

INVARIANCE OF SYMPLECTIC COHOMOLOGY AND TWISTED COTANGENT BUNDLES OVER SURFACES

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ABSTRACT. We prove that symplectic cohomology for open convex symplectic manifolds is invariant when the symplectic form undergoes deformations which may be non-exact and non-compactly supported, provided one uses the correct local system of coefficients in Floer theory. As a sample application beyond the Liouville setup, we describe in detail the symplectic cohomology for disc bundles in the twisted cotangent bundle of surfaces, and we deduce existence results for periodic magnetic geodesics on surfaces. In particular, we show the existence of geometrically distinct orbits by exploiting properties of the BV-operator on symplectic cohomology.

1. INTRODUCTION

Symplectic cohomology is an invariant of non-compact symplectic manifolds, whose importance both in dynamical applications and in homological mirror symmetry has become increasingly clear in recent literature. This invariant is constructed using Hamiltonian Floer cohomology, which is surveyed in Salamon's lecture notes [58] for closed symplectic manifolds: in that case, the invariant recovers the quantum cohomology, whilst in the non-compact setup the invariant is much richer due to the Hamiltonian dynamics at infinity. The surveys by Oancea [49], Seidel [60] and Abouzaid [2] review many of the developments relating to symplectic cohomology.

Its origins in the work of Cieliebak-Floer-Hofer-Wysocki [17] and Viterbo [62] were motivated especially by dynamical applications, specifically existence theorems for closed Hamiltonian orbits. Later on, the relationship between this invariant and the study of fillings of contact manifolds was explored, starting from the groundbreaking work of Bourgeois-Oancea [15, 16, 41, 34].

We will postpone to Section 1.1 the discussion of cotangent bundles, in which case there is an abundance of literature on how symplectic cohomology has been used to prove the existence of closed geodesics and magnetic geodesics.

Symplectic cohomology has also been used effectively to obtain obstructions on the existence of exact Lagrangian submanifolds, a very difficult problem in symplectic topology, via Viterbo's functoriality theorem [62]. In homological mirror symmetry, the crucial role played by symplectic cohomology goes back to Seidel's ICM talk [59], in particular the open-closed string map which relates the Hochschild homology of the wrapped Fukaya category to the symplectic cohomology has become a crucial tool to prove theorems about generators for Fukaya categories due to the work of Abouzaid [3], which was extended also to the non-exact setup by Ritter-Smith [55].

Much of the symplectic literature on non-compact symplectic manifolds is focused on the case where the symplectic form is globally exact. This is because it simplifies the Floer theory considerably, and cotangent bundles $(T^*N, d\theta)$ were a driving motivating example. Interest in the non-exact setting stems not only from twisted cotangent bundles $(T^*N, d\theta + \pi^*\sigma)$ and magnetic geodesics (see Section 1.1), but also from the fact that non-compact Kähler manifolds arising in algebraic geometry are very rarely exact as this would force closed holomorphic curves to be constant. We also wish to avoid the weaker assumption, often encountered in Floer theory, that ω is *aspherical*, meaning ω vanishes on $\pi_2(M)$, as this would rule out any Kähler manifold that contains a non-trivial holomorphic sphere. Non-exactness allows Gromov-Witten theory to play an interesting role in Floer theory [52, 54, 55] and one interpretation of symplectic cohomology is as a generalisation of the quantum cohomology $QH^*(M, \omega)$ to non-compact settings [56].

Even in situations where the non-compact symplectic manifold $(M, d\theta)$ is exact, one can obtain substantial applications in symplectic topology by considering how the invariants change upon deforming the symplectic form $d\theta$. For instance, in the work of the second author [51, 52] such a deformation gave rise to new obstructions to the existence of exact Lagrangian submanifolds in cotangent bundles and in ALE spaces. In situations where (M, ω) is non-exact, it can also be beneficial to deform ω . For instance, for non-compact Fano varieties in [56] a deformation of the monotone toric Kähler form forced symplectic cohomology to become a semi-simple algebra, and this combined with the use of the open-closed string map gave rise to generation theorems for the wrapped Fukaya category.

To avoid making the introduction too technical, Section 2 will be a summary of the precise definitions and deformation theorems which we now summarise in looser terms. Our paper is concerned with the setup of (typically non-exact) symplectic manifolds which are exact at infinity. We will say (M, ω, θ) is a *convex manifold* (Definition 2.1) to mean that (M, ω) is an open symplectic manifold admitting an exhausting (namely proper and bounded below) function $h : M \rightarrow \mathbb{R}$, where θ is a 1-form defined on $M^{\text{out}} := \{h \geq 0\}$ with

$$\omega = d\theta \quad \text{and} \quad \theta(X_h) > 0 \quad \text{holds on } M^{\text{out}}. \quad (1.1)$$

Here X_h is the Hamiltonian vector field, $\omega(\cdot, X_h) = dh$. An *isomorphism of convex manifolds* is a symplectomorphism which preserves the 1-form at infinity. On M^{out} there is a Liouville vector field Z via $\theta = \iota_Z \omega$, and positivity in (1.1) is equivalent to Z pointing out of the domains $\{h \leq a\}$ for all $a \geq 0$. Below we use the notation

$$\Sigma := \{h = 0\}, \quad M^{\text{in}} := \{h \leq 0\}.$$

We do not impose that Z is positively integrable since one can always embed (M, ω, θ) via the Z -flow into the *completion* $(\hat{M}, \hat{\omega}, \hat{\theta})$, which at infinity is identifiable with $(\Sigma \times [0, \infty), d(e^r \alpha), e^r \alpha)$ for some contact form α on Σ and r -coordinate on $[0, \infty)$.

The symplectic cohomology of a convex manifold is the direct limit

$$SH^*(M, \omega, \theta) = \varinjlim HF^*(H) \quad (1.2)$$

via Floer continuation maps of the Floer cohomologies computed for Hamiltonians $H : M \rightarrow \mathbb{R}$ which at infinity are “linear” of larger and larger slopes. Linearity refers

to a radial coordinate $R = e^r$ determined by the choice of h (r is the time flown in direction Z starting from Σ). The coefficients of the Floer chain complexes are taken in the Novikov field Λ involving series in a formal variable t , see (4.2). For any cohomology class $\zeta \in H^1(\mathcal{LM})$ on the free loop space \mathcal{LM} of M , one can use ζ to twist the differential in the Floer chain complexes to define the so-called twisted symplectic cohomology $SH^*(M, \omega, \theta)_\zeta$. However the twisted differential, and thus $SH^*(M, \omega, \theta)_\zeta$, may not always be well-defined due to a lack of convergence in the Novikov field Λ (Remark 4.6).

It is crucial to understand invariance of symplectic cohomology under certain natural operations. For example, as shown by the second author (following the proof in the Liouville case [60]) symplectic cohomology is invariant under isomorphism.

Theorem 1.1 ([52], Theorem 8¹). *Any isomorphism $\varphi : (M_0, \omega_0, \theta_0) \rightarrow (M_1, \omega_1, \theta_1)$ naturally induces an isomorphism $\varphi_* : SH^*(M_0, \omega_0, \theta_0) \rightarrow SH^*(M_1, \omega_1, \theta_1)$. The same holds for twisted symplectic cohomology if it is well-defined.*

Remark 1.2. *We always tacitly assume that (M, ω) is weakly monotone (see Section 2), which ensures $HF^*(H)$ is well-defined by the methods of Hofer-Salamon [37] rather than having to appeal to more advanced machinery, such as Kuranishi structures or Polyfolds.*

We emphasize that the function h is not fixed in the definition of convex manifold, but it enters crucially in the construction of symplectic cohomology, since it determines the class of Hamiltonians. An implicit consequence of Theorem 1.1 is that the symplectic cohomology is independent of h . Such statements will be familiar to experts from the exact setup, but some care is required in the non-exact setup as the surprisingly strong invariance result of the exact case [4] was only possible due to the fact that Z was globally defined and that the Floer action functional was single-valued, both of which fail in the non-exact setting.

By “exact setting”, in which case M is called a *Liouville manifold*, we mean that θ additionally extends to a global primitive of ω on the whole M . This is stronger than asking that ω is globally exact, as θ may fail to extend. When this fails, M is called *Quasi-Liouville*; such examples arise from $T^*\mathbb{T}^2$ endowed with certain twisted symplectic form (see Theorem 1.9). We show in Lemma 3.1 that the obstruction for a convex manifold to be Liouville is the *relative class*

$$[\omega, \theta] \in H^2(M, M^{\text{out}}) \cong H_c^2(M). \quad (1.3)$$

That isomorphism is canonical by Equation (3.1) which implies that the class is in fact independent of the choice of exhausting function h (and thus the choice of M^{out}).

As we will see next, this class plays a special role, when we consider a *deformation* of a convex manifold, which we now define. By deformation we mean a family $s \mapsto (M, \omega_s, \theta_s)$, $s \in [0, 1]$ of convex manifolds such that the corresponding family of exhausting functions h_s is given by $h_s := H(s, \cdot)$ for some exhausting function

¹ There is an erratum in the proof [52, Lemma 7] (analogously to [60, (3c)(3.21)]). The correct inequality is $|d(\partial_s f_s) \cdot \partial_s u| \leq \rho^{-1/2} C |\partial_s u|$. By Cauchy-Schwarz, $\rho^{-1/2} C |\partial_s u| \leq C(\varepsilon^{-1} \rho + \varepsilon |\partial_s u|^2)$ (for any $\varepsilon > 0$). The next line is: $\Delta \rho + (\text{first order terms}) \geq (1 - C\varepsilon) |\partial_s u|^2 - \rho(\partial_s h'_s + h'_s C + C + C\varepsilon^{-1})$. To make the first term non-negative, we pick $\varepsilon = 1/C$. The rest of the proof holds as written.

$H : [0, 1] \times M \rightarrow \mathbb{R}$. We will denote by M_s^{out} and M_s^{in} the outer and inner part with respect to h_s and by $(\hat{M}_s, \hat{\omega}_s, \hat{\theta}_s)$ the completion of (M, ω_s, θ_s) . By (1.3), we can identify the cohomology classes $[\omega_s, \theta_s]$ as elements of the same vector space $H_c^2(M)$. The next result shows that if, under this identification, the relative class is constant along the deformation, all convex manifolds in the deformation (and hence their symplectic cohomologies) are isomorphic. This fact is used, for example, in the applications of Theorem 1.7.

Theorem 1.3. *Let (M, ω_s, θ_s) be a deformation such that $[\omega_s, \theta_s] \in H_c^2(M)$ is independent of s . Then there is an isomorphism*

$$\varphi : (\hat{M}_0, \hat{\omega}_0, \hat{\theta}_0) \rightarrow (\hat{M}_1, \hat{\omega}_1, \hat{\theta}_1).$$

In particular, $SH^(M, \omega_0, \theta_0) \cong SH^*(M, \omega_1, \theta_1)$, and the same holds for twisted symplectic cohomology if it is well-defined.*

The proof is in two steps. In the first step, we assume that the deformation is C^0 -small and we construct the map in the statement as a composition $\varphi = \varphi_2 \circ \varphi_1$. To define $\varphi_2 : \hat{M}_0 \rightarrow \hat{M}_1$, we use a trick from Seidel-Smith [61]: on \hat{M}_0^{out} , φ_2 is obtained by applying Gray stability to the deformation of contact manifolds $(\Sigma, \theta_s|_\Sigma)$ and then extending it in a canonical way to M^{in} . As a result, $\varphi_2^* \hat{\theta}_1 = \hat{\theta}_0$ and $\varphi_2^* \hat{\omega}_1 = \hat{\omega}_0 + \beta$, where β is a closed 2-form with compact support. The class $[\beta] \in H_c^2(M_0)$ vanishes if and only if the relative class $[\omega_s, \theta_s]$ is constant. In this case, we build an isomorphism $\varphi_1 : (\hat{M}_0, \hat{\omega}_0, \hat{\theta}_0) \rightarrow (\hat{M}_0, \hat{\omega}_0 + \beta, \hat{\theta}_0)$ as the time-one map of a flow obtained by a Moser argument.

In the second step, we break the interval $[0, 1]$ into subintervals on which the deformation is C^0 -small, and we obtain the map φ as a composition of the maps obtained from the first step. The subdivision of the deformation into small ones (so-called “adiabatic steps”) is an idea which is frequently used in establishing isomorphisms of symplectic cohomology along deformations and we learned it from [9, Section 2.4].

We now want to go beyond Theorem 1.3 by considering deformations in which the relative class varies. In this case, the existence of an isomorphism φ_1 isotopic to the identity is obstructed (see Lemma 3.2). If the transgression $\tau(\omega_s) \in H^1(\mathcal{L}M)$ also varies (where $\mathcal{L}M$ is the free loop space of M), one cannot even expect $SH^*(M, \omega_0, \theta_0)$ to be isomorphic to $SH^*(M, \omega_1, \theta_1)$ as they involve different Novikov fields. To off-set that, one must use twisted coefficients induced by $\tau(\omega_1 - \omega_0) \in H^1(\mathcal{L}M)$. However, as observed before, the corresponding twisted symplectic cohomology may not be well-defined. The crucial point that we use to address this problem is to show that when β is a closed 2-form on M with sufficiently small norm on M^{in} , such that $\beta = d\lambda$ is exact on M^{out} , then twisted symplectic cohomology $SH^*(M, \omega, \theta)_{\tau(\beta)}$ is well-defined and satisfies

$$SH^*(M, \omega, \theta)_{\tau(\beta)} \cong SH^*(M, \omega + \beta, \theta + \lambda). \quad (1.4)$$

Moreover, the twisted symplectic cohomology is a unital Λ -algebra admitting a canonical unital Λ -algebra homomorphism

$$c^* : QH^*(M, \omega)_\beta \rightarrow SH^*(M, \omega, \theta)_{\tau(\beta)}. \quad (1.5)$$

Here, $QH^*(M, \omega)_\beta$ is the twisted quantum cohomology of (M, ω) , so holomorphic spheres $u : \mathbb{C}P^1 \rightarrow M$ are counted with Novikov weight t^k where $k = \int u^* \omega + \int u^* \beta$, and as a Λ -vector space $QH^*(M, \omega)_\beta = H^*(M; \Lambda)$.

The precise quantitative statement of the above claim is Theorem 2.7, which is a mouthful, but it implies the following memorable result, which is our main theorem and which proves invariance under “short deformations”.

Theorem 1.4. *Let (M, ω_s, θ_s) be a deformation of convex manifolds. Then for all sufficiently small $s \geq 0$, there is a unital Λ -algebra isomorphism*

$$SH^*(M, \omega_s, \theta_s) \cong SH^*(M, \omega_0, \theta_0)_{\tau(\omega_s - \omega_0)}, \quad (1.6)$$

which commutes via the c^* -maps from (1.5) with the unital Λ -algebra isomorphism

$$QH^*(M, \omega_s) \cong QH^*(M, \omega_0)_{\omega_s - \omega_0}. \quad (1.7)$$

We remark that equation (1.7) is much simpler than (1.6), because both vector spaces equal $H^*(M; \Lambda)$ and the moduli spaces defining the quantum product only depend on an almost complex structure J which can be simultaneously tamed by both ω_s and ω_0 , when $\|\omega_s - \omega_0\| < 1$. Thus the twist on the right in (1.7) just ensures that J -holomorphic spheres are counted with the correct Novikov weight. Explicit examples of twisted quantum cohomology can be described for closed Fano toric manifolds (by Batyrev [10] and Givental [28, 29]) in terms of the Landau-Ginzburg superpotential suitably twisted, and similarly in non-compact settings [56, Section 5].

The proof of Theorem 1.4 uses two key ideas. The first is to use the map φ_1 mentioned under Theorem 1.3 to reduce to the case where one modifies ω only on a compact subset. This approach bypasses the difficulty of proving a maximum principle for an s -dependent θ_s . The second, is a new energy estimate (4.16) which allows us to run a continuation argument whilst varying the symplectic form on a compact subset. A new and unexpected feature compared to deforming Liouville manifolds [52] is that even the formal twisting in Theorem 1.4 requires such an energy estimate to obtain convergence of counts of moduli spaces.

Remark 1.5. *The energy estimate (4.16) first appeared in the 2014 PhD thesis [11], which the first author used to prove Corollary 1.6 (these results were hitherto not published in a journal, which this paper rectifies). In retrospect (unknown to the author at the time) the idea involved is similar to estimates in Le-Ono [40, Lemma 5.4]: they do not deform ω , but [40, Theorem 5.3] builds a continuation map arising from a deformation of the symplectic vector field, similar to the one we construct in Section 5. This energy estimate has since appeared independently in the work of Zhang on spectral invariants for aspherical closed symplectic manifolds [64, Section 4]. The key idea of the estimate is also used at the heart of the recent work of Groman-Merry on the symplectic cohomology of twisted cotangent bundles [32, Theorem 6.2] (compare with Theorem 4.8).*

In addition to how this energy estimate played a role in these papers, we should also illustrate the non-triviality of Theorem 1.4 by comparing it with invariance theorems in the existing literature. A simpler version of Theorem 1.4 was proved by the second author in [52] for compactly supported deformations of Liouville manifolds, where $(M, \omega_0 = d\theta_0)$ is Liouville and $\omega_s = d\theta$ at infinity. Even this simple case at the time

required a complicated bifurcation argument and the use of the exact action functional A_H to control energy, which would not generalise to non-exact settings. As another example, consider the invariance result [62, Theorem 1.7] stated in the seminal paper by Viterbo, where ω is only allowed to vary amongst aspherical symplectic forms. Upon closer inspection, filling in the details of the proof of [62, Theorem 1.7] does not appear to be straightforward. Indeed notice that the proof does not address the non-trivial issue of obtaining a priori energy estimates needed for compactness of moduli spaces of continuation solutions, and proving an s -dependent version of the maximum principle. Bae-Frauenfelder [9] explain such an invariance proof for closed aspherical symplectic manifolds M , provided one assumes in addition that the aspherical symplectic forms ω_s have primitives with at most linear growth on the universal cover of M . Our Corollary 1.6 (a consequence of Theorem 1.4) yields a proof of [62, Theorem 1.7] that bypasses all of these concerns, and it also immediately implies [52, Theorem 8].

To prove Theorem 1.4 for “long deformations”, so for all $s \in [0, 1]$, we require an additional condition. Given a deformation (M, ω_s, θ_s) of convex manifolds together with a choice of twisting class $\zeta_0 \in H^1(\mathcal{L}M)$, we call it a *transgression-invariant deformation* if

$$\tau(\omega_s) \in \mathbb{R}_{\geq 0} \cdot (\tau(\omega_0) + \zeta_0) \subset H^1(\mathcal{L}M). \quad (1.8)$$

This condition essentially ensures that the local system of Novikov coefficients is constant in s . We can then break up a long deformation into “short” pieces and apply Theorem 1.4 appropriately twisted.

Corollary 1.6. *Let (M, ω_s, θ_s) be a transgression-invariant deformation of convex manifolds for $\zeta_0 \in H^1(\mathcal{L}M)$. Then the following twisted symplectic cohomologies are well-defined for all $s \in [0, 1]$ and they are related by a unital Λ -algebra isomorphism:*

$$SH^*(M, \omega_s, \theta_s)_{\zeta_s} \cong SH^*(M, \omega_0, \theta_0)_{\zeta_0}, \quad (1.9)$$

where $\zeta_s = \tau(\omega_0 - \omega_s) + \zeta_0 \in H^1(\mathcal{L}M)$.

As a special case, if (M, ω_s, θ_s) are convex and $\tau(\omega_s) \in H^1(\mathcal{L}M)$ is constant, then

$$SH^*(M, \omega_1, \theta_1) \cong SH^*(M, \omega_0, \theta_0).$$

A simple application of Corollary 1.6 is the case of deformations of convex manifolds starting from a Liouville manifold $(M, d\theta_0, \theta_0)$ such that $\tau(\omega_s) = a(s) \cdot \tau(\omega_1)$ for some function $a : [0, 1] \rightarrow [0, \infty)$ with $a(0) = 0$ and $a(1) = 1$. Taking $\zeta_0 = \tau(\omega_1)$, we get

$$SH^*(M, \omega_1, \theta_1) \cong SH^*(M, d\theta_0, \theta_0)_{\tau(\omega_1)},$$

which so far was known only for compactly supported deformations [52].

One can restate the above results for *convex domains*, namely compact symplectic manifolds with contact type boundary, instead of convex manifolds. Such domains have completions which are convex manifolds. We explain these details in Section 2.

1.1. Introduction: Applications. We decided to focus our applications on twisted cotangent bundles over surfaces, as these already display many interesting features. These are convex manifolds arising from non-compactly supported deformations of Liouville manifolds. However, the more general setup of Theorem 1.4 is relevant in many applications which do not arise from deforming Liouville manifolds such as

- (i) negative complex line bundles, see [54];
- (ii) the non-compact Fano toric manifolds described in [56], in which the deformations of the canonical monotone toric Kähler form to generic nearby toric Kähler forms played a crucial role in mirror symmetry applications [56, Section 5];
- (iii) the non-exact convex symplectic manifolds arising as crepant resolutions of isolated quotient singularities, described in the work on the McKay correspondence by McLean-Ritter [46].

Within the last class of examples there are the toric ones, which have been analysed, as far as the existence of multiple periodic Reeb orbits is concerned, in recent work of Abreu-Gutt-Kang-Macarini [5], where our invariance result played a role.

Magnetic geodesics on a closed manifold N are solutions of a second-order ODE determined by a Riemannian metric g and a closed 2-form σ on N . The natural lifts of magnetic geodesics to the cotangent bundle $\pi : T^*N \rightarrow N$ of N using the metric g are the integral lines of the Hamiltonian flow on T^*N for the symplectic form

$$\omega_\sigma = d\theta + \pi^*\sigma$$

and the Hamiltonian $H(q, p) = \frac{1}{2}g_q(p, p)$, where $\theta = p dq$ is the canonical 1-form. We use this Hamiltonian description to study magnetic geodesics which are periodic.

By now, there is a rich literature on such curves, inspired by work from the early 1980s by Arnol'd [7], Novikov and Taimanov [47, 48]. For an extensive survey and references on this literature, we refer to Contreras-Macarini-Paternain [20], Ginzburg-Gürel [27] and Benedetti's 2014 PhD thesis [11]. The current paper stems from the latter, namely Theorems 1.7 and 1.9 on the existence of magnetic geodesics (we rectified and expanded the proofs of the existence of multiple orbits), and Corollary 1.6 on transgression-invariant deformations (which we strengthened to Theorem 2.7). Since 2014 the field has moved on fast and it is now known in general that a periodic magnetic geodesic exists for almost all energy levels (see Asselle-Benedetti [8] and Groman-Merry [32]).

Our note originated from trying to relate the existence of periodic magnetic geodesics in the free-homotopy class $\nu \in [S^1, N]$ with the symplectic cohomology $SH_\nu^*(D_r^*N, \omega_\sigma)$, where D_r^*N is the co-disc bundle of radius $r > 0$. This symplectic invariant is well-defined if the co-sphere bundle S_r^*N is of positive contact-type [52] (see Remark 2.6), and is generated by the periodic magnetic geodesics of energy $\frac{1}{2}r^2$ together with, if $\nu = 0$, the cohomology of N . Therefore, the existence result would follow if this invariant turned out not to be zero, for $\nu \neq 0$, or not to coincide with the usual cohomology of N , for $\nu = 0$. That such a line of argument holds for standard geodesics, where $\sigma = 0$, for any closed manifold N , dates back to Viterbo [62] and has become a standard tool in symplectic topology [60].

