

# Robustness of coupled networks with multiple support from functional components at different scales

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Robustness is an essential component of modern network science. Here, we investigate the robustness of coupled networks where the functionality of a node depends not only on its connectivity, here measured by the size of its connected component in its own network, but also the support provided by at least  $M$  links from another network. We here develop a theoretical framework and investigate analytically and numerically the cascading failure process when the system is under attack, deriving expressions for the proportion of functional nodes in the stable state, and the critical threshold when the system collapses. Significantly, our results show an abrupt phase transition and we derive the minimum inner and inter-connectivity density necessary for the system to remain active. We also observe that the system necessitates an increased density of links inside and across networks to prevent collapse, especially when conditions on the coupling between the networks is more stringent. Finally, we discuss the importance of our results in real-world settings and their potential use to aid decision-makers design more resilient infrastructure systems.

**The relationships among diverse systems have become increasingly intertwined, and the coupling relationships among system elements often exhibit multiple effective edges supporting dependency. Therefore, the functionality the system depend not only on the internal connectivity within the system itself but also on the support provided by multiple effective edges between systems. To fill the gap within the pre-existing model frameworks, we here define the cascading failure mechanism of the system when subjected to initial failure and propose a theoretical framework. Within this framework, functional nodes are defined as belonging to local connective components of size greater than or equal to  $s$ , while requiring effective support from at least  $M$  links from another system. We find that there exists the minimum inner and inter-connectivity density that sustain the survival of the system when system occurs the first-order phase transition. Furthermore, it is noted that the system necessitates strengthened connection densities within both intra- and inter-links to prevent collapse, especially when a larger number of effective supporting links are necessary. The framework proposed within this study holds promise for aiding decision-makers by facilitating the design of more resilient reality-dependent systems and formulating optimal protection strategies.**

of complex systems that intricately interconnects various sub-networks. As a result, infrastructure networks, encompassing essential systems like the power grid and communication networks, have become increasingly dependent on each other. By integrating sophisticated technologies like sensors, communication systems, and advanced control mechanisms, contemporary electrical infrastructure networks are transitioning into cyber-physical systems<sup>1–5</sup>. Their relations ensure a dependable and optimal supply of electricity and services, but they also raise technological and mathematical challenges to ensure their stability, reliability, and efficiency, in particular due to their increased vulnerability to potential attacks. For these reasons, there has been an increased research interest in the mechanisms ensuring their robustness<sup>6–9</sup>.

When communication and power networks are interdependent, a system failure has the potential to trigger cascading failures on a significant scale, which could ultimately result in catastrophic consequences. As an example, in 2015, Ukraine encountered a blackout instigated by hackers who employed malware to disable circuit breakers within the nation's power system via its communication network. This malicious act led to thousands of customers losing access to electricity<sup>10–12</sup>. This incident underscores the necessity to protect the security and resilience of power networks, crucial not only for national safety but also for the welfare of citizens<sup>13–15</sup>. Furthermore, this event underlines the importance of understanding the mechanisms leading to cascading failure in interconnected networks<sup>16–21</sup>.

Network robustness captures a system's ability to maintain essential functions when some of its components, typically the nodes or the links, are under attack<sup>22</sup>. There exist multiple approaches to quantify the robustness of a system as a function of its network properties. For example, percolation theory provides a angle to explain the network robustness<sup>23–25</sup>. Related to this work, Dong et al. delineated network robustness through an examination of percolation within interconnected networks that possess feedback-dependent links<sup>26</sup>. Buldyrev

## I. INTRODUCTION

Over the past few decades, advanced communication, computing, and control technologies have facilitated the formation

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et al. introduced a theoretical framework model to analyze the robustness of one-to-one dependent networks subjected to cascading failures based on the power network and an Internet network in Italy, from the perspective of complex networks<sup>27</sup>. Chen et al. integrated game theory with network science to explore the impact of heterogeneous interdependent networks on system robustness<sup>28</sup>. Dong et al. improved the robustness of complex systems by analysing the optimal level of interaction between sub-networks<sup>29</sup>. Yuan et al. studied the robustness of dependent networks through one-to-one interdependence between nodes<sup>30</sup>. Furthermore, the presence of assortative and disassortative patterns, along with local interdependence across diverse networks, also significantly influences the structural robustness of the system<sup>31–33</sup>. In a real-world setting, networks often exhibit multiple forms of mutual support among themselves rather than a one-to-one dependency relationship. In cyber-physical systems, communication network base stations necessitate multiple sources of stable and reliable power supply from electrical distribution sites to sustain their normal operation. In addition, power network sites rely on monitoring and information transmission from various sites in the communication network for remote control and failure monitoring. When a power network initially experience failure, communication base stations lose their power supply, resulting in communication interruptions, which may affect the stable supply of electricity and disrupts the normal operation of the entire system<sup>34–39</sup>. For this reason, the functionality of nodes relies not only on the connectivity of the network itself but also on the multiple supports from its coupled system after undergoing the initial failure<sup>40–47</sup>. For example, in the corporate and investment network, mutual support between companies and various financial institutions is crucial for ensuring smooth operations. Companies typically require financial support from various financial institutions for their regular operations. Similarly, the smooth functioning of financial institutions depends on transactions with multiple companies. It is essential to emphasise that a major crisis at a financial institution in the network could lead to a break in the financial chain of the company, preventing it from functioning normally, or even risking its insolvency. And when the company goes bankrupt, the financial investment institutions are likely to be unable to recover the principal and interest of their loans or investments, which will lead to huge economic losses for the financial investment institutions<sup>48</sup>. Another example, within the supply chain network, maintaining support stable logistics support between core enterprises and various suppliers is imperative for the overall functionality of the network. For example, an interruption in energy supply may some enterprises experience declining profitability or even bankruptcy. And if multiple businesses go bankrupt, it can lead to a decrease in profits for energy suppliers and eventually force them to either change their business focus or cease operations altogether<sup>49,50</sup>.

