n$, $X_U(\infty) - X_{U'}(\infty)$ is Gaussian with mean 0 and variance at least

$$\operatorname{Var}(X_U(\infty) - X_{U'}(\infty)) \geq \int_{nK}^{\infty} Y_U(s)^2 + Y_{U'}(s)^2 ds \geq 2\exp\left(-nK(1+o_n(1))\right),$$

where we used that $Y_u(s) \geq \exp(-(s/2)(1+o_1(s)))$ for typical particles for $s \geq K$. In particular this implies that for $\eta < 1$,

$$\begin{aligned}
&\mathbb{E}\left[|X_U(\infty) - X_{U'}(\infty)|^{-\eta}\middle|\mathcal{T}(K), \tau = n\right] \\
&\qquad\qquad = \mathbb{E}[|Z|^{-\eta}]\mathbb{E}\left[\operatorname{Var}(X_U(\infty) - X_{U'}(\infty))^{-\eta/2}\middle|\mathcal{T}(K), \tau = n\right] \\
&\qquad\qquad \leq c\exp(\eta nK/2),
\end{aligned} \tag{4.9}$$

for an independent standard Gaussian Z and some constant $c > 0$.

Next, we need to understand the distribution of τ. Conditional on $\mathcal{T}$ and $(U_n)_n$ we have that

$$\mathbb{P}\left(\tau > k|\mathcal{T}, (U_n)_n\right) = \mathbb{P}\left(\forall i \leq k : U'_j = U_j\middle|\mathcal{T}, (U_n)_n\right) = \prod_{j=1}^{n} \frac{1}{N_j},$$

where $N_j = \#\{u \in \mathcal{T}_{jK} : U_{j-1} \preceq u\}$, the number of descendants of U_{j-1} in $\mathcal{T}_{jk}(K)$. By Lemma 4.2.6 below

$$\lim_{n\to\infty}\left(\prod_{j=1}^{n}\frac{1}{N_j}\right)^{1/n} = \exp(-\beta K),$$

almost surely on the event that $\mathcal{T}$ is non-empty. Combining this with (4.9) yields

$$\begin{aligned}
&\mathbb{E}\left[|X_U(\infty) - X_{U'}(\infty)|^{-\eta}\middle|\mathcal{T}(K)\right] \\
&\qquad \leq \sum_{n=1}^{\infty} \mathbb{E}\left[|X_U(\infty) - X_{U'}(\infty)|^{-\eta}\middle|\mathcal{T}(K), \tau = n\right] \mathbb{P}\left(\tau \geq n|\mathcal{T}(K)\right) \\
&\qquad \leq c\sum_{n=1}^{\infty} \exp(\eta nK/2)\exp(-\beta Kn(1+o_n(1)) \\
&\qquad < \infty,
\end{aligned}$$

for some $c > 0$ and where in the last step we used that $\eta < 2\beta$. By Frostman's Lemma, Lemma 4.2.5, this now implies that $\Upsilon(K)$ has dimension at least η for any η that satisfies $\eta < 2\beta$ and $\eta < 1$. □

Lemma 4.2.6. *In the setting on the previous proof,*

$$\lim_{n\to\infty}\left(\prod_{j=1}^{n}\frac{1}{N_j}\right)^{1/n} = \exp(-\beta K),$$

almost surely on the event that $\mathcal{T}$ is non-empty.

Proof. We sketch a proof of this fact, this proof can be made rigorous by carefully applying the strong law of large numbers. We provide only a sketch because Lemma 4.2.4 will not be used to show the lower bound of Theorem 4.1.1. We comment on this at the end of Section 4.3.

The key idea is that we can think of $\mathcal{T}(K)$ as a branching Brownian motion with space and time dependent branching rate. Let

$$\mathcal{W}(K) = \left\{(x,y,t) \in \mathbb{H}\times[0,\infty) : \forall s \geq K : \log(y) + t/2 \in [-t^{2/3}, t^{2/3}]\right\},$$

the space-time envelope of the definition of $\mathcal{T}(K)$. We also let

$$\phi(x,y,t) = \mathbb{P}_{(x,y,t)}\left(\forall s \geq t : (X_u(s), Y_u(s), s) \in \mathcal{W}(K)\right),$$

the probability that a particle started from (x, y) at time t stays in $\mathcal{W}(K)$ forever. Importantly, $\phi(0, 1, 0) > 0$, that is with positive probability the initial particle stays in $\mathcal{W}(K)$. We now describe a new BBM:

1. At time 0, we start with one particle at $(0, 1)$.
2. All particles move as independent hyperbolic Brownian motions conditioned to stay in $\mathcal{W}(K)$.
3. Particles branch into two at rate $\beta\phi(x, y, t)$.

One can show that this modified BBM has the same law as the homogenous hyperbolic BBM restricted to $\mathcal{T}(K)$. Now look at a marked particle in the modified BBM, that is the initial particle is marked and, when it splits, the mark follows one of the offspring particles chosen uniformly. This is similar to the construction of $(U_n)_n$ in the previous proof. Let $(X_t^*, Y_t^*)_{t\geq 0}$ be the path of the marked particle. In fact, we have that

$$\lim_{n\to\infty}\left(\prod_{j=1}^{n}\frac{1}{N_j}\right)^{1/n} = \lim_{t\to\infty}\exp\left(-\frac{1}{t}\int_0^{Kt}\beta\phi(X_s^*, Y_s^*, s)ds\right) = \exp\left(-K\beta\right).$$

The reason for this is that along the marked path, we have that $\phi(X_s^*, Y_s^*, s) \to 1$ almost surely. This is because $\log(Y_s^*) + s/2$ will be of order $s^{1/2} \ll s^{2/3}$ so it is very likely that for large s a particle started from (X_s^*, Y_s^*, s) will stay in $\mathcal{W}$ forever. □