When σ is exact, it is a classical result that S_r^*N is of positive contact-type for $\frac{1}{2}r^2 > c_0(g, \sigma)$, where $c_0(g, \sigma)$ is the Mañé critical value of the universal abelian cover of N . In this case, (D_r^*N, ω_σ) is a Liouville domain (see Example 2.5) and the invariance of symplectic cohomology for this class of manifolds (see e.g. [60, 61]) yields

$$SH_\nu^*(D_r^*N, \omega_\sigma) \cong SH_\nu^*(T^*N, d\theta) \cong H_{n-*}(\mathcal{L}_\nu N)_{\tau(w_2(N))}, \quad (1.10)$$

where on the right one obtains the singular homology of the space of free loops in the class ν with coefficients twisted by the transgression of the second Stiefel-Whitney

class. The latter is the Viterbo isomorphism [62] (see Abouzaid [2] for a survey). The twist can be ignored if N is spin or if one works with coefficients in characteristic two.

We are therefore interested in the situation in which (D_r^*N, ω_σ) is not a Liouville domain. However, if σ is not exact and either $\dim N \geq 3$ or $N = \mathbb{T}^2$, then none of the S_r^*N can be of contact-type since $\pi^*\sigma$ is not exact on S_r^*N .

Our paper will only consider the case of surfaces N , as the purpose of the application is only to illustrate the deformation theorem for convex manifolds. These lead to results about the existence of closed magnetic geodesics in surfaces which are by now classical thanks to the work of Cristofaro-Gardiner and Hutchings [21], which used embedded contact homology to prove the existence of two periodic magnetic geodesics, for $\dim N = 2$ and S_r^*N of contact-type, without non-degeneracy assumptions.

Theorem 1.7. *Let $N \neq \mathbb{T}^2$ be a closed orientable surface with a Riemannian metric g and a non-exact 2-form σ . If $r > 0$ is large, or for $N = S^2$ if $r > 0$ is small and σ is nowhere vanishing, then S_r^*N is of positive contact-type. Under those assumptions,*

$$SH^*(D_r^*S^2, \omega_\sigma) \cong SH^*(T^*S^2, d\theta)_{\tau(\pi^*\sigma)} = 0, \quad (1.11)$$

and there is a prime periodic magnetic geodesic of energy $\frac{1}{2}r^2$. Unless one of the iterates of that orbit is transversally degenerate, there are at least two such geodesics. If N has genus ≥ 2 ,

$$SH_\nu^*(D_r^*N, \omega_\sigma) \cong SH_\nu^*(T^*N, d\theta)_{\tau(\pi^*\sigma)} = \begin{cases} H_{2-*}(N) & \text{if } \nu = 0, \\ H_{2-*}(S^1) & \text{if } \nu \neq 0, \end{cases}$$

and there is at least one periodic magnetic geodesic in each free homotopy class $\nu \neq 0$.

Remark 1.8. *A neighbourhood of the zero-section in (T^*S^2, ω_σ) and a neighbourhood of the zero section in the line bundle $\mathcal{O}(-2) \rightarrow \mathbb{CP}^1$ are symplectomorphic by classical results (see, for instance, [44, Theorem 3.4.10] or [23, Section 2.4A]). In Appendix A, we construct an explicit global symplectomorphism between (T^*S^2, ω_σ) and $\mathcal{O}(-2) \rightarrow \mathbb{CP}^1$, using the round metric g and the area form σ . Thus, the vanishing (1.11) is consistent with the fact that $SH^*(\mathcal{O}_{\mathbb{CP}^1}(-2)) = 0$ by [52, 54].*

Now suppose that $N = \mathbb{T}^2$, σ is exact, $\frac{1}{2}r^2 \leq c_0(g, \sigma)$. Contreras, Macarini and Paternain showed in [20] that $(D_r^*\mathbb{T}^2, \omega_\sigma)$ cannot be a Liouville domain; however they also list a simple class of examples (which we refer to as QL-magnetic-tori) for which $S_r^*\mathbb{T}^2$ is of positive contact-type for $\frac{1}{2}r^2$ close to $c_0(g, \sigma)$.

Theorem 1.9. *Let (g, σ) be a QL-magnetic-torus as in Section 6.4, where σ is an exact 2-form on \mathbb{T}^2 . Then there exists an $\epsilon > 0$ such that $S_r^*\mathbb{T}^2$ is of positive contact-type for all $\frac{1}{2}r^2 \in (c_0(g, \sigma) - \epsilon, c_0(g, \sigma)]$ and*

$$SH_\nu^*(D_r^*\mathbb{T}^2, \omega_\sigma) \cong SH_\nu^*(T^*\mathbb{T}^2, d\theta) \cong H_{2-*}(\mathbb{T}^2) \quad \text{for all } \nu \in [S^1, \mathbb{T}^2].$$

In particular, there is at least one periodic magnetic geodesic of energy $\frac{1}{2}r^2$ in every non-trivial free homotopy class (and two in the non-degenerate case). If the contact form on $S_r^\mathbb{T}^2$ is non-degenerate, there are infinitely many contractible periodic magnetic geodesics with energy $\frac{1}{2}r^2$.*

To prove the passage from the existence of one to two (respectively infinitely many) closed orbits in Theorem 1.7 (respectively 1.9), we use a new general scheme which is applicable in theory to many other situations. We exploit the properties of the BV-operator $\Delta : SH^*(M) \rightarrow SH^{*-1}(M)$ on symplectic cohomology (see Section 4.3). To our knowledge, this approach has not appeared elsewhere in the literature. The method more familiar to experts is to prove such results using the S^1 -equivariant symplectic cohomology (we also sketch the proof using that method), for example see Kang [39] and more recently [5]. However using Δ is more economical as the S^1 -equivariant differential involves infinitely many correction terms to the ordinary differential, of which Δ is the first correction term. The BV-operator method is based on the interplay at the chain level arising from $\partial\Delta + \Delta\partial = 0$, between the degree +1 Floer differential ∂ and the degree -1 BV-operator Δ . Together with some non-trivial filtration properties by McLean-Ritter [46, Appendix D] and work of Zhao [65, Equation (6.1)] (see Section 4.4), we then deduce the multiplicity results for magnetic geodesics stated above.

Remark 1.10. *We mention four other Floer theories, which have been defined on twisted cotangent bundles and which are also generated by certain periodic magnetic geodesics. The first is the Rabinowitz Floer Homology constructed by Merry [42] when σ has a bounded primitive on the universal cover of N . This “RFH” involves a combination of the homology and cohomology of the free loop space. Moreover, Bae and Frauenfelder [9] established a continuation isomorphism between RFH of the twisted cotangent bundle and RFH of the ordinary cotangent bundle. The second, developed by Frauenfelder, Merry and Paternain in [25, 26] uses quadratic Hamiltonians satisfying the Abbondandolo-Schwarz growth condition and it is defined for forms σ having at most linear growth on the universal cover of N . In this case, periodic orbits of a given period, instead of a given energy, are detected. The third, due to Gong [30], assumes σ admits a primitive of at most linear growth on the universal cover, but uses compactly supported Hamiltonians which are large enough over the zero section. Periodic magnetic geodesics for almost every level in some energy range are detected.*

*In all three theories, the assumptions imply that ω_σ is aspherical, meaning ω_σ integrates to zero on $\pi_2(T^*N)$. This in particular implies global exactness of ω_σ when N is simply connected, so it does not apply to T^*S^2 . More specifically, their assumptions ensure that the Floer action functional is single-valued and that no twisted coefficients appear, whereas our setup endeavours to overcome such restrictive conditions.*

*After the appearance of the third arXiv version of the present paper, a fourth theory due to Groman and Merry [32] appeared. The theory works for higher dimensional N and non-exact σ , in which case the twisted T^*N is not convex but still geometrically bounded at infinity. By a careful choice of class of Hamiltonians, they proved that $SH_\nu^*(T^*N, \omega_\sigma)$ is well-defined and that the first isomorphism in*

$$SH_\nu^*(T^*N, \omega_\sigma) \cong SH_\nu^*(T^*N, d\theta)_{\tau(\pi^*\sigma)} \cong H_{n-*}(\mathcal{L}_\nu N)_{\tau(\sigma) \otimes \tau(w_2(N))} \quad (1.12)$$

holds. Here the middle term and the second isomorphism were constructed by the second author [51, 53], who also showed that

$$H_*(\mathcal{L}N)_{\tau(\beta)} = 0$$

if $\tau(\beta) \neq 0 \in H^1(\mathcal{L}_0 N)$ and $\pi_m(N)$ is finitely generated for each $m \geq 2$ (e.g. if N is simply connected and $\beta \neq 0 \in H^2(N)$). More refined conditions for vanishing were proved in [6]. The vanishing result then implies the existence of periodic magnetic geodesics in this setting.

1.2. Structure of the paper. In Section 3 we prove foundational results about convex manifolds, and Theorem 1.3 in Subsection 3.5. In Section 4 we construct (twisted) symplectic cohomology for convex manifolds, and prove Theorem 2.7.(1) in Subsection 4.6. In Section 5 we prove Theorem 2.7.(2) and Corollary 1.6. Section 6 deals with twisted cotangent bundles and proves Theorems 1.7 and 1.9. Appendix A constructs an explicit symplectomorphism between the twisted T^*S^2 and $\mathcal{O}_{\mathbb{CP}^1}(-2)$. Appendix B recalls iteration formulae for Conley-Zehnder indices in $\dim = 3$ used in Section 6.

1.3. Acknowledgements. G.B. wishes to thank his PhD advisor Gabriel Paternain for suggesting to compute the symplectic cohomology of twisted cotangent bundles of surfaces and for his valuable suggestions and insights on this problem. We are very grateful to the referee for the careful reading of the manuscript and for the valuable comments, which helped us improve the paper considerably.

2. CONVEX MANIFOLDS AND THEIR DEFORMATIONS: PRECISE DEFINITIONS

One often constructs symplectic cohomology as an invariant associated to a closed symplectic manifold D with contact-type boundary $\Sigma = \partial D$ (see [49, 60] and Remark 2.6). One then builds a non-compact symplectic manifold M by attaching a conical end $\Sigma \times [0, \infty)$. For example $D = D^*N$ completes to $M = T^*N$ with $\Sigma = S^*N$. When $\omega = d\theta$ is globally exact one blurs the distinction between $SH^*(D)$ and $SH^*(M)$ because a surprisingly strong invariance result applies [4]. This relies on the existence of a global compressing Liouville flow and a single-valued action functional, which are not available in the non-exact setting. Our paper could be entirely phrased in terms of closed manifolds D [11] but we decided to instead start with a given non-compact symplectic manifold (M, ω) , as this is increasingly the practical setup one encounters (in the exact setup, this point of view is discussed in Seidel-Smith [61]).

Definition 2.1. A convex manifold is a triple (M, ω, θ) where (M, ω) is an open symplectic manifold admitting some exhausting (namely proper and bounded below) function $h : M \rightarrow \mathbb{R}$ such that θ is a 1-form defined on $M^{\text{out}} := \{h \geq 0\}$ satisfying $\omega = d\theta$ and $\theta(X_h) > 0$. We call h a Liouville function for (M, ω, θ) . On M^{out} there exists a Liouville vector field Z defined by $\theta = \iota_Z \omega$. Thus $\theta(X_h) = dh(Z)$. Therefore, $\theta(X_h) > 0$ if and only if Z points out of the domains $M_{\leq a} := \{h \leq a\}$ for all $a \geq 0$. We set

$$\Sigma := \{h = 0\}, \quad M^{\text{in}} := \{h \leq 0\}.$$

We call (M, ω, θ) complete if the flow of Z is positively integrable.²

² By flowing via $\frac{1}{dh(Z)} \cdot Z$ starting from the regular level set $\Sigma = \partial M^{\text{in}} = h^{-1}(0)$ we obtain a foliation $M^{\text{out}} = \sqcup_{x \in [0, \infty)} h^{-1}(x) \cong \Sigma \times [0, \infty)$ by diffeomorphic regular level sets of positive contact type for $\theta|_{h^{-1}(x)}$, using $x = h$ as the second coordinate. Using this, one finds that Z is integrable for all positive time \Leftrightarrow for some choice of h we have $\int_0^\infty \frac{1}{\ell(x)} dx = \infty$ where $\ell(x) := \max_{p \in h^{-1}(x)} dh(Z)|_p$.

Definition 2.2. Given convex $(M_0, \omega_0, \theta_0)$, $(M_1, \omega_1, \theta_1)$, an isomorphism is a symplectomorphism $\varphi : (M_0, \omega_0) \rightarrow (M_1, \omega_1)$ which at infinity satisfies $\varphi^* \theta_1 = \theta_0$.

Definition 2.3. A deformation of a convex manifold is a family (M, ω, θ_s) with $s \in [0, 1]$ together with a family of exhausting functions $h_s : M \rightarrow \mathbb{R}$ such that $H : [0, 1] \times M \rightarrow \mathbb{R}$ given by $H(s, \cdot) = h_s$ is also exhausting. For $0 \leq a \leq b$, we will use the notation

$$M_s^{\geq a} := \{h_s \geq a\}, \quad M_s^{[a, b]} := \{a \leq h_s \leq b\}.$$

For convex (M, ω, θ) , the form $\alpha = \theta|_\Sigma$ is a positive contact form on $\Sigma = h^{-1}(0)$, meaning $\alpha \wedge (d\alpha)^{\dim M - 1} > 0$ with respect to the boundary orientation for ∂M^{in} . Moreover, there is a smooth function $\sigma : \Sigma \rightarrow (0, \infty]$ and a symplectomorphism j , called *conical parametrisation*,

$$j : (\{(y, r) \in \Sigma \times [0, \infty) : r < \sigma(y)\}, d(r\alpha)) \longrightarrow (M^{\text{out}}, \omega) \quad (2.1)$$

defined by the time r flow of Z , so

$$j(y, r) = \text{Flow}_r^Z(y).$$

In particular $j^* \theta = e^r \alpha$. Note that (M, ω, θ) is complete if and only if we can pick $\sigma \equiv \infty$ in (2.1).

Any convex (M, ω, θ) can always be embedded into the completion $(\hat{M}, \hat{\omega}, \hat{\theta})$ of M^{in} obtained by gluing $\Sigma \times [0, \infty)$ and M^{in} via the map j above. Since we will complete M with respect to different pairs (ω, θ) , we will also sometimes use the more precise notation

$$(M, \omega, \theta)^\wedge := (\hat{M}, \hat{\omega}, \hat{\theta}).$$

If (M, ω_s, θ_s) is a deformation, we write $(\hat{M}_s, \hat{\omega}_s, \hat{\theta}_s)$ to indicate the corresponding completions.

Remark 2.4. A choice of Liouville function h on M determines a positive contact hypersurface

$$\Sigma = \{h = 0\} = \partial M^{\text{out}} \subset M, \text{ with contact form } \alpha := \theta|_\Sigma, \quad (2.2)$$

but Σ , M^{out} , h are not fixed in Definition 2.1. For instance, in view of (2.1), the function r is one of many examples of a Liouville function on \hat{M} . Changing the choice of h corresponds to modifying $\Sigma = \Sigma \times \{0\} \subset \Sigma \times \mathbb{R}$ to a “graph”

$$\{(y, f(y)) : y \in \Sigma\}$$

of a smooth function $f : \Sigma \rightarrow \mathbb{R}$ (see Section 3.2). The condition $\varphi^* \theta_1 = \theta_0$ in Definition 2.1 implies $\varphi_* Z_1 = Z_0$ at infinity, therefore for large r in the coordinates (2.1) we have

$$\varphi(y, r) = (\psi(y), r - f(y)) \quad (2.3)$$

where $\psi : \Sigma_0 \rightarrow \Sigma_1$ is a contactomorphism, namely a diffeomorphism satisfying $\psi^* \alpha_1 = e^f \alpha_0$ for a smooth function $f : \Sigma_0 \rightarrow \mathbb{R}$. Thus the contactomorphism class of Σ and the contact structure $\xi = \ker \alpha \subset T\Sigma$ are invariants under isomorphism, but the positive contact form α is not. We show in Section 3.2 how α can be varied arbitrarily subject to those invariants (Remark 3.8).

Example 2.5. *As we do not require completeness, if $\theta(X_h) > 0$ only holds near $\Sigma = h^{-1}(0)$, we still obtain a convex submanifold: $(h^{-1}(-\infty, \epsilon), \omega, \theta)$ for small $\epsilon > 0$.*

Recall M is Liouville if in addition to (1.1), θ extends to a global primitive of ω . However, there are convex (M, ω, θ) for which ω is globally exact, but the given θ does not extend to a global primitive; we call these Quasi-Liouville. By Lemma 3.1, the obstruction for a convex manifold to be Liouville is the relative class (1.3). If $[\omega] = 0 \in H^2(M)$, this obstruction becomes $[j^\lambda - e^r\alpha] \in H^1(M^{\text{out}})$ where λ is the given global primitive of ω . The $T^*\mathbb{T}^2$ of Theorem 1.9 yield Quasi-Liouville examples.*

The relative class (1.3) is preserved under isomorphisms, meaning $\varphi^[\omega_1, \theta_1] = [\omega_0, \theta_0]$ viewed as classes in $H_c^2(M)$. This is an obstruction to the existence of an isomorphism (Example 3.17) which did not appear for Liouville manifolds as $[\omega, \theta] = 0$ in that case.*

The symplectic cohomology of a convex manifold is defined by the direct limit (1.2) over Floer continuation maps of the Floer cohomologies computed for Hamiltonians $H : M \rightarrow \mathbb{R}$ which at infinity are linear in $R = e^r$ of larger and larger slopes. This class of Hamiltonians depends on j, h and Σ in (2.1) and (2.2), as they determine the radial coordinate R , and these choices are not unique given (M, ω, θ) . A different choice corresponds to changing R to $e^{f(y)}R$ for a smooth function $f : \Sigma \rightarrow \mathbb{R}$. The proof of Theorem 1.1 constructs an isomorphism between the two symplectic cohomologies computed for Hamiltonians that are linear for the respective radial coordinate.

We will always tacitly assume that (M, ω) is *weakly monotone*⁴ as that ensures the Floer cohomology groups are well-defined [37] without appealing to advanced machinery, such as Kuranishi structures or Polyfolds. Magnetic (T^*N, ω_σ) are always weakly monotone as their first Chern class vanishes. Following [54, 56] we work over coefficients in the Novikov field Λ involving ‘series’ in a formal variable t (see Section 4.2).

Remark 2.6. *A convex domain (D, ω, α) is a compact symplectic manifold (D, ω) such that α is a positive contact form on the boundary $\Sigma = \partial D$ with $d\alpha = \omega|_{T\Sigma}$ (see Lemma 3.12). Given (D, ω) , the possible such choices of α determine a convex set. Convex domains arise as “sublevel sets” of convex manifolds:*

$$D = M^{\text{in}} \cup \{(y, r) \in \Sigma \times [0, \infty) : y \in \Sigma, r \leq f(y)\} \subset \hat{M}$$

for any smooth $f : \Sigma \rightarrow [0, \infty)$. By convention, $SH^(D) := SH^*(M)$ for any completion $M = \hat{D}$ of D (this is well-defined by Theorem 1.1, Remark 3.11). We abusively speak of isomorphisms of such D when we mean isomorphisms of their completions. Also, $SH^*(D)$ is invariant under deformations of the contact form by Corollary 1.6.*

We can now state the quantitative version of Theorem 1.4.

³It will hold in the relative de Rham cohomology $H^2(M_0, M_0^{\text{out}})$ if we choose M_0^{out} and $M_1^{\text{out}} = \varphi(M_0^{\text{out}})$ so that $\varphi^*\theta_1 = \theta_0$ holds on M_0^{out} , see Lemma 3.2.

⁴At least one of the following holds: (i) c_1 vanishes on $\pi_2(M)$, (ii) ω vanishes on $\pi_2(M)$, (iii) (M, ω) is monotone (so $c_1|_{\pi_2(M)} = \lambda\omega|_{\pi_2(M)}$ for some $\lambda > 0$), or (iv) the minimal Chern number $|N| \geq \dim_{\mathbb{C}} M - 2$, where $\langle c_1(TM), \pi_2(M) \rangle = N\mathbb{Z}$. This is equivalent to requiring that for each $A \in H_2(M, \mathbb{Z})$, if $3 - \dim_{\mathbb{C}} M \leq c_1(A) < 0$ then $\omega(A) \leq 0$.

Theorem 2.7. *Let (M, ω, θ) be convex. For $a > 0$, consider the convex manifold $M^{<a} := \{h < a\} \subset M$, and let $M^{(0,a)} := \{0 < h < a\}$. There are $\epsilon, \epsilon' > 0$ depending on a , such that for all closed two-forms β on $M^{<a}$ exact on $M^{(0,a)}$ with $\|\beta\|_{C^1(M^{<a})} < \epsilon$,*

- (1) *The twisted symplectic cohomology $SH^*(M, \omega, \theta)_{\tau(\beta)}$ can be constructed using a suitable cofinal subfamily of radial Hamiltonians (see Section 4.6). It is a unital Λ -algebra admitting a canonical unital Λ -algebra homomorphism*

$$c^* : QH^*(M, \omega)_\beta \rightarrow SH^*(M, \omega, \theta)_{\tau(\beta)}.$$

- (2) *For any representative $[\mu, \lambda] \in H^2(M^{<a}, M^{(0,a)})$ of the class $[\beta] \in H^2(M^{<a})$, with $\|(\mu, \lambda)\|_{C^1(M^{<a}, M^{(0,a)})} < \epsilon'$, the triple $(M^{<a}, \omega + \mu, \theta + \lambda)$ is convex and admits a unital Λ -algebra isomorphism*

$$SH^*(M^{<a}, \omega + \mu, \theta + \lambda) \cong SH^*(M, \omega, \theta)_{\tau(\beta)} \quad (2.4)$$

commuting with the canonical c^ maps from $QH^*(M, \omega + \mu) \cong QH^*(M, \omega)_\beta$. Thus, the group on the right in (2.4) is independent of the choice of the cofinal family of Hamiltonians.*

As mentioned in the Introduction, to obtain (1.6) for all s , one breaks down the “long” deformation into “short” pieces and uses a twisted version of Theorem 1.4. Unfortunately convergence issues in the Novikov field prevent such twistings in general, so we introduce a good notion of local systems on \mathcal{LM} which work.

Definition 2.8. *For convex (M, ω, θ) , call $\zeta \in H^1(\mathcal{LM})$ transgression-compatible if*

$$\zeta \in \mathbb{R}_{>-1} \cdot [\tau(\omega)] \quad \text{or} \quad [\tau(\omega)] = 0 \in H^1(\mathcal{LM})$$

Equivalently, choosing representative 1-forms τ_ω and η in the classes $\tau(\omega)$ and ζ , for some $c \geq 0$ there is a function $\mathcal{K} : \mathcal{LM} \rightarrow \mathbb{R}$, with

$$\tau_\omega = c(\tau_\omega + \eta) + d\mathcal{K}. \quad (2.5)$$

A family $(M, \omega_s, \theta_s; \zeta_s)$ is transgression-compatible if $s \mapsto \tau(\omega_s) + \zeta_s \in H^1(\mathcal{LM})$ is constant and for each $s \in [0, 1]$, ζ_s is transgression-compatible with ω_s . This implies that, after twisting by ζ_s , the system of Novikov coefficients is constant in s .

We prove that symplectic cohomology is always defined for transgression-compatible twists, and we prove that Theorem 1.4 implies Theorem 1.6. A simple example is if $\tau(\omega) = 0 \in H^1(\mathcal{LM})$: in that case any twisting $\zeta \in H^1(\mathcal{LM})$ is allowed.

Finally, we remark that deformations of convex domains (D, ω_s) reduce to the problem of deformations of convex manifolds. Observe that the completions of (D, ω_s) , as s varies, are all diffeomorphic to the completion $M = D \cup (\partial D \times [0, \infty))$ of (D, ω_0) . Pulling back the data via this identification we obtain a corresponding deformation (M, ω_s, θ_s) of complete convex manifolds. Thus Theorems 1.3 and 2.7 apply.

