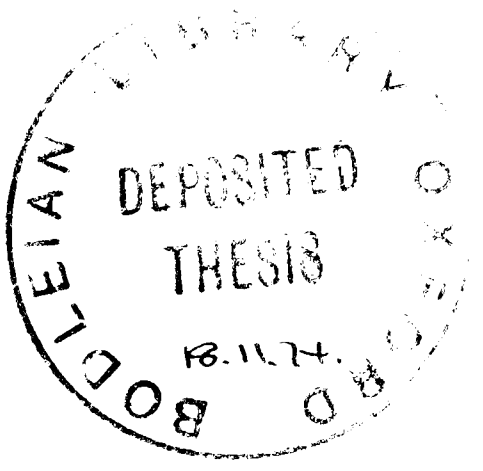
DECENTRALISATION OF NATIONAL PLANNING

DECISIONS OVER TIME

Pradeep K. Mitra



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The extent of my intellectual indebtedness to pioneering contributions in the field of decentralised planning will be evident from the references in the text. But, above all, I should like to thank Professor James A. Mirrlees most gratefully (a) for agreeing very kindly to supervise my research, and (b) for his invaluable guidance, help and encouragement at all times. There is no doubt that he has been the ideal thesis supervisor and I consider myself privileged for having had the opportunity of being one of his graduate students.

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DECENTRALISATION OF NATIONAL PLANNING DECISIONS OVER TIME :

ABSTRACT

This thesis examines the decentralisation of planning exercises in an economy where market behaviour alone is known not to be sufficient to support optimum allocations. Chapter 1 contains an overview of the problem, a justification for undertaking this investigation and a synopsis of results. Chapter 2 evaluates some iterative planning procedures that have been proposed to locate optima in the presence of constant and increasing returns to scale and presents some new results. Chapters 3 to 5 are concerned exclusively with answering some of the theoretical questions that arise in decentralising intertemporally optimal planning decisions.

The reason this problem deserves attention is most conveniently discovered by referring to the existing literature on planning. Most models of decentralised planning, following Oskar Lange, are based on a systematic exploitation of the analogy between planning procedures and market adjustment processes. While this recognition of the formal similarity between free enterprise and socialist systems has produced some formidably impressive literature, it has, unfortunately, tended to focus attention too narrowly on planning in environments where the fundamental theorem of welfare economics is valid, i.e., where the market performs satisfactorily provided the authorities can control the distribution of income. The urge to undertake planning at all in these circumstances is difficult to explain.

Weaker, "local" versions of the fundamental theorem are, however, available: correspondingly, economists have constructed planning procedures which generate satisfactory outcomes when there are increasing returns to scale. Three principal approaches may be distinguished: (a) price-decentralisation with non-competitive behaviour by economic agents, (b) the use of quantity indices by the central planning authorities, and (c) mixed price-quantity procedures. Some of the more significant contributions are briefly surveyed and assessed in Chapter 2, mainly from the point of view of informational efficiency and incentives given to economic agents to follow the rules of the allocation game, the latter consideration being particularly important in non-convex environments. Interest also centres on the case of constant returns to scale: it is well-known that gradient methods exhibit endless oscillations when the technology has this form. New results are then presented. These include an iterative decentralised procedure for the allocation of government contracts with firms setting prices to honour such contracts; in certain cases this converges to a market-clearing equilibrium under decreasing, constant and increasing returns to scale. The chapter also contains a proposal to modify producer behaviour in a way which permits an extension of Malinvaud's algorithm on plan formulation to constant-returns-to-scale technologies.

The attainment of allocations that are intertemporally optimal raises a host of serious problems. Its importance is undeniable, for a society where investment decisions are ineptly taken is not one of whose development we should normally approve. Equally, the market cannot be relied upon to generate optimum consumption and investment for it was never designed to fulfil such long-term functions. It is clear then that the rationale for intertemporal planning derives from market failure; the question arises whether such planning ought to

be of the centralised or decentralised variety. Centralised optimum planning in a dynamic economy is fraught with numerous difficulties. The optimum investment decision today cannot be determined without reference to the optimum plan over the entire planning horizon. This problem requires for its solution technological information to which a central planner typically will not have access. Thus, decentralised exchanges need to be organised to acquire information and to help the authorities form an overall view about technological possibilities. Secondly, lack of complete technological information implies that pursuit of the "local optimality" conditions without periodic recalculation will lead the economy into saddle-point difficulties standard in the theory of optimum growth; hence the informationally decentralised planning procedure must be iterative as well. It is reasonable to suppose that information accumulates in response to the development of the economy and therefore that the iterations occur in real as opposed to computational time. Thirdly, the computation of a multi-period optimum plan is likely to be quite difficult. It would therefore be desirable if the exchanges between central planners and producers took account of this problem and pointed to computationally convenient routines. These considerations establish a presumption in favour of decentralised as opposed to centralised planning. They are explored at some length in Chapter 3. That chapter develops a planning procedure which is informationally decentralised in that it postulates the exchange of price and quantity information at every stage. It is computationally very simple, it does not require memory and it operates in real time. The essential feature of the method is that it ties the pattern of market expectations about prices to the process of learning about the underlying technology. This ensures that the policies followed by the economy

at any stage of the iterations are optimal in the light of the information currently available; the decentralised economy ultimately converges to the optimum balanced growth path.

Informational and computational considerations, although important, are not the only reasons for decentralising planning exercises. The term "decentralisation" in everyday usage connotes delegation of authority. While there are important social, political and administrative reasons in favour of such delegation, there seem to be no economic ones; hence the relative neglect of this aspect of decentralisation in the literature. The most conspicuously economic point that can be made here is that if participants in the planning process feel they lack the power to take decisions, they may well refuse to engage in iterative exchanges of information, thereby jeopardising any attempt at planning. Chapter 4 examines the consequences of allowing producers a limited amount of autonomy in a two-sector model with non-shiftable capital goods. The planners announce the relative profitability of the sectors at every stage of the real-time iterations while producers decide on the sectoral allocation of new investment. But producers are not profit-maximisers. The decentralisation of the authority to take decisions is partially captured by allowing (a) cautious adjustment on their part in the direction of increased profitability and (b) implementation of their proposals at every stage - the latter, of course, being characteristic of any real-time procedure. The operation of this planning algorithm causes the economy to exhibit a rich pattern of behaviour; however, it is possible to demonstrate convergence to the optimum balanced growth path.

Nothing has been said so far about decentralisation of consumption decisions. Much of the planning literature is concerned with specifying the most favourable use of the economy's production apparatus. But no account of decentralised planning would be complete if it did not include an analysis of consumer behaviour as well. Accordingly, Chapter 5 introduces an infinite sequence of finitely long-lived overlapping generations, each of which is not interested in the future of society beyond its own lifetime. The inclusion of price-taking consumers and producers allows one to exploit the fundamental theorem of welfare economics and to regard the problem of attaining the Ramsey optimum as being one of choosing the correct intergenerational distribution of wealth. The government can control the intergenerational distribution by resorting to tax, subsidy, debt and public capital formation policies; this permits it to optimise a basically decentralised system in accordance with social objectives. A characterisation of optimum taxation is obtained, together with a plausible reason for instituting debt and public capital formation policies. The analysis is extended to cover the cases of differential discounting of the future by the government and the private sector and of uncertain lifetime of individuals allowing, in the latter case, for the existence of annuities. The prospects for controlling the Ramsey policy without quantitative directions on consumption look favourable in all cases. These results are worth having, for modern governments exercise a great deal of control over tax, debt and capital formation policies. Finally, the informationally decentralised procedure of Chapter 3 is examined in the sequence-of-generations model and convergence to optimum balanced growth established.

The question of decentralising optimum dynamic planning, important though it is, has received very little attention in the literature. Consequently, the approach and results of Chapters 3 to 5 outlined above are new: they provide preliminary answers to some important questions. (The new results of Chapter 2 are contained within a familiar framework). The main problems in this area appear to be difficult to tackle even in the simplest of cases. This is the reason simple models have been used to obtain insights and to derive some interesting results. In what is a preliminary excursion into uncharted territory, such an approach is easily defended.

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Chapter 1. Introduction

Decentralisation of optimum national planning decisions is widely held to be a virtue. Many formal studies of the question, however, assume environments where the market - arguably the most decentralised of economic institutions - performs quite adequately, so that the impulse to undertake planning (whether it be of the centralised or decentralised variety) is not readily explained. The present contribution examines the possibility of decentralised planning in situations where the market is known not to function satisfactorily; in particular, it is predominantly addressed to the problem of decentralising the optimum investment decision in a model extending over several time periods.

This introductory chapter is devoted to the three remarks made in the opening paragraph. Section I points to the desirability of decentralising national planning exercises, particularly in an intertemporal framework. Section II justifies such investment planning by arguing that the market cannot be relied upon to support optimal allocations in a dynamic context. Thus, Sections I and II taken together explain the motivation for engaging in decentralised investment planning and, therefore, provide a rationale for undertaking the investigation whose results are reported in subsequent chapters. Section III delineates the scope of the inquiry and briefly summarises the subject matter of the remaining chapters.

I. THE DESIRABILITY OF DECENTRALISATION.

1. Uses of "Decentralisation"

The debate on centralisation versus decentralisation has been conducted for a long time.[1] A variety of definitions of the terms "centralised" and "decentralised" have been bandied about in the literature,[2] but no very satisfactory line of division seems capable of being drawn and the debate has been inconclusive. The absence of precise definitions has worried those economists who regard the problem mainly as one of designing alternative resource allocation mechanisms possessing a host of agreeable properties,[3] for the categorisation implicit in clear-cut definitions helps introduce order into chaos. It is reasonable to assert that any discussion of planning procedures along welfare-theoretic lines is also a discussion about design; this point will be examined in Chapter 5. Nevertheless, the unavailability of unambiguously defined categories is no great loss for those who are solely concerned with national planning procedures. There are two senses in which the term "decentralised" is used by planning theorists, and the context makes clear which one a particular author has in mind. Each provides an argument in favour of decentralisation of planning, thus answering the question on which this section is based.

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[1] Hurwicz (1971), for example, has reminded economists that Plato and Aristotle, Thomas More and the anarchist Proudhon concerned themselves with the question.

[2] The relevant papers are those by Hurwicz [(1960a), 1960b), (1969), (1971) and (1972)], T. Marschak [(1959), (1969) and (1972)], together with the conference discussion following Hurwicz (1969) and Marschak (1969).

[3] See the works cited in Footnote 2 and the analysis of J. Marschak and Radner (1971).

(1) Informational Decentralisation : It is reasonable to regard the planning problem as being one of maximising social welfare subject to technological feasibility. Typically, however, no single agent will have enough technological information to undertake this task and one needs an iterative procedure which allows for successive exchanges of messages among economic units. Planning procedures which (a) assume an initial dispersion of information in that a producer, for example, knows his own technological set but not that of anyone else, and (b) permit only limited communication of information among agents are called "informationally decentralised".[1] Such procedures may be used either to mimic the approach of the ideal competitive market to an optimum in Walrasian tatonnement fashion[2] (the market socialism of Lange, Arrow and Hurwicz), or to help the authorities formulate an optimum national plan. This latter approach uses what is essentially an informationally decentralised procedure[3] to transmit technological information from producers to a Central Planning Board (henceforth the CPB) in response to price or quantity signals to enable the latter to

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[1] For a systematic treatment, together with an attempt to formalise the notion of limited communication, see Hurwicz [(1960a), (1969)].

[2] This is why the informational requirements of such a procedure are thought to be akin to those of the competitive market mechanism.

[3] The qualification is necessary (a) because the CPB initially needs to know a feasible plan, and (b) because the pioneering work of Malinvaud (1967) requires the CPB to remember all past proposals submitted to it. The former violates the requirement of initial dispersal of information but the discussion of Malinvaud's work in Chapter 2 shows this is not a serious restriction. It is argued by Hurwicz (1972) that the latter gets around the constraint imposed by limited communication if memory is sufficiently long. But, as mentioned earlier, no completely satisfactory definition of "decentralisation" is available, and it is not unreasonable to regard the Malinvaud procedure as being informationally decentralised.

solve for a feasible plan. Here the CPB progressively forms a clearer idea of underlying technological possibilities : the feasible plan at every stage of the iterations is optimal in the light of centrally available information and is, in the limit, fully optimal.

The preparation of the optimum plan is therefore seen to be equivalent to the iterative solution of the allocation game in both approaches. But it is necessary to pay some attention to the limited capacity of economic agents to process information that they are required to handle during the course of the iterations. In the plan formulation approach, the computations performed by the CPB will get more and more complex as information accumulates and the burden of solving the constrained maximisation problem, if imposed entirely on the CPB, may well be excessive. It is worth reminding oneself that great advantages have been claimed for the market with respect to the information handled: for example, it has been estimated that the amount done in only a part of the United States wheat market would involve 20 or 30 IBM 7090's.[1] Thus, despite spectacular advances in computer technology, no central planning board could solve the millions of excess demand equations, even if all the necessary information were made available centrally. The problem becomes almost impossible when one is considering investment planning necessarily extending over several time periods. Consequently, decentralisation might be undertaken to reduce complicated central computations: in the intertemporal case this usually involves real-time algorithms (so that the CPB can afford to forget what has already been

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[1] See the discussion of Malinvaud's paper in Malinvaud and Bacharach (1967).

implemented) and manageable, memory-less procedures. All this is examined in Chapter 3.

The question of informational decentralisation is further discussed in Chapters 2 and 3. But one can also be interested in (2) Decentralisation of Control : This refers to delegation of the authority to take and implement decisions. Producers may not be willing to follow the rules of the iterative game outlined in (1) above and act as passive purveyors of technological secrets without being allowed authority over certain spheres of economic activity. This is an area which, by contrast with (1), has received virtually no attention in the literature. The omission is regrettable because this is the sense in which the term "decentralised" is employed in everyday usage. Economists have been keenly aware of this deficiency,[1] but perhaps one reason not much progress has been made is that there seem to be no purely economic. as opposed to social and political arguments in favour of decentralisation of authority. The issue is taken up in Chapter 4, where a simple procedure which allows producers some latitude in decision-making is analysed. The most conspicuously economic point that can be made here is that, in the absence of this kind of decentralisation, (a) informationally decentralised procedures may break down because of local resistance to following the rules of the game, and (b) the whole process of plan-implementation as opposed to plan-formulation might be in jeopardy, even if there were no problems with (a). It is noteworthy that discussions of plan-formulation implicitly assume that an optimum plan only needs to be announced for it to be fulfilled.[2]

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[1] See, for example, Ellman (1968), Hurwicz (1971) and Heal (1973).

[2] See Malinvaud (1967) which is the most significant contribution to the literature on the preparation of optimum plans. Hurwicz (1971) has drawn attention to this difficulty.

One would, ideally, like planning procedures to be decentralised in both ways. There is a limited sense in which the distinction between the two could become less tenable in dynamic investment planning. For a real-time informationally decentralised procedure, unlike a tatonnement, allows producers to implement their current proposals; this particular feature is a concession (albeit an imperfect one) to the idea of decentralisation of authority.

2. Intertemporal problems

The presumption so far established in favour of decentralisation is valid in both a timeless as well as in a dynamic world. But the problems surrounding any attempt at completely centralised investment planning make the case for intertemporal decentralisation compelling. The difficulties arise precisely because investment decisions link successive time periods. To begin with, the literature on optimum investment paths demonstrates that investment today cannot be determined unless the entire investment policy up to the end of the planning horizon is known. The latter requires for its solution technological information to which a planner usually will not have access - hence the need for policies which explore the production set. Secondly, the information necessary for informed investment planning is, at best, likely to accrue over real as opposed to computational time in direct response to realised investment plans which map out relevant portions of the production set. This suggests that one will have to move away from the usual gradient methods based on the Walrasian tatonnement in favour of algorithms which operate in real time. Thirdly, in the postulated absence of perfect foresight at the beginning of the planning period, one has to make assumptions about the structure and optimality of expectations

entertained by agents in planning for the future. Finally, such "market uncertainty" apart, it is often the case that the investment planner is subject to increasing technological uncertainty as he peers into the future. This consideration not only makes central planning no easier, it raises the following problem for any attempt at market-like decentralisation as well : even if economic agents succeed in choosing the correct volume of initial investment (by itself a non-trivial problem, as mentioned above) the usual local optimality conditions alone will not generate the optimum policy. The numerous issues to which intertemporal planning gives rise seem to make the basic problem different almost in kind from its counterpart in the static case.

The difficulties outlined in the previous paragraph lead one to ask the author of any investigation into intertemporally decentralised planning why the task is worth attempting at all. By way of justification, it is enough to indicate the desirability of dynamic investment planning; once this is established, the case for doing it in a decentralised fashion follows from the remarks above. The answer is straightforward: the economic welfare of a society will depend in large measure on whether it takes well-informed investment decisions. It is the ability of the planners to strike a balance between their conflicting responsibilities to different generations, the skill with which they avoid bottlenecks in key industrial sectors and, above all, the prescience with which they forecast the future circumstances in which present-day acts of private investment will come to fruition, that ultimately decide the success or failure of a society's experiment in planning. This is by no means an easy task: it requires that planners announce a forecast of the most likely course of events that is firm enough on

the one hand to enable producers to decide on their investment plans, and flexible enough on the other to permit the exercise of entrepreneurial flair and initiative.[1] For the successful entrepreneur is often one who is better informed about the state of the market than his peers; the essence of imaginative planning consists in utilising and developing such scarce skills in the pursuit of socially desirable objectives. If the planners succeed in ensuring that the investment decision is judiciously taken, they may well be able to leave the current allocation of goods and services to the market without causing serious problems. But if no attempt is made to influence investment plans in a changing and uncertain world, especially when it is known that the market fails to function satisfactorily in these areas, I doubt if we should have much reason to approve of the pattern of that society's development.

These remarks point to (a) the desirability of dynamic investment planning and (b) the need for decentralising that form of planning. It is natural to inquire how far, if at all, the market can be relied upon to generate the optimum amount of saving and investment. The possibility of market failure was mentioned in the previous paragraph; it is worth looking more closely at this question. This, it will be remembered from the introductory remarks, is the subject matter of Section II.

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[1] A particularly good account of the scope and purpose of indicative planning is to be found in Meade (1970).

II. THE LITERATURE

The point of departure for the early literature on market socialism[1] was the recognition that both capitalist and socialist economies face the same problems when it comes to securing an optimum allocation of resources. Consider two different theoretical constructions: (a) the ideal free-enterprise market economy and (b) the ideal socialist economy.[2] It is clear that both mechanisms have to solve the same marginal equivalences characterising Paretian states. Secondly, these simultaneous equations are solved iteratively in each case. The method most extensively analysed for the free-enterprise economy is the Walrasian tatonnement: this is a reliable computing device for market equilibrium provided certain restrictions are imposed[3]; its analogue in the socialist economy is the gradient method for locating the saddlepoint of the Kuhn-Tucker kernel the planner is concerned to maximise. These methods converge provided that the underlying environment is strictly convex.[4, 5]

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[1] I have in mind the work of Lange (1936) and Taylor (1929). Lange's ideas have been formalised by Arrow and Hurwicz (1960); Taylor's by Malinvaud (1967). See also Lerner (1946) and Bergson (1948).

[2] In a stimulating contribution, Bliss (1972) refers to discussions centred around these economies "The Economics of Fairyland". The general point being made in the text though, is best brought out in terms of such stark contrasts, as Bliss readily recognizes.

[3] An early account is Hicks (1946); a fuller analysis is to be found in Arrow and Hahn (1971). That it is not perfectly reliable is illustrated in Gale (1963) and Scarf (1960).

[4] A compendium of gradient methods is Arrow, Hurwicz and Uzawa (1958).

[5] I return to this question in Chapter 2.

This recognition of the formal similarity between the two systems is an impressive leap of intuition for it provides preliminary guidance to what needs to be done when some of the circumstances making for an ideal operation of the market do not obtain. Recall that the fundamental theorem of welfare economics asserts that the full optimum for a continuous, individualistic social welfare function can be realised in decentralised fashion by price-taking utility-and profit-maximisers for some distribution of income provided that preference structures and technology satisfy certain convexity requirements. One can readily conceive of two ways in which the ideal socialist economy can improve on the market. To begin with, it is well known that there is no guarantee that the optimum distribution of income implied by the fundamental theorem will emerge unaided in the market. The market-socialist literature has not directly addressed itself to this difficulty but it has always implicitly assumed that the State uses taxes-cum-subsidies to control the distribution of income among consumers.[1] Secondly, the assumption of price-taking is reasonable in a free-enterprise economy only when there are a large number of traders in the market. In the ideal socialist economy, however, it is possible to argue that the CPB can instruct enterprises to be price-takers.

What the planner is supposed to do when convexity conditions are not satisfied is less clear. It is reasonable to suppose that the urge to plan occurs precisely in those circumstances where optimal

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[1] See the comments by Lange (1936), Lerner (1946) and Dobb (1933). An evaluation of some of these ideas appears in Sen (1973).

allocations cannot be decentralised through the market owing to externalities, indivisibilities and increasing returns to scale. The last two properties are especially characteristic of investments, as a casual glance at the development economics literature will confirm. Weaker, "local" versions of the fundamental theorem are available here; equally, theorists have developed gradient methods which locate local optima in the presence of production non-convexities.[1] Decentralisation algorithms in the presence of externalities have also been worked out;[2] the same is true of public goods.[3]

But the area where the market performs not at all satisfactorily is the area of investment planning. This was certainly clear to Lange, the progenitor of market socialism: he had occasion to observe [Lange (1967)] that the market never did fulfil such long-term functions. The reasons for this are well-known. Indivisibilities and increasing returns are one and have been mentioned. Secondly, there are externalities of the Prisoners' Dilemma type analysed by Sen [(1961), (1967)] and Marglin [(1963b), (1967)] which call for collective as opposed to individual action to generate Paretian outcomes. Thirdly, the non-existence of future generations today will, in general, cause the private sector not to make the optimum intergenerational transfers implied by the fundamental theorem. Fourthly, the market, typically, will not have the necessary technological information to determine the Ramsey optimum and, therefore, the correct initial investment decision. It is clear then that a society cannot rely upon the unaided operation of the free market to realise intertemporally optimal outcomes:

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[1] See Arrow and Hurwicz (1960), Heal [(1969), (1971), (1973)] and Aoki (1971). Their work is examined in Chapter 2.

[2] Aoki (1970), Camacho (1970).

[3] Malinvaud [(1971), (1972)], Drèze and de la vallée Poussin (1971).

this establishes the need for undertaking planning exercises.

III. THE SEQUEL

1. The Approach

What follows in the rest of the investigation is a theoretical analysis of some of the problems any investment planner would face in formulating and implementing an optimum programme. The reader may expect to find some discussion of the practical difficulties likely to be encountered in intertemporally decentralised planning. In this, I fear, he will be disappointed. The justification for limiting the scope of the inquiry in this way is a simple one. We have seen that there are considerable theoretical problems that arise in investment decision-making. Since no planner can wish these difficulties away, I believe it is premature to go into questions of practical detail at this stage. To adopt this position is to establish a set of priorities for research in this field; it is not to deny the importance of other issues, e.g., an appreciation of the socio-political environment in which the planning exercise is supposed to operate, the effect on participants' morale and performance of the power to take decisions at grass-roots level or the resistance of organised groups adversely affected by central attempts to implement the consequences of a particular social valuation.

2. Synopsis of Contents

It is convenient to separate the subject matter of each chapter into assessment of existing literature and new results.

A. Assessment of Existing Literature

Chapter 2 is concerned with the decentralisation of production decisions when returns to scale are non-decreasing. Competitive price-decentralisation of optima is no longer feasible and there are three

possible approaches: (a) price decentralisation with non-competitive behaviour by agents, used by Arrow and Hurwicz (1960), (b) the use of quantity indices by the CPB, suggested by Heal (1969) and (c) mixed price-quantity procedures, due to Heal (1971) and Aoki (1971). Parts of the chapter examine these ideas and assess the literature in the light of certain desirable properties theorists would like decentralised procedures to possess.

Chapter 3 looks at some problems in optimum dynamic planning and includes an evaluation of the contributions by Arrow and Kurz (1970), Mirrlees and Stern (1972) and Infante and Stein (1973).

B. Catalogue of New Results

Chapter 2 suggests one way of extending Malinvaud's (1967) informationally decentralised procedure for planning to the case of constant-returns-to-scale technologies. This is done by postulating that firms minimise costs and alter scale in the direction of increased unit profitability, thereby enabling one to determine the size of the enterprise. Secondly, the chapter includes a simple planning procedure which is feasible within reasonable limits and convergent without in general being monotonic. Finally, I also present a new iterative procedure where the CPB announces targets it wishes to see fulfilled and where firms have the power to set prices at which they can honour these contracts. The CPB then revises its quotas and the process goes on as before. This method has some appealing properties and is shown to converge to market-clearing equilibrium under decreasing, constant or increasing returns to

scale in certain special cases.

Chapter 3, as mentioned above, is concerned with the main difficulties in decentralising the investment decision. It highlights the problems of information, expectations and central computation that arise in this area. There follows an examination of the use of market-like price and quantity signals in investment planning - a subject which has received virtually no attention in the literature. I develop a planning procedure which is informationally decentralised in that it requires the exchange of price and quantity information at every stage. It is computationally very simple, it does not require memory and it operates in real time. The essential feature of the method is that it ties the pattern of expectations entertained by the CPB to the process of learning about the underlying technology. This ensures that the policies followed by the economy at any stage are optimal in the light of the information currently available; the decentralised economy ultimately converges to the optimum balanced growth path.

Chapter 4, unlike Chapter 3, assumes that the CPB possesses complete technological information. But it does not exercise complete control over producer behaviour. The setup is, admittedly, rather special: it is a simple two-sector model with non-shiftable capital goods. The CPB announces the relative profitability of the sectors at every stage of the (real-time) iterations while producers decide on the sectoral allocation of investment. But producers are not profit-maximisers. The decentralisation of the authority to take decisions is captured (imperfectly no doubt) by allowing (a) cautious adjustment

on their part in the direction of increased profitability, and (b) implementation of their proposals at every stage - the latter, of course, being characteristic of any real-time procedure. The operation of this planning algorithm causes the economy to exhibit a rich pattern of behaviour but it is nevertheless possible to demonstrate convergence to the optimum balanced growth path.

Finally, Chapter 5 adopts a slightly different approach to the problem. The analysis of Chapters 2, 3 and 4 is in the spirit of Hurwicz and Malinvaud in that certain desirable properties are laid down a priori; alternative planning procedures are then presented and evaluated in the light of these criteria. This is the design-oriented view of decentralisation to which reference was made in Section I. (I have called it the "systems engineering" approach in Chapter 5). As an alternative, it is worth pursuing the following route. Let us suppose that, from the point of view of the planner, the network of economic relationships in the private sector is a datum. In particular, I adopt the paradigm most widely used by economists, viz., decentralised competitive behaviour by economic agents. Is there a way in which this plausible and workable system of decentralised behaviour can be optimised? The problem of optimum accumulation is viewed as one of optimum distribution among (overlapping) generations; the CPB uses taxes, subsidies, debt and public capital formation policies to select the intergenerational distribution implied by the fundamental theorem of welfare economics.

The formulation has several advantages. To begin with, this way of posing the problem establishes it firmly within the standard welfare-theoretic framework and allows the application of familiar arguments to growth paths. Secondly, it introduces the decentralisation

of consumption decisions. And thirdly, it can be (and is) combined with the informationally decentralised procedure of Chapter 3 to handle both production and consumption decisions.

A sequence of models is analysed under various assumptions. It is found that certain patterns of intertemporal lump sum taxation needed to attain the Ramsey optimum are particularly simple; furthermore, these taxes have interesting consequences for debt and public capital formation policies. The analysis is extended to cover the cases of differential discounting of the future by the CPB and the private sector and of uncertain lifetime of individuals allowing, in the latter case, for the existence of annuities. The prospects for controlling the Ramsey policy in decentralised fashion (i.e., in the absence of quantitative controls on consumption) look favourable in all cases.

3. Concluding Remark

The reader will notice that some highly simplified models are used in the following chapters. Further work in this area will, no doubt, be concerned with the considerable problems raised by multi-sector optimising models, the effects of introducing more satisfactory assumptions about the formation of expectations, the prospects for controllability of the Ramsey policy when agents' foresight about the future course of taxes-cum-subsidies is less than perfect and systematic attempts at decentralisation of the authority to take decisions. But the results available (as outlined in the catalogue above) suggest that the use of simplified models highlights the important problems and helps point towards these directions for further research.

Chapter 2 : Planning When Returns to Scale are Non-Decreasing.

1. THE PROBLEM

It was pointed out in the introduction that one reason societies embark on planning is non-convexity of the production set. It is well known that, under these circumstances, the market cannot be relied upon to support an optimum allocation[1]: consequently, price-guided planning procedures which call for competitive price-taking behaviour are unlikely to possess agreeable welfare implications. Three approaches then become possible, (1) price-decentralisation with non-competitive rules of behaviour for economic agents, (2) the setting of production targets by the CPB, and (3) mixed price-quantity planning procedures.

This chapter is concerned exclusively with ways of surmounting some problems posed by constant and increasing returns to scale. Section II begins with the Arrow - Hurwicz contribution on decentralised decision-taking under increasing returns to scale. It then examines the desirable properties theorists would like decentralised methods of plan-formulation to possess, presents a simple iterative procedure which helps ensure successive proposals in "tatonnement" routines are mutually consistent (and therefore, in principle, implementable) and, finally, explores a modification which extends

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[1] It is being assumed that non convexities are "substantial" when there are increasing returns to scale. For a discussion of these matters, see Arrow (1971a), Arrow and Hahn (1971).

Malinvaud's plan formulation algorithm to a constant-returns-to-scale environment. Section III turns to quantity-guided methods and evaluates the debate regarding the use of these methods as against the price-guided approach. It presents a model where the CPB sets quotas and firms bid for contracts by setting prices and then discusses the planning routines due to Heal and Weitzman. Section IV attempts to see what lessons may be learnt from mixed price-quantity procedures due to Heal, Aoki and Portes and Section V contains a brief summing-up.

II. PRICE - GUIDED PROCEDURES

The problem of resource allocation is the familiar one of maximising an economywide utility function subject to technological constraints. Necessary conditions for a constrained maximum are easily written down; if, furthermore, one is prepared to assume concavity in preference and production, it is possible to appeal to the Kuhn-Tucker theorem and identify the constrained maximum with the saddlepoint of the appropriate Lagrangian. This has two advantages: (a) the constrained maximum is supported by price-taking economic agents and (b) with strict concavity, the saddlepoint can be approached in tatonnement fashion using the gradient methods elaborated, for example, in Arrow, Hurwicz and Uzawa (1958). This constitutes the basis for Lange's optimism with regard to the possibility of market socialism.

The results (a) and (b) mentioned above fail to hold when there are increasing returns to scale; result (b) is no longer true even when there are constant returns to scale. Firstly, economic agents are faced with the possibility of multiple equilibria.

Secondly, the gradient methods might not converge to equilibria. Thirdly, one can hope to do no better than design procedures which converge to local optima. Fourthly, the local optimum might occur at a point of minimum private profit, so that an incentive problem arises. The Arrow-Hurwicz solution to these difficulties is examined below.

1. The Arrow - Hurwicz model

The Arrow - Hurwicz (1960) contribution to the decentralisation problem is the earliest and one of the most significant. It is based on a simple, yet ingenious, idea. During the famous debate on competitive organisation and increasing returns, Sraffa (1926) argued that equilibrium under increasing returns was compatible only with imperfectly competitive rules of behaviour. The idea is, on reflection, unsurprising. Firstly, competitive behaviour supports optimal outcomes only in convex environments and there is no necessary reason why it should carry over without modification to other situations.[1] Secondly, one could argue that monopolies often arise in industry because of increasing returns to scale; therefore, monopolistic behaviour may well be appropriate when the technology exhibits this particular property. Accordingly, Arrow and Hurwicz demand imperfectly competitive behaviour of agents in their model.

Formally, the equivalence between (local) constrained maxima and (local) saddle points is established by concavifying the Lagrangian in two different ways; to each there corresponds a gradient method which has an appealing economic interpretation. The problem is to maximise (with respect to choice of x and y)

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[1] Profit maximising competitive behaviour by firms is compatible with increasing returns if economies of scale are external to firms and internal to the industry, provided that each firm believes it is operating under constant returns to scale. This approach has been explored by Chipman (1970) who talks about "parametric external economies of scale

(1) $U(x)$, where x is an n-vector of commodities and U is the economy-wide utility function

subject to

$$(2) \qquad 0 \leqq x \leqq \sum_{j=1}^{m} g(y_j) + w,$$

$$(3) \qquad y \geqq 0$$

where $g(y_j)$ is the n-vector of commodities produced by the j^{th} process operated at scale y_j $(j = 1,\ldots,m)$ and w is the n-vector of initial resources.

The Lagrangian is concavified (a) by the method of transformed constraints and then (b) by the method of quadratic modification.

Denoting

$$(4) \qquad g = \sum_{j=1}^{m} g(y_j) + w - x$$

$$(5) \qquad q_i = 1 - (1-g_i)^{1+\eta_i}, \ (i = 1,\ldots,m), \text{ where } g_i \text{ is the } i^{th}$$

component of the vector g in (4),

we define the Lagrangian with transformed constraints as

$$(6) \qquad L^*(x,y,p|\eta) = U(x) + p.q, \text{ where a dot denotes inner product.}$$

Then

Theorem 1: Under certain regularity conditions,[1] for all η_i's chosen sufficiently large, a necessary and sufficient condition for a locally optimal allocation is that (bars denote equilibrium values)

(a) $\bar{x}$, $\bar{y}$ maximise $U(x) + \bar{p}.q$

(b) $\left.\begin{matrix} \bar{g} \geqq 0 \\ \bar{p} \geqq 0 \end{matrix}\right\}$ complementary slackness

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[1] The Kuhn-Tucker constraint qualification is appropriate here. This may be stated as follows. Let $\bar{z}$ maximise $U(z)$ subject to $Gz \geqq 0$, $z \geqq 0$. Let $\bar{z}$ belong to the boundary of the constraint set of points z satisfying $Gz \geqq 0$, $z \geqq 0$. Let the inequalities $G\bar{z} \geqq 0$, $I\bar{z} \geqq 0$ (where I is the identity matrix of order equal to the dimensionality of z) be separated into $G_1\bar{z} = 0$, $I_1\bar{z} = 0$ and $G_2\bar{z} > 0$, $I_2\bar{z} > 0$. Then it is assumed that, for each $\bar{z}$ of the boundary of the constraint set, any vector differential dz satisfying the homogeneous linear inequalities $\bar{G}_1\, dz \geqq 0$, $\bar{I}_1\, dz \geqq 0$ (where $\bar{G}_1$ and $\bar{I}_1$ denote partial derivatives evaluated at $\bar{z}$) is tangent to an arc contained in the constraint set; that is, to any dz satisfying $\bar{G}_1 dz \geqq 0$, $\bar{I}_1\, dz \geqq 0$, there corresponds a differentiable arc $z = a(\theta)$, $0 \leq \theta \leq 1$, contained in the constraint set, with $\bar{z} = a(0)$, and some positive scalar λ such that $\left[\frac{da}{d\theta}\right] = \lambda.dz$. For a proof of Theorem 1, see Arrow and Hurwicz (1956).

If one defines a new variable

(7) $r_i = (1 + \eta_i)\, p_i\, (1-g_i)^{\eta_i}$, $(i = 1,\ldots n)$, it is clear from (6) that (subscripts after L denote partial derivatives)

(8) $L^*_x = U_x - r$

(9) $L^*_{y_j} = r \cdot g'(y_j)$ $(j = 1,\ldots,m)$ where $g'(y_j) = \left[\frac{dg^1(y_j)}{dy_j}, \ldots, \frac{dg^n(y_j)}{dy_j}\right]$

and $g^i(y_j)$ is the amount of good i $(i = 1,\ldots,n)$ produced by the j^{th} process operated at scale y_j.

The corresponding gradient method is given by

Theorem 2: Under certain regularity conditions,[1] for all η_i's chosen sufficiently large, the dynamic system (dots over the variables denote derivatives with respect to time):

$$\text{(a)} \quad \dot{x}_i = \begin{cases} 0 \text{ if } U_{x_i} < r_i \text{ and } x_i = 0 \\ U_{x_i} - r_i, \text{ otherwise} \end{cases} \qquad (i = 1,\ldots,n)$$

$$\text{(b)} \quad \dot{y}_j = \begin{cases} 0 \text{ if } r \cdot g'(y_j) < 0 \text{ and } y_j = 0 \\ r \cdot g'(y_j), \text{ otherwise} \end{cases} \qquad (j = 1,\ldots,m)$$

$$\text{(c)} \quad \dot{p}_i = \begin{cases} 0 \text{ if } g_i > 0 \text{ and } p_i = 0 \\ -q_i, \text{ otherwise} \end{cases} \qquad (i = 1,\ldots,n)$$

converges to a local optimum of allocation.

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[1] The regularity conditions are stronger than the constraint qualification to permit the saddle-point to be approached by the gradient method. For a complete spelling out of these conditions, see Arrow and Hurwicz (1956).

Thus, the agent who is in charge of determining consumer demands, x ("helmsman") together with those responsible for determining process scales, y ("process managers") are asked to behave as they would in the concave case: they move towards increased benefits at specially defined prices, r. There are two complications. Firstly, the CPB has to choose the vector η and communicate it to the price-adjusting auctioneer ("custodian"). Secondly, the custodian has to determine r from the definition (7). Notwithstanding these complications, the elements of decentralisation are quite pronounced.

The Lagrangian with quadratic modification is

$$(10) \quad \hat{L}(x, y, v, p \mid \lambda) = L - p.v - \lambda . (g - v),$$

where L is the unmodified Lagrangian and λ is a scalar.

This leads to

Theorem 3: Under certain regularity conditions[1] and for λ sufficiently large, a necessary and sufficient condition for a locally optimal allocation is that

(a) $\bar{x}, \bar{y}$ maximise $U(x) + \bar{p}. (g - v) - \lambda (g - v)^2$

(b) $\bar{g} = \bar{v}$; $\left.\begin{matrix} \bar{v} \geq 0 \\ \bar{p} \geq 0 \end{matrix}\right\}$ complementary slackness

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[1] The Kuhn - Tucker constraint qualification. For a proof of Theorem 3, see Arrow and Solow (1958), section 4, who base their analysis on a lemma due to Debreu (1952).

