

Model Theory and rough approximate subgroups



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A mi familia.

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Abstract

The aim of this thesis is to use model theory to study rough approximate subgroups, generalising various known results about approximate subgroups.

Firstly, a systematic study of the structure of piecewise hyperdefinable sets is developed. In particular, we show the most significant properties of their logic topologies. Then, we study piecewise hyperdefinable groups, generalising and improving two of the main model-theoretic results of [Hru11] to this context. The first one is the existence of Lie models. The second one is the Stabilizer Theorem. In the process, we define the model theoretic components, G^0 , G^{00} and G^{000} , for piecewise hyperdefinable groups, and introduce a new component G^{ap} . We use these results to generalise the Lie Model Theorem, one of the main applications of model theory to study approximate subgroups, to the case of rough approximate subgroups.

Secondly, we focus on the case of metric approximate subgroups, i.e. rough approximate subgroups in metric groups. In this particular case, we show that ultraproducts of metric approximate subgroups satisfying some discretisation conditions have Lie models up to an infinitesimal thickening. We conclude using this result to get various combinatorial consequences.

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Introduction

Approximate subgroups are basic combinatorial structures modelling objects similar to subgroups up to a constant error. Although they had been already studied for abelian settings by other authors (e.g. [Fre92] and [Ruz94]), the current definition was definitely established in [Tao08].

Two subsets of a group are *k-commensurable* if k many left translates of each one suffice to cover the other. A *k-approximate subgroup* is a symmetric subset which is k -commensurable to its set of pairwise products. The parameter k may be omitted if it is irrelevant or clear from the context.

Example (Generalised Geometric Progressions). Let G be abelian and $u_1, \dots, u_d \in G$. The set P of words $w(\bar{u})$ in G with at most N_i occurrences of u_i is a 2^d -approximate subgroup. Here, d is called the dimension of P .

Example (Nilprogressions). Let G be nilpotent of nilpotent length s and $u_1, \dots, u_d \in G$. The set of words $w(\bar{u})$ with at most N_i occurrences of u_i is an $k(s, d)$ -approximate subgroup.

The study of approximate subgroups was originally motivated by number theory and combinatorics in the context of abelian groups, more specifically \mathbb{Z} . The study of the non-abelian case was started by Tao in [Tao08]. The original techniques used to study them mostly come from combinatorics and harmonic analysis.

Most of the work in the area has focused on finding a classification of approximate subgroups. In 1966, Freiman gave a complete classification of the finite approximate subgroups of \mathbb{Z} [Fre92]. In 2007, Ruzsa and Green extended Freiman's Theorem to the general case of abelian groups [GR07].

In 2009, using model theory, Hrushovski found a fundamental connexion between approximate subgroups and Lie groups [Hru11]. This result, now known as the Lie Model Theorem, was the starting point used to finally give in 2012 a complete classification of finite approximate subgroups by Breuillard, Green and Tao [BGT12].

Hrushovski's work in [Hru11] introduced approximate subgroups into model theory. Since then, interest in approximate subgroups has steadily grown among the model theory community. In particular, important connections between approximate subgroups and other notions of model theory have been discussed [HKP22].

Fix a first-order language and a sufficiently saturated structure. Morally, the Lie Model Theorem [Hru11, Theorem 4.2] says that every *definably amenable* definable approximate subgroup is commensurable to a compact neighbourhood of the identity of some (finite-dimensional real) Lie group:

Theorem (Hrushovski's Lie Model Theorem). *Let G be a definable group and X a definably amenable definable k -approximate subgroup. Consider the piecewise definable group \tilde{G} generated by X and assume that \tilde{G} is definably amenable. Then, \tilde{G} has a connected Lie model $\pi : H \leq \tilde{G} \rightarrow L = H/K$, i.e. a surjective group homomorphism onto a connected Lie group L defined on a subgroup $H \leq \tilde{G}$ which is continuous and proper from the logic topology using enough parameters. Furthermore, $K = \ker \pi \subseteq X^4$ and*

- (1) $H \cap X^2$ is a definable approximate subgroup commensurable to X ,
- (2) $H \cap X^4$ generates H , and
- (3) $\pi[H \cap X^2]$ is a compact neighbourhood of the identity in L .

The main step in the proof of Hrushovski's Lie Model Theorem is the application of a general version of the Stabilizer Theorem [Hru11, Theorem 3.5] to show that there is an \wedge -definable (read infinitely-definable or type-definable) subgroup of \tilde{G} of bounded index contained in X^4 . The amenability hypothesis is used precisely to apply the Stabilizer Theorem and it holds, in particular, for ultraproducts of finite approximate subgroups, which was the main case of interest in [Hru11]. The rest of the theorem is a direct consequence of Gleason-Yamabe Theorem [Gle51] and [Yam53].

Actually, the Stabilizer Theorem only uses the ideal of zero measure definable subsets of X^3 . Hence, the amenability hypothesis can be replaced by the assumption that X is a *near-subgroup*, i.e. that there is an ideal of definable subsets of X^3 invariant under left translations that does not contain X and that satisfies a stability condition replacing the finiteness of the measure. We mostly work with this near-subgroup hypothesis instead of the amenability hypothesis as it is more flexible.

The notion of approximation used in the definition of approximate subgroups is fundamentally discrete. This discreteness can be eliminated by allowing also a small

thickening of the cosets. By doing so, approximate subgroups are further generalised to rough approximate subgroups.

The T -thickening of a subset X of a group G is the set XT , where $1 \in T \subseteq G$. Two subsets of a group are T -rough k -commensurable if k many left translates of the T -thickening of each one suffice to cover the other. A T -rough k -approximate subgroup is a symmetric subset which is T -rough k -commensurable to its set of pairwise products. As previously, the parameters T and k may be omitted if they are irrelevant or clear from the context.

The main examples of rough approximate subgroups occur in the context of metric groups. A *metric group* is a group together with a left invariant metric. In this context, two subsets are δ -metric k -commensurable if they are $\overline{\mathbb{D}}_\delta(1)$ -rough k -commensurable, where $\overline{\mathbb{D}}_\delta(1)$ denotes the closed ball of radius δ at the identity. A δ -metric k -approximate subgroup is a $\overline{\mathbb{D}}_\delta(1)$ -rough k -approximate subgroup. Some interesting applications of metric approximate subgroups to other combinatorial objects has recently been found [GL20].

Example. Let G be a metric group and X a k -approximate subgroup. Assume G is (*right*) l -Lipschitz, i.e. every right translation is l -Lipschitz (in other words $d(xz, yz) \leq l \cdot d(x, y)$ for any $x, y, z \in G$). Let Y be δ -metric 1-commensurable to X . Then, Y is a $\delta(l + 2)$ -metric k -approximate subgroup.

The main aim of this thesis is to generalise the model theoretic study of approximate subgroups to the case of rough approximate subgroups and, in particular, to the case of metric approximate subgroups. More specifically, we aim to adapt Hrushovski's Lie Model Theorem to these cases.

Fix a definable group G and a definable symmetric subset $X \subseteq G$. Assume that X is a T_n -rough k -approximate subgroup for a sequence $(T_n)_{n \in \mathbb{N}}$ of definable subsets containing the identity such that

- (a) it decreases in doubling scales, i.e. $T_{n+1}T_{n+1}^{-1} \subseteq T_n$ for each $n \in \mathbb{N}$, and
- (b) it is asymptotically normalised by X , i.e. $x^{-1}T_{n+1}x \subseteq T_n$ for each $x \in X$ and $n \in \mathbb{N}$.

Using compactness, $T = \bigcap T_n$ is a subgroup of G , X is a T -rough k -approximate subgroup and X normalises T . Therefore, we work modulo T with the k -approximate subgroup X/T .

Here, T is \wedge -definable, so X/T is a hyperdefinable set and the group generated by X/T is a piecewise hyperdefinable group. Thus, our first intention is to generalise the model theoretic results of [Hru11], originally proved for definable and piecewise

definable sets, to the case of hyperdefinable and piecewise hyperdefinable sets. We focus on two of the main results of [Hru11]: the Stabilizer Theorem and the existence of Lie models.

The Stabilizer Theorem [Hru11, Theorem 3.5], subsequently improved in [MOS18, Theorem 2.12], was originally itself a generalisation of the classical Stabilizer Theorem for stable and simple groups (see for example [Wag10, Section 4.5]) changing the stability and simplicity hypotheses by some kind of measure-theoretic ones. Here, we extend that theorem to piecewise hyperdefinable groups in Chapter 3. Once one has the right definition of dividing and forking for piecewise hyperdefinable sets, the original proof of [Hru11], and its improved version of [MOS18], can be naturally adapted. However, we also manage to simplify the proof in such a way that we get a slightly stronger result.

In [Hru11], Hrushovski studied piecewise definable groups generated by near-subgroups, i.e. approximate subgroups satisfying a kind of measure-theoretic condition. In particular, he worked with ultraproducts of finite approximate subgroups. In that context, Hrushovski proved that there exist some Lie groups, named *Lie models*¹, deeply connected to the model-theoretic structure of the piecewise definable group [Hru11, Theorem 4.2]. Furthermore, among all these Lie groups, Hrushovski focused on the minimal one, showing its uniqueness and its independence of expansions of the language.

Here, we improve these results by defining the more general notion of *Lie core*, with *Lie models* being the particular case of Lie cores with compact (i.e. \wedge -definable) kernel. Then, in Chapter 2, we prove the existence of Lie cores for any piecewise hyperdefinable group with a generic piece and the uniqueness of the minimal Lie core. In the process, we adapt the classical model-theoretic components (with parameters) G^0 , G^{00} and G^{000} to piecewise hyperdefinable groups — our definitions extend some particular cases already studied (e.g. [HKP22]). We also introduce a new component G^{ap} ; G^{ap} is the smallest possible kernel of a continuous projection to a locally compact topological group without non-trivial compact normal subgroups. We prove that G^{ap} exists and is independent of parameters and expansions of the language when G has a generic piece. We use these components to show that the minimal Lie core is precisely G^0/G^{ap} . Using this canonical presentation, we conclude that the minimal Lie core is piecewise 0-hyperdefinable and independent of expansions of the language.

¹In [Hru11] they are simply called *associated Lie groups*. The term *Lie model* was later introduced in [BGT12].

Hyperdefinable sets, originally introduced in [HKP00], are quotients of \wedge -definable sets over \wedge -definable equivalence relations. Hyperdefinable sets have been already well studied by different authors (e.g. [Wag10] and [Kim13]). Here we extend this study to piecewise hyperdefinable sets.

A piecewise hyperdefinable set is a strict direct limit of hyperdefinable sets. We are interested in the piecewise hyperdefinable sets as elementary objects in themselves that inherit an underlying model-theoretic structure. In particular, we are principally interested in studying the natural logic topologies that generalise the usual Stone topology and can be defined for any piecewise hyperdefinable set.

We discuss the general theory of piecewise hyperdefinable sets in Chapter 1, elaborating the basic theory that we need for the rest of the thesis. Mainly, we study their basic topological properties such as compactness, local compactness, normality, Hausdorffness and metrizability, and also their relations with the quotient, product and subspace topologies. All this study leads naturally to the definition of *locally hyperdefinable sets*, which is in fact one of the main notions of this document. The most fundamental results of this chapter are given after the introduction of locally hyperdefinable sets in Section 1.4.

Chapter 2 is the core of this thesis and is devoted to the general study of piecewise hyperdefinable groups. The first fundamental result of the chapter is Theorem 2.2.9, in which we show that piecewise hyperdefinable groups satisfying a natural combinatorial condition are locally hyperdefinable. In Section 2.3 we define the model-theoretic components for piecewise hyperdefinable groups, proving the existence of G^{ap} in Theorem 2.3.4. Finally, we focus on the study of Lie cores, proving their existence (Theorem 2.4.2), the uniqueness of minimal one (Theorem 2.4.4), giving a canonical representation of the minimal one in terms of the model-theoretic components (Theorem 2.4.10) and showing its independence of expansions of the language (Corollary 2.4.11).

Chapter 3 is devoted to the Stabilizer Theorem for piecewise hyperdefinable groups, which is divided over Theorem 3.5.9, Corollary 3.5.10 and Theorem 3.5.11 — the latter being the standard statement of the Stabilizer Theorem. We conclude the chapter stating the Rough Lie Model Theorem 3.6.4 which generalises Hrushovski's Lie Model Theorem to the case of rough approximate subgroups.

In the final chapter, we study the particular case of metric approximate subgroups. The main aim is to apply the Rough Lie Model Theorem 3.6.4, replacing the near-subgroup assumption (which is purely model theoretic) by a combinatorial condition using discretisations.

By a discretisation of a space we understand a method to optimally replace it by a discrete set approximating the whole space up to some small scale error. Many well-known facts of additive combinatorics can be easily adapted to the case of metric groups using some discretisation techniques. In [Tao08] (see also [Tao14b]), using discretisation techniques, the author adapted many classical results of additive combinatorics to the case of metric groups, like the Ruzsa Triangular Inequality, the Ruzsa Covering Lemma or the Balog-Szemerédi-Gowers Theorem.

We use the same idea to adapt the Rough Lie Model Theorem 3.6.4 to the case of metric groups in Theorem 4.2.2. This theorem, roughly speaking, states that an infinitesimal thickening of any ultraproduct of symmetric subsets of metric groups having *nice enough discretisations* has a Lie model. To rigorously measure how discrete a set is we use the discretisation numbers N_r .

Here, to simplify the exposition, we consider first *bi-invariant metric groups*, i.e. metric groups where also right translations are isometries. We can then give the following rigorous statement.

Theorem (Metric Lie Model Theorem). *Let $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$ be a sequence such that, for some fixed $k \in \mathbb{N}$,*

- (a) G_m is a bi-invariant metric group,
- (b) X_m is a symmetric subset,
- (c) $(r_{i,m})_{i \leq m}$ is a sequence of positive reals with $2r_{i,m} \leq r_{i-1,m}$ and

$$N_{r_{i,m}}(X_m^9) \leq k \cdot N_{9r_{i,m}}(X_m) < \infty.$$

Let (G^*, X, \dots) be a non-principal ultraproduct in the associated language \mathbb{L} , G the subgroup generated by X and $o_r(1) := \bigcap_i \overline{\mathbb{D}}_{r_i}(1)$, where $\overline{\mathbb{D}}_{r_i}(1)$ is the ultraproduct of the balls of radius $r_{i,m}$. Then, $G \cdot o_r(1) \leq G$ has a connected Lie model $\pi : H \rightarrow L = H/K$ with $o_r(1) \trianglelefteq K \subseteq X^8 \cdot o_r(1)$ such that

- (1) $H \cap X^4 \cdot o_r(1)$ and $X^2 \cdot o_r(1)$ are commensurable,
- (2) $H \cap X^8 \cdot o_r(1)$ generates H , and
- (3) $\pi[H \cap X^4]$ is a compact neighbourhood of the identity of L .

In the first section we study those discretisation techniques previously described. Section 4.2 is the proof of the Metric Lie Model Theorem 4.2.2. In the final section, we use Theorem 4.2.2 to prove some combinatorial consequences: Corollary 4.3.7, Corollary 4.3.8 and Corollary 4.3.9. These corollaries adapt to the metric case some of the combinatorial applications of the original Lie Model Theorem obtained in [Hru11].

Notations and conventions: Throughout this thesis, fix a many sorted first order language L and a κ -saturated and strongly κ -homogeneous L -structure \mathfrak{M} with $\kappa > |L|$ a strong limit cardinal. A set is called *small* if its cardinality is smaller than κ . We now specify some general details on the notations, terminology and conventions that we follow throughout this thesis:

■ By a sort of L we mean a (possibly infinite) tuple of single sorts of L . For a sort $s = (s_i)_{i \in I}$, we denote by $M^s = \prod_{i \in I} M^{s_i}$ the universe of sort s . An element of \mathfrak{M} of (possibly infinite) sort s is an element of M^s . The arity of a sort is its length; the arity of an element is the arity of its sort. We may omit the sort if there is no confusion.

■ From now on, except when otherwise stated, we use a, b, \dots and A, B, \dots to denote hyperimaginary elements and small sets of hyperimaginaries respectively, while a^*, a^{**}, \dots and A^*, A^{**}, \dots denote associated representatives. If we use a^* or A^* without mention of a or A , we mean real elements. Often, we silently identify sets and tuples by picking enumerations.

■ By a variable we mean a (possibly infinite) set of single variables; we use x, y, z, \dots to denote variables. We assume that L already contains countably many variables for each sort, but, nevertheless, we allow ourselves to use as many extra variables as we want. For a variable x , an evaluation of x is a map a^* assigning to each single variable in x a single element of the structure of the corresponding sort; we denote by M^x the set of evaluations of x in M . Often, we silently identify evaluations of x and elements via $a^* \mapsto (a_i^*)_{i \in I}$, by prefixing an order $(x_i)_{i \in I}$ of the variable x . For a sorted subset N of M , we write N^x for the subset of M^x of evaluations with values in N .

■ We denote by $\text{For}^x(L(A^*))$ the set of formulas of L with parameters in A^* and free variables in x . The cardinality of the language is the cardinality of its set of formulas (without extra variables); $|L| := |\text{For}(L)|$. In particular, $|L|$ is always at least \aleph_0 . A partial x -type $\Sigma(x)$ over A^* is a finitely satisfiable subset of $\text{For}^x(L(A^*))$. A (complete) x -type $p(x)$ over A^* is a maximal partial x -type over A^* . By κ -saturation, every partial type over a small set of parameters on a small variable is realised in M^x . If there is no confusion, we may omit the variable x . For $a^* \in \mathfrak{M}^x$, we write $\text{tp}(a^*/A^*)$ for the x -type of a^* over A^* . For a sorted subset N , the Stone space of x -types with parameters A^* finitely satisfiable in N is denoted by $\mathbf{S}_N^x(A^*)$; write $\mathbf{S}^x(A^*)$ for $\mathbf{S}_M^x(A^*)$. We usually omit the variable if there is no confusion.

■ An A^* -definable subset D of M^x is a set of realisations of some formula with parameters in A^* ; a subset is definable if it is A^* -definable for some set of parameters

A^* . After indicating the parameters, we write $\underline{D}(x)$ to denote a formula defining D . The arity of D is the arity of x . For a sorted subset N of M , we say that a subset D of N^x is relatively A^* -definable if it is the intersection of an A^* -definable subset of M^x with N^x ; a subset is relatively definable if it is relatively A^* -definable for some set of parameters A^* . After indicating the parameters, we write $\underline{D}(x)$ to denote a formula relatively defining D in N , i.e. such that $D = \underline{D}(N^x) := \underline{D}(\mathfrak{M}) \cap N^x$. The arity of D is the arity of x . An \bigwedge_{A^*} -definable subset V of M^x is a set of realisations of some partial x -type over A^* ; a subset is \bigwedge -definable if it is \bigwedge_{A^*} -definable for some small set of parameters A^* . After indicating the parameters, we write $\underline{V}(x)$ to denote a partial type defining V . For an element $a^* \in M^x$, we write $\text{tp}(a^*/A^*)$ for the smallest \bigwedge_{A^*} -definable subset of M^x containing a^* , which is obviously defined by $\underline{\text{tp}}(a^*/A^*)$. The arity of D is the arity of x . A \bigvee_{A^*} -definable subset U of M^x is the complement of an \bigwedge_{A^*} -definable subset; a subset is \bigvee -definable if it is \bigvee_{A^*} -definable for some small set of parameters A^* . The arity of D is the arity of x . We also use cardinals and cardinal inequalities. In that case, the subscript should be read as an anonymous set of parameters whose size satisfies the indicated condition. For example, $\bigwedge_{<\omega}$ -definable means \bigwedge_A -definable for some subset A with $|A| < \omega$. The same notation will be naturally used for hyperdefinable, piecewise hyperdefinable and piecewise \bigwedge -definable sets.

■ Write $\text{Aut}(\mathfrak{M}/A^*)$ for the subgroup of automorphisms of \mathfrak{M} pointwise fixing A^* . As \mathfrak{M} is strongly κ -homogeneous, $\underline{\text{tp}}(a^*/A^*) = \underline{\text{tp}}(b^*/A^*)$ with $|a^*|, |b^*|, |A^*| < \kappa$ if and only if there is $\sigma \in \text{Aut}(\mathfrak{M}/A^*)$ such that $\sigma(a^*) = b^*$.

■ Let $R \subseteq X \times Y$ be a set-theoretic binary relation. For $x \in X$, we write $R(x) := \{y \in Y : (x, y) \in R\}$. For a subset $V \subseteq X$, we write $R[V] := \{y \in Y : \exists x \in V (x, y) \in R\}$. We also write $R^{-1} := \{(y, x) : (x, y) \in R\}$, so $R^{-1}(y) := \{x \in X : (x, y) \in R\}$ for $y \in Y$ and $R^{-1}[W] := \{x \in X : \exists y \in W (x, y) \in R\}$ for $W \subseteq Y$. We denote the image and preimage functions between the power sets by $\text{Im } R : V \mapsto R[V]$ and $\text{Im}^{-1}R : W \mapsto R^{-1}[W]$. Most of the time, this notation is used for partial functions, which are always identified with their graphs — note that f^{-1} is only a function when f is invertible.

■ Cartesian projections are denoted by \mathfrak{p} , quotient maps are denoted by \mathfrak{q} , quotient group homomorphisms are denoted by π , inclusion maps are denoted by ι and identity maps are denoted by id .

■ A lattice of sets is a family of sets closed under finite unions and finite intersections. A complete algebra on a set X is a family of subsets of X closed under complements and arbitrary unions.

■ The class of ordinals is denoted by \mathbb{On} . The cardinality of a set X is written $|X|$. The set of natural numbers with 0 is denoted by \mathbb{N} .

■ We use product notation for groups. Also, unless otherwise stated, we consider the group acting on itself on the left. In particular, by a coset we mean a left coset. A subset X of a group is called symmetric if $1 \in X = X^{-1}$. For subsets X and Y of a group, we write XY for the set of pairwise products, and abbreviate $X^n := XX^{n-1}$ and $X^{-n} := (X^{-1})^n$ for $n \in \mathbb{N}$. We say that X normalises Y if $x^{-1}Yx \subseteq Y$ for every $x \in X$. We write $[X, Y] := \{[x, y] : x \in X, y \in Y\}$ where $[x, y] := x^{-1}y^{-1}xy$.

■ By a Lie group we always mean here a finite-dimensional real Lie group.

■ Recall that, for $l \in \mathbb{R}_{\geq 0}$, an l -Lipschitz function $f : X \rightarrow Y$ between two metric spaces (X, d_X) and (Y, d_Y) is a function satisfying $d_Y(f(x), f(x')) \leq l \cdot d_X(x, x')$ for any $x, x' \in X$.

Authorship note: The first three chapters are mostly purely original work. Nevertheless, I want to thank my supervisor, Prof. Ehud Hrushovski, for his fundamental contributions for some clearly specified parts of Chapter 2. The final chapter is in full joint work with my supervisor, Prof. Ehud Hrushovski.

Chapter 1

Piecewise hyperdefinable sets

1.1 Hyperdefinable sets

Let A^* be a small set of parameters. An A^* -hyperdefinable set is a quotient $P = X/E$ where X is a non-empty \bigwedge_{A^*} -definable set of small arity and E is an \bigwedge_{A^*} -definable equivalence relation on X . If we do not indicate the set of parameters, we mean that it is hyperdefinable for some small set of parameters. Write $\mathfrak{g} := \mathfrak{g}_P : X \rightarrow P$ for the *quotient map* given by $x \mapsto [x]_E := x/E := E(x) = \{x' \in X : (x, x') \in E\}$. Elements of A^* -hyperdefinable sets are called A^* -hyperimaginaries, while elements of the structure will be called *real*. Given a hyperimaginary element $a \in P$, we call P the (*hyperimaginary*) *sort* of a . A *representative* of a is a real element a^* such that $\mathfrak{g}_P(a^*) = a$. We write $a^* \in a$ to mean that a^* is a representative of a . The *arity* of a is its length $|a|$; the *arity* $\text{ary}(P)$ of P is the arity of its elements, i.e. the arity of X .

An \bigwedge_{A^*} -definable subset, $V \subseteq P$, is a subset such that $\mathfrak{g}^{-1}[V]$ is \bigwedge_{A^*} -definable in X . We will say that a partial type defines $V \subseteq P$ if it defines $\mathfrak{g}^{-1}[V]$. If $V \subseteq P$ is a non-empty \bigwedge -definable set, after declaring the parameters, we will write \underline{V} to denote a partial type defining $\mathfrak{g}^{-1}[V]$. The following basic proposition is the starting point to study hyperdefinable sets.

Lemma 1.1.1 (Correspondence Lemma). *Let $P = X/E$ be an A^* -hyperdefinable set. Then, the image by \mathfrak{g} of any \bigwedge_{A^*} -definable subset of X is an \bigwedge_{A^*} -definable subset of P . Moreover, the preimage function $\text{Im}^{-1}\mathfrak{g}$ is an isomorphism, whose inverse is $\text{Im}\mathfrak{g}$, between the lattice of \bigwedge_{A^*} -definable subsets of P and the lattice of \bigwedge_{A^*} -definable subsets of X closed under E .*

Proof. Let $V \subseteq X$ be \bigwedge_{A^*} -definable. Then, $\mathfrak{g}^{-1}[\mathfrak{g}[V]] = E[V] = \{y : (x, y) \in E \text{ for some } x \in V\}$. Now, by κ -saturation, this set is \bigwedge_{A^*} -defined by the partial type $\exists x (\underline{V}(x) \wedge \underline{E}(x, y)) := \{\exists x (\varphi(x) \wedge \psi(x, y)) : \varphi \in \underline{X}, \psi \in \underline{E}\}$. Therefore, $\mathfrak{g}[V]$ is \bigwedge_{A^*} -definable.

On the other hand, it is a general fact in set theory that, for a function $f : X \rightarrow Y$, the preimage function $\text{Im}^{-1}f$ is an isomorphism, whose inverse is the image function $\text{Im}f$, between the lattice of subsets of the image and the lattice of subsets of X closed under the equivalence relation $E(x, y) \Leftrightarrow f(x) = f(y)$. Therefore, as $\text{Im}^{-1}\mathfrak{g}$ and $\text{Im}\mathfrak{g}$ map \bigwedge_{A^*} -definable subsets to \bigwedge_{A^*} -definable subsets, we conclude. Q.E.D.

The main part of this lemma can be further generalised to Lemma 1.1.3. For this, note firstly that, given two A^* -hyperdefinable sets $P = X/E$ and $Q = Y/F$, the Cartesian product $P \times Q$ is canonically identified with the hyperdefinable set $X \times Y/E \hat{\times} F$ via $([x]_E, [y]_F) \mapsto [x, y]_{E \hat{\times} F}$, where $E \hat{\times} F := \{((x, y), (x', y')) : (x, x') \in E, (y, y') \in F\}$. More generally, if $(P_i)_{i \in I}$ is a sequence of A^* -hyperdefinable sets with $|I|$ small, then $\prod P_i = \prod_i X_i/E_i$ is canonically identified with the hyperdefinable set X/E via $([x_i]_{E_i})_{i \in I} \mapsto [(x_i)]_E$, where $X = \prod X_i$ and $E = \hat{\prod}_i E_i := \{((x_i)_{i \in I}, (y_i)_{i \in I}) : (x_i, y_i) \in E_i \text{ for each } i \in I\}$. Then, we can talk about \bigwedge_{A^*} -definable relations and partial functions.

Examples 1.1.2. The inclusion $\iota : V \rightarrow P$ of an \bigwedge_{A^*} -definable set V is \bigwedge_{A^*} -definable. The Cartesian projections $\mathfrak{p}_j : \prod_i P_i \rightarrow P_j$ are \bigwedge_{A^*} -definable. Also, the quotient map $\mathfrak{g}_P : X \rightarrow P$ is \bigwedge_{A^*} -definable.

Lemma 1.1.3. *Let $P = X/E$ and $Q = Y/F$ be two A^* -hyperdefinable sets and f an \bigwedge_{A^*} -definable partial function from P to Q . Then, for any \bigwedge_{A^*} -definable sets $V \subseteq P$ and $W \subseteq Q$, $f[V]$ and $f^{-1}[W]$ are \bigwedge_{A^*} -definable.*

Proof. Note that $f[V] = \mathfrak{p}_Q[f \cap (V \times Q)]$ and $f^{-1}[W] = \mathfrak{p}_P[f \cap (P \times W)]$, where $V \times Q = \mathfrak{p}_P^{-1}[V]$ and $P \times W = \mathfrak{p}_Q^{-1}[W]$. Hence, it is enough to check that \mathfrak{p}_P satisfies the lemma. It is trivial that $\mathfrak{p}_P^{-1}[V] = V \times Q$ is \bigwedge_{A^*} -definable for any \bigwedge_{A^*} -definable subset V . On the other hand, if $V \subseteq P \times Q$ is \bigwedge_{A^*} -definable, by compactness, $\Sigma(x) = \{\exists y \bigwedge \Delta(x, y) : \Delta \subseteq \underline{V} \text{ finite}\}$ defines $\mathfrak{g}_P^{-1}[\mathfrak{p}_P[V]]$. Q.E.D.

Remark 1.1.4. Let $f : P \rightarrow Q$ and $g : Q \rightarrow R$ be functions. As subsets of $P \times R$, we have $g \circ f = \mathfrak{p}_{P \times R}[(f \times R) \cap (P \times g)]$, where $\mathfrak{p}_{P \times R} : P \times Q \times R \rightarrow P \times R$ is the natural projection. Thus, compositions of \bigwedge_{A^*} -definable partial functions are also \bigwedge_{A^*} -definable partial functions.

A *type over A^** , or *A^* -type*, in P is an \bigwedge_{A^*} -definable subset of P which is \subset -minimal in the family of non-empty \bigwedge_{A^*} -definable subsets. For $a \in P$, we write $\text{tp}(a/A^*)$ for the type over A^* containing a . As the lattice of \bigwedge_{A^*} -definable sets is closed under arbitrary intersections, the type of a hyperimaginary element always exists.

Lemma 1.1.5. *Let P be an A^* -hyperdefinable set, $a \in P$ and $a^* \in a$. Then, $\text{tp}(a/A^*) = \mathfrak{g}[\text{tp}(a^*/A^*)]$. In particular, $b \in \text{tp}(a/A^*)$ if and only if there is $b^* \in b$*

such that $\text{tp}(a^*/A^*) = \text{tp}(b^*/A^*)$. In other words, $\text{tp}(a/A^*)$ is the orbit of a under the action of $\text{Aut}(\mathfrak{M}/A^*)$ on P .

Proof. By definition $\mathfrak{g}^{-1}[\text{tp}(a/A^*)]$ is \bigwedge_{A^*} -definable and contains a^* , so $\text{tp}(a^*/A) \subseteq \mathfrak{g}^{-1}[\text{tp}(a/A^*)]$, concluding $\mathfrak{g}[\text{tp}(a^*/A)] \subseteq \text{tp}(a/A^*)$. On the other hand, by the Correspondence Lemma 1.1.1, $\mathfrak{g}[\text{tp}(a^*/A)]$ is \bigwedge_{A^*} -definable. As it contains a , by minimality, $\text{tp}(a/A^*) \subseteq \mathfrak{g}[\text{tp}(a^*/A)]$, concluding $\text{tp}(a/A^*) = \mathfrak{g}[\text{tp}(a^*/A^*)]$. In other words, $b \in \text{tp}(a/A^*)$ if and only if $b = \mathfrak{g}(b^*)$ with $b^* \in \text{tp}(a^*/A^*)$. Now, by strong κ -homogeneity of \mathfrak{M} , $b^* \in \text{tp}(a^*/A^*)$ if and only if there is $\sigma \in \text{Aut}(\mathfrak{M}/A^*)$ with $\sigma(a^*) = b^*$. Hence, $b \in \text{tp}(a/A^*)$ if and only if $b = \sigma(a)$ for some $\sigma \in \text{Aut}(\mathfrak{M}/A^*)$, where the action of $\text{Aut}(\mathfrak{M}/A^*)$ on P is defined by $\sigma(\mathfrak{g}(a^*)) = \mathfrak{g}(\sigma(a^*))$ — well-defined since P is A^* -hyperdefinable. Q.E.D.

Let P be A^* -hyperdefinable. A subset $V \subseteq P$ is A^* -invariant if $\text{tp}(a/A^*) \subseteq V$ for any $a \in V$.

Corollary 1.1.6. *Let P be A^* -hyperdefinable and $V \subseteq P$. Then, V is A^* -invariant if and only if it is setwise invariant under the action of $\text{Aut}(\mathfrak{M}/A^*)$ on P , if and only if $\mathfrak{g}^{-1}[V]$ is A^* -invariant.*

Proof. As $\text{tp}(a/A^*)$ is the orbit of a under $\text{Aut}(\mathfrak{M}/A^*)$, it is obvious from the definition that V is A^* -invariant if and only if it is setwise invariant under the action of $\text{Aut}(\mathfrak{M}/A^*)$. On the other hand, suppose V is A^* -invariant and $a^* \in \mathfrak{g}^{-1}[V]$. Then, $\text{tp}(a^*/A^*) \subseteq \mathfrak{g}^{-1}[\text{tp}(a/A^*)] \subseteq \mathfrak{g}^{-1}[V]$, so $\mathfrak{g}^{-1}[V]$ is A^* -invariant. Conversely, suppose $\mathfrak{g}^{-1}[V]$ is A^* -invariant and $a \in V$. Then, $\text{tp}(a/A^*) = \mathfrak{g}[\text{tp}(a^*/A^*)] \subseteq \mathfrak{g}[\mathfrak{g}^{-1}[V]] = V$. Q.E.D.

Corollary 1.1.7. *Let P be A^* -hyperdefinable and $V \subseteq P$ an \bigwedge -definable subset. Then, V is \bigwedge_{A^*} -definable if and only if it is A^* -invariant.*

Proof. Take $b^* \notin \mathfrak{g}^{-1}[V]$. As V is A^* -invariant, by Corollary 1.1.6, $\mathfrak{g}^{-1}[V]$ is A^* -invariant. Therefore, $\text{tp}(b^*/A^*)$ and $\mathfrak{g}^{-1}[V]$ are disjoint. By saturation, $\Sigma \wedge \Sigma'(x)$ is not finitely satisfiable for any partial types $\Sigma(x)$ and $\Sigma'(x)$ respectively defining $\text{tp}(b^*/A^*)$ and $\mathfrak{g}^{-1}[V]$. Thus, there is an A^* -definable subset D_{b^*} with $b^* \notin D_{b^*}$ such that $\mathfrak{g}^{-1}[V] \subseteq D_{b^*}$. Consequently, $\mathfrak{g}^{-1}[V] = \bigcap_{b^* \notin \mathfrak{g}^{-1}[V]} D_{b^*}$, concluding that V is \bigwedge_{A^*} -definable. Q.E.D.

We also want to be able to use hyperimaginary parameters. We say that a subgroup $G \leq \text{Aut}(\mathfrak{M}/A)$ naturally acts on $P = X/E$ if X and E are setwise invariant under G . In that case, the natural action of G on P is given by $g(a) = \mathfrak{g}(g(a^*))$ for any $a^* \in a$.

Let A be a small set of hyperimaginaries. By recursion on $\alpha \in \mathbb{O}n$, we define $\text{Aut}(\mathfrak{M}/A)$ when A is a set of parameters of (hyperimaginary) complexity α .

- For $\alpha = 0$: we say that A is a set of parameters of complexity 0 if it is a set of real elements. We define $\text{Aut}(\mathfrak{M}/A)$ as the subgroup of automorphisms pointwise fixing A .
- For $\alpha + 1$: we say that A is a set of parameters of complexity $\alpha + 1$ if there is a subset $A_\alpha \subseteq A$ such that A_α is a set of parameters of complexity α and, for every $a \in A \setminus A_\alpha$, $\text{Aut}(\mathfrak{M}/A_\alpha)$ naturally acts on the sort of a . We define $\text{Aut}(\mathfrak{M}/A)$ as the subgroup of $\text{Aut}(\mathfrak{M}/A_\alpha)$ pointwise fixing every element $a \in A$ under the natural action of $\text{Aut}(\mathfrak{M}/A_\alpha)$ on the sort of a .
- For λ limit: we say that A is a set of parameters of complexity λ if there is an increasing sequence $(A_\alpha)_{\alpha < \lambda}$ with $A = \bigcup A_\alpha$ such that, for each $\alpha < \lambda$, A_α is a set of parameters of complexity α and, for every element in $a \in A_{\alpha+1} \setminus A_\alpha$, $\text{Aut}(\mathfrak{M}/A_\alpha)$ naturally acts on the sort of a . We take $\text{Aut}(\mathfrak{M}/A) = \bigcap_{\alpha < \lambda} \text{Aut}(\mathfrak{M}/A_\alpha)$.

Let A be a set of hyperimaginary parameters. A set of real elements X is *A-invariant* if it is invariant under $\text{Aut}(\mathfrak{M}/A)$. We say that a hyperdefinable set $P = X/E$ is *A-hyperdefinable* if X and E are A -invariant; we say that a hyperimaginary is *A-hyperimaginary* if its sort is A -hyperdefinable. A subset of an A -hyperdefinable set is *A-invariant* if it is A -invariant under the natural action of $\text{Aut}(\mathfrak{M}/A)$, equivalently if its preimage by the quotient map is A -invariant. An \bigwedge_A -definable subset is an A -invariant \bigwedge -definable subset.

Lemma 1.1.8. *Let A be a small set of hyperimaginary parameters, A^* a set of representatives of A and P an A -hyperdefinable set.*

(1) $\text{Aut}(\mathfrak{M}/A) = \{\sigma \in \text{Aut}(\mathfrak{M}) : \sigma(A^*) \text{ is a set of representatives of } A\}$. In particular, $\text{Aut}(\mathfrak{M}/A^*) \leq \text{Aut}(\mathfrak{M}/A)$.

(2) P is A^* -hyperdefinable and every A -invariant subset of P is A^* -invariant. In particular, every \bigwedge_A -definable subset of P is \bigwedge_{A^*} -definable.

(3) The family of A -invariant subsets of P is a complete algebra of subsets of P . The family of \bigwedge_A -definable subsets of P is a lattice of subsets of P closed under arbitrary intersections.

Proof. (1) If $\sigma \in \text{Aut}(\mathfrak{M}/A)$, then $\sigma(a)$ is well-defined for any $a \in A$ and $\sigma(a) = a$. In particular, $\sigma(a^*)$ is a representative of a , so $\sigma(A^*)$ is a set of representatives of A . Let α be the hyperimaginary complexity of A . We prove the other inclusion by induction on α . For $\alpha = 0$ the situation is trivial. Suppose that $\alpha = \beta + 1$ and it holds for β . Take $\sigma \in \text{Aut}(\mathfrak{M})$ such that $\sigma(A^*)$ is a set of representatives of A . Let

$A_\beta \subseteq A$ be a subset of parameters of complexity β such that every element in $A \setminus A_\beta$ is A_β -hyperimaginary. As $\sigma(A^*)$ is a set of representatives of A^* , in particular, $\sigma(A_\beta^*)$ is a set of representative of A_β^* . Thus, by induction hypothesis, $\sigma \in \text{Aut}(\mathfrak{M}/A_\beta)$ and $\sigma(a)$ is well-defined for any $a \in A$. Since $\sigma(a^*)$ is a representative of a , we get that $\sigma(a) = a$. As $a \in A$ is arbitrary, we conclude that $\sigma \in \text{Aut}(\mathfrak{M}/A)$. For α limit, take $(A_\beta)_{\beta < \alpha}$ increasing sequence of sets of parameters with $A = \bigcup A_\beta$ such that, for each $\beta < \alpha$, A_β has complexity β and every element in $A_{\beta+1}$ is A_β -hyperimaginary. We have that $\{\sigma : \sigma(A^*) \text{ representatives of } A\} = \bigcap \{\sigma : \sigma(A_\beta^*) \text{ representatives of } A_\beta\} \subseteq \bigcap \text{Aut}(\mathfrak{M}/A_\beta) = \text{Aut}(\mathfrak{M}/A)$.

(2) Obvious using Corollary 1.1.6, as we have $\text{Aut}(\mathfrak{M}/A^*) \leq \text{Aut}(\mathfrak{M}/A)$ by (1).

(3) The family of A -invariant subsets of P is the family of setwise invariant subsets under a group action, so it is a complete algebra of subsets of P . By (2), the family of \bigwedge_A -definable subsets of P is the intersection of the lattice of \bigwedge_{A^*} -definable subsets of P , which is closed under arbitrary intersections, and the complete algebra of A -invariant subsets of P , so it is a lattice of subsets of P closed under arbitrary intersections. Q.E.D.

Remark 1.1.9. In Lemma 1.1.8(1), the fact that A is a set of hyperimaginary parameters guarantees that the definition of $\text{Aut}(\mathfrak{M}/A)$ does not depend on the choice of representatives.

Lemma 1.1.10. *Let $P = X/E$ and $Q = Y/F$ be two A -hyperdefinable sets, f a partial function from P to Q and $V \subseteq P$ and $W \subseteq Q$.*

(1) *If f , V and W are A -invariant, then $f[V]$ and $f^{-1}[W]$ are A -invariant.*

(2) *If f , V and W are \bigwedge_A -definable, then $f[V]$ and $f^{-1}[W]$ are \bigwedge_A -definable.*

Proof. (1) As f is A -invariant, $\sigma(f(a)) = f(\sigma(a))$ for any $\sigma \in \text{Aut}(\mathfrak{M}/A)$, so $\sigma[f[V]] = f[\sigma[V]]$ and $\sigma[f^{-1}[W]] = f^{-1}[\sigma[W]]$. Hence, as V and W are A -invariant, $f[V]$ and $f^{-1}[W]$ are A -invariant.

(2) By Lemma 1.1.3, $f[V]$ and $f^{-1}[W]$ are \bigwedge -definable. By (1), they are also A -invariant, so $f[V]$ and $f^{-1}[W]$ are A -invariant. Q.E.D.

A *type over A* , or *A -type*, is a \subset -minimal non-empty \bigwedge_A -definable subset. For $a \in P$, we write $\text{tp}(a/A)$ for the type over A containing a . As the lattice of \bigwedge_A -definable sets is closed under arbitrary intersections (Lemma 1.1.8(3)), the type of a hyperimaginary element always exists.

Lemma 1.1.11. *Let P be an A -hyperdefinable set and $a \in P$. Then, $\text{tp}(a/A)$ is the orbit of a under the action of $\text{Aut}(\mathfrak{M}/A)$ on P . In other words, for any $a^* \in a$ and A^* representatives of A , we have $b \in \text{tp}(a/A)$ if and only if there are $b^{**} \in b$ and A^{**} representatives of A such that $\text{tp}(a^*, A^*) = \text{tp}(b^{**}, A^{**})$.*

Consequently, V is A -invariant if and only if $\text{tp}(a/A) \subseteq V$ for any $a \in V$.

Proof. Let $\text{orb}(a/A)$ be the orbit of a under $\text{Aut}(\mathfrak{M}/A)$. As $\text{tp}(a/A)$ is A -invariant, we have that $\text{orb}(a/A) \subseteq \text{tp}(a/A)$. By minimality of $\text{tp}(a/A)$, it suffices to show that $\text{orb}(a/A)$ is \bigwedge_A -definable. Say $P = X/E$ and $A = \{a_i\}_{i \in I}$ with $a_i \in Q_i = Y_i/F_i$. Consider $\Sigma(x)$ expressing $\exists x' \exists y \underline{E}(x, x') \wedge \underline{F}(y, A^*) \wedge \Gamma(x', y)$ where $\underline{F} = \bigwedge \underline{F}_i$ and $\Gamma = \text{tp}(a^*, A^*)$. By saturation, $b^* \in b \in P$ realises Σ if and only if there are $b^{**} \in b$ and \overline{A}^{**} representatives of A such that $\text{tp}(a^*, A^*) = \text{tp}(b^{**}, \overline{A}^{**})$. Thus, by strong homogeneity, $b^* \in b \in P$ realises Σ if and only if there is $\sigma \in \text{Aut}(\mathfrak{M})$ such that $\sigma(a^*) \in b$ and $\sigma(A^*)$ is a set of representatives of A . By Lemma 1.1.8(1), we conclude that $b^* \in b \in P$ realises Σ if and only if $b \in \text{orb}(a/A)$, so $\text{orb}(a/A)$ is \bigwedge -definable. As it is trivially A -invariant, it is \bigwedge_A -definable. Q.E.D.

Lemma 1.1.12. *Let P, Q be A -hyperdefinable sets, $f : P \rightarrow Q$ an A -invariant function and $a \in P$. Then, $f[\text{tp}(a/A)] = \text{tp}(f(a)/A)$.*

Proof. By Lemma 1.1.10(1), we have that $f[\text{tp}(a/A)]$ and $f^{-1}[\text{tp}(f(a)/A)]$ are A -invariant. Then, $\text{tp}(a/A) \subseteq f^{-1}[\text{tp}(f(a)/A)]$ and $\text{tp}(f(a)/A) \subseteq f[\text{tp}(a/A)]$ by Lemma 1.1.11. Thus, $f[\text{tp}(a/A)] = \text{tp}(f(a)/A)$. Q.E.D.

We explain now how to substitute hyperimaginary parameters. Let P be A -hyperdefinable, b a small set of hyperimaginaries over A and $V \subseteq P$ an $\bigwedge_{A,b}$ -definable subset. Let c be such that $\text{tp}(b/A) = \text{tp}(c/A)$. The set $V(c)$ given from V by replacing b by c is the set $\sigma[V]$ for $\sigma \in \text{Aut}(\mathfrak{M}/A)$ such that $\sigma(b) = c$. Note that this does not depend on the choice of $\sigma \in \text{Aut}(\mathfrak{M}/A)$.

Alternatively, we present a more explicit methodology using *uniform definitions* that does not require the use of automorphisms. Let $P = X/E$ be an A -hyperdefinable set. We say that a partial type $\Sigma(x, A^*)$ is *weak uniform on P over A* if $\Sigma(x, A^{**})$ defines the same set on P for any set of representatives A^{**} of A , i.e. $\Sigma(\mathfrak{M}, A^*)/E = \Sigma(\mathfrak{M}, A^{**})/E$. We say that $\Sigma(x, A^*)$ is *uniform on P over A* if $\Sigma(x, A^*) \cap \text{For}^x(\mathbb{L}(B^*))$ is weak uniform on P over B for any $B^* \subseteq A^*$ such that P is still B -hyperdefinable and B is still a set of hyperimaginary parameters. Let $V \subseteq P$ be \bigwedge_A -definable. A *uniform definition of V over A* is a partial type \underline{V} defining V which is uniform on P over A .

Lemma 1.1.13. *Let $P = X/E$ be an A -hyperdefinable set and $V \subseteq P$ a non-empty \bigwedge_A -definable set. Then, there is a uniform definition of V over A .*

Proof. Write $A = \{a_i\}_{i \in I}$ and $a_i \in Q_i = Y_i/F_i$. Pick A^* representatives of A and let $\Sigma(x, A^*)$ be any partial type defining V over A^* . Write $\underline{F} = \bigwedge_i \underline{F}_i$ and $\Gamma = \text{tp}(A^*)$, where \underline{F}_i is the maximal partial type over A^* defining F_i . Using saturation, take the partial type $\underline{V}(x, A^*)$ expressing $\exists y \Sigma(x, y) \wedge \underline{F}(y, A^*) \wedge \Gamma(y)$. For any $B \subseteq A$ such that P is still B -hyperdefinable and B is still a set of parameters, take $\underline{W}(x, B^*) = \underline{V}(x, A^*) \cap \text{For}^x(\mathbb{L}(B^*))$. From the definition and saturation, $a^* \models \underline{W}(x, B^*)$ if and

only if there are B^{**}, C^* with $\text{tp}(B^{**}, C^*) = \text{tp}(A^*)$ such that $a^* \models \Sigma(x, C^*)$ and B^{**} set of representatives of B . By strong homogeneity, we conclude that $a^* \models \underline{W}(x, B^*)$ if and only if there is then an automorphism $\sigma \in \text{Aut}(\mathfrak{M}/B)$ such that $\sigma(a^*) \models \Sigma(x, A^*)$. As P is still B -hyperdefinable, that means that $a^* \models \underline{W}(x, B^*)$ if and only if $\sigma(a) \in V$ for some $\sigma \in \text{Aut}(\mathfrak{M}/B)$. As the latter is independent of the choice of representatives B^* , we conclude that $\underline{W}(x, B^*)$ is weakly uniform on P over B . Q.E.D.

Lemma 1.1.14. *Let P be an A -hyperdefinable set and $V \subseteq P$ be a non-empty \bigwedge_b -definable subset with b a small set of hyperimaginaries over A and $A \subseteq b$. Let $\underline{V}(x, b^*)$ be a uniform definition of V . Let c be such that $\text{tp}(b/A) = \text{tp}(c/A)$ and c^* be representatives. Then, $\underline{V}(x, c^*)$ is a uniform definition of $V(c)$.*

Proof. Take $\sigma \in \text{Aut}(\mathfrak{M}/A)$ with $\sigma(b) = c$, so $b^{**} = \sigma^{-1}(c^*)$ is a representative of b . As $\underline{V}(x, b^*)$ is a uniform definition of V on P over b , we have that $\underline{V}(x, b^{**})$ also defines V . Consequently, $V(c)$ is defined by $\underline{V}(x, c^*)$.

Now, take $c_0 \subseteq c$ such that P is still c_0 -hyperdefinable and c_0 is still a set of parameters. Say $c_0 = \sigma(b_0)$ with $\sigma \in \text{Aut}(\mathfrak{M}/A)$. Note that P is still b_0 -hyperdefinable and b_0 is a set of parameters. As $\underline{V}(x, b^*)$ is uniform on P over b , we have that $\underline{W}(x, b_0^{**}) = \underline{V}(x, b^{**}) \cap \text{For}(\mathbf{L}(b_0^{**}))$ is weakly uniform on P over b_0 . Therefore, as $\text{tp}(c_0^*/A) = \text{tp}(b_0^{**}/A)$, we conclude that $\underline{W}(x, c_0^*)$ is weakly uniform on P over c_0 . Q.E.D.

Lemma 1.1.15. *Let P and Q be A -hyperdefinable sets and $b \in Q$. Then, for any $\bigwedge_{A,b}$ -definable subset $V \subseteq P$ there is an \bigwedge_A -definable set $W \subseteq Q \times P$ such that $V(c) = W(c) := \mathfrak{p}_P[W \cap (\{c\} \times P)]$ for any $c \in \text{tp}(b/A)$.*

Proof. Take $\Sigma(A^*, b^*, x)$ a uniform definition of V on P over A . Then, by Lemma 1.1.14, $\Sigma(A^*, y, x)$ defines an \bigwedge_A -definable subset W of $Q \times P$ such that $V(c) = W(c) := \mathfrak{p}_P[W \cap (\{c\} \times P)]$ for any $c \in \text{tp}(b/A)$. Q.E.D.

Lemma 1.1.16. *Let P be an A -hyperdefinable set. Then,*

$$\Delta_P(A) = \{(a, b) \in P \times P : \text{tp}(a/A) = \text{tp}(b/A)\}$$

is an \bigwedge_A -definable equivalence relation. Furthermore, it has a uniform definition $\underline{\Delta}_P(A^)$ such that $\underline{\Delta}_P(A^*) \cap \text{For}(\mathbf{L}(B^*))$ defines $\Delta_P(B)$ for any subset $B \subseteq A$ such that P is B -hyperdefinable and B is a set of parameters.*

Proof. Clearly, $\Delta_P(A)$ is an A -invariant equivalence relation. Let us prove that it is \bigwedge -definable. Write $A = \{a_i\}_{i \in I}$ and $a_i \in Q_i = Y_i/F_i$. Take A^* representatives. Take $\underline{E}(x, x')$ defining E over A^* and $\underline{F}(y, y') = \bigwedge \underline{F}_i(y_i, y'_i)$ where \underline{F}_i maximal defining F_i over A^* . Let $\Gamma(x, y; x', y') = \{\varphi(x, y) \leftrightarrow \varphi(x', y') : \varphi \in \text{For}^{xy}(\mathbf{L})\}$ be the usual partial type defining $\text{tp}(x, y) = \text{tp}(x', y')$. Take $\underline{\Delta}_P(A^*)(x_1, x_2) = \exists x' \exists y \underline{E}(x_2, x') \wedge \underline{F}(y, A^*) \wedge \Gamma(x_1, A^*; x', y)$. By Lemma 1.1.11, $(b^*, c^*) \in \mathfrak{g}_P^{-1}(\Delta_P(A))$ if and only if there

is c^{**} with $E(c^*, c^{**})$ and A^{**} representatives of A such that $\text{tp}(b^*, A^*) = \text{tp}(c^{**}, A^{**})$. Therefore, using saturation, $\underline{\Delta}_P(A^*)$ defines $\Delta_P(A)$. For any $B \subseteq A$ such that P is still B -hyperdefinable and B a set of parameters, $\underline{\Delta}_P(B^*) = \underline{\Delta}_P(A^*) \cap \text{For}(\mathbf{L}(B^*))$. Thus, $\underline{\Delta}_P(A^*)$ is in fact a uniform definition of $\Delta_P(A)$. Q.E.D.