3. CONVEX MANIFOLDS AND THEIR DEFORMATIONS: THE PROOFS

3.1. Relative cohomology. Let (M, ω, θ) be convex (Definition 2.1). Following Bott-Tu [13, Section I.7, p.78], we define the relative de Rham cohomology $H^*(M, M^{\text{out}})$ via the mapping cone $C^*(M, M^{\text{out}}) = \Omega^*(M) \oplus \Omega^{*-1}(M^{\text{out}})$ with differential

$$D(x, y) := (dx, j^*x - dy),$$

where j^*x is the pull-back of x to M^{out} via (2.1). So $[\omega, \theta] \in H^2(M, M^{\text{out}})$.

Lemma 3.1. *A convex manifold (M, ω, θ) is Liouville if and only if the relative class $[\omega, \theta] \in H^2(M, M^{\text{out}})$ vanishes.*

Proof. For Liouville $(M, d\theta, \theta)$, $[d\theta, \theta] = [D(\theta, 0)] = 0 \in H^2(M, M^{\text{out}})$. Conversely, assume $[\omega, \theta] = 0$. Then there is a 1-form $\lambda \in \Omega^1(M)$ and a function $f \in \Omega^0(M^{\text{out}})$ such that $\omega = d\lambda$ on M , and $\theta - \lambda = df$ on M^{out} . After extending f to a smooth function on M , we obtain an extension $\theta := \lambda + df \in \Omega^1(M)$ with $\omega = d\theta$ on M . \square

Relative cohomology is isomorphic to compactly supported cohomology $H_c^*(M)$. Indeed, we have maps $H_c^*(M \setminus M^{\text{out}}) \rightarrow H^*(M, M^{\text{out}}) \rightarrow H_c^*(M)$ given by

$$[x] \mapsto [x, 0], \quad [x, y] \mapsto [x - d(\rho y)], \quad (3.1)$$

where $\rho : M \rightarrow [0, 1]$ is any function with $\rho = 1$ at infinity and $\rho = 0$ near M^{in} . Up to the isomorphism $H_c^*(M \setminus M^{\text{out}}) \cong H_c^*(M)$ induced by the inclusion $M \setminus M^{\text{out}} \subset M$, the maps in (3.1) are isomorphisms that are inverse to each another.

Lemma 3.2. *If φ_t is an isotopy of M , $\varphi_0 = \text{id}$, then $[\varphi_t^*x] = [x]$ in $H_c^*(M)$ for any $[x] \in H_c^*(M)$. If in addition $\varphi_t(M^{\text{out}}) = M^{\text{out}}$, then $[\varphi_t^*x, \varphi_t^*y] = [x, y]$ in $H^*(M, M^{\text{out}})$ for any $[x, y] \in H^*(M, M^{\text{out}})$.*

Proof. This follows from the functorial properties of H_c^* [13, p. 26] (with respect to proper maps). The second claim is the relative analogue of [13, Cor. 4.1.2, p. 35]. \square

Given a convex manifold (M, ω, θ) , two different choices of exhausting function give rise to different conical ends, thus two choices of decomposition

$$M = M_1^{\text{in}} \cup M_1^{\text{out}} = M_2^{\text{in}} \cup M_2^{\text{out}}.$$

However, there is an isotopy $\varphi_t : M \rightarrow M$ supported near the conical end such that $\varphi_t(M_1^{\text{out}}) = M_2^{\text{out}}$ (by simply isotopying the radial coordinate, i.e. moving along flowlines of Z). The class $[\omega, \theta]$ viewed in $H_c^2(M)$ via (3.1) therefore does not depend on the choice of exhausting function, nor the decomposition $M = M^{\text{in}} \cup M^{\text{out}}$.

Given a family of convex manifolds $(M, \omega_s, \theta_s)_{s \in [0, 1]}$, observe that we may redefine any given exhausting function for (M, ω_s, θ_s) so that $M^{\text{out}} \subset M_s^{\text{out}}$, where $M^{\text{out}}, M_s^{\text{out}}$ are respectively conical ends for $(M, \omega_0, \theta_0), (M, \omega_s, \theta_s)$. We can thus view the family of classes $[\omega_s, \theta_s] \in H^2(M, M_s^{\text{out}})$ as elements in $H^2(M, M^{\text{out}})$ via pull-back, and it is meaningful to consider the case when this is a “constant” class. Equivalently:

Definition 3.3. *For a family of convex manifolds (M, ω_s, θ_s) we say that the family of relative classes $[\omega_s, \theta_s]$ is constant if their images in $H_c^2(M)$ are constant via the isomorphism $H^2(M, M_s^{\text{out}}) \cong H_c^2(M)$ of (3.1).*

Lemma 3.4. *Let (M, ω_s, θ_s) be a family of convex manifolds with constant class $[\omega_s, \theta_s]$. Let $\varphi_s : M \rightarrow M$ be an isotopy, $\varphi_0 = \text{id}$, and $\varphi_s^* \theta_s = \theta_0$ at infinity. Then, for a smooth family of compactly supported 1-forms λ_s ,*

$$\varphi_s^* \omega_s - \omega_0 = d\lambda_s. \quad (3.2)$$

Proof. By (3.1), $[\omega_s - d(\rho\theta_s)] \in H_c^*(M)$ is constant. By Lemma 3.2, $[\varphi_s^* \omega_s - d(\varphi_s^*(\rho\theta_s))]$ is constant in $H_c^*(M)$. It follows that $\varphi_s^* \omega_s - \omega_0 = d(\varphi_s^*(\rho\theta_s) - \rho\theta_0) + d\lambda'_s$ for some compactly supported one-forms⁵ λ'_s . By construction $\varphi_s^*(\rho\theta_s) - \rho\theta_0$ vanishes at infinity (since $\varphi_s^* \theta_s = \theta_0$ there), so we may take $\lambda_s = \lambda'_s + \varphi_s^*(\rho\theta_s) - \rho\theta_0$ in (3.2). \square

Lemma 3.5. *Let (M, ω_s, θ_s) be a family of convex manifolds with constant class $[\omega_s, \theta_s]$. Let $(\hat{M}_s, \hat{\omega}_s, \hat{\theta}_s)$ denote the completion of (M, ω_s, θ_s) . Suppose we are given an isomorphism $\varphi_s : (\hat{M}_0, \hat{\omega}_0 + \beta_s, \hat{\theta}_0) \rightarrow (\hat{M}_s, \hat{\omega}_s, \hat{\theta}_s)$ for compactly supported 2-forms β_s on \hat{M}_0 , satisfying $\varphi_s|_{M_0^{\text{in}}} = \text{id}_{M_0^{\text{in}}}$ and $\varphi_0 = \text{id}_{\hat{M}_0}$. Then there is a family of compactly supported 1-forms λ_s on \hat{M}_0 , with $\lambda_0 = 0$, such that $\beta_s = d\lambda_s$.*

Proof. The manifolds \hat{M}_s are all diffeomorphic to M through a family of proper diffeomorphisms ψ_s , with $\psi_0 = \text{id}$, which are equal to the identity on M_0^{in} (indeed the conical ends attached to M in the completion construction can be radially isotoped towards M). Composing yields an isotopy $\psi_s \circ \varphi_s : \hat{M}_0 \rightarrow \hat{M}_0$ which equals the identity on M_0^{in} . Lemma 3.4 therefore implies that $\beta_s = (\hat{\omega}_0 + \beta_s) - \hat{\omega}_0 = d\lambda_s$ for some compactly supported 1-forms λ_s on \hat{M}_0 , with $\lambda_0 = 0$. \square

Remark 3.6. *If $H^1(\Sigma) = 0$ over \mathbb{R} , $[\omega_s, \theta_s] \in H^2(M, M^{\text{out}})$ is constant in s precisely if $[\omega_s] \in H^2(M)$ is. This also holds if $H^1(M) \rightarrow H^1(M^{\text{out}}) \cong H^1(\Sigma)$ is surjective, by the long exact sequence $H^1(M) \rightarrow H^1(M^{\text{out}}) \rightarrow H^2(M, M^{\text{out}}) \rightarrow H^2(M)$.*

3.2. Deformations at infinity: varying the contact form. Let (M, ω, θ) be convex and complete (Definition 2.1). We often blur the distinction between the domain and image of j in (2.1) so $\Sigma \equiv \partial M^{\text{in}}$ and $\alpha = \theta|_{\Sigma}$. Before proving Theorem 1.3, we will prove a technical result which shows that any deformation of contact forms $(\alpha_s)_{0 \leq s \leq 1}$ on Σ with $\alpha_0 = \alpha$ can be recovered by a deformation i_s of the conical parametrisation (2.1). Applying Gray's stability theorem to the family α_s yields a smooth family of functions $f_s : \Sigma \rightarrow \mathbb{R}$ and a contact isotopy $\psi_s : \Sigma \rightarrow \Sigma$, $\psi_0 = \text{id}$, with $\psi_s^* \alpha_s = e^{f_s} \alpha_0$. Given $\epsilon > 0$, for any constant c with $c > \epsilon$ and $c > \epsilon + \max f_s$, we obtain the subset

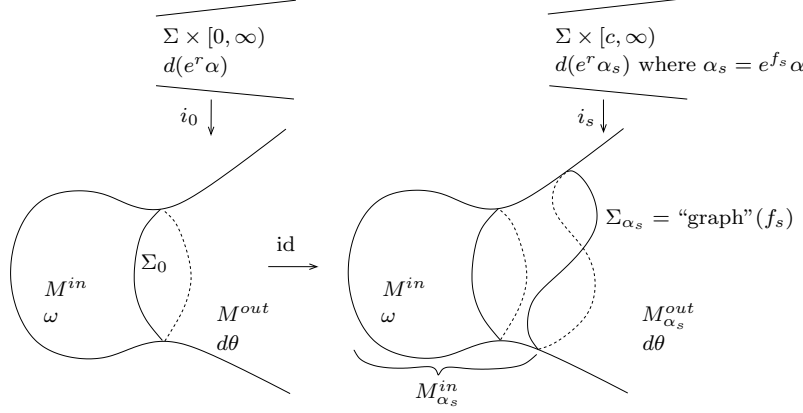
$$M_{\alpha_s}^{\text{out}} = \{(\psi_s(y), r) : y \in \Sigma, r \geq c - f_s(y)\} \subset \Sigma \times (\epsilon, \infty)$$

having as boundary

$$\Sigma_{\alpha_s} = \{(\psi_s(y), c - f_s(y)) : y \in \Sigma\}.$$

⁵That λ'_s can be chosen smoothly in s follows because a smooth path $[0, 1] \rightarrow \Omega_{\text{exact}}^2(M, M^{\text{out}})$ can be lifted to $[0, 1] \rightarrow \Omega^1(M, M^{\text{out}})$ via the smooth surjective linear map $d : \Omega^1(M, M^{\text{out}}) \rightarrow \Omega_{\text{exact}}^2(M, M^{\text{out}}) \subset \Omega^2(M, M^{\text{out}})$. Locally this corresponds to choosing a smooth family of orthogonal complements to $\ker d$, which can be achieved by taking orthogonal complements with respect to a choice of Riemannian metric.

Lemma 3.7. *The contact forms α_s determine conical parametrisations i_s of the end $M_{\alpha_s}^{\text{out}}$ of M , for the contact manifold (Σ, α_s) (see the figure below). Namely,*



$$i_s : (\Sigma \times [c, \infty), e^{r_s} \alpha_s, d(e^{r_s} \alpha_s)) \rightarrow (M_{\alpha_s}^{\text{out}}, \omega|_{M_{\alpha_s}^{\text{out}}}, \theta),$$

$$i_s(y, r_s) = \text{Flow}_{r_s - c}^Z(\psi_s(y), c - f_s(y)) = (\psi_s(y), r_s - f_s(y)).$$

In particular $i_s^* \theta = i_s^*(e^r \alpha_0) = e^{r_s} \alpha_s$ and $i_s(\Sigma \times \{c\}) = \Sigma_{\alpha_s}$. The new data induced by i_s is $Z_s = Z$ with radial coordinate $r_s = r + f_s(y) \in [c, \infty)$.

Proof. We readily compute

$$i_s^*(e^r \alpha_0)|_{(y, r_s)} = e^{r_s - f_s(y)} \psi_s^* \alpha_0 = e^{r_s - f_s(y)} e^{f_s(y)} \alpha_s = e^{r_s} \alpha_s. \quad \square$$

Remark 3.8. *The family i_s yields a family $M_{\alpha_s}^{\text{in}} := \overline{M} \setminus \overline{M_{\alpha_s}^{\text{out}}} \subset M$ of diffeomorphic manifolds. This yields an isotopy $\varphi_s : M \rightarrow M$, $\varphi_0 = \text{id}$, such that $\varphi_s(M^{\text{in}}) = M_{\alpha_s}^{\text{in}}$, $\varphi_s(\Sigma \times \{c\}) = \Sigma_{\alpha_s}$. Now $(M, \varphi_s^* \omega, \varphi_s^* \theta)$ is a deformation of (M, ω, θ) admitting the conical parametrisation $\varphi_s^{-1} \circ i_s$ on M^{out} modeled on (Σ, α_s) . Thus, up to isomorphisms of convex manifolds, we can arbitrarily vary the contact form α_s on Σ subject to fixing the contact structure $\xi = \ker \alpha$ (using Lemma 3.2, Theorem 1.3).*

3.3. Deformations are compactly supported up to isomorphisms.

Proposition 3.9. *Let (M, ω_s, θ_s) be a family of convex manifolds with completions $(\hat{M}_s, \hat{\omega}_s, \hat{\theta}_s)$. There is a family of compactly supported closed 2-forms β_s on \hat{M}_0 with $\beta_0 = 0$ such that $(\hat{M}_0, \hat{\omega}_0 + \beta_s, \hat{\theta}_0)$ is convex and admits a family of isomorphisms*

$$\varphi_s : (\hat{M}_0, \hat{\omega}_0 + \beta_s, \hat{\theta}_0) \rightarrow (\hat{M}_s, \hat{\omega}_s, \hat{\theta}_s), \quad \varphi_s|_{M_0^{\text{in}}} = \text{id}_{M_0^{\text{in}}}, \quad \varphi_0 = \text{id}_{\hat{M}_0}.$$

Note that $\beta_s = \varphi_s^ \hat{\omega}_s - \hat{\omega}_0$.*

Proof. Let us fix some $b > 0$ such that $M_s^{\geq b} \subset M_0^{\text{out}}$ for all $s \in [0, 1]$. As the statement of the proposition does not depend on the choice of exhausting function used for (M, ω_s, θ_s) , we may assume $b = 0$, so $M_s^{\text{out}} \subset M_0^{\text{out}}$. Let $s' \in [0, 1]$ be arbitrary. We claim that there exists $\epsilon_{s'} > 0$ such that if $s'' \in (s' - \epsilon, s' + \epsilon)$, then there exists a closed 2-form $\beta_{s', s''}$ on $\hat{M}_{s'}$ with compact support and an isomorphism $\varphi_{s', s''} : (\hat{M}_{s'}, \hat{\omega}_{s'} + \beta_{s', s''}, \theta_{s'}) \rightarrow (\hat{M}_{s''}, \hat{\omega}_{s''}, \theta_{s''})$ such that $\varphi_{s', s''}|_{M_0^{\text{in}}} = \text{id}_{M_0^{\text{in}}}$. If the claim holds, then we can extract a finite subcover of the compact interval $[0, 1]$ from

the cover $\{(s' - \epsilon_{s'}, s' + \epsilon_{s'})\}_{s' \in [0,1]}$. The isomorphism φ_s in the statement of the present proposition is then obtained by a finite composition of isomorphisms of the type $\varphi_{s',s''}$. We now prove the claim. For ease of notation we just do the case $s' = 0$.

As $M_s^{\text{out}} \subset M_0^{\text{out}}$, we may assume that Z_s is defined on M^{out} , in particular it is non-vanishing there. Since Z is transverse to $\Sigma = \partial M^{\text{out}}$ and outward pointing, the same will hold for Z_s for small s . By positively integrating Z_s starting from Σ we obtain a family of conical parametrisations $j_s : \{(y, r) : 0 \leq r < \sigma_s(y)\} \rightarrow M$ for (M, ω_s, θ_s) (compare (2.1)). Furthermore, one can construct an isotopy $F_s : M \rightarrow M$ with $F_0 = \text{id}$ such that $F_s \circ j_s|_{\Sigma \times [0, \epsilon]} = j_0|_{\Sigma \times [0, \epsilon]}$ for some $\epsilon > 0$. As F_s can be chosen to be compactly supported near Σ , we can ensure that $F_s = \text{id}$ on the original M^{in} before enlarging and $F_s = \text{id}$ at infinity. Observe that if we can prove the proposition for $(M, F_s^* \omega_s, F_s^* \theta_s)$ then it will also follow for (M, ω_s, θ_s) by conjugating the isomorphism by F_s . Thus we may now assume that

$$j_s|_{\Sigma \times [0, \epsilon]} = j_0|_{\Sigma \times [0, \epsilon]}. \quad (3.3)$$

The family j_s determines contact forms α_s on Σ , with $j_s^*(\theta_s) = e^\rho \alpha_s$ where $\rho \in [0, \infty)$ plays the same role as r in (2.1). By Lemma 3.7, we obtain a new family of conical parametrisations $i_s : (\Sigma \times [c, \infty)) \rightarrow M_{\alpha_s}^{\text{out}} \subset \hat{M}$, with $i_s^* \theta_0 = e^\rho \alpha_s$ where $\rho \in [c, \infty)$. By construction, $i_s^{-1}(M_{\alpha_s}^{\text{out}}) \subset \hat{M}$ does not intersect M^{in} . The composition

$$\varphi_s := j_s \circ i_s^{-1} : M_{\alpha_s}^{\text{out}} \rightarrow \hat{M}$$

satisfies $\varphi_s^* \theta_s = \theta_0$. We will now extend φ_s to $\hat{M} \setminus M^{\text{in}}$ so that φ_s equals the identity near ∂M^{in} (the proposition will then follow by further extending via $\varphi_s|_{M^{\text{in}}} = \text{id}$).

By construction, $M_{\alpha_s}^{\text{out}} \subset \Sigma \times (\epsilon, \infty) \subset \hat{M}$ (where we have identified the domain and the image of j_0 to simplify the notation). By the definition of i_s in Lemma 3.7, φ_s satisfies the equation

$$\varphi_s(y, r) = j_s(\psi_s^{-1}(y), r + f_s(\psi_s^{-1}(y))) \in \hat{M}_s \quad (3.4)$$

for $(y, r) \in M_{\alpha_s}^{\text{out}} \subset \Sigma \times (\epsilon, \infty)$. By construction those values satisfy the inequalities $r + f_s(\psi_s^{-1}(y)) \geq c > \epsilon$. Thus we can extend φ_s to $\Sigma \times [0, \infty)$ by

$$\varphi_s(y, r) = j_s(\psi_{a_s(r)}^{-1}(y), b_s(y, r))$$

where we now explain the interpolations $a_s : [0, \infty) \rightarrow [0, 1]$, $b_s : \Sigma \times [0, \infty) \rightarrow [0, \infty)$. We want $a_s(r) = s$ if $(r, y) \in M_{\alpha_s}^{\text{out}}$ for some y , $a_s(r) = 0$ for $r \in [0, \epsilon/2]$. We want $b_s(y, r)$ to smoothly interpolate between the value required on $M_{\alpha_s}^{\text{out}}$ by (3.4) and the function $b(y, r) = r$ for $r \in [0, \epsilon/2]$ (e.g. smoothen a linear interpolation). This ensures by (3.3) that for $r \in [0, \epsilon/2]$ we have $\varphi_s(y, r) = (y, r)$, the identity map. \square

Remark 3.10. *If the (M, ω_s, θ_s) are complete, then the proof simplifies as one can work on a fixed manifold $M^{\text{in}} \cup (\Sigma \times [0, \infty))$ throughout, instead of using \hat{M}_s .*

3.4. Convex domains. Convex domains (D, ω, α) are defined in Rmk.2.6. Observe ω is exact in a collar neighbourhood C of ∂D . A choice of primitive θ on C with $\theta|_\Sigma = \alpha$ yields a Liouville vector field Z by $\iota_Z \omega = \theta$. Integrating Z backwards in time from Σ , we may assume C is identified with $\Sigma \times [-\epsilon, 0]$. The completion \hat{D} extends this via (2.1) to a conical end $\Sigma \times (-\epsilon, \infty)$, yielding a complete convex manifold.

Remark 3.11. Note that $M = D \setminus \partial D$ is a convex manifold (M, ω, θ) , as we obtained a (non-complete) conical parametrisation by $\Sigma \times [-\epsilon, 0)$ above. Suppose θ_0, θ_1 are two choices of primitive as above. As C, Σ are homotopy equivalent, and the closed form $\theta_1 - \theta_0$ pulls back to zero on Σ , we have $\theta_1 - \theta_0 = df$ for some $f : C \rightarrow \mathbb{R}$. Thus $[\omega, \theta_0 + d(sf)]_{0 \leq s \leq 1}$ is a constant family in $H^2(M, M^{\text{out}})$. So Theorem 1.3 implies that the isomorphism class of (D, ω, α) does not depend on the chosen primitive on the collar, as there is an isomorphism of the completions $(M, \omega, \theta_0)^\wedge \cong (M, \omega, \theta_1)^\wedge$.

Lemma 3.12. Let (D, ω) be a closed symplectic manifold with boundary $\Sigma := \partial D$. Suppose that $h : D \rightarrow \mathbb{R}$ is a smooth function such that Σ is a regular level set, h strictly increases in the outward normal direction, and $\omega|_{T\Sigma} = d\alpha$ for some $\alpha \in \Omega^1(\Sigma)$. Then (D, ω, α) is a convex domain if and only if $\alpha(X_h) > 0$.

Proof. By definition, $X_h \in T\Sigma$ so that $\omega|_{T\Sigma}(\cdot, X_h) = dh|_\Sigma = 0$. As ω is symplectic, $\dim \ker \omega|_{T\Sigma} = 1$ and $\ker \omega|_{T\Sigma} = \mathbb{R} X_h$. This readily implies that the condition $\alpha \wedge (d\alpha)^{\dim_{\mathbb{C}} D - 1} > 0$ is equivalent to $\alpha(X_h) > 0$. \square

Arguing as in Proposition 3.9, we obtain the following.

Proposition 3.13. Let (D, ω_s, α_s) be a deformation of convex domains. After choosing a family of primitives θ_s for ω_s near ∂D with $\theta_s|_\Sigma = \alpha_s$, by completion we obtain a family of complete convex manifolds $(M_s, \omega_s, \theta_s)$. Then there is a family of convex manifolds $(M_0, \omega_0 + \beta_s, \theta_0)$ where the forms β_s are compactly supported, $\beta_0 = 0$, together with a family of isomorphisms

$$\varphi_s : (M_0, \omega_0 + \beta_s, \theta_0) \rightarrow (M_s, \omega_s, \theta_s), \quad \varphi_s|_D = \text{id}, \quad \varphi_0 = \text{id}. \quad \square$$

For small perturbations of a convex manifold which is the interior of some convex domain, Proposition 3.9 can be adapted to the following quantitative statement.