Robustness is an essential component of network science and has been explored theoretically and numerically in a variety of contexts. Yet, the robustness of the coupled systems with multiple support remains poorly investigated, especially in relation to real-world scenarios. To bridge this research gap,

we propose an analytical framework that describes the cascading failure of a system through both theoretical and numerical simulations. The framework models several mechanisms that may affect nodes in a network, emphasising the importance of belonging to a component of a certain size after an attack, and of multiple effective external support edges. Furthermore, we examine in detail the effects of both network component size (connectivity size) and external multiple support edges on the system robustness.

The paper is organized in five sections. Section II presents our model description. Section III focuses on the simulation and theoretical analysis. Finally, we present our conclusions in Section IV.

## II. MODEL AND THEORETICAL METHODS

### A. Model

We consider two networks referred to as network  $A$  and network  $B$ , characterized by the degree distributions  $P_A(k)$  and  $P_B(k)$ , respectively. The two networks mutually support each other, as network  $A$  provides multiple support to network  $B$  following the degree distribution  $\tilde{P}_A(\tilde{k}_A)$ . Network  $B$  reciprocally offers multiple support to network  $A$  with the degree distribution,  $\tilde{P}_B(\tilde{k}_B)$ . In the following, we consider the case where edges inside each network are undirected whereas the edges between the networks are directed.

Due to the multiple dependencies of the nodes between the networks, we assume that functional nodes require support from at least  $M$  supporting links from another network. Moreover, the connectivity within their own network also plays a role, and a node remains active if the size of its connected component is above a predetermined threshold. More precisely, the conditions for a node to be functional within a network are as follows: (i) belonging to a connected component with a size no smaller than  $s$ ; (ii) receiving at least  $M$  effective links from the other network. Fig. 1 illustrates this model with the size of network  $N_A = N_B = 7$ ,  $M = 2$ , and  $s = 2$  during the cascading failure process, and the network eventually reaches a stable state, as shown in Fig. 1(d).

### B. Theoretical methods

The networks  $A$  and  $B$  undergo random attacks, resulting in the failure of the fractions  $(1 - p^A)$  and  $(1 - p^B)$  of nodes in each network. To consider the cascading failures in the coupled networks, we first define the fraction of nodes in the network possessing a minimum of  $M$  effective supporting links at step  $n$  as  $y^A$  and  $y^B$

$$\begin{cases} y^A = p^A(1 - r_{n,M}^A), \\ y^B = p^B(1 - r_{n,M}^B), \end{cases} \quad (1)$$

where  $(1 - r_{n,M}^A)$  and  $(1 - r_{n,M}^B)$  denote the probabilities of possessing at least  $M$  effective supporting links from network  $B$