Now define

$$(11) \qquad p^f = p + 2\lambda(v - g)$$

Then, it is clear that

$$(12) \qquad \hat{L}_x = U_x - p^f$$

$$(13) \qquad \hat{L}_{y_j} = p^f \cdot g'(y_j), \quad (j = 1,\ldots,m)$$

and we have the gradient method described by

Theorem 4: Under certain regularity conditions[1] and for λ sufficiently large, the system of differential equations

$$(a) \qquad \dot{x}_i = \begin{cases} 0 \text{ if } U_{x_i} < p^f \text{ and } x_i = 0 \\ U_{x_i} - p_i^f, \text{ otherwise} \end{cases} \qquad (i = 1,\ldots,n)$$

$$(b) \qquad \dot{y}_j = \begin{cases} 0 \text{ if } p^f \cdot g'(y_j) < 0 \text{ and } y_j = 0 \\ p^f \cdot g'(y_j), \text{ otherwise} \end{cases} \qquad (j = 1,\ldots,m)$$

$$(c) \qquad \dot{v}_i = \begin{cases} 0 \text{ if } p_i^f > 0 \text{ and } w_i = 0 \\ -p_i^f, \text{ otherwise} \end{cases} \qquad (i = 1,\ldots,n)$$

$$(d) \qquad \dot{p}_i = v_i - g_i \qquad (i = 1,\ldots,n)$$

$$(e) \qquad p_i^f = p_i + 2\lambda \dot{p}_i \qquad (i = 1,\ldots n)$$

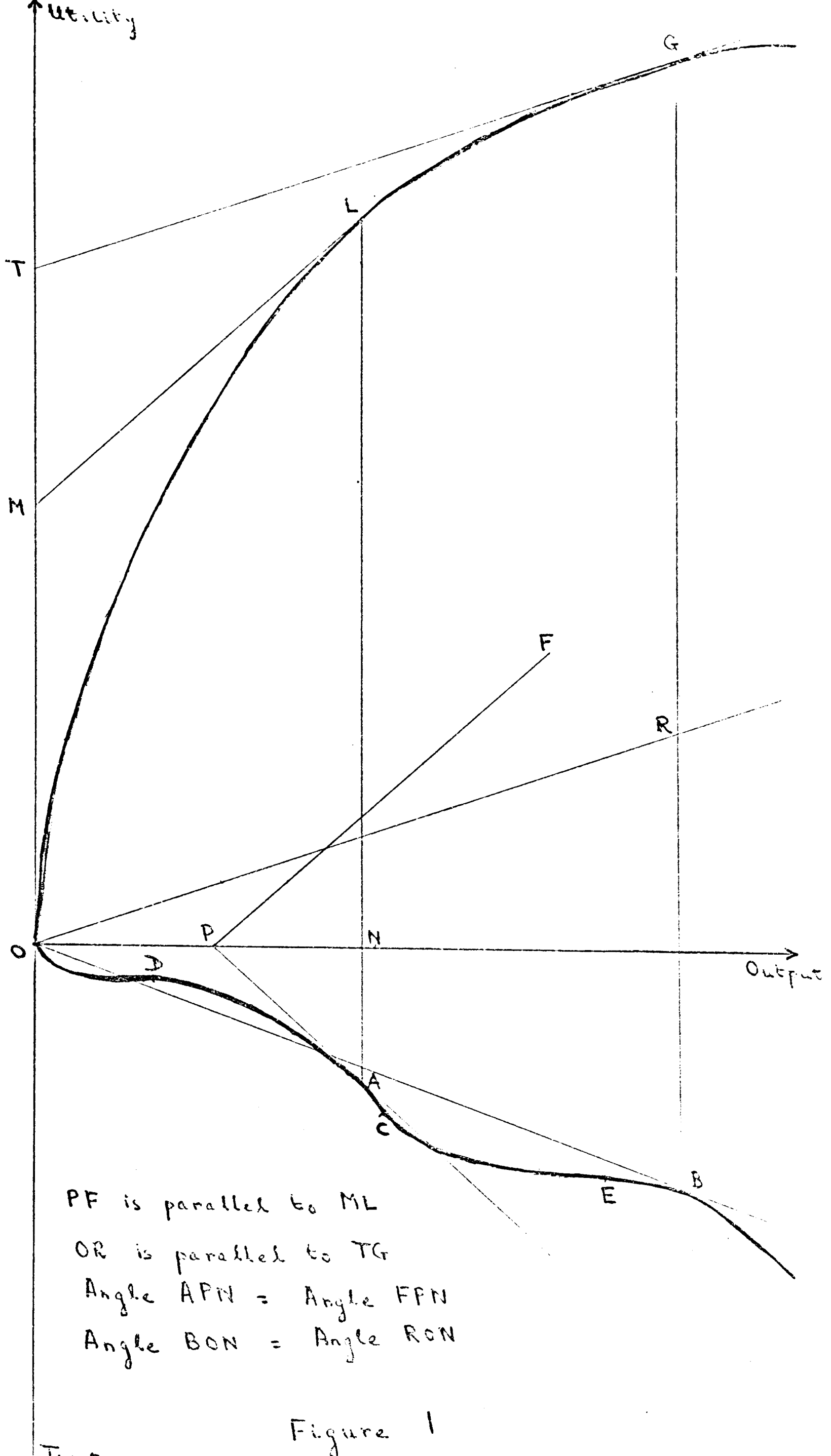
converges to a local optimum of allocation.

Remark: p^f may be considered as an expectation of prices in the near future formed by extrapolation of current rates of change.

As before, the elements of decentralisation are clear. The helmsman and process managers move towards increased benefits computed at the expected prices defined by (d) and (e) of Theorem 4. The complications, once again, are that λ has to be centrally chosen and that the custodian has to follow complex rules of price formation according to the rules laid down above. It should be noted that the theorems are true for large λ; this implies that expected price must be sufficiently sensitive to current rates of change.

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[1] The Kuhn-Tucker constraint qualification. See Arrow and Solow (1958) and, also, the paper by Arrow and Hurwicz (1958) cited there.



PF is parallel to ML

OR is parallel to TG

Angle APN = Angle FPN

Angle BON = Angle RON

Figure 1

Before moving on to an assessment of the Arrow - Hurwicz contribution, it is worth looking at the solution diagrammatically.[1] Figure 1 sketches the case where there are two goods: one input used to produce one output with a non-convex production technology. The upper part of the figure shows how utility varies with output; the lower part shows the aggregate production function concavified (locally) by either method discussed above. (A, L) and (B, G) are the two possible equilibria; the slope of the utility function at L (respectively G) is equal to the reciprocal of the slope of the production function at A (respectively B), as the necessary conditions of the theorems above require. Secondly, this slope is the price of the output in terms of the input, where prices are to be interpreted in the sense of definitions (7) and (11). Thirdly, it is to be noted that the transformed production function ODACEB is locally strictly concave at both A and B: this is ensured by either method of concavification. Fourthly, it is easily checked that the production sector (a) will not reach A unless the gradient method is started within the range CD and (b) will not reach B if it is started to the left of E. This illustrates the danger of the economy's failing to reach the global optimum G. Nothing better than local convergence can be hoped for within the framework of gradient methods using local information alone.

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[1] This figure has also been used by Heal (1973) who, however, restricts his attention to the strictly concave case.

Assessment

A reading of the Arrow - Hurwicz paper suggests the following comments.

(1) Their results are true provided that η_i (i = 1,..., n) and λ are very large. These values have to be chosen centrally before the gradient method can be initiated; one would therefore wish to have more precise guidelines on this particular choice. While from the point of view of formal rules of operation any very high value will suffice, it is possible to doubt if economic agents would take consequent extreme values of r (in (7)) and p_i^f (in (11)) seriously. It is reasonable to suppose that decentralised decision-takers doubt the credibility of centrally provided information which is, in their opinion, exaggerated in one direction or another; the introduction of such doubts must cause serious difficulties for the operation of any iterative procedure. A more complete model would take account of such considerations and suggest optimum values for η_i and λ; equally, it is hard to see how, in the absence of complete technological information, the planner could discover what these values were.

(2) A problem not unrelated to the divergences between true prices and the special prices used in the gradient methods is one of incentives. The fact that the Lagrangian is made locally strictly concave ensures that, at the local optimum, the marginal profitability of a process in use evaluated at the specially defined prices is not changing from negative to positive values. This implies that the adjustment processes of Theorems 2 and 4 above are satisfactory incentivewise at the specially defined prices. But, given that there is a large divergence between actual and specially defined prices on account of high values of η_i's and λ the processes may not at all be satisfactory incentivewise at

actual prices. In other words, hill-climbing moves at specially defined prices may actually be unprofitable at actual prices and the local optimum may well correspond to minimum private profit at actual prices.

(3) It can be argued that the convergence property enunciated in the theorems is not completely satisfactory. What Arrow and Hurwicz establish is that prices converge to a price vector associated with the optimum programme and not necessarily that the utility sequence does likewise.[1] Furthermore, Uzawa (1958) has shown that if the adjustment procedure is formulated as a discrete-time tatonnement, the convergence result is further weakened to the following: provided the coefficients of adjustment at every stage are small, the system stays within an ε - neighbourhood of the optimum after a large number of iterations. But a small coefficient of adjustment could delay convergence considerably.

(4) A further difficulty with the Arrow-Hurwicz gradient method has been pointed out by Malinvaud (1967), who attributes the finding to Gorman. The rule followed by the helmsman in Theorems 2 and 4, part (a) yields a value of x which is not invariant with respect to a monotonic transformation of the utility function. This is unsatisfactory: it implies, for example, that the speed of convergence may be low for certain choices of the utility function. Malinvaud suggests that in the limiting version of the gradient method, where utility and profits are maximised instantaneously with respect to prices, the latter being adjusted gradually as before, the helmsman should be required to maximise utility subject to a national income budget constraint at the

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[1] I return to this point on pp. 30-31.

ruling prices. This gets rid of the Gorman problem and formally makes the procedure equivalent to a price tatonnement in a one-consumer economy.

(5) The Arrow-Hurwicz contribution is one of the first attempts at applying gradient methods to the resource allocation problem[1]. There is no doubt that in this field it has been singularly influential, for all subsequent contributions of any importance have employed gradient methods as well, as we shall see later in the chapter. It is also fair to say that, as far as increasing returns to scale is concerned, there has not been a great deal of progress beyond the point where Arrow and Hurwicz left the subject nearly a decade and a half ago. Thus, no more than local convergence results are available[2]. The problem of devising a structure of incentives which might support social optima under increasing returns is yet to be satisfactorily solved[3]. The reason for this would seem to be that welfare economics under increasing returns to scale is still a relatively unexplored area. The authors are right to state that increasing returns call for imperfectly competitive behaviour[4]; corresponding to such behaviour, one has locally convergent gradient methods. But one must await careful analysis of monopolistic competition in a general equilibrium framework before beginning to understand what sort of information, communication among agents and the like will be needed to decentralise optima as monopolistically competitive equilibria. This would provide one with some sort of guidance to the decentralised attainment of <u>full</u> optima under increasing returns to scale.

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[1] Earlier attempts are to be found in Samuelson (1950) and the sequence of papers throughout the 'fifties by Arrow, Hurwicz and Uzawa mentioned earlier.

[2] Heal's [(1969), (1971)] and Aoki's [(1971)] gradient methods are, in addition, feasible and monotonic in the sense of Malinvaud.

[3] Heal (1971) and Aoki(1971) have given some thought to this problem.

[4] That this is not always true is pointed out in footnote 1 on page 19.

Such preliminary investigations as are available[1] suggest that the problem is a difficult one. Equally, without such theoretical developments, it is not easy to see how the Sraffa-Arrow-Hurwicz idea can be pushed much further.

2. The Malinvaud approach

Like the Arrow-Hurwicz analysis, the work of Malinvaud (1967) attempts to harness gradient methods in planning exercises, but there the similarity ends. Malinvaud is concerned exclusively with the formulation of optimum plans and, in particular, with the use of decentralised procedures as an aid to the preparation of such plans. This subsection evaluates the desirable properties Malinvaud would like decentralised planning procedures to possess, examines one way in which his analysis may be extended to the case of constant returns to scale and, finally, assesses the merits of this particular approach to the planning problem.

Properties of Planning Procedures

A decentralised method of plan-formulation is a procedure where the CPB transmits price or quantity signals ("prospective indices") and where firms respond with production plans or other technological information ("proposals") which the CPB then uses to prepare a plan. Typically, the CPB will not possess all the technological information necessary to determine the optimum plan and will therefore need to question firms several times along lines suggested above: this accounts for the iterative nature of the procedure. Malinvaud proposes that these planning procedures should possess the following desirable properties: (1) feasibility (2) monotonicity and (3) convergence.

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[1] Negishi (1960), Arrow (1971b), Arrow and Hahn (1971).

A feasible procedure is one where the plan formulated at every stage of the iterations is technologically feasible and mutually consistent. The importance of this property derives from the fact that planners in actual economies are far more concerned with finding mutually consistent plans which stand a chance of being successfully implemented, than with locating the full optimum: if consistent plans have agreeable welfare implications as well, so much the better. This has the advantage that should the planner wish to terminate the procedure after a few iterations (in practice, a likely possibility), he is left with a plan which is coherent and workable. Notice that this is not the case with the Arrow-Hurwicz procedure except in the limit.[1]

Secondly, it needs to be recognised that practical planners cannot afford to play the iterative game more than a few times. The cost of each iteration is considerable in terms of both financial and human resources; furthermore, many developing countries for whom discussions of planning are particularly important must surely lack the skilled manpower needed to perform the iterations successfully. The organisational costs of implementing alternative resource allocation mechanisms is a point to which several economists[2] have drawn attention. One would therefore like the sequence (with respect to computational time) of feasible plans formulated by the planners to be monotonic non-decreasing in utility space ("monotonicity"). This ensures that the addition of an extra round of iterations does not decrease and may

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[1] Arrow and Solow (1958) propose a modification of the gradient method which generates a sequence of feasible plans. Unfortunately, there seems to be no way of implementing that method in decentralised fashion.

[2] Informal remarks are to be found in Hurwicz [(1971), (1972)]. Marglin (1963a) has developed stopping rules for iterations based on the relationship between marginal gains and costs.

well increase the utility of the plan so formulated. If costs are not excessive in relation to the benefits gained, this might be an argument for increasing the number of iterations.

The two requirements of feasibility and monotonicity would no doubt be adjudged satisfactory by most planners and Malinvaud ensures these for his procedures by ingeniously exploiting a variant of the Dantzig-Wolfe [(1961), (1963)] decomposition principle. There is, however, a third property on which the theorist may nevertheless insist. The sequence of utilities generated by a procedure satisfying the first two properties is monotonic non-decreasing. If, furthermore, one assumes (what is quite reasonable) that production possibilities are bounded by the availability of finite amounts of primary resources, one would expect the utility sequence to be bounded above. A sequence satisfying these two properties is known to converge: a procedure leading to such a sequence is called "convergent" if the limit of the sequence is the supremum of utility over the set of economy-wide feasible programmes.

The desirability of convergence is clear when one is contemplating a very large number of iterations. But it may well be unreasonable to insist on its satisfaction otherwise. The reason is that there may be non-convergent planning procedures which yield a higher utility sequence for a small number of iterations; they are clearly superior if the planner cannot afford more than a few exchanges of information.

There is one technical point which belongs here. Malinvaud's convergence property is more demanding than its analogue in the Arrow-Hurwicz analysis. For, while, in the strictly concave case, one can

deduce convergence of prices and plans to their optimum values, one is, in general, unable to assert convergence in utility space. The difficulty seems to be related to that encountered in proofs of stability of general equilibrium based on Lyapunov's second method - thus, one might succeed in establishing convergence using a price norm, but fail to do so using a utility norm. This suggests straightaway that for the limiting form of the Arrow-Hurwicz gradient method with the Malinvaud modification mentioned in the previous sub-section, one has the usual tatonnement process for a one-consumer, "Hicksian" economy and in those circumstances the utility norm is known to do the job.[1] But this is clearly a special case.

Before moving on to a modification of the Malinvaud algorithm to fit the case of constant returns to scale, it is worth reminding ourselves that the enumeration of the above properties has drawn attention to the important features of any theoretical attempt at plan-formulation. Some economists might wish to add to the catalogue of desirable properties. Thus, for example, it would be instructive to have information on the rate of convergence of alternative procedures, the structure of incentives needed to ensure that firms play the iterative game and the costs of alternative allocation mechanisms. Unfortunately, no progress can be reported in this area.

Finally, it is worth noting that there seems to be no decisive presumption in favour of satisfying one property rather than another

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[1] See Arrow and Hahn (1971). Furthermore, the convergence is monotonic in the sense of the text.

although I imagine many economists may well put feasibility on the top of their list of priorities. This is part of the general problem of one's inability to assign weights to their relative importance. To illustrate this kind of difficulty, I present an example where the planning procedure exhibits feasibility within reasonable limits and convergence but where monotonicity could be violated and the speed of convergence quite low.

Let us suppose that all plans which are proposed can be rendered feasible by overtime legislation but that overtime is costly. The latter consideration is captured by the assumption

(A) $U(l_t)$ is maximised at $l_t = 0$, where

U : economywide utility function

l_t : excess demand for labour at time t.

Thus, when the plan at time t is proposed on the basis of the ruling wage W_t, the CPB discovers the extent of the excess labour demand l_t. Consider the following adjustment process for the wage rate:

$$(14) \qquad W_{t+1} = \frac{l_r W_s - l_s W_r}{l_r - l_s} \quad , \text{ where}$$

r : largest τ for which $l_\tau > 0$

s : largest τ for which $l_\tau < 0$

(14) is known as "regula falsi", or the method of false position and is often used in numerical analysis to locate the root of an unknown nonlinear function by successive linear approximation based on local information.

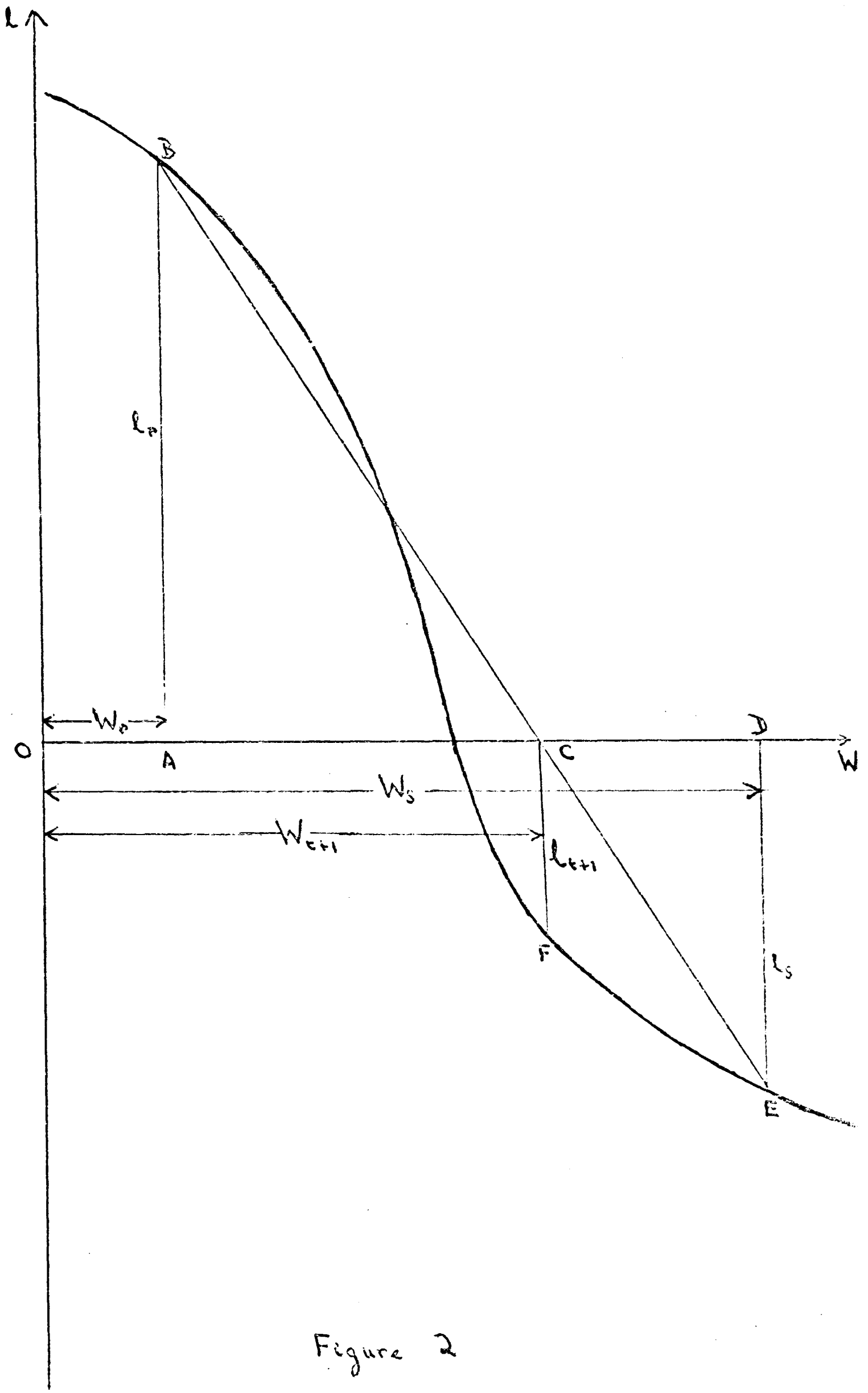


Figure 2

The method proceeds by taking the latest recorded values $(W_t, 1_t)$ together with the most recent reading where the sign of 1 is opposite that of 1_t and approximates the unknown function linearly by the chord joining those two points. Figure 2[1] shows BE as the approximating chord and OC as the new proposal for the wage rate, W_{t+1}. Excess demand is 1_{t+1} which is opposite in sign to 1_r ; hence the approximating chord is BF, and so on. It is easy to check that OC satisfies (14). That the procedure is a successive approximation is obvious; if s = t, then

$$(15) \qquad W_{t+1} = W_t - \frac{1_s}{1_r}(W_r - W_{t+1})$$

The procedure is called iterated linear inverse interpolation in the numerical analysis literature and is known to converge[2] to G. Equally, there is no guarantee that convergence is monotonic; furthermore, the rate of convergence can be quite slow. And yet the procedure is not unattractive from the point of view of simplicity and administrative convenience. It indicates that more sophisticated criteria are needed to choose among procedures possessing certain desirable properties.

The Malinvaud model with constant returns to scale.

The essentials of the model are as follows: There are (m + 1) agents : m firms (indexed by k) and the planning bureau. There are n commodities indexed by i. x_i denotes final consumption of good i .

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[1] The true function could have a fairly complicated shape; it has been drawn "well-behaved" in Figure 2 for simplicity.

[2] For a full discussion see Hildebrand (1956).

y_{ik} is the net output of good i by firm k : it is positive if firm k produces the good, negative if it uses the good as an input. The net demand for good i by consumers and firms is z_i. Thus,

$$(16) \qquad z = x - \sum_{k=1}^{m} y_k$$

where z, x and y_k are n-vectors with components z_i, x_i and y_{ik} respectively. P^T denotes the plan at stage T. Feasibility requires that

$$(17) \qquad z \leqq w$$

where w is the vector of primary resources available to the economy and where it is assumed that free disposal is possible. Formally, we make the following assumptions:

(1) The consumption set, X, which is given a priori, is closed and convex. We require that $x \in X$. The utility function of the CPB, U(x), is defined on the set X and is assumed to be continuous and concave.

(2) There exists a vector x_o such that $x \geqq x_o$ for all x in X.

(3) The production sets of the individual firms, Y_k, $(k = 1,\ldots,m)$ are closed, convex cones, i.e., technology exhibits constant returns to scale.

(4) The CPB knows a feasible programme, P^o. This could be a programme implemented in a previous year.

Assumptions (1), (2) and (3) are standard in the theory of general equilibrium and do not need comment. Assumption (4) might be regarded as being extremely demanding by some. Kornai[1] has, however,

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[1] See his remarks in the conference discussion on Malinvaud's paper in Malinvaud and Bacharach (1967). See also footnote 3 on p.3.

suggested that the CPB need not know an economically feasible plan to begin with. It is sufficient that the plan p^o be mathematically feasible : this is done in Hungary by introducing artificial variables whose levels are subsequently driven to zero by assigning them large negative values in the objective function.

The decentralised planning procedure may be defined as follows:

(i) The CPB at stage t sends out a price vector, p^t, which is normalised to lie in the unit simplex.

(ii) Firms perform the following operations:

(a) they select an optimising set of techniques at the announced prices and their current scales of operation, i.e. they choose A_k^t to maximise

$$(18) \qquad p^t . A_k s_k^{(t-1)} \text{ , subject to } A_k \in B_k$$

where B_k : set of matrices of net production coefficients, A_k, available to firm k $(k = 1,\ldots,m)$

A_k : a matrix of net production coefficients available to firm k $(k = 1,\ldots,m)$, with typical element $a_{j\ell}$ denoting the amount of commodity j needed to produce one unit of commodity ℓ. $(j,\ell = 1,\ldots,n)$

A_k^t : optimum A_k chosen by firm k $(k = 1,\ldots,m)$ at stage t when faced with prices p^t and given scale of production

s_k^{t-1}: Scale vector of firm k $(k = 1,\ldots,m)$ when the CPB announces prices, p^t, at stage t.

(b) they vary existing scale of production in proportion to unit profitability computed at the optimising matrix of net production coefficients

$$(19) \qquad \Delta s_k^t = s_k^t - s_k^{t-1} = p^t \circ A_k^t \qquad (k = 1,\ldots,m)$$

(c) Firms then report to the CPB

$$(20) \qquad A_k^t \circ s_k^t = y_k^t \qquad (k = 1,\ldots,m)$$

where y_k^t is the resulting vector of net production firm k is willing to undertake at stage t, using the techniques selected in (a) above. Clearly, $y_k^t \in Y_k$, the production set characterising firm k. (k=1,...,m)

Rule (ii) invites comment. The major problem with constant returns to scale is that one is left without a rule for determining the scale of operations of the firm. Competitive profit-maximisation in this context implies that the scale of production will oscillate greatly for minor changes in price. It is not a recipe for encouraging confidence on the part of producers in the reliability of signals sent out by the CPB. Accordingly, we postulate that firms minimise costs and alter scale cautiously in response to unit profitability. The behavioural rule is undeniably ad hoc. There are two reasons for employing it here. Firstly, casual empiricism suggests that enterprises do not undertake large changes in response to alternative forecasts made by the authorities. They react, not perversely, but conservatively, especially when it comes to making investment decisions (about which more in Chapter 4). Secondly, this adjustment rule enjoys the backing of tradition and has been used by economists since Walras' day.[1]

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[1] See the account of the Walrasian system in Morishima (1964).

(iii) The CPB learns about the production sets of different firms by generating them out of all previous production proposals submitted to it. Thus, at stage t, the CPB knows a subset Y_k^t of Y_k given by

$$(21) \qquad Y_k^t = \left\{ y_k : y_k = \sum_{r=0}^{t-1} \lambda_k^r y_k^r \; ; \; \lambda_k^r \geq 0 \right\} \qquad (k = 1,\ldots,m)$$

Equation (21) assumes the CPB knows that production sets are closed, convex cones.

Rule (iii) asserts that the CPB has "memory" and forms an increasingly accurate picture of production possibilities at successive rounds of the planning procedure. It is a significant innovation in the literature on plan-formulation and clearly distinguishes the Malinvaud approach from pure market simulation of the Lange-Arrow-Hurwicz kind.

Before describing step (iv), it needs to be mentioned that Malinvaud outlines the above procedure for a decreasing returns-to-scale technology. Firms transmit profit-maximising proposals in step (iii) and the CPB builds up its picture of technological possibilities according to equation (21) but with the coefficients λ_k^r adding up to unity. In a footnote, he proposes an extension to the case of constant returns to scale. Firms could then transmit optimum techniques at ruling prices rather than production proposals; the CPB would form linear combinations of these techniques to approximate the true production set.[1] But the procedure analysed here is not an unattractive one, for reasons mentioned earlier, and is worth exploring as well.

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[1] For a diagrammatic exposition of the Malinvaud model in both the decreasing-and constant - returns-to- scale versions, see Heal (1973).

(iv) At stage t, the CPB has to find a feasible programme $\bar{P}^t$ which is the solution to the following programming problem: maximise

(22) $U(x)$

subject to the constraints

(23) $x \in X$

(24) $y_K \in Y_k^t \qquad (k = 1,\ldots,m)$

(25) $z = x - \Sigma y_k \leq w$

It is easily checked that the "no-worse-than-the-optimum" and the production sets satisfy the hypotheses of separation theorems for convex sets[1]. The CPB therefore determines a non-negative price vector p^t belonging to the unit simplex and satisfying the following conditions:

(α) $p^t.x > p^{t-t}\bar{x}^t$ for all $x \in X$ such that $U(x) > U(\bar{x}^t)$

(β) $p^t.y_k \leq p^{t-t}\bar{y}_k^t$ for all $y_k \in Y_k^t$ and all $(k = 1,\ldots,m)$

(γ) $\left.\begin{array}{r} \bar{x}(p^t) - \bar{y}(p^t) - w \leq 0 \\ p^t \geq 0 \end{array}\right\}$ complementary slackness

where $\bar{x}$ and $\bar{y}$ denote the solution to the CPB's mathematical programme, $\bar{P}^t$.

(v) The programme $\bar{P}^T$ at the ultimate stage T of the iterations constitutes the plan.

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[1] For a particularly good account, see Eggleston (1958).

Finally, before analysing the properties of the planning procedure, it is necessary to introduce Assumption

(5) The attainable set

$$E = \left\{ (x, y_1, \ldots, y_m) \;\middle|\; x \leq w + \sum_{k=1}^{m} y_k,\ x \in X,\ y_k \in Y_k : (k=1,\ldots,m) \right\}$$

is bounded[1].

Properties of the procedure

The present subsection shows a la Malinvaud that the procedure outlined above possesses some of the desirable properties discussed earlier.

Property 1: The procedure is "well-defined", i.e. (a) there exist solutions to all operations and (b) the plan is feasible.

Since p^t has been normalised and since A_k^t is defined as a matrix of net production coefficients, a solution to rule (ii) always exists.

If the plan $\bar{P}^t$ exists, a non-negative normalised price vector p^t satisfying conditions (α), (β) and (γ) must also exist. This follows from the separation theorems. The plan $\bar{P}^t$ is clearly feasible. It is therefore necessary to show that a solution $\bar{P}^t$ to the CPB's mathematical programme in rule (iv) exists.

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[1] A catalogue of sufficient conditions ensuring the validity of Assumption 5 is (a) the set of technologically feasible production plans is bounded above whenever resources available to the economy are bounded and (b) the vector, w, of initial resources is bounded, and that Assumption 2 holds.

Let X^t be the set of vectors x included in the programme satisfying the feasibility conditions (23) - (25). Since the determination of $\bar{P}^t$ involves the maximisation of a continuous function U(x) over X^t, it is sufficient to show that the latter is a closed and bounded set.

Boundedness of X^t follows from Assumption 5. Consider now a sequence of vectors x^s contained in X^t and converging to a vector $\hat{x}$. Then, one can find y_k^s in Y_k^t and $z^s \leq w$ such that $x^s = z^s + \sum_k y_k^s$. The sets Y_k^t are closed. The sequence $\left[x^s, y_1^s, \ldots, y_k^s, \ldots y_m^s\right]$ is, from Assumption 5, bounded. The vectors z^s are included in a closed and bounded set which is independent of s, since

$$x_o - \sum_k y_k^s \leq z^s \leq w,$$

where x_o is the vector whose existence is postulated in Assumption 2. Therefore, the bounded sequences y_k^s and z^s contain subsequences which converge to the limits $\hat{y}_k$ and $\hat{z}$ satisfying the following relationships:

$$\hat{y}_k \in Y_k^t, \quad \hat{z} \leq w, \quad \text{and}$$

$$\hat{x} = \hat{z} + \sum_k \hat{y}_k$$

The set X is closed, by Assumption 1. Hence it must contain its limit points and, particularly, $\hat{x}$. Since $\hat{x}$, $\hat{y}_k$ and $\hat{z}$ satisfy (25), $\hat{x}$ indeed belongs to X^t. This shows that X^t is closed.

Property 2: The procedure is monotonic, i.e. $U^t \geq U^{t-1}$, for all t.

From (21), $Y_k^{t-1} \subset Y_k^t$ by construction and since this constitutes the only difference between successive programmes solved by

the CPB, $U^t \geq U^{t-1}$. This underlines the obvious fact that if the CPB stores past technological information all the time and uses it to construct an increasingly larger approximation to the true production set, it can do no worse as the number of iterations is increased.

Finally, we prove

Theorem 5 : The procedure is convergent.

Proof: Any feasible programme satisfies

$$x_o \leq x \leq w + \sum_k y_k \quad \text{with } x \in X,\ y_k \in Y_k,\ (k = 1,\ldots,m)$$

We know from Assumption 5 that x belongs to a bounded set. Therefore, the continuous function U(x) is bounded above for all feasible programmes. Let $\bar{U}$ be the supremum. Since the U - sequence is monotonically non-decreasing, it must converge to a limit U* which is less than or equal to $\bar{U}$. The Malinvaud proof begins by supposing that U* is strictly less than $\bar{U}$ and then generates a contradiction.

The sequence p^t of normalised prices, the sequence $p^t \bar{x}^t$ of the value of consumption in plan P^t, the sequence $p^t \bar{y}_k^t$ of the value of firm k's net production in plan P^t and the sequence $p^t A_k^t s_k^t = p^t y_k^t$ (from (20)) of the value of firm k's net production proposal all belong to bounded sets and therefore contain simultaneously convergent subsequences. We denote subsequential limits by asterisks (*).

From the complementary slackness property of (γ) in rule (iv), we have that

$$(27) \qquad p^* x^* - \sum_k p^* \bar{y}_k^* = p^* w$$

From condition (β), $p^t y_k^r \leq p^t \bar{y}_k^t$ for all r less than t.

Letting $t \to \infty$ with fixed r gives

(28) $\quad p^* y_k^r \leq p^* \bar{y}_k^*$ for all k and all r.

Now, $(p^* - p^r) y_k^r \to 0$ as $r \to \infty$, or $p^* y_k^r \to p^* y_k^*$. From (28), $p^* y_k^* \leq p^* \bar{y}_k^*$ for all k, or

(29) $\quad p^* A_k^* s_k^* \leq p^* \bar{y}_k^*$

Now suppose that $U^* < \bar{U}$. Then there exists a feasible programme $\hat{p}$ such that $U(\hat{x}) > U^* \geq U(\bar{x}^t)$, all t. By continuity of $U(x)$, there exists a positive number ε such that $U(x) > U(\bar{x}^t)$ for all t and all x whose distance from $\hat{x}$ is less than ε.

From (α) of rule (iv), $p^t x > p^t \bar{x}^t$ and, at the limit, $p^* x \geq p^* \bar{x}^*$ for all x whose distance from $\hat{x}$ is less than ε. This inequality is true, in particular, for those vectors x whose components are strictly smaller than the corresponding components of $\hat{x}$ and therefore

(30) $\quad p^* \hat{x} > p^* \bar{x}^*$

From rule (ii)

(31) $\quad p^* A_k^* s_k^* \geq p^* \hat{y}_k = p^* \hat{A}_k \hat{s}_k$

Thus, we have

$$p^* \hat{z} = p^* \hat{x} - \sum_k p^* \hat{y}_k > p^* \bar{x}^* - \sum_k p^* A_k^* s_k^* \geq p^* \bar{x}^* - \sum p^* \bar{y}_k^* = p^* w$$

But $p^*(\hat{z} - w) > 0$ contradicts $p^* \geq 0$ and $\hat{z} \leq w$. The contradiction proves Theorem 5.

Malinvaud - type models: an assessment

It is clear from the subsequent literature[1] on plan formulation that Malinvaud's contribution has defined the terms of reference for discussion in this area. The following comments on his approach seem worth noting.

(1) The informational and computational demands are considerably more severe than in any market simulation of the Lange-Arrow-Hurwicz kind. First, except in a special case noted in (3) below, the CPB needs to know the source of every production proposal it receives; knowledge of just overall market totals will not suffice.

Secondly, rule (21) shows that the CPB needs to possess "memory"[2]. This is not surprising in a model where the CPB is expected to learn progressively more about true production possibilities. Thirdly, step (iv) of the planning procedure and equations (22) - (25) require the CPB to solve a large non-linear programming problem at every stage. These considerations help illustrate (a) that the Malinvaud approach is quite different from traditional market socialism, a point to which attention has been drawn before, and (b) that the resource costs of administering decentralised planning procedures may well be considerable.

(2) Malinvaud's analysis depends heavily on convexity of the production sets of firms. The CPB's learning process consists in generating the convex hull of firms' previous proposals: without convexity of the true production sets, the sets Y_k^t mapped out by (21) would not necessarily

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[1] Heal [(1969), (1971), (1973)], Weitzman (1970), Aoki (1971).

[2] See Footnote 3 on page 3.

be subsets of the underlying feasible set. Here again it differs from the Arrow-Hurwicz analysis which holds (admittedly with weaker results) under increasing returns to scale.

(3) The analysis can be simplified considerably in the case where the two following requirements are satisfied:

(a) production possibility sets of firms are strictly convex and

(b) rule (ii) above is replaced by profit-maximisation.

In that case, it is enough for the CPB to consider aggregate production proposals and build up a picture of the aggregate production set by generating the convex hull of its extreme points. For it can decentralise the implementation of a plan drawn up at any stage by announcing the supporting shadow prices : these prices evoke unique profit-maximising responses by firms which ensure overall market clearance in all commodities. This is the special case mentioned in (1) above which permits substantial economies in the transmission of information and preserves anonymity of messages flowing from producers to the CPB[1].

(4) The Malinvaud contribution and the literature which it has spawned pay no formal attention to decentralisation of the authority to take decisions. Two difficulties are raised by this disregard. Firstly, the fact that producers play no role other than that of passive purveyors of technological information according to profit-maximisation or cost-minimisation rules may render them unwilling to participate in any decentralised procedure. This could jeopardise the entire process of plan-formulation and it is reasonable to expect that this consideration should be taken into account in contributions to the problem of drawing

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[1] Notice that the conditions defining this special case are really no more restrictive than those under which gradient methods converge in the Arrow-Hurwicz analysis, for the strictly concave case.

up an optimum plan. Secondly, except in the special case noted in (3) above, the problem of implementing the plan may be fraught with difficulties[1]. For producers may refuse to have anything to do with a plan imposed from above and in consequence one in which they might feel no sense of participation. This suggests once again the need for an analysis of the structure of incentives needed both to formulate and to implement an optimum plan.