4.3 From typical particles to empirical measure

In the previous section we analysed the accumulation set of typical particles whose definition we recall from (4.2). We slightly modify this. For any $t, K > 0$, let

$$\mathcal{T}_t^{\leq}(K) = \left\{ u \in \mathcal{N}(t) : \forall s \in [K, t] : \log Y_u(s) + s/2 \in [-s^{2/3}, s^{2/3}] \right\}.$$

The advantage of this modification is that $\mathcal{T}_t^{\leq}(K)$ is $\mathcal{F}_t$–measurable where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of the BBM. Similar to μ_t, we define the empirical measure of typical particles at time t,

$$\mu_t^K = \frac{1}{|\mathcal{N}(t)|} \sum_{u \in \mathcal{T}_t^{\leq}(K)} \delta_{(X_u(t), Y_u(t))}.$$

Note that we chose to normalise this by $|\mathcal{N}(t)|$ which means that μ_t^K is a sub-probability measure. The following proposition states that μ_∞ is determined by typical particles. In some sense this is a refinement of [87, Theorem 6.13] which states that a typical sample of μ_t moves at velocity $1/2$ in the hyperbolic metric.

Proposition 4.3.1. *Almost surely, there exists a family of sub-probability measures* $(\mu_\infty^K)_{K>0}$ *such that for every* $K > 0$

$$\mu_t^K \to \mu_\infty^K,$$

weakly, as $t \to \infty$*. Furthermore, for* $K < K'$ *we have* $\mu^K \leq \mu^{K'}$ *and as* $K \to \infty$*,*

$$\mu_\infty^K \to \mu_\infty.$$

Proof. Let $h : \mathbb{H} \to \mathbb{R}$ be a non-negative, bounded, $\mathcal{C}^\in$ function which is harmonic for hyperbolic Brownian motion. That is, for all $(x, y) \in \mathbb{H}$, we have $\mathbb{E}_{(x,y)}[h(X_t, Y_t)] = h(x, y)$.

It suffices to check weak convergence only on harmonic functions as these are dense in $\mathcal{C}_b(\overline{\mathbb{D}}, \mathbb{R})$ with respect to the uniform topology. This is because if h is harmonic for the hyperbolic Laplacian, it is also harmonic for the Euclidean Laplacian on $\mathbb{D}$. The harmonic functions for the Euclidean Laplacian are well understood, for example any h can be written as $\Re(f)$ where f is a holomorphic function. This fact is also discussed in the proof of [87, Theorem 6.7].

We define

$$M_h^K(t) = \frac{|\mathcal{N}(t)|}{e^{\beta t}} \left\langle h, \mu_t^K \right\rangle = \frac{1}{e^{\beta t}} \sum_{u \in \mathcal{T}_t^{\leq}(K)} h(X_u(t), Y_u(t)).$$

Then $(M_h^K(t))_{t\geq 0}$ is a non-negative supermartingale with respect to the natural filtration of hyperbolic BBM. Indeed,

$$\begin{aligned}\mathbb{E}\left[M_h^K(t)\middle|\mathcal{F}_s\right] &= \frac{1}{e^{\beta s}}\sum_{u\in\mathcal{T}_s^{\leq}(K)}\mathbb{E}_{(X_u(s),Y_u(s))}\left[\frac{1}{e^{\beta(t-s)}}\sum_{\substack{u\in\mathcal{T}_t^{\leq}(K)\\ u\preceq v}}h(X_v(t),Y_v(t))\middle|\mathcal{F}_s\right]\\ &= \frac{1}{e^{\beta s}}\sum_{u\in\mathcal{T}_s^{\leq}(K)}\mathbb{E}_{(X_u(s),Y_u(s))}\left[h(X_t,Y_t)\mathbb{1}_{\{\forall r\in[s\vee K,t]:\log(Y_r)+r/2\in[-r^{2/3},r^{2/3}]\}}\right],\end{aligned}$$

where we used the Markov property and the many–to–one formula (4.1). Next we use that $h\geq 0$ and that h is harmonic,

$$\mathbb{E}\left[M_h^K(t)\middle|\mathcal{F}_s\right]\leq\frac{1}{e^{\beta s}}\sum_{u\in\mathcal{T}_s^{\leq}(K)}\mathbb{E}_{(X_u(s),Y_u(s))}\left[h(X_t,Y_t)\right]=M_h^K(s).$$

Because $M_h^K(t)$ is a non-negative, uniformly integrable supermartingale, it converges almost surely and in L^1 to a limit, call it $M_h^K(\infty)$. Furthermore, let $W=\lim_{t\to\infty}e^{-\beta t}|\mathcal{N}(t)|$, where the limit is almost sure and $0<W<\infty$ almost surely. Combining these limits gives us the almost sure limit as $t\to\infty$,

$$\lim_{t\to\infty}\left\langle h,\mu_t^K\right\rangle = W^{-1}M_h^K(\infty).$$

Because h is arbitrary, this implies that there is μ_∞^K such that almost surely μ_t^K converges weakly to μ_∞^K. Next, for $K<K'$ and any h we have that $M_h^K(t)\leq M_h^{K'}(t)$ and consequently $M_h^K(\infty)\leq M_h^{K'}(\infty)$ almost surely. This implies that $\mu_\infty^K\leq\mu_\infty^{K'}$.

