

# LARGE SUMSETS FROM SMALL SUBSETS

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ABSTRACT

In this paper we start to investigate a new body of questions in additive combinatorics. The fundamental Cauchy–Davenport theorem gives a lower bound on the size of a sumset  $A + B$  for subsets of the cyclic group  $\mathbb{Z}_p$  of order  $p$  ( $p$  prime), and this is just one example of a large family of results. Our aim in this paper is to investigate what happens if we restrict the number of elements of one set that we may use to form the sums. Here is the question we set out to answer: given two subsets,  $A$  and  $B$ , does  $B$  have a subset  $B'$  of bounded size such that  $A + B'$  is large, perhaps even comparable to the size of  $A + B$ ? In particular, can we get close to the lower bound of the Cauchy–Davenport theorem?

Our main results show that, rather surprisingly, in many circumstances it is possible to obtain not merely an asymptotic version of the usual sumset bound, but even the exact bound itself. For example, in  $\mathbb{Z}$ , we show that if  $A$  and  $B$  have size  $n$  then there are three elements  $b_1, b_2, b_3 \in B$  such that  $|(A + b_1) \cup (A + b_2) \cup (A + b_3)| \geq 2n - 1$ . And for  $\mathbb{Z}_p$  itself, we show the following: if  $A$  and  $B$  have size  $n$ , where  $n \leq p/3$ , then there is a subset  $B'$  of  $B$  of size  $c$  such that  $|A + B'| \geq 2n - 1$ . Here  $c$  is an absolute constant. In the inverse direction for this result, we show that if for every subset  $B'$  of  $B$  of size  $c$  we have  $|A + B'| \leq 2n - 1 + r$ , where  $r \leq \varepsilon n$  for some absolute constant  $\varepsilon$ , then  $B$  is contained in an arithmetic progression of size  $n + r$ . We also prove ‘unbalanced’ forms of our results, when the sizes of  $A$  and  $B$  may differ.

As an application, we prove some considerable extensions of the Erdős–Heilbronn problem. We also present versions in the continuous setting, and give several open problems.

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## 1. Introduction

The aim of this paper is to introduce a new direction in the study of sumset sizes, by asking what happens when we may only use a bounded number of terms from one of the sets. Our main interest is in sumsets in  $\mathbb{Z}_p$ , where, as usual,  $p$  is a prime number; in fact, throughout this paper,  $p$  will always stand for an arbitrary prime. We start with some background.

Cauchy [14] was the first to study sums of subsets in  $\mathbb{Z}_p$ ; over one hundred years later, his result was rediscovered by Davenport [15, 16]. Their result asserts that if  $\emptyset \neq A, B \subset \mathbb{Z}_p$  and  $|A| + |B| \leq p + 1$  then

$$(1) \quad |A + B| \geq |A| + |B| - 1.$$

Proving this for sets of integers, without any restrictions on their sizes, is entirely trivial, and may be viewed as an analogue of the one-dimensional case of the Brunn–Minkowski inequality.

The Cauchy–Davenport theorem was followed by important contributions concerning sums of subsets of groups, including  $\mathbb{Z}$  itself, by Mann [46, 47], Kneser [39], Vosper [65, 66], Erdős and Heilbronn [19], Freiman [21, 22, 23], Plünnecke [53], Ruzsa [56], and others until the 1990s, when the subject really took off (see e.g., [2, 3, 34, 45, 52, 57, 60, 62, 63]). For various discrete analogues of the Brunn–Minkowski inequality, see [8, 9, 27, 28, 31, 35, 37] and many other results; for surveys of the inequality itself, see [5, 26].

One should distinguish between ‘direct’ results, giving lower bounds for subset sums, and ‘inverse’ results, that characterise the cases when the subset sum is close to its minimum (where ‘close’ may be interpreted in several different ways). Starting with Vosper [65] and Freiman [21, 22, 23, 24], much research has been done on inverse results: see Nathanson [48, 49, 50], Bilu, Lev and Ruzsa [7], Bilu [6], Nathanson and Tennenbaum [51], Breuillard, Green and Tao [13] and Serra and Zémor [60], among others.

Considerable attention has also been given to ‘restricted’ sumsets, meaning sumsets in which we do not consider the sum of every pair. The best-known example of this is the Erdős–Heilbronn conjecture [19], proved by Dias da Silva and Hamidoune [17], and then by Alon, Nathanson and Ruzsa [1]. Restricted sumsets sit at the core of the Balog–Szemerédi–Gowers theorem [4, 29], which is a fundamental tool in additive combinatorics. This notion has also been explored by several authors including Freiman, Low and Pitman [25], Lev [41, 42, 43], Károlyi [38], Vu and Wood [67], Tao [63], Griesmer [32] and Shao [61].

In this paper we wish to find out what happens if in (1) or its analogues we are allowed to use only boundedly many elements of  $B$ . This question is of interest not only in  $\mathbb{Z}_p$ , but also in additive groups and even in  $\mathbb{Z}$ .

In the rest of this section we describe our main results.

1.1. DIRECT RESULTS IN  $\mathbb{Z}$ . We start in the simplest place,  $\mathbb{Z}$  itself. If we are summing just one set  $A$  with itself, then of course with  $a_1 = \min A$  and  $a_2 = \max A$  we have that the sets  $A + a_1$  and  $A + a_2$  meet only at  $a_1 + a_2$ , so that

$$|A + \{a_1, a_2\}| = 2|A| - 1.$$

But with a sumset  $A + B$  of two sets then it is no longer the case that for some two elements  $b_1, b_2$  of  $B$  we have that  $|A + \{b_1, b_2\}| = |A| + |B| - 1$ , even when the sets are the same size. For example, taking  $n$  to be a multiple of 3, if  $A$  is the set  $[1, 2n/3] \cup [n, 4n/3]$  then it is easy to see that any translate of  $A$  by a distance  $d$ , where  $0 \leq d \leq n$ , meets  $A$  in at least  $n/3$  elements. Hence if  $B$  is the interval  $[0, n]$  then  $A$  and  $B$  have size  $n + 1$  and for any  $b_1, b_2 \in B$  we have that

$$|A + \{b_1, b_2\}| \leq |A| + |B| - n/3.$$

In light of this simple example, it is very surprising that, if we move up to three elements of  $B$ , then in fact we can recover the exact lower bound in (1). (Of course, we will be taking  $|A| \geq |B|$ , since if  $B$  is much larger than  $A$  then the lower bound in (1) is greater than  $3|A|$ , which is the most one could ever obtain by taking the union of three translates of  $A$ ). This is our first result.

**THEOREM 1:** *Let  $A$  and  $B$  be finite non-empty subsets of  $\mathbb{Z}$  with  $|A| \geq |B|$ . Then there exist elements  $b_1, b_2, b_3 \in B$  such that*

$$|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1.$$

Actually, if  $B$  is allowed to be larger than  $A$ , but only by a given ratio, then a bounded number of elements of  $B$  does still suffice to obtain the bounds in (1).

**THEOREM 2:** *For every  $\alpha > 0$  there exists  $c$  such that, whenever  $A$  and  $B$  are finite non-empty subsets of  $\mathbb{Z}$  with  $|A| \geq \alpha|B|$ , there exist elements*

$$b_1, b_2, \dots, b_c \in B$$

such that

$$|A + \{b_1, \dots, b_c\}| \geq |A| + |B| - 1.$$

1.2. DIRECT RESULTS IN  $\mathbb{Z}_p$ . Let us now turn to the deeper question of what happens in the cyclic group  $\mathbb{Z}_p$  of prime order. Can we again recover the exact bounds, this time in the Cauchy–Davenport theorem, if we insist that only a given number of terms of  $B$  may be used? The same examples as in  $\mathbb{Z}$  show that two translates do not suffice. However, a bounded number of translates is enough. This is again very surprising, and is one of the main results of our paper. In a certain sense, it is going far beyond results like the Erdős–Heilbronn conjecture, where we restrict the allowed sums only by forbidding  $a$  to equal  $b$  in a sum  $a + b$  we use: here on the contrary we only allow a fixed number of members of  $B$  to appear in the sums.

**THEOREM 3:** *There exists a universal constant  $c$  such that the following holds. Whenever  $A$  and  $B$  are subsets of  $\mathbb{Z}_p$  with  $|A| = |B| \leq p/3$ , there exist  $b_1, \dots, b_c \in B$  such that*

$$|A + \{b_1, \dots, b_c\}| \geq |A| + |B| - 1.$$

It would be fascinating to know if the number  $c$  of summands can be taken to be 3, if the sizes of  $A$  and  $B$  are a sufficiently small multiple of  $p$ .

More generally, we may allow the sizes of  $A$  and  $B$  to approach  $p/2$ , and also their sizes need not be the same. Note that we have to let the number of allowed summands in  $B$  increase as the sum of the sizes of  $A$  and  $B$  gets closer to  $p$ : this is easily seen if we choose  $A$  to be a random subset. Indeed, if  $A$  is a random subset of  $\mathbb{Z}_p$  of size  $(1/2 - \beta)p$  and  $c$  is fixed, then with high probability we have that for any  $b_1, \dots, b_c$  the union  $\bigcup_{1 \leq i \leq c} (A + b_i)$  has size about  $(1 - (1/2 + \beta)^c)p$ , and this is smaller than  $(1 - 3\beta)p$  if  $\beta$  is small. In other words, for such an  $A$ , and for any  $B$  at all of size  $(1/2 - \beta)p$ , we have that there do not exist  $c$  points of  $B$  whose sumset with  $A$  has size even close to  $|A| + |B|$ .

**THEOREM 4:** *For all  $\alpha, \beta > 0$  there exists a constant  $c$  such that the following holds. Whenever  $A$  and  $B$  are subsets of  $\mathbb{Z}_p$  with  $\alpha|B| \leq |A| \leq \frac{1}{\alpha}|B|$  and  $|A| + |B| \leq (1 - \beta)p$ , there exist  $b_1, \dots, b_c \in B$  such that*

$$|A + \{b_1, \dots, b_c\}| \geq |A| + |B| - 1.$$

Interestingly, if  $A$  is larger than a fixed (sufficiently large) multiple of the size of  $B$  then we do have that three translates are enough.

**THEOREM 5:** *For every  $\beta > 0$  there exists  $\alpha > 0$  such that the following holds. Whenever  $A$  and  $B$  are non-empty subsets of  $\mathbb{Z}_p$  with  $|B| \leq \alpha|A|$  and  $|A| + |B| \leq (1 - \beta)p$ , there exist elements  $b_1, b_2, b_3 \in B$  such that*

$$|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1.$$

(We mention in passing that this result actually means that, in Theorem 4, the condition that  $|A| \leq \frac{1}{\alpha}|B|$  may be removed. Indeed, if  $A$  is much larger than  $B$  then Theorem 5 tells us that actually we can take  $c = 3$ .)

An important case of Theorem 5 is when  $B$  is an interval, i.e., a set of the form  $[x, y] = \{x, x + 1, \dots, y\} \subset \mathbb{Z}_p$ . One is tempted to imagine that in this situation everything is much easier, but this is not the case, and indeed a substantial part of our approach to the various results in the paper consists of first proving the result when  $B$  is close to an interval (which is usually a large part of the overall work) and then seeing how, if at all, the proof can be modified for general  $B$ . When  $B$  is an interval we also recover the minimal possible number of translates of  $A$ .

**THEOREM 6:** *Let  $A$  and  $B$  be non-empty subsets of  $\mathbb{Z}_p$  with  $|B| \leq |A| \leq 2^{-20}p$  and  $B$  an interval. Then there exist elements  $b_1, b_2, b_3 \in B$  such that*

$$|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1.$$

**1.3. INVERSE RESULTS IN  $\mathbb{Z}_p$ .** We also prove inverse theorems for our results in  $\mathbb{Z}_p$ —where by ‘inverse’ we as usual mean statements of the form ‘what happens if the inequality is close to being tight’. (These are also often known as ‘stability’ results.) These show that, unless the lower bounds hold ‘with room to spare’, our sets must be highly structured, and in fact be very close to arithmetic progressions.

One of our main results is the following, which may be viewed as an extension of Freiman’s  $3k - 4$  theorem in this group where we demand that only a bounded number of summands from  $B$  are used.

**THEOREM 7:** *For all  $\alpha, \beta > 0$  there exist constants  $c$  and  $\varepsilon > 0$  such that the following holds. Let  $A$  and  $B$  be subsets of  $\mathbb{Z}_p$  of size at least 2, with  $\alpha|B| \leq |A| \leq \alpha^{-1}|B|$  and  $|A| + |B| \leq (1 - \beta)p$ . Suppose that for any  $c$  elements  $b_1, \dots, b_c \in B$  we have*

$$|A + \{b_1, \dots, b_c\}| \leq |A| + |B| - 1 + r$$

where  $r \leq \varepsilon|B|$ . Then  $B$  is contained in an arithmetic progression of size  $|B| + r$ .

This is actually the key to proving several of our direct results in  $\mathbb{Z}_p$ .

We remark that, in the form stated, if  $A \neq B$  then this is not really a ‘true’ inverse result to our results above, as it does not describe the structure of  $A$ . In fact, we cannot hope to insist that  $A$  is contained in a short arithmetic progression, since we may always add to  $A$  a small ‘sprinkling’ of faraway points without affecting the relevant properties of the sumsets with a bounded number of points from  $B$ . But one can obtain that  $A$  has small symmetric difference with some short arithmetic progression—we will say a few words about this when we come to prove Theorem 7.

1.4. ABELIAN GROUPS. There is an important tool in many of our results, that will often allow us to make progress with ‘arbitrary’ sets. This may be of independent interest.

**THEOREM 8:** *For all  $K$  and  $\varepsilon > 0$  there is an integer  $c$  such that the following holds. Let  $A$  and  $B$  be finite subsets of an abelian group. Then there are subsets  $A^* \subset A$  and  $B^* \subset B$ , with  $|A^*| \geq (1 - \varepsilon)|A|$  and  $|B^*| \geq (1 - \varepsilon)|B|$ , such that if we select points  $b_1, \dots, b_c$  uniformly at random from  $B$  then*

$$\mathbb{E}_{b_1, \dots, b_c \in B} |A + \{b_1, \dots, b_c\}| \geq \min((1 - \varepsilon)|A^* + B^*|, K|A|, K|B|).$$

One could view Theorem 8 as an approximate form of the sharp results we wish to prove in  $\mathbb{Z}$  and  $\mathbb{Z}_p$ . We remark that simple examples show that the dependence of  $c$  in terms of  $\varepsilon$  and  $K$  is necessary.

Amusingly, one can use a variant of Theorem 8 (namely Theorem 8’ below) to deduce Roth’s theorem [55] on three-term arithmetic progressions. This is not surprising, since (as we shall explain later) our result builds on work of Shao [61], which itself builds on the arithmetic regularity lemma of Green [30]—and the arithmetic regularity lemma has Roth’s theorem as an immediate consequence. But it is interesting that the deduction from Theorem 8’ is direct.

1.5. RESTRICTED SUMS. Although our aim in this paper is not so much to give applications, we do expect that our results will prove to be useful tools. As an example, as an application of our methods, we turn our attention to the Erdős–Heilbronn problem. Our results turn out to yield, very easily, some considerable extensions of this. For example, we consider ‘restricted’ sums, where for each  $a \in A$  there are some  $b \in B$  that we are not allowed to use when forming the ‘restricted sumset’ of  $A$  with  $B$ .

**THEOREM 9:** *For each  $\beta > 0$  and integer  $d$  there is an  $n_0$  such that following holds. Let  $A$  be a subset of  $\mathbb{Z}_p$  with  $n_0 \leq |A| = n < (1 - \beta)p/2$ , and suppose that we form all sums  $a + b$ , where  $a, b \in A$ , except that for each  $a \in A$  there is a set of  $d$  values in  $A$  that we are not allowed to take as  $b$ . Then the resulting ‘restricted sumset’ has size at least  $2n - 1 - 2d$ .*

Note that the Erdős–Heilbronn problem corresponds to the case when  $d = 1$  and for each  $a \in A$  the one element of  $A$  that we cannot sum with  $a$  is  $a$  itself.

In fact, we prove an extension of this where the number of summands from  $A$  that we take is bounded, as in the spirit of the rest of this paper. Indeed, this actually helps us: because we are already in the situation where the second summands are from a bounded set, the additional constraint that certain pairs are not allowed turns out to be much easier to handle than it would be in general. So it turns out that our results here will follow directly from Theorem 7.

**1.6. SUMS IN THE CONTINUOUS SETTING.** Our results tend to have consequences in the continuous setting, about sums of compact sets in the reals or in the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . We mention here two typical examples. In the statements we shall use  $|\cdot|$  to denote Lebesgue measure on the Euclidean space  $\mathbb{R}$  or Haar measure (normalized Lebesgue measure) on the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . As we will see, the deductions from the corresponding discrete results (namely Theorem 1 for the first and Theorems 4 and 5 for the second) will be fairly straightforward.

**COROLLARY 10:** *Let  $A$  and  $B$  be non-empty compact subsets of  $\mathbb{R}$  with  $|A| \geq |B|$ . Then there are elements  $b_1, b_2, b_3 \in B$  such that*

$$|A + \{b_1, b_2, b_3\}| \geq |A| + |B|.$$

**COROLLARY 11:** *For all  $\alpha, \beta > 0$  there exists a constant  $c$  such that the following holds. Whenever  $A$  and  $B$  are non-empty compact subsets of  $\mathbb{T}$  with  $\alpha|B| \leq |A|$  and  $|A| + |B| \leq 1 - \beta$ , there are  $b_1, \dots, b_c \in B$  such that*

$$|A + \{b_1, \dots, b_c\}| \geq |A| + |B|.$$

It is tempting to believe that these corollaries are equivalent to the corresponding results in the discrete case, but this does not appear to be the case: it does not seem easy to prove the reverse implications, essentially because the points we sample (the  $b_i$ ) in the continuous case may not ‘line up’ with the way we are embedding our discrete sets into the continuous world.

1.7. STRUCTURE OF THE PAPER. The plan of the paper is as follows. In Section 2 we mention various results from the literature that we will make use of. In Section 3 we prove the results in  $\mathbb{Z}$ , in the regime where  $A$  and  $B$  have the same size. We also determine the cases of equality. Then in Section 4 we prove Theorem 8, which is a key component in several of later proofs. Section 5 contains the result in  $\mathbb{Z}_p$  when  $B$  is an interval, and also deals with the situation when  $A$  is much larger than  $B$ , and then Section 6 deals with the general situation in  $\mathbb{Z}_p$  (including Theorem 7). In Section 7 we give the applications to the Erdős–Heilbronn problem mentioned above. Then Section 8 has the continuous versions of our statements. We finish in Section 9 with some open problems.

It is worth pointing out that a reader who is only interested in what happens in the integers will still need Section 4 since, although the proof of Theorem 1 is direct, the proof of Theorem 2 is much more involved and does make use of Theorem 8. Indeed, we will actually deduce Theorem 2 from Theorem 3, which is a particular case of Theorem 4. The latter will turn out to follow from a form of Theorem 7 itself.

1.8. NOTATION. Our notation is standard. To make our paper more readable, we often omit integer-part signs when these do not affect the argument. For example, given  $x > 0$ , we use the notation  $[x]$  for  $\{1, 2, \dots, [x]\}$ .

Sometimes we write ‘ $x \pmod d$ ’ as shorthand for the infinite arithmetic progression

$$\{y \in \mathbb{Z} : y \equiv x \pmod d\},$$

and refer to it as a **fibre** mod  $d$ . When  $S$  is a subset of  $\mathbb{Z}$  we often write  $S^x$  for the intersection of this fibre with  $S$ —when the value of  $d$  is clear. (We sometimes write  $S^x$  as  $S_d^x$  when we want to stress the value of  $d$ .) Thus  $S^x = S \cap \pi^{-1}(x)$ , where  $\pi = \pi_d$  denotes the natural projection from  $\mathbb{Z}$  to  $\mathbb{Z}_d$ . We also write  $\tilde{S}$  for  $\pi_d(S)$ .