Proposition 3.14. Let (D, ω, α) be a convex domain. Fix a primitive θ on a collar C of $\Sigma = \partial D$, $\theta|_\Sigma = \alpha$, making (M, ω, θ) convex as in Remark 3.11 where $M = D \setminus \partial D$. Then there is an $\epsilon > 0$ and a $K > 0$ such that for all closed $(\mu, \lambda) \in \Omega^2(D, C)$ with $\|(\mu, \lambda)\|_{C^1(D, C)} < \epsilon$, the triple $(M, \omega + \mu, \theta + \lambda)$ is a convex manifold. Moreover, there is a compactly supported closed 2-form β on M satisfying the bound $\|\beta\|_{C^0(M)} \leq K\|(\mu, \lambda)\|_{C^1(D, C)}$ and admitting an isomorphism

$$\varphi : (M, \omega + \beta, \theta)^\wedge \rightarrow (M, \omega + \mu, \theta + \lambda)^\wedge \quad \text{with} \quad \varphi|_{M^{\text{in}}} = \text{id}.$$

In particular, $[\beta] \mapsto [\mu, \lambda]$ via $H_c^2(M) \cong H^2(D, C)$. \square

Remark 3.15. We can ensure β is compactly supported in M , not just \hat{M} , since for small deformations we can ensure Σ_s is close to $\partial M^{\text{in}} = \Sigma \times \{-\epsilon\}$ (see Lemma 3.7).

3.5. Proof of Theorem 1.3. By Proposition 3.9, we have a family of isomorphisms

$$\varphi_s : (\hat{M}, \omega_0 + \beta_s, \theta_0) \rightarrow \hat{M}_s := (M, \omega_s, \theta_s)^\wedge.$$

By Lemma 3.5, there is a family of compactly supported 1-forms λ_s , $\lambda_0 = 0$, with

$$\beta_s = d\lambda_s. \quad (3.5)$$

We now run Moser's argument. Let V_s be the compactly supported vector field on \hat{M} determined by $(\omega_0 + \beta_s)(V_s, \cdot) = -\partial_s \lambda_s$. Let $F_s : \hat{M} \rightarrow \hat{M}$ be the isotopy generated

by V_s , so $\partial_s F_s = V_s \circ F_s$, $F_0 = \text{id}$. Then $F_s^*(\omega_0 + \beta_s) = \omega_0$ on \hat{M} , and $F_s^*\theta_0 = \theta_0$ at infinity since F_s is compactly supported. So we obtain the isomorphism

$$\varphi := \varphi_1 \circ F_1 : (M, \omega_0, \theta_0)^\wedge \rightarrow (M, \omega_1, \theta_1)^\wedge$$

claimed in Theorem 1.3. \square

Remark 3.16 (Liouville isomorphisms). *For Liouville manifolds (so θ_s extends to M^{in} with $\omega_s = d\theta_s$ on all of M), the $(\omega_s, \theta_s) = D(\theta_s, 0)$ are automatically a constant class. Let $\bar{\theta}_s := \varphi_s^*\theta_s$. Then $\bar{\theta}_0 = \theta_0$, $d\bar{\theta}_s = \omega_0 + \beta_s$, and $\bar{\theta}_s = \theta_0$ at infinity. So $\lambda_s := \bar{\theta}_s - \theta_0$ satisfies (3.5). Then, for V_s, F_s as above, apply Cartan's formula:*

$$\partial_s(F_s^*\bar{\theta}_s) = F_s^*[\partial_s\bar{\theta}_s + d\iota_{V_s}(\bar{\theta}_s) + \iota_{V_s}d\bar{\theta}_s] = d(F_s^*\iota_{V_s}(\bar{\theta}_s)).$$

Now define the compactly supported function $g := \int_0^1 (F_s^*i_{V_s}(\bar{\theta}_s)) ds$. Then

$$\varphi^*\theta_1 - \theta_0 = dg. \quad (3.6)$$

Such maps φ are called Liouville isomorphisms. This is a proof of [60, Lemma 2.2].

Example 3.17. Let $(\omega_s)_{0 \leq s \leq 1}$ be cohomologous symplectic forms making M^{in} a convex domain (Remark 2.6). Extending to a completion $W = M^{\text{in}} \cup (\Sigma \times [0, \infty))$ we may assume $\omega_s = d\theta_s$ on the conical end where $\theta_s = e^r\alpha_s$ on $\Sigma \times [0, \infty)$, and (Σ, α_s) is of positive contact type. By assumption, $\frac{d}{ds}\omega_s = d\sigma_s$ for some 1-forms σ_s on W .

If $H^1(\Sigma) = 0$ (which implies $[\omega_s, \theta_s]$ is constant), one can pick σ_s with $\sigma_s = e^r\alpha_s$ on the end. One can construct φ_s as a (typically non-compactly supported) flow of a vector field V_s by Moser's argument, so $\sigma_s = -\omega_s(V_s, \cdot)$. Thus, on the end,

$$\frac{d\alpha_s}{ds} = -[d\alpha_s + dr \wedge \alpha_s](V_s, \cdot)$$

after canceling out e^r factors. So V_s is integrable because the radial component $-\frac{d\alpha_s}{ds}(Y_s)Z$ (where Y_s is the Reeb vector field for α_s) is harmless. This is the argument in Harris [35, Lemma 6.1].⁶ In Section 6.4 we consider $(T^*T^2, d\theta)$ but we use a different primitive α at infinity making the relative class $[d\theta, \alpha]$ non-trivial. In this case, the Moser argument cannot yield an isomorphism, due to the obstructed relative class. Nevertheless we prove that symplectic cohomology is invariant.

4. SYMPLECTIC COHOMOLOGY FOR CONVEX MANIFOLDS

4.1. Symplectic cohomology. Symplectic cohomology for Liouville manifolds was constructed by Viterbo [62], see also the surveys [49, 53, 60]. Symplectic cohomology for complete convex manifolds was constructed by the second author [52], so we will only make some remarks here. We mention some finer points in Section 4.2.

Let (M, ω, θ) be convex. From now on, we use the *radial coordinate* $R = e^r \in [1, \infty)$, so $j^*\theta = R\alpha$. Recall the *Reeb vector field* Y on Σ is determined by $\alpha(Y) = 1$, $d\alpha(Y, \cdot) = 0$. By *Reeb periods* we mean the periods of closed orbits of Y . The contact form α is always assumed to have been perturbed generically (using Remark 3.8), so that the Reeb orbits are transversally non-degenerate and the Reeb periods form a

⁶The statement [35, Lemma 6.1] is missing the assumption $H^1(\Sigma) = 0$, otherwise there may be an obstruction to extending the closed form $\sigma_s - e^r\alpha_s \in H^1(W^{\text{out}})$ to W . In Harris' applications, $\Sigma \cong ST^*S^3 \cong S^3 \times S^2$ has $H^1(\Sigma) = 0$ as required.

discrete subset of \mathbb{R}^+ . The choice of perturbation does not affect $SH^*(M, \omega, \theta)$ up to isomorphism, by Theorems 1.1 and 1.3.

Recall $SH^*(M, \omega, \theta)$ is the direct limit in (1.2), and we now describe the class of Hamiltonians $H : M \rightarrow \mathbb{R}$ more precisely. Recall we identify M^{out} with the image of j in (2.1). We always assume that H is *radial* at infinity, meaning $H = h(R)$ only depends on the radial coordinate R , thus $X_H = h'(R)Y$. This yields a one-to-one correspondence between 1-periodic Hamiltonian orbits $x : S^1 \rightarrow M$ lying in a slice $\Sigma \times \{R\}$ with $h'(R) = T \neq 0$ and Reeb orbits $y : [0, T] \rightarrow M$ of period T , via $y(t) = x(t/T)$. The 1-orbits of X_H will be transversally non-degenerate, so non-degeneracy will be ensured by a generic one-periodic time-dependent perturbation of H (which we suppress from the notation – the choice of perturbations will not affect the Floer cohomology groups up to isomorphism).

Pick some R_∞ such that $j(\Sigma \times \{R_\infty\}) \subset M^{\text{out}}$ (for example R_∞ close to 1). Call $M_\infty \subset M$ the region $R \geq R_\infty$. Then assume that the Hamiltonian $H = h(R)$ is linear in R on M_∞ , with slope $m = h'(R)$ different from the Reeb periods. By the previous two paragraphs, this implies that there are no 1-orbits of X_H in M_∞ .

The Floer complex is generated by the 1-orbits of X_H , but the differential depends on a choice of almost complex structure J on M *compatible* with ω . This means:

$$\omega(Ju, Jv) = \omega(u, v), \quad \omega(v, Jv) > 0, \quad \forall u, v \in TM, \quad u, v \neq 0.$$

This yields a Riemannian metric $g = \omega(\cdot, J\cdot)$ on M . The data (H, J) yields the differential, which counts *Floer trajectories*, i.e. solutions $u = u(s, t) : \mathbb{R} \times S^1 \rightarrow M$ of the equation $\partial_s u + J(\partial_t u - X_H) = 0$ that are isolated up to \mathbb{R} -translation.

We must ensure that these trajectories do not escape to infinity so that moduli spaces of Floer trajectories have well-behaved compactifications by broken trajectories. A maximum principle will hold, i.e. $R \circ u$ cannot attain a local maximum in M_∞ , if we choose J to be of *contact type* on M_∞ , meaning

$$JZ = Y \quad (\text{equivalently } J^*\theta = dR). \quad (4.1)$$

The possible structures J as above form a non-empty contractible space, which is used to show that the Floer cohomology groups $HF^*(H)$ in (1.2) are independent up to isomorphism on the choice of J . We recall that a generic time-dependent perturbation of J is needed on $M \setminus M_\infty$ to ensure that moduli spaces of Floer trajectories are smooth manifolds (we suppress the perturbation from the notation – the choice of perturbation will not affect the Floer cohomology groups up to isomorphism).

The Floer cohomology group $HF^*(H) = HF^*(M; H, J)$ will be independent of the choices of H, J, R_∞ , in fact it is isomorphic to $HF^*(\hat{M}; H, J)$ computed for the completion \hat{M} for any generic time-dependent (H, J) subject to H having eventually slope m at infinity and J being of contact type at infinity. This is because continuation isomorphisms can be constructed provided the slope m at infinity is constant.

Continuation homomorphisms $HF^*(H_+, J_+) \rightarrow HF^*(H_-, J_-)$ for different choices of the data (H, J) can only be constructed if the slopes satisfy $m_+ \leq m_-$ (only then a maximum principle holds). These maps count isolated solutions $u : \mathbb{R} \times S^1 \rightarrow M$ of $\partial_s u + J_s(\partial_t u - X_{H_s}) = 0$ where $(H_s, J_s) = (H_\pm, J_\pm)$ for s close to $\pm\infty$.

As $HF^*(H)$ only depends on m up to isomorphism, the direct limit in (1.2) can therefore be taken for any sequence H_k with increasing slopes $m_k \rightarrow \infty$, using the continuation homomorphisms. The direct limit will be independent up to isomorphism on the choices. The above discussion implies that

$$SH^*(M, \omega, \theta) \cong SH^*((M, \omega, \theta)^\wedge),$$

in particular Theorem 1.1 implies that this group up to isomorphism only depends on the isomorphism class of (M, ω, θ) . By the same arguments as in [53], $SH^*(M, \omega, \theta)$ admits a pair-of-pants product and a unit, and the unital algebra $SH^*(M, \omega, \theta)$ only depends on the isomorphism class of (M, ω, θ) .

One typically chooses H to be Morse and C^2 -small in the compact region where H is not radial (in the above sense), and one perturbs J time-independently on this region so that (H, g) is a Morse-Smale pair. This ensures that the Floer complex on this region reduces to the Morse complex for H, g (the 1-orbits of X_H become non-degenerate constant orbits, and the Floer trajectories become time-independent $-\nabla H$ flow lines). If $m > 0$ is smaller than all Reeb periods, one can ensure that one globally obtains a Morse complex for M . This implies that there is a canonical map

$$c^* : QH^*(M, \omega) \rightarrow SH^*(M, \omega, \theta),$$

where $QH^*(M, \omega)$ is the quantum cohomology (the quantum product is constructed using J as above, but the unital algebra $QH^*(M, \omega)$ only depends on ω up to isomorphism). In particular, in the quantum product, holomorphic spheres $u : \mathbb{CP}^1 \rightarrow M$ are counted with weight $t^{\int u^* \omega}$. The same argument as in [53] shows that c^* is a unital algebra homomorphism.

Remark 4.1 (Viterbo's trick). *We remark that a generalisation of a key idea due to Viterbo [62] still applies here: if c^* is not a unital algebra isomorphism, then there must exist a closed Reeb orbit in Σ . Indeed, if for all choices of H there never existed a non-constant 1-orbit of X_H in the region where H is radial, then one could easily construct a family H_k that forces c^* to be an isomorphism.*

4.2. Novikov field, Action 1-form, Energy. The groups HF^*, SH^*, QH^* above are all defined over the Novikov field Λ in a formal variable t over a base field \mathbb{K} ,

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i t^{n_i} : a_i \in \mathbb{K}, n_i \in \mathbb{R}, n_i \rightarrow \infty \right\}. \quad (4.2)$$

For any H as above, there is a (typically non-exact) action 1-form $d\mathcal{A}_H$ on the space of free loops $\mathcal{LM} = C^\infty(S^1, M)$,

$$d\mathcal{A}_H := d\mathcal{H} - \tau_\omega$$

where we define the function $\mathcal{H} : \mathcal{LM} \rightarrow \mathbb{R}$ by

$$\mathcal{H}(x) := \int_0^1 H(x(t)) dt,$$

and τ_ω is the transgression 1-form on \mathcal{LM} defined by

$$\tau_\omega(\xi) := \int_0^1 \omega(\xi(t), \partial_t x) dt$$

for $\xi \in T_x \mathcal{LM} = C^\infty(x^* TM)$. Thus $dA_H(\xi) = - \int_0^1 \omega(\xi(t), F(x)) dt$ where

$$F = F_H : \mathcal{LM} \rightarrow \bigcup_{x \in \mathcal{LM}} x^* TM, \quad F(x) = \partial_t x - X_H. \quad (4.3)$$

Thus 1-orbits of X_H are the zeros of F , equivalently the zeros of dA_H , and Floer trajectories are maps $u : \mathbb{R} \rightarrow \mathcal{LM}$ satisfying *Floer's equation*: $F(u) = J \partial_s u$.

Let $\mathcal{M}(x, y; H, J)$ be the space of rigid Floer trajectories $u : \mathbb{R} \times S^1 \rightarrow M$ from x to y , modulo shift in the s -variable. Then the energy is

$$E(u) := \int_{\mathbb{R} \times S^1} |\partial_s u|^2 ds \wedge dt = \int_{\mathbb{R} \times S^1} |F(u)|^2 ds \wedge dt. \quad (4.4)$$

The differential ∂ on the Floer complex

$$CF^*(H) = \oplus \{\Lambda x : F(x) = 0\}$$

is explicitly

$$\partial y = \sum_{u \in \mathcal{M}(x, y; H, J)} \epsilon(u) t^{-dA_H(u)} x = \sum_{u \in \mathcal{M}(x, y; H, J)} \epsilon(u) t^{\tau_\omega(u)} t^{\mathcal{H}(x) - \mathcal{H}(y)} x \quad (4.5)$$

where $\epsilon(u) \in \{\pm 1\}$ are orientation signs (which we will not discuss), and $dA_H(u)$ and $\tau_\omega(u)$ are evaluations of these 1-forms on the 1-chain u in \mathcal{LM} . In particular,

$$\tau_\omega(u) = \int_{\mathbb{R} \times S^1} u^* \omega. \quad (4.6)$$

The maximum principle and Gromov compactness imply that the coefficient of x in (4.5) belongs to Λ if for every $C > 0$ there is an $E_C(x, y; H, J) > 0$, such that

$$\forall u \in \mathcal{M}(x, y; H, J), \quad \tau_\omega(u) \leq C \implies E(u) \leq E_C(x, y; H, J). \quad (4.7)$$

From Floer's equation, we see that this condition is satisfied since

$$E(u) = \tau_\omega(u) + \mathcal{H}(x) - \mathcal{H}(y). \quad (4.8)$$

From $(CF^*(H, J), \partial)$ one obtains the $HF^*(H)$ mentioned in Section 4.1. The continuation maps $\phi : HF^*(H_+, J_+) \rightarrow HF^*(H_-, J_-)$ are explicitly

$$\phi y = \sum_{u \in \mathcal{M}(x, y; H_s, J_s)} \epsilon(u) t^{\tau_\omega(u)} t^{\mathcal{H}_-(x) - \mathcal{H}_+(y)} x \quad (4.9)$$

where $\mathcal{M}(x, y; H_s, J_s)$ is the moduli space of rigid maps $u : \mathbb{R} \rightarrow \mathcal{LM}$ satisfying $F_{H_s}(u) = J_s \partial_s u$ for (H_s, J_s) as in Section 4.1. In this case,

$$E(u) = \tau_\omega(u) + \mathcal{H}_-(x) - \mathcal{H}_+(y) + \int_{\mathbb{R} \times S^1} (\partial_s H_s)(u(s, t)) ds \wedge dt.$$

If we require $\partial_s H_s \leq 0$ (which forces the slopes m_s at infinity to decrease), then the maximum principle holds and we have the estimate

$$E(u) \leq \tau_\omega(u) + \mathcal{H}_-(x) - \mathcal{H}_+(y),$$

which implies (4.7) for the new moduli space and so ϕ is well-defined. More generally, ϕ is well-defined if we just require m_s to decrease, since on the right above we get a harmless additional term $+c \cdot \max \{|\partial_s H_s(p)| : p \in M \setminus M_\infty\}$, since there is an M_∞

independent of s that works for all H_s in the notation of Section 4.1. The constant c is the measure of the bounded set of $s \in \mathbb{R}$ for which H_s is s -dependent.

4.3. The BV-operator. We will apply symplectic cohomology to prove the existence of more than one closed magnetic geodesic with given energy in Section 6.4 and 6.5 (see Theorem 1.7 and 1.9). To this purpose, we need to define an additional piece of structure, the *BV-operator*

$$\Delta : CF^*(H) \rightarrow CF^{*-1}(H)$$

constructed by Seidel [60]. Following [2, 16], we can describe Δ as follows.

Let $\nu \mapsto (H_s^\nu, J_s^\nu)$ be a family of continuation pairs depending on a parameter $\nu \in S^1$ in such a way that $(H_s^\nu, J_s^\nu) \equiv (H, J)$ for large r , $H_-^\nu = H(\cdot, \cdot + \nu)$ and $H_+^\nu = H$, where, as before, we identify H with a small one-periodic time-dependent perturbation of an autonomous Hamiltonian, so that H_+^ν is obtained from H by shifting the time by ν . The BV-operator is given by

$$\Delta y = \sum_{u \in \mathcal{M}(x, y; H_s^\nu, J_s^\nu)} \epsilon(u) t^{\tau_\omega(u)} t^{\mathcal{H}(x) - \mathcal{H}(y)} x,$$

where $\mathcal{M}(x, y, H_s^\nu, J_s^\nu)$ is the moduli space of rigid cylinders solving $F_{H_s^\nu}(u) = J_s^\nu \partial_s u$ and going from $x(\cdot + \nu)$ to y . The BV-operator is a chain map, namely,

$$\partial \Delta + \Delta \partial = 0. \quad (4.10)$$

In particular, it preserves the set of cocycles and of coboundaries.

4.4. Filtration by the radial coordinate. As the transgression form τ_ω might not be exact, the Hamiltonian action is multivalued and can not be used to filter the symplectic cohomology. However, as first observed by Bourgeois-Oancea in [15, p.654] and refined later by McLean-Ritter in [46, Appendix D], if the Hamiltonian $H : M \rightarrow \mathbb{R}$ is radial and convex on M^{in} , we still get a geometric filtration of $CF^*(H)$ by the radial coordinate, or equivalently by the period of the associated Reeb orbit. This filtration is preserved under ∂ and Δ , since both operators count solutions of a small perturbation of Floer's equation $F_H(u) = J \partial_s u$. More precisely, a one-periodic orbit x of H on M^{out} , yields two generators x_+ and x_- of $CF^*(H)$ after perturbation. Then,

$$\begin{cases} \partial x_- = ax_+ + y_1, \\ \partial x_+ = y_2, \end{cases} \quad \begin{cases} \Delta x_- = y_3, \\ \Delta x_+ = bx_- + y_4, \end{cases} \quad (4.11)$$

where $a, b \in \mathbb{Z}$ and y_1, y_2, y_3, y_4 are generated by orbits having r -component strictly less than that of x . The values of a and b depend on whether x is a good or bad orbit. Let z be the primitive Reeb orbit from which x is obtained by iteration and denote by k the order of iteration. Recall that an orbit is *good* if $\bar{\mu}(x) \equiv \bar{\mu}(z) \pmod{2}$ and *bad* otherwise. Here $\bar{\mu}$ is the transverse Conley-Zehnder index of an orbit. Results of Bourgeois-Oancea [14, Proposition 3.9] (see also [17, Proposition 2.2]), respectively of Zhao [65, Equation (6.1)], gives us the values of a and b :

$$a = \begin{cases} 0 & \text{if } x \text{ is good,} \\ \pm 2 & \text{if } x \text{ is bad,} \end{cases} \quad b = \begin{cases} k & \text{if } x \text{ is good,} \\ 0 & \text{if } x \text{ is bad.} \end{cases} \quad (4.12)$$

4.5. Twisted symplectic cohomology. Twisted symplectic cohomology was first constructed in [51, 52] for Liouville manifolds. We now adapt this to convex manifolds. Let $\zeta \in H^1(\mathcal{L}(M))$ be a cohomology class represented by a closed 1-form η on $\mathcal{L}(M)$. The twisted group $HF^*(H, J)_\eta$ (and respectively $SH^*(M, \omega, \theta)_\eta$) is defined by replacing τ_ω by $\tau_\omega + \eta$ in (4.5) (resp. (4.9)). Notice this changes the weights in the count, but not the moduli spaces. That the twisted differential and twisted continuation maps are well-defined requires the analogues of (4.7) with $\tau_\omega + \eta$ in place of τ_ω . Explicitly, abbreviating $\mathcal{M}(x, y) = \mathcal{M}(x, y; H, J)$ (resp. $\mathcal{M}(x, y; H_s, J_s)$), for every $C > 0$ we need a constant $E_C(x, y; H, J, \eta) > 0$ such that

$$\forall u \in \mathcal{M}(x, y), \quad \tau_\omega(u) + \eta(u) \leq C \implies E(u) \leq E_C(x, y; H, J, \eta). \quad (4.13)$$

Giving an upper bound C on $\tau_\omega(u) + \eta(u)$ is the same as assuming an upper bound on the total exponent $\int u^* \omega + \int u^* \eta + \int H_-(x) dt - \int H_+(y) dt$ appearing in the twisted differential (for $H_\pm = H$) resp. twisted continuation map. By (4.7), the implication (4.13) is equivalent to saying that given x, y , the following holds:

$$\forall C > 0, \exists C' > 0, \forall u \in \mathcal{M}(x, y), \quad \tau_\omega(u) + \eta(u) \leq C \implies \tau_\omega(u) \leq C'. \quad (4.14)$$

Note (4.14) can fail in general, e.g. if $|\mathcal{M}(x, y)| = \infty$ and $\eta = -\tau_\omega$.

Definition 4.2. Call $SH^*(M, \omega, \theta)_\eta$ well-defined if (4.14) holds.

Lemma 4.3. If there is a constant $c < 1$ such that

$$-\eta(u) \leq c E(u) \quad \text{for all } u \in \mathcal{M}(x, y),$$

then $SH^*(M, \omega, \theta)_\eta$ is well-defined.

Proof. By (4.8), $-\eta(u) \leq c \tau_\omega(u) + p_{x,y}$ where $p_{x,y} = c(\mathcal{H}(x) - \mathcal{H}(y))$ only depends on the asymptotics (a similar argument holds for continuation solutions). Then,

$$\tau_\omega(u) + \eta(u) \leq C \implies \tau_\omega(u) \leq c \tau_\omega(u) + p_{x,y} + C \implies \tau_\omega(u) \leq \frac{p_{x,y} + C}{1 - c}. \quad \square$$

Lemma 4.4. $SH^*(M, \omega, \theta)_\eta$ is well-defined if η is transgression-compatible (Def.2.8). In particular, if $\tau(\omega)$ is exact (e.g. when ω is exact), then any twist η is allowed.