Lastly, to show that $\mu_\infty^K\to\mu_\infty$, it suffices to show that

$$\langle 1,\mu_\infty^K\rangle\to 1,$$

almost surely as $t\to\infty$. Because $\mathbb{1}(x,y)=1$ for all $(x,y)\in\mathbb{H}$ is harmonic, this is equivalent to showing that $M_{\mathbb{1}}^K(\infty)\to W$ as $K\to\infty$. We know that $M_{\mathbb{1}}^K(\infty)\leq W$, therefore it is enough to show that $\lim_{K\to\infty}\mathbb{E}[M_{\mathbb{1}}^K(\infty)]=\mathbb{E}[W]=1$. By L_1–convergence and the many–to–one lemma (4.1) we have

$$\begin{aligned}\lim_{K\to\infty}\mathbb{E}\left[M_{\mathbb{1}}^K(\infty)\right] &= \lim_{K\to\infty}\lim_{t\to\infty}\mathbb{E}\left[M_{\mathbb{1}}^K(t)\right]\\ &= \lim_{K\to\infty}\lim_{t\to\infty}\mathbb{P}\left(\forall s\in[K,t]:\log(Y_s)+s/2\in[-s^{2/3},s^{2/3}]\right)\\ &= \lim_{K\to\infty}\mathbb{P}\left(\forall s\geq K:\log(Y_s)+s/2\in[-s^{2/3},s^{2/3}]\right)\\ &= 1,\end{aligned}$$

where we recalled that $(\log(Y_s)+s/2)_{s\geq 0}$ is a standard Brownian motion and hence the complement of the last probability decays like $\exp(-cK^{1/3})$. □

We can combine this with Proposition 4.2.1 to obtain the upper bound in Theorem 4.1.1.

Corollary 4.3.2. *For any* $\beta < 1/2$, *we almost surely have that* $\dim \operatorname{supp} \mu_\infty \leq 2\beta$.

Proof. By Proposition 4.3.1 we have that almost surely

$$\operatorname{supp} \mu_\infty = \bigcup_{K=1}^{\infty} \operatorname{supp} \mu_\infty^K \subseteq \bigcup_{K=1}^{\infty} \Upsilon(K),$$

where $\Upsilon(K)$ is the set of accumulation points on $\partial\mathbb{H}$ of particles counted in μ_t^K, see (4.3). This implies

$$\dim \operatorname{supp} \mu_\infty \leq \dim \bigcup_{K=1}^{\infty} \Upsilon(K) = \sup_{K\in\mathbb{N}} \dim \Upsilon(K) = 2\beta,$$

where we used that the Hausdorff dimension of a countable union is the supremum of the Hausdorff dimensions, and that $\dim \Upsilon(K) = 2\beta$ almost surely on the event that $\Upsilon(K)$ is non-empty by Proposition 4.2.1. Moreover, we almost surely have that for K large enough $\Upsilon(K)$ is non-empty. □

We could do a similar proof for the lower bound in Theorem 4.1.1. Here we have the bound

$$\dim \operatorname{supp} \mu_\infty \geq \dim \operatorname{supp} \mu_\infty^K.$$

The issue is that it should hold that $\dim \operatorname{supp} \mu_\infty^K = \dim \Upsilon(K)$, although this is not obvious. Nevertheless, this is true: in the proof of Lemma 4.2.4 we construct a probability measure ν on $\Upsilon(K)$ to then apply Frostman's Lemma for a lower bound on the Hausdorff dimension of $\Upsilon(K)$. One can see that ν is actually supported on $\operatorname{supp} \mu_\infty^K$, hence the lower bound on the Hausdorff dimension also applies to μ_∞^K. We leave the details of this to the reader and instead provide a proof of the lower bound using different methods in the next section.

4.4 More properties of μ_∞

In this section we show the lower bound of Theorem 4.1.1 as well as the other properties of μ_∞ which we claimed in Theorem 4.1.2. The key tool is a second moment computation.

Lemma 4.4.1 (many-to-two). *Fix $r \geq 0$, under $\mathbb{P}^r$, let $(X_s^1, Y_s^1)_{s\geq 0}$ and $(X_s^2, Y_s^2)_{s\geq 0}$ be two hyperbolic Brownian motions that move together until time r and afterwards move independently. Then we have for any interval $I \subseteq \mathbb{R}$ that*

$$\mathbb{E}\left[\mu_\infty(I)^2\right] = 2\beta \int_0^\infty \mathbb{P}^r\left(X_\infty^1, X_\infty^2 \in I\right) e^{-\beta r} dr.$$

Similarly for any $K > 0$,

$$\mathbb{E}\left[\mu_\infty^K(I)^2\right] = 2\beta \int_0^\infty \mathbb{P}^r\Bigg(X_\infty^1, X_\infty^2 \in I, \\ \forall s \geq K : \log(Y_s^1) + s/2, \log(Y_s^2) + s/2 \in [-s^{2/3}, s^{2/3}]\Bigg) e^{-\beta r} dr.$$

Proof. This is a variant of the classical many–to–two lemma. See [56] for the statement and proof in the general setting. Applying the many–to–two lemma to μ_t yields

$$\mathbb{E}\left[\mu_t(I \times \mathbb{R})^2\right] = 2\beta \int_0^t \mathbb{P}^r\left(X_t^1, X_t^2 \in I\right) e^{-\beta r} dr + e^{-\beta t}\mathbb{P}\left(X_t^1 \in I\right).$$

The dominated convergence theorem now completes the proof. We proceed analogously for μ_∞^K. ☐

To compute $\mathbb{E}\left[\mu_\infty(I)^2\right]$, we first need an estimate on hyperbolic Brownian motion. This is related to [67, Lemma 6] where the authors used geometric arguments to show that X_∞ has a bounded density with respect to the Lebesgue measure. We improve on this by determining some dependence on the starting position.

Lemma 4.4.2. *There is $c > 0$ such that for any $x \in \mathbb{R}$, $y \in (0, \infty)$ and any interval $I \subseteq \mathbb{R}$ we have*

$$\mathbb{P}_{(x,y)}(X_\infty \in I) \leq \left(c\frac{|I|}{y}\right) \wedge 1,$$

where $|I|$ is the length of the interval.