When we write a probability or an expectation over a finite set, we always assume that the elements of the set are being sampled uniformly. Thus, for example, for a finite set  $X \subset \mathbb{Z}$  we denote the expectation and probability when we sample uniformly over all  $x \in X$  by respectively  $\mathbb{E}_{x \in X}$  and  $\mathbb{P}_{x \in X}$ . Similarly, an expectation when we sample uniformly over a product space  $X_1 \times \dots \times X_n$  may be written as  $\mathbb{E}_{\substack{x_j \in X_j \\ j \in \{1, \dots, n\}}}$ .

We also often sample uniformly over all  $c$ -sets of a given set  $X$ . In most of those cases, we could instead sample  $c$  elements uniformly and independently, but the notation would tend to get unwieldy, and this is why we use the sampling over all  $c$ -sets instead.

For a set  $X$  and integer  $c \in \mathbb{N}$  we as usual denote by  $X^{(c)}$  the family of all subsets of  $X$  of size  $c$ , and by  $X^{(\leq c)}$  the family of all subsets of size at most  $c$ . However, in several places it is very useful, when we want to specify that a set is small, to also allow  $X$  to have size smaller than  $c$ , and in that case we make the convention that  $X^{(c)}$  denotes the singleton  $\{X\}$ . We hope that the reader will not mind this convention, which will allow arguments to flow without having to deal with many (unimportant) special cases.

For more general background on sumsets, or for background on any of the results mentioned in the next section, see the survey of Breuillard, Green and Tao [13] or the books of Nathanson [50] or Tao and Vu [64].

Before we end this Introduction, we mention two further questions of a seemingly similar flavour, although really there is little connection with our topic. First, given sets  $A$  and  $B$  in an abelian group, is there a large set of disjoint translates  $A + b$  with  $b \in B$ ? The answer is given by a simple yet fundamental tool in additive combinatorics, Ruzsa's covering lemma [58] (see also Lemma 2.14 in Tao and Vu [64]), which states that there is a set  $X \subset B$  such that  $X + A - A$  contains  $B$  and  $|X + A| = |A||X|$ , i.e., the translates of  $A$  through elements of  $X$  are disjoint and the translates of  $A - A$  through  $X$  cover  $B$ . In particular, this shows that there are at least  $|B|/|A - A|$  disjoint translates of  $A + b$  with  $b \in B$ .

Second, one might also wonder about the following related question. Given sets  $A$  and  $B$  in an Abelian group, are there small subsets  $A' \subset A$  and  $B' \subset B$  such that  $A + B = (A + B') \cup (B + A')$ ? A result of Ellenberg [18] answers this question in  $\mathbb{Z}_p^n$ , showing that we can always take  $|A'|, |B'| \leq c_p^n$  where  $c_p < p$  (and in fact  $c_p \leq \alpha p$  for some absolute constant  $\alpha < 1$ ). Of course, this result and our main result consider very different ranges. In [18] the union  $(A + B') \cup (B + A')$  is required to be much bigger, potentially of size  $|A||B|$ , but to achieve this  $A'$  and  $B'$  are allowed to be very large, of size  $c_p^n$ .

### 2. Prerequisites

As mentioned above, we will make use of the following recent result of Shao [61], which can be viewed as an ‘almost all’ version of the Balog–Szemerédi–Gowers theorem [4, 29]. The main ingredient in this result is an application of the arithmetic removal lemma (an adaptation of the graph removal lemma to groups) of Green (see [30]). We shall actually use a version of this result for unequal size sets that can be proved in the same way. Here as usual we write  $A +_\Gamma B$ , where  $\Gamma$  is a subset of  $A \times B$ , to denote the set of all sums  $a + b$  where  $(a, b) \in \Gamma$ . (And later we will also use  $A -_\Gamma B$  to denote the set of all sums  $a - b$  with  $(a, b) \in \Gamma$ .) Here is the (variant of the) result of Shao [61] that we shall need.

**THEOREM 12:** *For all  $\varepsilon, K > 0$  there exist  $\delta > 0$  such that the following holds. Let  $G$  be an abelian group and let  $N \in \mathbb{N}$ . Let  $A, B \subset G$  be two subsets with  $|A|, |B| \geq N$ , and let  $\Gamma \subset A \times B$  be a subset with  $|\Gamma| \geq (1 - \delta)|A||B|$ . If  $|A +_\Gamma B| \leq KN$ , then there exist  $A_0 \subset A, B_0 \subset B$  such that*

$$|A_0| \geq (1 - \varepsilon)|A|, \quad |B_0| \geq (1 - \varepsilon)|B|, \quad |A_0 + B_0| \leq |A +_\Gamma B| + \varepsilon N.$$

We will need the extension of Freiman’s  $3k - 4$  theorem [21, 22] by Lev and Smeliansky [45] and Stanchescu [62].

**THEOREM 13:** *Let  $A$  and  $B$  be finite non-empty subsets of  $\mathbb{Z}$  with*

$$|A + B| = |A| + |B| - 1 + r,$$

where  $r \leq \min(|A|, |B|) - 3$ . Then there are arithmetic progressions  $P_A \supset A$  and  $P_B \supset B$ , having the same common difference, such that

$$|P_A \setminus A|, |P_B \setminus B| \leq r.$$

We shall also make use of the following analogue of Freiman’s  $3k - 4$  theorem in  $\mathbb{Z}_p$  (see Gryniewicz [33, Theorem 21.8]).

**THEOREM 14:** *There is an absolute constant  $\eta > 0$  such that the following holds. Let  $A$  and  $B$  be finite nonempty subsets of  $\mathbb{Z}_p$ , let  $C = -(A + B)^c$  and let  $r$  be an integer with  $0 \leq r \leq \eta p - 2$ . Suppose that*

$$|A + B| = |A| + |B| - 1 + r$$

and

$$|A| \geq r + 3, \quad |B| \geq r + 3, \quad |C| \geq r + 3.$$

Then there exist arithmetic progressions  $P_A, P_B$  and  $P_C$  with the same common difference containing  $A, B$  and  $C$ , respectively, such that

$$|P_A \setminus A| \leq r, \quad |P_B \setminus B| \leq r, \quad |P_C \setminus C| \leq r.$$

Finally, we recall Vosper’s theorem [65, 66].

**THEOREM 15:** *Let  $A$  and  $B$  be finite subsets of  $\mathbb{Z}_p$  with at least two elements such that  $|A + B| = |A| + |B| - 1 \leq p - 2$ . Then  $A$  and  $B$  are arithmetic progressions with the same common difference.*

### 3. Sums in $\mathbb{Z}$

We start by proving Theorem 1 that if  $A$  and  $B$  are finite non-empty sets of integers with  $|A| \geq |B|$ , then there exist elements  $b_1, b_2, b_3 \in B$  such that  $|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1$ . One might imagine that a random argument cannot be helpful in proving this, since in the case when  $A$  and  $B$  are intervals then two of our elements must be the first and last elements of  $B$ . But, remarkably, what we will find is that if we fix those two elements and select only the third element at random, we do obtain exactly the bound we want.

*Proof of Theorem 1.* Let  $B$  have first element 0 and last element  $m$ , and write  $\pi_m : \mathbb{Z} \rightarrow \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$  for the canonical projection of  $\mathbb{Z}$  onto  $\mathbb{Z}_m$ . Then  $A + \{0, m\} = A \cup (A + m)$  satisfies  $\pi_m(A) = \pi_m(A \cup (A + m)) = \tilde{A}$ . Although it is easy to bound the size of this set  $A \cup (A + m)$ , we record it as a lemma because we shall make use of it several times later on.

**LEMMA 16:** *We have*

$$|A \cup (A + m)| \geq |A| + |\tilde{A}|.$$

*Proof.* Note that  $(A + m)^x = A^x + m$  for every  $x \in \mathbb{Z}$ . Hence, if  $x \in \tilde{A} = \pi_m(A)$  then  $(A + m)^x = A^x + m \neq A^x$ , so  $|[A \cup (A + m)]^x| = |A^x \cup (A + m)^x| \geq |A^x| + 1$ . Consequently,

$$|A \cup (A + m)| \geq \sum_{x \in \tilde{A}} |[A \cup (A + m)]^x| \geq \sum_{x \in \tilde{A}} (|A^x| + 1) = |A| + |\tilde{A}|,$$

completing the proof. ■

We now turn to the contribution from the ‘random’ term. It does seem very fortunate that this meshes so well with the bound above. Again, we record it as a separate lemma for future use.

LEMMA 17: *If we choose an element  $b$  from  $B \setminus \{m\}$ , uniformly at random, then*

$$\mathbb{E}_{b \in B \setminus \{m\}} |(A + b) \setminus \pi_m^{-1}(\tilde{A})| \geq |A| \max \left( 0, \frac{|\tilde{B}| - |\tilde{A}|}{|\tilde{B}|} \right).$$

*Proof.* Clearly,  $\pi_m : B \setminus \{m\} \rightarrow \tilde{B}$  is a bijection. Also, for  $a \in A$  we have

$$\mathbb{P}_{b \in B \setminus \{m\}} (a + b \notin \pi_m^{-1}(\tilde{A})) = \mathbb{P}_{\tilde{b} \in \tilde{B}} (\tilde{a} + \tilde{b} \notin \tilde{A}) \geq \max \left( 0, \frac{|\tilde{B}| - |\tilde{A}|}{|\tilde{B}|} \right).$$

so

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{m\}} |(A + b) \setminus \pi_m^{-1}(\tilde{A})| &= \sum_{a \in A} \mathbb{P}_{b \in B \setminus \{m\}} (a + b \notin \pi_m^{-1}(\tilde{A})) \\ &\geq |A| \max \left( 0, \frac{|\tilde{B}| - |\tilde{A}|}{|\tilde{B}|} \right), \end{aligned}$$

as claimed. ■

To prove Theorem 1, we combine these two lemmas. We have

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| &\geq |A| + |\tilde{A}| + |A| \max \left( 0, \frac{|\tilde{B}| - |\tilde{A}|}{|\tilde{B}|} \right) \\ &= |A| + |\pi_m(A)| + |A| \max \left( 0, \frac{|B| - 1 - |\pi_m(A)|}{|B| - 1} \right). \end{aligned}$$

If  $|\pi_m(A)| \geq |B| - 1$  then

$$\mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq |A| + |\pi_m(A)| \geq |A| + |B| - 1.$$

On the other hand, if  $|\pi_m(A)| \leq |B| - 1$  then

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| &\geq |A| + |\pi_m(A)| + |A| \frac{|B| - 1 - |\pi_m(A)|}{|B| - 1} \\ &\geq |A| + |\pi_m(A)| + |B| - 1 - |\pi_m(A)| = |A| + |B| - 1, \end{aligned}$$

completing our proof of Theorem 1. ■

In fact, the proof of Theorem 1 above shows the following stronger result, which we record here. Let  $A$  and  $B$  be finite non-empty subsets of integers with  $|A| \geq |B|$  and  $B$  having smallest element 0 and greatest element  $m$ . Then we have

$$(2) \quad \max_{b_1, b_2, b_3 \in B} |A + \{b_1, b_2, b_3\}| \geq \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq |A| + |B| - 1.$$

We now collect together some variants of the results above that we shall need at later points in the paper. We urge the reader to skip these (rather pedestrian) variations, returning to them only when they are actually needed in the sequel. To avoid clutter and some repetition, we leave the proofs for the Appendix.

First, we investigate when equality is attained in inequality (2) assuming  $|A| = |B|$ , that is when  $|A + \{b_1, b_2, b_3\}| \leq |A| + |B| - 1$  for all  $b_1, b_2, b_3 \in B$ .

**THEOREM 18:** *Let  $A$  and  $B$  be finite non-empty subsets of  $\mathbb{Z}$  such that  $|A| = |B|$ , with  $\min B = 0$  and  $\max B = m$ . Suppose that when we choose an element  $b$  of  $B \setminus \{0, m\}$  uniformly at random we have*

$$(3) \quad \mathbb{E}_{b \in B \setminus \{0, m\}} |A + \{0, b, m\}| \leq |A| + |B| - 1.$$

*Then  $A$  and  $B$  are arithmetic progressions with the same common difference.*

*In particular, suppose that when we choose any three elements  $b_1, b_2, b_3$  of  $B$  we have*

$$|A + \{b_1, b_2, b_3\}| \leq |A| + |B| - 1.$$

*Then  $A$  and  $B$  are arithmetic progressions with the same common difference.*

Next we have a version of Theorem 1.

**THEOREM 19:** *Let  $A$  and  $B$  be finite non-empty subsets of  $\mathbb{Z}$ , with  $\min B = 0$  and  $\max B = m$ . Then*

$$\mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq |A| + |\pi_m(A)| + |A| \max \left( 0, \frac{|B| - 1 - |\pi_m(A)|}{|B| - 1} \right).$$

*In particular, if  $|A| \geq |B| - 1$  we have*

$$\mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq |A| + |B| - 1$$

*and if  $|A| \leq |B| - 1$  we have*

$$\mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq 2|A| + |\pi_m(A)| \frac{|B| - 1 - |A|}{|B| - 1}.$$

We also have a simple variant of Lemma 17.

LEMMA 20: *Let  $A, B$  and  $A_1$  be finite non-empty subsets of  $\mathbb{Z}$ , with  $\min B = 0$  and  $\max B = m$ . Then with  $\tilde{A} = \pi_m(A)$  and  $\tilde{B} = \pi_m(B)$  we have*

$$\mathbb{E}_{b \in B \setminus \{m\}} |(A_1 + b) \setminus \pi_m^{-1}(\tilde{A})| \geq |A_1| \max \left( 0, \frac{|\tilde{B}| - |\tilde{A}|}{|\tilde{B}|} \right). \quad \blacksquare$$

Our next lemma follows from Lemma 20 by simple manipulations.

LEMMA 21: *Let  $A$  and  $B$  be finite non-empty subsets of  $\mathbb{Z}$ . Let  $\min(B) = 0$  and  $\max(B) = m$ , and suppose that  $|B| - 1 = |\tilde{B}| \geq 8|\tilde{A}|$ , where  $\tilde{A} = \pi_m(A)$  and  $\tilde{B} = \pi_m(B)$ . Then*

$$\mathbb{E}_{b_2, b_3 \in B \setminus \{m\}} |A + \{0, b_2, b_3\}| \geq 2.5|A|. \quad \blacksquare$$

Finally, we have a strengthening of Theorem 1.

THEOREM 22: *Let  $A$  and  $B$  be finite non-empty subsets of  $\mathbb{Z}$  with  $|A| = |B|$ , with  $\min B = 0$  and  $\max B = m$ . Then*

$$\begin{aligned} &\mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \\ &\geq |A| + |B| - 1 + \max \left( 0, \frac{(2|\pi_m(A)| - m)(m - (|B| - 1)) - 1}{|B| - 1} \right). \end{aligned}$$

All of the above results are proved in the Appendix.

To end this section, we return to two translates. As pointed out above, even if  $|A| = |B|$ , there may not be two points in  $B$  whose sum with  $A$  gives at least  $|A| + |B| - 1$  points. When would two translates suffice? In the first version of this paper we proved the following result.

THEOREM 23: *There is a constant  $K > 0$  such that for any subsets  $A$  and  $B$  of  $\mathbb{Z}$  with  $|B| = n \geq 2$  and  $|A| \geq Kn \log n$  there exist elements  $b_1, b_2$  in  $B$  such that*

$$|A + \{b_1, b_2\}| \geq |A| + |B| - 1.$$

One of the referees was kind enough to point out that this result is an immediate consequence of a theorem of Konyagin and Lev [40] about the Erdős–Heilbronn–Olson function  $f_A(s) = |(A + s) \setminus A|$ . Writing  $f_A(S)$  for  $\max\{f_A(s) : s \in S\}$ , Konyagin and Lev proved that there is an absolute constant  $c > 0$  such that  $f_A(S) \geq |S|$  holds for all finite sets  $A \subset \mathbb{Z}$ ,  $S \subset \mathbb{N}$  with  $|A| > 1$  and  $|S| < c|A|/\log |A|$ .

Now, trivially,  $|A + \{b_1, b_2\}| = |A| + f_A(b_2 - b_1)$ , so

$$\max_{b_1, b_2 \in B} |A + \{b_1, b_2\}| = |A| + f_A(S),$$

where  $S = \{b_2 - b_1 : b_1, b_2 \in B, b_1 < b_2\}$ . As  $|S| \geq n - 1$ , we can fix a subset  $S' \subset S$  with  $|S'| = n - 1 \leq c|A|/\log(|A|)$ . Since  $f_A(S) \geq f_A(S') \geq n - 1$ , Theorem 23 follows.

In fact, Lev [44] and Huicochea [36] proved analogues of the Konyagin–Lev theorem for  $\mathbb{Z}_p$ , which imply the analogue of Theorem 23 in  $\mathbb{Z}_p$ . We are grateful to the referee for drawing our attention to these results of Konyagin, Lev and Huicochea.

#### 4. Proof of Theorem 8

Our aim in this section is to prove Theorem 8. The main ingredient of the proof is Theorem 12, and our strategy is to apply it several times in succession. In fact, we shall prove Theorem 8 in the following slightly stronger form; the easy deduction of Theorem 8 is by changing the values of  $\varepsilon$  and  $c$  slightly, as required.

**THEOREM 8':** *For all  $K$  and  $\varepsilon > 0$  there is an integer  $c$  such that the following holds. Let  $A$  and  $B$  be finite subsets of an abelian group. Then there are subsets  $A^* \subset A$  and  $B^* \subset B$ , with  $|A^*| \geq (1 - \varepsilon)|A|$  and  $|B^*| \geq (1 - \varepsilon)|B|$ , such that if we select points  $b_1, \dots, b_c$  uniformly at random from  $B^*$  then*

$$\mathbb{E}_{b_1, \dots, b_c \in B^*} |A^* + \{b_1, \dots, b_c\}| \geq \min((1 - \varepsilon)|A^* + B^*|, K|A^*|, K|B^*|).$$

Note that Theorem 8' asserts the existence of a pair of large subsets  $A^*$  and  $B^*$  with a property that is ‘intrinsic’ to them.

*Proof of Theorem 8'.* Fix  $\varepsilon > 0$  and  $K > 0$ , where we assume that  $\varepsilon$  is sufficiently small and  $K$  is sufficiently large. Pick  $s = \lfloor \frac{50K}{\varepsilon} \rfloor$ . Let  $\delta$  be given by Theorem 12 with parameters  $\frac{\varepsilon}{s}, 10K$ . Also pick  $\alpha$  sufficiently small such that  $(1 - t)^{\frac{10K}{\delta}} \leq 1 - \frac{10K}{2\delta}t$  for  $t \leq \alpha$ . Finally pick  $c \geq \frac{10K}{\delta}$  sufficiently large such that  $1 - (1 - \alpha)^c \geq \frac{1 - \varepsilon}{1 - \varepsilon/2} \geq \frac{1}{2}$ . Let  $n_A$  and  $n_B$  be the sizes of  $A$  and  $B$  respectively.

We shall examine a process in which we repeatedly apply Theorem 12 in order to construct a decreasing sequence of  $s + 1$  pairs of sets

$$(A, B) = (A_0, B_0), (A_1, B_1), \dots, (A_s, B_s),$$

satisfying  $A_i \subset A_{i-1}, B_i \subset B_{i-1}, |A_i| \geq (1 - \varepsilon/s)^i n_A$  and  $|B_i| \geq (1 - \varepsilon/s)^i n_B$ . Fix  $i < s$  and assume that the pair of sets  $(A_i, B_i)$  has already been constructed. We shall either stop the process at step  $i$  or construct the pair of sets  $(A_{i+1}, B_{i+1})$ .