1.2 The logic topologies of hyperdefinable sets

Let $P = X/E$ be an A -hyperdefinable subset. The A -logic topology of P is the one given by taking as closed sets the \bigwedge_A -definable subsets of P . In particular, the A -logic topology of P is the quotient topology of the A -logic topology of X .

Proposition 1.2.1. *Let P and Q be A -hyperdefinable sets and $f : P \rightarrow Q$ an \bigwedge_A -definable function. Then, f is a continuous and closed function between the A -logic topologies.*

Proof. By Lemma 1.1.10. Q.E.D.

Recall that a topological space is normal if any two disjoint closed sets can be separated by open sets.

Lemma 1.2.2. *Let (X, \mathcal{T}) and (Y, \mathcal{T}') be topological spaces and $f : X \rightarrow Y$ be a continuous and closed onto map. If X is normal, so is Y .*

Proof. Let $C_1, D_1 \subseteq Y$ be closed such that $C_1 \cap D_1 = \emptyset$. Consider $C_0 = f^{-1}[C_1]$ and $D_0 = f^{-1}[D_1]$. As f is continuous, C_0 and D_0 are closed. Since f is onto, $C_0 \neq \emptyset$ and $D_0 \neq \emptyset$. Since $C_1 \cap D_1 = \emptyset$, $C_0 \cap D_0 = \emptyset$. By normality of X , there are closed sets $E_0, F_0 \subseteq X$ such that $C_0 \cap E_0 = \emptyset$, $D_0 \cap F_0 = \emptyset$ and $E_0 \cup F_0 = X$. Consider $E_1 = f[E_0]$ and $F_1 = f[F_0]$. It follows that $Y \setminus E_1$ and $Y \setminus F_1$ separates C_1 and D_1 . Indeed, as f is closed, they are closed subsets of Y . Also, as f is onto, $E_1 \cup F_1 = Y$. Finally, $C_1 \cap E_1 = \emptyset$ and $D_1 \cap F_1 = \emptyset$. Otherwise, either there is $y \in C_1 \cap E_1$ or $y \in D_1 \cap F_1$. Without loss of generality, suppose $y \in C_1 \cap E_1$, so there is $x \in E_0$ such that $x \in f^{-1}[C_1] = C_0$, getting a contradiction. Q.E.D.

Proposition 1.2.3. *Let P be an A -hyperdefinable set. Then, the A -logic topology of P is compact and normal. The closure of a point $a \in P$ is $\text{tp}(a/A)$, so the properties T_0 (i.e. Kolmogorov), T_1 (i.e. Fréchet) and T_2 (i.e. Hausdorff) are equivalent to $\text{tp}(a/A) = \{a\}$ for all $a \in P$.*

Proof. Compactness follows from saturation. The closure of a point is its type by definition, so T_1 is equivalent to $\text{tp}(a/A) = \{a\}$ for every $a \in P$. By Corollary 1.1.11, $a \in \text{tp}(b/A)$ if and only if $\text{tp}(a/A) = \text{tp}(b/A)$, so $b \in \overline{\{a\}}$ if and only if $a \in \overline{\{b\}}$ — this topological property is sometimes called R_0 . Therefore, T_0 is also equivalent to $\text{tp}(a/A) = \{a\}$ for every $a \in P$. If it is normal, then T_1 and T_2 are equivalent. Thus, it remains to show that it is normal. Take A^* representatives of A .

Trivially, the A^* -logic topology of X is normal. Indeed, if $V, W \subseteq X$ are closed disjoint sets, by compactness, there is a formula φ over A^* such that $V \subseteq \varphi(\mathfrak{M})$ and $W \subseteq \neg\varphi(\mathfrak{M})$, so V and W are separated by the open sets $\varphi(\mathfrak{M})$ and $\neg\varphi(\mathfrak{M})$. By the Correspondence Lemma 1.1.1, \mathfrak{g}_P is continuous and closed between the A^* -logic topologies of X and P , so P is normal with the A^* -logic topology by Lemma 1.2.2.

Consider $P/\Delta_P(A)$ with the quotient topology from the A -logic topology. By definition $\mathfrak{g}_{\Delta_P(A)}$ is continuous from the A -logic topologies. For any subset $V \subseteq P$, we have that $a \in \mathfrak{g}_{\Delta_P(A)}^{-1}[\mathfrak{g}_{\Delta_P(A)}[V]]$ if and only if there is $b \in V$ with $\text{tp}(a/A) = \text{tp}(b/A)$. Thus, if V is A -invariant, $\mathfrak{g}_{\Delta_P(A)}^{-1}[\mathfrak{g}_{\Delta_P(A)}[V]] = V$. Consequently, $\mathfrak{g}_{\Delta_P(A)}$ is also open and closed from the A -logic topology.

Since the A -logic topology is coarser than the A^* -logic topology, we conclude that $\mathfrak{g}_{\Delta_P(A)}$ is continuous from the A^* -logic topology too. Let $V \subseteq P$ be an \bigwedge_{A^*} -definable set, and consider $V' = \mathfrak{g}_{\Delta_P(A)}^{-1}[\mathfrak{g}_{\Delta_P(A)}[V]]$. By definition, $b \in V'$ if and only if there is $a \in V$ such that $\text{tp}(a/A) = \text{tp}(b/A)$. Therefore, V' is A -invariant. On the other hand, if $\mathfrak{g}_P^{-1}[V']$ is defined by $\underline{V}'(x) = \exists y \underline{V}(y) \wedge \underline{\Delta}_P(x, y, A^*)$, so it is \bigwedge -definable. Thus, V' is \bigwedge_A -definable, concluding that $\mathfrak{g}_{\Delta_P(A)}[V]$ is closed from the A^* -logic topology. Thus, by Lemma 1.2.2, we conclude that $P/\Delta_P(A)$ is normal. Take $V_1, V_2 \subseteq P$ disjoint and \bigwedge_A -definable. Then, $\mathfrak{g}_{\Delta_P(A)}[V_1], \mathfrak{g}_{\Delta_P(A)}[V_2] \subseteq P/\Delta_P(A)$ are disjoint and closed. By normality of $P/\Delta_P(A)$, there are $U_1, U_2 \subseteq P/\Delta_P(A)$ open separating them. Then, $\mathfrak{g}_{\Delta_P(A)}^{-1}[U_1]$ and $\mathfrak{g}_{\Delta_P(A)}^{-1}[U_2]$ are open in the A -logic topology of P and separate V_1 and V_2 . Therefore, the A -logic topology of P is normal. Q.E.D.

Proposition 1.2.4. *Let P be an A -hyperdefinable set and $A \subseteq A'$. If the A -logic topology of P is Hausdorff, then the A -logic topology and the A' -logic topology are equal. Thus, there is at most one Hausdorff logic topology and, if it exists, it will be called the global logic topology.*

Proof. As the $\bigwedge_{A'}$ -logic topology of P is compact and finer than the \bigwedge_A -logic topology of P , which is Hausdorff, we conclude that both topologies are the same; the identity map is a continuous bijection from a compact topology to a Hausdorff topology. Q.E.D.

Remark 1.2.5. Furthermore, the global logic topologies are preserved by expansion of the language. Indeed, let P be a hyperdefinable set whose A -logic topology is Hausdorff and \mathfrak{M}' a κ -saturated and strongly κ -homogeneous expansion of \mathfrak{M} . Then, obviously, $\text{id} : P \rightarrow P$ is a continuous bijection from the A -logic topology in \mathfrak{M}' to the A -logic topology in \mathfrak{M} . As both are compact and the second one is Hausdorff, we conclude that both are the same.

Proposition 1.2.6. *Let P be an A -hyperdefinable set. Either $|P| \geq \kappa$ (i.e. P is unbounded) or $|P| \leq 2^{|A|+|L|+\text{ary}(P)}$.*

Proof. Say $P = X/E$ and $\tau = |A| + |L| + \text{ary}(P)$. Write $E = \bigcap_{i \in \tau} E_i$, where E_i is a definable symmetric binary relation for each i .

Suppose $|P| > 2^\tau$ and write $P = \{a_\alpha\}_{\alpha \in |P|}$. Choose representatives $a_\alpha^* \in a_\alpha$ for each $\alpha \in |P|$ and write $[P]^2$ for the set of pairs in P . Consider

$$f : \begin{array}{ccc} [P]^2 & \rightarrow & \tau \\ \{a_\alpha, a_\beta\} & \mapsto & \min\{i \in \tau : (a_\alpha^*, a_\beta^*) \notin E_i\}. \end{array}$$

Note that it is well-defined as $E = \bigcap_{i \in \tau} E_i$. By Erdős-Rado generalisation of Ramsey's Theorem [TZ12, Theorem C.3.2], there is an infinite subset $D_0 \subseteq P$ such that $f(\bar{d})$ is a constant $i_0 \in \tau$ for any $\bar{d} \in [D_0]^2$. Consider $\mathcal{K} = \{D \subseteq P : D_0 \subseteq D, f \text{ constant in } [D]^2\}$. Using Zorn's Lemma [Jec02, Theorem 5.4], there is $D \in \mathcal{K}$ maximal. Say $D = \{a_{\sigma(\xi)}\}_{\xi \in |D|}$ with $\sigma : |D| \rightarrow |P|$ injective. Aiming a contradiction, suppose $|D| < \kappa$. Consider

$$\Sigma(x) = \left\{ \neg \underline{E}_{i_0}(x, a_{\sigma(\xi)}^*) \wedge \bigwedge_{j=0}^{i_0-1} \underline{E}_j(x, a_{\sigma(\xi)}^*) \right\}_{\xi \in |D|}.$$

Then, as $D \in \mathcal{K}$ is infinite, Σ is finitely satisfiable. Since we are assuming that $|D| < \kappa$, by κ -saturation of \mathfrak{M} , there is a^* realising Σ . Let $a = \mathfrak{g}_P(a^*)$. Then, $a \notin D$ as $(a^*, a_{\sigma(\xi)}^*) \notin E$ for any $\xi \in |D|$. Also, for any $\xi \in |D|$, $(a^*, a_{\sigma(\xi)}^*) \in \bigcap_{j=0}^{i_0-1} E_j$ and $(a^*, a_{\sigma(\xi)}^*) \notin E_{i_0}$. That means $f(\{a, a_{\sigma(\xi)}\}) = i_0$ for any $\xi \in |D|$. Therefore, $D \cup \{a\} \in \mathcal{K}$ and $D \subset D \cup \{a\}$, getting a contradiction with the maximality of D .
Q.E.D.

For A set of parameters, the *bounded closure* of A is

$$\text{bdd}(A) := \{a \text{ } A\text{-hyperimaginary unary} : |\text{tp}(a/A)| < \kappa\}.$$

Let P be an A -hyperdefinable set. Assuming that $\kappa > 2^{|A|+|L|+\text{ary}(P)}$, either $|P| \geq \kappa$ or $|P| \leq 2^{|A|+|L|+\text{ary}(P)}$. Then, P has a global logic topology if and only if P is small, if and only if $|P| \leq 2^{|A|+|L|+\text{ary}(P)}$. Furthermore, note that $|\text{bdd}(A)| \leq 2^{|A|+|L|}$. It follows that, for any A -hyperdefinable set P , it has a global logic topology if and only if the $\text{bdd}(A)$ -logic topology is the global logic topology.

Proposition 1.2.7. *Let $(P_i)_{i \in I}$ be A -hyperdefinable sets with I small. Then, the A -logic topology in $P = \prod_{i \in I} P_i$ is at least as fine as the product topology of the A -logic topologies. Furthermore, P has a global logic topology if and only if each P_i have so, and then the global logic topology of P is the product topology of the global logic topologies of P_i .*

Proof. The projection maps $\mathfrak{p}_i : P \rightarrow P_i$ are \bigwedge_A -definable, so continuous between the A -logic topologies. Consequently, the A -logic topology of P is at least as fine as its product topology. If each P_i has a global logic topology, then each P_i is small. Since I is also small, we conclude that P is small too. Therefore, P has a global logic topology. The global logic topology of P is compact and Hausdorff and the product topology of P is also compact and Hausdorff. Therefore, they coincide. Q.E.D.

1.3 Piecewise hyperdefinable sets

A *piecewise A -hyperdefinable* set is a strict direct limit of A -hyperdefinable sets with \bigwedge_A -definable inclusions. In other words, a piecewise A -hyperdefinable set is a direct limit

$$P := \varinjlim_{(I, \prec)} (P_i, \varphi_{ji})_{j \succeq i} = \bigsqcup_I P_i / \sim_P$$

of a direct system of A -hyperdefinable sets and 1-to-1 \bigwedge_A -definable functions $\varphi_{ji} : P_i \rightarrow P_j$.

Recall that (I, \prec) is a direct ordered set and, for each $i, j, k \in I$ with $i \preceq j \preceq k$, $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ and $\varphi_{ii} = \text{id}$. Also, recall that \sim_P is the equivalence relation on $\bigsqcup_I P_i$ defined by $x \sim_P y$ for $x \in P_i$ and $y \in P_j$ if and only if there is some $k \in I$ with $i \preceq k$ and $j \preceq k$ and $\varphi_{ki}(x) = \varphi_{kj}(y)$. Note that, in fact, as we consider only direct systems where the functions φ_{ji} are 1-to-1, the equivalence relation is given by $x \sim_P y$ with $x \in P_i$ and $y \in P_j$ if and only if $\varphi_{ki}(x) = \varphi_{kj}(y)$ for any $k \in I$ such that $i \preceq k$ and $j \preceq k$.

The *pieces* of P are the subsets P_i / \sim_P . The *canonical inclusions* are the maps $\iota_{P_i} : P_i \rightarrow P_i / \sim_P \subseteq P$ given by $a \mapsto [a]_{\sim_P}$ for $a \in P_i$. The *arity* of P is the supremum $\text{ary}(P) = \sup_{i \in I} \text{ary}(P_i)$ of the arities of its pieces. From now on, we always assume that $\text{ary}(P) < \kappa$.

The *cofinality* $\text{cf}(P)$ of P is the cofinality of I , which is the minimal ordinal α from which there is a function $f : \alpha \rightarrow I$ such that for every $i \in I$ there is $\xi \in \alpha$ with $i \preceq f(\xi)$. We say that P is *countably* piecewise hyperdefinable if it has countable cofinality. From now on, we always assume that $\text{cf}(P) < \kappa$.

A *piecewise \bigwedge_A -definable* subset of P is a subset $V \subseteq P$ such that $\iota_{P_i}^{-1}[V]$ is \bigwedge_A -definable in P_i for each $i \in I$. An *\bigwedge_A -definable* subset of P is a piecewise \bigwedge_A -definable subset contained in some piece. If $V \subseteq P$ is a non-empty \bigwedge -definable set, after fixing a piece P_i / \sim_P containing it and declaring the parameters, we will write \underline{V} to denote a partial type defining it in P_i . In that case, we will also say that \underline{V} defines V . If $a \in P$ is an element, after fixing a piece P_i / \sim_P containing it, we will say that a^* is a representative of a if $\iota_{P_i}(\mathfrak{g}_{P_i}(a^*)) = a$, where \mathfrak{g}_{P_i} is the quotient map of P_i as hyperdefinable set.

Remark 1.3.1. The set of piecewise \bigwedge_A -definable subsets is a lattice of sets closed under arbitrary intersections, and the collection of \bigwedge_A -definable subsets is the ideal of that lattice generated by the pieces.

Note that, in the previous definitions, piecewise hyperdefinable sets are not only sets but sets together with a particular structure. This structure is given by the

lattices of piecewise \bigwedge -definable subsets and the ideals of \bigwedge -definable subsets. It is very important to remember this as the same set could be represented as a piecewise hyperdefinable set in several different ways — see Example 1.7.1.

By the strictness condition we get the following fundamental lemma.

Lemma 1.3.2 (Correspondence Lemma). *Let P be piecewise A -hyperdefinable and P_i/\sim_P a piece of P . Then, $\iota_{P_i} : P_i \rightarrow P_i/\sim_P$ is a bijection. Furthermore, $V \subseteq P_i/\sim_P$ is \bigwedge_A -definable as subset of P if and only if $\iota_{P_i}^{-1}[V]$ is \bigwedge_A -definable as subset of P_i .*

Proof. Obviously $\iota_{P_i} : P_i \rightarrow P_i/\sim_P$ is a bijection by the strictness condition. Say $V \subseteq P_i/\sim_P$. Using again the strictness condition, for any $j, k \in I$ with $i, j \preceq k$, we have that $\iota_{P_j}^{-1}[V] = \varphi_{kj}^{-1}[\iota_{P_k}^{-1}[V]] = \varphi_{kj}^{-1}[\varphi_{ki}[\iota_{P_i}^{-1}[V]]]$. Therefore, by \bigwedge_A -definability of the maps and Lemma 1.1.10, we conclude that V is \bigwedge_A -definable if and only if $\iota_{P_i}^{-1}[V]$ is \bigwedge_A -definable as subset of P_i . Q.E.D.

The Correspondence Lemma 1.3.2 says that ι_{P_i} is a true identification between P_i and P_i/\sim_P in terms of the model theoretic structure. In other words, it says that pieces of piecewise hyperdefinable sets are, indeed, hyperdefinable. From now on, slightly abusing of the notation, we make no distinction between P_i and P_i/\sim_P , i.e. $P_i := P_i/\sim_P$ and $\mathfrak{g}_{P_i} := \iota_{P_i} \circ \mathfrak{g}_{P_i}$.

Corollary 1.3.3. *Let $P = \varinjlim_I P_i$ be piecewise A -hyperdefinable. Then, a subset $V \subseteq P$ is piecewise \bigwedge_A -definable if and only if $V \cap P_i$ is \bigwedge_A -definable as subset of P for each $i \in I$.*

Remark 1.3.4. As every piece is \bigwedge -definable, the lattice of piecewise \bigwedge_A -definable subsets can be recovered from the ideal of \bigwedge_A -definable subsets. In other words, the structure of P is completely determined by the ideals of \bigwedge -definable sets. However, in general, the lattice of piecewise \bigwedge_A -definable subsets does not determine the ideal of \bigwedge_A -definable subsets — see Example 1.7.1.

A *type over A* , or *A -type*, is a \subset -minimal non-empty piecewise \bigwedge_A -definable subset. The *type of $a \in P$ over A* , $\text{tp}(a/A)$, is the minimal piecewise \bigwedge_A -definable subset of P containing a . Since $a \in P_i$ for some $i \in I$ and pieces are \bigwedge_A -definable, $\text{tp}(a/A)$ is actually \bigwedge_A -definable.

Let $R \subseteq P \times Q$ be a binary relation between two piecewise hyperdefinable sets. We say that R is *piecewise bounded* (or *piecewise continuous*) if the image of any piece of P is contained in some piece of Q . We say that it is *piecewise proper* if the preimage of any piece of Q is contained in some piece of P . We use this terminology in particular for partial functions. To simplify the terminology, we often omit reiterative uses of

“piecewise” when they happen. So, for example, we say “a piecewise bounded and proper \bigwedge -definable function” instead of “a piecewise bounded and piecewise proper piecewise \bigwedge -definable function”.

Given two piecewise A -hyperdefinable sets $P = \varinjlim_I P_i$ and $Q = \varinjlim_J Q_j$, the Cartesian product $P \times Q$ is canonically identified with $\varinjlim_{I \times J} P_i \times Q_j$ via $([x]_P, [y]_Q) \mapsto [x, y]_{P \times Q}$, where $(x, y) \sim_{P \times Q} (x', y')$ if and only if $x \sim_P x'$ and $y \sim_Q y'$. Thus, we say that a binary relation R between P and Q is piecewise \bigwedge_A -definable if it is so as subset of the Cartesian product.

Lemma 1.3.5. *Let P and Q be piecewise A -hyperdefinable sets, $f : P \rightarrow Q$ a piecewise \bigwedge_A -definable function and $a \in P$. Then, $f[\text{tp}(a/A)] = \text{tp}(f(a)/A)$.*

Proof. Clear from Lemma 1.1.12. Q.E.D.

Proposition 1.3.6. *Let P and Q be two piecewise A -hyperdefinable sets and f a piecewise \bigwedge_A -definable partial function from P to Q . Then:*

(1) *If f is piecewise bounded, images of \bigwedge_A -definable sets are \bigwedge_A -definable, and preimages of piecewise \bigwedge_A -definable sets are piecewise \bigwedge_A -definable.*

(2) *If f is piecewise proper, images of piecewise \bigwedge_A -definable sets are piecewise \bigwedge_A -definable, and preimages of \bigwedge_A -definable sets are \bigwedge_A -definable.*

Proof. Both are quite similar, so let us show only (1). For $i \in I$ and $j \in J$ with $f[P_i] \subseteq Q_j$, consider $f_{ji} : P_i \rightarrow Q_j$ given by the restriction of f . As f is piecewise \bigwedge_A -definable, each f_{ji} is \bigwedge_A -definable. Given an \bigwedge_A -definable subset $V \subseteq P_i$, we have that $f[V] = f_{ji}[V]$, concluding that the image of V is \bigwedge_A -definable in Q by Lemma 1.1.10 and the Correspondence Lemma 1.3.2. On the other hand, given a piecewise \bigwedge_A -definable subset $W \subseteq Q$, we have that $f^{-1}[W] \cap P_i = f_{ji}^{-1}[W \cap Q_j]$, so $f^{-1}[W]$ is piecewise \bigwedge_A -definable in P by Lemma 1.1.10. Q.E.D.

An *isomorphism of piecewise A -hyperdefinable sets* is a piecewise bounded and proper \bigwedge_A -definable bijection. In that case, we will say that P and Q are *isomorphic over A* .

1.4 The logic topologies of piecewise hyperdefinable sets

The *A -logic topology* of P is the respective direct limit topology. In other words, a subset of P is closed if and only if it is piecewise \bigwedge_A -definable. By the Correspondence Lemma 1.3.2, each piece is compact and, further, every \bigwedge_A -definable subset is compact.

As in the case of hyperdefinable sets, $\overline{\{a\}} = \text{tp}(a/A)$ for any $a \in P$. Thus, the properties T_0 and T_1 are equivalent. If the A -logic topology is T_1 , for any other small set of hyperimaginary parameters A' containing A , the logic topologies over A and over A' are the same. Thus, there is at most one T_1 logic topology on P and, if it exists, it is called the *global logic topology*. It follows that P has a global logic topology if and only if every piece has size at most $2^{|A|+|L|+\text{ary}(P)}$, if and only if the $\text{bdd}(A)$ -logic topology is the global logic topology. In particular, if P has a global logic topology, then $|P| \leq 2^{|A|+|L|+\text{ary}(P)} + \text{cf}(P)$. Thus, assuming $2^{|A|+|L|+\text{ary}(P)} + \text{cf}(P) < \kappa$, we conclude that P has a global logic topology if and only if it is small.

There are still some topological properties that we want to extend from hyperdefinable sets to piecewise hyperdefinable sets. Ideally, we would like to show that these topologies are locally compact, normal and satisfy $T_1 \Leftrightarrow T_2$. Also, we would like to show that they are closed under taking finite products. In general, these properties may fail — see Example 1.7.4, Example 1.7.6 and Example 2.2.6 for some counterexamples. The rest of the section is dedicated to give sufficient natural conditions for these properties.

In general, for a topological space X , a *covering* of X is a set $\mathcal{C} \subseteq \mathcal{P}(X)$ such that $\bigcup \mathcal{C} = X$, whose elements are called *pieces*. A covering \mathcal{C} is *coherent* when, for every $U \subseteq X$, U is open in X if and only if $U \cap P$ is open in P for each $P \in \mathcal{C}$. Equivalently, \mathcal{C} is coherent when, for every $V \subseteq X$, V is closed in X if and only if $V \cap P$ is closed in P for each $P \in \mathcal{C}$. For example, X is *compactly generated* if the family of all the compact subsets is a coherent covering.

We say that a covering is *local* if for every point of X there is a piece that is a neighbourhood of it. The following topological results are straightforward:

Lemma 1.4.1 (Local Coverings Lemma). *Let X be a topological space and \mathcal{C} a local covering. Then, \mathcal{C} is coherent.*

Proof. Let U be such that $U \cap P$ is open in P for any $P \in \mathcal{C}$. We want to prove that U is open in X . Take $x \in U$ arbitrary. Find $P \in \mathcal{C}$ such that P is a neighbourhood of x and $B \subseteq P$ open in X such that $x \in B$. As $B \subseteq P$ is open in X , it is also open in P . Since $U \cap P$ is open in P , we have that $U \cap B$ is open in P . As $B \subseteq P$, $U \cap B$ is also open in B . As B is open in X , we conclude that $U \cap B$ is open in X . Thus, $x \in U \cap B \subseteq U$ with $U \cap B$ open. Since x was arbitrary, we conclude that U is open. Therefore, \mathcal{C} is coherent. Q.E.D.

The previous lemma has the following immediate consequence in general topology. Recall that a covering is *locally finite* if for every point there is a neighbourhood

that only meets finitely many pieces. Recall that a topological space is *weakly locally compact* if every point has a compact neighbourhood.

Corollary 1.4.2. (1) *Open coverings and locally finite closed coverings are coherent.*

(2) *Weakly locally compact spaces are compactly generated.*

Lemma 1.4.3. *Let X be a topological space and \mathcal{C} a covering. Let $Y \subseteq X$ be a topological subspace and $\mathcal{C}|_Y = \{P \cap Y : P \in \mathcal{C}\}$. Then, if \mathcal{C} is local, $\mathcal{C}|_Y$ is also local, and so coherent.*

Proof. Trivial. For $x \in Y$, if C is a neighbourhood of x in X , then $C \cap Y$ is a neighbourhood of x in Y . Q.E.D.

Lemma 1.4.4. *Let X and Y be topological spaces and \mathcal{C} and \mathcal{D} respective coverings. Let $\mathcal{C} \times \mathcal{D} := \{P \times Q : P \in \mathcal{C}, Q \in \mathcal{D}\}$. Then, if \mathcal{C} and \mathcal{D} are local, $\mathcal{C} \times \mathcal{D}$ is also a local covering of $X \times Y$, and so coherent.*

Proof. Trivial. If C is a neighbourhood of x in X and D is a neighbourhood of y in Y , then $C \times D$ is a neighbourhood of (x, y) in $X \times Y$. Q.E.D.

Recall that Tietze's Extension Theorem [Mun15, Theorem 35.1] states that any continuous function to $[-1, 1]$ defined on a closed subset of a normal topological space can be extended to a continuous function to $[-1, 1]$ defined on the whole space. A *perfectly normal* topological space is a topological space where any two closed disjoint sets can be perfectly separated by a continuous function to $[-1, 1]$, i.e. there is a continuous function to $[-1, 1]$ such that each one of the two sets is the fibre of -1 or 1 . There is a natural variation of Tietze's Extension Theorem for perfectly normal spaces:

Lemma 1.4.5 (Tietze's Perfect Extension Theorem). *Let X be a perfectly normal topological space, $A \subseteq X$ a closed subset and $f : A \rightarrow [-1, 1]$ a continuous function. There is then a continuous function $F : X \rightarrow [-1, 1]$ such that $f \subseteq F$ and $F^{-1}(-1) = f^{-1}(-1)$ and $F^{-1}(1) = f^{-1}(1)$.*

Proof. Define by recursion two sequences of continuous functions $(h_n)_{n \in \mathbb{N}}$ with $h_0 = f$ and $(g_n)_{n \in \mathbb{N}_{>0}}$ such that

$$h_n : A \rightarrow \left[-\frac{2^n}{3^n}, \frac{2^n}{3^n}\right] \quad , \quad g_n : X \rightarrow \left[-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}\right] ,$$

$$h_{n+1} = h_n - g_{n+1}|_A ,$$

$$h_n^{-1} \left(-\frac{2^n}{3^n}\right) = h^{-1}(-1) \text{ and}$$

$$h_n^{-1}(2^n/3^n) = h^{-1}(1).$$

Write $a_n = 2^n/3^n$. Given h_n , take $I = [-a_n, -a_n/3]$ and $J = [a_n/3, a_n]$. Then, $h_n^{-1}(I)$ and $h_n^{-1}(J)$ are disjoint closed sets of X . Take a continuous function $g_{n+1} : X \rightarrow [-a_n/3, a_n/3]$ perfectly separating them. Define $h_{n+1} = h_n - g_{n+1}|_A$.

$$\text{For } x \in h_n^{-1}[I], 0 \geq h_{n+1}(x) \geq -a_n + a_n/3 = -2a_n/3 = -a_{n+1}.$$

$$\text{For } x \in h_n^{-1}[J], 0 \leq h_{n+1}(x) \leq a_n - a_n/3 = a_{n+1}.$$

$$\text{For } x \in A \setminus h_n^{-1}[I \cup J], |h_n(x)| \leq 2a_n/3 = a_{n+1}.$$

Trivially, we have $h^{-1}(-1) = h_n^{-1}(-a_n) \subseteq h_{n+1}^{-1}(-a_{n+1})$ and $h^{-1}(1) = h_n^{-1}(a_n) \subseteq h_{n+1}^{-1}(a_{n+1})$. On the other hand, if $h_{n+1}(x) = -a_{n+1}$, then $h_n(x) = g_{n+1}(x) - a_{n+1} \leq -a_n/3$, and so $x \in h_n^{-1}(I)$, concluding that $g_{n+1}(x) = -a_n/3$ and $h_n(x) = -a_n$. Similarly, if $h_{n+1}(x) = a_{n+1}$, $h_n(x) = a_n$.

Now, consider $G_n = \sum_{i=1}^n g_i : X \rightarrow [-1 + a_n, 1 - a_n]$. It follows that $h - G_n|_A = h_n$ and

$$f^{-1}(-1) \subseteq G_n^{-1}(-1 + a_n) = \bigcap_{i=1}^n g_i^{-1}(-a_{i-1}/3) \subseteq A,$$

$$f^{-1}(1) \subseteq G_n^{-1}(1 - a_n) = \bigcap_{i=1}^n g_i^{-1}(a_{i-1}/3) \subseteq A.$$

Note that $(G_n)_{n \in \mathbb{N}}$ is a Cauchy sequence as $|G_n - G_m| \leq a_{n-1}$. Thus, it converges uniformly to a continuous function $F : X \rightarrow [-1, 1]$. As $(G_n|_A)_{n \in \mathbb{N}}$ converges uniformly to f , we conclude that $f \subseteq F$. Also, if $x \notin A$, then $g_{n+1}(x) \in (-a_n/3, a_n/3)$ and $F(x) \notin \{-1, 1\}$. Hence, if $F(x) = 1$, $x \in A$ and $f(x) = F(x) = 1$. Similarly, if $F(x) = -1$, $x \in A$ and $f(x) = F(x) = -1$. Thus, $F^{-1}(-1) = f^{-1}(-1)$ and $F^{-1}(1) = f^{-1}(1)$. Q.E.D.

Lemma 1.4.6. *Let X be a topological space and $\{P_n\}_{n \in \mathbb{N}}$ a closed coherent covering with $P_n \subseteq P_{n+1}$ for each $n \in \mathbb{N}$. Suppose that P_n is normal for each $n \in \mathbb{N}$. Then, X is normal. Similarly, if P_n is perfectly normal for each $n \in \mathbb{N}$, then X is perfectly normal.*

Proof. Let V and W be closed disjoint subsets of X . Take the continuous map $g : V \sqcup W \rightarrow \{-1, 1\}$ given by $g|_V = 1$ and $g|_W = -1$. By recursion, using Tietze's Extension Theorem [Mun15, Theorem 35.1], we construct a chain $(f_n)_{n \in \mathbb{N}}$ of continuous maps $f_n : P_n \rightarrow [-1, 1]$, each one extending $g|_{P_n}$. Taking $f = \bigcup f_n$, we get a continuous map separating V and W .

Similarly we get the version for perfectly normal by using Tietze's Perfect Extension Theorem (Lemma 1.4.5) in place of [Mun15, Theorem 35.1]. Q.E.D.

In the case of a piecewise hyperdefinable set P , we have a directed closed and compact coherent covering $\{P_i\}_{i \in I}$. In particular, it follows that logic topologies are

always compactly generated. If P is countably piecewise hyperdefinable, then it is σ -compact (i.e. a countable union of compact sets). Furthermore, by Lemma 1.4.6, we get normality:

Proposition 1.4.7. *Let P be a countably piecewise A -hyperdefinable set. Then, P is normal with the A -logic topology. In particular, global logic topologies of countably piecewise hyperdefinable sets are Hausdorff.*

Say that a piecewise hyperdefinable set P is *locally A -hyperdefinable* if its covering is local in the A -logic topology, i.e. if for every point of P there is an \bigwedge_A -definable set which is a neighbourhood of it in the A -logic topology. Say that P is locally hyperdefinable if it is so for some small set of parameters.

Remark 1.4.8. *Piecewise definable* sets are the special case of piecewise hyperdefinable sets when we only consider strict direct limits of definable sets with definable maps. Definable sets are always open in the logic topology so, following the terminology of this thesis, every piecewise definable set is trivially locally (hyper)definable. The distinction between these two notions only appears in the general context of piecewise hyperdefinable sets. In particular, our terminology is consistent with the typical use of the terms “piecewise definable” and “locally definable” as synonyms in the literature.

Proposition 1.4.9. *Let P and Q be piecewise A -hyperdefinable sets. Assume that P is locally A -hyperdefinable. Let $f : P \rightarrow Q$ be a piecewise bounded and proper \bigwedge_A -definable onto map. Then, Q is locally A -hyperdefinable.*

Proof. Pick $y \in Q$. By Proposition 1.3.6, $f^{-1}[\text{tp}(y/A)]$ is \bigwedge_A -definable, so compact in the A -logic topology of P . As P is locally hyperdefinable, we find an \bigwedge_A -definable set V such that $f^{-1}[\text{tp}(y/A)] \subseteq U \subseteq V$, where $P \setminus U$ is piecewise \bigwedge_A -definable. By Proposition 1.3.6, $f[V]$ is \bigwedge_A -definable, and $f[P \setminus U]$ is piecewise \bigwedge_A -definable. As $f^{-1}[\text{tp}(y/A)] \subseteq U$, $\text{tp}(y/A) \cap f[P \setminus U] = \emptyset$. Also, as f is onto, $Q = f[V \cup (P \setminus U)] = f[V] \cup f[P \setminus U]$. Therefore, $y \in \text{tp}(y/A) \subseteq Q \setminus f[P \setminus U] \subseteq f[V]$, concluding that $f[V]$ is an \bigwedge_A -definable neighbourhood of y in the A -logic topology. As y is arbitrary, we conclude that Q is locally A -hyperdefinable. Q.E.D.

For locally hyperdefinable sets we have really good control of the logic topology.

Proposition 1.4.10. *Let P be a locally A -hyperdefinable set. Then, P is locally closed compact in the A -logic topology, i.e. every point has a local base of closed compact neighbourhoods.*

Proof. Say $P = \varinjlim P_i$. Pick $x \in P$ and U open neighbourhood of x in the A -logic topology. Take a piece P_i and U_0 such that $x \in U_0 \subseteq P_i$ with U_0 open in the A -logic topology of P . Then, $U_1 := U \cap U_0$ is an open neighbourhood of x in the A -logic topology of P . Note that $\text{tp}(x/A) \subseteq U_1 \subseteq P_i$, so $\text{tp}(x/A)$ and $P_i \setminus U_1$ are disjoint

closed subsets of P_i . By Proposition 1.2.3, P_i is normal, so there are $\text{tp}(x/A) \subseteq U'$ and $P_i \setminus U_1 \subseteq P_i \setminus V'$ with $U' \cap (P_i \setminus V') = \emptyset$ such that $P_i \setminus U'$ and V' are \bigwedge_A -definable in P_i . Therefore, $x \in \text{tp}(x/A) \subseteq U' \subseteq V' \subseteq U_1 \subseteq P_i$. Now, since U_1 is open in the A -logic topology of P and $U' \subseteq U_1$ is open in the subspace topology of U_1 , we conclude that U' is open in the A -logic topology of P . Hence, V' is an \bigwedge_A -definable neighbourhood of x in the A -logic topology contained in U . As x and U are arbitrary, we conclude that P is locally closed compact. Q.E.D.

Proposition 1.4.11. *Let P be a locally A -hyperdefinable set. Then, every compact subset of P in the A -logic topology is contained in the interior of some piece.*

Proof. Clear from the definition. Q.E.D.

Proposition 1.4.12. *Let P be a locally A -hyperdefinable set. Then, any two closed compact disjoint subsets in the A -logic topology are separated by open sets. In particular, it is T_1 if and only if it is T_2 .*

Proof. Say $P = \varinjlim P_i$. Let K_1 and K_2 be two disjoint closed compact subsets of P in the A -logic topology. By Proposition 1.4.11, there is a piece P_i and an open subset U of P such that $K_1, K_2 \subseteq U \subseteq P_i$. By normality of P_i , there are disjoint open subsets U_1 and U_2 in the A -logic topology of P_i such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$. Thus, $U \cap U_1$ and $U \cap U_2$ are disjoint open subsets in the subspace topology of U separating K_1 and K_2 . As U is open in the A -logic topology of P , we conclude that $U \cap U_1$ and $U \cap U_2$ are disjoint open subsets in the A -logic topology of P separating K_1 and K_2 . Q.E.D.

Proposition 1.4.13. *Let P be a piecewise A -hyperdefinable set. Suppose either that P is countably piecewise hyperdefinable or it is a locally A -hyperdefinable set. Then, the closure of a compact set in the A -logic topology is the smallest A -invariant subset containing it.*

Proof. Let K be a compact subset and \overline{K} its closure in the A -logic topology. Then, \overline{K} is the smallest A -invariant set containing K , i.e. $\overline{K} = \bigcup_{a \in K} \text{tp}(a/A)$. Indeed, as \overline{K} is A -invariant and $K \subseteq \overline{K}$, $\bigcup_{a \in K} \text{tp}(a/A) \subseteq \overline{K}$. Conversely, if $b \notin \bigcup_{a \in K} \text{tp}(a/A)$, then $\text{tp}(a/A) \cap \text{tp}(b/A) = \emptyset$ for every $a \in K$. Either by Proposition 1.4.7 or by Proposition 1.4.12, there are disjoint open sets U_a and V_a such that $\text{tp}(a/A) \subseteq U_a$ and $\text{tp}(b/A) \subseteq V_a$. As K is compact, there are finitely many a_1, \dots, a_n such that $K \subseteq U = \bigcup U_{a_i}$ and $\text{tp}(b/A) \subseteq V = \bigcap V_{a_i}$ with U and V open and disjoint. Therefore, $V \cap K \neq \emptyset$, so $b \notin \overline{K}$. Q.E.D.

Recall that a function between topological spaces is *proper* if the preimage of every compact set is compact. For instance, every closed function with compact fibres is proper [Eng89, Theorem 3.7.2].

Proposition 1.4.14. *Let P and Q be piecewise A -hyperdefinable sets and $f : P \rightarrow Q$ a piecewise \bigwedge_A -definable function.*

(1) *If f is piecewise bounded, then f is continuous between the A -logic topologies.*

(2) *If f is piecewise proper, then f is closed and has compact fibres between the A -logic topologies. In particular, it is proper.*

(3) *If f is an isomorphism of piecewise A -hyperdefinable sets, then f is a homeomorphism between the A -logic topologies.*

(4) *If Q is locally A -hyperdefinable, then f is continuous between the A -logic topologies if and only if it is piecewise bounded.*

(5) *If P is locally A -hyperdefinable, then f is proper between the A -logic topologies if and only if it is piecewise proper.*

(6) *If P and Q are locally A -hyperdefinable, then f is an isomorphism of piecewise A -hyperdefinable sets if and only if it is a homeomorphism between the A -logic topologies.*

Proof. Point (1) is given by Proposition 1.3.6(1).

Point (2): closedness is given by Proposition 1.3.6(2). On the other hand, for any point $a \in Q$, $f^{-1}(a) \subseteq f^{-1}[\text{tp}(a/A)]$ and $f^{-1}[\text{tp}(a/A)]$ is \bigwedge_A -definable, so compact. As f is A -invariant and $\text{tp}(a/A)$ is the orbit of a under $\text{Aut}(\mathfrak{M}/A)$ by Lemma 1.1.11, it follows that $f^{-1}[\text{tp}(a/A)]$ is the orbit of $f^{-1}(a)$ under $\text{Aut}(\mathfrak{M}/A)$, i.e. the smallest A -invariant set containing $f^{-1}(a)$. Consequently, any open covering of $f^{-1}(a)$ in the A -logic topology is also a covering of $f^{-1}[\text{tp}(a/A)]$, so $f^{-1}(a)$ is compact too.

Point (3) is given by points (1) and (2). Point (4) is given by (1) and Proposition 1.4.11. Point (5) is given by (2) and Proposition 1.4.11. Point (6) is given by points (4) and (5). Q.E.D.

Proposition 1.4.15. *Let P be a piecewise A -hyperdefinable set and Q a locally A -hyperdefinable set whose A -logic topology is its global logic topology. Let $f : P \rightarrow Q$ be a function. Then, f is continuous between the A -logic topologies if and only if f is a piecewise bounded \bigwedge_A -definable function.*

Proof. By the Closed Graph Theorem [Mun15, Exercise 8, Section 26], Proposition 1.4.14 and Proposition 1.4.11. Q.E.D.

By Proposition 1.4.14(1), Cartesian projection maps are continuous between the logic topologies. Therefore, the A -logic topology of a finite product of piecewise A -hyperdefinable sets is at least as fine as the product topology of the A -logic topologies. In the case of local hyperdefinable sets with global logic topologies, we have that they coincide.

Proposition 1.4.16. *Let P and Q be locally hyperdefinable sets with global logic topologies. Then, $P \times Q$ is a locally hyperdefinable set and its product topology is its global logic topology.*

Proof. It suffices to note that, for any two topological spaces X and Y with respective local coverings \mathcal{C} and \mathcal{D} , $\mathcal{C} \times \mathcal{D} := \{P \times Q : P \in \mathcal{C}, Q \in \mathcal{D}\}$ is a local covering of the product topology. Q.E.D.

Similarly, in the case of countably piecewise hyperdefinable sets with global logic topologies, we also conclude that they coincide.

Proposition 1.4.17. *Let P and Q be two countably piecewise hyperdefinable sets with global logic topologies. Then, $P \times Q$ is a countably piecewise hyperdefinable set and its product topology is its global logic topology.*

Proof. Say $P = \varinjlim P_n$ and $Q = \varinjlim Q_n$. Let $\Gamma \subseteq P \times Q$ be closed in the global logic topology and $(a, b) \notin \Gamma$ arbitrary. We recursively define a sequence $(U_n, V_n)_{n \in \mathbb{N}}$ such that $a \in U_0, b \in V_0, U_n \subseteq U_{n+1}, V_n \subseteq V_{n+1}, U_n \subseteq P_n$ is open in $P_n, V_n \subseteq Q_n$ is open in Q_n and $(\bar{U}_n \times \bar{V}_n) \cap \Gamma = \emptyset$.

For any $n \in \mathbb{N}$, note that the product topology and the global logic topology in $P_n \times Q_n$ coincide. As (a, b) and $\Gamma \cap (P_0 \times Q_0)$ are disjoint and closed, by normality in $P_0 \times Q_0$, we can find $U_0 \subseteq P_0$ open in P_0 and $V_0 \subseteq Q_0$ open in Q_0 such that $(a, b) \in U_0 \times V_0$ and $(\bar{U}_0 \times \bar{V}_0) \cap \Gamma = \emptyset$. Now, suppose U_n and V_n are defined. By normality of $P_{n+1} \times Q_{n+1}$, for any $(x, y) \in \bar{U}_n \times \bar{V}_n$, we can find $U_{n+1}^{xy} \subseteq P_{n+1}$ open in P_{n+1} and $V_{n+1}^{xy} \subseteq Q_{n+1}$ open in Q_{n+1} such that $x \in U_{n+1}^{xy}, y \in V_{n+1}^{xy}$ and $(\bar{U}_{n+1}^{xy} \times \bar{V}_{n+1}^{xy}) \cap \Gamma = \emptyset$. As \bar{V}_n is compact, for each $x \in \bar{U}_n$ there is $F_x \subseteq \bar{V}_n$ finite such that $\bar{V}_n \subseteq \bigcup_{y \in F_x} V_{n+1}^{xy}$. Take $V_{n+1}^x = \bigcup_{y \in F_x} V_{n+1}^{xy}$ and $U_{n+1}^x = \bigcap_{y \in F_x} U_{n+1}^{xy}$. Then, $U_{n+1}^x \subseteq P_{n+1}$ is open in $P_{n+1}, V_{n+1}^x \subseteq Q_{n+1}$ is open in $Q_{n+1}, x \in U_{n+1}^x, \bar{V}_n \subseteq V_{n+1}^x$ and $(\bar{U}_{n+1}^x \times \bar{V}_{n+1}^x) \cap \Gamma = \emptyset$. As \bar{U}_n is compact, there is $F \subseteq \bar{U}_n$ finite such that $\bar{U}_n \subseteq \bigcup_{x \in F} U_{n+1}^x$. Take $U_{n+1} = \bigcup_{x \in F} U_{n+1}^x$ and $V_{n+1} = \bigcap_{x \in F} V_{n+1}^x$. Then, $U_{n+1} \subseteq P_{n+1}$ is open in $P_{n+1}, V_{n+1} \subseteq Q_{n+1}$ is open in $Q_{n+1}, \bar{U}_n \subseteq U_{n+1}, \bar{V}_n \subseteq V_{n+1}$ and $(\bar{U}_{n+1} \times \bar{V}_{n+1}) \cap \Gamma = \emptyset$.

Let $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$. For $n \in \mathbb{N}$, we have $U \cap P_n = \bigcup_{m \geq n} (U_m \cap P_n)$ and $V \cap Q_n = \bigcup_{m \geq n} (V_m \cap Q_n)$. As $P_n \subseteq P_m$ and $Q_n \subseteq Q_m$ for $m > n$, we get that $U_m \cap P_n$ is open in P_n and $V_m \cap Q_n$ is open in Q_n , so $U \cap P_n$ is open in P_n and $V \cap Q_n$ is open in Q_n for each $n \in \mathbb{N}$. Therefore, $U \times V$ is open in the product topology and $(a, b) \in U \times V$ with $(U \times V) \cap \Gamma = \emptyset$. As $(a, b) \notin \Gamma$ is arbitrary, Γ is closed in the product topology. As Γ is arbitrary, we conclude that the global logic topology in $P \times Q$ is the same as the product topology. Q.E.D.

Let $P = \varinjlim P_i$ be piecewise A -hyperdefinable set and $V \subseteq P$ a non-empty piecewise \bigwedge_A -definable subset. Note that the subspace topology of V inherited from the A -logic topology of P is the A -logic topology of V given as the piecewise A -hyperdefinable set $\varinjlim V \cap P_i$. We conclude showing that local hyperdefinability is hereditary.

Proposition 1.4.18. *Let $P = \varinjlim P_i$ be a locally A -hyperdefinable set and $V \subseteq P$ a piecewise \bigwedge_A -definable subset. Then, V with the induced piecewise hyperdefinable substructure $\varinjlim V \cap P_i$ is a locally A -hyperdefinable set.*

Proof. It suffices to note that, for any topological space X with local covering \mathcal{C} and any $Y \subseteq X$, $\mathcal{C}|_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a local covering of the subspace topology. Q.E.D.

1.5 Spaces of types

Let $P = X/E$ be an A -hyperdefinable set and $F \subseteq P \times P$ an \bigwedge_A -definable equivalence relation in P . Then, P/F is actually identified with the A -hyperdefinable set $X/\mathfrak{g}_{P \times P}^{-1}[F]$ via the canonical bijection

$$\eta : [x]_{\mathfrak{g}_{P \times P}^{-1}[F]} \mapsto [[x]_P]_F.$$

Furthermore, by the definition of the quotient topologies, this map is a homeomorphism. Indeed, it is continuous if and only if the function $x \mapsto [[x]_P]_F$ is continuous, which is so by definition. Conversely, the inverse function $[[x]_P]_F \rightarrow [x]_{\mathfrak{g}_{P \times P}^{-1}[F]}$ is continuous if and only if $x \mapsto [x]_{\mathfrak{g}_{P \times P}^{-1}[F]}$ is continuous, which is so by definition.

Let $P = \varinjlim_{(I, \prec)} (P_i, \varphi_{ji})_{j \succeq i}$ be a piecewise A -hyperdefinable set. Write $P_i = X_i/E_i$. Let $F \subseteq P \times P$ be a piecewise \bigwedge_A -definable equivalence relation. Write $F_i := F|_{P_i \times P_i}$. Define $\tilde{\varphi}_{ji} : P_i/F_i \rightarrow P_j/F_j$ by $\tilde{\varphi}_{ji}([x]_{F_i}) = [\varphi_{ji}(x)]_{F_j}$. Clearly, they are well-defined \bigwedge_A -definable and 1-to-1 functions and $\tilde{\varphi}_{ii} = \text{id}$ and $\tilde{\varphi}_{ki} = \tilde{\varphi}_{kj} \circ \tilde{\varphi}_{ji}$. Then, we have canonically the piecewise A -hyperdefinable structure in P/F given by

$$P/F := \varinjlim_{(I, \prec)} (P_i/F_i, \tilde{\varphi}_{ji})_{j \succeq i},$$

via the map $\eta : [\iota_{P_i}(x)]_F \mapsto \iota_{P_i/F_i}([x]_{F_i})$ for i such that $x \in P_i$. Furthermore, topologically, by the definitions of the quotient and the direct limit topologies, this bijection is clearly a homeomorphism. Indeed, by the final properties of the quotient and the direct limit, it is continuous if and only if, for each $i \in I$, $x \mapsto \iota_{P_i/F_i}([x]_{F_i})$ is continuous from P_i to $\varinjlim P_i/F_i$, which is clear by definition. On the other hand, the inverse is continuous if and only if, for each $i \in I$, $x \mapsto [\iota_{P_i}(x)]_F$ is continuous from P_i to P/F , which is clear by definition.

Let $P = X/E$ be an A^* -hyperdefinable set. Consider the space of types $\mathbf{S}_X(A^*) = \{\text{tp}(x/A^*) : x \in X\}$ with the usual topology. We define the equivalence relation \sim_E in $\mathbf{S}_X(A^*)$ as $p \sim_E q$ if and only if there are E -equivalent realisations of $p(x)$ and $q(y)$.

The *space of types* of P over A^* is the space $\mathbf{S}_P(A^*) := \mathbf{S}_X(A^*)/\sim_E$ with the quotient topology.

On the other hand, by Lemma 1.1.16, $P/\Delta_P(A^*) = \{\text{tp}(a/A^*) : a \in P\}$ is an A^* -hyperdefinable set and has its A^* -logic topology.