Proof. Because X_∞ is Gaussian with mean 0 and variance $\int_0^\infty Y_s^2 ds$ (conditional on $(Y_s)_{s\geq 0}$) we may restrict to the case of $x = 0$ and $I = [-L, L]$ for $L > 0$ without loss of generality. Let $\mathcal{N}(0, \sigma^2)$ denote a generic centered Gaussian with variance σ^2. We have that

$$\mathbb{P}_{(x,y)}(X_\infty \in I) = \mathbb{P}_{(0,y)}\left(x + \mathcal{N}\left(0, \int_0^\infty Y_s^2 ds\right) \in I\right) \leq \mathbb{P}_{(0,y)}\left(\mathcal{N}\left(0, \int_0^1 Y_s^2 ds\right) \in I\right), \tag{4.10}$$

because decreasing the variance increases the probability to be in I. Further, for any continuous function $h : [0,1] \to (0,\infty)$,

$$\mathbb{P}\left(\mathcal{N}\left(0, \int_0^1 h(s)^2 ds\right) \in I\right) = \mathbb{P}\left(\mathcal{N}(0,1) \in \left(\int_0^1 h(s)^2 ds\right)^{-1/2} \times I\right)$$
$$\leq |I| \left(\int_0^1 h(s)^2 ds\right)^{-1/2},$$

where we used that the density of $\mathcal{N}(0,1)$ is bounded by 1. By Jensen's inequality,

$$\left(\int_0^1 h(s)^2 ds\right)^{-1/2} \leq \int_0^1 h(s)^{-1} ds.$$

We apply this to (4.10) by conditioning on $(Y_s)_{s\geq 0}$,

$$\mathbb{P}_{(0,y)}(X_\infty \in I) \leq |I| \mathbb{E}_{(0,y)} \left[\int_0^1 Y_s^{-1} ds\right] = \frac{|I|}{y} \mathbb{E}_{(0,1)} \left[\int_0^1 Y_s^{-1} ds\right] = c\frac{|I|}{y}.$$

The penultimate equality follows from the representation under $\mathbb{E}_{(0,y)}$ that $Y_s = y\exp(-s/2 + B_s)$ where $(B_s)_{s\geq 0}$ is a Brownian motion. In the regime where $c\frac{|I|}{y} > 1$, we use the trivial bound $\mathbb{P}_{(x,y)}(X_\infty \in I) \leq 1$. □

Proposition 4.4.3.

(i) For any $\beta > 0$ *and* $\beta \neq 3$ *there is* $C_1 < \infty$ *such that*

$$\limsup_{\varepsilon\to 0} \sup_{\substack{I\subseteq\mathbb{R}\\ |I|=\varepsilon}} \frac{\mathbb{E}\left[\mu_\infty(I)^2\right]}{\varepsilon^{2\wedge(1+\beta/3)}} \leq C_1 < \infty.$$

For $\beta = 3$, *replace* ε^2 *above by* $\varepsilon^2 \log(\varepsilon^{-1})$.

(ii) For any $\beta > 0$, *any* $K > 0$, *and any* $\delta > 0$, *there is* $C_2 = C_2(\beta, K, \delta) < \infty$ *such that*

$$\limsup_{\varepsilon\to 0} \sup_{\substack{I\subseteq\mathbb{R}\\ |I|=\varepsilon}} \frac{\mathbb{E}\left[\mu_\infty^K(I)^2\right]}{\varepsilon^{2\wedge(1+2\beta-\delta)}} \leq C_2 < \infty.$$

Note that there is a discrepancy between μ_∞^K and μ_∞: the exponent for μ_∞^K is $2\wedge(1+2\beta-\delta)$ for any arbitrary $\delta > 0$ whereas for μ_∞ it is $2\wedge(1+\beta/3)$. We believe that the exponent $2\wedge(1+2\beta-\delta)$ should also apply to μ_∞ but this would require better estimates, for example a uniform control in $C_2(K)$. For μ_∞ and $\beta = 3$, the extra logarithmic factor is an artifact of suboptimal estimates in the proof.

Proof of Proposition 4.4.3 (i). Assume for now that $\beta \neq 3$. Without loss of generality, we consider $I = [-\varepsilon, \varepsilon]$.

We use Lemma 4.4.1 and subdivide the integral into three parts, $J_1 = [0,1]$, $J_2 = [1, \log(\varepsilon^{-2})]$ and $J_3 = [\log(\varepsilon^{-2}), \infty)$. For $i \in \{1,2,3\}$, let

$$T_i = \int_{J_i} \mathbb{P}^r \left(X^1_\infty, X^2_\infty \in I\right) e^{-\beta r} dr.$$

We start with T_1. Here we bound $e^{-\beta r} \leq 1$ and then condition on the splitting position at time r,

$$T_1 = \int_0^1 \int_{\mathbb{H}} \mathbb{P}_{(r,x,y)} \left(X_\infty \in I\right)^2 \mathbb{P}\left((X_r, Y_r) \in (dx, dy)\right) dr.$$

We apply Lemma 4.4.2 to obtain

$$T_1 \leq c_1 \varepsilon^2 \int_0^1 \mathbb{E}\left[Y_r^{-2}\right] dr \leq c_2 \varepsilon^2, \tag{4.11}$$

for some constants $c_1, c_2 > 0$.