III. NON PRICE - GUIDED METHODS

The previous section studied some ways in which competitive behaviour has to be modified when planners resort to price-decentralisation in non-strictly convex environments. This section is devoted to an examination and assessment of procedures where the CPB sets production targets and firms respond with price-like information, e.g., marginal productivities of proposed input allocations. It has been suggested[2] that many actual planning procedures are of the latter as opposed to the former type. This, by itself, is good reason for looking at this family of procedures; furthermore, there is now available a considerable literature on this subject[3]. As before, the discussion is restricted to situations where returns to scale are non-decreasing.

Before launching into quantity-guided procedures, it is worth noting an attempt made in the literature to compare them with price-guided procedures. In an interesting study based on an analysis of

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[1] Footnote 1 on Page 44 is appropriate here as well.

[2] Montias (1959), Portes (1971)

[3] Kornai and Liptak (1965), Heal [(1969), (1971). (1973)] Weitzman (1970), Aoki [(1970), (1971)], Portes (1971).

three gradient methods of plan-formulation in convex environments, Marglin (1969) concluded that although prices may well be considerably superior to quantities in devising an incentive structure for supporting the optimum[1], this is not the case when it comes to (a) informational efficiency and (b) respecting privacy. Each of the three gradient methods has a price version and a quantity or "command" version: each version requires the same kind of informational exchange with the qualification that while the CPB announces only one price applicable to all firms in the first case, it has to make separate announcements for every firm in the second. Secondly, so long as the CPB knows that entrepreneurs react in a specified way, it can solve the necessary marginal conditions to deduce the same information irrespective of whether the price or the command version of an algorithm is employed. Thus, for example, with price-guided algorithms the CPB knows that the profit-maximising proposals are arrived at by solving the usual marginal conditions; it can therefore utilise this information to deduce marginal productivities of inputs quite easily. Since the command version of the algorithm calls upon firms to report these marginal productivities, this demonstration suggests that price-guided procedures do not allow firms to guard their technological secrets any more jealously than their "command" counterparts. Two qualifications are in order. Firstly, the conclusions are based on an analysis of only three procedures. Secondly, it is not at all clear if a general presumption in favour of equivalence continues to be true in non-convex environments, as we shall see.

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[1] Recall the case when there are many producers of each commodity and the CPB need only know aggregate production plans to announce prices.

1. An Allocation - of - Contracts Model

Government agencies and central planners (sometimes acting on behalf of consumers) often decide on certain targets which have to be fulfilled; these targets are then communicated to producers in the form of government requirements. Producers bid for contracts, the CPB revises its quota in the light of these prices, and the process continues. In this subsection, I propose a simple framework which captures the essential features of this story and which allows for decreasing, constant or increasing returns to scale. The analysis will be concerned with the attainment of market-clearing equilibria rather than with optimality, for reasons which will become clear.

Basic Model:

There are n commodities each of which is produced by one firm: both are indexed by i. It is assumed that firm i sets price equal to average cost of production in bidding for government contracts. Thus, the price-setting rule is

$$(32) \qquad p_i = C_i = f_i(S_i) \qquad (i = 1,\ldots,n)$$

where p_i : price of good i

C_i : average cost of producing good i

S_i : production of good i $\qquad (i = 1,\ldots,n)$

It might be thought unnecessarily restrictive to assume that p_i does not depend on other prices. It is possible to allow for this dependence provided that one's own supply conditions affect the price set more significantly than prices prevailing elsewhere, an assumption of the quasi-diagonal dominance variety. But I prefer to ignore this complication here.

It is assumed that

(1) $$\left|f_i'(S_i)\right| \equiv \left|\frac{dp_i}{dS_i}\right| < \varepsilon, \quad \text{where } \varepsilon \text{ is a small positive number.}$$

$$(i = 1,\ldots,n)$$

Given the price-setting rule (32), assumption (1) permits small deviations from constant returns to scale : thus, returns to scale are ε - varying. I shall comment on this assumption later.

(2) The CPB has a demand function $D(p) \equiv \left[D_1(p_1 \ldots p_n),\ldots,D_n(p_1,\ldots,p_n)\right]$ which could, for example, be derived from utility-maximisation.

We are now in a position to describe the planning procedure.

(i) The CPB announces some production target, $S_i (i = 1,\ldots,n)$, to firms. This could be a set of targets drawn up in some previous period.

(ii) Firm i bids for quota S_i $(i = 1,\ldots,n)$ by calculating the prices, p_i, at which it can honour the contract, i.e. by setting the price equal to the unit cost of producing S_i. This is, of course,

(33) $$p_i = f_i(S_i) \qquad (i = 1,\ldots,n)$$

(iii) The CPB, taking these prices as given, calculates from its demand function the amounts D_i $(i = 1,\ldots,n)$ it would like to order at the ruling prices. It then announces revised quotas according to the difference between these demands and the quotas firms bid for last period, i.e.,

(34) $$S_i^* = D_i - S_i \qquad (i = 1,\ldots,n)$$

Firms bid for the new quotas as in rule (ii) and the process goes on as before. It is easy to prove

Theorem 6 : The quota-allocation procedure converges to a market-clearing equilibrium.

Proof: Consider the function

$$(35) \qquad V(t) = \tfrac{1}{2} \sum_{j=1}^{n} \dot{S}_j^2$$

$$(36) \qquad \dot{V}(t) = \sum_{j=1}^{n} \dot{S}_j \ddot{S}_j$$

But (34) may be written as

$$\dot{S}_i = D_i \left[f_1(S_1), \ldots, f_n(S_n) \right] - S_i$$

$$\text{and} \qquad \ddot{S}_i = \sum_{k=1}^{n} D_{ik} f_k'(S_k) \dot{S}_k - \dot{S}_k, \quad \text{where } D_{ik} = \frac{\partial D_i}{\partial p_k}$$

which, upon substitution into (36), gives

$$(37) \qquad \dot{V}(t) = \sum_{j=1}^{n} \dot{S}_j \left[\sum_{k=1}^{n} D_{jk} f_k'(S_k) \dot{S}_k - \dot{S}_j \right]$$

Since, by Assumption (1), $\left| f_n' \right| < \varepsilon$ where ε is a small positive number for all $(k = 1,\ldots,n)$ the second term will dominate the first to give $\dot{V}(t) < 0$. In the limiting case of constant returns to scale, $f_k' = 0$, all k and $\dot{V}(t) = -\sum_{j=1}^{n} \dot{S}_j^2 < 0$.

Let us assume that $\lim_{t \to \infty} V(t) = V^* > 0$. Since $V(t)$ converges to V^* in decreasing fashion, we have

$2V^* \geq V(t) \geq V^*$ for sufficiently large t. Since $\dot{V}(t)$ is strictly negative except at the singular point of the equation of motion (i.e., at full equilibrium), it must assume a negative maximum on the closed and bounded subset defined above, i.e., $\dot{V}(t) \leq -k < 0$ for sufficiently large t, say $t \geq T$. Therefore it follows that

$$V(t) \leq -k(t-T) + V(T) \quad \text{for } t \geq T$$

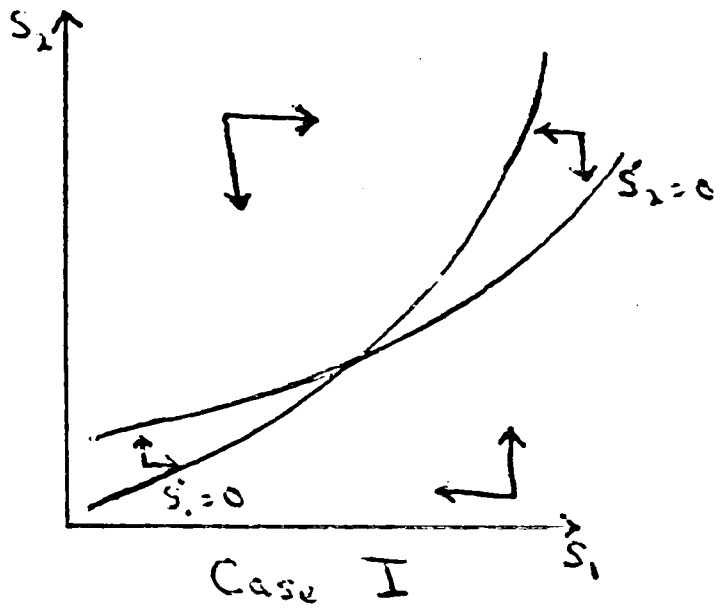
and $\lim_{t \to \infty} V(t) = -\infty$, which contradicts the supposition that V^* is strictly positive. Therefore, $V^* = 0$, i.e., $\lim_{t \to \infty} V(t) = 0$.

Since $V(t) = 0$ if and only if $\dot{S}_i = 0$ $(i = 1,\ldots,n)$, the allocation-of-contracts procedure is globally stable under the conditions postulated above.

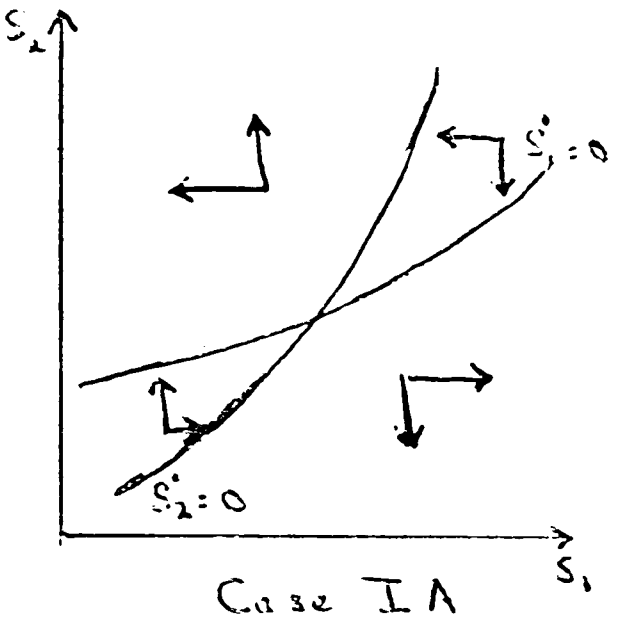
Before commenting on various aspects of the model, it might be illuminating to look at a diagrammatic portrayal of the two-good case. The relevant equations are given by

$$(38) \quad \begin{cases} \dot{S}_1 = D_1\left[f_1(S_1), f_2(S_2)\right] - S_1 \\ \dot{S}_2 = D_2\left[f_1(S_1), f_2(S_2)\right] - S_2 \end{cases}$$

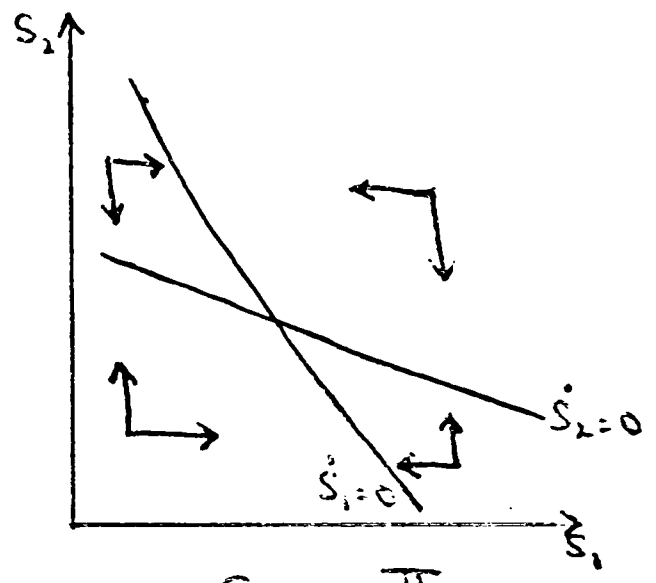
Cases I and IA assume normal increasing costs in both industries, i.e., $f_1', f_2' > 0$. Assume $D_{11}, D_{22} < 0$ and, from consumer theory for the two-good case, we know that $D_{12}, D_{21} > 0$. If we assume that price is not very sensitive to changes in supply (i.e., ε - decreasing returns), than f_1', f_2' are small and we get Case I. If f_1', f_2' are large, Case IA results and the system is not globally stable.



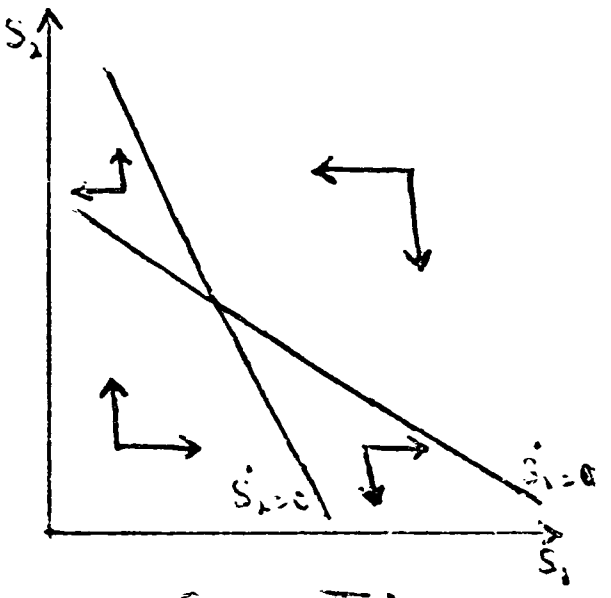
Case I



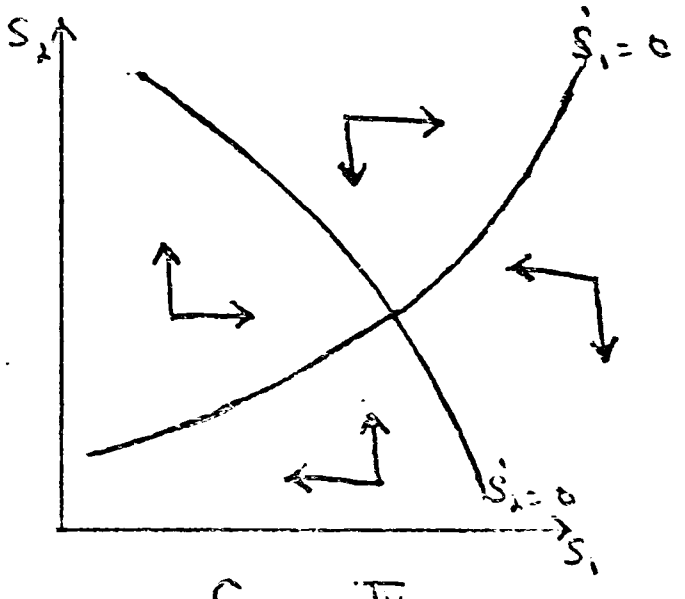
Case IA



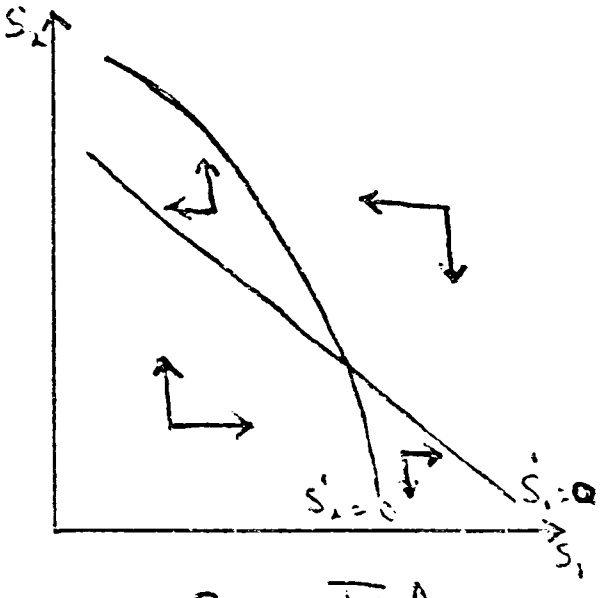
Case II



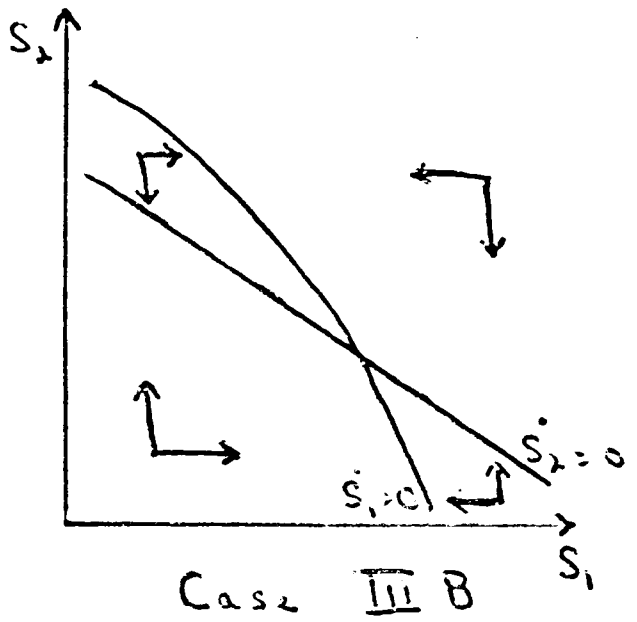
Case IIA



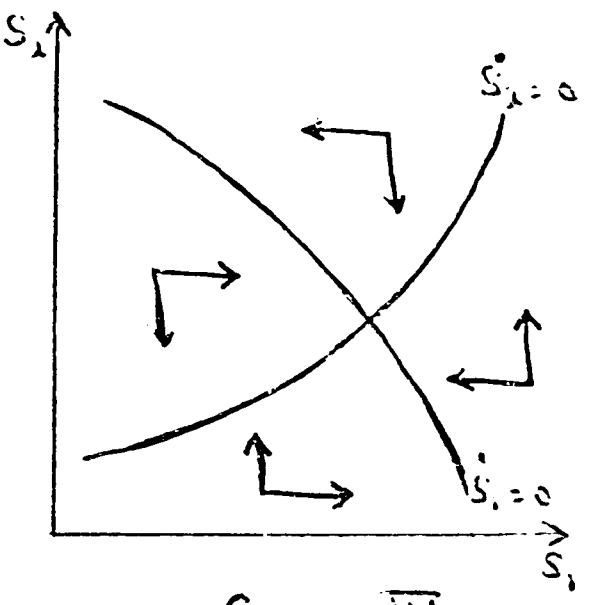
Case III



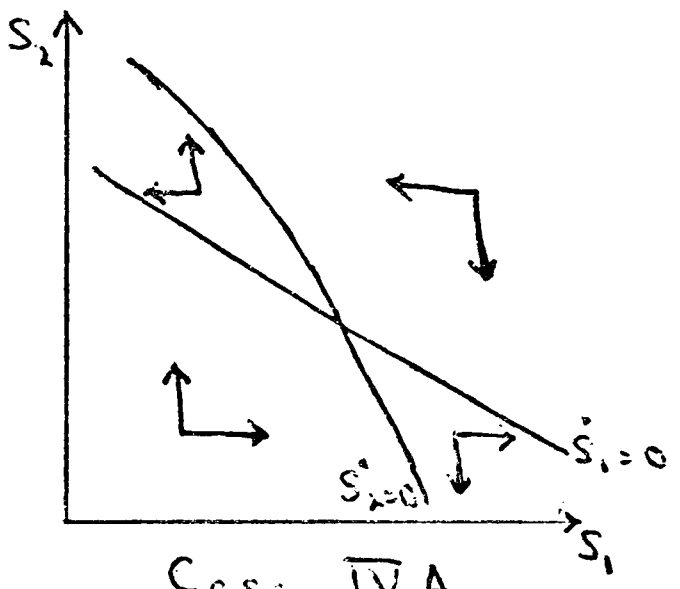
Case IIIA



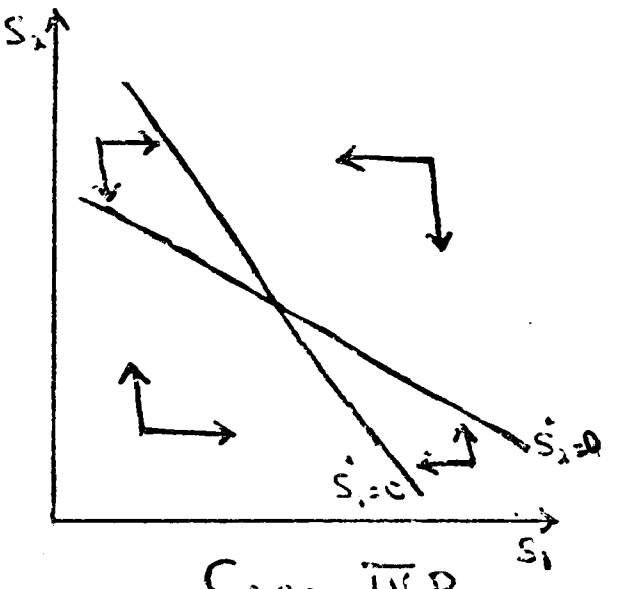
Case III B



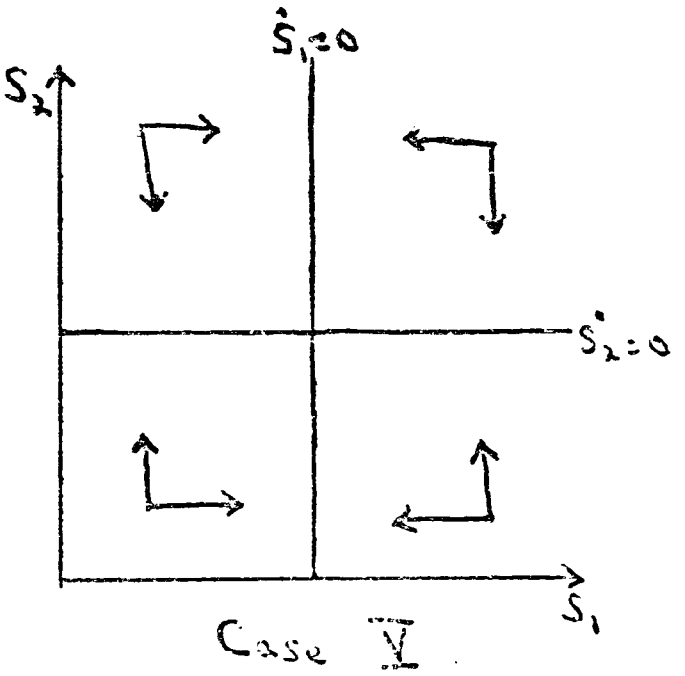
Case IV



Case IV A



Case IV B



Case V

We now assume increasing returns (decreasing costs) in both industries, i.e., $f_1' , f_2' > 0$. If we have $\mathcal{E}$ - increasing returns, we get Case II (stability); if not, Case IIA may easily result.

The next class of cases looks interesting. We assume increasing returns ($f_1' < 0$) in the first industry and decreasing returns ($f_2' > 0$) in the second. If Assumption (1) holds for both industries, the system is stable. (Case III) If, however, returns are changing sharply in both industries, the ($\dot{S}_2 = 0$) curve becomes somewhat steeper and the ($\dot{S}_1 = 0$) curve swings into a negatively sloping position, yielding Case IIIA (instability). One might be tempted to hope that this process would, after a time, yield Case IIIB and hence global stability. But this cannot occur, as can be easily checked: once the ($\dot{S}_1 = 0$) curve has become negatively sloped, it cannot get any steeper without making the ($\dot{S}_2 = 0$) curve more steep. Hence the ($\dot{S}_1 = 0$) curve can never get (negatively) steeper than the ($\dot{S}_2 = 0$) curve and so Case IIIB never holds.

The next group of cases produces the same kind of results. Here we assume decreasing returns in the first industry ($f_1' > 0$) and increasing returns in the second industry ($f_2' < 0$). Analogous comments apply : the satisfaction of Assumption (1) yields IV, its violation IVA; and IVB, predictably, is impossible.

Constant returns in both industries ($f_1' = f_2' = 0$) make the analysis particularly easy : the phase diagram is labelled Case V.

Assessment

Several comments on this approach are worth making.

(1) The present simple formulation has the advantage of allowing one to see what happens in a situation where governments set firms physical targets and ask them what prices they will charge to fulfil them. But the available data for British industry[1] do not support the assumption that returns to scale are ε - varying: economies of scale are more substantial than that. Accordingly, I present below a specialisation of the model which demonstrates global stability of the quota-allocation procedure under more satisfactory assumptions about economies of scale.

(2) The model is, in some ways, close in spirit to traditional models of stability of tatonnement processes used by economic theorists. In these models prices adjust in response to excess demands in markets. The emphasis here is on quantity adjustments. The competitive assumption of price-taking is abandoned; firms have the power to set their own prices while the onus is on the government to manipulate quantities. In fact, the government gives firms some idea of what they can sell at various prices - a not unappealing assumption remembering that imperfectly competitive behaviour is called for under increasing returns to scale. And yet we shall see in one special case examined below that the quota-allocation procedure will converge under the same circumstances as the so-called Marshallian adjustment process[2] where the quantity transacted

---

[1] I have in mind the work of Silberston (1972) who uses estimates provided by Pratten (1971). The figures need to be interpreted with caution, as Silberston emphasizes, but a reasonable estimate of the elasticity of unit cost with respect to scale of output would seem to be around - 0.25.

[2] See Negishi (1972).

in the market is adjusted in response to the difference between the demand price and the supply price.

(3) The average - cost pricing rule adopted in the model leads to overall suboptimality from the point of view of welfare: things are optimal when constant returns to scale prevail everywhere, rendering average costs equal to marginal costs[1]. The obvious corollary is that the same set of conditions which are sufficient for global stability, viz., slight departures from constant returns to scale in either direction, also ensure that the extent of suboptimality resulting from average cost pricing is not very great.

A "Marshall - Chipman"[2] general equilibrium model

Next, I examine the quota-allocation model in a somewhat special model which allows one to make more satisfactory assumptions about economies of scale.

(A. 1) There are n goods, each of which is produced by one of n firms. Both goods and firms are indexed by i. The i^{th} firm produces good i according to the production function

$$(39) \qquad y_i = \lambda_i \ell_i^{\rho_i} \quad (i = 1,\ldots,n) \; ; \; 0 < \rho_i < \infty$$

where y_i is the output of good i
ℓ_i is employment in the i^{th} firm.

---

[1] Aoki (1970) has examined some problems arising from the necessity to impose taxes and subsidies to supplement marginal cost pricing in industries characterised by different degrees of returns to scale. A locally convergent decentralised planning procedure is also proposed there.

[2] The reference is to the stimulating paper by Chipman (1970). The model used here is basically similar to Chipman's with one important proviso. No distinction is made between the firm and the industry here whereas the distinction is, according to Chipman, the most conspicuously Marshallian feature of his analysis. "Marshall" is retained in the present title because of the use to which the Marshallian stability criterion will be put.

Thus, there are no intermediate goods; furthermore, labour is indispensable in the production process. The restriction $0 < \rho_i < \infty$ ensures that the marginal product of labour is positive and finite.

(A.2) The CPB has a linear - logarithmic utility function

$$(40) \quad U = b_o \log (x_o + \frac{1}{b_o}) + \sum_{i=1}^{n} b_i \log x_i ; \; b_i > 0 \; (i = 0,\ldots,n)$$

$$\text{and} \quad \sum_{i=0}^{n} b_i = 1$$

where x_i is the consumption of good i $(i = 1,\ldots,n)$ and

x_o is the consumption of good 0.

and where units of labour are chosen such that the maximum possible supply of labour is $\frac{1}{b_o}$ units. Hence,

$$(41) \quad x_i \geq 0 \; (i = 1,\ldots,n) ; \quad -\frac{1}{b_o} \leq x_o \leq 0$$

The budget constraint facing the CPB is

$$(42) \quad p_o x_o + \sum_{i=1}^{n} p_i x_i = 0$$

where p_i is the price of the i^{th} good $(i = 0,\ldots,n)$

We are now in a position to describe the decentralised planning procedure

(i) The CPB announces some production target, y_i $(i = 1,\ldots,n)$, to firms.

(ii) Firm i bids for quota y_i $(i = 1,\ldots n)$ by calculating the price p_i at which it can honour the contract; it is postulated that firm i $(i = 1,\ldots,n)$ announces prices which cover its unit costs of production. From (39),

$$(43) \qquad p_i = \frac{p_o}{y_i} \left(\frac{y_i}{\lambda_i}\right)^{\frac{1}{\rho_i}} \qquad (i = 1,\ldots,n)$$

(iii) (a) The CPB, taking these prices as given (the determination of p_o will be discussed presently), maximises (40) subject to (41) and (42). Form the Lagrangian

$$L = b_o \log \left(x_o + \frac{1}{b_o}\right) + \sum_{i=1}^{n} b_i \log x_i - \mu\left[p_o x_o + \sum_{i=1}^{n} p_i x_i\right]$$

This leads to the demand functions

$$(44) \qquad x_o = 1 - \frac{1}{b_o}, \quad \text{a constant}$$

$$(45) \qquad x_i = \left(\frac{b_i}{b_o} p_o\right) \cdot \frac{1}{p_i} \qquad (i = 1,\ldots,n)$$

(b) The CPB announces revised quotas $(y_1,\ldots,y_n)$ according to the gap between the x_i in (45) and the y_i in Steps (i) and (ii) that firms bid for. Specifically, it follows the rule

$$(46) \qquad \frac{d}{dt}(\log y_i) = \log x_i - \log y_i \qquad (i = 1,\ldots,n)$$

I shall comment on rule (46) presently. Before that, it is necessary to complete the story by indicating how p_o is determined. It is assumed that p_o is adjusted instantaneously to clear the labour market. Since, by (44), the supply of labour is constant, this involves adjustment on the production side alone. From (39) and (43), the demand for labour in firm i is

$$(47) \qquad \ell_i = \frac{p_i y_i}{p_o} \qquad (i = 1,\ldots,n)$$

Summing over firms, we get

$$\sum_{i=1}^{n} \ell_i = \frac{1}{p_o} \sum_{i=1}^{n} p_i y_i$$

Factor - market equilibrium requires that

$$\frac{1}{p_o} \sum_{i=1}^{n} p_i y_i = \frac{1}{b_o} - 1$$

so that

$$(48) \qquad p_o = \left(\frac{b_o}{1 - b_o}\right) \sum_{i=1}^{n} p_i y_i$$

The form of the utility function (40) indicates that the CPB will always desire every produced good in positive quantities; otherwise utility goes to $-\infty$. This has three desirable consequences: (a) the constraints (41), so far disregarded, are always satisfied, (b) the adjustment equation (46) will be well-behaved in that the quotas y_i $(i = 1,\ldots,n)$ will have all their components positive, bounding the right hand side away from minus infinity and (c) p_o as defined in (48) is strictly positive. Consequence (a) is not particularly interesting as $x_i \geq 0$ $(i = 1,\ldots,n)$ is really a definition. Consequence (b) ensures (46) is a sensible rule to adopt. It will also be seen that this logarithmic version of the adjustment rule is the appropriate one to use when stability conditions are given in terms of elasticities. Consequence (c) allows one to conduct the subsequent discussion in terms of relative prices with respect to a labour numeraire.

Now define

$$(49) \qquad q_i = \frac{p_i}{p_o} \; ; \quad r_i = \log q_i \; , \; S_i = \log y_i, \; d_i = \log x_i \quad (i = 1,\ldots,n)$$

Then (43) and (45) may be written respectively as

$$(50) \qquad r_i = \left(\frac{1}{\rho_i} - 1\right) S_i - W_i \quad ; \quad \text{where } W_i = \frac{1}{\rho_i} \log \lambda_i$$

$$(51) \qquad d_i = a_i - r_i \quad ; \quad \text{where } a_i = \log \frac{b_i}{b_o}$$

The adjustment rule (46) becomes

$$(52) \qquad \dot{S}_i = d_i - S_i$$

It is now straightforward to prove

Theorem 7 : The quota - allocation procedure converges to a market-clearing equilibrium.

Proof: Consider the function

$$(53) \qquad V(t) = \tfrac{1}{2} \sum_{i=1}^{n} \dot{S}_i^2$$

$$(54) \qquad \dot{V}(t) = \sum_{i=1}^{n} \dot{S}_i \ddot{S}_i$$

From (52),

$$\ddot{S}_i = \left[d_i'(r_i) \frac{dr_i}{dS_i} - 1 \right] \dot{S}_i \qquad (i = 1,\ldots,n)$$

(where $d_i' = \frac{dd_i}{dr_i}$)

which, upon substitution in (54), gives

$$(55) \qquad \dot{V}(t) = \sum_{i=1}^{n} \left[d_i'(r_i) \cdot \frac{dr_i}{dS_i} - 1 \right] \dot{S}_i^2$$

which, from (50) and (51) becomes

$$(56) \qquad \dot{V}(t) = \sum_{i=1}^{n} \left[\left(1 - \frac{1}{\rho_i}\right) - 1 \right] \dot{S}_i^2 < 0, \text{ because } 0 < \rho_i < \infty$$

As in the proof of Theorem 6, it remains to show that $V(t)$ is a Lyapunov norm. Let us assume that $\lim_{t \to \infty} V(t) = V^* > 0$. Since $V(t)$ converges to V^* in decreasing fashion, we have $2V^* \geq V(t) \geq V^*$ for sufficiently large t.

Since $\dot{V}(t)$ is strictly decreasing except at full equilibrium, it must assume a negative maximum on the closed and bounded subset defined above, i.e., $\dot{V}(t) \leq -k < 0$ for sufficiently large t. Therefore, it follows that

$$V(t) \leq -k(t - T) + V(T) \quad \text{for} \quad t \geq T$$

and $\lim_{t \to \infty} V(t) = -\infty$, which contradicts the supposition that V^* is strictly positive. Therefore, $V^* = 0$, i.e., $\lim_{t \to \infty} V(t) = 0$.

Since $V(t) = 0$ if and only if $\dot{S}_i = 0$ $(i = 1,\ldots,n)$, the procedure is globally stable.

Remark : The elasticity of the inverse or Marshallian supply function is $(\frac{1}{\rho_i} - 1)$, which is greater than (-1), the elasticity of the inverse or Marshallian demand curve. This is the Marshallian stability condition, viz., that the supply curve cuts the demand curve from below. This is drawn against logarithmic axes in Figure 3 below.

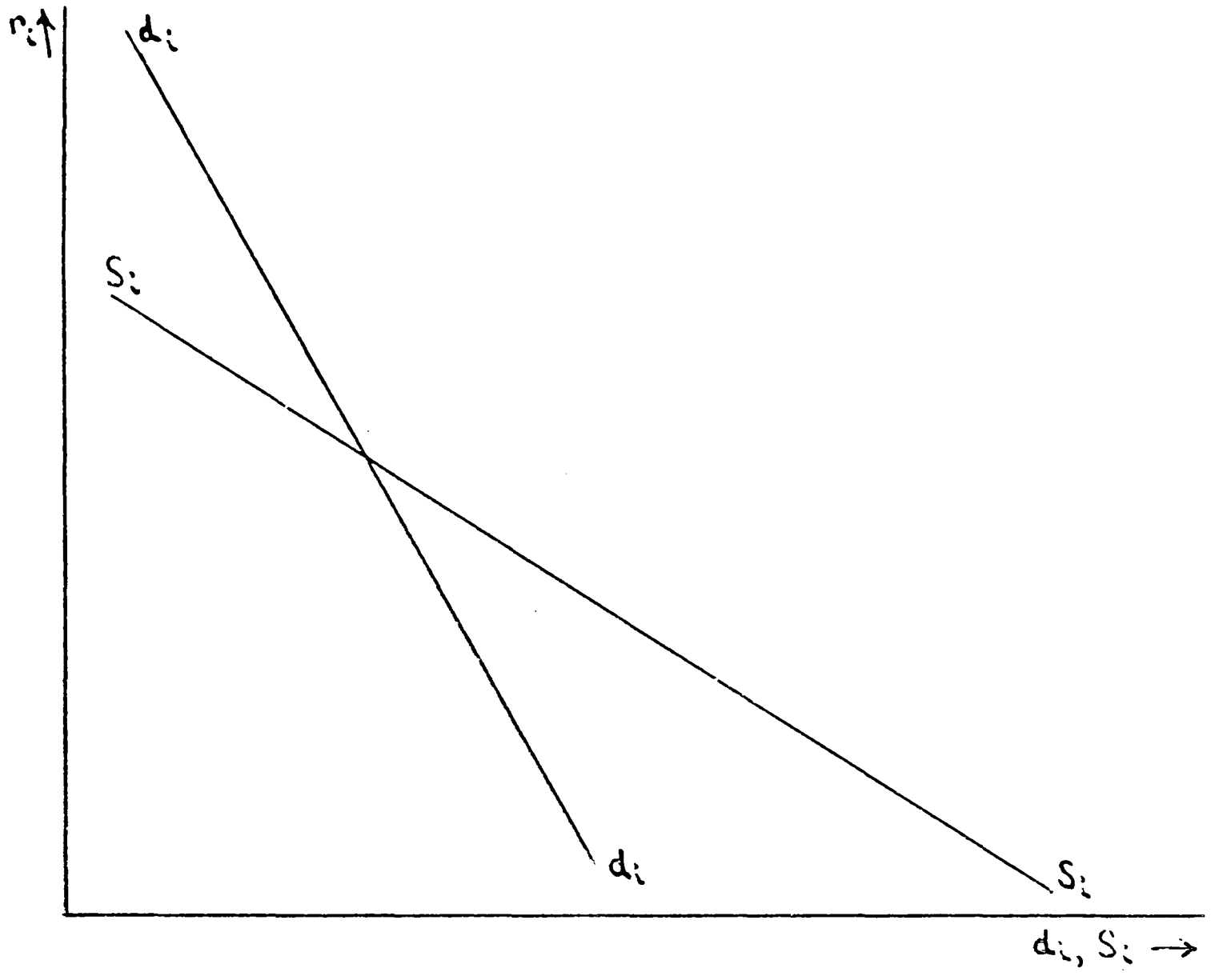


Figure 3

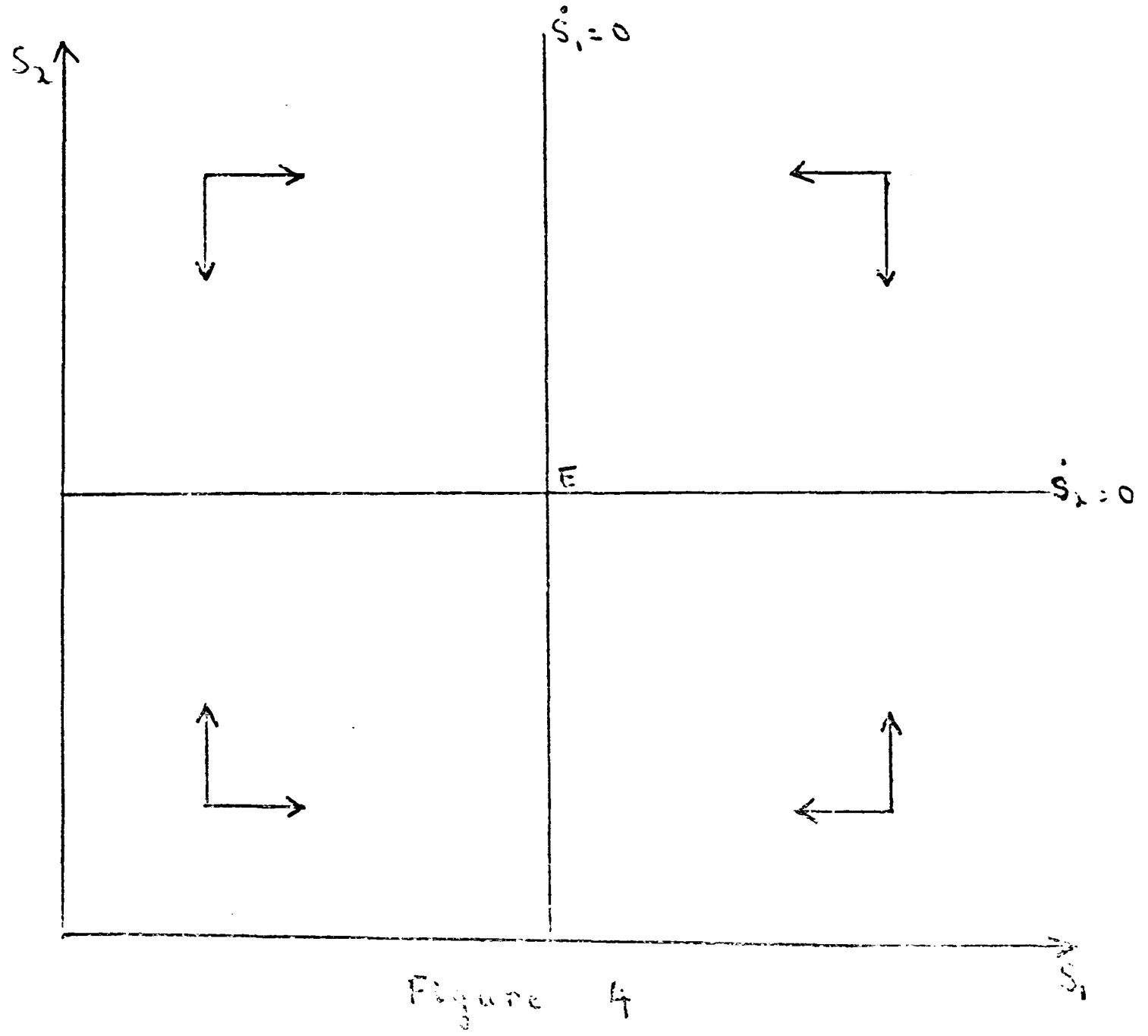


Figure 4

The phase diagram for the adjustment rule (46) is particularly simple since $\dot{S}_i$ depends only on S_i $(i = 1,...,n)$: the two-good case is shown in Figure 4.