Proof. If τ_ω is exact then C' in (4.14) is determined a priori by the data x, y . For the transgression-compatible case, (2.5) implies $C' = cC + \mathcal{K}(y) - \mathcal{K}(x)$ works. \square

Lemma 4.5. If $SH^*(M, \omega, \theta)_\eta$ is well-defined then for any function $\mathcal{K} : \mathcal{LM} \rightarrow \mathbb{R}$, $SH^*(M, \omega, \theta)_{\eta+d\mathcal{K}}$ is well-defined and there is a natural isomorphism $SH^*(M, \omega, \theta)_\eta \cong SH^*(M, \omega, \theta)_{\eta+d\mathcal{K}}$ induced by the chain-level change of basis isomorphism sending a 1-orbit x to $t^{-\mathcal{K}(x)}x$. So we may write $SH^*(M, \omega, \theta)_\zeta$ for a class $\zeta \in H^1(\mathcal{LM})$. \square

Remark 4.6. An alternative approach, is to distinguish two formal variables t, b ,

$$\partial y = \sum_{u \in \mathcal{M}(x, y; H, J)} \epsilon(u) b^{\eta(u)} t^{\tau_\omega(u)} t^{\mathcal{H}(x) - \mathcal{H}(y)} x.$$

One can work for example over $\Lambda \hat{\otimes} B$, where B consists of finite sums $\sum c_j b^{m_j}$ where $c_j \in \mathbb{K}$, $m_j \in \mathbb{R}$. The tensor product is completed, meaning $\Lambda \hat{\otimes} B = \{\sum a_i t^{n_i} : a_i \in B, n_i \in \mathbb{R}, \text{ with } n_i \rightarrow \infty\}$. Then $SH^*(M, \omega, \theta)_\zeta$ always exists. However, for the purposes of proving a deformation theorem like (1.6), one would need to specialise the twisted group by evaluating $b \mapsto t$, leading again to convergence issues in the Novikov field.

4.6. Proof of Theorem 2.7.(1): existence of twisted symplectic cohomology.

We now restrict ourselves to twisting by classes in $H^1(\mathcal{L}M)$ arising by transgression from classes in $H^2(M, M^{\text{out}}) \cong H_c^2(M)$ (if $H^1(\Sigma; \mathbb{R}) = 0$, this means any $H^2(M)$ class). Explicitly, we twist by any closed two-form β on M exact at infinity. By Lemma 4.5 we may assume β is supported in a compact region $N \subset \text{int}(M^{\text{in}})$. We will prove that $SH^*(M, \omega, \theta)_{\tau(\beta)}$ is well-defined when $\|\beta\|$ is sufficiently small.

Consider the pairs (H, J) in the construction of $SH^*(M, \omega, \theta)$ satisfying:

- (H1) $\forall x \in \mathcal{L}(M), \quad x(S^1) \cap N \neq \emptyset \Rightarrow \|F_H(x)\| := (\int_0^1 |\partial_t x - X_H|^2 dt)^{1/2} > \delta,$
- (H2) $\|H\|_{C^1(N)} \leq C.$

Lemma 4.7. *There exist $\delta, C > 0$ admitting a cofinal family (H_k, J_k) of such pairs (H, J) and monotone interpolating homotopies $H_{k,s}, J_{k,s}$ belonging to such pairs.*

Proof. For small $r \geq 0$, abbreviate $M_r = M^{\text{in}} \cup (\Sigma \times [0, r]) \subset M$ and $M_r^{\text{out}} = M \setminus M_r$, so $M_0 = M^{\text{in}}$. We will use the regions

$$N \subset M_0 \subset M_\epsilon \subset M_{2\epsilon}$$

for a small $\epsilon > 0$. Fix a C^2 -small Morse function $H_0 : M \rightarrow \mathbb{R}$, such that the only 1-periodic orbits of X_{H_0} in M_0 are critical points of H_0 , and $H_0 = h_0(R) = h_0(e^r)$ is radial and strictly convex on M_0^{out} of slope $h'(R)$ less than all Reeb periods. By composing H_0 with an isotopy of M supported in M_0 , we may assume the critical points $\text{Crit}(H_0)$ lie in $M_0 \setminus N$. By construction, the zeros of F_{H_0} on $\mathcal{L}M$ are precisely $\text{Crit}(H_0)$. An Arzelà-Ascoli argument⁷ implies that for $x \in \mathcal{L}M_\epsilon$ the value $F_{H_0}(x)$ is small only if x is C^0 -close to $\text{Crit}(H_0)$. Thus there is a $\delta_0 > 0$ such that if $x \in \mathcal{L}M_\epsilon$ and $\|F_{H_0}(x)\| \leq \delta_0$ then $x(S^1) \cap N = \emptyset$. Define

$$c := h_0(e^\epsilon) - h_0(e^0) > 0, \quad C := \|H_0\|_{C^1(M_\epsilon)}, \quad \delta := \min\{\delta_0, \frac{c}{C}\}.$$

Let $H_k : M \rightarrow \mathbb{R}$ equal H_0 on M_ϵ , and let H_k be radial on M^{out} , with fixed slope m_k on $M_{2\epsilon}^{\text{out}}$ not equal to a Reeb period, such that the slopes m_k strictly increase to infinity as $k \rightarrow \infty$ and the linear interpolations $H_{k,s}$ from H_{k+1} to H_k are monotone: $\partial_s H_{k,s} \leq 0$. The J_k can be chosen to be small generic perturbations of a fixed J .

These functions satisfy (H2). To establish (H1), it suffices to show $\|F_H(x)\| > c/C$ for $x : S^1 \rightarrow M$ with $x(a) \in N$ and $x(b) \in M_\epsilon^{\text{out}}$, for some $a, b \in S^1 = \mathbb{R}/\mathbb{Z}$, where $H = H_k$ or $H = H_{k,s}$. We may assume $b > a$ in \mathbb{R} with $|b - a| \leq 1$. By shrinking $[a, b]$ we may assume $x([a, b]) \subset M_\epsilon$, so the path lies in the region where $H_k = H_{k,s} = H_0$. Abbreviate $H = H_0$, $F = F_H$, $X = X_H$ and the restriction $y = x|_{[a,b]}$. By Cauchy-Schwarz:

$$\|F(x)\| \sqrt{b-a} \geq \int_a^b |F(x)(t)| dt \geq \int_a^b \frac{|d_{x(t)}H|}{\|dH \circ y\|} |\partial_t x - X| dt \geq \frac{|\int_a^b \partial_t (H \circ x) dt|}{\|dH \circ y\|} \geq \frac{c}{C},$$

where we used that $dH(X) = 0$. As $|b - a| \leq 1$, we deduce $\|F_H(x)\| \geq c/C$. \square

Theorem 4.8 (Energy Estimate). *For β, N, δ, C as above,*

$$|\tau_\beta(u)| \leq \|\beta\|_{C^0(N)} \cdot (1 + \frac{C}{\delta}) \cdot E(u),$$

where u is any Floer trajectory or continuation solution for the data from Lemma 4.7.

⁷This relies on the Sobolev embedding $W^{1,2}(S^1, \mathbb{R}^m) \hookrightarrow C^0(S^1, \mathbb{R}^m)$ [58, Exercise 1.22].

Proof. Let u be a Floer trajectory for the given data (H, J) (the proof for continuation maps is analogous). Denote $u_s = u|_{\{s\} \times S^1}$ for $s \in \mathbb{R}$, then $u_s \in \mathcal{LM}_{2\epsilon}$ as the maximum principle applies on $M_{2\epsilon}^{\text{out}}$ by the construction of the data in Lemma 4.7. Let

$$\mathcal{S}_u := \left\{ s \in \mathbb{R} \mid u_s \notin \mathcal{L}(M_{2\epsilon} \setminus N) \right\} \subset \mathbb{R} \quad (4.15)$$

be the values $s \in \mathbb{R}$ for which $u(s, t) \in N$ for some $t \in S^1$. Using **(H1)** and the definition $E(u) = \int_{\mathbb{R}} \|F_H(u_s)\|^2 ds$, Chebyshev's inequality implies

$$|\mathcal{S}_u| \leq \frac{E(u)}{\delta^2}, \quad (4.16)$$

where $|\mathcal{S}_u|$ is the Lebesgue measure of \mathcal{S}_u . We note that

$$\forall (s, t) \in \mathbb{R} \times S^1, \quad \beta|_{u(s, t)} \neq 0 \implies s \in \mathcal{S}_u, \quad u(s, t) \in N \text{ and } X_H|_{u(s, t)} = X_{H_0}|_{u(s, t)},$$

since $H = H_0$ on M_ϵ by construction. Using (4.16) and Cauchy-Schwarz:

$$\begin{aligned} |\tau_\beta(u)| &= \left| \int_{\mathbb{R} \times S^1} u^* \beta \right| = \left| \int_{\mathcal{S}_u \times S^1} \beta(\partial_s u, J \partial_s u + X_H) ds dt \right| \\ &\leq \|\beta\| \left(\int_{\mathcal{S}_u \times S^1} |\partial_s u|^2 ds dt + \|dH_0\|_{C^1(N)} \int_{\mathcal{S}_u \times S^1} |\partial_s u| ds dt \right) \\ &\leq \|\beta\| \left(E(u) + C \sqrt{E(u)} \cdot \sqrt{|\mathcal{S}_u|} \right) \\ &\leq \|\beta\| \left(1 + \frac{C}{\delta} \right) E(u). \quad \square \end{aligned}$$

Corollary 4.9. *If a class in $H_c^2(M) \cong H^2(M, M^{\text{out}})$ has a representative β compactly supported in $N \subset \text{int}(M^{\text{in}})$ with*

$$\|\beta\|_{C^0(N)} < (1 + \frac{C}{\delta})^{-1},$$

then twisted symplectic cohomology $SH^(M, \omega, \theta)_{\tau(\beta)} = \varinjlim HF^*(H_k, J_k)$ is well-defined using the data from Lemma 4.7. So for any class $\beta \in H_c^2(M) \cong H^2(M, M^{\text{out}})$, the group $SH^*(M, \omega, \theta)_{\tau(s\beta)}$ is defined for all sufficiently small $s \geq 0$.*

Proof. Let $c := \|\beta\|(1 + \frac{C}{\delta}) < 1$. By Theorem 4.8, the claim follows by Lemma 4.3. \square

Remark 4.10. *It is not yet clear whether $SH^*(M, \omega, \theta)_{\tau(\beta)}$ is independent of the chosen cofinal family H_k , as we cannot a priori control δ for general monotone homotopies between two given cofinal families. Nevertheless independence on this choice follows a posteriori from the isomorphism with $SH^*(M, \omega + \beta, \theta)$ in Theorem 2.7.(2).*

As part of the direct limit, we have the canonical Λ -linear homomorphism (1.5) since $HF^*(H_0, J_0; \omega, \theta) \cong QH^*(M, \omega)$ (as a vector space, $QH^*(M, \omega) = H^*(M) \otimes \Lambda$).

4.7. Product structure. To conclude the proof of Theorem 2.7.(1) we need to explain why $SH^*(M, \omega, \theta)_{\tau(\beta)}$ admits a unital ring structure given by the pair-of-pants product, under the assumptions in Corollary 4.9. We refer to [53] for the detailed construction of the product. This uses an auxiliary 1-form γ defined on the pair-of-pants P , satisfying $d\gamma \leq 0$ (this ensures the maximum principle), and γ is equal to a positive constant multiple of dt near each end. We may choose dt at the two positive ends, and $2dt$ at the negative end. The product involves the moduli space

$\mathcal{M}(x; y, z; H, J)$ of rigid solutions $u : P \rightarrow M$ of the equation $(du - X \otimes \gamma)^{0,1} = 0$ asymptotic to x at the negative end and y, z at the positive ends, where $X = X_H$. We obtain a Λ -linear map $HF^*(H) \otimes HF^*(H) \rightarrow HF^*(2H)$ which on 1-orbits is:

$$y \otimes z \quad \longmapsto \quad \sum_{u \in \mathcal{M}(x; y, z; H, J, \gamma)} t^{\tau_\omega(u)} t^{2\mathcal{H}(x) - \mathcal{H}(y) - \mathcal{H}(z)} x. \quad (4.17)$$

Here we abuse notation slightly, $\tau_\omega(u) = \int_P u^* \omega = \tau_\omega(P_-) + \tau_\omega(P_{+,1}) + \tau_\omega(P_{+,2})$ where we decompose $P = P_- \cup P_{+,1} \cup P_{+,2}$ as the union of three cylinders (whose images via u yield three 1-chains in \mathcal{LM}) asymptotic to the three ends, such that the positive boundary of P_- is the figure eight-loop consisting of the two negative boundaries of $P_{+,1}, P_{+,2}$. The choice of decomposition will not affect the weights (here it is crucial that we are twisting by a class in $H^1(\mathcal{LM})$ that arises as the transgression of a class in $H^2(M)$). The exponent of t in (4.17) is precisely the *topological energy*

$$E_{top}(u) := \int_P u^* \omega - d(u^* H \wedge \gamma),$$

which is a homotopy invariant that bounds from above the (geometric) energy

$$E(u) := \frac{1}{2} \int_P \|du - X \otimes \gamma\|^2 \text{vol}_P = \int_P u^* \omega - d(u^* H) \wedge \gamma,$$

since $d\gamma \leq 0$. Explicitly: $E(u) \leq \tau_\omega(u) + 2\mathcal{H}(x) - \mathcal{H}(y) - \mathcal{H}(z)$. This ensures that the above map is well-defined. By considering continuation maps as in [53] a direct limit of these maps defines a Λ -bilinear homomorphism $SH^*(M; \omega, \theta)^{\otimes 2} \rightarrow SH^*(M; \omega, \theta)$ called the pair-of-pants product. The same argument as in [53] shows that the element $c^*(1) \in SH^*(M; \omega, \theta)$ is a unit, so the map in (1.5) is a unital Λ -algebra homomorphism, using the quantum product on $QH^*(M, \omega)$.

Fix a compact subregion $P' \subset P$ independent of u such that $P \setminus P'$ is the disjoint union of the three cylindrical ends. Abbreviate $A := \text{Area}(P')$. We claim that

$$E(u) \geq A\delta^2 \quad \implies \quad |\tau_\beta(u)| \leq \|\beta\|_{C^0(N)} C' E(u),$$

for some constant $C' > 0$ independent of u . Let u_1, u_2, u_3 be the restriction of u to the three cylindrical ends with corresponding sets $\mathcal{S}_{u_1}, \mathcal{S}_{u_2}, \mathcal{S}_{u_3}$, as in (4.15). Let

$$P_u := P' \cup (\mathcal{S}_{u_1} \times S^1) \cup (\mathcal{S}_{u_2} \times S^1) \cup (\mathcal{S}_{u_3} \times S^1).$$

Assuming $E(u) \geq A\delta^2$, we obtain the following generalisation of (4.16):

$$\text{Area}(P_u) \leq A + \frac{1}{\delta^2} E(u) \leq \frac{2}{\delta^2} E(u).$$

We estimate $-\tau_\beta(u)$ from above as in Theorem 4.8 substituting $\mathcal{S}_u \times S^1$ with P_u :

$$\begin{aligned} \left| \int_{P_u} u^* \beta \right| &= \left| \int_{P_u} \beta \circ (du - X \otimes \gamma)^{\otimes 2} + \beta(du - X \otimes \gamma, X) \gamma + \beta(X, du - X \otimes \gamma) \gamma \right| \\ &\leq \|\beta\|_{C^0(N)} \left(E(u) + 2C \|\gamma\|_{C^0(P)} \sqrt{E(u)} \cdot \frac{1}{\delta} \sqrt{2E(u)} \right), \end{aligned}$$

where we used that $\beta(X, X) = 0$. This proves the claim. Corollary 4.9 together with the construction of the product conclude the proof of Theorem 2.7.(1). \square

5. INVARIANCE OF SYMPLECTIC COHOMOLOGY

5.1. Small perturbations: proof of Theorem 2.7.(2). To simplify notation, we may assume that $M = M_a = D \setminus \partial D$, where D is some convex domain with a collar C and $M^{\text{in}} = \overline{D} \setminus \overline{C}$. By Proposition 3.14, for $(\mu, \lambda) \in \Omega^2(D, C)$ which is closed and C^1 -small, we obtain a convex domain $(D, \omega + \mu, \theta + \lambda)$ and a closed form $\beta \in \Omega_c^2(M)$ (in a class corresponding to $[\mu, \lambda]$) with $\|\beta\|_{C^0(M)} \leq K\|(\mu, \lambda)\|_{C^1(D, C)}$, and admitting an isomorphism

$$(M, \omega + \beta, \theta)^\wedge \xrightarrow{\cong} (M, \omega + \mu, \theta + \lambda)^\wedge.$$

By Theorem 1.1,

$$SH^*(M, \omega + \beta, \theta) \cong SH^*(M, \omega + \mu, \theta + \lambda).$$

To prove Theorem 2.7.(2) it remains to show $SH^*(M, \omega + \beta, \theta) \cong SH^*(M, \omega, \theta)_{\tau(\beta)}$, for $\|\beta\|_{C^0(M)}$ small enough, where the right-hand side is defined by Theorem 2.7.(1). The twisting is needed so that the groups on both sides have the same system of local coefficients. To build the isomorphism, it suffices to construct a sequence of commutative diagrams for the twisted Floer cohomologies of M :

$$\begin{array}{ccc} HF^*(\omega, \theta; H_{k+1}, J_{0,k+1})_{\tau(\beta)} & \xleftarrow{\psi_{k+1}} & HF^*(\omega + \beta, \theta; H_{k+1}, J_{1,k+1}) \\ \varphi_{0,k} \uparrow & & \uparrow \varphi_{1,k} \\ HF^*(\omega, \theta; H_k, J_{0,k})_{\tau(\beta)} & \xleftarrow{\psi_k} & HF^*(\omega + \beta, \theta; H_k, J_{1,k}) \end{array} \quad (5.1)$$

where $(H_k, J_{0,k})$ and the continuation maps $\varphi_{0,k}$ are defined by data as in Lemma 4.7, whilst $(H_k, J_{1,k})$ and the continuation maps $\varphi_{1,k}$ are defined by data used in the construction of $SH^*(M, \omega + \beta, \theta)$. Thus, the direct limits over the vertical maps respectively define $SH^*(M, \omega, \theta)_{\tau(\beta)}$ and $SH^*(M, \omega + \beta, \theta)$. We now construct the horizontal maps ψ_k . To simplify the notation, we will drop all subscripts k . The map ψ is a continuation map, where for $s \in \mathbb{R}$ we vary the pair

$$(\omega_s := \omega + \rho(s)\beta, J_s)$$

but keep the Hamiltonian fixed. Here $\rho : \mathbb{R} \rightarrow [0, 1]$ is a function and J_s is an ω_s -compatible almost complex structure of contact type at infinity, satisfying $\rho_s = 0$ and $J_s = J_0$ for $s \leq 0$; and $\rho_s = 1$, $J_s = J_1$ for $s \geq 1$.

Let $\mathcal{M}(x, y; \omega_s, H, J_s)$ be the set of rigid solutions $u : \mathbb{R} \times S^1 \rightarrow M$ from x to y of the equation $F_H^s(u) = J_s \partial_s u$, where $F_H^s(x) := \partial_t x - X_s$ and X_s is the Hamiltonian vector field of H with respect to ω_s , so $\omega_s(\cdot, X_s) = dH$. Transversality is standard, since we allow J_s to vary and we assumed (M, ω) to be weakly monotone. At the chain level, $\psi : CF^*(\omega + \beta, H, J_1) \rightarrow CF^*(\omega, H, J_0)_{\tau(\beta)}$ is defined on 1-orbits as follows:

$$\psi(y) = \sum_{u \in \mathcal{M}(x, y; \omega_s, H, J_s)} \epsilon(u) t^{(\tau_\omega + \tau_\beta)(u)} t^{\mathcal{H}(x) - \mathcal{H}(y)} x. \quad (5.2)$$

Provided the ψ maps are well-defined, standard Floer theory arguments imply:

- (1) Diagram (5.1) commutes at the chain level up to chain homotopy.
- (2) The ψ maps are isomorphisms, their inverse being the continuation maps $\bar{\psi}$ obtained from the reverse deformation $(\bar{\omega}_s, \bar{J}_s) := (\omega_{-s}, J_{-s})$. Indeed, $\bar{\psi}\psi$ and $\psi\bar{\psi}$ correspond (up to chain homotopy) to a deformation where the symplectic

form and the almost complex structure are both held constant, and thus the map is the identity for dimension reasons (moduli spaces are never rigid, due to an s -translation symmetry, unless Floer continuation solutions are constant).

- (3) The isomorphism $\psi_\infty = \varinjlim \psi_k : SH^*(M, \omega, \theta)_{\tau(\beta)} \rightarrow SH^*(M, \omega + \beta, \theta)$ is compatible with the unital product structure (i.e. the ψ_k maps fit into commutative diagrams similar to those used in [53] to construct the product). This requires an energy estimate for pairs-of-pants, but just as in Section 4.7 this estimate will follow once one has the energy estimate for Floer cylinders (Theorem 5.1). This yields Theorem 2.7.(2).

We now prove that ψ is well-defined, if $\|\beta\|_{C^0(N)}$ is small. As ω_s only varies on M^{in} , we can keep J_s independent of s on M^{out} and the maximum principle applies on M^{out} . So we only need to bound the energy $E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u|_s^2 ds dt$, where $|\cdot|_s$ is the norm associated to the Riemannian metric $\omega_s(\cdot, J_s \cdot)$. Thus

$$E(u) = \int_{\mathbb{R} \times S^1} \omega_s(\partial_s u, \partial_t u - X_s) ds dt = (\tau_\omega + \tau_\beta)(u) + \int_{\mathbb{R} \times S^1} (\rho(s) - 1) u^* \beta + \mathcal{H}(x) - \mathcal{H}(y).$$

By Lemma 4.3, the following theorem implies that ψ is well-defined.

Theorem 5.1. *For δ, C as above, there is a constant $c'' > 0$ such that for any sufficiently small closed 2-form β compactly supported on M ,*

$$\left| \int_{\mathbb{R} \times S^1} (\rho(s) - 1) u^* \beta \right| \leq \int_{\mathbb{R} \times S^1} |u^* \beta| \leq \|\beta\|_{C^0(M)} \cdot c'' \left(1 + \frac{C}{\delta}\right) \cdot E(u),$$

for all $u \in \mathcal{M}(x, y; \omega_s, H, J_s)$.

Proof. To bound $\int |u^* \beta|$, we argue as in the proof of Theorem 4.8 except we now work with norms $|\cdot|_s$ depending on s . But since ω and ω_s differ only on a compact set, these norms are equivalent. So there is a constant $c'' > 0$ such that $\frac{1}{c''} |\cdot|_s \leq |\cdot| \leq c'' |\cdot|_s$ for all $s \in \mathbb{R}$. With this observation, the argument in Theorem 4.8 goes through. \square

Remark 5.2 (Technical Remark about Gromov Compactness). *The energy estimate of Theorem 5.1 is sufficient for the standard arguments of Gromov compactness to go through [37, Thm 3.3]. Indeed, the standard removal of singularities argument (e.g. see McDuff-Salamon [45]) involves considering bubbling that occurs at a specific value of s when energy concentrates, and that argument applies in our setup because our form ω_s is closed. Moreover, in our argument we need a uniform \hbar -bound (the minimal energy represented by a non-constant J -holomorphic sphere) that works for ω_s for all $s \in \mathbb{R}$, which is crucial for Hofer-Salamon's argument [37, Theorem 3.3] to apply. A clean approach would be to separately show that \hbar varies continuously in the metric $\omega_s(\cdot, J_s \cdot)$. A simpler but weaker argument goes as follows. We need to rule out the possibility of the vanishing of the infimum of \hbar_s (the optimal \hbar -value for J_s), taking the infimum over the compact interval C of values of s for which ω_s is s -dependent. If this infimum were zero, it would imply the existence of a sequence u_n of non-constant J_{s_n} -holomorphic spheres such that the energy $E(u_n)$ converges to zero. By passing to a subsequence we may assume s_n converges to some value $s^* \in C$. The usual Gromov compactness argument then says that u_n will converge to a J_{s^*} -holomorphic sphere, possibly with a bunch of bubbles, if the energy concentrates at*

certain points. Part of the proof is that the energy of this limit curve is the limit of the energies $E(u_n)$, if one remembers to take into account all the bubbles arising in the limit curve. In our case, this would imply that the limit curve has zero energy, so the limit is a point with no bubbles. By continuity this would imply that the u_n eventually lie inside a contractible neighbourhood of that point, and therefore these spheres u_n are homologically trivial, which in turn implies that their energy is zero, and thus the u_n are constant for large n . Contradiction.