Proposition 1.5.1. *Let $P = X/E$ be an A^* -hyperdefinable set. Then, $P/\Delta_P(A^*)$ and $\mathbf{S}_P(A^*)$ are homeomorphic.*

Proof. Consider the map

$$f : \begin{array}{ccc} \mathbf{S}_P(A^*) & \rightarrow & P/\Delta_P(A^*) \\ [\text{tp}(a^*/A^*)]_{\sim_E} & \mapsto & \text{tp}([a^*]_E/A^*). \end{array}$$

By saturation, it is well-defined. Clearly, it is onto. It is 1-to-1 by Lemma 1.1.5. By the definition of the quotient topology, it is clear that f is continuous. As $\mathbf{S}_X(A^*)$ is a compact topological space, $\mathbf{S}_P(A^*)$ is also compact. On the other hand, $P/\Delta_P(A^*)$ is a Hausdorff space. Then, as the domain is compact, the image is Hausdorff and f is a continuous bijection, we conclude that f is a homeomorphism. Q.E.D.

Therefore, for hyperimaginary parameters, we define the space of types $\mathbf{S}_P(A)$ of P over A as $P/\Delta_P(A)$ with its global logic topology.

Remark 1.5.2. By Lemma 1.1.10 and Lemma 1.1.12, any \bigwedge_A -definable function $f : P \rightarrow Q$ induces a continuous closed map between the spaces of types.

Similarly, let $P = \varinjlim_I P_i$ be a piecewise A^* -hyperdefinable set. For each $i \in I$, we have the space of types $\mathbf{S}_{P_i}(A^*) = \{\text{tp}(a/A^*) : a \in P_i\}$. Given $i, j \in I$ with $i \preceq j$, the map $\varphi_{ji} : P_i \rightarrow P_j$ induces a continuous and closed 1-to-1 function $\text{Im } \varphi_{ji} : \mathbf{S}_{P_i}(A^*) \rightarrow \mathbf{S}_{P_j}(A^*)$ given by $\text{tp}(a/A^*) \mapsto \text{tp}(\varphi_{ji}(a)/A^*)$. Clearly, $\text{Im } \varphi_{kj} \circ \text{Im } \varphi_{ji} = \text{Im } \varphi_{ki}$ and $\text{Im } \varphi_{ii} = \text{id}$. Therefore, we have a topological direct sequence $(\mathbf{S}_{P_i}(A^*), \text{Im } \varphi_{ji})_{j \succeq i}$. The *space of types* of P over A^* is then the direct limit topological space

$$\mathbf{S}_P(A^*) := \varinjlim \mathbf{S}_{P_i}(A^*).$$

By Proposition 1.5.1, we have

$$P/\Delta_P(A^*) = \varinjlim P_i/\Delta_{P_i}(A^*) \cong \varinjlim \mathbf{S}_{P_i}(A^*) = \mathbf{S}_P(A^*).$$

Therefore, for a piecewise hyperdefinable set P , like in the case of hyperdefinable sets, we also define the *space of types* of P over hyperimaginary parameters A , $\mathbf{S}_P(A)$, as $P/\Delta_P(A)$ with its global logic topology. Even more, we say that a piecewise hyperdefinable set S is a *space of types* if it has a global logic topology.

Let X be a topological space. Recall that two points are *topologically indistinguishable* if they have the same neighbourhoods. A *Kolmogorov map* is an onto continuous and closed map $k : X \rightarrow Y$ between topological spaces such that $k(a)$ and $k(b)$ are topologically indistinguishable if and only if a and b are topologically indistinguishable.

Lemma 1.5.3. *Let X and Y be topological spaces and $k : X \rightarrow Y$ a function. Then, the following are equivalent:*

(1) k is a Kolmogorov map.

(1') k is an onto continuous and open map such that $k(a)$ and $k(b)$ are topologically indistinguishable if and only if a and b are topologically indistinguishable.

(2) $\text{Im } k$ is a lattice isomorphism between the topologies with inverse $\text{Im}^{-1}k$.

(2') $\text{Im } k$ is a lattice isomorphism between the closed topologies with inverse $\text{Im}^{-1}k$.

Proof. (1) \Rightarrow (2') As k is continuous and closed, $\text{Im } k$ and $\text{Im}^{-1}k$ are lattice homomorphisms between the closed topologies. As k is onto, we have $k[k^{-1}[V]] = V$ for any $V \subseteq Y$. Finally, note that $k^{-1}[k[V]] = V$ for any closed subset $V \subseteq X$. Indeed, $V \subseteq k^{-1}[k[V]]$ trivially and, for any $g \notin V$, g and h are topologically distinguishable for any $h \in V$. Therefore, $k(g)$ and $k(h)$ are topologically distinguishable for any $h \in V$, concluding that $k(g) \notin k[V]$, so $g \notin k^{-1}[k[V]]$.

(2') \Rightarrow (1') By (2'), we have that k is continuous and onto. Take $U \subseteq X$ open. As $X \setminus U$ is closed, we conclude that $k^{-1}[k[X \setminus U]] = X \setminus U$. Therefore, $k[U] = Y \setminus k[X \setminus U]$. As $X \setminus U$ is closed, $k[X \setminus U]$ is closed, so $k[U]$ is open. Therefore, k is also an open map. Take $g, h \in X$. Suppose $k(g)$ and $k(h)$ are topologically indistinguishable. Take U open such that $g \in U$. As $k[U]$ is open neighbourhood of $k(g)$, then $h \in k^{-1}[k[U]] = U$. Thus, g and h are topologically indistinguishable. Suppose $k(g)$ and $k(h)$ are topologically distinguishable. Without loss of generality, say there is U open neighbourhood of $k(g)$ that does not contain $k(h)$. Then, $g \in k^{-1}[U]$ and $h \notin k^{-1}[U]$. Therefore, g and h are topologically distinguishable.

(1') \Rightarrow (2) Dual to (1) \Rightarrow (2').

(2) \Rightarrow (1) Dual to (2') \Rightarrow (1').

Q.E.D.

Remark 1.5.4. When Y is T_0 , we call a Kolmogorov map $k : X \rightarrow Y$ the *Kolmogorov quotient* of X . In that case, up to a homeomorphism, Y is the quotient space X/\sim , where \sim is the topologically indistinguishable equivalence relation, and k is the respective quotient map.

It is useful to note that the quotient map $\text{tp}_A : P \rightarrow \mathbf{S}_P(A)$ given by $\text{tp}_A : a \mapsto \text{tp}(a/A) = a/\Delta_P(A)$ is the Kolmogorov quotient between the A -logic topologies. By definition, it also satisfies that $\text{tp}_A^{-1}[\text{tp}_A[V]] = V$ for any A -invariant set V . Thus, when studying P , it is typically possible to map the discussion via tp_A , argue in $\mathbf{S}_P(A)$ and then lift the conclusions to P via tp_A^{-1} . Following this procedure, one can usually

assume without loss of generality that P is T_0 . This technique will be illustrated in the following section — see Theorem 1.6.4 and the Metrisation Theorem 1.6.7.

Remark 1.5.5. By Proposition 1.4.14(1) and Lemma 1.3.5, any piecewise bounded \bigwedge_A -definable function induces a continuous map between the spaces of A -types.

1.6 Metrisation results ¹

A *uniform space* is a pair (X, Φ) formed by a non-empty set X and a filter Φ of binary relations in X satisfying that, for every $U \in \Phi$,

- (i) $\Delta \subseteq U$,
- (ii) $U^{-1} \in \Phi$ and
- (iii) there is $V \in \Phi$ such that $V \circ V \subseteq U$;

where $\Delta := \{(x, x) : x \in X\}$ is the diagonal (equality) relation, $U^{-1} := \{(y, x) : (x, y) \in U\}$ and $W \circ V := \{(x, z) : \exists y (x, y) \in V, (y, z) \in W\}$. The filter Φ is called the *uniform structure* of the uniform space (X, Φ) .

A *uniformity base* of (X, Φ) is a filter base of Φ . Note that a filter base \mathcal{B} of reflexive binary relations on X is a uniformity base of some uniform structure if and only if, for any $U \in \mathcal{B}$, there are $V, W \in \mathcal{B}$ such that $V \circ V \subseteq U$ and $W \subseteq U^{-1}$.

Uniform spaces generalise (pseudo-)metric spaces. In other words, every pseudo-metric ρ induces a uniform space given by the uniformity base $\{\rho^{-1}[0, \varepsilon) : \varepsilon \in \mathbb{R}_{>0}\}$. We say then that a uniform structure Φ is *pseudo-metrisable* if it arises in this way; that means there is a pseudo-metric ρ such that $\{\rho^{-1}[0, \varepsilon) : \varepsilon \in \mathbb{R}_{>0}\}$ is a uniformity base of Φ .

As in the case of metric spaces, every uniform space has a topology given by the system of local bases of neighbourhoods $\{U(a) : U \in \Phi\}_{a \in X}$, where $U(a) := \{b : (a, b) \in U\}$. We say that a topology \mathcal{T} *admits a uniform structure* if there is a uniform structure Φ on X with uniform topology \mathcal{T} . Note that a topological space could admit many different uniform structures. Uniform structures inducing the same topology are called *equivalent*.

Remark 1.6.1. Uniform structures are the natural abstract context to study uniform continuity, uniform convergence, Cauchy sequences and completeness. It is important to note that two equivalent uniform structures may differ on these aspects. For example, a uniformly continuous function with respect to one uniform structure might not be uniformly continuous in another equivalent uniform structure.

¹This section is inspired on results from [Ben05].

Recall that a topological space X is *functionally regular* if, for any point $x \in X$ and any closed set $V \subseteq X$ such that $x \notin V$, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f|_V = 1$. It is a well-known fact from the theory of uniform spaces that a topological space admits a uniform structure if and only if it is functionally regular — see [Wil70, Theorem 38.2] for a proof. Hence, we get the following general result for countably piecewise A -hyperdefinable sets.

Proposition 1.6.2. *Every countably piecewise A -hyperdefinable set P with its A -logic topology admits a uniform structure.*

Proof. We know that P is normal by Proposition 1.4.7. Let V be piecewise \bigwedge_A -definable and $a \in P \setminus V$. As $\overline{\{a\}} = \text{tp}(a/A)$, we have that $\overline{\{a\}}$ and V are disjoint. By normality, using Urysohn’s Lemma [Mun15, Theorem 33.1], we conclude that P is functionally regular. Thus, we conclude that it admits a uniform structure. Q.E.D.

Our aim now is to give a better description of the uniform structure in each piece. In other words, we want to give an actual uniformity base.

Let P be an A -hyperdefinable set. We say that an \bigwedge_A -definable binary relation $\varepsilon \subseteq P \times P$ is *positive* if $\Delta_P(A) \subseteq \overset{\circ}{\varepsilon}$, where the interior is taken in the product topology of the A -logic topology. Write $\mathcal{E}_P(A)$ for the set of all positive \bigwedge_A -definable binary relations of P . It could seem odd the fact that we take the interior in the product topology. The following lemma explains why.

Lemma 1.6.3. *Let P be an A -hyperdefinable set and $\tau : P \times P \rightarrow \mathbf{S}_P(A) \times \mathbf{S}_P(A)$ the quotient map $(a, b) \mapsto (\text{tp}(a/A), \text{tp}(b/A))$. Then, for any $\varepsilon \in \mathcal{E}_P(A)$, $\tau[\varepsilon] \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$, and for any $\varepsilon' \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$, $\tau^{-1}[\varepsilon'] \in \mathcal{E}_P(A)$.*

Proof. First, $\tau^{-1}[\varepsilon] \in \mathcal{E}_P(A)$ for $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$. Indeed, it is clear that $\tau^{-1}[\varepsilon]$ is \bigwedge_A -definable, as τ is \bigwedge_A -definable. Now, as the product topology and the A -logic topology of $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$ are the same, we have that, for any $a \in P$, there is $U \subseteq \mathbf{S}_P(A)$ open such that $(\text{tp}_P(a/A), \text{tp}_P(a/A)) \in U \times U \subseteq \varepsilon$. Therefore, $\text{tp}(a/A) \times \text{tp}(a/A) \subseteq \text{tp}_A^{-1}[U] \times \text{tp}_A^{-1}[U] \subseteq \tau^{-1}[\varepsilon]$, so $\Delta_P(A)$ lies in the interior of $\tau^{-1}[\varepsilon]$ in the product topology of $P \times P$, concluding that it is positive.

On the other hand, given $\varepsilon \in \mathcal{E}_P(A)$, $\tau[\varepsilon]$ is \bigwedge_A -definable, as τ is \bigwedge_A -definable. Now, for $(a, b) \in \Delta_P(A)$ there are U_a and U_b open in P with the A -logic topology such that $(a, b) \in U_a \times U_b \subseteq \varepsilon$. Then, $(\text{tp}_P(a/A), \text{tp}_P(a/A)) \in \text{tp}_A[U_a] \times \text{tp}_A[U_b] \subseteq \tau[\varepsilon]$. That concludes that $\tau[\varepsilon] \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$. Q.E.D.

When the A -logic topology is the global topology, $\Delta_P(A)$ is precisely the diagonal Δ and $\mathcal{E}_P(A)$ is the set of closed neighbourhoods of Δ in the global logic topology of $P \times P$, which coincides with the product topology by Proposition 1.2.7. This in particular applies to $\mathbf{S}_P(A)$.

We now prove the main result of this section:

Theorem 1.6.4. *Let P be an A -hyperdefinable set. Then, P with the A -logic topology admits a unique uniform structure and $\mathcal{E}_P(A)$ is a uniformity base of it.*

Proof. We start by proving the theorem for $\mathbf{S}_P(A)$ rather than P , so consider the set $\mathcal{E}_{\mathbf{S}_P(A)}(A)$ of positive \bigwedge_A -definable binary relations on $\mathbf{S}_P(A)$. By Proposition 1.6.2, $\mathbf{S}_P(A)$ with the global logic topology admits a uniform structure and, as it is compact and Hausdorff, it only admits one [Wil70, Theorem 36.19]. Therefore, in the case of $\mathbf{S}_P(A)$, the only thing that we need to check is that $\mathcal{E}_{\mathbf{S}_P(A)}(A)$ is a base of this uniform structure.

First of all, we show that $\mathcal{E}_{\mathbf{S}_P(A)}(A)$ is a uniformity base of some uniform structure on $\mathbf{S}_P(A)$. It suffices to show that for any $\varepsilon' \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$ there is $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$ such that $\varepsilon \circ \varepsilon \subseteq \varepsilon'$. We start by noting that, by normality of $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$ and because the product topology and the global logic topology coincide (by Proposition 1.2.7), we have $\bigcap_{\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)} \varepsilon = \Delta$, where Δ is the diagonal (equality) relation. Indeed, by normality of $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$ in the A -logic topology, for any $p, q \in \mathbf{S}_P(A)$ with $p \neq q$, there is an \bigwedge_A -definable binary relation ε on $\mathbf{S}_P(A)$ such that $\Delta \subseteq U \subseteq \varepsilon$ and $(p, q) \notin \varepsilon$, where U is open in the A -logic topology of $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$. By Proposition 1.2.7, the A -logic topology and the product topology on $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$ are the same, so $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$. As $p, q \in \mathbf{S}_P(A)$ are arbitrary, we conclude that $\bigcap_{\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)} \varepsilon = \Delta$.

Now, by compactness on $\mathbf{S}_P(A)$, we get $\bigcap_{\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)} \varepsilon \circ \varepsilon = \Delta$. Indeed, if $(p, q) \in \bigcap_{\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)} \varepsilon \circ \varepsilon$, then $\varepsilon(p) \cap \varepsilon^{-1}(q) \neq \emptyset$ for every $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$. As $\mathcal{E}_{\mathbf{S}_P(A)}(A)$ is closed under finite intersections, we conclude that there is $r \in \mathbf{S}_P(A)$ such that $r \in \varepsilon(p) \cap \varepsilon^{-1}(q)$ for any $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$. Thus, as $\bigcap_{\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)} \varepsilon = \Delta$, we conclude that $p = r = q$. As p and q are arbitrary, we get $\bigcap_{\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)} \varepsilon \circ \varepsilon = \Delta$. Finally, for any $\varepsilon' \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$, we have that $\bigcap_{\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)} \varepsilon \circ \varepsilon = \Delta \subseteq U \subseteq \varepsilon'$ with U open in the product topology of $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$. Therefore, by compactness and Proposition 1.2.7, there is $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$ such that $\varepsilon \circ \varepsilon \subseteq \varepsilon'$.

Now, we need to check that the uniform topology given by $\mathcal{E}_{\mathbf{S}_P(A)}(A)$ is the A -logic topology in $\mathbf{S}_P(A)$. Take $V \subseteq \mathbf{S}_P(A)$ closed for the uniform topology given by $\mathcal{E}_{\mathbf{S}_P(A)}(A)$. For each $a \notin V$, there are $\varepsilon_a \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$ and $U_a \subseteq \mathbf{S}_P(A) \times \mathbf{S}_P(A)$ open in the product topology of $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$ such that $\Delta \subseteq U_a \subseteq \varepsilon_a$ and $\varepsilon_a(a) \cap V = \emptyset$. Then, $V = \bigcap_{a \notin V} (\mathbf{S}_P(A) \setminus U_a(a))$. By Proposition 1.2.7, $\mathbf{S}_P(A) \setminus U_a(a)$ is \bigwedge_A -definable. As each $a \in \mathbf{S}_P(A)$ is definable over A , we get that $\mathbf{S}_P(A) \setminus U_a(a)$ is in fact \bigwedge_A -definable, so V is \bigwedge_A -definable. On the other hand, take an \bigwedge_A -definable subset $V \subseteq \mathbf{S}_P(A)$ and $a \notin V$. Therefore, $R = \{a\} \times V$ is an \bigwedge_A -definable binary relation such that $R \cap \Delta = \emptyset$. As the A -logic topology is normal and R and Δ are disjoint closed sets, we find some \bigwedge_A -definable binary relation ε on $\mathbf{S}_P(A)$ disjoint to R such that $\Delta \subseteq U \subseteq \varepsilon$ with U open in the A -logic topology of $\mathbf{S}_P(A) \times \mathbf{S}_P(A)$. In particular, $\varepsilon(a) \cap V = \emptyset$. By Proposition 1.2.7, $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$. Since $a \notin V$ is arbitrary, it concludes that V is closed in the uniform topology given by $\mathcal{E}_{\mathbf{S}_P(A)}(A)$.

Therefore, the uniform structure defined by the base $\mathcal{E}_{\mathbf{S}_P(A)}(A)$ is the unique uniform structure admitted by the global logic topology on $\mathbf{S}_P(A)$, concluding the proof in the case of $\mathbf{S}_P(A)$.

We now use the quotient maps $\text{tp}_A : P \rightarrow \mathbf{S}_P(A)$ and $\tau : P \times P \rightarrow \mathbf{S}_P(A) \times \mathbf{S}_P(A)$ given by $\text{tp}_A(a) = \text{tp}(a/A)$ and $\tau(a, b) = (\text{tp}(a/A), \text{tp}(b/A))$ to extend the result to P .

Let Φ be any uniform structure on P inducing the A -logic topology. Note that at least one Φ exists by Proposition 1.6.2. Consider $\tau\Phi := \{\tau[U] : U \in \Phi\}$. Obviously, $\tau\Phi$ is a filter. Also, $\tau[U^{-1}] = \tau[U]^{-1}$ and $\tau[U] \circ \tau[U] \subseteq \tau[U \circ U \circ U]$ for any $U \in \Phi$, so $\tau\Phi$ is a uniform structure on $\mathbf{S}_P(A)$. On the other hand, for any $a \in P$, $\text{tp}_A[U(a)] \subseteq \tau[U](\text{tp}(a/A)) \subseteq \text{tp}_A[U \circ U(a)]$. Thus, by the properties of tp_A , $\tau\Phi$ induces the global logic topology on $\mathbf{S}_P(A)$. As $\mathbf{S}_P(A)$ with the global logic topology only admits one uniform structure, it follows that $\mathcal{E}_{\mathbf{S}_P(A)}(A)$ is a uniformity base of $\tau\Phi$. In particular, we have $\tau^{-1}\mathcal{E}_{\mathbf{S}_P(A)}(A) \subseteq \Phi$, where $\tau^{-1}\mathcal{E}_{\mathbf{S}_P(A)}(A) := \{\tau^{-1}[\varepsilon] : \varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)\}$. On the other hand, for $U \in \Phi$, take $V \in \Phi$ such that $V \circ V \circ V \subseteq U$ and find $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$ such that $\varepsilon \subseteq \tau[V]$. Therefore, $\tau^{-1}[\varepsilon] \subseteq \tau^{-1}[\tau[V]] = \Delta_P(A) \circ V \circ \Delta_P(A) \subseteq V \circ V \circ V \subseteq U$. Hence, $\tau^{-1}\mathcal{E}_{\mathbf{S}_P(A)}(A)$ is a uniformity base of Φ , concluding uniqueness.

We now show that $\mathcal{E}_P(A)$ is a uniformity base of Φ . Since $\tau^{-1}[\varepsilon] \in \mathcal{E}_P(A)$ for any $\varepsilon \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$, it is enough to show that for any $\varepsilon \in \mathcal{E}_P(A)$ there is $\varepsilon' \in \mathcal{E}_{\mathbf{S}_P(A)}(A)$ such that $\tau^{-1}[\varepsilon'] \subseteq \varepsilon$. Take $\varepsilon'' \in \mathcal{E}_P(A)$ such that $\varepsilon'' \circ \varepsilon'' \circ \varepsilon'' \subseteq \varepsilon$ and set $\varepsilon' = \tau[\varepsilon'']$. Then, $\tau^{-1}[\varepsilon'] = \Delta_P(A) \circ \varepsilon'' \circ \Delta_P(A) \subseteq \varepsilon$. Q.E.D.

Using compactness, we can get an even smaller uniformity base:

Lemma 1.6.5. *Let P be an A -hyperdefinable set and $\lambda = |A| + |L| + \text{ary}(P)$. There is a family $\{\varepsilon_i\}_{i \in \lambda}$ of \bigwedge_A -definable positive binary relations on P which is a uniformity base of (the uniformity given by) $\mathcal{E}_P(A)$.*

Proof. Write $\Delta_P(x, y, A^*) = \bigwedge_{i \in \lambda} \varphi_i(x, y)$ with $\varphi_i(x, y)$ formula over A^* . Write $E_i := \mathcal{G}_{P \times P}[\varphi_i(\mathfrak{M})]$ and $U_i := P \times P \setminus \mathcal{G}_{P \times P}[\neg \varphi_i(\mathfrak{M})]$. Then, $\Delta_P(A) \subseteq U_i \subseteq E_i$ where E_i is \bigwedge_A -definable and U_i is open in the A -logic topology. Then, by compactness of $P \times P$ in the A -logic topology, there is $\varepsilon_i \in \mathcal{E}_P(A)$ such that $\varepsilon_i \subseteq U_i \subseteq E_i$. It follows that $\{\varepsilon_i\}_{i \in \lambda}$ is a sequence of \bigwedge_A -definable positive binary relations on P such that $\Delta_P(A) = \bigcap_{i \in \lambda} \varepsilon_i$. Indeed, take $\varepsilon \in \mathcal{E}_P(A)$ and $U \subseteq \varepsilon$ open in the product topology of $P \times P$ such that $\bigcap_{i \in \lambda} \varepsilon_i = \Delta_P(A) \subseteq U \subseteq \varepsilon$. Now, U is open in the product topology, so it is also open in the A -logic topology. Then, by compactness, there is $i \in \lambda$ such that $\varepsilon_i \subseteq U \subseteq \varepsilon$, concluding that $\{\varepsilon_i\}_{i \in \lambda}$ is a base of (the uniformity given by) $\mathcal{E}_P(A)$.

Recall that a uniform structure is pseudo-metrisable if and only if it has a countable uniformity base — see [Wil70, Theorem 38.3]. Therefore, we get the following metrisation results:

Corollary 1.6.6. *Let P be an A -hyperdefinable set. Then, P with its A -logic topology is pseudo-metrisable if and only if there is a countable family $\{\varepsilon_n\}_{n \in \mathbb{N}}$ of \bigwedge_A -definable positive binary relations on P such that $\Delta_P(A) = \bigcap_{n \in \mathbb{N}} \varepsilon_n$. In particular, every A -hyperdefinable set of countable arity with the A -logic topology is pseudo-metrisable when L and A are countable.*

Theorem 1.6.7 (Metrisation Theorem). *Let P be a locally A -hyperdefinable set of countable cofinality. Then, P with the A -logic topology is pseudo-metrisable if and only if each piece is pseudo-metrisable. In particular, every locally A -hyperdefinable set of countable cofinality and countable arity with the A -logic topology is pseudo-metrisable when L and A are countable.*

Proof. Assume that each piece is pseudo-metrisable. Taking the quotient map $a \mapsto \text{tp}(a/A)$, we get that $\mathbf{S}_P(A)$ is locally metrisable and σ -compact. By σ -compactness, $\mathbf{S}_P(A)$ is trivially Lindelöf (i.e. every open cover has a countable subcover). By Proposition 1.4.7, $\mathbf{S}_P(A)$ is normal and Hausdorff, so it is in particular regular (i.e. any closed set and any point outside it can be separated by open sets). Therefore, by [Mun15, Theorem 41.5], $\mathbf{S}_P(A)$ is paracompact (i.e. every open cover has a locally finite open refinement). By Smirnov's Metrisation Theorem [Mun15, Theorem 42.1], we conclude that it is metrisable. Taking the composition with the quotient map $a \mapsto \text{tp}(a/A)$, we conclude that P with the A -logic topology is pseudo-metrisable. Q.E.D.

Using uniformities, it is easy to find small dense subsets of the logic topologies:

Corollary 1.6.8. *Let P be a piecewise A -hyperdefinable set. Then, there is a subset $D \subseteq P$ with $|D| \leq |A| + |L| + \text{ary}(P) + \text{cf}(P)$ which is dense in the A -logic topology. In particular, when A and L are countable, every countably piecewise A -hyperdefinable set of countable arity is separable (i.e. has a countable dense subset) with the A -logic topology.*

Proof. Say $P = \varinjlim_{i \in I} P_i$ with $|I| = \text{cf}(P)$. By Lemma 1.6.5, there is $\mathcal{B}_i \subseteq \mathcal{E}_{P_i}(A)$ with $|\mathcal{B}_i| \leq |A| + |L|$ such that $\{\varepsilon(a) : \varepsilon \in \mathcal{B}_i\}$ is a local base of neighbourhoods at a for every $a \in P_i$ in the A -logic topology of P_i . By compactness, for each $\varepsilon \in \mathcal{B}_i$, there is $D_{\varepsilon,i} \subseteq P_i$ finite such that $P_i \subseteq \bigcup_{a \in D_{\varepsilon,i}} \varepsilon(a)$. Take $D = \bigcup_{i \in I} \bigcup_{\varepsilon \in \mathcal{B}_i} D_{\varepsilon,i}$, so $|D| \leq |A| + |L| + \text{ary}(P) + \text{cf}(P)$. Let U be open in P and pick $a \in U$ arbitrary. Let $i \in I$ such that $a \in P_i$, so $P_i \cap U$ is a neighbourhood of a in the A -logic topology of P_i . Then, there is $\varepsilon \in \mathcal{B}_i$ such that $\varepsilon(a) \subseteq P_i \cap U$. Find ε_0 such that $\varepsilon_0^{-1} \subseteq \varepsilon$ and take $d \in D_{\varepsilon_0,i} \subseteq D$ such that $a \in \varepsilon_0(d)$. Then, $d \in \varepsilon_0^{-1}(a) \subseteq \varepsilon(a)$, so $D \cap U \neq \emptyset$. As U is arbitrary, we conclude that D is dense. Q.E.D.

Arguing similarly in the case of global logic topologies, we can reduce in general the number of parameters from $2^{|A|+|L|}$ to $|A| + |L| + \text{ary}(P) + \text{cf}(P)$.

Proposition 1.6.9. *Let P be a piecewise A -hyperdefinable set with a global logic topology. Then, there is $B \subseteq P \cup A$ with $|B| \leq |A| + |L| + \text{ary}(P) + \text{cf}(P)$ such that the B -logic topology of P is its global logic topology.*

Proof. Say $P = \varinjlim_{i \in I} P_i$ and $P_i = X_i/E_i$ with $|I| = \text{cf}(P)$. Write $B_0 = \text{bdd}(A^*)$, so the B_0 -logic topology of P is its global logic topology. Then, for each $i \in I$, $\Delta_i := \Delta_{P_i}(B_0) = \{(x, x) : x \in P_i\}$, so $\underline{\Delta}_i = \underline{E}_i$ is a partial type over A^* . Say $\underline{\Delta}_i = \bigwedge_{j \in |A| + |\mathbb{L}| + \text{ary}(P)} \varphi_j(x, y)$ with $\varphi_j(x, y)$ formula over A^* . Write $V_j := \mathfrak{g}_{P_i \times P_i}[\varphi_j(\mathfrak{M})]$ and $U_j := P_i \times P_i \setminus \mathfrak{g}_{P_i \times P_i}[\neg \varphi_j(\mathfrak{M})]$. Then, $\Delta_i \subseteq U_j \subseteq V_j$ where V_j is \bigwedge_{B_0} -definable and U_j is open in the B_0 -logic topology of $P_i \times P_i$. By compactness of $P_i \times P_i$ in the B_0 -logic topology, there is $\varepsilon_j \in \mathcal{E}_{P_i}(B_0)$ such that $\varepsilon_j \subseteq U_j \subseteq V_j$. It follows that $\mathcal{B}_i := \{\varepsilon_j\}_{j \in |A| + |\mathbb{L}| + \text{ary}(P)}$ is a uniformity base of (the uniform structure given by) $\mathcal{E}_{P_i}(B_0)$. By compactness of P_i , for each $\varepsilon \in \mathcal{B}_i$, there is $D_{\varepsilon, i} \subseteq P_i$ finite such that $P_i \subseteq \bigcup_{a \in D_{\varepsilon, i}} \varepsilon(a)$. Take $D_i = \bigcup_{\varepsilon \in \mathcal{B}_i} D_{\varepsilon, i}$ and $D = \bigcup_{i \in I} D_i$, so $|D| \leq |A| + |\mathbb{L}| + \text{ary}(P) + \text{cf}(P)$. It follows that D_i is dense in the global logic topology of P_i . Take $B = D \cup A$ with $|B| \leq |A| + |\mathbb{L}| + \text{ary}(P) + \text{cf}(P)$. We claim that the B -logic topology of P is its global logic topology. Indeed, take $a \in P$ and $\sigma \in \text{Aut}(\mathfrak{M}/B)$ arbitrary and, aiming a contradiction, suppose $\sigma(a) \neq a$, so $\sigma^{-1}(a) \neq a$. Pick $i \in I$ such that $a \in P_i$. Note that $\sigma[P_i] = P_i$. By Hausdorffness of P_i in the global logic topology, there are U and V such that $a \in U \subseteq V \subseteq P_i$ with $\sigma^{-1}(a) \notin V$, where U is open and V is closed in the global logic topology of P_i . Now, $D_i \cap U = \sigma[D_i \cap U] \subseteq \sigma[V]$. As U is open in P_i and D_i is dense in P_i , $U \cap D_i$ is dense in U . As $\sigma[V]$ is closed in the global logic topology, we conclude that $a \in U \subseteq \overline{D_i \cap U} \subseteq \sigma[V]$. Therefore, $\sigma^{-1}(a) \in V$, getting a contradiction. Q.E.D.

1.7 Examples

Most of the examples of locally hyperdefinable sets come from the following two basic remarks, which were, in fact, already known. First, note that any piecewise A -definable set is trivially locally A -definable. Secondly, note that if P is locally A -hyperdefinable and E is a piecewise bounded \bigwedge_A -definable equivalence relation on P , then P/E is locally hyperdefinable by Proposition 1.4.9.

Example 1.7.1. A classical example is the field of real numbers \mathbb{R} with its usual topology, which is a locally hyperdefinable set of countable cofinality in the theory of real closed fields. It is explicitly given as $O(1)/o(1)$ with $O(1) = \bigcup_{n \in \mathbb{N}} [-n, n]$ and $o(1) = \bigcap_{n \in \mathbb{N}} [-1/n, 1/n]$. Furthermore, up to isomorphism of piecewise hyperdefinable sets, $O(1)/o(1)$ is the unique representation of \mathbb{R} as a locally hyperdefinable set. Indeed, by Proposition 1.4.14(8), any two locally hyperdefinable sets homeomorphic with the logic topologies are isomorphic as piecewise hyperdefinable sets.

However, note that the real numbers with the usual topology can be represented as a piecewise hyperdefinable set (non-locally hyperdefinable) in other non-isomorphic ways. For instance, consider the direct system of all compact subsets of $O(1)/o(1)$ with empty interior in the global logic topology with the natural inclusion maps. Using that the topology is first-countable, it is easy to note that this is a coherent covering. Therefore, the resulting direct limit with the global logic topology is a piecewise

hyperdefinable set homeomorphic to \mathbb{R} with the usual topology. However, it is not locally hyperdefinable, so it is not isomorphic to $O(1)/o(1)$.

Example 1.7.2. More generally, any topological manifold X (i.e. a locally euclidean Hausdorff topological space) is a locally hyperdefinable set in the usual theory of real closed fields. Indeed, for any m , \mathbb{R}^m is locally hyperdefinable, so every compact subset of \mathbb{R}^m is hyperdefinable, concluding that the compact charts of X are hyperdefinable. Using now Proposition 1.4.15, the chart changing maps are \wedge -definable, so the finite unions of compact chart neighbourhoods are hyperdefinable. It follows then that the whole manifold is locally hyperdefinable. If it is also second countable, then it has countable cofinality too.

Example 1.7.3. As \mathbb{Q} with the usual metric topology is not locally compact, it cannot be given as a locally hyperdefinable set. However, it is possible to give it as a piecewise hyperdefinable set in the theory of real closed fields. Indeed, using that \mathbb{Q} is first countable, it is clear that \mathbb{Q} is compactly generated. Every compact subset of \mathbb{Q} is a hyperdefinable subset being \wedge -definable in $\mathbb{R} = O(1)/o(1)$. In other words, \mathbb{Q} is the direct limit of all the \wedge -definable subsets of $\mathbb{R} = O(1)/o(1)$ contained in \mathbb{Q} with the standard inclusion maps. Note that it has uncountable cofinality.

Example 1.7.4. More generally, any first countable Hausdorff topological space X can be given as a piecewise hyperdefinable set with a global logic topology in the theory of real closed fields. Indeed, let \mathcal{C} be the family of countable compact subsets of X . As X is first countable, \mathcal{C} is coherent. Now, by Mazurkiewicz-Sierpiński Theorem [Mil11, Theorem 4], every countable compact Hausdorff topological space is homeomorphic to a countable successor ordinal with the order topology. On the other hand, by induction, we easily see that every countable ordinal with the order topology is homeomorphic to a subset of \mathbb{Q} . Therefore, for every $A \in \mathcal{C}$, there is a compact subset $P_A \subseteq \mathbb{Q}$ such that A with the subspace topology is homeomorphic to P_A with the subspace topology. For each $A \in \mathcal{C}$, pick $\eta_A : A \rightarrow P_A$ a homeomorphism. For $A, B \in \mathcal{C}$ with $A \subseteq B$, take $\varphi_{BA} = \eta_B \circ \eta_A^{-1} : P_A \rightarrow P_B$. Now, as noted in the previous example, every compact subset of \mathbb{Q} is homeomorphic to a hyperdefinable set with a global logic topology in the theory of real closed fields. Also, for $A, B \in \mathcal{C}$ with $A \subseteq B$, φ_{BA} is continuous, so it is \wedge -definable by Proposition 1.4.15. Then, we conclude that X is homeomorphic to $\varinjlim_{A \in \mathcal{C}} P_A$ with the global logic topology.

Example 1.7.5. For a countably piecewise hyperdefinable set that is not locally hyperdefinable, consider the infinite countable rose, i.e. the infinite countable bouquet of circles. This is \mathbb{R}/\sim with the equivalence relation $x \sim y \Leftrightarrow x, y \in \mathbb{Z} \vee x = y$. Note that \mathbb{R} is countably piecewise hyperdefinable and the relation \sim is piecewise \wedge -definable in the theory of real closed fields. Thus, the infinite countable rose is a countably piecewise hyperdefinable set. It is not locally hyperdefinable and not first countable, so it is not pseudo-metrisable.

Example 1.7.6. For a piecewise hyperdefinable set which is not normal, consider \mathbb{R} with the rational sequence topology. For $i \in \mathbb{R}$, pick a sequence $(i_n)_{n \in \mathbb{N}}$ of rational

numbers converging to i . When $i \in \mathbb{Q}$, take $i_n = i$ for every $n \in \mathbb{N}$. Take $P_i = \{i_n : n \in \mathbb{N}\} \cup \{i\}$ for $i \in \mathbb{R}$. For a finite subset $I \subseteq \mathbb{R}$, take $P_I = \bigcup_{i \in I} P_i$. Note that, for any finite $I \subseteq \mathbb{R}$, P_I is compact in \mathbb{R} with the usual topology, so each P_i is hyperdefinable in the language of real closed fields. Take $P = \varinjlim P_I$ with the usual inclusion maps. We now check that P with the global logic topology is homeomorphic to \mathbb{R} with the rational sequence topology, i.e. the topology given by the local bases of open neighbourhoods $U_n(i) := \{i_k : k \geq n\} \cup \{i\}$ for $i \in \mathbb{R}$.

Note first that $U_n(i)$ is open in P for each $i \in \mathbb{R}$ and $n \in \mathbb{N}$. Obviously, $U_n(i) \cap P_i$ is open in P_i . For $j \neq i$, as $(i_n)_{n \in \mathbb{N}}$ converges to i and $(j_n)_{n \in \mathbb{N}}$ converges to j , we conclude that there is $m \in \mathbb{N}$ such that $U_n(i) \cap P_j \subseteq \{i_n : n \leq m\}$. Thus, $U_n(i) \cap P_j$ is open in P_j .

On the other hand, suppose $U \subseteq P$ is open in P . Take $i \in U$. As $U \cap P_i$ is open in P_i , there is $n \in \mathbb{N}$ such that $U_n(i) \subseteq U \cap P_i$, so $U_n(i) \subseteq U$. As $i \in U$ is arbitrary, we conclude that U is open in the rational sequence topology.

By Jone's Lemma [Wil70, Lemma 15.2], P is not normal. Indeed, \mathbb{Q} is dense, $\mathbb{R} \setminus \mathbb{Q}$ is discrete and closed and $|\mathbb{R} \setminus \mathbb{Q}| \geq 2^{|\mathbb{Q}|}$.

For a counterexample where the product topology of global logic topologies is not the global logic topology on the product, see Example 2.2.6. We have no counterexample of a piecewise hyperdefinable set with a non-Hausdorff global logic topology. We have no counterexample of a countably piecewise A -hyperdefinable set that is not locally A -hyperdefinable but has a locally compact A -logic topology.

Chapter 2

Piecewise hyperdefinable groups

In this chapter we study the particular case of piecewise hyperdefinable groups. Our main aim is to find sufficient and necessary conditions to conclude when they are locally compact topological groups with the logic topology. Then, we will discuss how to extend the classical Gleason-Yamabe Theorem and some related results to piecewise hyperdefinable groups. We start with an introduction recalling some fundamental facts about topological groups.

2.1 Preliminaries on topological groups

Recall that a *topological group* is a Hausdorff topological space with a group structure whose group operations are continuous. The following lemma summarises all the basic facts about topological groups that we need.

Lemma 2.1.1. *Let G be a topological group.*

(1) *Let $H \trianglelefteq G$ be closed. Then, G/H is a topological group and $\pi_{G/H} : G \rightarrow G/H$ is a continuous and open surjective homomorphism. Furthermore, if H is compact, then $\pi_{G/H}$ is also a closed map and has compact fibres. In particular, it is proper by [Eng89, Theorem 3.7.2].*

(2) *Let $H \leq G$ be an open subgroup. Then, H is also closed.*

(3) *The connected component G^0 of the identity is a normal closed subgroup of G . If G is locally connected (e.g. a Lie group), then G^0 is also open.*

(4) **(Closed Isomorphism Theorem)** *Let $f : G \rightarrow H$ be a continuous and closed surjective homomorphism between topological groups. Then, for any closed subgroup $S \trianglelefteq K := \ker(f)$ with $S \trianglelefteq G$, the map $f_S : G/S \rightarrow H$ defined by $f = f_S \circ \pi_{G/S}$ is a continuous, closed and open homomorphism. In particular, $f_K : G/K \rightarrow H$ is an isomorphism of topological groups and f is an open map.*

In the theory of topological groups, a *Yamabe pair* of a topological group G is a pair (K, H) with $K \trianglelefteq H \leq G$ such that K is compact, H is open and $L = H/K$ is a finite dimensional Lie group. We say that H is the *domain*, K is the *kernel* and L is the *Lie core*. We write $\pi_{H/K} : H \rightarrow L$ for the quotient map. A Lie group is a *Lie core* of G if it is isomorphic, as topological group, to the Lie core of some Yamabe pair of G .

Remark 2.1.2. Let G be a topological group and suppose that it has a Yamabe pair (K, H) with Lie core L . By Lemma 2.1.1(1), $\pi_{H/K} : H \rightarrow L$ is a continuous, open, closed and proper surjective group homomorphism. In particular, as L is locally compact, H must be a locally compact topological group too. Since H is open in G , we conclude that G is locally compact as well.

The following celebrated theorem, claiming that every locally compact topological group has Lie cores, is usually considered the solution to Hilbert's fifth problem.

Theorem 2.1.3 (Gleason-Yamabe Theorem). *Let G be a locally compact topological group and $U \subseteq G$ a neighbourhood of the identity. Then, G has a Yamabe pair (K, H) with $K \subseteq U$. In particular, a topological group has a Lie core if and only if it is locally compact.*

Proof. The original papers are [Gle51] and [Yam53]. A complete proof can be found in [Tao14a]. Model-theoretic treatments can be found in [Hir90] and [DG15]. Q.E.D.

In this work we mainly use this classical version of Gleason-Yamabe Theorem 2.1.3. Alternatively, we can use the following variation proved in [Car15, Theorem 1.25] which provides some extra control over some parameters. Recall that two subsets of a group are *k-commensurable* if k many left translates of each one cover the other. Recall that a *k-approximate subgroup* is a symmetric subset X which is k -commensurable with its set of pairwise products X^2 .

Theorem 2.1.4 (Gleason-Yamabe-Carolino Theorem). *There are functions $c : \mathbb{N} \rightarrow \mathbb{N}$ and $d : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds:*

Let G be a locally compact topological group and $U \subseteq G$ an open precompact k -approximate subgroup for some $k \in \mathbb{N}$. Then, G has a Yamabe pair (K, H) with $K \subseteq U^4$ such that H/K is a Lie group of dimension at most $d(k)$ and $H \cap U^4$ generates H and is $c(k)$ -commensurable to U .

A Yamabe pair (K', H') is *smaller than or equal to* (K, H) if $K \trianglelefteq K' \trianglelefteq H' \leq H$. A Yamabe pair is *minimal* if it has no smaller ones. A Lie core is *minimal* if it is the Lie core of some minimal Yamabe pair. In other words, let us define an

aperiodic topological group to be a topological group without non-trivial compact normal subgroups. Then, by Lemma 2.1.1(1), a Lie core is minimal if and only if it is an aperiodic connected Lie core. The following basic proposition implies that every Yamabe pair has a smaller or equal minimal Yamabe pair.

Lemma 2.1.5. *Every connected Lie group has a unique maximal compact normal subgroup.*

Proof. By Cartan-Iwasawa-Malcev Theorem [Hoc65, Chapter XV Theorem 3.1], there is a maximal compact subgroup T , and every compact subgroup is contained in a conjugate of it. Hence, $\bigcap_{g \in G} gTg^{-1}$ is the unique maximal compact normal subgroup. Q.E.D.

Remark 2.1.6. A different proof of the previous result, without using Cartan-Iwasawa-Malcev Theorem, was explained in [Hru11].

Corollary 2.1.7. *Let G be a topological group and (K_1, H_1) a Yamabe pair. Then, there is a minimal Yamabe pair (K, H) smaller than or equal to (K_1, H_1) . Furthermore, for any clopen subset U containing K_1 , we have $H \subseteq U^2$.*

Proof. Write $\pi_1 := \pi_{H_1/K_1} : H_1 \rightarrow L_1$ for the quotient map to the Lie core of (K_1, H_1) . Let $\tilde{L} \subseteq L_1$ be the topological connected component of the identity. As Lie groups are locally connected, \tilde{L} is open by Lemma 2.1.1(3). Let $\tilde{K} \leq \tilde{L}$ be its maximal compact normal subgroup, given by Lemma 2.1.5. Take $H := \pi_1^{-1}[\tilde{L}]$ and $K := \pi_1^{-1}[\tilde{K}]$. Then, by Lemma 2.1.1(1) and the Closed Isomorphism Theorem (Lemma 2.1.1(4)), (K, H) is a minimal Yamabe pair of G smaller than or equal to (K_1, H_1) . Finally, if U is clopen and $K_1 \subseteq U$, $\pi_1[U]$ is clopen with $1 \in \pi_1[U]$ as π_1 is open and closed. Thus, $\tilde{L} \subseteq \pi_1[U]$ as \tilde{L} is connected, concluding that $H \subseteq UK_1 \subseteq U^2$. Q.E.D.

Two Yamabe pairs (K, H) and (K', H') of G with Lie cores $\pi := \pi_{H/K} : H \rightarrow L$ and $\pi' := \pi_{H'/K'} : H' \rightarrow L'$ are *equivalent* if the map $\eta : \pi(h) \mapsto \pi'(h)$ for $h \in H \cap H'$ is a well-defined isomorphism of topological groups between L and L' . Equivalently, by the Closed Isomorphism Theorem (Lemma 2.1.1(4)), (K, H) and (K', H') are equivalent if and only if $H \cap K' \subseteq K$, $H' \cap K \subseteq K'$, $(H \cap H')K = H$ and $(H \cap H')K' = H'$. It follows that minimal Yamabe pairs are unique up to equivalence:

Lemma 2.1.8. *Let G be a locally compact topological group and (K_1, H_1) and (K_2, H_2) two minimal Yamabe pairs with Lie cores $\pi_1 := \pi_{H_1/K_1} : H_1 \rightarrow L_1$ and $\pi_2 := \pi_{H_2/K_2} : H_2 \rightarrow L_2$:*

(1) *Let $H' \leq H_1$ be an open subgroup. Then, $(K_1 \cap H', H')$ is a minimal Yamabe pair of G equivalent to (K_1, H_1) and $[K_1 : K_1 \cap H']$ is finite.*

(2) $K_1 \subseteq K_2 \Leftrightarrow H_1 \subseteq H_2 \Leftrightarrow K_2 \cap H_1 = K_1$. In particular, $K_1 = K_2$ if and only if $H_1 = H_2$.

(3) $(K_1 \cap K_2, H_1 \cap H_2)$ is a minimal Yamabe pair with $K_1 \cap H_2 = K_1 \cap K_2 = K_2 \cap H_1$. In particular, $[K_1 : K_1 \cap K_2]$ and $[K_2 : K_1 \cap K_2]$ are finite.

Proof. (1) By connectedness of L_1 and closedness and openness of π_1 , $\pi_1[H'] = L_1$. Thus, by the Closed Isomorphism Theorem (Lemma 2.1.1(4)), we conclude that $(K_1 \cap H', H')$ is a minimal Yamabe pair of G equivalent to (K_1, H_1) . Finally, as $K_1 \cap H'$ is an open subset of K_1 , by compactness, $[K_1 : K_1 \cap H']$ is finite.

(2) We already have $H_1 \subseteq H_2 \Rightarrow K_2 \cap H_1 = K_1 \Rightarrow K_1 \subseteq K_2$ by (1). On the other hand, by connectedness of L_1 and openness and closedness of π_1 , $\pi_1[H_1 \cap H_2] = L_1$. If $K_1 \subseteq K_2$, then $K_1 \leq H_1 \cap H_2$, so $H_1 = \pi_1^{-1}[\pi_1[H_1 \cap H_2]] = H_1 \cap H_2 \subseteq H_2$.

(3) By (1), $(K_1 \cap H_2, H_1 \cap H_2)$ and $(K_2 \cap H_1, H_2 \cap H_1)$ are minimal Yamabe pairs with $[K_1 : K_1 \cap H_2]$ and $[K_2 : H_1 \cap H_2]$ finite. By point (2), $K_1 \cap K_2 = K_1 \cap H_2 = K_2 \cap H_1$. Q.E.D.

Corollary 2.1.9. *Every locally compact topological group has a unique minimal Yamabe pair up to equivalence.*

The previous uniqueness statement implies that the minimal Lie core L is unique up to isomorphism of topological groups, but it is far stronger than that. Indeed, it also says that there is a *global minimal Lie core map* extending all the minimal Yamabe pairs which is unique up to isomorphisms of L :

Proposition 2.1.10. *Let G be a locally compact topological group and L its minimal Lie core. Let D_L be the union of all the domains of minimal Yamabe pairs of G . Then, there is a map $\pi_L : D_L \rightarrow L$ such that, for any minimal Yamabe pair (K, H) of G , $\pi_{L|H}$ is a continuous, closed, open and proper surjective homomorphism with kernel K . Furthermore, π_L is unique up to isomorphisms of L .*

Proof. Let \mathcal{Y} be the set of all minimal Yamabe pairs of G and fix any minimal Yamabe pair $(K_0, H_0) \in \mathcal{Y}$ and $L = H_0/K_0$. Now, for any Yamabe pair (K, H) , we define $\pi_{L|H} := \eta_{(K,H)} \circ \pi_{H/K} : H \rightarrow L$ where $\eta_{(K,H)} : H/K \rightarrow L$ is the canonical isomorphism given by the equivalence between (K, H) and (K_0, H_0) . Take any $(K, H), (K', H') \in \mathcal{Y}$. By Lemma 2.1.8 and Corollary 2.1.9, $(K \cap K', H \cap H')$ is a minimal Yamabe pair and equivalent to $(K, H), (K', H')$ and (K_0, H_0) . For any $h \in H \cap H'$, as $(K \cap K', H \cap H')$ is equivalent to (K_0, H_0) , there is $h_0 \in H \cap H' \cap H_0$ such that $h^{-1}h_0 \in K \cap K' \cap K_0$. Thus, $\pi_{L|H}(h) = \pi_{L|H}(h_0) = \pi_{H_0/K_0}(h_0) = \pi_{L|H'}(h_0) = \pi_{L|H}(h)$. Take $D_L = \bigcup_{\mathcal{Y}} H$ and define $\pi_L = \bigcup_{\mathcal{Y}} \pi_{L|H}$. By Lemma 2.1.1(1), we get a global map $\pi_L : D_L \rightarrow L$ such that $\pi_{L|H} : H \rightarrow L$ is a continuous, closed, open and proper surjective homomorphism with kernel K for each minimal Yamabe pair (K, H) .

Suppose $\pi'_L : D_L \rightarrow L$ is any other map such that $\pi'_{L|H}$ is a continuous, closed, open and proper surjective group homomorphism with kernel K for any minimal

Yamabe pair (K, H) . Then, by the Closed Isomorphism Theorem (Lemma 2.1.1(4)), we get an isomorphism $\eta : H_0/K_0 \rightarrow L$ such that $\pi'_{L|H_0} = \eta \circ \pi_{L|H_0}$. Now, for any $(K, H) \in \mathcal{Y}$ and $g \in H$, there is $g_0 \in H \cap H_0$ such that $g \in g_0K$. Then, $\pi'_L(g) = \pi'_L(g_0) = \eta \circ \pi_L(g_0) = \eta \circ \pi_L(g)$, concluding that $\pi'_L = \eta \circ \pi_L$. Q.E.D.

Note that $D_L = \text{Dom}(\pi_L)$ is the union of all the domains of minimal Yamabe pairs of G and $\ker(\pi_L) := \pi_L^{-1}(1)$ is the union of all the kernels of minimal Yamabe pairs of G . Consequently, D_L and $\ker(\pi_L)$ are invariant by any automorphism of G as topological group. In particular, both are normal sets (i.e. conjugate invariant sets).

Among all the minimal Yamabe pairs, it could be natural to look for the ones with maximal domain. We have the following criterion.