Finally, it needs to be pointed out that the model allows for substantial economies of scale and, therefore, accords better with the empirical evidence referred to in Footnote 1 on Page 52.

It remains to assess the planning procedures based on quantitative targets proposed by Heal (1969) and mixed price-quantity procedures suggested by Heal (1971), Aoki (1971) and Portes (1971). A convenient account of the two procedures due to Heal is to be found in Heal (1973).

2. Quantity Targets and Price-like Responses

The basic idea underlying the Heal model is simple. The CPB proposes a feasible trial allocation of inputs while firms respond by informing the planners of their output proposals and the marginal productivities of the inputs at the suggested allocation. Since the CPB possesses, as usual, an economywide utility function, the flow of information back from firms enables it to calculate the marginal social values of inputs in different firms. In the spirit of gradient methods, the CPB proposes a new trial allocation by shifting inputs away from firms where their marginal social values are low to those firms where their marginal social values are high. Firms respond as before and the process goes on.

The procedure embodies certain features of both the Arrow-Hurwicz and Malinvaud approaches to decentralised planning. The fact that a quantity of input is continually being reallocated among a number of different uses makes for feasibility; the principle that the reallocation should use information about the difference made to the overall utility function ensures that the procedure is monotonic in utility (in fact, strictly so, except of course at the optimum). A Heal procedure can therefore be terminated in a finite number of steps and still generate an overall feasible plan that is strictly superior to the one from which the iterations started assuming, as above, that the initial allocation is feasible. Finally, since, as in the Malinvaud model, production possibilities are bounded by the availability of a finite quantity of inputs, one is dealing with a monotonically increasing sequence on a bounded set: this hill-climbing prccedure of the Arrow-Hurwicz kind will, in general, converge to the local maximum of allocation over the set of feasible programmes. However, the algorithm works best if the CPB, using econometric models, possesses in advance a rough idea of whereabouts the global optimum lies. It can then begin a Heal routine to discover its exact position, starting within a suitably chosen neighbourhood of it. Alternatively, one might be satisfied with a local maximum if one's social philosophy were adequately captured by the logic underlying the gradient method, viz., that one ought to be content with small improvements vis-a-vis the status quo. In that case, the routine's inability to be certain of locating a global maximum may not be considered a serious drawback.

The procedure outlined above carries over to more complex situations. Heal considers the complications caused by the introduction of intermediate goods and, especially, the difficulties raised for the procedure by the fact that the volume of intermediate goods to be re-allocated among alternative uses changes at every stage - these are resolved in simple, if ad hoc, ways. Thus, for example, the marginal social valuation of a good in a firm is calculated on the assumption that the output of that firm will continue to be distributed among alternative uses in existing proportions. It is fair to say that the essential principles of the model remain unchanged in the face of these minor technical modifications.

It is worth making some comments on this particular approach to the planning problem. To begin with, the Heal model, like the model described in subsection 1, is simple. And yet, like that model, it seems to capture certain essential features of resource allocation in the real world. The structure of production, for example, is particularly simple (Heal assumes no joint production), although more complicated structures have been successfully analysed [Heal (1971), Aoki (1971)] and will be examined in Section IV. But it is possible that planners concern themselves with straightforward abstractions of reality when they are engaged in the allocation of certain scarce resources, e.g., foreign exchange in many less developed countries. Viewed in that light, the Heal procedure advises planners on the mechanics of constructing an optimal licensing policy when manufacturing industry is characterised by economies of scale.

Secondly, the results obtained by Heal might be taken to offer theoretical support for actual planning procedures which use

quantitative targets as opposed to price signals, especially when returns to scale are increasing. In the context of the foreign exchange example mentioned above, the Heal contribution could shed light on the controversy over whether the allocation of foreign exchange in developing countries is better achieved through the price mechanism or via quantitative controls. These are theoretically interesting issues which are not without substantial practical implications for ways in which an economy should allocate resources.

But the attractive properties of the Heal procedure are purchased not without cost. The informational exchanges are considerable: furthermore, the CPB has to perform a large number of computations to calculate marginal social values at each stage of the iterations, especially when there is a changing volume of intermediate goods. This aspect of the procedure compares unfavourably with the Arrow-Hurwicz gradient methods under increasing returns to scale. But, unlike the Malinvaud process, it does not require the CPB to possess memory: hence, no straightforward comparison between the two procedures (Heal's and Malinvaud's) seems possible.

Finally, it is worth reflecting on the validity of Marglin's remarks in the present context. It seems clear that informational requirements of decentralised procedures in non-convex environments exceed those of the convex, textbook case. But whether price and command procedures are equivalent from the point of view of informational efficiency and regard for privacy in such environments is a question to which no clear answer is yet available.

3. A Digression on the Weitzman procedure

No account of decentralised planning using quantitative targets would be complete if it did not mention Weitzman's (1970) highly stimulating contribution. The procedure is analogous to Malinvaud's model except that the CPB uses quantitative targets while firms respond with prices. The essential feature a Malinvaud planner shares with a Weitzman planner is the knowledge that the underlying production set is convex ; the former builds up this convex set as a convex hull of its extreme points while the latter chips away his zone of ignorance by constructing the intersection of its supporting halfspaces. A Weitzman planner, typically, has an over-optimistic idea of production possibilities. He solves a non-linear programme and announces the quantities thus arrived at to firms. Firms quote the hyperplane that supports the true production set at that point which represents the best firms can do in response to the CPB's quantity targets. The CPB realises that the true production set lies in the halfspace bounded by the supporting hyperplane quoted: this enables a closer approximation to the true production set. The procedure monotonically converges (from above) to the supremum of utility over the set of feasible programmes. The Weitzman method, like the Malinvaud method, is not applicable to non-convex production sets. But it is valid of course when there are constant returns to scale. The procedure would then converge faster because a supporting hyperplane would coincide with a flat segment of the true production set and therefore give the CPB a better idea of production possibilities for any given number of iterations.

IV. MIXED PRICE - QUANTITY PROCEDURES

The urge to use quantity - guided procedures when returns to scale are non-decreasing has already been discussed. It is natural to ask if price signals have a useful role to play in non-convex environments, by helping to (a) reduce excessive informational requirements of purely quantity-guided procedures, and (b) create incentives ensuring adherence to the rules of the game, a problem to which attention was drawn in the assessment of the Arrow-Hurwicz analysis. The two procedures examined below give favourable answers on both counts.

A procedure which uses both prices and quantities has been proposed by Heal (1971). The setup is the same as that discussed in subsection 2 of the previous section except that firms are now allowed to produce a number of outputs instead of just one. There are two parts to the procedure:

(1) the allocation of inputs to production processes, and

(2) the selection of output mix in joint product firms.

The first part is done as in Heal (1969) above: for example, the marginal social valuation of an input is calculated on the assumption that a firm's outputs will continue to be produced in existing proportions. For the second problem, the CPB announces prices of produced goods equal to their marginal social worth; firms cautiously alter their output bundles in the direction of increased profitability. As before, the procedure is feasible, monotonic and, in general, locally convergent. The informational requirements are, once again, considerable, but they are substantially less than those of the corresponding quantity-guided procedure in a joint product world.

The latter would require the CPB to set both input and output targets for each firm. Heal considers alternative institutional interpretations of the model and discusses in particular a "revised market socialist" economy where the procedure can operate with slightly reduced informational requirements. Finally, there is an interesting discussion of monopolistic rules of behaviour which might induce producers not to move away from the socially optimal point even though that is, in the non-convex case, a point of minimum private profit with competitive behaviour. It is shown that, for a special category of social welfare functions, the losers from a movement away from the social optimum can profitably bribe the gainers not to cause such a departure.

The Heal contribution represents an ingenious and successful attempt to use mixed price-quantity methods when returns to scale are increasing. Another such procedure is due to Aoki (1971). In Aoki's model, increasing returns to scale occur owing to the presence of an essential input: for fixed quantities of that input, the production possibility set is convex. Thus, Arrow - Hurwicz type price-adjustment rules are used to allocate other resources optimally. One can then allow profit-maximising managers to calculate and report the marginal profitability of the fixed quantity of the essential input in each alternative use; the input is then reallocated to uses where marginal profitability is higher. The price-guided and quantity-guided procedures can be operated simultaneously and monotonically converge to a local maximum, preserving feasibility along the way[1].

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[1] The qualification that the initial allocation must be feasible is appropriate here as well.

The essential idea underlying the Aoki process is both simple and appealing. To begin with, the representation of technology used by him is not unrealistic. It is assumed that increasing returns to scale are caused by indivisibilities; however, economies of scale caused, for example, by large scale use of current inputs with fixed capital equipment are assumed away. Secondly, managers have a certain freedom of choice in that they are allowed to pursue profit-maximising behaviour within limits set by their quota of essential input; this enables one to devise an incentive structure related to realised profits. Finally, it needs to be pointed out that the Aoki process has informational requirements akin to those of the Heal price-quantity routine discussed earlier.

This section is most conveniently brought to a close by concentrating on the outstanding virtue of the Aoki process. It uses price-guided methods where the environment is convex and quantity-guided methods to allocate the essential input. This would seem to be a natural way of proceeding in such situations. This point has also been noted by Portes (1971) who has proposed a locally convergent algorithm which uses price signals to a decreasing returns sector and quantity signals to an increasing returns one. The moral to be drawn from the Aoki - Portes approach is clear: it might be useful to partition the economy, if possible, into sectors on the basis of convexity or otherwise of the underlying technology and to use different decentralised procedures in each of them.

V. CONCLUDING REMARKS

There are three distinct, though not unrelated, ways of approaching the problem of decentralisation when returns to scale are non-decreasing. One is to continue to use price-guided methods but to modify rules of behaviour (Arrow-Hurwicz, the extension of the Malinvaud model to constant returns to scale); a second is to rely on quantity-guided procedures (Heal, the allocation-of-contracts model), while a third approach is to allow prices to play a useful supporting role (Heal, Aoki).

Several concluding comments on the models outlined in this chapter are worth making.

(1) The gradient method is only locally convergent under increasing returns to scale. This is not surprising if one is prepared to provide only local information to agents. However, monopolistic and other imperfect competitors may well have access to more information than this. They might have knowledge of perceived demand curves: these may have to be exploited to locate a full optimum.

(2) Not a great deal is known about the structure of incentives needed to support an optimum plan. This comment applies to the whole of the literature on decentralisation but it is particularly important when returns to scale are increasing. For, in such cases, a social optimum may correspond to a point of minimum private profit. The question is clearly not unrelated to comment (1) above as the tentative suggestions of Arrow and Hurwicz and Heal involving non-competitive behaviour indicate. Finally, it is worth remembering that price-guided procedures may well be superior to quantity-guided ones from the point of view of incentives, as the models of Section IV suggest.

(3) The important problem of devising planning procedures which are decentralised in the sense of allowing producers authority to take decisions, remains to be explored: practically no progress can be reported in this area.

Chapter 3 : On Decentralising the Production Decision Over Time

I. INTRODUCTION

The literature on optimum economic growth constitutes a possible foundation for a theory of long-term planning.[1] But its applicability to problems of planning is limited by the fact that there have been very few attempts to devise procedures which enable the attainment of optimum policies in decentralised fashion,[2] for completely centralised investment planning is fraught with many difficulties.[3] This and the two remaining chapters suggest some procedures which take a preliminary step towards remedying this deficiency. The present chapter is devoted to problems of informational decentralisation and computation. Chapter 4 contains a suggestion which partially captures decentralisation of the authority to take decisions. Finally, Chapter 5 concentrates on decentralisation of intertemporal production and consumption plans through the use of taxes, subsidies and public debt as control variables.

1. Features of Optimum Policies

The present discussion of optimum policies is in terms of the one-good model with exogenously growing labour. What follows would

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[1] Ramsey (1928), Samuelson and Solow (1956), Koopmans (1965), von Weizsäcker (1965), Radner (1966), Mirrlees (1967), Gale (1967), McFadden [(1967), (1973)], Brock and Gale (1969), Sutherland (1970), Brock (1971).

[2] A brief examination of these attempts together with references is to be found in Section II.

[3] Some of these difficulties were mentioned in Section I.2 of Chapter 1.

therefore not be very interesting if it were applicable solely to one-good models;[1] however, optimal plans in many interesting multi-sector economies share some of the characteristics of those in the one-sector case.

The optimum development of the one-sector economy is described by the Euler - Ramsey "local optimality" conditions and the feasibility condition respectively:

$$\left.\begin{aligned} (1) \qquad \frac{u''(c)}{u'(c)}\dot{c} &= \delta - f'(k) \\ (2) \qquad \dot{k} &= f(k) - nk - c \end{aligned}\right\}$$

$$k_o \quad \text{given,}$$

where $u'(c)$: instantaneous marginal utility of consumption

n : rate of population growth

δ : planner's rate of utility discount

$f(k)$: production function written in intensive units by virtue of constant returns to scale.

It is assumed that

$$(3a) \qquad u'(c) > 0, \quad u''(c) < 0, \qquad \lim_{c \to 0} u'(c) = \infty$$

$$(3b) \qquad f'(k) > 0, \; f''(k) < 0, \; \lim_{k \to 0} f'(k) = \infty, \; \lim_{k \to \infty} f'(k) = 0$$

Harrod - neutral technical progress and radioactive depreciation can be included in the usual way but, for reasons of expository

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[1] The very restrictive conditions under which multi-sector economies allow one-good approximations are set out in Fisher (1969).

convenience, I assume them away.[1]

Three features of optimum policies are of interest to the decentralised planner.

(1) Optimum policies tend asymptotically to balanced growth states.[2]

From equations (1) and (2), this optimum balanced growth path (henceforth the OBGP) is given by

$$\left.\begin{aligned} f'(k^*) &= \delta \quad (4) \\ c^* &= f(k^*) - nk^* \quad (5) \end{aligned}\right\}$$

from which it is possible to solve for (c^*, k^*).

Equations (1) and (2) describe a path that seems competitively attainable in a private ownership economy through myopic two-period utility maximisation by consumption units and instantaneous profit-maximisation by producers. But this is true provided such a system is supplied with the correct initial condition. Notice that (1) and (2) constitute a pair of first-order differential equations in c and k : however $k = k_o$ supplies only one initial condition. The remaining "initial condition" is provided by the information about the OBGP contained in equations (4) and (5). Hence one needs to know the asymptotic state of the economy and then compute the entire optimum path backwards from there before being able to determine how much investment

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[1] One needs to be satisfied that equations (1) and (2) actually describe the optimum policy. A sufficient condition is that $\delta \geq n$. See Koopmans (1965).

[2] This is not true, for example, of economies with fixed exhaustible resources or with intertemporally dependent preferences. See the discussion in Heal (1973).

ought to be undertaken today.[1] The CPB, after supplying the private sector with information about the initial investment may, if it so chooses, allow competitive behaviour to generate the optimum policy. But it would do so not on grounds of ignorance of that policy but because, for example, decentralisation might encourage participation at grass-roots level or, being price-oriented, provide incentives to follow the optimum policy.

(2) The OBGP exhibits the saddle-point property. Unless the initial investment decision mentioned above is taken correctly, decentralised pursuit of the local optimality conditions without subsequent recalculation by the CPB will lead either to permanently inefficient development or cause the economy to run out of capital in finite time.

(3) Even if the technological information necessary to work backwards from the OBGP were available to the CPB, the computational burden of determining the initial investment decision may well be excessive;[2] this is particularly true of multi-sector economies where it verges on the impossible. Thus, if optimum growth theory is to be a guide to action, it is not enough to be assured that there exists a functional equation which the optimum policy satisfies; one needs to construct manageable routines which enable computation of that policy.

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[1] Mirrlees (1967) computes optimum policies as a power series around the OBGP and presents numerical results for certain utility and production functions. The computational algorithm used there can be readily handled on computers and has been found to converge quite rapidly in practice.

[2] For one-sector models one needs to add Footnote 1 above as a qualification to this statement.

2. Implications for Decentralised Procedures

It was mentioned in Chapter 1 that the desire to exploit informational and computational economies often constitutes the raison d'ętre of decentralisation. In this subsection I want to argue that the three features of optimum policies outlined above allow one to appreciate the need for undertaking the decentralisation of long-term planning and point to certain desirable properties decentralised procedures should ideally possess.

Decentralised procedures are characterised by iterative exchanges of price and quantity information. It is convenient to discuss such procedures in the context of the model outlined above. The underlying institutional framework is described by the following assumptions.

(A.1.) There is a representative, lifetime utility-maximising consumer in the private sector who has access to a perfect capital market; the myopic aspect of this behaviour is governed by the equation

$$(6) \qquad \frac{u''(c_t)}{u'(c_t)}\,\dot{c}_t = \delta - r_t$$

where $u(c_t)$ is the instantaneous utility of consumption at time t. and r_t is the rate of interest ruling in the capital market at time t.

(A.2) The representative consumer owns all labour and capital in the economy and supplies them to producers in return for wage and interest payments. Thus, his flow budget constraint is

$$(7) \qquad c_t + \dot{a}_t = w_t + (r_t - n)\,a_t$$

where a_t is the amount of assets per capita owned at time t,

and w_t is the wage rate at time t.

The economy's initial capital stock is owned by the representative consumer. Thus,

(8) $\quad a_o = k_o$

(6), (7) and (8) are of course not sufficient to describe and solve the consumer's allocation problem; I return to this question in the subsequent discussion on planning procedures.

(B) Producers are subject to the feasibility constraint

(9) $\quad c_t + \dot{k}_t = f(k_t) - nk_t$

(C) Factor markets clear instantaneously with factors being paid their marginal products.

(10) $\quad a_t = k_t$

(11) $\quad r_t = f'(k_t),\ w_t = f(k_t) - k_t f'(k_t)$

Some of these assumptions need comment. The reader will have noticed that (A.1) postulates the existence of consumers who are infinitely long-lived and who seek to maximise the same objective function as the CPB. What is the motivation for explicitly introducing consumers who are, from the economist's point of view, virtually indistinguishable from the CPB? First, there are the usual "general equilibrium" reasons. The economy under consideration has factor markets: it is therefore reasonable to assert the existence of entities who make up the supply side of factor markets and to introduce assumptions about ownership of factors of production (assumption A.2). Furthermore, it is desirable that this should be done without creating

complications that divert attention from the subject matter of the present chapter, viz., the intertemporal decentralisation of production decisions. All this is adequately captured by the "representative consumer" assumption.

Secondly, the principal planning procedure of this chapter (described in Section III) postulates decentralised development via a sequence of plans, each drawn up on the assumption of static expectations. While it is true (as we shall see) that the CPB estimates technological possibilities and makes forecasts which help sustain alternative structures of expectations, I find it convenient to suppose that these expectations are held ultimately by consumers. It is therefore useful to treat the CPB and consumers as separate entities. But in the interests of brevity of exposition, I shall frequently use terms like "expectations entertained by the CPB" to denote the forecasting activities of the CPB and the associated expectations which they help consumers entertain.[1]

Thirdly, the token introduction of consumers equipped with patterns of expectations serves as a convenient prelude to the more satisfactory analysis of Chapter 5 which brings in finitely long-lived individuals whose aims in general conflict with those of the CPB. These conflicts when they arise are resolved in favour of the CPB through tax-cum-subsidy policies (which are absent in this chapter) and make for a more interesting theory of consumer behaviour. The emphasis in that chapter is primarily on the intertemporal decentralisation of consumption plans; a theorem combining the result of that

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[1] This point is elaborated in page 91 when I discuss the intertemporally decentralised procedure.

investigation with our present one into informational and computational aspects of production decision-making is however presented.[1]

To summarise, although consumers and the CPB do not have conflicting aims and therefore do not need to be sharply distinguished, the former fulfil a useful role in expositions of the planning procedures of this chapter.

We are now in a position to describe a typical decentralised planning procedure. The story starts with the historically given amount of capital stock, $a_o = k_o$.

(i) The CPB observes the wage rate w_o and the interest rate r_o ruling in factor markets. It therefore discovers $f(k_o)$ and $f'(k_o)$ from

$$f(k_o) = w_o + r_o k_o \tag{12}$$

$$f'(k_o) = r_o \tag{13}$$

Two features of rule (i) need comment. First, we do not permit the entire production function to be transmitted to the CPB. Such restrictions are necessary if the procedure is to be informationally decentralised. Secondly, there is no direct transmission of information from producers to the CPB. Under the assumption of rapid (in our case, instantaneous) market clearance, the CPB can infer the values of technological variables from market data.[2]

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[1] On pages 177 - 179.

[2] Step (i) could be modified by allowing firms to report current output, $f(k_o)$, to the CPB with the latter continuing to deduce the marginal product of capital by observing the market as in equation (13). Such a modification will be necessary in attempting to formulate informationally decentralised planning procedures in two-sector models. See page 100.

The assumption that markets clear quickly relative to the time separating adjacent iterations would seem quite appropriate to a real time procedure.

Neither producers nor consumers have enough information to determine their present moves. Producers require guidelines on the rate of investment; the lifetime utility-maximiser needs to form expectations about the future course of wage and interest rates to determine the consumption - asset formation decision. Such expectations about wage and interest rate movements are, given competitive imputation (11), equivalent to expectations about the future course of capital accumulation and on this the representative consumer may have very little information. Central intervention might help resolve some of these problems; hence we describe step

(ii) The CPB announces a quantitative investment target which it expects the economy to fulfil.

What are the operations that the CPB has to perform to arrive at this investment target? The planner's task is most conveniently divided into three parts.

(a) The CPB possesses only local information about the technology, viz., the information available in step (i). It must on the basis of this knowledge estimate the shape of the entire production function to determine what policy the economy should follow today.[1] The problem of devising an estimation procedure is related to (1) the formation of expectations and (2) the optimal use of prior information, and requires for a satisfactory solution answers to unsolved problems responsible for the ad hoc nature of the theory of investment.[2] Accordingly,

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[1] Memory of past observations is admissible at second and later rounds of the planning procedure.

[2] A particularly good account of these difficulties is to be found in Nerlove (1972); see the relevant references cited there.

we shall be content with the more modest requirement that the CPB forms a view of future possibilities that makes some economic sense and is not completely ad hoc.

(b) The CPB's social valuation is the representative consumer's. Armed with the view of production possibilities formulated in (a), the CPB chooses (c_t) to maximise

$$(14) \qquad \int_0^{\infty} e^{-\alpha t} u(c_t)\, dt$$

subject to

$$(15) \qquad \dot{k}_t = g_o(k) - nk - c$$

$$k_o \quad \text{given}$$

where $\alpha = \delta - n$ (and is nonnegative, by assumption) [1]

and $g_o(k)$ is the production function estimated at time zero by the CPB in (a) above. In other words, the CPB computes the plan that is optimal relative to its current view of the technology.

(c) The CPB announces the investment target $\dot{a}_o = \dot{k}_o$ that it wishes the economy to fulfil currently.

(iii) Consumers and producers save $(\dot{a}_o)$ and invest $(\dot{k}_o)$ in accordance with the target laid down in (ii) (c). The CPB observes the new wage rate, w_1, and the interest rate, r_1, and the process goes on as before.

Under our assumptions, the iterative exchanges described above constitute an informationally decentralised procedure which generates a real time sequence of equilibria. One point about this sequence is worth noticing. The economy only implements the current portion of the CPB's computed optimum sequence at any time; next

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[1] See Footnote 1 on Page 71.

period, the CPB obtains new information and that portion of the old plan as yet unimplemented is scrapped in favour of a new one. In other words, every point in the real-time sequence embodies continuously revised assumptions about the future.[1]

It now remains to impose restrictions on the paths generated by decentralised procedures and to comment on the partially satisfactory nature of the rules outlined so far.

Property 1 : The procedure must be feasible, i.e. the directive issued by the planners at every iteration must be technologically capable of implementation. Feasibility must be satisfied for every planning procedure which operates in real time. Finally, it is worth noting that there would seem to be no advantages in employing a tatonnement procedure here for the extra technological information derived from market observation is most conveniently thought of as accruing over real as opposed to computational time in response to investment plans which map out relevant portions of the production set.

Property 2 : The procedure must be asymptotically optimal, i.e., the decentralised path should converge to the OBGP. This is because one wants decentralised paths to possess agreeable welfare implications; it is not unreasonable to use asymptotic approach

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[1] Alternatively, one may think of the consumer, given $\dot{a}_o$ in step (iii), as planning today's and tomorrow's consumption from equations (6) and (7) and being unable to implement the latter tomorrow because of a falsification of expectations.

to the terminal state of the first-best policy as a benchmark in this regard.

Certain aspects of rule (ii) require comment. Rule (ii.b) calls upon the CPB to recalculate the optimum plan at every stage as information accumulates through real-time iterations. A more complete model which took account of costs of information-handling and computation would, no doubt, call for less frequent revisions than this. We require

Property 3 : The procedure must be computationally manageable, i.e. finding the solution to problem (6) subject to (7) must not verge on the impossible. What does or does not satisfy this property is partly a matter of judgment and partly a matter of technical skill. It is perhaps easier to identify situations where it is satisfied rather than those where it is violated.

Finally, it needs to be noticed that in the absence of complete technological information available to the CPB, the decentralised path will in general be a second-best path. Whether this second-best path is optimal in any sense is the subject matter of rule (ii.a) and on this the rule is regrettably vague. As mentioned above, this involves unsolved problems in the expectations literature. But the following, more modest, route is not unreasonable. Let us suppose the CPB takes a particular view of technological possibilities and computes the corresponding optimum policy. When it announces an investment target in rule (ii)(c), it expects that target to be fulfilled at wage and interest rates computed according to the CPB's perceived production function, i.e.,at

$$\hat{r}_1 = g_o'(k_1)$$

$$\hat{w}_1 = g_o(k_1) - k_1 g_o'(k_1)$$

where k_1 is the new capital stock the economy is asked to acquire and $g_o(k)$ is defined in (15).

What it observes on the market is

$$r_1 = f'(k_1)$$

$$w_1 = f(k_1) - k_1 f'(k_1),$$

i.e., the true marginal products. Hence we postulate

Property 4 : The CPB's expectations about wage and interest rates must be asymptotically self-fulfilling.

Property 4 relates to the particular forecasting and learning mechanisms adopted by the CPB. Thus, the property is satisfied by an asymptotically optimal procedure where the CPB entertains static expectations about wage and interest rates at every stage of the iterations.

The rest of this chapter is organised as follows. In section III I present an intertemporally decentralised procedure which satisfies many of the properties listed and discussed in this section. As far as I am aware, similar attempts have not been made in the literature before. Some plausible ad hoc arrangements for attaining approximate optima are assessed in Section II in the light of the criteria outlined above. Section IV is a digression on efficiency prices; finally, Section V contains a concluding remark.

II. PLAUSIBLE ARRANGEMENTS

This section reviews some ad hoc arrangements which generate intertemporal policies and which satisfy some of the criteria listed in the previous section.

1. Naive Decentralisation

One's first thought in attempting to construct an intertemporal price-decentralisation procedure is to wonder whether the methods proposed in the static context might not be generalisable. Specifically, suppose the CPB knows the production function. Assume that it solves the optimum growth problem and announces the rate of interest and that firms respond with profit-maximising output proposals. In a one-sector model of the kind described by equations (1) to (5), this procedure is clearly unworkable. The rate of interest is the marginal product of the existing capital stock and is therefore given, quite irrespective of optimisation. What is needed is a way of determining the rate of investment which is the rate of change of the capital stock. This implies that the CPB has to announce not the rate of interest but the rate of change of the rate of interest, and provides confirmation (if that were needed) that it requires the entire future to determine the present allocation.

A natural ad hoc adjustment mechanism that provides producers with information about the future rate of interest is

$$(16) \qquad \dot{k} = \frac{\delta - f'(k)}{f''(k)}$$

δ is the asymptotic (OBGP) rate of interest while $f'(k)$ is the current rate; $f''(k)$ is brought in as the definitional link between investment and the change in the interest rate. Firstly, the CPB only need

announce δ once and for all; since producers know f' and f" the process described by (16) requires no further central intervention. Secondly, (16) describes a Pavlovian model of adjustment for the system, as we shall see, moves towards the OBGP: how quickly will depend on speed of adjustment. Thirdly, no optimisation is involved on the CPB's part: consequently the utility function does not enter in the adjustment process (16). This ensures that the CPB never has to acquire any technological information; equally, a process which does not depend on the utility function can hardly be described as a satisfactory contribution to the decentralisation of optimum policies. Finally, it is quite straightforward to satisfy ourselves of the following

Proposition: The policy generated by rule (16) converges to the OBGP.

Proof: Since $f''(k) < 0$, $\dot{k} \gtrless 0$ according as $k \lesssim k^*$. Furthermore, the rate of change of k is bounded away from zero outside any neighbourhood of k^*. Hence k converges to k^*.

2. Approximate Optimal Growth

Some authors have wondered whether judiciously chosen sub-optimal policies which are administratively and computationally more manageable than the full optimum might not yield tolerable approximations to the optimum policy. Thus, Mirrlees and Stern (1972) discovered that constant savings ratio paths chosen with care[1] in a one-sector model with a constant elasticity utility function and a Cobb-Douglas production function perform almost as well as the optimum policy. A substantially different approach is due to von Weizsäcker (1967) and Phelps (1967) who attempt to establish a presumption in

[1] A rough optimum growth analysis is needed to make this choice.

favour of constant savings ratio paths defined by investment-equals-profits in analyses of optimum growth "making only slight use of utility functions"[1]. These papers are not to be regarded as contributions to the problem of intertemporal decentralisation; instead they attempt to explore the worth of doing sophisticated optimum growth analysis as opposed to following rules of thumb. Decentralised pursuit of second best policies with income taxation is analysed by Arrow and Kurz (1970) in a model where the CPB is prevented from achieving the Ramsey optimum owing to imperfections in the capital market formally captured by consumers' choice of a constant savings ratio.[2]

3. A Procedure Due to Infante and Stein

Infante and Stein (1973) address themselves to the problem of using procedures characterised by informational and computational economies to generate policies which possess certain agreeable welfare implications. I present the essential features of their analysis and subsequently assess it in the light of the criteria developed in Section I.

It is convenient to look at the Infante - Stein (henceforth I-S) suggestion for solving the problem to which rule (ii) of the decentralised procedure in Section I is devoted.[3] The problem divides itself naturally into two parts. First, what should the planner do,

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[1] von Weizsäcker, op. cit.

[2] Arrow and Kurz also analyse many first-best situations. Their approach will be commented on in Chapter 5.

[3] I-S do not present their model in these terms but I find it illuminating to examine their contribution within the framework of Section I.

given that he has only local information about the production function and is therefore unable to determine (c^*, k^*) from equations (4) and (5) ? Secondly, given an estimate of (c^*, k^*), are there any computational shortcuts available to him? Define

$$(17) \qquad v(k) \equiv \dot{k} = f(k) - nk - c(k)$$

then

$$v'(k) = f'(k) - n - \frac{dc}{dk}$$

$$= f'(k) - n - \frac{1}{v(k)} \cdot \frac{dc}{dt}$$

and substitution from (1) yields

$$(18) \qquad v'(k) = f'(k) - n + \frac{f'(k) - \delta}{v(k)} \left[\frac{u'(c)}{u''(c)}\right]$$

A second-order approximation to the optimum policy $v(k)$ is given by

$$(19) \qquad v(k_o) = v'(k^*)\ (k_o - k^*) \quad \left[\text{since } v(k^*) \equiv \dot{k}^* = 0\right]$$

Thus, $v(k_o)$ is the investment target announced in rule (ii.c); the problem therefore is to determine the two terms on the right hand side of (19). This is done in three steps.

(a) I-S assume that the local technological information obtained by the CPB in rule (i) of Section I consists of the first and second derivatives of the production function, i.e. $f'(k_o)$ and $f''(k_o)$. This naturally leads to the approximation

$$(20) \qquad f'(k) = f'(k_o) + f''(k_o)(k - k_o)$$

where k_o is the current stock of capital. Putting $k = k^*$ and substituting from (4) one gets

$$\delta = f'(k_o) + f''(k_o)(k^* - k_o)$$

and

$$(21) \qquad k_o - k^* = \frac{f'(k_o) - \delta}{f''(k_o)},$$

which, the reader will recall, looks very similar to the ad hoc rule (16) analysed above.

(b) $v'(k^*)$ can be evaluated from (18). Using L'Hospital's rule, one gets a quadratic equation in $v'(k^*)$ and a moment's reflection on the shape of the optimum trajectory in the neighbourhood of the OBGP in two-dimensional phase plane confirms that one is interested in the negative root:

$$(22) \qquad v'(k^*) = -\frac{\delta}{2}\left[\left(1 + \frac{4u'(c^*)}{\delta^2} \cdot \frac{f''(k^*)}{u''(c^*)}\right)^{\frac{1}{2}} - 1\right]$$

(c) In the absence of global knowledge, neither c^* nor k^* is known; I-S consequently adopt the crude expedient of replacing c^* and k^* by their current values, c and k.

Then, rule (19) becomes

$$(23) \qquad \dot{k}_o = -\frac{\delta}{2}\left[\left(1 + \frac{4u'(c_o)}{\delta^2} \cdot \frac{f''(k_o)}{u''(c_o)}\right)^{\frac{1}{2}} - 1\right]\left[\frac{f'(k_o) - \delta}{f''(k_o)}\right]$$

Furthermore, c_o is approximated by $f(k) - nk$.[1]

Rule (23) is an investment rule which depends solely on information about the current state of the economy. Given the observed similarity of (21) with (16), it is not surprising that (23) generates a path which, given strict concavity, converges to the OBGP.

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[1] This fact, which is clearly rather important for the algorithm to get started, is not mentioned till the discussion of actual computations towards the end of the paper!

I-S also develop a slightly more sophisticated approximation to order $(k^* - k_o)^2$ for $v'(k^*)$; the corresponding investment rule is again asymptotically optimal.

The Infante-Stein procedure is feasible, asymptotically optimal and computationally manageable: these, as we have seen, are useful properties. Furthermore, the mechanism is informationally decentralised in the sense of Hurwicz.[1] I restrict myself to comments on the way in which the authors tackle the problems of using information and of computation.

(1) The I-S procedure needs information about the second derivative of the production function as well as the first. This cannot be discovered by observing wage and interest rates on the market and must therefore be transmitted by producers.

(2) Once the CPB knows $f'(k)$ and $f''(k)$, it is "natural" to try and develop the Taylor approximation described by (20). But this view of production possibilities cannot be justified by any appeal to economic sense.

(3) Now suppose that (20) is given to us. Rule (ii)(b) of section I above requires the CPB to compute the optimum policy subject to its view about technological possibilities. It is reasonable to demand such behaviour on the CPB's part in what is an attempt to generate optimum decentralised development. By contrast, a sequence of ad hoc operations precedes rule (23). For this reason, it seems more appropriate to regard the paper not at all as an attempt to attain optimum policies by decentralised methods, but rather as an exercise in developing simple control laws which are asymptotically optimal.

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[1] See page 3.

This greatly reduces the attraction of the procedure for the decentralised planner for the latter would wish to pay more attention to the derivation of the economy's intermediate development path and not merely be content with asymptotic optimality. In mitigation it needs to be pointed out that the iterative exchanges embodied in the I-S procedure make the analysis closer in spirit to a satisfactory decentralised resource allocation mechanism than some of the other plausible arrangements outlined in this section.

III. AN INTERTEMPORALLY DECENTRALISED PROCEDURE

The procedure to be presented in this section possesses the properties outlined in Section I. Recall that the main difficulties in this area arise in connection with the problems mentioned under step (ii.a). It is known that decentralised determination of the fully optimal policy requires long run perfect foresight: this, in turn, implies detailed knowledge of the production function. Perfect foresight imputes to economic agents an extreme degree of sophistication in predicting the future. Economic agents are, typically, less prescient than this; accordingly, it is worth going to the other extreme and to postulate static expectations. This will be step (ii.a) in the model to be described below.

The Planning Procedure

The story starts at $t = 0$ with a historically given capital stock, k_o, which is known to the CPB. The planning procedure consists of the following steps.