5.2. Long deformations: proof of Corollary 1.6. Let $(M, \omega_s, \theta_s; \zeta_s)$ be a transgression invariant family of convex manifolds (Definition 2.8). Thus,

$$\zeta_s - \zeta_{s'} = \tau(\omega_{s'}) - \tau(\omega_s) \in H^1(\mathcal{L}M), \quad \forall s, s' \in [0, 1].$$

By Theorem 2.7.(2), given any $s \in [0, 1]$, there is a relatively open interval I_s such that $s \in I_s \subset [0, 1]$ and for all $s' \in I_s$ we have

$$SH^*(M, \omega_s, \theta_s)_{\zeta_s - \zeta_{s'}} \cong SH^*(M, \omega_s, \theta_s)_{\tau(\omega'_s - \omega_s)} \cong SH^*(M, \omega_{s'}, \theta_{s'}).$$

By Lemma 4.4, we can twist the above isomorphism by $\zeta_{s'}$:

$$SH^*(M, \omega_s, \theta_s)_{\zeta_s} \cong SH^*(M, \omega_{s'}, \theta_{s'})_{\zeta_{s'}}, \quad \forall s' \in I_s. \quad (5.3)$$

As $[0, 1]$ is compact, there is a sequence $0 = s_0, \dots, s_m = 1$ with $s_{i+1} \in I_{s_i}$. Corollary 1.6 follows by composing the isomorphisms in (5.3) for $s = s_i$ and $s' = s_{i+1}$. \square

6. TWISTED COTANGENT BUNDLES OF SURFACES

6.1. Basic notation. We review some background in the following two sections, but for the sake of brevity we refer the reader to [20, 27, 11] for a more extensive survey and for references on the topic of twisted cotangent bundles.

Let (N, g) be a closed Riemannian manifold. Let $\pi : T^*N \rightarrow N$ be the footpoint projection and let $\theta = p dq$ be the canonical 1-form (so $\theta_{(q,p)} = p \circ d\pi$). We identify T^*N and TN via the musical isomorphism

$$\flat : T_q N \rightarrow T_q^* N, \quad v \mapsto p = g_q(v, \cdot).$$

For example, we have $\theta_{(q,v)} = g_q(v, d\pi \cdot)$. Write g also for the dual metric on T^*N and denote all norms by $|\cdot|$. The disc and sphere bundle of radius r are

$$D_r^g = \{(q, v) \in TN : |v| \leq r\} \quad \Sigma_r^g = \partial D_r^g = \{(q, v) \in TN : |v| = r\}.$$

The use of the letter r here is for notational convenience and is not to be understood as a radial coordinate in a conical parametrization of a convex manifold as in (2.1).

The connection determines a splitting:

$$T_{(q,p)}(TN) \cong T_q N \oplus T_q N, \quad \partial_t(q, v) \mapsto (\partial_t q, \nabla_t v) \quad (6.1)$$

so that the first component is the map $\xi \mapsto d\pi \cdot \xi$ and ∇ denotes the Levi-Civita connection. Via (6.1), on TN we get a Riemannian metric $g \oplus g$ and an almost complex structure J_g compatible with the symplectic form $d\theta$, where

$$J_g = \begin{pmatrix} 0 & -\text{id} \\ \text{id} & 0 \end{pmatrix}, \quad d\theta = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}. \quad (6.2)$$

Applying the inverse of the map in (6.1) to $T_q N \oplus 0$ and $0 \oplus T_q N$, we get the horizontal distribution $T_{(q,v)}^{\text{hor}} TN$ and the vertical distribution $T_{(q,v)}^{\text{vert}} TN$ on $T(TN)$:

$$w \mapsto w^h \in T_{(q,v)}^{\text{hor}} TN, \quad w \mapsto w^\nu \in T_{(q,v)}^{\text{vert}} TN, \quad \forall q \in N, \quad v, w \in T_q N.$$

In particular, $T^{\text{vert}} TN = \ker d\pi$. The tautological horizontal and vertical lifts yield two vector fields on TN :

$$X_{(q,v)} = v^h \quad Y_{(q,v)} = v^\nu.$$

We recall that X is the geodesic vector field of g , namely the Hamiltonian vector field for $(q, v) \mapsto \frac{1}{2}|v|^2$ using $d\theta$, and Y is the Liouville vector field for $(d\theta, \theta)$, so $Y = v \partial_v$ in local coordinates. We will later use that for 1-forms β on N , $\pi^* \beta(X)_{(q,v)} = \beta(v)$.

6.2. Twisted cotangent bundles. Let (N, g) be a closed Riemannian manifold of dimension $n > 1$. Let $\sigma \in \Omega^2(N)$ be a closed 2-form, called *magnetic form*. The Lorentz force $\mathbb{Y} : TN \rightarrow TN$ is the bundle map determined by

$$g_q(\mathbb{Y}_q(u), v) = \sigma_q(u, v), \quad \forall q \in N, \quad u, v \in T_q N.$$

A smooth curve $\gamma : I \rightarrow N$ is a *magnetic geodesic*, if it satisfies

$$\nabla_{\gamma'} \gamma' = \mathbb{Y}_\gamma(\gamma'), \quad (6.3)$$

where ∇ is the Levi-Civita connection for g . From the equation, it follows that γ has constant speed $r := |\gamma'|$. If we reparametrise γ by arc-length and denote by $\dot{\gamma}$ the derivative of γ with respect to this new parameter, (6.3) becomes

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \frac{1}{r} \mathbb{Y}_\gamma(\dot{\gamma}). \quad (6.4)$$

Closed magnetic geodesics $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow N$ with speed r are exactly the critical points of the possibly multi-valued free-period action functional \mathbb{S}_r defined on the space of all free loops of any period:

$$\mathbb{S}_r(\gamma) = \int_0^T \frac{1}{2} (|\gamma'(t)|^2 + r^2) dt - \int_{[0,1] \times \mathbb{R}/T\mathbb{Z}} \hat{\gamma}^* \sigma,$$

where $\hat{\gamma} : [0, 1] \times \mathbb{R}/T\mathbb{Z}$ is a connecting cylinder to a fixed reference loop in the free-homotopy class of γ .

Consider the twisted tangent bundle (TN, ω) , where

$$\omega := d\theta - \pi^* \sigma, \quad (6.5)$$

which is weakly monotone as $c_1(TN, \omega) = 0$. We now interpret magnetic geodesics as flow lines (up to reparametrization) for the Hamiltonian given by

$$\rho : TN \rightarrow \mathbb{R}, \quad \rho(q, v) := |v|. \quad (6.6)$$

Let W be the vertical vector field determined by the Lorentz force:

$$W_{(q,v)} = (\mathbb{Y}(v))^\nu, \quad \forall (q, v) \in TN.$$

Lemma 6.1. *The Hamiltonian vector field of ρ with respect to ω is*

$$X_\rho = \frac{1}{\rho}(X + W). \quad (6.7)$$

Its flow lines in Σ_r^g are the curves $(\gamma, r\dot{\gamma})$, where γ is any solution of (6.4) parametrised by arc-length, and these are integral curves for the distribution $\ker \omega|_{\Sigma_r}$.

Proof. As $d\theta(\cdot, X) = d(\frac{1}{2}\rho^2) = \rho d\rho$ and $\pi^*\sigma(\cdot, W) = 0$ (as W is vertical), (6.7) is equivalent to $d\theta(\cdot, W) = \pi^*\sigma(\cdot, X)$. Using (6.2), $d\theta(T^{\text{vert}}TN, W) = 0$ since W is also vertical and, for $w^h \in T_q^{\text{hor}}N$, we deduce the required equality:

$$d\theta(w^h, W) = -g(w, \mathbb{Y}(v)) = -\sigma(v, w) = \pi^*\sigma(w^h, X).$$

Abbreviate $x = (\gamma(s), r\dot{\gamma}(s)) \in TN$. Using (6.4), $\frac{1}{r}W_x = \frac{1}{r}\mathbb{Y}(r\dot{\gamma})^\nu = r(\nabla_{\dot{\gamma}}\dot{\gamma})^\nu$. Thus, $\partial_s(\gamma, r\dot{\gamma}) = \dot{\gamma}^h + r(\nabla_{\dot{\gamma}}\dot{\gamma})^\nu = \frac{1}{r}X_x + \frac{1}{r}W_x$. The final claim is immediate since $\omega(\cdot, X_\rho) = d\rho = 0$ as ρ is constant on Σ_ρ . \square

Let us now assume that N is an oriented surface. Let $j : TN \rightarrow TN$ be fibrewise rotation by $\frac{\pi}{2}$, and μ the Riemannian area form. Then (X, Y, H, V) is a positively oriented frame with respect to $d\theta \wedge d\theta$, where

$$H_{(q,v)} = (jv)^h, \quad V_{(q,v)} = (jv)^\nu.$$

Here V is generated by the fibrewise rotation $e^{it} : (q, v) \mapsto (q, e^{it}v)$. Following [33], one verifies the Lie bracket relations

$$\begin{aligned} [Y, X] &= X, & [Y, H] &= H, & [Y, V] &= 0, \\ [V, X] &= H, & [H, V] &= X, & [X, H] &= \rho^2 K V, \end{aligned} \quad (6.8)$$

where $K : M \rightarrow \mathbb{R}$ is the Gaussian curvature for g . The linear algebra dual coframe is $(\frac{1}{\rho^2}\theta, \frac{1}{\rho}d\rho, \frac{1}{\rho^2}\eta, \tau)$, where $\eta = j^*\theta$, and $\tau = \frac{1}{\rho^2}g((\cdot)^\nu, jv)$ is an S^1 -connection form on every Σ_r^g with curvature $K\mu$:

$$\tau(V) = 1, \quad d\tau = -K\pi^*\mu.$$

The Lorentz force has the expression $\mathbb{Y}(v) = fjv$, where $f : N \rightarrow \mathbb{R}$ is the unique function satisfying $\sigma = f\mu$. Then, $W = (f \circ \pi)V$ and (6.4) becomes

$$\kappa_\gamma = \frac{1}{r}f(\gamma), \quad (6.9)$$

where κ_γ is the geodesic curvature of γ , as follows from the identity $\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa_\gamma j\dot{\gamma}$.

6.3. Convexity for twisted cotangent bundles. We now investigate when (D_r^g, ω) , for ω as in (6.5), has boundary of positive contact-type, i.e. there is a positive contact form $\alpha_r \in \Omega^1(\Sigma_r^g)$ with $d\alpha_r = \omega|_{\Sigma_r^g}$. To this purpose, we recall the Gysin sequence

$$H^1(N) \longrightarrow H^1(\Sigma_r^g) \longrightarrow H^{2-n}(N; o(TN)) \xrightarrow{\wedge e} H^2(N) \quad (6.10)$$

where $e \in H^n(N; o(TN))$ is the Euler class of N and $o(TN)$ is the orientation line bundle. The last map is conjugated via the Thom isomorphism to the map

$$H^2(D_r^g, \Sigma_r^g) \rightarrow H^2(D_r^g), \quad [x, y] \mapsto [x]. \quad (6.11)$$

in the long exact sequence of the pair (D_r^g, Σ_r^g) .

We first investigate when (D_r^g, ω) can be a Liouville domain. By Lemma 3.1 this is equivalent to requiring that $[\omega, \alpha_r] = 0 \in H^2(D_r^g, \Sigma_r^g)$. Thus, the map (6.11) shows

that a necessary condition is to have $[\omega] = 0$, which is equivalent to asking that σ is an exact form. In this case, we define

$$r_0 := \inf_{d\beta=\sigma} \|\beta\|, \quad \|\beta\| := \max_{q \in N} |\beta_q|, \quad (6.12)$$

where we run over all primitives β for σ . Note that r_0 depends only on g, σ . For $r > r_0$ and any dimension $n = \dim N$, D_r^g admits the deformation through the Liouville domains $(D_r^g, d\theta - s\pi^*\sigma, (\theta - s\pi^*\beta)|_{\Sigma_r^g})$, where β is a primitive of σ such that $\|\beta\| < r$. For $s = 0$ we get the standard domain $(D_r^g, d\theta, \theta|_{\Sigma_r^g})$, and thus $SH^*(D_r^g, \omega) \cong SH^*(D_r^g, d\theta)$ (thus, it recovers the ordinary homology of the free loop space of N). For $r \leq r_0$, we have two cases according to the dimension of N :

- If $\dim N \geq 3$ determining if D_r^g is Liouville for $r \leq r_0$ is a hard question. We know that D_r^g is not Liouville for $r \in (r_u, r_0]$ where $\frac{1}{2}r_u^2$ is the Mañé critical value of the universal cover of N [18], due to the existence of null-homologous magnetic geodesics with negative action [19, Appendix B, Theorem B.1].
- If $\dim N = 2$, D_r^g is not a Liouville domain for $r \leq r_0$, thanks to [20].

Next, we assume that (D_r^g, ω) is a non-Liouville convex domain for some contact form α on Σ_r^g . This means that $[\omega, \alpha] \neq 0$. We have two cases:

- If $\dim N \geq 3$, we claim that this can happen only if σ is exact and hence (D_r^g, ω) would be Quasi-Liouville. To prove the claim, notice that if $n = \dim N \geq 3$, then the last map in (6.10) vanishes. Hence, the map in (6.11) vanishes as well. This implies that if (D_r^g, ω, α) is convex, then $[\omega, \alpha] \in H^2(D_r^g, \Sigma_r^g)$ is sent to $[\omega] = 0$, which is equivalent to σ being exact. All known examples of Quasi-Liouville disc bundles for $\dim N \geq 3$ have $r > r_0$ and contact form at the boundary given by $\alpha := (\theta - s\pi^*\beta)|_{\Sigma_r^g} + \epsilon\eta$ with $\|\beta\| < r_0$, η a closed 1-form on Σ_r^g such that $[\eta]$ is not in the image of the map $H^1(N) \cong H^1(D_r^g) \rightarrow H^1(\Sigma_r^g)$ (this ensures that $[\omega, \alpha] \neq 0$) and ϵ small enough. In particular, sending ϵ to 0, we see that these are deformations of the Liouville domains described above. It is an open problem to find examples such that (D_r^g, ω) is Quasi-Liouville for some $r \in (r_u, r_0]$.
- The situation looks more promising when $\dim N = 2$. We have two sub-cases:
 - Assume in addition that (D_r^g, ω) is a Quasi-Liouville manifold (see Example 2.5). This means that $[\omega] = 0$ (namely σ is exact). This can happen only if the map in (6.11) and hence the last map in (6.10) is not injective, since $[\omega, \alpha] \neq 0$. In this case N has to be the two-torus (recall that $H^0(N; o(TN)) = 0$ if N is not orientable and that for orientable surfaces different from \mathbb{T}^2 the Euler class does not vanish). Contreras, Macarini and Paternain gave examples in [20] of a pair (g, σ) on the two-torus, discussed in the next subsection, for which $D_{r_0}^g$ is Quasi-Liouville.
 - Finally, consider the case in which (D_r^g, ω) is not Quasi-Liouville, meaning that ω , or equivalently σ , is not exact. This forces N to be an orientable surface. Moreover, N is different from the two-torus since on the two-torus the map in (6.11) vanishes as it is represented by multiplication by the Euler class in (6.10). This yields $[\omega, \alpha] \mapsto [\omega] = 0$ contrary to what we assumed. On the other hand, on orientable surfaces different from the

two-torus there are examples of convex domains, which are not Liouville. These are discussed in Section 6.5.

We now simplify the notation: we will use the dilation $\delta_r(q, v) := (q, rv)$ to bring (D_r^g, Σ_r^g) to $(D^g, \Sigma^g) := (D_1^g, \Sigma_1^g)$. The pull-back symplectic form on D^g is

$$\delta_r^* \omega = r\omega_s, \quad \omega_s := d\theta - s\pi^* \sigma, \quad s := 1/r.$$

Therefore, we will consider below the symplectic manifold with boundary (D^g, ω_s) (as $r\omega_s$ and ω_s have the same Hamiltonian vector fields up to reparametrisation and the same almost complex structures). From Lemma 6.1, we get

$$X_\rho = X + sW \quad \text{on } \Sigma^g.$$

Its flow lines $(\gamma, \dot{\gamma})$ can either be interpreted as magnetic geodesics of $(g, s\sigma)$ with speed 1 via (6.3) or as magnetic geodesics of (g, σ) with speed $\rho = 1/s$ via (6.4).

6.4. Quasi-Liouville examples using \mathbb{T}^2 : proof of Theorem 1.9. We will construct an exact σ such that ω_s has a primitive α_s on Σ and $(D^g, \omega_s, \alpha_s)$ is a Quasi-Liouville domain. This is an explicit construction of the contact form for the kind of systems considered in Contreras-Macarini-Paternain [20, Section 5.1]. We first build an angular form $\psi \in T^*(T\mathbb{T}^2 \setminus \mathbb{T}^2)$ as follows. Pick a global non-vanishing section u of $\Sigma^g \rightarrow \mathbb{T}^2$. Then let $\varphi(q, v)$ denote the angle between $v \neq 0 \in T_q \mathbb{T}^2$ and $u(q)$. Finally, set $\psi := d\varphi$. One can check that $\psi(V) = 1$. Explicitly, using properties of the Levi-Civita connection, one can verify [11, Lemma 2.4] that $\psi(X)_{(q,v)} = -\nu_q(v)$ where $\nu \in \Omega^1(\mathbb{T}^2)$ is the curvature of the section u ,

$$\nu_q(v) = g_q(\nabla_v u, Ju).$$

We now construct σ . First, we fix a simple contractible loop of period T ,

$$\delta : [0, T] \rightarrow \mathbb{T}^2, \tag{6.13}$$

where δ is parametrised by arc-length. Suppose that its geodesic curvature satisfies

$$\kappa_\delta - |\nu_\delta| > \varepsilon, \tag{6.14}$$

for some $\varepsilon > 0$. Then, choose a vector field B on \mathbb{T}^2 such that

- (i) δ is an integral curve for B ;
- (ii) $|B_q| \leq 1$ for all $q \in \mathbb{T}^2$, with equality precisely on the image of δ .

From this vector field B on \mathbb{T}^2 we obtain a one-form

$$\beta = \flat B.$$

Finally, define

$$\sigma = d\beta.$$

We will call the magnetic system $(D^g, \omega_s, \alpha_s)$ on \mathbb{T}^2 a *QL-magnetic-torus*.

In this case, the free-period action functional \mathbb{S}_r is obtained by integrating the Lagrangian function $L + \frac{1}{2}r^2$, where

$$L(q, v) = \frac{1}{2}|v|^2 - \beta(v) = \frac{1}{2}|v - B|^2 - \frac{1}{2}|B|^2.$$

It follows that $L + \frac{1}{2} \geq 0$ with equality exactly for $(q, v) = (\delta, \dot{\delta})$. Therefore, δ and its iterates represent the set of global minimizers for \mathbb{S}_1 on the set of contractible closed curves. In particular, δ is a closed magnetic geodesic with speed 1.

Lemma 6.2. *The value r_0 defined in (6.12) equals 1 for (g, σ) as above.*

Proof. We have $r_0 \leq \max |\beta| = 1$. Let β' be any primitive of σ , and $\hat{\delta}$ any disc bounding δ . Then, since $\beta(\hat{\delta}) = \beta(B) = |B|^2 = 1$,

$$T = \int_{\delta} \beta = \int_{\hat{\delta}} \sigma = \int_{\hat{\delta}} \beta' \leq T \|\beta'\| \|\hat{\delta}\| = T \|\beta'\|. \quad \square$$

Define $\alpha_{s,a} \in \Omega^1(\Sigma^g)$ by

$$\alpha_{s,a} = (\theta - s\pi^*\beta)|_{\Sigma^g} + a\psi.$$

The relative class $[\omega_s, \alpha_{s,a}] \in H^2(D^g, \Sigma^g)$ is non-trivial for $a \neq 0$, as the form $a\psi$ is a non-exact closed 1-form on $\Sigma^g \cong \mathbb{T}^3$ which does not extend to D^g .

Remark 6.3. *In the exact setup, it is possible to study geodesics in a closed manifold N by applying Morse theory for appropriate Lagrangian functionals L to the free loop space $\mathcal{L}N$, see [1]. This can also be carried out replacing θ by $\theta - \pi^*\beta$ if it is a contact form for the sphere bundle (one then changes L to $L - b$, where $b(q, v) = \beta_q(v)$). However, this fails to be a contact form in the case $N = \mathbb{T}^2$, and that trick does not apply to $\alpha_{s,a}$ because the non-trivial $a\psi$ term cannot be reabsorbed into L .*

Theorem 6.4. *The set $A = \{(s, a) \in [0, \infty) \times [0, \infty) : \alpha_{s,a}(X + sW) > 0\}$ is an open set such that $[0, 1) \times \{0\} \subset A$ and the connected component A_* of $(0, 0)$ contains a non-empty interval $\{1\} \times (0, a_0)$. For any $(s, a) \in A_*$ in this connected component, $(D^g, \omega_s, \alpha_s := \alpha_{s,a})$ is a convex domain which can be deformed to the standard $(D^g, d\theta, \theta|_{\Sigma})$ and*

$$SH_c^*(D^g, \omega_s, \alpha_s) \cong H_{2-*}(\mathcal{L}_c \mathbb{T}^2) \cong H_{2-*}(\mathbb{T}^2),$$

where c is any free homotopy class of loops in \mathbb{T}^2 and the latter isomorphism uses the homotopy equivalence $\mathcal{L}_c \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $\gamma \mapsto \gamma(0)$ (whose fibres $\Omega_c \mathbb{T}^2$ are contractible).

Proof. We compute

$$\begin{aligned} \alpha_{s,a}(X + sW) &= 1 - s\beta_q(v) + a(sf(q) - \nu_q(v)) \\ &\geq 1 - s|\beta_q| + a(sf(q) - |\nu_q|). \end{aligned} \quad (6.15)$$

Thus $[0, 1) \times \{0\} \subset A$. We now show that $(1 - b_0, 1 + b_0) \times (0, a_0) \subset A$ for some small $a_0, b_0 > 0$. The right-hand side of (6.15) is the sum of:

- (i) $a(sf(q) - |\nu_q|)$. This is larger than $a\varepsilon$ for $s = 1$ and q belonging to the image of δ by (6.14) and the identity $f = \kappa_{\delta}$ in (6.9). Thus, $a(sf(q) - |\nu_q|)$ is larger than $\frac{1}{2}a\varepsilon$ on a neighbourhood U of the image of δ , if $|s - 1|$ is small enough;
- (ii) $1 - s|\beta_q|$. This is strictly positive everywhere for $s < 1$. For $s = 1$ it only vanishes on the image of δ . So $1 - s|\beta_q| \geq \epsilon' > 0$ on $\mathbb{T}^2 \setminus U$, if $|s - 1|$ is small.