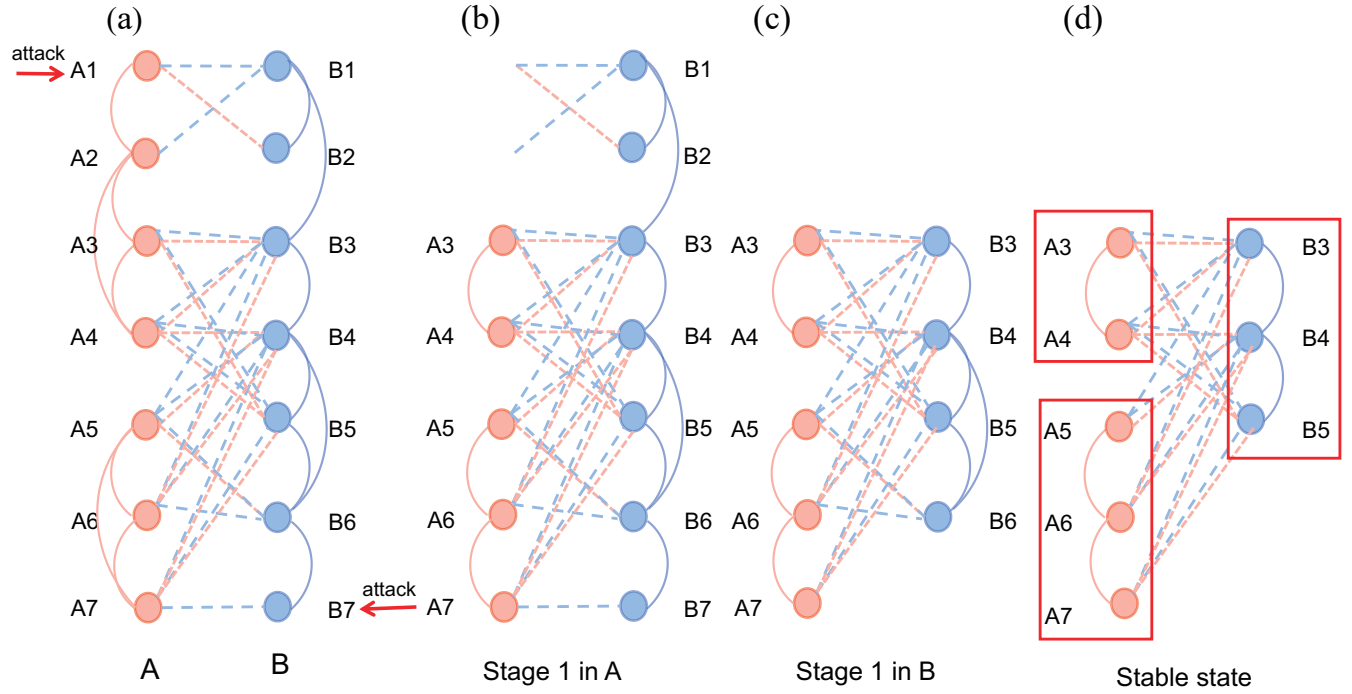


FIG. 1. Schematic illustration of cascading failure within a coupled network characterized by multiple-support dependent links, exemplified by two networks  $A$  and  $B$ , where  $N_A = N_B = 7$ . The solid curves and dashed lines denote the connectivity links within each network and multiple support links between networks, respectively. The red (blue) dashed lines represent the support links from network  $A$  ( $B$ ) to network  $B$  ( $A$ ). (a) Nodes  $A1$  and  $B7$  are initially attacked (red arrows) and become nonfunctional. (b) During stage 1 of network  $A$ , Node  $A2$  also experiences failure as it does not belong to a component with a size  $s \geq 2$ . (c) During the stage 1 of network  $B$ ,  $B7$  fails by attack in network  $B$ , and the failure of these nodes in the network  $A$  triggers a reduction of valid supporting edges for nodes  $B1$ ,  $B2$ ,  $B6$  since these nodes have only one effective supporting link from the network  $A$ , even though these three nodes belong to the component of size  $s \geq 2$ , causing these three nodes to fail. This cascading failure develops iteratively between networks and eventually reaches a stable state. At that stage, all nodes (surrounded by the red squares) within the networks concurrently satisfy the conditions for their functionality, as shown in Fig. 1(d).

and  $A$  during the stage  $n$ . These quantities can be calculated as

$$\begin{cases} 1 - r_{n,M}^A = 1 - r_{n,M-1}^A \\ \quad - \sum_{\tilde{k}_A=M-1}^{\infty} \binom{\tilde{k}_A}{M} \tilde{P}_A(\tilde{k}_A - M + 1) (1 - S_{n-1,M,s}^B)^{\tilde{k}_A} \\ \quad \times (S_{n-1,M,s}^B)^{(M-1)}, \\ 1 - r_{n,M}^B = 1 - r_{n,M-1}^B \\ \quad - \sum_{\tilde{k}_B=M-1}^{\infty} \binom{\tilde{k}_B}{M} \tilde{P}_B(\tilde{k}_B - M + 1) (1 - S_{n-1,M,s}^A)^{\tilde{k}_B} \\ \quad \times (S_{n-1,M,s}^A)^{(M-1)}, \end{cases} \quad (2)$$

where  $1 - r_{n,M-1}^A (1 - r_{n,M-1}^B)$  denotes the probability of node within network  $A$  ( $B$ ) with at least  $(M-1)$  effective support link. The last term gives the fraction of nodes in network  $A$  ( $B$ ) have only  $(M-1)$  effective supporting links. Here,  $S_{n,M,s}^A$  and  $S_{n,M,s}^B$  represent the fraction of functional nodes in network  $A$  and network  $B$  at the stage  $n$ , respectively.

The expressions for the giant component with the maximum connectivity in the remaining network after cascading failure can be obtained from the network's generating func-

tion

$$\begin{cases} g_n^A(y^A) = 1 - G_A[1 - y^A(1 - f_A)], \\ g_n^B(y^B) = 1 - G_B[1 - y^B(1 - f_B)], \end{cases} \quad (3)$$

where  $G_A(x) = \sum_{k=0}^{\infty} P_A(k)x^k$  and  $G_B(x) = \sum_{k=0}^{\infty} P_B(k)x^k$  are the generating functions of network  $A$  and  $B$  with degree distributions  $P_A(k)$  and  $P_B(k)$ , respectively<sup>46</sup>. Here,  $f_A = H_A[1 - y^A(1 - f_A)]$ ,  $f_B = H_B[1 - y^B(1 - f_B)]$  represent the probability that a functional node is not in the giant component of each network. And  $H_A(x) = \frac{G_A'(x)}{G_A'(1)}$  and  $H_B(x) = \frac{G_B'(x)}{G_B'(1)}$  are the generating function of excess degree distribution of network  $A$  and  $B$ .

Since a functional node requires not only multiple external support edges but also to belong to components of a given connectivity size, one also needs the generating function for the distribution of component size

$$C_i(x, y) = \sum_{s=1}^{\infty} \pi_{i,s}(y) x^s = x G_i[B_i(x, y)y + 1 - y], \quad (4)$$

where  $B_i(x, y)$  satisfies the recursive equation  $B_i(x, y) = x H_i[B_i(x, y)y + 1 - y]$ ,  $i = A$  and  $B$ , and  $\pi_{i,s}(y)$  represents the probability of a node existing within a functional component