Let  $A_i + B_i = C_i^+ \sqcup C_i^-$  be the partition into ‘popular’ and ‘unpopular’ elements given by

$$C_i^+ = \{c \in A_i + B_i : |(c - A_i) \cap B_i| \geq \alpha|B_i|\},$$

and

$$C_i^- = \{c \in A_i + B_i : |(c - A_i) \cap B_i| < \alpha|B_i|\}.$$

Also, let the partition  $A_i \times B_i = \Gamma_i \sqcup \Gamma_i^c$  be given by

$$\Gamma_i = \{(a, b) \in A_i \times B_i : a + b \in C_i^+\} \subset A_i \times B_i$$

and

$$\Gamma_i^c = \{(a, b) \in A_i \times B_i : a + b \in C_i^-\} \subset A_i \times B_i,$$

so that  $A_i +_{\Gamma_i} B_i = C_i^+$  and  $A_i +_{\Gamma_i^c} B_i = C_i^-$ . Finally, for each  $x \in A_i + B_i$  set

$$r_i(x) = |(x - A_i) \cap B_i| = |\{(a, b) \in A_i \times B_i : x = a + b\}|,$$

so that  $\sum_x r_i(x) = |A_i||B_i|$ . We stop this process ‘early’, at step  $i$ , if

$$|\Gamma_i| < (1 - \delta)|A_i||B_i| \quad \text{or} \quad |A_i +_{\Gamma_i} B_i| > 10K \min(|A_i|, |B_i|).$$

Otherwise, we apply Theorem 12 with parameters  $\varepsilon/s, 10K$  to the pair of sets  $(A_i, B_i)$ . Thus, we produce a pair of sets  $(A_{i+1}, B_{i+1})$ , satisfying  $A_{i+1} \subset A_i, B_{i+1} \subset B_i$ , and

$$\begin{aligned} |A_{i+1}| &\geq (1 - \varepsilon/s)|A_i|, & |B_{i+1}| &\geq (1 - \varepsilon/s)|B_i|, \\ |A_{i+1} + B_{i+1}| &\leq |A_i +_{\Gamma_i} B_i| + \frac{\varepsilon}{s} \min(|A_i|, |B_i|). \end{aligned}$$

We shall analyse separately the cases in which the process continues until the end and in which the process stops before that.

CLAIM A: *Suppose the process stops early, say at step  $j < s$ . Then the pair of sets  $(A_j, B_j)$  has the desired properties.*

*Proof.* CASE 1: Consider first the case when

$$|C_j^+| = |A_j +_{\Gamma_j} B_j| > 10K \min(|A_j|, |B_j|).$$

For  $x \in C_j^+$ , by construction we have that

$$r_j(x) = |(x - A_j) \cap B_j| \geq \alpha|B_j|.$$

If we choose elements  $b_i$  uniformly at random from  $B_j$ , for  $1 \leq i \leq c$ , then

$$\begin{aligned} \mathbb{E}_{b_1, \dots, b_c \in B_j} |A_j + \{b_1, \dots, b_c\}| &\geq \sum_x \mathbb{P}_{b_1, \dots, b_c \in B_j} (x \in A_j + \{b_1, \dots, b_c\}) \\ &\geq \sum_{x \in C_j^+} 1 - [1 - \mathbb{P}_{b \in B_j} (x \in A_j + b)]^c \\ &= \sum_{x \in C_j^+} [1 - [1 - r_j(x)/|B_j|]^c] \\ &\geq \sum_{x \in C_j^+} (1 - (1 - \alpha)^c) \geq (1 - (1 - \alpha)^c)|C_j^+| \\ &\geq (1 - (1 - \alpha)^c)10K \min(|A_j|, |B_j|) \\ &\geq 5K \min(|A_j|, |B_j|) \geq K \min(|A|, |B|). \end{aligned}$$

CASE 2: Consider now the case when

$$|\Gamma_j| < (1 - \delta)|A_j||B_j|.$$

By construction,

$$\sum_{x \in C_j^-} r_j(x) \geq \delta|A_j||B_j|.$$

Moreover, for  $x \in C_j^-$  we have

$$r_j(x) = |(x - A_j) \cap B_j| \leq \alpha|B_j|.$$

In particular, we deduce that

$$1 - [1 - r_j(x)/|B_j|]^c \geq 1 - [1 - r_j(x)/|B_j|]^{\frac{10K}{\delta}} \geq \frac{5K}{\delta} r_j(x)/|B_j|.$$

Again, if we choose elements  $b_i$  uniformly at random from  $B_j$ , for  $1 \leq i \leq c$ , then

$$\begin{aligned} \mathbb{E}_{b_1, \dots, b_c \in B} |A_j + \{b_1, \dots, b_c\}| &\geq \sum_x \mathbb{P}_{b_1, \dots, b_c \in B} (x \in A_j + \{b_1, \dots, b_c\}) \\ &\geq \sum_{x \in C_j^-} [1 - [1 - \mathbb{P}_{b \in B} (x \in A_j + b)]^c] \\ &= \sum_{x \in C_j^-} [1 - [1 - r_j(x)/|B_j|]^c] \\ &\geq \sum_{x \in C_j^-} \frac{5K}{\delta} r_j(x)/|B_j| \\ &\geq \frac{5K}{\delta} \delta |A_j| |B_j| / |B_j| \geq K|A|. \end{aligned}$$

We conclude that the pair of sets  $(|A_j|, |B_j|)$  has the desired properties, so Claim A is proved. ■

We now turn to the case when the process does not stop early.

CLAIM B: *Suppose that the process continues until the terminal step  $s$ . Then there is an index  $j \leq s$  such that the pair of sets  $(A_j, B_j)$  has the desired properties.*

*Proof.* Note that

$$|A_1 + B_1| \leq |A_0 +_{\Gamma_0} B_0| + \frac{\varepsilon}{s} \min(|A_0|, |B_0|) \leq 11K \min(|A_0|, |B_0|).$$

Moreover,

$$11K \min(|A|, |B|) = 11K \min(|A_0|, |B_0|) \geq |A_1 + B_1| \geq \dots \geq |A_s + B_s| \geq 0.$$

Therefore

$$|A_{j+1} + B_{j+1}| \geq |A_j + B_j| - \frac{11K}{s-1} \min(|A|, |B|)$$

for some index  $j$ ,  $1 \leq j \leq s-1$ . We shall show that the pair  $(A_j, B_j)$  has the desired properties. Indeed, by construction,

$$|A_j +_{\Gamma_j} B_j| + \frac{\varepsilon}{s} \min(|A|, |B|) \geq |A_j +_{\Gamma_j} B_j| + \frac{\varepsilon}{s} \min(|A_j|, |B_j|) \geq |A_{j+1} + B_{j+1}|.$$

It follows that

$$|A_j +_{\Gamma_j} B_j| \geq |A_j + B_j| - \left( \frac{11K}{s-1} + \frac{\varepsilon}{s} \right) \min(|A|, |B|) \geq \left( 1 - \frac{20K}{s} \right) |A_j + B_j|.$$

If we choose elements  $b_i$  uniformly at random from  $B_j$ , for  $1 \leq i \leq c$  then

$$\begin{aligned} \mathbb{E}_{b_1, \dots, b_c \in B} |A_j + \{b_1, \dots, b_c\}| &\geq \sum_x \mathbb{P}_{b_1, \dots, b_c \in B} (x \in A_j + \{b_1, \dots, b_c\}) \\ &\geq \sum_{x \in C_j^+} [1 - [1 - \mathbb{P}_{b \in B} (x \in A_j + b)]^c] \\ &= \sum_{x \in C_j^+} [1 - [1 - r_j(x)/|B_j|]^c] \\ &\geq \sum_{x \in C_j^+} (1 - (1 - \alpha)^c) = (1 - (1 - \alpha)^c) |C_j^+| \\ &= (1 - (1 - \alpha)^c) |A_j +_{\Gamma_j} B_j| \\ &\geq (1 - (1 - \alpha)^c) \left(1 - \frac{20K}{s}\right) |A_j + B_j| \\ &\geq (1 - \varepsilon) |A_j + B_j|. \end{aligned}$$

Thus the pair  $(|A_j|, |B_j|)$  has the desired properties, so Claim B is proved. ■

This concludes the proof of Theorem 8': whether the process stops early or does not, the pair of sets  $(A_j, B_j)$  has the desired properties. ■

We now digress briefly to point out that Roth's theorem is a direct consequence of Theorem 8'. Recall that Roth's theorem [55] states that for every  $\alpha > 0$  there is an integer  $k_0$  such that if  $k \geq k_0$  then every set  $A \subset [k]$  of size at least  $\alpha k$  contains 3 terms in arithmetic progression.

*Proof of Roth's theorem.* Fix  $\alpha > 0$  (where we may assume that  $\alpha$  is sufficiently small). Pick  $c$  as given by Theorem 8' with parameters  $\frac{\alpha}{4}, \frac{4}{\alpha}$ , and set  $k_0 = 2c[\alpha(1 - \alpha)]^{-1} + 1$ .

Suppose for a contradiction that there exists a set  $A$  that is free of three-term arithmetic progressions, of density  $\alpha$  in  $[1, k]$  for some  $k \geq k_0$ . We apply Theorem 8' with parameters  $\frac{\alpha}{4}, \frac{4}{\alpha}$  to construct sets  $A_1, A_2 \subset A$  of size at least  $(1 - \frac{\alpha}{4})|A|$  with the property that if we select  $a_1, \dots, a_c$  uniformly at random from  $A_2$  then

$$\begin{aligned} \mathbb{E}_{a_1, \dots, a_c \in A_2} |A_1 + \{a_1, \dots, a_c\}| &\geq \min \left( \left(1 - \frac{\alpha}{4}\right) |A_1 + A_2|, \frac{4}{\alpha} |A_1|, \frac{4}{\alpha} |A_2| \right) \\ &\geq \min \left( \left(1 - \frac{\alpha}{4}\right) |A_1 + A_2|, \frac{2}{\alpha} |A| \right). \end{aligned}$$

On the one hand, note that, as the only solutions to the equation  $x + y = 2z$  in  $A$  are  $x = y = z \in A$ , the left-hand side is bounded from above by

$$|A_1 + A_2| - |A_1 \cap A_2| + c \geq \mathbb{E}_{a_1, \dots, a_c \in A_2} |A_1 + \{a_1, \dots, a_c\}|.$$

On the other hand, as  $|A| \geq \alpha k$  and  $|A_1 + A_2| \leq 2k$ , the right-hand side is bounded from below by

$$\left(1 - \frac{\alpha}{4}\right) |A_1 + A_2| \geq |A_1 + A_2| - \frac{\alpha}{2} k \geq |A_1 + A_2| - \frac{|A|}{2}.$$

Combining the two inequalities above we obtain

$$2c \geq 2|A_1 \cap A_2| - |A|.$$

Finally, as  $A_1, A_2 \subset A$  satisfy  $|A| \geq \alpha k$  and  $|A_1|, |A_2| \geq (1 - \frac{\alpha}{4})|A|$ , we conclude that

$$2c \geq \alpha(1 - \alpha)k$$

which gives the desired contradiction. ■

In later sections we shall make considerable use of Theorem 8.

### 5. Sums in $\mathbb{Z}_p$ : the case of an interval

Our aim in this section is to prove Theorem 6. Although we are ‘only’ considering the case when  $B$  is an interval, nevertheless the proof is not nearly as simple as one would expect. Actually, in this section we also treat the case of general  $B$  when  $B$  is much smaller than  $A$ , Theorem 5, as this seems to go hand-in-hand with the case when  $B$  is an interval. When in the next section we turn to the ‘most interesting’ case, of general  $B$  of comparable size to  $A$ , the proofs will have their genesis in the proofs in this section, although several more ideas will still be needed.

We need two technical results. In proving the first one we will make use of the following corollary of Theorem 14.

**COROLLARY 24:** *Given  $\beta, \gamma > 0$  there is an  $\varepsilon > 0$  such that the following holds. Let  $A$  and  $B$  be subsets of  $\mathbb{Z}_p$ . Suppose that*

$$(4) \quad 2 \leq \min(|A|, |B|) \quad \text{and} \quad |A| + |B| \leq (1 - \beta)p$$

and

$$(5) \quad |A + B| \leq |A| + |B| - 1 + \varepsilon \min(|A|, |B|).$$

Then there are arithmetic progressions  $P$  and  $Q$  with the same common difference, containing  $A$  and  $B$ , respectively, such that

$$|P\Delta A| \leq \gamma \min(|A|, |B|) \quad \text{and} \quad |Q\Delta B| \leq \gamma \min(|A|, |B|).$$

*Proof.* Fix  $2^{-10} > \beta > \gamma > 0$ . Let  $\eta$  be as given in Theorem 14, and let  $\varepsilon = 2^{-10}\eta \min(\beta, \gamma)$ . Finally, write  $r = \lfloor \varepsilon \min(|A|, |B|) \rfloor$ .

Theorem 15, together with (4) and (5), implies that we may assume that

$$(6) \quad 1 \leq r \leq \varepsilon p.$$

In particular, we have

$$(7) \quad r \leq \eta p - 2,$$

and also

$$(8) \quad r \leq 2^{-10} \min(|A|, |B|) \leq \min(|A|, |B|) - 3.$$

If we let

$$C = -(A + B)^c$$

then by (4) and (5) we see that

$$|C| = p - |A + B| \geq p - (1 - \beta + \varepsilon)p = (\beta - \varepsilon)p \geq 2^{-1}\beta p,$$

and by (6) we obtain

$$(9) \quad r \leq 2^{-9}|C| \leq |C| - 3.$$

Theorem 14 and (7), (8) and (9) now imply that there exist arithmetic progressions  $P$  and  $Q$  with the same common difference, containing  $A$  and  $B$ , respectively, such that

$$|A\Delta P| \leq r \quad \text{and} \quad |B\Delta Q| \leq r.$$

So we obtain

$$|A\Delta P| \leq \gamma \min(|A|, |B|) \quad \text{and} \quad |B\Delta Q| \leq \gamma \min(|A|, |B|).$$

This concludes the proof of Corollary 24. ■

Here then is the first technical result that we shall need in the proof of Theorem 6. The reader should note that here the set  $A$  has small intersection with a relevant interval, in contrast to the usual requirement.

**THEOREM 25:** *For all  $\beta, \gamma > 0$  there exist  $\varepsilon, \alpha > 0$  such that the following holds. Let  $A$  and  $B$  be subsets of  $\mathbb{Z}_p$ . Suppose that*

$$(10) \quad 2 \leq |B| \leq \alpha|A| \quad \text{and} \quad |A| + |B| \leq (1 - \beta)p$$

and

$$(11) \quad \max_{B' \in B^{(2)}} |A + B'| \leq |A| + |B| - 1 + \varepsilon|B|.$$

*Then there exist arithmetic progressions  $P$  and  $Q$  with the same common difference, and sizes at least  $2^{10}|B|$  and at most  $\lfloor (1 + \gamma)|B| \rfloor$ , respectively, such that  $|A \cap P| \leq \gamma|B|$  and  $B \subset Q$ .*

*Proof.* Fix  $2^{-10} > \beta > \gamma > 0$ . Choose  $\varepsilon'$  to be the output of Corollary 24 with input  $\beta, \gamma$ . Choose  $\varepsilon = 2^{-3}\varepsilon'$  and  $\alpha = \min(2^{-5}\beta\varepsilon, 2^{-23}\beta\gamma)$ .

Define

$$A' = \begin{cases} A & \text{if } |A| < p/2, \\ A^c & \text{if } |A| \geq p/2. \end{cases}$$

By (10) and (11) we deduce

$$(12) \quad 2 \leq |B| \leq \frac{\alpha}{\beta}|A'| \leq |A'| \leq p/2.$$

and

$$(13) \quad \max_{B' \in B^{(2)}} |A' + B'| \leq |A'| + |B| - 1 + \varepsilon|B|.$$

In particular, we have

$$p - 1 \geq |B - B| - 1 \geq 2.$$

Define

$$m = \left\lfloor \frac{p}{|B - B| - 1} \right\rfloor \geq 1$$

and note that

$$(14) \quad m + 1 \geq \frac{2|A'|}{|B - B| - 1}.$$

By the Cauchy-Davenport theorem, for  $1 \leq n \leq m$  we have

$$(15) \quad |n(B - B)| \geq n(|B - B| - 1) \quad \text{and} \quad |(m + 1)(B - B)| = p.$$

Therefore, there exists a partition

$$(16) \quad \mathbb{Z}_p = C_1 \sqcup \dots \sqcup C_m \sqcup C_{m+1}$$

such that for  $1 \leq n \leq m$  we have

$$(17) \quad |C_n| \geq |B - B| - 1$$

and for  $1 \leq n \leq m + 1$  we have

$$(18) \quad C_n \subset n(B - B).$$

By (13), for every  $c \in B - B$  there exists a set  $A'_c$  of size

$$|A'_c| \geq |A'| - ((1 + \varepsilon)|B| - 1)$$

such that

$$A'_c \subset A' \quad \text{and} \quad A'_c + c \subset A'.$$

It follows that for every  $n \in \mathbb{N}$  and every  $c \in n(B - B)$  there is a set  $A'_{n,c}$  of size

$$(19) \quad |A'_{n,c}| \geq |A'| - n((1 + \varepsilon)|B| - 1)$$

such that

$$(20) \quad A'_{n,c} \subset A' \quad \text{and} \quad A'_{n,c} + c \subset A'.$$

By (16), (18) and (20) we have

$$\bigsqcup_{n \in [m+1]} \bigsqcup_{c \in C_n} \bigsqcup_{a \in A'_{n,c}} \{(a, a + c)\} \subset A' \times A'.$$

In particular, we deduce that

$$(21) \quad |A'|^2 \geq \sum_{n \in [m+1]} \sum_{c \in C_n} \sum_{a \in A'_{n,c}} 1.$$

On the one hand, (16), (19) and (21) imply

$$|A'|^2 \geq p(|A'| - (m + 1)((1 + \varepsilon)|B| - 1))$$

and (12) further implies

$$(22) \quad m + 1 \geq \frac{\beta}{4\alpha}.$$

On the other hand, (17), (19) and (21) imply

$$(23) \quad |A'|^2 \geq \sum_{n \in [m]} (|B - B| - 1) \max(0, |A'| - n((1 + \varepsilon)|B| - 1)) \\ \sum_{n \in [m]} ((1 + \varepsilon)|B| - 1)(|B - B| - 1) \max\left(0, \frac{|A'|}{(1 + \varepsilon)|B| - 1} - n\right).$$

We distinguish two cases.

CASE A: Suppose

$$(24) \quad m + 1 \leq \frac{|A'|}{(1 + \varepsilon)|B| - 1}.$$

In this case, from inequalities (14), (22) and (24) we deduce that

$$(25) \quad \sum_{n \in [m]} \max \left( 0, \frac{|A'|}{(1 + \varepsilon)|B| - 1} - n \right) \geq \left( 2 - \frac{8\alpha}{\beta} \right) \frac{|A'|}{|B - B| - 1} \left( \frac{|A'|}{(1 + \varepsilon)|B| - 1} - \frac{|A'|}{|B - B| - 1} \right).$$

Combining (23) and (25), we obtain

$$(26) \quad 1 \geq \left( 2 - \frac{8\alpha}{\beta} \right) \left( 1 - \frac{(1 + \varepsilon)|B| - 1}{|B - B| - 1} \right).$$

CASE B: Suppose

$$(27) \quad m + 1 > \frac{|A'|}{(1 + \varepsilon)|B| - 1}.$$

In this case, by (12) and (27) we deduce

$$(28) \quad \sum_{n \in [m]} \max \left( 0, \frac{|A'|}{(1 + \varepsilon)|B| - 1} - n \right) = \sum_{n \in \mathbb{N}} \max \left( 0, \frac{|A'|}{(1 + \varepsilon)|B| - 1} - n \right) \geq 2^{-1} \frac{|A'|}{(1 + \varepsilon)|B| - 1} \left( \frac{|A'|}{(1 + \varepsilon)|B| - 1} - 1 \right) \geq \left( 2^{-1} - \frac{4\alpha}{\beta} \right) \left( \frac{|A'|}{(1 + \varepsilon)|B| - 1} \right)^2.$$

By combining (23) and (28), after simplifying we obtain

$$(29) \quad 1 \geq \left( 2^{-1} - \frac{4\alpha}{\beta} \right) \frac{|B - B| - 1}{(1 + \varepsilon)|B| - 1}.$$

In both cases, by (26) and (29), after rearranging we get

$$(30) \quad |B - B| - 1 \leq \left( 2 + \frac{32\alpha}{\beta} \right) ((1 + \varepsilon)|B| - 1) \leq \left( 2 + \frac{32\alpha}{\beta} + 4\varepsilon \right) |B| - 2.$$

By Corollary 24, together with (12) and (30), it follows that there is an arithmetic progression  $Q$  containing  $B$  such that

$$|Q \setminus B| \leq \gamma|B|.$$

We may assume that  $Q$  is an interval. It is now easy to see that the interval  $J = [-|B| + 1, |B| - 1]$  of size  $2|B| - 1$  is contained inside  $B - B$ . Indeed, suppose instead that some  $l$ , with  $0 < l < |B|$ , does not belong to  $B - B$ . If  $l > |B|/2$  then at most one of each pair  $(t, t + l)$ , as  $t$  ranges from  $\min Q$  to  $\max Q - l$ , can belong to  $B$ , and this contradicts the fact that  $|B| \geq |Q|/(1 + \gamma)$ . On the other hand, if  $l \leq |B|/2$  then we consider the fibres of  $Q \bmod l$ : each fibre has size at least 2, and no fibre can have two consecutive points in  $B$ . It follows that each fibre meets  $B$  in at most  $2/3$  of its points, and again this contradicts the fact that  $|B| \geq |Q|/(1 + \gamma)$ .