Thus, the sum is positive in U if $a > 0$, and it is positive on $\mathbb{T}^2 \setminus U$ for $s \in (1 - b_0, 1 + b_0)$ and $a < a_0$, where $b_0 > 0$ is sufficiently small and

$$a_0 := \frac{\epsilon'}{\max\{0, c_0\}}, \quad c_0 := \sup_{q \notin U, s \in [1 - b_0, 1 + b_0]} sf(q) - |\nu_q|$$

So, $(1 - b_0, 1 + b_0) \times (0, a_0) \subset A$. Thus, the sets $[0, 1) \times \{0\}$ and $\{1\} \times (0, a_0)$ belong to the same path-connected component A_* . By Lemma 3.12, $(D^g, \omega_s, \alpha_{s,a})$ is a convex

domain for all $(s, a) \in A_*$. Therefore, the deformation in the claim arises from a path connecting (s, a) to $(0, 0)$ within A_* . Applying Corollary 1.6 and Viterbo's theorem [62], we deduce the isomorphisms in the statement. \square

We will now use Theorem 6.4 to infer existence results about magnetic geodesics. We clarify that ρ is not the radial coordinate R determined by Σ^g for the convex domain $(D^g, \omega_s, \alpha_{s,a})$ and, more generally, ρ is not a radial Hamiltonian. However, to prove our results we do not need to find R , it suffices to exploit the fact that chain level generators x for $SH^*(D^g, \omega_s, \alpha_{s,a})$ at infinity correspond under projection to Σ^g to closed Reeb orbits, which in turn correspond to closed magnetic geodesics γ of speed 1.

Observe that after a time-dependent perturbation of a radial Hamiltonian h , the Floer chain complex $CF^*(h)$ (where we suppress $D^g, \omega_s, \alpha_{s,a}$ from the notation) is generated by elements that can be labeled x_-^k and x_+^k , where x^k is the k -th iterate of a prime magnetic geodesic in Σ^g for $k \in \mathbb{N}$ (the labeling uses the above comments about projection to Σ^g). Following Appendix B, if x has transverse Conley-Zehnder index $\bar{\mu}(x)$, then x_- and x_+ have degrees $|x_-| = 1 - \bar{\mu}(x)$ and $|x_+| = 2 - \bar{\mu}(x)$, respectively (using that $n = \dim_{\mathbb{C}} D^g = 2$).

Lemma 6.5. *The transverse Conley-Zehnder index of x equals the Morse index of the corresponding magnetic geodesic γ for the free-period action functional \mathbb{S}_1 .*

Proof. Let (s_0, a_0) be a pair in A_* and consider a path (s, a) in A_* joining (s_0, a_0) to $(0, 0)$. Let $Y_{s,a}$ be the Liouville vector field of $(D^g, \omega_s, \alpha_{s,a})$ defined at Σ^g , so that $Y_{0,0} = Y$. Let $V_{s,a}$ be a nowhere vanishing vector field contained in $\ker \alpha_{s,a}$ such that $V_{0,0} = V$. It is not difficult to see that $Y_{s,a}$ and $V_{s,a}$ can be chosen to depend continuously on (s, a) . Let $L_{s,a}$ be the Lagrangian distribution for ω_s generated by $Y_{s,a}$ and $V_{s,a}$ and observe that $L_{0,0} = T^{\text{vert}}(T\mathbb{T}^2)$ is also a Lagrangian distribution for ω_s . It follows that the relative Maslov index of $L_{s,a}$ with respect to $L_{0,0}$ vanishes. Since $h'' > 0$ we have that the full Conley-Zehnder index $\mu(x)$ computed with respect to the distribution $L_{s,a}$ is equal to $\mu(x) = \bar{\mu}(x) + \frac{1}{2}$. Since the relative Maslov index vanishes, $\mu(x)$ is also the Conley-Zehnder index computed with respect to the vertical distribution $L_{0,0}$. By a classical result of Duistermaat [22, 63], $\mu(x) - \frac{1}{2}$ is the Morse index of γ for the *fixed*-period action functional, and as $h'' > 0$ this is equal to the Morse index $m(\gamma)$ for the *free*-period action functional, so $m(\gamma) = \bar{\mu}(x)$ (see Merry-Paternain [43] for details). \square

Corollary 6.6. *For all closed Reeb orbits x , we have $\bar{\mu}(x) \geq 0$, and thus $|x_-| \leq 1$ and $|x_+| \leq 2$. If the corresponding γ is a local minimizer of the free-period action functional \mathbb{S}_1 , then $\bar{\mu}(x) = 0$.*

Corollary 6.7. *The following hold for every QL-magnetic-torus $(D^g, \omega_s, \alpha_s)$:*

- (1) *There exists at least one periodic magnetic geodesic with speed 1 in every non-trivial free homotopy class;*
- (2) *If the magnetic geodesic in (1) is non-degenerate, then there exists at least 2 such magnetic geodesics;*

- (3) *If the contractible periodic magnetic geodesics of unit speed are non-degenerate, then there are infinitely many contractible magnetic geodesics of Morse index one.*

Proof. (1) For free homotopy classes $c \neq 0$, if there are no such geodesics then $SH_c^*(D^g, \omega_1, \alpha_1) = 0$, contradicting Theorem 6.4.

(2) If there is only one non-degenerate geodesic in the class $c \neq 0$ in (1), then $2 \geq \text{rank } SH_c^*(D^g, \omega_1, \alpha_1)$ (recall that each such geodesic contributes two generators to the chain complex after time-perturbation, and we remark that iterates lie in different free homotopy classes). This contradicts Theorem 6.4 (we expect rank 4).

(3) Suppose by contradiction that there are only finitely many prime magnetic geodesics with index 1. By the iteration formula (B.1), if $\bar{\mu}(x) = 1$, then $\bar{\mu}(x^k)$ eventually grows for large k . So there is a minimal Reeb period $T > 0$ such that all prime and non-prime magnetic geodesics with index 1 have Reeb period $\leq T$ and we denote by A the number of such orbits.

Sub-claim: $\Delta : CF^2(h) \rightarrow CF^1(h)$ is injective.

Proof: Let $0 \neq w \in CF^2(h)$, we want $\Delta w \neq 0$. Let $x_+ \in CF^2(h)$ be a generator appearing in w with maximal Reeb period. After rescaling if necessary, we may assume $w = w' + x_+$ where w' does not involve x_+ . From (4.11), it follows that $\langle \Delta w', x_- \rangle = 0$. Thus $\langle \Delta w, x_- \rangle = \langle \Delta x_+, x_- \rangle \neq 0$ by (4.11) and (4.12) (x is a good orbit: if $x = x_*^k$ for a prime Reeb orbit x_* , then (B.1) implies $\bar{\mu}(x_*) = 0$ since $\bar{\mu}(x) = 0$, so the $\bar{\mu}$ -values of x, x_* have the same parity). So $\Delta w \neq 0$. \checkmark

We take a Hamiltonian function h on the symplectisation of D^g so that

- its Morse complex has generators in degrees 2, 1, 1, 0, computing $H^*(\mathbb{T}^2)$,
- the slope τ of h at infinity is bigger than T , so that all generators of the Floer chain complexes involve Reeb periods $\leq \tau$, as will their images under ∂ and Δ (see 4.4).

We use the abbreviation $C_d = CF_0^d(h)$ (notice we restricted to contractible orbits), $\partial_d = \partial|_{C_d} : C_d \rightarrow C_{d+1}$ and $\Delta_d = \Delta|_{C_d} : C_d \rightarrow C_{d-1}$. There are no generators in degree 3 by Corollary 6.6. In particular, $\partial_2 = 0$. Observe that for each geodesic x_0 with $\bar{\mu}(x_0) = 0$ we have $|x_{0-}| = 1$, $|x_{0+}| = 2$, and for each geodesic x_1 with $\bar{\mu}(x_1) = 1$ we have $|x_{1-}| = 0$, $|x_{1+}| = 1$. Since h has two Morse generators in degree 1 and one in degree 2, we see that $c(\tau) := \dim C_1 - \dim C_2 - 1$ equals the number of geodesics x_1 with $\bar{\mu}$ -index 1 and period less than τ , since the counts of the $x_{0\pm}$ cancel out in $c(\tau)$. Therefore, by assumption $c(\tau) \leq A$ is bounded independently of τ . Since Δ_2 is injective by the sub-claim, the dimension of $\text{coker } \Delta_2$ is equal to $\dim C_1 - \dim C_2 = c(\tau) + 1$ and, hence, is bounded by $A + 1$. Since Δ is a chain map (see 4.10) and $\partial_2 = 0$, the map Δ_2 sends C_2 into $\ker \partial_1$, namely $\text{Im } \Delta_2 \subset \ker \partial_1$. Therefore, we obtain $\dim \text{Im } \partial_1 = \dim C_1 - \dim \ker \partial_1 \leq \dim \text{coker } \Delta_2 \leq A + 1$.

When we increase the slope τ , we modify h to h_1 by only increasing h in the region at infinity where $h' = \tau$. By the maximum principle, this implies that the continuation map $CF^*(h) \rightarrow CF^*(h_1)$ for the linear interpolation will be an inclusion of a subcomplex (non-constant continuation solutions lying in the region where $h = h_1$ cannot be rigid as they would admit an \mathbb{R} -reparametrization action). Thus $\text{Im } \partial_1$ computed for $CF^*(h)$ is contained in the $\text{Im } \partial_1$ computed for $CF^*(h_1)$. Using such Hamiltonians, it follows from the bound $\dim \text{Im } \partial_1 \leq A + 1$ that $\text{Im } \partial_1$ eventually stabilises as a vector

subspace, independently of τ . Finally, observe that $\dim C_2 \rightarrow \infty$ as $\tau \rightarrow \infty$, because if y is the closed Reeb orbit corresponding to δ , then all the iterates y_+^k have degree 2, as $\bar{\mu}(y^k) = 0$ by Corollary 6.6. Since $\partial_2 = 0$, it now follows that $SH_0^2(D^g, \omega_1, \alpha_1)$ is infinite dimensional, contradicting Theorem 6.4. \square

Remark 6.8 (Alternative Proof). *Corollary 6.7.(3) can also be proved using the more elaborated machinery of S^1 -equivariant symplectic cohomology $ESH^*(D^g, \omega_1, \alpha_1)$ (we use the conventions from [46]). Using the Morse-Bott spectral sequence from McLean-Ritter [46, Cor.7.2], aside from the Morse complex of h , the E_1 -page has generators labeled by the unperturbed magnetic geodesics with grading $2 - \bar{\mu}$. There are infinitely many orbits δ^k in degree 2, which are cycles as there are no generators in degree 3. The Morse-Bott spectral sequence converges in degree 2 to the finite dimensional group $ESH_0^2(D^g, \omega_1, \alpha_1)$, so for dimension reasons there cannot be only finitely many orbits in degree 1. Here we used that the analogue of Theorem 6.4 yields $ESH_c^*(D^g, \omega_s, \alpha_s) \cong ESH_c^*(D^g, \omega_0, \alpha_0) \cong H_{2-*}^{S^1}(\mathcal{L}_c \mathbb{T}^2)$ where the latter is the S^1 -equivariant Viterbo theorem [62], and we used that $\dim H_0^{S^1}(\mathcal{L}_c \mathbb{T}^2) = 1 < \infty$.*

6.5. Convex domains for $N \neq \mathbb{T}^2$: proof of Theorem 1.7. Let $N = S^2$ or a surface of genus ≥ 2 . We now work with non-exact magnetic forms. Define

$$\mathcal{N} := \left\{ (g, \sigma) \mid \int_N \sigma = 2\pi\chi(N) \right\}. \quad (6.16)$$

This is not restrictive, since, up to changing orientation of N , and rescaling σ to $c\sigma$ and s to s/c , we can assume that the normalisation above holds. By the Gauss-Bonnet theorem, the form $\sigma' := \sigma - K\mu$ is exact, and for every primitive β we get a primitive $\theta_{s,\beta}$ of ω_s outside of the zero section:

$$\theta_{s,\beta} := \theta - s\pi^*\beta + s\tau, \quad (6.17)$$

where τ is the S^1 -connection form. We let

$$\alpha_{s,\beta} := \theta_{s,\beta}|_{\Sigma^g}.$$

Recall the definition of the function $f : N \rightarrow \mathbb{R}$ as the density of σ with respect to the area form μ : $\sigma = f\mu$. We define for every $(g, \sigma) \in \mathcal{N}$,

$$s_-(g, \sigma) = \sup_{d\beta=\sigma'} \left\{ s_* \geq 0 \mid \forall s \in [0, s_*], \quad 1 - \|\beta\|s + (\min f)s^2 > 0 \right\}.$$

More explicitly, let $s_-(g, \sigma, \beta)$ be the smallest positive real root of the polynomial $1 - \|\beta\|x + (\min f)x^2$ if it exists, and otherwise let $s_-(g, \sigma, \beta) = +\infty$. Then,

$$s_-(g, \sigma) = \sup_{d\beta=\sigma'} s_-(g, \sigma, \beta).$$

Finally, let A be the set of triples (g, σ, s) such that $(g, \sigma) \in \mathcal{N}$ and $s < s_-(g, \sigma)$.

Lemma 6.9. *The set A is connected, and $(D^g, \omega_s, \alpha_{s,\beta})$ is a convex domain for any $(g, \sigma, s) \in A$, where β is any primitive of σ' with $s < s_-(g, \sigma, \beta)$.*

Proof. To see that A is connected, we just observe that if we have an interpolation (g_u, σ_u) with $u \in [0, 1]$, then we can take a small s such that $1 - \|\beta_u\|s + (\min f_u)s^2$

is positive for all u . To prove that $\alpha_{s,\beta}$ is a positive contact form, we use Lemma 3.12. Indeed, the formulae in Section 6.2 yield for $(q, v) \in \Sigma^g$

$$\alpha_{\beta,s}(X + sW)_{(q,v)} = 1 - \beta_q(v)s + s^2 f(q) \geq 1 - \|\beta\|s + s^2 \min_{q \in N} f(q). \quad \square$$

When $N = S^2$, we can prove that $(D^g, \omega_s, \alpha_{s,\beta})$ is also convex when σ is symplectic and s is large enough. Indeed, let $\mathcal{N}^+ \subset \mathcal{N}$ be the subset of those (g, σ) for which σ is symplectic, equivalently $f > 0$. Let

$$s_+(g, \sigma) = \inf_{d\beta=\sigma'} \left\{ s_* \geq 0 \mid \forall s \geq s_*, \quad 1 - \|\beta\|s + (\min f)s^2 > 0 \right\}.$$

More explicitly, let $s_+(g, \sigma, \beta)$ be the largest positive real root of the polynomial $1 - \|\beta\|x + (\min f)x^2$ if it exists, and otherwise let $s_+(g, \sigma, \beta) = 0$. Then,

$$s_+(g, \sigma) = \inf_{d\beta=\sigma'} s_+(g, \sigma, \beta).$$

Finally, let A^+ be the set of triples (g, σ, s) such that $(g, \sigma) \in \mathcal{N}^+$ and $s > s_+(g, \sigma)$.

Theorem 6.10. *Let $N = S^2$. The set A^+ is connected, and $(D^g, \omega_s, \alpha_{s,\beta})$ is a convex domain for all $(g, \sigma, s) \in A^+$, where β is any primitive of σ' with $s > s_+(g, \sigma, \beta)$. \square*

Having found large sets A and A^+ for which the domain is convex, we proceed to find in this class some symmetric examples, for which, we can compute the symplectic cohomology. For this purpose, we pick a metric \bar{g} on N with $|K| = 1$ and let $\bar{\sigma} = K\mu$. We consider the *symmetric twisted symplectic form*

$$\bar{\omega}_s := d\theta - s\pi^*\bar{\sigma}$$

and the speed Hamiltonian ρ associated to the metric \bar{g} .

For $N = S^2$, we have that $(\bar{g}, \bar{\sigma}) \in \mathcal{N}^+$. Moreover, $s_-(\bar{g}, \bar{\sigma}) = s_-(\bar{g}, \bar{\sigma}, 0) = +\infty$ and $s_+(\bar{g}, \bar{\sigma}) = s_+(\bar{g}, \bar{\sigma}, 0) = 0$. Thus, $(\bar{g}, \bar{\sigma}, s) \in A \cap A^+$, for all $s > 0$.

For N a surface of genus ≥ 2 , we have $(\bar{g}, \bar{\sigma}) \in \mathcal{N}$ and $s_-(\bar{g}, \bar{\sigma}) = s_-(\bar{g}, \bar{\sigma}, 0) = 1$. Thus, $(\bar{g}, \bar{\sigma}, s) \in A$, for all $s \in (0, 1)$.

Corollary 6.11. *Let $N = S^2$. For every $(g, \sigma, s) \in A \cup A^+$, the domain $(D^g, \omega_s, \alpha_{s,\beta})$ can be deformed through convex domains to $(D^{\bar{g}}, \bar{\omega}_{\bar{s}}, \alpha_{\bar{s},0})$ for any $\bar{s} > 0$.*

Let N be a surface of genus ≥ 2 . For every $(g, \sigma, s) \in A$, the domain $(D^g, \omega_s, \alpha_{s,\beta})$ can be deformed through convex domains to $(D^{\bar{g}}, \bar{\omega}_{\bar{s}}, \alpha_{\bar{s},0})$ for any $\bar{s} \in (0, 1)$.

In both cases, the relative class (1.3) is constant during the deformation up to a positive factor and up to identifying domains by a fibrewise rescaling. So,

$$SH^*(D^g, \omega_s, \alpha_{s,\beta}) \cong SH^*(D^{\bar{g}}, \bar{\omega}_{\bar{s}}, \alpha_{\bar{s},0}).$$

Proof. A deformation $(D^s, \omega^s, \alpha^s)$ from $(D^{\bar{g}}, \bar{\omega}_{\bar{s}'}, \bar{\alpha}_{\bar{s}',0})$ to $(D^g, \omega_s, \alpha_{s,\beta})$ exists since $A \cup A^+$ (respectively A) is connected. Performing a rescaling $\psi_s : D^{\bar{g}} \rightarrow D^s$ of the form $\psi_s(q, p) = (q, \lambda_s(q, p)p)$, for some suitable function $\lambda_s : D^{\bar{g}} \rightarrow [0, +\infty)$ we can pull-back all the objects to $D^{\bar{g}}$. Since $H^2(D^{\bar{g}}, \Sigma^{\bar{g}}) \cong H^2(N) \cong \mathbb{R}$ and ω_s is non-exact, it follows that there exists also some constant $c_s > 0$ such that $(D^{\bar{g}}, c_s \psi_s^* \omega_s, c_s \psi_s^* \alpha_{s,\beta})$ is a deformation with constant relative class, from $(D^{\bar{g}}, \bar{\omega}_{\bar{s}}, \bar{\alpha}_{\bar{s},0})$ to $(D^{\bar{g}}, c_1 \psi_1^* \omega_s, c_1 \psi_1^* \alpha_{s,\beta})$. Since the factor c_1 does not affect the symplectic cohomology, the isomorphism in the statement follows from Theorem 1.1 and Theorem 1.3. \square

We compute the symplectic cohomology of the symmetric cases in Lemma 6.12 and 6.13 by a direct and geometric approach which uses that Lemma 6.5 and Corollary 6.6 hold also in the present setting, as their proof can be readily adapted. Alternatively, the computation can be done using the deformation invariance of symplectic cohomology proved in Corollary 1.6 as follows. We decrease s to 0 to deform $(D^{\bar{g}}, \bar{\omega}_s, \bar{\alpha}_{s,0})$ to $(D^{\bar{g}}, d\theta, \theta|_{\Sigma^{\bar{g}}})$. Corollary 1.6 with $\zeta_0 = 0$ and $\zeta_1 = -s\tau(\pi^*\bar{\sigma})$ yields

$$SH^*(D^{\bar{g}}, \bar{\omega}_s, \bar{\alpha}_{s,0}) \cong SH^*(D^{\bar{g}}, d\theta, \theta|_{\Sigma^{\bar{g}}})_{-s\tau(\pi^*\bar{\sigma})} \cong H_{2-*}(\mathcal{L}N)_{-s\tau(\sigma)},$$

where we used the twisted Viterbo isomorphism [51]. The twisted homology of $\mathcal{L}N$ vanishes for $N = S^2$ since σ is not exact [51]. For a surface of genus ≥ 2 , we recover the untwisted homology of $\mathcal{L}N$ since σ is atoroidal, meaning $\tau(\sigma) = 0 \in H^1(\mathcal{L}N)$.

Lemma 6.12. *Consider the symmetric twisted tangent bundle $(TN, \bar{\omega}_s)$, where N is a surface of genus ≥ 2 and $s < 1$. The periodic Reeb orbits on $\Sigma^{\bar{g}}$ are as follows.*

- *There is no periodic orbit in the trivial free homotopy class of $D^{\bar{g}}$.*
- *In every non-trivial free homotopy class, there is exactly one periodic orbit, it is transversally non-degenerate, and its transverse Conley-Zehnder index is 0.*

It follows that $SH_c^(D^{\bar{g}}, \bar{\omega}_s, \bar{\alpha}_{s,0}) \cong H_{2-*}(N)$ if $c = 0$ is the trivial free homotopy class, and $SH_c^*(D^{\bar{g}}, \bar{\omega}_s, \bar{\alpha}_{s,0}) \cong H_{2-*}(S^1)$ if $c \neq 0$.*

Proof. The magnetic geodesics with speed 1 correspond to curves in N with geodesic curvature s . Following Hedlund [36], such curves have an explicit description, when lifted to the universal cover \mathbb{H} of N (where $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is the hyperbolic upper half-plane). The lifted curves are oriented segments of circles that form an angle $\theta \in (0, \frac{\pi}{2})$ between the exit direction and the boundary at infinity $\{y = 0\}$ oriented by ∂_x , given by $\cos \theta = s$. As for standard geodesics, we know that there are no contractible trajectories and exactly one trajectory in every non-trivial free homotopy class. After reparametrisation, the lifted curves are genuine geodesics for a Finsler metric on \mathbb{H}^2 with negative flag curvature. In particular, each non-contractible periodic orbit is transversally non-degenerate and length-minimizing in its class. Therefore, by Corollary 6.6, the transverse Conley-Zehnder index of the associated Reeb orbit is zero. In particular, each closed orbit is *good* (as the primitive orbits have even index). Thus, after a small time-dependent perturbation of the Hamiltonian it yields a Floer subcomplex with the homology of S^1 . \square

Lemma 6.13. *Consider the symmetric twisted tangent bundle $(TS^2, \bar{\omega}_s)$ and the primitive $\bar{\theta}_{s,0}$ as in (6.17), where $s > 0$. The radial coordinate induced by integrating the Liouville flow of $\theta_{s,0}$ starting from $\Sigma^{\bar{g}}$ is defined globally on T^*S^2 via*

$$R_s(q, v) = \sqrt{\frac{|v|^2 + s^2}{1 + s^2}},$$

$(TS^2, \bar{\omega}_s, \bar{\theta}_{s,0})$ is the completion of $(D^{\bar{g}}, \bar{\omega}_s, \bar{\alpha}_{s,0})$ and $SH^(D^{\bar{g}}, \bar{\omega}_s, \bar{\alpha}_{s,0}) = 0$.*

Proof. Let Z_s denote the Liouville vector field of $\bar{\theta}_{s,0} = \theta + s\tau$, which means that $\iota_{Z_s}\omega_s = \theta + s\tau$. Denote r_s the coordinate defined on the complement of the zero section by the flow of Z_s with $r_s = 0$ along $\Sigma^{\bar{g}}$. By definition $R_s = e^{r_s}$. We differentiate the

function ρ along a flow line of Z_s , using Lemma 6.1,

$$\frac{d\rho}{dr_s} = d\rho(Z_s) = -\omega_s\left(\frac{1}{\rho}(X + sW), Z_s\right) = \frac{1}{\rho}(\theta + s\tau)(X + sW) = \frac{1}{\rho}(\rho^2 + s^2).$$

Multiplying both sides by $\frac{\rho}{\rho^2 + s^2}$ and integrating from 0 to r_s yields the claimed formula for $r_s = \log R_s$. Note $r_s \rightarrow \infty$ as $\rho \rightarrow \infty$, so the flow of Z_s is positively complete. We now compute the symplectic cohomology. The closed Reeb orbits on Σ^g correspond to curves on S^2 with geodesic curvature s . An explicit computation in geodesic polar coordinates shows that all such trajectories are periodic with common minimal period $T = 2\pi/\sqrt{1 + s^2}$. We consider the sequence of Hamiltonians, for $k \in 1 + 2\pi\mathbb{N}$,

$$h_k : TS^2 \rightarrow \mathbb{R}, \quad h_k(q, p) = k\sqrt{1 + s^2} \cdot R_s(q, v) = k\sqrt{\rho^2 + s^2}.$$

The Hamiltonian vector field is $X_{h_k} = k(\rho^2 + s^2)^{-1/2}(X + sW)$. The associated flow defines a Hamiltonian S^1 -action on (T^*S^2, ω_s) with minimal period $2\pi/k$. Hence, the only 1-periodic orbits of the flow are the constant orbits, which lie in the zero section. One could now compute the Conley-Zehnder indices explicitly. One can bypass this, by mimicking the argument in [52] (compare also [46, Section 2.6]): changing the slope k to $k + 2\pi$ will decrease the indices by 2 (one looks at how the linearized flow for h_k acts on a trivialisation of the anti-canonical bundle, and one notices that it has winding number one). Finally by considering the direct limit, one concludes that symplectic cohomology vanishes in each degree. \square

Proof of Theorem 1.7. Let $(g, \sigma, s) \in A \cup A^+$ for $N = S^2$, or $(g, \sigma, s) \in A$ for N of genus ≥ 2 . The computation of $SH^*(D^g, \omega_s, \alpha_{s,\beta})$ in the statement follows from Corollary 6.11 and Lemmas 6.12-6.13. We now prove the statements about the existence of closed magnetic geodesics with speed $1/s$ for the pair (g, σ) .