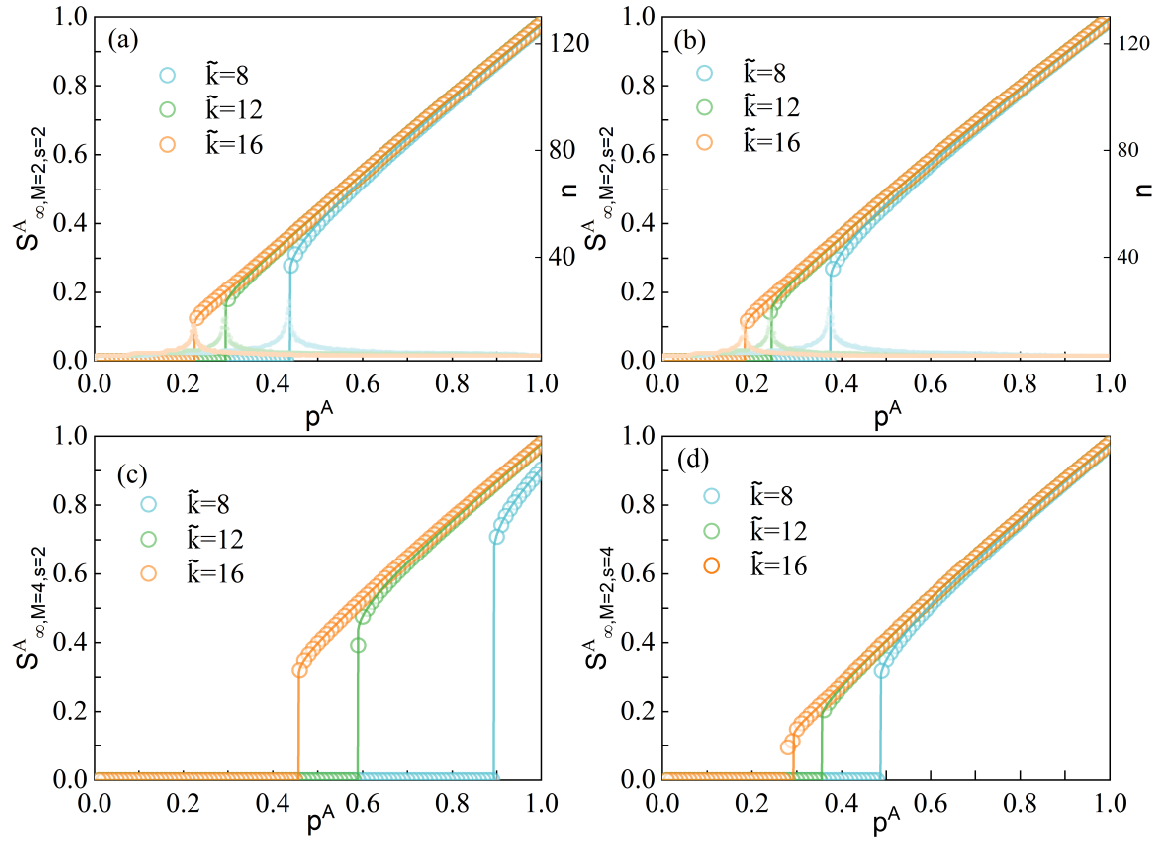


FIG. 2. For coupled homogeneous networks with multiple dependency relationship,  $S_{M,s}^A$  as a function with  $p^A$  for different parameters of  $M$ ,  $s$ , and  $\tilde{k}$  from Eq. 10. Here, left y-axis is the fraction of effective nodes at stable state of network A, right y-axis is the number of iterative failures. (a)  $M = 2$ ,  $s = 2$ ,  $\tilde{k} = 4$ ; (b)  $M = 2$ ,  $s = 2$ ,  $\tilde{k} = 6$ ; (c)  $M = 4$ ,  $s = 2$ ,  $\tilde{k} = 4$ ; (d)  $M = 2$ ,  $s = 4$ ,  $\tilde{k} = 4$ . Here, the network size is  $N_A = N_B = 10^6$  and  $1 - p^A = 2(1 - p^B)$ . Simulation results (symbols) agree well with theoretical results (solid lines). Simulation results are averaged over 100 independent realisations with  $N = 10^5$ .

of size  $s$  in a network for the fraction  $y$  of surviving nodes<sup>46</sup>. It is noteworthy that when  $x = 1$ , Eq. 4 can be written as

$$C_i(1, y) = \sum_{s=1}^{\infty} \pi_{i,s}(y) = 1 - g^i(y), \quad (5)$$

and by employing the Lagrange inversion formula, one can get

$$\pi_{i,s}(y) = \frac{y\tilde{k}}{(s-1)!} \frac{d^{s-2}}{dx^{s-2}} [H_i(xy + 1 - y)]^s \Big|_{x=0}, \quad (6)$$

Where  $s \geq 2$ , and  $H_i(xy + 1 - y)$  represent excess degree distribution of the remaining network. For  $s=1$ ,  $\pi_1(y) = G(1 - y)$ . Note that one can use  $g_{n,s}^i(y^i)$ , the fraction of nodes in functional component with size not smaller than  $s$ , to substitute  $g^i(y^i)$ , so that Eq. 3 can be expressed as

$$\begin{cases} g_{n,s}^A(y^A) = 1 - \sum_{r=1}^{s-1} \pi_{A,s}(y^A), \\ g_{n,s}^B(y^B) = 1 - \sum_{r=1}^{s-1} \pi_{B,s}(y^B). \end{cases} \quad (7)$$

The fraction of effective nodes at step  $n$  of each network can thus be describe as

$$\begin{cases} S_{n,M,s}^A = y^A g_{n,s}^A(y^A), \\ S_{n,M,s}^B = y^B g_{n,s}^B(y^B). \end{cases} \quad (8)$$

### III. THEORETICAL RESULTS

In this section, we apply the above theoretical framework to investigate the robustness of different families of networks, homogeneous and heterogeneous, and their coupling. As networks can face varying attack scenarios, we assume here that the attack on network A is twice more intense than on network B, thus yielding  $(1 - p^A) = 2(1 - p^B)$ . For simplicity, we make this assumption that the size of both networks  $N_A = N_B = N$ . Additionally, The simulation results are obtained by averaging the outcomes of 100 independent runs, in order to mitigate errors originating from stochastic fluctuations.



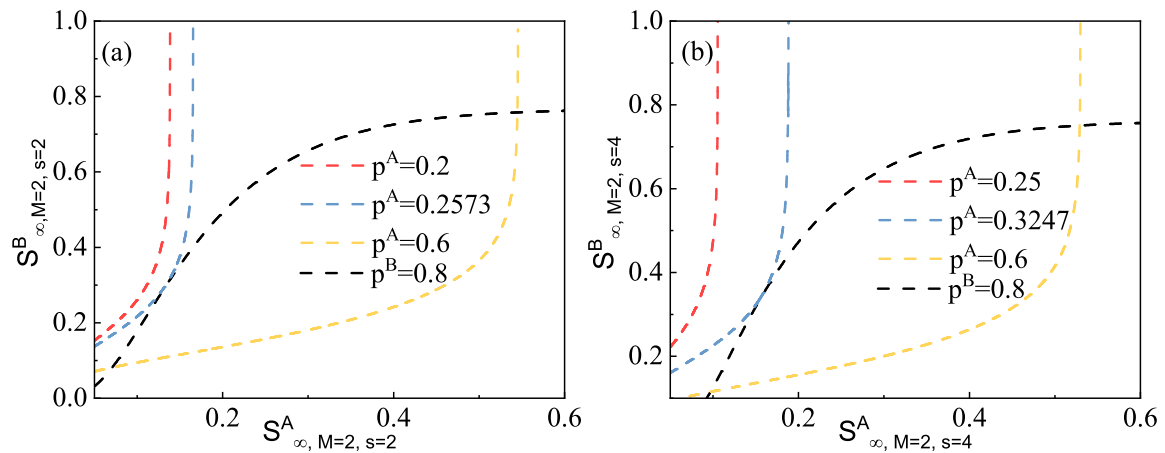


FIG. 3. The relationship between  $S_{n,M,s}^A$  and  $S_{n,M,s}^B$  for Eq.10 with different  $k_A = k_B = \bar{k} = 4$ ,  $\tilde{k}_A = \tilde{k}_B = \tilde{k} = 12$  demonstrates that the critical threshold occurs when the functional components of the system transition from a finite value to zero.

