SIAM J. COMPUT.
Vol. 54, No. 4, pp. 1021–1064

ON THE NISAN–RONEN CONJECTURE FOR SUBMODULAR VALUATIONS*

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Abstract. We consider mechanisms for scheduling n unrelated machines when the valuations of the players (i.e., machines) are submodular. We give a lower bound of $\Omega(\sqrt{n})$ on the approximation ratio of incentive compatible deterministic mechanisms. This lower bound holds for supermodular valuations and also when all players, except for one, have additive valuations. This is an information-theoretic impossibility result on the approximation ratio of mechanisms that provides strong evidence for the Nisan–Ronen conjecture, which states that the approximation ratio is n when the valuations of all machines are additive. Our approach is based on a novel multiplayer characterization of appropriately selected instances that allows us to focus on a particular type of algorithm, linear mechanisms, and it is a potential stepping stone towards the full resolution of the Nisan–Ronen conjecture.

Key words. algorithmic mechanism design, algorithmic game theory, truthfulness

MSC code. 91B03

DOI. 10.1137/22M1483141

1. Introduction.

The design of protocols that provide appropriate incentives to participants, entice them to cooperate, and behave in a way that is socially beneficial has a long and celebrated history.[1] It is the realm of mechanism design that is one of the most researched branches of game theory and microeconomics. It studies the design of algorithms, called *mechanisms,* and it has numerous applications to many situations in modern societies, whenever a protocol of conduct of selfish participants is required. The mechanism asks each participant to bid their preferences over the different social outcomes and implements one of them (e.g., the one that is most socially beneficial). The challenge is that the preferences of the participants are *private,* and they are either unmotivated to report them correctly or strongly motivated to report them erroneously, if a false report is profitable. An incentive compatible (a.k.a. truthful) mechanism provides incentives in such a way that it is in the best interest of each participant to bid *truthfully.*

The algorithmic nature of mechanism design and the associated computational issues were brought to light in the seminal 20-year-old paper by Nisan and Ronen [36], which essentially established the area of algorithmic mechanism design. They proposed the scheduling problem on unrelated machines, a fundamental, extensively studied from the algorithmic perspective, optimization problem as a representative

*Received by the editors March 8, 2022; accepted for publication (in revised form) May 2, 2025; published electronically August 11, 2025.
https://doi.org/10.1137/22M1483141
Funding: This work has been partially supported by project MIS 5154714 of the National Recovery and Resilience Plan Greece 2.0 funded by the European Union under the NextGenerationEU Program.
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[1] An extended abstract of this work appeared in [9].

specimen to study the limitations of truthful mechanisms. The objective is to incentivize n machines to execute m tasks, so that the maximum completion time of the machines, i.e., the makespan, is minimized. Traditional algorithms, such as the best-known approximation algorithm [30], are not incentive-compatible.

Nisan and Ronen applied the famous Vickrey–Clarke–Groves (VCG) mechanism [39, 15, 26], which is a general machinery that truthfully computes the outcome that maximizes the *social welfare,* which for the case of scheduling is the allocation that minimizes the sum of completion times. The VCG is truthful and polynomial-time for scheduling, but with respect to the makespan minimization it has a rather poor approximation ratio, equal to the number of machines n. Despite this, they conjectured that the VCG is the mechanism with the best approximation ratio for this problem.

CONJECTURE 1. *There is no deterministic truthful mechanism with approximation ratio better than n for the problem of scheduling n unrelated machines.*

The bound in the conjecture is information-theoretic, in the sense that it should hold for all mechanisms (algorithms), polynomial-time or not. The Nisan–Ronen conjecture has developed into one of the central problems in algorithmic game theory, and despite intensive efforts, very sparse progress has been made towards its resolution. The original Nisan–Ronen paper showed that no truthful deterministic mechanism can achieve an approximation ratio better than 2. This was improved to 2.41 [13], and later to 2.61 [27], which leaves a huge gap[2] with the known upper bound of n. The most general and interesting result, by Ashlagi et al. [3], resolved a restricted version of the conjecture, for the special yet natural class of *anonymous* mechanisms. However, the original conjecture (for nonanonymous mechanisms) is widely open.

The original conjecture was posed for the case of additive valuations, where the total cost of each machine equals the sum of its individual costs of each task. However, most mechanism design settings consider more general valuations such as submodular, subadditive, etc. In these valuations, the cost of a machine that takes a set of tasks S is a function of S that satisfies some natural properties. For example, for subadditive valuations the cost of a bundle of tasks S can be any value bounded above by the sum of the costs of the individual tasks in S.

It is natural to pose the Nisan–Ronen question for these extended valuation classes. Despite the importance of the problem, the conjecture has been widely open for all such valuations. The VCG mechanism provides a trivial n-approximation for these classes of valuations, while the best known lower bounds are still the aforementioned constants. In this work, we focus on *submodular cost functions,* where the marginal contribution of a task to the total cost of a machine is a nonincreasing function. The class of submodular valuations contains all additive valuations, it is a proper subset of subadditive valuations, and it is one of the most restrictive natural classes of valuations for the scheduling question. In the corresponding maximization problem of combinatorial auctions, submodular valuations is the most studied class in algorithmic mechanism design (see, for example, [29, 22, 18]). Our results hold also for supermodular valuations.

1.1. Our result. We give the first nonconstant lower bound that works for *all* deterministic truthful mechanisms when the cost functions are submodular. Our lower

[2]Since the conference version of this work [9], there was further improvement of the lower bound to 2.755 by Giannakopoulos, Hamerl, and Poças [24], to 3 by Dobzinski and Shaulker [20], and, more recently, to $1 + \sqrt{n-1}$ [11] by the authors of the current paper.

bound assumes *no restriction on the mechanism side,* but an expanded class of valuations. The Nisan–Ronen conjecture has been shown to hold for only *special classes of mechanisms* (local [36], strongly monotone [35], and anonymous [3] mechanisms).

The importance of the Nisan–Ronen conjecture is that it captures in a crisp way the difficulties that incentive compatibility adds to algorithm theory. It is a happy coincidence for the development of algorithmic mechanism design that the optimization problem they chose to study, the unrelated machines scheduling problem, turned out to be the most challenging problem in the area. This work is one of the few that convincingly verify the motivating assumption of Nisan and Ronen that incentive compatibility adds insurmountable difficulties for minmax optimization. In order to establish our result, we develop new techniques, with wider applicability:[3] a partial multiplayer characterization and lower bounds for linear mechanisms (section 4.3).

THEOREM 1. *There is no deterministic truthful mechanism with approximation ratio better than $\sqrt{n-1}$ for the problem of scheduling n unrelated machines with submodular cost functions.*

Actually we show a stronger result: Let $\mathcal{M}$ be the class of scheduling mechanisms which are deterministic and truthful *when all machines are additive except for one machine, which is submodular.* Then no mechanism in $\mathcal{M}$ has an approximation ratio better than $\sqrt{n-1}$ on the set of instances of scheduling unrelated additive machines.

Other valuation classes. Our results hold also for *supermodular* valuations. There is no difficulty in translating, and in fact carbon-copying the proof of the characterization theorem (see the next subsection) for this case. Interestingly, the class of additive valuations of the original Nisan–Ronen conjecture is exactly the intersection of submodular and supermodular valuations.

Actually, we prove a stronger version of Theorem 1 by considering submodular (or supermodular, or any superclass of these) functions which are also ϵ-additive, in the sense that the execution time of a set of tasks is within an arbitrarily small ϵ from the sum of the execution times of its tasks (see section 3 for precise definitions and for proofs).

It remains an open problem to close the gap between this lower bound of $\sqrt{n-1}$ and the best known upper bound n.

1.2. Overview of the techniques. We provide an overview of our approach for the lower bound. We consider instances in which every task has a fixed large value, which is practically infinite, for all except for two machines; one of the two machines is always the submodular player (player 0), and the other is an additive player. We can assume that each task is allocated to one of its two machines with the non-“infinite” values; otherwise the approximation ratio is sufficiently high. Such restrictions of the allocation to only two players per task have been previously used (e.g., [13, 27]). The main difference from previous approaches is that we use properties of mechanisms that involve at least three players, the submodular player and two other players. Obtaining such multiplayer statements is the bottleneck for a complete characterization of mechanisms in multiplayer domains.

Two-player characterization. We first focus on the tasks that can be allocated to a particular additive player (or to player 0) and fix the values of the remaining tasks. For each additive player, there are only two such tasks, and the situation is very similar to the two-player and two-task special case in which the valuation of one

[3]Indeed, our follow-up work [11] builds on some of these techniques to establish a lower bound of $1+\sqrt{n-1}$ for additive valuations.

player is submodular. A core part of our proof, which may be of independent interest, is a characterization of the allocation functions of all truthful mechanisms for this case. We provide a complete characterization for the case of two players with nonnegative, submodular valuations, which, for one player, are bounded above by a constant. The latter is essential in our construction to guarantee that the fixed large value of the other machines does not play any role. Since this is a minimization problem, it is the lower bound (i.e., the restriction that the values are nonnegative) that creates complications rather than the upper bound on the domain. We also provide a characterization for additive valuations, the actual scheduling domain. Although this part of the characterization is not needed for the lower bound of this work, it is an essential ingredient of our follow-up work on the Nisan–Ronen conjecture [11].

We note that similar two-player characterizations have been provided by previous work [21, 12], for auctions and scheduling domains, but none of these can be used in our approach. In particular, the characterization of [21] for scheduling relies extensively on the bounded approximation for two players, but we need a characterization without this assumption. The reason is that the approximation ratio of any two players is in general unrelated to the approximation ratio of the whole multiplayer instance. In the same work, a characterization for auctions with subadditive valuations is provided, but this is also of no use to the minimization we consider here. Finally, the characterization of [12] cannot be used because it allows negative values.

Indeed, as we show in section 3, the scheduling domain admits truthful mechanisms not present in the previous characterizations of [21, 12], which we call *relaxed affine minimizers.* Such mechanisms are essentially affine minimizers for large values, but for small values they can have *nonlinear* boundaries (see Definitions 6 and 7 and Figure 2). We summarize here the characterization result; see Theorem 4 for the precise statement.

Theorem 2 (informal). *Every truthful allocation for two tasks and two players where one player is submodular and the other is additive and bounded from above, and both tasks are always allocated, is one of these three types:* (1) *relaxed affine minimizer,* (2) *one-dimensional mechanism, or* (3) *constant mechanism.*

Gluing two-player mechanisms to multiplayer linear mechanisms. From the characterization of two-player two-task mechanisms and using the fact that we are interested in mechanisms with small approximation ratio for the whole multiplayer instances, we are able to exclude all mechanisms except of those that have affine boundaries (roughly, critical values/payments linear in the other player's cost for the task; see Definition 18 and Lemma 16), that is, affine minimizers *whose coefficients may depend on the values of the other tasks.*

One of the main technical steps is that we use the truthfulness of the submodular player to show that the scaling coefficient of these linear boundary functions does not actually depend on the values of the other tasks (Lemma 17). The rest of the proof, which includes the most complicated technical steps of this work, is to analyze the properties of truthful linear allocation algorithms that facilitate the proof of the lower bound (Lemma 19).

Organization of the paper. After the preliminaries (section 2), the types of mechanisms appearing in the characterization (Theorem 4) are introduced and their truthfulness is shown in section 3. Based on Theorem 4, we prove the lower bound result in section 4.

1.3. Related work. The problem of scheduling unrelated machines is a typical multidimensional mechanism design problem. In multidimensional mechanism design, the valuation of each player for different outcomes is expressed via a vector (one value for every outcome). In the case of unrelated scheduling, this vector expresses the processing times of a machine for each subset of tasks and can be succinctly represented by an m-valued vector, one value for each task.

An interesting special case, which is well understood, is the single-dimensional mechanism design in which the values of the vector are expressions of a single parameter. The principal representative is the problem of scheduling *related* machines, where the cost of each machine can be expressed via a single parameter, its *speed.* This was first studied by Archer and Tardos [2], who showed that, in contrast to the unrelated machines version, an algorithm that minimizes the makespan can be truthfully implemented—albeit in exponential time. It was subsequently shown that truthfulness has essentially no impact on the computational complexity of the problem. Specifically, a randomized truthful-in-expectation[4] PTAS was given in [17] and a deterministic PTAS was given in [14, 23]; a PTAS is the best possible algorithm even for the pure algorithmic problem (unless $P = NP$).

The main obstacle in resolving the Nisan–Ronen conjecture is the lack of clear algorithmic understanding of truthfulness for many players in multidimensional domains. In contrast, we understand better truthfulness for a single player. Saks and Yu [38] gave a nice, complete characterization of deterministic truthful mechanisms for convex domains which was later extended to truthful-in-expectation randomized mechanisms in [1]. This characterization states that the class of allocations of truthful mechanisms is the class of *weakly monotone* algorithms [5]. This is an elegant characterization, but it has not been proved very useful for mechanisms of many players, because it is difficult to combine monotonicity of each individual player into a single global condition. Its direct applications have provided only constant lower bounds for makespan minimization [13, 35, 27, 36]. What would be more useful is a characterization similar to the one provided by the seminal work of Roberts [37] for *unrestricted* domains. It essentially states that the only truthful mechanisms are affine extensions of VCG. Similar characterizations have been provided in [12, 21, 19] for settings with only two players. Extending these characterizations to multiple players for scheduling and combinatorial auctions is notoriously hard, mainly due to the lack of externalities in these settings: the valuation of a player for an allocation depends only on the subset of tasks it receives and is indifferent on how the remaining tasks are assigned to the other players.[5]

Scheduling is related to combinatorial auctions, where multiple items need to be assigned to a set of buyers. This is a broad and successful area, and the setting shares both aforementioned features of multidimensionality and lack of externalities; therefore insights and techniques can be transferred from one problem to the other. However the difference is that the objective for combinatorial auctions is *social welfare maximization,* and this is known to be achieved by the VCG mechanism, albeit in exponential time. Hence the focus of this rich area is on what can be achieved by computationally efficient mechanisms (see, for example, [22]). But in the case of the scheduling with the min-max objective, the flavor is more information-theoretic, as

[4]This is one of the two main definitions of truthfulness for randomized mechanisms, where truth-telling maximizes the expected utility of each player.

[5]For two players, there exist implicit externalities as the tasks one player doesn't get determine what the other player gets.

we know that *not even exponential time* truthful mechanisms can achieve the optimal makespan.

1.3.1. Further related work. Lavi and Swamy [28] proposed an interesting approach to attack the Nisan–Ronen question, by restricting the input domain, but still kept the multidimensional flavor of the setting. They assumed that each entry in the input matrix can take only two possible values, "low" and "high," that are publicly known to the designer. In this case, they showed an elegant deterministic mechanism with an approximation factor of 2. Surprisingly, even for this special case there is a lower bound of 11/10. Yu [41] extended the results for a range of values, and Auletta et al. [4] studied multidimensional domains where the private information of the machines is a single bit.

Randomization has been explored and has slightly improved the known guarantees. There are two notions of truthfulness for randomized mechanisms. A mechanism is *universally truthful* if it is defined as a probability distribution over deterministic truthful mechanisms, while it is *truthful-in-expectation* if in expectation no player can benefit by lying. In [36], a universally truthful mechanism was proposed for the case of two machines, and was later extended to the case of n machines by Mu'alem and Schapira [35] with an approximation guarantee of $0.875n$, which was later improved to $0.837n$ by [33]. Lu and Yu [34] showed a truthful-in-expectation mechanism with an approximation guarantee of $(m+5)/2$. Mu'alem and Schapira [35] showed a lower bound of $2-1/m$ for both notions of randomization. Christodoulou, Koutsoupias, and Kovács [8] extended the lower bound for fractional mechanisms, where each task can be fractionally allocated to multiple machines. They also showed a fractional mechanism with a guarantee of $(m+1)/2$. A sequence of papers studied randomized mechanisms for the special case of two machines [32, 34] where a tight answer on the approximation factor is still unresolved. Currently, the best upper bound is 1.587 due to Chen, Du, and Zuluaga [7].

Truthful implementations of other objectives have been explored by Mu'alem and Schapira [35] for multidimensional problems and by Epstein, Levin, and van Stee [23] for single-dimensional ones, giving a PTAS for a wide range of objective functions. Leucci, Mamageishvili, and Penna [31] showed high lower bounds for other min-max objectives on some combinatorial optimization problems. Christodoulou, Koutsoupias, and Kovács [10] consider truthful mechanisms for graph settings where each task can only be assigned to two players. In the Bayesian setting, Daskalakis and Weinberg [16] showed a mechanism that is at most a factor of 2 from the *optimal truthful mechanism,* but not with respect to optimal makespan. Chawla et al. [6] provided bounds of prior-independent mechanisms (where the input comes from a probability unknown to the mechanism). Giannakopoulos and Kyropoulou [25] showed that the VCG mechanism achieves an approximation ratio of $O(\log n/\log\log n)$ under some distributional and symmetry assumptions.

2. Preliminaries.

There is a set N of n machines and a set M of m tasks that need to be scheduled on the machines. The processing time or cost that each machine i takes to process a subset S of tasks is described by a set function $t_i : 2^M \to \mathbb{R}_{\geq 0}$. In classic unrelated machines scheduling, and in the Nisan–Ronen model, the cost functions (valuations) are additive and the objective is to minimize the *makespan* (min-max objective). In our lower bound construction, we will assume that all cost functions are additive except for one machine, for which we consider more general normalized and monotone cost functions:[6] We mainly focus on *submodular* cost functions, which satisfy the following condition for every $S,T \subseteq M$:

$$t_i(S \cup T) + t_i(S \cap T) \leq t_i(S) + t_i(T).$$

In case of *supermodular* functions the inequality is reversed. We also consider valuations which are arbitrarily close to additive, which we call ϵ-additive, such that for every subset S, $\sum_{j\in S}(t_i(\{j\})) - \epsilon \leq t_i(S) \leq \sum_{j\in S}(t_i(\{j\})) + \epsilon$.[7]

We will assume that all cost functions are additive, except for one which is submodular, or ϵ-additive, or both submodular and ϵ-additive. The results carry over to supermodular valuations and classes of valuations that include these, like subadditive, superadditive, or arbitrary (normalized, monotone) valuations.

Mechanism design setting. We assume that each machine $i \in N$ is controlled by a selfish agent (player) that is reluctant to process the tasks and the cost function t_i is private information known only to her (also called the *type* of agent i). In the most general version of the problem, the set $\mathcal{T}_i$ of possible types of agent i consists of all vectors $b_i \in \mathbb{R}_+^{2^m}$. Let also $\mathcal{T} = \times_{i\in N}\mathcal{T}_i$ be the space of type profiles.

A mechanism defines for each player i a set $\mathcal{B}_i$ of available strategies the player can choose from. We will consider *direct revelation* mechanisms, i.e., $\mathcal{B}_i = \mathcal{T}_i$ for all i, meaning that the players' strategies are to simply report their types to the mechanism. A player may report a false cost function $b_i \neq t_i$ if this serves her interests. A mechanism (A,P) consists of two parts:

An allocation algorithm: The allocation algorithm A allocates the jobs to the machines depending on the players' bids $b = (b_1,\dots,b_n)$. Let $\mathcal{A}$ be the set of all possible partitions of m tasks to n machines. The allocation function $A : \mathcal{T} \to \mathcal{A}$ partitions the tasks into the n machines; we denote by $A_i(b)$ the subset of tasks assigned to machine i for bid profile b.

A payment scheme: The payment scheme $P = (P_1,\dots,P_n)$ determines the payments also depending on the bid values b. The functions $P_1,\dots,P_n$ stand for the payments that the mechanism hands to each agent, i.e., $P_i : \mathcal{T} \to \mathbb{R}$.

The *utility* u_i of a player i is the payment that he gets minus the *actual* time that he needs to process the set of tasks assigned to her, $u_i(b) = P_i(b) - t_i(A_i(b))$. We are interested in *truthful* mechanisms. A mechanism is truthful if for every player, reporting his true type is a *dominant strategy.* Formally,

$$u_i(t_i,b_{-i}) \geq u_i(t'_i,b_{-i}) \qquad \forall i \in [n],\ t_i,t'_i \in \mathcal{T}_i,\ b_{-i} \in \mathcal{T}_{-i},$$

where $\mathcal{T}_{-i}$ denotes the possible types of all players disregarding i.

We use as an objective to evaluate the performance of a mechanism's allocation algorithm the makespan, that is, the maximum completion time over all machines. The makespan of the allocation algorithm A with respect to a given input t is

$$Mech(t) \stackrel{\text{def}}{=} \max_{i\in N} t_i(A_i(t)).$$

We aim at minimizing the makespan; hence the optimum is

[6]A cost function is *normalized* if $t_i(\emptyset) = 0$ and *monotone* if $t_i(S) \leq t_i(T)$ for $S \subseteq T$.

[7]We are free to use any reasonable definition of being close to additive; e.g., we could have chosen $(1-\epsilon)\sum_{j\in S}(t_i(\{j\})) \leq t_i(S) \leq (1+\epsilon)\sum_{j\in S}(t_i(\{j\}))$, or even the intersection of the latter with the current (additive) ϵ-neighborhood.

$$Opt(t) = \min_{A \in \mathcal{A}} \max_{i \in N} t_i(A_i).$$

We are interested in the approximation ratio of the mechanism's allocation algorithm. A mechanism M is *c-approximate* if the allocation algorithm is c-approximate, that is, if $c \geq \frac{Mech(t)}{Opt(t)}$ for all possible inputs t.

We are looking for truthful mechanisms with low approximation ratio irrespective of the running time to compute A and P. In other words, our lower bounds do not make use of any computational assumptions.

Weak monotonicity. A useful characterization of truthful mechanisms in terms of the following monotonicity condition helps us to get rid of the payments and focus on the properties of the allocation algorithm.