Next we consider T_2. By Lemma 4.4.2,

$$\begin{aligned}
\mathbb{P}^r \left(X^1_\infty, X^2_\infty \in I\right) &= \int_{\mathbb{H}} \mathbb{P}((X_r, Y_r) \in (dx, dy)) \mathbb{P}_{(r,x,y)}(X_\infty \in I)^2 \\
&\leq \int_{\mathbb{H}} \mathbb{P}((X_r, Y_r) \in (dx, dy)) \mathbb{P}_{(r,x,y)}(X_\infty \in I) \mathbb{P}_{(r,0,y)}(X_\infty \in I) \\
&\leq \int_{\mathbb{H}} \mathbb{P}((X_r, Y_r) \in (dx, dy)) \mathbb{P}_{(r,x,y)}(X_\infty \in I) \left(C\frac{\varepsilon}{y} \wedge 1\right).
\end{aligned}$$

Hence

$$T_2 \leq \int_1^{\log(\varepsilon^{-2})} \mathbb{E}\left[\mathbb{1}_{X_\infty \in I} \left(c\frac{\varepsilon}{Y_r} \wedge 1\right)\right] e^{-\beta r} dr.$$

We split this integral into two parts: for $r \in [1, \log(\varepsilon^{-1/3})]$ we bound

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{X_\infty \in I} \left(c\frac{\varepsilon}{Y_r} \wedge 1\right)\right] &\leq c\varepsilon \mathbb{E}\left[\mathbb{1}_{X_\infty \in I} Y_r^{-1}\right] = c\varepsilon \mathbb{E}\left[\mathbb{P}(X_\infty \in I | Y_r) Y_r^{-1}\right] \\
&\leq c^2 \varepsilon^2 \mathbb{E}\left[Y_r^{-2}\right] = c^2 \varepsilon^2 e^{3r},
\end{aligned}$$

where we used Lemma 4.4.2 again, and where we recall that $Y_r^{-1} = \exp(r/2 - B_r)$. For $r \in [\log(\varepsilon^{-1/3}), \log(\varepsilon^{-2})]$, we estimate

$$\mathbb{E}\left[\mathbb{1}_{X_\infty \in I} \left(c\frac{\varepsilon}{Y_r} \wedge 1\right)\right] \leq \mathbb{P}(X_\infty \in I) \leq c\varepsilon,$$

where we also used Lemma 4.4.2. Then our estimate on T_2 becomes

$$\begin{aligned}
T_2 &\leq c^2 \varepsilon^2 \int_1^{\log(\varepsilon^{-1/3})} e^{3r} e^{-\beta r} dr + c\varepsilon \int_{\log(\varepsilon^{-1/3})}^{\log(\varepsilon^{-2})} e^{-\beta r} dr \\
&\leq c_3 \left(\varepsilon^2 + \varepsilon^{1+\beta/3} + \varepsilon^{1+2\beta}\right),
\end{aligned} \tag{4.12}$$

for some $c_3 > 0$. For T_3, we estimate using Lemma 4.4.2

$$\mathbb{P}^r(X^1_\infty, X^2_\infty \in I) \leq \mathbb{P}(X^1_\infty \in I) \leq c_4\varepsilon.$$

Therefore

$$T_3 \leq c\varepsilon \int_{\log(\varepsilon^{-2})}^{\infty} e^{-\beta r} dr = c_4 \varepsilon^{1+2\beta}, \tag{4.13}$$

for $c_4 > 0$. To complete the proof, we combine (4.11), (4.12) and (4.13).

Lastly, in the case where $\beta = 3$, the final estimate on T_2 is

$$T_2 \leq c^2\varepsilon^2 \int_1^{\log(\varepsilon^{-1/3})} e^{3r}e^{-3r} dr + c\varepsilon \int_{\log(\varepsilon^{-1/3})}^{\log(\varepsilon^{-2})} e^{-3r} dr \leq c_3 \varepsilon^2 \log(\varepsilon^{-1}).$$

□

Proof of Proposition 4.4.3 (ii). Without loss of generality, we consider $I = [-\varepsilon, \varepsilon]$. We proceed as in the proof for (i), splitting the integral of Lemma 4.4.1 into T_1, T_2, T_3. For ease of notation, assume that $K = 1$, otherwise set $T_1 = [0, K]$ and $T_2 = [K, \log(\varepsilon^{-1})]$. We use the same bounds on T_1 and T_3 but a different one on T_2. For $r \in [1, \log(\varepsilon^{-2})]$, we integrate over the splitting location

$$\begin{aligned}\mathbb{P}^r &\left(X^1_\infty, X^2_\infty \in I, \forall s \geq 1 : \log(Y^1_s) + s/2, \log(Y^2_s) + s/2 \in [-s^{2/3}, s^{2/3}] \right) \\ &\leq \int_{\mathbb{R}\times[e^{-s/2-s^{2/3}}, e^{-s/2+s^{2/3}}]} \mathbb{P}((X_r, Y_r) \in (dx, dy)) \mathbb{P}_{(r,x,y)}(X_\infty \in I)^2,\end{aligned} \tag{4.14}$$

where the inequality comes from the fact that we kept the path restriction for Y only for the splitting location. By Lemma 4.4.2 we have for any $y \in [e^{-s/2-s^{2/3}}, e^{-s/2+s^{2/3}}]$ that

$$\mathbb{P}_{(r,x,y)}(X_\infty \in I) \leq c\varepsilon e^{r/2+r^{2/3}}.$$

We apply this only to one factor of $\mathbb{P}_{(r,x,y)}(X_\infty \in I)$ in (4.14),