(a)  $M = 2, s = 2$ , (b)  $M = 2, s = 4$ .

#### A. Coupled homogeneous networks with a Poisson degree distribution

As a first step, we focus on homogeneous networks, where nodes and edges share similar characteristics and properties. For this purpose, we consider networks with a Poisson degree distribution, where nodes exhibit a comparable number of connections<sup>46</sup> and are statistically equivalent. We here assume that degree distributions within the networks  $A$  and  $B$  are uniform and following the Poisson distribution,  $P_A(k) = \frac{\tilde{k}_A^k}{k!} e^{-\tilde{k}_A}$  and  $P_B(k) = \frac{\tilde{k}_B^k}{k!} e^{-\tilde{k}_B}$ . Due to the multiple dependent connections between networks, nodes in each network have various multiple-support dependent links. Similarly, the dependency degree distribution of nodes in network  $B$  supported by network  $A$  is  $\tilde{P}_A(\tilde{k}_A) = \frac{\langle k_A \rangle^{\tilde{k}_A}}{\tilde{k}_A!} e^{-\langle k_A \rangle}$ , and the distribution of degrees in  $B$  supported by  $A$  is  $\tilde{P}_B(\tilde{k}_B) = \frac{\langle k_B \rangle^{\tilde{k}_B}}{\tilde{k}_B!} e^{-\langle k_B \rangle}$ . For the networks with a Poisson degree distribution, the generating function can be expressed as  $G(x) = H(x) = e^{\langle k \rangle(x-1)}$ <sup>46</sup>, so that Eq. 6 can be reformulated as

$$\pi_{ER,s}(y) = \frac{(sy\bar{k})^{s-1} e^{-sy\bar{k}}}{s!}. \quad (9)$$

By substituting this into Eq. 8, one can derive

$$\begin{cases} S_{\infty, M, s}^A = y^A (1 - \sum_{r=1}^{s-1} \frac{(sy^A \bar{k})^{s-1} e^{-sy^A \bar{k}}}{s!}), \\ S_{\infty, M, s}^B = y^B (1 - \sum_{r=1}^{s-1} \frac{(sy^B \bar{k})^{s-1} e^{-sy^B \bar{k}}}{s!}). \end{cases} \quad (10)$$

Fig. 2 shows that the fraction  $S_{M,s}^A$  of functional nodes as a function of  $p^A$  at stable state in network  $A$ , one can observe that the analytical results (lines) agree well with the simulations (symbols). Moreover, as  $p^A$  increases, functional components within network  $A$  exhibit a first-order phase transition behavior from finite value to zero at the critical threshold  $p_c^A$ . A low critical threshold denotes that the system preserves its

connectivity ( $S_{\infty, M, s}^A > 0$ ) even after enduring substantial attack strength, signifying superior robustness in withstanding large-scale failure. Conversely, a high critical threshold suggests that even minor attack intensities prompt system collapse,  $S_{\infty, M, s}^A = 0$  and that the system is vulnerable. From Fig. 2(a) and (b), it is evident that  $p_c^A$  increases with the average dependency degree  $\bar{k}$ . This result indicates that the system becomes more robust as the dependency strength between networks increases. Comparing Figs. 2(a) and 2(b), one observes that  $p_c^A$  decreases as  $\bar{k}$  increases, suggesting that the system becomes more robust for larger network connection density. Therefore, by increasing the internal connectivity density within the system and the density of dependent links between networks, the robustness of the system can be significantly enhanced. Moreover, comparing Fig. 2(a) and 2(c), it can be found that there is a discontinuity even in the case where the effective supporting links  $M$  is different for the same parameters. As  $M$  becomes larger, we find that  $p_c^A$  becomes smaller, which means that the more effective external dependent links required to sustain node survival, the more fragile the system becomes. It is also clear from Fig. 2(a) and 2(d) that an increase in the size of the functional component ( $s$ ) makes the system more vulnerable, even when keeping the same number of effective supporting links ( $M$ ). This indicates that when a system undergoes an attack, its capacity to maintain connectivity of more functional components of diverse scales (larger  $s$ ) shows greater robustness (smaller  $p_c^A$ ) when the system undergoes failure. From Figs. 2(a) and 2(b), the number  $n$  of iterative failures are shown for a homogeneous network with different  $p^A$ . At the first-order transition threshold, the number of iterative failures that the network undergoes before disintegrating displays a nature of dramatic increase. This number sharply drops as the distance from the transition is increased. Thus, plotting the number of iterations  $n$  as a function of  $p$  provides a useful method for identifying the transition point,  $p_c^A$ , at the first-order region.