DEFINITION 1. *An allocation algorithm A is called* weakly monotone *(WMON) if it satisfies the following property: for every two inputs $t = (t_i, t_{-i})$ and $t' = (t'_i, t_{-i})$, the associated allocations A and A' satisfy*

$$t_i(A_i) - t_i(A'_i) \leq t'_i(A_i) - t'_i(A'_i).$$

It is well known that the allocation function of every truthful mechanism is WMON [5], and also that this is a sufficient condition for truthfulness in convex domains (all our considered domains are convex) [38].

A useful tool in our proof relies on the following immediate consequence of WMON, which holds in additive domains as well as in the other domains that we consider. Intuitively, it states that when you fix the values of all players for a subset of tasks (focus on a cut of your domain), the *restriction* of the allocation to the rest of the tasks must remain weakly monotone.

LEMMA 1. *Let A be a WMON allocation. Let us fix an (S, T) partition of M and consider only valuations t_i of player i that are additive across S and T, i.e., for every $X \subseteq M$, $t_i(X) = t_i(X \cap S) + t_i(X \cap T)$. Then the restriction of the allocation A on S is weakly monotone for each valuation fixed on the subsets of T.*

Proof. A is weakly monotone; therefore

$$t_i(A_i) - t'_i(A_i) \leq t_i(A'_i) - t'_i(A'_i),$$

and additivity across S and T implies

$$t_i(A_i \cap S) - t'_i(A_i \cap S) \leq t_i(A'_i \cap S) - t'_i(A'_i \cap S),$$

as the values of subsets of T are fixed, $t_i(A_i \cap T) = t'_i(A_i \cap T)$ and $t_i(A'{}_i \cap T) = t'_i(A'{}_i \cap T)$. ∎

The following lemma was essentially shown in [36] and has been a useful tool to show lower bounds for truthful mechanisms for several variants (see, for example, [13, 35, 3]). Although this holds more generally, we only state it (and use it in section 4) for additive valuations.

LEMMA 2. *Let t be a bid vector of additive valuations, and let $S = A_i(t)$ be the subset assigned to player i by a truthful mechanism (A, P). For any bid vector $t' = (t'_i, t_{-i})$ such that only the bid of machine i has changed and in such a way that for every task in S it has decreased (i.e., $t'_i(\{j\}) < t_i(\{j\}), j \in S$) and for every other task it has increased (i.e., $t'_i(\{j\}) > t_i(\{j\}), j \in M \setminus S$). Then the mechanism does not change the allocation to machine i, i.e., $A_i(t') = A_i(t) = S$.*

The main challenge in multiplayer settings is that the allocation of the other machines may change and the above condition makes no promise about how this can happen.

3. Characterization for two players and two tasks. A core element of our lower bound proof is a characterization of truthful mechanisms for two tasks and two players (called t-player and s-player). We first provide a characterization for additive valuations $t=(t_1,t_2)$ and $s=(s_1,s_2)$, so that both s_1 and s_2 are bounded by an arbitrarily large but fixed value B (Theorem 3). (The theorem also holds over the usual domain with arbitrarily large s and t values. We give here the more general characterization with bounded s bids, because this will be needed for the lower-bound result.[8]) Then we extend it to the case when the t player has submodular valuations $t=(t_1,t_2,t_{12})$ (Theorem 4), which is the main element that we need in the lower bound proof in section 4. We note that both the characterization and the lower bound results hold analogously when the t-player has arbitrarily monotone, ϵ-additive (or submodular *and* ϵ-additive), or supermodular valuations.

3.1. Basic definitions. Let (A,P) be a truthful mechanism, where A is the WMON allocation function, and P denotes the payment function. For input (t,s) the allocation is $A(t,s)$. Since we have only tasks 1 and 2, in $A(t,s)$ we can denote the allocation to one of the players as $\alpha_t,\alpha_s\in\{12,1,2,\emptyset\}$. For simplicity, we often identify $A(t,s)$ by the allocation α_t of the t-player; e.g., we say that the allocation $A(t,s)$ is 12, meaning that $\alpha_t=12$, and $\alpha_s=\emptyset$.

For given $s\in[0,B)\times[0,B)$ the allocation for the t-player as a function of his *own* bids (t_1,t_2) (or (t_1,t_2,t_{12})) is denoted by $A[s]$, and symmetrically $A[t]$ is an allocation function for the s-player. For $\alpha_t\in\{12,1,2,\emptyset\}$, the allocation *regions* $R_{\alpha_t}(s)\subseteq\mathbb{R}^2_{\geq 0}$ (resp., $\mathbb{R}^3_{\geq 0}$) of $A[s]$ are defined to be the *interior*[9] of the set of all t values such that $A(t,s)=\alpha_t$. For the s-player we denote the respective regions by $R^s_{\alpha_s}(t)$.

We assume that for every $s\in[0,B)\times[0,B)$ there exist t values so that the t-player receives no task, i.e., that $R_\emptyset(s)\neq\emptyset$ for every s. This assumption is without loss of generality for mechanisms with finite approximation of the makespan (even if some 2D cut mechanism of a mechanism with more tasks and/or players is considered). Note that we do *not* assume the same for the s-player, because his bids are bounded. We remark that the assumption $R_\emptyset(s)\neq\emptyset$ does not restrict the *types* of possible WMON mechanisms (see the four possible types below), but it simplifies the characterization to a large extent.

3.2. Mechanisms for additive players. It is known that in the case of two tasks, the regions in a WMON allocation subdivide $R^2_{\geq 0}$ basically into three possible forms, which will turn out to be characteristic for the type of the whole allocation function A (see Figure 1). In fact, it is an *equivalent* condition with WMON that $A[t_{-i}]$ has this form of a geometric representation for every fixed i and t_{-i} [12, 40]. The regions and their boundaries determine the *critical value* for t_1 (as function of t_2), the smallest value above which the t-player cannot get task 1, and symmetrically for task 2. These critical value functions are determined by the payment functions $P_\emptyset(s),P_1(s),P_2(s),P_{12}(s)$ for the fixed s: For any given s, we assume w.l.o.g. *virtual*

[8]Although the lower bound of this work does not use the characterization for additive valuation, this has been used in the follow-up work [11], which provides a lower bound for the actual Nisan–Ronen conjecture.

[9]i.e., the topological interior w.r.t. $\mathbb{R}^2_{\geq 0}$ in the additive case, and w.r.t. $\mathbb{R}^3_{\geq 0}$ in the submodular case; these are bids with no ties.

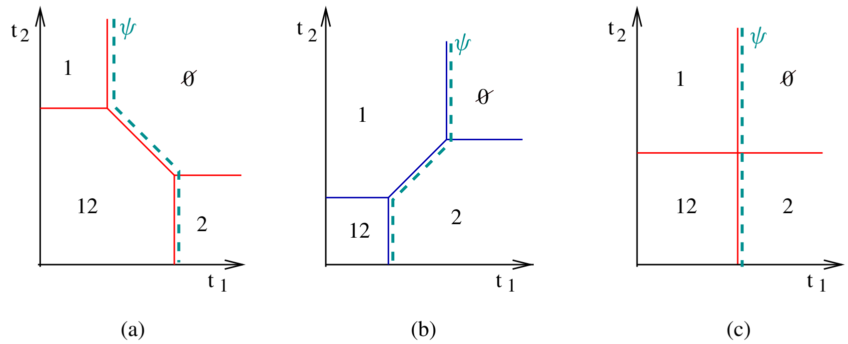

FIG. 1. *The allocation $A[s]$ to the t-player in the additive case, depending on his own bid vector* (t_1, t_2) : (a) *quasi-bundling allocation;* (b) *quasi-flipping allocation;* (c) *crossing allocation. The interiors of some regions might be empty, but $R_\emptyset \neq \emptyset$ can be assumed w.l.o.g. We marked the functions of critical values $\psi(t_2, s)$ (Definition* 17*) for receiving task* 1 *by broken lines.*

payments for the t-player, which are the unique normalized and monotone (as set functions) payments for a given allocation $A[s]$, i.e., payments can be defined in such a way that $0 = P_\emptyset \leq P_1 \leq P_{12}$, and $P_\emptyset \leq P_2 \leq P_{12}$. The main advantage of virtual payments $P_\alpha(s)$ is that they are uniquely determined by the allocation $A[s]$ even if the respective region R_α is empty. They can be considered here as simple technical (geometrical) tools. In the following definition s is fixed and is omitted from the notation.

DEFINITION 2. *Let $(A, \hat{P})$ be a truthful mechanism, with some normalized payment function $\hat{P}$, and let $s = (s_1, s_2)$ be fixed.*[10] *We define the* virtual payments *as*

$$
\begin{aligned}
P_\emptyset &= \hat{P}_\emptyset = 0,\\
P_1 &= \max\{\hat{P}_1, 0\}, \quad P_2 = \max\{\hat{P}_2, 0\},\\
P_{12} &= \max\{\hat{P}_{12}, P_1, P_2\}.
\end{aligned}
$$

The virtual payments are always well defined, nonnegative, and truthful payments to the t-player for the given allocation $A[s]$. We postpone their detailed treatment to the submodular case (section 3.7.1).

For fixed s, up to tie-breaking, the virtual payments determine uniquely the allocation and vice-versa. The value P_1 determines the position of the vertical boundary/critical value between $R_\emptyset$ and R_1. Since $R_\emptyset$ is nonempty, this boundary always exists; if $R_1 = \emptyset$, then $P_1 = 0$. Analogously, P_2 is the position of the horizontal boundary between $R_\emptyset$ and R_2; furthermore, $P_{12} - P_2$ (resp., $P_{12} - P_1$) is the position of the vertical (horizontal) boundary between R_{12} and R_2 (resp., R_{12} and R_1) if these regions are nonempty. For example, the t-player minimizes her total cost (negative utility) by truth-telling, so for all $t \in R_{12}$ it must hold that (say) $t_1 + t_2 - P_{12} \leq t_2 - P_2$, which yields the vertical boundary position $t_1 = P_{12} - P_2$ between R_{12} and R_2, etc.

DEFINITION 3. *For given s, we call the allocation $A[s]$*

- quasi-bundling *if there are at least two points $t \neq t'$ on the boundary of R_{12} and $R_\emptyset$;*

[10]From [38] it is known that since A is WMON, truthful payments $\hat{P}$ exist. Since $R_\emptyset \neq \emptyset$, we can assume w.l.o.g. that $\hat{P}_\emptyset = 0$; otherwise the use of $P_\emptyset = -\infty$ may be needed.

- quasi-flipping *if there are at least two points $t \neq t'$ on the boundary of R_1 and R_2;*
- crossing *otherwise (see Figure 1).*

Obviously, the allocation is quasi-bundling iff $P_1 < P_{12} - P_2$; it is quasi-flipping iff $P_1 > P_{12} - P_2$, and it is crossing iff $P_1 = P_{12} - P_2$.

Next we introduce the four different types of WMON allocations that can occur in the above setting. Observe that there exist mechanisms that adhere to more than one of the given types. After that, in section 3.3 we prove the characterization (Theorem 3).

3.2.1. Relaxed task-independent allocations. An allocation function A is *task-independent* if, for $i \in \{1,2\}$, the allocation of task i depends only on the input values s_i and t_i. For the t-player, the critical value of t_1, i.e., the lowest value, above which t_1 does not get task 1, is determined by an *arbitrary* nondecreasing function $\phi : [0,B) \to [0,\infty)$ of s_1. Analogously we can define the critical-value function $\eta(s_2)$ as the lowest value, depending on s_2, above which t_2 does not get task 2.[11] Geometrically, in a task-independent mechanism, the allocations $A[s]$ and $A[t]$ of both players are always crossing. In a *relaxed task-independent* mechanism the latter property is fulfilled in all but countably many s (resp., t) points, in which both ϕ and η (resp., ϕ^{-1} and η^{-1}) have a jump discontinuity.

DEFINITION 4. *An allocation is a* relaxed task-independent allocation *if there exist arbitrary nondecreasing critical-value functions for the t-player, $\phi(s_1)$ for task 1, and $\eta(s_2)$ for task 2, so that for every $s \in [0,B) \times [0,B)$*

- *if $\phi()$ is continuous in the point s_1* or *$\eta()$ is continuous in the point s_2, then the allocation of the two tasks is independent and adheres to the critical values $\phi(s_1)$ and $\eta(s_2)$, respectively; (note, however, that if, e.g., η has a jump discontinuity in s_2, then the critical value for task 2 can be any value between $\eta(s_2^-)$ and $\eta(s_2^+)$, and this value may depend arbitrarily on s_1);*
- *if $\phi()$ has a jump discontinuity in the point s_1* and *$\eta()$ has a jump discontinuity in s_2, then the allocation $A[s]$ of the t-player can be an arbitrary crossing, quasi-flipping, or quasi-bundling allocation, such that all critical values for task 1 are at least $\phi(s_1^-)$ and at most $\phi(s_1^+)$, and similarly for task 2.*

Symmetric statements hold for the s-player with the critical-value functions $\phi^{-1}(t_1)$ and $\eta^{-1}(t_2)$.[12] *In summary, every relaxed task-independent allocation is identical to a task-independent allocation on $t_i \in [0,\infty) \setminus T_i$, $s_i \in [0,B) \setminus S_i$, where the T_1, T_2, S_1, S_2 are countable sets.*

In (the continuous points of) such a mechanism the payments are additive, and identical to the functions ϕ and η; for example, for the t-player these are $P_1(s) = \phi(s_1)$, $P_2(s) = \eta(s_2)$, and $P_{12}(s) = \phi(s_1) + \eta(s_2)$.

3.2.2. One-dimensional mechanisms. In a *one-dimensional mechanism* at most two possible allocations are ever realized. The critical-value function between the two possible allocations is an arbitrary nondecreasing function of the respective input variables.

[11]The tie-breaking (i.e., when, e.g., $t_1 = \phi(s_1)$) may in all cases depend on the other variables (s_2 and t_2).

[12]Strictly speaking, these are functions from $[0,\infty)$ to $[0,B]$ because $R^s_\emptyset$ might be empty. Note also that where ϕ has a jump, there ϕ^{-1} is constant and vice versa.

Because of the assumption that $R_\emptyset(s) \neq \emptyset$ for every s, one of the occurring allocations of the t-player must be $\emptyset$. If the two occurring allocations (for the t-player) are $\emptyset$ and 12, we call the mechanism a *bundling mechanism.* In the other cases when the allocations to the t-player are $\emptyset$ and 1 (or $\emptyset$ and 2), the mechanism is identical to a degenerate task-independent mechanism, where $\eta \equiv 0$ (resp., $\phi \equiv 0$). Therefore, we define only the case of a bundling mechanism in detail.

DEFINITION 5. *In a* bundling mechanism *only the allocations* $\emptyset$ *and* 12 *can occur. There is an arbitrary, nondecreasing function* $\xi : [0, B) \to [0, \infty)$ *so that if* $t_{12} > \xi(s_1 + s_2)$*, then the t-player gets* $\emptyset$*, and if* $t_{12} < \xi(s_1 + s_2)$*, then the t-player gets* 12.

If ξ *has a jump discontinuity in some point* $s_1 + s_2$*, then the critical value may depend on the particular* (s_1, s_2) *with the given fixed sum and is an arbitrary value between* $\xi((s_1+s_2)^-)$ *and* $\xi((s_1+s_2)^+)$ *(a symmetric statement holds for the s-player if* ξ *is constant and therefore* ξ^{-1} *has a jump discontinuity).*

A bundling mechanism is truthful with the normalized payments $P_\emptyset(s) = P_1(s) = P_2(s) = 0$, and $P_{12}(s) = \xi(s_1 + s_2)$ for the t-player, and analogously, $P^s_{12}(t) = \xi^{-1}(t_1 + t_2)$ for the s-player. For every fixed bid s, the t-player incurs total cost of either $(t_1 + t_2) - P_{12} = (t_1 + t_2) - \xi(s_1 + s_2)$ or 0. Given that she bids t truthfully, her total cost will be $\min\{(t_1 + t_2) - \xi(s_1 + s_2), 0\}$, because the allocation of the mechanism minimizes among these two terms as well. The truthfulness for the s-player is analogous. A bundling mechanism can be considered as a relaxed affine minimizer having $\gamma_1 = \gamma_2 = \infty$ (see the next case).

The geometric figure of a one-dimensional mechanism consists of a boundary which is a single straight line (axis parallel in the task-independent case and slanted in the bundling case).

3.2.3. Relaxed affine minimizers.

DEFINITION 6. *An allocation A is an* affine minimizer *if there exist positive constants per player* μ_t *and* μ_s*, and constants* $\gamma_\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$ *per allocation (say, here* $\alpha = \alpha_t$*), so that for every input* (t, s) *the allocation* $A(t, s)$ *minimizes over*

$$\mu_t(t_1 + t_2) + \gamma_{12}, \quad \mu_t \cdot t_1 + \mu_s \cdot s_2 + \gamma_1, \quad \mu_s \cdot s_1 + \mu_t \cdot t_2 + \gamma_2, \quad \mu_s(s_1 + s_2) + \gamma_\emptyset.$$

Subtracting the last term and dividing by μ_t yields that the payments of the t-player in an affine minimizer can be $P_\emptyset = 0$, $P_1 = \max\{(\mu_s s_1 + \gamma_\emptyset - \gamma_1)/\mu_t, 0\}$, $P_2 = \max\{(\mu_s s_2 + \gamma_\emptyset - \gamma_2)/\mu_t, 0\}$, and $P_{12} = \max\{(\mu_s(s_1 + s_2) + \gamma_\emptyset - \gamma_{12})/\mu_t, P_1, P_2\}$. Then the allocation of the mechanism minimizes precisely the player's total cost if she bids truthfully (and analogously for the s-player).

An affine minimizer can have any type of allocation figure (say, of the t-player), but the figure keeps its shape and just gets translated as s is changed. Figures 1(a) and (b) are therefore characteristic for affine minimizers (they can also occur in singular points of relaxed task-independent mechanisms but not elsewhere).[13]

In some affine minimizers, if $s_1 + s_2$ and $t_1 + t_2$ are both "small," then the mechanism locally "looks like" a bundling mechanism, with only the regions $R_\emptyset$ and R_{12} for both players. If this is the case for *all* (s_1, s_2) and *all* (t_1, t_2) with sums $s_1 + s_2 < D$ and $t_1 + t_2 < C$ for some given $C, D > 0$, then the mechanism becomes locally a bundling mechanism. This is formulated more precisely in the next definition.

[13]The proof of the characterization (Theorem 3) identifies those affine minimizers that happen to be crossing (like VCG) as task-independent allocations.

Definition 7. *An allocation A is a* relaxed affine minimizer *if there exist positive constants per player μ_t and μ_s, constants γ_α per allocation $\alpha = \alpha_t$, and furthermore an arbitrary nondecreasing function $\xi : [0, \min(\gamma_1, \gamma_2) - \gamma_\emptyset] \to [0, \min(\gamma_1, \gamma_2) - \gamma_{12}]$ (assuming both intervals are nonempty) with $\xi(\min(\gamma_1, \gamma_2) - \gamma_\emptyset) = \min(\gamma_1, \gamma_2) - \gamma_{12}$, so that for every input (t, s)*

(1) *if $\mu_s \cdot (s_1 + s_2) \geq \min(\gamma_1, \gamma_2) - \gamma_\emptyset$, then the allocation $A(t, s)$ is that of an affine minimizer with the given constants;*
(2) *if $\mu_s \cdot (s_1 + s_2) \leq \min(\gamma_1, \gamma_2) - \gamma_\emptyset$, then if $\mu_t \cdot (t_1 + t_2) > \xi(\mu_s(s_1 + s_2))$, then the allocation for the t-player is $\emptyset$, and if $\mu_t \cdot (t_1 + t_2) < \xi(\mu_s \cdot (s_1 + s_2))$, then it is* 12.

The payments of the t-player $P_\emptyset(s)$, $P_1(s)$, and $P_2(s)$ are defined as for affine minimizers. The payment $P_{12}(s)$ is the same as in affine minimizers if $\mu_s \cdot (s_1 + s_2) \geq \min(\gamma_1, \gamma_2) - \gamma_\emptyset$, but it is $P_{12} = \xi(\mu_s(s_1 + s_2))/\mu_t$ if $\mu_s \cdot (s_1 + s_2) \leq \min(\gamma_1, \gamma_2) - \gamma_\emptyset$. Observe that in the latter case $P_1 = P_2 = 0$ must hold, and therefore the regions R_1 and R_2 are empty.[14]

The boundary conditions in the above definition are such that the affine minimizer in part (1) fits with the bundling mechanism in part (2), and the resulting mechanism is truthful by the same argument (for fixed s, resp., fixed t) as for bundling mechanisms or for affine minimizers. Furthermore, it is clear that the t-player cannot influence whether case (1) or case (2) holds. On the other hand, we could have defined the mechanism symmetrically, from the point of view of the s-player, so for her the mechanism is truthful as well.[15] See Figure 2 for an example of such a mechanism with $\xi(x) = \sqrt{x}$.