Hence we may also assume that  $B$  is an interval and that  $J = B - B$ . By (11) we have  $\max_{c \in J} |(A + c) \setminus A| \leq |B| - 1 + \varepsilon|B|$ , so

$$\max_{c \in 2^{10}J} |(A + c) \setminus A| \leq 2^{10}(|B| - 1 + \varepsilon|B|).$$

Set  $K = \{x \in A^c : |(x - 2^{10}J) \cap A| \leq 2^{-11}\gamma|2^{10}J|\}$ , and note that

$$\begin{aligned} 2^{11}|B| &\geq 2^{10}(|B| - 1 + \varepsilon|B|) \geq \mathbb{E}_{c \in 2^{10}J} |(A + c) \setminus A| = \sum_{x \in A^c} \mathbb{P}_{c \in 2^{10}J}(x \in A + c) \\ &= \sum_{x \in A^c} \frac{|(x - 2^{10}J) \cap A|}{|2^{10}J|} \geq 2^{-11}\gamma|A^c \setminus K|. \end{aligned}$$

By (10), we have

$$|K| \geq |A^c| - 2^{22}\gamma^{-1}|B| \geq \beta p - 2^{22}\gamma^{-1}\alpha p > 0.$$

Therefore  $K \neq \emptyset$ , so we can pick an element  $x \in K$ . By construction,  $P = x - 2^{10}J$  is an interval of size

$$|P| = 2^{10}(2|B| - 1) - 2^{10} + 1 \in (2^{10}|B|, 2^{11}|B|)$$

such that  $|A \setminus P^c| = |A \cap P| \leq \gamma|B|$ . This concludes the proof of Theorem 25. ■

Here is the second technical result that we shall need.

**THEOREM 26:** *Let  $A$  and  $B$  be subsets of  $\mathbb{Z}_p$  and let  $I = [p_l, p_r]$  and  $J = [q_l, q_r]$  be intervals of  $\mathbb{Z}_p$  satisfying*

$$(31) \quad 2 \leq |B| \leq \min(|A|, p/2^{11}), \quad |I| = 2^{10}|B|, \quad |J| \leq (1 + 2^{-10})|B|$$

and

$$(32) \quad |A \setminus I^c| \leq 2^{-10}|B|, \quad \{q_l, q_r\} \subset B \subset J.$$

Then there exists  $B' \in B^{(3)}$  such that  $|A + B'| \geq |A| + |B| - 1$ .

*Proof.* By (31), there is a partition  $I = I_{-2} \sqcup I_{-1} \sqcup I_0 \sqcup I_1 \sqcup I_2$  of interval  $I$  into five consecutive disjoint intervals such that  $p_l \in I_{-2}$  and  $p_r \in I_2$ ,  $|I_0| = |I_{-2}| = |I_2| = |J|$ , and

$$|I_0 \cap A| \leq 2^{-4}|(I_{-1} \sqcup I_0 \sqcup I_1) \cap A|.$$

Let  $I_\infty$  be the complement of  $I$ :  $I_\infty = I^c$ . For a subset  $S \subset \{-2, -1, 0, 1, 2, \infty\}$  write  $A_S = \bigsqcup_{x \in S} (A \cap I_x)$ . By construction and by (32), for  $x, y \in B$ , we have the following three assertions:

(33) the sets  $A_{-1} + q_r, A_1 + q_l$  and  $A_{-2, \infty, 2} + y$  are disjoint,

(34) the sets  $A_{-1, 0, 1} + x$  and  $A_\infty + y$  are disjoint,

and

(35) 
$$|A_0| \leq 2^{-4}|A_{-1, 0, 1}|.$$

We will need two claims.

CLAIM A: *If  $|A_{-2, \infty, 2}| \geq |B| - 1$  then*

$$\mathbb{E}_{b \in B \setminus \{q_r\}} |A_{-2, \infty, 2} + \{q_l, q_r, b\}| \geq |A_{-2, \infty, 2}| + |B| - 1.$$

*If  $|A_{-2, \infty, 2}| \leq |B| - 1$  then either*

$$\mathbb{E}_{b \in B \setminus \{q_r\}} |A_{-2, \infty, 2} + \{q_l, q_r, b\}| \geq 2|A_{-2, \infty, 2}| + 2^{-3}(|B| - 1 - |A_{-2, \infty, 2}|)$$

or

$$\mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |A_{-2, \infty, 2} + \{q_l, b_2, b_3\}| \geq 2.5|A_{-2, \infty, 2}|.$$

*Proof.* Claim A refers to subsets  $A_{-2, \infty, 2} \subset I_{-2} \sqcup I_\infty \sqcup I_2$  and  $B \subset J$ . Given that the intervals  $I_{-2} \sqcup I_\infty \sqcup I_2$  and  $J$  satisfy

$$|I_{-2} \sqcup I_\infty \sqcup I_2| + |J| < p,$$

we may assume without loss of generality that the ambient space is  $\mathbb{Z}$  rather than  $\mathbb{Z}_p$ .

First, suppose that  $|A_{-2, \infty, 2}| \geq |B| - 1$ . By Theorem 19, (31) and (32) we have

$$\mathbb{E}_{b \in B \setminus \{q_r\}} |A_{-2, \infty, 2} + \{q_l, q_r, b\}| \geq |A_{-2, \infty, 2}| + |B| - 1.$$

Second, suppose that  $|A_{-2,\infty,2}| \leq |B| - 1$  and  $8|\pi_{q_r-q_l}(A_{-2,\infty,2})| \geq |\pi_{q_r-q_l}(B)|$ . Again, from Theorem 19, (31) and (32) we see that

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{q_r\}} |A_{-2,\infty,2} + \{q_l, q_r, b\}| &\geq 2|A_{-2,\infty,2}| + |\pi_{q_r-q_l}(A_{-2,\infty,2})| \left(1 - \frac{|A_{-2,\infty,2}|}{|B| - 1}\right) \\ &= 2|A_{-2,\infty,2}| + (|B| - 1 - |A_{-2,\infty,2}|) \frac{|\pi_{q_r-q_l}(A_{-2,\infty,2})|}{|\pi_{q_r-q_l}(B)|} \\ &\geq 2|A_{-2,\infty,2}| + 2^{-3}(|B| - 1 - |A_{-2,\infty,2}|). \end{aligned}$$

Third, suppose that  $|A_{-2,\infty,2}| \leq |B| - 1$  and  $8|\pi_{q_r-q_l}(A_{-2,\infty,2})| \leq |\pi_{q_r-q_l}(B)|$ . Then Lemma 21, (31) and (32) imply that

$$\mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |A_{-2,\infty,2} + \{q_l, b_2, b_3\}| \geq 2.5|A_{-2,\infty,2}|,$$

proving Claim A. ■

CLAIM B: *We have*

$$\mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, q_r, b\}) \setminus (A_{-2,\infty,2} + \{q_l, q_r, b\})| \geq (2 - 2^{-3})|A_{-1,0,1}|$$

and

$$\mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, b_2, b_3\}) \setminus (A_{-2,\infty,2} + \{q_l, b_2, b_3\})| \geq (2 - 2^{-3})|A_{-1,0,1}|.$$

*Proof.* By (33) and (34), we have

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, q_r, b\}) \setminus (A_{-2,\infty,2} + \{q_l, q_r, b\})| \\ \geq |A_{-1} + q_r| + |A_1 + q_l| + \mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + b) \setminus (A_{-2,-1,1,2} + \{q_l, q_r\})|. \end{aligned}$$

Also, (31) and (32) imply that

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + b) \setminus (A_{-2,-1,1,2} + \{q_l, q_r\})| \\ \geq \sum_{a \in A_{-1,0,1}} \mathbb{P}_{b \in B \setminus \{q_r\}} (a + b \notin A_{-2,-1,1,2} + \{q_l, q_r\}) \\ \geq |A_{-1,0,1}| \frac{|B| - 1 - 2|A_{-2,-1,1,2}|}{|B| - 1} \geq (1 - 2^{-4})|A_{-1,0,1}|. \end{aligned}$$

Combining these two inequalities, we get

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, q_r, b\}) \setminus (A_{-2,\infty,2} + \{q_l, q_r, b\})| \\ \geq |A_{-1}| + |A_1| + (1 - 2^{-4})|A_{-1,0,1}|. \end{aligned}$$

From (35) we conclude that

$$\mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, q_r, b\}) \setminus (A_{-2,\infty,2} + \{q_l, q_r, b\})| \geq (2 - 2^{-3})|A_{-1,0,1}|.$$

By (33) and (34), we get

$$\begin{aligned} \mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, b_2, b_3\}) \setminus (A_{-2,\infty,2} + \{q_l, b_2, b_3\})| \\ \geq \mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |(A_{-1,0,1} + b_3) \setminus (A_{-2,-1,0,1,2} + \{q_l, b_2\})| \\ + \mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |(A_{-1,0,1} + b_2) \setminus (A_{-2,-1,0,1,2} + \{q_l, b_3\})|. \end{aligned}$$

By (31) and (32), for fixed  $b_3 \in B$  we have

$$\begin{aligned} \mathbb{E}_{b_2 \in B \setminus \{q_r\}} |(A_{-1,0,1} + b_2) \setminus (A_{-2,-1,0,1,2} + \{q_l, b_3\})| \\ \geq \sum_{a \in A_{-1,0,1}} \mathbb{P}_{b_2 \in B \setminus \{q_r\}} (a + b_2 \notin A_{-2,-1,0,1,2} + \{q_l, b_3\}) \\ \geq |A_{-1,0,1}| \frac{|B| - 1 - 2|A_{-2,-1,0,1,2}|}{|B| - 1} \geq (1 - 2^{-4})|A_{-1,0,1}|. \end{aligned}$$

Combining the last two inequalities, we conclude that

$$\mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, b_2, b_3\}) \setminus (A_{-2,\infty,2} + \{q_l, b_2, b_3\})| \geq (2 - 2^{-3})|A_{-1,0,1}|.$$

This proves Claim B. ■

Returning to the proof of Theorem 26, if  $|A_{-2,\infty,2}| \geq |B| - 1$ , then from Claims A and B we have

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{q_r\}} |A + \{q_l, q_r, b\}| &\geq \mathbb{E}_{b \in B \setminus \{q_r\}} |A_{-2,\infty,2} + \{q_l, q_r, b\}| \\ &\quad + \mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, q_r, b\}) \setminus (A_{-2,\infty,2} + \{q_l, q_r, b\})| \\ &\geq |A_{-2,\infty,2}| + |B| - 1 + (2 - 2^{-3})|A_{-1,0,1}| \\ &\geq |A| + |B| - 1. \end{aligned}$$

If  $|A_{-2,\infty,2}| < |B| \leq |A|$ , then from Claims A and B we conclude that either

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{q_r\}} |A + \{q_l, q_r, b\}| &\geq \mathbb{E}_{b \in B \setminus \{q_r\}} |A_{-2,\infty,2} + \{q_l, q_r, b\}| \\ &\quad + \mathbb{E}_{b \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, q_r, b\}) \setminus (A_{-2,\infty,2} + \{q_l, q_r, b\})| \\ &\geq 2|A_{-2,\infty,2}| + 2^{-3}(|B| - 1 - |A_{-2,\infty,2}|) + (2 - 2^{-3})|A_{-1,0,1}| \\ &= (2 - 2^{-3})|A| + 2^{-3}(|B| - 1) \geq |A| + |B| - 1, \end{aligned}$$

or

$$\begin{aligned} \mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |A + \{q_l, b_2, b_3\}| &\geq \mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |A_{-2,\infty,2} + \{q_l, b_2, b_3\}| \\ &\quad + \mathbb{E}_{b_2, b_3 \in B \setminus \{q_r\}} |(A_{-1,0,1} + \{q_l, b_2, b_3\}) \setminus (A_{-2,\infty,2} + \{q_l, b_2, b_3\})| \\ &\geq 2.5|A_{-2,\infty,2}| + (2 - 2^{-3})|A_{-1,0,1}| \geq |A| + |B| - 1. \end{aligned}$$

The last inequality follows from the fact that  $|A_{-1,0,1}| \leq |A \setminus I^c| \leq 2^{-1}|A|$ .

This finishes the proof of Theorem 26. ■

We are now ready to prove Theorems 5 and 6.

**THEOREM 5:** *For every  $\beta > 0$  there exists  $\alpha > 0$  such that the following holds. Whenever  $A$  and  $B$  are non-empty subsets of  $\mathbb{Z}_p$  with  $|B| \leq \alpha|A|$  and  $|A| + |B| \leq (1 - \beta)p$ , there exist elements  $b_1, b_2, b_3 \in B$  such that*

$$|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1.$$

*Proof.* Fix  $2^{-10} > \beta > 0$ . Choose  $\gamma = 2^{-10}$ . Let  $\varepsilon', \alpha'$  be the output of Theorem 25 with input  $\beta, \gamma$ . Choose  $\alpha = \min(2^{-11}, \alpha')$ . Assume for a contradiction that

$$(36) \quad \max_{B' \in B^{(3)}} |A + B'| < |A| + |B| - 1.$$

Theorem 25 and the conditions on  $|A|$  and  $|B|$  imply that there are arithmetic progressions  $I$  and  $J$  with the same common difference and sizes at least  $2^{10}|B|$  and at most  $\lfloor (1 + \gamma)|B| \rfloor$ , respectively, such that  $|A \setminus I^c| \leq 2^{-10}|B|$  and  $|B \setminus J| = 0$ . We may assume without loss of generality that  $I = [p_l, \dots, p_r]$  and  $J = [q_l, \dots, q_r]$  are actually intervals satisfying

$$|I| = 2^{10}|B|, \quad |J| \leq (1 + 2^{-10})|B|, \quad |A \setminus I^c| \leq 2^{-10}|B|, \quad \{q_l, q_r\} \subset B \subset J.$$

Theorem 26 and these conditions on  $|A|$  and  $|B|$  imply that

$$\max_{B' \in B^{(3)}} |A + B'| \geq |A| + |B| - 1. \quad \blacksquare$$

**THEOREM 6:** *Let  $A$  and  $B$  be non-empty subsets of  $\mathbb{Z}_p$  with  $|B| \leq |A| \leq 2^{-20}p$  and  $B$  an interval. Then  $B$  has three elements,  $b_1, b_2$  and  $b_3$ , such that*

$$|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1.$$

*Proof.* Consider the interval  $J = B$ . By hypothesis, there exists an interval  $I$  of size  $2^{10}|B|$  such that  $|A \setminus I^c| \leq 2^{-10}|B|$  (consider a random interval  $I$ ). By Theorem 26 we conclude that

$$\max_{B' \in B^{(3)}} |A + B'| \geq |A| + |B| - 1. \quad \blacksquare$$

In the next section we turn our attention to the general case.

**6. Sums in  $\mathbb{Z}_p$ : the general case**

We now focus on our main aim, Theorem 7. Actually, in the form (below) that we state Theorem 7 we will allow  $r$  to be negative, and since in that case the conclusion of Theorem 7 is trivially false Theorem 4 follows immediately. This avoids us writing down what is essentially the same long argument twice. Of course, Theorem 4 contains Theorem 3 and Theorem 2 as special cases.

What we actually prove is the following strengthening of Theorem 7.

**THEOREM 27:** *For every  $\beta > 0$  there exists  $\varepsilon > 0$  such that for every  $\alpha > 0$  there is a value of  $c$  for which the following holds. Let  $A$  and  $B$  be subsets of  $\mathbb{Z}_p$ . Suppose that*

$$(37) \quad 2 \leq \min(|A|, |B|), \quad \alpha|B| \leq |A| \leq \alpha^{-1}|B|, \quad |A| + |B| \leq (1 - \beta)p$$

and

$$(38) \quad \max_{B' \in B^{(c)}} |A + B'| = |A| + |B| - 1 + r \leq |A| + |B| - 1 + \varepsilon \min(|A|, |B|)$$

for some integer  $r$ . Then  $B$  is contained in an arithmetic progression of size  $|B| + r$ .

As remarked earlier, we cannot insist that  $A$  is also contained in a short arithmetic progression. However, using the same method as above, one can show that there is an arithmetic progression of size  $|A| + r$  such that  $A$  is contained in this progression except possibly for at most  $r_0 \leq \delta(c)r$  terms, where  $\delta(c) \rightarrow 0$  as  $c \rightarrow \infty$ . This result is optimal in the sense that  $r_0/r \rightarrow 0$  need not hold if  $c \not\rightarrow \infty$ .

As the proof of Theorem 27 is rather involved, we give an informal overview before proceeding to the proof itself. Some of the statements in this overview are somewhat imprecise, but they will give the rough picture.

Let us assume that  $\varepsilon > 0$  is sufficiently small and the sets that  $A$  and  $B$  have sizes that are say within a factor of 10 of  $n$ , where  $n$  is much smaller than  $p$ .

Let  $Q$  be a shortest arithmetic progression containing  $B$ . Our aim is to show that there is a subset  $B'$  of  $B$  of size  $c$  such that  $|A + B'|$  is bounded from below either by  $|A| + |Q| - 1$  or by  $|A| + |B| - 1 + \varepsilon n$ . The two cases depend upon whether or not both  $A$  and  $B$  are close to arithmetic progressions.

The arguments for  $A$  and  $B$  turn out to be very similar, so we consider first the case when say  $B$  is far from every arithmetic progression  $R$  (using the symmetric difference as our measure), meaning say that  $|B\Delta R| > 100\varepsilon n$ . Then, by applying Theorem 8, we have subsets  $A^*$  and  $B^*$  of  $A$  and  $B$ , of sizes at least  $(1 - \varepsilon)|A|$  and  $(1 - \varepsilon)|B|$  respectively, such that  $|A + B'| > (1 - \varepsilon)|A^* + B^*|$  for some subset  $B'$  of  $B$  of size  $c$ . Note that the property of being far from any arithmetic progression is robust under small perturbations of the set. Thus, for any arithmetic progression  $R$ , we have  $|B^*\Delta R| > 100\varepsilon n - \varepsilon|B| > 50\varepsilon n$ . By Freiman's theorem we have that

$$|A^* + B^*| \geq |A^*| + |B^*| + 50\varepsilon n > |A| + |B| + 30\varepsilon n$$

which implies that  $|A + B'| > |A| + |B| + \varepsilon n$ , as claimed.

Thus we have reduced our problem to the case when both  $A$  and  $B$  are close (in symmetric difference) to some progressions  $P$  and  $R$ , i.e.,  $|A\Delta P|$  and  $|B\Delta R|$  are both at most  $100\varepsilon n$ . A careful examination of the above argument actually shows that  $P$  and  $R$  have the same common difference, and so from now on we may assume that  $P$  and  $R$  are intervals.