From Fig. 2, one observes that at the critical threshold, the

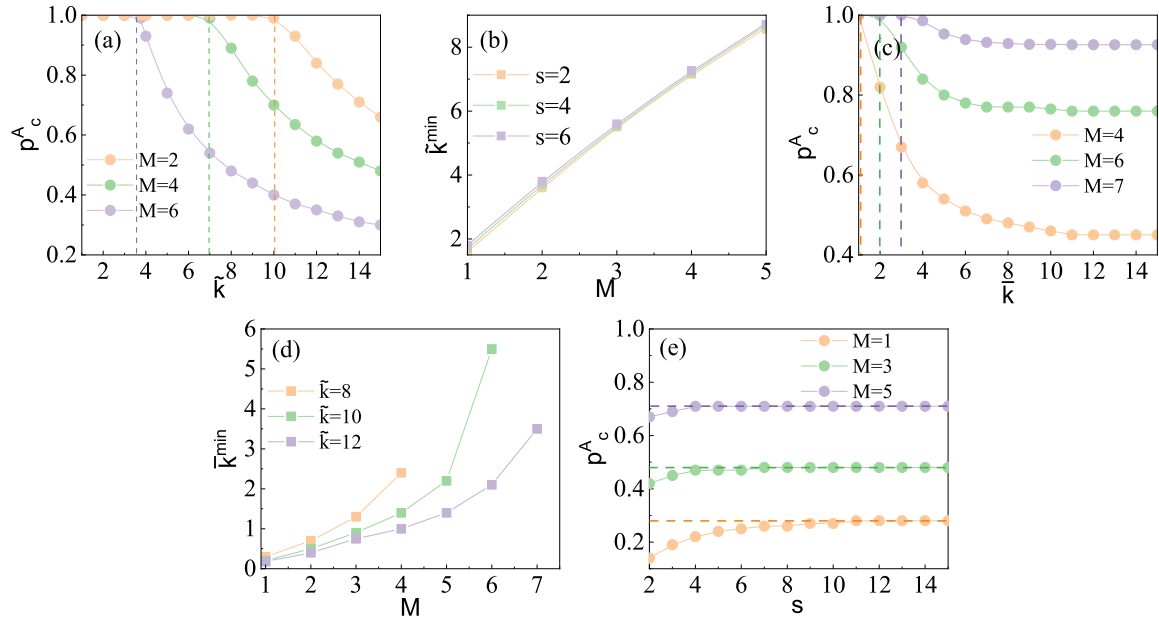


FIG. 4. (a) Critical points  $p_c^A$  as a function of  $\tilde{k}$  with  $s = 4$  and  $\tilde{k} = 4$ . (b) Minimal  $\tilde{k}^{min}$  as a function of  $M$  and  $\tilde{k} = 4$ . (c)  $p_c^A$  as a function of  $M$  with  $s = 4$  and  $\tilde{k} = 12$ . (d) Minimal  $\tilde{k}^{min}$  as a function of  $M$  and  $s = 4$ . (e)  $p_c^A$  as a function of  $s$  with  $\tilde{k} = 4$  and  $\tilde{k} = 12$ . The solid lines are theoretical predictions for the same parameters obtained from Eq. 10. The simulation results average over 100 independent realisations.

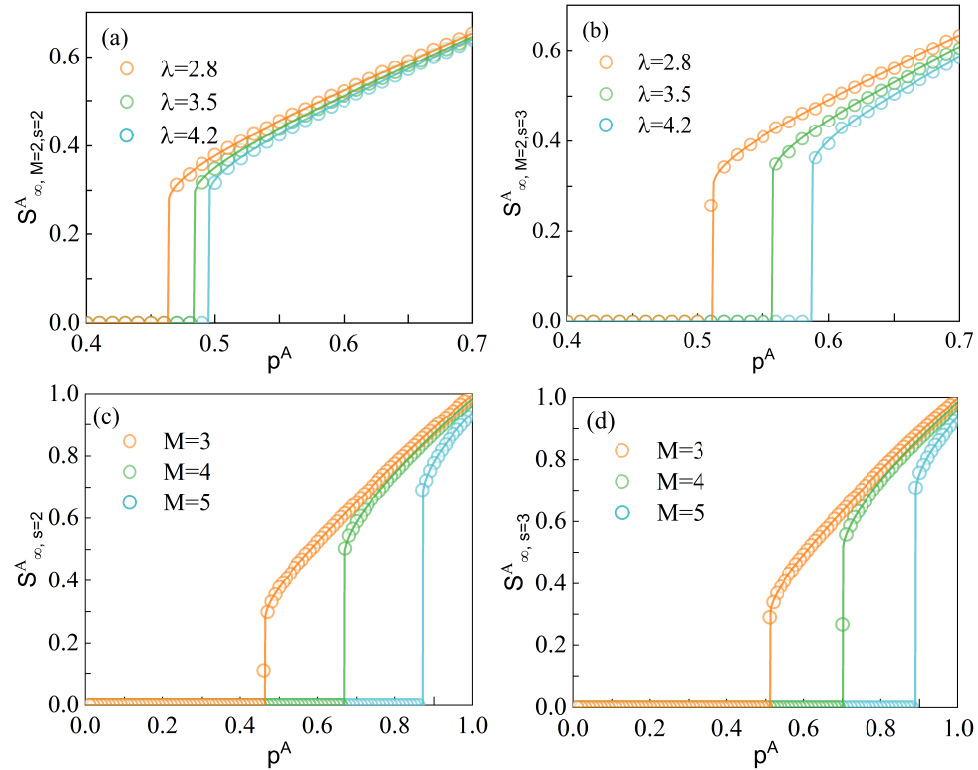


FIG. 5. For coupled heterogeneous networks with multiple dependency relationship,  $S_{\infty, M, s}^A$  as a function of  $p^A$  for different parameters of  $\lambda$  and  $M$  for Eq. 13. (a)  $M = 2$ ,  $s = 2$ ,  $\tilde{k}_A = \tilde{k}_B = \tilde{k} = 12$ ; (b)  $M = 4$ ,  $s = 2$ ,  $\tilde{k}_A = \tilde{k}_B = \tilde{k} = 12$ ; (c)  $s = 2$ ,  $\tilde{k}_A = \tilde{k}_B = \tilde{k} = 8$  and  $\lambda = 2.8$ ; (d)  $s = 3$ ,  $\tilde{k}_A = \tilde{k}_B = \tilde{k} = 8$  and  $\lambda = 2.8$ . Simulation results (symbols) agree well with theoretical results (solid lines). Simulation results are averaged over 100 independent realisations  $N = 10^5$ .