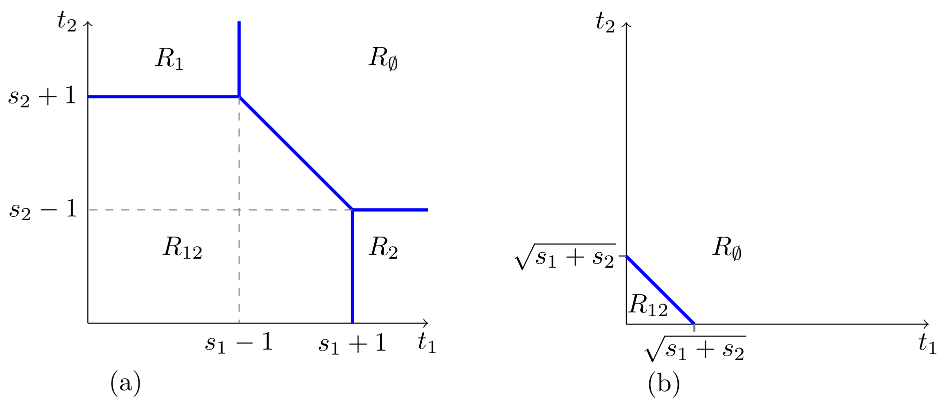

Fig. 2. *An example of a relaxed affine minimizer, which shows the allocation of the t-player for two distinct values of s. Figure* (a) *shows the allocation for some $s_1 \geq 1$ and $s_2 \geq 1$; in case $s_1 + s_2 \geq 1$, the boundaries are linear; for example, when $t_2 \in [s_2 - 1, s_2 + 1]$ the allocation for task* 1 *is determined by comparing t_1 to $\psi_1(t_2, s) = s_1 + s_2 - t_2$. Figure* (b) *shows the allocation at the bundling tail for some $s_1 + s_2 \leq 1$, in which case the boundary does not have to be linear; for example, for $t_2 \in [0, \sqrt{s_1 + s_2}]$, we have $\psi_1(t_2, s) = \sqrt{s_1 + s_2} - t_2$.*

[14]In turn, if $\mu_s \cdot x > \min(\gamma_1, \gamma_2) - \gamma_\emptyset$, then there exist s, so that $x = s_1 + s_2$ but $R_1 \neq \emptyset$ or $R_2 \neq \emptyset$, and this excludes a nonlinear ξ.

[15]The allocation is well defined even if (say) $\mu_s \cdot (s_1 + s_2) \leq \min(\gamma_1, \gamma_2) - \gamma_\emptyset$, but $\mu_t \cdot (t_1 + t_2) > \min(\gamma_1, \gamma_2) - \gamma_{12}$. Namely, for the t-player the allocation seems to be bundling, but $t_1 + t_2$ is large and he gets no task; for the s-player the allocation seems to be an affine minimizer, and $s_1 + s_2$ is so small that s gets both tasks.

3.2.4. Constant mechanisms. In a *constant mechanism* the allocation is independent of the bids of at least one of the players. Due to the assumption on $R_\emptyset$ we only need to consider constant mechanisms that are independent (at least) of the s-player. This property can also be interpreted as being an affine minimizer with multiplicative constant $\mu_s = 0$. Since we are interested in nonconstant critical value functions, it will be convenient to treat constant mechanisms separately from affine minimizers. Geometrically, a constant mechanism can have any type of (constant) allocation figure for the t-player, and for every given t the figure of the s-player consists of a single region.[16]

3.3. Characterization for additive valuations. Now we are ready to present the main characterization result, which provides a characterization of all mechanisms for the case of additive valuations.

THEOREM 3. *Let $B \in \mathbb{R}_+ \cup \{\infty\}$. For every WMON allocation for two tasks and two additive players with bids $t \in [0,\infty)^2$ and $s \in [0,B)^2$, if both tasks are always allocated, and for every s there exists a t high enough so that the t-player receives no task, then the restriction of the allocation to $t \in (0,\infty)^2$ and $s \in (0,B)^2$ is one of these four types:* (1) *relaxed affine minimizer,* (2) *relaxed task-independent mechanism,* (3) *one-dimensional mechanism, or* (4) *constant mechanism.*

Theorem 3 is a core element in the proof of the recent lower bound of $1+\sqrt{n-1}$ in [11] for the Nisan–Ronen problem.

3.4. Proof of Theorem 3. We start the proof by introducing a short notation for the positions of boundaries between the regions (see Figure 3) and making some basic observations about how these boundaries may depend on the bid s in a WMON allocation (Lemmas 3 to 6). After that we can provide some intuition about the proof of the theorem. We note that many of the following claims appeared in [12] in a similar form. An additional challenge here (leading to new mechanisms) was that

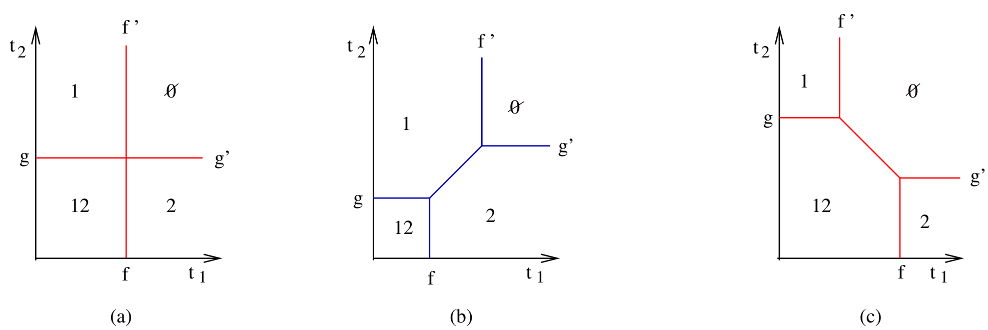

FIG. 3. *Illustration of the proof of Theorem 3: Allocation to the t-player in the* (a) *crossing,* (b) *quasi-flipping, and* (c) *quasi-bundling cases. For given fixed s, based on the virtual payments we will use the notation $f' = P_1$, $g' = P_2$, $f = P_{12} - P_2$, $g = P_{12} - P_1$ whenever these are positions of existing boundaries: f' and g' are always defined; f and g will be undefined, if at least one of their neighboring regions is empty.*

[16]If t is on a boundary, then the tie-breaking might depend on the s-player. For example, if the constant allocation $A = A[s]$ is crossing, and t is in the crossing point, then the allocation $A[t]$ to the s-player can be an arbitrary WMON allocation (this is also a relaxed task-independent allocation). (Also, $A[s]$ can change when $s_1 = 0$ or $s_2 = 0$, as with affine minimizers in general; see section 3.3.)

the domain of t and s values is bounded from below. This slight difficulty could be resolved by a sometimes technical case analysis.

3.4.1. Basic facts.

DEFINITION 8. *For the critical values (boundary positions) of the t-player for given $s \in [0,B) \times [0,B)$ we use the notation $f'(s) = P_1(s)$, $g'(s) = P_2(s)$ (s is omitted from the notation if it is clear from the context).*

Recall that by assumption there always exist high enough t values so that the t-player receives no task, i.e., so that $R_\emptyset(s) \neq \emptyset$ for every s. Therefore, the critical values $f', g' \in [0, \infty)$ are always defined and finite, since $f' = \infty$ or $g' = \infty$ would mean $R_\emptyset = \emptyset$. In particular, $f' = P_1 = 0$ (resp., $g' = 0$) if and only if $R_1 = \emptyset$ (resp., $R_2 = \emptyset$). Thus for every s, and for t high enough, a vertical boundary of $R_\emptyset$ at position f' exists and symmetrically a horizontal boundary at g' exists.

We prove next that f' is an increasing function of s_1, and furthermore it is continuous in almost all (i.e., except for countably many) points $s_1 \in [0,B)$, and in every s_1 where f' is continuous, it is independent of s_2.

LEMMA 3. *f' is monotone increasing in s_1 in the following strong sense: Let s_2, s_2' be arbitrary. If $s_1 < s_1'$, then $f'(s_1, s_2) \leq f'(s_1', s_2')$. A symmetric statement holds for g'.*

Proof. Let $s = (s_1, s_2)$, and $s' = (s_1', s_2')$, and assume for contradiction that $f'(s) > f'(s')$. Then there exists a $\hat{t}$ (with high enough $\hat{t}_2$) so that $\hat{t} \in R_1(s)$ but $\hat{t} \in R_\emptyset(s')$. In turn, in the allocation $A[\hat{t}]$ the s-player gets both tasks if he bids s', and he gets only task 2 with bid s, contradicting WMON for the s-player, because $s_1 < s_1'$ (cf. Figure 3). ∎

Since $f'(., s_2)$ is increasing as a function of s_1, it must be continuous in all but at most countably many points s_1. Furthermore, since f' is increasing *independently* of s_2, it must be independent of s_2 in all s_1 where it is continuous, as we show next.

LEMMA 4. *If f' is continuous as a function of s_1 in some point $(\bar{s}_1, \bar{s}_2)$, with $\bar{s}_1 > 0$, then it is independent of s_2, i.e., then $f'(\bar{s}_1, s_2) = f'(\bar{s}_1, \bar{s}_2)$ for every s_2; moreover f' is continuous in the point $\bar{s}_1$ for every s_2. Symmetric statements hold for g'.*

Proof. Let $f'(., \bar{s}_2)$ be continuous as a function of s_1 in the point $\bar{s}_1$. Assume for contradiction, w.l.o.g., that $f'(\bar{s}_1, \bar{s}_2) > f'(\bar{s}_1, s_2)$ for some $s_2 \neq \bar{s}_2$. Since $f'(., \bar{s}_2)$ is continuous in the point $\bar{s}_1$, we can take an $\bar{s}_1 - \delta$, so that $f'(\bar{s}_1 - \delta, \bar{s}_2) > f'(\bar{s}_1, s_2)$ also holds. However, this contradicts Lemma 3, because $\bar{s}_1 - \delta < \bar{s}_1$.

Since $f'(., \bar{s}_2)$ is increasing, we can take an infinite sequence $s_1^i \to \bar{s}_1$ (from left or right) so that f' is continuous in each of the points $(s_1^i, \bar{s}_2)$, and by the previous paragraph, independent of s_2. By continuity of f' in $(\bar{s}_1, \bar{s}_2)$, it holds that $f'(s_1^i, \bar{s}_2) \to f'(\bar{s}_1, \bar{s}_2)$, and by independence on s_2, and by monotonicity, f' must be continuous in the point $\bar{s}_1$, regardless of s_2. ∎

We introduce similar notation for other critical values of the t-player based on the virtual payments. We define f and g as the other two axis-parallel boundary positions, in case a piece of the corresponding boundary is visible. For technical reasons, we also define f (to be 0) if the nonempty region R_2 touches the t_2-axis (and symmetrically for g). The latter occurs when $P_{12} - P_2 = 0$ and $g' > 0$. (Like the virtual payments, the boundary positions f, g, f', g' (or their nonexistence) are uniquely determined by the allocation $A[s]$.)

Definition 9. *If ($R_{12}(s) \neq \emptyset$ and $R_2 \neq \emptyset$),* or *($P_{12} - P_2 = 0$ and $g' > 0$), then let $f(s) = P_{12} - P_2$. Symmetrically, if ($R_{12}(s) \neq \emptyset$ and $R_1 \neq \emptyset$),* or *($P_{12} - P_1 = 0$ and $f' > 0$), then let $g(s) = P_{12} - P_1$.*

As opposed to f' and g', the values f and g are not defined for every $s \in [0,B) \times [0,B)$; this is needed for the following lemmas to hold. The next lemmas about f and g are analogous to Lemmas 3 and 4.

Lemma 5. *f is monotone increasing in s_1 in the following sense: Assume that $f(s)$ is defined for $s = (s_1, s_2)$.*

i. *Let $s'_2 - s_2 > s'_1 - s_1 \geq 0$, then f is also defined for $s' = (s'_1, s'_2)$;*
ii. *Let $s_1 < s'_1$, and assume that f is also defined in (s'_1, s''_2). Then $f(s_1, s_2) \leq f(s'_1, s''_2)$.*

Symmetric statements hold for g.

Proof. We prove i first. Since f is defined for s, it holds that $R_2(s) \neq \emptyset$, implying $g'(s) > 0$. As $s'_2 > s_2$, and g' is nondecreasing in s_2 (Lemma 3), $g'(s') \geq g'(s) > 0$, and therefore $R_2(s') \neq \emptyset$. Assume now for contradiction that despite $g'(s') > 0$ the boundary f is not defined for s'; then it must be the case that $R_{12}(s') = \emptyset$, the figure $A[s']$ is quasi-flipping, and the point $\hat{t} = (\epsilon, 0)$ is in the region $R_1(s')$ for ϵ small enough; in turn, $s' \in R_2(\hat{t})$.

Since f is defined for s, either $\hat{t} \in R_{12}(s)$ (if $f > 0$) or $\hat{t} \in R_2(s)$ (if $f = 0$). In the first case $s \in R_\emptyset(\hat{t})$; in the second case $s \in R_1(\hat{t})$. Both cases contradict WMON, because $s' \in R_2(\hat{t})$, and $s'_2 - s'_1 > s_2 - s_1$.

Now we show the statement ii. Assume for contradiction that $0 \leq f(s'_1, s''_2) < f(s_1, s_2)$. Then in both cases f marks the position of a boundary between the regions R_{12} and R_2 (R_{12} might be empty when $f = 0$). Let $t = (t_1, t_2)$ be so that $f(s'_1, s''_2) < t_1 < f(s_1, s_2)$, and t_2 is infinitesimally small. Then, for this t, the t-player gets task 2 for the bid (s'_1, s''_2) and he gets 12 for the bid (s_1, s_2) of the s-player. In turn, the s-player gets $\emptyset$ if he bids (s_1, s_2), but he gets 1 if he bids (s'_1, s''_2), contradicting WMON, because $s_1 < s'_1$. ◻

Lemma 6. *Assume that $f(s_1, \bar{s}_2)$ exists over a neighborhood $s_1 \in (\bar{s}_1 - \Delta, \bar{s}_1 + \Delta)$ of $\bar{s}_1 > 0$ for some $\bar{s}_2$; moreover f is continuous as a function of s_1 in the point $(\bar{s}_1, \bar{s}_2)$. Then it is independent of s_2, i.e., then $f(\bar{s}_1, s_2) = f(\bar{s}_1, \bar{s}_2)$ for every s_2 where f is defined, in particular for every $s_2 > \bar{s}_2$. A symmetric statement holds for g.*

Proof. Assume first that $f(\bar{s}_1, s_2) < f(\bar{s}_1, \bar{s}_2)$; then by continuity in $\bar{s}_1$ a $0 < \delta < \Delta$ exists so that $f(\bar{s}_1, s_2) < f(\bar{s}_1 - \delta, \bar{s}_2)$. Assume, on the other hand, that $f(\bar{s}_1, s_2) > f(\bar{s}_1, \bar{s}_2)$; then again, a δ exists so that $f(\bar{s}_1, s_2) > f(\bar{s}_1 + \delta, \bar{s}_2)$. Both possibilities contradict Lemma 5(ii). ◻

We will sometimes abuse notation and consider f and f' as univariate functions of s_1. This is incorrect only for at most countably many s_1 where these functions have jump discontinuities. In these s_1 the above single-variate functions remain undefined, or can be identified with, e.g., $\lim_{x \to s_1^+} f(x)$. Analogously, we will often treat g' and g as univariate functions.

3.4.2. Intuition. The proof proceeds by analyzing three cases. CASE A treats the following scenario: for some s an allocation function of the t-player exists where at least three (interiors of) regions R are nonempty, the allocation is quasi-flipping or quasi-bundling, and one of the critical value functions f', g', f, g is continuous in s as a univariate function. In this case the WMON allocation function over the whole additive domain, with (possibly) bounded s, is a relaxed affine minimizer. The proof of this is based on three observations, roughly summarized as follows.

First, once an allocation of the t-player is quasi-bundling (quasi-flipping) in some s, where at least *one* of f, g, f', g' is continuous as a univariate function, and strictly positive, we can infinitesimally adjust s in a carefully selected direction, so that the allocation remains quasi-bundling (quasi-flipping), and *all* of these functions are continuous (when defined) in the adjusted s (Lemmas 7–10).

Second, if in any quasi-flipping or quasi-bundling allocation *all* the critical value functions f', g', f, g are continuous in some point s, then they all must have the same derivative in this point.[17] This will follow from the WMON property of the s-player (see Claims 1–3).

Third, in such continuity points f' and f (if defined) are independent of s_2, and g' and g are independent of s_1 (Lemmas 3–6), which eventually implies that the above derivatives must remain the same for *every* s, implying that each of these critical value functions is linear (see, e.g., Lemma 10).

In CASE B (quasi-)bundling allocation figures exist, but $f'(s) = 0$, $g'(s) = 0$ for *every* s, and therefore the regions $R_1(s)$ and $R_2(s)$ are empty for every s. In this case the allocation is a 1-dimensional monotone allocation over the sets of positive s and t, where the two tasks are bundled and always allocated to the same player.

Finally, in CASE C we consider those allocations where the t-player has a crossing figure whenever any of f', g' (or f, g) is continuous in s, and we show that in this case the mechanism is a relaxed task-independent mechanism.

If in CASE A the common derivatives of f, g, f', g' happen to be 0, or in CASE B or C the arbitrary increasing critical value functions turn out to be constant, then we obtain a constant mechanism. We do not discuss these cases separately in the rest of the proof.

3.4.3. General lemmas. We start the argument with three lemmas, altogether stating that the mechanism must be a (relaxed) affine minimizer if a single point $\hat{s}$ exists where *all* of f', g' and f, g (if defined) are continuous as univariate functions (i.e., f' and f in the point $\hat{s}_1$ and g' and g in the point $\hat{s}_2$), and the allocation $A[\hat{s}]$ is quasi-flipping or quasi-bundling.

LEMMA 7. *Assume that both f and g are undefined or zero for every $s \in [0, B) \times [0, B)$. If an $\hat{s} \in (0, B) \times (0, B)$ exists so that $f'(\hat{s}) > 0$ and $g'(\hat{s}) > 0$ for the allocation $A[\hat{s}]$ of the t-player, then the mechanism is an affine minimizer (with $\gamma_{12} = \infty$) for strictly positive t and s.*

Proof. In $A[\hat{s}]$ the allocation of the t-player is quasi-flipping, and the interior of $R_{12}(\hat{s})$ is empty; that is, $R_{12}(\hat{s})$ contains no $t > 0$. This follows directly from the fact that f and g are zero or undefined, and $f', g' > 0$. We claim first that similarly, $R_{12}(s) = \emptyset$ holds for *every* s. Assume by contradiction that for some s^* this is not the case. Since the boundaries f and g do not appear, this could happen only if the figure is bundling, and $f'(s^*) = g'(s^*) = 0$. Then, since f' and g' are increasing (Lemma 3), $s^*_1 < \hat{s}_1$ and $s^*_2 < \hat{s}_2$ must hold. Now picking any positive $t \in R_{12}(s^*)$, it holds that $s^* \in R_\emptyset(t)$, and by WMON $\hat{s} \in R_\emptyset(t)$ as well, because $s^* < \hat{s}$. Therefore, $t \in R_{12}(\hat{s})$, contradicting the first statement of the proof. As a consequence, for every s if $f' > 0$ and $g' > 0$, the allocation $A[s]$ is quasi-flipping without a region R_{12}, and if any of f', g' is zero, then there are only two (interiors of) regions $R_\emptyset$ and R_1 or $R_\emptyset$ and R_2.

[17]To be precise, the four difference quotients (for arbitrary fixed small difference $\delta > 0$) must be equal.

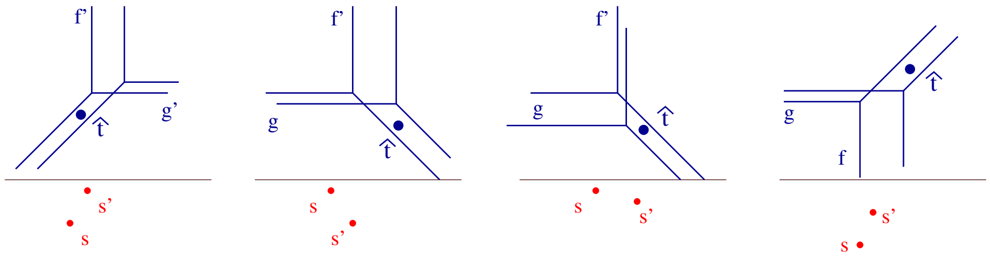

FIG. 4. *Illustrations of the proofs of Claims* 1, 2, *and* 3.

Now we prove the following general claim, which is the core observation implying the linearity of the payment functions whenever an allocation is noncrossing.

CLAIM 1. *Assume that f' is positive and continuous as a univariate function in the points s_1 and $s_1+\delta$, and g' is positive and continuous in the points s_2 and $s_2+\delta$; furthermore, the allocation of the t-player is quasi-flipping in a 2δ-neighborhood of $s=(s_1,s_2)$. Then $f'(s_1+\delta)-f'(s_1)=g'(s_2+\delta)-g'(s_2)$.*

Proof. Assume for contradiction w.l.o.g. that $f'(s_1+\delta)-f'(s_1)>g'(s_2+\delta)-g'(s_2)$. Since f' is continuous in $s_1+\delta$, it even holds that $f'(s_1+\delta')-f'(s_1)>g'(s_2+\delta)-g'(s_2)$ for some $\delta'<\delta$.

Let $s'=(s_1+\delta',s_2+\delta)$. Now there exists a point $\hat{t}$ so that $\hat{t}\in R_2(s)$ but $\hat{t}\in R_1(s')$ (see Figure 4(a)). In turn, for this bid $\hat{t}$ of the t-player the s-player receives task 1 if he bids s, and he receives task 2 if he bids s'. This contradicts WMON for the s-player because $s'_1-s_1=\delta'<\delta=s'_2-s_2$. ∎

Next we argue that the increasing functions f' and g' are continuous over $(0,B)$. Observe that since they are continuous in almost every point s_1, resp., s_2 (because they are increasing), we could use Claim 1 to show linearity on $s_i\in(0,B)$ whenever f' and g' are positive. Here we show continuity (and then linearity) of f',g' on $s_i\in(0,B)$ in any case.