We now want to argue that  $B$  is not only a subset of the arithmetic progression  $Q$ , the arithmetic progression chosen at the beginning of the argument, but is also close to it (again in symmetric difference), say  $|Q \setminus B| < 10000\varepsilon n$ . If this is not the case, then  $B$  cannot be contained in the interval obtained from  $R$  by blowing it up by  $5000\varepsilon n$ , i.e., there is a point  $b \in B$  at distance at least  $5000\varepsilon n$  from  $R$ .

Let  $A_1$  be the part of  $A$  inside  $P$  and  $B_1$  be the part of  $B$  inside  $R$ . The sets  $A_1$  and  $B_1$  are about the same size as  $|A|$  and  $|B|$ :  $|A_1|$  and  $|B_1|$  are at least  $(1 - \varepsilon)|A|$  and  $(1 - \varepsilon)|B|$  respectively. Applying Theorem 8 again, we obtain subsets  $A^*$  and  $B^*$  of  $A_1$  and  $B_1$ , of sizes at least  $(1 - \varepsilon)|A_1|$  and  $(1 - \varepsilon)|B_1|$  respectively, such that  $|A_1 + B'_1| > (1 - \varepsilon)|A^* + B^*|$  for some subset  $B'_1$  of  $B_1$  of

size  $c$ . It follows that  $|A_1 + B'_1| > |A| + |B| - 1000\epsilon n$  and moreover that  $A_1 + B_1$  is a subset of  $P + R$ . Now using the additional point  $b$  which is far from  $R$  we get that  $P + b$  sticks out from  $P + R$  by many points, at least  $5000\epsilon n$ , and hence  $A_1 + b$  also sticks out from  $P + R$  by many points, namely at least  $4000\epsilon n$ . Therefore  $|A_1 + (B'_1 \cup b)| > |A| + |B| + 3000\epsilon n$ , say.

Hence both  $|A\Delta P|$  and  $|Q \setminus B|$  are small, namely at most  $10000\epsilon n$ . It is now that the most significant obstacle comes into play: it could be that  $|Q \setminus B|$  is small, like say  $\sqrt{n}$ .

For simplicity assume that  $A$  is contained in  $P$ , and recall that  $Q$  is the shortest arithmetic progression containing  $B$ . In this situation, in the sum  $A+B$  there is no wraparound, since  $|P| + |Q| < (1 + 10^5\epsilon)(|A| + |B|) < p$ . Hence we can view  $A$  and  $B$  as being subsets of  $\mathbb{Z}$  rather than  $\mathbb{Z}_p$ .

When  $|A| \geq |B|$ , we take a random subset  $B'$  of  $B$  formed from the leftmost and rightmost points of  $B$  and an additional random point of  $B$ . Recalling Theorem 1, we find that  $\mathbb{E}|A + B'| \geq |A| + |B| - 1$ .

We adopt an analogous strategy in general: let say  $|B|/100 \leq |A| \leq |B|$ . Consider the following ideal case. Suppose that there is an arithmetic progression  $M$  with step size  $d$ , containing the endpoints of  $B$ , such that  $d$  is much smaller than  $|A|$  and with  $B \cap M = Q \cap M$ . Set  $B_x = B \cap (x \pmod d)$  and assume  $B_0 = B \cap M$ . In this case, an analysis similar to that in the proof of Theorem 1 shows that, for a random  $x \in \mathbb{Z}_d$ , if we set  $B' = B_0 \cup B_x$  then  $\mathbb{E}|A + B'| \geq |A| + |Q| - 1$ . For this argument to work, it is important that  $A$  is close to  $P$  and  $B$  is close to  $Q$ . In general there need not be an arithmetic progression  $M$  with step size  $d$  (containing the endpoints of  $B$ ) such that  $d$  is much smaller than  $|A|$  and  $B \cap M = Q \cap M$ . However, as it turns out, we can always divide  $A$  and  $B$  into three consecutive blocks such that in each block we can perform this construction. For this argument to work we do also need that  $B$  is close to  $Q$ .

We now embark on the actual proof of Theorem 27. First we need a couple of straightforward estimates for which we omit the proofs.

LEMMA 28: *For any  $\mu > 0$  there exists  $c_0$  such that for all  $c \geq c_0$  the following holds. Let  $A, B$  be finite subsets of an abelian group. If  $C$  is a subset of  $B$  of size  $|C| \geq 2|B|/3$  then*

$$\mathbb{E}_{B' \in B^{(c)}} |A + (B' \cap C)| \geq (1 - \mu) \mathbb{E}_{C' \in C^{(\frac{c}{2})}} |A + C'|. \quad \blacksquare$$

LEMMA 29: For all  $m \in \mathbb{N}$  and  $\mu > 0$  there exists  $c_0$  such that for all  $c \geq c_0$  the following holds. Let  $A, B$  and  $C$  be finite subsets of an abelian group. If  $B_1, B_2, \dots, B_m$  are subsets of  $B$  of size  $|B_i| \geq \mu|B|$  then

$$\mathbb{E}_{B' \in B^{(c)}} |(A + B') \setminus C| \geq \frac{3}{4} \mathbb{E}_{\substack{b_j^1, b_j^2 \in B_j \\ j \in [m]}} \left| \left( A + \bigcup_{j \in [m]} \{b_j^1, b_j^2\} \right) \setminus C \right|. \quad \blacksquare$$

We also need the following weak stability result.

THEOREM 30: For all  $\beta, \gamma > 0$  there exists  $\varepsilon > 0$  such that for all  $\alpha > 0$  there is a value of  $c$  for which the following holds. Let  $A$  and  $B$  be subsets of  $\mathbb{Z}_p$  with

$$(39) \quad 2 \leq \min(|A|, |B|), \quad \alpha|B| \leq |A| \leq \frac{1}{\alpha}|B|, \quad |A| + |B| \leq (1 - \beta)p.$$

Then there exists an element  $b \in B$  such that if

$$(40) \quad \mathbb{E}_{B' \in B^{(c)}} |A + (B' \cup \{b\})| \leq |A| + |B| - 1 + \varepsilon \min(|A|, |B|)$$

then there exist arithmetic progressions  $P$  and  $Q$  with the same common difference such that

$$|A\Delta P| \leq \gamma \min(|A|, |B|), \quad |B\Delta Q| \leq \gamma \min(|A|, |B|), \quad B \subset Q.$$

*Proof of Theorem 30.* Fix  $2^{-10} > \beta > \gamma > \alpha > 0$ . Let  $\varepsilon_1$  be the output of Theorem 24 with input  $\beta, 2^{-11}\gamma$ . Let  $\varepsilon = \min(2^{-2}\varepsilon_1, 2^{-6}\gamma)$ . Let

$$\mu = 2^{-12} \min(\alpha\varepsilon, \alpha\gamma).$$

Let  $c_0$  be the output of Lemma 28 with input  $\mu$ . Let  $c_1$  be the output of Theorem 8 with input  $(4\alpha^{-1}, \mu)$ . Let  $c = 2 \max(c_0, c_1)$ . This time we partition our proof into three Claims.

CLAIM A: There exist arithmetic progressions  $U, V$  with the same common difference and

$$(41) \quad |A\Delta U| \leq \frac{\gamma}{2^{10}} \min(|A|, |B|) \quad \text{and} \quad |B\Delta V| \leq \frac{\gamma}{2^{10}} \min(|A|, |B|).$$

*Proof of Claim A.* By Theorem 8, there are subsets  $A^* \subset A$  and  $B^* \subset B$  with

$$(42) \quad |A^*| \geq \lceil (1 - \mu)|A| \rceil \quad \text{and} \quad |B^*| \geq \lceil (1 - \mu)|B| \rceil$$

such that

$$(43) \quad \mathbb{E}_{B' \in B^{(c)}} |A + B'| \geq \min \left( (1 - \mu)|A^* + B^*|, \frac{4}{\alpha}|A|, \frac{4}{\alpha}|B| \right).$$

By (39) we have

$$(44) \quad \min\left(\frac{4}{\alpha}|A|, \frac{4}{\alpha}|B|\right) \geq 4 \max(|A|, |B|) > |A| + |B| - 1 + \varepsilon \min(|A|, |B|),$$

and by (39) and (42) we find that

$$(45) \quad 2 \leq \min(|A^*|, |B^*|), \quad \frac{\alpha}{2}|B^*| \leq |A^*| \leq \frac{2}{\alpha}|B^*|, \quad |A^*| + |B^*| \leq (1 - \beta)p.$$

Hence (40), (43) and (44) imply

$$(46) \quad |A| + |B| - 1 + \varepsilon \min(|A|, |B|) \geq (1 - \mu)|A^* + B^*|,$$

and (42), (45) and (46) imply

$$(47) \quad \begin{aligned} |A^* + B^*| &\leq (1 - \mu)^{-1}(|A| + |B| - 1 + \varepsilon \min(|A|, |B|)) \\ &\leq (1 - \mu)^{-2}|A^*| + (1 - \mu)^{-2}|B^*| - 1 + 2\varepsilon \min(|A^*|, |B^*|) \\ &\leq |A^*| + |B^*| - 1 + 8\mu \max(|A^*|, |B^*|) + 2\varepsilon \min(|A^*|, |B^*|) \\ &\leq |A^*| + |B^*| - 1 + 4\varepsilon \min(|A^*|, |B^*|). \end{aligned}$$

Corollary 24, together with (45) and (47), now guarantees that there exist arithmetic progressions  $U$  and  $V$  with the same common difference such that

$$(48) \quad |A^* \Delta U| \leq \frac{\gamma}{2^{11}} \min(|A^*|, |B^*|) \quad \text{and} \quad |B^* \Delta V| \leq \frac{\gamma}{2^{11}} \min(|A^*|, |B^*|)$$

Claim A follows from (42) and (48):

$$\begin{aligned} |A \Delta U| &\leq |A^* \Delta U| + |A \Delta A^*| \leq \frac{\gamma}{2^{11}} \min(|A^*|, |B^*|) + \mu|A| \\ &\leq \frac{\gamma}{2^{11}} \min(|A|, |B|) + \mu|A| \leq \frac{\gamma}{2^{10}} \min(|A|, |B|). \end{aligned}$$

and

$$\begin{aligned} |B \Delta V| &\leq |B^* \Delta V| + |B \Delta B^*| \leq \frac{\gamma}{2^{11}} \min(|A^*|, |B^*|) + \mu|B| \\ &\leq \frac{\gamma}{2^{11}} \min(|A|, |B|) + \mu|B| \leq \frac{\gamma}{2^{10}} \min(|A|, |B|). \quad \blacksquare \end{aligned}$$

CLAIM B:

$$(49) \quad \mathbb{E}_{B' \in B^{(c)}} |(A \cap U) + (B' \cap V)| \geq |A| + |B| - 1 - \frac{\gamma}{2^8} \min(|A|, |B|).$$

*Proof of Claim B.* By Claim A and Lemma 28 we have

$$(50) \quad \mathbb{E}_{B' \in B^{(c)}} |(A \cap U) + (B' \cap V)| \geq (1 - \mu) \mathbb{E}_{C' \in (B \cap V)^{(c/2)}} |(A \cap U) + C'|.$$

By Theorem 8, we may find subsets  $A^* \subset A \cap U$  and  $B^* \subset B \cap V$  with

$$(51) \quad |A^*| \geq \lceil (1 - \mu)|A \cap U| \rceil \quad \text{and} \quad |B^*| \geq \lceil (1 - \mu)|B \cap V| \rceil$$

such that

$$(52) \quad \mathbb{E}_{C' \in (B \cap V)^{(c/2)}} |(A \cap U) + C'| \geq \min \left( (1 - \mu)|A^* + B^*|, \frac{4}{\alpha}|A \cap U|, \frac{4}{\alpha}|B \cap V| \right).$$

By (39) and (41) we have

$$(53) \quad \min \left( \frac{4}{\alpha}|A \cap U|, \frac{4}{\alpha}|B \cap V| \right) \geq 2 \max(|A \cap U|, |B \cap V|) \geq |A^*| + |B^*|.$$

By the Cauchy–Davenport theorem, together with (39), we have

$$(54) \quad |A^* + B^*| \geq \min(p, |A^*| + |B^*| - 1) = |A^*| + |B^*| - 1.$$

By (50), (52), (53) and (54) this yields

$$(55) \quad \mathbb{E}_{B' \in B^{(c)}} |(A \cap U) + (B' \cap V)| \geq (1 - \mu)^2(|A^*| + |B^*| - 1).$$

Finally, from (41), (51) and (55) we deduce that

$$\begin{aligned} \mathbb{E}_{B' \in B^{(c)}} |(A \cap U) + (B' \cap V)| &\geq (1 - \mu)^2(|A^*| + |B^*| - 1) \\ &\geq (1 - \mu)^3(|A \cap U| + |B \cap V|) - 1 \\ &\geq (1 - \mu)^3(|A| + |B| - 2^{-9}\gamma \min(|A|, |B|)) - 1 \\ &\geq |A| + |B| - 1 - 2^{-9}\gamma \min(|A|, |B|) - 8\mu \max(|A|, |B|) \\ &\geq |A| + |B| - 1 - 2^{-8}\gamma \min(|A|, |B|). \quad \blacksquare \end{aligned}$$

Let  $Q$  be the arithmetic progression which extends  $V$  on each side by exactly  $\lfloor 2^{-2}\gamma \min(|A|, |B|) \rfloor$ .

CLAIM C:

$$|B \Delta Q| \leq \gamma \min(|A|, |B|) \text{ and } |B \setminus Q| = 0.$$

*Proof of Claim C.* For the first part, by (41) we have

$$(56) \quad \begin{aligned} |B \Delta Q| &\leq |B \Delta V| + |V \Delta Q| \\ &\leq 2^{-10}\gamma \min(|A|, |B|) + 2 \lfloor 2^{-2}\gamma \min(|A|, |B|) \rfloor \\ &\leq \gamma \min(|A|, |B|). \end{aligned}$$

For the second part, we assume for a contradiction that we have some  $b \in B \setminus Q$ . This implies  $\lfloor 2^{-2}\gamma \min(|A|, |B|) \rfloor \geq 1$ , which gives

$$(57) \quad \lfloor 2^{-2}\gamma \min(|A|, |B|) \rfloor \geq 2^{-3}\gamma \min(|A|, |B|).$$

By (39), (41) and (56) we have

$$|U| + |Q| \leq (1 + \gamma)(|A| + |B|) \leq (1 + \gamma)(1 - \beta)p < p$$

and

$$|U| \geq 2^{-1}|A| \geq \lfloor 2^{-2}\gamma \min(|A|, |B|) \rfloor,$$

which yields

$$(58) \quad |(U + b) \setminus (U + V)| \geq \lfloor 2^{-2}\gamma \min(|A|, |B|) \rfloor.$$

By (41), (57), (58) we have

$$(59) \quad \begin{aligned} |(A + b) \setminus (U + V)| &\geq -|A\Delta U| + |(U + b) \setminus (U + V)| \\ &\geq -2^{-10}\gamma \min(|A|, |B|) \\ &\quad + \lfloor 2^{-2}\gamma \min(|A|, |B|) \rfloor \frac{\gamma}{16} \min(|A|, |B|). \end{aligned}$$

Finally, by (40), (49) and (59) we obtain

$$\begin{aligned} |A| + |B| - 1 + \varepsilon \min(|A|, |B|) &\geq \mathbb{E}_{B' \in B^{(c)}} |A + (B' \cup \{b\})| \\ &\geq |(A + b) \setminus (U + V)| + \mathbb{E}_{B' \in B^{(c)}} |(A \cap U) + (B' \cap V)| \\ &\geq 2^{-4}\gamma \min(|A|, |B|) + |A| + |B| - 1 - 2^{-8}\gamma \min(|A|, |B|) \\ &\geq |A| + |B| - 1 + 2^{-5}\gamma \min(|A|, |B|). \quad \blacksquare \end{aligned}$$

The conclusion of Theorem 30 now follows from Claims A and C. ■

We need one final result.

**THEOREM 31:** *For all  $\beta > 0$  there exists  $\gamma > 0$  such that for every  $\alpha > 0$  there is a value of  $c$  for which the following holds. Let  $A$  and  $B$  be subsets of  $\mathbb{Z}_p$  and let  $I = [p_l, p_r]$  and  $J = [q_l, q_r]$  be intervals of  $\mathbb{Z}_p$  satisfying*

$$(60) \quad \alpha|J| \leq |I| \leq \alpha^{-1}|J| \quad \text{and} \quad |I| + |J| \leq (1 - \beta)p$$

and

$$(61) \quad \max(|A\Delta I|, |B\Delta J|) \leq \gamma \min(|I|, |J|) \quad \text{and} \quad \{q_l, q_r\} \subset B \subset J.$$

*Then there is a family  $\mathcal{F} \subset B^{(c)}$ , depending only on  $I, J$  and  $B$  (but not on  $A$ ), such that*

$$\mathbb{E}_{B' \in \mathcal{F}} |A + B'| \geq |A| + |J| - 1 \geq |A| + |B| - 1.$$

*Proof.* Fix  $2^{-10} > \beta > 0$ . Let  $\gamma = 2^{-1000}\beta$ . Fix  $2^{-10} > \alpha > 0$ . Let  $s = \lfloor 2^{10} \log(\alpha^{-1}) \rfloor$ . Let  $c_1 = (18^{13}\alpha^{-1} + 1)(s + 1)$ . Let  $c_0$  be the output of Lemma 29 with input  $m = \lfloor 4\beta^{-1} \rfloor$  and  $\mu = 2^{-5}\alpha\beta$ . Let  $c' = \max(c_0, c_1)$  and  $c = 5c' + 2$ . The proof will be divided into three lemmas.

First, note that if  $A = I$  and  $B = J$  then trivially there exists a set  $B_0 \subset B$  (depending only on  $I$  and  $J$ ) such that  $|A + B_0| \geq |A| + |J| - 1$  and  $|B_0| \leq c'$ . Hence, for the rest of the proof we may assume that  $A \neq I$  or  $B \neq J$ , and so

$$(62) \quad \min(|I|, |J|) \geq 1/\gamma \quad \text{and} \quad \beta \geq 2^{1000}\gamma.$$

Moreover, by passing to a subinterval  $I'$  of  $I$  and changing  $\alpha$  to  $2^{-1}\alpha$  and  $\gamma$  to  $2\gamma$  if needed, we may assume that the size of  $|I|$  is a multiple of 6.