functional components of the system transition from a finite value to zero. To understand this behaviour, we show graphically Eq. 10 in Fig. 3. As  $p_c^A$  increases, the intersection between  $S_{n,M,s}^A$  and  $S_{n,M,s}^B$  gradually evolves from nonexistence to a singular tangent intersection and then bifurcates into two intersections. For Fig. 3(a), this implies that when  $p^A = 0.2 < 0.2573$ , the solution to Eq.10 is zero and  $S_{n,M,s}^A = 0$ . However, when  $p^A > 0.2573$ , there exist two solutions. The largest solution is the physical solution, and the iterative process starting from  $S_{0,M,s}^A = S_{0,M,s}^B = 1$  converges only to the larger of these two solutions. When  $p^A = 0.2573$ ,  $S^A$  abruptly transitions from zero to a finite value, indicating that  $p^A = p_c^A = 0.2573$  is the critical threshold of the system. The critical threshold can thus be identified from the condition

$$\frac{dS_{\infty,M,s}^A}{dS_{\infty,M,s}^B} \times \frac{dS_{\infty,M,s}^B}{dS_{\infty,M,s}^A} = 1. \quad (11)$$

When  $s = 4$ , one can also employ this method to determine the critical threshold, as illustrated in Fig. 3(b).

Furthermore, Fig. 4 illustrates the relationship between the critical threshold and sub-network's average degree  $\bar{k}$ , the average dependency degree  $\tilde{k}$ , the effective supporting links  $M$  and the size of effective finite functional component. The robustness of the network increases as the number of effective supporting edges  $M$  decreases, as observed by Fig.4(a). Moreover, as the average dependency  $\tilde{k}$  decreases, the critical threshold  $p_c^A$  gradually increases. Furthermore, we find that there exists a minimum average dependency degree  $\tilde{k}^{min}$  for the critical threshold  $p_c^A = 1$ , as in Fig.4(b), where the system is extremely fragile, and the whole system collapses even if one node is failed. It can be seen that the minimum average dependency  $\tilde{k}^{min}$  that supports the survival of the system's functions becomes larger as  $M$  increases. This suggests that when a greater number of effective supporting links ( $M$ ) is required, the system needs to increase the minimum degree  $\tilde{k}^{min}$  to preserve the connectivity of the system. It is also noted that for different  $s$ , there is no significant change in  $\tilde{k}^{min}$ , indicating that the functional component size  $s$  has little effect on the  $\tilde{k}^{min}$ .

Similarly, as the density of internal connections  $\bar{k}$  increases, the system becomes stronger and shows more resilience to external damage, as seen in Fig.4(c). It can be seen that  $p_c^A$  decreases as  $\bar{k}$  increases. In addition,  $\tilde{k}^{min}$  as a function of  $M$  is shown in Fig.4(d), and the results indicate that  $\tilde{k}^{min}$  increases with increasing  $M$ .

These findings highlight that the system requires a larger  $\tilde{k}^{min}$  to avoid system breakdown. As  $M$  increases, it is particularly noteworthy that the system also necessitates a higher density of inner connections to prevent the system from collapsing after suffering severe damage. Additionally, from Fig. Fig. 4(e), one can observe that as  $s$  increases,  $p_c$  gradually increases to the stable value. The increase in larger component sizes, consequently resulting in an increase in  $p_c^A$ , which gradually approaches the critical point of the giant component  $s \rightarrow \infty$ , as indicated by the dashed line in Fig. 4(e).

## B. Coupled heterogeneous networks with a power-law degree distribution

We now turn to the analysis of heterogeneous networks that exhibit fat-tailed degree distributions, and provide a more realistic framework for the modeling of real-world systems. Here, we will assume that the degree distribution inside each network follows the same power-law degree distribution  $P(k) = \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}}$ , where  $r$  and  $R$  denote the minimum and maximum degree values,  $\lambda$  is the power-law exponent. The degree distributions for the dependency degree distributions across the networks are Poissonian and the same as in the previous section.

The generating function of the degree distribution and the excess degree distribution of the scale-free (SF) network are shown in as follows, respectively<sup>26</sup>

$$\begin{cases} G_0(x) = \sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}} x^k, \\ H_l(x) = \frac{\sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}} k x^{k-1}}{\sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}} k}. \end{cases} \quad (12)$$

From Eqs. (6), (7) and (8), one finds

$$\begin{cases} S_{\infty,M,s}^A = y^A \left( 1 - \sum_{r=1}^{s-1} \frac{y^A \bar{k}}{(s-1)!} \frac{d^{s-2}}{dx^{s-2}} \left[ \frac{\sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}} k x^{k-1}}{\sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}} k} \right] \right)_{x=0}, \\ S_{\infty,M,s}^B = y^B \left( 1 - \sum_{r=1}^{s-1} \frac{y^B \tilde{k}}{(s-1)!} \frac{d^{s-2}}{dx^{s-2}} \left[ \frac{\sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}} k x^{k-1}}{\sum_{k=0}^{\infty} \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(R+1)^{1-\lambda} - r^{1-\lambda}} k} \right] \right)_{x=0}. \end{cases} \quad (13)$$

Fig. 5 shows the proportion of different functional component as a function of  $p_c^A$  across various parameters, showing a good alignment between analytical results (lines) and the simulations (symbols) for the coupled SF network. Furthermore, as the fraction of remaining nodes ( $p^A$ ) increases, network  $A$  undergoes a first-order phase transition behavior for these finite components. The critical threshold point  $p_c^A$  gradually decreases with a reduction in the power-law exponent  $\lambda$  of the coupled networks, as shown in Figs. 5(a). This indicates that the higher the heterogeneity in the connectivity of the network, the stronger the robustness of the system. From comparing Fig. 5(a) with Fig. 5(b), we can find that the vulnerability of the system will increase with the number of functional connected functional components increases (larger  $s$ ). Increasing the number of effective supporting links  $M$ , it can be seen that the critical threshold of the network gradually increases, as shown in Fig. 5(c). This observation aligns with the intuitive understanding that an increase in the number of effective supported edges imposes greater demands on the node effective condition, consequently undermining the system's robustness after occurring a failure. Fig. 5(c) comparing to Fig. 5(d), one can observe that as  $s$  increases leads to a corresponding increase in the critical threshold  $p_c^A$  within the network. This implies that an increase in the functional component size  $s$  will lead to a decrease in the robustness of the system. For different values of  $M$  and  $\lambda$ ,  $p_c^A$  becomes larger as