Proposition 2.1.11. *Let G be a locally compact topological group and (K, H) a minimal Yamabe pair of G . Let K' be a compact subgroup of G with $K \leq K'$ such that H normalises K' (i.e. $hK' = K'h$ for any $h \in H$). Then, $K = H \cap K'$, $[K' : K]$ is finite and (K', H') is a minimal Yamabe pair of G with $H' = K'H$. Furthermore, H' is a finite union of cosets of H .*

In particular, (K, H) is a minimal Yamabe pair with maximal domain if and only if there is no compact subgroup $K' \leq G$ normalised by H with $K < K'$.

Proof. Clearly, $K \leq K' \cap H \trianglelefteq H$ is compact. Then, as K is the maximal compact normal subgroup of H , we conclude $K' \cap H = K$ and $H/K' \cap H = H/K = L$. As $K = K' \cap H$ is open in K' compact, $[K' : K]$ is finite. Take $\Delta \subseteq K'$ finite such that $K' = \Delta K$. Note that $\Delta H = K'H$ is a clopen subgroup and $K' \trianglelefteq K'H$. Write $H' = \Delta H$. Then, $\pi_{H'/K'|_H} : H \rightarrow H'/K'$ is a continuous and closed onto homomorphism. Therefore, by the Closed Isomorphism Theorem (Lemma 2.1.1(4)), H'/K' and $H/K' \cap H = L$ are isomorphic. That concludes that (K', H') is also a minimal Yamabe pair of G . Using also Lemma 2.1.8(1,2), we conclude that this gives a necessary and sufficient condition for the maximality of the domain. Q.E.D.

Similarly, it is natural to look at minimal Yamabe pairs with minimal kernel. In this case, this question is related to the connected component of G .

Recall that the *quasicomponent* of a point in a topological space is the intersection of all its clopen neighbourhoods. By definition, quasicomponents are closed sets containing the connected components. In locally connected spaces, connected components are clopen, so quasicomponents and connected components coincide. Similarly, in every compact Hausdorff space, connected components and quasicomponents coincide [Wil70, Lemma 29.6]. In general, however, they may be different — even for locally compact Hausdorff topological spaces.

In a topological group G , as the inversion, the conjugations and the translations are homeomorphisms, the connected component G^0 and the quasicomponent G^{qs} of the identity are both normal closed subgroups of G . When G is locally compact, we have the following well-known result.

Lemma 2.1.12. *Let G be a locally compact topological group. Then, $G^0 = G^{\text{qs}}$ is the intersection of all the open subgroups of G .*

Proof. As the quotient map $\pi_{G/G^0} : G \rightarrow G/G^0$ is open and continuous by Lemma 2.1.1(1), G/G^0 is locally compact. On the other hand, note that G/G^0 is also totally disconnected. Indeed, let $C \subseteq G$ with $CG^0 = C$ such that C/G^0 is a connected subset of G/G^0 . Take U and V open disjoint subsets of C such that $U \cup V = C$. For any $a \in C$, $aG^0 \subseteq U \cup V$ and aG^0 is connected. Thus, $aG^0 \subseteq U$ for any $a \in U$ and $bG^0 \subseteq V$ for any $b \in V$. Hence, U/G^0 and V/G^0 are open disjoint subsets of C/G^0 such that $C/G^0 = U/G^0 \cup V/G^0$. As C/G^0 is connected, either $U/G^0 = \emptyset$ or $V/G^0 = \emptyset$. Hence, $U = \emptyset$ or $V = \emptyset$. As U and V are arbitrary, C is connected, concluding that $C \subseteq cG^0$ for any $c \in C$, i.e. C/G^0 is trivial.

Hence, G/G^0 is a locally compact totally disconnected space. Thus, every point has a local base of compact open neighbourhoods [Wil70, Theorem 29.7], getting that the quasicomponents of G/G^0 are trivial. Now, if $U \subseteq G$ is a clopen neighbourhood of the identity, $UG^0 \subseteq U$ by connectedness of G^0 , so U/G^0 is a clopen neighbourhood of the identity with $\pi_{G/G^0}^{-1}[U/G^0] = U$. Thus, we conclude that $G^{\text{qs}} = G^0$.

Finally, note that every compact open neighbourhood of the identity in G/G^0 contains an open subgroup. Indeed, suppose F is a compact open neighbourhood of the identity. As it is open, for any $x \in F$, there is V_x open neighbourhood of the identity such that $xV_x \subseteq F$. Take U_x open and symmetric such that $U_xU_x \subseteq V_x$. Now, $\bigcup_{x \in F} xU_x$ is an open cover of F , so there is a finite subset $F_0 \subseteq F$ such that $F = \bigcup_{x \in F_0} xU_x$. Take $U = \bigcap_{x \in F_0} U_x$. As $F \subseteq \bigcup_{x \in F_0} xU_x$, we have $FU \subseteq \bigcup_{x \in F_0} xU_xU \subseteq \bigcup_{x \in F_0} xV_x \subseteq F$. As $U \subseteq F$, we inductively get that $U^n \subseteq F$ for every $n \in \mathbb{N}$. Therefore, the subgroup generated by U is an open subgroup contained in F .

In particular, the intersection of all the open subgroups of G/G^0 is trivial. As π_{G/G^0} is a continuous homomorphism, we conclude that $G^0 = G^{\text{qs}}$ is the intersection of all the open subgroups of G . Q.E.D.

Hence, we conclude the following criterion for the existence of a minimal Yamabe pair with minimal kernel.

Proposition 2.1.13. *Let G be a locally compact topological group. Then, there is a minimal Yamabe pair with minimal kernel if and only if G^0 is open (i.e. G is locally connected). Furthermore, in that case, for any other minimal Yamabe pair (K, H) , $(K \cap G^0, G^0)$ is the minimal Yamabe pair of G with minimal kernel.*

Proof. Suppose that (K, H) is a minimal Yamabe pair with minimal kernel. As H is clopen by Lemma 2.1.1(2), we have that $G^0 \subseteq H$. On the other hand, for any

other open subgroup $H' \leq G$, by Lemma 2.1.8(1), we have that $(K \cap H', H \cap H')$ is a minimal Yamabe pair. As (K, H) is the one with minimal kernel, it follows that $H \cap H' = H$, so $H \subseteq H'$. As G^0 is the intersection of all the open subgroups, we conclude that $G^0 = H$. Conversely, suppose that G^0 is open. Then, by Lemma 2.1.8(1), for any minimal Yamabe pair (K, H) , we have that $(K \cap G^0, G^0)$ is a minimal Yamabe pair. Thus, for any minimal Yamabe pairs (K, H) and (K', H') , we have that $(K \cap G^0, G^0)$ and $(K' \cap G^0, G^0)$ are minimal Yamabe pairs, and so $K \cap G^0 = K' \cap G^0$ by Proposition 2.1.8(2). Therefore, $(K \cap G^0, G^0)$ is the minimal Yamabe pair with minimal kernel. Q.E.D.

In general, even if G^0 is not open, a similar conclusion is “asymptotically” true:

Proposition 2.1.14. *Let G be a locally compact topological group and L its minimal Lie core. Then, the restriction to G^0 of the global minimal Lie core map $\pi_{L|G^0} : G^0 \rightarrow L$ is a continuous, open, closed and proper surjective group homomorphism.*

Proof. Take (K, H) minimal Yamabe pair and $\pi_{L|H} : H \rightarrow L$. By Proposition 2.1.10, $\pi_{L|H}$ is a continuous, open, closed and proper surjective group homomorphism. By definition, $G^0 \leq H$. Thus, consider the restriction $\pi_{L|G^0} : G^0 \rightarrow L$. As G^0 is a closed subgroup, $\pi_{L|G^0}$ is also a continuous, closed and proper group homomorphism. It remains to show that it is onto and open. Let $b \in L$. We want to show that $\pi_{L|H}^{-1}(b) \cap G^0 \neq \emptyset$. Let $H' \leq G$ be a clopen subgroup such that $H' \leq H$. Then, $\pi_{L|H}[H']$ is clopen in L . As L is connected, we get that $\pi_{L|H}^{-1}(b) \cap H' \neq \emptyset$. Since $\pi_{L|H}^{-1}(b)$ is compact and H' is arbitrary, we conclude that $\pi_{L|H}^{-1}(b) \cap G^0 \neq \emptyset$. Therefore, $\pi_{L|G^0} : G^0 \rightarrow L$ is onto. We conclude that it is also open by the Closed Isomorphism Theorem (Lemma 2.1.1(4)). Q.E.D.

2.2 Local compactness and generic pieces

A *piecewise A -hyperdefinable group* is a group whose universe is piecewise A -hyperdefinable and whose operations are piecewise bounded \bigwedge_A -definable.

Example 2.2.1. Let G be a definable group and $X \subseteq G$ a symmetric definable subset. Then, the subgroup $H \leq G$ generated by X is a countably piecewise definable group. If $K \trianglelefteq H$ is a piecewise \bigwedge -definable normal subgroup, the quotient $H/K = \varinjlim X^n/K$ is a countably piecewise hyperdefinable group too. If $K \subseteq X^n$ for some n , then H/K is also locally hyperdefinable. This corresponds to the case studied in [Hru11].

Remark 2.2.2. Piecewise \bigwedge -definable subgroups of piecewise hyperdefinable groups are piecewise hyperdefinable groups. The quotient of a piecewise hyperdefinable group by a normal piecewise \bigwedge -definable subgroup is a piecewise hyperdefinable group.

Note that the group operations are continuous between the logic topologies by Proposition 1.4.14(1). However, the product topology and the logic topology may

differ, so piecewise hyperdefinable groups with the logic topologies do not need to be topological groups.

Proposition 2.2.3. *Let G be a piecewise hyperdefinable group with a global logic topology. Then, every translation is a homeomorphism in its global logic topology.*

Proof. Trivial by Proposition 1.4.14(1) and Proposition 1.2.4. Q.E.D.

Remark 2.2.4. Groups with a Fréchet topology such that every translation is continuous are called *semitopological groups* — see [Hus18] for an introduction to semitopological groups.

Theorem 2.2.5. *Let G be a countably piecewise hyperdefinable group with a global logic topology. Then, G is a topological group with the global logic topology.*

Proof. Clear by Proposition 1.4.14(1) and Proposition 1.4.17. Q.E.D.

Example 2.2.6. We show now an example of a piecewise hyperdefinable group with a global logic topology that is not a topological group. We simply adapt the fundamental example given in [TSH98, Example 1.2] to the case of piecewise hyperdefinable groups.

First, recall that \mathbb{Q}^n with the usual topology is piecewise hyperdefinable with the global logic topology in the theory of real closed fields by Example 1.7.4. Now, the inclusion $\psi_{n,m} : \mathbb{Q}^m \rightarrow \mathbb{Q}^n$ given by $\psi(x) = (x, 0, \dots, 0)$ for $n > m$ is a piecewise bounded \wedge -definable 1-to-1 map. Also, the set of pairwise sums of two compact countable subsets of \mathbb{Q}^n is a compact countable subset of \mathbb{Q}^n , so $+$ is a piecewise bounded \wedge -definable map. Then, $\bigoplus_{\mathbb{N}} \mathbb{Q} = \varinjlim \mathbb{Q}^n$ is a piecewise hyperdefinable group. Now, $\bigoplus_{\mathbb{N}} \mathbb{Q}$ is not a topological group. Consider the set $U = \{x : |x_j| < |\cos(jx_0)| \text{ for } j \in \mathbb{N}_{>0}\}$. As $x_0 \in \mathbb{Q}$ for any $x \in \bigoplus_{\mathbb{N}} \mathbb{Q}$, we have $\cos(jx_0) \neq 0$, so U is an open neighbourhood of 0. However, there is no open neighbourhood V of 0 such that $V + V \subseteq U$, concluding that $\bigoplus_{\mathbb{N}} \mathbb{Q}$ is not a topological group. Aiming a contradiction, suppose otherwise; take V an open neighbourhood of 0 such that $V + V \subseteq U$. As V is an open neighbourhood of 0, there is $\varepsilon_0 \in \mathbb{R}_{>0}$ such that $\{x : |x_0| < \varepsilon_0 \text{ and } x_i = 0 \text{ for } i \in \mathbb{N}_{>0}\} \subseteq V$. Take $n \in \mathbb{N}_{>0}$ such that $2n\varepsilon_0 > \pi$. There is then $\varepsilon_1 \in \mathbb{R}_{>0}$ such that $\{x : |x_n| < \varepsilon_1 \text{ and } x_i = 0 \text{ for } i \neq n\} \subseteq V$. Hence, $\{x : |x_0| < \varepsilon_0, |x_n| < \varepsilon_1 \text{ and } x_i = 0 \text{ for } i \in \mathbb{N} \setminus \{0, n\}\} \subseteq V + V \subseteq U$. In particular, $(-\varepsilon_0, \varepsilon_0)_{\mathbb{Q}} \times (-\varepsilon_1, \varepsilon_1)_{\mathbb{Q}} \subseteq \{(x_0, x_1) \in \mathbb{Q} \times \mathbb{Q} : |x_1| < |\cos(nx_0)|\}$. However, this is impossible when $2n\varepsilon_0 > \pi$, getting a contradiction.

Theorem 2.2.7. *Let G be a locally hyperdefinable group with a global logic topology. Then, G is a locally compact topological group with this topology.*

Furthermore, a countably piecewise hyperdefinable group G is a locally compact topological group with some logic topology if and only if this logic topology is the global logic topology and G is locally hyperdefinable.

Proof. We know that the global logic topology of G is locally compact by Proposition 1.4.10 and Hausdorff by Proposition 1.4.12. By Proposition 1.4.16 and Proposition 1.4.14(1), we conclude that G is a locally compact topological group.

By definition, a logic topology is T_1 if and only if it is the global logic topology. On the other hand, assuming $G = \varinjlim_{n \in \mathbb{N}} G_n$, by Baire's Category Theorem [Mun15, Theorem 48.2], if G is locally compact Hausdorff, there are h and $n \in \mathbb{N}$ such that G_n is a neighbourhood of h . Thus, for any $g \in G$, $gh^{-1}G_n$ is an \wedge -definable neighbourhood of g . Q.E.D.

Example 2.2.8. We give an example of a countably piecewise hyperdefinable group with a global logic topology which is not locally hyperdefinable; this is the infinite countable direct sum of circles with the inductive topology. Denote the unit circle by $\mathbb{S} := \{x \in \mathbb{C} : |x| = 1\}$, which is hyperdefinable in the theory of real closed fields as quotient of the common definable circle by the infinitesimals. Then, for $n > m$, we take the map $\psi_{n,m} : \mathbb{S}^m \rightarrow \mathbb{S}^n$ by $\psi_{n,m}(x) = (x, 1, \dots, 1)$. Then, $\bigoplus_{\mathbb{N}} \mathbb{S} := \varinjlim \mathbb{S}^n$ is a countably piecewise hyperdefinable group with a global logic topology that is not locally hyperdefinable.

A local base of open neighbourhoods of the identity in the global logic topology of $\bigoplus_{\mathbb{N}} \mathbb{S}$ is the family of subsets $U_\varepsilon := \{x : d_{\mathbb{S}}(x_i, 1) < \varepsilon_i \text{ for } i \in \mathbb{N}\}$ for sequences $\varepsilon = (\varepsilon_i)_{i \in \mathbb{N}}$ with $\varepsilon_i \in (0, 1]$, where $d_{\mathbb{S}}$ is the normalised usual distance in the unit circle. Indeed, suppose U is an open neighbourhood of 1 in $\bigoplus_{\mathbb{N}} \mathbb{S}$. By using compactness in \mathbb{S}^n for each $n \in \mathbb{N}$, we recursively find a sequence $\varepsilon = (\varepsilon_i)_{i \in \mathbb{N}}$ with $\varepsilon_i \in (0, 1]$ such that $\{x : d_{\mathbb{S}}(x_i, 1) \leq \varepsilon_i \text{ for } i \leq n\} \subseteq U \cap \mathbb{S}^n$.

Unfortunately, proving that a particular piecewise hyperdefinable set is locally hyperdefinable may be truly hard, as it requires to check a property about a topological space that we understand only vaguely. Until now, the only method available to show that a piecewise hyperdefinable set is locally hyperdefinable relies on Proposition 1.4.9 and the fact that piecewise definable sets are trivially locally hyperdefinable (i.e. piecewise definable and locally definable are the same). Sometimes that is not enough. To solve this problem we introduce generic sets.

Let G be a piecewise hyperdefinable group. A *generic* subset is an \wedge -definable subset V such that, for any other \wedge -definable subset W , $[W : V] := \min\{|\Delta| : W \subseteq \Delta V\}$ is finite, i.e. there is a finite $\Delta \subseteq G$ with $W \subseteq \Delta V$. Obviously, if V is a generic set and $V \subseteq W$ for \wedge -definable W , then W is also a generic set. In other words, if there are generic sets, the generic sets form an upper set of the family of \wedge -definable subsets. Hence, if there is a generic set, then there is in particular a generic piece (i.e. a piece which is a generic set).

The following theorem is a generalisation of an unpublished example due to Hrushovski.

Theorem 2.2.9 (Generic Set Lemma). *Let G be a piecewise hyperdefinable group and V a symmetric generic set. Then, for each $n \in \mathbb{N}$, V^{n+2} is a neighbourhood of V^n in some logic topology. In particular, if G has a generic piece, then G is locally hyperdefinable. Furthermore, when G is small, G is locally hyperdefinable if and only if it has a generic piece.*

Proof. Using that V is generic, find a well-ordered sequence $(a_\xi)_{\xi \in \alpha}$ in G such that, for every \wedge -definable subset $W \subseteq G$, there is $\Delta_W \subseteq \alpha$ finite with $W \subseteq \bigcup_{\xi \in \Delta_W} a_\xi V$. Let A be a set of parameters containing $\{a_\xi\}_{\xi \in \alpha}$ and such that $G = \varinjlim G_i$ is a piecewise A -hyperdefinable group and V is \wedge_A -definable. From now on, we work on the A -logic topology. We want to show that V^{n+2} is a neighbourhood of V^n . Let $\Delta \subseteq \alpha$ be finite and minimal such that $V^{n+4} \subseteq V^{n+2} \cup \bigcup_{\xi \in \Delta} a_\xi V$. Let $U = V^{n+4} \setminus \bigcup_{\xi \in \Delta} a_\xi V$. Obviously, $U \subseteq V^{n+2}$ and U is open in V^{n+4} . Note also that $V^n \subseteq U$; otherwise, taking $a \in V^n \setminus U$, there is $\xi \in \Delta$ such that $a \in a_\xi V$, so $a_\xi \in V^{n+1}$ and $a_\xi V \subseteq V^{n+2}$, contradicting minimality of Δ . Similarly, for each piece G_i such that $V^{n+4} \subseteq G_i$, pick a finite and minimal subset $\Delta_i \subseteq \alpha$ such that $G_i \subseteq V^{n+2} \cup \bigcup_{\xi \in \Delta_i} a_\xi V$ and $\Delta \subseteq \Delta_i$. Define $U_i = G_i \setminus \bigcup_{\xi \in \Delta_i} a_\xi V$. Again, it is clear that $U_i \subseteq V^{n+2} \subseteq V^{n+4}$ and U_i is open in G_i . Also, by minimality of Δ_i , it follows that $V^n \subseteq U_i$.

We claim that $U_i = U$ for any $i \in I$ such that $V^{n+4} \subseteq G_i$. It is clear by definition that $U_i \subseteq U$. On the other hand, take $a \in V^{n+2} \setminus U_i$. As $a \notin U_i$ and $a \in V^{n+2} \subseteq V^{n+4} \subseteq G_i$, there is $\xi \in \Delta_i$ such that $a \in a_\xi V$. As $a \in V^{n+2}$, $a_\xi \in a \cdot V \subseteq V^{n+3}$, concluding $a_\xi V \subseteq V^{n+4}$. Then, by minimality of Δ_i , it follows that $\xi \in \Delta$, so $a \notin U$. That shows that U is open in G , so V^{n+2} is a neighbourhood of V^n .

In particular, V^3 is a neighbourhood of V . As $G = \bigcup_{\xi \in \alpha} a_\xi V$, for every $a \in G$ there is $\xi \in \alpha$ such that $a \in a_\xi V \subseteq a_\xi U \subseteq a_\xi V^3$ with $a_\xi U$ open in the A -logic topology. Thus, $a_\xi V^3$ is an \wedge_A -definable neighbourhood of a in the A -logic topology, concluding that G is locally A -hyperdefinable.

On the other hand, if G is locally hyperdefinable and small, it is a locally compact topological group by Theorem 2.2.7. Therefore, the identity is in the interior of some piece of G . Every \wedge -definable subset W of G is compact, and so covered by finitely many translates of this piece. As W is arbitrary, this piece is generic. Q.E.D.

Remark 2.2.10. The previous theorem can be adapted to general topological groups. See Theorem A.1 on the appendix for more details.

If G has a generic subset, so has G/K for $K \trianglelefteq G$ piecewise \wedge -definable. Therefore, by the Generic Set Lemma (Theorem 2.2.9), when G has a generic subset, G/K is a locally hyperdefinable group for any piecewise \wedge -definable normal subgroup $K \trianglelefteq G$.

Let G be a group and $T \subseteq G$ a subset containing the identity. For a subset $X \subseteq G$, the T -thickening of X is the subset XT . Two subsets $X, Y \subseteq G$ are T -rough k -commensurable if $X \subseteq \Delta_1 Y T$ and $Y \subseteq \Delta_2 X T$ with $|\Delta_1|, |\Delta_2| \leq k \in \mathbb{N}$ and $1 \in T \subseteq G$. A T -rough k -approximate subgroup of a group G is a symmetric subset

$X \subseteq G$ which is T -rough k -commensurable to its set of pairwise products X^2 . In particular, two subsets are k -commensurable if they are 1-rough k -commensurable, and a k -approximate subgroup is a 1-rough k -approximate subgroup.

It is clear from the definitions that every symmetric generic set is in particular an approximate subgroup. Conversely, any \wedge -definable approximate subgroup is a generic set of the piecewise hyperdefinable group that it generates. Thus, we can understand generic sets as a strengthening of \wedge -definable approximate subgroups.

Corollary 2.2.11. *Let $G = \varinjlim X^n$ be a piecewise hyperdefinable group generated by an \wedge -definable symmetric set X and $T \trianglelefteq G$ be a normal piecewise \wedge -definable subgroup of small index. Then, G/T is a locally compact topological group if and only if X^n is a T -rough approximate subgroup for some n . In particular, if T is \wedge -definable, G/T is a locally compact topological group if and only if X^n is an approximate subgroup for some n .*

An *isomorphism* of piecewise A -hyperdefinable groups is an isomorphism of groups which is also an isomorphism of piecewise A -hyperdefinable sets.

Theorem 2.2.12 (Isomorphism Theorem). *Let $f : G \rightarrow H$ be an onto piecewise bounded and proper \wedge_A -definable homomorphism of piecewise A -hyperdefinable groups. Then, for each \wedge_A -definable subgroup $S \leq K := \ker(f)$ with $S \trianglelefteq G$, there is a unique map $\tilde{f}_S : G/S \rightarrow H$ such that $f = \tilde{f}_S \circ \pi_{G/S}$. This map \tilde{f}_S is a piecewise bounded and proper \wedge_A -definable homomorphism of groups with kernel K/S . In particular, $\tilde{f}_K : G/K \rightarrow H$ is a piecewise \wedge_A -definable isomorphism. Furthermore, if G has a global logic topology, then f is an open map between the global logic topologies.*

Proof. Existence and uniqueness are given by the usual Isomorphism Theorem [Hun96, Theorem 5.6]. Say $G = \varinjlim G_i$ and $H = \varinjlim H_j$. We get that \tilde{f}_S is obviously piecewise \wedge_A -definable as, for any pieces G_i and H_j , $\mathfrak{g}_{(G_i/S) \times H_j}^{-1}[\tilde{f}_S] = \mathfrak{g}_{G_i \times H_j}^{-1}[f]$. It is trivially piecewise bounded and proper as f and $\pi_{G/S}$ are so. Assuming that G has a global logic topology, one-side translations are continuous. Since, by Proposition 1.4.14(3), \tilde{f}_K is a piecewise \wedge_A -definable homeomorphism, it is in particular an open map. Since $\pi_{G/K}^{-1}[\pi_{G/K}[U]] = KU = \bigcup_{x \in K} xU$, we see that $\pi_{G/K}$ is open. Therefore, $f = \tilde{f}_K \circ \pi_{G/K}$ is open too. Q.E.D.

2.3 Model-theoretic components

We define now some model-theoretic components for piecewise hyperdefinable groups. Let G be a piecewise A -hyperdefinable group.

The *invariant component* of G over A is

$$G_A^{000} := \bigcap \{H \leq G : H \text{ is } A\text{-invariant with } [G : H] \text{ small}\}.$$

The *infinitesimal component* of G over A is

$$G_A^{00} := \bigcap \{H \leq G : H \text{ is p/w. } \bigwedge\text{-def. with } G_A^{000} \leq H\};$$

The *connected component* of G over A is

$$G_A^0 := \bigcap \{H \leq G : H \text{ and } G \setminus H \text{ are p/w. } \bigwedge\text{-def. with } G_A^{000} \leq H\}.$$

Obviously, $G_A^{000} \leq G_A^{00} \leq G_A^0 \leq G$.

Lemma 2.3.1. *Let G be a piecewise A -hyperdefinable group and $T \leq G$ be an A -invariant subgroup of small index. Then, there is a unique maximal normal subgroup $\tilde{T} \trianglelefteq G$ contained in T . Furthermore, \tilde{T} is A -invariant and has small index. Moreover, \tilde{T} is piecewise \bigwedge -definable when T is so.*

Proof. Take $\tilde{T} = \bigcap_{i \in [G:T]} T^{g_i}$ with $\{g_i\}_{i \in [G:T]}$ set of representatives. Q.E.D.

Let B be a set of parameters with $A \subseteq B$. The B -logic topology in G/G_A^{000} is the one such that $V \subseteq G/G_A^{000}$ is closed if and only if $\pi_{G/G_A^{000}}^{-1}[V]$ is piecewise \bigwedge_B -definable. The global logic topology in G/G_A^{000} is the one such that $V \subseteq G/G_A^{000}$ is closed if and only if $\pi_{G/G_A^{000}}^{-1}[V]$ is piecewise \bigwedge -definable.

Theorem 2.3.2. *Let G be a piecewise A -hyperdefinable group. Then:*

(1) G_A^{000} is an A -invariant normal subgroup of G and $[G : G_A^{000}]$ is small. In fact, $[V : G_A^{000}] \leq 2^{|A|+|L|+\text{ary}(G)}$ for any \bigwedge -definable subset $V \subseteq G$.

(2) Let B be a small set of parameters with $A \subseteq B$. The inversion map is continuous on G/G_A^{000} with the B -logic topology.

(3) The global logic topology on G/G_A^{000} coincides with the B -logic topology for every small B containing A and a set of representatives of G/G_A^{000} . Every translation map is continuous on G/G_A^{000} with the global logic topology.

(4) G_A^{00} is a piecewise \bigwedge_A -definable normal subgroup of G . Furthermore, G_A^{00}/G_A^{000} is the closure of the identity in the global logic topology of G/G_A^{000} .

(5) Let $\pi : G/G_A^{000} \rightarrow G/G_A^{00}$ be the natural quotient map given by $\pi_{G/G_A^{00}} = \pi \circ \pi_{G/G_A^{000}}$. Then, π is a Kolmogorov map between the B -logic topologies for any B small with $A \subseteq B$. In particular, it is the Kolmogorov quotient between the global logic

topologies. If G/G_A^{00} is a topological group, then the group operations in G/G_A^{000} are continuous.

(6) G_A^0 is a piecewise \wedge_A -definable normal subgroup of G .

(7) G/G_A^{00} is a locally compact topological group whenever G has a generic piece modulo G_A^{00} . In that case, G_A^0/G_A^{00} is the connected component of G/G_A^{00} in the global logic topology and G_A^0/G_A^{000} is the connected component of G/G_A^{000} in the global logic topology.

Proof. (1) Trivially, G_A^{000} is A -invariant and $[G : G_A^{000}]$ is small. Using Lemma 2.3.1, it is obvious that G_A^{000} is a normal subgroup. Finally, recall that, for real elements, an A^* -invariant equivalence relation on an \wedge_{A^*} -definable set of arity α with a small amount of equivalence classes has at most $2^{|A^*|+|\mathbb{L}|+\alpha}$ equivalence classes — indeed, such a relation is coarser than having the same type over an elementary substructure containing A^* (see [TZ12, Exercise 7.4.8]). Then, the same holds for A -invariant equivalence relations on hyperdefinable sets. Consequently, we actually have $[V : G_A^{000}] \leq 2^{|A|+|\mathbb{L}|+\text{ary}(G)}$ for any \wedge -definable V .

(2) Trivial. The B -logic topology on G/G_A^{000} is the quotient topology from the B -logic topology on G , so it is a well-defined topology. As the inversion map is piecewise bounded \wedge_A -definable, it follows that, for any $V \subseteq G$, V^{-1} is piecewise \wedge_B -definable whenever V is piecewise \wedge_B -definable. Since $\pi_{G/G_A^{000}}^{-1}[V^{-1}] = \pi_{G/G_A^{000}}^{-1}[V]^{-1}$, we conclude that the inversion map is continuous on G/G_A^{000} with the B -logic topology.

(3) Take B small containing A and a set of representatives of G/G_A^{000} . We have that, for any $V \subseteq G/G_A^{000}$, $\pi_{G/G_A^{000}}^{-1}[V]$ is B -invariant. Indeed, take $\sigma \in \text{Aut}(\mathfrak{M}/B)$. For any $g \in \pi_{G/G_A^{000}}^{-1}[V]$, we have $gG_A^{000} \subseteq \pi_{G/G_A^{000}}^{-1}[V]$. By choice of B , there is $h \in B$ such that $gG_A^{000} = hG_A^{000}$. Then, $\sigma(g) \in hG_A^{000} \subseteq \pi_{G/G_A^{000}}^{-1}[V]$. As $g \in \pi_{G/G_A^{000}}^{-1}[V]$ and $\sigma \in \text{Aut}(\mathfrak{M}/B)$ are arbitrary, we conclude that $\pi_{G/G_A^{000}}^{-1}[V]$ is B -invariant. In particular, the global logic topology on G/G_A^{000} is the B -logic topology, so it is a well-defined topology. Finally, as $\pi_{G/G_A^{000}}$ is a homomorphism and every translation map is piecewise bounded \wedge -definable, we conclude, by Proposition 1.4.14(1), that every translation map is continuous in G/G_A^{000} with the global logic topology.

(4) Let \widehat{V} be the closure of the identity on the global logic topology of G/G_A^{000} and $V = \pi_{G/G_A^{000}}^{-1}[\widehat{V}]$. Obviously, $V \subseteq G_A^{00}$. On the other hand, as translations are continuous by point (3), it follows that $\widehat{V}^{-1}\widehat{V} \subseteq \widehat{V}$ and $\bar{g}\widehat{V}\bar{g}^{-1} \subseteq \widehat{V}$ for any $\bar{g} \in G/G_A^{000}$. Thus, V is a normal subgroup, concluding $G_A^{00} = V$. In particular, G_A^{00} is a piecewise \wedge_A -definable normal subgroup.

(5) As translations are continuous, for any $\bar{g} \in G/G_A^{000}$, the closure of \bar{g} in the global logic topology is $\bar{g} \cdot G_A^{00}/G_A^{000}$. Thus, π is the Kolmogorov quotient between the global logic topologies. Indeed, $\pi(\bar{g}) = \pi(\bar{h})$ if and only if \bar{g} and \bar{h} are topologically indistinguishable. Now, suppose $\widehat{V} \subseteq G/G_A^{000}$ is closed in the global logic topology and

write $V = \pi_{G/G_A^{000}}^{-1}[\widehat{V}]$. Then, $VG_A^{00} \subseteq V$, concluding $\pi_{G/G_A^{00}}^{-1}[\pi_{G/G_A^{00}}[V]] = V$. Therefore, $\pi_{G/G_A^{00}}[V] = \pi[\widehat{V}]$ is closed in the global logic topology. On the other hand, suppose $\widehat{V} \subseteq G/G_A^{00}$ is closed in the global logic topology and write $V = \pi_{G/G_A^{00}}^{-1}[\widehat{V}]$. We have that V is piecewise \wedge -definable and $VG_A^{00} \subseteq V$. As $G_A^{000} \leq G_A^{00}$, we get $VG_A^{000} \subseteq V$. Thus, $\pi_{G/G_A^{000}}^{-1}[\pi_{G/G_A^{000}}[V]] = V$, concluding that $\pi^{-1}[\widehat{V}] = \pi_{G/G_A^{000}}[V]$ is closed in the global logic topology. By Lemma 1.5.3, $\text{Im } \pi$ is a lattice isomorphism between the global logic topologies with inverse $\text{Im}^{-1}\pi$. Now, as π is A -invariant, it follows that $\text{Im } \pi$ is a lattice isomorphism between the B -logic topologies with inverse $\text{Im}^{-1}\pi$. Therefore, by Lemma 1.5.3, we conclude that π is a Kolmogorov map between the B -logic topologies.

Now, suppose G/G_A^{00} is a topological group. Take $U \subseteq G/G_A^{000}$ open and $\bar{g}_1\bar{g}_2 \in U$. Then, $\pi(\bar{g}_1)\pi(\bar{g}_2) \in \pi[U]$. As $\pi[U]$ is open, there are \widehat{U}_1 and \widehat{U}_2 open in G/G_A^{000} such that $\pi(\bar{g}_1) \in \widehat{U}_1$ and $\pi(\bar{g}_2) \in \widehat{U}_2$ and $\widehat{U}_1\widehat{U}_2 \subseteq \pi[U]$. Write $U_1 = \pi^{-1}[\widehat{U}_1]$ and $U_2 = \pi^{-1}[\widehat{U}_2]$. By Proposition 1.5.3, $\pi^{-1}[\pi[U]] = U$. Then, $U_1U_2 \subseteq U$ with $\bar{g}_1 \in U_1$, $\bar{g}_2 \in U_2$ and U_1 and U_2 open. As U , \bar{g}_1 and \bar{g}_2 are arbitrary, we conclude that the product operation is continuous.

(6) For any $H \leq G$ with $G_A^{000} \leq H$ such that H and $G \setminus H$ are piecewise \wedge -definable, we have that H/G_A^{00} is a clopen subgroup in the global logic topology. Hence, G_A^0/G_A^{00} is the intersection of all the clopen subgroups of G/G_A^{00} in the global logic topology. It follows that G_A^0/G_A^{00} is closed in the global logic topology, so G_A^0 is piecewise \wedge -definable. As it is A -invariant, we conclude that G_A^0 is piecewise \wedge_A -definable. Finally, as every translation is a homeomorphism in G/G_A^{00} , we conclude that G_A^0/G_A^{00} is a normal subgroup of G/G_A^{00} . Thus, $G_A^0 \trianglelefteq G$.

(7) By the Generic Set Lemma (Theorem 2.2.9) and Theorem 2.2.7, G/G_A^{00} with the global logic topology is a locally compact topological group if it has a generic piece. In that case, by Lemma 2.1.12, G_A^0/G_A^{00} is the connected component of G/G_A^{00} . By point (5), G_A^0/G_A^{000} is the connected component of G/G_A^{000} . Q.E.D.

We define now a final special model-theoretic component which has no analogues in the definable or hyperdefinable case. I want to thank Hrushovski for all his help via private conversations in relation with this result. Recall that an aperiodic topological group is a topological group that has no non-trivial compact normal subgroups.

Let G be a piecewise hyperdefinable group with a generic piece. The *aperiodic component* G^{ap} of G is the smallest piecewise \wedge -definable normal subgroup of G with small index such that G/G^{ap} is an aperiodic locally compact topological group with its global logic topology.

Lemma 2.3.3. *Let G be an aperiodic locally compact topological group which is the union of λ compact subsets with $\lambda \geq \aleph_0$. Then, $|G| \leq (\lambda + 2^{\aleph_0})^\lambda$.*

Proof. By Gleason-Yamabe Theorem 2.1.3 and Proposition 2.1.7, there is an open subgroup $H \leq G$ and a compact subgroup $K \trianglelefteq H$ such that H/K is an aperiodic connected Lie group. Thus, $[H : K] \leq 2^{\aleph_0}$. On the other hand, as G is a union of λ compact subsets, $[G : H] \leq \lambda$, so $[G : K] \leq \lambda + 2^{\aleph_0}$. Now, take $\{a_i\}_{i \in \lambda}$ a set of representatives of G/H and write $K_i := K^{a_i} = a_i K a_i^{-1}$ for each i . As $K \trianglelefteq H$, for any $g \in G$, we have $K^g = K_i$ for some $i \in \lambda$. Note that $[G : K_i] \leq \lambda + 2^{\aleph_0}$, so $[G : \bigcap_{i \in \lambda} K_i] \leq (\lambda + 2^{\aleph_0})^\lambda$. As G is aperiodic, $\bigcap_{i \in \lambda} K_i = 1$, so we conclude $|G| \leq (\lambda + 2^{\aleph_0})^\lambda$. Q.E.D.

Theorem 2.3.4. *Let G be a piecewise 0-hyperdefinable group with a generic piece. Then, G^{ap} exists, is piecewise \bigwedge_0 -definable and does not change by expansions of the language.*

Proof. Let $\lambda = \text{cf}(G)$ and $\tau = (\lambda + 2^{\aleph_0})^\lambda$. By assumption $\kappa > \lambda + 2^{\aleph_0}$ is a strong limit cardinal, and so $\kappa > \tau^+$. For any small subset of parameters A , let \mathcal{H}_A be the family of normal subgroups $H \trianglelefteq G$ of small index which are piecewise $\bigwedge_{\leq \tau}$ -definable with parameters from A such that G/H , with its global logic topology, is an aperiodic locally compact topological group.

Claim: For any A small, \mathcal{H}_A is closed under arbitrary intersections. Furthermore, for any $\mathcal{F} \subseteq \mathcal{H}_A$ there is $\mathcal{F}_0 \subseteq \mathcal{F}$ with $|\mathcal{F}_0| \leq \tau$ and $\bigcap \mathcal{F}_0 = \bigcap \mathcal{F}$.

Proof of claim: Take $\mathcal{F} \subseteq \mathcal{H}_A$. Then, $H := \bigcap \mathcal{F}$ is a piecewise \bigwedge_A -definable normal subgroup of G of small index. As G has a generic piece, G/H has a generic piece as well, so G/H is a locally compact topological group with its global logic topology by the Generic Set Lemma (Theorem 2.2.9) and Theorem 2.2.7. Take $K \leq G$ such that $K/H \trianglelefteq G/H$ is a compact normal subgroup in its global logic topology. Take $F \in \mathcal{F}$ arbitrary. We know that $\pi : G/H \rightarrow G/F$ is a continuous homomorphism between the global logic topologies, so K/F is a compact normal subgroup of G/F . As G/F is aperiodic, we conclude that $K \leq F$. As $F \in \mathcal{F}$ is arbitrary, we conclude that $K \subseteq \bigcap \mathcal{F} = H$, so G/H is aperiodic. Since G/H is an aperiodic locally compact topological group, by Lemma 2.3.3, $[G : H] \leq \tau$.

If $|\mathcal{F}| \leq \tau$, then H is piecewise $\bigwedge_{\leq \tau}$ -definable, so $H \in \mathcal{H}$. Now, we claim that there is $\mathcal{F}_0 \subseteq \mathcal{F}$ with $|\mathcal{F}_0| \leq \tau$ such that $\bigcap \mathcal{F}_0 = H$. Indeed, suppose otherwise, then we can find recursively a sequence $(F_i)_{i \in \tau^+}$ of elements in \mathcal{F} such that $H < \bigcap_{i \in \alpha} F_i \subset \bigcap_{i \in \beta} F_i$ for $\beta < \alpha < \tau^+$. Thus, we can find a sequence $(g_i)_{i \in \tau^+}$ such that $g_i \in F_j/H$ for $j \leq i$ and $g_i \notin F_{i+1}/H$; contradicting that $[G : H] \leq \tau$. □

Take a τ^+ -saturated elementary substructure $\mathfrak{N} \preceq \mathfrak{M}$. Set $G^{\text{ap}} = \bigcap \mathcal{H}_N$, so $G^{\text{ap}} \in \mathcal{H}_N$. By τ^+ -saturation of \mathfrak{N} , we have that G^{ap} is 0-invariant, so it is piecewise \bigwedge_0 -definable. As G^{ap} is 0-invariant and \mathfrak{N} is τ^+ -saturated, we conclude that $G^{\text{ap}} = \bigcap \mathcal{H}_A$ for any A . Therefore, G^{ap} is the smallest piecewise $\bigwedge_{\leq \tau}$ -definable normal subgroup of G with small index such that G/G^{ap} is an aperiodic locally compact topological group with its global logic topology.

Now, we show that G^{ap} does not change by expansions of the language. In any κ -saturated and strongly κ -homogeneous \mathcal{L}'/\mathcal{L} -expansion \mathfrak{M}' of an elementary extension of \mathfrak{M} , we find a piecewise \bigwedge_0 -definable normal subgroup $G_{\mathcal{L}'}^{\text{ap}} \trianglelefteq G$ which is

the smallest $\bigwedge_{\leq \tau}$ -definable normal subgroup of G of small index such that G/G^{ap} is an aperiodic locally compact topological group with its global logic topology. Since $[G : G^{\text{ap}}] \leq \tau$ by Lemma 2.3.3, there is a large enough κ -saturated and strongly κ -homogeneous $\mathcal{L}_0/\mathcal{L}$ -expansion \mathfrak{M}_0 of an elementary extension of \mathfrak{M} such that, for any further κ -saturated and strongly κ -homogeneous $\mathcal{L}_1/\mathcal{L}_0$ -expansion \mathfrak{M}_1 of an elementary extension of \mathfrak{M}_0 , we have that $G_0^{\text{ap}}(\mathfrak{M}_1) = G_1^{\text{ap}}$ — where G_1^{ap} is computed in \mathfrak{M}_1 , G_0^{ap} is computed in \mathfrak{M}_0 and $G_0^{\text{ap}}(\mathfrak{M}_1)$ is the realisation of G_0^{ap} in $\mathfrak{M}_1|_{\mathcal{L}_0}$.

Take this base expansion \mathfrak{M}_0 of an elementary extension of \mathfrak{M} , with language \mathcal{L}_0 . By replacing \mathfrak{M}_0 by an elementary extension, we can assume that the \mathcal{L} -reduct $\mathfrak{M}_0|_{\mathcal{L}}$ of \mathfrak{M}_0 is a κ -saturated and strongly κ -homogeneous elementary extension of \mathfrak{M} . Let G_0^{ap} be computed in \mathfrak{M}_0 . Take $\alpha \in \text{Aut}(\mathfrak{M}_0|_{\mathcal{L}})$. Consider the language \mathcal{L}_1 expanding \mathcal{L}_0 by adding a new symbol αx of the same sort that x for each symbol x of \mathcal{L}_0 which is not in \mathcal{L} . Take the \mathcal{L}_1 -expansion \mathfrak{M}'_1 of \mathfrak{M}_0 given by interpreting $(\alpha x)^{\mathfrak{M}'_1} = \alpha(x^{\mathfrak{M}_0})$. Let \mathfrak{M}_1 be a κ -saturated and strongly κ -homogeneous elementary extension of \mathfrak{M}'_1 . Let $G_0^{\text{ap}}(\mathfrak{M}_1)$ be the realisation of G_0^{ap} in $\mathfrak{M}_1|_{\mathcal{L}_0}$, G_1^{ap} be computed in \mathfrak{M}_1 and G_α^{ap} be computed in $\mathfrak{M}_1|_{\mathcal{L}_1 \setminus (\mathcal{L}_0 \setminus \mathcal{L})}$. By the choice of \mathfrak{M}_0 , we know that $G_0^{\text{ap}}(\mathfrak{M}_1) = G_1^{\text{ap}} \subseteq G_0^{\text{ap}} \cap G_\alpha^{\text{ap}}$. Since $G_\alpha^{\text{ap}}(\mathfrak{M}_0)$ is precisely $\alpha(G_0^{\text{ap}})$, we get that $G_0^{\text{ap}} \subseteq \alpha(G_0^{\text{ap}})$. Therefore, as α is arbitrary, we conclude that G_0^{ap} is $\text{Aut}(\mathfrak{M}_0|_{\mathcal{L}})$ -invariant. By Beth's Definability Theorem [TZ12, Exercise 6.1.4], we conclude that G_0^{ap} is already piecewise \bigwedge_0 -definable in \mathcal{L} . Therefore, $G_0^{\text{ap}} = G^{\text{ap}}$ and G^{ap} does not change by expansions of the language.

Finally, by invariance under arbitrary expansions of the language, we have that G^{ap} is the smallest \bigwedge -definable normal subgroup of G of small index such that G/G^{ap} is an aperiodic locally compact topological group with its global logic topology. Q.E.D.

2.4 Lie cores

Now, we adapt the results about Lie cores to piecewise hyperdefinable groups. Let me thank again my supervisor, Ehud Hrushovski, for his suggestions for this section.

Let G be a piecewise A -hyperdefinable group. An A -Yamabe pair of G is a pair (K, H) of subgroups $K \trianglelefteq H \leq G$ which is a Yamabe pair modulo G_A^{00} for the global logic topology of G/G_A^{00} and K/G_A^{00} is \bigwedge -definable. In other words, it is a pair satisfying the following three properties:

- (i) $H \leq G$ is a piecewise \bigwedge -definable subgroup whose complement is also piecewise \bigwedge -definable.
- (ii) $K \trianglelefteq H$ is a piecewise \bigwedge -definable normal subgroup of H such that $G_A^{00} \leq K$ and K/G_A^{00} is \bigwedge -definable.
- (iii) $L := H/K$ with the respective global logic topology is a finite dimensional Lie group.

We say that H is the *domain*, K is the *kernel* and L is the *Lie core*. Write $\pi := \pi_{H/K} : H \rightarrow L$ and $\tilde{\pi} := \tilde{\pi}_{H/K} : H/G_A^{00} \rightarrow L$ for the quotient maps with $\pi = \tilde{\pi} \circ \pi_{G/G_A^{00}}$ where $\pi_{G/G_A^{00}} : G \rightarrow G/G_A^{00}$. We say that a Lie group is an *A-Lie core* of G if it is isomorphic, as Lie group, to the Lie core of some *A-Yamabe pair* of G .

Remarks 2.4.1. Let G be a piecewise *A*-hyperdefinable group. Let (K, H) be an *A-Yamabe pair* of G and $\pi := \pi_{H/K} : H \rightarrow L$ its Lie core and $\tilde{\pi} := \tilde{\pi}_{H/K} : H/G_A^{00} \rightarrow L$ such that $\pi = \tilde{\pi} \circ \pi_{G/G_A^{00}}$.

(1) The fact that $G^{00} \leq K$ means that $[G : H] < \kappa$. Indeed, as H/K is Hausdorff, we already have that $[H : K] < \kappa$. Therefore, if $[G : H] < \kappa$, we conclude that $[G : K] < \kappa$, so $G^{00} \leq K$ (for some set of parameters). Thus, the condition of working modulo G^{00} is saying that H is large in some sense.

On the other hand, the condition that K/G_A^{00} is \wedge -definable may seem superfluous. Indeed, as $(K/G_A^{00}, H/G_A^{00})$ is a Yamabe pair, K/G_A^{00} is compact. If G/G_A^{00} has a generic piece (as we will assume in the rest of the section), this is enough to conclude that K/G_A^{00} is \wedge -definable. However, in general, G/G_A^{00} may have compact sets that are not \wedge -definable, so we need to add this condition.

(2) Note that $\tilde{\pi}$ is a piecewise bounded and proper \wedge -definable map. Thus, by Proposition 1.4.14, $\tilde{\pi}$ is continuous and closed between the global logic topologies. By the Isomorphism Theorem 2.2.12, since $\tilde{\pi}$ is an onto homomorphism, we have that $(H/G_A^{00})/(K/G_A^{00})$ and H/K are isomorphic as piecewise hyperdefinable groups and $\tilde{\pi}$ is also an open map between the global logic topologies. Finally, as it has compact fibres, $\tilde{\pi}$ is also proper [Eng89, Theorem 3.7.2]. In sum, $\tilde{\pi}$ deeply connects the logic topology of G/G_A^{00} and the geometry of L .

When G_A^{00} is \wedge_A -definable, π is also a piecewise bounded and proper \wedge -definable surjective homomorphism. Consequently, π is a continuous, closed and proper map between the logic topologies (with enough parameters), concluding that the relation between G/G_A^{00} and L can be mostly lifted into a relation between G and L . In that special case, we say that $\pi : H \rightarrow L$ is a *Lie model* of G . In general, however, G_A^{00} is only piecewise \wedge_A -definable and π is only piecewise bounded — we could even have $G = G_A^{00}$ (e.g. take $G = \bigcup_{n \in \mathbb{N}} [-a_n, a_n]$ with $a_n \in o(a_{n+1})$ for each $n \in \mathbb{N}$ in the theory of real closed fields). In Chapter 3, we give sufficient conditions to show that G_A^{00} is \wedge_A -definable.

(3) L is a minimal Lie core if and only if it is an aperiodic connected Lie group. In that case, K/G_A^{00} is the maximal compact normal subgroup of H/G_A^{00} , so it is its maximal \wedge -definable normal subgroup. In particular, every automorphism over A leaving H invariant leaves K invariant.

(4) As L is locally compact and $\tilde{\pi}$ is proper and continuous, we conclude that H/G_A^{00} is locally compact. Since H/G_A^{00} is open, we conclude that G/G_A^{00} is locally compact with the global logic topology. Thus, if G is a countably piecewise hyperdefinable group, by Theorem 2.2.7 and the Generic Set Lemma (Theorem 2.2.9), we conclude that G has a generic piece modulo G_A^{00} . For that reason, we will assume from now on that G has a generic piece modulo G_A^{00} .

(5) Note that our definitions of Lie core and Lie model extend the notion of Lie model used in [Hru11]. Suppose that G is a piecewise A -definable group and G_A^{00} is \bigwedge_A -definable. Then, π is proper and continuous so, for any $\Gamma \subseteq U \subseteq L$ with Γ compact and U open, $\pi^{-1}[\Gamma]$ is \bigwedge -definable and $\pi^{-1}[U]$ is \bigvee -definable. Hence, there is a definable subset $D \subseteq G$ such that

$$\pi^{-1}[\Gamma] \subseteq D \subseteq \pi^{-1}[U].$$

As L is first countable, it is metrisable by Birkhoff-Kakutani Theorem [Tao14a, Theorem 1.5.2]. Thus, every closed set in L is G_δ [Mun15, Example 2, page 249]. In particular, it follows that the preimage of any compact set is \bigwedge_ω -definable. When L is connected, it is also second countable, so the preimage of every open set is \bigvee_ω -definable.

Theorem 2.4.2. *Let G be a piecewise A -hyperdefinable group with a generic piece modulo G_A^{00} . Then, G has an A -Lie core. If G is countably piecewise hyperdefinable, this condition is also necessary.*

Furthermore, for any U such that U/G_A^{00} is an open neighbourhood of the identity in the global logic topology of G/G_A^{00} , there is an A -Yamabe pair (K, H) of G with $K \subseteq UG_A^{00}$.

Proof. (\Rightarrow) Clear from Gleason-Yamabe Theorem 2.1.3, using the Generic Set Lemma (Theorem 2.2.9), Theorem 2.2.7 and Proposition 1.4.11. (\Leftarrow) We have also explained the necessity of the generic piece assumption in Remark 2.4.1(4). Q.E.D.

Theorem 2.4.3. *Let G be a piecewise A -hyperdefinable group with a generic piece modulo G_A^{00} and (K_1, H_1) an A -Yamabe pair of G . Then, G has a minimal A -Yamabe pair (K, H) smaller than or equal to (K_1, H_1) . Furthermore, $H \subseteq U^2$ for any U containing K_1 such that U/G_A^{00} is a clopen neighbourhood of K_1/G_A^{00} in the global logic topology.*

Proof. Clear from Corollary 2.1.7, using the Generic Set Lemma (Theorem 2.2.9), Theorem 2.2.7 and Proposition 1.4.11. Q.E.D.