W.l.o.g., we prove the continuity of f'. The proof is essentially the same as that of Claim 1. Assume the contrary, that for some fixed s_2 the $f'(.,s_2)$ has a jump discontinuity of height Δ as a function of s_1 in the point $\bar{s}_1\in[0,B)$. We disprove this first in the simpler case when $g'(s_2)>0$ and g' is continuous in the point s_2. Let $s_1\le\bar{s}_1<s_1+\delta$ so that f' has the assumed jump discontinuity (for the fixed s_2) between s_1 and $s_1+\delta$ and so $f'(s_1)+\Delta<f'(s_1+\delta)$; moreover, δ can be chosen so small that $g'(s_2+2\delta)-g'(s_2)<\Delta/2$ holds for every s_1. The latter is possible, since g' is continuous in the point s_2 and by Lemma 4 independent of s_1. Now again, since $g'(s_2)>0$, a point $\hat{t}$ exists (see Figure 4(a)) so that for this $\hat{t}$ and bid $s=(s_1,s_2)$ the s-player gets task 1, but with bid $s'=(s_1+\delta,s_2+2\delta)$ the s-player gets task 2, contradicting WMON.[18]

If the $g'(s_2)=0$ or g' is not continuous in s_2, then we pick an arbitrary $\bar{s}_2>s_2$ that fulfills both requirements. By the previous paragraph, for this fixed $\bar{s}_2$ the f' must be continuous in the point $\bar{s}_1$. By Lemma 4, if $\bar{s}_1>0$, then f' is independent of s_2 and continuous in the point $\bar{s}_1$ for every s_2.

[18]This will eventually imply continuity of f' even in the point $\bar{s}_1=0$ in case $g'(s_2)>0$, because $s_1=\bar{s}_1$ was allowed. The only exceptions in this lemma can occur when $\bar{s}_1=0$ and $g'(s_2)=0$, or $\bar{s}_2=0$ and $f'(s_1)=0$.

Having shown that $f'(s_1)$ and $g'(s_2)$ are continuous over $(0,B)$, Claim 1 implies that they are linear with the same slope whenever they are positive. Consider w.l.o.g. g', and let us fix $\bar{s}_1 > 0$ so that $f'(\bar{s}_1) > 0$, and $\delta = 1$. For the constant $0 \le \Delta = f'(\bar{s}_1 + 1) - f'(\bar{s}_1)$ Claim 1 implies

$$g'(s_2+1) = g'(s_2) + \Delta$$

for *every* s_2 such that $g'(s_2) > 0$. Further, by taking $\delta = 1/2$, we also obtain $g'(s_2 + 1/2) = g'(s_2) + \Delta/2$, and analogously

$$g'(s_2 + 1/2^r) = g'(s_2) + \Delta/2^r$$

over all s_2. By taking the limit $r \to \infty$, we obtain that g' is linear where it is positive. Then, using that g' is continuous, we obtain that g' must be truncated linear over all $s_2 > 0$.

By analogous argument we obtain that f' is truncated linear over all $s_1 > 0$. Moreover, by Claim 1 f' has the same multiplicative constant μ as g'. In summary, for $s > 0$, for the virtual payments of the t-player we obtain $P_1(s) = f'(s) = \max(\mu \cdot s_1 + \gamma_1, 0)$, and $P_2(s) = g'(s) = \max(\mu \cdot s_2 + \gamma_2, 0)$ (and $P_{12}(s) = \max(P_1(s), P_2(s))$ follows from $f = 0$ or $g = 0$).

The described virtual payments define an affine minimizer over $t \in (0,\infty)\times(0,\infty)$ and $s \in (0,B)\times(0,B)$. If $\mu = 0$, then the mechanism is a constant mechanism (the allocation is independent of s). ◻

Example 1. If $t_i = 0$ or $s_i = 0$, then the allocation can be different from that of an affine minimizer. As a simple example consider a minimizer over

$$t_1 + s_2, \qquad s_1 + t_2, \qquad s_1 + s_2 + 1,$$

with the modification that for $s = (0,0)$ both tasks are given to the s-player. The allocation is different from the original minimizer for $s = (0,0)$ when $t_1 < 1$ or $t_2 < 1$. For fixed s, the allocation to the t-player is obviously WMON; for fixed t, the allocation of the s-player differs from the affine minimizer at most in the point $(0,0)$, which is now put into R^s_{12}; this is always consistent with WMON. f' and g' both have a jump in the point $s = (0,0)$.

The example can be made less trivial if, e.g., for $s = (0,0)$ both tasks are given to the s-player only when $t_1 > 1/2$, and $t_2 > 1/2$, or if in the minimizer we replace $s_1 + t_2$ by $s_1 + t_2 + 2$, and give both tasks to the s-player for $s_1 = 0$ and $g'(s_2) = 0$ (i.e., when $s_1 = 0$ and $s_2 \le 1$). In summary, it might happen to be consistent with WMON if an affine minimizer (or constant mechanism) has a jump discontinuity of the payments in points "at the border," that is, in points with $s_i = 0$ or $t_i = 0$. In these singular point(s) the mechanism can even look like a different type, similarly to the case of a relaxed task-independent mechanism.

LEMMA 8. *Assume that $g'(s_2) = 0$ for every $s_2 \in [0,B)$. If an $\hat{s} \in (0,B)\times(0,B)$ exists so that the allocation $A[\hat{s}]$ of the t-player is quasi-bundling, $f' > 0$ is continuous in the point $\hat{s}_1$, and $g > 0$ is continuous in the point $\hat{s}_2$, then the mechanism is a relaxed affine minimizer (with $\gamma_2 = \infty$) over strictly positive t and s.*

For the symmetric case, assume that $f'(s_1) = 0$ for every $s_1 \in [0,B)$. If an $\hat{s} \in (0,B)\times(0,B)$ exists so that the allocation $A[\hat{s}]$ of the t-player is quasi-bundling, $g' > 0$ is continuous in the point $\hat{s}_2$, and $f > 0$ is continuous in the point $\hat{s}_1$, then the mechanism is a relaxed affine minimizer (with $\gamma_1 = \infty$) over strictly positive t and s.

Proof. Assume w.l.o.g. that the conditions of the first statement of the lemma hold. We use a claim analogous to that in the previous lemma.

CLAIM 2. *Assume that f' is continuous as a univariate function in the points s_1 and $s_1+\delta$, and g is positive and continuous in the points s_2 and $s_2-\delta$; furthermore, the allocation of the t-player is quasi-bundling in a 2δ-neighborhood of $s=(s_1,s_2)$; then $f'(s_1+\delta)-f'(s_1)=g(s_2)-g(s_2-\delta)$.*

Proof. Assume for contradiction first that $f'(s_1+\delta)-f'(s_1)>g(s_2)-g(s_2-\delta)$ (see Figure 4(b)). Since f' is continuous in $s_1+\delta$, it even holds that $f'(s_1+\delta')-f'(s_1)>g(s_2)-g(s_2-\delta)$ for some $\delta'<\delta$.

Let $s'=(s_1+\delta', s_2-\delta)$. Now there exists a point $\hat{t}$ so that $\hat{t}\in R_\emptyset(s)$ but $\hat{t}\in R_{12}(s')$. In turn, for this bid $\hat{t}$ of the t-player the s-player receives both tasks $1,2$ if he bids s, and he receives no task if he bids s'. This contradicts WMON for the s-player because $s_1+s_2>s_1+s_2+\delta'-\delta=s'_1+s'_2$.

Assume now that $f'(s_1+\delta)-f'(s_1)<g(s_2)-g(s_2-\delta)$ (see Figure 4(b')). Since f' is continuous in $s_1+\delta$, it even holds that $f'(s_1+\delta')-f'(s_1)<g(s_2)-g(s_2-\delta)$ for some $\delta'>\delta$.

Let $s'=(s_1+\delta', s_2-\delta)$. Now there exists a point $\hat{t}$ so that $\hat{t}\in R_{12}(s)$ but $\hat{t}\in R_\emptyset(s')$. In turn, for this bid $\hat{t}$ of the t-player the s-player receives no task if he bids s, and he receives both tasks if he bids s'. This contradicts WMON for the s-player because $s_1+s_2<s_1+s_2+\delta'-\delta=s'_1+s'_2$. ∎

Since $g'=0$ for every s, the allocation is quasi-bundling whenever $f'(s)>0$ and $g(s)>0$, and g is defined if and only if $f'(s)>0$. We prove first that g is continuous (and thus independent of s_1 by Lemma 6) in every $s_2\in[0,B)$, where g exists, i.e., when $f'(s)>0$ (except maybe for $g(0,0)$). Let us fix $\hat{s}_1$ from the statement of the lemma. Observe that g is defined for this $\hat{s}_1$, because $f'(\hat{s}_1)>0$, independently of s_2 since f' is continuous in $\hat{s}_1>0$. Assume for contradiction that g has a jump discontinuity of height Δ in some $\bar{s}_2\ge 0$. Let $s_2\le\bar{s}_2<s_2+\delta$ so that $g(s_2+\delta)-g(s_2)>\Delta$, g is continuous in the point $s_2+\delta$, and moreover, $f'(\hat{s}_1)-f'(\hat{s}_1-\delta)<\Delta/2$, and f' is continuous in the point ($\hat{s}_1$ and) $\hat{s}_1-\delta$. Now $f'(\hat{s}_1)-f'(\hat{s}_1-\delta)<\Delta/2<\Delta<g(\hat{s}_1,s_2+\delta)-g(\hat{s}_1,s_2)=g(\hat{s}_1-\delta,s_2+\delta)-g(\hat{s}_1,s_2)$, and we are in a situation analogous to that in the second case of Claim 2, and we obtain a contradiction the same way (see Figure 4(b')). (This argument works even for $s_2=0$, and the independence of g from s_1 can be shown, with the only exception that $g(0,0)<g(\hat{s}_1,0)$ is possible.)

An analogous argument shows that f' is continuous over $s_1\in[0,B)$ and independent of s_2, except that $f'(0,s_2)<f'(0,\hat{s}_2)$ is possible when $g(s_2)=0$. This case is basically the same as the first case in Claim 2 (Figure 4(b)) and leads to a contradiction.

Having that g is continuous over $s_2\in[0,B)$ and independent of s_1 (except in $s=(0,0)$), and symmetrically for f', with the help of Claim 2 the same argument as in Lemma 7 proves that both are linear with the same slope μ whenever they are strictly positive. Altogether the functions $f'=\max(\mu s_1+\gamma_1,0)$ and g are truncated linear with common slope μ, as corresponds to an affine minimizer. These functions uniquely determine the allocation (up to tie-breaking) whenever $f'>0$, because then both f' and g exist, and by assumption $g'=0$.

Assume finally, that $f'(s_1)=0$ iff $s_1\le D$ for some $D>0$, and $g(s_2)=\mu s_2+C$ for an $C>0$. Then for $(s_1+s_2)\in[0,D)$ the $P_{12}(s_1+s_2)$ is an arbitrary increasing function $P_{12}:[0,D)\to[0,C)$, because there exists no $s>0$ such that $s_1+s_2<D$ but $f'(s_1)>0$ would hold, and g would be defined (and similarly for $t_1+t_2<C$), so P_{12}

need not be linear over this interval. These s and t values, having a sum of coordinates below the given constants D and C, respectively, are the only exceptions where the mechanism is a 1-dimensional bundling mechanism and not an affine minimizer. $\square$

LEMMA 9. *Assume that an $\hat{s} \in (0,B) \times (0,B)$ exists so that $f' > 0$ and $f > 0$ are defined and both are continuous in the point $\hat{s}_1$; and $g' > 0$ and $g > 0$ are defined and both are continuous in the point $\hat{s}_2$. If the allocation $A[\hat{s}]$ of the t-player is quasi-flipping, then the restriction of the mechanism to positive bids is an affine minimizer; if $A[\hat{s}]$ is quasi-bundling, then it is a relaxed affine minimizer.*

Proof. Assume first that the allocation $A[\hat{s}]$ is quasi-flipping, i.e., $0 < f(\hat{s}) < f'(\hat{s})$ and $0 < g(\hat{s}) < g'(\hat{s})$. A claim analogous to Claim 1 says that f and g must have equal difference quotients.

CLAIM 3. *Assume that f is continuous as a univariate function in the points s_1 and $s_1 + \delta$, and g is continuous in the points s_2 and $s_2 + \delta$; furthermore the allocation of the t-player is quasi-flipping in a 2δ-neighborhood of $s = (s_1, s_2)$. Then $f(s_1 + \delta) - f(s_1) = g(s_2 + \delta) - g(s_2)$.*

Proof. Assume w.l.o.g. that $f(s_1 + \delta) - f(s_1) > g(s_2 + \delta) - g(s_2)$. Since f is continuous in the point $s_1 + \delta$, there is a $\delta' < \delta$ so that even $f(s_1 + \delta') - f(s_1) > g(s_2 + \delta) - g(s_2)$ (see Figure 4(c)). Let $s = (s_1, s_2)$ and $s' = (s_1 + \delta', s_2 + \delta)$. There exists a $\hat{t}$ so that for this $\hat{t}$ the s-player receives task 1 with bid s and task 2 with bid s', contradicting WMON. $\square$

Let us now fix $\hat{s}_1$ from the conditions of the lemma. Recall that f (when defined) and f' are independent of s_2 for this fixed $\hat{s}_1$, by Lemmas 4 and 6. By a proof analogous to that of Lemma 7, $g'(\hat{s}_1, .)$ is continuous over $s_2 \in (0,B)$, and right continuous in $s_2 = 0$. The same holds for g analogously to the proof of Lemma 8. Since both are continuous in every point $s_2 > 0$, g and g' are independent of s_1 over all $s_2 \in (0,B)$ by Lemmas 4 and 6.

By a symmetric argument, f (where defined) and f' are continuous and independent of s_2 over $s_1 \in (0,B)$. By changing only s_1 or only s_2, the figure remains quasi-flipping (because $g' - g$, resp., $f' - f$ does not change).

Using Claims 1 and 3, it can be shown that f, f', g, and g' are linear with the same μ, as long as these boundaries are positive (see also the proof of Lemma 7). Altogether each of them must be a truncated linear function.

The allocation remains quasi-flipping as long as $f' > 0$ and $g' > 0$. At least one of f or g is defined for every $s \in [0,B) \times [0,B)$, unless $f' = g' = 0$. In the latter case all $t > 0$ are in $R_\emptyset$. In every other case at least three of f, g, f', g' exist and are always uniquely defined (by the respective truncated linear functions). They determine the allocation of the t-player (up to tie-breaking), which determines the allocation of the s-player. The mechanism must be an affine minimizer over the positive bids.

Second, assume that $A[\hat{s}]$ is quasi-bundling. We can use Claim 2 to prove that f, g (when defined) and f' and g' are truncated linear, each with the same multiplicative constant μ on $(0,B)$. Also, $A[s]$ remains quasi-bundling as long as $f' > 0$ or $g' > 0$ holds. This determines the allocation as an affine minimizer if $f' > 0$ or $g' > 0$, because then at least three of f, g, f', and g' are defined. This is not the case if $f' = g' = 0$, i.e., when the allocation is fully bundling. $P_{12}(x)$ is determined only if an $s = (s_1, s_2)$ with $x = s_1 + s_2$ exists so that $f'(s_1) > 0$ or $g'(s_2) > 0$. If (for $x < D$ for a given constant $D > 0$) no such s exists, then $P_{12}(x)$ is some arbitrary increasing function $P_{12} : [0,D) \to [0,C)$ and the mechanism becomes 1-dimensional. C is here a constant playing the same role for t as D for s. All in all, concerning only positive bids s and t, the mechanism is a relaxed affine minimizer. ☐

3.4.4. CASE A. In CASE A the following conditions are assumed: $\exists \bar{s} = (\bar{s}_1, \bar{s}_2) \in (0, B) \times (0, B)$ so that $f'(\bar{s}) > 0$, f' as a function of s_1 is continuous in the point $\bar{s}_1$, and the allocation of the t-player is quasi-flipping or quasi-bundling (or the symmetric case holds for g').

LEMMA 10. *In CASE A the conditions of at least one of Lemmas* 7, 8, *and* 9 *hold, and so the mechanism is a (relaxed) affine minimizer when restricted to strictly positive s and t.*

Proof. We proceed by a case analysis that depends on the type of the allocation function of the t-player.

Case 1. *In $\bar{s}$ the allocation of the t-player is quasi-flipping.* Since the allocation is quasi-flipping, it follows that $f'(\bar{s}) > 0$ and $g'(\bar{s}) > 0$. We change $\bar{s}$ slightly to an appropriate s^* (when necessary) so that (in addition to f') also g' is continuous in the point (s_1^*, s_2^*) as a univariate function $g'(s_1^*, .)$ of the variable s_2. If in $\bar{s}$ this is not the case, then we modify $\bar{s}$ as follows: first, we increase $\bar{s}_2$ to some $s_2^* > \bar{s}_2$ so that in the point s_2^* the $g'(\bar{s}_1, .)$ is continuous. Note that f' is still continuous as a univariate function, and $f'(\bar{s}_1, s_2^*) = f'(\bar{s})$ by Lemma 4. Now, to be able to use WMON, we slightly decrease $\bar{s}_1$ to $s_1^* < \bar{s}_1$ so that f' is continuous in s_1^*, and $0 < f'(\bar{s}_1) - \epsilon < f'(s_1^*) \leq f'(\bar{s}_1)$. Note that the latter is possible, since f' was continuous in the point $\bar{s}_1$. Again by Lemma 4, g' did not change by reducing $\bar{s}_1$, and $g'(s_1^*, .)$ is still continuous in the point s_2^*. Altogether, in the new point s^* both f' and g' are strictly positive and continuous. We show that the allocation $A[s^*]$ is still quasi-flipping.

Assume for contradiction that for s^* the allocation is crossing or quasi-bundling. Consider the point $\hat{t} = (f'(s^*) - \epsilon', \epsilon')$. Since for $\bar{s}$, the allocation is quasi-flipping, for ϵ and ϵ' small enough, $\hat{t} \in R_2(\bar{s})$ holds. If for s^* the allocation is crossing or quasi-bundling, then $\hat{t} \in R_{12}(s^*)$. Consequently, $\bar{s} \in R_1^s(\hat{t})$ and $s^* \in R_\emptyset^s(\hat{t})$, contradicting WMON, since $s_1^* < \bar{s}_1$. Thus, $A[s^*]$ is quasi-flipping, and both f' and g' are positive and continuous as univariate functions in the point s^*.

Due to symmetry, in the rest of Case 1 *we can assume w.l.o.g. that* $f'(s^*) \leq g'(s^*)$. By assumption $0 < f'(s^*) \leq g'(s^*)$ holds, and the figure is quasi-flipping; this implies that f is defined for the point s^* (see Definition 9). Like before, we can slightly decrease s_1^* so that the continuity of *all* these functions f', g', f holds. We claim first that f is still defined if we reduce s_1^* to some $s_1' := s_1^* - \delta$, so that f' just slightly changes: Since g' is continuous in the point s_2^*, reducing s_1^* to s_1' does not change g', and it could not increase f'. So, it still holds that $0 < g'(s^*) = g'((s_1', s_2^*))$, and also $f'((s_1', s_2^*)) \leq g'(s^*) = g'((s_1', s_2^*))$. These imply that f is defined in (s_1', s_2^*) no matter if the shape of the allocation changed.

Thus we can reduce s_1^* to an appropriate s_1' so that f' remains nearly the same and is continuous in s_1', and f (exists and) is also continuous in s_1'. Since f' does not change much, and f cannot increase, the allocation is still quasi-flipping; moreover, g' is continuous in the new point, since we did not change s_2^*.

Set $s_2' := s_2^*$. In summary, both $f' > 0$ and $f \geq 0$ are continuous in the point s_1', and $g' > 0$ is continuous in the point s_2'; moreover the allocation $A[s']$ is quasi-flipping.

Case 1.1. *Both f and g are either zero or undefined for every $s = (s_1, s_2)$.* By Lemma 7, in this case the mechanism is an affine minimizer.

Case 1.2. *$\exists s$ so that $f(s)>0$ or $g(s)>0$.* We show that also a point $\hat{s}$ exists where the allocation is quasi-flipping, and each of f, f', g, g' are positive *and* continuous in $\hat{s}$ as univariate functions. Then the allocation is an affine minimizer by Lemma 9.

Assume first that Case 1.2 holds because $f(s)>0$ for some s. By Lemma 5, for every $\Delta>0$, and $\hat{s}_1=s_1+\Delta$ and $\hat{s}_2>s_2+\Delta$, the boundary $f(\hat{s})$ is also defined, and $f(\hat{s})>0$ must be the case. Therefore we can pick such an $\hat{s}$, and so that in addition $\hat{s}_1>s_1'$ and $\hat{s}_2>s_2'$ holds. Since $f(\hat{s}), f'(\hat{s})$, and $g'(\hat{s})$ are all positive, $g(\hat{s})$ must be defined and be positive, too. Finally, we can pick $\hat{s}$ so that the positive f' and f are continuous in $\hat{s}_1$, and g' and g are continuous in the point $\hat{s}_2$, because the (univariate) continuity holds in almost every $\hat{s}_1$ and $\hat{s}_2$ (say, we select $\hat{s}_1$ first appropriately for f and f', and then increase $\hat{s}_2$ properly for g' and g).

Second, if Case 1.2 holds due to $g(s)>0$ for some s, then for every $\hat{s}$ with $\hat{s}_1>s_1+\Delta$ and $\hat{s}_2=s_2+\Delta$, the boundary $g(\hat{s})$ is also defined, and $g(\hat{s})>0$ must be the case. Therefore we can pick such an $\hat{s}$ so that that even $\hat{s}_1>s_1'$ and $\hat{s}_2>s_2'$ hold, and since $g(\hat{s}), f'(\hat{s})$ and $g'(\hat{s})$ are all positive, $f(\hat{s})$ must be defined and positive; moreover w.l.o.g. they are all continuous in $\hat{s}$ as univariate functions.