(i) The CPB observes the wage rate, w_o, and the interest rate, r_o, ruling in factor markets. It therefore discovers $f(k_o)$ and $f'(k_o)$

from

(24) $$f(k_o) = w_o + r_o k_o$$

(25) $$f'(k_o) = r_o$$

(ii)(a) The CPB uses the information obtained in (i) above to construct a "perceived" production function, $g_o(k)$, given by

(26) $$g_o(k) = f(k_o) + f'(k_o)(k-k_o) = w_o + r_o k ,$$

where use has been made of (24) and (25)

(b) The CPB then solves the problem

(27) $$\underset{(c_t)}{\text{Maximise}} \int_o^{\infty} e^{-\alpha t} u(c_t)dt$$

subject to

(28) $$\dot{k}_t = g_o(k_t) - nk_t - c_t$$

k_o given

(c) The CPB announces the investment target $\dot{a}_o = \dot{k}_o$ that it wishes the economy to fulfil.

(iii) Consumers and producers save $(\dot{a}_o)$ and invest $(\dot{k}_o)$ in accordance with the target laid down in (ii)(c). The CPB observes wage and interest rates in factor markets to enable it to form a new linear approximation tc the production function and the process goes on as before.[1]

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[1] Note that k_1 is the result of the economy's fulfilling the investment target announced in (ii)(c) and is therefore known to the CPB.

Rule (ii) followed by the CPB requires comment. Equation (26) shows that only currently observed market information is used to estimate the production set. Two interrelated features of this formulation are of interest. First, in addition to the local information of step (i), all the global information the CPB has at the beginning of the procedure is that the production set is convex. Therefore (26) is equivalent to approximating a convex set by its supporting hyperplane at that point on its frontier which is currently known. This is analogous to the Weitzman planning procedure discussed in the previous chapter,[1] except that a Weitzman planner has "memory" and uses all past information about supporting hyperplanes to progressively whittle down his overoptimistic ideas about production possibilities. Here, the CPB could use all previous supporting hyperplanes in conjunction with the currently estimated one but it will turn out that lack of memory never imposes a welfare loss on society. Secondly, the use of such an approximation to the production set by the CPB is equivalent to postulating static expectations on the part of consumers with regard to the future course of wage and interest rates. Since the CPB computes the optimum policy on the basis of the production function (26) and announces the saving consumers ought to do, we have a procedure where consumers expect w and r to remain at w_o and r_o and implement the current consequences of such a belief. This belief is, of course, wrong and is repeatedly demonstrated to be so because a new set of wage and interest rates is observed in factor markets in the

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[1] See page 63.

next round. At that stage, consumers again implement the current portion of a recomputed plan with static expectations, and so on. In this formulation, the CPB is not permitted to learn and modify its naive estimation rule, therefore consumers do not entertain increasingly sophisticated expectations. There are difficulties associated with learning on the basis of past experience: we shall have occasion to mention some of these later[1]. The important point to note is that our way of setting up the problem ties the method of estimation of the production set by the CPB to a particular class of expectations, i.e. to a particular view of the future held by consumers.[2] The assumption of static expectations, simple and unsatisfactory though it is, makes some economic sense. It is an assumption often made in the economic literature. It therefore represents a modest advance on completely ad hoc ways of forming views about the future. Furthermore, we have seen that it is equivalent in this model to a natural mathematical procedure for estimating a convex set.

Thus, at a typical time t, the CPB communicates an investment target based on the view of the future by solving the following problem:

(29) Maximise $\{c_s\}$ $\int_t^\infty e^{-\alpha(s-t)} u(c_s)ds$

subject to

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[1] On pp. 95-98.

[2] The notion that the CPB performs the job of estimating the production set by communicating with producers while consumers hold the appropriate expectations was given as one reason for introducing consumers explicitly into the analysis. See the remarks on page 75.

$$(30) \qquad \frac{dk}{ds} = g_t(k) - nk - c$$

$$k_t \quad \text{given}$$

where

$$(31) \qquad g_t(k) = f(k_t) + f'(k_t)(k - k_t)$$

The solution to this problem is simple. Denoting $u'(c)$ by q, it is given by

$$\frac{1}{q} \cdot \frac{dq}{ds} = \delta - g_t'(k)$$

or, from (31),

$$(32) \qquad \frac{1}{q} \cdot \frac{dq}{ds} = \delta - f'(k_t)$$

Since $f'(k_t)$ is a constant at time t, (32) is a linear differential equation which, together with equation (30), constitute a pair of equations which is quite easy to solve. The CPB only has to work out the initial q_t (and, therefore, c_t) that is consistent with feasibility. Specifically,

$$(33) \qquad q(s) = q(t)e^{[\delta - f'(k_t)](s-t)}$$

and the CPB chooses the lowest (highest) q_t consistent with feasibility when δ is less (greater) than $f'(k_t)$. The procedure is therefore computationally manageable.

The procedure described in this section is feasible. It is based on iterative exchanges of price and quantity information. Furthermore, economic agents are not required to transmit any information directly to the CPB for the latter can observe market-clearing prices in what is a real-time development of the economy. It remains to show that the procedure is asymptotically optimal. This has the additional advantage that a forecasting mechanism based on static expectations is

self-fulfilling at balanced growth. We therefore prove the following

Theorem: The procedure converges monotonically to the OBGP.

Proof: The problem (29) - (31) allows one to see that whenever $k_t \lesseqgtr k^*$, $g_t'(k) = f'(k_t) \gtreqless \delta$, by strict concavity of the true production function. From the properties of optimum policies in one-sector models,[1] it is known that the CPB's computed optimum plan at typical time t will, from (32), call for q to fall (rise) and therefore the capital stock in the economy to rise (fall) when $g_t'(k) \gtreqless \delta$. Since the economy, by rule (iii), always fulfils the CPB's investment directives, we know that the actual capital stock in the economy, k_t, rises (falls) whenever $k_t < (>) k^*$.

Consider the Lyapunov norm

$$V(t) = \tfrac{1}{2}\left(\frac{1}{q}\cdot\frac{dq}{dt}\right)^2$$

The function is positive everywhere except at the OBGP when it is zero.

$$\dot{V}(t) = \left(\frac{1}{q}\frac{dq}{dt}\right)\frac{d}{dt}\left[\frac{1}{q}\cdot\frac{dq}{dt}\right]$$

$$= \left\{\delta - f'(k_t)\right\}\left\{-f''(k_t)\right\}\cdot\frac{dk_t}{dt}$$

It has been argued in the previous paragraph that $\frac{dk_t}{dt} \gtreqless 0$ according as $\delta \lesseqgtr f'(k_t)$. Since $f''(k_t) < 0$, $\dot{V}(t) < 0$.

A routine argument now shows that $\dot{V}(t)$ must actually go to zero. Assume $\lim_{t \to \infty} V(t) = V^* > 0$. $V(t)$ has been shown to converge to V^* in decreasing fashion: therefore, for sufficiently large t, $2V^* \geq V(t) \geq V^*$.

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[1] See, for example, Cass (1965).

Since $\dot{V}(t) < 0$ except at the OBGP, it assumes a negative maximum on the above closed and bounded set. That is, there exists a positive number ε such that

$$\dot{V}(t) \leq -\varepsilon < 0 \quad \text{for} \quad t \geq T.$$

and $V(t) \leq -\varepsilon (t-T) + V(T)$ for $t \geq T$.

But then $\lim_{t \to \infty} V(t) = -\infty$, which contradicts the supposed positivity of V^*. Hence $V^* = 0$ and the system must therefore converge to the OBGP.

Since $g_t'(k) = f'(k_t)$ falls (rises) monotonically as k_t rises (falls), we have that $\frac{\dot{q}}{q}$ is a monotonically increasing (decreasing) sequence for $k_o < (>) k^*$; it has been proved already that the supremum (infimum) of the sequence is zero. Thus $\frac{\dot{q}}{q}$ never changes sign, which implies that c_t and k_t converge monotonically to c^* and k^*, their OBGP values.

Remark: The monotonicity of the decentralised policy confirms our earlier observation that the CPB need not store past hyperplanes and use them in conjunction with the currently generated one to compute the policy; the latest one is enough.

The logic of the above proof is very simple. The planning procedure pushes the system monotonically towards the balanced growth state every time static expectations are frustrated: the proof argues that the rate at which it is pushed is fast enough to guarantee convergence. The economy ultimately enters balanced growth where, of course, static expectations are entirely justified.

Commentary

(1) The static expectations procedure analysed above has been found to possess many desirable properties. Static expectations, however, although widely discussed in the economic literature ought to be capable of being improved upon. Ideally, one would like to have a procedure which starts from static expectations as a rough-and-ready view of the future and eventually attains perfect foresight through successive improvements. Given that we have tied the expectations structure to the method of estimating the production function, an answer to this clearly depends on how efficiently planners utilise information transmitted to them by producers.

The above procedure used linear approximations to a convex set. An instinctive attempt at generalisation is to have the CPB start with linear approximations and store all past observations of $f'(k)$ to generate higher - order approximations to the production function of the form

$$(34)\quad g(k) = f(k_o) + f'(k_o)(k-k_o) + \tfrac{1}{2}f''(k_o)(k-k_o)^2 + \ldots + \frac{1}{n!}f^{(n)}(k_o)(k-k_o)^n + \ldots.$$

where each of the higher derivatives is computed successively, i.e. $f^{(n)}(k_o)$ is computed at the n^{th} iteration with the help of all observed values of $f'(k)$ in the meantime. But this procedure is, clearly, unsatisfactory. All one knows globally about the true production function is that it is concave, i.e., $f'' < 0$: in the absence of any information about the higher derivatives, higher degree polynomials of the kind illustrated by (34) will generate poor approximations and bad policies.

Static expectations policies might be naive but they have very satisfactory convergence properties.

Recall that the convergence theorem is valid provided one uses an approximation to the production function such that the marginal product of existing capital stock according to the approximating function is equal to the marginal product of existing capital stock according to the true production function.[1] Hence a planning procedure which moves one step from the static expectations procedure towards the one outlined in the previous paragraph may be described as follows.

The story starts at $t = 0$ with a historically given capital stock, k_o, which is known to the CPB.

(i)(a) The CPB observes w_o and r_o in factor markets to discover $f(k_o)$ and $f'(k_o)$.

(b) Producers communicate $f''(k_o)$ to the CPB.

(ii)(a) The CPB uses the information in (i) to construct a perceived production function, $g_o(k)$, given by

$$(35) \qquad g_o(k) = f(k_o) + f'(k_o)(k-k_o) + \tfrac{1}{2}f''(k_o)(k-k_o)^2.$$

(b) This step is the same as that described by (27) and (28).

(c) The CPB announces the investment target $\mathring{a}_o = \dot{k}_o$.

(iii)(a) Consumers and producers save $(\mathring{a}_o)$ and invest $(\dot{k}_o)$ in accordance with the target laid down in (ii)(c). The CPB observes wage and interest rates in factor markets to determine $f(k_1)$ and $f'(k_1)$.

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[1] Thus, in (31), $g_t'(k) = f'(k_t)$ for all k : in particular, therefore, $g_t'(k_t) = f'(k_t)$, where k_t is the economy's actual capital stock at t.

(b) Producers communicate $f''(k_1)$ to the CPB.

Equation (35) ensures that

$$(36) \qquad g_o'(k) = f'(k_o) + f''(k_o)(k-k_o)$$

and that

$$(37) \qquad g_t'(k_t) = f'(k_t)$$

where g_t is the perceived production function at stage t of the iterations and where k_t is the actual stock of capital in the economy at stage t of the iterations.

(37) shows that the marginal product condition stated just before the description of this planning procedure is satisfied; hence the planning procedure is asymptotically optimal.[1]

Three comments need to be made about this procedure. First, the CPB can no longer rely on market observations alone to compute the optimum policy. Thus, an attractive property of impersonality is lost. Secondly, the fact that a quadratic as opposed to a linear approximation to the production function is used does not, in the absence of anything more than strict concavity, imply that the path of development is better

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[1] The marginal product condition is not satisfied in (34) where the marginal product of the current stock of capital reported by producers is used by the CPB to estimate higher derivatives of f at k_o, where the story started. Thus, at the t^{th} stage of the iterations, $f'(k_t)$, where k_t is the actual capital stock in the economy at stage t, is used to help calculate $f^{(t)}(k_o)$. At this stage, the perceived production function is

$$g_t(k) = f(k_o) + f'(k_o)(k-k_o) + \ldots + \frac{1}{t!} f^{(t)}(k_o)(k-k_o)^t$$

Hence, $g_t'(k_t) \neq f'(k_t)$.

than that generated by the static expectations procedure. Thirdly, assuming that $\frac{dr}{dk}$ will be constant, unlike static expectations, lacks an economic rationale. These points ought to be borne in mind before deciding to use any more than linear approximations to the production function.

(2) It has already been noted that the rate of interest is not sufficient to determine the investment policy; it is necessary to have information about changes in the rate of interest. Thus, one might expect the CPB in a typical planning exercise to announce the rate of interest expected to prevail tomorrow and allow profit-maximising producers to undertake investment in response to such a signal. It is clear that this method is not open to the CPB in the static-expectations procedure analysed above for, at every stage of the iterations, the rate of interest is expected to remain constant forever at its currently observed value. Setting annual investment targets and inferring production possibilities from their fulfilment is, no doubt, an attractive procedure. Nevertheless, it might be thought that price-guided procedures are incentivewise superior to quantity-guided ones.

The quadratic approximation procedure described by equation (35) does allow the CPB to communicate with firms by announcing the future rate of interest. Thus, in step (ii)(c), the CPB announces the rate of interest that supports the investment $\dot{k}_o$ it wishes producers to undertake according to the perceived production function (35). The actual investment forthcoming will be different from $\dot{k}_o$: producers in step (iii) report the amount of investment, $\dot{k}_o$, and (what is necessary in the quadratic approximation procedure) $f''(k_1)$, while the

CPB deduces $f(k_1)$ by observing wage and interest rates in the capital market.

Two brief comments are in order.

(a) The price version of the procedure has not specified consumer behaviour so far. We continue to assume as before that consumers save the amount necessary to clear the capital market: this involves making available the funds producers decide to invest in response to the centrally announced rate of interest.

(b) The reservations expressed about the quadratic approximation procedure in (1) above are applicable here as well.

(3) The static-expectations procedure analysed in this section performs satisfactorily in the one-sector model with which this chapter has so far been concerned. It is worth reflecting on how this method or, at any rate, an identifiable variant of it would work in a model with more than one sector. I shall discuss the main issues involved in the standard two-sector model where there is a consumption good industry and an investment good industry, each of which uses capital and labour in production.[1]

Perhaps the most important decision that has to be taken along the optimum path in a two-sector model is the allocation of the economy's stock of capital between the two sectors; this helps determine the output of consumption and investment goods in the next period. The planning procedure described so far encounters considerable difficulties

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[1] Discussions of optimum policies in two-sector models are to be found in Srinivasan (1964), Uzawa (1964) and Hadley and Kemp (1971).

with this kind of allocation for two reasons, the second being rather more serious than the first.

(a) The planning exercise starts with given amounts of total capital stock and labour. Suppose that the CPB announces an arbitrary allocation of capital between the two sectors. Given that the value marginal productivities of both factors have to be the same in both uses,[1] this decision determines the allocation of labour, the wage and interest rates and the price ratio between the two outputs. I introduce the following notation:

a_i : marginal product of capital in sector i $(i = 1, 2)$

b_i : marginal product of labour in sector i $(i = 1, 2)$

w : wage rate

r : rate of interest

p : price of the second output in terms of the first.

The CPB observes w, r and p in factor and product markets given by

$$(38) \quad \begin{cases} r = p\, a_2 = a_1 \\ w = p\, b_2 = b_1 \end{cases}$$

Note that the planning algorithm has to be modified by allowing producers to report outputs to the CPB: observation of factor and product markets alone is not, in the absence of knowledge about labour allocation, sufficient to permit their deduction.

As before, the CPB expects w, r and p to remain constant in the future. It would then wish to compute the policy which is optimal

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[1] I am ignoring for the purposes of this exposition the complication caused by inequalities when the economy is completely specialised in one good.

in the light of its current expectations, propose a different allocation of capital, and so on. But this procedure is unworkable. For, given that the production functions perceived by the planning authority are linear and not strictly concave, the CPB does not have enough equations to determine the optimum policy, the allocation of capital and, therefore, its investment directives. The problem here is that, as far as the CPB is concerned, value marginal productivities in each sector are not only equal but constant for all allocations; hence (38) leaves factor allocation indeterminate.

This observation, though perhaps unsurprising, is discouraging because the static-expectations procedure is quite appealing for reasons outlined before. A way round this difficulty is to use nonlinear approximations to the production function; the quadratic approximation procedure of equation (35) will suffice for this purpose.

(b) The second difficulty is more troubling. All one had to do with quadratic approximation in the one-sector case was to read off first and second derivatives of the production function at a given level of capital stock; strict concavity was then sufficient to ensure that the decentralised procedure had satisfactory convergence properties. Things are rather different in the two-sector model. Here, deciding on the intersectoral allocation is an important part of the optimisation problem; however, the CPB can get no information on derivatives of the production function in each sector unless it first proposes an arbitrary initial allocation. And such an arbitrary initial allocation, if sufficiently far removed from the optimum allocation, could lead the CPB to compute and recommend a sequence of subsequent allocations which generate non-convergent policies. The question has nothing to do with whether the CPB uses price or quantity proposals; either version of the

procedure requires it to propose an initial allocation. It is tempting to suppose that if somehow the initial allocation were correctly chosen, the quadratic approximation procedure would see the planners through. There are two difficulties with this. First, the CPB does not possess the information necessary to determine the optimum initial allocation. And secondly, correct choice of the initial policy allows myopic competitive behaviour to trace out the full Ramsey optimum; hence to assume that this situation obtains is to assume away much of the problem on which this chapter is based.

(4) One way of appreciating the above difficulty is to consider a special example where the initial intersectoral allocation does not depend on the solution to the overall optimisation problem. Let us suppose, following Arrow and Kurz (1970)[1], that there is one good but that it has three uses: consumption, private investment and public investment. In that case, retaining most of our earlier notation, the problem in a centrally planned economy is to maximise

$$(39) \qquad \int_0^{\infty} \bar{e}^{\delta t} U(c)\, dt$$

subject to

$$(40) \qquad c + \dot{k}_p + \dot{k}_g = f(k_p, k_g) - nk$$

$$(41) \qquad k_p + k_g = k$$

$$k_o \quad \text{given}$$

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[1] This is a special case of the Arrow-Kurz model, as we shall see. See Footnote 2 on page 106.

where k_p is private capital per capita

k_g is public capital per capita

k is total stock of capital per capita

and where, following Arrow and Kurz[1], I assume that

(A.1) U and f are strictly concave;

(A.2) $\lim_{c \to 0} U'(c) = \infty$.

(A.3) "Diagonal Dominance", i.e., $f_{pp} < f_{pg}$, $f_{gg} < f_{gp}$

(A.4) Inada conditions: (i) $\lim_{k_p \to 0} f_p(k_p,k_g) = \infty$, $\lim_{k_p \to \infty} f_p(k_p,k_g) = 0$,

for all k_g

(ii) $\lim_{k_g \to 0} f_g(k_p,k_g) = \infty$, $\lim_{k_g \to \infty} f_g(k_p,k_g) = 0$, for all k_p

The first-order conditions are:

(42a) $U'(c) = q$

(42b) $f_g = f_p$,

which requires that capital be allocated to equalise marginal rates of return in its two uses.

(42c) $k = k_g + k_p$

and k and q satisfy the differential equations

(43) $\dot{k} = f(k_p, k_g) - nk - c$

(44) $\frac{\dot{q}}{q} = \delta - f_p(k_p, k_g)$ [2]

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[1] I merely sketch the outline of the model in what is intended to be no more than an example. For full details the reader is referred to the original work.

[2] It is assumed as before that $\delta \geq n$

Our assumptions ensure a unique solution to (42a-c)

$$(45) \qquad c = c(q)$$

$$(46) \qquad k_g = k_g(k)$$

$$(47) \qquad k_p = k_p(k)$$

whence (43) and (44) may be written as

$$(48) \qquad \dot{k} = f\left[k_p(k),\ k_g(k)\right] - nk - c(q)$$

$$(49) \qquad \frac{\dot{q}}{q} = \delta - f_p\left[k_p(k),\ k_g(k)\right]$$

It is seen from (46) and (47) that the allocation between private and public capital depends on k alone and not on marginal utility of consumption. It is therefore easy to confirm the conjecture that the static expectations procedure is asymptotically optimal, which we now proceed to do. Define, after Arrow and Kurz, the function

$$(50) \qquad h(k) = \max_{k_p + k_g = k} f(k_p,\ k_g)$$

The assumptions made above ensure that the above maximisation problem possesses a solution; furthermore, $h'(k) > 0$, $h''(k) < 0$. This reduces the planning problem (39) to (41) to the familiar one of choosing c to maximise

$$(51) \qquad \int_0^\infty e^{-\alpha t}\ U(c)\ dt$$

subject to

$$(52) \qquad c + \dot{k} = h(k) - nk$$

$$k_o \text{ given}$$

where h(k) is given by (50).

This problem is nothing other than the optimum growth problem in the one-sector case andthis because the allocation of output among c, k_p and k_g can be subdivided into the allocation of output between c and k (in (51) and (52)) and the allocation of k between k_p and k_g (in (50)).

We are now in a position to describe the planning procedure. The story starts at $t = 0$ with a historically given capital stock which is known to the CPB. The planning procedure consists of the following steps

(i)(a) Producers perform the maximisation in (50).

(b) The CPB observes the wage rate, w_o, and the interest rate, r_o, ruling in factor markets. It therefore discovers $h(k_o)$ and $h'(k_o)$ from

$$(53) \qquad h(k_o) = w_o + r_o k_o$$

$$(54) \qquad h'(k_o) = r_o$$

(ii)(a) The CPB constructs a "perceived" production function, $g_o(k)$, given by

$$(55) \qquad g_o(k) = h(k_o) + h'(k_o)\,(k - k_o)$$

(b) It solves the problem:

$$(56) \qquad \underset{(c_t)}{\text{maximise}} \int_o^{\infty} e^{-\alpha t}\, U(c_t)\,dt$$

subject to

$$(57) \qquad \dot{k}_t = g_o(k_t) - nk_t - c_t$$

$$k_o \text{ given}$$

(c) It announces the investment target $\dot{a}_o = \dot{k}_o$ that it wishes the economy to fulfil.

(iii)(a) Consumers save $\dot{a}_o$ in accordance with the CPB's wishes.

(b) Producers consider the new level of capital stock, k_1, that would satisfy the quoted investment target in (ii)(c) and perform the maximisation in (50) for this k_1.

(c) The CPB observes wage and interest rates in factor markets to enable it to form a new linear approximation to the function h(k), and the process goes on.

The proof that the planning procedure is asymptotically optimal is exactly the same as before[1] with f replaced by h everywhere.[2]

(5) It should be pointed out that the procedure analysed in this section has certain superficial similarities with the Infante-Stein algorithm discussed in Section II : compare equation (20) with the quadratic approximation procedure (36). But there is a substantial difference between what rule (ii) of our procedure attempts to do and

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[1] See pp. 93 - 94.

[2] Arrow and Kurz suppose that one way in which public capital differs from private capital is that it confers direct consumption benefits as well : formally, k_g is an argument of the instantaneous utility function. In that case, (42b) in the text has to be replaced by

(42b') $U_g + q(f_g - f_p) = 0$

and the allocation of k between k_p and k_g depends on k and q, the marginal utility of consumption and, therefore, on the overall optimisation problem. The allocation of output can no longer be divided into two sub-problems and the static expectations procedure runs into the same difficulties outlined in the two-sector discussion in (3) in the text.

the steps taken by I - S in equations (17) to (23). The I - S contribution is an attempt to develop informationally and computationally economical ways of approximating optimal controls. The procedure of this section, on the other hand, solves an optimisation problem at every stage; furthermore, it attempts to do so consistent with a view of the future that makes some economic sense.

IV. A DIGRESSION ON EFFICIENCY PRICES

Discussions of price-decentralisation have dominated the theoretical literature on decentralisation. The reasons for the economic appeal of this variety of decentralisation have been discussed in Chapter 2, and need not be repeated here. Mathematically, efficient and optimal programmes in convex environments can be supported by hyperplanes which admit of a natural interpretation as "price systems"; hence their popularity. In fact, the rigorous way of stating the Euler-Ramsey local optimality conditions is in terms of such supporting prices. There is however a difficulty here to which Malinvaud (1953) and Radner (1967) have drawn attention.

The difficulty arises for maximal (i.e., efficient) infinite programmes of the kind one encounters, for example, in optimum growth. For while in certain function spaces within which all programmes are embedded (and the usual one is ℓ_∞ , the linear space of all bounded sequences of real numbers endowed with the "sup-norm" topology), maximal programmes continue to be supported by linear functionals, such functionals cannot in general be characterised as a sequence of prices yielding finite present value over the set of feasible

programmes. Under these circumstances, price-decentralisation of efficient and optimal programmes is no longer possible.

Considerable theoretical ingenuity has been addressed to this problem : a particularly good account of the results available is to be found in Kurz and Majumdar (1972). It has been shown that under certain restrictions on the class of admissible programmes, a decentralising price-sequence yielding finite present value exists. Secondly, even when such restrictions are not satisfied, the supporting linear functional can be closely approximated by price systems having the above properties.

It is reassuring and important to learn that price systems can be defined in general function spaces : that indeed is the rationale for including this section. But the literature is not at all concerned with the informational requirements for computing efficiency prices; nor is the computation discussed in a way that helps the central planners. It is consequently no help as far as the intertemporal decentralisation problem is concerned. No more will be said about this approach.

V. A CONCLUDING REMARK

The unsatisfactory features of the planning procedure analysed in Section III ultimately rest on the lack of a satisfactory micro-economics of expectations - formation and the optimal use of information. Any systematic attempt at real-time informational decentralisation of intertemporal optima must concentrate on models which generate developments that are optimal relative to the information available to the CPB at every stage of the iterations. The strategy of tying estimation of production possibilities to expectations might continue to prove useful here. It seems progress on this front will have to await progress in the expectations literature.

Chapter 4 : Decentralisation in a Two - Sector Model with Non-Shiftable Capital

I. INTRODUCTION

The previous chapter analysed some ways in which inter-temporal decentralisation might help exploit informational and computational economies in the planning process. It was pointed out in the introductory chapter,[1] however, (a) that the term "decentralisation" is popularly understood to refer to the delegation of authority to take decisions and (b) that the (regrettable) relative neglect of this aspect of decentralisation in the literature could partly be explained by the lack of any compelling economic, as opposed to social, political and administrative reasons in favour of dispersing decision - making among the numerous participants in a planning exercise. This chapter looks at the delegation of authority by analysing the consequences of allowing producers to follow a plausible, if ad hoc, rule of adjustment which differs from the usual postulate of profit-maximisation. In so doing it permits itself the simplification of assuming away informational and computational problems altogether; its concerns are therefore quite different from those of Chapter 3.

Specifically, I shall examine the price-decentralisation of production decisions in a two-sector model, i.e., a model with a consumption - good industry and an investment - good industry, where capital goods are specific to each sector and where, for that reason, decision-makers may influence only the allocation of new investment.

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[1] On page 5.

It is well known from the theory of optimum growth (a) that only in the terminal phase of the optimum policy for such models does the non-shiftability restriction cease to be binding, and (b) that unless the economy starts off with initial conditions appropriate to those of the terminal phase, the investment allocation exhibits "flip-flop" behaviour during the (finite) time it takes the economy to enter this phase.[1, 2] In other words, there is in general a finite time interval during which the optimum policy calls for new investment at every instant to be allocated entirely to the consumption-good or the capital-good industry depending on which sector happens to be more profitable at that particular instant. Such behaviour is, of course, extreme and is optimal because of the excessive rigidity implicit in the assumption of non-shiftability. And it might lead to difficulties with decentralised attainment of the optimum policy for a reason outside the model: a profit - maximising rule which calls for "flip-flop" behaviour is unlikely to be followed by producers who may instead opt for less

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[1] Characterisations of the optimum policy for a fixed-coefficients technology without labour are to be found in Bose (1968) and Dasgupta (1969), for a fixed-coefficients technology with labour in Bose (1970) and, for technologies permitting factor substitution of the kind standard in neoclassical growth theory in Ryder (1969) and Hadley and Kemp (1971).

[2] The terminal phase mentioned in the text is a non-degenerate interval in the capital stocks phase plane. The optimally developing economy in its terminal phase travels along this interval to a point within it which can be termed the OBGP. Two points need to be noted here. First, which end of the interval the terminal phase begins from in any particular case depends on initial conditions. Secondly, while the terminal phase is approached in finite time, the OBGP is attained only asymptotically, as in the two-sector model with shiftable capital.

extreme variations in the allocation ratio. This kind of conservative response looks more plausible and may well be optimal in a model incorporating certain features intermediate between the rigid strait-jacket of non-shiftability on the one hand andthe extreme flexibility consistent with free transferability of capital between sectors on the other. However, no such model has been proposed: in its absence, it is useful to analyse the consequences of conservative adjustment in the direction of greater profitability in the non-shiftable capital model where, as we shall see, it generates a rich pattern of behaviour.

Finally, it needs to be stated how this analysis has a bearing on the question of the decentralisation of control. First, the behavioural rule mentioned in the previous paragraph is different from profit-maximisation which is the usual rule laid down for producers in discussions of informational decentralisation. Thus, producers are not bound to follow adjustment rules prescribed by the CPB. Secondly, a real-time planning procedure of the kind we are concerned with allows producers the authority to implement every production proposal they arrive at by following their own rule of adjustment. Here it differs from tatonnement routines where producers merely transmit information without being permitted to implement any proposal till the planning procedure is terminated by the CPB. An analysis of decentralised policies satisfying these two features is, of course, not an adequate substitute for a systematic discussion of the delegation of authority. But it seems to capture in a simple way some of the issues to which a more complete investigation would have to address itself.

II. THE MODEL

I assume that technology in both sectors is of the fixed - coefficients type and, furthermore, that there is no primary factor of production. These simplifications, which are made in the interests of analytical tractability, allow one to deal with the simplest model that retains the "flip-flop" characteristic of optimum policies : it will be recalled that it is precisely this feature of the optimum policy which makes the present problem interesting.[1]

It is also assumed that the CPB possesses complete knowledge of technological possibilities. Since this chapter is, unlike Chapter 3, not concerned with informational decentralisation at all, the assumption is a natural one to make. In what follows, we will not have occasion to refer to wages, factor markets or expectations; hence, consumers are not explicitly introduced into the model.

The Planning Procedure

A typical stage of the intertemporally decentralised planning procedure comprises the following steps.

(i)(a) At time t, the CPB maximises (with respect to investment allocation)

$$(1) \qquad \int_t^{\infty} e^{-\alpha(s-t)} U\left[\bar{\underline{c}}(s)\right] ds$$

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[1] The terminal phase of the optimum policy in such a model shrinks to the OBGP. The relevant papers are those by Bose (1968) and Dasgupta (1969).

subject to

$$(2) \qquad \dot{\bar{k}}_1(s) = \bar{\lambda}(s)\beta\,\bar{k}_1(s)$$

$$(3) \qquad \dot{\bar{k}}_2(s) = \left[1 - \bar{\lambda}(s)\right]\beta\,\bar{k}_1(s)$$

$$(4) \qquad 0 \leq \bar{\lambda}(s) \leq 1 \text{ , where}$$

δ = rate of utility discount

β = output-capital ratio in sector 1, the capital goods sector.

$\bar{\lambda}(s)$ = proportion of new investment going into sector 1 at s

$U\left[\bar{C}\ (s)\right]$ = instantaneous utility function

$\bar{k}_i(s)$ = capital stock in sector i at s $(i = 1, 2)$

and

$$(5) \qquad \bar{C}\ (s) = \bar{k}_2(s)$$

where units are chosen such that the output - capital ratio in sector 2, the consumer-goods sector, is unity.

I adopt the convention that bars over variables denote magnitudes in the CPB's computed optimum plan at every stage, as opposed to actual realisations in the economy.

The problem (1) to (5) is not completely posed, for initial and boundary conditions have not been specified. Initial conditions are provided by

$$(6) \qquad \bar{k}_i(t) = k_i(t), \text{ given } (i = 1,2)$$

Thus, at time t, the CPB computes the optimum plan subject to the actual capital stocks in the economy. This leaves one with the problem of "end-conditions". This is clearly related to the existence of an optimum policy. I shall return to this question in section III

below in analysing the optimum policy. Meanwhile, in the rest of this section, I assume that an optimum policy exists.

Equations (1) to (5) are straightforward and do not require comment. (4) requires $\bar{\lambda}$ to be restricted to the unit interval : this is the condition that capital, once installed, may not be shifted between sectors. Denote

$$(7) \qquad W\left[\bar{k}_1(t),\ \bar{k}_2(t)\right] \equiv \max_{\lambda(s)} \int_t^{\infty} e^{-\delta(s-t)} U\left[\bar{C}(s)\right]\, ds$$

subject to (2) - (6).

(b) The CPB announces the shadow prices $W_i(t) \left[\equiv \frac{\partial \bar{W}(t)}{\partial \bar{k}_i(t)}\right]$ $(i = 1,2)$ emerging from its optimisation exercise in (a).

(ii)(a) Producer behaviour at t is given by

$$(8) \qquad \dot{\lambda}(t) = f\left[\frac{W_1(t)}{W_2(t)} - 1\right]$$

where f is sign-preserving and where unbarred magnitudes represent actual realisations.

Rule (8) states that producers have to decide the allocation of new investment between the two sectors and that they vary the existing allocation λ in the direction of the sector which is more profitable at shadow prices currently announced by the CPB. Several points about (8) are worth noting. First, the way λ is being varied is not the way it should be varied according to the optimum policy. We shall see that the optimum policy requires λ to assume its extreme values of 0 or 1

according as W_1 is less than or greater than W_2 and not to be changed when $W_1 = W_2$; (8) asserts that producers adjust more cautiously than that. A more complete model would, no doubt, derive the optimum degree of caution from some pattern of expectations entertained by producers. In the meantime, rule (8) is no more unsatisfactory for being stated a priori rather than being derived by resorting to the crude expedient of suitably tailored adjustment costs. Secondly, it will not do to suggest that, given that the CPB knows equation (8), it can always greatly exaggerate the announced difference between W_1 and W_2 to offset the sluggish response of producers. There is no doubt that, within the strict framework of the model, the CPB could declare the greater shadow price to be infinite : this would drive producers' λ to the appropriate extreme value. The reason this is not allowed is that, for reasons outside the model, a CPB which resorted to such extreme pronouncements would not be believed by producers who would continue to adjust cautiously in the interior of the set of definition of λ. Thirdly, we shall see that conservative response on the part of producers leads to a pattern of development of the economy qualitatively rather different from the fully optimal development; more rapid adjustment by producers, provided it is not "flip-flop" has the same consequences. Since the freedom of producers to act in this way and not according to the dictates of profit-maximisation is the principal problem of this chapter, this result is of considerable interest. And, finally, rule (8) embodies a simple, if uneasy, compromise between the fact that capital goods in the real world are, once installed, specific in some ways to different uses and the feeling that the non-shiftable capital model used here postulates a degree of specificity that is clearly excessive.

(b) Producers report to the CPB the new amounts of the capital stocks arrived at in the two sectors by following rule (8).

The CPB, as in step (i), recomputes the optimum policy subject to these new initial conditions, and the procedure continues as before.

Properties of the Procedure

Recall that we are not now interested in problems of informational and computational decentralisation. Hence we require that the planning procedure be feasible and asymptotically optimal. For a discussion of the desirability of these properties, the reader is referred to Chapter 3.[1]

It is easily checked from (8) that feasibility is satisfied - the amount of investment available at any time is allocated between the two sectors and infeasibility cannot occur. The question of asymptotic optimality next has to be examined but for this it is necessary to have some idea of what the optimum policy looks like in a two-sector model with nonshiftable capital.

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[1] See page 79.

III. THE OPTIMUM POLICY

I shall find it convenient to assume that the utility function is of the constant elasticity type :

$$(9) \qquad U(C) = -\frac{1}{n} C^{-n} ; \quad n > 0$$

This assumption makes no real difference to the characteristics and shape of the optimum policy and turns out to be a great help in the subsequent analysis of the development of the decentralised economy.

The full optimum is the solution to the problem :

$$(10) \qquad \max_{\lambda(s)} \; -\frac{1}{n} \int_t^{\infty} e^{-\delta(s-t)} \left[C(s)\right]^{-n} . ds$$

subject to

$$(11) \qquad \dot{k}_1(s) = \lambda(s)\beta \, k_1(s)$$

$$(12) \qquad \dot{k}_2(s) = \left[1 - \lambda(s)\right] \beta \, k_1(s)$$

$$(13) \qquad 0 \leq \lambda(s) \leq 1$$

$$(14) \qquad C(s) = k_2(s)$$

$$k_1(t), \quad k_2(t) \quad \text{given}$$

(where bars over variables are dropped since one is now dealing with a centrally planned economy).

Finally, it is assumed that

$$(15) \qquad \beta > \delta > 0$$

The first inequality ensures that consumption grows rather than declines along the optimum path, while the second ensures that an optimum policy exists.

As before, define

$$(16) \qquad W\left[k_1(t),\, k_2(t)\right] \equiv \max_{\lambda(s)} \; -\frac{1}{n}\int_t^{\infty} e^{-\delta(s-t)} \left[C(s)\right]^{-n} \cdot ds$$

subject tc (11) to (14). (15) ensures that W is a well-defined function of initial capital stocks.

A simple and direct argument may now be used to characterise the optimum policy. (16) may be rewritten as

$$(17) \quad W\left[k_1(t),k_2(t)\right] = \max_{\lambda(s)}\left\{-\frac{1}{n}\int_t^{t'} e^{-\delta(s-t)}\left[C(s)\right]^{-n} ds + e^{-\delta(t'-t)} W\left[k_1(t'),k_2(t')\right]\right\}$$

which is Bellman's "principle of optimality".[1] Dividing by (t'-t) and transposing,

$$\frac{W\left[k_1(t),k_2(t)\right]}{t'-t} - \max_{\lambda(s)}\left\{ e^{-\delta(t'-t)} \frac{W\left[k_1(t'),k_2(t')\right]}{t'-t} \right\}$$

$$= \max_{\lambda(s)}\left\{ \frac{1}{t'-t}\left(-\frac{1}{n}\right)\int_t^{t'} e^{-\delta(s-t)} \left[C(s)\right]^{-n} \cdot ds \right\}$$

---

[1] Bellman (1957).