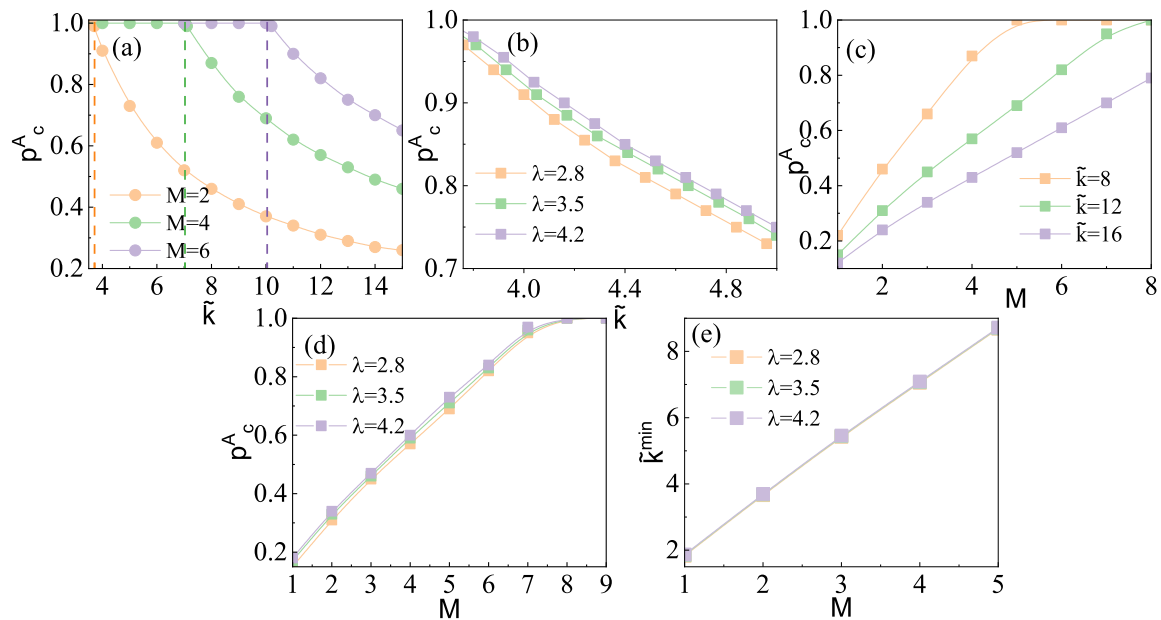


FIG. 6. Critical points  $p_c^A$  as a function of different parameters  $\tilde{k}$  and  $M$  for Eqs. 11 and 13. (a)  $s = 2$ ,  $\lambda = 2.8$ , (b)  $s = 2$ ,  $M = 2$ , (c)  $s = 2$ ,  $\lambda = 2.8$ , (d)  $s = 2$ ,  $\tilde{k} = 12$ . (e) Minimal  $\tilde{k}^{\min}$  as a function of  $M$  and  $s = 2$ . The solid lines are theoretical predictions for the same parameters obtained from Eq.(13). The simulation results average over 50 independent realisations.

$\tilde{k}$  decreases, and the system becomes more fragile, as shown in Figs. 6(a) and (b). This phenomenon suggests that reducing  $\lambda$  and the number of effective supporting links  $M$  can significantly enhance the robustness of the system. And, when  $\tilde{k}$  is decreased to a  $\tilde{k}^{\min}$  (dash lines in Fig. 6(a)) the critical threshold  $p_c^A = 1$ , the system exhibits extreme fragility. And, this also suggests that reducing  $\lambda$  and the number of effective supporting links  $M$  can enhance the robustness of the system.

Fig. 6(c) demonstrates an approximate linear correlation between  $p_c$  when  $M$  is less than the average dependency degree  $\tilde{k}$ . This implies that as the parameters  $\tilde{k}$  and  $\lambda$  are gradually strengthened,  $p_c^A$  also gradually increase. And, as depicted in Fig. 6(d), one can notice when the value of  $M$  approaches  $\tilde{k}$  and exceeds  $M = 9$ , the initial network collapses upon attack, highlighting the significance of constraining the count of effective supporting links  $M$ . One can also notice that there exists a minimum average dependency degree  $\tilde{k}^{\min}$  corresponds to the critical threshold  $p_c^A = 1$ , as in Fig. 6(e), where the system is extremely fragile. It can be seen that the minimum average dependency  $\tilde{k}^{\min}$  that supports the survival of the system's functions becomes larger as  $M$  increases.

#### IV. CONCLUSION

In this study, we have introduced a novel model for the robustness of coupled networks. The model describes realistic scenarios where maintaining the normal functioning of nodes in a system requires not only the support of a given number of links from an external network, but also that the connectivity inside the node's network remains above a certain value in or-

der to resist cascade failures caused by external failures. We have investigated the effect of the model parameters, essentially the coupling inside and across networks on robustness. Through numerical simulations performed for coupled homogeneous and heterogeneous network systems, we have validated our analytical predictions and shown a rich behaviour where cascade failure may lead to the collapse of the system. We have found that after initial failure and cascading failure, the system exhibits discontinuous behaviours. We have found that when the system requires more effective supporting links, the system also requires more inner and inter-connectivity density to avoid collapse. It is also found that increasing the inner and inter-connectivity density while decreasing the size of functional components and the number of effective supporting links can significantly enhance the robustness of the system. The model assumes that the functional dependence on multiple (at least  $M$ ) support links from the coupled network plays the same role. This will help us to better understand and design more resilience coupled complex systems. Furthermore, mutual support in real coupled systems may involve different resources or functionalities. For the sake of convenience, we assume in this paper that these impacts are consistent. For the case of inconsistency, it will be discussed in our future research.

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## DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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