$$\begin{aligned}\mathbb{P}^r &\left(X^1_\infty, X^2_\infty \in I, \forall s \geq 1 : \log(Y^1_s) + s/2, \log(Y^2_s) + s/2 \in [-s^{2/3}, s^{2/3}] \right) \\ &\leq c\varepsilon e^{r/2+r^{2/3}} \int_{\mathbb{R}\times[e^{-s/2-s^{2/3}}, e^{-s/2+s^{2/3}}]} \mathbb{P}((X_r, Y_r) \in (dx, dy)) \mathbb{P}_{(r,x,y)}(X_\infty \in I) \\ &\leq c\varepsilon e^{r/2+r^{2/3}} \mathbb{P}(X_\infty \in I) \\ &\leq c^2\varepsilon^2 e^{r/2+r^{2/3}},\end{aligned}$$

where we applied Lemma 4.4.2 again, this time with $(x, y) = (0, 1)$. This then yields the following bound on T_2,

$$T_2 \leq c^2 \varepsilon^2 \int_1^{\log(\varepsilon^{-2})} e^{r/2 + r^{2/3}} e^{-\beta r} dr = \varepsilon^{-2} \leq C(\delta) \left(\varepsilon^2 + \varepsilon^{1+2\beta-\delta}\right),$$

for any $\delta > 0$ and a constant $C(\delta)$. Here we used that $\exp(\log(\varepsilon^{-1})^{2/3})$ grows slower than $\varepsilon^{-\delta}$ for any δ as $\varepsilon \to 0$. Combining this with (4.11) and (4.13) completes the proof. □

We can use the same methods to derive bounds on the expected k–th moment of $\mu_\infty^K(I)$.

Lemma 4.4.4. *For any $\beta > 0$, any $K > 0$, any $k \in \mathbb{N}_{\geq 2}$, and any $\delta > 0$, there is $C_3 = C_3(\beta, K, k, \delta) < \infty$ such that*

$$\limsup_{\varepsilon \to 0} \sup_{\substack{I \subseteq \mathbb{R} \\ |I| = \varepsilon}} \frac{\mathbb{E}\left[\mu_\infty^K(I)^k\right]}{\varepsilon^{k \wedge (1+2\beta(k-1)-\delta)}} \leq C_3 < \infty.$$

Proof. Due to considering the k–th moment we now need the many–to–few lemma. This is tedious to state, so we only present the consequences that we need, the precise formulation can be found in [56]. To state the many–to–few lemma, we need to describe the joint law of k hyperbolic Brownian motions. We do this by describing the behaviour of k marks $1, \ldots, k$.

1. We start with one particle carrying all marks.
2. All particles move as independent hyperbolic Brownian motions, branching at rate β.
3. For a particle carrying j marks, at a branching event, each mark is independently attached to one of the two offspring particles with equal probability.

Let $(X_t^i, Y_t^i)_{t \geq 0}^{1 \leq i \leq k}$ denote the positions of the marks. The many–to–few lemma states that there is an explicit function $g((X_t^i, Y_t^i); 1 \leq i \leq k)$ such that

$$\mathbb{E}\left[\mu(I)^k\right] = \mathbb{E}\Bigg[\mathbb{1}_{\bigcap_{i=1}^k \{X_\infty^i \in I\}} \mathbb{1}_{\bigcap_{i=1}^k \{\forall s \geq K : \log(Y_s^i) + s/2 \in [-s^{2/3}, s^{2/3}]\}} \\ \times \exp\left(\int_0^\infty g((X_s^i, Y_s^i); 1 \leq i \leq k) ds\right)\Bigg]. \qquad (4.15)$$

For $i \geq 2$, let s_i be the last time that the mark i is carried by the same particle as a mark j with $j < 1$. Set $s_1 = 0$. Let $\mathcal{G} = \sigma\left(\left\{s_i, Y^i_{s_i}, s \geq 0, 1 \leq i \leq k\right\}\right)$. Conditional on $\mathcal{G}$, we have

$$\mathbb{P}\left(\forall i \leq k : X_\infty \in I | \mathcal{G}\right) \leq \prod_{i=1}^{k} \mathbb{P}_{(s_i, 0, Y^i_{s_i})}\left(X_\infty \in I\right) \leq C \prod_{i=1}^{k}\left(\left(\frac{\varepsilon}{Y^i_{s_i}}\right) \wedge 1\right),$$

where we used Lemma 4.4.2. Assume that for all $i \geq 2$ we have that $s_i \geq K$. Then on this event we have control on $Y^i_{s_i}$, therefore

$$\mathbb{P}\left(\forall i \leq k : X_\infty \in I | \mathcal{G}\right) \leq C\varepsilon \prod_{i=2}^{k}\left(\left(\varepsilon e^{s_i/2 + s_i^{2/3}}\right) \wedge 1\right).$$

In fact, this still holds if $s_i \leq K$ by changing C. Using the explicit representation of g this becomes

$$\begin{aligned}
\mathbb{E}\left[\mu(I)^k\right] &\leq C\varepsilon \int_{\mathbb{R}_+^{k-1}} \left[\prod_{i=2}^{k}\left(\left(\varepsilon e^{s_i/2 + s_i^{2/3}}\right) \wedge 1\right) e^{-\beta s_i}\right] ds_2 \dots ds_k \\
&= C\varepsilon \left(\int_0^\infty \left(\varepsilon e^{s/2 + s^{2/3}} \wedge 1\right) e^{-\beta s} ds\right)^{k-1} \\
&\leq C\varepsilon \left(\varepsilon + \varepsilon^{2\beta - \delta/(k-1)}\right)^{k-1},
\end{aligned}$$

for any $\delta > 0$ and some C. □

From Proposition 4.4.3 we derive the following corollary which is Theorem 4.1.2 (i) and (ii). We stated Theorem 4.1.2 (ii) for μ_∞, and consequently also F, defined on $\partial\mathbb{D}$. The following corollary is stated for μ_∞ viewed on $\partial\mathbb{H}$. This is equivalent because the isometry that maps $\mathbb{H}$ to $\mathbb{D}$ is a diffeomorphism and thus preserves Hölder–continuity.