We need to show that the allocation is quasi-flipping in $\hat{s}$. We first consider the allocation in the point $(s_1', \hat{s}_2)$. Since for s_1' the f and f' are independent of s_2, they do not change as s_2' gets changed to $\hat{s}_2$, and the allocation remains quasi-flipping. Now let us also increase s_1' to $\hat{s}_1$. The same argument might not work here, since $g(s_1', \hat{s}_2)$ could be undefined. Assume for contradiction that $A[\hat{s}]$ is crossing or quasi-bundling, meaning that $g(\hat{s})>g'(\hat{s})$. Since in $\hat{s}$ each of f, f', g, g' are defined and continuous, we can decrease $\hat{s}_2$ to $\hat{s}_2-\delta$, by an arbitrarily small δ, the f and f' do not change, and the allocation is still crossing or quasi-bundling. Now there is a point $\hat{t}=(0, g'(\hat{s}_2)-\epsilon)$ so that $\hat{t}\in R_1(s_1', \hat{s}_2)$ but $\hat{t}\in R_{12}(\hat{s}_1, \hat{s}_2-\delta)$. This contradicts WMON for the s-player for this $\hat{t}$, because for $(s_1', \hat{s}_2)$ he gets task 2, but with bid $(\hat{s}_1, \hat{s}_2-\delta)$ he gets no task. Finally, Lemma 9 can be applied to show that the mechanism is an affine minimizer.

Case 2. *In $\bar{s}$ the allocation $A[\bar{s}]$ of the t-player is quasi-bundling.* This will turn out to be the simpler case. Since $f'(\bar{s})>0$, and the allocation is quasi-bundling, $g(\bar{s})$ is defined and positive.

Case 2.1. *$g'(\bar{s}_1, s_2)=0$ for all $s_2\in[0,B)$.* Note that by Lemma 4, $g'(s)=0$ for every s, because for $\bar{s}_1$ it is continuous (constant) in every s_2, and therefore independent of s_1. Furthermore, $g(\bar{s}_1, s_2)$ is defined and positive for every $s_2>\bar{s}_2$. Otherwise the allocation would have to become quasi-flipping, implying $g'>0$, a contradiction (we used that f' is independent of s_2 and therefore constant and positive).

Now we can increase s_2 to $\bar{s}_2+\delta$ so that in $(\bar{s}_1, \bar{s}_2+\delta)$ both g and f' are continuous as univariate functions, and since $g'(s)=0$ for every s, Lemma 8 can be applied.

Case 2.2. *$\exists s_2$ so that $g'(\bar{s}_1, s_2)>0$.* Assume that Case 2.2 holds by $g'(\bar{s}_1, s_2)>0$. In a first step, we increase s_2 to $s_2^*>\max\{\bar{s}_2, s_2\}$ so that in $(\bar{s}_1, s_2^*)$ the g' is strictly positive, and continuous as a univariate function. This will not change f', since f' is continuous in the point $\bar{s}_1$.

Also by the continuity of f', for every small enough $\epsilon>0$ there is a δ, so that for every $\bar{s}_1<s_1^*<\bar{s}_1+\delta$ it holds that $f'((s_1^*, s_2^*))<f'(\bar{s}_1)+\epsilon<f(\bar{s}_1)$.

Consider now only those s_1^* points where f' is continuous, and therefore independent of s_2. We claim that with the fixed s_2^*, and $s^*=(s_1^*, s_2^*)$, the allocation $A[s^*]$ is quasi-bundling for all such $s_1^*\in[\bar{s}_1, \bar{s}_1+\delta)$. Otherwise we can pick a $\hat{t}=(f'(s_1^*)+\epsilon, \epsilon)$ so that $\hat{t}\in R_{12}(\bar{s})$, but $\hat{t}\in R_2(s^*)$. This is possible, since $A[\bar{s}]$ is quasi-bundling, but by indirect assumption $A[s^*]$ is crossing or quasi-flipping. Now, for this $\hat{t}$, $\bar{s}$ gets $\emptyset$, but s^* gets task 1, contradicting WMON, since $s^*>\bar{s}$.

So, we can choose an $s_1^* > \bar{s}_1$ so that both of f, f' are continuous in the point s_1^* as univariate functions, and $A[s^*]$ is quasi-bundling. Let $\hat{s}_1$ be this chosen value. Finally, we slightly increase s_2^* to an $\hat{s}_2$ so that both of g and g' are continuous in $\hat{s}_2$ (now f and f' are fixed, so g exists). Finally, Lemma 9 with $\hat{s} = (\hat{s}_1, \hat{s}_2)$ implies that the restriction of A to positive s and t is a relaxed affine minimizer. ∎

3.4.5. CASE B. In CASE B we assume that the conditions of CASE A do not hold, but $\exists \bar{s} = (\bar{s}_1, \bar{s}_2) \in (0, B) \times (0, B)$ so that $f'(\bar{s}) = 0$, f' as a function of s_1 is continuous in the point $\bar{s}_1$, and the allocation of the t-player is quasi-bundling (or the analogous case holds for g').

LEMMA 11. *In CASE B the restriction of the WMON allocation to positive bids is a bundling mechanism.*

Proof. The next claim proves for every $s \in [0, B) \times [0, B)$ that $f'(s) = g'(s) = 0$, implying $R_1(s) = R_2(s) = \emptyset$; the allocation $A[s]$ to the t-player can only be bundling since only the (interiors of) regions $R_{12}(s)$ and $R_\emptyset(s)$ can be nonempty.

CLAIM 4. *In CASE B* $f'(s) \equiv 0$ *and* $g'(s) \equiv 0$ *for every* $s \in [0, B) \times [0, B)$.

Proof. In $\bar{s}$ the f' is independent of s_2, whereas g' is increasing in s_2. Assume first for contradiction that $g' > 0$ for some s_2, and take an $s_2^* > \bar{s}_2$ point where g' is continuous, and strictly positive. Since CASE A does not hold (for g'), in $(\bar{s}_1, s_2^*)$ the allocation of the t-player must be crossing, while $f' = 0$ still holds.

Next, by continuity of f' in $\bar{s}_1$ (see Lemma 4) it is possible to slightly increase $\bar{s}_1$ to $s_1^* > \bar{s}_1$ so that $f'((s_1^*, s_2^*)) < \epsilon$. Increasing $\bar{s}_1$ did not change g', i.e., $g'((s_1^*, s_2^*)) = g'((\bar{s}_1, s_2^*)) > 0$, and the allocation must still be crossing, since CASE A does not hold.

Consider now the point $\hat{t} = (\epsilon, \epsilon)$. For $\bar{s}$ the allocation $A[\bar{s}]$ was bundling, so $\hat{t} \in R_{12}(\bar{s})$ for ϵ small enough. On the other hand, since $A[s^*]$ is crossing, $f'(s^*) < \epsilon$, and $g'(s^*) > 0$, it follows that $\hat{t} \in R_2(s^*)$. For this $\hat{t}$ the allocation of the s-player $\bar{s} \in R_\emptyset^s$ but $(s^*) \in R_1^s$, contradicting WMON, because $\bar{s} < s^*$.

We obtained that $g'(s) = 0$ for every s (independently of s_1 by Lemma 4). So, g' is continuous also in the point $\bar{s}$, and $A[\bar{s}]$ is quasi-bundling by assumption. So, we can apply the same argument of the proof symmetrically, to show that $f' = 0$ for every s. ∎

For any given t, the allocation $A[t]$ to the s-player can have s points receiving exactly one of the tasks only if in $A[s]$ the point t receives exactly one of the tasks, and this can occur only if $t_1 = 0$, or $t_2 = 0$, since $f'(s) = g'(s) = 0$.[19] In the restriction of the WMON allocation to positive bids, only the regions R_{12} and $R_\emptyset$ can exist for both players, and the boundary between them must be a nondecreasing function. We obtain bundling mechanisms. ∎

3.4.6. CASE C. Finally, in CASE C it remains to consider that for every $s \in (0, B) \times (0, B)$ where either f' or g' is continuous (as univariate functions), the allocation of the t-player is crossing.

In this case we obtain relaxed task-independent mechanisms. Note that some one-dimensional, constant, or affine minimizer mechanisms can be at the same time (relaxed) task-independent and captured as such by the following lemma.

LEMMA 12. *In CASE C the mechanism is a relaxed task-independent mechanism.*

[19]For such t, $R_1^s(t) \neq \emptyset$ or $R_2^s(t) \neq \emptyset$ might occur, but only if the domain of s is bounded above.

Proof. Fix an arbitrary $s_2^* > 0$. Since f' is nondecreasing, $f'(s_1, s_2^*)$ is continuous in almost every $s_1 \in [0, B)$. Fix a point $\bar{s}_1 > 0$, where $f'(., s_2^*)$ is continuous. Now $f'((\bar{s}_1, s_2^*)) = f'((\bar{s}_1, s_2))$ for every s_2 (by Lemma 4), and for all s_2 in the points $(\bar{s}_1, s_2)$ the allocation is crossing, that is, $g' = g$ (here g exists if $f' > 0$). The function $g'(\bar{s}_1, s_2)$ is nondecreasing in s_2, and it defines the critical value for task 2. Moreover, this critical value is independent of s_1 in every $\bar{s}_2$ point, where $g'(\bar{s}_1, .)$ is continuous. Thus, we define the nondecreasing critical value function η as this function $\eta(s_2) := g'(\bar{s}_1, s_2)$.

Symmetrically, we fix an arbitrary $s_1^* > 0$ and define $\phi(s_1) := f'(s_1, \bar{s}_2)$ for some $\bar{s}_2$ point where $g'(s_1^*, .)$ is continuous. ϕ and η may have jumps, and only then may the critical value depend on the other bid.

When can the allocation $A[s]$ be noncrossing? $f'(\hat{s}) \neq f(\hat{s})$ is possible only if f' has a discontinuity in $\hat{s}_1$, but since $f' - f = g' - g$ must hold, g' must have a discontinuity in $\hat{s}_2$ as well. Due to WMON, even in jump discontinuities the monotonicity of f', f in s_1 and the monotonicity of g', g in s_2 must hold. The allocation corresponds to a relaxed task-independent mechanism. How did we miss the jump discontinuities of ϕ^{-1} and η^{-1}? In the allocations $A[s]$ to the t-player these are intervals of s_1 (resp., s_2), where ϕ or η is constant, and t is on a boundary, so the (non–task-independent) allocation to the s-player is just a tie-breaking matter from the point of view of the t-player. In particular, if ϕ and η are locally constants, and t is the "crossing point" ($t_1 = \phi(s_1)$ and $t_2 = \eta(s_2)$), then the allocation to the s-player (the tie-breaking) might look noncrossing. ∎

3.5. Mechanisms for submodular players. In this section we relax the additivity of the t-player in different ways and show in Theorem 4 that for all these extensions of the additive domain the only remaining truthful mechanisms are relaxed affine minimizers (or constant mechanisms) and 1-dimensional mechanisms. This result holds for (i) arbitrary (normalized, monotone) valuations, (ii) submodular or subadditive valuations,[20] (iii) supermodular or superadditive valuations, (iv) ϵ-additive valuations (see V_ϵ below), and (v) certain intersections of the above domains. In essence, we characterize mechanisms for two tasks and two players in case (i), where the t-player has arbitrary monotone valuation (t_1, t_2, t_{12}), and the s-player has additive valuation $(s_1, s_2, s_1 + s_2)$ with s_1 and s_2 both bounded by an arbitrarily large fixed value B. We use the characterization for two additive players from section 3.2 and show that relaxed task-independent mechanisms are not extendable to the domain with nonadditive t-values, unless they are task-independent affine minimizers, or one-dimensional. Along the way we take care that our proofs work in cases (ii) to (v) as well. In particular, task-independent mechanisms cannot be extended even to an arbitrarily "narrow" 3-dimensional superset V_ϵ of the (2-dimensional) additive domain.

3.5.1. Basic concepts and notation. We consider any bid of either of the players (also for the additive s-player) as a vector of three entries. We summarize the notation for the considered domains of valuations (cost functions) as follows.

DEFINITION 10. *$V \subset \mathbb{R}^3_{\geq 0}$ is the 3D set of all monotone valuations:*

$$V = \{(x_1, x_2, x_{12}) \in \mathbb{R}^3_{\geq 0} \mid x_{12} \geq x_1 \geq 0,\, x_{12} \geq x_2 \geq 0\};$$

[20]For two tasks subadditive and submodular valuations are the same.

$V_+ \subset V$ *is the* additive plane*:*

$$V_+ = \{(x_1, x_2, x_{12}) \in \mathbb{R}^3_{\geq 0} \,|\, x_1, x_2 \geq 0, x_{12} = x_1 + x_2\};$$

$V_{+,B} \subset V_+$ *is the additive plane, restricted to* $[0, B) \times [0, B)$ *for some arbitrarily large fixed* B

$$V_{+,B} = \{(x_1, x_2, x_{12}) \in \mathbb{R}^3_{\geq 0} \mid 0 \leq x_1, x_2 < B, x_{12} = x_1 + x_2\};$$

$V_\epsilon \subset V$ *is the* ϵ*-neighborhood*[21] *of the additive plane for some arbitrary fixed* $\epsilon > 0$*:*

$$V_\epsilon = \{(x_1, x_2, x_{12}) \in V \mid x_{12} \in (x_1 + x_2 - \epsilon, x_1 + x_2 + \epsilon)\};$$

$V_{sub} \subset V$ *is the* 3*D set of all submodular valuations; for two tasks this is equivalent with subadditivity:*

$$V_{sub} = \{(x_1, x_2, x_{12}) \in V \mid x_{12} \leq x_1 + x_2\};$$

$V_{sup} \subset V$ *is the* 3*D set of all supermodular valuations; for two tasks this is equivalent with superadditivity:*

$$V_{sup} = \{(x_1, x_2, x_{12}) \in V \mid x_{12} \geq x_1 + x_2\}.$$

Let $V^* \in \{V, V_\epsilon, V_{sub}, V_{sup}, V_{sub} \cap V_\epsilon, V_{sup} \cap V_\epsilon\}$, and let $(A, \hat{P})$ denote the allocation and payment functions of a truthful mechanism on $V^* \times V_{+,B}$. For the valuations $t = (t_1, t_2, t_{12}) \in V^*$ and $s = (s_1, s_2, s_{12}) \in V_{+,B}$ the allocation is $A(t, s) = (\alpha_t, \overline{\alpha}_t)$, where $\alpha_t \in \{12, 1, 2, \emptyset\}$. Since both tasks are always allocated, we can identify $A(t, s)$ with α_t. For given s, $A[s]$ denotes the WMON allocation to the t-player depending on t. The interiors of allocation regions $R_{\alpha_t}(s)$ are now open sets w.r.t. V^*, and similarly for $R^s_{\alpha_s}(t) \subseteq V_{+,B}$.

The payments to the t-player for taking the tasks $12, 1$, and 2, respectively, depending on the bid s, are denoted by $\hat{P}_{12}(s)$, $\hat{P}_1(s)$, $\hat{P}_2(s)$. Since by assumption $R_\emptyset \neq \emptyset$, we can assume w.l.o.g. that the payment function to the t-player $\hat{P}$ is normalized, that is, $\hat{P}_\emptyset(s) = 0$ for every s. We omit the argument s if it is clear from the context. The payments to the s-player depending on the bid t are denoted by $\hat{P}^s_{12}(t)$, $\hat{P}^s_1(t)$, $\hat{P}^s_2(t)$, $\hat{P}^s_\emptyset(t)$.

Observe that a truthful mechanism $(A, \hat{P})$ over $V^* \times V_{+,B}$ is a truthful mechanism when restricted to $V_+ \times V_{+,B}$. We denote this restricted mechanism by $(A|_{V_+ \times V_{+,B}}$, $\hat{P}|_{V_+ \times V_{+,B}})$.

We introduce the types of WMON allocations that can occur on $V^* \times V_{+,B}$. As we prove later in this section, relaxed task-independent mechanisms do not extend to $V^* \times V_{+,B}$. Relaxed affine minimizers and constant and one-dimensional mechanisms do extend; these have the same definition as before; for the sake of completeness we repeat these definitions here for V^* (see sections 3.2.2 to 3.2.4 for more details).

Relaxed affine minimizers.

Definition 11. *An allocation* A *is a relaxed affine minimizer if there exist positive constants per player* μ_t *and* μ_s*, and constants* $\gamma_\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$ *per allocation* α*, and possibly an arbitrary increasing function* $\xi : [0, \min(\gamma_1, \gamma_2) - \gamma_\emptyset] \to [0, \min(\gamma_1, \gamma_2) - \gamma_{12}]$ *with* $\min(\gamma_1, \gamma_2) - \gamma_{12} = \xi(\min(\gamma_1, \gamma_2) - \gamma_\emptyset)$*, so that for every input* (t, s) *the following hold:*

[21] We are quite free to use any reasonable definition of such a convex neighborhood; e.g., we could have chosen $x_{12} \in ((1 - \epsilon)(x_1 + x_2), (1 + \epsilon)(x_1 + x_2))$, or even the intersection of the latter with the current (additive) neighborhood. The characterization is not very sensitive to this definition.

(a) *If* $\mu_s \cdot (s_1 + s_2) \geq \min(\gamma_1, \gamma_2) - \gamma_\emptyset$*, the allocation* $A(t, s)$ *minimizes over*

$$\mu_t \cdot t_{12} + \gamma_{12}, \quad \mu_t \cdot t_1 + \mu_s \cdot s_2 + \gamma_1, \quad \mu_s \cdot s_1 + \mu_t \cdot t_2 + \gamma_2, \quad \mu_s(s_1 + s_2) + \gamma_\emptyset.$$

(b) *If* $\mu_s \cdot (s_1 + s_2) \leq \min(\gamma_1, \gamma_2) - \gamma_\emptyset$*, then if* $\mu_t \cdot t_{12} > \xi(\mu_s(s_1 + s_2))$ *then the allocation for the t-player is* $\emptyset$ *and if* $\mu_t \cdot t_{12} < \xi(\mu_s \cdot (s_1 + s_2))$ *then it is* 12. *The payments are defined as in the additive case.*

One-dimensional mechanisms. In a *one-dimensional mechanism* only at most two possible allocations can occur. If the two occurring allocations are $\emptyset$ and 12, we call the mechanism a *bundling mechanism*: There is an arbitrary, increasing function $\xi : [0, B) \to [0, \infty)$ so that if $t_{12} > \xi(s_1+s_2)$, then the t-player gets $\emptyset$, and if $t_{12} < \xi(s_1+s_2)$, then the t-player gets 12. The payments are $P_1 = P_2 = 0$ and $P_{12} = \xi(s_1 + s_2)$.

In the other cases the only occurring allocations are $\emptyset$ and 1 (or $\emptyset$ and 2, respectively). The mechanism is in this case identical with a degenerate task-independent mechanism where task 2 is always received by the s-player. The critical value between getting/losing task 1 is determined by an arbitrary increasing function $\phi : [0, B) \to [0, \infty)$. The t-player receives task 1 if $t_1 < \phi(s_1)$. The payments are $P_2 = 0$ and $P_{12} = P_1 = \phi(s_1)$. Such mechanisms can also be extended to $V \times V_{+,B}$.

Constant mechanisms. In a *constant mechanism* the allocation is independent of the bids of the s-player, except for tie-breaking points. For the t-player it is a fixed WMON allocation (see also section 3.7.3). The payments to the t-player P_{12}, P_1, P_2 are constants.

3.6. Submodular characterization. The characterization that we use for the lower bound is captured by the following theorem.

THEOREM 4. *Let* $V^* \in \{V, V_{sub}, V_{sup}, V_\epsilon, V_{sub} \cap V_\epsilon, V_{sup} \cap V_\epsilon\}$*, and let* $B \in \mathbb{R}_{>0} \cup \{\infty\}$.

Consider an arbitrary WMON allocation A for two tasks and two players with bids $t \in V^*$ *and* $s \in V_{+,B}$*, where both tasks are always allocated, and* $R_\emptyset(s) \neq \emptyset$ *for every s. The restriction of A to strictly positive bids is one of these three types: relaxed affine minimizer, one-dimensional mechanism, or constant mechanism.*

Remark 1.

(a) We note that for $t \in V$ or $t \in V_{sub}$, Theorem 4 has a direct proof not using the characterization for additive players of section 3.2. However, to the best of our knowledge this direct proof does not carry over to the almost additive domain V_ϵ for the t-player.

(b) Arguments analogous to the following proof show that the same characterization holds, e.g., for player valuations (t, s) in $V^* \times V_+$, or in $V \times V$, or in any combination $V^* \times V^{**}$. Since this is not the central topic of this paper, and similar characterizations have been known, we do not consider these cases here; our primary goal is to show that *even* if the s-player remains additive and *even* bounded, and the t-player is *almost* additive, the disturbing task-independent allocations disappear.

3.7. Proof of Theorem 4.

3.7.1. Virtual payments. Let $(A, \hat{P})$ be a truthful mechanism for (t, s) over $V^* \times V_{+,B}$ (since A is WMON, some truthful payments $\hat{P}$ exist by [38]). For every s, since by assumption the $R_\emptyset(s) \neq \emptyset$, w.l.o.g. the payments are normalized: $\hat{P}_\emptyset(s) = 0$. We will use *virtual payments* for the t-player defined as follows.

Definition 12. *For every truthful mechanism* $(A,\hat{P})$ *over* $(t,s)\in V^*\times V_{+,B}$, *or over* $(t,s)\in V_+\times V_{+,B}$, *we define the* virtual payments *iteratively. Let* s *be an arbitrary fixed valuation from* $V_{+,B}$. *The payments of the* t*-player occurring in this definition are all defined for this fixed* s.

$$P_\emptyset = 0,$$

$$P_1 = \max\{\hat{P}_1, 0\}, \quad P_2 = \max\{\hat{P}_2, 0\},$$

$$P_{12} = \max\{\hat{P}_{12}, P_1, P_2\}.$$

Technically, P_α is simply the boundary position between $R_\emptyset$ and R_α of the WMON allocation $A[s]$ (see also section 3.7.3). The virtual payments to the s-player can be defined analogously to the t-player: either $B=\infty$, and therefore $R^s_\emptyset \neq \emptyset$ is assumed by finite approximation; or B is finite, and therefore $\hat{P}^s_\emptyset > -\infty$ (say, $\hat{P}^s_\emptyset \geq -2B$) is w.l.o.g. Altogether, to assume normalized payments $\hat{P}^s_\emptyset(t)=0$ is also w.l.o.g., so the virtual payments can be defined uniquely.