LEMMA 32: *We have*

$$\mathbb{E}_{B' \in B(c')} |((A \cap I^c) + (B' \cup \{q_l, q_r\})) \cap (I + J)^c| \geq |A \cap I^c|.$$

*Proof.* Construct intervals  $I_l$  and  $I_r$  of size  $\lfloor 2^{-1}\beta \min(|I|, |J|) \rfloor$  to the left and right of the interval  $I$ :  $I_l = p_l - \lfloor 2^{-1}\beta \min(|I|, |J|) \rfloor$  and  $I_r = p_r + \lfloor 2^{-1}\beta \min(|I|, |J|) \rfloor$ . By (60), we can construct the disjoint unions

$$K = I_l \sqcup I \sqcup I_r \quad \text{and} \quad L = (I_l + q_l) \sqcup (I + J) \sqcup (I_r + q_r).$$

By (60), there exists a collection  $\mathcal{J}$  of  $\lfloor 4\beta^{-1} \rfloor$  intervals  $\mathcal{J} = \{J_1, \dots, J_{\lfloor 4\beta^{-1} \rfloor}\}$  such that for every  $J_j \in \mathcal{J}$  we have  $J_j \subset J$  and  $|J_j| = \lfloor 2^{-1}\beta \min(|I|, |J|) \rfloor \geq 2^{10}$ , and for every  $x \in K^c$  there exists  $j_x \in \mathcal{J}$  such that

$$(63) \quad (x + J_{j_x}) \cap (I + J) = \emptyset.$$

For  $J_j \in \mathcal{J}$  set  $B_j = J_j \cap B$ . Recalling (60) and (61), we find that

$$(64) \quad |B_j| \geq 2^{-3}\beta \min(|I|, |J|) \geq 2^{-5}\alpha\beta|B|.$$

It is time to put together what we have learned about our sets. By construction we have

$$(65) \quad \begin{aligned} & \mathbb{E}_{B' \in B(c')} |((A \cap I^c) + (B' \cup \{q_l, q_r\})) \cap (I + J)^c| \\ & \geq |A \cap I_l| + |A \cap I_r| \\ & \quad + \mathbb{E}_{B' \in B(c')} |((A \cap I^c) + B') \setminus ((A \cap I_l) + q_l) \cup [I + J] \cup ((A \cap I_r) + q_r)|. \end{aligned}$$

By Lemma 29 combined with (64) we have

$$\begin{aligned}
 & \mathbb{E}_{B' \in B(c')} |((A \cap I^c) + B') \setminus ((A \cap I_l) + q_l) \cup [I + J] \cup [(A \cap I_r) + q_r]| \\
 (66) \quad & \geq \frac{3}{4} \mathbb{E}_{\substack{b_j^1, b_j^2 \in B_j \\ j \in [4\beta^{-1}]}} |((A \cap I^c) + \cup_j \{b_j^1, b_j^2\}) \setminus ((A \cap I_l) + q_l) \\
 & \qquad \qquad \qquad \cup [I + J] \cup [(A \cap I_r) + q_r]|.
 \end{aligned}$$

By (63) we have

$$\begin{aligned}
 & \mathbb{E}_{\substack{b_j^1, b_j^2 \in B_j \\ j \in [4\beta^{-1}]}} |((A \cap K^c) + \cup_j \{b_j^1, b_j^2\}) \setminus ((A \cap I_l) + q_l) \cup [I + J] \cup [(A \cap I_r) + q_r]| \\
 (67) \quad & \geq \mathbb{E}_{\substack{b_j^1, b_j^2 \in B_j \\ j \in [4\beta^{-1}]}} \left| \left( \bigcup_{a \in A \cap K^c} a + \{b_{j_a}^1, b_{j_a}^2\} \right) \setminus ((A \cap I_l) + q_l) \cup [(A \cap I_r) + q_r] \right|.
 \end{aligned}$$

A trivial bound gives

$$\begin{aligned}
 & \mathbb{E}_{\substack{b_j^1, b_j^2 \in B_j \\ j \in [4\beta^{-1}]}} \left| \left( \bigcup_{a \in A \cap K^c} a + \{b_{j_a}^1, b_{j_a}^2\} \right) \setminus ((A \cap I_l) + q_l) \cup [(A \cap I_r) + q_r] \right| \\
 (68) \quad & \geq \sum_{\substack{a \in A \cap K^c \\ i \in \{1,2\}}} \mathbb{E}_{\substack{b_j^1, b_j^2 \in B_j \\ j \in [4\beta^{-1}]}} \left| (a + b_{j_a}^i) \setminus \left( [(A \cap I_l) + q_l] \cup [(A \cap I_r) + q_r] \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \cup \left[ \bigcup_{\substack{a' \in A \cap K^c \\ i' \in \{1,2\}: \\ j_{a'} \neq j_a \text{ or } i' \neq i}} a' + b_{j_{a'}}^{i'} \right] \right) \right|.
 \end{aligned}$$

Continuing, by (61), given  $b_j^i \in B_j$  for  $i = 1, 2$  and  $j \in [4\beta^{-1}]$ , we have

$$\begin{aligned}
 & \left| [(A \cap I_l) + q_l] \cup [(A \cap I_r) + q_r] \cup \left[ \bigcup_{\substack{a \in A \cap K^c \\ i \in \{1,2\}}} a + b_{j_a}^i \right] \right| \leq 2|A \cap I^c| \\
 & \qquad \qquad \qquad \leq 2\gamma \min(|I|, |J|).
 \end{aligned}$$

By (64) and the last inequality we have

$$\begin{aligned}
 (69) \quad & \sum_{\substack{a \in A \cap K^c \\ i \in \{1,2\}}} \mathbb{E}_{b_j^1, b_j^2 \in B_j} \left| (a + b_{j_a}^i) \setminus \left( [(A \cap I_l) + q_l] \cup [(A \cap I_r) + q_r] \right. \right. \\
 & \left. \left. \cup \left[ \bigcup_{\substack{a' \in A \cap K^c \\ i' \in \{1,2\}: \\ j_{a'} \neq j_a \text{ or } i' \neq i}} a' + b_{j_{a'}}^{i'} \right] \right) \right| \\
 & \geq \frac{3}{4} 2|A \cap K^c| = \frac{3}{2} |A \cap K^c|.
 \end{aligned}$$

By (66), (67), (68) and (69) we get

$$(70) \quad \mathbb{E}_{B' \in \mathcal{B}(c')} |((A \cap I^c) + B') \setminus ((A \cap I_l) + q_l] \cup [I + J] \cup [(A \cap I_r) + q_r])| \geq \frac{9}{8} |A \cap K^c|.$$

Finally, by (65) and (70) we obtain

$$\begin{aligned}
 \mathbb{E}_{B' \in \mathcal{B}(c')} |((A \cap I^c) + (B' \cup \{q_l, q_r\})) \cap (I + J)^c| & \geq |A \cap I_l| + |A \cap I_r| + \frac{9}{8} |A \cap K^c| \\
 & \geq |A \cap I^c|.
 \end{aligned}$$

This concludes the proof of Lemma 32. ■

In the rest of the proof of Theorem 31 we shall only be interested in the subsets  $A \cap I$  and  $B = B \cap J$ . Given that the intervals  $I$  and  $J$  satisfy  $|I| + |J| < p$ , we may assume from now on that the ambient space is  $\mathbb{Z}$  rather than  $\mathbb{Z}_p$ .

Here is the second lemma we shall need.

LEMMA 33: *For each  $i = 1, 2, 3$  there exists a parameter  $d_i \in \mathbb{N}$  and there exist consecutive intervals  $I_i = [p_{i-1} + 1, p_i]$  and  $J_i = [q_{i-1}, q_i]$ , depending only on  $I, J$  and  $B$  (but not on  $A$ ), such that the sets  $A_i = A \cap I_i$  and  $B_i = B \cap J_i$  satisfy*

- (1)  $I = I_1 \sqcup I_2 \sqcup I_3$  and  $J = J_1 \cup J_2 \cup J_3$ ,
- (2)  $6^{-13} \alpha |J_i| \leq |I_i| \leq 6^{13} \alpha^{-1} |J_i|$ ,
- (3)  $\max(|I_i \Delta A_i|, |J_i \Delta B_i|) \leq 6^{13} \gamma \min(|I_i|, |J_i|)$ ,
- (4)  $3d_i \leq \min(|I_i|, |J_i|) \leq (3^{10} + 1)d_i$ ,
- (5)  $d_i | (q_i - q_{i-1})$ ,
- (6)  $J_i \cap \{q_i \bmod d_i\} = B_i \cap \{q_i \bmod d_i\}$ .

*Proof.* Fix parameters  $q_0 = q_l, q_3 = q_r, k_1 = 2^{10}$  and  $k_3 = 3^{10}$  and

$$(71) \quad k_2 = 6 \left\lfloor \frac{|J|}{\min(|I|, |J|)} \right\rfloor - 1 \quad \text{and} \quad d_2 = \lfloor 6^{-1} \min(|I|, |J|) \rfloor.$$

Since (62) holds and  $|I|$  is a multiple of 6, we have

$$(72) \quad |J| - 6^{-1} \min(|I|, |J|) \geq k_2 d_2 \geq \max(|J| - 2 \min(|I|, |J|), 2^{-1} \min(|I|, |J|)).$$

By the Chinese remainder theorem, there exist parameters  $q_1, q_2, d_1, d_3 \in \mathbb{Z}$  such that  $q_1 = q_l + k_1 d_1, q_2 = q_r - k_3 d_3$  and  $q_2 - q_1 = k_2 d_2$ . For  $t \in \mathbb{Z}$  construct parameters  $q_{0,t} = q_l, q_{1,t} = q_1 + tk_1 k_3, q_{2,t} = q_2 + tk_1 k_3, q_{3,t} = q_r, d_{1,t} = d_1 + tk_3, d_{2,t} = d_2$  and  $d_{3,t} = d_3 - tk_1$  such that  $q_{1,t} = q_{0,t} + k_1 d_{1,t}, q_{2,t} = q_{3,t} - k_3 d_{3,t}$  and  $q_{2,t} - q_{1,t} = k_2 d_{2,t} = k_2 d_2$ .

Then the set  $T = \{t \in \mathbb{Z} : q_l = q_{0,t} < q_{1,t} < q_{2,t} < q_{3,t} = q_r\}$  is an interval of size

$$|T| \geq \frac{|J| - 2 - k_2 d_2}{k_1 k_3} - 1.$$

Using our earlier inequalities, we find that  $|T| \geq 6^{-12} \min(|I|, |J|) \geq 6$ . Therefore, for  $i = 1, 2, 3$  there are disjoint consecutive intervals  $T_i = [t_{i-1} + 1, t_i]$  of sizes satisfying  $|T_2| = \lfloor 6^{-13} \min(|I|, |J|) \rfloor \geq 1$  and  $\min(|T_1|, |T_3|) \geq 6^{-13} \min(|I|, |J|)$ , which form a partition  $T = T_1 \sqcup T_2 \sqcup T_3$ . We also have  $|I| \geq 4 \lfloor 2^{-2} \min(|I|, |J|) \rfloor \geq 4$ . Therefore, for  $i = 1, 2, 3$  there are disjoint consecutive intervals

$$I_i = [p_{i-1} + 1, p_i]$$

of sizes  $|I| \geq |I_2| \geq 2^{-1}|I|$  and  $|I_1| = |I_3| = \lfloor 2^{-2} \min(|I|, |J|) \rfloor \geq 1$ , forming a partition  $I = I_1 \sqcup I_2 \sqcup I_3$ . Finally, for  $i = 1, 2, 3$  and  $t \in T_2$  we take the overlapping consecutive intervals  $J_{i,t} = [q_{i-1,t}, q_{i,t}]$ , which form a cover  $J = J_{1,t} \cup J_{2,t} \cup J_{3,t}$ .

Now define functions

$$(73) \quad g_1(i_1, t) = q_{0,t} + i_1 d_{1,t}, \quad g_2(i_2, t) = q_{1,t} + i_2 d_{2,t}, \quad g_3(i_3, t) = q_{3,t} - i_3 d_{3,t}.$$

Form the partition  $T_2 = S \sqcup S^c$ , where

$$(74) \quad S = \{t \in T_2 : g_i(j_i, t) \in B \text{ for all } i = 1, 2, 3 \text{ and } j_i \in [k_i]\}.$$

It turns out that  $S$  is non-empty, and that if we set, for any fixed  $t \in S, J_i = J_{i,t}$ , then the given parameters and intervals do have the required properties (1)–(6). The proof of this is rather lengthy and tedious, and not very instructive. For this reason we present the details in the Appendix. ■

The final lemma we shall need in the proof of Theorem 31 is as follows.

LEMMA 34: *Under the conditions of Lemma 33, for  $i = 1, 2, 3$  we have*

$$\mathbb{E}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| A_i + \left( B_i^{q_i} \bigcup_{j \in [s]} B_i^{w_j} \right) \right| \geq |A_i| + |J_i| - 1 \quad \text{and} \quad \max_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| B_i^{q_i} \bigcup_{j \in [s]} B_i^{w_j} \right| \leq c'.$$

*Proof.* Fix  $i$ . The proof is divided into two claims, but before we begin we need a simple remark. By condition (5) we can write  $q_i - q_{i-1} = k_i d_i$  for some  $k_i \in \mathbb{N}$ . Moreover, by conditions (2) and (4) we have  $k_i \leq 18^{13} \alpha^{-1}$ .

CLAIM A: *We have  $|A_i + B_i^{q_i}| \geq |A_i| + k_i |\widetilde{A}_i|$  and  $\pi_{d_i}(A_i + B_i^{q_i}) = q_i + \widetilde{A}_i$ .*

*Proof.* By condition (6) we have  $|B_i^{q_i}| = k_i + 1$ . Also, if  $x \in q_i + \widetilde{A}_i$  then  $|A_i^{x - q_i}| \geq 1$ . By Theorem 13 it follows that

$$|(A_i + B_i^{q_i})^x| \geq |A_i^{x - q_i}| + |B_i^{q_i}| - 1 = |A_i^{x - q_i}| + k_i.$$

Hence

$$|A_i + B_i^{q_i}| \geq \sum_{x \in q_i + \widetilde{A}_i} |(A_i + B_i^{q_i})^x| \geq \sum_{y \in \widetilde{A}_i} (|A_i^y| + k_i) = |A_i| + k_i |\widetilde{A}_i|$$

completing the proof of Claim A. ■

CLAIM B: *For  $z \in \mathbb{Z}_{d_i}$ , we have  $\mathbb{E}_{w_j \in \mathbb{Z}_{d_i}, j \in [s]} |(A_i + \bigcup_{j \in [s]} B_i^{w_j})^z| \geq k_i$ .*

*Proof.* By condition (3) we have

$$\begin{aligned} \max \left( \sum_{w \in \mathbb{Z}_{d_i}} |I_i^w \setminus A_i^w|, \sum_{w \in \mathbb{Z}_{d_i}} |J_i^w \setminus B_i^w| \right) &\leq \max(|I_i \Delta A_i|, |J_i \Delta B_i|) \\ &\leq 6^{13} \gamma \min(|I_i|, |J_i|), \end{aligned}$$

and condition (4) implies that

$$\max \left( \sum_{w \in \mathbb{Z}_{d_i}} |I_i^w \setminus A_i^w|, \sum_{w \in \mathbb{Z}_{d_i}} |J_i^w \setminus B_i^w| \right) \leq 18^{13} \gamma d_i.$$

By condition (4), if  $w \in \mathbb{Z}_{d_i}$  then  $|I_i^w| \geq 2$  and  $|J_i^w| \geq k_i$ , so

$$\max \left( \sum_{w \in \mathbb{Z}_{d_i}} \max(2 - |A_i^w|, 0), \sum_{w \in \mathbb{Z}_{d_i}} \max(k_i - |B_i^w|, 0) \right) \leq 18^{13} \gamma d_i.$$

We conclude that the sets

$$X = \{w \in \mathbb{Z}_{d_i} : |A_i^w| \leq 1\} \quad \text{and} \quad Y = \{w \in \mathbb{Z}_{d_i} : |B_i^w| \leq k_i - 1\}$$

are not too large:  $\max(|X|, |Y|) \leq 18^{13} \gamma d_i \leq 3^{-1} d_i$ . Thus

$$\mathbb{P}_{w \in \mathbb{Z}_{d_i}} (w \notin Y \text{ and } z - w \notin X) \geq 1/3$$

and

$$\mathbb{P}_{w_j \in \mathbb{Z}_{d_i}, j \in [s]} (\exists j; w_j \notin Y \text{ and } z - w_j \notin X) \geq 1 - 2^s 3^{-s}.$$

Equivalently, the event

$$E = \{(w_1, \dots, w_d) \in \mathbb{Z}_{d_i}^s; \exists j; w_j \notin Y \text{ and } z - w_j \notin X\}$$

has measure

$$\mathbb{P}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} ((w_1, \dots, w_s) \in E) \geq 1 - 2^s 3^{-s}.$$

By construction and by Theorem 13, if  $(w_1, \dots, w_s) \in E \subset \mathbb{Z}_{d_i}^s$  then

$$\begin{aligned} \left| \left( A_i + \bigcup_{j \in [s]} B_i^{w_j} \right)^z \right| &\geq \max_{j \in [s]} |A_i^{z-w_j} + B_i^{w_j}| \\ &\geq 2 + k_i - 1 = k_i + 1. \end{aligned}$$

From the last two inequalities it follows that

$$\begin{aligned} \mathbb{E}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| \left( A_i + \bigcup_{j \in [s]} B_i^{w_j} \right)^z \right| &\geq \mathbb{E}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left( \left| \left( A_i + \bigcup_{j \in [s]} B_i^{w_j} \right)^z \right|; E \right) \\ &\quad \times \mathbb{P}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} ((w_1, \dots, w_s) \in E) \\ &\geq (k_i + 1)(1 - 2^s 3^{-s}) \geq k_i, \end{aligned}$$

where the last inequality follows from the estimates  $k_i \leq 18^{13} \alpha^{-1}$ ,  $s \geq 2^9 \log \alpha^{-1}$  and  $\alpha < 2^{-10}$ . This finishes the proof of Claim B. ■

Returning to the proof of Lemma 34, we see that by combining Claim A and Claim B we obtain the first part of what we want:

$$\begin{aligned}
 & \mathbb{E}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| A_i + \left( B_i^{q_i} \bigcup_{j \in [s]} B_i^{w_j} \right) \right| \\
 & \geq \sum_{z \in \mathbb{Z}_{d_i}} \mathbb{E}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| \left( A_i + \left( B_i^{q_i} \bigcup_{j \in [s]} B_i^{w_j} \right) \right)^z \right| \\
 & \geq \sum_{z \in q_i + \widetilde{A}_i} |(A_i + B_i^{q_i})^z| + \sum_{z \in (q_i + \widetilde{A}_i)^c} \mathbb{E}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| \left( A_i + \bigcup_{j \in [s]} B_i^{w_j} \right)^z \right| \\
 & \geq |A_i + B_i^{q_i}| + \sum_{z \in (q_i + \widetilde{A}_i)^c} \mathbb{E}_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| \left( A_i + \bigcup_{j \in [s]} B_i^{w_j} \right)^z \right| \\
 & \geq |A_i| + |\widetilde{A}_i|k_i + (d_i - |\widetilde{A}_i|)k_i = |A_i| + d_i k_i = |A_i| + |J_i| - 1.
 \end{aligned}$$

Note also that for each  $w \in \mathbb{Z}_{d_i}$ , by construction we have  $|B_i^w| \leq k_i + 1$ . Hence we have the second part

$$\max_{\substack{w_j \in \mathbb{Z}_{d_i} \\ j \in [s]}} \left| B_i^{q_i} \bigcup_{j \in [s]} B_i^{w_j} \right| \leq (s + 1)(k_i + 1) \leq c',$$

where the last inequality follows from the bounds  $k_i \leq 18^{13}\alpha^{-1}$ ,  $s \leq 2^{10} \log \alpha^{-1}$  and  $\alpha < 2^{-10}$ . This concludes the proof of Lemma 34. ■

To finish the proof of Theorem 31 we shall construct the desired family of sets  $\mathcal{F}$ . We remark that we allow these families to have repeated sets. For each  $i = 1, 2, 3$  define

$$\mathcal{F}_i = \left\{ B_i^{q_i} \bigcup_{j \in [s]} B_i^{w_j}; (w_1, \dots, w_s) \in \mathbb{Z}_{d_i}^s \right\}.$$

In addition, put  $\mathcal{F}_0 = \{B_0\}$  and  $\mathcal{F}_4 = \{B' \cup \{q_l, q_r\}; B' \in B^{(c')}\}$ . Finally, let

$$\mathcal{F} = \{F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4; F_i \in \mathcal{F}_i\}.$$

By construction (see Lemmas 33 and 34, and the remark before Lemma 32), the family  $\mathcal{F}$  depends only on the sets  $I, J$  and  $B$  (but not  $A$ ), and any set  $F \in \mathcal{F}$  has size  $|F| \leq 3c' + c' + c' + 2 \leq c$ . In the case  $A = I$  and  $B = J$ , we have  $\mathbb{E}_{B' \in \mathcal{F}} |A + B'| \geq |A + B_0| \geq |A| + |J| - 1$ .