Letting $t' \to t$, and substituting from (14),

$$\max_{\lambda(t)} \left\{ \frac{\partial W\left[k_1(t),k_2(t)\right]}{\partial k_1(t)} \dot{k}_1(t) + \frac{\partial W\left[k_1(t),k_2(t)\right]}{\partial k_2(t)} \dot{k}_2(t) - \delta W\left[k_1(t),k_2(t)\right] \right\}$$

$$= \frac{1}{n} \left[k_2(t)\right]^{-n}$$

Writing $W_i(t) \equiv \dfrac{\partial W\left[k_1(t),k_2(t)\right]}{\partial k_i(t)}$, $(i = 1,2)$, and substituting from (11) and (12), one gets

$$(18) \quad \max_{\lambda(t)} \left\{ W_1(t)\lambda(t)\beta k_1(t) + W_2(t) \left[1-\lambda(t)\right] \beta k_1(t) - \delta W(t) \right\}$$

$$= \frac{1}{n} \left[k_2(t)\right]^{-n}$$

whence the maximisation leads to the following:

$$(19) \quad \begin{cases} \text{in Phase I, defined by } W_1 > W_2, & \text{choose } \lambda = 1 \\ \text{in Phase II, defined by } W_1 < W_2, & \text{choose } \lambda = 0 \\ \text{in Phase III, defined by } W_1 = W_2, & \text{choose } \lambda \in [0, 1] \end{cases}^{1}.$$

Since the partial derivatives of W have a natural price interpretation, the conditions above state that all new investment must go into the sector where the shadow price of investment is higher; the only "interior" allocation (and we shall examine this later) occurs when the two shadow prices are equal.

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[1] I have omitted explicit reference to time-dependence of the variables in (19) for convenience; there seems to be no risk of confusion.

The next step is to discover what sequence of phases comprises the optimum policy. We now state the obvious

Lemma 1 : The maximum valuation, W, is homogeneous of degree (-n) in the capital stocks, k_1 and k_2.

Proof : Follows from the production relations (11), (12) and (14), the utility function (9) and the definition of W, (16).

Lemma 2 : The maximum valuation, W, is strictly concave in k_1 and k_2.

Proof : Follows from (9), (11), (12) and (14).

Lemma 1 enables one to write

$$(20) \qquad W = -(k_2)^{-n} \Phi\left(\frac{k_1}{k_2}\right)$$

Since W_1 and W_2 are homogeneous of degree (-n-1) in k_1 and k_2, we may write

$$(21) \qquad \frac{W_1}{W_2} = \Psi\left(\frac{k_1}{k_2}\right)$$

Furthermore, strict concavity of W ensures that Ψ is monotonic and strictly decreasing.

Denote $\frac{k_1}{k_2} = x$. (20) implies that there exists a unique x^* for which $W_1 = W_2$. As long as λ is kept at the appropriate "interior" value, which may be determined from (11) and (12), x may be kept at x^* with $W_1 = W_2$; this is Phase III according to the conditions (19).

Again,

$$(22) \qquad W_1 = -(k_2)^{-n-1}\,\tilde{\Phi}'\left(\frac{k_1}{k_2}\right)$$

Since $\frac{k_1}{k_2}$ is constant in Phase III, relations (11), (12) and (22) may be seen to imply that the value of capital tends to zero along Phase III. Clearly, if the economy is in Phase III, following it forever is optimal.

Three cases may be distinguished:

(a) the initial $x^0 < x^*$. Clearly, $W_1 > W_2$, $\lambda = 1$ and the economy follows Phase I till its capital stock ratio catches up with the desired figure, x^*. This goal is achieved in finite time which may easily be calculated (in terms of x^*) from (11) and (12). $\frac{W_1}{W_2}$ falls monotonically along Phase I till x^* is reached when $W_1 = W_2$. The economy is in Phase III and remains there forever with $0 < \lambda = \frac{x^*}{1 + x^*} < 1$, since x^* is finite.

(b) the initial $x^0 > x^*$. Clearly, $W_1 < W_2$, $\lambda = 0$ and the economy follows Phase II till its capital stock ratio falls to the desired figure, x^*. Again, this goal is achieved in finite time which may easily be calculated (in terms of x^*) from (11) and (12). $\frac{W_1}{W_2}$ grows monotonically along Phase II till x^* is reached when $W_1 = W_2$. The economy is in Phase III and remains there forever with $0 < \lambda = \frac{x^*}{1 + x^*} < 1$.

(c) the initial $x^0 = x^*$. The economy remains in Phase III.

x^* is naturally called the optimum balanced growth path (OBGP) capital stock ratio.

The optimum policy may be summarised as being one of getting to x^* as soon as possible and of staying there forever. We have therefore arrived at

Theorem 1 : The optimum policy consists of a finite-time Phase I followed by Phase III,

a finite time Phase II followed by Phase III, or just Phase III forever according as the initial ratio of capital stocks in the two sectors is less than, greater than or equal to the OBGP capital stock configuration.[1]

Thus, the qualitative description of the optimum policy is straightforward. The values of x^*, λ^* and the valuation during the different phases may be computed by substituting (9), (11), (12), (14) and (20) into (18), using (19).

IV. THE DECENTRALISED ECONOMY

We return now to an examination of the plan generated by the decentralised procedure. Investment behaviour is given by

$$(23) \qquad \dot{\lambda} = f\left[\frac{W_1}{W_2} - 1\right]$$

and the resulting actual behaviour of the economy may be summarised in

$$(24) \qquad \dot{k}_1(t) = \lambda(t)\,\beta\,k_1(t)$$

$$(25) \qquad \dot{k}_2(t) = \left[1 - \lambda(t)\right]\beta\,k_1(t)$$

Clearly, $\frac{W_1}{W_2}$ depends on $\frac{k_1}{k_2}$ $(=x)$ at every stage ; furthermore, the relationship is monotonic and strictly decreasing. Thus, (23) may be rewritten

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[1] See the papers by Bose and by Dasgupta for this kind of result.

(26) $\lambda^{\bullet} = \eta(x) \; ; \quad \eta' < 0$

(24) and (25) may be used to write

(27) $\dot{x} = \beta x \left[\lambda(1 + x) - x\right]$

which reflects the actual behaviour of the capital stock ratio when both the CPB and producers play the game described by equation (26). Thus, one obtains a pair of first-order, autonomous differential equations (26) and (27) in two variables to characterise the behaviour of the decentralised economy completely.

An examination of equations (23) and (26), together with our earlier discussion on the relationship between $\frac{W_1}{W_2}$ and x shows that $\dot{\lambda} = 0$ if and only if $x = x^*$, the OBGP capital stock configuration in the completely controlled economy. And manipulation of equation (27) shows that $\dot{x} = 0$ if and only if $\lambda = \frac{x}{1+x}$. If we recall from the discussion of the fully optimal policy of the previous section that ($\lambda = \frac{x^*}{1+x^*}$, $x = x^*$) represents Phase III, or the terminal phase of the optimum policy, it is clear that the singular point of equations (26) and (27) is the point where the decentralised economy gets on to the fully optimal path. If it can be shown that any trajectory of (26) and (27) starting from arbitrary initial conditions eventually tends to this singularity, then the decentralised plan tends to the OBGP over time. The property of asymptotic optimality is therefore established if the solutions to equations (26) and (27) are globally stable.

The proof, unfortunately, is not quite straightforward: difficulties arise, as we shall see, with equations (27).

Accordingly, I begin by examining the phase diagram for the pair of differential equations (26) and (27) and then use suitably transformed variables to establish global stability.

I bring the two equations together for ease of reference:

$$(26) \qquad \dot{\lambda} = \eta(x)$$

$$(27) \qquad \dot{x} = \beta x \left[\lambda(1+x) - x\right]$$

Put

$$(28) \qquad z = \frac{x}{1+x}$$

Then, $z \to 0$ as $x \to 0$

$z \to 1$ as $x \to \infty$

Substituting (28) into (26) and (27), we get

$$(29) \qquad \dot{z} = \beta z(\lambda - z)$$

$$(30) \qquad \mathring{\lambda} = \alpha(z) \; ; \quad \alpha' < 0$$

as the new pair of equations in transformed variables. The phase diagram in (z, λ) space is drawn below.

Since z and λ are both restricted to lie in the unit interval, the phase plane is the unit square. The diagonal, $\lambda = z$, represents the singular point of equation (29), while the vertical line at z^* represents the singular point of equation (30). Their intersection, E, is the equilibrium solution. The arrows indicate the course of a typical trajectory.

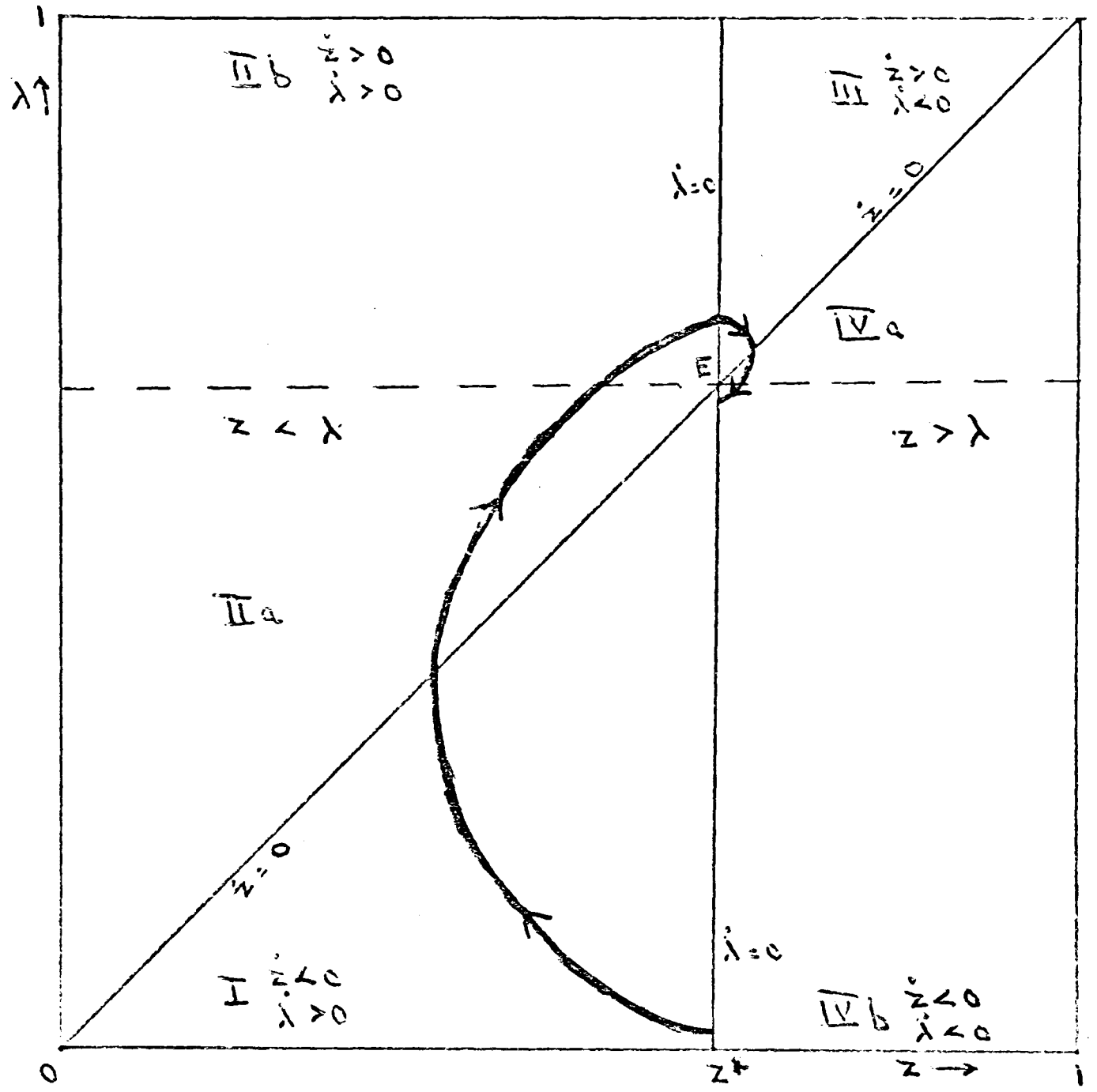


Figure 1

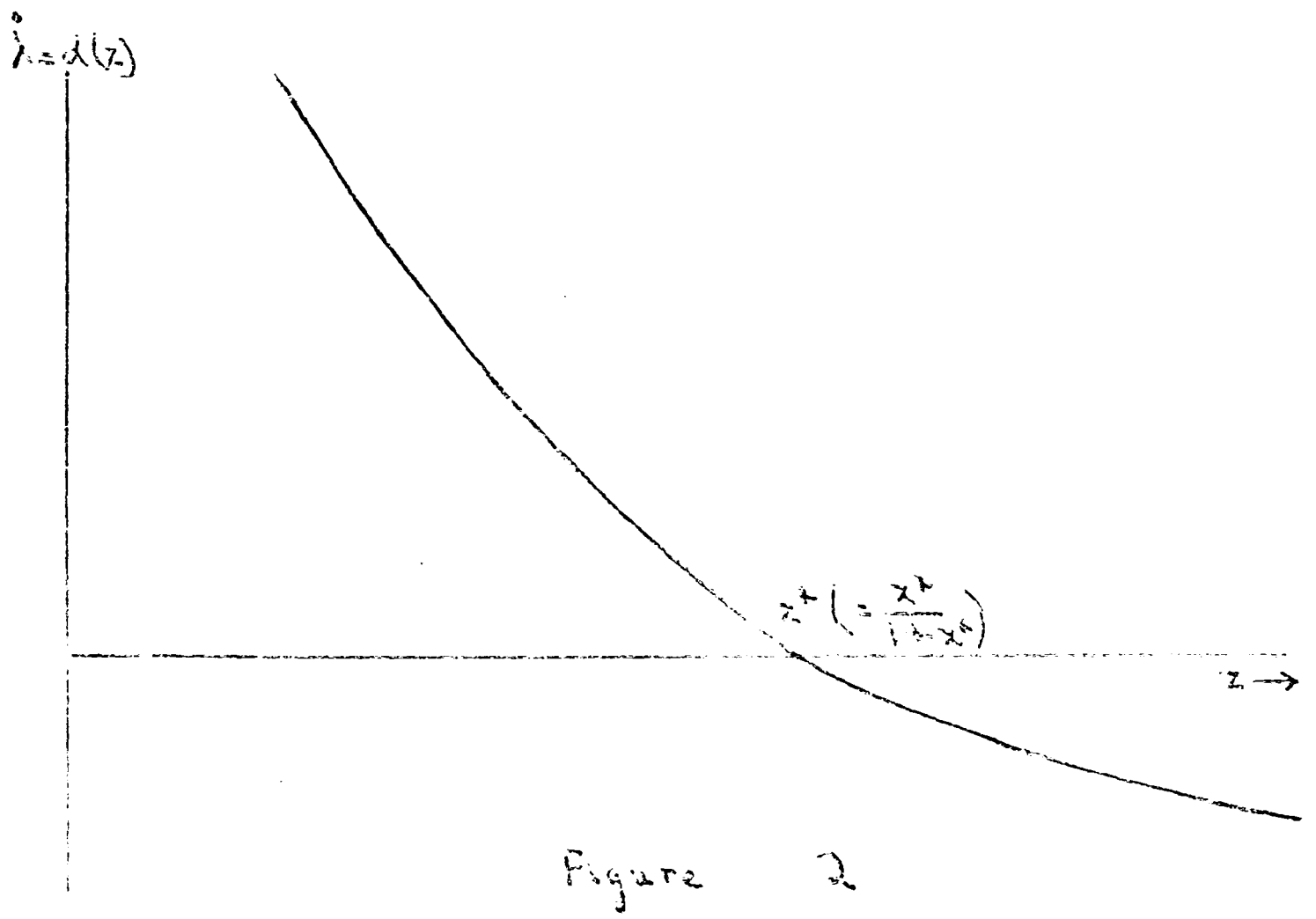


Figure 2

A casual glance at Figure 1 enables one to deduce nothing about stability, but it is worth enunciating in economic terms what is going on in each of the different sectors. Consider a trajectory starting in Sector I where $z < z^*$, $\lambda < \lambda^*$. The allocation ratio, λ, though rising, is too low to maintain the existing capital stock ratio which therefore falls. But the farther the capital stock ratio moves away from the balanced growth value, the greater the increase in λ (see Figure 2); eventually one hits the $\dot{z} = 0$ curve where the capital stock ratio is just maintained. But λ continues to increase and, being above the value of z, pulls z up with it. Since z is moving towards z^* from the moment the trajectory leaves sector I, λ must necessarily increase at a decreasing rate ($\dot{\lambda}$ is falling in Figure 2). After some time, λ reaches and surpasses λ^*, the value of λ which is sufficient to maintain a capital stock ratio of z^*. This is sector IIb. z keeps on increasing till z^* is reached. But λ is higher than λ^* and z overshoots z^*, which implies λ must start falling (sector III). Since z continues to increase away from z^*, λ begins to fall at an increasing rate till it reaches z, at which point the capital stock ratio can increase no further. z begins to fall but λ is below it and this drags z down even lower (sector IVa). λ gradually falls to λ^* and goes below it (sector IVb). All throughout sector IV, the rate of fall of λ decreases but λ must necessarily fall till z^* is reached. The value of λ is, of course, too low to stem the downward trend and the capital stock ratio continues to fall below the balanced growth value. At this stage, λ starts rising again: the trajectory is back in sector I and the story starts all over again.

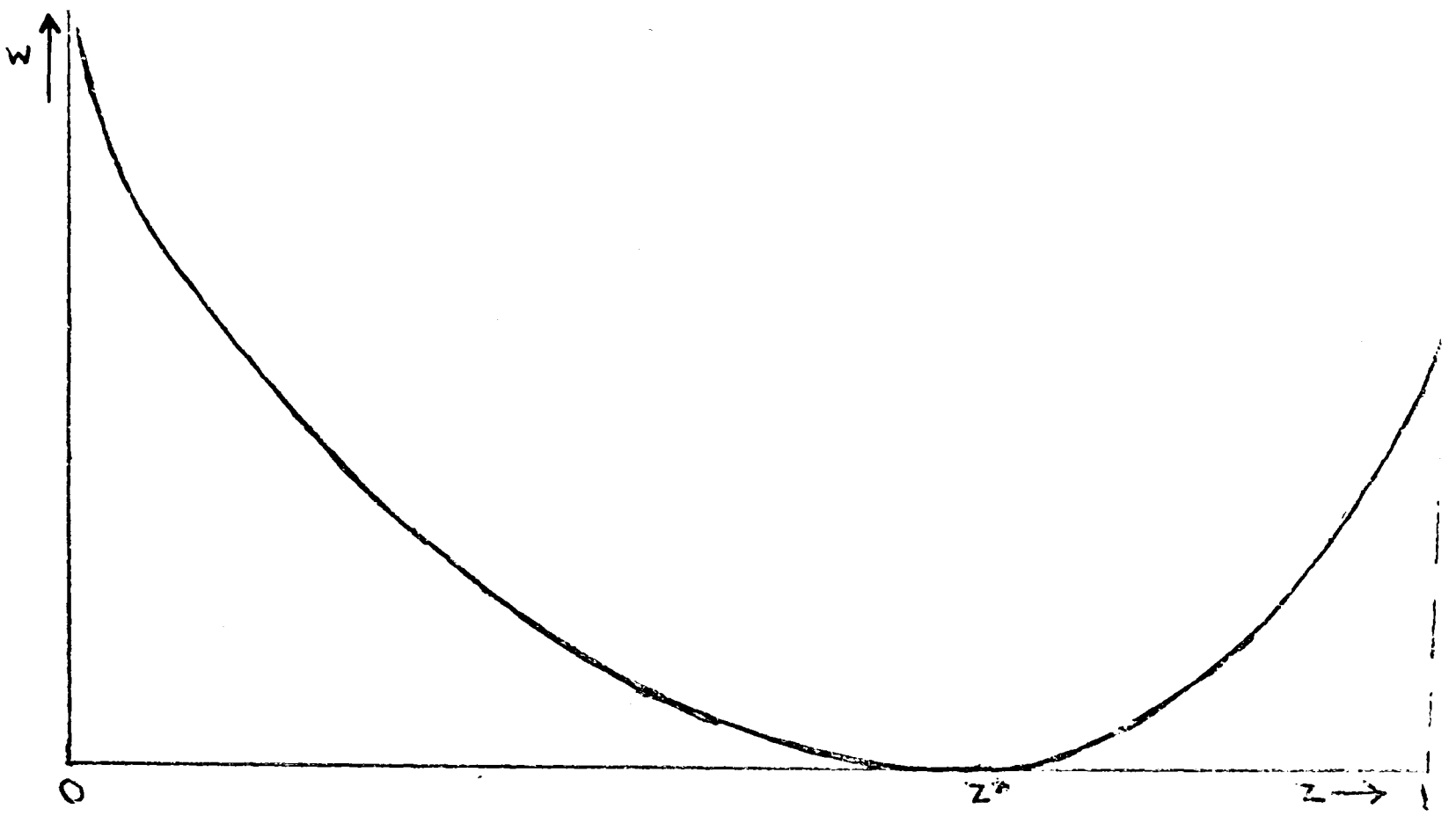


Figure 3

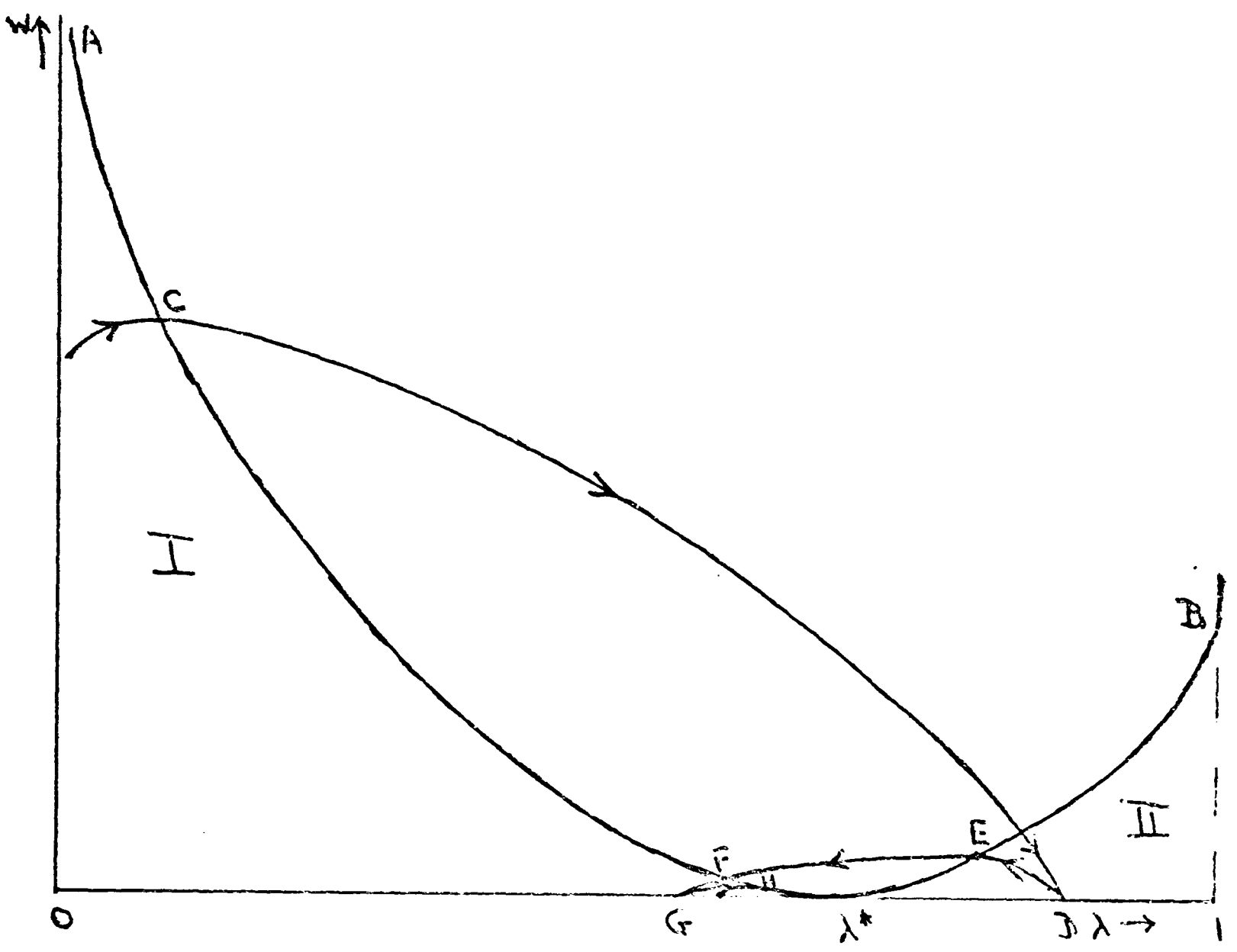


Figure 4

It would appear that the behaviour of λ around the OBGP capital stock ratio, together with the fact that changes in λ depend only on z, might bring about stability. The proof - a geometric one in transformed variables - uses precisely this idea. I shall return to Figure 1 later and demonstrate how the behaviour of λ is stabilising.

I put

$$(31) \qquad w = -\frac{1}{\beta}\int_{z^*}^{z} \alpha(y)\, dy$$

w represents (apart from the coefficient) the area between the α- curve and the z - axis in Figure 2, and is drawn in Figure 3. Clearly, z is a two - valued function of w. I examine the behaviour of the decentralised economy in (w, λ) space.

$$(32) \qquad \frac{dw}{d\lambda} = \frac{dw}{dz} \cdot \frac{dz}{d\lambda}$$

Substitution from (29), (30) and (31) yields

$$\frac{dw}{d\lambda} = -\frac{1}{\beta}\,\alpha(z)\,\frac{\beta z(\lambda - z)}{\alpha(z)}$$

$$(33) \qquad = z(z - \lambda)$$

Hence, $\frac{dw}{d\lambda} = 0$, i.e., $z = \lambda$ always lies on the curve in Figure 4. This is the singular point of (29). However it is important to remember that z is two-valued : therefore, all points on the curve in Figure 4 do not correspond to $\frac{dw}{d\lambda} = 0$. The singular point of (30) is given by $z = z^*$ or $w = 0$.

We now trace the course of an ordinary trajectory in (w, λ) space. It is instructive to tell the story in terms of the behaviour of $\frac{dw}{d\lambda}$. It follows from the two - valued nature of the (w, z)

correspondence that only in sectors I and II in Figure 4 is it possible to state unconditionally what the sign of $\frac{dw}{d\lambda}$ is : it is positive in I and negative in II. The sign of $\frac{dw}{d\lambda}$ for any trajectory in the middle area depends on its source, as we shall see.

Consider a trajectory starting in sector I in Fig.1 with $z < z^*$, $\lambda < \lambda^*$, $z > \lambda$, z falling and λ rising. $\lambda < \lambda^*$ and $z > \lambda$ implies that this corresponds to region I in Fig.4. $\frac{dw}{d\lambda}$ is positive but falling and becomes zero when the heavy curve is reached. Note that w is rising here since z is falling away from z^*. The trajectory now enters sector IIa in Fig.1 with $z < \lambda$ and $z > 0$, $\dot{\lambda} > 0$. This implies that $\frac{dw}{d\lambda}$ is negative; w is clearly falling, since z is rising with $z < z^*$. λ gradually reaches λ^* and crosses over into sector IIb in Fig.1 : nothing else changes. Eventually, the trajectory reaches $\dot{\lambda} = 0$ (maximum λ) with $z = z^*$. Here, $w = 0$ and this corresponds to point D in Fig.4. The story reaches its most important point here. In sector III in Fig. 1, $z > z^*$ and z is increasing. Hence w must increase, λ begins to decrease and $\frac{dw}{d\lambda}$ is negative. But - and this is crucial - since, for any particular λ, z is now greater than before, the slope of $\frac{dw}{d\lambda}$ must be flatter in absolute terms than CD : this is represented by section DE in Fig.4. The next stage arises when z and λ become equal again : this is point E in Fig. 4. We next enter sector IVa in Fig.1 with $z > \lambda$ and both z and λ falling. $\frac{dw}{d\lambda}$ is clearly positive and w must fall since z starts moving back towards z^*. Sector IVa and IVb after it correspond to EFG in Fig. 4, divided by the point where EFG crosses a vertical through λ^*. G corresponds once again to $\dot{\lambda} = 0$ (minimum λ) with $z = z^*$ (i.e., $w = 0$). The subsequent behaviour of the trajectory at G is analogous to its behaviour subsequent to D. It is easily seen that $\frac{dw}{d\lambda}$ is positive;

furthermore, for any particular λ, z is now less than before and the trajectory is flatter along GH than along FG. The behaviour of the trajectory at H is the same as that at C and the story repeats itself.

It is clear that the values of maximum (minimum) λ , i.e. λ_D - values (λ_G - values) for which z hits z* from above (below) form a monotonic sequence which is continuously strictly decreasing (strictly increasing) and which is bounded below (above) by λ^*. Furthermore, its rate of decrease (increase) is bounded away from zero outside any neighbourhood of λ^* ; hence the λ_D (λ_G) sequence converges to λ^* . Thus, every trajectory converges to $w = 0$, $\lambda = \lambda^*$.

The argument set out above is sufficient to establish convergence of the decentralised plan to the optimum plan. But it is worth going back briefly to the phase plane of Figure 1 to see how, in economic terms, the behaviour of λ on either side of $z = z^*$ leads to global stability. Nothing new is thereby added to the proof of stability, but I believe useful insight is gained into what is really going on. Figure 5 merely reproduces the details in Figure 1. Investigation centres around the basic equation:

$$(34) \qquad \frac{dz}{d\lambda} = \frac{\beta z(\lambda - z)}{\alpha(z)}$$

Once a trajectory has hit the OBGP capital stock ratio z* at a point like J, z continues to rise because λ is too high but λ keeps falling at an increasing rate. Thus $\frac{dz}{d\lambda}$ is negative and rising in sector III; this exercises a restraining effect on the overshooting of z which finally hits its maximum at C. If one compares this with a backward

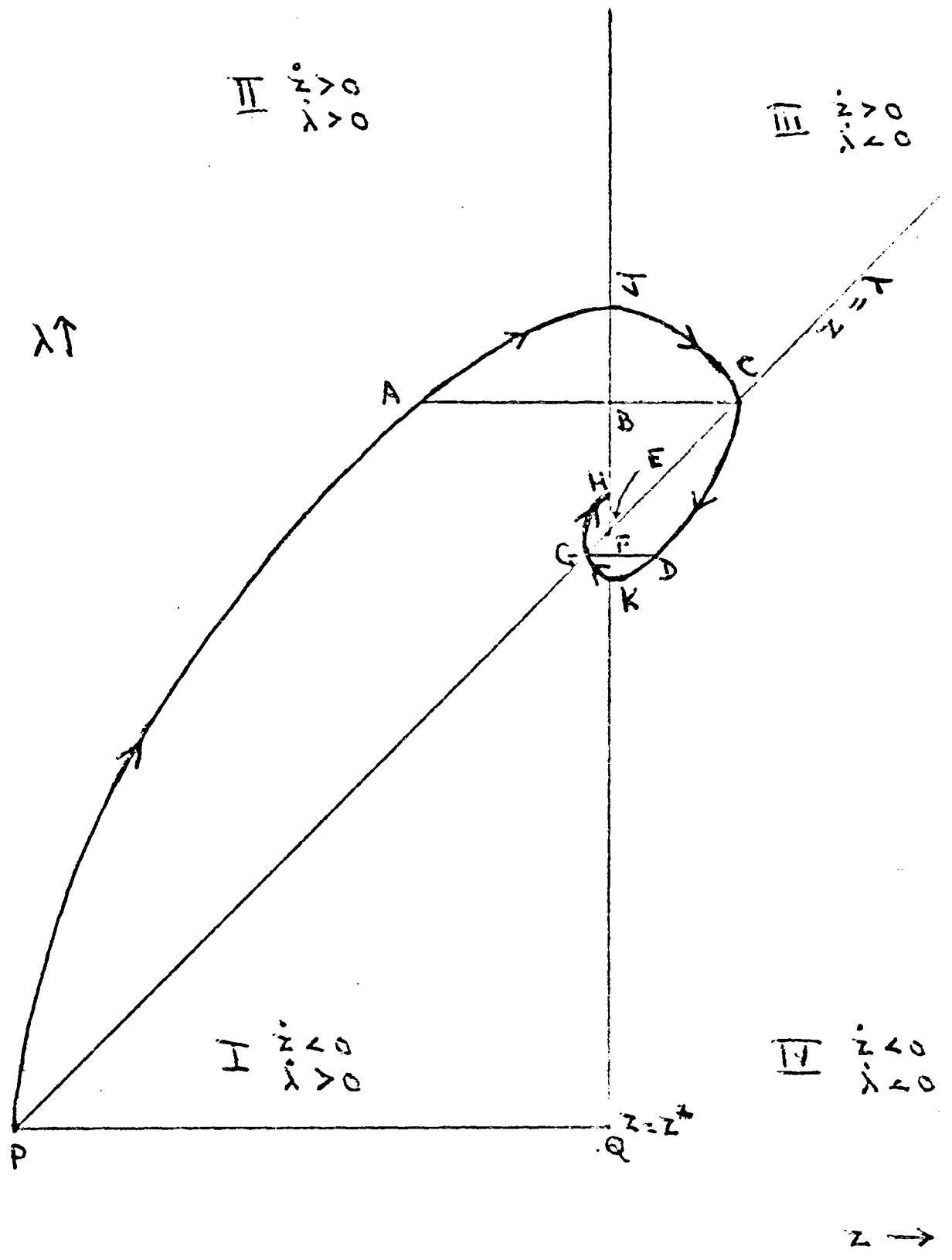

II $\dot{z}>0$ $\dot{\lambda}>0$
III $\dot{z}>0$ $\dot{\lambda}<0$
$\lambda\uparrow$
A
B
C
H
E
F
G
D
K
$z=z^*$
P
Q
I $\dot{z}<0$ $\dot{\lambda}>0$
IV $\dot{z}<0$ $\dot{\lambda}<0$
$z\rightarrow$

Figure 5

trajectory from J towards A in sector II where z and λ are both moving in the same direction, i.e., decreasing, and λ is decreasing at an increasing rate, it is easy to see that the z value in sector II corresponding to the λ value which secures the $(z - \lambda)$ equality at C is farther away from z^* than is the value of z at C, i.e., AB > BC[1]. Both z and λ fall in sector IV; hence BC > DF. The behaviour of λ in sector I is analogous to that in sector III. z overshoots K but λ rises at an increasing rate to make $\frac{dz}{d\lambda}$ negative and rising as in sector III. A similar argument as before shows that DF > FG. Both z and λ are rising in sector I; hence PQ > AB. Combining the above string of inequalities yields AB > GF. Since trajectories of differential equations may not intersect, this implies that FJ > FH, i.e., the maximum λ - value corresponding to $z = z^*$ monotonically decreases during every period of the cycle. Similarly, the minimum λ - value corresponding to $z = z^*$ monotonically increases during every period of the cycle. Therefore, an argument identical to the one used in the previous paragraph shows that these two monotonic sequences converge to $\lambda = \lambda^*$.

It is clear that the adjustment rule followed by producers in response to prices announced by the CPB, as represented by equation (30), is primarily responsible for convergence. It is the strictly monotonic decreasing relationship which suffices for the qualitative property of asymptotic optimality; obviously, speed of adjustment by producers would affect speed of convergence to the optimum plan. We have therefore established

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[1] This is the counterpart of DE being flatter than CD in Figure 4.

Theorem 2 : The intertemporally decentralised procedure is asymptotically optimal.

Remark : The convergence is monotonic in the sense that the plan generated by the decentralised procedure gets closer and closer to the OBGP configuration over time.

Assessment

Theorem 2 on asymptotic optimality of the decentralised procedure suggests the following comments.

(1) Most writers on the two-sector model with non-shiftable capital have taken that model to capture (albeit imperfectly) a noticeable feature of developing economies with unpromising trade prospects, viz., the constraint imposed on the economy's ability to save by lack of investment capacity.[1] The optimum policy in the simple version of that model is to remedy this deficiency by a finite time initial specialisation phase. The analysis of the present chapter shows that if producers in such an economy adopt a cautious but plausible rule of adjustment, the capital stock configuration exhibits (damped) oscillations around the OBCP configuration. This implies that the economy alternates between phases during which its ability to save is constrained by available investment capacity and those during which its desire to consume is similarly constrained by capacity in the consumer-good industry.[2]

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[1] See, in particular, the papers by Bose (1968) and Dasgupta (1969).

[2] This slightly understates the true extent of the problem in the decentralised economy, for in sectors I and III of Figure I, the capacity of the "wrong" industry is actually being enlarged.

(2) Such oscillatory behaviour in a continuous - time model is of interest. It is worth recalling that this result is not due to fixed coefficients but to sluggish response by producers and, in particular, to their unwillingness to reduce (increase) the allocation ratio, λ, discontinuously to its Phase III value, λ^*, the moment the OBGP capital stock ratio is reached from below (above). It might be supposed that oscillations are inherently implausible as a description of economic development. In that case it is possible either to postulate a learning process which, after a preliminary round of cycling, so modifies agents' behaviour as to damp the oscillations heavily or, alternatively, to conjecture that the two-sector non-shiftable capital model with its immutable commitment to capital-specificity overstates the structural rigidities to which a developing economy is subject.[1]

(3) This rigidity is, for our present purposes, not without pedagogic value. The decentralisation of authority, i.e., the genuine freedom of producers to follow courses of action not always in accordance with the CPB's wishes, becomes interesting when it generates patterns of development rather different from the fully optimal development. The rigidity of the two-sector non-shiftable capital model allows one to demonstrate such extravagant patterns even with a non-perverse adjustment rule : equation (8), after all, pushes the investment allocation towards the more profitable sector. This indeed was the rationale

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[1] In passing I draw the reader's attention to the fact that the optimum policy in a two-sector non-shiftable capital model with labour as a primary factor exhibits oscillations around the OBGP for finite time, but for rather different reasons from the one discussed here in the text. See the papers by Ryder (1969), Bose (1970) and Hadley and Kemp (1971).

for resorting to this part of the two-sector literature, for the application of rules analogous to (8) in a one-sector malleable capital model or a two-sector model with freely transferable capital would in general cause the economy to do more or less what the CPB wanted, where more or less depended on speeds of adjustment.[1] The analysis of this chapter suggests that formulating an interesting problem of authority-decentralisation in these, more usual, growth models might be rather hard.

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[1] See, for example, the discussion surrounding rule (16) in Chapter 3.