Definition 13. *We call a payment function* P monotone *if for every* s, *and sets of tasks* $\beta\subset\alpha$, *it holds that* $P_\beta(s)\leq P_\alpha(s)$.

Claim 5. *The virtual payments are normalized and monotone and* (A,P) *is a truthful mechanism.*

Proof. The virtual payment function P is obviously normalized and monotone. We show that P is truthful for the t-player for arbitrary s. The proof is simple, by exploiting the truthfulness of $\hat{P}(s)$.

We need to prove that the given allocation $A(t)$ maximizes the profit $P-t$ for every $t\in V^*$ (resp., for every $t\in V_+$). Let $A(t)=\alpha$. We show next that $P_\alpha - t_\alpha \geq P_\beta - t_\beta$ for all $\beta\in\{12,1,2,\emptyset\}$. Notice that by the truthfulness of the payments $\hat{P}$, and since $P_\alpha\geq\hat{P}_\alpha$, it holds for every β that $P_\alpha - t_\alpha \geq \hat{P}_\alpha - t_\alpha \geq \hat{P}_\beta - t_\beta$. This solves the cases in which $P_\beta=\hat{P}_\beta$, in particular the case $\beta=\emptyset$, because $P_\emptyset=\hat{P}_\emptyset=0$. This will imply the proof for $\beta=1$, and subsequently for $\beta=12$, as follows.

Assume that $\beta=1$ and $P_1>\hat{P}_1$ (the case $\beta=2$ is analogous). This happens only if $P_1=0$. Now, $P_\alpha - t_\alpha \geq P_\emptyset - t_\emptyset = 0-0=0\geq 0-t_1 = P_1-t_1$. Assume that $\beta=12$, and let w.l.o.g. $P_{12}=P_1$. Then $P_\alpha - t_\alpha \geq P_1 - t_1 = P_{12}-t_1 \geq P_{12}-t_{12}$. This concludes the proof. ∎

In fact, for any given WMON allocation function $A(t)$ as a function of the bid t of the t-player (in particular for the allocation function $A[s]$ for arbitrary fixed s), the virtual payments are the unique payments which are truthful, normalized, and monotone, as we show next.

Claim 6. *For every WMON allocation function* $A(t)$ *of the* t*-player (i.e., for fixed* s*) over* $t\in V^*$ *or over* $t\in V_+$, $A(t)$ *uniquely determines the virtual payments as the only truthful, normalized, and monotone payments. Conversely, the virtual payments determine the allocation* $A(t)$ *up to tie-breaking.*

Proof. For every V^* it is well known that the payments $P_{12},P_1,P_2,P_\emptyset$ uniquely determine the truthful allocation up to tie-breaking, since the allocation maximizes $P_\alpha - t_\alpha$ over $\alpha\in\{12,1,2,\emptyset\}$.

We prove the uniqueness of monotone, normalized, truthful payments for a given allocation function $A(t)$. By definition, $P_\emptyset=0$ must hold. We claim that if $R_1=\emptyset$, then $P_1 = 0$ must be the case. Assume the contrary; then by monotonicity, $P_1 > 0$ must hold. However, if $P_1 > 0$, then the points $t = (t_1 < P_1, t_2 = L, t_{12} = t_1 + L) \in V_+ \subset V^*$ (for large enough L) are in R_1 (by truthfulness), contradicting $R_1 = \emptyset$. If $R_1 \neq \emptyset$, then $P_1 = \sup\{t_1 \,|\, (t_1, L, t_1 + L) \in R_1\}$ is uniquely determined by $A(t)$. Note that the supremum cannot be infinite; otherwise $R_\emptyset = \emptyset$ would hold.[22] By a symmetric argument, P_2 is also uniquely determined by $A(t)$.

Consider now P_{12}. We claim that, if $R_{12} = \emptyset$, then $P_{12} = \max\{P_1, P_2, 0\}$ must be the case (these are unique by the paragraph above). Assume the contrary, that $P_{12} > \max\{P_1, P_2, 0\}$. Then, for every small enough $\epsilon > 0$ the point $t = (\epsilon, \epsilon, 2\epsilon) \in V_+ \subset V^*$ must be a point in R_{12}, contradicting $R_{12} = \emptyset$. Finally, if $R_{12} \neq \emptyset$, then $(0, 0, 0) \in R_{12}$ and R_{12} must have boundary points t with at least one of $R_\emptyset$, or R_1 or R_2, given that $R_\emptyset \neq \emptyset$. If R_{12} has a boundary with (say) R_1, then $P_{12} - P_1 = \sup\{t_2 \,|\, (0, t_2, t_2) \in R_{12}\}$, because this point (i.e., with the supremum t_2 value) must be on the boundary with R_1. If R_{12} has no common boundary with R_1 or R_2, then $P_{12} = \sup\{t_2 \,|\, (0, t_2, t_2) \in R_{12}\}$, because this point must be on the boundary with $R_\emptyset$. ∎

3.7.2. One-to-one correspondence of allocation and virtual payments. The virtual payments determine the allocation so that critical values for a bid getting/losing a task (for fixed other bids) are always nonnegative. In geometric terms, axis-parallel boundaries always have position at least 0. The next definition makes explicit how the virtual payments determine the allocation. $\overline{R}_\alpha$ is the region of t bids, where $A(t) = \alpha$ is *possible*. Intuitively, it is the closure of R_α; however, it is not necessarily the topological closure, e.g., if R_α is empty.

DEFINITION 14.

$$\begin{aligned}
\overline{R}_\emptyset &= \{t \in V^* \,|\, t_1 \geq P_1, t_2 \geq P_2, t_{12} \geq P_{12}\},\\
\overline{R}_1 &= \{t \in V^* \,|\, t_1 \leq P_1, t_1 \leq t_{12} + P_1 - P_{12}, t_1 \leq t_2 + P_1 - P_2\},\\
\overline{R}_2 &= \{t \in V^* \,|\, t_2 \leq P_2, t_2 \leq t_{12} + P_2 - P_{12}, t_2 \leq t_1 + P_2 - P_1\},\\
\overline{R}_{12} &= \{t \in V^* \,|\, t_{12} \leq P_{12}, t_{12} \leq t_1 + P_{12} - P_1, t_{12} \leq t_2 + P_{12} - P_2\}.
\end{aligned}$$

Importantly, as opposed to the regions R_α, for every fixed s, and every α, the $\overline{R}_\alpha$ is never empty. This holds for V_+, and since $V_+ \subset V^*$, also for every V^*. For example, $(0, 0, 0) \in \overline{R}_{12}$ and $(0, L, L) \in \overline{R}_1$ for large enough L always hold, because P_{12} is the largest payment, and P_1 is nonnegative. This property is crucial for the regions to uniquely determine the payments. Since the closures $\overline{R}$ correspond to possible allocations only, in the proofs we always pick bids t from the *interior* of some allocation region (there the allocation is uniquely determined).

For any fixed s, Claim 6 has a crucial, and maybe surprising, implication concerning a one-to-one relation between WMON allocations $A(t)$ over $t \in V^*$ and their restrictions to the additive plane $A|_{V_+}$.

LEMMA 13. *For every WMON allocation function $A(t)$ over $t \in V^*$ (in particular for the allocation function $A[s]$ for arbitrary fixed s) and its restriction to the additive plane $A_+ := A|_{V_+}$ it holds that A uniquely determines A_+, and conversely, up to tie-breaking, A_+ uniquely determines A over V^*. The virtual payments $P_{12}, P_1, P_2, P_\emptyset$ are the same for A and for A_+.*

[22] The infinite supremum would mean $\hat{P}_\emptyset = -\infty$; the assumption $R_\emptyset \neq \emptyset$ lets us exclude the use of (negative) infinite payments right from the start. Positive infinite payments should (and can) be excluded anyway.

Proof. The first statement, that A determines $A|_{V_+}$, is obvious. Since the payments for A are normalized, monotone, and truthful for A_+, by uniqueness these are the only possible virtual payments for A_+.

We show how the other implication (that A_+ determines A) also follows from the uniqueness of the normalized and monotone payments. Suppose that there exist two different WMON allocation functions over V^*, $A'(t)$ and $A''(t)$, such that $A_+(t) = A'|_{V_+}(t) = A''|_{V_+}(t)$ for every $t \in V_+$. By Claim 6 there exist unique virtual payments $P' = (P'_{12}, P'_1, P'_2, P'_\emptyset)$ and $P'' = (P''_{12}, P''_1, P''_2, P''_\emptyset)$, i.e., truthful, normalized, and monotone payments for A' and A'', respectively. Obviously, both are truthful, normalized, and monotone for the allocation function $A_+(t)$, when restricted to V_+. Since A_+ has unique truthful, normalized, and monotone payments, by Claim 6, it must be the case that $P'_{12} = P''_{12}$, $P'_1 = P''_1$, $P'_2 = P''_2$, and $P'_\emptyset = P''_\emptyset$. This, in turn, implies by the same claim applied on V^* that $A' \equiv A''$ up to tie-breaking, which concludes the proof. ∎

3.7.3. Illustration. We provide a geometric illustration to Lemma 13. For fixed s, the allocation function to the t-player $A(t_1, t_2, t_{12})$ has a very simple geometric representation in $\mathbb{R}^3_{\geq 0}$.

Consider the four (possible) allocation regions of the t-player in V as determined by the virtual payments: $\overline{R}_{12}, \overline{R}_1, \overline{R}_2, \overline{R}_\emptyset$. The boundaries in V^* or in V_+ between pairs of these regions, if they exist, must be *subsets* of the following planes, respectively:

$$\{t \in V \,|\, t_1 = P_1\} \qquad (R_1|R_\emptyset),$$

$$\{t \in V \,|\, t_2 = P_2\} \qquad (R_2|R_\emptyset),$$

$$\{t \in V \,|\, t_{12} = P_{12}\} \qquad (R_{12}|R_\emptyset),$$

$$\{t \in V \,|\, t_{12} - t_1 = P_{12} - P_1\} \qquad (R_{12}|R_1),$$

$$\{t \in V \,|\, t_{12} - t_2 = P_{12} - P_2\} \qquad (R_{12}|R_2),$$

$$\{t \in V \,|\, t_1 - t_2 = P_1 - P_2\} \qquad (R_2|R_1).$$

All six of these planes have a common intersection point in V, namely the point $t = (P_1, P_2, P_{12})$. By monotonicity of the virtual payments, this common point is in the domain V of monotone valuations, and it is a boundary point of all four regions in V (the point may not be in V^*, and for noncrossing allocations it is not in V_+). Moreover, in V these boundaries always exist (not only their planes). Obviously, they can be degenerate (think of $(P_1, P_2, P_{12}) = (0, 0, 0)$ or $(P_1, P_2, P_{12}) = (P, P, P)$), but the common point exists and uniquely determines the boundary planes. Their positions determine the allocation and vice-versa. The four regions in $V \subset \mathbb{R}^3_{\geq 0}$, of a general WMON allocation have thus a rather simple and visible structure.

Let us now consider the additive plane V_+; this plane always intersects the first two boundaries ($R_1|R_\emptyset$ and $R_2|R_\emptyset$), so that the allocation on the additive plane uniquely determines P_1 and P_2. Finally, always at least one of the third, fourth, and fifth boundaries intersects (as a boundary, which is possibly degenerate) the additive plane V_+, which determines P_{12}.

In particular, the interior of a region R_α of the t-player is empty in V_+ if and only if it is empty in V (and consequently, if and only if it is empty in V^*, since $V_+ \subset V^* \subseteq V$). This might seem surprising, since it could have happened that the additive plane V_+ does not intersect some nonempty region $R_\alpha \subset V$. On the other hand, it is obvious by the facts that ($R_1 \neq \emptyset \Leftrightarrow P_1 > 0$), ($R_2 \neq \emptyset \Leftrightarrow P_2 > 0$), and ($R_{12} \neq \emptyset \Leftrightarrow P_{12} > \max(P_1, P_2, 0)$) hold in each of the domains.

3.7.4. Extending the additive characterization to $V^*\times V_{+,B}$. In what follows, we characterize all truthful mechanisms for two tasks and two bidders with valuations from the domains $V^* \times V_{+,B}$, based on the characterization on $V_+ \times V_{+,B}$. According to Theorem 3, every truthful mechanism on $V_+ \times V_{+,B}$ is one of the following four types: (1) relaxed affine minimizer, (2) relaxed task-independent mechanism, (3) 1-dimensional mechanism, (4) constant mechanism. Let A be some WMON allocation on $V^* \times V_{+,B}$. Its restriction $A|_{V_+\times V_{+,B}}$ to additive valuations of the t-player is a WMON allocation for additive valuations and therefore is of one of the types (1) to (4) above.

LEMMA 14. *Let A be some WMON allocation on $V^* \times V_{+,B}$. If $A|_{V_+\times V_{+,B}}$ is a mechanism of type* (1) *relaxed affine minimizer,* (3) *one-dimensional, or* (4) *constant, then A over $V^*\times V_{+,B}$ must be the same mechanism, with the same virtual payments.*

Proof. Let $A|_{V_+\times V_{+,B}}$ be, for instance, a (relaxed) affine minimizer. Then, for every fixed $s \in V_{+,B}$ the allocation $A[s](t)$ and the virtual payments for the t-player on $t \in V_+$ are those of this affine minimizer. By uniqueness (Lemma 13), they extend to $t \in V^*$ uniquely to be the allocation and payments of the same affine minimizer (i.e., we know that the extended affine minimizer allocation with the same parameters is truthful for the t-player over V^*, so this must be the unique extension of $A[s]$). This, in turn, means that for every fixed $t \in V^*$ the s-player on $s \in V_{+,B}$ has allocation and payments according to this affine minimizer over $V^* \times V_{+,B}$. In detail, for every s, the affine-minimizer payments $P_\alpha(s)$ over V^* determine the allocation to the t-player, and this, in turn, determines the allocation to the given s for every single $t \in V^*$— all these necessarily according to the particular affine minimizer's parameters. Once the allocation $A[t](s)$ is determined, this uniquely yields the virtual payments to the s-player, according to the same affine minimizer. Altogether the mechanism on $V^*\times V_{+,B}$ is the same affine minimizer, with the same parameters, and the same virtual payments as on $V_+ \times V_{+,B}$. The argument is analogous for 1-dimensional mechanisms and for constant mechanisms. ◻

On the other hand, for relaxed task-independent mechanisms we show next that these are *not* extendable to $V^* \times V_{+,B}$, except for those that are at the same time affine minimizers, constant mechanisms, or 1-dimensional mechanisms.

LEMMA 15. *Let A be some WMON allocation on $V^* \times V_{+,B}$. Then $A|_{V_+\times V_{+,B}}$ cannot be a relaxed task-independent mechanism unless it is at the same time also of type* (1)*,* (3)*, or* (4)*.*

Proof. A relaxed task-independent mechanism allocates the two tasks by independent monotone allocations, except for (s_1, s_2) (resp., (t_1, t_2)) points where the critical value functions $\phi()$ and $\eta()$ of both of these mechanisms have a jump discontinuity. Since there are at most countably many s_1, s_2, t_1, and t_2 where these increasing functions of the 1-dimensional mechanisms can have a jump, it will be easy to handle these s and t values in the following discussion.

Assume that on $V_+ \times V_{+,B}$ the payments to the t-player, according to the task-independent mechanism, are

$$P_1(s) = \phi(s_1), \quad P_2(s) = \eta(s_2), \quad P_{12}(s) = \phi(s_1) + \eta(s_2),$$

where ϕ and η are arbitrary nonnegative increasing functions over $s_1, s_2 \in [0, B)$. We can assume that the values of ϕ and η are finite, because $R_\emptyset = \emptyset$ would hold for the t-player in every point (s_1, s_2) where $\phi(s_1) = \infty$ or $\eta(s_2) = \infty$.

We assume that the inverse ϕ^{-1} of an increasing function ϕ is constant over intervals where ϕ has a jump, and conversely, the inverse ϕ^{-1} has a jump discontinuity in c, and $\phi^{-1}(c)$ is defined in an arbitrary monotone way, when the original function is constant c over some interval. If $\phi(s_1)$ has a jump in some $\bar{s}_1$ point, there $\phi(\bar{s}_1, s_2) \neq \phi(\bar{s}_1, s_2')$ is possible, so in such points the mechanism is not task-independent.

By Lemma 13, the same payments to the t-player hold over $t \in V^*$.

In what follows, we consider the allocation and payments to the s-player on $V^* \times V_{+,B}$ for some carefully chosen $t \in V_\epsilon \cap V_{sub} \subset V^*$ (resp., $t \in V_\epsilon \cap V_{sup} \subset V^*$) values and will conclude that the mechanism is truthful only if $\phi \equiv 0$ or $\eta \equiv 0$ (implying a 1-dimensional mechanism), or if ϕ and η are both linear by the same multiplicative constant, which corresponds to a task-independent affine minimizer.[23]

We use the notation $\phi(B) = \lim_{s_1 \to B^-} \phi(s_1)$ and $\eta(B) = \lim_{s_2 \to B^-} \eta(s_2)$, and furthermore $\phi(0) = \lim_{s_1 \to 0^+} \phi(s_1)$ and $\eta(0) = \lim_{s_2 \to 0^+} \eta(s_2)$.[24] The case when $\phi(0) = \phi(B)$ *and* $\eta(0) = \eta(B)$ yields a constant mechanism. The cases when $\phi(0) = \phi(B) = 0$ or $\eta(0) = \eta(B) = 0$ correspond to 1-dimensional mechanisms. The following general claim implies that mechanisms with $\phi(0) = \phi(B) = C > 0$ (or $\eta(0) = \eta(B) = C > 0$) do *not* extend to $V^* \times V_{+,B}$ when $C > 0$ (unless both ϕ and η are constants). In particular the claim excludes that $\phi(s_1)$ is a strictly positive constant C over some nonzero interval of s_1 values if $\eta(s_2)$ is not a constant.

CLAIM 7. *Assume that in a relaxed task-independent allocation A_+ over additive t and s, $\phi(s_1) = \phi(s_1') = C > 0$ for some $s_1 < s_1'$, but $\eta(s_2)$ takes at least two different values. Then $A_+ \neq A|_{V_+ \times V_{+,B}}$.*

Proof. We assume the contrary and obtain a contradiction separately over the subadditive and the superadditive domains for t. Let $s_1 < s_1'$ be as in the statement of the lemma.

Consider the subadditive case $V_{sub} \cap V_\epsilon \subseteq V^*$ first. Since η is nondecreasing, and by the indirect assumption, we can find arbitrarily close values $s_2' < s_2$, and so that $\eta(s_2') < \eta(s_2)$ (otherwise $\eta(s_2') = \eta(s_2)$ would hold whenever $s_2 < s_2' + \delta$ for some positive δ, and η would be constant over $(0, B)$). Thus $s_2' < s_2$ exist so that $0 < s_2 - s_2' < s_1' - s_1$, and $\eta' := \eta(s_2') < \eta(s_2) =: \eta$. We show that a $\hat{t} \in V_{sub} \cap V_\epsilon$ exists, so that for $s' = (s_1', s_2')$ the point $\hat{t} \in R_\emptyset(s')$, but for $s = (s_1, s_2)$ it is $\hat{t} \in R_{12}(s)$; however, then $s' \in R^s_{12}(\hat{t})$, and $s \in R^s_\emptyset(\hat{t})$, contradicting WMON, since $s_1 + s_2 < s_1' + s_2'$. We pick $\hat{t}$ so that

$$\hat{t}_1 = C + \epsilon', \qquad \eta' < \hat{t}_2 < \eta, \qquad \hat{t}_{12} = \hat{t}_1 + \hat{t}_2 - \epsilon'',$$

where $\epsilon' < \epsilon'' < \hat{t}_2 - \eta'$ are small enough so that $\hat{t}$ fits into the domain.[25] Recall that $P_1(s) = C$, $P_2(s) = \eta$, $P_{12}(s) = C + \eta$, and analogously for s' and η'. Due to $\hat{t}_1 > C$, the allocation $\alpha = 1$ will not occur for $\hat{t}$. For s' we have $0 > \eta' - \hat{t}_2$ and $0 > -\epsilon' + \eta' - \hat{t}_2 + \epsilon'' = C - \hat{t}_1 + \eta' - \hat{t}_2 + \epsilon'' = C + \eta' - \hat{t}_{12}$, which yields $\hat{t} \in R_\emptyset(s')$. For s we have $\eta + C - \hat{t}_{12} = \eta + C - \hat{t}_1 - \hat{t}_2 + \epsilon'' = \eta - \hat{t}_2 - \epsilon' + \epsilon'' > 0$, and $\eta + C - \hat{t}_{12} = \eta + C - \hat{t}_1 - \hat{t}_2 + \epsilon'' > \eta - \hat{t}_2$, which yields $\hat{t} \in R_{12}(s)$.

Consider now the superadditive case $V_{sup} \cap V_\epsilon \subseteq V^*$. Since η is nondecreasing, and by the indirect assumption, $s_2 < s_2'$ exist so that $0 < s_2' - s_2 < s_1' - s_1$, and

[23] A task-independent affine minimizer minimizes among t_{12}, $t_1 + \mu s_2 + \gamma_1$, $\mu s_1 + t_2 + \gamma_2$, $\mu(s_1 + s_2) + \gamma_1 + \gamma_2$ for some given constants $\mu \geq 0$, γ_1, γ_2. If $\mu = 0$, we obtain a constant mechanism.

[24] Since in $s_i = 0$ even affine minimizers might (change and) have jumps, we don't consider the original $\phi(0)$ and $\eta(0)$ values.

[25] Here we exploit $C > 0$; otherwise $\hat{t}_1 = \epsilon'$ and $\hat{t}_{12} < \hat{t}_2$ would occur.