Applying also Corollary 2.1.9, we conclude the following result.

Theorem 2.4.4. *Let G be a piecewise hyperdefinable group with a generic piece modulo G_A^{00} . Then, G has a unique minimal A -Yamabe pair up to equivalence.*

By Proposition 2.1.10, we get a *global minimal Lie core map* $\tilde{\pi}_L : D_L/G_A^{00} \rightarrow L$ extending all the minimal A -Yamabe pairs, which is unique up to isomorphisms of L . Let $\pi_L : D_L \rightarrow L$ be the map given by $\pi_L = \tilde{\pi}_L \circ \pi_{G/G_A^{00}}|_{D_L}$. Here, D_L is the union of all the domains of minimal A -Yamabe pairs and $\ker(\pi_L) := \pi_L^{-1}(1)$ is the union of all the kernels of minimal A -Yamabe pairs. Consequently, D_L and $\ker(\pi_L)$ are A -invariant.

Remark 2.4.5. As noted in [Hru11], the uniqueness of the minimal Lie core is achieved at a price. Indeed, while Gleason-Yamabe Theorem 2.1.3 gives us Yamabe pairs of arbitrarily small kernel, we have lost the control over the kernel in Theorem 2.4.3. If we do not care about uniqueness (as in [BGT12] or [MW15]), it could be better just to apply Theorem 2.4.2 to find a Yamabe pair (K, H) with arbitrarily small kernel. Also, it may be preferable to apply Gleason-Yamabe-Carolino Theorem 2.1.4 rather than Gleason-Yamabe Theorem 2.1.3 to get some extra control on some parameters.

In Proposition 2.1.11, we gave a criterion to find minimal Yamabe pairs with maximal domain in topological groups. Applying it modulo G_A^{00} , this result can be easily adapted to piecewise hyperdefinable groups:

Proposition 2.4.6. *Let G be a piecewise hyperdefinable group and (K, H) a minimal A -Yamabe pair of G . Let K' be an \wedge -definable subgroup of G with $K \leq K'$ and $K' \cap H \trianglelefteq H$. Then, $K = H \cap K'$, $[K' : K]$ is finite and there is H' , finite union of cosets of H , such that (K', H') is a minimal A -Yamabe pair of G . In particular, (K, H) is a minimal A -Yamabe pair with maximal domain if and only if there is no \wedge -definable subgroup $K' \leq G$ with $K < K'$ such that $K' \cap H \trianglelefteq H$.*

Similarly, we can adapt Proposition 2.1.13 by applying it modulo G_A^{00} :

Proposition 2.4.7. *Let G be a locally compact topological group. Then, there is a minimal Yamabe pair with minimal kernel if and only if G^0 is open (i.e. G is locally connected). Furthermore, in that case, for any other minimal Yamabe pair (K, H) , $(K \cap G^0, G^0)$ is the minimal Yamabe pair of G with minimal kernel.*

Adapting Proposition 2.1.14 to the context of piecewise hyperdefinable groups, it follows that the minimal A -Lie core is piecewise A -hyperdefinable:

Proposition 2.4.8. *Let G be a piecewise A -hyperdefinable group with a generic piece modulo G_A^{00} and let L be the minimal A -Lie core of G . Then, the restriction to G_A^0 of the global minimal Lie core map $\pi_{L|G_A^0} : G_A^0 \rightarrow L$ is a piecewise bounded \wedge -definable surjective group homomorphism. Furthermore, we conclude that $L \cong G_A^0 / \ker(\pi_{L|G_A^0})$.*

The previous Proposition 2.4.8 gives us a canonical presentation of the minimal A -Lie core of G as the piecewise A -hyperdefinable group G_A^0/K with $K := \ker(\pi_{L|G_A^0})$. Similarly, we get a canonical A -invariant presentation of the global minimal Lie core map $\pi_L : D_L \rightarrow L$ by taking $\pi_{L|G_A^0} = \pi_{G_A^0/K}$. We now give a more precise description of $K := \ker(\pi_{L|G_A^0})$ using G^{ap} .

Lemma 2.4.9. *Let G be a piecewise 0-hyperdefinable group with a generic piece and G^{ap} the aperiodic component of G . Let (K, H) be a minimal A -Yamabe pair of G with Lie core $\pi_{H/K} : H \rightarrow L$. Then, $G^{\text{ap}} \cap H \leq K$.*

Proof. Let B be the small set of parameters with $A \subseteq B$ such that the B -logic topology of G/G_A^{00} is its global logic topology. Denote by cl_B the closure in the B -logic topology of G . We define by recursion the sequence $(J_\alpha)_{\alpha \in \mathbb{O}_n}$ of piecewise \bigwedge_B -definable normal subgroups of G given by $J_0 = G_A^{00}$, $J_\gamma = \text{cl}_B(\bigcup_{i \in \gamma} J_i)$ for γ limit and $J_{\alpha+1} = \text{cl}_B(\bigcup \mathcal{K}_\alpha)$ where \mathcal{K}_α is the family of piecewise \bigwedge -definable normal subgroups which are \bigwedge -definable modulo J_α .

Claim: For $\alpha_0 \in \mathbb{O}_n$ large enough, $J_\alpha = G^{\text{ap}}$ for any $\alpha \geq \alpha_0$.

Proof of claim: We prove inductively that $J_\alpha \subseteq G^{\text{ap}}$ for any $\alpha \in \mathbb{O}_n$. Obviously, $J_0 \subseteq G^{\text{ap}}$. For γ limit, assuming that $J_i \subseteq G^{\text{ap}}$ for each $i \in \gamma$, as G^{ap} is piecewise \bigwedge_B -definable, we get that $J_\gamma \subseteq G^{\text{ap}}$. Finally, assuming that $J_\alpha \subseteq G^{\text{ap}}$, we have a piecewise bounded \bigwedge -definable onto homomorphism $\pi : G/J_\alpha \rightarrow G/G^{\text{ap}}$. As G/G^{ap} is aperiodic, we have that every piecewise \bigwedge -definable normal subgroup of G which is \bigwedge -definable modulo J_α must be contained in G^{ap} . Therefore, $\bigcup \mathcal{K}_\alpha \subseteq G^{\text{ap}}$, concluding that $J_{\alpha+1} \subseteq G^{\text{ap}}$.

On the other hand, as there is a small amount of piecewise \bigwedge_B -definable normal subgroups of G , for some large enough α_0 , we must have $J_{\alpha_0+1} = J_{\alpha_0}$. In particular, that means that G/J_{α_0} is an aperiodic locally compact topological group with its global logic topology, so $G^{\text{ap}} \leq J_{\alpha_0}$. Thus, $G^{\text{ap}} = J_\beta$ for $\beta \geq \alpha_0$. \square

Now, we prove by induction that, for any $\alpha \in \mathbb{O}_n$, we have $J_\alpha \cap H \leq K$. Obviously, $J_0 \cap H \leq K$. For γ limit, assuming that $J_i \cap H \leq K$ for any $i \in \gamma$, we have $H \cap \bigcup_{i \in \gamma} J_i \leq K$. Therefore, $H \cap J_\gamma = \text{cl}_B(H \cap \bigcup_{i \in \gamma} J_i) \subseteq K$, as H is clopen and K is closed in the B -logic topology. Finally, assuming $J_\alpha \cap H \leq K$, we have a piecewise bounded \bigwedge -definable onto homomorphism $\pi : H/J_\alpha \rightarrow L = H/K$. As L is aperiodic, we have that $H \cap \bigcup \mathcal{K}_\alpha \subseteq K$. Therefore, $H \cap J_{\alpha+1} = \text{cl}_B(H \cap \bigcup \mathcal{K}_\alpha) \subseteq K$, as H is clopen and K is closed in the B -logic topology. In particular, by the claim, we conclude that $G^{\text{ap}} \cap H \leq K$. Q.E.D.

Theorem 2.4.10. *Let G be a piecewise A -hyperdefinable group with a generic piece. Let L be the minimal A -Lie core of G and $\pi_{L|G_A^0} : G_A^0 \rightarrow L$ the restriction to G_A^0 of the global minimal Lie core map of L . Then, $G_A^0 \cap G^{\text{ap}} = \ker(\pi_{L|G_A^0})$. In particular, $L \cong G_A^0/G^{\text{ap}}$.*

Proof. By Lemma 2.4.9, we know that $G^{\text{ap}} \cap G_A^0 \leq \ker(\pi_{L|G_A^0})$. On the other hand, $\ker(\pi_{L|G_A^0}) = G_A^0 \cap \pi_L^{-1}(1) \trianglelefteq G$ is \bigwedge -definable modulo $G_A^{00} \leq G^{\text{ap}}$. Therefore, $\ker(\pi_{L|G_A^0})/G^{\text{ap}}$ is a compact normal subgroup of G/G^{ap} . As it is aperiodic, we conclude that $\ker(\pi_{L|G_A^0}) \leq G^{\text{ap}}$, so $\ker(\pi_{L|G_A^0}) = G^{\text{ap}} \cap G_A^0$. Q.E.D.

In sum, for any piecewise A -hyperdefinable group G with a generic piece, we have

the following structure in terms of the components G_A^0 , G_A^{00} and G^{ap} :

1. $G^{\text{ap}} \cap G_A^0/G_A^{00}$ is a compact topological group.
2. G_A^0/G^{ap} is an aperiodic connected Lie group.
3. G/G_A^0 is a totally disconnected locally compact topological group, i.e. a *locally profinite group*.

Unfortunately, if $G_A^{00} = G^{\text{ap}} = G_A^0 = G$, all the previous results say nothing about G . In the following chapter, we extend the Stabilizer Theorem to the context of piecewise hyperdefinable groups. This theorem gives sufficient conditions to conclude that, with enough parameters, G^{00} is \wedge -definable. As we pointed out at the beginning of the section, in this particular situation, the minimal Lie core gives very precise information about G since the quotient homomorphism is piecewise bounded and proper.

We now note that the minimal Lie core is independent of the parameters. Furthermore, as G^{ap} is independent of expansions of the language, so is the minimal Lie core.

Corollary 2.4.11. *Let G be a piecewise 0-hyperdefinable group with a generic piece. Then, the minimal A -Lie core of G is isomorphic to the minimal 0-Lie core of G . Furthermore, the minimal Lie core of G does not change by expansions of the language.*

Proof. By Lemma 2.4.9, we have that the minimal A -Lie core of G is isomorphic to the minimal Lie core of G/G^{ap} . As the latter does not depend on parameters or expansions of the language, we conclude. Q.E.D.

We have shown that the minimal A -Lie core is A -invariant, using the canonical presentation as G_A^0/K with $K = \ker(\pi_{L|G_A^0})$. In general, (K, G_A^0) is not an A -Yamabe pair, so it may be also interesting to try to find A -invariant minimal A -Yamabe pairs. That was the original strategy followed in [Hru11] to show independence of the parameters. In the following proposition, we adapt the arguments used in [Hru11]. Unfortunately, the hypothesis seems very restrictive; we hope that better results are true.

Proposition 2.4.12. *Let G be a piecewise A -hyperdefinable group with a generic piece modulo G_A^{00} . Suppose that there are an A -hyperdefinable group \widehat{G} and a piecewise \wedge_A -definable 1-to-1 group homomorphism $f : G/G_A^{00} \rightarrow \widehat{G}$ which is open between the A -logic topologies. Then, G has an A -invariant minimal A -Yamabe pair (K, H) with $K \trianglelefteq G$ and $H \subseteq H_0$.*

Proof. Let (K_0, H_0) be a minimal A -Yamabe pair of G , and let c be a small set of parameters such that K_0 and H_0 are c -invariant. Write $\text{Aut}_A := \text{Aut}(\mathfrak{M}/A)$, $K_1 := f[K_0]$ and $H_1 := f[H_0]$. Consider $\widehat{G} \leq \text{Aut}_A \times \widehat{G}$. For any $(\sigma, g) \in \text{Aut}_A \times \widehat{G}$, we get $H_1^{(\sigma, g)} = gH_1^\sigma g^{-1}$. Note that H_1 is open and K_1 is compact in the $A, \sigma(c)$ -logic topology. Since H_1 has small index in $f[G]$, it follows that $[K_1 : K_1 \cap H_1^{(\sigma, g)}] \in \mathbb{N}$ for any $\sigma \in \text{Aut}_A$ and $g \in \widehat{G}$. Now, as $K_1 \cap H_1^{(\sigma, g)}$ is uniformly $\bigwedge_{\sigma(c), g}$ -definable, we conclude by compactness that there is a uniform bound $n \in \mathbb{N}$ such that $[K_1 : K_1 \cap H_1^{(\sigma, g)}] \leq n$ for any $\sigma \in \text{Aut}_A$ and $g \in \widehat{G}$. In particular, for any $\sigma \in \text{Aut}_A$ and $g \in G$, $[K_1 : K_1 \cap H_1^{(\sigma, f(g))}] = [K_0 : K_0 \cap H_0^{(\sigma, g)}] \leq n$. By Lemma 2.1.8, for $g \in G$ and $\sigma \in \text{Aut}_A$, we have $K_0 \cap H_0^{(\sigma, g)} = K_0 \cap K_0^{(\sigma, g)}$. Thus, applying Bergman-Lenstra Lemma [BL89, Theorem 3, Section 2], we find $K' \trianglelefteq \text{Aut}_A \times G$ and a finite set $S \subseteq \text{Aut}_A \times G$ containing the identity with $K_0^S \leq K'$ such that $[K' : K_0^S], [K_0 : K_0^S] \leq m := n^{|S|+1} \in \mathbb{N}$. Write $K := K' \cap G$ and note that $K \trianglelefteq G \trianglelefteq \text{Aut}_A \times G$. By Lemma 2.1.8, (K_0^S, H_0^S) is a minimal A -Yamabe pair of G . Obviously, K is \bigwedge -definable being a finite union of translates of K_0^S . Also, $K_0^S \leq K$ and $K \cap H_0^S \trianglelefteq H_0^S$. Thus, by Lemma 2.1.11, (K, H) is a minimal A -Yamabe pair of G for some $H \leq G$. As $K \trianglelefteq \text{Aut}_A \times G$, we get that K is A -invariant, concluding that (K, H) is A -invariant. Q.E.D.

Remark 2.4.13. If \widehat{G} is a definable group and G is the \bigvee -definable subgroup generated by a definable subset X , then G is an open subgroup of \widehat{G} , being the inclusion an open and continuous piecewise \bigwedge -definable map. Therefore, if G^{00} is \bigwedge -definable, after quotienting by it, the hypotheses in Proposition 2.4.12 correspond to the already known case studied in [Hru11] when we work with definable groups instead of hyperdefinable ones.

The A -Lie rank $\text{Lrank}_A(G)$ of a piecewise hyperdefinable group G with a generic piece modulo G_A^{00} is the dimension of its minimal A -Lie core. As a consequence of Theorem 2.4.4, $\text{Lrank}_A(G)$ is a well-defined invariant. Note that, by Corollary 2.4.11, we have that $\text{Lrank}(G) := \text{Lrank}_A(G)$ does not depend on the parameters when G has a generic piece.

Proposition 2.4.14. *Let G be a piecewise A -hyperdefinable group with a generic piece modulo G_A^{00} and $N \trianglelefteq G$ a piecewise \bigwedge_A -definable normal subgroup of small index. Then,*

$$\text{Lrank}_A(G) \geq \text{Lrank}_A(G/N) + \text{Lrank}_A(N).$$

More precisely, let L and $L_{G/N}$ be the minimal A -Lie cores of G and G/N respectively, presented with the canonical piecewise A -hyperdefinable structures given by Proposition 2.4.8. Let π_L and $\pi_{L_{G/N}}$ be their canonical global minimal Lie core maps. Let (K, H) be a minimal A -Yamabe pair of G . Then:

(1) $(K \cap N, H \cap N)$ is an A -Yamabe pair of N with Lie core $H \cap N / K \cap N \cong L_N := \pi_{L|H}[N]$. The connected component of L_N is aperiodic so, in particular, $\text{Lrank}_A(N) = \dim(L_N)$.

(2) There is an \wedge -definable normal subgroup $T \trianglelefteq H/N$ with $K/N \trianglelefteq T$ such that $(T, H/N)$ is a minimal A -Yamabe pair of G/N .

(3) There is a piecewise bounded \wedge_A -definable surjective group homomorphism $\psi : L \rightarrow L_{G/N}$ such that $\psi \circ \pi_L = \pi_{L_{G/N}} \circ \pi_{G/N}$ (on the domain of π_L).

Proof. (1) Firstly, note that $G_A^{00} \leq N$, so we get $N_A^{00} = G_A^{00}$. We have that $H \cap N / K \cap N \cong H \cap N / K \cong L_N$. Note that $H \cap N / G_A^{00}$ is a closed normal subgroup of H / G_A^{00} . As $\tilde{\pi}_{H/K} : H / G_A^{00} \rightarrow H / K$ is closed between the global logic topologies, we get that $L_N \cong H \cap N / K \cong ((H \cap N) / G_A^{00}) / (K / G_A^{00})$ is closed in $L \cong H / K \cong (H / G_A^{00}) / (K / G_A^{00})$. Therefore, L_N is a closed normal subgroup of L , concluding that L_N is a Lie group. Then, $(K \cap N, H \cap N)$ is an A -Yamabe pair of N with Lie core (isomorphic to) L_N . Let L_N^0 be the connected component of L_N . By Lemma 2.1.5, L_N^0 has a unique maximal compact normal subgroup K_N . Thus, by uniqueness, K_N is characteristic in L_N^0 , which is characteristic in L_N . Therefore, K_N is a compact normal subgroup of L , so it is trivial by aperiodicity of L . In other words, L_N^0 is aperiodic. Thus, L_N^0 is the minimal A -Lie core of N . In particular, $\text{Lrank}_A(N) = \dim(L_N)$.

(2) By (1), we know that $L_N \trianglelefteq L$ is a closed subgroup, so $L_0 := L / L_N$ is a Lie group too. As L is connected, L_0 is connected too. By Lemma 2.1.5, there is a maximal compact normal subgroup $T_0 \trianglelefteq L_0$. Then, L_0 / T_0 is a connected aperiodic Lie group.

Since G / G_A^{00} is a topological group, we know that the quotient homomorphism $\pi_{(G / G_A^{00}) / (N / G_A^{00})} : G / G_A^{00} \rightarrow (G / G_A^{00}) / (N / G_A^{00}) \cong G / N$ is an open map. Therefore, H / N is an open subgroup of G / N . Now, G / N is small and contains a generic set, so G / N is a locally compact topological group by the Generic Set Lemma (Theorem 2.2.9) and Theorem 2.2.7. Thus, H / N is clopen. Consider, $\phi_0 : H / N \rightarrow L_0$ given by $\phi_0 \circ \pi_{(G/N)|H} = \pi_{L/L_N} \circ \pi_{L|H}$. It is clear that ϕ_0 is a piecewise bounded \wedge -definable surjective group homomorphism with kernel K / N , which is \wedge -definable. Therefore, it is also piecewise proper. By the Isomorphism Theorem 2.2.12, it follows that $(K / N, H / N)$ is an A -Yamabe pair of G / N with Lie core (isomorphic to) L_0 .

As L_0 is locally hyperdefinable, T_0 is \wedge -definable. Thus, $\pi_{L_0/T_0} : L_0 \rightarrow L_0 / T_0$ is piecewise bounded and proper \wedge -definable surjective group homomorphism. Take $\phi = \pi_{L_0/T_0} \circ \phi_0 : H / N \rightarrow L_0 / T_0$. Then, ϕ is a piecewise bounded and proper \wedge -definable surjective group homomorphism. By the Isomorphism Theorem 2.2.12, we conclude that $L_0 / T_0 \cong (H / N) / T$ where $T := \ker(\phi)$ is an \wedge -definable normal subgroup of H / N with $K / N \trianglelefteq T$. Consequently, $(T, H / N)$ is a minimal A -Yamabe pair of G / N with Lie core (isomorphic to) $L_0 / T_0 \cong L_{G/N}$.

(3) By the Isomorphism Theorem 2.2.12, take $\eta : L_0 / T_0 \rightarrow L_{G/N}$ isomorphism such that $\pi_{L_{G/N}|(H/N)} = \eta \circ \phi$. Consider $\psi = \eta \circ \pi_{L_0/T_0} \circ \pi_{L/L_N} : L \rightarrow L_{G/N}$ — note that,

a priori, the definition of ψ depends on (K, H) . Obviously, ψ is a piecewise bounded \wedge -definable onto group homomorphism. Also, $\psi \circ \pi_{L|H} = \pi_{L_{G/N}|(H/N)} \circ \pi_{(G/N)|H}$.

Let (K', H') be any other minimal A -Yamabe pair of G . For $h' \in H'$, there is $h \in H \cap H'$ such that $\pi_L(h') = \pi_L(h)$, i.e. $h^{-1}h' \in K'$. By point **(2)**, it follows that $\pi_{L_{G/N}}(\pi_{G/N}(h)) = \pi_{L_{G/N}}(\pi_{G/N}(h'))$. Thus, $\psi(\pi_L(h')) = \psi(\pi_L(h)) = \pi_{L_{G/N}}(\pi_{G/N}(h)) = \pi_{L_{G/N}}(\pi_{G/N}(h'))$. Therefore, $\psi \circ \pi_L = \pi_{L_{G/N}} \circ \pi_{G/N}$ on the domain of π_L .

It remains to show that ψ is A -invariant. Take $\sigma \in \text{Aut}(\mathfrak{M}/A)$ and $x \in L$. Take h such that $\pi_L(h) = x$. Using that $\pi_L, \pi_{L_{G/N}}$ and $\pi_{G/N}$ are A -invariant, we get that $\psi(\sigma(x)) = \psi(\pi_L(\sigma(h))) = \pi_{L_{G/N}}(\pi_{G/N}(\sigma(x))) = \sigma(\pi_{L_{G/N}} \circ \pi_{G/N}(x)) = \sigma(\psi(h))$.

Finally, putting everything together, we conclude that

$$\text{Lrank}_A(G/N) = \dim(L_{G/N}) \leq \dim(L_0) = \text{Lrank}_A(G) - \text{Lrank}_A(N).$$

Q.E.D.

The following proposition is a direct consequence of the results of [JTZ21]. I am very grateful to Chieu-Minh for telling me about it.

Proposition 2.4.15 (An-Jing-Tran-Zhang Bound). *Let G be a piecewise hyperdefinable group and X a symmetric generic \wedge -definable set of G . Then, $\text{Lrank}(G) \leq 12 \log_2(k)^2$ where $k = [X^2 : X]$.*

Proof. By working modulo G^{00} , we may assume that G is small. Let (K, H) be a minimal Yamabe pair of G and $\pi := \pi_{H/K} : H \rightarrow L$ the minimal Lie core. By the Generic Set Lemma (Theorem 2.2.9), we have that X^2 is a symmetric compact neighbourhood of the identity in the global logic topology. Thus, as H is open, $Y = H \cap X^2$ is also a symmetric compact neighbourhood of the identity in the global logic topology. By [Mac21, Lemma 2.3], Y is a k^3 -approximate subgroup. As π is a continuous and open homomorphism, $\pi[Y]$ is a compact neighbourhood of the identity and a k^3 -approximate subgroup in L . As it is a neighbourhood of the identity, it has positive Haar measure, so the general Brunn-Minkowski Inequality [JTZ21, Theorem 1.1] applies and we get that $2 \leq k^{3/\alpha}$ with $\alpha := d - m - h$, where $d = \dim(L)$, $m = \max\{\dim(\Gamma) : \Gamma \leq L \text{ compact}\}$ and h is the helix dimension of L . On the other hand, by [JTZ21, Corollary 2.15], $h \leq n/3$ with $n := d - m$, so $2n \leq 9 \log_2(k)$. Finally, as L has no compact normal subgroups, by [An+21, Fact 3.6, Lemma 3.9], we conclude that $d \leq \frac{n(n+1)}{2}$, so

$$d \leq \frac{1}{8} [9 \log_2(k)]^2 + \frac{1}{4} [9 \log_2(k)] \leq 12 \log_2(k)^2.$$

Q.E.D.

Breillard-Green-Tao Theorem. Let \mathcal{A} be a particular class of piecewise hyperdefinable groups and L a finite dimensional Lie group. It is then natural to study the *direct problem* for \mathcal{A} or (dually) the *inverse problem* for L :

What are the Lie cores of elements of \mathcal{A} ?

Is L a Lie core of an element of \mathcal{A} ?

The classification theorem by Breuillard, Green and Tao for finite approximate subgroups [BGT12] can be interpreted as an answer to the direct problem for the class of piecewise definable subgroups generated by pseudo-finite definable approximate subgroups. Indeed, it can be restated to say that Lie cores of piecewise definable subgroups generated by pseudo-finite definable approximate subgroups are nilpotent [BGT12, Proposition 9.6]. We consider it interesting to discuss the direct and inverse problems in general. In fact, one may expect that solutions of these questions would yield classification results similar to the one of [BGT12].

The following easy examples show that the inverse problem is trivially solved for some basic classes of piecewise hyperdefinable groups. In particular, these examples show that no general classification result for the Lie cores analogous to the one of [BGT12] could be found in those cases.

Examples 2.4.16. Any Lie group L is a Lie core of some piecewise hyperdefinable group in the theory of real closed fields. Indeed, L is in particular a second countable manifold, so it is a locally hyperdefinable subset of countable cofinality. By Proposition 1.4.15, the group operations are piecewise bounded \wedge -definable, so L is a locally hyperdefinable group. Clearly, L is its own Lie core.

For a slightly more explicit construction, note that any connected Lie group L is the Lie core of some piecewise definable group generated by a definable approximate subgroup. Indeed, connected Lie groups are metrisable by Birkhoff-Kakutani Theorem [Tao14a, Theorem 1.5.2], so take a left invariant metric d for L . As L is locally compact, we may assume that the closed unit ball $\overline{\mathbb{D}}$ is a symmetric compact neighbourhood of the identity. Consider the structure of L with the language of groups, a sort for \mathbb{R} with the language of ordered rings and a function symbol for the metric d . Let L' be an $|L|$ -saturated elementary extension of it. Consider the subgroup $H \leq L'$ generated by the closed unit ball and $E = \{(a, b) : d(a, b) < 1/n \text{ for any } n \in \mathbb{N}\}$. Clearly, $\overline{\mathbb{D}}$ is a definable approximate subgroup and H is the piecewise definable group generated by it. Then, L is a Lie core of H , as, in fact, we have $H/E \cong L$.

In the case of a linear connected Lie group L , we can combine both examples. Indeed, in that case, in the real numbers, L is piecewise definable and its metric and group operations are definable, so, after saturation, we just need to take the piecewise definable group generated by the closed unit ball and quotient out by the infinitesimals as in the previous example.

Chapter 3

Stabilizer Theorem

In this chapter, we aim to extend the Stabilizer Theorem [Hru11, Theorem 3.5] to piecewise hyperdefinable groups. The main point of this theorem is that it provides sufficient conditions to conclude that G^{00} is \bigwedge -definable. As we already noted in the previous chapter, this result has significant consequences for the projection π_L of the minimal Lie core.

To prove the Stabilizer Theorem, we need first to extend the model-theoretic notions of dividing, forking and stable relation to piecewise hyperdefinable sets. Once these model-theoretic notions have been properly defined for piecewise hyperdefinable groups, adapting the original proof of the Stabilizer Theorem is straightforward.

Forking and dividing for hyperimaginaries have already been well studied by many authors (e.g. [HKP00], [Wag10] and [Kim13]). Here, in the first section, we rewrite the definitions of dividing and forking for hyperdefinable sets in a slightly different way which we find more natural from the point of view of this thesis. After that, it is trivial to extend the definition to the context of piecewise hyperdefinable sets.

3.1 Dividing and forking

Let $(I, <)$ be a linear order and P an A -hyperdefinable set. An A -indiscernible sequence in P indexed by $(I, <)$ is a sequence $a = (a_i)_{i \in I} \in P^I$ such that, for any \bar{i} and \bar{j} with the same quantifier free type in $(I, <)$, $\text{tp}(a_{\bar{i}}/A) = \text{tp}(a_{\bar{j}}/A)$.

For a sequence $a = (a_i)_{i \in I} \in P^I$, the *Ehrenfeucht-Mostowski type* $\text{EMtp}_I(a/A)$ of a over A (with respect to $(I, <)$) is the set of all sequence $b = (b_i)_{i \in I} \in P^I$ such that $b_{\bar{i}} \in W$ for any \bar{i} in I and \bigwedge_A -definable subset $W \subseteq P^{\bar{i}}$ such that $a_{\bar{j}} \in W$ for any \bar{j} with the same quantifier free type as \bar{i} in $(I, <)$.

Remark 3.1.1. The Ehrenfeucht-Mostowski type $\text{EMtp}(a/A)$ is an intersection of \bigwedge_A -definable subsets of P^I , so it is \bigwedge_A -definable. Note that, by definition, a is A -indiscernible if and only if $\text{tp}(a/A) = \text{EMtp}(a/A)$.

Lemma 3.1.2 (Standard Lemma). *Let P be an A -hyperdefinable set and $(I, <)$ a small infinite linear order. Then, for any sequence $a = (a_i)_{i \in I} \in P^I$ and any set of representatives A^* , there is a sequence $b = (b_i)_{i \in I} \in P^I$ with an A^* -indiscernible sequence of representatives $b^* = (b_i^*)_{i \in I}$ such that $b \in \text{EMtp}(a/A)$.*

Proof. Trivial from the classic Ehrenfeucht-Mostowski Standard Lemma proved using Ramsey's Theorem [TZ12, Theorem 5.1.5]. Q.E.D.

Corollary 3.1.3. *Let P be an A -hyperdefinable set and $(I, <)$ a small infinite linear order. Then, a sequence $b = (b_i)_{i \in I} \in P^I$ is A -indiscernible if and only if, for some set of representatives A^* of A , there is an A^* -indiscernible sequence of representatives $b^* = (b_i^*)_{i \in I}$.*

Furthermore, if b is A -indiscernible, for any set of representatives A^ there is an A^{**} -indiscernible sequence of representatives b^* of b where A^{**} is another set of representatives of A with $\text{tp}(A^*) = \text{tp}(A^{**})$.*

Proof. By the Standard Lemma 3.1.2 and Corollary 1.1.11. Q.E.D.

Let $P = X/E$ be an A -hyperdefinable set and $\mathfrak{g}_P : X \rightarrow P$ its quotient map. An \bigwedge -definable subset $V \subseteq P$ divides over A if and only if $\mathfrak{g}_P^{-1}[V]$ divides over A^* for some set of representatives of A .

Lemma 3.1.4. *Let P be an A -hyperdefinable set and $V \subseteq P$ an $\bigwedge_{A,B}$ -definable subset where B is a set of hyperimaginaries over A . Then, V divides over A if and only if there are a finite tuple b from B and an $\bigwedge_{A,b}$ -definable subset $W \subseteq P$ such that $V \subseteq W$ and W divides over A .*

Proof. One direction is trivial. Let us check the other. Take a uniform definition \underline{V} of V , which exists by Lemma 1.1.13. Take representatives A^* of A such that $\underline{V}(x, A^*, B^*)$ divides over A^* . There is then b finite such that $\underline{W}(x, A^*, b^*) := \underline{V}(x, A^*, B^*) \cap \text{For}^x(\mathbb{L}(A^*, b^*))$ divides over A^* . Now, as \underline{V} is a uniform definition, W is $\bigwedge_{A,b}$ -definable. Obviously, $V \subseteq W$ and W divides over A . Q.E.D.

Lemma 3.1.5. *Let P and Q be A -hyperdefinable sets. Let $V \subseteq Q \times P$ be an \bigwedge_A -definable set and $b \in Q$. Then, $V(b)$ divides over A if and only if there is an A -indiscernible sequence $(b_i)_{i \in \omega}$ of hyperimaginaries from Q such that $\text{tp}(b_0/A) = \text{tp}(b/A)$ and $\bigcap_{i \in \omega} V(b_i) = \emptyset$.*

Furthermore, $V(b)$ divides over A if and only if for any set of representatives A^ of A there is another set of representatives A^{**} such that $\text{tp}(A^*) = \text{tp}(A^{**})$ and $V(b)$ divides over A^{**} .*

Proof. Assume $V(b)$ divides over A . Take a uniform definition \underline{V} of V , which exists by Lemma 1.1.13. Take representatives A^* of A such that $\underline{V}(x, A^*, b^*)$ divides over A^* . There is then an A^* -indiscernible sequence $(b_i^*)_{i \in \omega}$ such that $\text{tp}(b_0^*/A^*) = \text{tp}(b^*/A^*)$ and $\bigcup_{i \in \omega} \underline{V}(x, A^*, b_i^*)$ is not finitely satisfiable. Then, $(b_i)_{i \in \omega}$ given by $b_i = \mathfrak{g}_Q(b_i^*)$ is A -indiscernible and $\text{tp}(b_0/A) = \text{tp}(b/A)$. Also, $\mathfrak{g}_P^{-1}[\bigcap_{i \in \omega} V(b_i)] = \bigcap_{i \in \omega} \mathfrak{g}_P^{-1}[V(b_i)] = \bigcap_{i \in \omega} \underline{V}(\mathfrak{M}, A^*, b_i^*) = \emptyset$, so $\bigcap_{i \in \omega} V(b_i) = \emptyset$ by surjectivity of \mathfrak{g}_P .

On the other hand, assume there is an A -indiscernible sequence $(b_i)_{i \in \omega}$ in Q such that $\text{tp}(b_0/A) = \text{tp}(b/A)$ and $\bigcap_{i \in \omega} V(b_i) = \emptyset$. Take a uniform definition \underline{V} of V , which exists by Lemma 1.1.13. By Corollary 3.1.3, there is an A^* -indiscernible sequence $(b_i^*)_{i \in \omega}$ of representatives of $(b_i)_{i \in \omega}$ with A^* representatives of A . Now, as $\text{tp}(b_0/A) = \text{tp}(b/A)$, by Corollary 1.1.11, there is a representative $b^{**} \in b$ and a set of representatives A^{**} of A such that $\text{tp}(b_0^*, A^*) = \text{tp}(b^{**}, A^{**})$. Take $\sigma \in \text{Aut}(\mathfrak{M})$ mapping (b_0^*, A^*) to (b^{**}, A^{**}) , and write $b'_i := \sigma(b_i)$ and $b_i'^* := \sigma(b_i^*)$ for $i \in \omega$. Then, $(b_i'^*)_{i \in \omega}$ is an A^{**} -indiscernible sequence with $b_0'^* = b^{**}$. For this sequence, it follows that $\emptyset = \mathfrak{g}_P^{-1}[\bigcap_{i \in \omega} V(b_i)] = \bigcap_{i \in \omega} \mathfrak{g}_P^{-1}[V(b_i)] = \bigcap_{i \in \omega} \underline{V}(\mathfrak{M}, A^{**}, b_i'^*)$, concluding that \underline{V} divides over A^{**} . In particular, V divides over A .

For the “furthermore part”, note that in the previous paragraph $\text{tp}(A^*)$ can be chosen arbitrarily by Proposition 3.1.3. Q.E.D.

Combining both propositions we conclude that the definition given here is equivalent to the one studied previously by other authors (e.g. [HKP00], [Wag10] and [Kim13]). It is now straightforward to prove the following basic lemma.

Lemma 3.1.6. *Let P and Q be A -hyperdefinable sets. Let $V \subseteq P$ be \wedge -definable and $f : P \rightarrow Q$ a 1-to-1 \wedge_A -definable function. Then, $f[V]$ divides over A provided that V divides over A .*

Proof. Let V be an \wedge -definable subset of P dividing over A . Take B^* such that V is \wedge_{A, B^*} -definable. By Lemma 3.1.4 and 1.1.15, there is $b^* \subseteq B^*$ finite and an \wedge_A -definable subset $V_0 \subseteq X \times P$, with X the sort of b^* , such that $V \subseteq V_0(b^*)$ and $V_0(b^*)$ divides over A . By Lemma 3.1.5, there is an A -indiscernible sequence $(b_i^*)_{i \in \omega}$ starting at b^* such that $\bigcap_{i \in \omega} V_0(b_i^*) = \emptyset$. Then, $\bigcap f[V_0](b_i^*) = f[\bigcap_{i \in \omega} V_0(b_i^*)] = \emptyset$, where $f[V_0] = \{(x, f(a)) \in X \times P : (x, a) \in V_0\}$ is \wedge_A -definable as f is \wedge_A -definable. Thus, by Lemma 3.1.5, $f[V_0](b^*)$ divides over A . Since $f[V] \subseteq f[V_0](b^*)$, we conclude that $f[V]$ divides over A . Q.E.D.

Let P be an A -hyperdefinable set. The *forking ideal* $\mathfrak{f}_P(A)$ of P over A is the ideal of \wedge -definable subsets of P generated by the ones dividing over A . An \wedge -definable subset of P *forks over A* if it is in that forking ideal.

Remark 3.1.7. Trivially, we have that V forks over A implies that \underline{V} forks over A . In the case of simple theories, the converse also holds as forking and dividing are the same [Wag10, Proposition 3.2.7]. In general, however, the converse is not true. For instance, let \mathfrak{M} be the monster model of the theory of dense circular orders in the

usual language, let M be the whole 1-ary universe of \mathfrak{M} and E the trivial equivalence relation given by xEy for every $x, y \in M$. Obviously, M/E is a singleton, so it does not fork over \emptyset . However, M forks over \emptyset .

Let P be a piecewise A -hyperdefinable set. An \bigwedge -definable set $V \subseteq P$ *divides* over A if it divides over A as subset of some/any piece P_i containing V . Note that, by Lemma 3.1.6, this is well-defined. An \bigwedge -definable subset V of P forks over A if and only if V forks over A as subset of some/any piece P_i containing V . The *forking ideal* $\mathfrak{f}_P(A)$ of P over A is the family of \bigwedge -definable subsets of P forking over A . Clearly, $\mathfrak{f}_P(A)$ is the ideal of \bigwedge -definable subsets of P generated by the ones dividing over A .

3.2 Ideals

Throughout this section, λ is a cardinal with $\kappa \geq \lambda > |\mathbf{L}| + |A|$. Also, we assume that all arities are smaller than λ .

Let P be a piecewise A -hyperdefinable set and μ an ideal of $\bigwedge_{<\lambda}$ -definable subsets in P . We say that μ is *A -invariant* if it is invariant under $\text{Aut}(\mathfrak{M}/A)$, i.e. $W(b) \in \mu$ implies $W(b') \in \mu$ for any b' with $\text{tp}(b/A) = \text{tp}(b'/A)$ and any \bigwedge_b -definable subset $W(b) \subseteq P$ with $|b| < \lambda$. We say that an $\bigwedge_{<\lambda}$ -definable subset is *wide* if it is not in μ . We say that μ is *locally atomic* if, for any wide \bigwedge_B -definable subset V with $|B| < \lambda$, there is $a \in V$ such that $\text{tp}(a/B)$ is wide. We say that μ is *compact* if any intersection of less than λ many $\bigwedge_{<\lambda}$ -definable subsets of P lies in μ if and only if there is a finite sub-intersection lying in μ .

Remark 3.2.1. For $X \subseteq P$, write $\mu|_X := \{W \in \mu : W \subseteq X\}$. Clearly, $\mu|_X$ is an ideal of $\bigwedge_{<\lambda}$ -definable subsets of X . Say $P = \varinjlim P_i$, then write $\mu_i := \mu|_{P_i}$ for each piece. We have from the definitions that μ is A -invariant if and only if each μ_i is so. Similarly, it is locally atomic if and only if each μ_i is so.

Let R be a piecewise bounded \bigwedge_A -definable reflexive binary relation in P . We say that an \bigwedge_A -definable subset V is *R -rough A -medium* if, for any $\bigwedge_{A,b}$ -definable subset $W(b) \subseteq V$ with $|b| < \lambda$, we have

$$R[W(b_0)] \cap R[W(b_1)] \in \mu \Rightarrow W(b) \in \mu$$

for any A -indiscernible sequence $(b_i)_{i \in \mathbb{N}}$ realizing $\text{tp}(b/A)$. We say that an $\bigwedge_{<\lambda}$ -definable subset V is *A -medium* if there is an R -rough A -medium \bigwedge_A -definable subset V_0 with $V \subseteq V_0$. We say that V is *strictly R -rough A -medium* if it is wide and R -rough A -medium. Write $\mathfrak{m}_\mu^R(A)$ for the A -invariant ideal of all $\bigwedge_{<\lambda}$ -definable R -rough

A -medium subsets. We say that V is (strictly) A -medium if it is (strictly) $=$ -rough A -medium and write $\mathfrak{m}_\mu(A)$ for $\mathfrak{m}_\mu^-(A)$.

Note that, from the definition, $\mathfrak{m}_\mu^{R_1}(A) \subseteq \mathfrak{m}_\mu^{R_0}(A)$ whenever $R_1 \subseteq R_0$. In particular, $\mathfrak{m}_\mu(A) \subseteq \mathfrak{m}_\mu^R(A)$ for any R .

We say that μ is R -rough A -medium if every $\bigwedge_{<\lambda}$ -definable subset is R -rough A -medium. Similarly, we say that μ is A -medium¹ if it is $=$ -rough A -medium, i.e. if every $\bigwedge_{<\lambda}$ -definable subset is A -medium. Note that, by definition, if μ is A -medium, μ is in particular A -invariant.

Examples 3.2.2. (1) The forking ideal over A is an A -invariant ideal of \bigwedge -definable subsets. In simple theories, the forking ideal over A is locally atomic and A -medium. Indeed, it is locally atomic by the extension property of forking [Wag10, Theorem 3.2.8]. On the other hand, suppose $V(b)$ is \bigwedge_b -definable and $V(b_0) \cap V(b_1)$ forks over A with $(b_i)_{i \in \omega}$ an A -indiscernible sequence realising $\text{tp}(b/A)$. Then, by simplicity, $V(b_0) \cap V(b_1)$ divides over A , so $\bigcap_{i=0}^n V(b_{2i}) \cap V(b_{2i+1}) = \emptyset$ for some $n \in \mathbb{N}$. Thus, $V(b)$ divides over A , so it forks over A . As V is arbitrary, we conclude that the forking ideal over A is A -medium.

(2) If we have an A -invariant measure on the lattice of $\bigwedge_{<\lambda}$ -definable sets, the ideal of zero measure $\bigwedge_{<\lambda}$ -definable subsets is an A -invariant ideal. In this case, every \bigwedge_A -definable subset of finite measure is A -medium.

Lemma 3.2.3. [Hru11, Lemma 2.9] *Let P be a piecewise A -hyperdefinable set and μ an ideal of $\bigwedge_{<\lambda}$ -definable subsets on P . Let $V \subseteq P$ be a strictly A -medium $\bigwedge_{<\lambda}$ -definable subset. Then, V does not fork over A .*

Proof. Take an A -medium \bigwedge_A -definable set V_0 such that $V \subseteq V_0$. Suppose that V forks over A . Then, there are W'_1, \dots, W'_n \bigwedge -definable subsets dividing over A such that $V \subseteq \bigcup_i W'_i$. Applying Lemma 3.1.4, we may assume that each W'_i is $\bigwedge_{<\lambda}$ -definable. Now, $V \subseteq \bigcup(W'_i \cap V_0)$ and $W'_i \cap V_0$ divides over A . Write $W_i(b) = W'_i \cap V_0$. By Lemma 3.1.5, there is $(b_j)_{j \in \omega}$ A -indiscernible such that $\bigcap_{j \in \omega} W_i(\bar{b}_j) = \emptyset$. Hence, $\bigcap_{j=0}^k W_i(\bar{b}_j) = \emptyset \in \mu$ for some k , concluding that $W_i(b) \in \mu$ by A -mediumness of V_0 . This concludes that $V \in \mu$, contradicting that V is wide. Q.E.D.

Lemma 3.2.4. *Let $\mathfrak{N} \preceq \mathfrak{M}$ with $|N| < \lambda$ and $G = \varinjlim G_k$ be a piecewise N -hyperdefinable group. Let μ be an N -invariant ideal of $\bigwedge_{<\lambda}$ -definable subsets of G . Let W and U be non-empty \bigwedge_N -definable subsets of G . If μ is invariant under left translations and $U \cdot W$ is N -medium, then W is N -medium too.*

Proof. Let $X(b) \subseteq W$ be \bigwedge_b -definable with $|b| < \lambda$. Take an N -indiscernible sequence $(b_i^*)_{i \in \omega}$ of representatives of elements in $\text{tp}(b/N)$ and write $X_i := X(b_i)$. Fix k such that $U \subseteq G_k$ and take \underline{U} defining U . As $\mathfrak{N} \prec \mathfrak{M}$, \underline{U} is finitely satisfiable in N , so \underline{U}

¹We use the terminology of [MOS18]. In [Hru11], it is said that the ideal has the S_1 property.

does not fork over N [TZ12, Lemma 7.1.10]. Thus, \underline{U} has a non-forking extension to a complete type p over N, b_0^* . As p does not fork over N , it does not divide over N . By [TZ12, Lemma 7.1.5], there is a^* realising p such that $(b_i^*)_{i \in \omega}$ is N, a^* -indiscernible. As a^* realises p , in particular, $a \in U$.

Suppose μ is invariant under left translations and $U \cdot W$ is N -medium. Then, we get that $a \cdot X_0 \cap a \cdot X_1 \in \mu$ if and only if $a \cdot X_0 \in \mu$. Therefore, provided left translational invariance,

$$X_0 \cap X_1 \in \mu \Leftrightarrow a \cdot X_0 \cap a \cdot X_1 \in \mu \Leftrightarrow a \cdot X_0 \in \mu \Leftrightarrow X_0 \in \mu.$$

We similarly prove the case with right translations.

Q.E.D.

3.3 Building ideals

Throughout this section, λ is a cardinal with $\kappa \geq \lambda > |\mathbf{L}| + |A|$. Also, we assume that all arities are smaller than λ .

Making functions definable: Let X be a definable set and $f : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$. We say that f is A -invariant if $f(a) = f(b)$ when $\text{tp}(a/A) = \text{tp}(b/A)$. From the tradition of continuous logic, f is called A -definable when it is continuous using the A -logic topology in X . The *standard expansion* \mathfrak{M}_f making f definable is the one given by adding the predicates $f \leq \alpha$, for $\alpha \in \mathbb{Q}$, with the natural interpretations.

Let \mathcal{A} be a family of definable subsets of X closed under substitution of parameters. A function ν on \mathcal{A} defines a family $\varphi^*\nu(y)$ of functions given by $\varphi^*\nu(a) = \nu(\varphi(\mathfrak{M}, a))$ for the formulas $\varphi \in \mathcal{A}$. We say then that ν is A -invariant if each $\varphi^*\nu$ is A -invariant. We say that ν is A -definable if $\varphi^*\nu$ is A -definable for each formula φ . The *standard expansion* \mathfrak{M}_ν making ν definable is the one given by making definable each $\varphi^*\nu$.

For example, a *Keisler measure* is a *probability content* (also called a finitely additive probability measure) defined on the whole boolean algebra of definable subsets. Usually, as soon as we add new predicates to the language to make it definable, the Keisler measure is not longer a Keisler measure. Fortunately, we can in general apply the following basic remark:

Remark 3.3.1. Let \mathcal{B} be an algebra of subsets of X and $\mathcal{B}_0 \subseteq \mathcal{B}$ a subalgebra. Let ν_0 be a probability content on \mathcal{B}_0 . Then, there is a content ν on \mathcal{B} extending ν_0 .

Proof. Let V be the vector space of simple function on \mathcal{B} and $V_0 \leq V$ the subspace of simple functions on \mathcal{B}_0 . Let $C \subseteq V$ be the convex cone of positive simple functions of \mathcal{B} and $C_0 = V_0 \cap C \subseteq V_0$ the convex cone of positive simple function of \mathcal{B}_0 . Define the linear functional $E_0 : V_0 \rightarrow \mathbb{R}$ given by $E_0(\sum a_i \mathbb{1}_{A_i}) = \sum_i a_i \nu_0(A_i)$, where $\mathbb{1}_A$ denotes the indicator function of A . Since it is positive in C_0 , by Marcel Riesz's Extension Theorem [Akh21, Theorem 2.6.2], there is a linear functional $E : V \rightarrow \mathbb{R}$ positive in

C extending E_0 . Take $\nu(A) = E(\mathbb{1}_A)$. Then, $\nu : \mathcal{B} \rightarrow \mathbb{R}_{\geq 0}$ extends ν_0 . By linearity of E , ν is a content. As $\nu(X) = \nu_0(X) = 1$, it is a probability content. Q.E.D.

In that case, for a given Keisler measure ν_0 , we can recursively build a sequence $(\nu_n)_{n \in \mathbb{N}}$ where ν_{n+1} is a Keisler measure on \mathfrak{M}_{ν_n} extending ν_n . Then, $\nu = \bigcup \nu_n$ is a definable Keisler measure on the expansion $\mathfrak{M}_\nu = \bigcup \mathfrak{M}_{\nu_n}$ extending the original Keisler measure ν_0 . Note that, however, when extending the Keisler measure, it may happen that some important properties of the original one (e.g. invariance under the action of some specific group) get lost.

Ultralimits: Let $\{\mathfrak{M}_i\}_{i \in I}$ be a family of L-structures and $f_i : X(\mathfrak{M}_i) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ functions. Let \mathfrak{u} be an ultrafilter on I and $\mathfrak{M} = \prod \mathfrak{M}_i / \mathfrak{u}$ the respective ultraproduct. We define $f = \lim_{\mathfrak{u}} f_i$ by $f([a_i]_{\mathfrak{u}}) = \text{st}[f_i(a_i)]_{\mathfrak{u}}$ for any $a = [a_i]_{\mathfrak{u}} \in X(\mathfrak{M})$, where st is the standard part function. If each f_i is invariant, so is f .

Suppose that the functions f_i are uniformly definable. In other words, suppose that, for any close subset $C \subseteq \mathbb{R} \cup \{\pm\infty\}$, there is a common partial type Σ_C such that $f_i^{-1}[C] = \Sigma_C(\mathfrak{M}_i)$ for each $i \in I$. Then, by Łoś's Theorem [CK90, Theorem 4.1.9], we get $f^{-1}[C] = \Sigma_C(\mathfrak{M})$ for each $C \subseteq \mathbb{R} \cup \{\pm\infty\}$ close, so f is definable too.

For functions ν_i on the boolean algebras of definable subsets of $X(\mathfrak{M}_i)$, we take $\nu = \lim_{\mathfrak{u}} \nu_i$ defined by $\varphi^* \nu = \lim_{\mathfrak{u}} \varphi^* \nu_i$. If each ν_i is invariant, so is ν . If all the ν_i are uniformly definable, ν is definable. If each ν_i is a Keisler measure, so is ν .

Liminf of ideals: Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of ideals of definable subsets. Then, $\mu = \liminf \mu_n := \bigcup_{n_0} \bigcap_{n > n_0} \mu_n$ is an ideal of definable subsets of X . Clearly, if each μ_n is invariant, so is μ .

Now, take a definable reflexive binary relation R on X . Then,

$$\liminf \mathfrak{m}_{\mu_n}^R(A) \subseteq \mathfrak{m}_{\mu}^R(A).$$

Indeed, take $Y \subseteq X$ definable. Suppose that there is n_0 such that μ_n is R -rough A -medium in Y for every $n > n_0$. Pick $Z(b) \subseteq Y$ such that $R[Z(b_0)] \cap R[Z(b_1)] \in \mu$ where $(b_i)_{i \in \mathbb{N}}$ is A -indiscernible at b . Then, for some $k > n_0$, $R[Z(b_0)] \cap R[Z(b_1)] \in \mu_n$ for any $n > k$, so $Z(b) \in \mu_n$ for any $n > k$, concluding $Z(b) \in \mu$. As $Z(b)$ was arbitrary, we conclude that μ is R -rough A -medium in Y .

Compactifications of ideals of definable sets: Let μ be an ideal of definable subsets of X . We extend μ to an ideal $\hat{\mu}$ of $\bigwedge_{< \lambda}$ -definable subsets of X by compactification as

$$\hat{\mu} := \{V \subseteq X \text{ } \bigwedge_{< \lambda}\text{-def.} : \exists D \in \mu \ V \subseteq D\}.$$

If μ is invariant, so is $\widehat{\mu}$.