Chapter 5 : On Optimising Decentralised Systems

I. INTRODUCTION

The two previous chapters have been principally concerned with the intertemporal decentralisation of production decisions.[1] The present chapter examines (a) the simultaneous determination of consumption as well as production decisions and (b) the ability of the planners to optimise a basically decentralised system with respect to certain control variables at their disposal. Each of these two aspects requires some explanation.

1. Consumers

The elements of choice in intertemporal consumption decisions have so far been ignored; it has been implicitly assumed that aggregate consumption is satisfactorily distributed in accordance with certain rules. No discussion of decentralisation can, however, be complete without an explicit introduction of consumers and an account of the range of choices open to them. Utility - maximising decision-making by consumers does, as we shall see later, make the analysis easier in some ways: in particular, it permits a straightforward application of the fundamental theorem of welfare economics to decentralise the optimum

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[1] The introduction of the infinitely long-lived "representative consumer" in Chapter 3 gave the model of that chapter a convenient institutional interpretation. The analysis there was not concerned with the decentralisation of consumption decisions. See pp. 73-76.

policy of accumulation. The Ramsey-optimal path of accumulation tells one what the optimum distribution of consumption among generations is; it is then natural to inquire how the planners can manipulate the inter-generational distribution of income to ensure that decentralised price-taking behaviour leads the economy to the full optimum.

2. Optimising a Decentralised System

The standard theory of decentralisation is essentially an exercise in economic "engineering". It views the whole business of the allocation of resources as a game, it regards economic agents (consumers, managers of enterprises and central planners) as players and defines a "decentralised procedure" as a computational routine for finding the solution of the game. What makes the procedure decentralised is that the computational algorithm is solved by a multiplicity of agents and not solely by the central planners. The relative merits and demerits of alternative procedures (feasibility, monotonicity, incentive-compatibility and the like) are then discussed, usually on a fairly ad hoc basis : the question of which procedure in a given set of circumstances constitutes an optimal procedure has not been satisfactorily formulated, far less settled.

Such investigations of selected procedures are, no doubt, valuable and the accomplished technical work in this area is evidence that the economist, should he be called upon to do so, can offer useful advice on the design, performance and efficiency of resource allocation systems. But it can be argued quite persuasively that governments engaged in planning in most mixed economies often do not have the power to design decentralised procedures afresh by altering the pattern of economic relationships along "systems engineering" lines. It is commonplace that

the prevailing rules which regulate economic relationships in a society are a product of tradition, character and historical accident. This is not to claim that the standard theory of decentralisation is irrelevant nor, incidentally, is it to deny the possibility that the sum total of human welfare might be much increased if rational discourse were allowed to dictate the network of economic relationships. Indeed, to make the former claim would be an attempt to form normative judgments from purely "is" propositions - a procedure widely regarded as illegitimate by moral philosophers and, more recently, by welfare economists as well. But it does follow that the economist ought to be able to advise planners on ways of influencing the outcomes of existing resource allocation mechanisms through the use of monetary, fiscal and other instruments over which modern governments exercise a great deal of control. In sum, planners are often not in a position to demand behaviour in accordance with centrally prescribed rules, as the successors of the Lange-Lerner theory suggest. On the contrary, they have to accept the decentralised system of resource allocation represented by the private sector together with its pattern of economic relationships as a datum of the problem and use traditional instruments to generate outcomes that are optimal in the light of social objectives. The intention, then, is to focus attention away from a "systems analysis" view of decentralisation of planning, an approach which proceeds by enumeration of different rules of behaviour and concentrate instead on the issues raised by optimising a given decentralised model of growth and planning.

This chapter analyses how a government can use tax and debt policies to generate the optimum intergenerational distribution of income implied by the fundamental theorem of welfare economics. In other words,

it addresses itself to the question of whether the government in a mixed economy can induce a basically decentralised private sector to achieve the Ramsey-optimal path.

3. The Analysis in Outline

The economy comprises an infinite sequence of mortals. Individuals have access to a perfect capital market and are assumed to maximise lifetime utility subject to wage, interest and tax rates; the government chooses tax rates to maximise a Benthamite social valuation which is additive in lifetime utilities of generations from the present to an open-ended future. Production is neo-classical.

It is clear from this brief description that, in the absence of central intervention of one kind or another, the sequence economy will not follow a Ramsey-optimal path. A Ramsey planner is (rightly) concerned with the welfare of society beyond the lifetime of the youngest individual alive today and while unborn generations stake their claim on resources today through expectations, there is no guarantee that the private sector will make the right intergenerational transfers unaided. In other words, there is no reason to suppose that the optimum inter-generational wealth distribution will automatically emerge in the private sector.

In the sequel, I examine the efficacy of intertemporal lump-sum taxation and commodity taxation in controlling the mixed economy. The analysis is also extended to the case of a divergence between private and public objectives in the form of differential discounting of future utilities and to the case of uncertain individual lifetime with insurance opportunities. Finally, I embed the discussion of optimisation using

fiscal policy in a model where the central planners follow an iterative procedure to acquire information about the production technology. In all cases, the results available augur well from the point of view of controllability. In particular, the possibility of implementing intertemporal lump sum taxation looks rather more promising than in the static case.

I believe the novelty of the approach adopted here to lie in the simultaneous inclusion and synthesis of the following: (a) life-time utility maximisation and profit maximisation within an essentially competitive framework to determine consumption and saving decisions in the private sector, (b) an analysis of how a government could optimise a plausible and workable decentralisation of growth, (c) the use of standard welfare - theoretic arguments in discussing optimal redistributive policies making for intergenerational equity along growth paths,(d) a plausible reason for the government's debt policies and the undertaking of public investment even in the absence of externalities and increasing returns; and (e) some insight into the simultaneous operation of two policies to attain the full optimum when the sole application of one leads to non-optimality. Therefore, the basic model, though quite simple, provides the starting point for discussion over a wide range of interesting theoretical issues.

4. The Literature

The problem outlined above has, in one form or another, received considerable attention in the theoretical literature. The idea that the unaided evolution of a competitive economy with overlapping generations might not be efficient, not to say optimal, was first formulated by Samuelson (1958) who also suggested that the introduction of money as a social contrivance might restore efficiency. The analysis was confined

to an exchange economy; Diamond (1965), however, showed that the problem remains even when production is introduced. Samuelson's ideas and the phenomenon of dynamic inefficiency have been taken up by a number of authors, e.g., Cass and Yaari [(1966), (1967)], Shell (1971), Starrett (1972), who demonstrates the possibility of inefficiency on the distribution side; and Gale (1973). These writers concentrate on the role of monetary and debt policies in rescuing the competitive economy from inefficiency. Gale's paper argues that inefficiency cannot arise if the rate of impatience for future consumption is sufficiently high, and the rewards for waiting in the form of investment opportunities attractive. Kurz and Starrett (1970) explore the restrictions on technology that prevent inefficient development and, predictably, find them to be stringent. And, finally, Grandmont and Laroque (1973) have used the recent techniques of temporary equilibrium analysis to examine the existence of intertemporal equilibrium in a Samuelsonian economy with money.

The preoccupation of the literature cited above has been with the dynamic inefficiency of a competitive economy. The contributors are therefore concerned with the first basic theorem of welfare economics which asserts the Pareto optimality of a competitive economy in the absence of local non-satiation in the timeless case. The central concern of the present chapter is related to this literature in just the same way as is the second (or fundamental) theorem of welfare economics to the first. Are there policies which ensure that the full optimum can be realised by a decentralised competitive system? Not a great deal has been written on this question. Arrow and Kurz [(1969), (1970)] have made a thorough study of the controllability of the Ramsey

policy in a mixed economy using taxation, borrowing and public capital formation. Several different models have been analysed by the authors, the most sophisticated being one where the private sector is modelled as an infinitely long-lived utility-maximising "representative individual". Government intervention is needed because of the existence of public capital or a volume of debt historically given. The question of dynamic optimisation with fiscal policy has also been explored by Diamond (1973) who attempts to extend the theorems of Diamond and Mirrlees (1971) on optimal taxation and public production to a growth model with overlapping generations.

II. THE INSTITUTIONAL SET UP

All individuals of all generations are assumed to have an equal lifespan of "one" period. They have identical tastes which may be represented by a utility-of-consumption function. Since we are concerned with intergenerational equity, it is natural to assume that the circumstances of all individuals within a generation are identical. Specifically, every individual of every generation (i) receives a stream of wage income throughout his lifetime, (ii) can borrow and lend on a perfect capital market and (iii) is born without and dies without any physical assets. Bequests and inherited wealth could be accommodated as an index of altruism for future generations. They are clearly important when one is dealing with differences in tastes among individuals within a generation in a discussion of intra-generational equity. But they do not add much to our analysis and I prefer to ignore them here.

1. The Private Sector

It was noted in the introduction that the government must take the pattern of economic organisation in the private sector as a datum of the problem. Alternative paradigms are available but, unlike the standard theory of decentralisation to which reference has been made above, the interest of the present discussion does not lie in its quest for alternative rules of behaviour and I restrict myself to the one used most widely by economists. I assume that the private sector is completely described by a competitive model of general equilibrium. The deficiencies of this paradigm are well-known and I do not want to comment on that here. Its adoption implies that individuals maximise utility and producers maximise profits subject to prices which they accept as given; prices adjust to clear markets. Stated in this form, the assumptions look no more drastic than those commonly employed by economists. But it must be remembered that time is of the essence in the problem at hand. Not very much is available in the theory of intertemporal general equilibrium. One can follow either of two routes here. One has the option of generalising the static notion of perfect knowledge on the part of all economic agents to its dynamic analogue of perfect foresight about the course of future prices. This, strictly speaking, makes sense if one subscribes to the Arrow - Debreu notion of the existence of markets and commodities dated in time.[1] The alternative is to postulate either static expectations about market prices or some other rule of expectations-formation which depends on current prices. This is the spirit in which recent work on the existence and optimality of temporary

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[1] Even a complete set of markets is not quite enough. See Section II.3 on page 148.

equilibria has been conceived. There are two very difficult problems in this area, to neither of which has a satisfactory solution yet been found. The first is the absence of a microeconomic theory of expectations, a shortcoming elaborated in Chapter 3. Let us now ignore this first problem and assume that a particular rule of expectations has been selected. Anything short of perfect foresight requires one to describe how ex-ante plans and ex-post occurrences with unfulfilled expectations are reconciled in any sequence of events unfolding over real time. Preliminary analysis suggests that this latter problem is rather hard. Results are available on the existence of temporary equilibria but it is not known how one moves from one temporary equilibrium to its successor along a sequence.

In what follows, I shall present models incorporating alternative extreme assumptions : the cases of perfect foresight and static expectations respectively. While the latter model allows both government and private individuals to entertain static expectations about the future course of wage and interest rates, it does not make a similarly weak assumption with regard to the private sector's views about government policy.[1] In all cases we postulate perfect foresight about the future course of tax rates. Although it is true that individuals hold expectations about the probable shape of government policies and allow their current behaviour to be influenced by them, perfect foresight is, regrettably, a strong assumption and imputes to agents an extreme degree of sophistication in forecasting fiscal policy. The main reason that

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[1] This model is presented in Section V.

may be advanced in support of the assumption is that dynamic analysis of fiscal policy with foresight is a relatively unexplored area; it is therefore instructive to begin by examining the consequences of the easiest assumption available here.

Furthermore, I shall present one particular theorem where the optimum policy is time-invariant : in this case, perfect foresight is clearly a good assumption.[1]

I keep things simple by concentrating on a one - good model. An individual of generation v is one who is born at time v and dies at (v + 1) ; therefore, in the absence of central intervention, his task is to maximise

$$(1) \qquad \int_{v}^{v+1} e^{-\lambda(s-v)} U\left[C(s,v)\right] ds$$

-- subject to

$$(2) \qquad \int_{v}^{v+1} C(s,v)\, p(s,v)\, ds = \int_{v}^{v+1} W(s)p(s,v)\, ds$$

and

$$(3) \qquad C(s,v) \geq 0, \quad \text{where}$$

$$(4) \qquad p(s,v) = \exp\left[-\int_{v}^{s} r(x)dx\right]$$

λ : individual's rate of utility discount

r : rate of interest

W : wage rate.

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[1] See Theorem 6 and the remark on page 177.

Equation (2) says that the present value of the consumption stream discounted at the market rate of interest is equal to the present value of contractual income similarly discounted. It is natural to regard the expression on the right hand side as lifetime wealth. If we break in during his lifetime, the budget constraint looks like

$$(5) \qquad \int_{t}^{v+1} C(s,v)p(s,t)ds = \int_{t}^{v+1} W(s)p(s,t)ds + K(t,v)$$

where $v \leq t \leq v+1$, i.e., we break in at time t and $K(t,v)$: amount of physical asset owned by individual v at time t. The expression on the right hand side is the individual's wealth at time t and is composed of "human" assets and physical assets. The postulate that an individual is born without and dies without any physical assets may be written:

$$(6) \qquad K(v,v) = K(v+1, v) = 0$$

Differentiation of (5) with respect to t yields

$$r(t)\int_{t}^{v+1} C(s,v)p(s,t)ds - C(t,v) = r(t)\int_{t}^{v+1} W(s)p(s,t)ds - W(t) + \frac{\partial K(t,v)}{\partial t}$$

which may be rewritten, on substitution from (5), as

$$(7) \qquad C(t,v) + \frac{\partial K(t,v)}{\partial t} = W(t) + r(t)\ K(t,v)$$

which is the more familiar flow constraint equating contractual plus interest income to the sum of consumption and investment.

The budget constraints (5) and (7) have to be modified with the introduction of taxes. Thus, for a lump sum tax $L(t,v)$, (7) becomes

$$(8) \qquad C(t,v) + \frac{\partial}{\partial t} K(t,v) = W(t) + r(t)\, K(t,v) - L(t,v)$$

I assume exponential growth of the labour force (here identified with the population). Formally, generation v consists of $e^{nv}dv$ individuals; hence, one has for the population at time t:

$$(9) \qquad N(t) = \int_{t-1}^{t} e^{nv}\, dv$$

Production is neoclassical. There is one good which is used for both consumption and investment and which is produced by itself and labour. Individuals supply labour andthe capital good in return for wage and interest payments. The production function exhibits constant returns to scale and strictly diminishing returns to each factor; profit-maximising behaviour by producers implies that factors are paid their marginal products.[1] As usual, I measure variables in intensive units and summarise the above propositions in

$$(10) \qquad \begin{cases} r = f'(k) \; ; \; f''(k) < 0 \\ w = f(k) - kf'(k) \end{cases}$$

I also assume that

$$(11) \qquad \lim_{k \to 0} f'(k) = \infty \; , \quad \lim_{k \to \infty} f'(k) = 0, \quad \lim_{C \to 0} U'(C) = \infty$$

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[1] Similar assumptions were made in Section I of Chapter 3.

2. Government

The government chooses tax rates to maximise a social welfare functional while taking the behaviour of the private sector as given. Denoting by $\bar{C}(s,v)$ the solution to the problem (1) to (3), we have that the government maximises, in the overtaking sense,

$$(12)\quad \int_{-1}^{\infty} e^{-\delta v}\left\{\int_{v_+}^{v+1} e^{-\lambda(s-v)} U\left[\bar{C}(s,v)\right]\, ds\right\} e^{nv} dv, \text{ where } v_+ = \max(0,v)$$

The maximand needs some explanation. The story starts at $s = 0$; at this point there exists a continuum of pre-plan generations v, where $-1 \leq v < 0$. The additively separable valuation implies that nothing can be done about the accumulated utility (before time zero) of pre-plan generations. From time zero onwards, the government takes into account the maximised lifetime utility of every member of a typical generation (the expression in curly brackets); this is weighted by the size of each generation, discounted back to birth at a planners' rate δ and, finally, added across generations.

The government, like the private sector, has access to a perfect capital market : this allows its budget constraint to be written as

$$(13)\quad \int_{-1}^{\infty} p(v_+, 0)\, \theta(v) e^{nv} dv = A^1$$

where $\theta(v) = \int_{v_+}^{v+1} p(s,v_+)\, L(s,v)\, ds$, the discounted value of lifetime lump-sum taxes paid by a member of generation v.

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[1] I assume here that such infinite integrals are well-defined. I shall comment on the validity of this assumption in section III below.

The government's budget constraint may be expressed as

$$(14) \qquad Y(s) = \frac{d}{ds}\left[K_g(s) - D(s)\right] - r(s)\left[K_g(s) - D(s)\right]$$

where $Y(s) \equiv$ tax revenue at $s = \int_{s-1}^{s} L(s,v)\, e^{nv} dv$

$K_g(s) \equiv$ public capital at s

$D(s) \equiv$ government debt at s

Notice that

$$\int_0^\infty p(s,o) \int_{s-1}^{s} L(s,v) e^{nv} dv ds = \int_{-1}^{\infty} p(v_+, o) \int_{v_+}^{v+1} p(s,v_+) L(s,v) ds\, e^{nv} dv$$

$$= \int_{-1}^{\infty} p(v_+, 0)\, \theta(v)\, e^{nv} dv = A, \quad \text{from (13)}$$

where the first equality is obtained by interchanging the order of integration and the second by using the definition of $\theta(v)$ above. Using these equalities, (14) and the definition of $Y(s)$, we see that

$$(15) \quad A = \int_0^\infty p(s,o)\, Y(s) ds = \int_0^\infty p(s,o) \left\{ \frac{d}{ds}\left[K_g(s) - D(s)\right] - r(s)\left[K_g(s) - D(s)\right] \right\} ds$$

In section III I shall have occasion to remark upon some reasons for instituting the policies summarised by **A**. Meanwhile it is easily checked that Walras' Law is satisfied for our model.

The government's problem therefore is to select $L(s,v)$ to maximise

$$\int_{-1}^{\infty} e^{-\alpha v} \left\{ \int_{v_+}^{v+1} e^{-\lambda(s-v)} U\left[\bar{C}(s,v)\right] ds \right\} dv, \text{ where } \alpha = \delta - n$$

subject to

$$\int_{-1}^{\infty} \left\{ p(v_+,o) \int_{v_+}^{v+1} p(s,v_+)\, L(s,v)\, ds \right\} e^{nv} dv \quad = \quad A$$

which may be rewritten, upon interchanging the order of integration, as

maximise

$$(16) \begin{cases} \int_{o}^{\infty} e^{-\alpha s} \left\{ \int_{s-1}^{s} e^{(\alpha-\lambda)(s-v)}\, U\left[\bar{C}(s,v)\right] \quad dv \right\} ds \quad {}^{1} \\ \text{subject to} \\ \int_{o}^{\infty} \left\{ p(s,o) \int_{s-1}^{s} L(s,v) e^{nv} dv \right\} ds \quad = \quad A. \end{cases}$$

This way of writing the problem shows that the planner is being invited to establish the optimal intergenerational welfare distribution at every instant s ; this is then added over the entire time-horizon.

To sum up, there is a two-stage maximisation going on in this economy. The private sector maximises utility and profits subject to equilibrium market prices and tax rates; the government manipulates tax rates to ensure that private maximisation leads to the social optimum. It is an identifiable variation on the traditional theme of decentralisation.

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[1] This makes clear that an optimum policy exists provided that $\alpha \geq 0$, i.e., $\delta \geq n$, as in Footnote 1 on page 71.

3. Existence of Solutions

Before going on to the detailed analysis of solutions, it is worth commenting on their existence. Consider the private sector in the absence of government intervention. The demographic setup implies that not all traders can transact with each other, notably with the dead and the unborn. Hence, given the peculiar time - structure of trades, no intertemporal general equilibrium need exist even with a complete set of futures markets today. In that case, the perfect foresight assumption is logically impossible to sustain because there is no path of self-fulfilling consistent expectations which economic agents can foresee perfectly. And any attempt to characterise non-existent solutions is seriously misleading. Various writers on Samuelson's "consumption loans" model have recognised this possibility[1] and have suggested money or financial intermediation as a way of ensuring that active markets with equilibrium prices actually emerge. The cited paper by Grandmont and Laroque (1973) attempts a rigorous investigation of precisely this question using the techniques of temporary equilibrium analysis.

The existence problem in our model is simpler, precisely because of the two-stage maximisation. One can, for the purposes of this argument, regard all variables as government control variables and therefore the path followed by the economy as the outcome of optimum policies pursued by the government. If a solution to the government's problem exists, and one can reasonably conjecture that it does, given

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[1] See, for example, the account in Cass and Yaari (1966).

the usual compactness arguments in the theory of optimal control, one is entitled to claim that equilibrium wage rates, interest rates and tax rates exist corresponding to the optimal intergenerational distribution brought about through the operation of government policy. This is a substantially weaker result than a theorem on the existence of intertemporal competitive equilibrium in the private sector but, by the fundamental theorem of welfare economics, that is all one needs to be satisfied of before proceeding with the characterisation of that solution. It is substantially weaker because an existence theorem on competitive equilibria would have to demonstrate the result without recourse to any redistributive measures. It would therefore have to work for an entire class of admissible wealth distributions and not, as in our case, just for the distribution implied by the fundamental theorem. The two-stage maximisation approach permits one to exploit theorems on the existence of optimum policies to answer the relevant questions on the existence of competitive equilibria. And such existence theorems on optimum policies are available in the one-good neoclassical laboratory being considered here.[1]

The economy under consideration has been completely described. We should, ideally, like to discover the optimum development of the economy and the properties of optimum tax structures.

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[1] An optimum policy exists when $\delta \geq n$.

III. THE ARGUMENT

The analysis is most conveniently performed by describing first of all the optimum policy in the centrally controlled economy and then by examining whether the optimum can be attained through the two-stage maximisation outlined in Section II above.

1. The Optimum Policy

The government's problem in the centrally planned economy is to choose C(s,v) to maximise (in the overtaking sense)

$$(17) \qquad \int_{-1}^{\infty} e^{-\alpha v} \left\{ \int_{v_+}^{v+1} e^{-\lambda(s-v)} U\left[C(s,v)\right] \, ds \right\} dv$$

subject to

$$(18) \qquad \int_{-1}^{\infty} \left\{ \int_{v_+}^{v+1} p(s,o)\, C(s,v) ds \right\} e^{nv} dv \quad = \quad B$$

Equation (18) needs comment. It may be rewritten as

$$(19) \qquad \int_{o}^{\infty} p(s,o)\, C(s)\, ds \quad = \quad B,$$

where $C(s) = \int_{s-1}^{s} C(s,v) e^{nv} dv$, i.e., total consumption at time s.

B represents the total assets of the economy at time zero, i.e., the assets that can support a consumption stream given by C(s). Since the only sources of income in this economy are wage income and interest income, one might regard B as the sum of physical assets (i.e., capital stock) and human assets (i.e., the discounted value of the wage stream in the economy).

In the rest of this chapter, I shall assume that $\delta > n$, i.e., $\alpha > 0$. This is more than one needs for the existence of an optimum policy ($\alpha \geq 0$ will do for that). But, given that the price and quantity variables corresponding to an optimum policy converge to bounded OBGP values, the condition ensures that the integrals in (18) and (19) are convergent and, therefore, that B is well-defined.

We are now in a position to state

Lemma 1 : Assume that $U\left[C(s,v)\right]$ is concave. Then the optimum policy is characterised by

$$(20) \qquad e^{-\delta v}\left\{ e^{-\lambda(s-v)}\, U'\left[C(s,v)\right]\right\} = \mu\, p(s,o)$$

where μ is a constant and where, therefore, the right hand side of (20) is independent of v.

Proof: Let $C^*(s,v)$ be a policy satisfying (20) and let $C(s,v)$ be any other feasible policy, i.e., one which, like $C^*(s,v)$, satisfies the constraint (18). Then, by a well-known property of concave functions

$$\int_{-1}^{\infty}\int_{v_+}^{v+1}(e^{-\alpha v})\; e^{-\lambda(s-v)}\left\{ U\left[C^*(s,v)\right] - U\left[C(s,v)\right]\right\} ds.\, dv \geq$$

$$\int_{-1}^{\infty}\int_{v_+}^{v+1}(e^{-\alpha v})\; e^{-\lambda(s-v)}\, U'\left[C^*(s,v)\right]\left\{ C^*(s,v) - C(s,v)\right\} ds.\, dv$$

$$= \mu\int_{-1}^{\infty}\int_{v_+}^{v+1}\left[p(s,o)\left\{ C^*(s,v) - C(s,v)\right\} ds\right] e^{nv}dv = 0,$$

from (18) and (20).

Therefore, any policy satisfying (20) is optimal.

2. Individual Maximisation

I now return to the decentralised economy described in section II. Characterisation of the individual's maximum is straightforward. A typical individual chooses $C(s,v)$ to maximise

$$(21) \quad \int_{v_+}^{v+1} e^{-\lambda(s-v)} U\left[C(s,v)\right] ds$$

subject to

$$(22) \quad \int_{v_+}^{v+1} C(s,v)\, p(s,v_+)\, ds = \int_{v_+}^{v+1} W(s)p(s,v_+)ds + K(v_+,v) - \int_{v_+}^{v+1} L(s,v)p(s,v_+)ds$$

where $p(s,v_+) = \exp\left[-\int_{v_+}^{s} r(x)\, dx\right]$

Denoting $\int_{v_+}^{v+1} p(s,v_+)\, L(s,v)\, ds$ by $\theta(v)$, as before, one can write the Lagrangian for this problem as

$$(23) \quad R = \int_{v_+}^{v+1} \left\{ e^{-\lambda(s-v)} U\left[C(s,v)\right] - \wp C(s,v)\, p(s,v_+) \right\} ds + \wp \left[\int_{v_+}^{v+1} W(s)\, p(s,v_+)\, ds + K(v_+,v) - \theta(v) \right]$$

Assume that a solution to the individual's problem exists. The individual performs this maximisation with respect to the function $C(s,v)$; since (11) ensures that the solution to the problem is interior, we should like R to remain stationary for all admissible variations in $C(s,v)$ around the optimum. The strategy, then, is to discover

$$\delta R = \int_{v_+}^{v+1} \left\{ e^{-\lambda(s-v)} U' \left[C(s,v)\right] - \oint p(s,v_+) \right\} ds. \ \delta C(s,v)$$

and to want $\delta R = 0$ for all $\delta C(s,v)$ satisfying the constraint (22). This implies that

$$(24) \qquad e^{-\lambda(s-v)} U' \left[C(s,v)\right] = \oint p(s,v_+)$$

is a necessary condition for the maximisation (21) - (22).

One would expect concavity of the instantaneous utility function $U\left[C(s,v)\right]$ to ensure that (24) is sufficient as well. I summarise all this in

Lemma 2: Assume that $U\left[C(s,v)\right]$ is concave. Then, a necessary and sufficient condition that $C(s,v)$ is the individual's best consumption plan is given by

$$e^{-\lambda(s-v)} U' \left[C(s,v)\right] = \oint p(s,v_+)$$

Proof: Necessity has been established above. Sufficiency is easily shown. Let $\bar{C}(s,v)$ be a policy satisfying (24) and let $C(s,v)$ be any other feasible policy, i.e., one which, like $\bar{C}(s,v)$, satisfies the constraint (22).

Then,

$$\int_{v_+}^{v+1} e^{-\lambda(s-v)} \left\{ U\left[\bar{C}(s,v)\right] - U\left[C(s,v)\right] \right\} ds \geq$$

$$\int_{v_+}^{v+1} e^{-\lambda(s-v)} U'\left[\bar{C}(s,v)\right] \left\{ \bar{C}(s,v) - C(s,v) \right\} ds$$

$$= \int_{v_+}^{v+1} p(s,v_+) \left\{ \bar{C}(s,v) - C(s,v) \right\} ds$$

$$= 0, \quad \text{from (22).}$$

Thus, any policy satisfying (24) is optimal for the individual.

Remark: Note that $\int$ is the private marginal utility of income.

3. Controllability

I bring the relevant equations together for ease of reference. They are

$$(20) \qquad e^{-\delta v} \left\{ e^{-\lambda(s-v)} U'\left[C^*(s,v)\right] \right\} = \mu\, p(s,o)$$

$$(24) \qquad e^{-\lambda(s-v)} U'\left[\bar{C}(s,v)\right] = \int p(s,v_+)$$

The fundamental theorem of welfare economics leads one to expect that the optimum can be realised in decentralised fashion by assigning to each individual a budget which purchases the consumption vector he is entitled to have along the optimum path. (20) and (24) reveal that this can be done if lump-sum taxation is arranged such that

$$(25) \qquad \mathcal{F} = \mu e^{\delta v} p(v_+;0)$$

We have therefore arrived at

Theorem 1 : The full optimum is controllable with intertemporal lump-sum taxation.

At this stage it is worth returning briefly to the existence problem referred to in section II. We have assumed that $\alpha > 0$; therefore, a Ramsey optimum exists. Theorem 1 then helps allay one's doubts about the existence of general equilibrium in the private sector in the presence of intertemporal lump-sum taxation. This continues to be true whenever the government uses policy tools which enable the economy to attain the full optimum.

Some light is thrown on Theorem 1 by returning to the basic optimality condition (20) which I spell out completely as

$$(26) \qquad e^{-\delta v}\left\{ e^{-\lambda(s-v)} U'\left[C(s,v)\right] \right\} = \mu \exp\left[-\int_0^s r(x)dx\right]$$

Logarithmic differentiation of (26) with respect to s and v respectively yields

$$\frac{U''\left[C(s,v)\right]}{U'\left[C(s,v)\right]} \frac{\partial C(s,v)}{\partial s} = \lambda - r(s)$$

$$\frac{U''\left[C(s,v)\right]}{U'\left[C(s,v)\right]} \frac{\partial C(s,v)}{\partial v} = \delta - \lambda$$

which may be rewritten, upon defining $\eta\left[C(s,v)\right] = -\frac{U''\left[C(s,v)\right]}{U'\left[C(s,v)\right]} C(s,v)$, the elasticity of marginal utility of consumption, as

$$(27) \qquad \frac{\partial C(s,v)}{\partial s} = \frac{r(s) - \lambda}{\eta\left[C(s,v)\right]} \cdot C(s,v)$$

$$(28) \qquad \frac{\partial C(s,v)}{\partial v} = \frac{\lambda - \delta}{\eta\left[C(s,v)\right]} C(s,v)$$

Equations (27) and (28) describe the "time-series" and "cross-sectional" behaviour of consumption arising out of the simultaneous operation of the two-stage maximisation carried out by the government and the private sector. (28) allows one to calculate the optimum post-tax distribution of consumption among generations at every point along a Ramsey-optimal path. The redistributive policy is easily interpreted. Suppose, for example, that the government discounts the future less heavily than the private sector, so that $\delta < \lambda$. (28) then asks the government to redistribute consumption in favour of younger generations alive at s. Both this, and the fact that the redistribution is inversely related to the elasticity of marginal utility, are intuitively satisfactory. An interesting special case occurs when $\delta = \lambda$, for then $C(s,v)$ is the same for all v in $[s-1, s]$, i.e., the optimum post-tax distribution is perfectly egalitarian.

4. Optimum Taxation

Theorem 1 has established the path of development of the sequence-of-generations economy. We know an optimum tax policy exists; can we say anything about its shape? In particular, we should, ideally, like to know if the time path of lump sum taxation is well-behaved, whether it converges asymptotically as the economy tends to balanced growth and whether its behaviour is sufficiently "predictable" for the perfect foresight assumption to be a reasonably valid one.

To investigate some of these issues, I return to the definition of taxation:

$$(29) \qquad \theta(v) = \int_{v_+}^{v+1} L(s,v)\, p(s,v_+)\, ds$$

$\theta(v)$ is the lifetime tax burden on an individual of generation v discounted back to birth or the start of the planning exercise, whichever is the later. There is an asymmetry in the treatment of generations who are alive when the planning exercise starts because they cannot be taxed at birth and therefore have a historically given distribution of capital assets outstanding which will, in general, call for special treatment.

From the individual's budget constraint (8), we know that

$$(30) \qquad \theta(v) = \int_{v_+}^{v+1} \left[W(s) - C(s,v)\right] p(s,v_+)\, ds + K(v_+, v)$$

Equations (29) and (30) suggest that one might choose $L(s,v)$ to be the difference between $W(s)$ and $C(s,v)$ along an optimum path, taking care to treat the $K(v_+,v)$ term suitably. The optimum $C(s,v)$ can be calculated from (28) knowing the value of $C(s)$, the aggregate consumption at time s along the optimum path. (29) and (30) lead to

$$(31) \qquad \int_{v_+}^{v+1} \left[L(s,v) - \left\{W(s) - C(s,v)\right\}\right] p(s,v_+)\, ds - K(v_+, v) = 0$$

Differentiation of (31) with respect to v yields

$$\left[L(v+1,v) - \left\{ W(v+1) - C(v+1,v)\right\} \right]p(v+1,v_+) - \left[L(v,v) - \left\{ W(v) - C(v,v)\right\} \right]$$

$$p(v,v_+) \quad \frac{dv_+}{dv} \quad + \int_{v_+}^{v+1} \frac{\partial L(s,v)}{\partial v} \quad p(s,v_+) \; ds + \int_{v_+}^{v+1} \frac{\partial C(s,v)}{\partial v} \; p(s,v_+) \; ds$$

$$= \quad \frac{\partial K(v_+, v)}{\partial v} \quad - \quad r(v_+) \quad K(v_+,v) \quad \frac{dv_+}{dv}$$

Since $K(v,v) = 0$, $K(v_+,v) \quad \frac{dv_+}{dv} = 0$ and we may write

$$(32)\left[L(v+1,v)-\left\{W(v+1)- C(v+1,v)\right\}\right]p(v+1,v_+)-\left[L(v,v)-\left\{W(v)-C(v,v)\right\}\right]p(v,v_+)\frac{dv_+}{dv}$$

$$+ \int_{v_+}^{v+1} \frac{\partial L(s,v)}{\partial v} \; p(s,v_+)ds + \int_{v_+}^{v+1} \frac{\partial C(s,v)}{\partial v} \; p(s,v_+)ds = \frac{\partial K(v_+,v)}{\partial v}$$

I first examine the ideas following equation (30) to the effect that L may be taken to be the difference between wage and consumption, in a simple yet instructive case. Assume that the government and the private sector discount the future at the same rate, i.e., $\delta = \lambda$. Then, from (28), $\frac{\partial C(s,v)}{\partial v} = 0$ and the optimum post-tax distribution of consumption is perfectly egalitarian. Define $C(s,v) = z(s)$, the aggregate consumption per head at time s. This allows L to be a function of s alone. Remembering that pre-plan generations need special treatment, one defines

$$(33) \qquad L(s) \quad = \quad T(s) \quad \text{for } v \geq 0$$

$$(34) \qquad \quad = \quad T(s) + E(s) \quad \text{for} \quad -1 \leq v < 0$$

Thus, $E(s) = 0$ for $v \geq 0$, i.e., no tax over and above T is imposed on individuals born no earlier than the start of the planning process.

Substitution of (33) and (34) into (32) allows one to write

$$(35)\left[T(v+1)-\left\{w(v+1)-z(v+1)\right\}\right] = \frac{1}{p(v+1,v)}\left[T(v)-\left\{w(v)-z(v)\right\}\right] \quad ; \quad v \geq 0$$

$$(36)\left[T(v+1)-\left\{w(v+1)-z(v+1)\right\}\right] = \frac{1}{p(v+1,0)} \cdot \frac{\partial K(0,v)}{\partial v} - E(v+1); \quad -1 \leq v < 0$$

Inspection of the above equations shows that T = w - z is a solution to (35); this suggests that initial conditions be arranged in such a way that T = w - z is feasible. Accordingly, the procedure is to

$$\text{(i) choose } E(v+1) \begin{cases} = \dfrac{1}{p(v+1,0)} \dfrac{\partial K(0,v)}{\partial v} & \text{for } -1 \leq v < 0 \\ = 0, \text{ by definition elsewhere} \end{cases}$$

(ii) choose $T(v+1) = w(v+1) - z(v+1)$, always.

Step (i) describes precisely how generations who are born before time zero should be treated differently. In addition to T, they have to pay an extra E which depends on $\frac{\partial K(0,v)}{\partial v}$, the intergenerational distribution of physical assets historically given at the start of the planning process. Step (ii) shows that all other generations always pay the difference between wage rate and consumption along the optimum path. The discriminatory treatment of generations obviously stops when s = 1, when nobody who was alive when the planning exercise started is present any more.

Formally, define a new variable M = T - (w - z). Then (35) may be written as

$$(37) \qquad M(v+1) = \frac{1}{p(v+1,v)} M(v)$$

One can, on remembering that $\frac{1}{p(v+1,v)}$ is $\exp\left[\int_v^{v+1} r(x)dx\right]$,

write the solution to this difference equation as

$$(38) \qquad M(s) = \frac{1}{p(s,\{s\})} M\left[\{s\}\right]$$

where $\{s\} = s \mod 1$, or the fractional part of s. The value of M over the fractional part of s , i.e., for s between zero and one is made zero by the application of rule (i) above and equation (36). But then, by (38), $M(s) = 0$ for all s.

In the general case ($\delta \neq \lambda$), one can similarly define

$$(39) \qquad L(s,v) = T(s,v) \quad \text{for} \quad v \geq 0$$

$$(40) \qquad = T(s,v) + E(s) \quad \text{for} \quad -1 \leq v < 0$$

and obtain the solution

$$(41) \qquad L(s,v) = W(s) - C(s,v) \quad \text{for} \quad v \geq 0$$

$$(42) \qquad = W(s) - C(s,v) + \frac{1}{p(v+1,0)} \frac{\partial K(0,v)}{\partial v} \text{, for } -1 \leq v < 0$$

where C(s,v) along an optimum path may be calculated for every s and v $\in$ [s - 1,s] from equation (28).

All this is worth summarising as

Theorem 2 : One way of doing optimum lump-sum taxation has the following features:

(a) it requires all generations born no earlier than the start of the planning process to pay to the State the difference between the wage rate and their consumption along the optimum path; and

(b) it requires pre-plan generations to pay the same as in (a) plus an extra, solely date-related, term which discriminates between them and the generations referred to in (a) above according to rule (i).

The conclusions of Theorem 2 suggest the following comments.

(1) The tax rate depends on age alone as the economy tends asymptotically to the optimum balanced growth path. This is because, in balanced growth, $C(s,v)$ depends on age $(s-v)$ alone. Therefore,

$$(43) \quad \lim_{s\to\infty} T(s,v) = \lim_{s\to\infty} \left[W(s) - C(s,v)\right]$$

$$= W^* - C^*(s-v) = T(s-v), \text{ (say).}$$

Hence, the lump-sum tax in balanced growth is age-related and therefore easy to foresee.

(2) The theorem shows one way of doing optimum lump-sum taxation. Alternative ways are, no doubt, possible. Thus, for example, one would like to be able to describe the behaviour of a tax related to vintage alone, $L(s,v) = L(v)$. Such a tax would be particularly easy to foresee, not only in balanced growth, but outside it as well. Unfortunately, no analytical results on this can be reported here; it seems one would have to compute such a policy numerically with specific utility and production functions.