Corollary 4.4.5.

1. *Consider $F(x) = \mu_\infty((-\infty, x])$, the (random) cumulative distribution function of μ_∞. Then F is almost surely Hölder–continuous for every exponent $\gamma < (1/2) \wedge (\beta/3)$. In particular, μ_∞ has no atoms almost surely.*

2. *Consider $F^K(x) = \mu^K_\infty((-\infty, x])$, the (random) cumulative distribution function of μ^K_∞. Then F^K is almost surely Hölder–continuous for every exponent $\gamma < 1 \wedge 2\beta$.*

Proof. The key idea of this proof is to look at $(x \mapsto F(x))$ as a stochastic process to which we can apply Kolmogorov's continuity theorem. By Proposition 4.4.3 we have that for ε small enough, uniformly in x,

$$\mathbb{E}\left[|F(x+\varepsilon) - F(x)|^2\right] \leq C\varepsilon^{2 \wedge (1+\beta/3)}.$$

For $\beta = 3$ we use $\varepsilon^{2-\delta}$ for arbitrarily small $\delta > 0$. It now follows immediately from Kolmogorov's continuity theorem (see for example [61, Theorem 4.23]) that F is almost surely continuous because F is non-decreasing and has càdlàg paths. Further, F is Hölder continuous for any $\gamma < (1/2) \wedge (\beta/3)$.

We can improve on the bound on γ if we consider F^K. The same reasoning applies but if we use Lemma 4.4.4 instead of Proposition 4.4.3 then Kolmogorov's continuity theorem provides us with Hölder continuity for any γ with

$$\gamma < \frac{k-1}{k} \wedge \frac{2\beta(k-1)-\delta}{k}.$$

Because $k \geq 2$ and $\delta > 0$ are arbitrary, we have Hölder continuity for any $\gamma < 1 \wedge 2\beta$. $\square$

We also complete the proof of Theorem 4.1.1 by showing a lower bound on the Hausdorff dimension.

Corollary 4.4.6. *For any $\beta > 0$, we almost surely have that* $\dim \operatorname{supp} \mu_\infty \geq 2\beta \wedge 1$.

Proof. By Proposition 4.3.1, we have that for any K

$$\operatorname{supp} \mu_\infty^K \subseteq \operatorname{supp} \mu_\infty,$$

and hence

$$\dim \operatorname{supp} \mu_\infty \geq \dim \operatorname{supp} \mu_\infty^K.$$

Choose K large enough so that μ_∞^K is a non-trivial measure. This is almost surely possible. Let F^K be the cumulative distribution function for μ_∞^K as in Corollary 4.4.5. It is a basic fact of Hausdorff dimension that F^K being Hölder continuous with exponent γ implies that

$$\dim \operatorname{supp} \mu_\infty^K \geq \gamma.$$

This is essentially a consequence of Frostman's Lemma, Lemma 4.2.5, which is called the mass distribution principle [45, Principle 4.2]. This completes the proof as we can choose any $\gamma < 2\beta \wedge 1$ by Corollary 4.4.5. $\square$

Lastly, we prove Theorem 4.1.2 (iii).

Corollary 4.4.7. *If $\beta > 1/2$, then μ_∞ almost surely has a density with respect to the Lebesgue measure.*

Proof. We first show that for any K, μ_∞^K has a density with respect to the Lebesgue measure. In the regime $\beta > 1/2$, we can choose δ small enough so that we have $1 + 2\beta - \delta \geq 2$, hence Proposition 4.4.3 states

$$\sup_{|I|=\varepsilon} \mathbb{E}\left[\mu_\infty^K(I)^2\right] \leq C\varepsilon^2, \tag{4.16}$$

uniformly in ε small.

We construct a density of μ_∞^K by approximations. Let $R, n \in \mathbb{N}$. Define

$$\rho_n^{K,R}(x) = \sum_{k=-nR}^{nR-1} \frac{\mu_\infty^K([k/n,(k+1)/n))}{1/n} 1_{\{x\in[k/n,(k+1)/n)\}}.$$

We think of $\rho_n^{K,R}$ as an approximation to the density of μ_∞^K on $[-R, R]$. We compute the expected L^2 norm of $\rho_n^{K,R}$,

$$\mathbb{E}\left[\|\rho_n^{K,R}\|_2^2\right] = \sum_{k=-nR}^{nR-1} \frac{1}{n}\mathbb{E}\left[\frac{\mu_\infty^K([k/n,(k+1)/n))^2}{1/n^2}\right] \leq \sum_{k=-nR}^{nR-1} \frac{C}{n} = 2RC,$$

where we used (4.16). Note that this bound is uniform in n. This also means that for every $L > 0$, by Markov's inequality,

$$\mathbb{P}\left(\|\rho_n^{K,R}\|_2 > L\right) \leq \frac{(2RC)^2}{L^2} \xrightarrow{L\to 0} 0. \tag{4.17}$$

By the Banach–Alaoglu theorem, sets of the form $\{f \in L^2([-R, R]) : \|f\|_2 \leq L\}$ are compact in the weak topology. This means by (4.17) the sequence $(\rho_n^{K,R})_n$ is tight in $L^2([-R, R])$ and by Prokhorov's theorem there exists a weakly convergent subsequence, call its limit $\rho^{K,R}$.

On the other hand, let $\mu_\infty^{R,n}$ be the measure induced by the density $\rho_n^{K,R}$. Clearly, $\mu_\infty^{K,R,n}$ converges almost surely weakly to $\mu_\infty^K|_{[-R,R]}$. Hence $\rho^{K,R}$ is a density for $\mu^K|_{[-R,R]}$. As $R \to \infty$, $\mu_\infty^K|_{[-R,R]}$ converges weakly to μ_∞^K. By a diagonal argument one can see that $\rho^{K,R}$ converges weakly to a function ρ^K which is a density for μ^K.