For N of genus ≥ 2 , if there were no such curve in a free homotopy class $\nu \neq 0$, we would obtain the contradiction $H_{2-*}(S^1) \cong SH_\nu^*(D^g, \omega_s, \alpha_{s,\beta}) = 0$.

Now let $N = S^2$. If there were no closed magnetic geodesics of speed $1/s$, we would obtain the contradiction $H^*(S^2) \cong SH^*(D^g, \omega_s, \alpha_{s,\beta}) = 0$. We now prove that there are at least two prime periodic magnetic geodesics with speed $1/s$, assuming all the periodic orbits are transversally non-degenerate. Suppose by contradiction that x is the only such geodesic. We do a case-by-case analysis of indices, using (B.1):

- a) $\bar{\mu}(x) \leq 0$. Then $\bar{\mu}(x^k) \leq 0$, for all $k \in \mathbb{N}$. Thus, the non-constant orbits in the Floer chain complexes have grading $|x_\pm^k| \geq 1$, which would imply that $SH^0(T^*S^2, \omega_s, \alpha_{s,\beta}) \cong H^0(T^*S^2)$, contradicting Theorem 1.7.
- b) $\bar{\mu}(x) \geq 3$ with x hyperbolic. Then $\bar{\mu}(x^{k+1}) - \bar{\mu}(x^k) \geq 3$. It follows for grading reasons that x_+^k is a cycle, and it is not a boundary unless it arises from the Floer differential applied to x_-^k . But in the local Floer complex for x^k , we have $\partial x_-^k = 0$ whenever x^k is a good orbit by (4.12), and we can always ensure that x^k is good (if x^k is a bad hyperbolic orbit, we replace k by $k + 1$).
- c) $\bar{\mu}(x) \geq 3$ with x elliptic. Here $\bar{\mu}(x) \geq 3$ forces $\tilde{\Delta} \geq 1$ and non-degeneracy implies $\tilde{\Delta} \notin \mathbb{Q}$, so $\tilde{\Delta} > 1$. So, for some sufficiently large k , $\bar{\mu}(x^{k+1}) - \bar{\mu}(x^k) \geq 4$. The proof follows as in the previous case (using that elliptic orbits are always good).

- d) $\bar{\mu}(x) = 2$. Then all iterates x^k are good hyperbolic orbits. The x_{\pm}^k and the two generators of the Morse complex for S^2 , give generators in gradings $2, 0, 0, -1, -2, -3, \dots$ which cannot be acyclic in degrees 2 and 0, contradicting Theorem 1.7.
- e) $\bar{\mu}(x) = 1$ with x hyperbolic. Generators' gradings: $2, 1, 0, 0, 0, -1, -1, -2, \dots$, which by rank-nullity cannot be acyclic either in degree 2 or 0 (or both).
- f) $\bar{\mu}(x) = 1$ with x elliptic. Then $0 < \tilde{\Delta} < 1$. Suppose $\tilde{\Delta} < \frac{1}{2}$. Then the $\bar{\mu} = 1$ orbits are x^1, x^2, \dots, x^a for some $a \geq 2$, thus $\bar{\mu}(x^{a+1}) = 3$, and recall iterates of an elliptic orbit are good. Let m_2, m_0 denote the Morse critical points in degrees 2, 0. The restriction of the differential to $\Lambda x_+^1 \oplus \Lambda x_+^a \rightarrow \Lambda m_2$ must have non-trivial kernel by rank-nullity, thus we obtain a cycle $y = \lambda_1 x_+^1 + \lambda_2 x_+^a \neq 0$, for some $\lambda_1, \lambda_2 \in \Lambda$. As symplectic cohomology vanishes, there is a chain z with $\partial z = y$. By (4.11) and (4.12), this can happen only if $\lambda_2 = 0$, as x^a is good and has maximal period among the orbits with $\bar{\mu} = 1$. Moreover, $\Delta z = 0$ by (4.11) as all orbits in grading -1 have Reeb period strictly larger than those with grading 0. By (4.10), $\Delta y = \Delta \partial z = -\partial \Delta z = 0$. However, $y = \lambda_1 x_+^1$ and, therefore, $\langle \Delta y, x_-^1 \rangle = \lambda_1 \neq 0$ by (4.12), contradiction. Now suppose $\tilde{\Delta} > \frac{1}{2}$. There are $m, k \in \mathbb{Z}$ such that x^{k-1} is the only orbit with $\bar{\mu} = m$ and x^k and x^{k+1} are the only orbits with $\bar{\mu} = m + 2$. The previous argument applies with x_-^{k-1} in place of m_2 , x_+^k in place of x_+^1 , and x_+^{k+1} in place of x_+^a . \square

Remark 6.14 (Alternative Proof). *The last case above can be proved using ESH^* as in Remark 6.8. Suppose x^1, x^2, \dots, x^{a_1} have $\bar{\mu} = 1$, and $x^{1+a_k-1}, \dots, x^{a_k}$ have $\bar{\mu} = 2k - 1$ for $k \geq 1$. Recalling the two Morse critical points, the number of generators in degrees $(2, 1, 0, -1, \dots)$ after perturbation is $(1, a_1, a_1 + 1, a_2, a_2, a_3, a_3, \dots)$. Consider the E_1 -page of the Morse-Bott spectral sequence for ESH^* [46, Cor.7.2]. The Morse complex for S^2 contributes generators in degrees $2 + 2\mathbb{Z}_{\leq 0}, 0 + 2\mathbb{Z}_{\leq 0}$ due to the formal variables u^{-m} in degree $-2m$. Each non-constant S^1 -orbit with index $1 - \bar{\mu}$ contributes one copy of $H^{*-1}(S^1/S^1) = H^{*-1}(\text{pt})$ in grading $* + 1 - \bar{\mu}$ (using [46, Thm.4.1]). The total number of generators in degrees $(2, 1, 0, -1, \dots)$ is $(1, a_1, 2, a_2, 2, a_3, 2, \dots)$. We use two facts explained in [46]: the vanishing of symplectic cohomology implies the vanishing of the S^1 -equivariant symplectic cohomology; and the equivariant Morse complex for S^2 constitutes a subcomplex. Thus, by Theorem 1.7, the spectral sequence converges to zero, and the E_1 -page considered with total gradings must satisfy the same rank-nullity conditions as an acyclic complex. This implies $a_1 = 1$ (the degree 0 generators in the subcomplex cannot kill a non-constant orbit, and the degree 1 orbit will eventually kill the Morse index 2 critical point) and thus $a_2 = a_3 = \dots = 2$. Using (B.1), $2m\tilde{\Delta}, (2m+1)\tilde{\Delta}$ must lie in the open interval $(m, m+1)$ for all $m \geq 1$. So $\tilde{\Delta} \in (\frac{m}{2m}, \frac{m+1}{2m+1})$. Letting $m \rightarrow \infty$ yields the contradiction $\frac{1}{2} = \tilde{\Delta} \notin \mathbb{Q}$.*

APPENDIX A. FROM THE MAGNETIC T^*S^2 TO THE HYPERKÄHLER $T^*\mathbb{CP}^1$

The tangent bundle $T^*\mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is isomorphic as a complex line bundle to $\mathcal{O}(-2) \rightarrow \mathbb{CP}^1$. After picking a Hermitian metric on $\mathcal{O}(-2)$, we can ensure that this identification is S^1 -equivariant (where $S^1 \subset \mathbb{C}^*$ acts naturally by rotation in the complex fibres) and preserves the norm $\rho = |p|$. The curvature form σ on $T^*\mathbb{CP}^1$

then satisfies $\frac{1}{2\pi}[\sigma] = c_1(\mathcal{O}(-2)) = -2\omega_{FS} \in H^2(\mathbb{CP}^1)$ where $\int_{\mathbb{CP}^1} \omega_{FS} = 1$, and let τ be the associated angular form for $\mathcal{O}(-2)$.

Fix a metric g on $S^2 \cong \mathbb{CP}^1$ of constant Gaussian curvature one and identify the real vector bundle T^*S^2 with TS^2 as in Section 6. Note however that, since $\int_{S^2} \sigma = -4\pi$, the induced rotation $j : TS^2 \rightarrow TS^2$ is rotation by $-\frac{\pi}{2}$ compared to the usual orientation for \mathbb{CP}^1 , and $\sigma = \mu$ where $\mu = g(\cdot, j\cdot)$ is the area form of g with respect to j in the notation of Section 6.

Following the conventions in [54, Section 7.3], we can construct a symplectic form $\omega = d\tau + \varepsilon d(\rho^2\tau)$ on TS^2 for $\varepsilon > 0$, where $d(\rho^2\tau)$ is fiberwise the area form and we have $d\tau = -\pi^*\sigma$. On the zero section, ω restricts to $-\pi^*\sigma$, therefore $[\omega] = -\pi^*\sigma \in H^2(TS^2)$. Thus, away from the zero section, $\omega = d((1+\varepsilon\rho^2)\tau)$. By replacing $\varepsilon = 1/2s$ for $s > 0$, and rescaling the symplectic form by s , we redefine the symplectic form by

$$\tilde{\omega}_s = d\left(\left(\frac{\rho^2}{2} + s\right)\tau\right).$$

Thus $\tilde{\omega}_s$ restricts to $-s\pi^*\sigma$ on S^2 , just like the magnetic symplectic form $\omega_s = d\theta - s\pi^*\sigma$, and so $[\tilde{\omega}_s] = [\omega_s] \in H^2(TS^2)$.

The form $\tilde{\omega}_s$ can be identified with the Hyperkähler form ω_I for $TS^2 \cong T^*\mathbb{CP}^1$ viewed as an asymptotically locally Euclidean manifold [52], for which the zero section and the fibres are holomorphic submanifolds. So $\tilde{\omega}_s$ makes the zero section and the fibres of TS^2 both symplectic submanifolds; $d\theta$ makes them both Lagrangian; and ω_s makes the zero section symplectic but keeps the fibres Lagrangian.

Theorem A.1. *There exists a diffeomorphism $F_s : TS^2 \rightarrow TS^2$ preserving the zero section (but not the fibres) such that*

$$F_s^*\left(\left(\frac{\rho^2}{2} + s\right)\tau\right) = \theta + s\tau. \quad (\text{A.1})$$

In particular, we can identify the magnetic $(TS^2, \omega_s = d\theta - s\pi^\sigma)$ with the negative line bundle $(\mathcal{O}_{\mathbb{CP}^1}(-2), \tilde{\omega}_s)$ and $SH^*(TS^2, \omega_s) \cong SH^*(\mathcal{O}_{\mathbb{CP}^1}(-2), \tilde{\omega}_s)$. The latter is known to vanish by [54], consistently with Theorem 1.7.*

Proof. We follow the ideas in [12, Section 2] and refer to Section 6.1 and 6.2 for the notation. We denote by Σ_ρ an arbitrary level set of ρ . From (6.8) and the general formula $d\alpha(U_1, U_2) = U_1 \cdot \alpha(U_2) - U_2 \cdot \alpha(U_1) - \alpha([U_1, U_2])$, for a 1-form α and vector fields U_1, U_2 , we get

$$d\theta = \tau \wedge \eta, \quad d\eta = \theta \wedge \tau, \quad d\tau = \frac{1}{\rho^2} \eta \wedge \theta \quad \text{on } T\Sigma_\rho, \quad (\text{A.2})$$

where the last equality follows since $K = 1$. For $a > 0$, consider the rescaling

$$m_a : TS^2 \rightarrow TS^2, \quad m_a(q, v) = (q, av),$$

which satisfies

$$m_a^*\tau = \tau, \quad \partial_a m_a = \frac{1}{a}Y, \quad dm_a \cdot H = aH. \quad (\text{A.3})$$

Denote by Φ_t the flow at time t of the vector field $-H = (-jv)^h$. The integral curves of Φ_t are $t \mapsto (\gamma(t), j\gamma'(t))$, where $t \mapsto \gamma(t)$ is a geodesic in S^2 for g . Indeed, $j\gamma'$ is a parallel field along γ and $\gamma' = -jv$ with $v = j\gamma'$. We claim that

$$\Phi_b^*\tau = -\frac{\sin(b\rho)}{\rho} \theta + \cos(b\rho) \tau.$$

Proof of claim. Say $\Phi_b^* \tau = x\theta + y\eta + z\tau$ for smooth x, y, z depending on b (and write x' etc. to denote derivatives in b). Here we used that Φ preserves the tangent bundle of Σ_ρ , which is spanned by X, H, V and the dual space is spanned by θ, η, τ . Multiplying that equation by $(\Phi_b^*)^{-1} = \Phi_{-b}^*$, differentiating in b yields

$$x'\theta + y'\eta + z'\tau = x\mathcal{L}_H\theta + y\mathcal{L}_H\eta + z\mathcal{L}_H\tau = -\rho^2 x\tau + z\theta,$$

where we used Cartan's formula together with (A.2). Thus y is constant in b , and $x' = z$, $z' = -\rho^2 x$. This implies $z'' = -\rho^2 z$ and $x = z'$. Since $\Phi_1^* = \text{Id}$, we have $x_1 = y_1 = 0$, $z_1 = 1$, and therefore $x = \sin(b\rho)/\rho$, $y = 0$ and $z = \cos(b\rho)$. \checkmark

We now claim that the following diffeomorphism satisfies (A.1) for some smooth functions $a_s : [0, \infty) \rightarrow [0, \infty)$, $b_s : [0, \infty) \rightarrow \mathbb{R}$ that we must determine:

$$F_s = m_{a_s(\rho)} \circ \Phi_{b_s(\rho)}.$$

Proof of claim. Abbreviate $c_s(\rho) = a_s(\rho)^2 \cdot \frac{\rho^2}{2} + s$. The pull-back $F_s^*((\frac{\rho^2}{2} + s)\tau)$ equals

$$\Phi_{b_s(\rho)}^* m_{a_s(\rho)}^* ((\frac{\rho^2}{2} + s)\tau) = c_s(\rho) \left(-\frac{\sin(b_s(\rho)\rho)}{\rho} \theta + \cos(b_s(\rho)\rho) \tau \right).$$

Observe indeed that

$$dF_s = \frac{1}{b_s(\rho)} Y \otimes d(a_s(\rho)) - a_s(\rho) dm_{a_s(\rho)} \cdot H \otimes d(b_s(\rho)) + dm_{a_s(\rho)} d\Phi_{b_s(\rho)}$$

and τ vanishes on the first two terms. To satisfy (A.1) we want $c_s(\rho) \cos(b_s(\rho)\rho) = s$ and $c_s(\rho) \sin(b_s(\rho)\rho) = -\rho$. Squaring and adding gives $c_s(\rho) = \sqrt{\rho^2 + s^2}$, whereas taking the quotient implies $\tan(b_s(\rho)\rho) = -\frac{\rho}{s}$. Therefore, we find

$$a_s(\rho) = \frac{1}{\rho} \sqrt{2(\sqrt{\rho^2 + s^2} - s)} = \sqrt{\frac{2}{\sqrt{\rho^2 + s^2} + s}}, \quad b_s(\rho) = -\frac{1}{\rho} \tan^{-1} \left(\frac{\rho}{s} \right) = -\frac{1}{s} u \left(\frac{\rho^2}{s^2} \right),$$

where $u : [0, +\infty) \rightarrow (0, 1]$ is the unique strictly decreasing smooth function such that $u(x^2) = \frac{\tan^{-1}(x)}{x}$. Thus, the functions $a_s : [0, +\infty) \rightarrow (0, \sqrt{2/s}]$ and $b_s : [0, +\infty) \rightarrow [-1/s, 0)$ are strictly monotone and $a_s \circ \rho : TS^2 \rightarrow (0, \sqrt{2/s}]$ and $b_s \circ \rho : TS^2 \rightarrow [-1/s, 0)$ are globally smooth. Therefore, the map F_s is a diffeomorphism satisfying (A.1) and the claim is established. \square

APPENDIX B. ITERATION FORMULA FOR CZ-INDICES IN DIMENSION 4

Let (M, ω) be convex and let $n = \frac{1}{2} \dim M$. For autonomous (i.e. time-independent) Hamiltonians $H : M \rightarrow \mathbb{R}$, any non-constant 1-periodic orbit x will be degenerate as there is at least an S^1 -family of such orbits obtained by time-translation $x(\cdot + \text{constant})$. Such an orbit x is called *transversally non-degenerate* if the 1-eigenspace of $d_{x(0)}\varphi_H^1$ is one-dimensional, i.e. equals $\mathbb{R} \cdot X_H$. Assume now that x is transversally non-degenerate. Then the family of 1-orbits near x is parametrised by S^1 . Following the conventions⁸ of Salamon's notes [58] one can define a Conley-Zehnder index⁹ $\mu(x)$ associated to the linearisation $d_{x(0)}\varphi_H^t$ of the Hamiltonian flow,

⁸Except, we declare $\omega(\cdot, X_H) = dH$, which means that our grading by $n - \mu_{\text{CZ}}$ will agree with the Morse index of H when H is a time-independent, C^2 -small Morse function.

⁹more precisely, the Robbin-Salamon index [57].

written as a path of symplectic matrices ψ_t by symplectically trivializing the symplectic vector bundle x^*TM compatibly with a trivialisation of the canonical bundle.¹⁰

Assume $x(t) = (y(t), R) \in M^{\text{out}} = \Sigma \times [1, \infty)$ lies in the conical end, and that $H = h(R)$ is radial there with $h'' \geq 0$ (with equality for large R , where h is linear). Note $X_H = h'(R)Y$ is a multiple of the Reeb field Y , so $\bar{y}(t) = y(t/T)$ is a Reeb orbit of period $T = h'(R)$. Decompose $TM^{\text{out}} \cong \xi \oplus \mathbb{R}Z \oplus \mathbb{R}Y$, where ξ is the contact structure and $Z = R\partial_R$ is the Liouville field (noting that the oriented basis for ξ^\perp is Z, Y , not Y, Z). We assume $n \geq 2$ (i.e. $\xi \neq 0$), so one can pick a basis of independent sections for $x^*\xi$ which together with Y, Z yield a trivialisation of x^*TM^{out} compatible with the trivialisation of $x^*\mathcal{K}$. In that basis,

$$\psi_t = \bar{\psi}_t \oplus \begin{pmatrix} {}_{tc}h''(R) & 0 \\ 1 & 1 \end{pmatrix}$$

where $c > 0$ is a positive constant, and where $\bar{\psi}_t$ is the path of symplectic matrices obtained by trivializing the contact distribution ξ along the orbit. As $h'' > 0$, the latter symplectic shear contributes $+\frac{1}{2}$ to the CZ-index [57]. Thus

$$\mu(x) = \bar{\mu}(x) + \frac{1}{2},$$

where $\bar{\mu}$ denotes the CZ-index of $\bar{\psi}_t$.

By [17], a suitable time-dependent perturbation of the Hamiltonian, supported near the orbits, will break the S^1 -family of orbits into two orbits x_-, x_+ corresponding to the minimum and maximum of a Morse function on the S^1 -parameter space. In our convention, this affects our grading by $-\frac{1}{2} \dim S^1 + (\text{corresponding Morse index})$. So

$$|x_-| = n - \mu(x) - \frac{1}{2} \dim S^1 = n - \bar{\mu}(x) - 1, \quad |x_+| = |x_-| + 1.$$

Denote $x^k : S^1 \rightarrow M$ the Hamiltonian orbit corresponding to the k -th iterate of x (which appears when $h' = kT$, and corresponds under projection to the k -th iterate $y^k : [0, kT] \rightarrow \Sigma$). For x^k the shear part still only contributes $+\frac{1}{2}$, whereas we need an iteration formula for the $\bar{\psi}_t$ -summand.

Now assume $\dim M = 4$, so $\dim \xi = 2$, $\bar{\psi}_t \in \text{Sp}(2)$, and assume all iterates of x are transversally non-degenerate. The following is a consequence of [38, Appendix 8.1]¹¹

$$\begin{aligned} \bar{\mu}(x^k) &= \begin{cases} 2\lfloor k\tilde{\Delta} \rfloor + 1 & \text{for some } \tilde{\Delta} \in \mathbb{R} \setminus \mathbb{Q} \quad \text{if } x \text{ is elliptic,} \\ k\bar{\mu}(x) & \text{if } x \text{ is hyperbolic.} \end{cases} \\ |x_-^k| &= \begin{cases} -2\lfloor k\tilde{\Delta} \rfloor & \text{if } x \text{ is elliptic,} \\ 1 - k\bar{\mu}(x), & \text{if } x \text{ is hyperbolic.} \end{cases} \\ |x_+^k| &= |x_-^k| + 1. \end{aligned} \tag{B.1}$$

¹⁰We work under the assumption that $c_1(M) = 0$. Thus the canonical bundle $\mathcal{K} = \Lambda_c^{\text{top}} T^*M$ is trivial, using $c_1(\mathcal{K}) = -c_1(M)$. A trivialisation of \mathcal{K} is specified by a choice of global non-vanishing section Ω of \mathcal{K} , whereas a trivialisation of x^*TM is specified by n linearly independent sections $v_1(t), \dots, v_n(t) \in T_{x(t)}M$. Let $f(t) = \Omega(v_1(t), \dots, v_n(t))$. The obstruction for the two trivializations to be compatible, is the homotopy class of the phase map $f/|f| : S^1 \rightarrow S^1$.

¹¹In our case, all iterates of x are transversally non-degenerate, so the elliptic case involves a rotation matrix by an irrational angle $\tilde{\Delta}$ (times 2π) and the first iterate has index $2k + 1$ where $k = \lfloor \tilde{\Delta} \rfloor$, whereas the k -th iterate involves a rotation matrix with angle $k\tilde{\Delta}$ and so has index $2\lfloor k\tilde{\Delta} \rfloor + 1$.

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