$\eta := \eta(s_2) < \eta(s_2') =: \eta'$. We show that a $\tilde{t} \in V_{sup} \cap V_\epsilon$ exists, so that for $s' = (s_1', s_2')$ the point $\tilde{t} \in R_2(s')$, but for $s = (s_1, s_2)$ it is $\tilde{t} \in R_1(s)$; however, then $s' \in R_1^s(\tilde{t})$, and $s \in R_2^s(\tilde{t})$, contradicting WMON, since $s_2' - s_1' < s_2 - s_1$. We pick $\tilde{t}$ so that

$$\tilde{t}_1 = C - \epsilon', \qquad \eta < \tilde{t}_2 < \eta', \qquad \tilde{t}_{12} = \tilde{t}_1 + \tilde{t}_2 + \epsilon'',$$

where $\epsilon' < \epsilon'' < \eta' - \tilde{t}_2$ are small enough so that $\tilde{t}$ fits into the domain. Due to $\tilde{t}_1 < C$, the allocation $\alpha = \emptyset$ will not occur for $\tilde{t}$. For s' we have $\eta' - \tilde{t}_2 > \epsilon' = C - \tilde{t}_1$ and $\eta' - \tilde{t}_2 > \eta' - \tilde{t}_2 + \epsilon' - \epsilon'' = \eta' - \tilde{t}_2 + C - \tilde{t}_1 - \epsilon'' = \eta' + C - \tilde{t}_{12}$, which yields $\tilde{t} \in R_2(s')$. For s we have $C - \tilde{t}_1 > 0 > \eta - \tilde{t}_2$, and $C - \tilde{t}_1 > \eta - \tilde{t}_2 + C - \tilde{t}_1 - \epsilon'' = \eta + C - \tilde{t}_{12}$, which yields $\tilde{t} \in R_1(s)$. ∎

Note that since ϕ is increasing, it is continuous in almost every point s_1. In the next claim, basically the same argument as in Claim 7 proves that η cannot have jump discontinuities. A symmetric statement results in the same for ϕ.

CLAIM 8. *Assume that in a relaxed task-independent allocation A_+ over additive t and s, the function ϕ is continuous in the point s_1, and η has a jump discontinuity in $s_2 > 0$. Then $A_+ \neq A|_{V_+ \times V_{+,B}}$.*

Proof. The proof is almost the same as for the previous claim, and we show here the necessary changes only for the subadditive case $V_{sub} \cap V_\epsilon \subseteq V^*$.

We assume for contradiction w.l.o.g. that η has a left-jump in s_2 (otherwise we change the roles of s_2' and s_2). Let $\eta(s_2) = \eta > \bar{\eta} = \eta(s_2^-)$, where the latter is the limit of η in s_2 from the left. We choose $0 < \epsilon' < \epsilon'' < (\eta - \bar{\eta})/2$. By the continuity of ϕ in s_1 we can choose $s_1' > s_1$ such that $\phi(s_1') < \phi(s_1) + \epsilon'$. Finally, we select $s_2' < s_2$ so that $0 < s_2 - s_2' < s_1' - s_1$. We use the notation $\eta' = \eta(s_2') \leq \bar{\eta} < \eta$, as well as $C = \phi(s_1)$ and $C' = \phi(s_1')$. Recall that $C \leq C' < C + \epsilon'$. The remaining part of the proof carries over almost completely.

We show that a $\hat{t} \in V_{sub} \cap V_\epsilon$ exists, so that for $s' = (s_1', s_2')$ the point $\hat{t} \in R_\emptyset(s')$, but for $s = (s_1, s_2)$ it is $\hat{t} \in R_{12}(s)$; however, then $s' \in R_{12}^s(\hat{t})$, and $s \in R_\emptyset^s(\hat{t})$, contradicting WMON, since $s_1 + s_2 < s_1' + s_2'$. We can pick $\hat{t}$ so that

$$\hat{t}_1 = C + \epsilon' = C' + \epsilon^\circ, \qquad \eta' + \epsilon'' < \hat{t}_2 < \eta, \qquad \hat{t}_{12} = \hat{t}_1 + \hat{t}_2 - \epsilon'',$$

where $0 < \epsilon^\circ < \epsilon' < \epsilon'' < \hat{t}_2 - \eta'$ hold.

Recall that $P_1(s) = C$, $P_2(s) = \eta$, $P_{12}(s) = C + \eta$, and analogously $P_1(s') = C'$, $P_2(s') = \eta'$, $P_{12}(s') = C' + \eta'$. Due to $\hat{t}_1 > C' > C$, the allocation $\alpha = 1$ will not occur for $\hat{t}$. For s' we have $0 > \eta' - \hat{t}_2$ and $0 > -\epsilon^\circ + \eta' - \hat{t}_2 + \epsilon'' = C' - \hat{t}_1 + \eta' - \hat{t}_2 + \epsilon'' = C' + \eta' - \hat{t}_{12}$, implying that $\hat{t} \in R_\emptyset(s')$. For s we have $\eta + C - \hat{t}_{12} = \eta + C - \hat{t}_1 - \hat{t}_2 + \epsilon'' = \eta - \hat{t}_2 - \epsilon' + \epsilon'' > 0$, and $\eta + C - \hat{t}_{12} = \eta + C - \hat{t}_1 - \hat{t}_2 + \epsilon'' = \eta - \epsilon' - \hat{t}_2 + \epsilon'' > \eta - \hat{t}_2$, which yields $\hat{t} \in R_{12}(s)$. ∎

In the rest of the proof of the lemma we assume $\phi(0) < \phi(B)$ and $\eta(0) < \eta(B)$. For this case the previous two claims imply that ϕ and η can have neither jump discontinuities nor a positive constant value over any interval of nonzero length; that is, they are basically strictly increasing and continuous. This implies that over $t_1 \in (\phi(0), \phi(B))$, resp., $t_2 \in (\eta(0), \eta(B))$, and $s_1, s_2 \in (0, B)$, it holds that $\phi(s_1) = t_1$ iff $\phi^{-1}(t_1) = s_1$, and analogously for η. The proof of the above claims would even show (after some fine-tuning) that ϕ and η must have the same derivatives in every point. We show here the linearity of ϕ and η using a more global point of view.

DEFINITION 15. *We will say that*
- $t_1 > 0$ *is* valid *if* $t_1 \in (\phi(0), \phi(B))$*;*
- $t_2 > 0$ *is* valid *if* $t_2 \in (\eta(0), \eta(B))$*;*
- $t = (t_1, t_2, t_{12}) \in V$ *is* valid *if each of* t_1, $t_1' = t_{12} - t_2$, t_2, *and* $t_2' = t_{12} - t_1$ *is valid.*

Observe that if t_1 and t_2 are valid, then the point $(t_1, t_2, t_1 + t_2) \in V_+$ is valid, and for all $\epsilon' < \epsilon$, the points $(t_1, t_2, t_1 + t_2 - \epsilon')$ are in $V_\epsilon \cap V_{sub}$ (resp., $(t_1, t_2, t_1 + t_2 + \epsilon')$ are in $V_\epsilon \cap V_{sup}$) and they are valid if $t_1 \in (\phi(0) + \epsilon', \phi(B) - \epsilon')$ and $t_2 \in (\eta(0) + \epsilon', \eta(B) - \epsilon')$ and if $\epsilon' < \min(\epsilon, (\phi(B) - \phi(0))/4, (\eta(B) - \eta(0))/4)$.

Our goal is to show that the restriction of ϕ^{-1} to $(\phi(0) + 2\epsilon', \phi(B) - 2\epsilon')$ is identical to a linear function, and analogously for t_2 and η^{-1}; moreover ϕ^{-1} and η^{-1} have the same slope. Then, by taking $\epsilon' \to 0$, we obtain the linearity of ϕ^{-1} and η^{-1} over $(\phi(0), \phi(B))$ and $(\eta(0), \eta(B))$, respectively. This implies directly that $\phi : (0, B) \to (\phi(0), \phi(B))$ (resp., $\eta : (0, B) \to (\eta(0), \eta(B))$) are piecewise linear functions with three parts: two constant parts of value $\phi(0)$ and $\phi(B)$ (resp., $\eta(0)$ and $\eta(B)$) with a linear part in between. However, by Claim 7 constant pieces with *strictly positive* value can be excluded: either the function ϕ has no constant part, or it starts with a constant part of value $\phi(0) = 0$ followed by a linear part, and similarly for η. This will prove that (restricted to positive s and t) the allocation $A|_{V_+ \times V_{+,B}}$ is a task-independent affine minimizer, and A is necessarily the same affine minimizer on $V^* \times V_{+,B}$.

First we argue that for all valid t the regions $R^s_\emptyset(t) \subset V_+$ and $R^s_1(t) \subset V_+$ are nonempty; in particular there exists $s \in V_{+,B}$ on the boundary of these two regions. Such an s on the boundary will help us to obtain a formula for $P^s_1(t)$. Showing the existence of such an s is somewhat technical.

Consider an arbitrary fixed valid $\bar{t} \in V^*$. Since $\bar{t}_2 < \eta(B)$, we have $\eta^{-1}(\bar{t}_2) < B$. For every $s \in V_{+,B}$ so that $\eta^{-1}(\bar{t}_2) < s_2$, it holds that $\bar{t}_2 < \eta(s_2)$. Call every such s *good.* For every good s and the fixed $\bar{t}$, the t-player receives task 2. Moreover, since η^{-1} is continuous in t_2, for every fixed good s the same holds even for *all* points in some neighborhood of $\bar{t}$.

Now, for every good s, the s is on the boundary of $R^s_\emptyset(\bar{t})$ and $R^s_1(\bar{t})$ iff $s_1 = P^s_1(\bar{t})$ (here we use that s cannot get task 2). On the other hand, s is on the boundary iff at the same time $\bar{t}$ is on the boundary of $R_{12}(s)$ and $R_2(s)$, which happens when $\bar{t}_{12} - \bar{t}_2 = P_{12}(s) - P_2(s)$. We know that $P_{12}(s) - P_2(s) = \phi(s_1)$. Therefore, we have to choose s_1 such that $\bar{t}_{12} - \bar{t}_2 = \phi(s_1)$.[26]

In summary, for the fixed valid $\bar{t}$ and the selected s it must hold that

$$P^s_1(\bar{t}) = s_1 = \phi^{-1}(\bar{t}_{12} - \bar{t}_2).$$

This yields the formula for $P^s_1(t)$ for any valid t independently of s. Analogously, we obtain for every valid t that

$$P^s_2(t) = \eta^{-1}(t_{12} - t_1).$$

Next we determine P^s_{12} in two different ways for every valid t. First we claim that if t is valid, then $R^s_{12}(t)$ is nonempty. Indeed, by the validity of t, there exist s such that $s_1 < \phi^{-1}(t_1)$, and $s_2 < \eta^{-1}(t_2)$, and $s_2 < \eta^{-1}(t_{12} - t_1)$. For all such s it holds that $t_1 > \phi(s_1) = P_1(s)$, and $t_2 > \eta(s_2) = P_2(s)$, and $t_{12} > \eta(s_2) + t_1 > \eta(s_2) + \phi(s_1) = P_{12}(s)$. Therefore, $t \in R_\emptyset(s)$. This implies that the (open) set of all such s must belong to $R^s_{12}(t)$.

[26]Here we use time and again that all boundary functions are strictly increasing, with well-defined inverse.

Second, we claim that there exists an s' on the boundary of $R^s_{12}(t)$ and $R^s_2(t)$, and another s'' on the boundary of $R^s_{12}(t)$ and $R^s_1(t)$. Indeed, let s' be such that $s'_1 = \phi^{-1}(t_1)$, and $s'_2 < \eta^{-1}(t_2)$, and $s'_2 < \eta^{-1}(t_{12} - t_1)$. Then, $t_2 > \eta(s'_2) = P_2(s')$, and $t_{12} > \eta(s'_2) + t_1 = \eta(s'_2) + \phi(s'_1) = P_{12}(s')$, so the t-player does not get task 2 even with a bid in a small enough neighborhood of t, and because of $t_1 = \phi(s'_1)$, t must be on the boundary of $R_1(s')$ and $R_\emptyset(s')$, so s' is on the boundary of $R^s_2(t)$ and $R^s_{12}(t)$ (here we use that s'_1 is uniquely defined as $\phi^{-1}(t_1)$). The proof is analogous for s''. By the boundary position of s',

$$P^s_{12} - P^s_2 = s'_1 = \phi^{-1}(t_1).$$

Plugging in P^s_2 we obtain

$$P^s_{12} = \phi^{-1}(t_1) + \eta^{-1}(t_{12} - t_1).$$

On the other hand, using s'' we obtain by an analogous argument that

$$P^s_{12} = \eta^{-1}(t_2) + \phi^{-1}(t_{12} - t_2).$$

This yields that

$$\phi^{-1}(t_1) + \eta^{-1}(t_{12} - t_1) = \eta^{-1}(t_2) + \phi^{-1}(t_{12} - t_2)$$

for *every* valid t.

Let $\epsilon' < \epsilon$. Consider a $\bar{t} \in V_\epsilon \cap V_{sub}$ (resp., $V_\epsilon \cap V_{sup}$) such that $t_1 \in (\phi(0) + 2\epsilon', \phi(B) - 2\epsilon')$ and $t_2 \in (\eta(0) + 2\epsilon', \eta(B) - 2\epsilon')$. Increasing now only $\bar{t}_1$ by an arbitrary positive $\delta < \epsilon'$, the right-hand side of the last equality does not change. This is possible only if on the left-hand side $\phi^{-1}(\bar{t}_1)$ and $\eta^{-1}(\bar{t}_{12} - \bar{t}_1)$ increases (resp., decreases) by the same value $\Delta := \phi^{-1}(\bar{t}_1 + \delta) - \phi^{-1}(\bar{t}_1) = \eta^{-1}(\bar{t}_{12} - \bar{t}_1) - \eta^{-1}(\bar{t}_{12} - \bar{t}_1 - \delta)$.

Now fix $\bar{t}_1, \bar{t}_2$, and δ (thereby fixing Δ), and consider *every* $t_{12} \in (\bar{t}_1 + \bar{t}_2 - \epsilon', \bar{t}_1 + \bar{t}_2)$. For every such t_{12} it holds by the same argument that $\eta^{-1}(t_{12} - \bar{t}_1) - \eta^{-1}(t_{12} - \bar{t}_1 - \delta) = \Delta$ with the fixed Δ. The last statement can be rewritten as $\eta^{-1}(x) - \eta^{-1}(x - \delta) = \Delta$ for every $x \in (\bar{t}_2 - \epsilon', \bar{t}_2)$. By using $\delta/2$ instead of δ, the same argument necessarily yields $\eta^{-1}(x) - \eta^{-1}(x - \delta/2) = \Delta/2$. Taking $\delta/2^r$ we obtain $\eta^{-1}(x) - \eta^{-1}(x - \delta/2^r) = \Delta/2^r$, and letting $r \to \infty$ implies—by the monotonicity of η^{-1}—that η^{-1} must be *linear* on the interval $(\bar{t}_2 - \epsilon', \bar{t}_2)$. (The proof of this is similar to the proof of Lemma 7.)

As a final step, change $\bar{t}_2$ to an arbitrary other valid t'_2. Here again, by the same argument η^{-1} is linear over $(t'_2 - \epsilon', t'_2)$. Since such overlapping open intervals cover the interval $(\eta(0) + 2\epsilon', \eta(B) - 2\epsilon')$, the function η^{-1} must be a single linear function over $(\eta(0) + 2\epsilon', \eta(B) - 2\epsilon')$. By taking the limit $\epsilon' \to 0$ we obtain that ϕ^{-1} is linear over $(\eta(0), \eta(B))$. By a symmetric argument, ϕ^{-1} is linear, too, and ϕ^{-1} and η^{-1} have the same multiplicative constant Δ/δ as follows from $\Delta = \phi^{-1}(\bar{t}_1 + \delta) - \phi^{-1}(\bar{t}_1) = \eta^{-1}(\bar{t}_{12} - \bar{t}_1) - \eta^{-1}(\bar{t}_{12} - \bar{t}_1 - \delta)$. This concludes the proof that the allocation function is a task-independent affine minimizer on $V^* \times V_{+,B}$. ∎

This concludes the proof that any truthful mechanism (A, P) on $V^* \times V_{+,B}$ is a relaxed affine minimizer, a one-dimensional mechanism, or a constant mechanism.

4. Lower bound. In this section, we give a proof of our main theorem (Theorem 1). First in section 4.1 we describe the domain of instances that we use, and in section 4.2 we use the characterization for two machines and two tasks (Theorem 4) to establish that the only interesting mechanisms are linear (see section 4.2 for a precise definition); then in section 4.3 we explore the linearity property of mechanisms with bounded approximation ratio to establish some useful locality lemmas that are eventually used in section 4.4 to complete the proof.

4.1. The construction. To prove the lower bound, we focus on the domain of $2(n-1)$ tasks and n players. Player 0 is special and for convenience we use the symbol t for its values; sometimes we refer to it as the t-player. We use the symbol s for the values of the remaining players $1,\dots,n-1$, and sometimes we refer to them as the s-players.

The set of tasks $M=\{1,1',2,2',\dots,n-1,(n-1)'\}$ is partitioned into pairs and each pair $\{i,i'\}$ is associated with player i, $i=1,\dots,n-1$. We call the two tasks of each pair $\{i,i'\}$ *twin tasks.*

Let $v_i(S)$ denote the cost (valuation) of player i when it takes the subset $S\subseteq M$ of tasks.

- the cost of the s-players is *additive*: $v_i(S)$ is additive for $i\geq 1$.
- the cost of the s-players for tasks not in their associated pair is a sufficiently large fixed constant $\Theta\gg 4n^2$: for distinct $i,j\geq 1$, $v_i(\{j\})=v_i(\{j'\})=\Theta$.
- the cost of the t-player for twin tasks is submodular: for every $i=1,\dots,n-1$, the restriction of $v_0(S)$ to $S\subseteq\{i,i'\}$ is submodular.[27]
- the cost of the t-player is *additive across pairs*: $v_0(S)=\sum_{i=1}^{n-1}v_0(S\cap\{i,i'\})$. Therefore $v_0(S)$ is *submodular,* as the sum of submodular functions.

To simplify the notation we will denote an instance that satisfies the above conditions by

$$(t_i,\, s_i,\, t_{i'},\, s_{i'},\, t_{i,i'})_{i=1}^{n-1}, \tag{4.1}$$

where

- s_i and $s_{i'}$ are the costs of the ith s-player for the tasks in its associated pair $\{i,i'\}$,
- t_i or $t_{i'}$, resp., is the cost of the t-player when it takes *only one* of the twin tasks $\{i,i'\}$,
- $t_{i,i'}$ is the cost of the t-player when it takes *both twin tasks* $\{i,i'\}$.

With the exception of costs $t_{i,i'}$ an instance is captured by the following matrix indexed by players and tasks, which shows the cost of each when a player gets no other task:

$$\begin{bmatrix} t_1 & t_{1'} & t_2 & t_{2'} & \cdots & t_{n-1} & t_{(n-1)'} \\ s_1 & s_{1'} & \Theta & \Theta & \cdots & \Theta & \Theta \\ \Theta & \Theta & s_2 & s_{2'} & \cdots & \Theta & \Theta \\ & & & \vdots & & & \\ \Theta & \Theta & \Theta & \Theta & \cdots & s_{n-1} & s_{(n-1)'} \end{bmatrix}.$$

If the valuations of all players were additive, this matrix would be sufficient to determine the cost for all bundles. The instances we consider have more general cost functions and include the submodular valuation of twin tasks for the t-player. Since for two tasks, a function is submodular if and only if it is subadditive, values $t_{i,i'}$ satisfy

$$t_{i,i'}\leq t_i+t_{i'}.$$

It is useful to think of the value Θ as practically infinite, since it is much larger than the other values. On the other hand, to prove our main theorem in its generality, we need this value to be finite, and this complicates the characterization of truthful mechanisms.

[27]The construction is analogous in the case of supermodular valuations.

We focus on instances that satisfy $s_i \in [0,1]$, $s_{i'} = n$, $t_{i'} = 0$, $t_{i,i'} = t_i + t_{i'}$ for all $i \in [n-1]$. Note the subtlety here: while the domain contains instances with $t_{i,i'} \leq t_i + t_{i'}$, for the proof of the lower bound, we consider the subclass of additive instances. This should not be surprising in the sense that lower bound proofs usually employ a subclass of instances. Still, it raises the question of whether we could carry out the same proof in the additive domain. The answer is negative, because the sets of mechanisms for additive and submodular (subadditive) domains for two tasks are different; for example, task-independent mechanisms are not truthful for submodular domains.

Definition 16 (restricted (t,s) instance). *Instances of form* $(t_i, s_i, t_{i'}, s_{i'}, t_{i,i'})_{i=1}^{n-1}$ (4.1) *that satisfy*

$$s_i \in [0,1], \quad s_{i'} = n, \quad t_{i'} = 0, \quad t_{i,i'} = t_i + t_{i'} \qquad \text{for all } i \in [n-1], \tag{4.2}$$
$$(t,s) = ((t_i)_{i=1}^{n-1}, (s_i)_{i=1}^{n-1})$$

will be called restricted (t,s) instances.

Note that, unlike the other values, the t_i values can be arbitrarily high. This will be useful later (Lemma 16). Note also that the optimum makespan of every restricted instance is at most 1. For these instances, any algorithm with approximation ratio less than n must allocate all the i' tasks to the t-player and every i task either to the associated s_i-player or to the t-player.