(3) The fact that the optimum lump sum tax is, in the special case $\delta = \lambda$, a poll tax, makes evasion in that case difficult for, with one exception, no one has any incentive to claim that he is not what the

tax authorities suppose him to be. The exception is anyone liable for the tax E as well. When E is positive (a tax), a member of a pre-plan generation would wish to claim otherwise; when E is negative (a subsidy), a member of a post-plan generation would wish to claim otherwise. The possibility of such claims may be thought a fanciful one - the assumption is that the government can easily discover who is present at time 0 : only these people are subject to E and no one else. Secondly, on reasonable assumptions about the initial intergenerational asset distribution $K(0,v)$ $(-1 \leq v < 0)$, one might suppose from rule (i) that E will initially be negative and then become positive - this implies that a member of a post-plan generation would wish to claim he was of pre-plan vintage and subsequently wish to assert he really was post-plan after all! This establishes a slight presumption against evasion of E.

(4) The model we have used shows that the optimum tax policy could have a reasonably simple form. This is comforting for a planner who is concerned with Ramsey-type issues of intergenerational equity but is faced with constituents who are not interested at all in what happens beyond their own lifetimes. For the model quite adequately captures two very important features of reality: (a) an extreme degree of impatience by private individuals and (b) the inability of the planner to impose quantitative directions on consumption.

But the above comments should be taken as being no more than tentative, for they are based on a simple, exploratory model. In particular, one might reasonably suppose that the absence of bequests,

variations in intertemporal preference patterns and considerations pertaining to intragenerational equity is a serious shortcoming even in a preliminary analysis of intergenerational questions.

5. Complementary Government Policy

In this section, I concentrate on the effect of pursuing optimum taxation on the government's debt and public capital formation policies. Recall that the budget constraint of the typical individual is

$$(44) \qquad C(s,v) + \frac{\partial K(s,v)}{\partial s} = W(s) + r(s)\,K(s,v) - L(s,v)$$

The optimum tax is $L(s,v) = T(s,v) = W(s) - C(s,v)$ for $v \geq 0$

Substitution into (44) leads to

$$(45) \qquad \frac{\partial K(s,v)}{\partial s} = r(s)\,K(s,v) \quad \text{for} \quad v \geq 0$$

But $K(v,v) = 0$ by assumption; hence $K(s,v) = 0$ for all s in $\left[v,\ v+1\right]$. Thus we have the result that no member of a post-plan generation will ever use the capital market to which he theoretically has access. This implies that, by time $s = 1$, all capital in the economy is publicly owned.

The result is confirmed by recalling the government's budget constraint

$$(14) \qquad Y(s) = \frac{d}{ds}\left[K_g(s) - D(s)\right] - r(s)\left[K_g(s) - D(s)\right]$$

No distinction need be made, for our purposes, between government capital and debt; accordingly, it is convenient to define $Q = K_g - D$ or, equivalently, to assume an absence of debt and concentrate on

government capital alone. (14) becomes

$$(46) \qquad Y(s) = \frac{dQ}{ds} - rQ$$

where

$$(47) \qquad Y(s) = \begin{cases} \int_{s-1}^{s} T(s,v)e^{nv}dv + E(s)\int_{s-1}^{0} e^{nv}dv, & \text{for } 0 \leq s \leq 1 \\ \int_{s-1}^{s} T(s,v)\, e^{nv}dv, & \text{for } s \geq 1 \end{cases} \qquad (48)$$

One may as well assume without, from our point of view, any loss of generality that $Q(0) = 0$. This has the added advantage of demonstrating how this particular aspect of government policy develops, starting from scratch. Thus equation (46) may be written

$$(49) \quad \frac{dQ}{ds} - rQ = \int_{s-1}^{s} \left[W(s) - C(s,v)\right] e^{nv}dv + E(s)\int_{s-1}^{0} e^{nv}dv; \quad 0 \leq s < 1$$

with $Q(0) = 0$

$$(50) \quad \frac{dQ}{ds} - rQ = \int_{s-1}^{s} \left[W(s) - C(s,v)\right] e^{nv}dv \; ; \quad s \geq 1$$

Equation (50) may be written

$$(51) \quad C(s) + \frac{dQ(s)}{ds} = W(s) + r(s)\, Q(s), \quad s \geq 1$$

$$\left\{ \text{where } C(s) = \int_{s-1}^{s} C(s,v)\, e^{nv}dv \text{ and } W(s) = \int_{s-1}^{s} W(s)\, e^{nv}dv \right\}$$

which is equivalent, by virtue of the constant returns to scale and marginal productivity assumptions summarised in (10) to

$$(52) \qquad C(s) + \frac{dQ(s)}{ds} = f\left[Q(s)\right] , \quad s \geq 1$$

But this is the aggregate feasibility relation for the economy, re-affirming that $Q(s)$ is the aggregate capital stock for $s \geq 1$. This shows that the optimum tax policy requires the government to take all capital into public ownership at $s = 1$; it then has to assume sole responsibility for ensuring that capital accumulation is carried out according to the Ramsey-optimal policy. We therefore have

Theorem 3 : The optimum tax policy outlined in Theorem 2 requires the government to take all capital into public ownership at $s = 1$. [1]

Remark : The analysis provides one with a reason for the government's wishing to engage in public investment. In this respect, it differs from the work of Arrow and Kurz (1970) and Diamond (1973) cited in the introduction, for these authors tend to postulate the existence of a public production sector a priori.

IV. EXTENSIONS

1. Differential Discounting

The analysis of the previous section, while allowing individuals and government to discount the future at different rates, remained within the Paretian tradition of welfare economics by ensuring

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[1] It ought to be pointed out that other forms of optimum lump sum taxation need not call for public ownership of capital for their implementation.

that the government respected individuals' notions of their own lifetime utilities. It is instructive to examine the case where individuals suffer from Pigovian telescopic faculty and discount future satisfactions within their lifetimes for that reason [1] but where the government considers such discounting unethical and consequently refuses to regard individuals' own notions of lifetime utilities as being the relevant arguments in the social valuation.

The individual's maximand continues to be

$$(53) \qquad \int_{v_+}^{v+1} e^{-\lambda(s-v)} U\left[C(s,v)\right] ds$$

but (53) is, from the government's point of view, without normative significance. The social valuation is

$$(54) \qquad \int_{-1}^{\infty} e^{-\alpha v} \left\{ \int_{v_+}^{v+1} U\left[C(s,v)\right] ds \right\} dv$$

where $\alpha = \delta - n$ and the government discounts, not because of telescopic faculty, but in order to allow for extinction.

(54) is a non-Paretian valuation : consequently, one would not expect the full optimum to be controllable with lump-sum taxation alone.

As before, the full optimum is given by

$$(55) \qquad e^{-\delta v} U'\left[C(s,v)\right] = \mu\, p(s,0)$$

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[1] As before, individuals cease to care about the future of society altogether beyond their own lifetimes.

But the individual's best plan is characterised by

$$(56) \qquad e^{-\lambda(s-v)}\, U'\left[C(s,v)\right] \;=\; \int p(s,v_+)$$

The difficulty for controllability is caused by the term $e^{-\lambda(s-v)}$ in (56). It is natural to suppose that an interest subsidy of λ in the capital market would exactly offset the consequences of individual discounting. The individual maximum in the presence of such a subsidy is characterised by

$$(57) \qquad U'\left[C(s,v)\right] \;=\; c\, p(s,v_+)\,, \quad \text{where } c = \int e^{-\lambda(v-v_+)}$$

This allows one to assert

Theorem 4 : The full optimum with a non-Paretian valuation is controllable with a combination of intertemporal lump-sum taxation and an interest subsidy equal to the individual discount rate applicable to all transactions in the capital market.

It would be interesting to enquire into the properties and shape of the second-best optimum policy that emerges when an interest subsidy is not available. While it is relatively straightforward to derive, from (56), an equation that the policy must satisfy, it has not proved possible to obtain an explicit form for that policy.

2. Uncertain Lifetime

The analysis so far has assumed that the lifespan of every generation is equal and completely known. It is, however, a reasonably straightforward matter to incorporate uncertainty of lifetime into the model, provided one is permitted to introduce a market in

annuities. The analysis of this subsection owes much to the pioneering work of Yaari (1965); where convenient, I have also taken the liberty of using his notation.

There are two essential ways in which uncertainty enters the picture. First, it changes the individual's maximand : specifically, it is reasonable to suppose that he maximises expected lifetime utility over a probability distribution of the age at which death occurs. Secondly, it enters the budget constraint by affording the individual the opportunity of trading in annuities. We examine each of these aspects in turn before going on to the optimisation problem facing the government.

I assume that T represents the age at which an individual dies - a random variable defined in the interval $[0,1]$. T obeys a probability law which is specified by a probability density function, π. Thus, if y is a variable representing age, we have that

$$(58) \qquad \pi(y) \geq 0, \quad \text{all } y$$

$$(59) \qquad \int_0^1 \pi(y)\, dy = 1$$

It is also convenient to assume that π is positive for all y in the open interval $(0,1)$.

Define a function M and a family of conditional probability density functions $\{ \pi_y, 0 \leq y \leq 1 \}$ as follows:

$$(60) \qquad M(y) = \int_y^1 \pi(x)\, dx \; ; \quad 0 \leq y \leq 1$$

and

$$(61) \qquad \pi_y(x) = \frac{\pi(x)}{M(y)}, \quad 0 \leq y \leq x \leq 1, \quad y \neq 1.$$

Thus, $M(y)$ is the probability that the individual will be alive at age y, and $\pi_y(x)$ is the conditional density of death at age x, given that the individual is alive at age y.

Therefore, the expected utility maximand for an individual of vintage v at time v_+ is (assuming for notational convenience that $\lambda = 0$) :

$$(62) \qquad \int_0^{v+1-v_+} \left\{ \pi_{v_+-v}(t) \int_{v_+}^{v_++t} U\left[C(s,v)\right] ds \right\} dt$$

where, in this context,

(63) $M(v_+ - v)$ = $\int_{v_+-v}^{1} \pi(y)\, dy$, the probability of being alive at age $(v_+ - v)$, i.e., at time v_+, and

$$(64) \qquad \pi_{v_+-v}(t) = \frac{\pi(v_+-v+t)}{M(v_+-v)},$$ the conditional probability of the individual's death at age $\left[(v_+-v) + t\right]$, i.e., at time (v_++t), given that he is alive at age $(v_+ - v)$, i.e., at time v_+.

Upon interchanging the order of integration in (62), one gets

$$(65) \qquad \int_{v_+}^{v+1} \left\{ U\left[C(s,v)\right] \int_{s-v_+}^{v+1-v_+} \pi_{v_+-v}(t)\, dt \right\} ds$$

It is convenient to evaluate the inner integral separately as

$$(66)\qquad \int_{s-v_+}^{v+1-v_+} \pi_{v_+-v}(t)\,dt \;=\; \frac{1}{M(v_+-v)} \int_{s-v_+}^{v+1-v_+} \pi(v_+-v+t)\,dt$$

Putting the age variable $y = v_+ - v + t$ in (66) gives

$$(67)\qquad \frac{1}{M(v_+-v)} \int_{s-v}^{1} \pi(y)\,dy \;=\; \frac{M(s-v)}{M(v_+-v)}\,, \quad \text{from (60)}$$

Thus, the maximand (65) becomes

$$(68)\qquad \frac{1}{M(v_+-v)} \int_{v_+}^{v+1} M(s-v)\;\; U\left[C(s,v)\right]\;\; ds$$

The maximand is very similar to that in the no-uncertainty case; M can be interpreted as a rate of utility discount.

Next, I introduce the notion of an actuarial note. The buyer (seller) of an actuarial note receives (pays) a stream of income which continues as long as the individual's death, after which it is automatically cancelled. Since this particular asset becomes valueless on the individual's death, it is clear that the rate of return on an actuarial note must be higher than the rate of return on physical capital which does not share this property. Furthermore, it is intuitively plausible that the difference between the two rates of return should be related to the probability density π.

Specifically, let us suppose that actuarial trades are carried out between individuals and an insurance company. It is not

unreasonable to assume that the insurance company is not risk-averse and therefore that it charges a rate of interest that is fair in the actuarial sense. This implies that when an actuarial note is bought at age y and redeemed with interest at age $y + c$ unless the individual dies in the meanwhile, we have that

$$(69) \qquad \left[1 + c \cdot g(y)\right] \cdot \frac{M(y+c)}{M(y)} = 1 + c \cdot r(y) + 0(c)$$

where $g(y)$ is the actuarial rate of interest and

$r(y)$ is the market rate of interest

Letting $c \to 0$ in (69) yields

$$g(y) = r(y) - \frac{1}{M(y)} \cdot \frac{d}{dt} M(y), \quad \text{or}$$

$$(70) \qquad g(y) = r(y) + \pi_y(y), \quad \text{from (60) and (61).}$$

The assumption that g is actuarially fair therefore implies that the actuarial rate of interest at age y exceeds the market rate of interest by π_y ; π_y can then be regarded as the premium paid on an actuarial note.

We turn next to an examination of the individual's budget constraint. In the no-uncertainty case, this required him to die with non-negative terminal assets outstanding :

$$\int_{v_+}^{v+1} C(s,v)\, p(s,v_+)\, ds \leq \int_{v_+}^{v+1} W(s)\, p(s,v_+)\, ds + K(v_+,v)$$

(where we concentrate for the moment on the case where there is no fiscal policy). It is natural to generalise this condition to read that terminal assets should be nonnegative with probability one. This is equivalent (up to a set of π -measure zero) to the requirement that outstanding assets be non-negative at all points of the individual's lifetime. It is clear therefore that the individual will never plan to have negative net assets at any time, for this violates the constraint. He then has the choice of holding positive net assets either in the form of actuarial notes or in the form of physical capital. Actuarial notes yield a higher return than physical capital; furthermore, since the individual is not concerned about the future beyond his lifetime, the fact that actuarial assets are valueless after his death is a source of no concern to him. Accordingly, the individual will hold actuarial notes alone.

It is reasonable to suppose that the insurance company will demand a settlement of accounts at some age at which the individual will be alive with positive probability. This requirement may be written (for an individual of vintage v) as

$$(71) \qquad \int_{v_+}^{v+1-\beta} \left[W(s) - C(s,v)\right] q(s,v_+)\, ds + K(v_+,v) \geq 0$$

where $q(s,v_+) = \exp\left[-\int_{v_+}^{s} g(x)\, dx\right]$

As $\beta \to 0$, (71) tends to

$$(72) \qquad \int_{v_+}^{v+1} \left[W(s) - C(s,v)\right] q(s,v_+)\, ds + K(v_+, v) \geq 0$$

faster than $\beta \longrightarrow 0$. On introducing taxation, the constraint (72)

may be written as

$$(73) \qquad \int_{v_+}^{v+1} \left[W(s) - C(s,v) \right] \; q(s,v_+) \, ds \; + \; K(v_+, v) \; - \; \theta(v) \; = 0$$

where $\theta(v) \; = \int_{v_+}^{v+1} L(s,v) \; q(s,v_+) \; ds$

and where the inequality has been dropped because of consumer non-satiation.

Thus, the problem facing an individual of generation v at time v_+ is one of choosing C(s,v) to maximise

$$(74) \qquad \frac{1}{M(v_+ - v)} \int_{v_+}^{v+1} M(s-v) \; U\left[C(s,v) \right] \; ds$$

subject to

$$(75) \qquad \int_{v_+}^{v+1} \left[W(s) \; - \; C(s,v) \right] \; q(s,v_+) \; ds \; + \; K(v_+, v) - \theta(v) \; = \; 0$$

and

$$(76) \qquad K(v,v) \; = \; 0 \; .$$

As before, a necessary and sufficient condition characterising the individual's maximum is that

$$(77) \qquad \frac{M(s-v)}{M(v_+ - v)} \; U' \left[C(s,v) \right] \; = \int q(s,v_+)$$

The full optimum in such an economy is once again the solution to the problem of maximising

$$(78) \qquad \int_{-1}^{\infty} e^{-\alpha v} \int_{v_+}^{v+1} \frac{M(s-v)}{M(v_+-v)} \; U\left[C(s,v)\right] ds. \; dv$$

subject to

$$(79) \qquad \int_{-1}^{\infty} \left\{ p(v_+,0) \int_{v_+}^{v+1} C(s,v)\; q(s,v_+)\; ds \right\} e^{nv}\, dv \;=\; B$$

The optimum is characterised by

$$(80) \qquad e^{-\delta v} \; \frac{M(s-v)}{M(v_+-v)} \; U'\left[C(s,v)\right] \;=\; \mu\, p(v_+,0)\; q(s,v_+)$$

Equation (77) shows that the full optimum is controllable with lump-sum taxation if that is arranged to make

$$(81) \qquad \mathcal{S} \;=\; \mu\, e^{\delta v} p(v_+,\, 0)$$

We have therefore arrived at

Theorem 5 : The full optimum in the case where lifetime is uncertain and where there are insurance opportunities is controllable with intertemporal lump-sum taxation.

Remark : It should be noted that investment in physical capital needed along the Ramsey-optimal path has to be done by the government, for the private sector wishes to hold only actuarial notes.

3. Commodity Taxation

It is reasonable to enquire what commodity taxation means in a one-good model of the kind considered in this chapter. Here, I follow the standard general equilibrium convention of regarding commodities at different points in time as different commodities : accordingly, commodity taxation is the study of how consumption at different times ought to be taxed. Some authors prefer to call this a consumption tax.

The analysis reverts to the case of fixed (i.e., non-random) lifetime. Let

$$(82) \quad q(s,v_+) = \left[1 + j(s,v)\right] \exp\left[-\int_{v_+}^{s} r(x)\, dx\right] = \left[1 + j(s,v)\right] p(s,v_+)$$

where $j(s,v)$ is the tax rate at time s on generation v.

Thus, an individual of generation v at time v_+ has to select $C(s,v)$ to maximise (assuming $\lambda = 0$) :

$$(83) \quad \int_{v_+}^{v+1} U\left[C(s,v)\right] \, ds$$

subject to

$$(84) \quad \int_{v_+}^{v+1} C(s,v)\, q(s,v_+)\, ds = \int_{v_+}^{v+1} W(s)\, p(s,v_+)\, ds + K(v_+,v)$$

and

$$(85) \quad K(v,v) = K(v+1, v) = 0$$

The lifetime tax paid by him is given by

$$(86) \qquad \theta(v) = \int_{v_+}^{v+1} C(s,v)\, j(s,v)\, p(s,v_+)\, ds$$

As before, one has at the individual's maximum

$$(87) \qquad U'\left[C(s,v)\right] = \mathcal{S}\, q(s,v_+) = \mathcal{S}\left[1 + j(s,v)\right] p(s,v_+)$$

The full optimum is characterised by

$$(88) \qquad e^{-\delta v}\, U'\left[C(s,v)\right] = \mu\, p(s,0)$$

Equation (87) then allows us to see that the full optimum is not in general controllable with commodity taxation. But an interesting special case is where j is date-independent and is written as j(v), a function of the vintage alone. In that case, (87) reduces to

$$(89) \qquad U'\left[C(s,v)\right] = \mathcal{S}\left[1 + j(v)\right] p(s,v_+)$$

and the full optimum is controllable when one puts

$$(90) \qquad \mathcal{S} = \frac{\mu e^{\delta v}\, p(v_+, 0)}{1 + j(v)}$$

This is not a surprising result, for two reasons. First, j, being s - invariant, does not affect intertemporal marginal rates of substitution between consumption at adjacent dates. Secondly, j(v) is fully discriminatory according to generation. Nevertheless, the result tells the planner that his goals can be attained through the taxation of consumption. I summarise all this in

Theorem 6 : The full optimum is controllable with a time - invariant, vintage - related commodity tax.

Remark : The theorem shows that perfect foresight about the future course of tax rates over one's lifetime is, in this case, an eminently satisfactory assumption.

V. INFORMATIONAL DECENTRALISATION IN AN ECONOMY WITH OVERLAPPING GENERATIONS

1. The Problem

The models presented in this chapter have assumed that the Central Planning Board is completely informed about the underlying technological possibilities open to the economy : the problem therefore has been one of choosing the intergenerational distribution that optimises a decentralised private sector. The analysis can, however, be readily extended to situations where the CPB possesses only local knowledge about the production function, along lines suggested in Chapter 3.[1] Here, I sketch very briefly the details of an iterative procedure in which the private sector entertains static expectations about wage and interest rates and perfect foresight about the course of government fiscal policy, given that the government has a "static expectations" view of the technology at each stage of the iterations.

2. The Procedure in Outline

The story starts at $s = 0$ with a historically given capital stock which is known to the CPB. The informationally decentralised

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[1] See Section III of Chapter 3.

planning procedure comprises the following steps.

(1) The CPB observes the wage rate and the interest rate ruling in factor markets. Under the assumption of competitive imputation, these variables allow the CPB to deduce the current values of output and the marginal product of capital.

(2) It uses this information to construct a "perceived" production function which is the approximating tangent to the true production function at the current output - capital configuration.

(3) It computes the Ramsey - optimal policy on the basis of the perceived production function constructed in (2); this only involves solving a linear differential equation.

(4) It communicates the amount of investment to be undertaken currently to producers in the form of a quantitative investment target.

(5) It distributes optimum consumption at every instant among existing generations using intertemporal lump-sum taxes along lines analysed in Section III of this chapter.

(6) Producers fulfil the CPB's investment target; the resulting factor - market clearance allows the system to return to step (1) of the planning procedure.

Step (5) of the planning procedure corresponds to the problem analysed in most of the present chapter. We know from the theorems on controllability that step (5) is feasible, i.e., the consumption policy that is deemed optimal by the CPB at every stage is controllable with intertemporal lump-sum taxation.

Furthermore, it was proved in Chapter 3[1] that the procedure converges to the optimum balanced growth path. We are therefore in a position to assert

Theorem 7 : The informationally decentralised procedure with
(a) "static expectations" approximations on the production side, and
(b) intertemporal lump-sum taxation on the consumption side converges to the OBGP monotonically.

The theorem is an amalgam of two different results : the reader is referred to the comments accompanying each of them.[2] But one additional comment needs to be made here. The dialogue between the CPB and producers is, after the disappearance of pre-plan generations, an exchange between the CPB and managers of public investment corporations alone in those cases where the authorities resort to the tax policy analysed in Section III. This is because the private sector will not hold capital stocks in the face of that particular policy.

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[1] See pp. 93-94.

[2] See Section III of Chapter 3 and Section III of the present chapter.

VI CONCLUDING COMMENTS

The important points to emerge from the analysis have been summarised in the theorems. We have seen that it is useful to regard the problem of optimum accumulation as being one of distribution and illuminating to formulate it explicitly with overlapping finitely - lived generations. This allows one to apply the basic theorems of welfare economics to decentralise the optimum. Recall that the fundamental result on decentralisation in the standard theory is one which invites governments to choose the optimum distribution of income; our analysis in this chapter is therefore firmly within the traditional framework of thinking about decentralisation in welfare economics.

Secondly, most of our results are favourable from the point of view of controllability and indicate that a planner committed to Ramsey-type intergenerational equity need not despair if he is not able to exercise direct controls on consumption.

Thirdly, the controllability result carries over to various extended versions of the model. It is, however, reasonable to ask what happens if the taxes we have resorted to in this chapter are not available to the planner. The answer to this question must await a dynamic analysis of second-best paths.

BIBLIOGRAPHY

Aoki, Masahiko (1970) "A Note on Marshallian Process Under Increasing Returns," Quarterly Journal of Economics, 84.

Aoki, Masahiko (1971) "An Investment Planning Process for an Economy with Increasing Returns," Review of Economic Studies, XXXVIII.

Arrow, K.J. (1971a) "Political and Economic Evaluation of Social Effects and Externalities," in M.D.Intriligator (ed.) Frontiers of Quantitative Economics. (North-Holland, Amsterdam-London).

Arrow, K.J. (1971b) "The Firm in General Equilibrium Theory," in R.Marris and A.Wood (eds.), The Corporate Economy: Growth, Competition and Innovative Potential. (Macmillan, London and Harvard University Press, Cambridge, Mass.).

Arrow, K.J. and F.H.Hahn (1971) General Competitive Analysis (Holden-Day, San Francisco and Oliver and Boyd, Edinburgh).

Arrow, K.J. and L.Hurwicz (1956) "Reduction of Constrained Maxima to Saddle-point Problems," in J.Neyman (ed.), Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Vol V. (University of California Press, Berkeley and Los Angeles).

Arrow, K.J. and L.Hurwicz (1958) "Gradient Method for Concave Programming, I : Local Results," in K.J.Arrow, L.Hurwicz and H.Uzawa, Studies in Linear and Nonlinear Programming (Stanford University Press, Stanford).

Arrow, K.J. and L.Hurwicz (1960) "Decentralisation and Computation in Resource Allocation," in R.W. Pfouts (ed.), Essays in Economics and Econometrics (University of North Carolina Press, Chapel Hill, N.C.).

Arrow, K.J., L.Hurwicz and H.Uzawa (1958) Studies in Linear and Nonlinear Programming (Stanford University Press, Stanford).

Arrow K.J. and M.Kurz (1969) "Optimal Consumer Allocation over an Infinite Horizon," Journal of Economic Theory, 1.

Arrow, K.J. and M.Kurz (1970) Public Investment, the Rate of Return and Optimal Fiscal Policy (The Johns Hopkins Press, Baltimore and London).

Arrow, K.J. and R.M.Solow (1958) "Gradient Methods for Constrained Maxima, with Weakened Assumptions," in K.J.Arrow, L.Hurwicz and H. Uzawa Studies in Linear and Non-linear Programming. (Stanford University Press, Stanford).

Bellman, R. (1957) — Dynamic Programming. (Princeton University Press, Princeton, N.J.)

Bergson, A.(1948) — "Socialist Economics", in H.S.Ellis (ed.), A Survey of Contemporary Economics, Vol.I (Richard D.Irwin Inc., Homewood, Ill.)

Bliss, C.J. (1972) — "Prices, Markets and Planning," Economic Journal, 82

Bose, S. (1968) — "Optimal Growth and Investment Allocation," Review of Economic Studies, XXXV

Bose, S. (1970) — "Optimal Growth in a Nonshiftable Capital Model," Econometrica, 38.

Brock, W. A. (1971) — "On Existence of Weakly Maximal Programmes in a Multi-Sector Economy," Review of Economic Studies, XXXVIII

Brock, W.A. and D. Gale (1969) — "Optimal Growth Under Factor-Augmenting Progress", Journal of Economic Theory, 1

Camacho, A. (1970) — "Externalities, Optimality and Informationally Decentralised Resource Allocation Processes," International Economic Review, 11

Cass, D. (1965) — "Optimum Growth in an Aggregative Model of Capital Accumulation," Review of Economic Studies, XXXII

Cass, D. and M.E. Yaari (1966) — "A Re-Examination of the Pure Consumption Loans Model," Journal of Political Economy, 74

Cass, D. and M.E. Yaari (1967) — "Individual Saving, Aggregate Capital Accumulation, and Efficient Growth," in K. Shell (ed.), Essays in the Theory of Optimal Economic Growth. (The M.I.T. Press, Cambridge, Mass. and London)

Chipman, J.S. (1970) — "External Economies of Scale and Competitive Equilibrium," Quarterly Journal of Economics,84

Dantzig, G. (1963) — Linear Programming and Extensions. (Princeton University Press, Princeton, N.J.)

Dantzig, G. and P.Wolfe (1961) — "The Decomposition Algorithm for Linear Programmes," Econometrica, 29

Dasgupta, P.S. (1969) — "Optimum Growth When Capital is Non-Transferable," Review of Economic Studies, XXXVI

Debreu, G. (1952) — "Definite and Semidefinite Quadratic Forms," Econometrica, 20

Diamond, P.A. (1965) — "National Debt in a Neoclassical Growth Model," American Economic Review, 55

Diamond, P.A. (1973) "Taxation and Public Production in a Growth Setting," in J.A.Mirrlees and N.H.Stern (eds.), Models of Economic Growth. (Macmillan, London).

Diamond, P.A. and Mirrlees, J.A.(1971) "Optimal Taxation and Public Production," American Economic Review, 61

Dobb, M.H. (1933) "Economic Theory and the Problems of a Socialist Economy," Economic Journal, 43

Drèze, J.H. and D. de la Vallée Poussin (1971) "A Tatonnement Process for Public Goods," Review of Economic Studies, XXXVIII

Eggleston, H.G. (1958) Convexity (Cambridge University Press, Cambridge)

Ellman, M. (1968) "Optimal Planning", Soviet Studies, XX

Fisher, F.M. (1969) "The Existence of Aggregate Production Functions," Econometrica, 37

Gale, D (1963) "A Note on Global Instability of Competitive Equilibrium," Naval Research Logistics Quarterly,

Gale, D. (1967) "On Optimal Development in a Multi-Sector Economy," Review of Economic Studies, XXXIV

Gale, D. (1973) "Pure Exchange Equilibrium of Dynamic Economic Models," Journal of Economic Theory, 6

Grandmont, J-M. and G.Laroque (1973) "Money in the Pure Consumption Loan Model," Journal of Economic Theory, 6

Hadley, G.and M. C. Kemp (1971) Variational Methods in Economics (North-Holland, Amsterdam-London and American Elsevier, New York).

Heal, G.M. (1969) "Planning without Prices", Review of Economic Studies,XXXVI

Heal, G.M. (1971) "Planning, Prices and Increasing Returns," Review of Economic Studies, XXXVIII

Heal, G.M. (1973) The Theory of Economic Planning (North Holland, Amsterdam-London and American Elsevier, New York)

Hicks, J.R. (1946) Value and Capital (Clarendon Press, Oxford)

Hildebrand, F.B. (1956) Introduction to Numerical Analysis (McGraw-Hill, New York).

Hurwicz, L (1960a) "Optimality and Informational Efficiency in Resource Allocation Processes," in K.J.Arrow, S.Karlin and P.Suppes (eds.), Mathematical Methods in the Social Sciences (Stanford University Press, Stanford).

Hurwicz, L. (1960b) "Conditions for Economic Efficiency of Centralised and Decentralised Structures," in G. Grossman (ed.), Value and Plan (University of California Press, Berkeley).

Hurwicz, L. (1969) "On the Concept and Possibility of Informational Decentralisation," American Economic Review, 59

Hurwicz, L. (1971) "Centralisation and Decentralisation in Economic Processes," in A. Eckstein (ed.), Comparison of Economic Systems: Theoretical and Methodological Approaches (University of California Press, Berkeley).

Hurwicz, L. (1972) "On Informationally Decentralised Systems," in C.B. McGuire and R. Radner (eds.), Decision and Organisation (North-Holland, Amsterdam-London)

Infante, E.F. and J.L.Stein (1973) "Optimal Growth with Robust Feedback Control," Review of Economic Studies, XL

Koopmans, T.C. (1965) "On the Concept of Optimal Economic Growth," in The Econometric Approach to Development Planning (North-Holland, Amsterdam and Rand McNally, Chicago).

Kornai, J. (1967) Mathematical Planning of Structural Decisions (North-Holland, Amsterdam-London).

Kornai, J. and T. Liptak (1965) "Two-Level Planning," Econometrica, 33

Kurz, M. and M.Majumdar (1972) "Efficiency Prices in Infinite Dimensional Spaces: a Synthesis," Review of Economic Studies, XXXIX

Kurz, M. and D. A. Starrett (1970) "On the Efficiency of Competitive Programmes in an Infinite-Horizon Model," Review of Economic Studies, XXXVII

Lange, O. (1936) "On the Economic Theory of Socialism," Review of Economic Studies.IV. Reprinted in B.Lippincott (ed.), On the Economic Theory of Socialism (McGraw-Hill Paperbacks, 1964).

Lange, O. (1967) "The Computer and the Market", in C.H.Feinstein (ed.), Capitalism, Socialism and Economic Growth (Cambridge University Press, Cambridge).

Lerner, A.P. (1946) The Economics of Control (Macmillan, London and New York).

Lippincott, B. (1938) On the Economic Theory of Socialism (University of Minnesota Press, Minneapolis). Republished by McGraw Paperbacks, 1964.

Malinvaud, E. (1953) "Capital Accumulation and Efficient Allocation of Resources," Econometrica, 21, and "Efficient Capital Accumulation: A Corrigendum," Econometrica, 30, 1962.

Malinvaud, E. (1967) "Decentralised Procedures for Planning," in E.Malinvaud and M.O.L.Bacharach (eds.), Activity Analysis in the Theory of Growth and Planning (Macmillan, London and St.Martin's Press, New York).

Malinvaud, E. (1971) "A Planning Approach to the Public Good Problem," Swedish Journal of Economics, 11.

Malinvaud, E. (1972) "Prices for Individual Consumption, Quantity Indicators for Collective Consumption," Review of Economic Studies, XXXIX

Malinvaud, E. and M.O.L. Bacharach (eds.), (1967) Activity Analysis in the Theory of Growth and Planning (Macmillan, London and St. Martin's Press, New York).

Marglin, S. A. (1963a) Approaches to Dynamic Investment Planning (North-Holland, Amsterdam-London).

Marglin, S. A. (1963b) "The Social Rate of Discount and the Optimal Rate of Investment" Quarterly Journal of Economics, 77

Marglin, S. A. (1967) Public Investment Criteria (Allen and Unwin, London).

Marglin, S. A. (1969) "Information in Price and Command Systems of Planning," in J.Margolis and H. Guitton (eds.), Public Economics (Macmillan, London and St. Martin's Press, New York).

Marschak, J. and R. Radner (1971) The Economic Theory of Teams (Yale University Press, New Haven).

Marschak, T. (1959) "Centralisation and Decentralisation in Economic Organisations," Econometrica, 27.

Marschak, T. (1969) "On the Comparison of Centralised and Decentralised Economies," American Economic Review, 59.

Marschak, T. (1972) "Computation in Organisations: the Comparison of Price Mechanisms and other Adjustment Processes' in C.B.McGuire and R.Radner (eds.), Decision and Organisation (North-Holland, Amsterdam-London).

McFadden, D. (1967) "The Evaluation of Development Programmes," Review of Economic Studies, XXXIV

McFadden, D. (1973) "On the Existence of Optimal Development Programmes in Infinite-Horizon Economies," in J. A. Mirrlees and N. H. Stern (eds.), Models of Economic Growth (Macmillan, London)

Meade, J. E. (1970) The Theory of Indicative Planning (Manchester University Press, Manchester).

Mirrlees, J. A. (1967) "Optimum Growth When Technology is Changing," Review of Economic Studies, XXXIV

Mirrlees, J. A. and N. H. Stern (1972) "Fairly Good Plans," Journal of Economic Theory, 4

Montias, J. M. (1959) "Planning with Material Balances in Soviet-Type Economies," American Economic Review, 49.

Morishima, M. (1964) Equilibrium, Stability and Growth (Clarendon Press, Oxford).

Negishi, T. (1960) "Monopolistic Competition and General Equilibrium," Review of Economic Studies, XXVIII

Negishi, T. (1972) General Equilibrium Theory and International Trade (North-Holland, Amsterdam-London)

Nerlove, M. (1972) "Lags in Economic Behaviour," Econometrica, 40.

Phelps, E. S. (1967) Golden Rules of Economic Growth: Studies of Efficient and Optimal Investment (North-Holland, Amsterdam and W. W. Norton, New York)

Portes, R. D. (1971) "Decentralised Planning Procedures and Centrally Planned Economies," American Economic Review, 61

Pratten, C. F. (1971) Economies of Scale in Manufacturing Industries, Department of Applied Economics Occasional Papers, No. 28 (Cambridge University Press, Cambridge).

Radner, R. (1966) "Optimal Growth in a Linear-Logarithmic Economy," International Economic Review, 7

Radner, R. (1967) "Efficiency Prices for Infinite Horizon Production Programmes," Review of Economic Studies, XXXIV

Ramsey, F. P. (1928) "A Mathematical Theory of Saving," Economic Journal, 38.

Ryder, H. E. (1969) "Optimal Accumulation in a Two-Sector Neoclassical Economy with Non-Shiftable Capital," Journal of Political Economy, 77

Samuelson, P. A. (1950) "Market Mechanisms and Maximisation," in J. E. Stiglitz (ed.), The Collected Scientific Papers of Paul A. Samuelson, Vol. I (The M.I.T. Press, Cambridge, Mass., 1966).

Samuelson, P. A. (1958) "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," Journal of Political Economy, 66

Samuelson, P. A. and R. M. Solow (1956) "A Complete Capital Model Involving Heterogeneous Capital Goods," Quarterly Journal of Economics, 7

Scarf, H. E. (1960) "Some Examples of Global Instability of the Competitive Equilibrium," International Economic Review, 1

Sen, A. K. (1961) "On Optimising the Rate of Saving," Economic Journal, 71

Sen, A. K. (1967) "Isolation, Assurance and the Social Rate of Discount," Quarterly Journal of Economics, 81

Sen, A. K. (1973) On Economic Inequality (Clarendon Press, Oxford).

Shell, K. (1971) "Notes on the Economics of Infinity," Journal of Political Economy, 79

Silberston, A. (1972) "Economies of Scale in Theory and Practice," Economic Journal, 82

Sraffa, P. (1926) "The Laws of Return Under Competitive Conditions," Economic Journal, 36

Srinivasan, T. N. (1964) "Optimal Savings in a Two-Sector Model of Growth," Econometrica, 32

Starrett, D. A. (1972) "On Golden Rules, the 'Biological Theory of Interest' and Competitive Inefficiency," Journal of Political Economy, 80

Sutherland W. R. S. (1970) "On Optimal Development in a Multi-Sector Economy," Review of Economic Studies, XXXVII

Taylor, F. M. (1929) "The Guidance of Production in a Socialist State," American Economic Review, 19. Reprinted in B. Lippincott (ed.), On the Economic Theory of Socialism. (McGraw-Hill Paperbacks, 1964).

Uzawa, H. (1958) "Iterative Methods for Concave Programming," in K. J. Arrow, L. Hurwicz and H. Uzawa, Studies in Linear and Nonlinear Programming (Stanford University Press, Stanford).

Uzawa, H. (1964) "Optimal Growth in a Two-Sector Model of Capital Accumulation," Review of Economic Studies, XXXI

von Weizsäcker, C. C. (1965) "Existence of Optimal Programmes of Accumulation for an Infinite Time Horizon," Review of Economic Studies, XXXII

von Weizsäcker, C. C. (1967) "Lemmas for a Theory of Approximate Optimal Growth," Review of Economic Studies, XXXIV

Weitzman, M. L. (1970) "Iterative Multi-Level Planning with Production Targets," Econometrica, 38

Yaari, M. E. (1965) "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," Review of Economic Studies, XXXII