We now turn to μ_∞. For $K < K'$ we have that $\rho^K \leq \rho^{K'}$ almost everywhere because $\mu_\infty^K \leq \mu_\infty^{K'}$ almost surely by Proposition 4.3.1. Define now $\rho = \lim_{K\to\infty} \rho^K$ taken along the sequence $K \in \mathbb{N}$, by monotonicity this limit exists almost everywhere and $\rho < \infty$ almost everywhere. Because μ_∞^K converges weakly to μ_∞ as $K \to \infty$ by Proposition 4.3.1, we get that ρ is a density for μ_∞. □

4.5 Open questions

Question 4.5.1. For $\beta = 1/2$, show that μ_∞ does not admit a density with respect to the Lebesgue measure. This is because we believe Proposition 4.4.3 to be sharp, that is we cannot achieve the exponent ε^2.

Question 4.5.2. In Proposition 4.3.1 we have shown that μ_∞ is determined by particles that satisfy $Y_u(t) \approx e^{-t/2}$. It would be interesting look at the empirical measure of particles that satisfy $Y_u(t) \approx e^{-t/2+\lambda t}$. This should be non–trivial for any λ with $|\lambda| < \sqrt{2\beta}$. In particular, if you look at $\lambda = 1/2$ there should be a phase transition when $\beta = 1/8$ because this is the threshold for local survival. For BBM on $\mathbb{R}$, the number of particles at a given speed is counted by the additive martingale; note that here in the hyperbolic setting

$$\sum_{u \in \mathcal{N}(t)} (Y_u(t))^\lambda e^{-t(\lambda^2+\lambda-2)/2}$$

has the same distribution as the regular additive martingale.

Question 4.5.3. It would be interesting to study Λ and supp μ_∞ in the presence of a drift: assume that there is a drift of strength $\lambda \in \mathbb{R}$ away from the origin. If $\lambda > -1/2$, μ_∞ should still be supported on the boundary. For $\lambda < -1/2$, we should almost surely have that μ_∞ is a Dirac mass at the origin, and for $\lambda = -1/2$, μ_t should not converge.

If we were to consider drift away from a point $\zeta \in \partial\mathbb{D}$, the analysis becomes easy. By isometry, we can choose $\zeta = 1$, which corresponds to the unique boundary point ∞ at infinity in $\partial\mathbb{H}$. Geodesics going through ∞ in $\mathbb{H}$ are straight vertical lines. This means drift away from ∞ is a simple vertical drift of λ (weighted by the hyperbolic metric). This means that the calculations in this work and in [67] still apply and we should get for $\lambda > -1/2$,

$$\dim \operatorname{supp} \mu_\infty(\lambda) = [2\beta/(2\lambda+1)] \wedge 1,$$

and

$$\dim \Lambda(\lambda) = \begin{cases} \frac{1}{2}\left(1 + 2\lambda - \sqrt{(1+2\lambda)^2 - 8\beta}\right) & \text{for } \beta \le \frac{(1+2\lambda)^2}{8}, \\ 1 & \text{else.} \end{cases}$$

We believe the same expressions should still hold for drift away from the origin rather than from the boundary. The reason for this is that for $z \in \mathbb{D}$ we can replace drift away from the origin with drift away from $-\frac{z}{|z|}$. If we start a hyperbolic Brownian motion Z_t at z where z is far away from the origin, then $-\frac{Z_t}{|Z_t|}$ should not vary much, hence we can replace drift away from the origin with drift away from a boundary point.

Question 4.5.4. In this paper, the underlying stochastic process from which we build the branching process is continuous in time and space. It is also natural to consider a discrete setting, that is a branching random walk on a hyperbolic group. Let Γ be a non–elementary hyperbolic group, generated by a finite set S. Given a probability measure ν on S with $\operatorname{supp}(\nu) = S$, we can then construct a random walk on Γ by setting $p(x, y) = \nu(x^{-1}y)$, and hence a branching random walk (BRW) with branching rate $\beta > 0$. Let $|x|$ be the norm of $x \in \Gamma$ in the word metric induced by S. Then for the random walk induced by ν, $(X_n, n \geq 0)$, there are $\sigma^2, v > 0$ such that

$$\frac{|X_n|}{n} \xrightarrow[n\to\infty]{a.s.} v \qquad \text{and} \qquad \frac{|X_n| - nv}{\sqrt{\sigma^2 n}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1),$$

see for example [23] and the references therein. That is, $(|X_n|)_{n\geq 0}$ satisfies a strong law of large numbers and a central limit theorem. Like the hyperbolic plane $\mathbb{H}$, Γ can be endowed with a natural boundary $\partial\Gamma$ with metric d_a where $a > 1$ is the visual parameter. In this setting, we can again consider Λ, the set of accumulation points of the BRW on $\partial\Gamma$, and μ_∞, the limit of the empirical measure. There have been multiple works studying Λ and its dimension, for example [82], but none studying μ_∞. We believe that the methods from this paper should transfer easily to this setting, in particular because the LLN and CLT guarantee that an equivalent of Proposition 4.3.1 still holds true. We believe the following should be true.

Conjecture 4.5.5. *In this setting,* $\dim \operatorname{supp} \mu_\infty = \left(\frac{\beta}{v \log(a)}\right) \wedge \dim \partial\Gamma$ *almost surely.*

Note that $\dim \partial\Gamma = \frac{\delta}{\log a}$ where δ is the exponential growth rate of the volume of Γ. This would be in contrast to the complicated expressions for $\dim \Lambda$, for example determined by [82] and [59].

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