By construction $\widehat{\mu}$ is compact. Furthermore, note that this is the unique compact ideal of $\bigwedge_{<\lambda}$ -definable subsets of X extending μ .

Note that $\widehat{\mu}$ is also locally atomic. Indeed, given any \bigwedge_B -definable subset $V \notin \widehat{\mu}$, we have $V \cap \bigcap_{D \in \mu|_B} D^c \neq \emptyset$ by compactness. Thus, for any $a \in V \cap \bigcap_{D \in \mu|_B} D^c$, $\text{tp}(a/B) \notin \widehat{\mu}$.

Let R be a definable reflexive binary relation in X . Then, clearly,

$$\mathbf{m}_\mu^R(A) = \{D \in \mathbf{m}_{\widehat{\mu}}^R(A) : D \text{ def.}\}.$$

Indeed, trivially, the left-hand side is contained in the right-hand side. On the other hand, suppose that μ is R -rough A -medium in D , where D is definable. Let $W(b) \subseteq D$ be an \bigwedge_b -definable set with $R[W(b_0)] \cap R[W(b_1)] \in \widehat{\mu}$, where $(b^i)_{i \in \mathbb{N}}$ is an A -indiscernible sequence at b and $|b| < \lambda$. By compactness, there is $W_0(b) \subseteq D$ definable with $W(b) \subseteq W_0(b)$ such that $R[W(b_0)] \cap R[W(b_1)] \subseteq R[W_0(b_0)] \cap R[W_0(b_1)] \in \mu$. Then, $W(b) \subseteq W_0(b) \in \mu$ as μ is R -rough A -medium in D . Since $W(b)$ is arbitrary, we conclude that $\widehat{\mu}$ is R -rough A -medium in D .

Finally, let $\{R_i\}_{i \in I}$ be a directed family of \bigwedge_A -definable reflexive binary relations on X and $R = \bigcap R_i$. Then, by compactness,

$$\bigcap \mathbf{m}_{\widehat{\mu}}^{R_i}(A) = \mathbf{m}_{\widehat{\mu}}^R(A).$$

Indeed, as $R \subseteq R_i$ for each $i \in I$, $\mathbf{m}_\mu^R(A) \subseteq \bigcap \mathbf{m}_\mu^{R_i}(A)$. On the other hand, suppose that $\widehat{\mu}$ is R_i -rough A -medium in V for each $i \in I$ and take $W(b) \subseteq V$ with $R[W(b_0)] \cap R[W(b_1)] \in \widehat{\mu}$ where $(b_i)_{i \in \mathbb{N}}$ is an A -indiscernible sequence at b with $|b| < \lambda$. Then, by compactness, $R_i[W(b_0)] \cap R_i[W(b_1)] \in \widehat{\mu}$ for some $i \in I$, so $W(b) \in \widehat{\mu}$ as $\widehat{\mu}$ is R_i -rough A -medium in V , concluding that $V \in \mathbf{m}_\mu^R(A)$.

Mapping ideals: Let P and Q be two piecewise hyperdefinable sets and μ an ideal of $\bigwedge_{<\lambda}$ -definable subsets of P . Let $f : P \rightarrow Q$ be a piecewise \bigwedge_A -definable function. We may then map μ via f as

$$f_*\mu = \{V \subseteq Q \text{ } \bigwedge_{<\lambda}\text{-def.} : f^{-1}[V] \in \mu\}.$$

Obviously, $f_*\mu$ is an ideal of \bigwedge -definable subsets of Q . It is also clear that $f_*\mu$ is invariant as long as μ is so.

Suppose now that f is piecewise proper. Then, if μ is locally atomic, so is $f_*\mu$. Indeed, take an \bigwedge_B -definable subset $V \notin f_*\mu$ with B bounded. As f is piecewise proper, $f^{-1}[V]$ is an \bigwedge -definable μ -wide subset. By local atomicity, there is $a \in f^{-1}[V]$ such

that $\text{tp}(a/B) \notin \mu$, so $\text{tp}(f(a)/B) \notin f_*\mu$ as $\text{tp}(a/B) \subseteq f^{-1}[\text{tp}(f(a)/B)]$, concluding that $f_*\mu$ is locally atomic too.

Similarly, when f is piecewise proper, we have that $f_*\mu$ is compact if μ is compact. Indeed, take a family $\{W_i\}_{i \in I}$ of $\bigwedge_{<\lambda}$ -definable subsets of Q with $|I| < \lambda$ such that $\bigcap_{i \in I} W_i \in f_*\mu$. Then, $\bigcap_{i \in I} f^{-1}[W_i] = f^{-1}[\bigcap_{i \in I} W_i] \in \mu$, where $\{f^{-1}[W_i]\}_{i \in I}$ is a family of $\bigwedge_{<\lambda}$ -definable subsets of P as f is piecewise proper. By compactness, there is a finite subset $I_0 \subseteq I$ such that $f^{-1}[\bigcap_{i \in I_0} W_i] = \bigcap_{i \in I_0} f^{-1}[W_i] \in \mu$, so $\bigcap_{i \in I_0} W_i \in f_*\mu$, concluding that $f_*\mu$ is compact too.

Finally, suppose that f is piecewise bounded and proper. Let R be a piecewise bounded \bigwedge_A -definable reflexive binary relation on Q . Then,

$$\mathbf{m}_{f_*\mu}^R(A) = \{V \subseteq Q : f^{-1}[V] \in \mathbf{m}_\mu^{f^*R}(A)\},$$

where $f^*R := \{(x, y) : (f(x), f(y)) \in R\}$. Indeed, it suffices to note that $f^{-1}[R[Y]] = f^*R[f^{-1}[Y]]$ for any $Y \subseteq Q$ and $f^*R[X] = f^{-1}[R[f[X]]]$ for any subset $X \subseteq P$. Hence, the first identity implies that $\{V : f^{-1}[V] \in \mathbf{m}_\mu^{f^*R}(A)\} \subseteq \mathbf{m}_{f_*\mu}^R(A)$ while the second implies that $\mathbf{m}_{f_*\mu}^R(A) \subseteq \{V : f^{-1}[V] \in \mathbf{m}_\mu^{f^*R}(A)\}$.

In particular, suppose that μ is an E -rough A -medium ideal of $\bigwedge_{<\lambda}$ -definable subsets, where E is an \bigwedge_A -definable equivalence relation on X . Take $\mathfrak{g} : X \rightarrow X/E$. Then, from the previous discussion, $\mathfrak{g}_*\mu$ is an A -medium ideal of $\bigwedge_{<\lambda}$ -definable subsets of X/E .

3.4 Stable relations

Let P and Q be piecewise A -hyperdefinable sets and $V, W \subseteq P \times Q$ disjoint A -invariant subsets. We say that V and W are *unstably separated over A* if there is an infinite A -indiscernible sequence $(a_i, b_i)_{i \in \omega}$ such that $(a_0, b_1) \in V$ and $(a_1, b_0) \in W$. We say that they are *stably separated over A* if they are not unstably separated. We say that an A -invariant binary relation $R \subseteq P \times Q$ is *stable over A* if R and R^c are stably separated over A .

Remarks 3.4.1. (1) Clearly, being stably separated over A is invariant under piecewise \bigwedge_A -definable isomorphisms.

(2) Note that V and W are stably separated over A if and only if $V \cap (P_i \times Q_j)$ and $W \cap (P_i \times Q_j)$ are stably separated over A for any pieces P_i and Q_j .

(3) Let P and Q be A -hyperdefinable. By Corollary 3.1.3, V and W are stably separated over A if and only if $\mathfrak{g}_{P \times Q}^{-1}[V]$ and $\mathfrak{g}_{P \times Q}^{-1}[W]$ are stably separated over A^* for any set of representatives A^* of A . In particular, if V and W are \bigwedge_A -definable, we have that V and W are stably separated over A if and only if \underline{V} and \underline{W} are stably

separated over A^* for any set of representatives A^* of A and any partial types \underline{V} and \underline{W} defining V and W .

(4) Note that, by the symmetry of stability, R is stable over A if and only if R^c is stable over A .

We need the following two lemmas from [Hru11] for the Stabilizer Theorem.

Recall that we say that a partial type $\Sigma(x, b^*)$ divides over A^* with respect to a global A^* -invariant type \widehat{q} extending $\text{tp}(b^*/A^*)$ if there is a sequence $(b_i^*)_{i \in \omega}$ such that $\text{tp}(b_i^*/A^*, b_0^*, \dots, b_{i-1}^*) \models \widehat{q}$ and $\bigwedge \Sigma(x, b_i^*)$ is inconsistent.

Lemma 3.4.2. [Hru11, Lemma 2.2] *Let $P = X/E$ and $Q = Y/F$ be A^* -hyperdefinable sets and \widehat{q} a global A^* -invariant type with $\underline{Y} \in \widehat{q}$. Let $V, V' \subseteq P \times Q$ be \bigwedge_{A^*} -definable sets stably separated over A^* . Assume that $a \in P$ is such that $\underline{V}'(a) \subseteq \widehat{q}$. Then, $W = V \cap (\text{tp}(a/A^*) \times Q)$ divides over A^* . Furthermore, \underline{W} divides over A^* with respect to \widehat{q} .*

Proof. Suppose \underline{W} does not divide over A^* with respect to \widehat{q} . Take a representative $a^* \in a$. Define by recursion a sequence $(a_i^*, c_i^*)_{i \in \omega}$ in $X \times Y$ such that

$$c_i^* \models \widehat{q}_{|A, a_0^*, \dots, a_{i-1}^*, c_0^*, \dots, c_{i-1}^*}$$

$$a_i^* \models \underline{\text{tp}}(a^*/A) \wedge \bigwedge_{j=0}^{i-1} \underline{W}(x, c_j^*).$$

It is well-defined by saturation. Indeed, since \underline{W} does not divide over A^* with respect to \widehat{q} , $\bigwedge_{j=0}^{i-1} \underline{W}(x, c_j^*)$ is satisfiable. Given b^* realising $\bigwedge_{j=0}^{i-1} \underline{W}(x, c_j^*)$, we have that $b = [b]_P \in \text{tp}(a/A)$. Then, by Corollary 1.1.5, we can find a_i^* .

As $\text{tp}(a_i^*/A) = \text{tp}(a^*/A)$ and \widehat{q} is A -invariant, $\underline{V}'(a_i^*, y) \subseteq \widehat{q}$. Then, for $a_i = [a_i^*]_P$ and $c_j = [c_j^*]_Q$, $(a_i, c_j) \in V$ for $i > j$ and $(a_i, c_j) \in V'$ for $i < j$, contradicting the stably separated hypothesis. Q.E.D.

For sets V and W , write

$$V \times_{\text{nf}(A)} W := \{(a, b) \in V \times W : \text{tp}(b/aA) \text{ does not fork over } A\}.$$

Similarly, $V \times_{\text{ndiv}(A)} W$, $V_{\text{nf}(A)} \times W$ and $V_{\text{ndiv}(A)} \times W$.

Lemma 3.4.3. [Hru11, Lemma 2.3] *Let P and Q be piecewise A^* -hyperdefinable sets and $R \subseteq P \times Q$ a stable binary relation over A^* . Let $p \subseteq P$ and $q \subseteq Q$ be types over A^* . Assume that there is an A^* -invariant global type \widehat{q} extending a partial type \underline{q} over A^* defining q .*

(1) *Take $a \in p$ and $b \in q$ such that $(a, b) \in R$. Suppose that there are representatives $a^* \in a$ and $b^* \in b$ such that $b^* \models \widehat{q}_{|A^*, a^*}$. Then, we have that $(a', b) \in R$ for any $a' \in p$ such that $\text{tp}(a'/A^*, b)$ does not divide over A^* .*

(2) Take $a', a \in p$ and $b \in q$. Suppose that $\text{tp}(a/A^*, b)$ and $\text{tp}(a'/A^*, b)$ do not divide over A^* . Then, $(a, b) \in R$ if and only if $(a', b) \in R$.

(3) Assume that there is also an A^* -invariant global type \widehat{p} extending a partial type \underline{p} over A^* defining p . Then, the following conditions are equivalent:

- | | | | |
|------|---|-------|---|
| i. | $p \times_{\text{ndiv}(A^*)} q \cap R \neq \emptyset$ | v. | $p_{\text{ndiv}(A^*)} \times q \cap R \neq \emptyset$ |
| ii. | $p \times_{\text{nf}(A^*)} q \cap R \neq \emptyset$ | vi. | $p_{\text{nf}(A^*)} \times q \cap R \neq \emptyset$ |
| iii. | $p \times_{\text{nf}(A^*)} q \subseteq R$ | vii. | $p_{\text{nf}(A^*)} \times q \subseteq R$ |
| iv. | $p \times_{\text{ndiv}(A^*)} q \subseteq R$ | viii. | $p_{\text{ndiv}(A^*)} \times q \subseteq R$ |

Proof. (1) Suppose $(a', b) \notin R$ with $a' \in p$ and $b \in q$. Fix $i_0 \in I$ and $j_0 \in J$ such that $p \subseteq P_{i_0}$ and $q \subseteq Q_{j_0}$. Write $V = \text{tp}(a, b/A^*)$ and $V' = \text{tp}(a', b/A^*)$. As R is stable, V and V' are stably separated. Let $\underline{V}(x, y)$ be a partial type defining $V \subseteq P_{i_0} \times Q_{j_0}$. Take $a^* \in a$ such that there is $b^* \in b$ realizing $\widehat{q}_{|A^*, a^*}$. Note that, by definition, $\underline{V}(a^*, y) = \underline{\text{tp}}(b/A^*, a)$. As b^* realizes $\underline{\text{tp}}(b/A^*, a)$, which is a partial type over A^*, a^* , we conclude that $\underline{V}(a^*, y) \subseteq \widehat{q}$.

Consider $W = V' \cap (\text{tp}(a/A^*) \times Q)$. By Lemma 3.4.2, we conclude that \underline{W} divides over A^* with respect to \widehat{q} . Now, $\text{tp}(a/A^*) = p$ since p is minimal. As V' is minimal and $V' \cap (p \times Q) \neq \emptyset$, we conclude $W = V' \cap (p \times Q) = V'$. Hence, \underline{V}' divides over A^* with respect to \widehat{q} . In particular, $\underline{V}'(x, b^*)$ divides over A^* , so $V'(b) = \text{tp}(a'/A^*, b)$ divides over A^* .

(2) Let $a^* \in a$ and $a'^* \in a'$ be representatives. Take b'^* realizing $\widehat{q}_{|A^*, a^*, a'^*}$ and $b^* \in b$ realizing $\widehat{q}_{|A^*}$. As $\text{tp}(b'^*/A^*) = \text{tp}(b^*/A^*)$, $b' \in W'$. Let $V = \text{tp}(a, b/A^*)$ and $V' = \text{tp}(a', b/A^*)$. We have that neither $V(b)$ nor $V'(b)$ divides over A^* . In other words, neither $\underline{V}(x, b^*)$ nor $\underline{V}'(x, b^*)$ divides over A^* . As $\text{tp}(b^*/A^*) = \text{tp}(b'^*/A^*)$, that means that neither $V(b')$ nor $V'(b')$ divides over A^* . We want to prove that

$$(a, b) \in R \Leftrightarrow (a, b') \in R \Leftrightarrow (a', b') \in R \Leftrightarrow (a', b) \in R.$$

By (1), we have already that $(a, b') \in R$ if and only if $(a', b') \in R$. On the other hand, let c^* be such that $\text{tp}(c^*, b^*/A^*) = \text{tp}(a^*, b^*/A^*)$. Let c be its projection. Therefore, $\text{tp}(c, b'/A^*) = \text{tp}(a, b/A^*) = V$. In particular, $\text{tp}(c/A^*, b') = V(b')$, which does not divide over A^* . Then, applying (1), $(a, b') \in R$ implies $(c, b') \in R$. Note that R^c is also a stable and A^* -invariant relation. Therefore, again by (1), $(a, b') \notin R$ implies $(c, b') \notin R$. By A^* -invariance of R , $(c, b') \in R$ if and only if $(a, b) \in R$. Therefore, summarizing,

$$(a, b) \in R \Leftrightarrow (c, b') \in R \Leftrightarrow (a, b') \in R.$$

Similarly, let d^* be such that $\text{tp}(d^*, b^*/A^*) = \text{tp}(a'^*, b^*/A^*)$. Let d be its projections. Therefore, $\text{tp}(d, b'/A^*) = \text{tp}(a', b/A^*) = V'$. In particular, $\text{tp}(d/A^*, b') = V'(b')$, which does not divide over A^* . Then, applying (1), $(a', b') \in R$ implies $(d, b') \in R$. Note that R^c is also a stable and A^* -invariant relation. Therefore, again by (1), $(a', b') \notin R$ implies $(d, b') \notin R$. By A^* -invariance of R , $(d, b') \in R$ if and only if $(a', b) \in R$. Therefore, summarizing,

$$(a', b') \in R \Leftrightarrow (d, b') \in R \Leftrightarrow (a', b) \in R.$$

(3) Clearly **iv.** \Rightarrow **iii.** \Rightarrow **ii.** \Rightarrow **i.** and **viii.** \Rightarrow **vii.** \Rightarrow **vi.** \Rightarrow **v.** since non-forking implies non-dividing. Also, as $R^{-1} = \{(b, a) : (a, b) \in R\}$ is an A -invariant stable relation, it suffices to prove that **i.** \Rightarrow **iv.** and **iv.** \Rightarrow **v.**

iv. \Rightarrow **v.** Let $a \in p$, $a^* \in a$, b^* realizing $\widehat{q}|_{A^*, a^*}$ and b the projection of b^* . Since \widehat{q} is A^* -invariant, \widehat{q} does not fork over A^* . As $\text{tp}(b/A^*, a) \subseteq \text{tp}(b^*/A^*, a^*) = \widehat{q}|_{A^*, a^*}$, $\text{tp}(b/A, a)$ does not fork over A^* . So $(a, b) \in p \times_{\text{nf}(A^*)} q$ with $\overline{b^*} \in b$ realizing $\widehat{q}|_{A^*, a^*}$ for $a^* \in a$. Assuming **iv.** we get **iii.**, and in particular $(a, b) \in R$. Then, by (1), $(a', b) \in R$ for any $(a', b) \in p_{\text{ndiv}(A^*)} \times q$. As there is $(a', b) \in p_{\text{ndiv}(A^*)} \times q$, we conclude that $R \cap p_{\text{ndiv}(A^*)} \times q \neq \emptyset$, i.e. **v.** holds.

i. \Rightarrow **iv.** Let $(a, b), (a', b') \in p \times_{\text{ndiv}(A^*)} q$. Take b'' such that $\text{tp}(a', b'/A^*) = \text{tp}(a, b''/A^*)$. Then, $\text{tp}(b''/A^*, a)$ does not divide over A^* . As R is A^* -invariant, $(a', b') \in R$ if and only if $(a, b'') \in R$. By (2) applied to $R^{-1} = \{(b, a) : (a, b) \in R\}$, we conclude that $(a, b) \in R$ if and only if $(a, b'') \in R$. Q.E.D.

Remark 3.4.4. Note that, in (3), **i.** \Leftrightarrow **ii.** \Leftrightarrow **iii.** \Leftrightarrow **iv.** only uses that there is an A^* -invariant extension \widehat{p} of a partial type \underline{p} defining p . Symmetrically, **v.** \Leftrightarrow **vi.** \Leftrightarrow **vii.** \Leftrightarrow **viii.** only uses that there is an A^* -invariant extension \widehat{q} of a partial type \underline{q} defining q .

3.5 Stabilizer Theorem

Throughout this section, λ is a cardinal with $\kappa \geq \lambda > |\mathbf{L}| + |A|$. Also, we assume that all arities are smaller than λ .

Let G be a piecewise A -hyperdefinable group and μ an ideal of $\bigwedge_{<\lambda}$ -definable subsets of G . Let $V, W \subseteq G$ be two $\bigwedge_{<\lambda}$ -definable subsets. The (left) μ -stabilizer of V with respect to W is the set $\text{St}_\mu(V, W) = \{a \in G : a^{-1} \cdot V \cap W \notin \mu\}$, and the (left) μ -stabilizer group of V with respect to W is the subgroup $\text{Stab}_\mu(V, W)$ of G generated by $\text{St}_\mu(V, W)$. Write $\text{St}_\mu(V) := \text{St}_\mu(V, V)$ and $\text{Stab}_\mu(V) := \text{St}_\mu(V, V)$. We omit the subscript μ if there is no confusion.

Lemma 3.5.1. *Let G be a piecewise A -hyperdefinable group, μ an A -invariant ideal of $\bigwedge_{<\lambda}$ -definable subsets of G and V and W two \bigwedge_A -definable wide subsets. Then:*

(1) *Take any $B \supseteq A$ with $|B| < \lambda$. If μ is locally atomic, $g \in \text{St}(V, W)$ if and only if there are $a \in V$ and $b \in W$ such that $g = ab^{-1}$ with $\text{tp}(b/B, g) \notin \mu$.*

(2) *If μ is invariant under left translations, then $\text{St}(V, W)^{-1} = \text{St}(W, V)$. In particular, $\text{St}(V)$ is a symmetric subset.*

We say that a subset $X \subseteq G$ is *stable over A* if the relation $\{(a, b) : a^{-1}b \in X\}$ is stable over A .

Example 3.5.2. [Hru11, Lemma 2.10] & [MOS18, Lemma 2.8] Let μ be an A -invariant ideal of $\bigwedge_{<\lambda}$ -definable subsets of a piecewise A -hyperdefinable group. If μ is invariant under left translations and V and W are A -medium \bigwedge_A -definable subsets, then $\text{St}(V, W)$ is stable over A .

Remark 3.5.3. Since $\bullet^{-1} : G \rightarrow G$ is a piecewise \bigwedge_A -definable isomorphism, X is stable over A if and only if X^{-1} is stable over A . Also, X is stable over A if and only if $\{(a, b) : ab \in X\}$ is stable over A , if and only if $\{(a, b) : ab^{-1} \in X\}$ is stable over A .

For subsets V and W of a piecewise hyperdefinable group G , write

$$V \cdot_{\text{nf}(A)} W := \{a \cdot b : (a, b) \in V \times_{\text{nf}(A)} W\}.$$

Similarly, $V \cdot_{\text{ndiv}(A)} W$, $V_{\text{nf}(A)} \cdot W$ and $V_{\text{ndiv}(A)} \cdot W$.

Lemma 3.5.4. *Let G be a piecewise A^* -hyperdefinable group and μ a locally atomic A^* -invariant ideal of $\bigwedge_{<\lambda}$ -definable subsets of G which is also invariant under left translations. Let p be an A^* -medium A^* -type and $X \subseteq G$ a stable subset over A^* containing p . Suppose that there is an A^* -invariant global type \hat{p} extending a partial type p over A^* defining p . Then, $\text{Stab}(p) \subseteq X \cdot X^{-1}$. Furthermore, $\text{Stab}(p) \cdot_{\text{nf}(A^*)} p \subseteq X$.*

Proof. We may assume that p is strictly A -medium; otherwise, $\text{St}(p) = \emptyset$ and $\text{Stab}(p) = \{1\}$, so the lemma holds trivially. Since μ is invariant under left translations, $\text{St}(p)^{-1} = \text{St}(p)$ by Lemma 3.5.1(2). Thus, by definition, $\text{Stab}(p) = \bigcup_{n=0}^{\infty} \text{St}(p)^n$. We prove by induction on n that $b \cdot c \in X$ for $b \in \text{St}(p)^n$ and $c \in p$ such that $\text{tp}(c/A^*, b)$ does not fork over A^* .

By hypothesis, $p \subseteq X$, so we are done for $n = 0$. Assume it is true for $n - 1$ with $n \in \mathbb{N}_{>0}$. Take $b = b_1 \cdots b_n$ with $b_i \in \text{St}(p)$ for each $i \in \{1, \dots, n\}$ and $c \in p$ such that $\text{tp}(c/A^*, b)$ does not fork over A^* . We want to prove that $b \cdot c \in X$. As $b_n \in \text{St}(p)$, by Lemma 3.5.1(1), there is $c' \in p$ such that $b_n \cdot c' \in p$ and $\text{tp}(c'/A^*, b_1, \dots, b_n) \notin \mu$. In particular, as p is A^* -medium, $\text{tp}(c'/A^*, b_1, \dots, b_n)$ does not fork over A^* by Lemma 3.2.3. Hence, $\text{tp}(c'/A^*, b)$ and $\text{tp}(b_n \cdot c'/A^*, b_1, \dots, b_{n-1})$ do not fork over A^* . By induction hypothesis, $b \cdot c' = (b_1 \cdots b_{n-1}) \cdot b_n \cdot c' \in X$. Since $(b, c') \in \text{tp}(b/A^*) \times_{\text{nf}(A^*)} p$, by Lemma 3.4.3(2), we conclude that $b \cdot c \in X$ too.

Now, for any $b \in \text{Stab}(p)$ we can find $c \in p$ such that $\text{tp}(c/A^*, b)$ does not fork over A^* — namely, choose $c^* \models \hat{p}|_{A^*, b^*}$. Therefore, we conclude that $\text{Stab}(p) \subseteq X \cdot p^{-1} \subseteq X \cdot X^{-1}$. Q.E.D.

Corollary 3.5.5. *Let $\mathfrak{N} \prec \mathfrak{M}$ with $|N| < \lambda$. Let G be a piecewise N -hyperdefinable group and μ a locally atomic N -invariant ideal of $\bigwedge_{<\lambda}$ -definable subsets invariant under left translations. Let p be an N -medium N -type with $p \subseteq \text{St}(p)$. Then, $\text{Stab}(p) = \text{St}(p)^2 = (p \cdot p^{-1})^2$.*

Proof. If $p \subseteq \text{St}(p)$, by Lemma 3.5.1, we get that $\text{St}(p) \subseteq p \cdot p^{-1} \subseteq \text{St}(p)^2$. By Example 3.5.2, $\text{St}(p)$ is a stable subset. Hence, by Lemma 3.5.4, taking \hat{p} a coheir of p , we conclude that $\text{Stab}(p) \subseteq \text{St}(p)^2 \subseteq (p \cdot p^{-1})^2 \subseteq \text{Stab}(p)$. Q.E.D.

Lemma 3.5.6. [MOS18, Lemma 2.11] *Let G be a piecewise A^* -hyperdefinable group and μ an A^* -invariant ideal of $\bigwedge_{<\lambda}$ -definable subsets of G which is also invariant under left translations. Let p be an A^* -type and W a strictly A^* -medium \bigwedge_{A^*} -definable subset such that $p^{-1}W$ is A^* -medium too. Suppose that there is an A^* -invariant global type \widehat{p} extending a partial type \underline{p} over A^* defining p . Then, $p \cdot_{\text{nf}(A^*)} p^{-1} \subseteq \text{St}(W)$.*

Proof. Take $(a_i^*)_{i \in \omega}$ such that $a_i^* \models \widehat{p}|_{A^*, a_0^*, \dots, a_{i-1}^*}$. Then, $(a_i)_{i \in \omega}$ is an A^* -indiscernible sequence of realisations of p with $\text{tp}(a_1/A^*, a_0)$ non-forking. As μ is invariant under left translations, $a_0^{-1}W$ is wide. Since $p^{-1}W$ is A^* -medium, we conclude that $a_0^{-1}W \cap a_1^{-1}W \notin \mu$. Thus, by invariance under left translations, $a_0 a_1^{-1} \in \text{St}(W)$. By Lemma 3.4.3(2) and Example 3.5.2, we conclude that $p \cdot_{\text{nf}(A^*)} p^{-1} \subseteq \text{St}(W)$. Q.E.D.

For two subsets X and Y of a group G , recall that $[X : Y] := \min\{|\Delta| : X \subseteq \Delta Y\}$. Note that $[X^{-1} : Y^{-1}] = \min\{|\Delta| : X \subseteq Y \Delta\}$.

Lemma 3.5.7. *Let G be a piecewise A^* -hyperdefinable group and $S \leq G$ an \bigwedge_{A^*} -definable subgroup. Let $p \subseteq G$ be an A^* -type such that there is an A^* -invariant global type \widehat{p} extending a partial type \underline{p} defining p over A^* .*

- (1) *If $[p : S]$ is small, then $[p : S] = 1$ and $p^{-1} \cdot p \subseteq S$.*
- (2) *If $[p^{-1} : S]$ is small, then $[p^{-1} : S] = 1$ and $p \cdot p^{-1} \subseteq S$.*

Proof. (1) Let $(a_i)_{i \in \alpha}$ be a sequence in p such that $\alpha = [p : S]$ and $a_j S \cap a_i S = \emptyset$ for $i \neq j$, and $p \subseteq \bigcup_{i \in \alpha} a_i S$. Suppose that there is no $i \in \alpha$ such that $\overline{(a_i S) \cap p}$ is contained in \widehat{p} . Then, there are formulas $\varphi_i \in \overline{(a_i S) \cap p}$ for each $i \in \alpha$ such that $\{\neg \varphi_i\}_{i \in \alpha} \subseteq \widehat{p}$. In particular, $\{\neg \varphi_i\}_{i \in \alpha} \cup \underline{p}$ is finitely satisfiable. Thus, by saturation, there is $c \in p$ such that $c \notin \bigcup_{i \in \alpha} a_i S$, getting a contradiction. That concludes that there is $i \in \alpha$ such that $\overline{(a_i S) \cap p} \subseteq \widehat{p}$. Since $a_i S \cap a_j S = \emptyset$ for any $i \neq j$, we conclude that there is just one such $i \in \alpha$. Since \widehat{p} and S are A^* -invariant, we get that $a_i S$ is A^* -invariant too. Indeed, take $\sigma \in \text{Aut}(\mathfrak{M}/A^*)$ arbitrary. Then, $\sigma[\overline{(a_i S) \cap p}] \subseteq \widehat{p}$ defines $(\sigma(a_i)S) \cap p$, concluding that $\sigma(a_i)S = a_i S$. As σ is arbitrary, we conclude that $a_i S$ is A^* -invariant. Thus, $a_i S$ is \bigwedge_{A^*} -definable by Corollary 1.1.7. As $(a_i S) \cap p \neq \emptyset$, by minimality of p , $p \subseteq a_i S$. So, $[p : S] = 1$ and $p^{-1} \cdot p \subseteq S$.

(2) Analogous to point (1), but now working with right cosets — as $[p^{-1} : S] = \min\{|\Delta| : p \subseteq S \Delta\}$. Q.E.D.

Lemma 3.5.8. *Let G be a piecewise A -hyperdefinable group, $S \leq G$ an \bigwedge_A -definable subgroup and $V \subseteq G$ an \bigwedge_A -definable set. Suppose $[V : S]$ is not small. Then, there is an A -indiscernible sequence $(a_i)_{i \in \omega}$ in V such that $a_i \cdot S \cap a_j \cdot S = \emptyset$ for $i \neq j$.*

Proof. Let \mathbf{G} , \mathbf{S} and \mathbf{V} be the interpretations in the monster model \mathfrak{C} . Since \mathfrak{C} is the monster model, $[\mathbf{V} : \mathbf{S}] \notin \text{On}$. Recall that there is $\tau \in \text{On}$ depending only on $|A^*|$ such that, for any sequence $(a_i^*)_{i \in \tau}$ of elements in \mathfrak{C} , there is an A^* -indiscernible sequence $(b_j^*)_{j \in \omega}$ of elements in \mathfrak{C} with the property that, for any $j_1 < \dots < j_n$,

$\text{tp}(a_{i_1}^*, \dots, a_{i_n}^*/A^*) = \text{tp}(b_{j_1}^*, \dots, b_{j_n}^*/A^*)$ for some $i_1 < \dots < i_n$ — see [TZ12, Lemma 7.2.12]. In particular, take $(a_i)_{i \in \tau}$ a sequence of hyperimaginaries of \mathbf{V} such that $a_i \cdot \mathbf{S} \cap a_j \cdot \mathbf{S} = \emptyset$ for each $i \neq j$. Let $(a_i^*)_{i \in \tau}$ be representatives of $(a_i)_{i \in \tau}$. Then, there is a sequence $(\tilde{b}_j^*)_{j \in \omega}$ of A^* -indiscernible elements such that $\text{tp}(\tilde{b}_0^*, \tilde{b}_1^*/A^*) = \text{tp}(a_{j'}^*, a_{j''}^*/A^*)$ for some $j' < j''$. Now, by κ -saturation of \mathfrak{M} , we can find $(b_j^*)_{j \in \omega}$ elements in \mathfrak{M} realising the same type that $(\tilde{b}_j^*)_{j \in \omega}$ over A^* . So, the projections $b_j \in V$ form an A -indiscernible sequence $(b_j)_{j \in \omega}$ in V such that $b_i \cdot S \cap b_j \cdot S = \emptyset$ for $i \neq j$. Q.E.D.

We now prove the Stabilizer Theorem for piecewise hyperdefinable groups. Below, Corollary 3.5.10 corresponds to [MOS18, Theorem 2.12 (B2)]; Theorem 3.5.11 corresponds to [Hru11, Theorem 3.5] and [MOS18, Theorem 2.12 (B1)]; and Theorem 3.5.11(c) corresponds to [MOS18, Proposition 2.14].

Theorem 3.5.9. *Let $\mathfrak{N} \prec \mathfrak{M}$ with $|N| < \lambda$. Let G be a piecewise N -hyperdefinable group and μ a locally atomic N -invariant ideal of $\bigwedge_{< \lambda}$ -definable subsets invariant under left translations. Let $p \subseteq G$ be an N -medium N -type. Assume that there is a wide N -type $q \subseteq \text{St}(p)$ such that $p^{-1} \cdot q$ is N -medium. Then, $\text{Stab}(p) = \text{Stab}(q) = \text{St}(p)^2 = \text{St}(q)^4 = (p \cdot p^{-1})^2$ is a wide \bigwedge_N -definable subgroup of G without proper \bigwedge_N -definable subgroups of small index. Furthermore:*

- (a) $\text{Stab}(q) \cdot_{\text{nf}(N)} q \subseteq \text{St}(p)$.
- (b) Every strictly N -medium N -type of $\text{Stab}(p)$ lies in $\text{St}(p)$.

Proof. We take \hat{p} and \hat{q} coheirs of p and q . By Lemma 3.2.4, q is N -medium. By Lemma 3.5.6, $p \cdot_{\text{nf}(N)} p^{-1} \subseteq \text{St}(q)$. Then, $\overline{\text{St}(p)} \subseteq p \cdot p^{-1} \subseteq \text{St}(q)^2$. Indeed, for any $a, b \in p$ we can find $c \in p$ such that $\text{tp}(c/N, a, b)$ does not fork over N — simply, take c^* realising $\hat{p}|_{N, a^*, b^*}$. Thus, $ac^{-1}, bc^{-1} \in \text{St}(q)$, concluding that $ab^{-1} \in \text{St}(q)^2$ by Lemma 3.5.1(2).

Since $q \subseteq \text{St}(p)$, $\text{Stab}(q) \cdot_{\text{nf}(N)} q \subseteq \text{St}(p)$ by Lemma 3.5.4 and Example 3.5.2. In particular, $\text{Stab}(q) \subseteq \text{St}(p)^2$ by Lemma 3.5.1(2). Thus, $\text{Stab}(p) = \text{Stab}(q) = \text{St}(p)^2 = (p \cdot p^{-1})^2 = \text{St}(q)^4$ is an \bigwedge_N -definable subgroup. Since $q \subseteq \text{St}(p) \subseteq \text{Stab}(p)$, we have that $\text{Stab}(p)$ is wide.

Take an \bigwedge_N -definable subgroup $T \leq \text{Stab}(p)$ such that $[\text{Stab}(p) : T]$ is small. Since $p \cdot p^{-1} \subseteq \text{Stab}(p)$, $[p^{-1} : T]$ is also small. By Lemma 3.5.7(2), $p \cdot p^{-1} \subseteq T$. Therefore, $\text{Stab}(p) = (p \cdot p^{-1})^2 \subseteq T$. In other words, $\text{Stab}(p)$ does not have proper \bigwedge_N -definable subgroups of small index.

Finally, we prove property (b). Take $r \subset \text{Stab}(p)$ a strictly N -medium N -type. Set $c \in q$. Since μ is locally atomic, there is $b \in r$ such that $\text{tp}(b/N, c) \notin \mu$. By Lemma 3.2.3, $\text{tp}(b/N, c)$ does not fork over N . Then, by Lemma 3.1.6 and Lemma 1.1.12, $\text{tp}(b^{-1}c^{-1}/N, c)$ does not fork over N . Since $c, b \in \text{Stab}(q)$, we have $b^{-1}c^{-1} \in \text{Stab}(q)$. Write $r' = \text{tp}(b^{-1}c^{-1}/N) \subseteq \text{Stab}(q)$. By (a), $r' \cdot_{\text{nf}(N)} q \subseteq \text{St}(p)$. By Lemma 3.4.3 and Example 3.5.2, we conclude that $b^{-1} = b^{-1} \cdot c^{-1} \cdot c \in \text{St}(p)$, so $b \in \text{St}(p)$. Since r is a type over N , we conclude $r \subseteq \text{St}(p)$. Q.E.D.

Corollary 3.5.10. *Let $\mathfrak{N} \prec \mathfrak{M}$ with $|N| < \lambda$ and G a piecewise N -hyperdefinable group. Let μ be an N -invariant locally atomic ideal of $\bigwedge_{<\lambda}$ -definable subsets of G invariant under left and right translations. Let p be a wide type over N and assume that $p^{-1} \cdot p \cdot p^{-1}$ is N -medium. Then, $\text{Stab}(p) = \text{St}(p)^2 = (p \cdot p^{-1})^2$ is a wide \bigwedge_N -definable subgroup of G without proper \bigwedge_N -definable subgroups of small index such that every strictly N -medium N -type of $\text{Stab}(p)$ lies in $\text{St}(p)$.*

Proof. Take $a \in p$ arbitrary. By the local atomic property, there is $b \in p$ such that $\text{tp}(b/N, a)$ is wide. By Lemma 3.2.4, $p \cdot p^{-1}$ is N -medium. Again by Lemma 3.2.4 but now using invariance under right translations, we get that p is N -medium. By Lemma 3.2.3, it follows that $\text{tp}(b/N, a)$ does not fork over N . Consider the N -type $q = \text{tp}(ba^{-1}/N) \subseteq p_{\text{nf}(N)} \cdot p^{-1}$.

By invariance under right translations, we know that q is a wide \bigwedge_N -definable set. As $q \subseteq p \cdot p^{-1}$ with $p \cdot p^{-1}$ N -medium, we have that q is N -medium too. Also, note that $p^{-1} \cdot q \subseteq p^{-1} \cdot p \cdot p^{-1}$, so $p^{-1} \cdot q$ is N -medium too. On the other hand, by Lemma 3.2.4 using invariance under right translations, $p^{-1} \cdot p$ is N -medium, so $p \cdot_{\text{nf}(N)} p^{-1} \subseteq \text{St}(p)$ by Lemma 3.5.6. By Example 3.5.2 and Lemma 3.4.3(3), $p_{\text{nf}(N)} \cdot p^{-1} \subseteq \text{St}(p)$ too, so $q \subseteq \text{St}(p)$. By Theorem 3.5.9, we conclude that $\text{Stab}(p) = \text{St}(p)^2 = (p \cdot p^{-1})^2$ is a wide \bigwedge_N -definable subgroup of G without proper \bigwedge_N -definable subgroups of small index such that every strictly N -medium N -type of $\text{Stab}(p)$ lies in $\text{St}(p)$. Q.E.D.

Theorem 3.5.11 (Stabilizer Theorem). *Let $\mathfrak{N} \prec \mathfrak{M}$ with $|N| < \lambda$ and $G = \varinjlim X^n$ a piecewise N -hyperdefinable group generated by a symmetric \bigwedge_N -definable subset X . Let μ be an N -invariant locally atomic ideal of $\bigwedge_{<\lambda}$ -definable subsets of G invariant under left translations. Let $p \subseteq X$ be a wide type over N and assume that X^3 is N -medium. Then, $\text{Stab}(p) = \text{St}(p)^2 = (p \cdot p^{-1})^2$ is a wide and N -medium \bigwedge_N -definable normal subgroup of small index of G without proper \bigwedge_N -definable subgroups of small index. Furthermore:*

- (a) $p \cdot p \cdot p^{-1} = p \cdot \text{Stab}(p)$ is a left coset of $\text{Stab}(p)$.
- (b) Every wide N -type of $\text{Stab}(p)$ is contained in $\text{St}(p)$.
- (c) Assume μ is also invariant under right translations. Then, $p \cdot p^{-1} \cdot p = \text{Stab}(p) \cdot p$ is a right coset of $\text{Stab}(p)$.

Proof. Take $a \in p$ arbitrary. By the local atomic property, we find $b \in p$ such that $\text{tp}(b/N, a)$ is wide. By Lemma 3.2.4, X is N -medium. As $\text{tp}(b/N, a) \subseteq p \subseteq X$, we conclude that $\text{tp}(b/N, a)$ is N -medium, so $\text{tp}(b/N, a)$ does not fork over N by Lemma 3.2.3. Also, $\text{tp}(a^{-1}b/N, a) = a^{-1} \cdot \text{tp}(b/N, a)$ is wide by left invariance of μ , so $q = \text{tp}(a^{-1}b/N) \subseteq p^{-1} \cdot_{\text{nf}(N)} p$ is wide. By Lemma 3.2.4, X^2 is N -medium, so p^2 is N -medium too. Using any coheir extending $\underline{p^{-1}}$, we get $p^{-1} \cdot_{\text{nf}(N)} p \subseteq \text{St}(p)$ by Lemma 3.5.6. In particular, $q \subseteq \text{St}(p)$. As X is symmetric, $p^{-1} \cdot q \subseteq p^{-1} \cdot p^{-1} \cdot p \subseteq X^3$, so $p^{-1} \cdot q$ is N -medium. Thus, by Theorem 3.5.9, $\text{Stab}(p) = \text{Stab}(q) = \text{St}(p)^2 = \text{St}(q)^4 = (p \cdot p^{-1})^2$ is a wide \bigwedge_N -definable subgroup without proper \bigwedge_N -definable subgroups of small

index and every strictly N -medium type of $\text{Stab}(p)$ is contained in $\text{St}(p)$. Also, $\text{Stab}(q) \cdot_{\text{nf}(N)} q \subseteq \text{St}(p)$.

We show that it is a normal subgroup of small index. First of all, note that $[X^2 : \text{Stab}(p)]$ is small. Otherwise, by Lemma 3.5.8, there is an N -indiscernible sequence $(b_j)_{j \in \omega}$ in X^2 such that $b_i \cdot \text{Stab}(p) \cap b_j \cdot \text{Stab}(p) = \emptyset$ for $i \neq j$. Take $d \in p^{-1}$ arbitrary. As $p \cdot p^{-1} \subseteq \text{Stab}(p)$, we get $(b_i \cdot p \cdot d) \cap (b_j \cdot p \cdot d) = \emptyset$ for $i \neq j$, so $b_i \cdot p \cap b_j \cdot p = \emptyset$ for $i \neq j$. As X^3 is N -medium, that implies $b_0 \cdot p \in \mu$, so $p \in \mu$ by invariance under left translations, contradicting our hypotheses. For any $c \in X$, we have $[\text{tp}(c/N) : \text{Stab}(p)] = 1$ by Lemma 3.5.7(1), so $\text{tp}(c/N) \cdot \text{Stab}(p) \cdot \text{tp}(c/N)^{-1} = c \cdot \text{Stab}(p) \cdot c^{-1}$ is \bigwedge_N -definable. As X is symmetric, we also get that $[p^{-1} : c \cdot \text{Stab}(p) \cdot c^{-1}] = [p^{-1} \cdot c : \text{Stab}(p)] \leq [X^2 : \text{Stab}(p)]$ is small, so $p \cdot p^{-1} \subseteq c \cdot \text{Stab}(p) \cdot c^{-1}$ by Lemma 3.5.7(2). Therefore, $\text{Stab}(p) \subseteq c \cdot \text{Stab}(p) \cdot c^{-1}$. As $c \in X$ is arbitrary and X is symmetric, $\text{Stab}(p) = c \cdot \text{Stab}(p) \cdot c^{-1}$. Thus, $X \subseteq N_G(\text{Stab}(p))$, concluding $\text{Stab}(p) \trianglelefteq G$. Since we have proved that $[X : \text{Stab}(p)]$ is small, we conclude that $\text{Stab}(p)$ has small index by normality.

We show now property **(a)**, i.e. $p \cdot p \cdot p^{-1} = p \cdot \text{Stab}(p) = y \cdot \text{Stab}(p)$ for any $y \in p$. As $(p \cdot p^{-1})^2 = \text{Stab}(p)$, we have $p \cdot p \cdot p^{-1} \subseteq p \cdot \text{Stab}(p)$. As $[p : \text{Stab}(p)] \leq [X : \text{Stab}(p)]$ is small, we get $[p : \text{Stab}(p)] = 1$ by Lemma 3.5.7(1). Thus, $p \cdot \text{Stab}(p) = y \cdot \text{Stab}(p)$. On the other hand, take $x \in y \cdot \text{Stab}(p)$ arbitrary, so $x^{-1} = c \cdot y^{-1}$ with $c \in \text{Stab}(p) = \text{Stab}(q)$. By definition of q , we know that $q = \text{tp}(a^{-1}b/N)$ with $\text{tp}(b/N, a)$ wide and $a, b \in p$. Take z_0 such that $\text{tp}(z_0, y/N) = \text{tp}(b, a/N)$. By N -invariance of μ , $\text{tp}(z_0/N, y)$ is wide. By the local atomic property, we can find $z \in \text{tp}(z_0/N, y)$ such that $\text{tp}(z/N, y, c)$ is wide. Thus, $\text{tp}(y^{-1}z/N, c)$ is wide by invariance under left translations. By Lemma 3.2.3, $\text{tp}(y^{-1}z/N, c)$ does not fork over N . Thus, $x^{-1} \cdot z = c \cdot y^{-1}z \in \text{Stab}(q) \cdot_{\text{nf}(N)} q \subseteq \text{St}(p)$. In particular, we conclude $x^{-1} \in \text{St}(p) \cdot p^{-1} \subseteq p \cdot p^{-1} \cdot p^{-1}$, so $x \in p \cdot p \cdot p^{-1}$.

As X is symmetric, $p \cdot \text{Stab}(p) \subseteq X^3$, so $p \cdot \text{Stab}(p)$ is N -medium. By Lemma 3.2.4, we conclude that $\text{Stab}(p)$ is N -medium. Thus, we conclude property **(b)**.

Finally, it remains to prove property **(c)**. In other words, assuming that μ is also invariant under right translations, we want to show that $p \cdot p^{-1} \cdot p = \text{Stab}(p) \cdot p = \text{Stab}(p) \cdot y$ for any $y \in p$. Take $a \in p$ arbitrary and, using the local atomic property, find $b \in p$ such that $\text{tp}(b/N, a)$ is wide. As p is N -medium, by Lemma 3.2.3, $\text{tp}(b/N, a)$ does not fork over N . Consider $q_2 = \text{tp}(ba^{-1}/N) \subseteq p \cdot_{\text{nf}(N)} p^{-1}$. Then, by invariance under right translations, q_2 is a wide \bigwedge_N -definable set. As $q_2 \subseteq X^2$ and $p^{-1} \cdot q_2 \subseteq X^3$, we conclude that q_2 and $p^{-1} \cdot q_2$ are N -medium. As $p \cdot_{\text{nf}(N)} p^{-1} \subseteq \text{St}(p)$ by Lemma 3.5.4, we get using Example 3.5.2 and Lemma 3.4.3(3) that $p \cdot_{\text{nf}(N)} p^{-1} \subseteq \text{St}(p)$ too, so $q_2 \subseteq \text{St}(p)$. By Theorem 3.5.9, we conclude that $\text{Stab}(p) = \text{Stab}(q_2)$ and $\text{Stab}(q_2) \cdot_{\text{nf}(N)} q_2 \subseteq \text{St}(p)$. Take $x \in \text{Stab}(p) \cdot y$ arbitrary, so $x = cy$ with $c \in \text{Stab}(p) = \text{Stab}(q_2)$ and $y \in p$. Find z_0 such that $\text{tp}(z_0, y/N) = \text{tp}(a, b/N)$, so $\text{tp}(y/N, z_0)$ is wide. Using the local atomic property, we may find $y_1 \in \text{tp}(y/N, z_0)$ such that $\text{tp}(y_1/N, z_0, c)$ is wide. Take z such that $\text{tp}(z, y/N, c) = \text{tp}(z_0, y_1/N, c)$. Thus, $yz^{-1} \in q_2$ and $\text{tp}(y/N, z, c)$ is wide. In particular, as X^2 is N -medium, by Lemma 3.2.3, $\text{tp}(yz^{-1}/N, c)$ does not fork over N . Thus, $x \cdot z^{-1} = c \cdot yz^{-1} \in \text{Stab}(q_2) \cdot_{\text{nf}(N)} q_2 \subseteq \text{St}(p)$, concluding $x \in p \cdot p^{-1} \cdot p$. As x is arbitrary, $\text{Stab}(p) \cdot y \subseteq p \cdot p^{-1} \cdot p$. As $p \cdot p^{-1} \cdot p \subseteq \text{Stab}(p) \cdot y$, we get

$p \cdot p^{-1} \cdot p = \text{Stab}(p) \cdot p = \text{Stab}(p) \cdot y$ for any $y \in p$. Q.E.D.

Remarks 3.5.12. (1) The main improvement of Theorem 3.5.11 with respect to the original formulations studied in [Hru11] and [MOS18] is that we do not require invariance of μ under right translations. In both papers, while they preferably considered ideals invariant under two-sided translations, the authors already anticipated that it should be possible to weaken the invariance under right translations hypothesis, perhaps losing a few properties of $\text{Stab}(p)$. We have shown that, in fact, it can be completely eliminated without significant consequences. Indeed, we only use invariance under right translations for **(c)**, but this is essentially replaced by **(a)**.

(2) Although in [MOS18, Theorem 2.12] it was assumed that X^4 is N -medium, we only need N -mediumness of X^3 .

(3) Note that the locally atomic property is mostly used only for subsets of p . The only time when we apply the locally atomic property for other subsets is in the proof of Theorem 3.5.9 to get property **(b)**. Thus, when μ only satisfies the locally atomic property for subsets of p , we only miss property **(b)**.

For property **(b)**, we have used the locally atomic property for subsets of $\text{Stab}(p) = (p \cdot p^{-1})^2$. However, in fact, it suffices to assume it only for subsets of $p \cdot p \cdot p^{-1}$. Indeed, by **(a)** and invariance under left translations, if $V \subseteq \text{Stab}(p)$ is \bigwedge_B -definable and wide, then $y \cdot V \subseteq p \cdot p \cdot p^{-1}$ is $\bigwedge_{B,y}$ -definable and wide for any $y \in p$. By the locally atomic property on $p \cdot p \cdot p^{-1}$, there is $b \in V$ such that $\text{tp}(yb/B, y)$ is wide, concluding that $\text{tp}(b/B)$ is wide by invariance under left translations.

(4) It suffices to have μ defined only on X^3 , as, in that case, we may extend the ideal by taking the one generated by the finite unions of left translates of elements of μ . As μ is invariant under left-translations within X^3 (i.e. $V \in \mu$ if and only if $gV \in \mu$ for any $V \subseteq X^3$ and $g \in G$ such that $gV \subseteq X^3$), this extension coincides with μ inside of X^3 , so we can apply the Stabilizer Theorem 3.5.11 using Remark 3.5.12(3).

Proposition 3.5.13. [Hru11, Corollary 3.11] & [MOS18, Proposition 2.13]

Let $\mathfrak{N} \prec \mathfrak{M}$ with $|N| < \lambda$ and $G = \varinjlim X^n$ a piecewise N -hyperdefinable group generated by a symmetric \bigwedge_N -definable subset X . Let μ be an N -invariant locally atomic ideal of $\bigwedge_{<\lambda}$ -definable subsets of G invariant under left translations such that X is wide and X^3 is N -medium. Then, μ is N -medium on G .