4.2. From affine minimizers for twin tasks to linear mechanisms. In this section, we use the characterization of mechanisms (Theorem 4) for *each pair of twin tasks* in an instance $(t_i, s_i, t_{i'}, s_{i'}, t_{i,i'})_{i=1}^{n-1}$ to derive an essential property of mechanisms for the restricted (t,s) instances. By the characterization, there are mechanisms for twin tasks, such as the one-dimensional, constant, or bundling tails of relaxed affine minimizers, that may have nonaffine boundaries between different allocations. In this section, we show that mechanisms with nonaffine boundaries are also excluded, when we additionally require that the mechanisms have small approximation ratio—on whole instances, not just on a pair of twin tasks.

First we define the notion of boundaries that we consider in the proof of the lower bound.

Definition 17. *For a given allocation algorithm, considered on restricted* (t,s) *instances, we call* $\psi_i(s_i, s_{-i}, t_{-i})$ *a* critical value *or* boundary *if the t-player receives task i when* $t_i < \psi_i(s_i, s_{-i}, t_{-i})$, *and does not receive task i when* $t_i > \psi_i(s_i, s_{-i}, t_{-i})$ *(for example, in Figure* 1, ψ_1 *is shown with dashed lines).*

We claim below that a truthful allocation with a reasonable approximation ratio (say, of at most n) for the restricted (t,s) instances satisfies the following important property.

Definition 18 (linear mechanisms). *An allocation algorithm for the restricted* (t,s) *instances* (4.2) *is called* linear *if the critical values for task i,* $i = 1, \dots, n-1$, *are truncated affine functions in* s_i. *In particular when*

$$\psi_i(s_i, s_{-i}, t_{-i}) = \max(0,\, \lambda_i(s_{-i}, t_{-i})\, s_i + \nu_i(s_{-i}, t_{-i})),$$

for some $\lambda_i(s_{-i}, t_{-i}) > 0$. *We call a mechanism linear if it uses a linear allocation algorithm.*

We emphasize that this paper considers (relaxed) affine minimizers only for 2 players and 2 tasks. In contrast, here we defined the more general *linearity* property for mechanisms for n machines and $2(n-1)$ tasks. We have no global characterization of all truthful mechanisms of this dimension. The following proof of linearity applies the 2×2 characterization for twin tasks when the rest of the instance is fixed as in a restricted instance.

LEMMA 16. *A mechanism for the instances* $(t_i, s_i, t_{i'}, s_{i'}, t_{i,i'})_{i=1}^{n-1}$ (4.1) *with approximation ratio less than n must be linear for the restricted* (t,s) *instances.*

Proof. By Lemma 1, the restriction of the mechanism for the twin tasks i, i' is weakly monotone, and therefore by the characterization theorem (Theorem 4), it is either a relaxed affine minimizer, a one-dimensional mechanism, or a constant mechanism. We will show that if the approximation ratio is less than n, the mechanism cannot be one-dimensional, constant, or the bundling part of a relaxed affine minimizer. The reason is that algorithms in all these cases are so inefficient that their makespan only for these two tasks is large enough to show a large approximation ratio for the entire instance.

Let's first argue that it cannot be a one-dimensional or a constant mechanism. We fix all values of the other tasks as in restricted instances. There are (not restricted) values for the twin tasks i, i' for which the algorithm has makespan $\mu\geq 2n$ and approximation ratio at least $2n$. (For each one-dimensional or constant allocation this holds for at least one of the inputs $\left[\begin{smallmatrix}2n & 0\\ 0 & 2n\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2n & 0\\ 4n^2 & 0\end{smallmatrix}\right]$.) Since the optimal makespan of the other tasks of a restricted instance is at most 1, the approximation ratio is at least $\mu/(1+\mathrm{opt}_{\{i,i'\}})\geq \mu/(1+\mu/2n)\geq 2n\mu/2\mu = n$.

As a result, the restriction to twin tasks i, i' is a relaxed affine minimizer. We now argue that if the instance is restricted, and in particular $s_i\leq 1$ and $s_{i'}=n$ hold, then the t-player should not be in the bundling part of the relaxed affine minimizer when the approximation ratio is less than n (t-player's allocation figure is shown by the solid lines in Figure 2, because $s_i+s_{i'}$ is large). Let us fix the values $s_j\in[0,1]$, $s_{j'}=n$ for *all* $j\in[n-1]$, and the values $t'_j=0$, $t_{j,j'}=t_j+t_{j'}$ for all $j\neq i$. We observe that the restriction of the mechanism in the cut $t_{-\{i,i'\}}$ (the 2-dimensional partition when we also fix all other values except t_i and t'_i) should definitely contain an area where the t-player gets task i' but not i (in particular, when $t_{i'}=\delta$, for some arbitrarily small δ, and t_i is very large). Otherwise the mechanism has approximation ratio at least n.

This shows that in the domain of restricted (t,s) instances, the mechanisms with approximation ratio less than n are affine minimizers (for the t-player) in every cut of pairs of twin tasks (even if in general, that is, for $s_{i'}<n$ they are relaxed affine minimizers). So, the only mechanisms with approximation ratio less than n are linear for the restricted (s,t) instances. ∎

4.3. Linearity. From now on we focus on linear truthful mechanisms for restricted (t,s) instances. Establishing linearity of the boundaries (Lemma 16) is not directly useful, because the linear coefficient of the boundary i *may depend on the other values of* t. The next lemma shows that this is not the case for the scaling factor λ_i, although the ν_i term may still depend on the other values of t. Its proof is based on the interaction of pairs of tasks i and j. Note that these are *not* twin tasks but involve at least three players, the t-player and the two associated s-players. Obtaining such multiplayer statements is the bottleneck for a complete characterization of mechanisms in multiplayer domains, and it is perhaps the most crucial part of the proof.

LEMMA 17. *For fixed i and j, and for fixed* s_{-i}, t_{-ij}, *assume that* $\psi_i(s_i,s_{-i},t_{-i})$ *is a truncated linear function of* s_i, *i.e.,* $\psi_i(s_i,s_{-i},t_{-i}) = \max(0,\lambda_i(s_{-i},t_{-i})s_i +$

$\nu_i(s_{-i},t_{-i}))$, *for some* $\lambda_i(s_{-i},t_{-i}) \geq 0$. *Then* $\lambda_i(s_{-i},t_{-i})$ *is the same for all* t_j *that satisfy* $\psi_i(s_i,s_{-i},t_{-i}) > 0$ *(for some* s_i*).*

Proof. To keep the notation simple, we drop all values except for s_i and t_j. Let $K(s_i) = \{t_j : \psi_i(s_i,t_j) > 0\}$ be the interval of interest. Note first that when we increase s_i, the interval of interest can only expand, because $\psi_i(s_i,t_j)$ is nondecreasing in s_i.

Within interval $K(s_i)$, function $\psi_i(s_i,t_j)$ is positive and therefore equal to $\lambda_i(t_j)s_i + \nu_i(t_j)$. Furthermore, weak monotonicity implies that as a function of t_j, $\psi_i(s_i,t_j)$ is a piecewise linear function with derivative (slope) in $\{0,1,-1\}$ (see, for example, Figure 1).

The piecewise linear function $\psi_i(s_i,t_j)$ is differentiable[28] everywhere except perhaps for the at most two break points. We have

$$\frac{\partial\psi_i(s_i,t_j)}{\partial t_j} = \frac{\partial\lambda_i(t_j)}{\partial t_j}s_i + \frac{\partial\nu_i(t_j)}{\partial t_j}.$$

If $\partial\lambda_i(t_j)/\partial t_j \neq 0$, then by varying s_i, the slope cannot stay in $\{0,1,-1\}$. We conclude that $\partial\lambda_i(t_j)/\partial t_j = 0$ in each piece, which shows that $\lambda_i(t_j)$ is independent of t_j within each piece and therefore independent of t_j everywhere. ◻

The following corollary is an essential tool for establishing the lower bound.

COROLLARY 1. *If the allocation restricted to tasks* i,j *is quasi-bundling or quasi-flipping, then the constant parts of the piecewise linear function* $\psi_i(s_i,s_{-i},t_{-i})$*, as a function of* t_j*, are independent of* s_j *for* $j \neq i$*.*

Proof. We fix all values except for tasks i and j. By the previous lemma, when we change s_j, the boundary $\psi_j(s_j,s_i,t_i)$ is translated rectilinearly and therefore its break points remain the same. Since the constant parts of $\psi_i(s_i,s_j,t_j)$ are determined by the break points of $\psi_j(s_j,s_i,t_i)$, they also remain the same. See Figure 5 for an illustration. ◻

COROLLARY 2. *If the allocation restricted to tasks* i,j *is quasi-bundling or quasi-flipping, then the piecewise linear function* $\psi_i(s_i,s_{-i},t_{-i})$ *is either nondecreasing in* t_j *or nondecreasing in* s_j*.*

Proof. The $t_{-\{i,j\}}$ cut is either quasi-flipping, and $\psi_i(s_i,s_j,t_j)$ is nondecreasing in t_j, or quasi-bundling, in which case 2D geometry shows that it is nondecreasing in s_j (when s_j decreases the diagonal part shifts downwards, and the rectilinear parts remain fixed). See Figure 5 for an illustration. ◻

4.4. The main theorem. We now have most of the ingredients to prove the next theorem, which directly implies the main lower bound (Theorem 1).[29]

THEOREM 5. *The approximation ratio of linear truthful algorithms on restricted* (t,s) *instances* (4.2) *with* n *machines is at least* $\sqrt{n-1}$*.*

To prove this theorem, we fix some linear truthful algorithm, and we focus on a particular set of instances, which we call s-inefficient. The set of s-inefficient instances depends on the linear algorithm, and each instance consists of two types of tasks:

[28] It is not hard to see that if $\psi_i(s_i,t_j)$ is differentiable, then so are $\lambda_i(t_j)$ and $\nu_i(t_j)$; alternatively, one could use small differences instead of derivatives.

[29] Observe that Lemma 17 and Theorem 5 hold independent of the domain of the t-player (whether additive or submodular), whereas Lemma 16 does not hold for all truthful mechanisms when the t-player is additive.

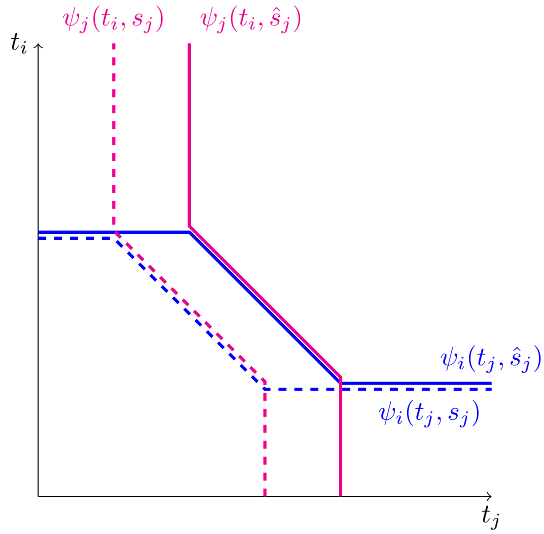

FIG. 5. *Consider the dashed lines that show $\psi_i(t_j, s_j)$ and $\psi_j(t_i, s_j)$ (these functions may depend on other values, but they play no role, and we can ignore them). The regular lines show $\psi_i(t_j, \hat{s}_j)$ and $\psi_j(t_i, \hat{s}_j)$ for some $\hat{s}_j > s_j$. Notice that linearity implies that $\psi_j(t_i, \hat{s}_j)$ is a shift to the right of $\psi_j(t_i, s_j)$; in particular, the break points stay at the same height. As a consequence, the blue horizontal parts remain at the same height, and the blue diagonal part shifts to the right. Therefore, $\psi_i(t_j, \hat{s}_j) \geq \psi_i(t_j, s_j)$. (Color available online.)*

either a task is unimportant (i.e., it has a very small value for one of the machines), or its s value is significantly higher than its t value, yet the algorithm allocates the task to the s-player.

DEFINITION 19 (s-inefficient instances). *Let's call a task i* trivial *when either $s_i = 0$ or $t_i \in (0, \delta_0]$ for some fixed (sufficiently small) δ_0. We call a restricted (t, s) instance* s-inefficient *for a mechanism, if it contains at least one nontrivial task, every nontrivial task i satisfies $s_i/t_i > \sqrt{n-1}$, and the mechanism allocates all nontrivial tasks to the s-player.*

The heart of the proof is to show that if the set of s-inefficient instances is nonempty, there exists an s-inefficient instance with exactly one nontrivial task. From this, it immediately follows that if s-inefficient instances exist, then the algorithm has approximation ratio at least $\sqrt{n-1}$.

However, it may be that for a given linear truthful algorithm there are no s-inefficient instances. But then we can use weak monotonicity to easily derive a $\sqrt{n-1}$ lower bound on the approximation ratio as the following lemma shows.

LEMMA 18. *If for a given truthful algorithm the set of s-inefficient instances is empty, then its approximation ratio is at least $\sqrt{n-1}$.*

Proof. Towards a contradiction, consider an algorithm that has approximation ratio $\sqrt{n-1} - \delta$, for some $\delta > 0$, for which the set of s-inefficient instances is empty. We consider the instance with $t_i = \alpha = 1/\sqrt{n-1} - \delta/n$ and $s_i = 1$ for all $i \in [n-1]$.

$$\begin{bmatrix} \alpha & \alpha & \cdots & \alpha \\ 1 & & & \\ & 1 & & \\ & & \vdots & \\ & & & 1 \end{bmatrix} \tag{4.3}$$

Note that at least one task is allocated to the s-players, because if all tasks are assigned to the t-player, the makespan is $(n-1)\alpha = \sqrt{n-1}-(n-1)\delta/n > \sqrt{n-1}-\delta$, the optimum makespan is 1, and the approximation ratio is strictly greater than $\sqrt{n-1}-\delta$.

Now for every task i which is allocated to the t-player, we lower its value from α to some small value in $(0,\delta_0]$ and increase the t value of every other task to $\alpha+\delta/(2n) < 1/\sqrt{n-1}$. By weak monotonicity (Lemma 2), the allocation of the tasks for the t-player remains the same. Since at least one task is allocated to the s-players, we end up with an s-inefficient instance, a contradiction. ∎

The proof of the next lemma (Lemma 19), which provides a very useful property of linear mechanisms, is the most critical part of the proof. It shows that linear weakly monotone algorithms satisfy a locality property, which bears some resemblance to the locality property in [36]. But unlike [36], our proof does not assume this property, but it derives it from weak monotonicity for the special class of instances that we consider.

Lemma 19. *If for a given linear truthful algorithm the set of s-inefficient instances is nonempty, then there is an s-inefficient instance with exactly one nontrivial task.*

Proof. Fix a linear truthful algorithm and consider an s-inefficient instance (t,s) with the minimum number of nontrivial tasks. If the number of nontrivial tasks is at least two, let us assume without loss of generality that tasks 1 and 2 are nontrivial. We will derive a contradiction by reducing the number of nontrivial tasks.

Consider the boundary function of the first task, $\psi_1(s_1,s_{-1},t_{-1}) = \max(0,\lambda_1(s_{-1},t_{-1})s_1 + \nu_1(s_{-1},t_{-1})$, for some positive λ_1. The crux of the matter is that we can reduce either the value of s_1 to 0, or the value of t_1 to at most δ_0, and guarantee that the t-player will keep not getting the second task. This guarantees that the second task is nontrivial and is given to the s-player, while the first task becomes trivial.

If $\psi_2(s_2,s_{-2},t_{-2})$ as a function of t_1 is nondecreasing (i.e., the $t_{-\{1,2\}}$ cut is quasi-flipping or crossing), we set $t_1^* \in (0,\delta_0]$. Since $\psi_2(s_2,s_{-2},t_{-2})$ is nondecreasing in t_1, in the new instance the second task is still allocated to the s-player (Figure 6).

Otherwise (i.e., the $t_{-\{1,2\}}$ cut is quasi-bundling), by Corollary 2, function $\psi_2(s_2,s_{-2},t_{-2})$ is nondecreasing in s_1. In this case, we change the instance as follows (see Figure 7 for an illustration of the first two cases):

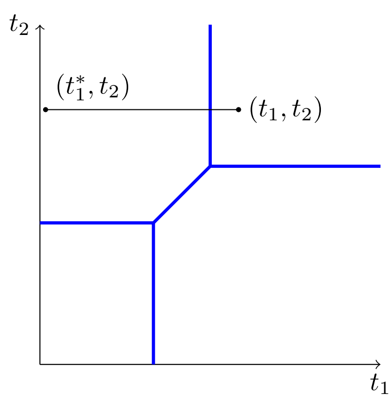

Fig. 6. *When the cut is quasi-flipping or crossing, and neither task is given to the t-player in the allocation for (t_1,t_2) (w.l.o.g. (t_1,t_2) is not on the boundary), then the second task is still not allocated to the t-player in the allocation for $(t_1^* \approx 0, t_2)$.*

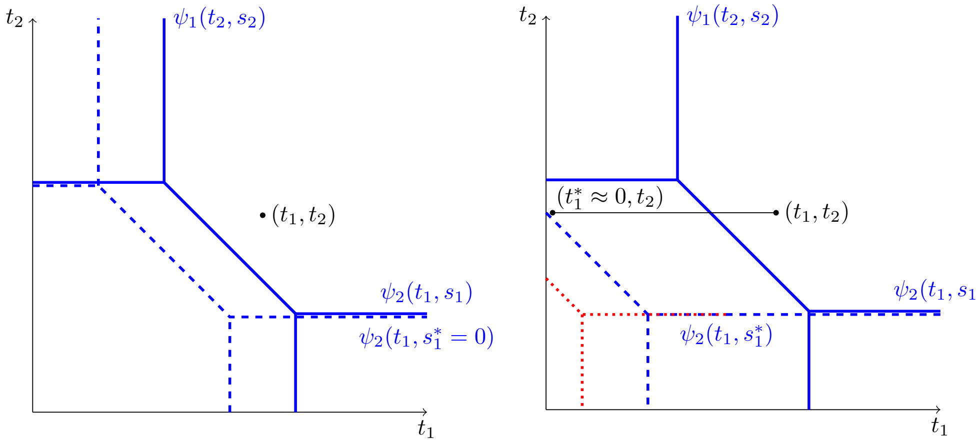

FIG. 7. *Left side, case* (a), *when* s_1^* *decreases to* 0*: The boundary* $\psi_2(t_1, s_1^*)$ *is lower than the boundary* $\psi_2(t_1, s_1^*)$*. Both tasks are still allocated to the s-player. Right side, case* (b), *when* s_i *decreases to* $s_1^* = -\nu_1(s_{-1}, t_{-1})/\lambda_1(s_{-1}, t_{-1})$ *and* t_1 *changes to* $t_1^* \approx 0$*. The new* (t_1^*, t_2) *remains in the region in which both tasks are allocated to the s-player.*

(a) if $\nu_1(s_{-1}, t_{-1}) \geq 0$, we set $s_1^* = 0$ and $t_1^* = t_1$;
(b) otherwise, if $\psi_1(s_1, s_{-1}, t_{-1}) > 0$, we set $s_1^* = -\nu_1(s_{-1}, t_{-1})/\lambda_1(s_{-1}, t_{-1})$ and $t_1^* \in (0, \delta_0]$ small enough to make task 1 trivial;
(c) and if $\psi_1(s_1, s_{-1}, t_{-1}) = 0$, we set $s_1^* = s_1$ and $t_1^* \in (0, \delta_0]$ small enough to make task 1 trivial.

In the first two cases, we lower the s_1 value to s_1^* until either s_1^* becomes 0 (case a), or $\psi_1(s_1^*, s_{-1}, t_{-1})$ becomes 0, and then we set $t_1 = t_1^* \approx 0$ (case b). The third case is when $\psi_1(s_1, s_{-1}, t_{-1})$ is already 0.

In all cases, the first task becomes a trivial task with the new values. Furthermore, in all cases the new value of s_1 is not greater than the original value: $s_1^* \leq s_1$. This is clearly true in the first and third cases. To see that this is true in the second case, observe that the boundary $\psi_1(s_1, s_{-1}, t_{-1})$, which is a nondecreasing function on s_1, moved from a positive value to $\psi_1(s_1^*, s_{-1}, t_{-1}) = 0$.

We now argue that *the second task is still given to the s-player after the change.* The argument is based on Corollary 1, which guarantees that the changes can only shift the boundary $\psi_2(s_2, s_{-2}, t_{-2})$ rectilinearly: in Figure 7, the slanted part moves only horizontally.

For case (a), by Corollary 2, the boundary $\psi_2(s_2, s_{-2}, t_{-2})$ did not increase, and therefore the second task is still given to the s-player (left part of Figure 7). For the other two cases, this is not sufficient because t_1 changed, and therefore t_{-2} changed as well. For case (b), the change shifts the slanted boundary (right part of Figure 7). In its new position, the slanted boundary meets the boundary of the positive orthant at $(0, t_2)$, which is dominated by the point (t_1^*, t_2). Therefore, the t-player gets neither task, so the s-player gets the second task. Case (c) is simpler and similar to case (b); the difference is that the slanted boundary starts at the leftmost position and it does not need to move at all (see the dotted lines in the right part of Figure 7).

The change of values of the first task did not change the allocation of the second task, but it may have changed the allocation of the remaining nontrivial tasks. But we can use weak monotonicity to further change the instance to obtain an s-inefficient instance. If some nontrivial tasks changed allocation and were given to the t player, we reduce their t values to 0 (this will make them trivial) and increase slightly the t values of the other nontrivial tasks without violating the constraint $s_i/t_i > \sqrt{n-1}$. By weak monotonicity (Lemma 2), this preserves the allocation of the first player for the other nontrivial tasks. The resulting instance is s-inefficient and has fewer nontrivial tasks, a contradiction. $\square$

The proof of the main result of this section follows directly from the last two lemmas.

Proof of Theorem 5. By the last two lemmas, either a linear truthful algorithm has approximation ratio at least $\sqrt{n-1}$, or there exists an s-inefficient instance with one nontrivial task. The approximation ratio of such an instance is trivially at least $\sqrt{n-1}$, and the proof is complete. $\square$

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