In the case  $A \neq I$  or  $B \neq J$ , combining Lemmas 32 and 34, we obtain

$$\begin{aligned}
 \mathbb{E}_{B' \in \mathcal{F}} |A + B'| &\geq \mathbb{E}_{\substack{B'_i \in \mathcal{F}_i \\ i \in [4]}} \left| A + \bigcup_{i \in [4]} B'_i \right| \\
 &\geq \mathbb{E}_{B' \in B(c')} |(A + (B' \cup \{q_l, q_r\})) \cap (I + J)^c| \\
 &\quad + \sum_{i \in [3]} \mathbb{E}_{B'_i \in \mathcal{F}_i} |(A + B'_i) \cap (I_i + J_i)| \\
 &\geq \mathbb{E}_{B' \in B(c')} |((A \cap I^c) + (B' \cup \{q_l, q_r\})) \cap (I + J)^c| \\
 &\quad + \sum_{i \in [3]} \mathbb{E}_{B'_i \in \mathcal{F}_i} |A_i + B'_i| \\
 &\geq |A \cap I^c| + \sum_{i \in [3]} |A_i| + |J_i| - 1 \\
 &\geq |A \cap I^c| + |A \cap I| + |J| - 1 \geq |A| + |J| - 1,
 \end{aligned}$$

where the second inequality follows by considering the partition

$$\mathbb{Z}_p = (I + J)^c \bigsqcup_{i \in [3]} (I_i + J_i).$$

This finishes the proof of Theorem 31. ■

We are now ready to prove Theorem 27.

*Proof of Theorem 27.* Fix  $2^{-10} > \beta > 0$ . Let  $\gamma$  be the output of Theorem 31 with input  $\beta$ , with the extra condition that  $2^{-10}\beta \geq \gamma$ . Let  $\varepsilon$  be the output of Theorem 30 with input  $\beta, \gamma$ . Let  $c_1$  be the output of Theorem 30 with input  $\beta, \gamma, \alpha$ . Let  $c_2$  be the output of Theorem 31 with input  $2^{-1}\beta, 2^{-1}\alpha$ . Finally, let  $c = \max(c_1, c_2)$ .

Combining Theorem 30 with inequalities (37) and (38), we see that there exist arithmetic progressions  $I$  and  $J$  with the same common difference such that  $|A \Delta I| \leq \gamma \min(|A|, |B|)$ ,  $|B \Delta J| \leq \gamma \min(|A|, |B|)$  and  $|B \setminus J| = 0$ . By scaling, we may assume that  $I$  and  $J$  are intervals  $I = [p_l, p_r]$  and  $J = [q_l, q_r]$ . Moreover, we may assume that  $\{q_l, q_r\} \subset B \subset J$ . Furthermore, we have

$$2^{-1}\alpha|J| \leq |I| \leq 2\alpha^{-1}|J| \quad \text{and} \quad |I| + |J| \leq (1 - 2^{-1}\beta)p.$$

Theorem 31 and the relations we have just found imply that there is a family  $\mathcal{F} \subset B^{(c)}$  such that

$$\mathbb{E}_{B' \in \mathcal{F}} |A + B'| \geq |A| + |J| - 1.$$

Finally, recalling (38), we obtain

$$|A| + |J| - 1 \leq |A| + |B| - 1 + r.$$

Thus  $J$  is an arithmetic progression of size at most  $|B| + r$  containing  $B$  (and so  $r \geq 0$ ). ■

We have thus proved our main results in  $\mathbb{Z}_p$  since, as we remarked at the start of the section, Theorem 27 contains both Theorems 4 and 7 as special cases.

### 7. Applications—restricted sums

We now turn to the applications mentioned earlier. In a slight variant of earlier notation, given sets  $A, B \subset \mathbb{Z}_p$  and a set  $\mathcal{F} \subset A \times B$ , the  **$\mathcal{F}$ -restricted sum** of  $A$  and  $B$  is

$$A +_{\mathcal{F}} B = \{a + b : (a, b) \notin \mathcal{F}\},$$

so that  $\mathcal{F}$  is the set of **forbidden pairs**. These general  $\mathcal{F}$ -restricted sums were introduced and studied by Lev [41, 42, 43] over twenty years ago. (We hope that the reader will not be too annoyed that the subscript now refers to the pairs we do not take, as opposed to the earlier notation of  $A +_{\Gamma} B$ : both of these notations are standard, unfortunately, and we will use a curly letter in this section to emphasise the difference.)

For  $b \in B$ , we call  $d(b) = |\{a : (a, b) \in \mathcal{F}\}|$  the **degree** of  $b \in B$ , and  $d(\mathcal{F}) = \max_{b \in B} d(b)$  the **degree** of  $\mathcal{F}$ . Note that if  $\mathcal{F} \subset A \times A$  then  $d(a)$  is the number of pairs  $(a', a) \in \mathcal{F}$ , with  $a$  in the second factor of  $A \times A$ .

We start with two questions. Firstly, can one extend the Erdős–Heilbronn conjecture along the lines of Theorem 3, our extension of the Cauchy–Davenport theorem? In other words, is there a constant  $c$  such that if  $p \geq 2n - 3$  and  $A$  is a subset of  $\mathbb{Z}_p$  of size  $n$ , then there is a set  $B' \in A^{(\leq c)}$  such that  $\{a + b : a \in A, b \in B', a \neq b\}$  has at least  $2n - 3$  elements?

Secondly, can one extend the Erdős–Heilbronn conjecture in another direction, replacing the sum  $A + A$  by an  $\mathcal{F}$ -restricted sum  $A +_{\mathcal{F}} A$  for a suitable family  $\mathcal{F}$ ? Is it true that if  $0 \leq d \leq n - 1$ ,  $p \geq 2n$  and we have  $A \subset \mathbb{Z}_p$  of

size  $n$  and  $\mathcal{F} \subset A \times A$  with  $d(\mathcal{F}) \leq d$ , then the  $\mathcal{F}$ -restricted sum  $A +_{\mathcal{F}} A$  has at least  $2n - 2d - 1$  elements? The Cauchy–Davenport theorem is the case  $d = 0$ , and the Erdős–Heilbronn conjecture [19], made in 1964, and proved thirty years later by Dias da Silva and Hamidoune [17], using tools from linear algebra and the representation theory of the symmetric group, is the case  $d = 1$  with the very special family  $\mathcal{F} = \{(a, a) : a \in A\}$ . Subsequently Alon, Nathanson and Ruzsa [1] gave a simpler proof of the Erdős–Heilbronn conjecture, which was based on the polynomial method. More recently, Vu and Wood [67] gave an entirely combinatorial proof under a weak additional condition.

Our next result shows that both extensions hold simultaneously, provided  $p - 2n$  is not too small and  $d = d(n)$  does not grow too fast with  $n$ . As we shall see, the proof is a straightforward application of Theorem 7. Note that of course this result implies Theorem 9 from the Introduction.

**THEOREM 35:** *Let  $(d(n))_1^\infty$  be a sequence of natural numbers with  $d = d(n) = o(n)$ , and let  $\beta > 0$ . Then there are integers  $c$  and  $n_0$  such that the following holds.*

*Let  $A \subset \mathbb{Z}_p$  with  $n_0 \leq |A| = n \leq (1 - \beta)p/2$ , and let  $\mathcal{F} \subset A \times A$  have degree  $d$ . Then there is a set  $B' \in A^{(\leq c)}$  such that  $|A +_{\mathcal{F}} B'| \geq 2n - 1 - 2d$ .*

*Proof.* Let  $c \geq 2$  and  $\varepsilon > 0$  be the constants guaranteed by Theorem 7 for  $\alpha = 1$  and our  $\beta > 0$ , and then let  $n_0$  be such that if  $n \geq n_0$  then  $cd < \min(\varepsilon n, 2^{-1}\beta n)$ . Finally, assume  $n \geq n_0$ .

If there is a set  $B' \subset A$  with  $|B'| = c$  such that  $|A + B'| \geq 2n + r = 2n + cd$ , then this  $B'$  will do for every  $\mathcal{F}$ , since

$$|A +_{\mathcal{F}} B'| \geq 2n + cd - \sum_{b \in B'} d(b) \geq 2n + cd - |B'|d = 2n.$$

Otherwise, assume that

$$\max_{B' \in A^{\leq c}} |A + B'| \leq 2n - 1 + cd \leq 2n - 1 + \varepsilon n.$$

Applying Theorem 7 to the sets  $A$  and  $B = A$ , we find that  $A$  is contained in an arithmetic progression of size  $n + cd$ . We may assume that the common difference of this arithmetic progression is 1, and we may also assume that  $A$  is contained in the interval  $J = \{1, \dots, n + cd\}$  of  $\mathbb{Z}_p$ . Then  $J + J$  has no wraparound, so neither has  $A + A$ . Hence, writing  $a_1$  for the first element of  $A$  and  $a_n$  for the last, the  $\mathcal{F}$ -restricted sum of  $A$  and  $B' = \{a_1, a_n\}$  has at least  $2n - 1 - 2d$  elements. ■

The bound  $2n - 1 - 2d$  in Theorem 35 cannot be improved: this is the best bound not only for  $|A +_{\mathcal{F}} B'|$  but also for  $|A +_{\mathcal{F}} A|$ , even if we demand that we have few forbidden pairs. Indeed, assuming, as we may, that  $n > 2d$ , this is shown by the set

$$\mathcal{F} = \{(i, j) : 1 \leq i, j \leq d \text{ or } n - d < i, j \leq n\}$$

of size only  $d^2$  and the set  $A = \{1, \dots, n\}$ . Indeed, in this case the  $\mathcal{F}$ -restricted sum  $A +_{\mathcal{F}} A$  of  $A$  and  $A$  is  $\{d + 2, \dots, 2n - d\}$ .

Let us give another application of Theorem 7 to a restricted sum. This time we shall assume that the size of  $\mathcal{F}$  is not too large. It is perhaps surprising that  $|A +_{\mathcal{F}} A|$  can decrease considerably as we increase the number of forbidden pairs.

**THEOREM 36:** *Let  $k = k(n) = o(\sqrt{n})$ ,  $k \geq 2$ , and  $0 < \beta < 1/2$ . Let  $c \geq 2$  and  $0 < \varepsilon < 1/2$  be the constants guaranteed by Theorem 7 for  $\alpha = 1$  and our  $\beta$ . Then, for large enough  $n$ , whenever  $A$  is a subset of  $\mathbb{Z}_p$  with  $|A| = n < (1 - \beta)p/2$  and  $\mathcal{F} \subset A \times A$  satisfies  $|\mathcal{F}| \leq k(k - 1)$ , then for some  $B' \in A^{(\leq c)}$  we have*

$$(75) \quad |A +_{\mathcal{F}} B'| \geq 2n + 1 - 2k.$$

*Proof.* Let  $n$  be large enough to guarantee  $k^2 < \varepsilon n$  and  $2k^2 < \beta p$ . If there is a set  $B' \subset A$  with  $|B'| = c$  such that  $|A + B'| \geq 2n + k^2$ , then (75) follows.

Otherwise, by Theorem 7, the set  $A$  is contained in an arithmetic progression of size  $n + k^2$ . Assume, as we may, that  $A = \{a_1, \dots, a_n\} \subset \{1, \dots, n + k^2\}$ , with  $a_i < a_{i+1}$ . Since  $2(n + k^2) < p$ , the sum  $A + A$  has no ‘wraparound’. Set  $d_i = d(a_i)$ ,  $s = \min_{1 \leq i \leq n} (i + d_i)$  and  $t = \min_{1 \leq i \leq n} (n + 1 - i + d_i)$ . Then, crudely,  $s, t < n/2$ ,

$$d_1 \geq s - 1, d_2 \geq s - 2, \dots, d_{s-1} \geq 1,$$

and

$$d_n \geq t - 1, d_{n-1} \geq t - 2, \dots, d_{n-t+2} \geq 1.$$

This implies that

$$s(s - 1)/2 + t(t - 1)/2 \leq \sum_1^n d_i = |\mathcal{F}| \leq k(k - 1),$$

so  $s + t \leq 2k$ .

Finally, if  $j$  and  $h$  are suffices where the minima are attained then, using  $\{a_j, a_h\}$  for the second summand,

$$\begin{aligned} |A +_{\mathcal{F}} \{a_j, a_h\}| &\geq |A + \{a_j, a_h\}| - d_j - d_h \\ &\geq 2n + 1 - j - d_j - (n + 1 - h) - d_h \\ &\geq 2n + 1 - s - t \geq 2n + 1 - 2k. \end{aligned}$$

This completes our proof. ■

The bound in Theorem 36 cannot be improved. Indeed, let  $A = [n]$  and

$$\mathcal{F} = \{(i, j) \in A \times A : i + j \leq k \text{ or } i + j \geq 2n - k + 2\}.$$

Then  $|\mathcal{F}| = k(k - 1)$  and  $A +_{\mathcal{F}} A = [k + 1, 2n - k + 1]$ .

It is not impossible that the bounds we have imposed on  $d(n)$  in Theorem 35 and on  $k(n)$  in Theorem 36 are unnecessary. We shall state these as conjectures in the last section.

### 8. Applications—sums of sets of reals

As we shall see now, our results in the discrete world have easy corollaries in the continuous world. Throughout this section we shall denote by  $|\cdot|$  the Lebesgue measure on the Euclidean space  $\mathbb{R}$  and Haar measure (normalized Lebesgue measure) on the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . Our starting point is the triviality that if  $A$  and  $B$  are non-empty compact sets of reals then  $|A + B| \geq |A| + |B|$ .

The deduction of these corollaries from the corresponding theorems is straightforward. First one reduces the statement from general compact sets to finite unions of closed intervals. Then one reduces the statement to discrete sets by taking the intersection with a sufficiently fine lattice. Let us show, for example, how one can deduce Corollary 10 from Theorem 1.

*Proof of Corollary 10.* We first reduce Corollary 10 to the case when  $A$  and  $B$  are finite unions of intervals and  $|A| > |B|$ .

Fix compact sets  $A$  and  $B$  in  $\mathbb{R}$  and construct nested sequences of sets  $A_n$  and  $B_n$  in  $\mathbb{R}$ , all of which are finite unions of closed intervals,  $|A_n| \geq |B_n|$ ,  $\bigcap_n A_n = A$  and  $\bigcap_n B_n = B$ .

To prove

$$\max_{b_1, b_2, b_3 \in B} |A + \{b_1, b_2, b_3\}| \geq |A| + |B|$$

given that

$$\max_{b_1^n, b_2^n, b_3^n \in B_n} |A_n + \{b_1^n, b_2^n, b_3^n\}| \geq |A_n| + |B_n| \geq |A| + |B|,$$

it suffices to show that

$$\max_{b_1, b_2, b_3 \in B} |A + \{b_1, b_2, b_3\}| \geq \lim_{n \rightarrow \infty} \max_{b_1^n, b_2^n, b_3^n \in B_n} |A_n + \{b_1^n, b_2^n, b_3^n\}|.$$

For this, note that the points  $b_1^n, b_2^n, b_3^n$  are contained inside the compact set  $B_1$ . Therefore, we can restrict to convergent subsequences  $b_1^n \rightarrow b_1^0, b_2^n \rightarrow b_2^0$  and  $b_3^n \rightarrow b_3^0$ . As the sets  $B_i$  are compact, it follows that  $b_1^0, b_2^0, b_3^0 \in \cap_i B_{n_i} = B$ . To conclude observe that the function  $f(x_1, x_2, x_3) = |A + \{x_1, x_2, x_3\}|$  is continuous and that for all  $x_1, x_2, x_3$  we have

$$|A + \{x_1, x_2, x_3\}| \geq |A_n + \{x_1, x_2, x_3\}| - 3|A_n \Delta A|.$$

Now we further reduce Corollary 10 to Theorem 1. Fix sets  $A$  and  $B$  in  $\mathbb{R}$  that are finite unions of intervals and  $|A| > |B|$ , and construct sets  $A_n = A \cap n^{-1}\mathbb{Z}$  and  $B_n = B \cap n^{-1}\mathbb{Z}$ .

To prove

$$\max_{b_1, b_2, b_3 \in B} |A + \{b_1, b_2, b_3\}| \geq |A| + |B|$$

given that

$$\max_{b_1^n, b_2^n, b_3^n \in B_n} |A_n + \{b_1^n, b_2^n, b_3^n\}| \geq |A_n| + |B_n| - 1,$$

it suffices to show that, for  $n$  large,  $|A_n + \{b_1^n, b_2^n, b_3^n\}| = n|A + \{b_1^n, b_2^n, b_3^n\}| - O(1)$ ,  $|A_n| = n|A| + O(1)$  and  $|B_n| = n|B| + O(1)$ . For this, recall that  $A$  is a finite union of intervals and for an interval  $X$  it is trivial that  $|X| = |X \cap n^{-1}\mathbb{Z}| + O(1)$ . This concludes the proof. ■

The proof of Corollary 11 from Theorems 4 and 5 follows identical lines. Finally, we mention that there is also a continuous version of Theorem 8, again proved by the same method.

**COROLLARY 37:** *For all  $K$  and  $\varepsilon > 0$  there is an integer  $c$  such that for all  $d$  the following holds. Let  $A$  and  $B$  be non-empty compact subsets of the torus  $\mathbb{T}^d$ . Then there are compact subsets  $A^* \subset A$  and  $B^* \subset B$ , with  $|A^*| \geq (1 - \varepsilon)|A|$  and  $|B^*| \geq (1 - \varepsilon)|B|$ , such that if we select points  $b_1, \dots, b_c$  uniformly at random from  $B^*$  then*

$$\mathbb{E} |A^* + \{b_1, \dots, b_c\}| \geq \min((1 - \varepsilon)|A^* + B^*|, K|A^*|, K|B^*|). \quad \blacksquare$$

**9. Open problems and conjectures**

In [10], the version of this paper we put on ArXiv on 12th May 2022, we stated the intriguing conjecture that in Theorem 4 the number of translates can be taken to be three provided  $|A| = |B|$  is at most a sufficiently small multiple of  $p$ , i.e., there is a constant  $\alpha > 0$  such that whenever  $A$  and  $B$  are subsets of  $\mathbb{Z}_p$  with  $|A| = |B| \leq \alpha p$ , there exist  $b_1, b_2, b_3 \in B$  such that  $|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1$ . This conjecture has been proved by Fox, Luo, Pham and Zhou [20].

As we said in the Introduction, the main purpose of this paper is to start a new direction in the theory of sumsets. The first paper in this new direction (in addition to our three papers so far, namely [10], [11], [12]) is precisely the beautiful paper of Fox, Luo, Pham and Zhou [20] we have just mentioned, which also poses several interesting questions. Not surprisingly, there is a huge supply of open problems and conjectures concerning the possibility of replacing a summand by an appropriate small subset: here we shall mention only a handful. We hope and expect that many further problems will be posed and solved by others.

Let us start with a very general question concerning restricted sumsets inspired by the Erdős–Heilbronn type results in Section 7.

*Problem 1:* For what quadruples  $(p, n, c, d)$  is it true that if  $A \subset \mathbb{Z}_p$  with  $|A| = n$ , and  $\mathcal{F} \subset A \times A$  has degree at most  $d$ , then there is a set  $B' \in A^{(\leq c)}$  such that  $|A +_{\mathcal{F}} B'| \geq 2n - 1 - 2d$ ?

Needless to say, in this generality Problem 1 is bound to be out of reach, so we shall state some special cases as problems and conjectures. An obvious question is whether the Erdős–Heilbronn type results in Section 7 hold without any restriction on  $d$ , the maximum degree of the set of forbidden pairs. We believe that they do.

**CONJECTURE 2:** *For every  $\beta > 0$  there is a constant  $c$  such that the following holds. If  $A$  is a subset of  $\mathbb{Z}_p$  with  $|A| = n < (1 - \beta)p/2$ , and  $\mathcal{F} \subset A \times A$  with  $d(\mathcal{F}) \leq d$ , then there is a set  $B' \in A^{(\leq c)}$  such that the  $\mathcal{F}$ -restricted sum  $A +_{\mathcal{F}} B'$  has at least  $2n - 2d - 1$  elements.*

Theorem 35 tells us that this conjecture holds for  $d = d(n) = o(n)$ , and pedestrian arguments can be used to prove it for  $d = n - 1, n - 2$  and  $n - 3$ . A very interesting case would be  $d = \lfloor n/2 \rfloor$ , say.