Proof. By the locally atomic property, there is a wide N -type $p \subseteq X$. By the Stabilizer Theorem 3.5.11, $S := \text{Stab}(p) = (p \cdot p^{-1})^2$ is an N -medium normal subgroup of small index. Let $Y(b) \subseteq G$ be $\bigwedge_{N,\bar{b}}$ -definable with $|b| < \lambda$ small. Let $(b_i)_{i \in \omega}$ be an N -indiscernible sequence with $\text{tp}(\bar{b}/N) = \text{tp}(b_i/N)$. Suppose that $Y(b) \notin \mu$. By the locally atomic property of μ , there is a wide N, b -type $q(b) \subseteq Y(b)$. By Lemma 3.5.7(1), there is a such that $q(b) \subseteq a \cdot S$. Let $(b'_i)_{i \in \omega}$ be an N, a^* -indiscernible sequence realising $\text{tp}(\bar{b}_i/N)$. By invariance under left translations, $a^{-1}q(b) \notin \mu$. As S is N -medium, $a^{-1}q(b'_0) \cap a^{-1}q(b'_1) \notin \mu$. Thus, by N -invariance and invariance under left translations, we get that $q(b_0) \cap q(b_1) \notin \mu$, concluding $Y(b_0) \cap Y(b_1) \notin \mu$. Q.E.D.

3.6 Near-subgroups

Let G be a piecewise A -hyperdefinable group and μ an ideal of $\bigwedge_{<\lambda}$ -definable subsets invariant under left translations with $\kappa \geq \lambda > |A| + |\mathbb{L}|$. A μ -near-subgroup of G over A is a wide \bigwedge_A -definable symmetric set X generating G such that $\mu|_{X^3}$ is locally atomic and A -medium. We say that X is a near-subgroup if it is a μ -near-subgroup for some ideal.

Theorem 3.6.1. *Let G be a piecewise A -hyperdefinable group, μ an ideal of $\bigwedge_{<\lambda}$ -definable subsets and X a μ -near-subgroup over A . Then, there is a wide $\bigwedge_{|A|+|\mathbb{L}|}$ -definable normal subgroup of small index S contained in X^4 and contained in every \bigwedge_A -definable subgroup of small index. Furthermore, $S = (p \cdot p^{-1})^2$ and $ppp^{-1} = pS = yS$ for any $y \in p$, where $p \subseteq V$ is a wide type over an elementary substructure of size $|A| + |\mathbb{L}|$.*

Proof. By Löwenheim-Skolem Theorem [TZ12, Theorem 2.3.1], we find an elementary substructure $\mathfrak{N} \preceq \mathfrak{M}$ containing A^* with $|N| = |A^*| + |\mathbb{L}| < \lambda$. As $A^* \subseteq N$, $\mu|_{X^3}$ is N -medium. By the locally atomic property, there is a wide N -type $p \subseteq X$. Applying the Stabilizer Theorem 3.5.11 (and Remark 3.5.12(3)), we conclude that there is a wide \bigwedge_N -definable normal subgroup $S \trianglelefteq G$ of small index which does not have proper \bigwedge_N -definable subgroups of small index. Furthermore, $S = \text{Stab}(p) = (p \cdot p^{-1})^2 \subseteq X^4$ and $ppp^{-1} = pS = yS$ for any $y \in p$. As $A^* \subseteq N$, we conclude. Q.E.D.

Theorem 3.6.2. *Let $G = \varinjlim X^n$ be a piecewise A -hyperdefinable group generated by a near-subgroup X over A . Assume that X^n is an approximate subgroup for some $n \in \mathbb{N}$. Then, G has a connected Lie model $\pi_{H/K} : H \rightarrow L = H/K$ with $K \subseteq X^{2n+4}$ such that*

- (1) $H \cap X^{2n}$ is $\bigwedge_{|A|+|\mathbb{L}|}$ -definable and commensurable to X^n ,
- (2) $\pi_{H/K}[H \cap X^{2n}]$ is a compact neighbourhood of the identity in L ,
- (3) H is generated by $H \cap X^{2n+4}$, and
- (4) $\pi_{H/K}$ is continuous and proper from the logic topology using $|A| + |\mathbb{L}|$ many parameters.

Proof. As X is a near-subgroup, by Theorem 3.6.1, $G_N^{00} \subseteq X^4$ with $|N| \leq |A| + |\mathbb{L}|$. As X^n is an approximate subgroup, X^n/G_N^{00} is an approximate subgroup. Thus, X^{2n}/G_N^{00} is a neighbourhood of the identity by the Generic Set Lemma (Theorem 2.2.9). Then, by Theorem 2.4.2, we can find a connected Lie core $\pi_{H/K} : H \rightarrow L = H/K$ with $K \subseteq X^{2n}G_N^{00} \subseteq X^{2n+4}$.

(1) Since X^n/G_A^{00} is compact and H/G_A^{00} is an open subgroup in the global logic topology, we conclude that $[X^n/G_A^{00} : H/G_A^{00}]$ is finite. As $G_A^{00} \leq H$, we get that

$[X^n : H]$ is finite. Thus, by [Mac21, Lemma 2.2, Lemma 2.3], $H \cap X^{2n}$ is an approximate subgroup commensurable to X^n .

(2) As X^{2n}/G_N^{00} is a compact neighbourhood of the identity and H/G_N^{00} is an open subgroup in the global logic topology, we conclude that $H \cap X^{2n}/G_N^{00}$ is a compact neighbourhood of the identity. Recall that $\tilde{\pi}_{H/K} : H/G_N^{00} \rightarrow L$ given by $\pi_{H/K} = \tilde{\pi}_{H/K} \circ \pi_{G/G_N^{00}}$ is a continuous, closed, open and proper onto group homomorphism. Thus, we conclude that $\pi_{H/K}[H \cap X^{2n}]$ is a compact neighbourhood of the identity.

(3) As L is connected, $\pi_{H/K}[H \cap X^{2n}]$ generates L . Therefore, we have that $\pi_{H/K}^{-1}[\pi_{H/K}[H \cap X^{2n}]] = (H \cap X^{2n}) \cdot K$ generates H . Now, $K \subseteq H \cap X^{2n+4}$, so $(H \cap X^{2n}) \cdot K \subseteq (H \cap X^{2n+4})^2$, concluding that $H \cap X^{2n+4}$ generates H .

(4) By Proposition 1.6.9, the global logic topology of G/G_N^{00} is given using $|A| + |\mathbb{L}|$ many parameters, so $H \cap X^{2n}$ is $\bigwedge_{|A|+|\mathbb{L}|}$ -definable and $\pi_{H/K}$ is continuous and closed using $|A| + |\mathbb{L}|$ many parameters. Q.E.D.

Alternatively, using Gleason-Yamabe-Carolino Theorem 2.1.4 rather than Gleason-Yamabe Theorem 2.1.3, we get the following variation which provides some extra control over some of the parameters.

Theorem 3.6.3. *There are functions $c : \mathbb{N} \rightarrow \mathbb{N}$ and $d : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds for any κ -saturated \mathbb{L} -structure \mathfrak{M} and any set of parameters A with $\kappa > |\mathbb{L}| + |A|$:*

Let $G = \varinjlim X^n$ be a piecewise A -hyperdefinable group generated by a near-subgroup X over A . Assume that X^n is a k -approximate subgroup for some n and k . Then, G has a Lie model $\pi_{H/K} : H \rightarrow L = H/K$ with $K \subseteq X^{12n+4}$ and $\dim(L) \leq d(k)$ such that

- (1) $H \cap X^{2n}$ is $\bigwedge_{|A|+|\mathbb{L}|}$ -definable and $c(k)$ -commensurable to X^n ,
- (2) $\pi_{H/K}[H \cap X^{2n}]$ is a compact neighbourhood of the identity in L ,
- (3) H is generated by $H \cap X^{12n+4}$, and
- (4) $\pi_{H/K}$ is continuous and proper from the logic topology using $|A| + |\mathbb{L}|$ many parameters.

Proof. As X is a near-subgroup, by Theorem 3.6.1, $G_N^{00} \subseteq X^4$ with $|N| \leq |A| + |\mathbb{L}|$. As X^n is a k -approximate subgroup, X^n/G_N^{00} is an approximate subgroup. Thus, X^{2n}/G_N^{00} is a neighbourhood of the identity by the Generic Set Lemma (Theorem 2.2.9), and, in particular, G/G_N^{00} is a locally compact topological group with the global logic topology. Thus, X^n/G_N^{00} is contained in the interior U/G_N^{00} of X^{3n}/G_N^{00} in the global logic topology. Now, we obviously have that $U^2/G_N^{00} \subseteq X^{6n}/G_N^{00}$, which is covered by k^5 left translates of $X^n/G_N^{00} \subseteq U/G_N^{00}$. Hence, U/G_N^{00} is an open precompact k^5 -approximate subgroup.

Let c_0 and d_0 be the functions provided by Gleason-Yamabe-Carolino Theorem 2.1.4. Applying Theorem 2.1.4, we get a Lie model $\pi_{H/K} : H \rightarrow L = H/K$ with $G_N^{00} \leq K \subseteq U^4 \subseteq X^{12n}G_N^{00} \subseteq X^{12n+4}$ and $\dim(L) \leq d(k) := d_0(k^5)$ such that $H \cap U^4$ generates H and is $c_0(k^5)$ -commensurable to U . Thus, $H \cap X^{12n+4}$ generates H and $c_0(k^5)$ left translates of $H \cap X^{12n+4}$ cover X^n . As X^n is a k -approximate subgroup, k^{15} many left translates of X^n cover $X^{16n} \supseteq H \cap X^{12n+4}$. By [Mac21, Lemma 2.2], we get that k^{15} many left translates of $H \cap X^{2n}$ cover $H \cap X^{12n+4}$, so $H \cap X^{2n}$ and X^n are $c(k)$ -commensurable, where $c(k) := c_0(k^5)k^{15}$.

Now, $H \cap X^{2n}/G_N^{00}$ is a compact neighbourhood of the identity in the global logic topology. Recall that $\tilde{\pi}_{H/K} : H/G_N^{00} \rightarrow L$ given by $\pi_{H/K} = \tilde{\pi}_{H/K} \circ \pi_{G/G_N^{00}}$ is a continuous, closed, open and proper onto group homomorphism. Thus, $\pi_{H/K}[H \cap X^{2n}]$ is a compact neighbourhood of the identity of L .

By Proposition 1.6.9, the global logic topology of G/G_N^{00} is given using $|A| + |L|$ many parameters, so $H \cap X^{2n}$ is $\bigwedge_{|A|+|L|}$ -definable and $\pi_{H/K}$ is continuous and proper using $|A| + |L|$ many parameters. Q.E.D.

We conclude applying Theorem 3.6.2 to the case of rough approximate subgroups. Recall that, for a subset $X \subseteq G$ of a group G , the T -thickening of X by a subset $T \subseteq G$ containing the identity is the set XT . We say that two subsets are T -rough k -commensurable if k many left translates of the T -thickening of each one suffice to cover the other. We say that a symmetric subset of a group is a T -rough k -approximate subgroup if it is T -rough k -commensurable to its set of pairwise products.

Let G be a group and $X \subseteq G$ a symmetric subset containing the identity. For some fixed n , assume that X^n is a T_i -rough k -approximate subgroup for a sequence $(T_i)_{i \in \mathbb{N}}$ of thickenings decreasing in doubling scales, i.e. $T_{i+1}T_{i+1}^{-1} \subseteq T_i$ for each $i \in \mathbb{N}$, and asymptotically normalised by X , i.e. $x^{-1}T_{i+1}x \subseteq T_i$ for each $x \in X$ and $i \in \mathbb{N}$.

Assuming saturation and \bigwedge -definability, we have that $T = \bigcap T_i$ is an \bigwedge -definable subgroup of G normalised by X and X^n is a T -rough k -approximate subgroup.

Theorem 3.6.4 (Rough Lie Model Theorem). *Let G be an A -definable group, $T \leq G$ an \bigwedge_A -definable subgroup and $X \subseteq G$ a symmetric \bigwedge_A -definable subset. Write \tilde{G} for the subgroup generated by X . Assume that X normalises T , X/T is a near-subgroup of \tilde{G}/T and X^n is a T -rough approximate subgroup for some n . Then, $\tilde{G}T \leq G$ has a connected Lie model $\pi_{H/K} : H \rightarrow L = H/K$ with $T \subseteq K \subseteq X^{2n+4}T$ such that*

- (1) $H \cap X^{2n}$ is $\bigwedge_{|A|+|L|}$ -definable and T -rough commensurable to X^n ,
- (2) $\pi_{H/K}[H \cap X^{2n}]$ is a compact neighbourhood of the identity in L ,
- (3) H is generated by $H \cap X^{2n+4}T$, and

(4) $\pi_{H/K}$ is continuous and closed from the logic topology using $|A| + |\mathbb{L}|$ many parameters.

Alternatively, using Theorem 3.6.3:

Theorem 3.6.5 (Rough Lie Model Theorem, version 2). *There are functions $c : \mathbb{N} \rightarrow \mathbb{N}$ and $d : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds for any κ -saturated \mathbb{L} -structure \mathfrak{M} and any set of parameters A with $\kappa > |\mathbb{L}| + |A|$:*

Let G be an A -definable group, $T \leq G$ an \bigwedge_A -definable subgroup and $X \subseteq G$ a symmetric \bigwedge_A -definable subset. Write \tilde{G} for the subgroup generated by X . Assume that X normalises T , X/T is a near-subgroup of \tilde{G}/T and X^n is a T -rough k -approximate subgroup for some n and k . Then, $\tilde{G}T \leq G$ has a Lie model $\pi_{H/K} : H \rightarrow L = H/K$ with $T \subseteq K \subseteq X^{12n+4}T$ and $\dim(L) \leq d(k)$ such that

- (1) $H \cap X^{2n}$ is $\bigwedge_{|A|+|\mathbb{L}|}$ -definable and T -rough $c(k)$ -commensurable to X^n ,
- (2) $\pi_{H/K}[H \cap X^{2n}]$ is a compact neighbourhood of the identity in L ,
- (3) H is generated by $H \cap X^{12n+4}T$, and

(4) $\pi_{H/K}$ is continuous and proper from a logic topology using $|A| + |\mathbb{L}|$ many parameters.

Chapter 4

Metric approximate subgroups

In this chapter we apply our previous results to rough approximate subgroups and, in particular, to metric approximate subgroups, concluding with some combinatorial consequences.

We mostly follow the same notations and conventions indicated at the end of the introduction. The main ones concerning this chapter are the following:

■ The cardinality of a set X is denoted by $|X|$. The set of natural numbers with 0 is denoted by \mathbb{N} .

■ We use product notation for groups. Also, unless otherwise stated, we consider the group acting on itself on the left. In particular, by a coset we mean a left coset. A subset X of a group is called symmetric if $1 \in X = X^{-1}$. For subsets X and Y of a group, we write XY for the set of pairwise products, and abbreviate $X^n := XX^{n-1}$ and $X^{-n} := (X^{-1})^n$ for $n \in \mathbb{N}$. We say that X normalises Y if $x^{-1}Yx \subseteq Y$ for any $x \in X$. We write $[X, Y] := \{[x, y] : x \in X, y \in Y\}$ where $[x, y] := x^{-1}y^{-1}xy$.

■ Recall that, for $l \in \mathbb{R}_{\geq 0}$, an l -Lipschitz function $f : X \rightarrow Y$ between two metric spaces (X, d_X) and (Y, d_Y) is a function satisfying $d_Y(f(x), f(x')) \leq l \cdot d_X(x, x')$ for any $x, x' \in X$.

4.1 Discretisations

For $r \in \mathbb{R}_{\geq 0}$, we define the *(closed) r -distance relation* as $\overline{\mathbb{D}}_r := \{(x, y) : d(x, y) \leq r\}$. For a subset X , the *(closed) r -thickening of X* is the set $\overline{\mathbb{D}}_r(X) := \{y : d(x, y) \leq r \text{ for some } x \in X\}$. In particular, $\overline{\mathbb{D}}_r(x)$ is the closed ball of radius r at x .

Recall that a set Z is *r -separated* if $d(z, z') > r$ for every $z, z' \in Z$ with $z \neq z'$. The *r -discretisation number of X* is the maximum size $N_r(X)$ of a finite r -separated

subset of X — write $N_r(X) = \infty$ if there are infinite r -separated subsets of X . An r -discretisation of X is an r -separated finite subset $Z \subseteq X$ with $|Z| = N_r(X)$.

The *covering r -discretisation number of X relative to Y* is the minimum size $N_r^{\text{cov}}(X/Y)$ of a finite subset Z of Y such that $X \subseteq \overline{\mathbb{D}}_r(Z)$ — write $N_r^{\text{cov}}(X/Y) = \infty$ if there are no finite subsets $Z \subseteq Y$ with $X \subseteq \overline{\mathbb{D}}_r(Z)$. A *covering r -discretisation of X relative to Y* is a finite subset $Z \subseteq Y$ with $|Z| = N_r^{\text{cov}}(X/Y)$ such that $X \subseteq \overline{\mathbb{D}}_r(Z)$.

Discretisations and covering discretisations are not unique but their numbers are indeed unique. Thus, a natural way of measuring the discreteness of a set consists on studying its (covering) discretisation numbers. Our first remark is that, although the functions $N_r(X)$ and $N_r^{\text{cov}}(X/Y)$ are not measures, they still enjoy one of the main properties of measures. Recall that a function ν on sets is *subadditive* if $\nu(A \cup B) \leq \nu(A) + \nu(B)$ for any A and B .

Lemma 4.1.1. *The functions $N_r(X)$ and $N_r^{\text{cov}}(X/Y)$ are subadditive on X . Also, $N_r(X)$ is increasing on X and decreasing on r , and $N_r^{\text{cov}}(X/Y)$ is increasing on X and decreasing on r and Y . Obviously, $N_r(X) = 0$ if and only if $N_r^{\text{cov}}(X/Y) = 0$, if and only if $X = \emptyset$.*

Proof. Let $Z \subseteq X \cup X'$ be r -separated, then $|Z| \leq |Z \cap X| + |Z \cap X'| \leq N_r(X) + N_r(X')$. As Z is arbitrary, $N_r(X \cup X') \leq N_r(X) + N_r(X')$.

Let $Z, Z' \subseteq Y$ such that $X \subseteq \overline{\mathbb{D}}_r(Z)$ and $X' \subseteq \overline{\mathbb{D}}_r(Z')$. Then, $X \cup X' \subseteq \overline{\mathbb{D}}_r(Z \cup Z')$, so $N_r^{\text{cov}}(X \cup X'/Y) \leq |Z \cup Z'| \leq |Z| + |Z'|$. As Z and Z' are arbitrary, $N_r^{\text{cov}}(X \cup X'/Y) \leq N_r^{\text{cov}}(X/Y) + N_r^{\text{cov}}(X'/Y)$.

Let $Z \subseteq X$ be r -separated. Then, for $r' \leq r$, Z is r' -separated too. Therefore, $|Z| \leq N_{r'}(X)$. As Z is arbitrary, $N_r(X) \leq N_{r'}(X)$ for $r \geq r'$.

Let $Z \subseteq X$ be r -separated and $X \subseteq X'$. Then, $Z \subseteq X'$ is r -separated, so $|Z| \leq N_r(X')$. As Z is arbitrary, $N_r(X) \leq N_r(X')$.

Let $Z \subseteq Y$ such that $X \subseteq \overline{\mathbb{D}}_r(Z)$ and $r \leq r'$. Then, $X \subseteq \overline{\mathbb{D}}_{r'}(Z)$ too, so $N_{r'}^{\text{cov}}(X/Y) \leq |Z|$. As Z is arbitrary, we conclude that $N_{r'}^{\text{cov}}(X/Y) \leq N_r^{\text{cov}}(X/Y)$ for $r' \geq r$.

Let $Z \subseteq Y$ such that $X \subseteq \overline{\mathbb{D}}_r(Z)$ and $X' \subseteq X$. Then, $X' \subseteq \overline{\mathbb{D}}_r(Z)$, so $N_r^{\text{cov}}(X'/Y) \leq |Z|$. As Z is arbitrary, we conclude $N_r^{\text{cov}}(X'/Y) \leq N_r^{\text{cov}}(X/Y)$ when $X' \subseteq X$.

Let $Z \subseteq Y$ such that $X \subseteq \overline{\mathbb{D}}_r(Z)$ and $Y \subseteq Y'$. Then, $Z \subseteq Y'$, so $N_r^{\text{cov}}(X/Y') \leq |Z|$. As Z is arbitrary, we conclude $N_r^{\text{cov}}(X/Y') \leq N_r^{\text{cov}}(X/Y)$ when $Y' \supseteq Y$.

Obviously, $N_r(\emptyset) = 0$ for any r . On the other hand, for $X \neq \emptyset$, $x \in X$ is itself r -separated for any $r \in \mathbb{R}_{\geq 0}$, so $N_r(X) \geq 1$. Finally, $N_r^{\text{cov}}(X/Y) = 0$ if and only if $X \subseteq \overline{\mathbb{D}}_r(\emptyset) = \emptyset$, so $N_r^{\text{cov}}(X/Y) = 0$ if and only if $X = \emptyset$. Q.E.D.

Furthermore, they are continuous according to their monotonicity:

Lemma 4.1.2. For any X, X_1, X_2, \dots and any $r \in \mathbb{R}_{\geq 0}$,

$$N_r(X) = \sup_{\varepsilon > 0} N_{r+\varepsilon}(X) \text{ and } N_r(\bigcup^\infty X_n) = \sup_{m \in \mathbb{N}} N_r(\bigcup^m X_n).$$

For any X, X_1, X_2, \dots , any $r \in \mathbb{R}_{\geq 0}$ and any Y compact in the metric topology,

$$N_r^{\text{cov}}(X/Y) = \sup_{\varepsilon > 0} N_{r+\varepsilon}^{\text{cov}}(X/Y) \text{ and } N_r^{\text{cov}}(\bigcup^\infty X_n/Y) = \sup_{m \in \mathbb{N}} N_r^{\text{cov}}(\bigcup^m X_n/Y).$$

Proof. By monotonicity, we know that $N_{r_0}(X) \geq \sup_{r > r_0} N_r(X)$. Now, suppose that $N_{r_0}(X) \geq k$ and take an r_0 -separated set $Z \subseteq X$ with $|Z| \geq k$. Taking a subset, we can assume $|Z| = k$. There is then $r = \min\{d(z, z') : z, z' \in Z, z \neq z'\}$ and $r > r_0$. Hence, Z is $\frac{1}{2}(r+r_0)$ -separated with $\frac{1}{2}(r+r_0) > r_0$, concluding that $\sup_{r > r_0} N_r(X) \geq k$. As $k \in \mathbb{N}$ is arbitrary, we conclude that $N_{r_0}(X) = \sup_{r > r_0} N_r(X)$.

By monotonicity, we know that $N_r(\bigcup_{n \in \mathbb{N}} X_n) \geq \sup_{m \in \mathbb{N}} N_r(\bigcup_{n \leq m} X_n)$. Now, suppose that $N_r(\bigcup_{n \in \mathbb{N}} X_n) \geq k$ and take an r -separated set $Z \subseteq \bigcup_{n \in \mathbb{N}} X_n$ with $|Z| \geq k$. Taking a subset, we can assume that $|Z| = k$. There is then some $m \in \mathbb{N}$ large enough such that $Z \subseteq \bigcup_{n \leq m} X_n$, concluding that $\sup_{m \in \mathbb{N}} N_r(\bigcup_{n \leq m} X_n) \geq k$. As $k \in \mathbb{N}$ is arbitrary, we conclude that $N_r(\bigcup^\infty X_n) = \sup_{m \in \mathbb{N}} N_r(\bigcup^m X_n)$.

By monotonicity, we know that $N_{r_0}^{\text{cov}}(X/Y) \geq \sup_{r > r_0} N_r^{\text{cov}}(X/Y)$. Now, suppose that $\sup_{r > r_0} N_r^{\text{cov}}(X/Y) \leq k$ and assume Y is compact for the metric topology. For each $r > r_0$, let $\mathcal{Z}_r = \{\bar{z} \in Y^k : X \subseteq \overline{\mathbb{D}}_r(\{z_1, \dots, z_k\})\}$. Note that $\mathcal{Z}_r = \bigcap_{x \in X} \bigcup_{i=1}^k \{\bar{z} \in Y^k : d(x, z_i) \leq r\}$, so \mathcal{Z}_r is a closed subset of Y^k in the metric topology. As $\sup_{r > r_0} N_r^{\text{cov}}(X/Y) \leq k$, $\mathcal{Z}_r \neq \emptyset$ for every $r > r_0$. Also, $\mathcal{Z}_r \subseteq \mathcal{Z}_{r'}$ when $r \leq r'$. Therefore, by compactness of Y^k , there is $\bar{z} = (z_1, \dots, z_k) \in \bigcap_{r > r_0} \mathcal{Z}_r$. Set $Z = \{z_1, \dots, z_k\}$, so $|Z| \leq k$ and $Z \subseteq Y$. Now, $X \subseteq \overline{\mathbb{D}}_r(Z)$ for any $r > r_0$. Pick $x \in X$. As Z is finite, it is compact, so there is $z_0 \in Z$ such that $d(x, z_0) = d(x, Z)$. Hence, if $x \in X$, $d(x, z_0) = d(x, Z) \leq r$ for any $r > r_0$. Thus, $d(x, z_0) \leq r_0$, concluding $x \in \overline{\mathbb{D}}_{r_0}(Z)$. As $x \in X$ is arbitrary, we conclude that $X \subseteq \overline{\mathbb{D}}_{r_0}(Z)$, so $N_{r_0}^{\text{cov}}(X/Y) \leq |Z| \leq k$. As $k \in \mathbb{N}$ is arbitrary, we conclude that $N_{r_0}^{\text{cov}}(X/Y) = \sup_{r > r_0} N_r^{\text{cov}}(X/Y)$.

By monotonicity, we know that $N_r^{\text{cov}}(\bigcup_{n \in \mathbb{N}} X_n/Y) \geq \sup_{m \in \mathbb{N}} N_r^{\text{cov}}(\bigcup_{n \leq m} X_n/Y)$. Now, suppose that $\sup_{m \in \mathbb{N}} N_r^{\text{cov}}(\bigcup_{n \leq m} X_n/Y) \leq k$ and assume Y is compact for the metric topology. For each $m \in \mathbb{N}$, let $\mathcal{Z}_m = \{\bar{z} \in Y^k : \bigcup_{n \leq m} X_n \subseteq \overline{\mathbb{D}}_r(\{z_1, \dots, z_k\})\}$. Note that $\mathcal{Z}_m = \bigcap_{x \in \bigcup_{n \leq m} X_n} \bigcup_{i=1}^k \{\bar{z} \in Y^k : d(x, z_i) \leq r\}$, so \mathcal{Z}_m is a closed subset of Y^k in the metric topology. As $\sup_{m \in \mathbb{N}} N_r^{\text{cov}}(\bigcup_{n \leq m} X_n/Y) \leq k$, $\mathcal{Z}_m \neq \emptyset$ for every $m \in \mathbb{N}$. Also, $\mathcal{Z}_{m+1} \subseteq \mathcal{Z}_m$ for each $m \in \mathbb{N}$. Therefore, by compactness of Y^k , there is $\bar{z} = (z_1, \dots, z_k) \in \bigcap_{m \in \mathbb{N}} \mathcal{Z}_m$. Set $Z = \{z_1, \dots, z_k\}$, so $|Z| \leq k$ and $Z \subseteq Y$. Now, $\bigcup_{n \leq m} X_n \subseteq \overline{\mathbb{D}}_r(Z)$ for any $m \in \mathbb{N}$, so $\bigcup_{n \in \mathbb{N}} X_n \subseteq \overline{\mathbb{D}}_r(Z)$. Thus, $N_r^{\text{cov}}(\bigcup_{n \in \mathbb{N}} X_n/Y) \leq |Z| \leq k$. As $k \in \mathbb{N}$ is arbitrary, we conclude that $N_r^{\text{cov}}(\bigcup_{n \in \mathbb{N}} X_n/Y) = \sup_{m \in \mathbb{N}} N_r^{\text{cov}}(\bigcup_{n \leq m} X_n/Y)$. Q.E.D.

Remark 4.1.3. Lemma 4.1.1 and Lemma 4.1.2 together imply in particular that $N_r(\bullet)$ and $N_r^{\text{cov}}(\bullet/Y)$ are outer-measures for any $r \in \mathbb{R}_{\geq 0}$ and Y compact in the metric topology.

The covering discretisations are extrinsic, i.e. they depend on the ambient space Y . Therefore, it is natural to work preferably with discretisations than with covering discretisations. Despite this consideration, the choice is still arbitrary. Indeed, the following lemma shows that they are in fact essentially equivalent up to a constant factor of 2 in the scale.

Lemma 4.1.4. *For any $X \subseteq Y$ and $r \in \mathbb{R}_{>0}$, $N_{2r}(X) \leq N_r^{\text{cov}}(X/Y) \leq N_r(X)$.*

Proof. If $N_r(X) = \infty$, we trivially get that $N_r^{\text{cov}}(X/Y)$. Suppose $N_r(X) < \infty$ and take an r -discretisation Z of X . For any other $x \in X$, $\{x\} \cup Z$ is not r -separated, so $d(x, z) \leq r$ for some $z \in Z$. In other words, $x \in \overline{\mathbb{D}}_r(Z)$. As $x \in X$ is arbitrary, $X \subseteq \overline{\mathbb{D}}_r(Z)$. Since $Z \subseteq X \subseteq Y$, we get $N_r^{\text{cov}}(X/Y) \leq |Z| = N_r(X)$.

If $N_r^{\text{cov}}(X/Y) = \infty$, trivially $N_{2r}(X) \leq N_r^{\text{cov}}(X/Y)$. Now, say $N_r^{\text{cov}}(X/Y) \leq n$, so there is $Z \subseteq Y$ such that $X \subseteq \overline{\mathbb{D}}_r(Z)$ and $|Z| \leq k$. Take any total order \prec in Z . Let $W \subseteq X$ be $2r$ -separated. For each $w \in W$, $d(w, z) \leq r$ for some $z \in Z$, i.e. $\overline{\mathbb{D}}_r(w) \cap Z \neq \emptyset$. Define $f : W \rightarrow Z$ given by $f(w) = \min_{\prec}(\overline{\mathbb{D}}_r(w) \cap Z)$. Take $w, w' \in W$ arbitrary and suppose $z := f(w) = f(w')$. By the triangular inequality, $d(w, w') \leq d(w, z) + d(w', z) \leq 2r$. As W is $2r$ -separated, we get that $w = w'$. Hence, $f : W \rightarrow Z$ is 1-to-1, concluding that $|W| \leq |Z| \leq k$. As W is arbitrary, $N_{2r}(X) \leq k$. As k is arbitrary, $N_{2r}(X) \leq N_r^{\text{cov}}(X/Y)$. Q.E.D.

A *(left) metric group* is a group together with a metric invariant under left translation. Note that, in this case, $\overline{\mathbb{D}}_r(X) = X\overline{\mathbb{D}}_r(1)$. For $X, Y \subseteq G$, we say that X and Y are *r -metric k -commensurable* if they are $\overline{\mathbb{D}}_r(1)$ -rough k -commensurable. We say that X is an *r -metric k -approximate subgroup* if it is a $\overline{\mathbb{D}}_r(1)$ -rough k -approximate subgroup.

We prove now some basic lemmas that we will need for the Metric Lie Model Theorem 4.2.2.

Lemma 4.1.5. *Let G be a metric group, $X \subseteq G$, $m \in \mathbb{N}_{\geq 2}$ and $r, k \in \mathbb{R}_{\geq 0}$. Assume that*

$$N_r(XX^{-1}X) \leq k \cdot N_{(2m+1)r}(X) < \infty.$$

Then:

- (1) *For every $b \in X$, $N_r(X \cap \overline{\mathbb{D}}_{mr}(b)) \leq k$.*
- (2) *For every $Y \subseteq X$, $N_r^{\text{cov}}(Y/X) \leq k \cdot N_{mr}^{\text{cov}}(Y/X)$.*

Proof. (1) Let Z be an $(2m+1)r$ -discretisation of X . For each $z \in Z$, let W_z be an r -discretisation of $\overline{\mathbb{D}}_{mr}(z) \cap XX^{-1}X$. For different $z, z' \in Z$, we have that

$$(2m+1)r < d(z, z') \leq d(z, w) + d(w, w') + d(w', z') \leq 2mr + d(w, w'),$$

for any $w \in W_z$ and $w' \in W_{z'}$. Therefore, $d(W_z, W_{z'}) > r$, so $W_z \cap W_{z'} = \emptyset$. Write $n_z := |W_z|$ and $n := |Z|$. Then,

$$\sum_{z \in Z} n_z = \left| \bigcup_{z \in Z} W_z \right| \leq N_r(XX^{-1}X).$$

In particular, $\min n_z \leq \frac{1}{n} \sum n_z \leq k$. Take $z_0 \in Z$ with $n_{z_0} = \min n_z$. For each $b \in X$, consider $f_b : X \rightarrow XX^{-1}X$ given by $f_b(x) = z_0 b^{-1}x$. Clearly, f_b is an isometry. Note $f_b[\overline{\mathbb{D}}_{mr}(b) \cap X] \subseteq \overline{\mathbb{D}}_{mr}(z_0) \cap XX^{-1}X$. Also, if $Y \subseteq X$ is r -separated, then $f_b[Y]$ is r -separated. Thus, $N_r(\overline{\mathbb{D}}_{mr}(b) \cap X) \leq N_r(\overline{\mathbb{D}}_{mr}(z_0) \cap XX^{-1}X) = n_{z_0} \leq k$.

(2) By definition, Y can be covered by $N_{mr}^{\text{cov}}(Y/X)$ balls of radius mr centred at points of X . By (1) and Lemma 4.1.4, each one of these balls restricted to $Y \subseteq X$ can be covered by k balls of radius r centred at points of X . Hence, we conclude that $N_r^{\text{cov}}(Y/X) \leq k \cdot N_{mr}^{\text{cov}}(Y/X)$. Q.E.D.

Lemma 4.1.6. *Let G be a metric group, $X \subseteq G$ and $r, k \in \mathbb{R}_{\geq 0}$. Assume that*

$$N_r(XX^{-1}X) \leq k \cdot N_{9r}(X) < \infty.$$

Let Z be a $2r$ -discretisation of X . Then, for any $Y \subseteq X$,

$$N_{2r}^{\text{cov}}(Y/X) \leq |Z \cap \overline{\mathbb{D}}_{2r}(Y)| \leq k \cdot N_{2r}^{\text{cov}}(Y/X).$$

Proof. We know that $X \subseteq \overline{\mathbb{D}}_{2r}(Z)$. Thus, $Y \subseteq \overline{\mathbb{D}}_{2r}(Z \cap \overline{\mathbb{D}}_{2r}(Y))$ for any $Y \subseteq X$. Therefore, $N_{2r}^{\text{cov}}(Y/X) \leq |Z \cap \overline{\mathbb{D}}_{2r}(Y)|$.

On the other hand, Z is $2r$ -separated, so there is no closed ball of radius r in which there are two elements of Z . In other words, $|Z_0| = N_r^{\text{cov}}(Z_0/X)$ for any $Z_0 \subseteq Z$. In particular, $|Z \cap \overline{\mathbb{D}}_{2r}(Y)| = N_r^{\text{cov}}(Z \cap \overline{\mathbb{D}}_{2r}(Y)/X)$. By Lemma 4.1.5(2) with $m = 4$, we have $N_r^{\text{cov}}(Z \cap \overline{\mathbb{D}}_{2r}(Y)/X) \leq k \cdot N_{4r}^{\text{cov}}(Z \cap \overline{\mathbb{D}}_{2r}(Y)/X)$. Finally, $N_{4r}^{\text{cov}}(Z \cap \overline{\mathbb{D}}_{2r}(Y)/X) \leq N_{4r}^{\text{cov}}(\overline{\mathbb{D}}_{2r}(Y)/X) \leq N_{2r}^{\text{cov}}(Y/X)$, concluding

$$N_{2r}^{\text{cov}}(Y/X) \leq |Z \cap \overline{\mathbb{D}}_{2r}(Y)| \leq k \cdot N_{2r}^{\text{cov}}(Y/X).$$

Q.E.D.

Let G be a metric group, $X \subseteq G$ and $l, r \in \mathbb{R}_{\geq 0}$. We say that X is (*right*) (l, r) -Lipschitz if every right translation by an element of X is l -Lipschitz when restricted to $\overline{\mathbb{D}}_r(1)$.

Lemma 4.1.7. *Let G be a metric group and $r_0, \dots, r_m \in \mathbb{R}_{\geq 0}$ such that $2r_{i+1} \leq r_i$. Assume that X is (l, r_0) -Lipschitz. Then, $\overline{\mathbb{D}}_{r_{i+1}}(1)\overline{\mathbb{D}}_{r_{i+1}}(1)^{-1} \subseteq \overline{\mathbb{D}}_{r_i}$ for every $i < m$ and $x^{-1}\overline{\mathbb{D}}_{r_{i+k}}(1)x \subseteq \overline{\mathbb{D}}_{r_i}(1)$ for every $x \in X$ and $i \leq m - k$, where $k = \lceil \log_2(l) \rceil$.*

Proof. Write $\|g\| := d(g, 1)$. By invariance under left translations, $\|g\| = \|g^{-1}\|$ for all $g \in G$, so $\overline{\mathbb{D}}_{r_i}(1) = \overline{\mathbb{D}}_{r_i}(1)^{-1}$. By the triangular inequality, $\|gh\| \leq \|g\| + \|h\|$, so $\overline{\mathbb{D}}_{r_{i+1}}(1)\overline{\mathbb{D}}_{r_{i+1}}(1) \subseteq \overline{\mathbb{D}}_{2r_{i+1}}(1) \subseteq \overline{\mathbb{D}}_{r_i}(1)$. By the Lipschitz condition, $\|x^{-1}gx\| \leq l\|g\|$ for any $g \in \overline{\mathbb{D}}_{r_0}(1)$ and $x \in X$, so $x^{-1}\overline{\mathbb{D}}_{r_{i+k}}(1)x \subseteq \overline{\mathbb{D}}_{lr_{i+k}}(1) \subseteq \overline{\mathbb{D}}_{r_i}(1)$. Q.E.D.

In the following lemma, to simplify the notation, we write $l^{[n]} := \sum_{i=0}^{n-1} l^i$ for $n \in \mathbb{N}$ and $l \in \mathbb{R}_{\geq 0}$.

Lemma 4.1.8. *Let G be a metric group and X a δ -metric k -approximate subgroup. Assume that X is (l, r) -Lipschitz and take $m \in \mathbb{N}_{>0}$ with $l^{[m-3]}\delta < r$. Take $\Delta \subseteq G$ such that $X^2 \subseteq \Delta \overline{\mathbb{D}}_\delta(X)$. Then, $X^n \subseteq \Delta^{n-1} \overline{\mathbb{D}}_{l^{[n-2]}\delta}(X)$ for each $n \leq m$. In particular, $N_{r+l^{[n-2]}\delta}^{\text{cov}}(X^n/Y) \leq k^{n-1} \cdot N_r^{\text{cov}}(X/Y)$ for any $r \in \mathbb{R}_{\geq 0}$, $Y \subseteq G$ and $n \leq m$.*

Proof. We prove it by induction on n . For $n = 2$, by assumption, $X^2 \subseteq \Delta \overline{\mathbb{D}}_\delta(X)$, so there is nothing to prove. Suppose it holds for $n - 1$. Then,

$$\begin{aligned} X^n &= X^{n-1}X \subseteq \Delta^{n-2}X \overline{\mathbb{D}}_{\delta l^{[n-3]}}(1)X \subseteq \Delta^{n-2}X X \overline{\mathbb{D}}_{\delta l \cdot l^{[n-3]}}(1) \\ &\subseteq \Delta^{n-1}X \overline{\mathbb{D}}_\delta(1) \overline{\mathbb{D}}_{\delta l \cdot l^{[n-3]}}(1) \subseteq \Delta^{n-1}X \overline{\mathbb{D}}_{\delta + \delta l \cdot l^{[n-3]}}(1) = \Delta^{n-1}X \overline{\mathbb{D}}_{\delta l^{[n-2]}}(1). \end{aligned}$$

Note that we have used that $d(x^{-1}\varepsilon x, 1) = d(\varepsilon x, x) \leq l \cdot d(\varepsilon, 1)$ for any $\varepsilon \in \overline{\mathbb{D}}_{\delta l^{[n-3]}}(1) \subseteq \overline{\mathbb{D}}_r(1)$ and $x \in X$ to get $\overline{\mathbb{D}}_{\delta l^{[n-3]}}(1)X \subseteq X \overline{\mathbb{D}}_{\delta l \cdot l^{[n-3]}}(1)$. Q.E.D.

Lemma 4.1.9. *Let G be a metric group and X an (l, r) -Lipschitz symmetric subset. Suppose that $N_r(X^5) \leq k \cdot N_r(X) < \infty$ with $k \in \mathbb{N}$. Then, X^2 is a $2lr$ -metric k -approximate subgroup.*

Proof. Let $\Delta \subseteq X^4$ be a maximal set such that $\{a \cdot \overline{\mathbb{D}}_r(X) : a \in \Delta\}$ is a family of pairwise disjoint sets. If Z is an r -discretisation of X , then $\Delta Z \subseteq X^5$ is also r -separated. Therefore, $|\Delta| \cdot |Z| = |\Delta Z| \leq k \cdot |Z|$, concluding $|\Delta| \leq k$. As Δ is maximal, for any $a \in X^4$ there is $b \in \Delta$ such that $a \overline{\mathbb{D}}_r(X) \cap b \overline{\mathbb{D}}_r(X) \neq \emptyset$. Then, $X^4 \subseteq \Delta X^2 \overline{\mathbb{D}}_{2lr}(1)$, concluding that X^2 is a $2lr$ -metric k -approximate subgroup. Q.E.D.

Remark 4.1.10. In the previous lemma, if we only assume $N_r(X^4) \leq k \cdot N_r(X) < \infty$, then we get that $X^3 \subseteq \Delta X^2 \overline{\mathbb{D}}_{2lr}(1)$ with $|\Delta| \leq k$. Thus, if X is $(l', 2lr)$ -Lipschitz, we get that $X^4 \subseteq \Delta X^3 \overline{\mathbb{D}}_{2l'r}(1) \subseteq \Delta^2 X^2 \overline{\mathbb{D}}_{2(l'+1)r}(1)$, concluding that X^2 is a $2l(l'+1)r$ -metric k^2 -approximate subgroup.

Lemma 4.1.11. *Let $k, n \in \mathbb{N}_{>0}$ and $l \in \mathbb{R}_{\geq 1}$. Take $m \geq 2n \log_2(18(1 + l^{[7]}))$. Let G be a metric group and X an $(l, 1)$ -Lipschitz symmetric subset. Suppose that X is a 2^{-m} -metric k -approximate subgroup and*

$$N_{2^{-m}}(X) \leq C \cdot N_1(X) < \infty.$$

Then, there are r_1, \dots, r_n with $2r_{i+1} \leq r_i \leq 1$ for each $i < n$ such that

$$N_{r_i}(X^9) \leq k^8 \sqrt[n]{C} \cdot N_{9r_i}(X).$$

Proof. Write $\alpha := 2^{m/2n}$, so $\alpha \geq 18(1 + l^{[7]})$. We have

$$\frac{N_{\alpha^{-2n}}(X)}{N_{\alpha^{-2n+1}}(X)} \cdots \frac{N_{\alpha^{-1}}(X)}{N_1(X)} \leq C,$$

where each factor is at least 1. Thus, there is $I \subseteq \{0, \dots, 2n-1\}$ with $|I| = n$ such that $N_{\alpha^{-i-1}}(X) \leq \sqrt[n]{C} \cdot N_{\alpha^{-i}}(X)$ for each $i \in I$. Take an enumeration r'_1, \dots, r'_n of $\{\alpha^{-i} : i \in I\}$. Then, using Lemma 4.1.4 and Lemma 4.1.8, we get

$$\begin{aligned} N_{2(l^{[7]}\delta + r'_i)}(X^9) &\leq N_{l^{[7]}\delta + r'_i}^{\text{cov}}(X^9/G) \leq k^8 \cdot N_{r'_i}^{\text{cov}}(X/G) \\ &\leq k^8 \cdot N_{r'_i}(X) \leq k^8 \sqrt[n]{C} \cdot N_{\alpha r'_i}(X). \end{aligned}$$

As $\alpha r'_i \geq 18(l^{[7]}\delta + r'_i)$, taking $r_i = 2(l^{[7]}\delta + r'_i)$, we have that $2r_{i+1} \leq r_i \leq 1$ and

$$N_{r_i}(X^9) \leq k^8 \sqrt[n]{C} \cdot N_{9r_i}(X).$$

Q.E.D.

4.2 Metric Lie Model Theorem

In this section we prove the main theorem of this chapter.

Fix $l \in \mathbb{N}$ and $k = (k_i)_{i \in \mathbb{N}}$ sequence in \mathbb{N} . An l -Lipschitz sequence of growth k in doubling scales is a sequence $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$ such that

1. G_m is a metric group,
2. X_m is an $(l, r_{0,m})$ -Lipschitz symmetric subset,
3. $r_m = (r_{i,m})_{i \leq m}$ is a sequence of positive reals with $2r_{i,m} \leq r_{i-1,m}$ and

$$N_{r_{i,m}}(X_m^9) \leq k_i \cdot N_{9r_{i,m}}(X_m) < \infty.$$

Remark 4.2.1. For each $r \in \mathbb{R}_{>0}$, write $M_r(X) := \frac{\ln N_r(X)}{\ln(1/r)}$. If it exists, the limit $\dim_{\text{MB}}(X) := \lim_{r \rightarrow 0^+} M_r(X)$ is called the *Minkowski-Bouligand dimension* of X . The number $M_r(X)$ is the r -approximation to the *Minkowski-Bouligand dimension* of X . We can rewrite condition 3 above as

$$M_{r_{i,m}}(X_m^9) - M_{9r_{i,m}}(X_m) \leq \frac{\ln(k_i)}{\ln(1/r_{i,m})}.$$

Now, to illustrate it, forget for a moment the parameter m and assume $k_i = k$ is constant. Then, this condition is saying that we can find a sequence $(r_i)_{i \in \mathbb{N}}$ converging to 0 such that

$$M_{r_i}(X^9) - M_{9r_i}(X) = O\left(\frac{1}{\ln(1/r_i)}\right).$$

In particular, that implies that $\dim_{\text{MB}}(X^9) = \dim_{\text{MB}}(X)$ if they exist. Hence, we can understand condition 3 above as a comparison between (the approximation sequences of) the Minkowski-Bouligand dimensions of X and X^9 .

On the other hand, the parameter m is the parameter of the ultraproduct construction. It allows us to translate the previous limit comparison into a comparison at finitely many scales.

Now, we describe the first-order language we are going to use. We consider the language of groups enriched with a predicate for X_m and predicates for $\overline{\mathbb{D}}_{r_{i,m}}$ for each $i \in \mathbb{N}$ — where $r_{i,m} = r_{m,m}$ for $i \geq m$. Write L_0 for this language. We expand L_0 by making definable, for each $i \in \mathbb{N}$, the normalised covering r_i -discretisation numbers on X^3 , i.e. the functions

$$\lambda_i : Y \mapsto \frac{N_{r_{i,m}}^{\text{cov}}(Y/X_m^3)}{N_{r_{i,m}}^{\text{cov}}(X_m^3/X_m^3)}.$$

Write L for this language; we call it the *associated language* for $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$.

Finally, pick a family $\{Z_{i,m}\}_{i,m \in \mathbb{N}}$ of $2r_{i,m}$ -discretisations of X_m^3 for each $i, m \in \mathbb{N}$ — where $Z_{i,m} = Z_{m,m}$ for $i \geq m$ — and consider the counting probability measures $\nu_{i,m}$ on $Z_{i,m}$ for each $i, m \in \mathbb{N}$. We expand L by adding a predicate for each $Z_{i,m}$ and making every $\nu_{i,m}$ definable — and still making each $\lambda_{i,m}$ definable. Write L' for this language; we call it the *auxiliary language* for the sequence $(G_m, X_m, r_{i,m}, Z_{i,m})_{i \leq m \in \mathbb{N}}$. Note that L' depends on the non-canonical choice of $\{Z_{i,m}\}_{i,m \in \mathbb{N}}$.

We now prove the main theorem of this chapter:

Theorem 4.2.2 (Metric Lie Model Theorem). *Let $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$ be an l -Lipschitz sequence of growth $k = (k_i)_{i \in \mathbb{N}}$ in doubling scales. Let L be its associated language and, for each $m \in \mathbb{N}$, consider G_m with its natural L -structure. Let $G^* = \prod G_m/\mathbf{u}$ be a non-principal ultraproduct. Write $G \leq G^*$ for the subgroup generated by $X = \prod X_m/\mathbf{u}$ and $o_r(1) := \bigcap_i \overline{\mathbb{D}}_{r_i}(1)$ where $\overline{\mathbb{D}}_{r_i} := \prod \overline{\mathbb{D}}_{r_{i,m}}/\mathbf{u}$. Then:*

- (1) $X/o_r(1)$ is a near-subgroup (with respect to L).
- (2) If k is constant, X^2 is an $o_r(1)$ -rough k -approximate subgroup.

In particular, if k is constant, $G \cdot o_r(1) \leq G^$ has a connected Lie model $\pi : H \rightarrow L = H/K$ with $o_r(1) \trianglelefteq K \subseteq X^8 \cdot o_r(1)$ such that*

- $H \cap X^4$ is an \wedge_ω -definable subset $o_r(1)$ -rough commensurable to X^2 ,
- $\pi[H \cap X^4]$ is a compact neighbourhood of the identity in L , and
- $H \cap X^8 \cdot o_r(1)$ generates H .

Remark 4.2.3. Let V be compact and U open in L with $V \subseteq U \subseteq L$. By definition of the Lie core, we have that $\pi^{-1}[V]$ is \wedge -definable, $\pi^{-1}[U]^c$ is piecewise \wedge -definable and $\pi^{-1}[V] \subseteq \pi^{-1}[U]$. Hence, there is a definable subset D such that $\pi^{-1}[V] \subseteq D \cdot o_r(1) \subseteq \pi^{-1}[U]$. As L is second countable and locally compact, we conclude that $\pi^{-1}[V]$ is \wedge_ω -definable and $\pi^{-1}[U] = U_0 \cdot o_r(1)$ for some \vee_ω -definable subset $U_0 \subseteq G$.

Proof. First of all, note that the ultraproduct is \aleph_1 -saturated by [TZ12, Exercise 5.2.3]. Also, by Lemma 4.1.7, we have that X normalises $o_r(1)$ and $o_r(1) \leq G^*$, so $G \cdot o_r(1)/o_r(1)$ is a well-defined piecewise 0-hyperdefinable group.

(1) We start by noting that, for $i \in \mathbb{N}$ and $Y \subseteq X^3$ definable in L' ,

$$\lambda_i(Y) = 0 \Leftrightarrow \nu_i(\overline{\mathbb{D}}_{r_i}(Y) \cap Z_i) = 0.$$

Write $Y_m := Y(G_m)$. By Lemma 4.1.1, Lemma 4.1.4 and Lemma 4.1.5, we have

$$N_{2r_{i,m}}^{\text{cov}}(Y_m/X_m^3) \leq N_{2r_{i,m}}(Y_m) \leq N_{r_{i,m}}^{\text{cov}}(Y_m/X_m^3) \leq k_i \cdot N_{2r_{i,m}}^{\text{cov}}(Y_m/X_m^3).$$

On the other hand, as $N_{r_{i,m}}(X_m^9) \leq k_i \cdot N_{9r_{i,m}}(X_m)$, by Lemma 4.1.1 and Lemma 4.1.4, we get

$$\frac{1}{N_{r_{i,m}}(X_m^3)} \leq \frac{1}{N_{r_{i,m}}^{\text{cov}}(X_m^3/X_m^3)} \leq k_i \cdot \frac{1}{N_{r_{i,m}}(X_m^3)}.$$

By Lemma 4.1.6, we conclude

$$\frac{1}{k_i} \cdot \lambda_{i,m}(Y_m) \leq \nu_{i,m}(Z_{i,m} \cap \overline{\mathbb{D}}_{r_{i,m}}(Y_m)) \leq k_i \cdot \lambda_{i,m}(Y_m).$$

Therefore, by Łoś's Theorem [CK90, Theorem 4.1.9], we conclude that

$$\lambda_i(Y) = 0 \Leftrightarrow \nu_i(\overline{\mathbb{D}}_{r_i}(Y) \cap Z_i) = 0.$$

For each $i \in \mathbb{N}$, let μ'_i be the ideal of definable subsets of X^3 in L' given by $Y \in \mu'_i \Leftrightarrow \lambda_i(Y) = 0$. Let μ_i be its L-reduct.

Write $\overline{\mathbb{D}}_{r_i}^{X^3} := \overline{\mathbb{D}}_{r_i} \cap (X^3 \times X^3) = \{(x, y) \in X^3 \times X^3 : d(x, y) \leq r_i\}$.

Claim: μ_i is an invariant under left translations $\overline{\mathbb{D}}_{r_i}^{X^3}$ -rough 0-medium ideal of definable subsets of X^3 with X wide, with respect to the language L . Similarly, μ'_i is an invariant under left translations $\overline{\mathbb{D}}_{r_i}^{X^3}$ -rough 0-medium ideal of definable subsets of X^3 with X wide, with respect to the language L' .

Proof of claim: μ_i and μ'_i are clearly ideals by subadditivity of the discretisation numbers (Lemma 4.1.1). Now, μ_i and μ'_i are 0-invariant by definability of λ_i . As λ_i is invariant under left translations, it is clear that μ_i and μ'_i are invariant under left translations. By Lemma 4.1.4 and Lemma 4.1.1, we get that $N_{r_{i,m}}^{\text{cov}}(X_m^3/X_m^3) \leq k_i \cdot N_{9r_{i,m}}(X_m) \leq k_i \cdot N_{r_{i,m}}^{\text{cov}}(X_m/X_m^3)$. Therefore, $\lambda_i(X) \geq 1/k_i$, so X is wide.

Finally, with respect to L' , let $Y(b) \subseteq X^3$ be b -definable and $(b_i)_{i \in \mathbb{N}}$ a 0-indiscernible sequence starting at b . Suppose

$$X^3 \cap \overline{\mathbb{D}}_{r_i}(Y(b_0)) \cap \overline{\mathbb{D}}_{r_i}(Y(b_1)) \in \mu'_i.$$

Then,

$$\nu_i(Z_i \cap \overline{\mathbb{D}}_{r_i}(X^3 \cap \overline{\mathbb{D}}_{r_i}(Y(b_0)) \cap \overline{\mathbb{D}}_{r_i}(Y(b_1)))) = 0,$$

so, in particular, $\nu_i(Z_i \cap \overline{\mathbb{D}}_{r_i}(Y(b_0)) \cap \overline{\mathbb{D}}_{r_i}(Y(b_1))) = 0$. As ν_i is a Keisler measure,

$$1 = \nu_i(Z_i) \geq \sum_{j=0}^{\infty} \nu_i(Z_i \cap \overline{\mathbb{D}}_{r_i}(Y(b_j))) \geq \infty \cdot \nu_i(Z_i \cap \overline{\mathbb{D}}_{r_i}(Y(b_0))).$$

Therefore, $\nu_i(Z_i \cap \overline{\mathbb{D}}_{r_i}(Y(b_0))) = 0$, so $Y(b_0) \in \mu'_i$. Since Y is arbitrary, we conclude that μ'_i is a $\overline{\mathbb{D}}_{r_i}^{X^3}$ -rough 0-medium.

Now, with respect to L , let $Y(b) \subseteq X^3$ be b -definable and $(b_i)_{i \in \mathbb{N}}$ a 0-indiscernible sequence starting at b . Suppose

$$X^3 \cap \overline{\mathbb{D}}_{r_i}(Y(b_0)) \cap \overline{\mathbb{D}}_{r_i}(Y(b_1)) \in \mu_i.$$

As μ_i is 0-invariant, we have $X^3 \cap \overline{\mathbb{D}}_{r_i}(Y(b_j)) \cap \overline{\mathbb{D}}_{r_i}(Y(b_{j'})) \in \mu_i$ for $j < j'$. By the Standard Lemma [TZ12, Theorem 5.1.5], we can find a 0-indiscernible sequence $(b'_i)_{i \in \mathbb{N}}$ starting at b with respect to L' such that

$$X^3 \cap \overline{\mathbb{D}}_{r_i}(Y(b'_0)) \cap \overline{\mathbb{D}}_{r_i}(Y(b'_1)) \in \mu_i \subseteq \mu'_i.$$

Then, by the rough mediumness of μ'_i , we conclude that $Y(b_0) \in \mu'_i$, so $Y(b_0) \in \mu_i$. As Y is arbitrary, we conclude that μ_i is a $\overline{\mathbb{D}}_{r_i}^{X^3}$ -rough 0-medium. \square

From now on we only work in L . We apply several steps explained in Section 3.3. Let $\mu_\infty = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \mu_i = \liminf \mu_i$. Then, μ_∞ is an invariant under left translations $\overline{\mathbb{D}}_{r_i}^{X^3}$ -rough 0-medium ideal of definable subsets of X^3 with $X \notin \mu_\infty$ for every $i \in \mathbb{N}$. Consider

$$\widehat{\mu}_\infty = \{W \wedge_\omega\text{-definable} : W \subseteq Y \in \mu_\infty \text{ for some } Y \text{ definable}\}.$$

Then, $\widehat{\mu}_\infty$ is an invariant under left translations compact locally atomic $o_r^{X^3}$ -rough 0-medium ideal of \wedge_ω -definable subsets of X^3 with $X \notin \widehat{\mu}_\infty$, where $o_r^{X^3} := \bigcap_i \overline{\mathbb{D}}_{r_i}^{X^3} = o_r \cap (X^3 \times X^3) = \{(x, y) \in X^3 \times X^3 : x^{-1}y \in o_r(1)\}$.

Consider the canonical projection $\mathfrak{g} : X^3 \rightarrow X^3/o_r^{X^3} \cong X^3/o_r(1)$. Clearly, \mathfrak{g} is a piecewise bounded and proper \wedge_0 -definable function. Define

$$\mu := \mathfrak{g}_* \widehat{\mu}_\infty = \{Y \subseteq X^3/o_r(1) \wedge_\omega\text{-definable} : \mathfrak{g}^{-1}[Y] \in \widehat{\mu}_\infty\}.$$

Hence, μ is an invariant under left translations compact locally atomic 0-medium ideal of \wedge_ω -definable subsets of $X^3/o_r(1)$ with $X/o_r(1)$ wide. In other words, $X/o_r(1)$ is a near-subgroup.

(2) Using Lemma 4.1.1,

$$N_{r_i, m}(X_m^5) \leq N_{r_i, m}(X_m^9) \leq k_i \cdot N_{9r_i, m}(X_m) \leq k \cdot N_{r_i, m}(X_m).$$

By Lemma 4.1.9, X_m^2 is a $2lr_{i,m}$ -metric k_i -approximate subgroup for each $i \leq m \in \mathbb{N}$. Now, assuming that $k := k_i$ is constant, by compactness, we get that X^2 is a $2lr_i$ -metric k -approximate subgroup for each $i \in \mathbb{N}$. Thus, we conclude that X^2 is a $o_r(1)$ -rough k -approximate subgroup.

Finally, applying the Rough Lie Model Theorem 3.6.4, we get the desired Lie model. Q.E.D.

Remark 4.2.4. (1) If $r_i/2 \leq 9r_{i+1}/4$, then $\mu_{i+1} \subseteq \mu_i$. In particular, if $r_i \leq 9r_{i+1}/2$ for every $i \in \mathbb{N}$, then $\mu_\infty = \bigcap \mu_i$. In this case, as each μ_i is \bigwedge_0 -definable, we conclude that μ_∞ is \bigwedge_0 -definable.

(2) If we get that X is an $o_r(1)$ -rough approximate subgroup, we can reduce the powers in the Metric Lie Model Theorem 4.2.2. Explicitly, we can get that $K \subseteq X^6 \cdot o_r(1)$, $H \cap X^2$ and X are $o_r(1)$ -rough commensurable and that H is generated by $H \cap X^6 \cdot o_r(1)$.

(3) Taking $Y = H \cap X^8$, we get a definable subset $o_r(1)$ -rough commensurable to X^2 such that $Y \cdot o_r(1)$ generates H .

(4) Without significant loss of generality, we may focus only on the case when $r_{0,m} \leq \text{diam}(X_m)$ for all $m \in \mathbb{N}$. Indeed, after the ultraproduct, up to taking a subsequence, this assumption is equivalent to imposing that there is $i \in \mathbb{N}$ such that $r_i \leq \text{diam}(X)$. Now, this only excludes the case $X \subseteq o_r(1)$, for which the Metric Lie Model Theorem 4.2.2 trivially holds.

Alternatively, using the second version of the Rough Lie Model Theorem 3.6.5 rather than the Rough Lie Model Theorem 3.6.4, we get the following variation:

Corollary 4.2.5 (Metric Lie Model Theorem, version 2). *There are functions $c : \mathbb{N} \rightarrow \mathbb{N}$ and $d : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds:*

Let $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$ be an l -Lipschitz sequence of constant growth k in doubling scales. Let \mathbb{L} be its associated language and, for each $m \in \mathbb{N}$, consider G_m with its natural \mathbb{L} -structure. Let $G^ = \prod G_m/\mathfrak{u}$ be a non-principal ultraproduct. Write $G \leq G^*$ for the subgroup generated by $X = \prod X_m/\mathfrak{u}$. Then, $G \cdot o_r(1) \leq G^*$ has a Lie model $\pi : H \rightarrow L = H/K$ with $o_r(1) \trianglelefteq K \subseteq X^{28} \cdot o_r(1)$ and $\dim(L) \leq d(k)$ such that*

- $H \cap X^4$ is an \bigwedge_ω -definable subset $o_r(1)$ -rough $c(k)$ -commensurable to X^2 ,
- $\pi[H \cap X^4]$ is a compact neighbourhood of the identity in L , and
- $H \cap X^{28} \cdot o_r(1)$ generates H .

One of the fundamental strengths of Theorem 4.2.2 is that $N_{r_i}(X_m)$ does not need to be bounded. In the following proposition we discuss the easy case when $r_i = 2^{-i}$ and $N_{2^{-i}}(X_m) \leq 2^{i\sigma}$:

Proposition 4.2.6. *There is a function $c : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds:*

Let $(G_m, X_m)_{i \leq m \in \mathbb{N}}$ be a sequence such that, for some fixed $k, l, C \in \mathbb{N}$ and $\sigma \in \mathbb{R}_{>0}$,

- (a) G_m is a metric group,*
- (b) X_m is an $(l, 1)$ -Lipschitz symmetric subset,*
- (c) for any $i \leq m \in \mathbb{N}$, we have $N_{2^{-i}}(X_m) \leq C \cdot 2^{i\sigma}$ and*

$$N_{2^{-i}}(X_m^4) \leq k \cdot N_{2^{-i}}(X_m).$$

Let (G^, X, \dots) be a non-principal ultraproduct in the language L_0 , G the subgroup generated by X and $o(1) := \bigcap_i \overline{\mathbb{D}}_{2^{-i}}(1)$. Then, $G \cdot o(1) \leq G^*$ has a connected Lie model $\pi : H \rightarrow L = H/K$ with $o(1) \trianglelefteq K \subseteq X^{24} \cdot o(1)$ and $\dim(L) \leq \sigma$ such that*

- $H \cap X^4$ and X^2 are $o(1)$ -rough $c(k)$ -commensurable,*
- $\pi[H \cap X^4]$ is a compact neighbourhood of the identity in L , and*
- $H \cap X^{24} \cdot o(1)$ generates H .*

Proof. By Lemma 4.1.7, $o(1) \trianglelefteq G \cdot o(1) \leq G^*$, so $G \cdot o(1)/o(1)$ is a piecewise hyperdefinable group. Since $N_{2^{-i}}(X_m) \leq C \cdot 2^{i\sigma}$, by Łoś's Theorem [CK90, Theorem 4.1.9], there is a finite 2^{-i} -separated subset $Z_i \subseteq X$ with $|Z_i| \leq C \cdot 2^{i\sigma}$ such that, for any $x \in X$ there is $a \in Z_i$ with $a \in \overline{\mathbb{D}}_{2^{-i}}(x)$. Take a function $\zeta : X \rightarrow \prod Z_i$ such that, for $x \in X$, we have $\zeta_i(x) \in \overline{\mathbb{D}}_{2^{-i}}(x)$ for each $i \in \mathbb{N}$. For any $x, y \in X$, if $\zeta(x) = \zeta(y) = (\zeta_i)_{i \in \mathbb{N}}$, then $x^{-1}y \in o(1)$. Consequently, $[X : o(1)] \leq |\prod Z_i| \leq 2^\omega$. Therefore, $G \cdot o(1)/o(1)$ is bounded, so it has a global logic topology.

By Lemma 4.1.9 (and Remark 4.1.10), X_m^2 is a $2^{-i+2}l^2$ -metric k^2 -approximate subgroup for each $i \leq m \in \mathbb{N}$ with $m > \lceil \log_2 l \rceil$. Then, X^2 is an $o(1)$ -rough k^2 -approximate subgroup, concluding that $X^2/o(1)$ is a k^2 -approximate subgroup. By the Generic Set Lemma (Theorem 2.2.9), we get that $X^4/o(1)$ is a neighbourhood of the identity in the global logic topology of $G \cdot o(1)/o(1)$. In particular, $G \cdot o(1)/o(1)$ is a locally compact topological group with the global logic topology.

Applying Gleason-Yamabe-Carolino Theorem 2.1.4 to the interior of $X^6 \cdot o(1)/o(1)$, we conclude that $G \cdot o(1)$ has a Lie model $\pi : H \rightarrow L = H/K$ with $o(1) \leq K \subseteq X^{24} \cdot o(1)$ such that $H \cap X^{24} \cdot o(1)$ generates H and is c_0 -commensurable to $X^2 \cdot o(1)$, where $c_0 = c_0(k)$ only depends on k . Using [Mac21, Lemma 2.2, Lemma 2.3], we conclude that $H \cap X^4 \cdot o(1)$ is c -commensurable to $X^2 \cdot o(1)$, where c only depend on k . Now, $H \cap X^4/o(1)$ is a compact neighbourhood of the identity in the global logic topology. Note that the homomorphism $\tilde{\pi} : H/o(1) \rightarrow L$ is a continuous, closed, open and proper map from the global logic topology, so $\pi[H \cap X^4]$ is a compact neighbourhood of the identity of L .

Consider the metric in $X^4/o(1)$ given by $\bar{d}(\bar{x}, \bar{y}) = \text{st } d(x, y)$, where st is the usual standard map. Since $N_{2^{-i}}(X_m^4) \leq kC \cdot 2^{i\sigma}$ for each $i \leq m \in \mathbb{N}$, by Łoś's Theorem

[CK90, Theorem 4.1.9] $X^4/o(1)$ is totally bounded with the metric \bar{d} . Suppose $(\bar{x}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $X^4/o(1)$. Take representatives $(x_n)_{n \in \mathbb{N}}$ in X^4 . Since $(\bar{x}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, for each $i \in \mathbb{N}$, there is $n_i \in \mathbb{N}$ such that $x_n \in \bar{\mathbb{D}}_{2^{-i}}(x_{n_i})$ for any $n \geq n_i$. Consequently, by compactness, there is $x \in \bigcap_i \bar{\mathbb{D}}_{2^{-i}}(x_{n_i}) \cap X^4$. Take $\bar{x} = x/o(1)$. Then, $\bar{d}(\bar{x}, \bar{x}_{n_i}) \leq 2^{-i}$ for each $i \in \mathbb{N}$, so $(\bar{x}_n)_{n \in \mathbb{N}}$ converges to \bar{x} . As $(\bar{x}_n)_{n \in \mathbb{N}}$ is arbitrary, we conclude that $X^4/o(1)$ is complete, so $X^4/o(1)$ is compact with the metric topology given by \bar{d} [Mun15, Theorem 45.1]. As \bar{d} is \wedge -definable, the global logic topology in $X^4/o(1)$ is finer than the metric topology given by \bar{d} . Being both compact and Hausdorff, we conclude that the metric topology given by \bar{d} is the global logic topology.

Since $N_{2^{-i}}(X_m^4) \leq kC \cdot 2^{i\sigma}$ for each $i \leq m \in \mathbb{N}$, by Łoś's Theorem [CK90, Theorem 4.1.9], we conclude that $M_{2^{-i}}(X^4/o(1)) \leq \sigma + \frac{\ln(kC)}{i \ln(2)}$ with respect to the metric \bar{d} . Consequently, $\liminf_{r \rightarrow 0^+} M_r(X^4/o(1)) \leq \sigma$. By [HW15, Theorem VII 2] and [Fal90, Eq.3.17 p.46], we get that $\dim(X^4/o(1)) \leq \sigma$, where \dim is the (large) inductive dimension of the topological space $X^4/o(1)$.

Claim: Let $f : X \rightarrow Y$ is a continuous, closed and open surjective map between compact metric topological spaces. Then, $\dim(Y) \leq \dim(X)$, where \dim is the (large) inductive dimension.

Proof of claim: We prove, by induction on n , that $\dim(X) \leq n$ implies $\dim(Y) \leq n$. If $\dim(X) = -1$, $X = \emptyset$, so $Y = \emptyset$ and $\dim(Y) = -1$. Suppose it is true for n and $\dim(X) \leq n + 1$. If $F \subseteq U$ with F closed and U open in Y , $f^{-1}[F] \subseteq f^{-1}[U]$ with $f^{-1}[U]$ open and $f^{-1}[F]$ closed. Since $\dim(X) \leq n + 1$, there is $f^{-1}[F] \subseteq V \subset \bar{V} \subseteq f^{-1}[U]$ with V open and $\dim(\bar{V} \setminus V) \leq n$. Thus, $F \subseteq f[V] \subseteq f[\bar{V}] \subseteq U$ where $f[V]$ is open and $f[\bar{V}] = f[V]$. Obviously, $f[\bar{V}] \setminus f[V] \subseteq f[\bar{V} \setminus V]$, so $\dim(f[\bar{V}] \setminus f[V]) \leq \dim(f[\bar{V} \setminus V]) \leq \dim(\bar{V} \setminus V) \leq n$ by induction hypothesis. As F and U are arbitrary, we get that $\dim(Y) \leq n + 1$. \square

Since $\tilde{\pi} : X^4/o(1) \rightarrow \pi[H \cap X^4]$ is a continuous, open and closed surjective map, we conclude that $\dim(\pi[H \cap X^4]) \leq \sigma$ from the claim. As $\pi[H \cap X^4]$ is a neighbourhood of the identity, there is a subset $U \subseteq \pi[H \cap X^4]$ open in L containing the identity and homeomorphic to an open subset of \mathbb{R}^d where $d = \dim(L)$. Therefore, $\dim(L) = \dim(U) \leq \sigma$. Q.E.D.

Remark 4.2.7. Alternatively, using Gleason-Yamabe Theorem 2.1.3 rather than Gleason-Yamabe-Carolino Theorem 2.1.4, we get a connected Lie model $\pi : H \rightarrow L = H/K$ with $o(1) \trianglelefteq K \subseteq X^4 \cdot o(1)$ and $\dim(L) \leq \sigma$ such that $H \cap X^4 \cdot o(1)$ generates H , $H \cap X^4$ and X^2 are $o(1)$ -rough commensurable, and $\pi[H \cap X^4]$ is a compact neighbourhood of the identity in L .

4.3 Applications

We conclude by showing some applications of the Metric Lie Model Theorem 4.2.2.

The first immediate application is the following Corollary 4.3.2. This corollary provides a metric version of one of the most basic applications of [Hru11], namely:

Lemma 4.3.1. *Fix $k \in \mathbb{N}$ and $n : \mathbb{N} \rightarrow \mathbb{N}$. Then, there is $c_0 \in \mathbb{N}$ such that the following holds:*

Let G be a group and X a finite k -approximate subgroup. Then, for some $c < c_0$, there is a c -commensurable subset $Y \subseteq X^4$ with $x^{-1}Y^n x \subseteq X^4$ for all $x \in X^n$.

Our result is the following:

Corollary 4.3.2. *Fix l, n, s and $k = (k_i)_{i \in \mathbb{N}}$ in \mathbb{N} . There is $m := m(k, l, n, s)$ such that the following holds:*

Let G be a metric group, X an (l, r_0) -Lipschitz symmetric subset and r_0, \dots, r_m positive reals with $2r_i \leq r_{i-1}$ such that

$$N_{r_i}(X^9) \leq k_i \cdot N_{9r_i}(X) < \infty.$$

Then, there is $I \subseteq \{0, \dots, m-1\}$ with $|I| = n$ and a symmetric subset Y with $x^{-1}Y^n x \subseteq \overline{\mathbb{D}}_{r_s}(X^4)$ for any $x \in X^n$ such that

$$N_{r_i}(Y) \geq \frac{1}{m} N_{r_i}(X) \text{ for all } i \in I.$$

Proof. Aiming for a contradiction, suppose otherwise. Thus, we get an l -Lipschitz sequence of growth k in doubling scales $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$ of counterexamples. Let (G^*, X, \dots) be a non-principal ultraproduct as in the Metric Lie Model Theorem 4.2.2. Write G for the \bigvee_0 -definable subgroup generated by X and consider $G \cdot o_r(1)/o_r(1)$.

We have that $X/o_r(1)$ is a near-subgroup by the Metric Lie Model Theorem 4.2.2. Moreover, recall that, in Theorem 4.2.2, the ideal μ of \bigwedge_ω -definable subsets of $X^3/o_r(1)$ is defined as $\mu = f_* \widehat{\mu}_\infty$, where $f = \mathfrak{g}|_{X^3} : X^3 \rightarrow X^3/o_r(1)$ is the canonical projection and $\widehat{\mu}_\infty$ is the compactification of $\mu_\infty = \liminf \mu_i$ with

$$Y \in \mu_i \Leftrightarrow \frac{N_{r_i}^{\text{cov}}(Y/X^3)}{N_{r_i}^{\text{cov}}(X^3/X^3)} = 0.$$

By the Stabilizer Theorem 3.6.1, we get an \bigwedge_ω -definable normal subgroup $S' \leq G \cdot o_r(1)/o_r(1)$ of bounded index contained in $X^4/o_r(1)$ such that $S' = (pp^{-1})^2$ for some wide type p in $X/o_r(1)$ and $p \cdot p \cdot p^{-1} = p \cdot S' = aS'$ for any $a \in p$.

Note that $\mathfrak{g} : G \cdot o_r(1) \rightarrow G \cdot o_r(1)/o_r(1)$ is a piecewise bounded and proper \bigwedge_0 -definable homomorphism. Take $S = \mathfrak{g}^{-1}(S')$. Then, $S \trianglelefteq G \cdot o_r(1)$ is \bigwedge_ω -definable symmetric subset with $x^{-1}S^n x = S \subseteq X^4 \cdot o_r(1) \subseteq \overline{\mathbb{D}}_{r_s}(X^4)$ for any $x \in X^n$ and $o_r(1) \leq S$. Now, by compactness, we find a definable symmetric set Y containing S such that $x^{-1}Y^n x \subseteq \overline{\mathbb{D}}_{r_s}(X^4)$ for any $x \in X^n$.

Pick $a \in p$, so we have $a \cdot S' \subseteq X^3/o_r(1)$. Write $S'_a := a^{-1}S' \cap X^3/o_r(1)$, $S_a := a^{-1}S \cap X^3 = \mathfrak{g}_{|X^3}^{-1}(S'_a)$ and $Y_a := a^{-1}Y \cap X^3$. As S' is wide, we know that S'_a is wide. Hence, $S_a \notin \widehat{\mu}_\infty$ and so $Y_a \notin \mu_\infty$. Then, $Y_a \notin \mu_i$ for infinitely many $i \in \mathbb{N}$. Choose $n < i_1 < i_2 < \dots < i_n$ such that $Y_a \notin \mu_{i_t}$ for each $t \in \{1, \dots, n\}$, and write $I = \{i_1, \dots, i_n\}$. Hence, for each $i \in I$, there is $m_i \in \mathbb{N}$ such that

$$\frac{N_{r_i}^{\text{cov}}(Y/X^3)}{N_{r_i}^{\text{cov}}(X^3/X^3)} \geq \frac{N_{r_i}^{\text{cov}}(Y_a/X^3)}{N_{r_i}^{\text{cov}}(X^3/X^3)} \geq \frac{1}{m_i}.$$

Using also Lemma 4.1.1 and Lemma 4.1.4, we get

$$\frac{N_{r_i}(Y)}{N_{r_i}(X)} \geq \frac{1}{k_i} \frac{N_{r_i}^{\text{cov}}(Y/X^3)}{N_{r_i}^{\text{cov}}(X^3/X^3)} \geq \frac{1}{k_i m_i}.$$

Taking $m_0 = \max\{k_i m_i : i \in I\}$, we conclude $N_{r_i}(Y)/N_{r_i}(X) \geq \frac{1}{m_0}$ for each $i \in I$.

By Łoś's Theorem [CK90, Theorem 4.1.9], we have some $m > \max\{m_0, i_1, \dots, i_n, s\}$ such that $I \subseteq \{0, \dots, m-1\}$ with $|I| = n$ and, for $Y_m = Y(G_m)$,

- (1) $1 \in Y_m = Y_m^{-1}$;
- (2) $x^{-1}Y_m^n x \subseteq \overline{\mathbb{D}}_{r_s, m}(X_m^4)$ for all $x \in X_m^n$;
- (3) $N_{r_i, m}(Y_m) \geq \frac{1}{m} N_{r_i, m}(X_m)$ for any $i \in I$.

That is a contradiction with the assumption that $G_m, X_m, r_{i, m}$ is a counterexample for m . Q.E.D.

For the next application, let us first recall the following result:

Lemma 4.3.3. [Hru11, Corollary 4.18] *Fix $k, N \in \mathbb{N}$. Then, there exists $c := c(k, N)$ such that the following holds:*

Let G be a group and X a finite k -approximate subgroup such that $x^N = 1$ for any $x \in X^2$. Then, there is a subgroup S of G with $S \subseteq X^4$ such that S and X are c -commensurable.

Our second corollary is a metric version of [Hru11, Corollary 4.18]:

Corollary 4.3.4. *Fix $k, l, N, n, s \in \mathbb{N}$. There are $m := m(k, l, N, n, s) \in \mathbb{N}$ and $c := c(k, N) \in \mathbb{N}$ such that the following holds:*

Let G be a metric group, X a (l, r_0) -Lipschitz symmetric subset and r_0, \dots, r_m positive reals with $2r_i \leq r_{i-1}$ such that $d(g^N, 1) \leq r_m$ for all $g \in X^8$ and

$$N_{r_i}(X^9) \leq k \cdot N_{9r_i}(X) < \infty.$$

Then, there is a symmetric subset $Y \subseteq X^{16}$ such that Y and X^2 are r_s -metric c -commensurable and $Y^n \subseteq \overline{\mathbb{D}}_{r_s}(Y)$.

Proof. Aiming for a contradiction, suppose otherwise. Thus, we get an l -Lipschitz sequence of growth k in doubling scales $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$ of counterexamples. Take a non-principal ultraproduct (G^*, X, \dots) as in the Metric Lie Model Theorem 4.2.5. Write G for the \bigvee_0 -definable subgroup generated by X . By Theorem 4.2.5, we have a Lie model $\pi : H \rightarrow L := H/K$ of $G \cdot o_r(1)$ with $o_r(1) \subseteq K \subseteq X^{28} \cdot o_r(1)$ such that $\pi[H \cap X^4]$ is a neighbourhood of the identity in L and $H \cap X^4$ is $o_r(1)$ -rough $c_0(k)$ -commensurable to X^2 , where $c_0(k)$ is a constant that only depend on k .

Now, since $d(g^N, 1) < r_{m,m} < r_{i,m}$ for every $g \in X_m^8$ and $i \leq m \in \mathbb{N}$, it follows by Łoś' Theorem [CK90, Theorem 4.1.9] that $g^N \in o_r(1)$ for every $g \in X^8$. Then, $\pi(g)^N = 1$ for every $g \in H \cap X^4$. Now, $\pi[H \cap X^4]$ is a neighbourhood of the identity of L . As L is a Lie group, it has a neighbourhood U_0 of the identity such that $g^2 = h^2$ if and only if $g = h$. Take a neighbourhood of the identity U in L such that $U^N \subseteq U_0$. Therefore, for any $g, h \in U$, $g^N = h^N$ if and only if $g = h$. Hence, $U \cap \pi[H \cap X^4] = \{1\}$, concluding that L is discrete. As $\pi[H \cap X^4]$ is compact, we have that it is finite. Note that $\pi[H \cap X^4]$ is a k^3 -approximate subgroup by [Mac21, Lemma 2.3] and $g^N = 1$ for any $g \in \pi[H \cap X^4]^2$. Applying Lemma 4.3.3, it follows that there is a finite subgroup $S_0 \subseteq \pi[H \cap X^4]^4$ which is $c_1(k, N)$ -commensurable to $\pi(H \cap X^4)$, where $c_1(k, N)$ is a constant that only depends on k and N . Since π is a group homomorphism, we get that $S := \pi^{-1}[S_0] \subseteq X^{16} \cdot o_r(1)$ is a subgroup of $G \cdot o_r(1)$ which is c -commensurable to $X^2 \cdot o_r(1)$, where $c := c_0(k)c_1(k, N)$ only depends on k and N . As π is continuous from the logic topology, S is open and compact, concluding that it is a relatively definable subgroup of $G \cdot o_r(1)$ contained in $X^{16} \cdot o_r(1)$.

Take $Y = S \cap X^{16}$. Hence, $Y \subseteq X^{16}$ is definable with $Y^n \subseteq S = Y \cdot o_r(1)$ and $o_r(1)$ -rough c -commensurable to X^2 . We have then $Y^n \subseteq \overline{\mathbb{D}}_{r_s}(Y)$ and Y is r_s -metric c -commensurable to X^2 . By Łoś' Theorem [CK90, Theorem 4.1.9], we conclude that there is some $m \in \mathbb{N}$ such that $Y_m \subseteq X_m^{16}$ is $r_{s,m}$ -metric c -commensurable to X_m^2 with $Y_m^n \subseteq \overline{\mathbb{D}}_{r_{s,m}}(Y_m)$, contradicting the assumption that $G_m, X_m, r_{i,m}$ is a counterexample. Q.E.D.

Finally, we also get a metric version of [Hru11, Corollary 4.13], which was one of the main applications of the original Lie Model Theorem in [Hru11]. Let us first recall the statement:

Lemma 4.3.5. [Hru11, Corollary 4.15] *Fix $k \in \mathbb{N}$ and $N : \mathbb{N} \rightarrow \mathbb{N}$. There is $c_0 \in \mathbb{N}$ such that the following holds:*

Let G be any group and X a finite k -approximate subgroup. Then, for some $c \leq c_0$, there is a sequence of symmetric subsets $X_N \subseteq \dots \subseteq X_1 \subseteq X^4$, with $N := N(c)$,

satisfying the following properties for each $n \leq N$:

- (1) X^2 and X_1 are c -commensurable.
- (2) $X_{n+1}X_{n+1} \subseteq X_n$.
- (3) X_n is covered by c cosets of X_{n+1} .
- (4) $x^{-1}X_{n+1}x \subseteq X_n$ for every $x \in X_1$.
- (5) $[X_{n_1}, X_{n_2}] \subseteq X_n$ whenever $n \leq n_1 + n_2$.
- (6) $\{x \in X_1 : x^4 \in X_n\} = X_{n+1}$.
- (7) If $x, y \in X_1$ with $x^2 = y^2$, then $y^{-1}x \in X_N$.

Our version has a few changes, significantly in point (6).

Corollary 4.3.6. *Fix $k, l, N \in \mathbb{N}$ and $s : \mathbb{N} \rightarrow \mathbb{N}$. There is $m := m(k, l, N, s) \in \mathbb{N}$ such that the following holds:*

Let G be a metric group, X an (l, r) -Lipschitz symmetric subset and r_0, \dots, r_m positive reals with $2r_i \leq r_{i-1}$ such that

$$N_{r_i}(X^9) \leq k \cdot N_{9r_i}(X) < \infty.$$

Then, for some $c \leq m$, there is a sequence of symmetric subsets $X_N \subseteq \dots \subseteq X_1 \subseteq X^8$ satisfying the following properties for each $n \leq N$ and $s := s(c)$:

- (1) X^2 and X_1 are r_s -metric c -commensurable.
- (2) $X_{n+1}X_{n+1} \subseteq \overline{\mathbb{D}}_{r_s}(X_n)$.
- (3) X_n is covered by c cosets of $\overline{\mathbb{D}}_{r_s}(X_{n+1})$.
- (4) $x^{-1}X_{n+1}x \subseteq \overline{\mathbb{D}}_{r_s}(X_n)$ for every $x \in X_1$.
- (5) $[X_{n_1}, X_{n_2}] \subseteq \overline{\mathbb{D}}_{r_s}(X_n)$ whenever $n \leq n_1 + n_2$.
- (6) $\{x \in X_1 : x^2, x^4 \in X_1 \text{ and } x^8 \in X_n\} \subseteq X_{n+1}$.
- (7) If $x, y \in X_1$ with $x^2 = y^2$, then $y^{-1}x \in \overline{\mathbb{D}}_{r_s}(X_N)$.

Proof. Aiming for a contradiction, suppose otherwise. Thus, we get an l -Lipschitz sequence of constant growth k in doubling scales $(G_m, X_m, r_{i,m})_{i \leq m \in \mathbb{N}}$ of counterexamples. Let (G^*, X, \dots) be a non-principal ultraproduct as in the Metric Lie Model Theorem 4.2.2. Write G for the \bigvee_0 -definable subgroup generated by X . By Theorem 4.2.2, we have a Lie model $\pi : H \rightarrow L := H/K$ of $G \cdot o_r(1)$ with $o_r(1) \subseteq K \subseteq X^8 \cdot o_r(1)$ such that H is generated by $H \cap X^8 \cdot o_r(1)$, $H \cap X^4$ is $o_r(1)$ -rough commensurable to X^2 and $\pi[H \cap X^8]$ is a neighbourhood of the identity in L .

As it was noted in [Hru11, Corollary 4.13], in any Lie group L , for any given compact neighbourhood of the identity U_{-1} , it is possible to find a sequence $(U_n)_{n \in \mathbb{N}}$ of symmetric compact neighbourhoods of the identity with $U_{n+1} \subseteq U_n^\circ$ and $U_0 \subseteq U_{-1}$

satisfying the following properties:

- (1) U_{-1} and U_0 are c_0 -commensurable.
- (2) $U_{n+1}U_{n+1} \subseteq U_n$.
- (3) U_n is covered by c_0 cosets of U_{n+1} .
- (4) $x^{-1}U_{n+1}x \subseteq U_n$ for every $x \in U_0$.
- (5) $[U_{n_1}, U_{n_2}] \subseteq U_n$ whenever $n < n_1 + n_2$.
- (6) $\{x \in U_0 : x^2 \in U_n\} = U_{n+1}$.
- (7) If $x, y \in U_0$ with $x^2 = y^2$, then $x = y$.

In our case, we pick $U_{-1} = \pi[H \cap X^4]$.

Now, as $\pi : H \rightarrow L$ is continuous from a logic topology using enough parameters, there is a symmetric relatively definable subset Y_0 such that $\pi^{-1}[U_1] \subseteq Y_0 \subseteq \pi^{-1}[U_0] \subseteq X^8 \cdot o_r(1)$. As in [Hru11, Corollary 4.13], taking $Y_{n+1} = \{y \in Y_0 : y^2 \in Y_n\}$, we get that $\pi^{-1}[U_{n+1}] \subseteq Y_n \subseteq \pi^{-1}[U_n]$. Thus, $(Y_n)_{n \in \mathbb{N}}$ is a sequence of relatively definable symmetric subsets with $Y_{n+1} \subseteq Y_n$ and $Y_0 \subseteq X^8 \cdot o_r(1)$ satisfying the following properties, where $c_1 = \max\{k, c_0^2\}$:

- (1) $X^2 \cdot o_r(1)$ and Y_0 are c_1 -commensurable.
- (2) $Y_{n+2}Y_{n+2} \subseteq Y_n$.
- (3) Y_n is covered by c_1 cosets of Y_{n+1} .
- (4) $y^{-1}Y_{n+2}y \subseteq Y_n$ for every $y \in Y_0$.
- (5) $[Y_{n_1}, Y_{n_2}] \subseteq Y_n$ whenever $n < n_1 + n_2$.
- (6) $\{y \in Y_0 : y^2 \in Y_n\} = Y_{n+1}$.
- (7) If $x, y \in Y_0$ with $x^2 = y^2$, then $x = y$.

Take $X'_{n+1} = Y_n \cap X^8$. Obviously, X'_n is a symmetric subset. As Y_n is relatively definable on $X^8 \cdot o_r(1)$ and $X^8 \subseteq X^8 \cdot o_r(1)$ is definable, we conclude that X'_{n+1} is definable. Now, note that $Y_n \subseteq X'_n \cdot o_r(1)$. Indeed, given $y \in Y_n$, we know $y \in \pi^{-1}[U_n]$. Thus, $y \cdot o_r(1) \subseteq \pi^{-1}[U_n] \subseteq Y_{n-1}$, so $X^8 \cap y \cdot o_r(1) \subseteq X'_n$. As $Y_n \subseteq X^8 \cdot o_r(1)$, we get that $y \in X^8 \cdot o_r(1)$, so there is $x \in y \cdot o_r(1) \cap X^8 \subseteq X'_n$. Then, $y \in x \cdot o_r(1) \subseteq X'_n \cdot o_r(1)$. Since y is arbitrary, we conclude $Y_n \subseteq X'_n \cdot o_r(1)$.

Set $X_n = X'_{3n}$, write $c = c_1^4$ and $s = s(c)$. We have a sequence $X_N \subseteq \dots \subseteq X_1 \subseteq X^8$ of symmetric definable subsets containing the identity satisfying the following properties:

- (1) X^2 and X_1 are r_s -metric c -commensurable.
- (2) $X_{n+1}X_{n+1} \subseteq \overline{\mathbb{D}}_{r_s}(X_n)$.
- (3) X_n is covered by c cosets of $\overline{\mathbb{D}}_{r_s}(X_{n+1})$.
- (4) $x^{-1}X_{n+1}x \subseteq \overline{\mathbb{D}}_{r_s}(X_n)$ for every $x \in X_1$.
- (5) $[X_{n_1}, X_{n_2}] \subseteq \overline{\mathbb{D}}_{r_s}(X_n)$ whenever $n \leq n_1 + n_2$.
- (6) $\{x \in X_1 : x^2, x^4 \in X_1 \text{ and } x^8 \in X_n\} \subseteq X_{n+1}$.
- (7) If $x, y \in X_1$ with $x^2 = y^2$, then $y^{-1}x \in \overline{\mathbb{D}}_{r_s}(X_N)$.

By Łoś's Theorem [CK90, Theorem 4.1.9], we get a contradiction with the initial assumption that $(G_m, X_m, r_{i,m})_{m \in \mathbb{N}}$ is a sequence of counterexamples. Q.E.D.

Using Lemma 4.1.11, we can rewrite the previous corollaries in the more specific case of metric approximate subgroups:

Corollary 4.3.7. *Fix $k, C, l, n, s \in \mathbb{N}$. There is $m := m(k^8 C, l, n, s) \in \mathbb{N}$ such that the following holds:*

Let G be a metric group and X an $(l, 1)$ -Lipschitz symmetric subset. Suppose that X is a 2^{-m} -metric k -approximate subgroup and

$$N_{2^{-m}}(X) \leq C^m \cdot N_1(X) < \infty.$$

Then, there is $I \subseteq \{0, \dots, m\}$ with $|I| = n$ and a symmetric subset Y with $x^{-1}Y^n x \subseteq \overline{\mathbb{D}}_{2^{-s}}(X^4)$ for all $x \in X^n$ such that

$$N_{2^{-i}}(Y) \geq \frac{1}{m} N_{2^{-i}}(X) \text{ for all } i \in I.$$

Corollary 4.3.8. *Fix $k, C, l, N, n, s \in \mathbb{N}$. There are $m := m(k^8 C, l, N, n, s) \in \mathbb{N}$ and $c := c(k^8 C, N) \in \mathbb{N}$ such that the following holds:*

Let G be a metric group and X an $(l, 1)$ -Lipschitz symmetric subset. Suppose that X is a 2^{-m} -metric k -approximate subgroup, $d(g^N, 1) < 2^{-m}$ for all $g \in X^8$ and

$$N_{2^{-m}}(X) \leq C^m \cdot N_1(X) < \infty.$$

Then, there is a symmetric subset $Y \subseteq X^{16}$ such that Y and X^2 are 2^{-s} -metric c -commensurable and $Y^n \subseteq \overline{\mathbb{D}}_{2^{-s}}(Y)$.

Corollary 4.3.9. *Fix $k, C, l, N, n \in \mathbb{N}$ and $s : \mathbb{N} \rightarrow \mathbb{N}$. There is $m := m(k^8 C, l, N, s)$ such that the following holds:*

Let G be a metric group and X an $(l, 1)$ -Lipschitz symmetric subset. Suppose that X is a 2^{-m} -metric k -approximate subgroup and

$$N_{2^{-m}}(X) \leq C^m \cdot N_1(X) < \infty.$$

Then, for some $c \leq m$, there is a sequence of symmetric subsets $X_N \subseteq \dots \subseteq X_1 \subseteq X^8$ satisfying the following properties for each $n \leq N$ and $s := s(c)$:

- (1) X^2 and X_1 are 2^{-s} -metric c -commensurable.
- (2) $X_{n+1}X_{n+1} \subseteq \overline{\mathbb{D}}_{2^{-s}}(X_n)$.
- (3) X_n is covered by c cosets of $\overline{\mathbb{D}}_{2^{-s}}(X_{n+1})$.
- (4) $x^{-1}X_{n+1}x \subseteq \overline{\mathbb{D}}_{2^{-s}}(X_n)$ for every $x \in X_1$.
- (5) $[X_{n_1}, X_{n_2}] \subseteq \overline{\mathbb{D}}_{2^{-s}}(X_n)$ whenever $n \leq n_1 + n_2$.
- (6) $\{x \in X_1 : x^2, x^4 \in X_1 \text{ and } x^8 \in X_n\} \subseteq X_{n+1}$.
- (7) If $x, y \in X_1$ with $x^2 = y^2$, then $y^{-1}x \in \overline{\mathbb{D}}_{2^{-s}}(X_N)$.

Appendix A

Digression: de Saxcé’s Product Theorem

The Generic Set Lemma (Theorem 2.2.9) can be rewritten in purely topological terms. Written in this way, it is likely that this result was already (partially) known in the theory of topological groups. Recall that a semitopological group is a group with a Fréchet topology such that left and right translations are continuous.

Theorem A.1. *Let G be a semitopological group with a coherent covering \mathcal{C} by closed symmetric subsets such that, for any $A, B \in \mathcal{C}$, there is $C \in \mathcal{C}$ with $AB \subseteq C$. Let V be a generic piece, i.e. an element $V \in \mathcal{C}$ such that $[W : V]$ is finite for every $W \in \mathcal{C}$. Then, V has non-empty interior. In particular, if V is compact Hausdorff with the subspace topology, then G is a locally compact topological group. Furthermore, if \mathcal{C} is a countable covering by compact Hausdorff subsets, then G is locally compact if and only if \mathcal{C} contains a generic piece.*

Proof. Mimicking the proof of Theorem 2.2.9, we show that V^2 is a neighbourhood of the identity, and so V has non-empty interior as finitely many translates of it cover V^2 . To show that G is a topological group, we use Ellis’s Theorem [Ell57, Theorem 2]. The “furthermore” part is an immediate consequence of Baire’s Category Theorem [Mun15, Theorem 48.2]. Q.E.D.

As a corollary we get the following notable Closed Approximate Subgroups Theorem, which was first proved by Machado in [Mac21, Theorem 1.4]. Recall that the *commensurator* in G of an approximate subgroup $\Lambda \subseteq G$ is the subset $\text{Comm}(\Lambda) = \{g \in G : g^{-1}\Lambda g \text{ and } \Lambda \text{ are commensurable}\}$. The commensurator was first introduced by Hrushovski in [Hru22]. The following is one of the most fundamental results about the commensurator:

Lemma A.2. [Hru22, Lemma 5.1] *Let G be a group and $\Lambda \subseteq G$ an approximate subgroup. Then, $\text{Comm}(\Lambda) = \bigcup\{\Lambda' \subseteq G \text{ approximate subgroup} : \Lambda' \text{ and } \Lambda \text{ are commensurable}\}$ and $\text{Comm}(G) \leq G$.*

Corollary A.3 (Machado's Closed Approximate Subgroups Theorem). *Let G be a locally compact topological group and X a closed approximate subgroup. Take $\Lambda = \overline{X^2 \cap K^2}$ where K is a compact symmetric neighbourhood of the identity. Then, $\text{Comm}(\Lambda)$, with the direct limit topology given by $\Omega = \{\Lambda' : \Lambda' \text{ compact approximate subgroup commensurable to } \Lambda\}$, is a locally compact topological group such that $\iota : \text{Comm}(\Lambda) \rightarrow G$ is a continuous 1-to-1 group homomorphism, $X \subseteq \text{Comm}(\Lambda)$, $\iota|_X$ is an homeomorphism and X has non-empty interior. Furthermore, if G is a Lie group, then $\text{Comm}(\Lambda)$ is a Lie group too.*

Proof. As Λ is compact, for any B commensurable to Λ , we have that Λ is commensurable to \overline{B} and \overline{B} is compact. Thus, Ω is a covering of $\text{Comm}(\Lambda)$ by Lemma A.2. By construction, Λ is generic in Ω . Also, by an easy computation, if Λ_1 and Λ_2 are commensurable approximate subgroups, then $\Lambda_1\Lambda_2 \cup \Lambda_2\Lambda_1$ is an approximate subgroup commensurable to Λ_1 and to Λ_2 . Thus, by Theorem A.1, $\text{Comm}(\Lambda)$ with the direct limit topology is a locally compact topological group such that $\iota : \text{Comm}(\Lambda) \rightarrow G$ is a continuous proper 1-to-1 group homomorphism and Λ has non-empty interior. As any two compact neighbourhoods of the identity are commensurable, by [Mac21, Lemma 2.2, Lemma 2.3], we get that $X^2 \cap (K')^2 \subseteq \text{Comm}(\Lambda)$ for any compact neighbourhood of the identity K' , concluding that $X \subseteq X^2 \subseteq \text{Comm}(\Lambda)$. Since $\Lambda \subseteq X^2$ has non-empty interior, we get that X^2 has non-empty interior and so X has non-empty interior. If $Y \subseteq X$ is closed in G , by continuity of ι , it is closed in $\text{Comm}(\Lambda)$. On the other hand, if $Y \subseteq X$ is closed in $\text{Comm}(\Lambda)$, then $Y \cap (K')^2 = Y \cap \overline{X^2 \cap (K')^2}$ is closed in $\overline{X^2 \cap (K')^2}$ (and so in G) for any compact symmetric neighbourhood of the identity K' of G . Hence, Y is closed in G by local compactness. As X is closed, this concludes that the subspace topologies of X in G and $\text{Comm}(\Lambda)$ coincide. Finally, if G is a Lie group, $\text{Comm}(\Lambda)$ is a Lie group by [Bou75, Chapter III, §8.2, Corollary 1].
Q.E.D.

Remark A.4. [Mac21, Theorem 4.1] If X is compact, we do not need to assume that G is locally compact; in that case, we can simply take $\Lambda = X$.

Theorem A.3 implies the following remarkable result by using Poguntke's Theorem [Pog94, Theorem 3.3]. Recall that a connected Lie group is *semi-simple* if it has no non-trivial connected solvable normal subgroups.

Corollary A.5. *Let G be a semi-simple Lie group and X a closed approximate subgroup with empty interior. Then, X is contained in a closed proper subgroup $H \subsetneq G$. In particular, the closure of the subgroup generated by X is a proper subgroup of G .*

Proof. By Machado's Closed Approximate Subgroups Theorem A.3, there is a subgroup $C \leq G$ containing X that admits a Lie group structure such that X has non-empty interior and the map $\iota: C \rightarrow G$ is a 1-to-1 continuous group homomorphism. By [Pog94, Theorem 3.3] and [Lee02, Proposition 6.5], as G is semi-simple, ι has dense image if and only if it is actually an isomorphism. Therefore, the closure of the group generated by X is a proper subgroup of G . Q.E.D.

As a consequence, by an easy ultraproduct argument, we get Theorem A.7, which provides a variation of de Saxcé's Product Theorem [Sax15, Theorem 1] valid for semi-simple Lie groups. Before stating and proving Theorem A.7, let us restate the original result by de Saxcé. Recall that a *simple* Lie group is a non-abelian connected Lie group without non-trivial connected normal subgroups:

Theorem A.6 (de Saxcé's Product Theorem). *Let G be a simple Lie group and take some left invariant metric. There is a neighbourhood U of the identity such that, for any $0 < \sigma < \dim(G)$, there are $\varepsilon = \varepsilon(\sigma) > 0$ and $\delta_0 = \delta_0(\sigma)$ with the following property: if $X \subseteq U$ and $0 < \delta < \delta_0$ satisfy*

- (1) $N_\delta^{\text{cov}}(X) \leq \delta^{-\sigma-\varepsilon}$,
- (2) $N_\rho^{\text{cov}}(X) \geq \delta^\varepsilon \rho^{-\sigma}$ for any $\rho \geq \delta$ and
- (3) $N_\delta^{\text{cov}}(X^3) \leq \delta^{-\varepsilon} N_\delta^{\text{cov}}(X)$,

then there is a closed connected subgroup $H \leq G$ with $X \subseteq \overline{\mathbb{D}}_{\delta^\varepsilon}(H)$.

Hypothesis (3) in [Sax15, Theorem 1] is very close to assuming that X is a metric approximate subgroup, while hypotheses (1) and (2) are morally saying that the dimension of X looks like $\sigma < d$. Our Theorem A.7 is based on these analogies:

Theorem A.7 (A Product Theorem for Semisimple Lie Groups). *Let G be a semi-simple Lie group and U a compact neighbourhood of the identity. Take some left invariant metric. Let $\sigma < \dim(G)$ and $C, k, s \in \mathbb{N}$. Then, there is $m \in \mathbb{N}$ such that, for any 2^{-m} -metric k -approximate subgroup $X \subseteq U$ satisfying $N_{2^{-i}}(X) \leq C2^{i\sigma}$ for each $i \leq m$, there is a closed subgroup $H \leq G$ with $X \subseteq \overline{\mathbb{D}}_{2^{-s}}(H)$.*

Proof. Aiming a contradiction, suppose otherwise. Then, we have a counterexample X_m for each $m \in \mathbb{N}$. Take an ultraproduct in the sense of (unbounded) continuous logic, i.e. take an ultraproduct, take the subgroup generated by U and quotient by the infinitesimals. By compactness of U , we end then with a subset $X \subseteq U$ of G . By Łoś's Theorem, X is a k -approximate subgroup and satisfies $N_{2^{-i}}(X) \leq C \cdot 2^{i\sigma}$ for every $i \in \mathbb{N}$. Then, $\dim(X) \leq \sigma < d$, where \dim denotes the (large) inductive dimension, by [HW15, Theorem VII 2] and [Fal90, Eq.3.17 p.46]. In particular, X has empty interior in G by [HW15, Corollary 1 of Theorem IV 3]. By Corollary A.5, as G is semi-simple, the closure of the subgroup generated by X is a proper closed subgroup $H \leq G$ containing X . Thus, by Łoś's Theorem, there is some m such that $X_m \subseteq \overline{\mathbb{D}}_{2^{-s}}(H)$, getting a contradiction with our initial assumption. Q.E.D.

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