Conjecture 2 has a good many variants, stronger forms and weaker.

Firstly, does it hold for two sets,  $A, B \in \mathbb{Z}_p$  with  $|A| = |B| = n$ , with  $\mathcal{F} \subset A \times B$  having  $d(\mathcal{F}) \leq d$ ? It is important to note that such questions are unknown even if we drop the condition that we are passing to a subset  $B'$  of  $B$ . Indeed, the case  $d = 1$  (for general sets  $A$  and  $B$  with  $|A| = |B| = n$ ) is a beautiful open problem of Lev [42]. Secondly, does Conjecture 2 hold for sets  $A$  and  $B$  of comparable sizes? And what happens if the sizes are incomparable? Thirdly, what is the minimal value of  $c$  for which Conjecture 2 and its relatives hold?

We mention that there are many questions about how the various parameters in the results in this paper relate to each other: it would be very nice to find out the correct dependencies. For example, how does  $c$  depend on  $\alpha$  and  $\beta$  in Theorem 4? How does  $\alpha$  depend on  $\beta$  in Theorem 5? How small is  $\varepsilon$  in Theorem 7? We do not see how to answer any of these questions.

In the same way in which it was possible to produce a restricted sumset version of Cauchy–Davenport’s theorem, it would be interesting to produce a restricted sumset version of Pollard’s theorem on popular sums in sumsets.

Given  $i \in \mathbb{N}$  and subsets  $X$  and  $Y$  of an abelian group, let

$$N_i(X, Y) = \{z : |(z - X) \cap Y| \geq i\}.$$

It would be interesting to obtain an extension of Pollard’s theorem [54] where we replace  $B$  by a subset  $B'$  of bounded size  $O_t(1)$ .

**CONJECTURE 3:** *For all  $\alpha, \beta, t > 0$  there exists a constant  $c$  such that the following holds. Whenever  $A$  and  $B$  are subsets of  $\mathbb{Z}_p$  with  $\alpha|B| \leq |A| \leq \frac{1}{\alpha}|B|$  and  $|A| + |B| \leq (1 - \beta)p$ , there exist  $B' \subset B$  with  $|B'| = c$  such that*

$$\sum_{1 \leq i \leq t} |N_i(A, B')| \geq t(|A| + |B| - 1).$$

Pollard’s theorem tells us that this inequality holds if  $B' = B$ .

Turning to  $\mathbb{Z}$ , it would also be interesting to know if the influence of the structure of  $B$  on our results is independent of scaling. For example, we suspect that the following is the case.

**CONJECTURE 4:** *Let integers  $c$  and  $n$  and a set  $B \subset \mathbb{Z}$  be fixed. Suppose for every set  $A \subset \mathbb{Z}$  of size  $n$  there are  $c$  elements  $b_1, \dots, b_c \in B$  such that  $|\bigcup_{i=1}^c (A + b_i)| \geq m$ . Then the same holds if  $B$  is replaced by  $\lambda B$  for any integer  $\lambda$ .*

Continuing with  $\mathbb{Z}$ , could it be that Theorem 1 holds even when  $A$  is somewhat smaller than  $B$ ?

*Question 5:* When  $A$  and  $B$  are subsets of  $\mathbb{Z}$  with  $|B| = n$ , how much smaller than  $n$  can  $|A|$  be if we can still guarantee that

$$\max_{b_1, b_2, b_3 \in B} |A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1?$$

Another interesting avenue of research would be to study how large a set  $A + B'$  can be in terms not of the minimum sumset bound but of the actual sumset size  $|A + B|$ . In other words, how much do our bounds improve if we happen to know that  $A + B$  is larger than its theoretical minimum value? There is no chance of a result in this precise form, since  $|A + B|$  may be large because  $A$  has a small subset scattered in the ground set, so that  $|A + B|$  is large because of a tiny subset of  $A$ . Note that Theorem 8 is a result in this direction. But could the following be true? If so, what is the dependence of  $c$  on  $\varepsilon$  and  $\lambda$ ?

*Question 6:* Given constants  $\varepsilon > 0$  and  $\lambda > 0$ , is there a constant  $c > 0$  such that the following assertion holds? If  $A, B \subset \mathbb{Z}_p$  with  $|A| = |B| = n$  and  $|A + B| \leq \lambda n$  then there are sets  $A' \subset A$  and  $B' \subset B$  such that  $|A'| \geq (1 - \varepsilon)n$ ,  $|B'| \leq c$  and  $|A' + B'| \geq (1 - \varepsilon)|A' + B|$ .

[Note added in proof: this has recently been answered in the negative by Fox, Luo, Pham and Zhou [20].]

Here is another interesting variant which takes into account that  $A + B$  may be large because of a few ‘outliers’.

*Question 7:* Does there exist an absolute constant  $c > 0$  such that the following holds. Let  $A$  and  $B$  be subsets of  $\mathbb{Z}$  of size  $n$ . Let

$$s = S(A, B) = \min\{|X + Y| : |X|, |Y| \geq n/2\}.$$

Then there exists a subset  $B'$  of  $B$  of size at most  $\frac{cs}{n}$  such that  $|A + B'| \geq S(A, B)$ .

The function  $S(A, B)$  in the question above was based on the constant  $1/2$ : clearly, similar questions can be asked when  $1/2$  is replaced by some  $\alpha$ ; even more, we could have  $S_{\alpha, \beta}(A, B)$  with the obvious definition.

Finally, there are many questions that one could ask in higher dimensions, so concerning the groups  $\mathbb{Z}_p^d$  or  $\mathbb{T}^d$ . One of the cleanest questions is in the continuous torus.

CONJECTURE 8: For each dimension  $d$  there is a constant  $c$  such that the following holds. Whenever  $A$  and  $B$  are non-empty compact subsets of  $\mathbb{T}^d$  with  $|A| = |B| \leq 1/3$ , there exist  $b_1, \dots, b_c \in B$  such that  $|A + \{b_1, \dots, b_c\}| \geq |A| + |B|$ .

There is no doubt that the interested reader can pose a host of further fascinating and potentially important questions and conjectures in this spirit.

### 10. Appendix 1: Proofs of some results from Section 3

We collect here the proofs of three results from Section 3.

*Proof of Theorem 18.* For  $|A| = |B| \leq 3$  this is trivial, so we may assume that  $|A| = |B| \geq 4$ . Since  $\min_{b \in B \setminus \{m\}} |A + \{0, b, m\}| = |A \cup (A + m)|$  is attained at  $b = 0$ , we have  $\mathbb{E}_{b \in B \setminus \{0, m\}} |A + \{0, b, m\}| \geq \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}|$ , with equality if and only if  $|A + \{0, b, m\}| = |A \cup (A + m)|$  for every  $b \in B$ . By (2),

$$(76) \quad \mathbb{E}_{b \in B \setminus \{0, m\}} |A + \{0, b, m\}| \geq \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq |A| + |B| - 1$$

and, by our assumption (3), both inequalities in (76) are equalities. Consequently,

$$|A + \{0, b, m\}| = |A \cup (A + m)| = |A| + |B| - 1$$

for every  $b \in B$  and so  $A + B \subset A \cup (A + m) \subset A + B$  and  $|A + B| = |A| + |B| - 1$ . Since  $|A| = |B| \geq 4$ , it follows, for example by Freiman’s  $3k - 4$  theorem (Theorem 13), that  $A$  and  $B$  are arithmetic progressions with the same difference. ■

*Proof of Theorem 19.* The proof is based on Lemma 16 and Lemma 17, which combine to give

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| &\geq |A| + |\tilde{A}| + |A| \max \left( 0, \frac{|\tilde{B}| - |\tilde{A}|}{|\tilde{B}|} \right) \\ &= |A| + |\pi_m(A)| + |A| \max \left( 0, \frac{|B| - 1 - |\pi_m(A)|}{|B| - 1} \right). \end{aligned}$$

If we assume that  $|A| \geq |B| - 1$  and  $|\pi_m(A)| \geq |B| - 1$  then we have

$$\mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq |A| + |\pi_m(A)| \geq |A| + |B| - 1.$$

If we assume that  $|A| \geq |B| - 1$  and  $|\pi_m(A)| \leq |B| - 1$  then we have

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| &\geq |A| + |\pi_m(A)| + |A| \frac{|B| - 1 - |\pi_m(A)|}{|B| - 1} \\ &\geq |A| + |\pi_m(A)| + |B| - 1 - |\pi_m(A)| = |A| + |B| - 1. \end{aligned}$$

Finally, if we assume  $|A| \leq |B| - 1$ , which implies  $|\pi_m(A)| \leq |B| - 1$ , then we obtain

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| &\geq |A| + |\pi_m(A)| + |A| \frac{|B| - 1 - |\pi_m(A)|}{|B| - 1} \\ &= 2|A| + |\pi_m(A)| \frac{|B| - 1 - |A|}{|B| - 1}. \end{aligned}$$

This concludes the proof of Theorem 19. ■

*Proof of Theorem 22.* We begin with a construction and continue with some preliminary estimates. Partition  $A$  into two parts,  $A = A_1 \sqcup A_2$ , such that  $\widetilde{A}_2 = \widetilde{A}$  and  $|A_2| = |\widetilde{A}|$  and  $|A_1| = |A| - |\widetilde{A}|$ . Lemma 16 tells us that  $\pi_m(A \cup (A + m)) = \widetilde{A}$  and  $|A \cup (A + m)| \geq |A| + |\widetilde{A}|$ , and Lemma 20 gives

$$\mathbb{E}_{b \in B \setminus \{m\}} |(A_1 + b) \setminus \pi_m^{-1}(\widetilde{A})| \geq (|A| - |\widetilde{A}|) \max\left(0, \frac{|\widetilde{B}| - |\widetilde{A}|}{|\widetilde{B}|}\right).$$

Noting the bijection  $\pi_m(B \setminus \{m\}) = \widetilde{B}$ , for  $z \in \mathbb{Z}_d$  we have

$$\begin{aligned} \mathbb{P}_{b \in B \setminus \{m\}} ((A_2 + b) \cap \pi_m^{-1}(z) \neq \emptyset) &= \mathbb{P}_{\widetilde{b} \in \widetilde{B}} (z \in \widetilde{b} + \widetilde{A}_2) = \mathbb{P}_{\widetilde{b} \in \widetilde{B}} (z \in \widetilde{b} + \widetilde{A}) \\ &\geq \max\left(0, \frac{|\widetilde{B}| + |\widetilde{A}| - m}{|\widetilde{B}|}\right). \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}_{b \in B \setminus \{m\}} |(A_2 + b) \setminus \pi_m^{-1}(\widetilde{A})| &= \sum_{z \notin \widetilde{A}} \mathbb{P}_{b \in B \setminus \{m\}} ((A_2 + b) \cap \pi_m^{-1}(z) \neq \emptyset) \\ &\geq (m - |\widetilde{A}|) \max\left(0, \frac{|\widetilde{B}| + |\widetilde{A}| - m}{|\widetilde{B}|}\right). \end{aligned}$$

Now, by Theorem 19, we may assume that

$$\frac{(2|\pi_m(A)| - m)(m - (|B| - 1)) - 1}{|B| - 1} > 0.$$

In particular, we have  $2|\pi_m(A)| > m$ , i.e.,  $2|\tilde{A}| > m$ . Since  $|\tilde{B}| + 1 \geq |\tilde{A}|$ , the last two inequalities imply

$$\frac{|\tilde{B}| + |\tilde{A}| - m}{|\tilde{B}|} \geq 0.$$

The inequalities in this proof can be put together to imply

$$\begin{aligned} & \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \\ & \geq |A| + |B| - 1 + \max \left( 0, \frac{(2|\pi_m(A)| - m)(m - (|B| - 1)) - 1}{|B| - 1} \right), \end{aligned}$$

completing the proof of Theorem 22. ■

### 11. Appendix 2: Proof of the conditions (1)–(6) in Lemma 33

Here we show that the construction given in the proof of Lemma 33 does indeed have the claimed properties, as well as showing that the set  $S$  is non-empty, as claimed.

CLAIM A: *Given  $t \in S$ , for  $i = 1, 2, 3$  the parameters  $d_{i,t}$  and the consecutive intervals  $I_i$  and  $J_{i,t}$  satisfy conditions (1)–(5).*

*Proof.* Condition (1) is immediate from the construction.

For  $i = 1$  or  $i = 3$ , we have  $|J_{i,t}| \leq |J| - |J_{2,t}| + 1$  and  $|J_{i,t}| \leq |J| - k_2 d_2$ . From this we find that  $|J_{i,t}| \leq 2 \min(|I|, |J|)$ , and so we can deduce the first part of condition (2) :  $|J_{i,t}| \leq 2^4 |I_i|$ .

For  $i = 2$ ,  $|J_{2,t}| \leq |J|$ , which leads to the first part of condition (2):

$$|J_{2,t}| \leq 2\alpha^{-1} |I_2|.$$

Returning to  $i = 1$  or  $i = 3$ , we find that the function  $q_i(x) = q_{i,x} : \mathbb{Z} \rightarrow \mathbb{Z}$  is strictly increasing. As  $|q_{i,t} - q_{i-1,t}| \geq |T_i|$ , we find that

$$|q_{i,t} - q_{i-1,t}| \geq 6^{-13} \min(|I|, |J|).$$

Since  $|J_{i,t}| = q_{i,t} - q_{i-1,t} + 1 \geq 6^{-13} \min(|I|, |J|)$ , we obtain the second part of condition (2):  $|J_{i,t}| \geq 6^{-13} |I_i|$ .

For  $i = 2$ , we have  $|J_{2,t}| = q_{2,t} - q_{1,t} + 1 = k_2 d_2 + 1 \geq 2^{-1} \min(|I|, |J|)$ , which implies the second part of condition (2) :  $|J_{2,t}| \geq 2^{-1} \alpha |I_2|$ .

For  $i = 1, 2, 3$ , we have  $|J_{i,t}| \geq 6^{-13} \min(|I|, |J|)$  and by  $|I_i| \geq 2^{-3} \min(|I|, |J|)$ . Furthermore,  $|I_i \Delta A_i| \leq |A \Delta I|$  and  $|J_{i,t} \Delta B_{i,t}| \leq |J \Delta B|$ , and so

$$\max(|I_i \Delta A_i|, |J_{i,t} \Delta B_{i,t}|) \leq \gamma \min(|I|, |J|).$$

From the last four inequalities we obtain condition (3):

$$\max(|I_i \Delta A_i|, |J_{i,t} \Delta B_{i,t}|) \leq 6^{13} \gamma \min(|I_i|, |J_i|).$$

We now turn to condition (4). For  $i = 1, 2, 3$ , we have  $q_{i,t} - q_{i-1,t} + 1 = k_i d_{i,t} + 1$  and  $d_{i,t} \geq 1$ . Consequently,

$$(k_i + 1)d_{i,t} \geq q_{i,t} - q_{i-1,t} + 1 \geq k_i d_{i,t},$$

which implies

$$(77) \quad (k_i + 1)d_{i,t} \geq |J_{i,t}| \geq k_i d_{i,t}.$$

Since  $k_1 = 2^{10}$  and  $k_3 = 3^{10}$ , for  $i = 1, 2, 3$  we have

$$(3^{10} + 1)d_{i,t} \geq |J_{i,t}| = 2^{10} d_{i,t},$$

and so  $|I_i| \geq 2^6 d_{i,t}$ . Also, if  $i = 2$  then  $k_2 d_{2,t} \geq 2^{-1} \min(|I|, |J|)$ . This tells us that, on the one hand,  $|J_{2,t}| \geq 2^{-1} \min(|I|, |J|)$ , and on the other,  $|I_2| \geq 2^{-1} \min(|I|, |J|)$ . Additionally,  $d_{2,t} = \lfloor 6^{-1} \min(|I|, |J|) \rfloor$ . From the last three inequalities we conclude that

$$\min(|I_2|, |J_{2,t}|) \geq 3d_{2,t}.$$

From (62) we may also deduce that  $6^2 d_{2,t} \geq \min(|I|, |J|)$ , and so

$$6^2 d_{2,t} \geq \min(|I_2|, |J_2|).$$

This completes the proof of Claim A. ■

Now we turn to the proof that  $S$  is non-empty, and that condition (6) holds.

We start by observing that the functions  $g_1, g_3$  are strictly increasing in the second coordinate, and also that  $g_2$  is injective. Indeed, recall that  $T_2$  is an interval of size  $|T_2| \leq 6^{-10} d_2 = k_1^{-1} k_3^{-1} d_2$ , and  $d_{2,t} = d_2 \neq 0$ . It is easily checked that for  $t, t' \in T_2$  we have  $q_{1,t} = q_{1,t'} \pmod{d_{2,t}}$  if and only if  $t = t'$ . Note that  $q_{1,t} = g_2(i_2, t) \pmod{d_{2,t}}$ . It follows that for  $t, t' \in T_2$  we have  $g_2(i_2, t) = g_2(i'_2, t') \pmod{d_{2,t}}$  if and only if  $t = t'$ . Moreover, if  $t = t'$  then, as  $d_{2,t} = d_2 \neq 0$ , it follows that  $g_2(i_2, t) = g_2(i'_2, t)$  if and only if  $i_2 = i'_2$ .

Let us note the following three inequalities. First,

$$|S^c| \leq \sum_{i \in [3]} |g_i^{(-1)}(J \setminus B)|,$$

second,

$$|g_i^{(-1)}(J \setminus B)| \leq k_i |J \setminus B|,$$

and third, for  $i = 2$  we have

$$|g_2^{(-1)}(J \setminus B)| \leq |J \setminus B|.$$

From these three inequalities we deduce that  $|S^c| \leq (k_1 + k_3 + 1)|J \setminus B|$ , and so, by (61), we have  $|S^c| \leq \gamma(k_1 + k_2 + 1) \min(|I|, |J|)$ . Finally, the choice of  $k_1$  and  $k_3$  implies that  $S$  is non-empty.

CLAIM B: Given  $t \in S$ , for each  $i = 1, 2, 3$  the parameters  $d_{i,t}$  and intervals  $J_{i,t}$  satisfy condition (6).

*Proof.* Our task is to check that

$$\bigcup_{i \in [3]} [q_{i-1,t}, q_{i,t}] \cap \{q_{i,t} \bmod d_{i,t}\} \subset B.$$

Note the following four identities:

$$\begin{aligned} [q_{0,t}, q_{1,t}] \cap \{q_{1,t} \bmod d_{1,t}\} &= \{q_{0,t} + id_{1,t}; i \in [0, k_1]\}, \\ [q_{1,t}, q_{2,t}] \cap \{q_{2,t} \bmod d_{2,t}\} &= \{q_{1,t} + id_{2,t}; i \in [0, k_2]\}, \\ [q_{2,t}, q_{3,t}] \cap \{q_{3,t} \bmod d_{3,t}\} &= \{q_{3,t} - id_{3,t}; i \in [0, k_3]\}, \end{aligned}$$

and

$$q_{0,t} + k_1 d_{1,t} = q_{1,t} + 0d_{2,t}.$$

By (73) and (74) it follows that

$$\bigcup_{i \in [3]} [q_{i-1,t}, q_{i,t}] \cap \{q_{i,t} \bmod d_{i,t}\} \setminus \{q_{0,t}, q_{3,t}\} \subset B.$$

Finally, by (61) and the definition of the parameters  $q_{i,t}$ , we have

$$\{q_{0,t}, q_{3,t}\} = \{q_l, q_r\} \subset B.$$

This establishes Claim B. ■

This finishes the proof of Lemma 33, since by Claim A, Claim B and the fact that  $S$  is non-empty we conclude that there exists  $t$  for which the parameters  $d_{i,t}$  and the consecutive intervals  $I_i$  and  $J_{i,t}$  satisfy conditions (1)–(6). Moreover, it is straightforward to check that each step of the construction is independent of the set  $A$ .

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